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Univ.Prof.in Elisa Davoli, PhD

Valerio Pagliari, PhD

eingereicht von

Jakob Deutsch, BSc.

Matrikelnummer: 01526885

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Jakob Deutsch

Elisa Davoli

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Kurzfassung

Diese Diplomarbeit ist den Korn Ungleichungen gewidmet, die eine wichtige Rolle in den Beweisen von zahlreichen Existenzresultaten von Variationsmodellen in der Theorie der Elastizität und Bruchmechanik innehaben. Auf der einen Seite befassen wir uns mit der klassischen Theorie der Korn-Ungleichungen in den L^p-Räumen, die, historisch gesehen, in enger Verbindung zur Theorie der elastischen Materialien steht. Auf der anderen Seite untersuchen wir Korn Ungleichungen in $GSBD^p(\Omega)$, die ein aktuelles Thema im Bereich der Variationsrechnung sind. In diesem Raum sind Korn Ungleichungen ein essenzielles Werkzeug um unterschiedliche Resultate wie Approximationssätze (vgl. [CCI19]), Existenz von Extensionsoperatoren (vgl. [Cag+21]) und Existenz von Minimierern der Griffith Energie (vgl. [CCI19]) zu zeigen. Aufbauend auf der Arbeit von F. Cagnetti, A. Chambolle and L. Scardia (vgl. [CCS22]) und vorausgegangenen Resultaten ist es unser Ziel die verschiedenen Aspekte dieser Ungleichungen zu beleuchten.

Abstract

This thesis is dedicated to Korn inequalities, which play a crucial role in the existence results of many variational models derived from elasticity theory and fracture mechanics. On the one hand, we are interested in the classical theory of Korn inequalities in L^p-spaces, which is historically deeply connected to the theory of elastic materials. We have gathered the results scattered over the literature and revised the originally intricate proofs through modern tools. On the other hand, we investigate Korn inequalities in $GSBD^p(\Omega)$, a very recent topic in calculus of variations. Korn inequalities in this space are essential for proving a variety of results like approximation theorems (cf. [CCI19]), existence of extension operators (cf. [Cag+21]) and existence of minimizers for the Griffith energy (cf. [CCI19]). We aim to elucidate such aspects by elaborating on the work of F. Cagnetti, A. Chambolle and L. Scardia (cf. [CCS22]) and related previous results.

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All of my friends and colleagues also deserve special thanks. They have continued to provide me with countless inspiring discussions over the last few years and share my never-ending love for mathematics. Without them, my studies would only be half the fun.

Lastly, I would like to thank my family, especially my parents. They have made it possible for me to solely concentrate on my studies during the last few years.

Eidestattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 25.10.2022

Jakob Deutsch

Contents

1 Introduction

The mathematical modelling of elastic bodies is a classical topic in continuum mechanics. However, the elasticity theory does not allow it to encompass a series of phenomena, such as plastic slips, damage, and fracture mechanics, which frequently occur in numerous settings. Despite being well known from physical observations that some materials exhibit non-linear behaviour, applications need to have linearized models since they are often easier to handle from a numerical point of view. In the variational formulation of such models, there is a differential operator which is indispensable to the theory. Consider a body $\Omega \subset \mathbb{R}^n$ and let $y : \Omega \to \mathbb{R}^n$ be a deformation of Ω. Denote with $u := y - \mathrm{id}_\Omega$ the displacement where id_Ω is the identity on Ω. Then, the *symmetric gradient* of u is defined by

$$e(u) := \nabla u + (\nabla u)^T = (\partial_i u_j + \partial_j u_i)_{i,j=1}^n.$$

Here, ∇ denotes the gradient and ∂_i the i-th partial derivative of u. In this setting, we usually require Ω to be open, bounded and with boundary of sufficient regularity. In elasticity and fracture mechanics the two main examples of variational problems associated with $e(u)$ are the minimization of the total elastic energy and the so-called Griffith energy. These are given by

$$\inf_{u \in X} \left(\int_\Omega W(e(u))\, dx \right), \tag{1.1}$$

and

$$\inf_{u \in Y} \left(\int_\Omega W(e(u))\, dx + \mathcal{H}^{n-1}(J_u) \right), \tag{1.2}$$

where J_u is the jump set of a function u which will be introduced in more detail in chapter 4 of the thesis, H^{n-1} is the $(n-1)$-dimensional Hausdorff measure, and X and Y are a space of (measurable) functions for which a suitable symmetric gradient can be defined. In these problems, a typical density $W : \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is given by

$$W(M) = \frac{1}{p}(M : \mathbb{C}M)^{p/2}$$

for some $p \in (1, \infty)$ where $\mathbb{C}$ is a symmetric fourth order tensor such that for a $C > 0$ the definiteness condition

$$|M|^2 \le C(M : \mathbb{C}M) \tag{1.3}$$

holds for all $M \in \mathbb{R}^{n\times n}$. We will discuss the connection between the minimization of the total elastic energy and the classical boundary value problem of elasticity in the first section of chapter 3.

To solve the variational problems above, the choice of the underlying spaces X and Y is crucial. Consider, for instance, the space X and the problem (1.1). In practice, we would prefer maps $u \in X$ to satisfy certain boundary conditions. Naturally, we would also like eligible functions from $C^\infty(\Omega)$ to be a subset of X. We can limit ourselves to functions $u \in X$ with $e(u) \in L^p(\Omega)$ since all other functions do not contribute to the minimization problem: indeed, if $e(u)$ is not p-integrable then condition (1.3) implies that $\int_\Omega W(e(u))\,dx = \infty$. Now, suppose that a minimizing sequence $\{u_k\} \subseteq C^\infty(\Omega) \cap X$ exists for problem (1.1). A priori, we only know that the L^p-norm of $\{e(u_k)\}$ is uniformly bounded due to (1.3). Here is where a *Korn inequality* comes into play. It asserts that if the symmetric gradient of a map u is L^p-integrable, the gradient of u is also L^p-integrable. In a more concrete form: there exists a constant $K > 0$ such that

$$\|\nabla u\|_{L^p(\Omega)} \leq K \|e(u)\|_{L^p(\Omega)} \tag{1.4}$$

for all $u \in C^\infty(\Omega)$ which satisfy some given boundary data. This has far-reaching consequences. For instance, L^p-integrability of the gradient ∇u implies that $u \in L^p(\Omega)$ because of the conditions imposed on Ω. So, we can infer $\{u_k\} \subseteq W^{1,p}(\Omega)$. At this point, one notices that $W^{1,p}(\Omega)$ has the right compactness property. Indeed, since the gradients of $\{u_k\}$ are uniformly bounded in L^p, the Poincaré inequality yields that $\{u_k\}$ is a uniformly bounded sequence in $W^{1,p}(\Omega)$. Then, one can extract a weakly converging subsequence and prove that problem (1.1) has a solution via the direct method of calculus of variations. The lower semi-continuity required to apply this argument is given due to the convexity of W. For more details, we refer to section 2.2 of the thesis. There, a summary of the direct method of calculus of variations can be found.

Korn inequalities are the central topic of interest in this thesis. Commonly, we refer to an inequality as a Korn inequality if it is of a similar form to

$$\|u\|_Z \leq \|e(u)\|_W ,$$

where Z, W are (different) normed function spaces. The main application of such inequalities is to transfer uniform bounds of the W-norm of minimization sequences to uniform bounds in the Z-norm. We have already seen an application of this in the last paragraph. The first mention of this inequality can be attributed to the physicist A. Korn around 1900. In particular, the two inequalities (1.4) and

$$\|\nabla u\|_{W^{1,p}(\Omega)} \leq K(\|u\|_{L^p(\Omega)} + \|e(u)\|_{L^p(\Omega)})$$

go back to this time. These are called *Korn's first inequality* and *Korn's second inequality* respectively. However, it was not until 1981 that both were rigorously proven by J.A. Nitsche (cf. [Nit81]). Generally, the inequalities that branched out from these two have been extensively studied over the second half of the last century. In the second part of chapter 3, we have gathered results related to Korn inequalities in Sobolev spaces which are scattered over the literature and presented them in a compact, modern form. Furthermore, we discuss the boundary cases $p \in \{1, \infty\}$ where most Korn inequalities fail, most notably (in the case of $p = 1$) due to Ornstein's famous non-inequality (cf. [Orn62]).

Up to this point, we have discussed Korn inequalities in conjunction with Sobolev spaces. However, such functional setting is unsuited for models that feature inelastic phenomena like the Griffith energy (1.2). Indeed, from a modelling perspective, Sobolev functions turn out to be too regular since they are absolutely continuous along almost every line parallel to the coordinate axes and they 'do not jump'. To be more precise, the second property amounts to the vanishing of the second term in (1.2). Another mathematical difficulty can be encountered when considering the boundary case $p = 1$ in the variational models (1.1) and (1.2). Neither Korn's first inequality nor Korn's second inequality hold in $L^1(\Omega)$ as mentioned above. Partly, the reason for this is the lack of reflexivity of $L^1(\Omega)$. This problem naturally carries over to $W^{1,1}(\Omega)$. In particular, since $W^{1,1}(\Omega)$ is not reflexive it does not have the right compactness properties that are usually sought after when choosing a function space. However, if we have a minimizing sequence $\{u_k\} \subseteq C^\infty(\Omega)$ for the variational problem (1.1), we can then derive additional information from the fact that $\{e(u_k)\}$ is uniformly bounded in $L^1(\Omega)$. Indeed, we can use that

$$\left(B \mapsto \int_B |e(u_k)| \; dx\right)$$

is a finite Radon measure to extract a subsequence $\{u_{k_j}\}$ such that for some finite vector valued Radon measure $E \in \mathcal{M}_b(\Omega, \mathbb{R}^{n\times n})$

$$e(u_{k_j}) \rightharpoonup^* E$$

holds in $\mathcal{M}_b(\Omega, \mathbb{R}^{n\times n})$. A short synopsis of Radon measures can be found in section 2.3. It turns out that this is sufficient to conclude that there exists a $u \in L^1(\Omega)$ such that

$$\int_\Omega u \odot \nabla\varphi \; dx = -\int_\Omega \varphi \; dE$$

holds for all $\varphi \in C^\infty(\Omega)$. So E can be seen as the distributional symmetric gradient of a L^1 function. With this in mind, we denote the measure E by $\mathrm{E}u$. The space of all such functions is called $BD(\Omega)$, the *space of functions with bounded deformation*. In contrast to Sobolev spaces, jumps occur in this larger space in a non-negligible way. Chapter 4 is dedicated to the space $BD(\Omega)$. We first discuss the results that can be carried over to this space from the Sobolev spaces. However, since L^1 is not reflexive, the classical Korn inequalities need to be handled by a different approach. To set up the discussion for Korn inequalities in this space the second part of this chapter deals with the fine properties of functions with bounded deformation. Most importantly, we will discuss the structure of the measure $\mathrm{E}u$.

Let us now observe problem (1.2) in this space. Notice that the Griffith energy can be split into an elastic part and a jump part. But we have a problem: W is only defined pointwise. We need to properly describe how the elastic part behaves with respect to the measure $\mathrm{E}u$. For this, we denote with $\mathcal{E}u$ the density of the absolutely continuous part of $\mathrm{E}u$ (with respect to the Lebesgue measure). $\mathcal{E}u$ is called the approximate symmetric gradient. One can then formulate the variational problem associated with the Griffith energy as follows:

$$\inf_{u\in SBD(\Omega)} \left(\int_\Omega W(\mathcal{E}u) \; dx + \mathcal{H}^{n-1}(J_u)\right), \tag{1.5}$$

where $SBD(\Omega)$ is a suitable subspace of $BD(\Omega)$. We will postpone the definition of $SBD(\Omega)$ and the discussion of this space to chapter 4 and 5, but we remark that we can reduce ourselves to the case of the approximate symmetric gradient being p-integrable with $p \in [1, \infty)$. It turns out that a Korn inequality again is essential to show the existence of minimizers. However, in this setting, it takes the form

$$\|\nabla u - \nabla a\|_{L^q(\Omega\setminus\omega)} \leq C\,\|\mathcal{E}u\|_{L^p(\Omega)} \tag{1.6}$$

where $\omega \subseteq \Omega$ is a 'relatively small' set and $a : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid displacement, i.e., it is of the form $a(x) = Ax + b$ with $A \in \mathbb{R}^{n\times n}$ being a skew-symmetric matrix and $b \in \mathbb{R}^n$. 'Relatively small' will be made precise in the last chapter of the thesis. Notice that ∇u on the left-hand side may not seem well justifiable a priori, since functions in $BD(\Omega)$ do not necessarily admit a weak derivative in the form of a L^p-function or a measure. But it turns out that the gradient of $u \in BD(\Omega)$ exists in a measure-theoretical sense. To be more precise, there exists $\nabla u \in L^1(\Omega, \mathbb{R}^{n\times n})$ such that

$$\lim_{r\to 0^+} \frac{1}{r^n} \int_{B_r(x)} \frac{|u(y) - u(x) - \nabla u(x)(y-x)|}{r}\, dy = 0$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$. Such gradients are called *approximate gradients*.

The proof to (1.6) is highly non-trivial. Over the last years, it has turned out that the right spaces to investigate results related to Korn inequalities in $BD(\Omega)$ are the inherently larger spaces $GBD(\Omega)$ and $GSBD(\Omega)$. The reason for this is the slicing method developed to show results for $BD(\Omega)$. It relies on the theory of one-dimensional functions of bounded variation. One can draw a comparison to Sobolev spaces. They are absolutely continuous along lines parallel to coordinate axes as mentioned before. In a similar spirit, one can think of $BD(\Omega)$ functions to be of bounded variation along lines (with a small modification). The space $GBD(\Omega)$ further generalizes the idea. It also has the right compactness properties. Indeed, historically, the space was developed since it turned out to be the natural relaxation for many variational problems (for instance (1.5)).

Chapter 5 is dedicated to Korn inequalities in $GSBD(\Omega)$. It deals with the recent developments in the theory related to these inequalities. To introduce the setting we first present a collection of important theorems regarding $GBD(\Omega)$, $GSBD(\Omega)$ and $GSBD^p(\Omega)$ from [Dal13]. In particular, we focus on the results needed to prove Korn inequalities. Then, we discuss a Korn inequality on the unit cube which will be the starting point to prove (1.6). This is based on the work of [CC18]. The Korn inequality that holds on cubes then implies several approximation results. These are the cornerstones in showing (1.6). In particular, the works of [CCI19] and [CCS22] are presented. This chapter aims to compactly present the proof to (1.6) and to expand on the theory related to the theorems involved.

We had two main goals with the thesis: Firstly, we wanted to gather the information available on Korn inequalities in L^p-spaces and BD over the many different sources and present them in a modern way. The focus is on delivering proofs of the classical Korn inequalities based around short high-level arguments which were, historically, developed in the 90s and illuminating the connections between the different known inequalities. Secondly, we want to contribute to the understanding of Korn inequalities in $GSBD(\Omega)$ which is a recent topic in calculus of variations. We have elucidated some aspects of the theorems. In particular, we used the work of F. Cagnetti , A. Chambolle and L. Scardia (cf. [CCS22]) and related previous results (cf. [Dal13], [CCF14], [CCI19]) as basis for the last part of the thesis.

2 Preliminaries

In this chapter, we will introduce the notation used throughout the thesis and recap some basic concepts in calculus of variations and measure theory.

2.1 Notation

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$. We denote the interior of Ω by $\text{int}(\Omega)$, the closure by $\overline{\Omega}$ and the topological boundary by $\partial\Omega$. We remark that in most cases throughout the thesis we assume Ω to be bounded and that $\partial\Omega$ is Lipschitz. We denote the ball with radius $r > 0$ and center $x \in \mathbb{R}^n$ by $B_r(x)$. The balls centered around 0 with radius $r > 0$ we will just denote by B_r, the unit sphere by $S^{n-1} = \partial B_1$ and the volume of the unit ball by γ_n. We denote a sequence with values in a set X and index set I by $\{x_i\}_{i \in I}$. For convenience, we abuse the notation and often write $\{x_i\}$ if the index set is clear.

For continuously differentiable functions the standard notation is used ($k, N \in \mathbb{N}$):

$$C^k(\Omega, \mathbb{R}^N) = \left\{\varphi : \Omega \to \mathbb{R}^N \mid \varphi \text{ is } k\text{-times continuously differentiable}\right\},$$
$$C^k(\overline{\Omega}, \mathbb{R}^N) = \left\{\varphi \in C^k(\Omega, \mathbb{R}^N) \mid \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k : \partial^\alpha \varphi \text{ has a continuous extension to } \overline{\Omega}\right\},$$
$$C_c^k(\Omega, \mathbb{R}^N) = \left\{\varphi \in C^k(\Omega, \mathbb{R}^N) : \text{supp}(\varphi) \text{ is compact}\right\}.$$

Here ∂^α denotes the usual differential operator $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ for a multi-index $\alpha \in \mathbb{N}^n$ and $\text{supp}(\varphi) = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$ is the support of a function $\varphi : \Omega \to \mathbb{R}^N$. If it is clear in which space the values are we just write $C^k(\Omega)$, etc. We further denote the space of test functions with

$$\mathcal{D}(\Omega) = C_c^\infty(\Omega).$$

The topology on $\mathcal{D}(\Omega)$ can be introduced over the convergence of sequences: For a sequence $\{\varphi_k\}$ we say $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$ if and only if there exists a compact $K \subseteq \Omega$ such that $\text{supp}(\varphi_k) \subseteq K$ for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$ and $\|\mathcal{D}^\alpha(\varphi_k - \varphi)\|_\infty \to 0$ for all $\alpha \in \mathbb{N}^\alpha$. Similarly, we say $\varphi_k \to \varphi$ in $C_c(\Omega, \mathbb{R}^m)$ if and only if there exists a compact $K \subseteq \Omega$ such that $\text{supp}(\varphi_k) \subseteq K$ for all $n \geq k_0$ for some $n_0 \in \mathbb{N}$ and $\|(\varphi_k - \varphi)\|_\infty \to 0$. The dual space of $\mathcal{D}(\Omega)$, i.e., the space of distributions, is denoted in typical fashion by $\mathcal{D}'(\Omega)$. Some general knowledge regarding distributions is assumed to be known. Otherwise, we refer to [Hör98] for an introduction to this topic.

Throughout the thesis, we denote with λ^n the Lebesgue measure on $\mathbb{R}^n$ and with $\mathcal{H}^s$ the s-fractional Hausdorff measure. For convenience, we sometimes write $|A| := \lambda^n(A)$ for a λ^n-measurable set $A \subseteq \mathbb{R}^n$. For a measure μ and a μ-measurable set A we write 'Statement S holds for μ-a.e. $x \in A$' if S holds for all $x \in A \setminus N$ with some $N \subset A$ and $\mu(N) = 0$. For the set of all Borel sets, we usually write $\mathcal{B}$. The Lebesgue spaces $L^p(\Omega, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$ are denoted as usual, and their norms by $\|\cdot\|_p$ respectively $\|\cdot\|_{k,p}$ with $1 \le p \le \infty$ and $k \in \mathbb{N}$. We set $W^{0,p}(\Omega,\mathbb{R}^N) := L^p(\Omega,\mathbb{R}^N)$. For $k \ge 1$ the Sobolev spaces are defined by

$$W^{k,p}(\Omega,\mathbb{R}^N) := \left\{u \in L^p(\Omega,\mathbb{R}^N) \mid \forall \alpha \in \mathbb{N}^k, |\alpha| \le k : \partial^\alpha u \in L^p(\Omega,\mathbb{R}^N)\right\}$$

where '$\partial^\alpha u \in L^p(\Omega,\mathbb{R}^N)$' means that the distributional derivative $\partial^\alpha u$ can be identified with a p-integrable function and $|\alpha| = \sum_{i=1}^n \alpha_i$ is called the order of the multi-index. The Hilbert spaces $W^{k,2}(\Omega,\mathbb{R}^N)$ are denoted by $H^k(\Omega,\mathbb{R}^N)$. We also remind the reader that

$$W^{k,p}(\Omega,\mathbb{R}^N) = \overline{C^k(\overline{\Omega},\mathbb{R}^N)}^{\|\cdot\|_{k,p}}$$
$$W_0^{k,p}(\Omega,\mathbb{R}^N) = \overline{C_c^k(\Omega,\mathbb{R}^N)}^{\|\cdot\|_{k,p}}$$

holds for $p \in [1,\infty)$. We recall that the inclusion

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

is compact by the Rellich-Kondrachov theorem. In particular, every bounded sequence in $W^{1,p}(\Omega)$ has a subsequence converging in L^p. Furthermore, we denote the usual trace operator for $p \in [1,\infty)$ by

$$|_{\partial\Omega} : W^{1,p}(\Omega) \to L^p(\partial\Omega,\mathcal{H}^{n-1})$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. For function $u : \Omega \to \mathbb{R}^N$, a set-valued function $\mu : \mathcal{P}(\Omega) \to \mathbb{R}^N$ and a set $\omega \subseteq \Omega$ we denote the restriction of u and μ to ω as usual also by $u|_\omega : \omega \to \mathbb{R}^N$ resp. $\mu|_\omega : \mathcal{P}(\omega) \to \mathbb{R}^N$. Here, $\mathcal{P}(X)$ denotes the power set of a set X. As it is well-known, in the case of $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ the two notions of trace and restriction coincide.

We denote the space of real $(n \times m)$-matrices by $\mathbb{R}^{n\times m}$. We write $\mathbb{R}^{n\times n}_{sym}$ resp. $\mathbb{R}^{n\times n}_{skew}$ for the subspace of symmetric and skew-symmetric matrices. Furthermore, the square matrices with positive determinants are denoted by $\mathbb{R}^{n\times n}_+$. We identify the tensor $a \otimes b$ with its matrix representation ab^T for $a, b \in \mathbb{R}^n$. The symmetric tensor product $\odot$ is defined by

$$a \odot b = \frac{1}{2}(a \otimes b + b \otimes a).$$

We define the indicator of a set ω

$$\chi_\omega(x) := \begin{cases} 1, & \text{if } x \in \omega, \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Overview of the direct method of Calculus of Variations

Let X be an arbitrary topological space and $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ be a functional on X. The basic problem in calculus of variations is the existence of minima of a functional. A way to guarantee the existence of a minimizer is to impose the following two requirements onto $\mathcal{F}$:

- *Compactness*: Sublevel sets are sequentially relatively compact. This means that for any sequence $\{x_k\} \in X$ such that $\{\mathcal{F}(x_k)\}$ is bounded from above there exists a subsequence that converges in X.

- *Semicontinuity*: $\mathcal{F}$ is sequentially lower semi-continuous.

Indeed, suppose that these conditions hold for $\mathcal{F}$ and that $\{x_k\}$ is a minimizing sequence of $\mathcal{F}$, i.e.,

$$\lim_{n\to\infty} \mathcal{F}(x_k) = \inf_{x\in X} \mathcal{F}(x).$$

If $\mathcal{F} \not\equiv +\infty$ then $\inf_{x\in X} \mathcal{F}(x) < +\infty$. We can assume that $\mathcal{F}(x_k) \neq \infty$ for all $k \in \mathbb{N}$. Furthermore, we notice that $\{F(x_k)\}$ is uniformly bounded from above. By the first condition we can now assume (after extracting a subsequence) that $\{x_k\}$ converges to some $x \in X$. By the lower semi-continuity of $\mathcal{F}$ we now know that

$$\inf_{x\in X} \mathcal{F}(x) \le \mathcal{F}(x) \le \liminf_{k\to\infty} F(x_k) = \inf_{x\in X} \mathcal{F}(x).$$

Therefore, $\mathcal{F}$ attains a minimum at x. This is called the direct method of calculus of variation.

A typical example where the direct method can be applied is the following. Let $X = \{u \in W^{1,p}(\Omega, \mathbb{R}^N) : u|_{\partial\Omega} = g|_{\partial\Omega}\}$ for some fixed $g \in W^{1,p}(\Omega, \mathbb{R}^N)$ ($p \in (1,\infty)$) be equipped with the weak topology. Suppose that

$$\mathcal{F}(u) = \int_\Omega F(x, u(x), \nabla u(x))dx$$

where $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is called the Lagrange function. To ensure that $\mathcal{F}$ is well defined on X we require $F(x, z, M)$ to be a Caratheodory integrand, i.e., Lebesgue measurable in x for all (z, M) and continuous in $(z, M) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ for λ^n-a.e. $x \in \Omega$, and to fulfil the growth condition

$$|F(x, z, M)| \le C(1 + |z|^p + |M|^p) \tag{2.1}$$

for some $C > 0$ and for all $x \in \mathbb{R}^n, z \in \mathbb{R}^N$ and $M \in \mathbb{R}^{n\times N}$.

To establish the compactness needed for the direct method we also require the condition

$$F(x, z, M) \ge \alpha|M|^p - \beta \tag{2.2}$$

to hold for some $\alpha, \beta > 0$ and all $x \in \mathbb{R}^n, z \in \mathbb{R}^N$ and $M \in \mathbb{R}^{n\times N}$. Indeed, suppose that $\{u_k\}$ is a sequence s.t. $\{\mathcal{F}(u_k)\}$ is bounded. Then (2.2) ensures the boundedness of the gradients $\{\nabla u_k\}$ in L^p. By the classical Poincaré inequality we then have:

$$\begin{aligned} \|u_k\|_{1,p} &\le \|u_k - g\|_{1,p} + \|g\|_{1,p} \\ &\le C_p \|\nabla(u_k - g)\|_p + \|g\|_{1,p} \\ &\le (C_p + 1)(\|\nabla(u_k)\|_p + \|g\|_{1,p}). \end{aligned}$$

This implies the boundedness of $\{u_k\}$ in $W^{1,p}(\Omega, \mathbb{R}^N)$. By the classical Banach-Alaoglu theorem, we know that u_k possesses a subsequence which converges weakly to some element $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. Since X is convex and closed in the norm of $W^{1,p}(\Omega, \mathbb{R}^N)$ we also know that X is weakly closed which implies $u \in X$.

The question of what conditions are suitable to guarantee the lower semi-continuity of $\mathcal{F}$ is closely related to the convexity of F in the last variable. More generally, we introduce a weaker notion of convexity:

Definition 2.2.1. A locally bounded Borel measurable function $f : \mathbb{R}^{n\times N} \to \mathbb{R}$ is called quasiconvex if

$$f(M) \leq \frac{1}{\lambda(B_1(0))} \int_{B_1(0)} (M + \nabla\varphi)\, dx$$

for all $\varphi \in W_0^{1,\infty}(B_1(0), \mathbb{R}^m)$ and $M \in \mathbb{R}^{n\times N}$.

A way to now set up the second condition for the direct method is the following result which essentially goes back to L. Tonelli (cf. Theorem 5.20 in [Rin18]).

Theorem 2.2.2. *Suppose that the Carathéodory integrand F fulfils (2.1) and is quasiconvex in M. Then the functional $\mathcal{F}$ is lower semi-continuous with respect to the weak topology on $W^{1,p}(\Omega, \mathbb{R}^N)$.*

So, the two growth conditions (2.1) and (2.2) together with quasiconvexity in the last variable are enough to ensure the existence of a minimizer in a minimization problem over X.

2.3 Short introduction to Radon measures

We start with the general definition of a Radon measure:

Definition 2.3.1. Let μ be a (signed) measure on the Borel sets $\mathcal{B}$ of an open $\Omega \subseteq \mathbb{R}^n$. We call μ:

- inner regular if
  $$\mu(B) = \sup\{\mu(K) : K \subseteq B,\ K \text{ compact}\}$$
  holds for every Borel set $B \subseteq \Omega$.
- outer regular if
  $$\mu(B) = \inf\{\mu(O) : B \subseteq O,\ O \text{ open}\}$$
  holds for every Borel set $B \subseteq \Omega$.
- locally finite if $\mu(K) < \infty$ for every compact $K \subseteq \Omega$.

We call μ regular if it is both inner and outer regular. If μ is regular and locally finite then we call μ a (signed) *Radon measure*. Furthermore, we denote the space of all Radon measures by $\mathcal{M}(\Omega)$ and the space of all finite Radon measures by $\mathcal{M}_b(\Omega)$.

It is a well-known fact that $\mathcal{M}_b(\Omega)$ is a Banach space if endowed with the *total variation norm*

$$\|\mu\|_{\mathcal{M}_b} := |\mu|(\Omega)$$

where

$$|\mu|(E) = \sup\left\{\sum_{i=1}^k |\mu(B_i)| : B_i \in \mathcal{B} \text{ pairwise disjoint},\ \bigcup_{i=1}^k B_i = E\right\} \tag{2.3}$$

is the *total variation measure* of a Radon measure μ. Similarly, one can introduce the spaces of *vector-valued (finite) Radon measures* $\mathcal{M}(\Omega, \mathbb{R}^m) := (\mathcal{M}(\Omega))^m$ (and $\mathcal{M}_b(\Omega, \mathbb{R}^m) := (\mathcal{M}_b(\Omega))^m$). By using (2.3) the total variation measure can also be introduced in these spaces.

Not surprisingly, $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ then becomes a Banach space if endowed with the total variation norm. It is immediately apparent that $\mathcal{M}(\Omega, \mathbb{R}^m) \subseteq C_c(\Omega, \mathbb{R}^m)'$ and $\mathcal{M}_b(\Omega, \mathbb{R}^m) \subseteq C_0(\Omega, \mathbb{R}^m)'$ holds via the identification

$$\mu \mapsto \left(f \mapsto \int_\Omega f \cdot d\mu\right)$$

where $\int f \cdot d\mu := \sum_{i=1}^m \int_\Omega f_i \; d\mu_i$. The famous representation theorem of Riesz, Markov and Kakutani then asserts that indeed every bounded functional on $C_c(\Omega, \mathbb{R}^m)$ can be written in terms of a Radon measure:

Theorem 2.3.2 (Riesz-Markov-Kakutani)**.** *Let $\Omega \subseteq \mathbb{R}^d$ open. Consider a bounded linear functional $T : C_c(\Omega, \mathbb{R}^m) \to \mathbb{R}$. Then there exists a unique Radon measure $|\mu|$ and $g \in L^\infty(\Omega, \mathbb{R}^m, |\mu|)$ such that*

- *$T(f) = \int_\Omega f \cdot g \; d|\mu|$ holds for all $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$,*
- *$|g| = 1$ holds $|\mu|$-a.e.*

We call $\mu := g|\mu|$ the polar decomposition of $|\mu|$.

The proof of this particular version can be found in Chapter 4 in [Mag12]. As an immediate consequence it can be shown that $C_0(\Omega, \mathbb{R}^m)' = \mathcal{M}_b(\Omega, \mathbb{R}^m)$ holds. Furthermore, by a typical characterisation of Radon measures we have

$$\|\mu\|_{\mathcal{M}_b} = \sup\left\{\int_\Omega f \cdot \mu : f \in C_c(\Omega, \mathbb{R}^m), \|\varphi\|_\infty \leq 1\right\}.$$

on $\mathcal{M}_b(\Omega, \mathbb{R}^m)$.

We now recap the notion of weak-* convergence for measures:

Definition 2.3.3. For a sequence $\{\mu_k\} \in \mathcal{M}(\Omega, \mathbb{R}^m)$ we say $\mu_k \rightharpoonup^* \mu$ if and only if

$$\int_\Omega f \cdot d\mu_k \to \int_\Omega f \cdot d\mu$$

for all $f \in C_c(\Omega)$. Analogously, for a sequence $\{\mu_k\} \in \mathcal{M}_b(\Omega, \mathbb{R}^m)$ we have $\mu_k \rightharpoonup^* \mu$ in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ if and only if

$$\int_\Omega f \cdot d\mu_k \to \int_\Omega f \cdot d\mu$$

for all $f \in C_0(\Omega)$.

If we have a sequence of finite vector-valued Radon measures $\mu_k \in \mathcal{M}_b(\Omega, \mathbb{R}^m)$ then the weak-* convergence in $\mathcal{M}(\Omega, \mathbb{R}^m)$ implies the weak-* convergence in $\mathcal{M}(\Omega, \mathbb{R}^m)$. The reverse is true under the additional assumption of uniform boundedness of the total variations of the sequence:

Lemma 2.3.4. *Let $\{\mu_k\}$ be a sequence in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$. Then $\mu_k \rightharpoonup^* \mu$ in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ if and only if $\mu_k \rightharpoonup^* \mu$ in $\mathcal{M}(\Omega, \mathbb{R}^m)$ and $\sup_{k\in\mathbb{N}} \|\mu_k\|_{\mathcal{M}_b} < \infty$.*

3 Korn inequalities in L^p-spaces

Around 1900 the physicist A. Korn first mentioned an inequality for $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$ on a bounded, connected subset $\Omega \subseteq \mathbb{R}^3$ of the type

$$\int_\Omega |\varphi|^2 dx + \int_\Omega |\nabla\varphi|^2 \leq C \left(\int_\Omega |\varphi|^2 dx + \int_\Omega |e(\varphi)|^2 dx \right),$$

where $e(\varphi)$ denotes the symmetric gradient $\frac{1}{2}(\nabla\varphi + \nabla\varphi^T)$. While some call the original proof of Korn doubtful (cf. [Nit81]), this marked the start of the research into this type of inequality. The inequality above is known today as *Korn's second inequality.*

In this chapter, we introduce the basic concepts related to this inequality. We will discuss an application of a Korn inequality (namely in the classical boundary problem of elasticity) in section 3.1. In section 3.2, we will then summarize the most important aspects of Korn-type inequalities in the L^p spaces and give short modern proofs based on the lemma of J. L. Lions for $p \in (1, \infty)$. Lastly, we will discuss why Korn inequalities fail in the boundary cases $p \in \{1, \infty\}$.

3.1 The boundary problem of elasticity and its relation to Korn inequalities

Continuum mechanics is a rich source for many mathematical theories. Especially nowadays, mathematical elasticity is a lively field with many researchers continuously contributing to its development. In this section, we will study a simple form of the boundary problem of elasticity. The aim is to observe the role of a Korn inequality which will guarantee solutions to the linearized problem. This section follows chapter 5 and 6 in [Cia88]. For a variational point of view, we refer to [DNP02].

Consider $\Omega \subseteq \mathbb{R}^3$ open, bounded, connected and with Lipschitz boundary. We call this the *reference configuration* of some body in the three-dimensional Euclidean space. We say that a *deformation* φ is admissible if $\varphi \in C^1(\bar{\Omega}, \mathbb{R}^3)$ and it is injective on Ω with $\det(\nabla\varphi) > 0$. One possible formulation of the boundary value problem of elasticity for an elastic material is the following: We search for an admissible deformation φ such that

$$\begin{cases} -\mathrm{div}\, T(x, \nabla\varphi(x)) = f(x, \varphi(x)) & \text{in } \Omega, \\ \varphi(x) = \varphi_0(x) & \text{on } \Gamma_0, \\ T(x, \nabla(\varphi(x)))\nu(x) = g(x, \nabla\varphi(x)) & \text{on } \Gamma_1, \end{cases} \tag{3.1}$$

where the following quantities are given:

- $\Gamma_0, \Gamma_1 \subseteq \partial\Omega$ are a relative open subsets of the boundary of Ω such that $\mathcal{H}^2(\partial\Omega \setminus (\Gamma_0 \cup \Gamma_1)) = 0$.

- $T : \bar{\Omega} \times \mathbb{R}^{3\times3}_{+} \to \mathbb{R}^{3\times3}$ is the response function for the *first Piola-Kirchhoff stress* of the material.
- $f : \bar{\Omega} \times \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \Gamma_1 \times \mathbb{R}^{3\times3}_{+} \to \mathbb{R}^3$ are density functions of applied body and surface forces.
- $\varphi_0 \in C^1(\Gamma_0, \mathbb{R}^3)$.
- ν is the outward unit normal on $\partial\Omega$.

This is called a *displacement-traction* problem. To proceed, we simplify the problem. We assume that f, g are dead loads (they do not depend on φ), the material is homogeneous (the response function T does only depend on $\nabla\varphi(x)$), frame-invariant, isotropic, compressible and $\varphi_0 = \mathrm{id}_{\Gamma_0}$ where id_{Γ_0} denotes the identity on Γ_0. For a comprehensible analysis of these notions, we refer to chapters 2 to 4 in [Cia88]. Writing the PDE (3.1) in terms of the displacement $u = \varphi - \mathrm{id}_\Omega$ and setting $\Sigma(\nabla\varphi) = \nabla\varphi^{-1}T(\nabla\varphi)$ (called the response function for the *second Piola-Kirchhoff stress*) paired with the fact that Σ can be written as function of the *Cauchy stress tensor* $\nabla\varphi^T\nabla\varphi$ (cf. Theorem 3.6-2 in [Cia88]) the above equations read as

$$\begin{cases} -\mathrm{div}\left[(I+\nabla u)\Sigma(E(u))\right] = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ (I+\nabla u)\Sigma(E(u))n = g & \text{on } \Gamma_1. \end{cases} \tag{3.2}$$

Here $E(u) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u)$ is the *Green-St. Venant strain tensor*. If the material is assumed to be in a natural state (meaning $\Sigma(0) = 0$ holds) one can compute the following Taylor expansion around 0 (cf. section 3.8 in [Cia88])

$$\Sigma(E) = \lambda(\mathrm{tr}(E))I + 2\mu E + o(\|E\|)$$

with $\lambda, \mu > 0$. Now, consider the non-linear operator

$$A(u) = -\mathrm{div}\left[(I+\nabla u)\Sigma(E(u))\right].$$

If some regularity assumptions are imposed onto Σ such that Σ can be considered as a function from $W^{1,p}(\Omega)$ to $L^p(\Omega)$ for $p > 3$ then A is well defined on $W^{2,p}(\Omega)$. As a remark, we note here that it is natural to observe the problem in the space $W^{2,p}(\Omega)$ for $p > 3$ since it can be continuously embedded into $C^1(\Omega)$ due to Sobolev embeddings. While injectivity of φ and the assumption $\det(\nabla\varphi) > 0$ are dropped at this point they can be verified a posteriori. (cf. section 5.6 in [Cia88]). We notice now that the Green-St. Venant strain tensor can be considered as a non-linear operator $E : W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ with

$$E(u) = e(u) + o(\|u\|_{1,p}).$$

As a consequence, we have that A is well-defined operator from $W^{2,p}(\Omega)$ into $L^p(\Omega)$. Furthermore, we compute

$$A(u) - A(0) = -\mathrm{div}(\lambda(\mathrm{tr}(e(u)))I + 2\mu(e(u))) + o(\|u\|_{2,p})$$

which implies that the Fréchet derivative of A at 0 is nothing else than

$$A'(0)v = -\mathrm{div}(\lambda(\mathrm{tr}(e(v)))I + 2\mu(e(v))).$$

Applying the same line of arguments to the boundary condition operator

$$B(u) = (I + \nabla u)\Sigma(E(u))n$$

on Γ_1 we can linearize the boundary value problem (3.2) the following way:

$$\begin{cases} -\mathrm{div}(\lambda(\mathrm{tr}(e(v)))I + 2\mu(e(v))) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ [\lambda(\mathrm{tr}(e(v)))I + 2\mu(e(v))]n = g. & \text{on } \Gamma_1. \end{cases} \tag{3.3}$$

Since the condition $\Sigma(0) = 0$ implies that (3.2) is solvable for f, g equal to 0 the idea is to apply the implicit function theorem to the non-linear problem or to be more precise to the operator

$$(A, B) : \begin{cases} \{u \in W^{2,p}(\Omega) : u|_{\Gamma_0} = 0\} \to (L^p(\Omega), C(\partial\Omega)) \\ u \mapsto (-\mathrm{div}(\lambda(\mathrm{tr}(e(v)))I + 2\mu(e(v))), [\lambda(\mathrm{tr}(e(v)))I + 2\mu(e(v))]n). \end{cases}$$

To apply the theorem one needs to show that the Frechet derivative in 0 is regular, i.e., the linearized problem has a unique solution for all f, g. If the boundary of Ω is now sufficiently regular we only look need to analyse the problem (3.3) in $H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$ which is a closed subspace in $H^1(\Omega)$. This is due to the Sobolev embeddings which hold under these assumptions (for a detailed analysis of this we refer to section 6.3 in [Cia88]).

The weak formulation of the linearized problem (3.3) then reads as follows:

$$\int_\Omega e(v) : \mathbb{C}e(w)dx = \int_\Omega f \cdot w \; dx + \int_{\Gamma_1} g \cdot w \; d\mathcal{H}^2 \text{ holds for all } w \in H^1_{\Gamma_0}(\Omega),$$

where $(\mathbb{C}_{ijkl})$ is the symmetric fourth order tensor associated with $M : \mathbb{C}N = \lambda\,\mathrm{tr}(M)\mathrm{tr}(N) + 2\mu(M : N)$ for all $M, N \in \mathbb{R}^{3\times3}_{sym}$. Now, define the following quadratic form by

$$B(v, w) = \int_\Omega e(v) : \mathbb{C}e(w)dx$$

and set

$$F(w) = \int_\Omega f \cdot w \; dx + \int_{\Gamma_1} g \cdot w \; d\mathcal{H}^2.$$

Observe that $M : \mathbb{C}M \geq 2|M|^2$ holds for arbitrary $M \in \mathbb{R}^{3\times3}_{sym}$. Therefore, we generalize our analysis and consider arbitrary symmetric fourth-order tensors $(\mathbb{C}_{ijkl})$ which are also positive definite, i.e., $(M : \mathbb{C}M) \geq K|M|^2$ holds for some constant $K > 0$. Assume now that Γ_0 has non-vanishing area. By a classical result, the weak problem is uniquely solvable if B is a continuous, symmetric bilinear form such that for all $v \in H^1_{\Gamma_0}(\Omega)$

$$B(v, v) \geq L \left\|\nabla v\right\|_2^2$$

holds for some $L > 0$ and F is a continuous linear form. At this point, *Korn's first inequality* comes into play. It guarantees that

$$\left\|e(u)\right\|_2 \geq \tilde{L} \left\|\nabla u\right\|_2.$$

for some $\tilde{L} > 0$. We will discuss the validity of this inequality thoroughly in the next section (cf. Theorem 3.2.5). Korn's first inequality now ensures the coercivity of B since

$$B(v,v) \geq K \|e(v)\|_2^2 \geq K\tilde{L}^2 \|\nabla u\|_2^2$$

holds for every $v \in H^1_{\Gamma_0}(\Omega)$. Therefore, we know the linearized problem is uniquely solvable which in turn implies we can employ the implicit function theorem which gives us local solutions around 0.

3.2 Korn inequalities in L^p-spaces

After our introductory example in the last section, we will now concentrate on the proof techniques which are used to show a variety of different Korn-type inequalities. We will first concentrate on Korn's second inequality in $W^{1,p}(\Omega)$ for $p \in (1,\infty)$. Furthermore, we are going to discuss how the validity of this inequality implies the existence of many different Korn-type inequalities. In the end, we discuss the edge cases $p = 1, \infty$ where we will observe the lack of such inequalities.

We start by formulating Korn's second inequality for Ω open, bounded with Lipschitz boundary. It states that there exists a constant $C > 0$ so that

$$\|u\|_{1,p} \leq C(\|u\|_p + \|e(u)\|_p) \tag{3.4}$$

holds for all $u \in W^{1,p}(\Omega,\mathbb{R}^n)$ with $p \in (1,\infty)$. Different approaches to the proof of this inequality can be found for instance in [OS92; Nit81; DD21] and the references therein. However, we choose to discuss the particularly elegant method presented by P. Ciarlet in [Cia10]. To start, we recap a well-known result from functional analysis:

Theorem 3.2.1. *Let X, Y be Banach spaces and $L : X \to Y$ a bijective, linear operator. If L is continuous then the inverse L^{-1} is also continuous.*

With this theorem in mind, we present the key steps in the proof of (3.4). We start with the following lemma:

Lemma 3.2.2. *Let*

$$X^p(\Omega) := \{u \in L^p(\Omega,\mathbb{R}^n) : e(u) \in L^p(\Omega,\mathbb{R}^{n\times n}_{sym})\} \tag{3.5}$$

where '$e(u) \in L^p(\Omega,\mathbb{R}^{n\times n}_{sym})$' means that the distributional symmetric gradient of u can be identified with a L^p-function. If we endow $X^p(\Omega)$ with the norm

$$\|u\|_{X^p} = \|u\|_p + \|e(u)\|_p\,,$$

then $X^p(\Omega)$ is a Banach space.

Proof. Indeed, to see that $(X^p(\Omega), \|\cdot\|_{X^p})$ is complete let $\{u_k\} \in X^p(\Omega)$ be a Cauchy sequence with respect to $\|\cdot\|_{X^p(\Omega)}$. By definition of $\|\cdot\|_{X^p(\Omega)}$ we have that $\{u_k\}$ and $\{e(u_k)\}$ are Cauchy sequences in $L^p(\Omega,\mathbb{R}^n)$. Therefore, they respectively converge to some $u \in L^p(\Omega,\mathbb{R}^n)$ and

$e \in L^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ with respect to the L^p-Norm. Since $e = \nabla + \nabla^T$ is a linear differential operator it is continuous on $\mathcal{D}'(\Omega, \mathbb{R}^n)$. We have

$$u_n \xrightarrow{L^p} u \;\Rightarrow\; u_n \xrightarrow{\mathcal{D}'} u \;\Rightarrow\; e(u_n) \xrightarrow{\mathcal{D}'} e(u)$$

and

$$e(u_n) \xrightarrow{L^p} e \;\Rightarrow\; e(u_n) \xrightarrow{\mathcal{D}'} e.$$

Hence, $e = e(u)$ in $\mathcal{D}'$. Since $e, e(u) \in L^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ we conclude that $e = e(u)$ has to hold in $L^p(\Omega, \mathbb{R}^{n\times n}_{sym})$. □

Now, observe that the inclusion $\iota : (H^1(\Omega, \mathbb{R}^n), \|\cdot\|_{1,p}) \to (X^p, \|\cdot\|_{X^p})$ is trivially injective and continuous. Assume that $\iota(H^1(\Omega, \mathbb{R}^n))$ is closed in $(X^p, \|\cdot\|_{X^p})$. Then, by Theorem 3.2.1 we have the continuity of $\iota^{-1} : (\iota(H^1(\Omega, \mathbb{R}^n)), \|\cdot\|_{X^p}) \to (H^1(\Omega, \mathbb{R}^n), \|\cdot\|_{1,p})$. This is equivalent to (3.4). We will see that the above assumption '$\iota(H^1(\Omega, \mathbb{R}^n))$ is closed' trivially holds since ι is surjective. To show this, we first observe the following:

Let $f \in L^p(\Omega, \mathbb{R}^n)$ and $q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Since

$$\int fu \, dx \leq \|f\|_p \|u\|_{1,q}$$

and

$$\int f\partial_i u \, dx \leq \|f\|_p \|u\|_{1,q}$$

hold for all $u \in W^{1,q}(\Omega)$ and all $i = 1, .., n$ we know that the distributions $f, \partial_i f$ can be extended to elements of $W^{-1,q}(\Omega)$. In short:

$$f \in L^p(\Omega) \;\Rightarrow\; f, \partial_i f \in W^{-1,q}(\Omega).$$

The astounding fact whose discovery is attributed to J. L. Lions is that above implication is an equivalence. More generally, the following holds (cf. also [GSN86]):

Lemma 3.2.3. *[CMM18, Theorem 1.1] For $m \geq 1$ and any $q \in (1, \infty)$ we have*

$$f \in \mathcal{D}'(\Omega) \text{ and } \nabla f \in W^{-m,q}(\Omega) \;\Rightarrow\; f \in W^{-m+1,q}(\Omega).$$

With these two facts we can now give a simple proof for (3.4):

Proof of Korn inequality (3.4). As discussed above, it is enough to show that the embedding

$$\iota : (H^1(\Omega, \mathbb{R}^n), \|\cdot\|_{1,p}) \to (X^p(\Omega), \|\cdot\|_{X^p})$$

is surjective. Therefore, let $u \in X^p(\Omega)$. By definition, we have $e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \in L^p(\Omega)$ for all $i, j = 1, ..., n$. Now, notice that

$$\partial_k u_i \in W^{-1,q}(\Omega)$$

for $i, k = 1, ..., n$ since $u_i \in L^p$. We want to show that $\partial_k u_i \in L^p(\Omega)$ holds. For this we are going to employ Lemma 3.2.3. We observe that

$$\partial_j(\partial_k u_i) = \partial_j e_{ik}(u) + \partial_k e_{ij}(u) - \partial_i e_{jk}(u)$$

holds in $\mathcal{D}'(\Omega)$ for all $j = 1, ..., n$. From $e_{ij}(u) \in L^p(\Omega)$ we infer $\nabla e_{ij}(u) \in W^{-1,q}(\Omega, \mathbb{R}^{n\times n})$ for all $i, j = 1, .., n$. But this implies

$$\nabla(\partial_k u_i) \in W^{-1,q}(\Omega, \mathbb{R}^{n\times n})$$

for all $i, k = 1, .., n$. Therefore, we can apply Lemma 3.2.3 to conclude $\partial_k u_i \in L^p(\Omega)$ for each $i, k = 1, ..., n$. Consequently, we deduce $u \in W^{1,p}(\Omega)$. □

Before we can talk about a variety of different types of Korn inequalities which are a consequence of this result we need a specific property of e, namely the structure of its kernel.

Theorem 3.2.4. *Let $e : W^{1,p}(\Omega, \mathbb{R}^n) \to L^p(\Omega, \mathbb{R}^{n\times n}_{sym})$ be the symmetric gradient $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. Then*

$$\ker e = \{u \in W^{1,p}(\Omega, \mathbb{R}^n) \mid \exists A \in \mathbb{R}^{n\times n}_{skew}, b \in \mathbb{R}^n : u(x) = Ax + b \textit{ holds for } \lambda^n\textit{-a.e.}x \in \Omega\}.$$

A skew-symmetric displacement is a mapping of the type $u(x) = Ax + b$ with $A \in \mathbb{R}^{n\times n}_{skew}$ and $b \in \mathbb{R}^n$.

The proof will be postponed to the next chapter (cf. Theorem 4.1.10). Similarly, one can also show that

$$\ker E = \{u \in W^{1,p}(\Omega, \mathbb{R}^n) : u + \mathrm{id}_\Omega \text{ is equal to a rigid deformation } \lambda^n\text{-a.e.}\}$$

holds, where $E(u) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u)$ is the Green St. Venant tensor (cf. section 3.1). A rigid deformation is a map of the form $Ox + b$ with $O \in O(n)$ and $b \in \mathbb{R}^n$. For proofs of this, see for instance Theorem 1.8-1 in [Cia88].

As a consequence of Theorem 3.2.4 and (3.4), we can now prove *Korn's first inequality* which we have already seen in section 3.1:

Theorem 3.2.5. *Let $\Gamma \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$*

$$W^{1,p}_\Gamma(\Omega, \mathbb{R}^n) = \{u \in W^{1,p}(\Omega, \mathbb{R}^n) : u|_\Gamma = 0\}.$$

Then

$$\|\nabla u\|_p \le C \|e(u)\|_p$$

holds for some constant $C > 0$ and all $u \in W^{1,p}_\Gamma(\Omega, \mathbb{R}^n)$.

Proof. The proof is a classical argument by contradiction. Suppose that there exists no such $C > 0$. Then, there exists a sequence $(u_m) \in W^{1,p}_\Gamma(\Omega, \mathbb{R}^n)$ with

$$\|\nabla u_m\|_p = 1 \text{ and } e(u_m) \xrightarrow{L^p} 0.$$

Since $W^{1,p}_\Gamma(\Omega, \mathbb{R}^n)$ is closed in $W^{1,p}(\Omega, \mathbb{R}^n)$ and $\|\nabla\cdot\|_p$ is equivalent to $\|\cdot\|_{1,p}$ due to Poincaré inequality, we have the boundedness of (u_m) in $W^{1,p}(\Omega, \mathbb{R}^n)$. Rellich-Kondrachov theorem now implies that (without renaming the particular subsequence)

$$u_m \xrightarrow{L^p} u$$

for a $u \in W^{1,p}_\Gamma(\Omega, \mathbb{R}^n)$. This in turn means

$$e(u_m) \xrightarrow{\mathcal{D}'} e(u).$$

But, by assumption we also have

$$e(u_m) \xrightarrow{\mathcal{D}'} 0.$$

Therefore, $e(u) = 0$ in $\mathcal{D}'$ which implies $e(u) = 0$ in L^p. By Theorem 3.2.4, we know that there exists $A \in \mathbb{R}^{n\times n}_{skew}, b \in \mathbb{R}^n$ such that $u(x) = Ax + b$ a.e. for $x \in \Omega$. We sloppily write $u(x)$ for $Ax + b$, i.e., we choose $Ax + b$ as the representative of the function class $u \in L^p$. Since the rank of a real skew symmetric matrix can only be even we have that the kernel of A either has dimension equal to n or lower than $n - 1$. In the case of $\dim \ker A = n$ we get $A = 0$ and $u|_\Gamma = b = 0$ since Γ has positive $\mathcal{H}^{n-1}$-measure. Hence $u = 0$.

If $\dim \ker A < n - 1$ we have that $u^{-1}(0)$ lies in an $(n - 2)$-plane and therefore its $\mathcal{H}^{n-1}$-measure equals 0. Notice, that $u|_\Gamma = 0$ implies $\Gamma \subseteq u^{-1}(0)$ since the trace of a continuous function (class) corresponds to the (class of the) usual restriction. This contradicts with $\mathcal{H}^{n-1}(u^{-1}(0)) = 0$, so $\dim \ker A < n - 1$ cannot happen. So we have shown that $e(u) = 0$ implies $u = 0$ in $W^{1,p}_\Gamma(\Omega, \mathbb{R}^n)$. In turn, this now contradicts

$$\|\nabla u\|_p = \lim_{n\to\infty} \|\nabla u_m\|_p = 1,$$

which concludes the proof. □

The proof to Theorem 3.2.5 also shows another way to formulate the above theorem in a more general way:

Theorem 3.2.6. *Let V be a closed, convex subset of $W^{1,p}(\Omega, \mathbb{R}^n)$ with*

$$V \cap \ker e = \{0\}.$$

Then

$$\|\nabla u\|_p \leq C \|e(u)\|_p$$

holds for some constant $C > 0$ and all $u \in V$.

With this we can formulate Korn's inequality in a different way:

Lemma 3.2.7. *Let*

$$\dot{W}^{1,p}(\Omega, \mathbb{R}^n) := W^{1,p}(\Omega, \mathbb{R}^n)/(\ker e).$$

Then there exists a constant $C > 0$

$$\|\dot{u}\|^{\sim}_{1,p} \leq C \|e(\dot{u})\|_p$$

for all $\dot{u} \in \dot{W}^{1,p}(\Omega, \mathbb{R}^n)$ where $\|\cdot\|^{\sim}_{1,p}$ denotes the usual factor norm on $W^{1,p}(\Omega, \mathbb{R}^n)/(\ker e)$.

Proof. First of all notice that e is well defined as a function from $\dot{W}^{1,p}(\Omega, \mathbb{R}^n)$ to $L^p(\Omega, M^n_{sym})$. Since $\ker e$ has finite dimension we know that there exists a closed subspace $M \subseteq W^{1,p}(\Omega, \mathbb{R}^n)$ with

$$W^{1,p}(\Omega, \mathbb{R}^n) = \ker e \oplus M.$$

By the last theorem, we know that there exists a constant $C > 0$ such that

$$\|\nabla v\|_p \leq C \|e(v)\|_p$$

for all $v \in M$. If we now write $u = r + (v - v_\Omega)$ with $r \in \ker e, v \in M$ and set

$$v_\Omega := \frac{1}{\lambda^n(\Omega)} \int_\Omega v dx$$

we observe with the Poincaré inequality that

$$\|\dot{u}\|^{\sim}_{1,p} \leq \|v - v_\Omega\|_{1,p} \leq C_p \|\nabla v\|_p \leq \tilde{C} \|e(v)\|_p = \tilde{C} \|e(\dot{u})\|_p$$

holds with Poincaré constant C_p and $\tilde{C} = C_p C$. □

As an immediate consequence we have

Lemma 3.2.8. *Let $P : W^{1,p}(\Omega) \to \ker e$ be the projection onto $\ker e$. Then*

$$\|u - Pu\|_{1,p} = \|\dot{u}\|^{\sim}_{1,p} \leq C \|e(u)\|_p$$

holds for all $u \in W^{1,p}(\Omega, \mathbb{R}^n)$.

After discussing the case of $p \in (1, \infty)$ it is natural to ask if Korn inequalities of the type presented in the above sections can hold for $p \in \{1, \infty\}$. In both cases, however, we lose the reflexivity of the Sobolev spaces which in turn implies that we cannot apply the compactness arguments used in the sections above. In fact, in neither of the boundary cases, we can hope to achieve Korn-type inequalities as one can construct explicit counterexamples which we are going to present in this section. For a discussion about which conditions are sufficient so that a Korn inequality has to hold in a broad class on 'L^p'-like function spaces including the usual Sobolev spaces (namely Orlicz spaces), we refer to [BD12].

We start with the discussion of the case $p = 1$. We first give the definition for homogeneous differential operators.

Definition 3.2.9. Let $p \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial in n variables. The associated linear differential operator $p(D) \in \mathcal{D}'(\Omega)$ is called *homogeneous* if p is homogenous. p is called the *symbol* of $p(D)$. Linear differential operators $p_1(D), ..., p_l(D)$ are *linearly independent* if their corresponding symbols are linearly independent as elements of $\mathbb{R}[x_1, ..., x_n]$.

The construction of a counterexample in the case of $p = 1$ goes back to 1962 when D. Ornstein proved that a variety of integral inequalities regarding homogeneous differential operators cannot hold (see [Orn62]). The idea for this proof came from the question if the Riesz inequalities

$$\|\partial_{ij}\varphi\|_{L^1} \leq C \|\Delta\varphi\|_{L^1} \tag{3.6}$$

for $i, j = 1, ..n$ can hold for $\varphi \in C_0^\infty((0,1)^n)$. Interestingly, one way to prove the Korn inequalities discussed in this section depends heavily on these (cf. for instance Chapter 7 in [DD21]). We now state the theorem which is generally refereed to by *Ornstein's L^1 non-inequalities*:

Theorem 3.2.10. *[Orn62, Theorem 1] Suppose $B, D_1, ..., D_L$ are homogeneous, linear differential operators on $\mathbb{R}^n$ of order m which are linearly independent. Then, for any $K > 0$ there exits a $f \in C_0^\infty((0,1)^n)$ such that*

$$\int_{[0,1]^n} |Bf| dx > K$$

and

$$\int_{[0,1]^n} |D_i f| dx < 1$$

for every $i = 1, .., n$.

Proof sketch. The proof of this result is rather constructive. Ornstein starts with a polynomial p which fulfils $Bp = 1$ and $D_i p = 0$, $i = 1, .., L$ which can be found due to the linear independence of the differential operators. He then sets

$$f(x) = \begin{cases} p(x) & \text{for } x \in [0,1]^n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly f fulfils

$$\frac{\int_{[0,1]^n} D_i f dx}{\int_{[0,1]^n} Bf dx} < \epsilon \tag{3.7}$$

for $i = 1, .., L$ and $\epsilon > 0$ fixed. But f does not vanish at the boundary by definition. To fix this Ornstein introduced algorithmic machinery that works as follows: It takes f and turns it and every partial derivative up to some order $m-1$ into some function in $C^\infty(\mathbb{R}^n \setminus P)$ where P is a collection of hyperplanes parallel to the n-th coordinate axis while retaining that f still vanishes outside of the unit cube and the inequality (3.7). Then this process can be repeated with respect to the $(n-1)$-th coordinate axis while preserving the differentiability along x_n gained in the first step and so on. After repeating this process n times we have a $C^\infty(\mathbb{R}^n)$-function which vanishes outside the unit cube and fulfils (3.7). □

Ornstein also gave an explicit construction of a counterexample to the aforementioned Riesz inequalities (3.6) in two dimension [Orn62]. Rather than using a polynomial as a starting point like in the more general case of Theorem 3.2.10 he constructs a more graphic example. For each $l \in (0,1)$ he introduces a sequence of weighted indicator functions p_n taking constant values on a square partition of $[-1,1]^2$ such that

$$P_n(x,y) := \int_{-1}^{x} \int_{-1}^{y} p_n(\tilde{x}, \tilde{y}) \, d\tilde{x} d\tilde{y}$$

for n large enough satisfies the conditions

$$\int_{-1}^{1} \mathrm{Var}_y(\partial_y P_n) \, d\lambda = C,$$

$$\int_{[-1,1]^2} |\partial_{xy} P_n| \, d\lambda^2 \geq nT$$

for a constant $T > 0$ and

$$\int_{-1}^{1} \mathrm{Var}_x(\partial_x P_n)\ d\lambda < 2l \int_{[0,1]^2} |\partial_{xy} P_n|\ d\lambda^2.$$

Here Var_x is the pointwise variation along the x-axis for fixed y and, analogously, Var_y is the pointwise variation along the y-axis for fixed x. After a suitable convolution/redefinition around the lines of discontinuity, P_n can be assumed to be differentiable enough while retaining the above inequalities that in turn imply that the inequality (3.6) cannot hold. Assume otherwise. We could choose $l < (4\tilde{C})^{-1}$ where $\tilde{C}$ is the constant in (3.6). On the one hand, we then have

$$\begin{aligned}\int_{[-1,1]^2} |\Delta P_n|\ d\lambda^2 &\leq \int_{[-1,1]^2} |\partial_{xx} P_n| + |\partial_{yy} P_n| d\lambda^2 \\ &\leq 2l \int_{[-1,1]^2} |\partial_{xy} P_n|\ d\lambda^2 + C \\ &\leq \frac{1}{2} \int_{[-1,1]^2} |\Delta P_n| + C.\end{aligned}$$

But on the other hand

$$nT \leq \int_{[-1,1]^2} |\partial_{xy} P_n|\ d\lambda^2 \leq \tilde{C} \int_{[-1,1]^2} |\Delta P_n|$$

holds. This is a contradiction.

To conclude this chapter we now shortly discuss Korn inequality for the case $p = \infty$. For simplicity, we consider an example on $\Omega = B_1(0)$. Suppose $M \in \mathbb{R}^{n\times n}_{skew}$ and let $u(x) := Mx \ln(|x|)$. Notice that u vanishes on the boundary of the unit ball and is bounded. We can compute the gradient and the symmetric gradient explicitly by

$$\nabla u(x) = \frac{1}{|x|^2} Mx \otimes x + M \ln(|x|)$$

and

$$e(u)(x) = \frac{1}{|x|^2}(Mx \odot x).$$

One immediately sees that ∇u is unbounded, but $e(u)$ is bounded. With the notation in (3.5) we therefore deduce that $W^{1,\infty}(\Omega)$ is a proper subspace of $X^\infty(\Omega)$. This means that embedding in Theorem 3.2 is not surjective. With a suitable argument (for instance by a series expansion of the logarithm) we can also approximate u by functions in $W^{1,\infty}(\Omega)$ with respect to the norm on $X^\infty(\Omega)$. In particular, we see that $W^{1,\infty}(\Omega)$ is not closed in $X^\infty(\Omega)$ which implies directly that a Korn inequality like (3.4) cannot happen.

4 The space $BD(\Omega)$

In the previous chapter, we discussed Korn inequalities in the setting of Sobolev spaces. To motivate this discussion we have drawn inspiration from the theory developed to investigate elastic materials. If however materials deform to the point where their internal structures are undergoing permanent changes (for instance if cracks appear) then a theory in Sobolev spaces is too 'smooth'.

Take for instance the space $W^{1,1}_{loc}(\mathbb{R})$. As it is well-known, this space only contains absolutely continuous functions. If one wants to model anything where jumps appear on a function space with values in $\mathbb{R}$ then this is not the right choice. Let us now have a look at the classical example of the Heaviside function. We set

$$H := \chi_{\mathbb{R}^+}.$$

Since H is not continuous (and cannot be made continuous if changed on a set of measure zero) it is no element of any Sobolev space. However, we have $\mathcal{H} \in \mathcal{D}'(\mathbb{R})$ and its distributional derivative can be computed by

$$H' = \delta_0,$$

where δ_0 is the Dirac delta distribution ($\varphi \mapsto \varphi(0)$). Observe that δ_0 can be identified as the probability measure with its mass concentrated at 0. With this heuristic example in mind, the 'natural' approach to model cracks or jumps would be to require the distributional derivative of a function to be a measure. In fact, due to Sobolev functions being absolutely continuous along lines (cf. section 4.9 in [EG15]) in higher dimensions, it is generally reasonable to relax assumptions made on the distributional derivatives so that discontinuities such as jumps can appear.

With this line of thought as a motivation, we will devote this chapter to those L^1-functions whose distributional symmetric gradient is a (finite) Radon measure, the so-called space of *functions of bounded deformation*. This space is denoted by $BD(\Omega)$. First, we will discuss some fundamental results in section 4.1 which can be carried over from the Sobolev setting. Afterwards, we present some of the fine properties of the distributional derivative; most notably, the structure theorem. These properties will set up the discussion about a Korn inequality in a larger space in chapter 5.

4.1 The fundamentals of $BD(\Omega)$

Before starting our discussion we will fix notation as follows: Until now, we denoted the (symmetric) gradient of some function u with ∇u ($e(u)$) and we did not differentiate if ∇u ($e(u)$) exists pointwise, as an L^p function or a distribution. To avoid confusion, we now denote the (symmetric) gradient with ∇u ($e(u)$) only if it exists pointwise. Furthermore, we denote the distributional gradient by $\mathrm{D}u$ and the distributional symmetric gradient $\frac{1}{2}(\mathrm{D}u+\mathrm{D}u^T)$ by $\mathrm{E}u$.

Now we turn our attention to u being in $L^1_{loc}(\Omega,\mathbb{R}^n) \subseteq \mathcal{D}'(\Omega,\mathbb{R}^n) = (C_c^\infty(\Omega,\mathbb{R}^n))'$. In this setting, $\mathrm{E}u$ is well defined as a distribution. As this space of distributions often lacks the properties required for applications, one wants to strengthen the assumptions on $\mathrm{E}u$. A natural approach is to require the distribution to be identifiable with a function, but, as seen in the introduction to this chapter, this approach does not necessarily lead to desired results. Another idea would be to require $\mathrm{E}u$ to be an element of $(C_c(\Omega,\mathbb{R}^{n\times n}))' \subseteq (C_c^\infty(\Omega,\mathbb{R}^{n\times n}))'$. By the Riesz-Markov-Kakutani Theorem 2.3.2 this means $\mathrm{E}u$ is identifiable with a matrix-valued measure. This will be the standing assumption throughout the rest of the chapter. For a short overview of the needed measure theoretic basics, we refer to section 2.3.

Definition 4.1.1. We define the space of *functions of bounded deformation* as

$$BD(\Omega) := \{u \in L^1(\Omega,\mathbb{R}^n) : \mathrm{E}u \in \mathcal{M}_b(\Omega,\mathbb{R}^{n\times n}_{sym})\}$$

and the space of *functions of locally bounded deformation* by

$$BD_{loc}(\Omega) := \{u \in L^1_{loc}(\Omega,\mathbb{R}^n) : \mathrm{E}u \in \mathcal{M}(\Omega,\mathbb{R}^{n\times n}_{sym})\}.$$

Moreover, we denote the (ij)-th component of $\mathrm{E}u$ by $\mathrm{E}_{ij}u$.

Notice that we denote with $\mathrm{E}u$ the measure, but also its corresponding distribution. Furthermore, we observe here two important properties:

- The usual divergence theorem holds for $\varphi \in C^1_c(\Omega,\mathbb{R}^{n\times n}_{sym})$ since

$$\langle \mathrm{E}u,\varphi\rangle = \frac{1}{2}\langle \mathrm{D}u+\mathrm{D}u^T,\varphi\rangle = \frac{1}{2}\langle \mathrm{D}u,(\varphi+\varphi^T)\rangle = \langle \mathrm{D}u,\varphi\rangle = \langle u,\mathrm{div}\varphi\rangle. \tag{4.1}$$

- $BD(\Omega)$ can be defined equivalently by requiring the distributions

$$x\cdot((\mathrm{E}u)x) := \sum_{i,j=1}^n x_i x_j \mathrm{E}_{ij}u$$

  to be in $\mathcal{M}_b(\Omega)$ for all $x \in \mathbb{R}^n$. Notice that $x\cdot Mx = x\cdot M^Tx$ holds for each $x\in\mathbb{R}^n$ and $M\in\mathbb{R}^{n\times n}$. In other words the kernel of the functional $M \mapsto x\cdot Mx$ includes $\mathbb{R}^{n\times n}_{skew}$. In particular, we have

$$x\cdot((\mathrm{E}u)x) = x\cdot((\mathrm{D}u)x). \tag{4.2}$$

  While this seems not important at first this serves as the starting point to the theory presented in section 4.2.

We now turn our discussion to the relevant topologies on $BD(\Omega)$. If $BD(\Omega)$ is endowed with the norm

$$\|u\|_{BD} = \|u\|_1 + \|\mathrm{E}u\|_{\mathcal{M}_b},$$

the corresponding normed space is complete, i.e., it is a Banach space. The proof of this is essentially the same as the one of Lemma 3.2.2.

Here, we also note that there is a strong analogy between the space $BD(\Omega)$ and the space of *functions of bounded variation*

$$BV(\Omega) := \{u \in L^1(\Omega, \mathbb{R}^m) : \mathrm{D}u \in \mathcal{M}_b(\Omega, \mathbb{R}^n)\},$$

which is endowed with the norm

$$\|u\|_{BV} = \|u\|_1 + \|\mathrm{D}u\|_{\mathcal{M}_b}.$$

For a detailed analysis of BV functions see [AFP00].

Most of the fundamental properties of BD functions are reminiscent of the ones that hold in Sobolev and BV spaces. We present an overview in the next paragraphs. We start our discussion with the embedding theorems which hold for BD. Similarly to Sobolev and BV spaces we attain higher integrability by knowing that the symmetric derivative is a measure. More specifically, the following theorem holds:

Theorem 4.1.2. *We have the following continuous embedding:*

$$BD(\mathbb{R}^n) \hookrightarrow L^{n'}(\mathbb{R}^n),$$

where $n' = \frac{n}{n-1}$ is the Hölder conjugate of n.

This theorem is essentially based on a Gagliardo-Nirenberg-Sobolev type inequality first shown by Strauss M.J. in 1973 (cf. [Str73]):

$$\|u\|_{L^{n'}(\mathbb{R}^n)} \leq C \, \|e(u)\|_{BD(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. In the case of a bounded Ω with Lipschitz we even have a Rellich-Kondrachov type result:

Theorem 4.1.3. *[Tem85, Theorem 2.4] Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open with $\partial\Omega$ being Lipschitz. Then the embedding*

$$BD(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous for $1 \leq p \leq n'$. For $1 \leq p < n'$ it is also compact.

In many application one often wants to specify the behaviour of u on the boundary of a bounded, open Ω with Lipschitz boundary. As for Sobolev and BV-functions this can be achieved in BD by means of a trace operator:

Theorem 4.1.4. *[Bab15] Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open with Lipschitz boundary. Then there exists a unique linear and continuous mapping*

$$\gamma : BD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^n; \mathcal{H}^{n-1})$$

such that

$$\gamma(u) = u|_{\partial\Omega}$$

holds for every $u \in C(\overline{\Omega}, \mathbb{R}^n)$. Moreover, the following Gauß-Green formula holds for every $\varphi \in C^1(\overline{\Omega})$:

$$\int_\Omega u \odot \nabla\varphi \; dx + \int_\Omega \varphi \; d(\mathrm{E}u) = \int_{\partial\Omega} \gamma(u) \odot \nu \; \varphi \; d\mathcal{H}^{n-1},$$

where ν is the outer unit normal of $\partial\Omega$.

As a consequence of the trace theorem we see that $BD(\Omega)$ can be continuously embedded into $BD(\mathbb{R}^n)$ if the boundary of Ω is well-behaved. Indeed, define the extension on $\mathbb{R}^n$ of some $u \in BD(\Omega)$ by

$$\hat{u} := \mathbb{1}_\Omega u.$$

Then, we have $\|\hat{u}\|_{L^1(\mathbb{R}^n)} = \|u\|_{L^1(\Omega)}$ and, since by Theorem 4.1.4

$$\int_\Omega (\hat{u} \odot \nabla\varphi)_{ij} dx = \int_\Omega (u \odot \nabla\varphi)_{ij} dx \leq |\mathrm{E}_{ij} u|(\Omega) + \|\gamma(u)\|_{L^1(\partial\Omega)}$$

for all $\varphi \in C_c^1(\mathbb{R}^n)$ with $\|\varphi\|_\infty \leq 1$, we infer that the total variation of every component of the distributional derivative is bounded. Therefore, $\mathrm{E}\hat{u}$ can be identified with a bounded measure, i.e., $\hat{u} \in BD(\mathbb{R}^n)$.

We now consider the relevant notions of convergence for a sequence $\{u_k\} \subseteq BD(\Omega)$:

- *Weak-* convergence:* We write $u_k \overset{*}{\rightharpoonup} u$ in $BD(\Omega)$ for the convergence with respect to the weak-* topology. By Lemma 4.1.9 this holds if and only if

$$u_k \overset{*}{\rightharpoonup} u \text{ in } L^1 \text{ and } \mathrm{E}u_k \overset{*}{\rightharpoonup} \mathrm{E}u \text{ in } \mathcal{M}_b(\Omega). \tag{4.3}$$

  We will later see that for an open, bounded Ω with Lipschitz boundary $BD(\Omega)$ can be characterized as a dual space (cf. Lemma 4.1.9) and therefore can be naturally endowed with a weak-* topology. The weak-* convergence introduced here corresponds with the natural one. By Theorem 4.1.6 the embedding of $BD(\Omega)$ into $L^1(\Omega, \mathbb{R}^n)$ is compact. Therefore, (4.3) is equivalent to

$$u_k \to u \text{ strongly in } L^1 \text{ and } \mathrm{E}u_k \overset{*}{\rightharpoonup} \mathrm{E}u \text{ in } \mathcal{M}_b(\Omega).$$

- *Strict convergence*: We say that $u_k \to u$ strictly in $BD(\Omega)$ if and only if

$$u_k \overset{*}{\rightharpoonup} u \text{ in } BD(\Omega) \text{ and } \|\mathrm{E}u_k\|_{\mathcal{M}_b} \to \|\mathrm{E}u\|_{\mathcal{M}_b}.$$

- *Strong convergence*: We write $u_k \to u$ in $BD(\Omega)$ if u_k converges to u in the norm topology associated with $\|\cdot\|_{BD}$.

As an immediate result we have that:

$$\text{strong convergence} \Rightarrow \text{strict convergence} \Rightarrow \text{weak-* convergence.}$$

Strict convergence can therefore be seen as an intermediate type of topology. One can verify that it corresponds to the convergence in BD endowed with the metric

$$d(u,v) = \|u - v\|_1 + \left| \|\mathrm{E}u\|_{\mathcal{M}_b} - \|\mathrm{E}v\|_{\mathcal{M}_b} \right|.$$

The reason we consider this type of convergence is that the convergence in the norm sense is often too strong and weak-* convergence too weak. For instance, while we can observe that $C^\infty(\overline{\Omega})$ densely embeds in $BD(\Omega)$ with respect to weak-* convergence we cannot hope to achieve this for the norm topology since the closure of $C^\infty(\overline{\Omega})$ with respect to the norm is

$W^{1,1}(\Omega)$. However, for many types of results, one wants to show the continuity of some linear operator $T : BD(\Omega) \to X$ with a normed space X. A typical approach is to show that

$$\|T(u)\|_X \leq C \|u\|_{BD}$$

holds for some constant $C > 0$ and all $u \in C^\infty(\overline{\Omega})$, and then the conclusion would follow by a density argument. But the norm $\| \ \|_{BD}$ is only lower semi-continuous with respect to weak-* convergence. Strict convergence solves this dilemma since we have the following result:

Theorem 4.1.5. *$C^\infty(\overline{\Omega}) \cap BD(\Omega)$ is dense in $BD(\Omega)$ with respect to the strict topology.*

The proof is done via a classical mollification argument (cf. Theorem 3.2 in [Tem85]). Most notably this method is applied in the proofs of Theorem 4.1.3 and Theorem 4.1.4 which also implies that the trace operator and embeddings are continuous with respect to the strict convergence.

Suppose now for a moment that we want to enlarge $BD(\Omega)$, in the sense, in wider generality, we consider distributions $u \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ such that $e(u) \in \mathcal{M}(\Omega, \mathbb{R}^{n\times n})$. For instance, an initial idea could be to study spaces with $u \in \mathcal{M}(\Omega, \mathbb{R}^n)$. From the following regularity result for distributions (cf. Theorem 2.3 [Tem85]), however, we infer that such an enlargement is not possible, in this sense, $BD(\Omega)$ is maximal.

Theorem 4.1.6. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open with Lipschitz boundary and $u \in \mathcal{D}'(\Omega, \mathbb{R}^n)$. If $\mathrm{E}u \in \mathcal{M}(\Omega, \mathbb{R}^{n\times n})$, then $u \in L^1(\Omega)$.*

Similarly we could also ask what conditions should a matrix valued (bounded) measure E fulfil so that it can be written as the symmetric gradient of some $u \in L^1(\Omega)$. The following two lemmas characterize some conditions.

Lemma 4.1.7. *Let $T = (T_{ij}) \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^{n\times n})$ be such that $T_{ij} = T_{ji}$ for all $i, j = 1, .., n$. Then the following statements are equivalent:*

i) There exists $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\mathrm{E}u = T.$$

ii) For any test function $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^{n\times n}_{sym})$ with $\operatorname{div} \varphi = 0$ we have

$$\langle T, \varphi \rangle = 0.$$

iii) T satisfies the Saint-Venant compatibility relations in $\mathcal{D}'(\mathbb{R}^n)$:

$$\partial_{jl} T_{ik} + \partial_{ik} T_{jl} - \partial_{il} T_{jk} - \partial_{jk} T_{il} = 0$$

for all $i, j, k, l = 1, .., n$.

Proof.

i) $\Rightarrow$ *ii)*: Suppose $T = \mathrm{E}u$ for a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$. From (4.1) we can deduce *ii)*.

ii) $\Rightarrow$ *iii)*: Take any $\tilde{\varphi} \in \mathcal{D}(\Omega)$ and fix $i, j, k, l = 1, .., n$. We set

$$\varphi_{ab} := \begin{cases} \partial_{jl}\tilde{\varphi} & \text{for } \{a,b\} = \{i,k\}, \\ \partial_{ik}\tilde{\varphi} & \text{for } \{a,b\} = \{j,l\}, \\ -\partial_{il}\tilde{\varphi} & \text{for } \{a,b\} = \{j,k\}, \\ -\partial_{jk}\tilde{\varphi} & \text{for } \{a,b\} = \{i,l\}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that per definition $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^{n\times n}_{sym})$. We also observe that div $\varphi = 0$ holds. By assumption we therefore derive

$$2\langle \partial_{jl}T_{ik} + \partial_{ik}T_{jl} - \partial_{il}T_{jk} - \partial_{jk}T_{il}, \tilde{\varphi}\rangle = \langle T, \varphi\rangle = 0.$$

iii) $\Rightarrow$ *i)*: For the last implication we follow the proof of Theorem 4.1 in [Cia10]. We set

$$h_{ijk} := \partial_j T_{ik} - \partial_i T_{jk}.$$

The assumptions imply

$$\partial_l h_{ijk} = \partial_k h_{ijl}$$

for all i, j, k, l. Therefore, the Poincaré lemma for distributions (cf. for instance Proposition 9 in [Hor66]) tells us that for each i, j there exist $p_{ij} \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\partial_k p_{ij} = h_{ijk}$$

holds for each k. Hence, we observe that

$$\mathrm{D}(p_{ij} + p_{ji}) = 0.$$

After changing p up to a constant we therefore can assume $p = -p^T$. We set now $q = T + p$. After computing

$$\partial_k q_{ij} = \partial_j q_{ik}$$

for all i, j, k we can again apply the Poincaré lemma to derive the existence of $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\mathrm{D}u = T + p.$$

By construction we have

$$\mathrm{E}u = \frac{1}{2}\left(T + T^T + p + p^T\right) = T.$$

This concludes the proof. □

Notice that the distribution associated to a finite measure $\mu \in \mathcal{M}_b(\Omega)$ is extendable to $\mathcal{D}'(\mathbb{R}^n)$. Therefore, in combination with Lemma 4.1.6 we can derive the following result:

Proposition 4.1.8. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open with Lipschitz boundary and $T = (T_{ij}) \in \mathcal{M}_b(\Omega, \mathbb{R}^{n\times n})$ such that $T_{ij} = T_{ji}$ for all $i, j = 1, .., n$. Then the following statements are equivalent:*

i) There exists $u \in L^1(\Omega, \mathbb{R}^n)$ such that

$$\mathrm{E}u = T,$$

i.e., $u \in BD(\Omega)$.

ii) For any test function $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^{n\times n}_{sym})$ with div $\varphi = 0$ *we have*

$$\langle T, \varphi \rangle = 0.$$

As an immediate consequence of either Theorem 4.1.6 or Proposition 4.1.8 one can now derive that $BD(\Omega)$ can be written as a dual if the boundary is Lipschitz.

Lemma 4.1.9. *$BD(\Omega)$ can be identified as the dual of X/X_0 endowed with the quotient topology where*

$$X := C_c(\Omega, \mathbb{R}^n) \times C_c(\Omega, \mathbb{R}^{n\times n}_{sym})$$

and

$$X_0 := \{(g, h) \in X : g = \operatorname{div} h\}$$

are endowed with their respective product topology. Here we set $\operatorname{div} h := (\sum_{j=1}^n \partial_j h_{ij})_i$.

Proof. We will show that $BD(\Omega)$ is isomorphic to the annihilator $X_0^\perp \cong X/X_0$. The isomorphism can be given explicitly by $T := (u \mapsto T_u)$ where for $u \in BD(\Omega)$ we set T_u as the map

$$\begin{cases} X \to \mathbb{R} \\ (g, h) \mapsto \langle u, g\rangle + \langle \mathrm{E}u, h\rangle. \end{cases}$$

Notice, naturally $T_u \in X'$ and $T_u \in X_0$ holds. Furthermore, notice that $T_u = 0$ implies $u = 0$ because of Riesz theorem, so T is one-to-one.

Conversely, let now be $(\mu, \eta) \in X_0^\perp \subseteq X' = \mathcal{M}_b(\Omega, \mathbb{R}^n) \times \mathcal{M}_b(\Omega, \mathbb{R}^{n\times n}_{sym})$. For $f \in C_c^\infty(\Omega, \mathbb{R}^{n\times n}_{sym})$ we have $(\operatorname{div} f, f) \in X_0$ and therefore

$$\langle \mathrm{E}\mu, f\rangle = -\langle \mu, \operatorname{div} f\rangle = \langle \eta, f\rangle.$$

In particular, the distributional symmetric gradient is a measure which implies $\mu \in L^1(\Omega, \mathbb{R}^n)$ by Theorem 4.1.6. We have shown: $T_\mu = (\mu, \eta)$, i.e., $(u \mapsto T_u)$ is onto. Naturally, we also have

$$\|T_u\|_{X'} \le \|u\|_{BD}.$$

This means T is a continuous bijection. From Theorem 3.2.1 we derive that T is also a isomorphism. □

In the last chapter we saw that a Korn inequality of the type

$$\|u - Pu\|_{1,p} \le C \|e(u)\|_p$$

holds for $u \in W^{1,p}(\Omega, \mathbb{R}^n)/\ker \mathrm{E}$ with P being the projection onto $\ker \mathrm{E} \subseteq W^{1,p}(\Omega, \mathbb{R}^n)$ (cf. Theorem 3.2.8). We have seen that the kernel of the symmetric gradient on Sobolev spaces can be characterized as the space of rigid displacements. More generally, we can identify the kernel of the symmetric gradient on the space of distributions:

Lemma 4.1.10. *Let Ω be open and connected. The kernel of the symmetric gradient on $\mathcal{D}'(\Omega, \mathbb{R}^n)$ is the space of distributions that can be identified with a rigid displacement.*

Proof. We present the elegant proof from Lemma 2.1 in [DR19]. Let $T \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ with $\mathrm{E}T = 0$. Define the skew-symmetric part of the gradient by

$$w(T) = \frac{1}{2}(\mathrm{D}T - (\mathrm{D}T)^T).$$

Notice now that

$$\begin{aligned} 2\partial_k(w(T))_{ij} &= \partial_{kj}T_i - \partial_{ki}T_j \\ &= \partial_j(\mathrm{E}T)_{ik} - \partial_i(\mathrm{E}T)_{kj} \\ &= 0. \end{aligned}$$

In particular, $\mathrm{D}(w(T))_{ij} = 0$. This now implies that $w(T)$ can be identified with a constant $C \in \mathbb{R}^{n\times n}$. Therefore, $\mathrm{D}T = \mathrm{E}T + w(T) \equiv C$. As a consequence, T can be identified with an affine transformation $Cx + b$ for some $b \in \mathbb{R}^n$. Since $0 = \mathrm{E}T \equiv \frac{1}{2}(C + C^T)$ holds we also have that $C = -C^T$, i.e., T can be identified with a rigid displacement. □

This now in turn implies that we can identify the kernel of the symmetric gradient on $BD(\Omega)$ with the space of rigid displacements like in the L^p case. With this we can now show a Korn-Poincaré type inequality.

Theorem 4.1.11. *Let Ω be bounded, open with Lipschitz boundary. Then we have*

$$\|u - Pu\|_{BD} \leq C \|\mathrm{E}u\|_{\mathcal{M}_b} \tag{4.4}$$

for all $u \in BD(\Omega)$. P denotes the projection onto $\ker \mathrm{E}$.

Proof. The proof of this is similar to Lemma 3.2.8. The only difference: we cannot show that the factor norm of $BD(\Omega)/\ker \mathrm{E}$ is equivalent to the total variation norm via Korn inequality. By Lemma 4.1.10, however, we know that

$$\|u\|_{BD}^{\sim} := \|\mathrm{E}u\|_{\mathcal{M}_b}$$

induces a norm on $BD(\Omega)/\ker \mathrm{E}$. We will therefore use a strategy similar to the original proof of Korn inequality (3.4). First we argue that $BD(\Omega)/\ker \mathrm{E}$ endowed with the total variation norm is complete.

Take a Cauchy sequence $\{u_m\}$ with respect to the total variation norm in $BD(\Omega)/\ker \mathrm{E}$. Then $\{\mathrm{E}u_m\}$ is a Cauchy sequence in $\mathcal{M}_b(\Omega, \mathbb{R}^{n\times n}_{sym})$ and by the completeness of $\mathcal{M}_b$ it converges to some E. Now we notice

$$\mathrm{E}u_m \xrightarrow{\mathcal{M}_b} E \;\Rightarrow\; \mathrm{E}u_m \rightharpoonup^* E \;\Rightarrow\; \mathrm{E}u_m \xrightarrow{\mathcal{D}'} E.$$

Notice that the distributions $\mathrm{E}u_m$ fulfil the condition $ii)$ from Proposition 4.1.8. These are preserved under convergence in $\mathcal{D}'$ and therefore also hold for E. From this we infer that there exists $u \in BD(\Omega)$ such that $\mathrm{E}u = E$. Notice that u is unique up to a rigid displacement, i.e., u is a unique element in $BD(\Omega)/\ker \mathrm{E}$.

We now notice that by definition of the factor norm $\|\ \|_{\sim}$ the inclusion

$$(BD(\Omega)/\ker \mathrm{E}, \|\ \|_{\sim}) \hookrightarrow (BD(\Omega)/\ker \mathrm{E}, \|\ \|_{BD}^{\sim})$$

is continuous. Therefore, we can apply Theorem 3.2.1 to deduce the continuity of the inverse. From this we can infer that there exists a constant $C > 0$ such that

$$\|u\|_{\sim} \leq C \|\mathrm{E}u\|_{\mathcal{M}_b}$$

which in turn implies (4.4). □

4.2 Fine properties of $BD(\Omega)$

All of the statements we have presented up to now are results which originated from the study of $BV(\Omega)$. The proof structures of their counterparts in $BD(\Omega)$ version are very similar to the original ones. We now want to discuss some of the fine properties of $BD(\Omega)$. The result that we are mainly interested in is the structure theorem. Its equivalent in $BV(\Omega)$ has been known for a long time. However, in this case, the approach heavily relies on the BV-coarea formula (cf. Theorem 3.40 in [AFP00]). This cannot be replicated since no coarea formula holds in $BD(\Omega)$. However, most of the parts of this famous theorem have been recovered by now via a different method, namely via a slicing argument. We will now first state the definitions of the terms used in the structure theorem for $BD(\Omega)$. Then, we shortly discuss the concepts behind the proof.

Definition 4.2.1. The *jump set* J_u is the set of all points $x \in \Omega$ such that (different) one-sided Lebesgue limits exists, i.e., there exist $u^{\pm}(x) \in \mathbb{R}^n$ with $u^+(x) \neq u^-(x)$ and $\nu_u(x) \in \mathcal{S}^{n-1}$ such that

$$\lim_{r\to 0^+} \frac{1}{r^n} \int_{B_r^{\pm}(x)} |u(y) - u^{\pm}(x)| = 0$$

where $B_r^{\pm}(x) = \{y \in B_r(x) : (y - x) \cdot \nu_u(x)\}$. We call $\nu_u(x)$ the (measure theoretic) unit normal at x. Furthermore, we set $[u] := u^+ - u^-$.

Notice that the triplet $(u^+(x), u^-(x), \nu_u(x))$ is unique up to a sign change of $\nu_u(x)$ and a permutation of $u^+(x)$ and $u^-(x)$. When talking about approximate normals and tangent planes we also need to introduce the concept of $\mathcal{H}^{n-1}$-rectifiable sets:

Definition 4.2.2. Let $B \in \mathcal{B}$ a Borel set. It is called countable $\mathcal{H}^{n-1}$-rectifiable if and only if there exist countable many $f_i \in C^1(\mathbb{R}^{n-1}, \mathbb{R}^n)$ such that

$$\mathcal{H}^{n-1}(B \setminus \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^{n-1})) = 0.$$

For more detailed analysis about rectifiable sets we refer to chapter 10 in [Mag12]. The most important feature about them is the existence of approximate tangent planes, i.e., if B is countable $\mathcal{H}^{n-1}$-rectifiable then for $\mathcal{H}^{n-1}$-a.e. $x \in B$ there exists a hyperplane π such that

$$\mathcal{H}^{n-1}|_{\frac{B-x}{r}} \rightharpoonup^* \mathcal{H}^{n-1}|_{\pi}$$

for $r \to 0^+$. As a consequence, we have that

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{n-1}(B \cap B_r(x))}{\gamma_{n-1} r^{n-1}} = 1, \tag{4.5}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in B$ (cf. Theorem 10.2 [Mag12]). We now state the structure theorem for $BD(\Omega)$ (cf. [ACD97] and Theorem 1.1. in [DR19]):

Theorem 4.2.3. *Let $u \in BD_{loc}(\Omega)$. Then we have*

$$\mathrm{E}u = \mathrm{E}^a u + \mathrm{E}^j u + \mathrm{E}^c u$$

where

- $\mathrm{E}^a u$ *is the absolutely continuous with respect to the Lebesgue measure λ^n. Its density with respect to λ^n is a L^1_{loc} function and we denote it by $\mathcal{E}u$. Furthermore, for λ^n-a.e. $x \in \Omega$ we have*

$$\lim_{r\to 0} \frac{1}{\lambda^n(B_r(x))} \int \frac{|(u(y)-u(x))\cdot(y-x) - \mathcal{E}u(x)(y-x)\cdot(y-x)|}{|y-x|^2} = 0. \tag{4.6}$$

- $\mathrm{E}^j u$ *is called the jump part of Eu. Moreover, $\mathrm{E}^j u$ is concentrated on the jump set J_u which is countable $\mathcal{H}^{n-1}$-rectifiable and we have*

$$\mathrm{E}^j u = [u] \odot \nu_{J_u} \mathcal{H}^{n-1}|_{J_u},$$

where $u^{\pm}$ and ν are the functions from Definition 4.2.1.

- $\mathrm{E}^c u$ *is called the Cantor part. It vanishes on every $\mathcal{H}^{n-1}$ σ-finite set.*

Sketch of the proof. The strategy is the following: by the Lebesgue-Besicovitch differentiation theorem the measure Eu can be decomposed into

$$\mathrm{E}u = \mathrm{E}^a u + \mathrm{E}^s u$$

where $\mathrm{E}^a u$ is the absolutely continuous part with respect to the Lebesgue measure and E^s is the singular part. Furthermore, the *approximate symmetric gradient* $\mathcal{E}u$ is well-defined λ^n-a.e. by

$$\mathcal{E}u(x) = \lim_{r\to 0} \frac{|\mathrm{E}u|(B_r(x))}{\lambda^n(B_r(x))}.$$

Formula (4.6) can then be shown for all $x \in \Omega$ which are Lebesgue points of both u and $\mathcal{E}u$ (cf. Theorem 4.3 in [ACD97]).

The singular part of Eu can now be further decomposed into

$$\mathrm{E}^s u = \mathrm{E}^j u + \mathrm{E}^c u$$

where we set $\mathrm{E}^j u := \mathrm{E}^s u|_{J_u}$ and $\mathrm{E}^c u := \mathrm{E}^s u|_{\Omega \setminus J_u}$. One can show that the jump set J_u is countable $\mathcal{H}^{n-1}$-rectifiable. To show this, one generally wants to prove that the set of points with $(n-1)$-dimensional density

$$\left\{ x \in \Omega : \limsup_{r\to 0^+} \frac{|\mathrm{E}u|(B_r(x))}{r^{n-1}} > 0 \right\}$$

is $\mathcal{H}^{n-1}$-rectifiable (cf. Proposition 3.5 in [ACD97]) since this set contains J_u.

Now one observes the following: For a Lipschitz (or C^1) hyper surface $S \subseteq \Omega$ we can consider $\Omega \setminus S$. Since the now discussed property is local we can assume that S cuts Ω into two distinct

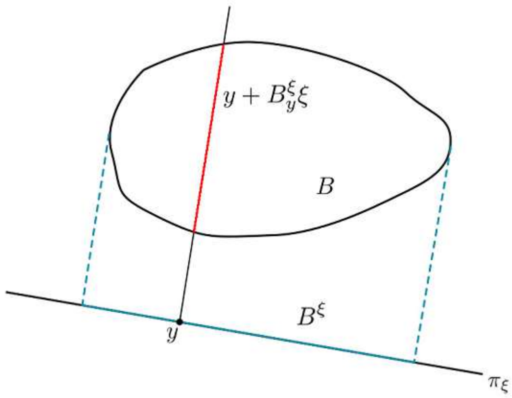

Figure 4.1: Slicing method (4.7) applied to a Borel set B.

open sets Ω^+, Ω^-. By the trace theorem there exist traces u^+ and u^- such that for every $\varphi \in C_c^1(\Omega)$

$$\int_{\Omega^+} u \odot \nabla\varphi \, dx + \int_{\Omega^+} \varphi \, d(\mathrm{E}u) = \int_S u^+ \odot \nu_S \, \varphi \, d\mathcal{H}^{n-1}$$

and

$$\int_{\Omega^-} u \odot \nabla\varphi \, dx + \int_{\Omega^-} \varphi \, d(\mathrm{E}u) = \int_S u^- \odot (-\nu_S) \, \varphi \, d\mathcal{H}^{n-1}$$

hold with ν_S being the unit normal pointing in Ω^+. Adding both terms together and since

$$\int_\Omega u \odot \nabla\varphi \, dx + \int_\Omega \varphi \, d(\mathrm{E}u) = 0$$

holds, we then have

$$-\int_S \varphi \, d(\mathrm{E}u) = \int_S (u^+ - u^-) \odot \nu_S \, \varphi \, d\mathcal{H}^{n-1}.$$

By a density argument we conclude $\mathrm{E}u|_S = [u] \odot \nu \, d\mathcal{H}^{n-1}$. Since J_u is countable $\mathcal{H}^{n-1}$-rectifiable a covering argument can be employed to see that

$$\mathrm{E}^s u|_{J_u} = [u] \odot \nu \, d\mathcal{H}^{n-1}.$$

Before discussing the last point we turn now to the slicing method generally used in the discussion around $BD(\Omega)$ in [ACD97]. For a Borel set $B \subseteq \Omega$, $y \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$ denote

$$\begin{aligned} B_y^\xi &:= \{t \in \mathbb{R} : y + t\xi \in B\} \\ B^\xi &:= \{x \in \pi_\xi : B_x^\xi \neq \emptyset\} \end{aligned} \tag{4.7}$$

where π_ξ denotes the hyperplane $\{\xi\}^\perp$ (cf. Figure 4.1). For a function $u : \Omega \to \mathbb{R}^n$ we then define

$$u_y^\xi(t) := u(x + t\xi) \text{ and } \hat{u}_y^\xi(t) := \xi \cdot u(x + t\xi)$$

on the segment Ω_y^ξ. Since $u \in BD(\Omega) \subseteq L^1(\Omega, \mathbb{R}^n)$ Fubini's theorem tells us that these restriction on segments are well defined as elements of $L^1(\Omega_y^\xi)$ at least for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$. The following theorem asserts that the structure of $\mathrm{E}u$ is inherited by these one-dimensional restrictions (cf. Proposition 3.2. and Theorem 4.5 in [ACD97]):

Theorem 4.2.4. *[ACD97] Let $u \in BD(\Omega)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Then the following holds:*

- *For $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ we have $\hat{u}_y^\xi \in BV(\Omega)$ and*
$$\xi \cdot (\mathrm{E}u)\xi = \int_{\Omega^\xi} \mathrm{D}\hat{u}_y^\xi \; d\mathcal{H}^{n-1}(y)$$
and
$$|\xi \cdot (\mathrm{E}u)\xi| = \int_{\Omega^\xi} |\mathrm{D}\hat{u}_y^\xi| \; d\mathcal{H}^{n-1}(y).$$

- *We have a one to one relation between absolute continuous, jump and Cantor parts of $\xi \cdot (\mathrm{E}u)\xi$ and Du_y^ξ*
$$\xi \cdot \mathrm{E}u^a\xi = \int_{\Omega^\xi} (\mathrm{D}\hat{u}_y^\xi)^a \; d\mathcal{H}^{n-1}(y),$$
$$|\xi \cdot \mathrm{E}u^a\xi| = \int_{\Omega^\xi} |(\mathrm{D}\hat{u}_y^\xi)^a| \; d\mathcal{H}^{n-1}(y),$$
etc.

- *For $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$*
$$\xi \cdot \mathcal{E}u(y + t\xi)\xi = \nabla \hat{u}_y^\xi(t)$$
holds for λ^1-a.e. $t \in \Omega_y^\xi$.

- *Let*
$$J_u^\xi := \{x \in J_u : \xi \cdot [u](x) \neq 0\}.$$
For $y \in \Omega^\xi$ $\mathcal{H}^{n-1}$-a.e. we have $(J_u^\xi)_y^\xi = J_{\hat{u}_y^\xi}$,
$$\xi \cdot u^+(y + t\xi) = (\hat{u}_y^\xi)^+(t)$$
and
$$\xi \cdot u^-(y + t\xi) = (\hat{u}_y^\xi)^-(t)$$
holds for $t \in (J_u^\xi)_y^\xi$ λ^1-a.e. The normals are oriented such that $\nu_u(y + t\xi) \cdot \xi > 0$ and $\nu_{\hat{u}_y^\xi}(t) = 1$.

Here,
$$\int_{\Omega^\xi} \mathrm{D}\hat{u}_y^\xi \; d\mathcal{H}^{n-1}(y)$$
denotes the measure
$$\mu(B) = \int_{\Omega^\xi} \mathrm{D}\hat{u}_y^\xi(B_y^\xi) \; d\mathcal{H}^{n-1}(y).$$

As an immediate consequence, we have that for a Borel set $B \subseteq \Omega$
$$\mathcal{H}^{n-1}(B) = 0 \;\Rightarrow\; |\xi \cdot (\mathrm{E}u)\xi|(B) = 0 \text{ for every } \xi \in \mathbb{R}^n \;\Rightarrow\; |\mathrm{E}u|(B) = 0$$
holds, i.e., $|\mathrm{E}u| \ll \mathcal{H}^{n-1}$. As a consequence the last point of the Theorem 4.2.3 holds true. (cf. Proposition 4.4 in [ACD97]). □

Note that it is well-known that the set of all singular points S_u for $u \in BV(\Omega)$, so the points which are not Lebesgue points can be seen to fulfil (cf. Theorem 3.78 [AFP00])

$$\mathcal{H}^{n-1}(S_u \setminus J_u) = 0.$$

If this result holds in $BD(\Omega)$ is still an open question. But it is known that

$$\mathcal{H}^{n-1+\epsilon}(S_u \setminus J_u) = 0$$

for any $\epsilon > 0$ (cf. Remark 6.2 in [ACD97]) and

$$|\mathrm{E}v|(S_u \setminus J_u) = 0$$

for all $v \in BD(\Omega)$.

To conclude this chapter, we will now shortly discuss the approximate differentiability of u. For each $\epsilon > 0$, $u \in L^1_{\text{loc}}(\Omega, \mathbb{R})$ and $z \in \mathbb{R}^n$ we set

$$\Omega^z_\epsilon := \{y \in \Omega : |u(y) - z| > \epsilon\}.$$

If the density of Ω^z_ϵ with respect to λ^n vanishes at $x \in \mathbb{R}^n$, i.e.,

$$\lim_{r \to 0^+} \frac{\lambda^n(\Omega^z_\epsilon \cap B_r(x))}{\lambda^n(B_r(x))} = 0$$

holds for all $\epsilon > 0$, we define the *approximate limit of u at x*

$$\operatorname{ap}\lim_{y \to x} u(y) = z.$$

Furthermore, we say that $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ is approximately differentiable at $x \in \mathbb{R}^n$ if there exists $M \in \mathbb{R}^{m \times n}$ with

$$\operatorname{ap}\lim_{y \to x} \frac{|u(y) - u(x) - M(y - x)|}{|y - x|} = 0.$$

$\nabla u(x) := M$ is called the *approximate gradient* at x. We abuse notation here and also denote the approximate gradient with ∇.

Observe that from Ornstein's non-inequality (cf. Theorem 3.2.10) we can infer that the inclusion $BV(\Omega) \hookrightarrow BD(\Omega)$ is strict. Fascinatingly, we can still say that a function in $BD(\Omega)$ admits an approximate gradient (cf. Theorem 7.4 in [ACD97]):

Theorem 4.2.5. *Let $u \in BD(\mathbb{R}^n)$. Then for λ^n-a.e. $x \in \mathbb{R}^n$ the approximate gradient exists. Moreover,*

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r(x)} \frac{|u(y) - u(x) - \nabla u(x)(y - x)|}{r} \, dy = 0$$

holds for λ^n-a.e. $x \in \mathbb{R}^n$. Additionally, for some $C > 0$ (which does not depend on u) the weak L^1 type estimate

$$\lambda^n(\{x \in \mathbb{R}^n : |\nabla u(x)| > t\}) \leq \frac{C}{t} |\mathrm{E}u|(\mathbb{R}^n)$$

holds for every $t > 0$.

5 Korn inequalities in $GSBD(\Omega)$

In the previous chapter, we have already seen that a Korn inequality of the type (cf. Theorem 4.1.11)

$$\|u - Pu\|_{L^1(\Omega)} \leq C\, \|u - Pu\|_{BD(\Omega)} \leq C\, \|\mathrm{E}u\|_{\mathcal{M}_b} \tag{5.1}$$

holds for all $u \in BD(\Omega)$. However, to apply such an inequality for existence results in some specific variational models which depend on so-called *Griffith* energy

$$\int_\Omega |\mathcal{E}u|^p dx + \mathcal{H}^{n-1}(J_u),$$

we would rather have the right side only depend on the absolutely continuous part of Eu. Since the Cantor part of the derivative is hard to control this naturally leads one to consider the space *special functions of bounded deformation*

$$SBD(\Omega) = \{u \in BD(\Omega) : \mathrm{E}u^c = 0\}.$$

It turns out that we can still derive such an inequality for $u \in SBD(\Omega)$ if we cut out a set $\omega \subseteq \Omega$ on the left-hand side of (5.1) which is relatively small compared to the jump set. However, while studying the particular setting surrounding the Korn inequality and $BD(\Omega)$ it turned out that the many results can be transferred to a larger space called the space of *generalized function of bounded deformation* $GBD(\Omega)$ and $GSBD(\Omega)$. This space is based on the slicing methods from section 4.2. In section 5.1, we will introduce the theory about these spaces where we will present results from [Dal13]. One important feature of a function $u \in GBD(\Omega)$ is the existence of an approximate symmetric gradient, i.e., a function $\mathcal{E}u \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$ such that

$$\lim_{r\to 0} \frac{1}{\lambda^n(B_r(x))} \int \frac{|(u(y)-u(x))\cdot(y-x) - \mathcal{E}u(x)(y-x)\cdot(y-x)|}{|y-x|^2} = 0. \tag{5.2}$$

holds for λ^n-a.e. $x \in \Omega$. This leads to consider inequalities of type

$$\|u - a\|_{L^p(\Omega\setminus\omega)} \leq C\, \|\mathcal{E}u\|_{L^p(\Omega)} \tag{5.3}$$

with $p \geq 1$ and a suitable rigid displacement a. We will discuss this inequality for Ω being a cube in section 5.2 where we will present the original proof from [CCF14] for $SBD(\Omega)$ adapted to $GSBD(\Omega)$. As a consequence of this Poincaré-Korn inequality, we will derive a approximation result (cf. Theorem 5.3.5) in section 5.3 based on the work of [CCI19]. This result will serve as the starting point when discussing the main theorem of this chapter which we will formulate for $SBD(\Omega)$ for now (cf. Theorem 4.5 in [CCS22]):

Theorem 5.0.1. *Let $n \in \mathbb{N}, n \geq 2$, $p \in (1,\infty)$, and $\Omega \subseteq \mathbb{R}^n$ be bounded, open, connected with Lipschitz boundary. Then there exists a constant $C > 0$ only dependent on n, p and Ω*

such that for any $u \in SBD(\Omega)$ with $\mathcal{E}u \in L^p(\Omega)$ exists a set of finite perimeter $\omega \subseteq \Omega$ with $\mathcal{H}^{n-1}(\partial^*\omega) \leq C\mathcal{H}^{n-1}(J_u)$ and an affine function a such that

$$\|\nabla u - \nabla a\|_{L^p(\Omega\setminus\omega)} \leq C\,\|\mathcal{E}u\|_{L^p(\Omega)}\,. \tag{5.4}$$

Here, ∇u denotes the approximate gradient of a BD function u.

In section 5.4 we will discuss the preliminaries to this theorem. More specifically, we follow along the work of F. Cagnetti, A. Chambolle and L. Scardia in [CCS22] where this Korn inequality is first proved.

5.1 The spaces $GBD(\Omega)$ and $GSBD(\Omega)$

At the end of the last chapter, we presented results which can be proven purely by means of slicing arguments. This raises the question of whether these results can be extended to a larger function space defined in terms of slicing. Suppose, for instance, that we have a Lebesgue measurable function $u : \Omega \to \mathbb{R}^n$. We now apply the slicing approach, i.e., we consider $\hat{u}_y^\xi$ on the ray Ω_y^ξ for a slicing direction $\xi \in \mathcal{S}^{n-1}$ and $y \in \pi^\xi$. Assume that $\hat{u}_y^\xi \in BV_{loc}(\Omega_y^\xi)$. The situation now is comparable to the one from theorem 4.2.3. At this point, the question is raised what additional conditions are sufficient to transfer and extend the results to functions of this type. It turns out that it is crucial that there exists a $\lambda \in \mathcal{M}_b^+(\Omega)$ such that for all directions $\xi \in \mathcal{S}^{n-1}$ and all Borel sets $B \subseteq \Omega$ we have

$$\int_{\pi^\xi} |D\hat{u}_y^\xi|(B_y^\xi \setminus J^1_{\hat{u}_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{\hat{u}_y^\xi})\; d\mathcal{H}^{n-1}(y) \leq \lambda(B)$$

where $J_v^1 := \{x \in J_v : |[v](x)| \geq 1\}$ with $|[v](x)| = |v^+(x) - v^-(x)|$ being the jump height of a Lebesgue measurable function v at its approximate jump points $x \in J_v$. Due to the structure theorem 4.2.3 $u \in BD(\Omega)$ naturally fulfils this condition with $\lambda = |\mathrm{E}u|$ (also compare remark 4.5 in [Dal13]). This line of thought is fundamentally the motivation for the definition of the spaces $GBD(\Omega)$ and $GSBD(\Omega)$. Before defining $GBD(\Omega)$ we state the following theorem :

Theorem 5.1.1. *[Dal13, Theorem 3.5] Let $\Omega \subseteq \mathbb{R}^n$ be open, $u : \Omega \to \mathbb{R}$ be Lebesgue measurable, $\xi \in \mathbb{R}^n \setminus \{0\}$. We set*

$$\mathcal{T} := \left\{\tau \in C^1\left(\mathbb{R}, \left[-\frac{1}{2}, \frac{1}{2}\right]\right) : \tau' \in C(\mathbb{R}, [0,1])\right\}.$$

For $\lambda \in \mathcal{M}_b^+(\Omega)$ the following two conditions are equivalent:

1. *For every $\tau \in \mathcal{T}$ we have $D_\xi(\tau(u)) \in \mathcal{M}_b(\Omega)$ and*

   $$|D_\xi(\tau(u))|(B) \leq \lambda(B)$$

   holds for every Borel set $B \subseteq \Omega$.

2. *For $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$ we have $u_y^\xi \in BV_{loc}(\Omega_y^\xi)$ and*

   $$\int_{\pi^\xi} |Du_y^\xi|(B_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{u_y^\xi})\; d\mathcal{H}^{n-1}(y) \leq \lambda(B)$$

   holds for every Borel set $B \subseteq \Omega$.

Definition 5.1.2. The space of *generalized functions of bounded deformation* $GBD(\Omega)$ is the space of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}^n$ for which there exists a positive, bounded Radon measure $\lambda \in \mathcal{M}_b^+(\Omega)$ such that for every directions $\xi \in \mathcal{S}^{n-1}$ one of the two equivalent conditions of theorem 5.1.1 holds for $\hat{u} := u \cdot \xi$ with direction $\xi \in \mathcal{S}^{n-1}$.
By the second condition of 5.1.1 we have $\hat{u}_y^\xi = u_y^\xi \cdot \xi \in BV_{loc}$ for every $\xi \in \mathcal{S}^{n-1}$. We can therefore define the space of *generalized special functions of bounded deformation* by

$$GSBD(\Omega) := \left\{ u \in GBD(\Omega) \mid \forall \xi \in \mathcal{S}^{n-1} : \hat{u}_y^\xi \in SBV_{loc}(\Omega_y^\xi) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \pi_\xi \right\}.$$

Similarly, one can define the spaces $GBV(\Omega)$ and $GSBV(\Omega)$ by requiring the conditions of theorem 5.1.1 of u to hold for every direction $\xi \in \mathcal{S}^{n-1}$. For completeness, note that $BD(\Omega)$ can also be defined similarly by a slicing argument:

Proposition 5.1.3. *[ACD97, Proposition 3.2] Let Ω be open. Suppose that $u \in L^1(\Omega)$ and that for all $\xi \in \mathcal{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$ we have $u_y^\xi \in BV_{loc}(\Omega_y^\xi)$ and*

$$\int_{\pi^\xi} |Du_y^\xi|(\Omega_y^\xi)\, d\mathcal{H}^{n-1}(y) < +\infty$$

holds. Then $u \in BD(\Omega)$.

We now list various results regarding the jump set, the existence of traces and approximate derivatives which are analogous to the ones from the last section. We first start with a well-known local trace theorem similar to the result for functions bounded deformation in section 4.1:

Theorem 5.1.4. *[Dal13, Theorem 5.1] Let $\xi \in \mathcal{S}^{n-1}$, $B \subseteq \pi_\xi$ be a relative open ball in the plane, $a, b \in \mathbb{R}$, $a < b$, and $\psi : \overline{B} \to (a, b)$ be Lipschitz continuous. Furthermore, define the open sets $U, V \subseteq \mathbb{R}^n$ by*

$$U := \{y + t\xi : y \in B, a < t < \psi(y)\},$$
$$V := \{y + t\xi : y \in B, t \in (a, b)\}.$$

Now let $v \in L^1(\Omega)$ with $D_\xi v \in \mathcal{M}_b(\Omega)$, set

$$M := \{y + \psi(y)\xi : y \in B\}$$

and let ν the outer unit normal to M. Then, there exists a trace $v_M \in L^1(M; \mathcal{H}^{n-1})$ such that for every $\varphi \in C_c^1(V)$ the generalized Gauss formula

$$\int_U v D_\xi \varphi\, dx + \int_U \varphi\, d(D_\xi v) = \int_M \varphi v_M \xi \cdot \nu\, d\mathcal{H}^{n-1}$$

holds. We also have for $\mathcal{H}^{n-1}$-a.e. $x \in M$

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r(x) \cap U} |v(z) - v_M(x)| dz = 0$$

and for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$

$$v_M(x + \psi(y)\xi) = \operatorname*{ap\,lim}_{t \to \psi(y)^-} v_y^\xi(t).$$

For $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary we can apply this theorem (locally) to $v = \tau(u \cdot \xi)$ with $u \in GBD(\Omega)$, $\xi \in \mathcal{S}^{n-1}$ and $\tau \in \mathcal{T}$. This guarantees the existence of a trace for v. Choosing a suitable $\tau \in \mathcal{T}$ such that τ is invertible this trace theorem can then be transferred to $GBD(\Omega)$ (cf. Theorem 5.4 in [Dal13] for more details regarding this argument):

Theorem 5.1.5. *[Dal13, Theorem 5.5] Let $\Omega \subseteq \mathbb{R}^d$ be bounded with Lipschitz boundary and ν be the corresponding outward unit normal. Then for every $u \in GBD(\Omega)$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ there exist a $\mathcal{H}^{n-1}$-measurable function*

$$u_{\partial\Omega} : \partial\Omega \to \mathbb{R}^n$$

such that

$$\operatorname*{ap\,lim}_{\substack{y \to x \\ y \in \Omega}} u(y) = u_{\partial\Omega}(x). \tag{5.5}$$

Moreover, fix any $\xi \in \mathcal{S}^{n-1}$ and denote with $\sigma : \partial\Omega \to \{-1, 0, 1\}$ the sign of $\xi \cdot \nu$. We have for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$ and every $t \in (\partial\Omega)^\xi_y$

$$\operatorname*{ap\,lim}_{\substack{s \to t \\ \sigma^\xi_y(t)(s-t)>0}} u^\xi_y(s) = u_{\partial\Omega}(y + t\xi) \cdot \xi.$$

As a direct consequence we will present a helpful lemma about glueing two $GBD(\Omega)$ functions together:

Lemma 5.1.6. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open, bounded with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\gamma \subseteq \partial\Omega_1 \cup \partial\Omega_2$ being Lipschitz. Set*

$$w := \begin{cases} u, & \text{in } \Omega_1, \\ v, & \text{in } \Omega_2, \\ \text{arbitrary}, & \text{on } \gamma. \end{cases}$$

Suppose the following two conditions hold:

- *For all $\xi \in \mathcal{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$ the set γ^ξ_y is discrete.*
- *$\mathcal{H}^{n-1}(\gamma) < +\infty$.*

Then for $u \in GBD(\Omega_1)$ and $v \in GBD(\Omega_2)$ we have $w \in GBD(\Omega)$ with $\Omega := \Omega_1 \cup \Omega_2 \cup \gamma$. This holds also true if GBD is substituted with $GSBD$.

Proof. Let $\xi \in \mathcal{S}^{n-1}$. By assumption we have $u^\xi_y \in (S)BV_{loc}((\Omega_1)^\xi_y)$ and $v^\xi_y \in (S)BV_{loc}((\Omega_2)^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$. Furthermore, notice that $(\Omega_1)^\xi_y$ and $(\Omega_2)^\xi_y$ are each a collection of open intervals with $(\Omega_1)^\xi_x \cap (\Omega_2)^\xi_y = \emptyset$. Also by assumption, for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_y$ we have that γ^ξ_y is discrete and by Theorem 5.1.5 $\hat{u}^\xi_y$ and $\hat{v}^\xi_y$ admit a trace at $x \in \gamma^\xi_y$. This means there only occur additional jumps when glueing $\hat{u}^\xi_y$ and $\hat{v}^\xi_y$ together along γ^ξ_y which implies that we have $w^\xi_y \in (S)BV_{loc}(\Omega^\xi_y)$ with $J_{\hat{w}^\xi_y} \subseteq J_{\hat{u}^\xi_y} \cup J_{\hat{v}^\xi_y} \cup \gamma^\xi_y$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$. Now define

$$\lambda_w := \lambda_u|_{\Omega_1} + \lambda_v|_{\Omega_2} + 2\mathcal{H}^{n-1}|_\gamma$$

with λ_u, λ_v being the respective measures for u and v from Definition 5.1.2. We observe now that for a Borel set B we have by the area formula

$$\begin{aligned}&\int_{\pi^\xi} \mathcal{H}^0(B^\xi_y \cap J^1_{w^\xi_y})\, d\mathcal{H}^{n-1}(y)\\ \leq &\int_{\pi^\xi} \mathcal{H}^0((B\cap\Omega_1)^\xi_y \cap J^1_{u^\xi_y}) + \mathcal{H}^0((B\cap\Omega_2)^\xi_y \cap J^1_{v^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap \gamma^\xi_y)\, d\mathcal{H}^{n-1}(y)\\ = &\int_{\pi^\xi} \mathcal{H}^0((B\cap\Omega_1)^\xi_y \cap J^1_{u^\xi_y}) + \mathcal{H}^0((B\cap\Omega_2)^\xi_y \cap J^1_{v^\xi_y})\, d\mathcal{H}^{n-1}(y) + \mathcal{H}^{n-1}(B\cap\gamma).\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_{\pi^\xi} |Dw^\xi_y|(B^\xi_y \setminus J^1_{w^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{w^\xi_y})\, d\mathcal{H}^{n-1}(y)\\ \leq &\int_{\pi^\xi} |Du^\xi_y|((B\cap\Omega_1)^\xi_y \setminus J^1_{u^\xi_y}) + |Dv^\xi_y|((B\cap\Omega_2)^\xi_y \setminus J^1_{v^\xi_y})\\ &+ \mathcal{H}^0(B^\xi_y \cap \gamma^\xi_y) + \mathcal{H}^0(B^\xi_y \cap J^1_{w^\xi_y})\, d\mathcal{H}^{n-1}(y)\\ \leq &\lambda_u(B\cap\Omega_1) + \lambda_v(B\cap\Omega_2) + 2\mathcal{H}^{n-1}(B\cap\gamma),\end{aligned}$$

i.e., we have shown (5.1.1) holds for λ_w and therefore $w \in (S)GBD(\Omega)$. □

With the trace theorem available one can generalize results for the jump set J_u from the previous chapter. At this point, we observe that for a Lebesgue measurable function the jump set is contained in

$$\{x \in \Omega \mid \exists \xi \in \mathcal{S}^{n-1} : \operatorname*{ap\,lim}_{\substack{y\to x\\ \pm(y-x)\cdot\xi>0}} u(y) \text{ exist and are not equal }\}.$$

We have a similar result to the results for the jump set in the structure theorem of $BD(\Omega)$ 4.2.3 (cf. Theorem 6.2 and 8.1 in [Dal13]):

Theorem 5.1.7. *The jump set of a function $u \in GBD(\Omega)$ is countably $\mathcal{H}^{n-1}$-rectifiable. Furthermore, for $\xi \in \mathcal{S}^{n-1}$ let*

$$J^\xi_u = \{x \in J_u : [u](x)\cdot\xi \neq 0\}.$$

For $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$ we have

$$(J^\xi_u)^\xi_y = J_{\hat{u}^\xi_y}$$

and for every $t \in (J_u)^\xi_y$

$$u^\pm(y+t\xi)\cdot\xi = (\hat{u}^\xi_y)^\pm(t).$$

The normals to J_u and $J_{\hat{u}^\xi_y}$ are oriented such that $\xi\cdot\nu_u \geq 0$ and $\nu_{\hat{u}^\xi_y} = 1$.

Analogously to the discussion in the previous section the $\mathcal{H}^{n-1}$-rectifiability of J_u guarantees the existence of traces $u^\pm\cdot\xi$ and a (measure-theoretic) unit normal ν along J_u. The remarkable statement of this theorem is that the traces can be restricted to the one-dimensional slices Ω^ξ_y and then still correspond to the one-dimensional traces along this slice of $u\cdot\xi$ (at least for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_y$) and vice versa.

We now turn our discussion to the approximate symmetric gradient. For $u \in BD(\Omega)$ we have seen that an approximate symmetric gradient $\mathcal{E}u \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$ exists and it coincides with the density of the absolutely continuous part of the measure Eu. It is a priori not clear in what form this carries over to the setting of $u \in GBD(\Omega)$. The following theorem, however, guarantees that not only does such an approximate symmetric gradient exist, but (analogously to the result in $BD(\Omega)$) it corresponds to the one-dimensional (approximate) gradient if u is restricted to any slice:

Theorem 5.1.8. *[Dal13, Theorem 9.1] Let $u \in GBD(\Omega)$. Then there exists a function $\mathcal{E}u \in L^1(\Omega, \mathbb{R}^{n\times n}_{sym})$ such that for λ^n-a.e. $x \in \Omega$*

$$\operatorname{ap}\lim_{y\to x} \frac{(u(y) - u(x) - \mathcal{E}u(x)(y-x)\cdot(y-x))}{|y-x|^2} = 0.$$

Additionally, for every $\xi \in \mathbb{R}^n \setminus \{0\}$ for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$ we have

$$(\mathcal{E}u)^\xi_y \xi \cdot \xi = \nabla \hat{u}^\xi_y \tag{5.6}$$

in $L^1(\Omega^\xi_y)$. Here, $\nabla \hat{u}^\xi_y$ is the density of the absolutely continuous part of the measure $D\hat{u}^\xi_y$.

To conclude this section we will talk about compactness in $GBD(\Omega)$ and $GSBD(\Omega)$. The next two results presented are the main reason why these spaces are relevant in calculus of variations. For instance, if minimizing sequences for the Griffith energy are considered in $SBD(\Omega)$ this space is too small to derive the existence of minimizers in this space. Similar, to the theory inspired by the Mumford-Sha functional, one would have to impose conditions like equiboundedness of the L^∞-norm of sequences to stay in this space (cf. chapter 7 and 8 in [AFP00]). These are rather strong conditions. It turned out that relaxing the Griffith energy to the larger space of $GSBD(\Omega)$ solved this problem.

To formulate the first compactness result we first introduce the measure $\hat{\mu}_u$ for a function $u \in GBD(\Omega)$ as the smallest measure λ which can be used in definition 5.1.2. Such a $\hat{\mu}_u$ does exist and can be written down explicitly by (cf. proposition 4.17 in [Dal13])

$$\hat{\mu}_u(B) = \sup_m \sup_{\substack{\xi\in(\mathcal{S}^{n-1})^m \\ B\in\mathcal{B}^m}} \sum_{i=1}^k \hat{\mu}^{\xi_i}_u(B_i)$$

where

$$\hat{\mu}^\xi_u(B) = \int_{\pi_\xi} |D\hat{u}^\xi_y|(B^\xi_y \setminus J^1_{u^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{u^\xi_y})\, d\mathcal{H}^{n-1}$$

for a $\xi \in \mathcal{S}^{n-1}$ and a Borel set $B \in \mathcal{B}$. We now state a very general compactness theorem for $GBD(\Omega)$:

Theorem 5.1.9. *[AT22, Theorem 1.1] Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded and $\{u_k\}_{k\in\mathbb{N}} \subseteq GBD(\Omega)$. Suppose now that*

$$\sup_{k\in\mathbb{N}} \hat{\mu}_{u_k} < +\infty$$

holds. Then, there exists a subsequence $\{u_{k_j}\}$ such that

- *$A := \{x \in \Omega : |u_k(x)| \xrightarrow{k\to\infty} +\infty\}$ has finite perimeter,*

- *$u_k \to u$ pointwise for λ^n-a.e. $x \in \Omega \setminus A$ and some $u \in GBD(\Omega)$ with $u = 0$ in A,*
- *we have*

$$\mathcal{H}^{n-1}(\partial^* A) \leq \lim_{\sigma \to \infty} \liminf_{k \to \infty} \mathcal{H}^{n-1}(J^\sigma_{u_k})$$

where $J^\sigma_{u_k} := \{x \in J_{u_k} : |[u_k(x)]| \geq \sigma\}$.

While the theorem is most certainly useful for extracting pointwise converging subsequences it does not necessarily tells us how the approximate symmetric derivative and $\mathcal{H}^{n-1}(J_u)$ behave. However, in $GSBD(\Omega)$ we can extract this information when additional conditions are imposed:

Theorem 5.1.10. *[CC18, Theorem 1.1] Let $\Omega \subseteq \mathbb{R}^n$ be bounded and $\{u_k\}_{k\in\mathbb{N}} \subseteq GSBD(\Omega)$. Suppose there exist $M > 0$, $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ non-decreasing with superlinear growth at infinity, i.e.,*

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = +\infty,$$

such that

$$\int_\Omega \psi(|\mathcal{E}u_k|)\, dx + \mathcal{H}^{n-1}(J_{u_k}) < M \tag{5.7}$$

for all $k \in \mathbb{N}$. Then, there exists a subsequence $\{u_{k_j}\}$ such that

- *$A := \{x \in \Omega : |u_{k_j}(x)| \to +\infty\}$ has finite perimeter,*
- *$u_{k_j} \to u$ pointwise for $x \in \Omega \setminus A$ λ^n-a.e. for some $u \in GSBD(\Omega)$ with $u = 0$ in A,*
- *we have*

$$\mathcal{E}u_{k_j} \rightharpoonup \mathcal{E}u \text{ in } L^1(\Omega \setminus A),$$

and

$$\mathcal{H}^{n-1}(J_u \cup \partial^* A) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}).$$

If we are in the situation of $\mathcal{E}u_k \in L^p(\Omega)$ with $p \in (1, \infty)$ and condition (5.7) of Theorem 5.1.10 is fulfilled with $\psi = ()^p$ then we can also assume

$$\mathcal{E}u_{k_j} \rightharpoonup \mathcal{E}u \text{ in } L^p(\Omega \setminus A). \tag{5.8}$$

Indeed, since $\mathcal{E}u_{k_j}$ is equibounded in $L^p(\Omega) \subseteq L^p(\Omega \setminus A)$ by the Banach Alaoglu theorem we can extract a subsequence (without renaming) such that $\mathcal{E}u_{k_j} \rightharpoonup E$ in $L^p(\Omega \setminus A)$ for some some $E \in L^p(\Omega)$. But since $L^\infty(\Omega \setminus A) \subseteq L^{p'}(\Omega \setminus A)$ we have $\mathcal{E}u_{k_j} \rightharpoonup E$ in $L^1(\Omega \setminus A)$ and we can deduce $E = \mathcal{E}u$.

5.2 The local Poincaré-Korn inequality for $GSBD^p((-1,1)^n)$

In this section we will consider the following Poincaré-Korn inequality:

$$\|u-a\|_{L^p(\Omega\setminus\omega)} \leq C\,\|\mathcal{E}u\|_{L^p(\Omega)}$$

for $u \in GSBD(\Omega)$, a suitable affine transformation and $\omega \subseteq \Omega$. Since we are mostly interested in the case of the right-hand side being finite we will introduce the following definition:

Definition 5.2.1. For $p \in [1,\infty)$ we define the space

$$GSBD^p(\Omega) := \{u \in GSBD(\Omega) : \mathcal{E}u \in L^p(\Omega)\}.$$

Notice that $GSBD^1(\Omega) = GSBD(\Omega)$ holds by theorem 5.1.8.

Before presenting the proof of this inequality we will state a result from convex geometry which we are going to need in the proof:

Lemma 5.2.2. *Let $Q = (-1,1)^n$ and $z_0 \in Q$. Suppose there exists $t \in \mathbb{R}^+$ such that*

$$z_i := z_0 + te_i \in Q.$$

For an arbitrary $y \in Q$ there exist $i_0 \in \{0,..,n\}$ such that for $I := \{0,..,n\} \setminus \{i_0\}$ the matrix $Y \in \mathbb{R}^{n\times n}$ with columns $(y-z_i)_{i\in I}$ fulfils

$$|\det Y| \geq \frac{t^n}{(n+1)!}.$$

Moreover, we have

$$|Y^{-1}| \leq \frac{C}{|\det Y|}$$

with a constant $C > 0$ only depending on n. As a consequence,

$$\frac{t^n}{C(n+1)!}|x| \leq \frac{|\det Y|}{C}|x| \leq |Yx| \leq \sum_{i=0}^{n} |(y-z_i)\cdot x| \tag{5.9}$$

for all $x \in \mathbb{R}^n$.

With this we will now present the precise formulation of the inequality with its proof.

Proposition 5.2.3. *[CCF14, Proposition 2] Let $p \in [1,\infty)$ and $u \in GSBD^p(Q)$. Then there exist*

- *a constant $C > 0$ (only depending on n and p),*
- *a set $\omega \subseteq Q$ with $\lambda^n(\omega) \leq C\mathcal{H}^{n-1}(J_u)$, and*
- *a rigid displacement $a : \mathbb{R}^n \to \mathbb{R}^n$*

such that

$$\int_{Q\setminus\omega} |u-a|^p\,dx \leq C\int_Q |\mathcal{E}u|^p\,dx. \tag{5.10}$$

Proof. Throughout the whole proof we denote with C a generic (changing) constant. First notice that we can assume

$$\mathcal{H}^{n-1}(J_u) \leq \frac{1}{32n^3} \tag{5.11}$$

if we take $C \geq 2^n 32n^3$ since for every $u \in GSBD(\Omega)$ with $\mathcal{H}^{n-1}(J_u) \geq \frac{1}{32n^3}$ we can just take $\omega = Q$ and an arbitrary rigid displacement a.
Suppose for a moment that we have an absolutely continuous $u \in AC(Q, \mathbb{R}^n)$. By the fundamental theorem of calculus and (4.2) we have for any direction $\xi \in S^{n-1}$

$$\hat{u}_x^\xi(t) - \hat{u}_x^\xi(0) = \xi \cdot (u(x+t\xi) - u(x)) = t\int_0^1 \xi \cdot \nabla u(x+st\xi)\xi \, ds = t\int_0^1 \xi \cdot \mathcal{E}u(x+st\xi)\xi \, ds. \tag{5.12}$$

For an arbitrary $u \in GSBD(Q)$ we say that (5.12) holds on the segment $[x, x+t\xi] \subseteq Q$ if $\hat{u}_x^\xi \in W^{1,1}[0,t]$ (so $J_{\hat{u}_x^\xi} \cap (0,t) = \emptyset$), formula (5.12) holds and we have (5.6) $\mathcal{H}^1$-a.e. along the line segment $(x + [0,t]\xi) \cap Q$. Notice that in this case $\xi \cdot \mathcal{E}u(x+st\xi)\xi$ exists for λ-a.e. $s \in [0,1]$ (cf. theorem 5.1.8). Now, we introduce an indicator on $\mathbb{R}^n \times \mathcal{S}^{n-1} \times \mathbb{R}$ by setting

$$T(x,\xi,t) := \begin{cases} 1, & \text{if } x \in Q,\ x+t\xi \in Q \text{ and (5.12) does not hold,} \\ 0, & \text{otherwise.} \end{cases} \tag{5.13}$$

The main idea for the proof is to choose a 'good' simplex $(z_0, (z_0 + t_* e_i)_{i=1}^n)$ with $t_* \in (0,1]$ and $z_0 \in (-1,0)^n =: q$ such that the following four conditions hold (we write $z_i := z_0 + t_* e_i$):

1. $t_* \in (1/2, 1)$ and (5.12) holds on all edges $[z_i, z_j]$, $0 \leq i < j \leq n$.

2. Set
$$F(z_0) := \sum_{i,j=0}^n \int_{[z_i,z_j]} |\mathcal{E}u| \, d\mathcal{H}^1$$
which is well defined by the first condition. Then
$$F(z_0) \leq 4\sqrt{2}(n+1)^2 \|\mathcal{E}u\|_{L^1(Q)}, \tag{5.14}$$
i.e., the L^1-norm along the simplex edges of the approximate symmetric gradient is comparable with the L^1-norm taking over the whole cube.

3. For each $i = 0, .., n$ we have
$$\int_{\mathcal{S}^{n-1}} \int_{\mathbb{R}} T(z_i, \xi, t) \, dt \, d\mathcal{H}^{n-1} \leq 16\mathcal{H}^{n-1}(\mathcal{S}^{n-1})(n+1)^2 \mathcal{H}^{n-1}(J_u). \tag{5.15}$$

4. Set $g = |\mathcal{E}u|\chi_Q$. Furthermore, define
$$H(z_0) = \sum_{i=0}^n \int_{(-2,2)^n} \int_{[y,e_i]} g(z_0 + t) \, d\mathcal{H}^1(t) dy.$$
We have
$$H(z_0) \leq 8\sqrt{n}(n+1)4^{n+1} \|\mathcal{E}u\|^p_{L^p(Q)}. \tag{5.16}$$

We first show that we can find a z_0 and suitable simplex fulfilling the sought-after requirements before choosing a and ω. In the end we bring everything together.

1st condition: Observe that for a chosen direction $\xi \in \mathcal{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$ we have $\xi \cdot u_y^\xi = \hat{u}_y^\xi \in SBV_{loc}(Q_y^\xi)$ and (5.6). We distinguish two cases:

- $J\hat{u}_y^\xi = \emptyset$: In this case $\hat{u}$ is λ^n-a.e. equal to an absolutely continuous function on $(y + \mathbb{R}\xi) \cap Q$, i.e., we have $T(y, t, \xi) = 0$ for λ^n-a.e. $t \in \mathbb{R}$.
- $J\hat{u}_y^\xi \neq \emptyset$: By Theorem 5.1.7 we have $(y + \mathbb{R}\xi) \cap J_u \neq 0$.

Now let P_ξ be the orthogonal projection onto π_ξ and set

$$\omega_\xi := \{x \in Q : x + \mathbb{R}\xi \cap J_u \neq 0\} = (P_\xi(J_u) + \mathbb{R}\xi) \cap Q.$$

By Fubini we have

$$\begin{aligned}
|\omega_\xi| &= \int_{\pi_\xi} \int_{\mathbb{R}} \chi_{\omega_\xi}(y + t\xi)\ dt\ d\mathcal{H}^{n-1}(y) \\
&= \int_{P_\xi(J_u)} \int_{\mathbb{R}} \chi_{\omega_\xi}(y + t\xi)\ dt\ d\mathcal{H}^{n-1}(y) \\
&\leq \int_{P_\xi(J_u)} \lambda((y + \mathbb{R}\xi) \cap Q)\ d\mathcal{H}^{n-1}(y) \\
&\leq 2\sqrt{n}\mathcal{H}^{n-1}(P_\xi(J_u)) \\
&\leq 2\sqrt{n}\mathcal{H}^{n-1}(J_u).
\end{aligned}$$

Notice also here that

$$\int_{\mathbb{R}} T(x, \xi, t) dt \leq \operatorname{diam}(Q).$$

With this and the aforementioned case distinction we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} T(x, \xi, t)\ dt\ dx = \int_{\omega_\xi} \int_{\mathbb{R}} T(x, \xi, t)\ dt\ dx \leq |\omega_\xi| 2\sqrt{n} \leq 4n\mathcal{H}^{n-1}(J_u). \tag{5.17}$$

To choose a suitable $z_0 \in q$ we now use the following trick: Define the function

$$G(z_0, t) = \sum_{i=1}^{n} T(z_0, e_i, t) + \sum_{1 \leq i < j \leq n} T\left(z_0 + te_i, \frac{e_j - e_i}{\sqrt{2}}, \sqrt{2}t\right).$$

Notice that $G(z_0, t) = 0$ implies

$$T(z_0, e_i, t) = 0 \text{ and } T\left(z_0 + te_i, \frac{e_j - e_i}{\sqrt{2}}, \sqrt{2}t\right) = 0$$

for all $1 \leq i < j \leq n$ which is exactly the 1. condition we want of our simplex. So we want to find a $t_* \in (1/2, 1)$ and a $z_0 \in q$ such that $G(z_0, t_*) = 0$. To argue that such values do exist we use a trick. We integrate G over $(1/2, 1) \times q$ and notice that:

$$\begin{aligned}
\int_{\frac{1}{2}}^{1} \int_q G(z_0, t)\ dz_0\ dt &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}} \sum_{i=1}^{n} T(x, e_i, t) + \frac{1}{\sqrt{2}} \sum_{1 \leq i < j \leq n} T\left(x, \frac{e_j - e_i}{\sqrt{2}}, t\right)\ dt\ dx \\
&\leq 4n^2\mathcal{H}^{n-1}(J_u) + \frac{4n^2(n-1)}{2\sqrt{2}}\mathcal{H}^{n-1}(J_u) \\
&\leq 4n^3\mathcal{H}^{n-1}(J_u)
\end{aligned}$$

due to (5.17). At this point we fix $t_* \in (1/2, 1)$ such that

$$\int_q G(z_0, t_*)\ dz_0 \leq 8n^3\mathcal{H}^{n-1}(J_u). \tag{5.18}$$

We can argue the existence of such a t_* by contradiction as follows: Suppose no such t_* exists. Then

$$\int_q G(z_0, t)\ dz_0 > 8n^3\mathcal{H}^{n-1}(J_u)$$

must hold for all $t \in (1/2, 1)$. Therefore, we have

$$4n^3\mathcal{H}^{n-1}(J_u) < \int_{\frac{1}{2}}^{1} \int_q G(z_0, t)\ dz_0\ dt \leq 4n^3\mathcal{H}^{n-1}(J_u),$$

which is a contradiction.

Now observe that the assumption (5.11) in combination with (5.18) implies

$$\int_q G(z_0, t_*)\ dz_0 \leq \frac{1}{4}.$$

Since $G(z_0, t_*) \neq 0$ implies $G(z_0, t_*) \geq 1$ we have

$$|\{z_0 \in q : G(z_0, t_*) \neq 0\}| \leq \frac{1}{4}.$$

The proof for this claim is exactly of the same method via contradiction as for the existence of the t_* in (5.18). Moreover, this means

$$|\{z_0 \in q : G(z_0, t_*) = 0\}| \geq \frac{3}{4}.$$

So we have 'three quarters of q' available to choose an eligible z_0 which then induces a simplex $((z_i)_{i=0}^n)$ (with $z_i = z_0 + t_* e_i$) that fulfils the first condition.

2nd condition: We now show:

$$|\{z_0 \in q : G(z_0, t_*) = 0 \text{ and } (5.14) \text{ holds }\}| \geq \frac{1}{2}$$

To be more precise, we show

$$|\{z_0 \in q : G(z_0, t_*) = 0 \text{ and } (5.14) \text{ does not hold }\}| \leq \frac{1}{4}. \tag{5.19}$$

The proof is again done via contraction: Suppose (5.19) does not hold. Set

$$\tilde{q} = \{z_0 \in q : G(z_0, t_*) = 0\}$$

which is the set where F is defined. Then, for each $z_0 \in \tilde{q}$ with $G(z_0, t_*) = 0$ we observe with Fubini

$$\begin{aligned}
\sqrt{2}(n+1)^2 \|\mathcal{E}u\|_{L^1(Q)} &< \int_{\tilde{q}} F(z_0)\ dz_0 \\
&\leq \sum_{i,j=0}^{n} \int_{\tilde{q}} \int_{[z_i,z_j]} |\mathcal{E}u|\ d\mathcal{H}^1\ dz_0 \\
&= \sum_{i,j=0}^{n} \int_{\tilde{q}} \int_{[0,t_*]} |\mathcal{E}u|(z_0 + t_* e_i + s(e_j - e_i))|e_i - e_j|\ ds\ dz_0 \\
&\leq \sqrt{2} \sum_{i,j=0}^{n} \int_{[0,t_*]} \int_{\tilde{q}+t_* e_i + s(e_j - e_i))} |\mathcal{E}u|(z_0)\ dz_0\ ds \\
&\leq \sqrt{2} \sum_{i,j=0}^{n} \int_{[0,t_*]} \int_{Q} |\mathcal{E}u|(z_0)\ dz_0\ ds \\
&\leq \sqrt{2}(n+1)^2 \int_{Q} |\mathcal{E}u|(z_0)\ dz_0.
\end{aligned}$$

This is a contradiction.

3rd condition: We combine the methods for the first and second condition. We show:

$$|\{z_0 \in q : (5.15) \text{ does not hold }\}| \leq \frac{1}{4}.$$

Notice we do not require $G(z_0, \xi, t) = 0$ here. Define the auxiliary function

$$\tilde{G}(z_0, \xi, t) := \sum_{i=0}^{n} T(z_i, \xi, t).$$

Notice that due to (5.17) and Fubini we have

$$\int_q \int_{S^{n-1}} \int_{\mathbb{R}} \tilde{G}(z_0, \xi, t)\ dt\ d\mathcal{H}^{n-1}(\xi)\ dz_0 \leq (n+1)\mathcal{H}^{n-1}(S^{n-1}) 4n\mathcal{H}^{n-1}(J_u).$$

Now we argue exactly like in the proof for the second condition to see that

$$\int_{S^{n-1}} \int_{\mathbb{R}} \tilde{G}(z_0, \xi, t)\ dt\ d\mathcal{H}^{n-1}(\xi) \leq 4(n+1)\mathcal{H}^{n-1}(S^{n-1}) 4n\mathcal{H}^{n-1}(J_u)$$

holds for at least three quarters of $z_0 \in q$. This then again implies (5.15) for these values since

$$T(z_i, \xi, t) \leq \tilde{G}(z_0, \xi, t)$$

for all $i = 0, .., n$ by construction.

4th condition: We argue again like before. This time, however, we want to show

$$|\{z_0 \in q : (5.16) \text{ does not hold }\}| \leq \frac{1}{8}.$$

We only observe

$$\int_q H(z_0)\ dz_0 \leq \sum_{i=0}^n \int_{(-2,2)^n} \int_{[y,e_i]} \int_{\mathbb{R}^n} g(x+t)\ dx\ d\mathcal{H}^1(t)dy \leq \sqrt{n}(n+1)4^{n+1} \|e(u)\|^p_{L^p(Q)}$$

and then argue by contradiction as before. Putting all these results together we have shown that

$$|\{z_0 \in q : \text{ all four conditions hold for the induced simplex } (z_i)_{i=0}^n \}| \geq \frac{1}{8},$$

i.e., we have shown that there are an abundance of choices for z_0. Therefore, we now fix a $z_0 \in q$ such that all conditions hold.

Choice of $\omega \subset Q$: For each $i = 0, .., n$ we define the set

$$\omega_i = \{y \in Q : y = z_i + t\xi \text{ and } T(z_i, \xi, t) = 1\}$$

which is the set of all points $y \in Q$ which are reachable from z_i, but the fundamental theorem of calculus (5.12) does not hold along $[z_i, y]$. Due to the third condition (5.15) we have

$$\begin{aligned} |\omega_i| &= \int_{\mathbb{R}^n} \chi_{\omega_i}(x)\ dx \\ &= \int_{S^{n-1}} \int_{(0,\infty)} t^{n-1}\chi_{\omega_i}(z_i + t\xi)\ dt\ d\xi \\ &= \int_{S^{n-1}} \int_{(0,2\sqrt{n})} t^{n-1}T(z_i, \xi, t)\ dt\ d\xi \\ &\leq C\mathcal{H}^{n-1}(J_u). \end{aligned}$$

We set

$$\omega = \bigcup_{i=1}^n \omega_i,$$

and by the above we have

$$|\omega| \leq C\mathcal{H}^{n-1}(J_u).$$

Choice of a: For a we first choose an affine interpolation of u along the simplex vertices, i.e., we define the affine mapping through

$$a(z_i) = u(z_i)$$

for each $i = 0, .., n$. Notice, that by the first condition

$$\begin{aligned} |(e_i - e_j) \cdot (u(z_i) - u(z_j))| &= \left| \frac{1}{\sqrt{2}} \int_{[z_i,z_j]} (e_i - e_j) \cdot (\mathcal{E}u)(e_i - e_j)\ d\mathcal{H}^1 \right| \\ &\leq \sqrt{2}F(z_0) \\ &\leq C \|\mathcal{E}u\|_{L^1(Q)} \end{aligned}$$

holds for all $i, j = 0, .., n$. This implies

$$|(\nabla a)_{ii}| \leq 2t_*|(\nabla a)_{ii}| = 2|e_i \cdot (a(z_0 + t_* e_i) - a(z_0)))| \leq C \|\mathcal{E}u\|_{L^1(Q)}$$

and consequently we have

$$\begin{aligned}|(\nabla a)_{ij} + (\nabla a)_{ji}| &\leq 2t_*(|(\nabla a)_{ij} + (\nabla a)_{ji} - (\nabla a)_{ii} - (\nabla a)_{jj}| + |(\nabla a)_{ii}| + |(\nabla a)_{jj}|) \\ &\leq 2t_*|(e_i - e_j) \cdot (a(z_i) - a(z_j))| + C \left\|\mathcal{E}u\right\|_{L^1(Q)} \\ &\leq C \left\|\mathcal{E}u\right\|_{L^1(Q)}.\end{aligned}$$

Putting this together, we derive

$$|e(a)| \leq C \left\|\mathcal{E}u\right\|_{L^1(Q)} \leq C \left\|\mathcal{E}u\right\|_{L^p(Q)}. \tag{5.20}$$

Notice here that $e(a)$ does not vanish. However, (5.20) guarantees that we can substitute a at the end with a suitable rigid displacement.

Combining everything: We set $w = u - a$. For $y \in \Omega \setminus \omega$ we observe that by construction, due to lemma 5.2.2 (more concretely (5.9)), (5.20) and the linearity of $\mathcal{E}$ we have:

$$\begin{aligned}|w(y)| &\leq C \sum_{i=0}^{n} |(y - z_i) \cdot w(y)| \\ &= C \sum_{i=0}^{n} |(y - z_i) \cdot (w(y) - w(z_i))| \\ &= C \sum_{i=0}^{n} |y - z_i| \int_{[y,z_i]} |\mathcal{E}w| d\mathcal{H}^1 \\ &\leq C \left(\sum_{i=0}^{n} \int_{[y,z_i]} |\mathcal{E}u| d\mathcal{H}^1 + \left\|\mathcal{E}u\right\|_{L^p(Q)} \right).\end{aligned}$$

Notice that C is not dependant on t_* since we have $t_* \geq 1/2$. In combination with Hölder's inequality, we can therefore derive

$$|w(y)|^p \leq C \left(\sum_{i=0}^{n} \int_{[y,z_i]} |\mathcal{E}u|^p \ d\mathcal{H}^1 + \left\|\mathcal{E}u\right\|_{L^p(Q)}^p \right).$$

Integrating over $Q \setminus \omega$ we observe

$$\int_{Q\setminus\omega} |w(y)|^p \ dy \leq C \left(\sum_{i=0}^{n} \int_{Q\setminus\omega} \int_{[y,z_i]} |\mathcal{E}u|^p \ d\mathcal{H}^1 \ dy + \left\|\mathcal{E}u\right\|_{L^p(Q)}^p \right).$$

We still need to estimate the first term on the right-hand side. With the function $g = |e(u)|\chi_Q$ introduced in forth condition we can derive

$$\begin{aligned}\sum_{i=0}^{n} \int_{Q\setminus\omega} \int_{[y,z_i]} |\mathcal{E}u|^p \ d\mathcal{H}^1 \ dy &= \sum_{i=0}^{n} \int_{(Q\setminus\omega)-z_0} \int_{[z_0+y,z_i]} |\mathcal{E}u|^p \ d\mathcal{H}^1 \ dy \\ &\leq \sum_{i=0}^{n} \int_{(-2,2)^n} \int_{[z_0+y,z_i]} g(x) \ d\mathcal{H}^1(x) \ dy \\ &= \sum_{i=0}^{n} \int_{(-2,2)^n} \int_{[y,e_i]} g(z_0 + x) \ d\mathcal{H}^1(x) \ dy \\ &= H(z_0).\end{aligned}$$

So by (5.16):

$$\int_{Q\setminus\omega} |w(y)|^p \, dy \le C(H(z_0) + \|\mathcal{E}u\|^p_{L^p(Q)}) \le C \|\mathcal{E}u\|^p_{L^p(Q)} .$$

Due to (5.20) we have

$$\|u - a + e(a)x\|_{L^p(Q\setminus\omega)} \le \|u - a\|_{L^p(Q\setminus\omega)} + |e(a)| \le C \|\mathcal{E}u\|^p_{L^p(Q)} .$$

Therefore, we can substitute a with $a - e(a)x$ to get $e(a) = 0$ while retaining the inequality (with a small increase of the constant C) to conclude the proof. □

Note that for $r > 0$ we can extend the result from Proposition 5.2.3 to $Q_r = (-r,r)^n$ by scaling. The condition for ω then reads as

$$\lambda^n(\omega) \le Cr\mathcal{H}^{n-1}(J_u)$$

and the right-hand side (5.10) additionally depends on r^p. We have

$$\int_{Q_r\setminus\omega} |u - a|^p \, dx \le Cr^p \int_{Q_r} \mathcal{E}u|^p \, dx.$$

Observe that the constant C obtained from 5.2.3 does not change if Q is dilated. We will now state an important consequence of this Korn-Poincaré inequality which states that if the jump set of a $GSBD^p(\Omega)$ function vanishes we have a Sobolev function.

Proposition 5.2.4. *Let $n \ge 2$, $r > 0$ and $u \in GSBD^p(Q_r)$. Suppose that $\mathcal{H}^{n-1}(J_u) = 0$. Then $u \in W^{1,p}(Q_r)$.*

Proof. Theorem 5.2.3 scaled to Q_r guarantees that $u \in L^p(\Omega)$ with $\Omega = Q_r$. Now, observe that in this case also

$$\int_{\pi^\xi} |D\hat{u}^\xi_y|(\Omega^\xi_y) \, d\mathcal{H}^{n-1}(y) = \int_{\pi^\xi} |D\hat{u}^\xi_y|(\Omega^\xi_y \setminus J^1_{u^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat{u}^\xi_y}) \, d\mathcal{H}^{n-1}(y) < +\infty$$

for every direction $\xi \in S^{n-1}$ so $u \in SBD(\Omega)$ by Proposition 5.1.3. Notice, that the approximate symmetric gradient coincides with the distributional symmetric gradient, i.e., we have $Eu \in L^p(\Omega)$ by definition of $GSBD^p(\Omega)$. We have already seen in section 3.2 that $u \in L^p(\Omega)$ and $\mathcal{E}u \in L^p(\Omega)$ then implies $u \in W^{1,p}(\Omega)$. □

At the end of this section, we note that in [CCF14] a Sobolev-type inequality is also shown. This provides higher integrability of $u - a$ over $Q \setminus \omega$. More specifically, we have the following result (cf. also Proposition 3 in [CCF14]):

Proposition 5.2.5. *[CCI19, Proposition 3.1] Let $0 < \theta'' < \theta' < 1$, $r > 0$. Let $Q = (-r,r)^n, Q' = (-\theta' r, \theta' r)^n, Q'' = (-\theta'' r, \theta'' r)^n, p \in [1,\infty), u \in GSBD^p(\Omega)$. There exist $c_* > 0$ only depending on p, n and $c > 0$ only depending on n, p and a given mollifier ρ with the following properties:*

- *Then there exists $\omega \subseteq Q'$ and an affine function $a : \mathbb{R}^n \to \mathbb{R}^n$ with $e(a) = 0$ such that*

$$|\omega| \subseteq c_* r \mathcal{H}^{n-1}(J_u) \tag{5.21}$$

and

$$\int_{Q'\setminus\omega} |u - a|^{\frac{np}{n-1}} \le c_* r^{\frac{n(p-1)}{n-1}} \left(\int_Q |\mathcal{E}u|^p \right)^{\frac{n}{n-1}} \tag{5.22}$$

- *If $p > 1$ then there exists $\bar{p} > 0$ such that for $\rho_r \in C_c^\infty(B_{(\theta'-\theta'')r})$, $\rho_r(x) = r^{-n}\rho\left(\frac{x}{r}\right)$ the function $v = u\chi_{Q'\setminus\omega} + a\chi_\omega$ fulfils*

$$\int_{Q''} |e(v * \rho_r) - e(u) * \rho_r|^p dx \leq C\left(\frac{\mathcal{H}^{n-1}(J_u)}{r^{n-1}}\right)^{\bar{p}} \int_Q |\mathcal{E}u|^p dx.$$

5.3 A local approximation result

In this section, we will discuss an approximation result for $GSBD^p(\Omega)$ functions. It essentially says that we can substitute a $u \in GSBD^p(\Omega)$ with a $\tilde{u} \in GSBD^p(\Omega) \cap \mathbb{C}^\infty(\tilde{\Omega})$ where $\tilde{\Omega} \subseteq \Omega$ and $\operatorname{dist}(\tilde{\Omega}, \partial\Omega)$ is reasonably small. In doing so we only increase the jump set and the L^p-norm of the approximate symmetric gradient by a small amount. The idea is to employ a classical Whitney covering theorem and then use the results from the previous section to derive a suitable approximation of u. We start by stating the precise covering theorem which we are going to use:

Lemma 5.3.1. *[Gra10, J.1] Let $\Omega \subseteq \mathbb{R}^n$ be an open, proper subset. Then there exists a countable family of closed (dyadic) cubes $(Q_j)_{j\in\mathbb{N}}$ such that*

1. $\bigcup_{j\in\mathbb{N}} Q_j = \Omega$.
2. $\operatorname{int}(Q_j) \cap \operatorname{int}(Q_k) = \emptyset$ *for all $j, k \in \mathbb{N}$.*
3. *We have for all $j \in \mathbb{N}$*
$$\operatorname{diam}(Q_j) \leq \operatorname{dist}(Q_j, \partial\Omega) \leq 4\operatorname{diam}(Q_j).$$
4. *If Q_j and Q_k touch each other for some $j, k \in \mathbb{N}$, i.e., $\partial Q_j \cap \partial Q_k \neq \emptyset$, then*
$$\frac{1}{4} \leq \frac{\operatorname{diam}(Q_j)}{\operatorname{diam}(Q_k)} \leq 4.$$
5. *Each Q_j only has at most $12^n - 4^n$ neighbours.*
6. *Let $\theta \in (1, 5/4)$. Denote with $Q'_j = \theta Q_j$ the cube with the same center as Q_j, but with its length scaled by θ. Then Q'_j only overlaps with finitely other cubes. In particular, we have*
$$\sum_{j\in\mathbb{N}} \chi_{Q'_j} \leq 12^n - 4^n + 1.$$

Before stating the main result of this section, we will now present some technical, helpful lemmas. The first one is based on [CFI15, Theorem 4.3], but we have adapted the proof for orthotopes:

Lemma 5.3.2. *Let $\omega \subseteq Q \subseteq \mathbb{R}^n$ where Q is an orthotope (i.e., $Q = \prod_{i=1}^n [a_i, b_i]$ for $a_i, b_i \in \mathbb{R}$) and $\epsilon \in (0, 1/2)$. Suppose that*

$$|\omega| \leq \epsilon|Q|. \tag{5.23}$$

Then for all affine functions φ we have

$$|Q| \, \|\varphi\|_{L^\infty(Q,\mathbb{R}^n)} \leq C \, \|\varphi\|_{L^1(Q\setminus\omega,\mathbb{R}^n)} \,. \tag{5.24}$$

for some constant $C > 0$ which only depends on n and ϵ.

Proof. Notice that (5.23) and (5.24) are invariant under translation and stretching in one coordinate. In particular, we can assume w.l.o.g. that Q is the unit cube. We write $\varphi(x) = Ax + b$ with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. First we choose $\delta \in (0, 1-2\epsilon)$. Notice, that for every $i = 1, .., n$ we have

$$(1-\delta)|Q| = |Q \cap (Q + \delta e_i)|.$$

In particular, this implies with $Q_i := ((Q \setminus \omega) \cap (Q \setminus \omega + \delta e_i))$

$$(1-\delta)|Q| - 2\epsilon|Q| = |Q \cap (Q + \delta e_i)| - 2|\omega| \leq |Q_i|.$$

We also have due to the triangle inequality

$$\delta|Q_i||Ae_i| = \int_{Q_i} |\varphi(x) - \varphi(x - \delta e_i)|dx \leq 2\int_{Q\setminus\omega} |\varphi|dx.$$

Putting these together we derive

$$(1-2\epsilon-\delta)\delta|Q| \, \|Ax\|_{L^\infty(Q)} \leq \delta|Q_i| \, \|Ax\|_{L^\infty(Q)} \leq \delta|Q_i| \max_{i=1,..,n} |Ae_i| \leq 2\, \|\varphi\|_{L^1(Q\setminus\omega)}.$$

Since $|Q| = 1$ we have

$$\|Ax\|_{L^\infty(Q)} \leq \frac{8}{(1-2\epsilon)^2} \|\varphi\|_{L^1(Q\setminus\omega)}$$

for $\delta = \frac{1-2\epsilon}{2}$. To conclude the proof, we just observe that

$$\|b\|_{L^\infty(Q)} = |b| = \frac{|Q|}{|Q\setminus\omega|}|Q\setminus\omega||b| = \frac{1}{|Q\setminus\omega|}\|b\|_{L^1(Q\setminus\omega)}$$
$$\leq \frac{1}{|Q\setminus\omega|}\left(\|\varphi - \varphi(0)\|_{L^1(Q\setminus\omega)} + \|\varphi\|_{L^1(Q\setminus\omega)}\right).$$

holds since $b = \varphi(0)$ and $Ax = \varphi(x) - \varphi(0)$. □

Lemma 5.3.3. *[CCI19, Lemma 3.3] Let $a_i \geq 0$ and $b_i \geq 0$ with*

$$\sum_{i=1}^k a_i \leq A$$

and

$$\sum_{i=1}^k b_i \leq B$$

for some $k \in \mathbb{N}$ and $A, B \geq 0$. Then there exists a $i_0 \in \{1, .., k\}$ such that

$$a_j \leq \frac{2}{k}A$$

and

$$b_j \leq \frac{2}{k}B.$$

Lemma 5.3.4. *[Per95, Theorem 6.2] Let $0 \leq s < \infty$, $A \subseteq \mathbb{R}^n$. Suppose that A is $\mathcal{H}^s$-measurable and $\mathcal{H}^s(A) < \infty$ holds. Then*

$$\limsup_{r\to 0^+} \frac{\mathcal{H}^s(A \cap B_r(x))}{r^s} = 0.$$

holds for $\mathcal{H}^s$-a.e. $x \in \mathbb{R}^n \setminus A$.

We now state the approximation theorem which is the main result of this section. The original theorem and proof [CCI19, Theorem 3]) were written for $\Omega = (-1,1)^n$. Via a Whitney covering we have adapted the arguments for arbitrary open, bounded sets with Lipschitz boundary. For convenience, we set

$$\Omega_R := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > Rd\}$$

where $d = \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$.

Theorem 5.3.5. *Let $n \geq 2$, $p \in [1,\infty)$ and $\Omega \subseteq \mathbb{R}^n$ be bounded, open with Lipschitz boundary. Then there exist two positive constants $\eta, C > 0$ such that for every $u \in GSBD^p(\Omega)$ with $\delta := \mathcal{H}^{n-1}(J_u)^{\frac{1}{n}} < \eta$ there exist $R \in (0, \sqrt{\delta})$ and $\tilde{u} \in GSBD^p(\Omega)$ with the following properties:*

1. *$\tilde{u} \in C^\infty(\Omega_{1-\sqrt{\delta}})$, $\tilde{u} = u$ in $\Omega \setminus \Omega_R$ and $\mathcal{H}^{n-1}(J_u \cap \partial\Omega_R) = \mathcal{H}^{n-1}(J_{\tilde{u}} \cap \partial\Omega_R)$.*
2. *$\mathcal{H}^{n-1}(J_{\tilde{u}} \setminus J_u) \leq C\sqrt{\delta}\mathcal{H}^{n-1}(J_u \cap (\Omega \setminus \Omega_{1-\sqrt{\delta}}))$.*
3. *We have:*
$$\int_\Omega |e(\tilde{u})|^p dx \leq (1 + C\delta^s) \int_\Omega |e(u)|^p dx.$$

Proof. We start by taking $\eta \leq (4\tilde{C}(8\sqrt{n})^n)^{-1}$ with $\tilde{C}$ being the maximum of all constants obtained from Proposition 5.2.3 resp. Proposition 5.2.5. Furthermore, choose η small so small that we can assume $\partial\Omega_R$ has Lipschitz boundary for every $R \in (0, \sqrt{\delta})$. For $\delta = \mathcal{H}^{n-1}(J_u)^{\frac{1}{n}}$ we set $N := \lfloor \frac{1}{K\delta} \rfloor$ for a fixed K which will be determined later. The strategy will be to find a strip such that the L^p-norm of the approximate symmetric gradient and the jump set along the strip are comparable to L^p-norm and jump set along $\Omega \setminus \Omega_{1-\sqrt{\delta}}$. Then we cover the strip with cubes and filter out the ones that contain a (to the diameter) disproportionate amount of the jump set. On the rest of the cubes, we then will apply the results from the previous sections.

Choosing R: Now, define

$$\Omega^i := \Omega_{(N-i)K\delta}$$

for $i = 1, ..., N-1$. Since $(\partial\Omega_R)_{R>0}$ covers Ω we have that $\mathcal{H}^{n-1}(\partial\Omega_R \cap J_u) \neq 0$ for only countable many $R > 0$. In particular, we we will assume $\mathcal{H}^{n-1}(\partial\Omega^i \cap J_u) = 0$. Otherwise we choose $\epsilon_i > 0$ small enough and substitute $(N-i)K\delta$ with $(N-i)K\delta + \epsilon_i$ in the definition of Ω^i. As a consequence, we have that $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega^i$ does not belong to the jump set of u. With an analogous argument, we can also assume that almost every point in $\partial\Omega^i$ is a Lebesgue point of $\mathcal{E}u$.

Now we set $C_i := \Omega^i \setminus \Omega^{i+1}$ and $C_{N-1} := \Omega^{N-1}$. Observe that

$$\sum_{i=1}^{\lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 3} \int_{C_i \cup C_{i+1}} |e(u)|^p dx \leq 2 \int_{\Omega^1 \setminus \Omega^{\lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 1}} |e(u)|^p dx \leq 2 \int_{\Omega \setminus \Omega_{1-\sqrt{\delta}}} |e(u)|^p dx \tag{5.25}$$

holds since $(N - (\lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 1))K\delta \geq 1 - \sqrt{\delta}$ and analogously

$$\sum_{i=1}^{\lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 3} \mathcal{H}^{n-1}(J_u \cap (C_i \cup C_{i+1})) \leq 2\mathcal{H}^{n-1}(J_u \cap (\Omega^1 \setminus \Omega^{\lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 1}) \leq 2\mathcal{H}^{n-1}(J_u \cap (\Omega \setminus \Omega_{1-\sqrt{\delta}})).$$

Hence, we can now apply Lemma 5.3.3 with $a_i = \int_{C_i \cup C_{i+1}} |e(u)|^p dx$, $b_i = \mathcal{H}^{n-1}(J_u \cap (C_i \cup C_{i+1}))$ and $k = \lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 3$ to derive the existence of an $i_0 \in \{1, .., \lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 3\}$ such that

$$\int_{C_{i_0} \cup C_{i_0+1}} |e(u)|^p dx \le \frac{4}{\lfloor \frac{1}{K\sqrt{\delta}} \rfloor - 3} \int_{\Omega \setminus \Omega^{1-\sqrt{\delta}}} |e(u)|^p dx \le 8\sqrt{\delta} \int_{\Omega \setminus \Omega^{1-\sqrt{\delta}}} |e(u)|^p dx \tag{5.26}$$

and similarly

$$\mathcal{H}^{n-1}(J_u \cap (C_{i_0} \cup C_{i_0+1})) \le 8\sqrt{\delta}\mathcal{H}^{n-1}(J_u \cap (\Omega \setminus \Omega_{1-\sqrt{\delta}})). \tag{5.27}$$

Notice here, that Ω^{i_0} will serve as a starting point to construct $\tilde{u}$, i.e., we will set $R := (N - i_0)\delta$.

Introducing a suitable covering of cubes: To define $\tilde{u}$ in Ω^{i_0} we first cover Ω^{i_0} with cubes $(Q_j)_{j\in\mathbb{N}}$ given by the Whitney Lemma 5.3.1 and construct a partition of unity with respect to these cubes. We choose $1 < \theta' < \theta'' < \theta''' = \frac{9}{8} < \frac{5}{4}$ (where θ'' and θ' are to be determined later) and define Q_j', Q_j'', Q_j''' as the cubes with the same center, but scaled by θ', θ'' and θ''' respectively. Now, set

$$S_k = \left\{ x \in \Omega^{i_0} : \frac{1}{2^{k+1}} K\delta d < \operatorname{dist}(x, \partial\Omega^{i_0}) \le \frac{1}{2^k} K\delta d \right\}$$

for $k \in \mathbb{N}$. Notice that by construction

$$\Omega^{i_0} = \dot{\bigcup}_{k\in\mathbb{N}} S_k \,\dot{\cup}\, \Omega^{i_0+1}.$$

We now observe the following: Assume that for any $k, j \in \mathbb{N}$

$$S_k \cap Q_j \neq \emptyset \text{ and } S_{k+2} \cap Q_j \neq \emptyset$$

holds. This immediately implies

$$\operatorname{diam}(Q_j) \le \operatorname{dist}(Q_j, \partial\Omega^{i_0}) \le \frac{1}{2^{k+2}} K\delta d.$$

Consequently, we see that for $x \in Q_j, y \in S_k \cap Q_j$ and $z \in S_{k+2} \cap Q_j$ we have

$$\begin{aligned} \operatorname{dist}(x, \partial\Omega^{i_0}) &\le |x - z| + \operatorname{dist}(z, \partial\Omega^{i_0}) \le \operatorname{diam}(Q_j) + \operatorname{dist}(z, \partial\Omega^{i_0}) \\ &\le K\delta d \left(\frac{1}{2^{k+2}} + \frac{1}{2^{k+2}} \right) = K\delta d \frac{1}{2^{k+1}}. \end{aligned}$$

But this contradicts $S_k \cap Q_j \neq \emptyset$. Hence, it cannot happen that Q_j intersects with more than two S_k. Moreover, denote with P the orthogonal projection onto the cube Q_j. We observe that for any $x \in Q_j'''$

$$\begin{aligned} \operatorname{dist}(x, \partial\Omega^{i_0}) &\ge \operatorname{dist}(Px, \partial\Omega^{i_0}) - |x - Px| \\ &\ge \operatorname{dist}(Q_j, \partial\Omega^{i_0}) - (\theta''' - 1) \operatorname{diam}(Q_j) \\ &\ge \frac{1}{8} \operatorname{dist}(Q_j, \partial\Omega^{i_0}) \end{aligned}$$

and

$$\begin{aligned}\operatorname{dist}(x,\partial\Omega^{i_0}) &\le \operatorname{dist}(Q_j,\partial\Omega^{i_0})+\operatorname{diam}(Q_j''')\le \operatorname{dist}(Q_j,\partial\Omega^{i_0})+\theta'''\operatorname{diam}(Q_j)\\ &\le (1+\theta''')\operatorname{dist}(Q_j,\partial\Omega^{i_0})<4\operatorname{dist}(Q_j,\partial\Omega^{i_0}).\end{aligned}$$

For a cube Q_j with $\frac{1}{2^{k+1}}K\delta d\le \operatorname{dist}(Q_j,\partial\Omega^{i_0})\le \frac{1}{2^k}K\delta d$ for some $k\in\mathbb{N}$ we can deduce from that that Q_j''' only intersects with at most seven S_k.

Now, we divide the cubes into two categories. We say Q_j is 'good' if

$$\mathcal{H}^{n-1}(Q_j'''\cap J_u)\le \eta d_j^{n-1}$$

with $d_j=\operatorname{diam}(Q_j)$ and 'bad' if this does not hold true. Notice here, that $Q_j\cap\Omega^{i_0+1}\neq\emptyset$ for some $j\in\mathbb{N}$ implies

$$4\operatorname{diam}(Q_j)\ge \operatorname{dist}(Q_j,\partial\Omega^{i_0})\ge \frac{K\delta d}{4}$$

since in this case only $Q_j\cap S_k\neq\emptyset$ can happen for $k=0,1$. Consequently,

$$\mathcal{H}^{n-1}(Q_j'''\cap J_u)\le \mathcal{H}^{n-1}(J_u)=\delta^n\le \eta\delta^{n-1}\le \left(\frac{16}{Kd}\right)^{n-1}\eta d_j^{n-1}. \tag{5.28}$$

At this point, we choose $K=16/d$ to ensure that all cubes which intersect Ω^{i_0+1} are good cubes. In particular, it follows that all bad cubes are in the strip C_{i_0}.

We now use the fact that the cubes given by the Whitney covering lemma are dyadic as follows: First observe that we have seen that for each cube Q_j $Q_j\cap\Omega^{i_0+1}\neq\emptyset$ we have

$$\delta\le d_j.$$

We now decompose every cube with diameter bigger than δ naturally into smaller dyadic cubes with all having the same side length $2\tilde r=2^{-\tilde k+1}$ with

$$\frac{2\sqrt{n}}{2^{\tilde k+1}}\le\delta\le\frac{2\sqrt{n}}{2^{\tilde k}}=2\sqrt{n}\tilde r=:\tilde d.$$

Notice that in doing so we only decompose good cubes into smaller good cubes since naturally (5.28) still holds for cubes with diameter greater or equal than δ. Moreover, although the new covering does not fulfil the third property in the Whitney lemma, we notice that all other properties still hold for the covering. Also, we now have that all cubes Q_j with $Q_j\cap\Omega^{i_0+1}\neq\emptyset$ have a constant diameter $\tilde d$. Lastly, since we did not change the bad cubes Q_j of the covering we still can compare their diameter to the distance of the cube to the boundary, i.e., we have

$$d_j\le\operatorname{dist}(Q_j,\partial\Omega^{i_0})\le 4d_j.$$

Furthermore, we observe that all bad Q_j''' are in $C_{i_0}\cup C_{i_0+1}$. Indeed, for $x\in Q_j'''$ with $Q_j\cap\Omega^{i_0+1}=\emptyset$ we have by triangle inequality

$$\operatorname{dist}(x,\partial\Omega)\le |x-Px|+\operatorname{dist}(Px,\partial\Omega^{i_0})+i_0K\delta\le((\theta'''-1)+1+i_0)K\delta<(i_0+2)K\delta.$$

Let A_k denote the number of bad cubes which have a non-empty intersection with S_k. Set $S := \bigcup_{\tilde{k}=k-6}^{k+6} S_{\tilde{k}} \subset C_{i_0} \cup C_{i_0+1}$ where we set $S_k := C_{i_0+1}$ for negative integers k. Observe that by the above, the second and the last point of lemma 5.3.1

$$\begin{aligned} A_k \eta \left(\frac{K\delta d}{2^{k+2}}\right)^{n-1} &\leq \sum_{\substack{Q_j \cap S_k \neq \emptyset \\ Q_j \text{ is bad}}} \eta (4d_j)^{n-1} \\ &< 4^{n-1} \sum_{\substack{Q_j \cap S_k \neq \emptyset \\ Q_j \text{ is bad}}} \mathcal{H}^{n-1}(Q_j''' \cap J_u) \\ &\leq 4^{n-1}(12^n - 4^n + 1)\mathcal{H}^{n-1}(S \cap J_u). \end{aligned}$$

For $k \in \mathbb{N}$ we set $\mathcal{B}_k$ as the union of all bad cubes with $S_k \cap \Omega^{i_0} \neq 0$. We can estimate the perimeter of $\mathcal{B}_k$ by

$$\begin{aligned} \mathcal{H}^{n-1}(\partial^* \mathcal{B}_k) &\leq \sum_{\substack{Q_j \cap S_k \neq \emptyset \\ Q_j \text{ is bad}}} \mathcal{H}^{n-1}(\partial Q_j) \\ &\leq C \sum_{\substack{Q_j \cap S_k \neq \emptyset \\ Q_j \text{ is bad}}} d_j^{n-1} \\ &\leq C A_k \left(\frac{K\delta d}{2^k}\right)^{n-1} \\ &\leq C \frac{\mathcal{H}^{n-1}(S \cap J_u)}{\eta}. \end{aligned}$$

Denote now the union of all bad cubes with $\mathcal{B}$. We observe

$$\begin{aligned} \mathcal{H}^{n-1}(\partial^* \mathcal{B}) &\leq \sum_{k=0}^{\infty} \mathcal{H}^{n-1}(\partial^* \mathcal{B}_k) \\ &\leq 13C \sum_{k=0}^{\infty} \frac{\mathcal{H}^{n-1}(S_k \cap J_u)}{\eta} \\ &\leq C \frac{\mathcal{H}^{n-1}(C_{i_0} \cup C_{i_0+1})}{\eta} \\ &\leq C \frac{\sqrt{\delta}}{\eta} \mathcal{H}^{n-1}(J_u \cap (\Omega \setminus \Omega_{1-\sqrt{\delta}})), \end{aligned} \tag{5.29}$$

where the last inequality follows from (5.25). Analogously, we can show a bound for the volume:

$$|\mathcal{B}_k| \leq C \frac{\delta}{\eta} \mathcal{H}^{n-1}(S \cap J_u).$$

Consequently,

$$|\mathcal{B}| \leq C \frac{\delta^{\frac{3}{2}}}{\eta} \mathcal{H}^{n-1}(J_u \cap (\Omega \setminus \Omega_{1-\sqrt{\delta}})).$$

For convenience, we enumerate the cubes so that all cubes Q_j with $Q_j \cap \Omega^{i_0} \neq \emptyset$ are in the index range $\{1, .., N_0\}$ for some $N_0 \in \mathbb{N}$ and all others have indices greater than N_0.

Constructing a partition of unity with respect to the Whitney cover: For each good cube Q_j we now first choose $\tilde{\varphi}_j \in C_c^\infty(Q'_j;[0,1])$ with

$$\tilde{\varphi}_j = 1 \text{ on } Q_j \text{ and } |\nabla\tilde{\varphi}_j| \leq \frac{C}{d_j} \text{ on } Q'_j$$

where C is independent of Q_j (only dependent on θ'). Now, we define

$$\varphi_j := \frac{\tilde{\varphi}_j}{\sum_{k\in\mathbb{N}}\tilde{\varphi}_k}$$

on $Q'_j \setminus \mathcal{B}$. Notice that the sums $\sum_{k\in\mathbb{N}}\tilde{\varphi}_k$ and consequently $\sum_{k\in\mathbb{N}}\varphi_k$ are locally finite due to Q'_j only intersecting with finitely many other cubes Q'_k. Furthermore, we have by construction $\varphi_j \in C^\infty(\Omega\setminus\overline{\mathcal{B}})$, $\sum_{j\in\mathbb{N}}\varphi_j = 1$ in $\Omega\setminus\mathcal{B}$ and $|\nabla\varphi_j| \leq \frac{C}{d_j}$ (with a larger C only depending on n.).

Application of local estimates onto good cubes: By applying Proposition 5.2.3 and 5.2.5 onto Q''_j and Q'''_j for each good cube Q_j we find a set $\omega_j \subseteq Q''_j$ and a rigid displacement a_j with

$$|\omega_j| \leq \tilde{C}d_j\mathcal{H}^{n-1}(J_u\cap Q'''_j) \leq \tilde{C}\eta d_j^n, \tag{5.30}$$

and with $r_j := \frac{d_j}{2\sqrt{n}}$

$$\int_{Q_j\setminus\omega_j}|u-a_j|^p dx \leq \tilde{C}r_j^p\int_{Q'''_j}|\mathcal{E}u|^p dx \tag{5.31}$$

$$\int_{Q''_j\setminus\omega_j}|u-a_j|^{\frac{np}{n-1}}dx \leq \tilde{C}r_j^{\frac{n(p-1)}{n-1}}\left(\int_{Q'''_j}|\mathcal{E}u|^p dx\right)^{\frac{n}{n-1}}. \tag{5.32}$$

Furthermore, let ρ be a symmetric mollifier with support in $B_{\frac{1}{18}}$ (here fix $\theta'' = \frac{10}{9}$ and $\theta' = \frac{19}{18}$). Notice that $\frac{1}{18}r_j = (\theta''-\theta')r_j = \frac{\theta''-\theta'}{\theta'''}\theta'''r_j$, $\theta'''r_j$ is half of the side length of Q'''_j and $\rho_{r_j} \in C^\infty(B_{(\theta''-\theta')r_j})$. In particular, Lemma 5.2.5 tells us that for

$$u_j := \rho_{r_j} * (u\chi_{Q''_j\setminus\omega_j} + a_j\chi_{\omega_j}) \in C^\infty(Q'_j,\mathbb{R}^n)$$

we have that

$$\int_{Q'_j}|e(u_j)-e(u)*\rho_{r_j}|^p dx \leq \tilde{C}\left(\frac{\mathcal{H}^{n-1}(J_u\cap Q'''_j)}{r_j^{n-1}}\right)^{\tilde{p}}\int_{Q'''_j}|\mathcal{E}u|^p dx \tag{5.33}$$

with $\tilde{C}$ being the maximum of all constants appearing in Proposition 5.2.3 and Proposition 5.2.5 which only depend on ρ, n and p and some $\tilde{p} > 0$. Observe that for any affine mapping $Ax+b$ we have

$$(\rho_{r_j}*(Ay+b))(x) = \int_{\mathbb{R}^n}(A(x-y)+b)\rho_{r_j}(y)dy = Ax+b-\int_{\mathbb{R}^n}A(y)\rho_{r_j}(y)dy = Ax+b \tag{5.34}$$

because of the symmetry of ρ. As a first consequence we infer

$$\begin{aligned}\|u_j-a_j\|_{L^p(Q'_j)} &\overset{(5.34)}{=} \left\|\rho_{r_j}*(u-a_j)\chi_{Q''_j\setminus\omega_j}\right\|_{L^p(Q'_j)}\\ &\leq \left\|(u-a_j)\chi_{Q''_j\setminus\omega_j}\right\|_{L^p(Q''_j)}\\ &\overset{(5.31)}{\leq} Cr_j\|\mathcal{E}u\|_{L^p(Q''_j)}.\end{aligned} \tag{5.35}$$

Suppose now that two good cubes Q_j and Q_k touch each other. We can derive

$$\begin{aligned}\|a_i - a_j\|_{L^p(Q_j''\cap Q_k''\setminus(\omega_j\cup\omega_k))} &\le \|u-a_j\|_{L^p(Q_j''\setminus\omega_j)} + \|u-a_k\|_{L^p(Q_k''\setminus\omega_k)}\\ &\overset{(5.35)}{\le} C(r_j\,\|\mathcal{E}u\|_{L^p(Q_j''')} + r_k\,\|\mathcal{E}u\|_{L^p(Q_k''')}).\end{aligned}$$

In particular, since the side lengths are comparable with a factor of 4 we have

$$\|a_i - a_j\|_{L^p(Q_j''\cap Q_k''\setminus(\omega_j\cup\omega_k))} \le Cr_j\,\|\mathcal{E}u\|_{L^p(Q_j'''\cup Q_k''')}\,. \tag{5.36}$$

Furthermore, we also can estimate

$$|Q_j''\cap Q_k''| \ge \frac{1}{8^n}\max\{|Q_j|,|Q_k|\}. \tag{5.37}$$

Indeed, observe that the intersection of Q_j'' and Q_k'' is an orthogon with side lengths greater of equal than $(\theta''-1)2(r_j+r_k) = \frac{2}{9}(r_j+r_k)$. It follows with $r_j \ge \frac{1}{4}r_k$ that

$$|Q_j''\cap Q_k''| \ge \left(\frac{2}{9}(r_j+r_k)\right)^n \ge \left(\frac{5}{36}(2r_j)\right)^n \ge \frac{1}{8^n}|Q_j|.$$

Swapping roles of j and k gives estimate (5.37). In particular, we derive

$$\begin{aligned}|\omega_j\cup\omega_k| &\le 2\max\{|\omega_j|,|\omega_k|\} \le 2\tilde{C}\eta\max\{d_j^n,d_k^n\}\\ &= 2\tilde{C}\eta(\sqrt{n})^n\max\{|Q_j|,|Q_k|\} \overset{(5.37)}{\le} \frac{1}{4}|Q_j''\cap Q_k''|.\end{aligned}$$

Notice that at this point the choice of the constant η is motivated. Now we apply Lemma 5.3.2 paired with Hölder inequality and (5.36) to derive

$$\begin{aligned}\|a_j-a_k\|_{L^{\frac{np}{n-1}}(Q_j''\cap Q_k'')} &\le |Q_j''\cap Q_k''|^{\frac{n-1}{np}}\,\|a_j-a_k\|_{L^\infty(Q_j''\cap Q_k'')}\\ &\overset{5.3.2}{\le} C|Q_j''\cap Q_k''|^{\frac{n-1}{np}-1}\,\|a_j-a_k\|_{L^1(Q_j''\cap Q_k''\setminus(\omega_j\cup\omega_k))}\\ &\le C|Q_j''\cap Q_k''|^{\frac{n-1}{np}-1+\frac{p-1}{p}}\,\|a_j-a_k\|_{L^p(Q_j''\cap Q_k''\setminus(\omega_j\cup\omega_k))}\\ &= Cr_j^{-\frac{1}{np}}\,\|a_j-a_k\|_{L^p(Q_j''\cap Q_k''\setminus(\omega_j\cup\omega_k))}\\ &\overset{(5.36)}{\le} Cr_j^{1-\frac{1}{np}}\,\|\mathcal{E}u\|_{L^p(Q_j'''\cup Q_k''')}\end{aligned} \tag{5.38}$$

since $\frac{n-1}{np} - 1 + \frac{p-1}{p} = -\frac{1}{np}$. Analogously, we have

$$\|a_j-a_k\|_{L^p(Q_j''\cap Q_k'')} \le Cr_j\,\|\mathcal{E}u\|_{L^p(Q_j'''\cup Q_k''')}\,. \tag{5.39}$$

Definition of $\tilde{u}$: Now we define $\tilde{u}$ the following way

$$\tilde{u} = \begin{cases}\sum_{j\in\mathbb{N}} u_j\varphi_j, & \text{in } \Omega^{i_0}\setminus\mathcal{B}\\ u & \text{in } \Omega\setminus\Omega^{i_0}\cup\mathcal{B}.\end{cases}$$

Since $\varphi_j \in C^\infty(Q'_j \backslash \overline{\mathcal{B}})$ and $u_j \in C^\infty(Q''_j, \mathbb{R}^n)$ we have $\tilde{u} \in C^\infty(\Omega^{i_0} \backslash \overline{\mathcal{B}}, \mathbb{R}^n)$ which, in particular, implies $\tilde{u} \in C^\infty(\Omega_{1-\sqrt{\delta}})$ (remember that $\Omega_{1-\sqrt{\delta}} \subseteq \Omega^{i_0+1}$). Also observe that $\gamma := \partial\mathcal{B} \cap \Omega^{i_0}$ has Lipschitz regularity with γ^ξ_y being discrete for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_y$ and every $\xi \in \mathcal{S}^{n-1}$. Additionally, we have $\mathcal{H}^{n-1}(\Omega^{i_0} \cap \gamma) < +\infty$ by (5.29). Therefore, we can apply Lemma 5.1.6 to infer $\tilde{u} \in GSBD(\Omega^{i_0})$. Furthermore, we can compute the symmetric gradient in $\Omega^{i_0} \setminus \overline{\mathcal{B}}$ by

$$e(\tilde{u}) = \sum_{j\in\mathbb{N}} e(u_j)\varphi_j + \sum_{j\in\mathbb{N}} u_j \odot (\nabla\varphi_j). \tag{5.40}$$

We will estimate each term separately on the right-hand side of (5.40). For this, we first introduce the following relation on the index set of good cubes:

$$j \sim k \;:\Leftrightarrow\; Q'_j \cap Q'_k \neq 0 \text{ and } j \neq k.$$

Since $\sum_{j\in\mathbb{N}} \varphi_j = 1$ in $\Omega^{i_0} \setminus \mathcal{B}$ we have

$$\nabla\varphi_j = -\sum_{j\sim k} \nabla\varphi_k$$

which implies

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} u_k \odot (\nabla\varphi_k) = u_j \odot \nabla\varphi_j + \sum_{j\sim k} u_k \odot (\nabla\varphi_k) \\
&= -\, u_j \odot (\sum_{j\sim k} \nabla\varphi_k) + \sum_{j\sim k} u_k \odot (\nabla\varphi_k) = \sum_{j\sim k} (u_k - u_j) \odot (\nabla\varphi_k)
\end{aligned}$$

in Q'_j.

Estimating $e(\tilde{u})$ on Ω^{i_0+1} (minus a small set): Let Q_j, Q_k be two good cubes with $j \sim k$, $j, k \le N_0$ (so $Q_j \cap \Omega^{i_0+1} \neq \emptyset$ and $Q_k \cap \Omega^{i_0+1} \neq \emptyset$). They then have the same diameter $\tilde{d}$ (and consequently $\tilde{r} = r_j = r_k$). We have due to the generalized Hölder inequality, (5.31) and (5.38)

$$\begin{aligned}
&\|u_j - u_k\|_{L^p(Q'_j\cap Q'_k)} \\
&= \left\|\rho_{r_j} * (u\chi_{Q''_j\backslash\omega_j} + a_j\chi_{\omega_j}) + \rho_{r_k} * (u\chi_{Q''_k\backslash\omega_k} - a_k\chi_{\omega_k})\right\|_{L^p(Q'_j\cap Q'_k)} \\
&\le \left\|u\chi_{Q''_j\backslash\omega_j} - u\chi_{Q''_k\backslash\omega_k} + a_k\chi_{\omega_k} - a_j\chi_{\omega_j}\right\|_{L^p(Q''_j\cap Q''_k)} \\
&= \left\|(u-a_j)\chi_{\omega_k\backslash\omega_j} + (u-a_k)\chi_{\omega_j\backslash\omega_k}) + (a_k - a_j)\chi_{\omega_j\cup\omega_k}\right\|_{L^p(Q''_j\cap Q''_k)} \\
&\le \|u - a_j\|_{L^{\frac{np}{n-1}}(Q''_j\backslash\omega_j)} |\omega_k|^{\frac{1}{np}} + \|u - a_k\|_{L^{\frac{np}{n-1}}(Q''_k\backslash\omega_k)} |\omega_j|^{\frac{1}{np}} \\
&+ \|a_j - a_k\|_{L^{\frac{np}{n-1}}(Q''_j\cap Q''_k)} |\omega_k \cup \omega_j|^{\frac{1}{np}} \\
&\overset{(5.31),(5.38)}{\le} c\tilde{r}(\|\mathcal{E}u\|_{L^p(Q'''_j)} |\omega_k|^{\frac{1}{np}} + \|\mathcal{E}u\|_{L^p(Q'''_k)} |\omega_j|^{\frac{1}{np}}) + C\tilde{r}^{1-\frac{1}{p}} \|\mathcal{E}u\|_{L^p(Q'''_j\cup Q'''_k)} |\omega_k \cup \omega_j|^{\frac{1}{np}} \\
&\le C\tilde{r}^{\frac{p-1}{p}} (\|\mathcal{E}u\|_{L^p(Q'''_j)} |\omega_k|^{\frac{1}{np}} + \|\mathcal{E}u\|_{L^p(Q'''_k)} |\omega_j|^{\frac{1}{np}})
\end{aligned} \tag{5.41}$$

Now we set for an open $U \subseteq \Omega$

$$U_\delta := \Omega \cap (U + (-3\delta, 3\delta)^n).$$

For any cube with $Q_j' \cap U \neq \emptyset$ we infer $Q_k''' \subseteq U_\delta$ for $k \sim j$ since $2\left(\frac{\theta'''}{\theta'}\right) < 3$ and $d_j < \delta$. Now, set $\tilde{\Omega} := \Omega^{i_0+1} \setminus \bigcup_{j>N_0} Q_j'$. We observe

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{N}} u_k \odot (\nabla\varphi_k) \right\|_{L^p(\tilde{\Omega}\cap U)} &\leq \sum_{k=1}^{N_0} \|u_k \odot (\nabla\varphi_k)\|_{L^p(\tilde{\Omega}\cap Q_k)} \\
&= \sum_{Q_k'''\subseteq U_\delta, k\leq N_0} \left\| \sum_{j\sim k} (u_k - u_j) \odot (\nabla\varphi_k) \right\|_{L^p(\tilde{\Omega}\cap Q_k)} \\
&\leq \sum_{Q_k'''\subseteq U_\delta, k\leq N_0} \|\nabla\varphi_k\|_\infty \sum_{j\sim k} \|(u_k - u_j)\|_{L^p(\tilde{\Omega}\cap Q_k)} \\
&\overset{(5.41)}{\leq} C \sum_{Q_k'''\subseteq U_\delta, k\leq N_0} \left(\frac{1}{\tilde{d}}\right) \sum_{j\sim k} |\omega_j|^{\frac{1}{np}} \tilde{r}^{\frac{p-1}{p}} \|\mathcal{E}u\|_{L^p(Q_j'''\cup Q_k''')} \\
&\overset{(5.30)}{\leq} C \sum_{Q_k'''\subseteq U_\delta, k\leq N_0} \left(\frac{1}{\tilde{d}}\right)^{\frac{1}{p}} \sum_{j\sim k} \tilde{d}^{\frac{n+1}{np}} \|\mathcal{E}u\|_{L^p(Q_j'''\cup Q_k''')} \\
&= C \sum_{Q_k'''\subseteq U_\delta, k\leq N_0} \tilde{d}^{\frac{1}{np}} \sum_{j\sim k} \|\mathcal{E}u\|_{L^p(Q_j'''\cup Q_k''')} \\
&= C\tilde{d}^{\frac{1}{np}} \|\mathcal{E}u\|_{L^p(U_\delta\cap\Omega^{i_0})} \qquad (5.42)
\end{aligned}$$

We also notice now that also $\sum_{j=1}^{N_0} \varphi_j = 1$ on $U \cap \tilde{\Omega}$. In combination with (5.33) we can estimate the L^p distance on $U \cap \tilde{\Omega}$ from $e(\tilde{u})$ to $\rho_{\tilde{r}} * \mathcal{E}u$ by

$$\begin{aligned}
&\|e(\tilde{u}) - \rho_{\tilde{r}} * \mathcal{E}u\|_{L^p(\tilde{\Omega}\cap U)} \\
&= \left\| \sum_{j=1}^{N_0} e(u_j)\varphi_j + \sum_{j\in\mathbb{N}} u_j \odot (\nabla\varphi_j) - \sum_{j=1}^{N_0} (\rho_{\tilde{r}} * \mathcal{E}u)\varphi_j \right\|_{L^p(U\cap\tilde{\Omega})} \\
&\leq \left\| \sum_{j=1}^{N_0} (e(u_j) - \rho_{\tilde{r}} * \mathcal{E}u)\varphi_j \right\|_{L^p(U\cap\tilde{\Omega})} + \left\| \sum_{j\in\mathbb{N}} u_j \odot (\nabla\varphi_j) \right\|_{L^p(U\cap\tilde{\Omega})} \\
&\overset{(5.42)}{\leq} \sum_{j=1}^{N_0} \|(e(u_j) - \rho_{\tilde{r}} * \mathcal{E}u)\varphi_j\|_{L^p(U\cap\tilde{\Omega})} + C\tilde{d}^{\frac{1}{np}} \|\mathcal{E}u\|_{L^p(U_\delta\cap\Omega^{i_0})} \\
&\leq \sum_{Q_k'''\subseteq U_\delta, j\leq N_0} \|(e(u_j) - \rho_{\tilde{r}} * \mathcal{E}u)\|_{L^p(Q_j')} + C\tilde{d}^{\frac{1}{np}} \|\mathcal{E}u\|_{L^p(U_\delta\cap\Omega^{i_0})} \\
&\overset{(5.33)}{\leq} C\left(\sum_{Q_k'''\subseteq U_\delta, j\leq N_0} \left(\frac{\mathcal{H}^{n-1}(J_u \cap Q_j''')}{\tilde{r}^{n-1}}\right)^{\frac{\bar{p}}{p}} \|\mathcal{E}u\|_{L^p(Q_j''')} + \tilde{d}^{\frac{1}{np}} \|\mathcal{E}u\|_{L^p(U_\delta\cap\Omega^{i_0})} \right) \\
&\overset{(5.30)}{\leq} C\left(\left(\delta^{\frac{\bar{p}}{p}} + \tilde{d}^{\frac{1}{np}}\right) \|\mathcal{E}u\|_{L^p(U_\delta\cap\Omega^{i_0})} \right) \leq C\delta^s \|\mathcal{E}u\|_{L^p(U_\delta\cap\Omega^{i_0})}
\end{aligned}$$

with $s := \min\{\frac{\bar{p}}{p}, \frac{1}{np}\}$. We used here that $\tilde{d} \le 4\sqrt{n}\delta$. As a consequence, we have

$$\begin{aligned}\|e(\tilde{u})\|_{L^p(U\cap\tilde{\Omega})} &\le \|\rho_{\tilde{r}} * e(u)\|_{L^p(U\cap\tilde{\Omega})} + \|e(\tilde{u}) - \rho_{\tilde{r}} * e(u)\|_{L^p(U\cap\tilde{\Omega})} \\ &\le \|e(u)\|_{L^p(U_\delta\cap\Omega^{i_0})} + C\delta^s \|e(u)\|_{L^p(U_\delta\cap\Omega^{i_0})}.\end{aligned}$$

Estimating the remaining terms: Now, we just need to control $\mathcal{E}\tilde{u}$ on $U \cap (\Omega^{i_0} \setminus (\tilde{\Omega} \cup \mathcal{B}))$ since

$$\|\mathcal{E}\tilde{u}\|_{L^p(U\cap\Omega)} = \|\mathcal{E}u\|_{L^p(U\cap\mathcal{B})} + \|e(\tilde{u})\|_{L^p(U\cap(\Omega^{i_0}\setminus(\tilde{\Omega}\cup\mathcal{B})))} + \|e(\tilde{u})\|_{L^p(U\cap\tilde{\Omega})} + \|\mathcal{E}u\|_{L^p(U\cap(\Omega\setminus\Omega^{i_0}))}.$$

Hence, due to (5.31),(5.39) and (5.40) we have

$$\begin{aligned}&\|e(\tilde{u})\|_{L^p(U\cap(\Omega^{i_0}\setminus(\tilde{\Omega}\cup\mathcal{B})))} \le \sum_{k>N_0}\left(\|e(u_k)\|_{L^p(Q'_k)} + \sum_{j\sim k}\|\nabla\varphi_j\|_{L^\infty(Q'_j\cap Q'_k)}\|u_j - u_k\|_{L^p(Q'_k\cap Q'_j)}\right) \\ &\le C\sum_{Q'''_k\subseteq U_\delta, k>N_0}\left(\|e(u_k - a_k)\|_{L^p(Q'_k)} + \sum_{j\sim k}\frac{1}{d_j}\|u_j - a_j\|_{L^p(Q'_j)}\right. \\ &\qquad\left. + \frac{1}{d_j}\|a_j - a_k\|_{L^p(Q'_k\cap Q'_j)} + \frac{1}{d_k}\|a_k - u_k\|_{L^p(Q'_k)}\right) \\ &\le C\sum_{Q'''_k\subseteq U_\delta, k>N_0}\left(\left\|e(\rho_{r_k} * ((u - a_k)\chi_{Q''_k\setminus\omega_k}))\right\|_{L^p(Q'_k\cap Q'_j)} + \|e(u)\|_{L^p(Q'''_k)}\right) \\ &\le C\sum_{Q'''_k\subseteq U_\delta, k>N_0}\left(\frac{\|\nabla\rho\|_{L^1(\mathbb{R}^n)}}{r_k}\|u - a_k\|_{L^p(Q''_k\setminus\omega_k)} + \|e(u)\|_{L^p(Q'''_k)}\right) \\ &\le C\sum_{Q'''_k\subseteq U_\delta, k>N_0}\|e(u)\|_{L^p(Q'''_k)} \\ &\le C\|e(u)\|_{L^p(U_\delta\cap(C_{i_0}\cup C_{i_0+1}))}.\end{aligned}$$

By the choice of i_0 and since $u = \tilde{u}$ on $\mathcal{B}$ we see the that

$$\|e(\tilde{u})\|_{L^p(U\cap(\Omega^{i_0}\setminus\tilde{\Omega}))} \le C\|e(u)\|_{L^p(C_{i_0}\cup C_{i_0+1})} \le C\sqrt{\delta}\|e(u)\|_{L^p(\Omega_{1-\sqrt{\delta}})}.$$

Consequently, we have for $U = \Omega$ and $s \le \frac{1}{2}$

$$\begin{aligned}\|e(\tilde{u})\|_{L^p(\Omega)} &= \|e(u)\|_{L^p(\Omega\setminus\Omega^{i_0})} + \|e(\tilde{u})\|_{L^p(\Omega^{i_0})} \\ &\le \|e(u)\|_{L^p(\Omega\setminus\Omega^{i_0})} + \|e(u)\|_{L^p(\Omega\cap\Omega^{i_0})} + C\delta^s\|e(u)\|_{L^p(\Omega^{i_0})} + C\sqrt{\delta}\|e(u)\|_{L^p(\Omega_{1-\sqrt{\delta}})} \\ &\le (1 + C\delta^s)\|e(u)\|_{L^p(\Omega)}.\end{aligned}$$

Now we notice that $(1 + C\delta^s)^p \le 1 + pC(1 + C)^{p-1}\delta^s$ as long as η is chosen small enough to derive the third point in the theorem.

Showing $\mathcal{H}^{n-1}(J_{\tilde{u}} \cap \partial\Omega_R \setminus J_u) = 0$: We are left with showing that the inner trace $\tilde{u}$ and u coincide on $\partial\Omega^{i_0} = \partial\Omega_R$ at least $\mathcal{H}^{n-1}$ almost everywhere. Notice that with Lemma 5.1.6 we can derive $\tilde{u} \in GSBD^p(\Omega)$ if $\tilde{u}$ is extended outside Ω_R with u. Therefore, due to (5.27) we are left with showing $J_{\tilde{u}} \setminus N$ is a subset of $J_u \cup \partial^* B$ with some $\mathcal{H}^{n-1}$-negligible set N to conclude the proof of the first and second point of the theorem. For this we first estimate the L^p-norm of the difference of u and $\tilde{u}$ on $\Omega \setminus \omega$ with $\omega := \bigcup_{Q_j \text{ is good}} \omega_j \setminus \mathcal{B}$

$$
\begin{aligned}
\|\tilde{u} - u\|_{L^p(\Omega\setminus\omega)} &\leq \left\| \sum_{Q_j \text{ is good}} (u_j\varphi_j - u\varphi_j) \right\|_{L^p(\Omega\setminus\omega)} \\
&\leq \sum_{Q_j \text{ is good}} \|u_j\varphi_j - u\varphi_j\|_{L^p(Q_j\setminus\omega_j)} \\
&\leq \sum_{Q_j \text{ is good}} \|u_j - u\|_{L^p(Q_j'\setminus\omega_j)} \\
&\leq \sum_{Q_j \text{ is good}} \left(\|u_j - a_j\|_{L^p(Q_j'\setminus\omega_j)} + \|a_j - u\|_{L^p(Q_j'\setminus\omega_j)} \right) \\
&\overset{(5.31),(5.35)}{\leq} C\delta \sum_{Q_j \text{ is good}} \|\mathcal{E}u\|_{L^p(Q_j''')} \\
&= C\delta \, \|\mathcal{E}u\|_{L^p(\Omega)}
\end{aligned}
$$

Analogously, we can estimate the difference on $B_\gamma(y)$ for $y \in \partial\Omega_R$ and $\gamma > 0$ small enough, i.e., so small that all cubes Q_j with $Q_j \cap B_\gamma(y) \neq \emptyset$ fulfil $d_j \leq \operatorname{dist}(Q_j, \partial\Omega^{i_0}) \leq \gamma$:

$$
\begin{aligned}
\|\tilde{u} - u\|_{L^p(B_\gamma(y)\setminus\omega)} &\leq \left\| \sum_{Q_j \text{ is good}} (u_j\varphi_j - u\varphi_j) \right\|_{L^p(B_\gamma(y)\setminus\omega)} \\
&\leq \sum_{\substack{Q_j \text{ is good} \\ B_\gamma(y)\cap Q_j' \neq \emptyset}} \|u_j\varphi_j - u\varphi_j\|_{L^p(Q_j'\setminus\omega_j)} \\
&\leq \sum_{\substack{Q_j \text{ is good} \\ B_\gamma(y)\cap Q_j' \neq \emptyset}} \left(\|u_j - a_j\|_{L^p(Q_j\setminus\omega_j)} + \|a_j - u\|_{L^p(Q_j\setminus\omega_j)} \right) \\
&\overset{(5.31),(5.35)}{\leq} Cd_j \sum_{\substack{Q_j \text{ is good} \\ B_\gamma(y)\cap Q_j \neq \emptyset}} \|\mathcal{E}u\|_{L^p(Q_j''')} \\
&\leq C\gamma \, \|\mathcal{E}u\|_{L^p(B_{3\gamma}(y))} .
\end{aligned}
$$

Analogously, we can derive

$$
\begin{aligned}
|\omega \cap B_\gamma(y)| &= \sum_{\substack{Q_j \text{ is good} \\ B_\gamma(y)\cap Q_j \neq \emptyset}} |\omega \cap Q_j| \\
&\leq C \sum_{\substack{Q_j \text{ is good} \\ B_\gamma(y)\cap Q_j'' \neq \emptyset}} |\omega_j| \\
&\leq C \sum_{\substack{Q_j \text{ is good} \\ B_\gamma(y)\cap Q_j'' \neq \emptyset}} d_j \mathcal{H}^{n-1}(J_u \cap Q_j''') \\
&\leq C\gamma \mathcal{H}^{n-1}(J_u \cap B_{3\gamma}(y)).
\end{aligned}
$$

From these last inequalities, we infer for an arbitrary $\epsilon > 0$

$$\begin{aligned}\frac{|\{|\tilde{u}-u|>\epsilon\}\cap B_\gamma(y)|}{\gamma^n} &= \frac{|\omega\cap\{|\tilde{u}-u|>\epsilon\}\cap B_\gamma(y)|}{\gamma^n} + \frac{|\{|\tilde{u}-u|>\epsilon\}\cap(B_\gamma(y)\setminus\omega)|}{\gamma^n} \\ &\le \frac{|\tilde{\omega}\cap B_\gamma(y)|}{\gamma^n} + \frac{1}{\epsilon\gamma^n}\|\tilde{u}-u\|_{L^p(B_\gamma(y)\setminus\omega)} \\ &\le C\left(\frac{\mathcal{H}^{n-1}(J_u\cap B_{3\gamma}(y))}{\gamma^{n-1}} + \frac{1}{\gamma^{n-1}}\|\mathcal{E}u\|_{L^p(B_{3\gamma}(y))}\right).\end{aligned}$$

By construction, $\mathcal{H}^{n-1}$-a.e. $y \in \partial\Omega_R$ is not a jump point of u and a Lebesgue point of $\mathcal{E}u$. Consequently, the first term of last line above vanishes for $\gamma \to 0^+$ because of Theorem 5.3.4 and the second since for $p = 1$

$$\limsup_{\gamma\to 0^+}\frac{1}{\gamma^n}\|\mathcal{E}u\|_{L^1(B_\gamma(y))} = \gamma_n \operatorname{ap}\lim_{x\to y}|\mathcal{E}u|.$$

Therefore, we have shown

$$\operatorname{ap}\lim_{x\to y}\tilde{u} = \operatorname{ap}\lim_{x\to y}u.$$

for $\mathcal{H}^{n-1}$ a.e. $y \in \partial\Omega_R$, i.e., the (inner) trace of $\tilde{u}$ and u coincide. This means there are no additional jumps (up to a negligible amount) occurring at $\partial\Omega_R$ for $\tilde{u}$ which concludes the proof. □

5.4 The Korn inequality in $GSBD^p(\Omega)$

In this section, we will discuss the main result of this chapter: the Korn inequality in $GSBD^p(\Omega)$ (5.4). For this will first present the approximation theorems which lead to this result. Similarly to Theorem 5.3.5, they are based on a covering argument in combination with applying the results from the previous sections. Heuristically, they state that functions in $GSBD^p(\Omega)$ which have a very small jump set are nearly Sobolev functions.

The coverings used to show these results are only partly based on the Whitney Lemma 5.3.1. In particular, for the first approximation theorem will need a consequence of the Besicovitch covering theorem (cf. for instance Theorem 5.1 in [Mag12]):

Lemma 5.4.1. *[Mag12, Corollary 5.2] Let $\mathcal{F}$ be a family of closed, non-degenerate balls in $\mathbb{R}^n$, $n \ge 1$. Denote their set of centers with C. Suppose C is bounded. Then, for each outer measure μ there exists a countable, disjoint subfamily $\mathcal{F}'$ of $\mathcal{F}$ with*

$$\mu(C) \le \xi \sum_{\bar{B}\in\mathcal{F}'}\mu(C\cap\bar{B})$$

where $\xi \in \mathbb{N}$ only depends on the dimension n.

Before giving the proof of the approximation theorem we will start with observing the following: Similar to section 4.2 where we obtained a result for the cube $(-1,1)^n$ and then extended the results to arbitrary cubes via scaling (and translation) we can do the same for Theorem 5.3.5. First, we apply the theorem to the unit ball B_1. Then, we scale and translate the results and assumptions to $B_\rho(x)$ for some arbitrary $x \in \mathbb{R}^n$. In doing so the constant obtained from the theorem for B_1 does not change. We will denote this constant by $\bar{\delta}$. Since we can assume $\bar{\delta}$ to be arbitrary small we require it to be smaller than $\gamma_{n-1} := \mathcal{H}^{n-1}(\mathcal{S}^{n-1})$. The only lines that changes are the involved with the term $\mathcal{H}^{n-1}(J_u)$ due to scaling, i.e., for instance we need $\mathcal{H}^{n-1}(J_u) \leq \bar{\delta}^n \rho^{n-1}$ to hold for $u \in GSBD^p(B_\rho(x))$. We write this requirements with $\leq$ instead of $<$ like in Theorem 5.3.5 by choosing $\bar{\delta}$ a little bit smaller. With this in mind, we state the approximation result:

Theorem 5.4.2. *[CCS22, Theorem 3.1] Let $n \in \mathbb{N}, n \geq 2$, $p \in (1,\infty)$, $\epsilon > 0$ and $\sigma \in (0,1)$. Then there exists a constants $\tau > 0$ only dependent on n, p, ϵ and σ and $C > 0$ only dependent on n, p and ϵ such that for any $\rho > 0$ and any $u \in GSBD^p(B_\rho)$ with $\mathcal{H}^{n-1}(J_u) < \tau\rho^{n-1}$ there exists a set of finite perimeter $\omega \subseteq B_\rho$ with $\mathcal{H}^{n-1}(\partial^*\omega) \leq C\mathcal{H}^{n-1}(J_u)$ and $w \in W^{1,p}(B_{(1-\sigma)\rho})$ such that $u = w$ in $B_\rho \setminus \omega$,*

$$\int_\Omega |\mathcal{E}w|^p dx \leq (1+\epsilon)\int_\Omega |\mathcal{E}u|^p dx$$

and

$$\mathcal{H}^{n-1}(J_w) \leq \mathcal{H}^{n-1}(J_u).$$

Proof. We will choose $\tau > 0$ smaller and smaller during the proof. The aim is to iteratively construct a sequence of $\{w_k\} \in GSBD^p(\Omega)$ by applying Theorem 5.3.5 on 'good' balls such that the jump set successively decreases and the sequence converges pointwise to a function w. With the help of the compactness Theorem 5.1.10 we will then derive $w \in GSBD^p(\Omega)$. For the start, we first fix $\alpha \in (0,1)$ where the exact value will be determined later. The role of α is to ensure the convergence of w_k. We define the sequence inductively such that the following properties hold for all $k \in \mathbb{N}$:

(i) $\mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) < \eta_k(s_k\rho_k)^{n-1}$,

(ii) $\mathcal{H}^{n-1}(J_{w_{k+1}}) \leq \mathcal{H}^{n-1}(J_{w_k})$,

(iii) $s_k = \lambda^k s_0$,

(iv) $\eta_k = \lambda^k \eta_0$,

(v) $\rho_k = \rho\prod_{j=1}^{k-1}(1-\lambda^j s_0)$.

$\lambda > 0$ will be determined at a later point in the proof and depends on the constant obtained from the classical Besicovitch Theorem, i.e., the constant ξ from Lemma 5.4.1, and s_0 which will be chosen at the induction start. We begin the induction by setting

$$w_0 := u, \qquad \eta_0 := (\alpha\bar{\delta})^n, \qquad \rho_0 = \rho.$$

Then we choose any

$$s_0 \in \left(\left(\frac{\mathcal{H}^{n-1}(J_u)}{\rho^{n-1}\eta_0}\right)^{\frac{1}{n-1}}, \left(\frac{\tau}{\eta_0}\right)^{\frac{1}{n-1}}\right).$$

Notice that choosing such a s_0 is possible by assumption and we then have

$$\mathcal{H}^{n-1}(J_{w_0} \cap B_{\rho_0}) = \mathcal{H}^{n-1}(J_u) < \eta_0(\rho_0 s_0)^{n-1}.$$

Constructing a suitable covering: For the induction step we are given $w_k \in GSBD^p(B_\rho)$, $s_k \in (0,1)$, $\rho_k \le \rho$ and $\eta_k \le \bar{\delta}^n$ such that properties (i) to (v) hold. We will now construct a covering $B_{r_i}(x_i)$ of a large part $J_{w_k} \cap B_{(1-s_k)\rho_k}$ which fulfils

$$\begin{cases} \mathcal{H}^{n-1}(J_{w_k} \cap B_{r_i}(x_i)) = \eta_k r_i^{n-1}, \\ \mathcal{H}^{n-1}(J_{w_k} \cap B_{r_i}(x_i)) \ge \eta_k r^{n-1} \text{ for all } r \le r_i, \\ \mathcal{H}^{n-1}(J_{w_k} \cap \partial B_{r_i}(x_i)) = 0. \end{cases} \tag{5.43}$$

Here, we notice that the jump set J_{w_k} is $\mathcal{H}^{n-1}$-rectifiable and therefore (4.5) holds for $\mathcal{H}^{n-1}$-a.e. $x \in J_{w_k}$. Let us denote the set where (4.5) holds with $C \subseteq J_{w_k} \cap B_{\rho_k(1-s_k)}$. Notice, that $\mathcal{H}^{n-1}(C) = \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k(1-s_k)})$. For $x \in C$, we define the function

$$\phi(r) := \frac{\mathcal{H}^{n-1}(J_{w_k} \cap B_r(x))}{r^{n-1}}$$

on $(0, s_k r_k)$. From (4.5) we know $\lim_{r\to 0^+} \phi(r) = \gamma_{n-1}$. However, also observe that $\phi(s_k\rho_k) < \eta_k$ by assumption. Now set $A := \{r \in (0, s_k\rho_k) : \phi(r) < \eta_k\}$ and

$$r_x := \inf A.$$

Since $\mathcal{H}^{n-1}|_{J_{w_k}}$ is inner regular we have the left continuity of ϕ. By

$$\lim_{r\to\tilde{r}^+} \phi(r) = \phi(\tilde{r}) + \frac{\mathcal{H}^{n-1}(J_{w_k} \cap \partial B_{\tilde{r}}(x))}{\tilde{r}^{n-1}} > \phi(\tilde{r}) \tag{5.44}$$

we know that ϕ is also lower semi-continuous. Being left continuous paired with $\phi(s_k\rho_k) < \eta_k$ means that $A \neq \emptyset$. Furthermore, the lower semi-continuity together with $\lim_{r\to 0^+} \phi(r) = \gamma_{n-1} > \eta_k$ implies $r_x > 0$. By construction $\phi(r) \ge \eta_k$ for all $r \le r_x$ and therefore $\phi(r_x) \ge \eta_k$ by left continuity. In particular, $r_x \notin A$ which means that r_x is an accumulation point of A. This implies

$$\eta_k \le \phi(r_x) \le \lim_{\substack{r\to r_x \\ r\in A}} \phi(r) \le \eta_k,$$

so $\phi(r_x) = \eta_k$. By (5.44) we also have

$$\frac{\mathcal{H}^{n-1}(J_{w_k} \cap \partial B_{r_x}(x))}{r_x^{n-1}} = 0.$$

Now, we apply Lemma (5.4.1) to the family $\mathcal{F} = \{B_{r_x} : x \in C\}$. We can therefore extract countable many $\{x_i\}_{i\in\mathbb{N}} \subseteq C$ such that with $r_i := r_{x_i}$ and $B_i := B_{r_i}(x_i)$ we have

$$\mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k(1-s_k)}) = \mathcal{H}^{n-1}(C) \le \xi \sum_{i=1}^{n} \mathcal{H}^{n-1}(C \cap \overline{B_i}) = \xi \sum_{i=1}^{n} \mathcal{H}^{n-1}(J_{w_k} \cap \overline{B_i})$$

for some $\xi \ge 1$. Observe that by (5.43) we also have

$$\mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k(1-s_k)}) \le \xi \sum_{i=1}^{n} \mathcal{H}^{n-1}(J_{w_k} \cap B_i). \tag{5.45}$$

Definition of w_{k+1}: Set

$$\theta := \frac{2\xi}{1+2\xi}.$$

When defining w_{k+1} we distinguish two cases. Either

$$\mathcal{H}^{n-1}(J_{w_k} \cap B_{(1-s_k)\rho_k}) < \theta \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k})$$

holds or the opposite inequality, which implies

$$\mathcal{H}^{n-1}(J_{w_k} \cap (B_{\rho_k} \setminus B_{(1-s_k)\rho_k})) \leq (1-\theta)\mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}). \tag{5.46}$$

In the first case, we just set $w_{k+1} := w_k$. Notice that by assumption

$$\mathcal{H}^{n-1}(J_{w_k} \cap B_{(1-s_k)\rho_k}) \leq \theta \eta_k (s_k \rho_k)^{n-1}.$$

In the second case, we have by construction

$$\mathcal{H}^{n-1}(J_{w_k} \cap B_i) = \eta_k r_i^{n-1} < \bar{\delta}^n r_i^{n-1}.$$

We can therefore apply Theorem 5.3.5 to B_i and $w_k|_{B_i}$ to get an approximation $\tilde{w}_{k,i} \in GSBD^p(B_i)$ with corresponding radius $R_{k,i} \in (\eta_k^{\frac{1}{2n}}, 1)$. At this point, also notice that the δ in Theorem 5.3.5 is nothing else than

$$\left(\frac{\mathcal{H}^{n-1}(J_{w_k} \cap B_i)}{r_i^{n-1}}\right)^{\frac{1}{n}} \leq \eta_k^{\frac{1}{n}}.$$

We now set

$$w_{k+1} := \begin{cases} w_k & , \text{ in } B_\rho \setminus (\bigcup_{i \in \mathbb{N}} B_i), \\ \tilde{w}_{k,i} & , \text{ in } B_i. \end{cases}$$

To see that $w_{k+1} \in GSBD^p(\Omega)$ we argue via the compactness theorem for $GSBD$. We define the approximation sequence

$$w_{k+1}^l := \begin{cases} w_k & , \text{ in } B_\rho \setminus (\bigcup_{i=0}^l B_i), \\ \tilde{w}_{k,i} & , \text{ in } B_i \text{ for } i = 0, .., l. \end{cases}$$

Since this piecewise defined function fulfils the assumption to apply Lemma 5.1.6 we have $w_{k+1}^l \in GSBD(B_\rho)$ for all $l \in \mathbb{N}$. Since $\bigcup_{i=0}^l B_i$ and $B_\rho \setminus \bigcup_{i=0}^l \overline{B_i}$ are open with

$$\lambda^n \left(\partial \left(\bigcup_{i=0}^l B_i\right)\right) = 0$$

we further have

$$\mathcal{E} w_{k+1}^l = \begin{cases} \mathcal{E} w_k & , \text{ in } B_\rho \setminus (\bigcup_{i=0}^l B_i), \\ \mathcal{E} \tilde{w}_{k,i} & , \text{ in } B_i \text{ for } i = 0, .., l. \end{cases}$$

Since by Theorem 5.3.5

$$\int_{B_i} |\mathcal{E} \tilde{w}_{k,i}|^p \, dx \leq \left(1 + C\left(\frac{\mathcal{H}^{n-1}(J_{w_k} \cap B_i)}{r_i^{n-1}}\right)^{\frac{s}{n}}\right) \int_{B_i} |\mathcal{E} w_k| \, dx \leq \left(1 + C\eta_k^{\frac{s}{n}}\right) \int_{B_i} |\mathcal{E} w_k|^p \, dx$$

holds we have $w_{k+1}^l \in GSBD^p(B_\rho)$ with

$$\int_{B_\rho} |\mathcal{E} w_{k+1}^l|^p \, dx \leq \left(1 + C\eta_k^{\frac{s}{n}}\right) \int_{B_\rho} |\mathcal{E} w_k|^p \, dx.$$

Notice that $J_{w_{k+1}} \cap B_i = J_{w^l_{k+1}} \cap B_i = J_{\tilde{w}_{k,i}} \cap B_i$ for $i = 1, .., l$ since B_i is open and $w_{k+1} = w^l_{k+1} = \tilde{w}_{k,i}$ on B_i. So we will use the terms interchangeably when talking about quantities in B_i.

Now, set $B'_i = B_{\left(1-\eta_k^{1/(2n)}\right)r_i}$ and $B''_i = B_{R_{k,i}r_i}$. By the second result of Theorem 5.3.5 we can now derive

$$\mathcal{H}^{n-1}((J_{w_{k+1}} \setminus J_{w_k}) \cap B_i) \le C\eta_k^{\frac{1}{2n}} \mathcal{H}^{n-1}(J_{w_k} \cap (B_i \setminus B'_i)). \tag{5.47}$$

By the first result of Theorem 5.3.5 we observe $\tilde{w}_{k,i} \in C^\infty(B'_i)$ and $(J_{\tilde{w}_{k,i}} \setminus J_{w_k}) \subseteq (B''_i \setminus B'_i)$ consequently

$$\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B'_i) = 0 \qquad \text{and} \qquad \mathcal{H}^{n-1}(J_{w_{k+1}} \setminus J_{w_k}) \cap (B_i \setminus B''_i)) = 0.$$

In particular, this means we can derive

$$\begin{aligned}
\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B_i) &= \mathcal{H}^{n-1}((J_{w_{k+1}}) \cap (B_i \setminus B'_i)) \\
&= \mathcal{H}^{n-1}((J_{w_{k+1}} \setminus J_{w_k}) \cap (B_i \setminus B'_i)) + \mathcal{H}^{n-1}(J_{w_k} \cap (B_i \setminus B'_i)) \\
&= \mathcal{H}^{n-1}((J_{w_{k+1}} \setminus J_{w_k}) \cap (B''_i \setminus B'_i)) + \mathcal{H}^{n-1}(J_{w_k} \cap (B_i \setminus B'_i)) \\
&\overset{(5.47)}{\le} (1 + C\eta_k^{\frac{1}{2n}})\mathcal{H}^{n-1}(J_{w_k} \cap (B_i \setminus B'_i)).
\end{aligned}$$

We can estimate further

$$\begin{aligned}
(1 + C\eta_k^{\frac{1}{2n}})\mathcal{H}^{n-1}(J_{w_k} \cap (B_i \setminus B'_i)) &= (1 + C\eta_k^{\frac{1}{2n}})(\mathcal{H}^{n-1}(J_{w_k} \cap B_i) - \mathcal{H}^{n-1}(J_{w_k} \cap B'_i)) \\
&\overset{(5.43)}{\le} (1 + C\eta_k^{\frac{1}{2n}})(\eta_k r_i^{n-1} - \eta_k((1 - \eta_k^{\frac{1}{2n}})r_i)^{n-1}) \\
&= (1 + C\eta_k^{\frac{1}{2n}})(1 - (1 - \eta_k^{\frac{1}{2n}})^{n-1})\eta_k r_i^{n-1} \\
&\overset{(5.43)}{=} (1 + C\eta_k^{\frac{1}{2n}})(1 - (1 - \eta_k^{\frac{1}{2n}})^{n-1})\mathcal{H}^{n-1}(J_{w_k} \cap B_i).
\end{aligned}$$

If $\bar{\delta}$ is chosen small enough such that

$$(1 + C\bar{\delta}^{\frac{1}{2n}})(1 - (1 - \bar{\delta}^{\frac{1}{2n}})^{n-1}) \le \frac{1}{2}$$

we can infer

$$\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B_i) \le \frac{1}{2}\mathcal{H}^{n-1}(J_{w_k} \cap B_i). \tag{5.48}$$

Notice now that

$$\mathcal{H}^{n-1}\left(J_{w_{k+1}} \cap \partial\left(\bigcup_{i=0}^{l} B_i \setminus B''_i\right)\right) = \mathcal{H}^{n-1}\left(J_{w_k} \cap \partial\left(\bigcup_{i=0}^{l} B_i \setminus B''_i\right)\right) = 0,$$

$w^l_{k+1} = w_k$ on $\bigcup_{i=0}^{l}(B_i \setminus B''_i)$ and

$$\operatorname{dist}\left(\{x \in B_\rho : w^l_{k+1}(x) \ne w_k(x)\}, \partial\left(\bigcup_{i=0}^{l} B_i\right)\right) \ge \min_{i=0,..,l} R_{k,i}.$$

This means approximate limits along the boundary of $\bigcup_{i=0}^{l} B_i$ only consider values of w_k and we can infer that no additional jump set is generated along $\partial\left(\bigcup_{i=0}^{l} B_i\right)$ for w_{k+1}^l. In particular, we can now derive

$$\mathcal{H}^{n-1}(J_{w_{k+1}^l}) \leq \mathcal{H}^{n-1}(J_{w_k}).$$

We have checked the assumptions that we can apply the compactness Theorem 5.1.10 to the sequence $\{w_{k+1}^l\}_l$. Since w_{k+1}^l converges pointwise to w_{k+1} we have $w_{k+1} \in GSBD^p(B_\rho)$ with $w_{k+1}^l \rightharpoonup w_{k+1}$ in $L^p(B_\rho)$ by (5.8). Notice that the set A obtained from the compactness theorem is empty. By Theorem 5.1.10 we also have

$$\mathcal{H}^{n-1}(J_{w_{k+1}}) \leq \liminf_{l\to\infty} \mathcal{H}^{n-1}(J_{w_{k+1}^l}) \leq \mathcal{H}^{n-1}(J_{w_k})$$

and from the weak convergence of the approximate symmetric gradients in L^p (cf. (5.8)) we infer

$$\int_{B_\rho} |\mathcal{E}w_{k+1}|^p \, dx \leq \liminf_{l\to\infty} \int_{B_\rho} |\mathcal{E}w_{k+1}^l|^p \, dx \leq \left(1 + C\eta_k^{\frac{s}{n}}\right) \int_{B_\rho} |\mathcal{E}w_k|^p \, dx.$$

Now, notice that

$$\begin{aligned}
\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B_{\rho_k}) &\leq \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) - \mathcal{H}^{n-1}((J_{w_{k+1}} \setminus J_{w_k}) \cap B_{\rho_k}) \\
&\leq \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) - \sum_{i\in\mathbb{N}} \mathcal{H}^{n-1}((J_{w_{k+1}} \setminus J_{w_k}) \cap B_i) \\
&\leq \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) - \sum_{i\in\mathbb{N}} (\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B_i) - \mathcal{H}^{n-1}(J_{w_k} \cap B_i)) \\
&\overset{(5.48)}{\leq} \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) - \frac{1}{2} \sum_{i\in\mathbb{N}} \mathcal{H}^{n-1}(J_{w_k} \cap B_i) \\
&\overset{(5.45)}{\leq} \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) - \frac{1}{2\xi} \mathcal{H}^{n-1}(J_{w_k} \cap B_{(1-s_k)\rho_k}) \\
&= \mathcal{H}^{n-1}(J_{w_k} \cap (B_{\rho_k} \setminus B_{(1-s_k)\rho_k})) + \left(1 - \frac{1}{2\xi}\right) \mathcal{H}^{n-1}(J_{w_k} \cap B_{(1-s_k)\rho_k}) \\
&\overset{(5.46)}{\leq} \left(1 - \frac{\theta}{2\xi}\right) \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) \\
&= \theta \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k})
\end{aligned} \tag{5.49}$$

As an immediate consequence, we have

$$\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B_{(1-s_k)\rho_k}) \overset{(5.49)}{\leq} \theta \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) < \theta \eta_k (s_k \rho_k)^{n-1}.$$

Now set

$$\lambda := \left(\frac{\theta}{(1-s_0)^{n-1}}\right)^{\frac{1}{n}} < \left(\frac{\theta}{(1-\tau^{\frac{1}{n-1}})^{n-1}}\right)^{\frac{1}{n}} \eta_0.$$

By choosing τ so small that

$$\left(\frac{1}{(1-\tau^{\frac{1}{n-1}})^{n-1}}\right)^{\frac{1}{n}} \alpha\bar{\delta} < 1$$

we can ensure $\lambda \leq \theta^{\frac{1}{n}} < 1$. Here, we define

$$\rho_{k+1} := (1 - s_k)\rho_k, \quad \eta_{k+1} := \lambda\eta_k, \quad \text{and} \quad s_{k+1} := \lambda s_k.$$

We observe that

$$\begin{aligned}\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B_{(1-s_k)\rho_k}) &< \theta\eta_k(s_k\rho_k)^{n-1} \\ &= \frac{\theta}{\lambda^n(1-s_k)^{n-1}}\eta_{k+1}(s_{k+1}\rho_{k+1})^{n-1} \\ &< \eta_{k+1}(s_{k+1}\rho_{k+1})^{n-1}.\end{aligned}$$

With this, we conclude the induction step.

Pointwise convergence of w_k: We will now show that w_k converges pointwise λ^n-a.e., more specifically, we will see that w_k is λ^n-a.e. pointwise eventually constant. Set $\omega_k := \bigcup_{i\in\mathbb{N}} \overline{B_i}$ (notice that B_i is dependent on k by the construction above). Due to the lower semi-continuity of the perimeter, we can estimate

$$\begin{aligned}\mathcal{H}^{n-1}(\partial^*\omega_k) \leq \sum_{i\in\mathbb{N}} \mathcal{H}^{n-1}(\partial B_i) &= \gamma_{n-1}\sum_{i\in\mathbb{N}} r_i^{n-1} \\ &= \frac{\gamma_{n-1}}{\eta_k}\sum_{i\in\mathbb{N}} \mathcal{H}^{n-1}(J_{w_k} \cap B_i) \\ &\leq \frac{\gamma_{n-1}}{\eta_k}\mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) \\ &< \gamma_{n-1}(s_k\rho_k)^{n-1}.\end{aligned}$$

In particular, we have for $l \in \mathbb{N}$

$$\begin{aligned}\mathcal{H}^{n-1}\left(\partial^*\left(\bigcup_{k\geq l}\omega_k\right)\right) &\leq \sum_{k\geq l}\mathcal{H}^{n-1}(\partial^* w_k) \\ &< \sum_{k\geq l}\gamma_{n-1}(s_k\rho_k)^{n-1} \\ &\leq \gamma_{n-1}\sum_{k\geq l}(\lambda^{k-l}s_l\rho_l)^{n-1} \\ &= \gamma_{n-1}(s_l\rho_l)^{n-1}\sum_{k\geq l}(\lambda^{k-l})^{n-1} \\ &= \gamma_{n-1}(s_l\rho_l)^{n-1}\frac{1}{1-\lambda^{n-1}}.\end{aligned}$$

Consequently, since ρ_l is a decreasing sequence and $s_l \xrightarrow{l\to\infty} 0$ we have

$$\mathcal{H}^{n-1}\left(\partial^*\left(\bigcup_{k\geq l}\omega_k\right)\right) \xrightarrow{l\to\infty} 0$$

which implies

$$\left|\bigcup_{k\geq l}\omega_k\right| \xrightarrow{l\to\infty} 0 \tag{5.50}$$

by the isoperimetric inequality (cf. Theorem 14.1 in [Mag12]). Observe now that

$$\{x \in B_\rho : w_k(x) \neq w_{k+1}(x)\} \subseteq \omega_k$$

which implies

$$\{x \in B_\rho : \text{ there exists some } k \in \mathbb{N} \text{ with } w_l(x) \neq w_k(x)\} \subseteq \bigcup_{k \geq l} \omega_k.$$

From this, we can infer

$$(\limsup_{k\to\infty} \omega_k)^c \subseteq \{x \in B_\rho : \ w_k(x) \text{ is eventually constant } \}.$$

Since $|\limsup_{k\to\infty} \omega_k| = 0$ by (5.50), i.e., $\{x \in B_\rho : \ w_k(x) \text{ is eventually constant } \}$ has full measure, we know that w_k is λ^n-a.e. eventually constant and therefore converges pointwise λ^n-a.e. to some function w.

Show $w \in GSBD^p(\Omega)$: We now want to again deploy Theorem 5.1.10. We have already seen that

$$\int_{B_i} |\mathcal{E} w_{k+1}| \ dx \leq \left(1 + C\eta_k^{\frac{s}{n}}\right) \int_{B_i} |\mathcal{E} w_k| \ dx$$

and consequently using induction we can infer

$$\begin{aligned}
\int_{B_\rho} |\mathcal{E} w_{k+1}| \ dx &\leq \prod_{j=1}^{k-1} \left(1 + C\eta_j^{\frac{s}{n}}\right) \int_{B_\rho} |\mathcal{E} u| \ dx \\
&\leq \prod_{j=1}^{k-1} \left(1 + C\lambda^{\frac{js}{n}}\eta_0\right) \int_{B_\rho} |\mathcal{E} u| \ dx \\
&\leq \prod_{j=1}^{\infty} \left(1 + C\lambda^{\frac{js}{n}}\eta_0\right) \int_{B_\rho} |\mathcal{E} u| \ dx \\
&\leq \prod_{j=1}^{\infty} \exp(C\lambda^{\frac{js}{n}}\eta_0) \int_{B_\rho} |\mathcal{E} u| \ dx \\
&= \exp(C\eta_0 \sum_{j=1}^{\infty} \lambda^{\frac{js}{n}}) \int_{B_\rho} |\mathcal{E} u| \ dx \\
&= \exp\left(\frac{C\eta_0}{1 - \lambda^{\frac{s}{n}}}\right) \int_{B_\rho} |\mathcal{E} u| \ dx.
\end{aligned}$$

Since $\eta_0 = \alpha\bar{\delta}$ and $\lambda \leq \theta^{\frac{1}{n}}$ at this point we can choose α small enough that for the given $\epsilon > 0$ we have

$$\exp\left(\frac{C\eta_0}{1 - \lambda^{\frac{s}{n}}}\right) \leq \exp\left(\frac{C\alpha\bar{\delta}}{1 - \theta^{\frac{s}{n^2}}}\right) < 1 + \epsilon.$$

Therefore, we have

$$\int_{B_\rho} |\mathcal{E} w_k|^p \ dx \leq (1 + \epsilon) \int_{B_\rho} |\mathcal{E} u|^p \ dx \tag{5.51}$$

for all $k \in \mathcal{H}^{n-1}$ and since by construction $\mathcal{H}^{n-1}(J_{w_{k+1}}) \leq \mathcal{H}^{n-1}(J_{w_k})$ holds we also have inductively

$$\mathcal{H}^{n-1}(J_{w_k}) \leq \mathcal{H}^{n-1}(J_u).$$

We have shown that the prerequisites of Theorem 5.1.10 are fulfilled with $\psi(t) = t^p$. We can therefore extract a subsequence w_{k_j} such that $w_{k_j} \to \tilde{w}$ pointwise λ^n-a.e. on $B_\rho \setminus A$ for a $\tilde{w} \in GSBD(\Omega)$ with

$$A = \{x \in \Omega : |w_{k_j}(x)| \to +\infty\}.$$

But since

$$\{x \in \Omega : |w_{k_j}(x)| \to +\infty\} \subseteq \limsup_{k\to\infty} \omega_k$$

we have $|A| = 0$, in particular, $w_{k_j} \to \tilde{w}$ pointwise λ^n-a.e. on B_ρ. But since also $w_k \to w$ pointwise λ^n-a.e. on B_ρ we have $\tilde{w} = w$ at least λ^n-a.e. which in particular implies that $w \in GSBD^p(\Omega)$. By Theorem 5.1.10, we also have

$$\mathcal{H}^{n-1}(J_w) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{w_k}) \leq \mathcal{H}^{n-1}(J_u)$$

and $\mathcal{E}w_k \rightharpoonup \mathcal{E}w$ in $L^p(B_\rho)$ from which we further derive

$$\int_{B_\rho} |\mathcal{E}w|^p dx \leq \liminf_{k\to\infty} \int_{B_\rho} |\mathcal{E}w_k|^p dx \leq (1+\epsilon) \int_{B_\rho} |\mathcal{E}u|^p dx.$$

Show $w \in W^{1,p}(B_{(1-\sigma)\rho})$: First, observe that in the first case of the induction step we have naturally

$$\mathcal{H}^{n-1}(J_{w_{k+1}} \cap B_{\rho_k}) \leq \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k})$$

Together with (5.49) this implies that

$$\mathcal{H}^{n-1}(J_{w_i} \cap B_{\rho_k}) \leq \mathcal{H}^{n-1}(J_{w_j} \cap B_{\rho_k}) \leq \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) \leq \eta_k (s_k \rho_k)^{n-1}$$

holds for all $i, j \in \mathbb{N}$ with $i \leq j \leq k$. We can use the same line of argumentation as for (5.51) to derive

$$\int_{B_{\rho_k}} |\mathcal{E}w|^p dx \leq (1+\epsilon) \int_{B_{\rho_k}} |\mathcal{E}u|^p dx.$$

Now, we can apply the same compactness method we used for w_k on B_ρ to derive

$$\mathcal{H}^{n-1}(J_w \cap B_{\rho_k}) \leq \eta_k (s_k \rho_k)^{n-1}. \tag{5.52}$$

Now let $\rho' = \lim_{k\to\infty} \rho_k$ which exists due to the monotonicity of the sequence $\{\rho_k\}$. By (5.52) derive

$$\mathcal{H}^{n-1}(J_w \cap B_{\rho'}) \leq \eta_k (s_k \rho_k)^{n-1}$$

for all $k \in \mathbb{N}$ and with $k \to \infty$ we have $\mathcal{H}^{n-1}(J_w \cap B_{\rho'}) = 0$. By covering B_ρ with cubes (for instance with Lemma 5.3.1) and applying Proposition 5.2.4 to each cube we derive $w \in W^{1,p}_{loc}(B_{\rho'})$. Since $\mathcal{E}w \in L^p$ we can infer $w \in W^{1,p}(B_{\rho'})$ with the help of Korn inequality (cf. Lemma 3.2.8).

We are left to show that $\rho' \geq (1-\sigma)\rho$ if τ is chosen suitable small. Now, we use the known inequality

$$(1-xy) \geq (1-y)^x$$

for $x, y \in (0,1)$ to show

$$\rho' = \rho \prod_{j\in\mathbb{N}} (1-\lambda^j s_0) \geq \rho \prod_{j\in\mathbb{N}} (1-s_0)^{\lambda^j} = \rho(1-s_0)^{\sum_{j\in\mathbb{N}}\lambda^j} = \rho(1-s_0)^{\frac{1}{1-\lambda}} \geq \rho(1-s_0)^{\frac{1}{1-\sqrt[n]{\theta}}}.$$

Since $s_0 \to 0$ for $\tau \to 0$ we can choose τ small enough to ensure

$$\rho(1-s_0)^{\frac{1}{1-\sqrt[n]{\theta}}} \geq \rho(1-\sigma)$$

which concludes the proof. □

Note at this point that B_ρ can be substituted with $(-\rho,\rho)^n$ and the proof would exactly work the same way. When comparing Theorem 5.3.5 and Theorem 5.4.2 the most notable feature of the approximation theorem in this section is that the jump set indeed decreases when using this kind of approximation. This can be used to prove a global version of this theorem:

Theorem 5.4.3. *[CCS22, Theorem 4.1] Let $n \in \mathbb{N}, n \geq 2$, $p \in (1,\infty)$ and let $\Omega \subseteq \mathbb{R}^n$ be open, bounded with Lipschitz boundary. Then there exists a constant $C > 0$ only dependent on n, p and Ω such that for any $u \in GSBD^p(\Omega)$ exists a set of finite perimeter $\omega \subseteq \Omega$ with $\mathcal{H}^{n-1}(\partial^*\omega) \leq C\mathcal{H}^{n-1}(J_u)$ and $v \in W^{1,p}(\Omega)$ such that $u = v$ in $\Omega \setminus \omega$ and*

$$\int_\Omega |\mathcal{E}v|^p dx \leq C \int_\Omega |\mathcal{E}u|^p dx \tag{5.53}$$

The proof of this theorem is very similar to the already discussed proof of Theorem 5.3.5. More concretely, one uses a Whitney-type covering and distinguishes the cubes into 'good' and 'bad' ones. Then, the approximation happens similarly by using a suitable partition of unity with respect to the cube covering. The only notable difference is that the 'bad' set ω is not only the union of the 'bad' cubes. For each cube one rather cuts out a cone-like set which depends on the cube and the Lipschitz boundary. In this sense, the proof relies more on the particular structure of a Lipschitz boundary than the proof of Theorem 5.3.5.

Now, an important consequence of Theorem 5.4.3 is the existence of approximate gradients in $\Omega \setminus \omega$. Indeed, for $u \in GSBD^p(\Omega)$ with the assumptions of Theorem 5.4.3 fulfilled and a corresponding $v \in W^{1,p}(\Omega)$ we have for $x \in \Omega \setminus \omega$ which is a Lebesgue point of ∇v and $\epsilon > 0$

$$\begin{aligned}
&\limsup_{r\to 0^+} \frac{1}{|B_r(x)|} \left| \left\{ y \in \Omega : \frac{|u(y)-u(x)-\nabla v(x)(y-x)|}{|y-x|} > \epsilon \right\} \cap B_r(x) \right| \\
\leq &\limsup_{r\to 0^+} \frac{1}{|B_r(x)|} \left| \left\{ y \in \Omega : \frac{|u(y)-u(x)-\nabla v(x)(y-x)|}{|y-x|} > \epsilon \right\} \cap B_r(x) \cap (\Omega\setminus\omega) \right| \\
&+ \limsup_{r\to 0^+} \frac{|B_r(x)\cap\omega|}{|B_r(x)|}.
\end{aligned}$$

Since $u = v$ λ^n-a.e. and x is a Lebesgue point of ∇v the first term of the last two lines vanishes (cf. Theorem 4.2.5) and Lemma 5.3.4 guarantees that the second term vanishes for λ^n-a.e. $x \in \Omega \setminus \omega$. In particular, we have that the approximate gradient of u does exist in $\Omega \setminus \omega$ and is equal to ∇v almost everywhere. We are now in the position to prove the main result of this chapter:

Theorem 5.4.4. *[CCS22, Theorem 4.5] Let $n \in \mathbb{N}, n \geq 2$, $p \in (1, \infty)$, and let $\Omega \subseteq \mathbb{R}^n$ be bounded, open, connected with Lipschitz boundary. Then there exists a constant $C > 0$ only dependent on n, p and Ω such that for any $u \in GSBD^p(\Omega)$ exists a set of finite perimeter $\omega \subseteq \Omega$ with $\mathcal{H}^{n-1}(\partial^* \omega) \leq C\mathcal{H}^{n-1}(J_u)$ and an affine function a such that*

$$\|\nabla u - \nabla a\|_{L^p(\Omega \setminus \omega)} \leq C \|\mathcal{E}u\|_{L^p(\Omega)}.$$

and

$$\|\nabla u - \nabla a\|_{L^q(\Omega \setminus \omega)} \leq C \|\mathcal{E}u\|_{L^p(\Omega)}$$

with $q \leq \frac{pn}{n-p}$ if $p < n$, $q < \infty$ if $p = n$, and $q \leq \infty$ if $p > n$. Here, ∇u denotes the approximate gradient of u in $\Omega \setminus \omega$.

Proof. We will apply Theorem 5.4.3 to u. Let $v \in W^{1,p}(\Omega)$ be the function provided by the approximation theorem. By Theorem 3.2.8 we find a rigid displacement a with

$$\|v - a\|_{W^{1,p}(\Omega)} \leq C \|\mathcal{E}v\|_{L^p(\Omega)} \overset{(5.53)}{\leq} C \|\mathcal{E}u\|_{L^p(\Omega)}.$$

Now, we have

$$\|\nabla u - \nabla a\|_{L^p(\Omega \setminus \omega)} = \|\nabla v - \nabla a\|_{L^p(\Omega \setminus \omega)} \leq \|v - a\|_{W^{1,p}(\Omega)} \leq C \|\mathcal{E}u\|_{L^p(\Omega)}$$

and due to the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ we infer

$$\|u - a\|_{L^q(\Omega \setminus \omega)} = \|v - a\|_{L^q(\Omega \setminus \omega)} \leq C \|v - a\|_{W^{1,p}(\Omega)} \leq C \|\mathcal{E}u\|_{L^p(\Omega)}$$

for q fulfilling one of the assumptions. □

As a conclusion to this section, we note that although the existence of the approximate gradient is ensured in $\Omega \setminus \omega$ by Theorem 5.4.3 this result can even be extended. As long as Ω is a set of finite perimeter and the conditions of Theorem 5.4.3 are fulfilled, then the approximate gradient of a function in $GSBD^p(\Omega)$ exists globally (cf. Theorem 5.2 in [CCS22]).

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