TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

D I P L O M A R B E I T

Policy Implications in Dynamic Models of Drug Supply and Demand

Ausgeführt am Institut für
Wirtschaftsmathematik
der Technischen Universität Wien

unter der Anleitung von
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Wien, Mai 2010

Abstract

In this thesis a three-state dynamic model of illicit drug consumption is analyzed. Not only the number of people who are susceptible to initiating into drug use and the active users but also the current throughput capacity of the supply network of drugs is taken into consideration. Furthermore, initiation into drug use is price dependent with the price itself depending on the relative sizes of supply and demand capacity. We run the evaluations for two different parameterizations, i.e., for the cocaine epidemic in the United States and the injection drug use (IDU) in Australia.

In the first part the uncontrolled model is described and analyzed and some sensitivity and bifurcation analyses are carried out. The second part is devoted to the effects of the control instrument "enforcement". We determine the optimal years in the epidemic for a supply reduction by 50% and we investigate the cost-efficiency of such a depression with respect to several initial values. In addition, we compare different ways to measure the social costs and discuss the diverse policy implications they imply. Moreover, we combine the different social cost functionals by weighting them appropriately and rerun our analyses. One anticipated conclusion is that the results strongly depend on the way we measure the social costs as well as on the initial values and the relation between drug users and the drug stock.

Finally, we look at the price elasticity of initiation because previous work tells us that the elasticity level has a strong impact on the policy conclusions.

In summary, the U.S. parameter case turned out to be more interesting presumably because of the convex initiation function contrary to the concave one describing the feedback effect between susceptibles and users in the Australian IDU epidemic.

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Chapter 1

Introduction

Illicit drug consumption and the adherent social costs, drug related deaths and problems are great challenges for societies and decision makers all over the world. Accordingly, several models in the field of optimal dynamic control of drug use have been implemented and studied in recent years. Among them we find the so-called SA model where the groups of susceptible non-users, S, and the active users, A, are the states; see, e.g., [Caulkins et al., 2009a], [Caulkins et al., 2009b], [Wallner, 2008].

My thesis is an extension of this model based on [Caulkins, 2008] using parameterizations for the U.S. cocaine epidemic and Australian injection drug use (IDU). The enhancement involves adding a third state, C, representing the current throughput capacity of the supply network of drugs which measures how much smugglers can bring in on an ongoing basis. One can think of C as the capital stock of the drug smuggling industry, except that it would primarily be social/relational capital and tacit knowledge not fixed assets as with capital stocks in a typical manufacturing context. Hence, this model is a dynamic three-state model of drug use and will be referred to as SAC model. Beyond that, the SAC model makes initiation into drug use price dependent with price itself depending on the relative sizes of demand and supply capacity.

In the first part of this thesis we will analyze the uncontrolled model, whereas the second part primarily deals with the strategic examination of enforcement and its effects.

Chapter 2 presents the mathematical formulation of our base model and the parameterization for the U.S. cocaine and the Australian IDU epidemics. Fur-

thermore, we will analyze the steady states and their stability properties, phase portraits around the equilibria and some time paths.

In Chapter 3 we discuss the results of a sensitivity analysis. We will focus on β, the exponent of the initiation function. In addition, we display several three-dimensional phase portraits around the steady state.

Bifurcation analyses with respect to several parameters dominate Chapter 4. We limit ourselves to two special cases of β. First, we set β equal to 2 to get a convex initiation function. Later, we simulate a concave initiation function by keeping β at the level of 1/2. Moreover, we will carry out two-dimensional bifurcation analyses.

In Chapter 5 we will add the control instrument "enforcement". We want to investigate if enforcement is more valuable earlier or later in an epidemic. Furthermore, we look at some trajectories with different values of enforcement to compare the epidemic process.

Chapter 6 discusses a supply reduction of the current drug stock by 50% at a certain time in the epidemic. On one hand, we will examine the most efficient year of such a supply shock and on the other hand, the benefit-cost ratio of a supply shock in the first year.

We dedicate Chapter 7 to different functional forms for the social costs resulting from a drug epidemic. Scenarios will be compared where the emphasis is laid on different types of cost, for example leaving out the current throughput capacity of the supply network of drugs, C, in the evaluation.

Chapter 8 deals with the question how the model behaves when the overall elasticity level is increased. We look at the new steady states and their stability properties and we compare the results obtained with the original model.

Finally, we'll summarize the most important results from this thesis and make suggestions for some possible extensions for further studies in Chapter 9.

Please note that all numerical calculations in this thesis were carried out with Matlab R2008b.

Chapter 2

The Original SAC Model

2.1 Model Formulation

My diploma thesis discusses the dynamic three state SAC model where $S(t)$ captures the number of people who are susceptible to initiating into drug use, $A(t)$ tracks the number of active drug users, and $C(t)$ represents the current throughput capacity of the supply network of drugs. $C(t)$ can be understood as the capital stock of the drug smuggling industry. For the sake of compactness, we omit the time argument t, if no confusion arises.

The system dynamics we consider is

$$\begin{aligned} \dot{S} &= k - \delta S - f(A)Sp(A,C)^a, \\ \dot{A} &= f(A)Sp(A,C)^a - \mu A, \\ \dot{C} &= g(p)C, \end{aligned} \tag{2.1}$$

where

$$p(A,C) = c\left(\frac{A}{C}\right)^{\frac{1}{\eta s - \eta d}}, \tag{2.2}$$

$$g(p) = c^{'}(\ln(A) - \ln(C)), \tag{2.3}$$

$$f(A) = \alpha A^{\beta}, \tag{2.4}$$

$$c^{'} = \frac{c}{\eta s - \eta d}. \tag{2.5}$$

The frequency of consumption or the degree of addiction are not taken into consideration explicitly. People enter the pool of susceptibles S with a constant rate k, which can be understood as reaching a certain age when you become interested in drugs. There is also a constant outflow rate δ from S interpreted as "maturing out". Furthermore, the initiation function $f(A)Sp(A,C)^a$ is price dependent. In the U.S. base parameter case we use a convex initiation function $f(A)$, while for the Australian IDU epidemic we have a concave initiation function. The exit from the active use rate, μ, can, for example, be interpreted as successful participation in a treatment program or death. The growth rate of the supply network is described with $g(p)$.

Table 2.1 summarizes the base case parameter values used in the analyses presented in the following sections.

Parameter	**Symbol**	**U.S. Cocaine**	**Australian IDU**
inflow into S state	k	1.3417	0.0526
exit from S state	δ	0.0605	0.0952
coefficient in initiation function	α	0.0090	0.5112
exponent in initiation function	β	1.5604	0.8622
price elasticity of supply	ηs	0.5	0.5
price elasticity of demand	ηd	-0.5	-0.5
elasticity of initiation	a	-0.25	-0.25
exit from active use	μ	0.1661	0.1136
coefficient in growth rate	c	0.15	0.15
annual discount rate	r	0.04	0.04

Table 2.1: Base case parameter values for the U.S. cocaine and the Australian IDU epidemics.

2.2 Steady States and Stability Behaviour

2.2.1 U.S. Cocaine Use

The steady states $(\hat{S}, \hat{A}, \hat{C})$ are given by the solution of the SAC model (2.1) where the three states $\dot{S}$, $\dot{A}$, and $\dot{C}$ are simultaneously set equal to zero which is visualized in Figure 2.1.

We find two different steady states: a small one, $\hat{E}_1$, relating to the number of active users, A, and an equilibrium with a relatively high number of

users, $\hat{E}_2$.

We can see that the number of active users, A, and the current throughput capacity of the supply network of drugs, C, are equal in the equilibrium.

$$\begin{aligned} \hat{E}_1 &= (\hat{S}_1, \hat{A}_1, \hat{C}_1) = (21.2619, 0.3332, 0.3332), \\ \hat{E}_2 &= (\hat{S}_2, \hat{A}_2, \hat{C}_2) = (3.9796, 6.6281, 6.6281). \end{aligned}$$

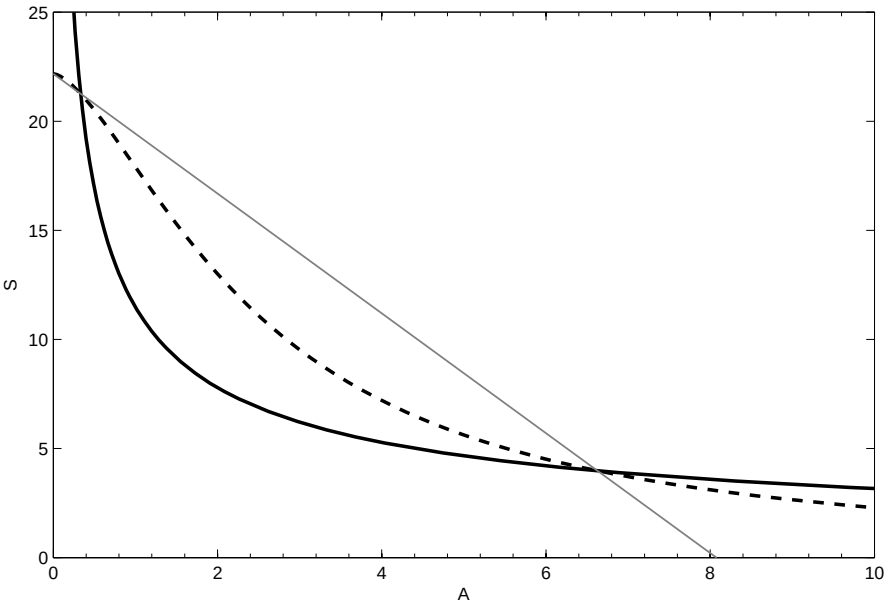


Figure 2.1: Isoclines and the linear relation between the steady state values for the U.S. base parameterization.

Looking at the phase portrait, Figure 2.2, one can see that some trajectories converge towards the point $(S, A, C) = (22.1769, 0, 0)$. By showing that $\lim_{A\to 0, C\to 0}(\ln(A) - \ln(C)) = 0$ we find a third equilibrium $\hat{E} = (\hat{S}, \hat{A}, \hat{C}) = (k/\delta, 0, 0)$.

Now we want to investigate the local stability behaviour of the system. We use the Jacobian matrix, the matrix of all first-order partial derivatives, with

$$\mathbf{J} = \begin{pmatrix} \dot{S}_S & \dot{S}_A & \dot{S}_C \\ \dot{A}_S & \dot{A}_A & \dot{A}_C \\ \dot{C}_S & \dot{C}_A & \dot{C}_C \end{pmatrix}$$

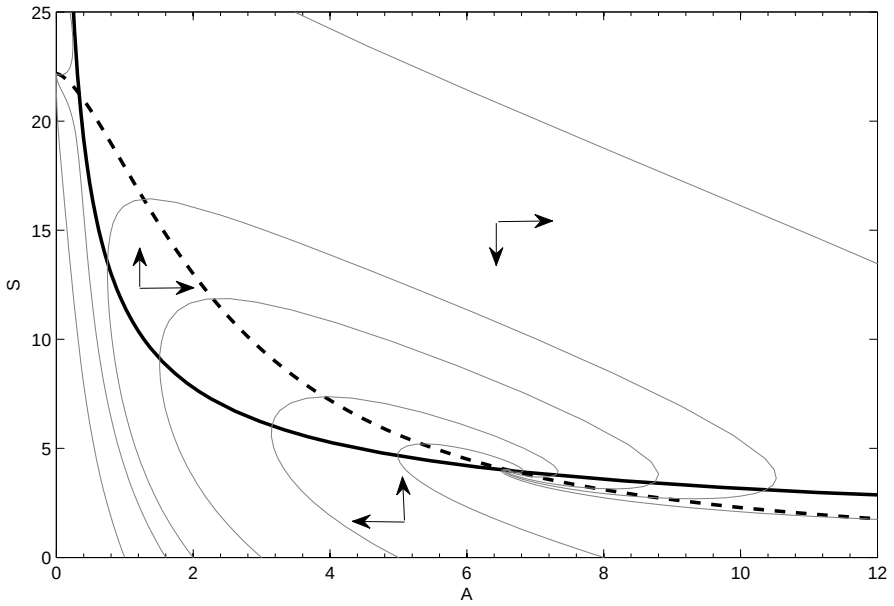


Figure 2.2: Phase portrait for the U.S. base parameter set within the (A, S)-plane.

$$
\begin{aligned}
\dot{S}_S &= -\delta - \alpha A^{\beta} \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}}, \\
\dot{S}_A &= -\beta\alpha A^{\beta-1} S \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}} - \alpha A^{\beta} S \frac{a}{\eta s - \eta d} \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}-1} \frac{c}{C}, \\
\dot{S}_C &= -\alpha A^{\beta} S \frac{a}{\eta s - \eta d} \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}-1} c\frac{A}{C^2}, \\
\dot{A}_S &= \alpha A^{\beta} \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}}, \\
\dot{A}_A &= \beta\alpha A^{\beta-1} S \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}} + \alpha A^{\beta} S \frac{a}{\eta s - \eta d} \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}-1} \frac{c}{C} - \mu, \\
\dot{A}_C &= \alpha A^{\beta} S \frac{a}{\eta s - \eta d} \left(c\frac{A}{C} \right)^{\frac{a}{\eta s - \eta d}-1} c\frac{A}{C^2}, \\
\dot{C}_S &= 0, \\
\dot{C}_A &= \frac{cC}{(\eta s - \eta d)A}, \\
\dot{C}_C &= \frac{c}{\eta s - \eta d}(\ln(A) - \ln(C)) - \frac{c}{\eta s - \eta d}.
\end{aligned}
$$

Using the parameters listed in Table 2.1 and evaluating at the fixed point $\hat{E}_1$, the low equilibrium relating to A, we get the eigenvalues

$$\begin{aligned}\lambda_1 &= 0.0747,\\ \lambda_2 &= -0.0589,\\ \lambda_3 &= -0.1773.\end{aligned}$$

All eigenvalues are real, and there are positive and negative values, so this equilibrium is a saddle point.

For the high steady state $\hat{E}_2$ we get the eigenvalues

$$\begin{aligned}\lambda_1 &= -0.1815,\\ \lambda_{2,3} &= -0.1271 \pm 0.1311i.\end{aligned}$$

Thus, we have a stable focus, because $Re(\lambda_1)$ and $Re(\lambda_{2,3})$ are negative and λ_2 and λ_3 are conjugate-complex numbers.

Next, we want to look at the stability of the artificial steady state $\hat{E} = (\hat{S}, \hat{A}, \hat{C}) = (22.1769, 0, 0)$. Since $C = A$, in this case we do get a two-dimensional system with its partial derivatives

$$\begin{aligned}\dot{S}_S &= -\delta - \alpha A^{\beta} c^{\frac{a}{\eta s - \eta d}},\\ \dot{S}_A &= -\beta\alpha A^{\beta-1} S c^{\frac{a}{\eta s - \eta d}},\\ \dot{A}_S &= \alpha A^{\beta} c^{a},\\ \dot{A}_A &= \beta\alpha A^{\beta-1} S c^{\frac{a}{\eta s - \eta d}}.\end{aligned}$$

If we let A converge towards 0, we obtain the Jacobian matrix

$$\mathbf{J} = \begin{pmatrix} -\delta & 0 \\ 0 & -\mu \end{pmatrix}.$$

Therefore, we can conclude that our system has two stable steady states for

the U.S. base parameterization. $\hat{E}_2$ is a stable focus, $\hat{E}$ is a stable node, and in between there is the saddle point $\hat{E}_1$.

2.2.2 Australian Injection Drug Use

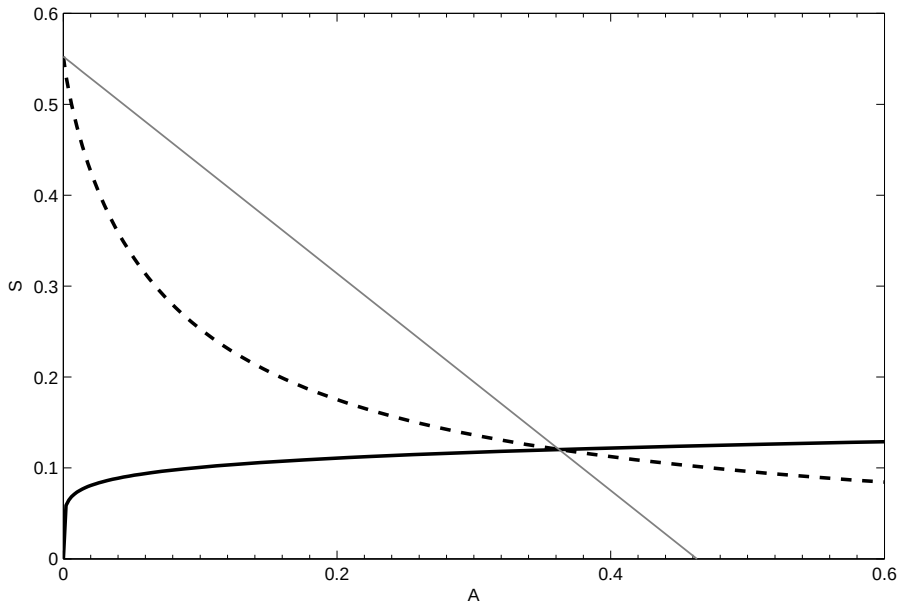


Figure 2.3: Isoclines and the linear relation between the steady state values $\hat{S}$ and $\hat{A}$ for the Australian IDU epidemic.

To determine the steady states $(\hat{S}, \hat{A}, \hat{C})$ for the Australian base parameter set, the system of equations (2.1) is set to zero simultaneously.

Here, we find only one steady state $\hat{E}$ with

$$\hat{E} \quad = \quad (\hat{S}, \hat{A}, \hat{C}) \quad = \quad (0.1202, 0.3623, 0.3623)$$

illustrated in Figure 2.3.

The eigenvalues of this steady state are given by

$$\lambda_1 = -0.3827,$$
$$\lambda_{2,3} = -0.1244 \pm 0.0186i.$$

Therefore, we have a stable focus for the Australian injection drug use epidemic as shown in Figure 2.4 with the phase portrait around the equilibrium in the (A, S)-plane.

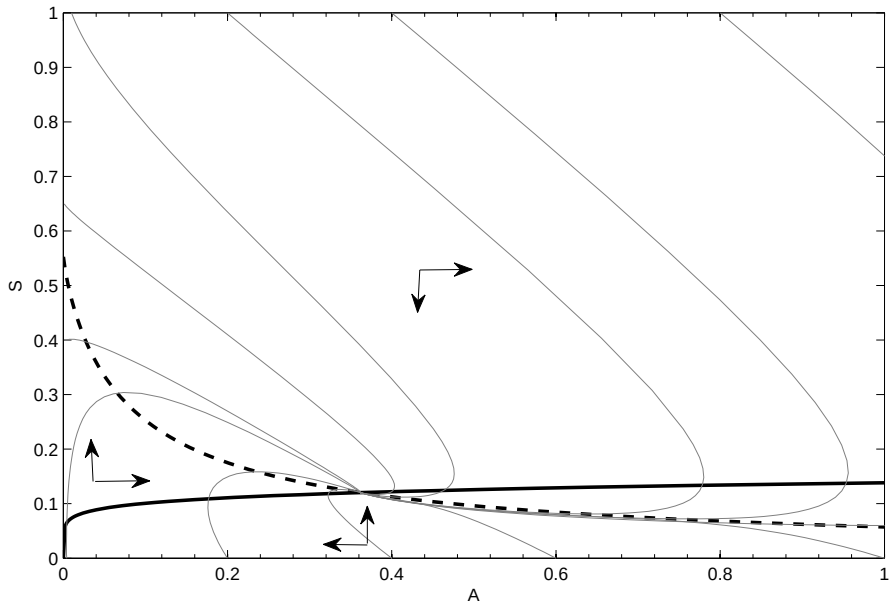


Figure 2.4: Phase portrait around the steady state for the Australian base parameter set within the (A, S)-plane.

2.3 Time Paths

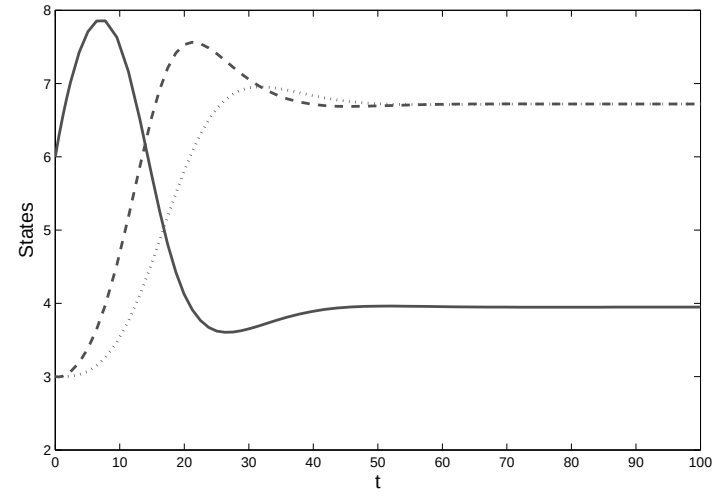
2.3.1 U.S.A.

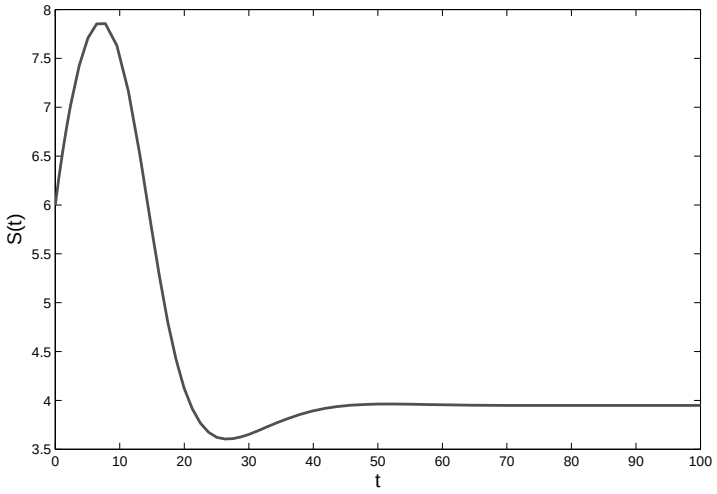
Next, we want to look at some trajectories to get a better understanding of the dynamics of our model.

The first Figure 2.5 shows trajectories where we start with the initial values $(S(0), A(0), C(0)) = (6, 3, 3)$. The second trajectories arise from starting with high initial values $(S(0), A(0), C(0)) = (22, 10, 10)$, and the last one refer to $(S(0), A(0), C(0)) = (1, 1, 1)$. Intentionally we choose $A(0) = C(0)$ to see the different progress from these two states A and C over a short time before converging towards the same corresponding equilibrium value.

Figures 2.5 - 2.7 show the evolutions of $S(t)$, $A(t)$, $C(t)$, and of the price function within the first 100 years. The number of people who are susceptible to initiating into drug use is increasing strongly in the beginning, then decreasing strongly in the first Figure 2.5. A and C are rising towards a high steady state.

Figure 2.6 visualizes the time paths where the number of susceptible people first declines and then slightly rises towards the equilibrium value. The number of active drug users is booming in the first 5 years and then declines towards the steady state. The current throughput capacity of the supply network of drugs increases in the beginning of the epidemic but decreases drastically after 10 years.





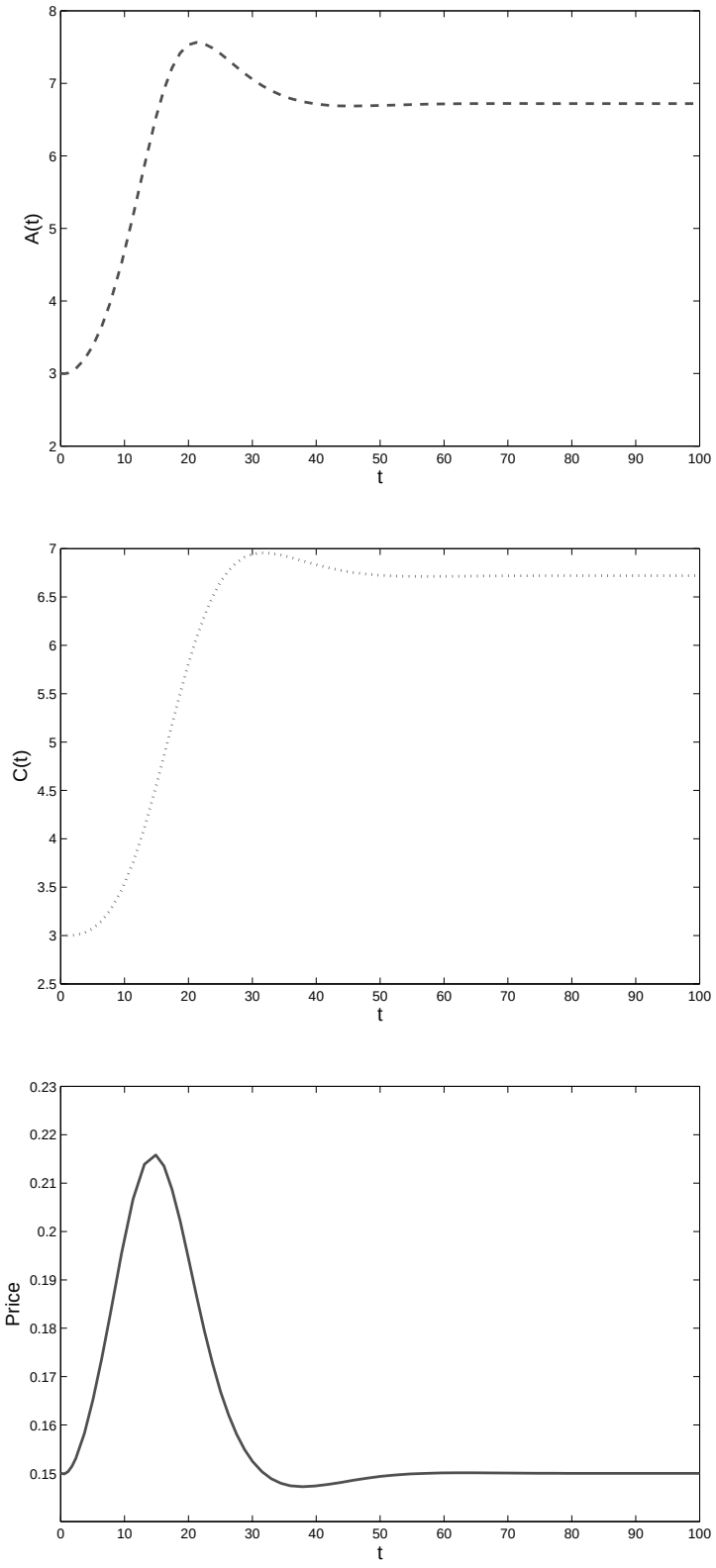
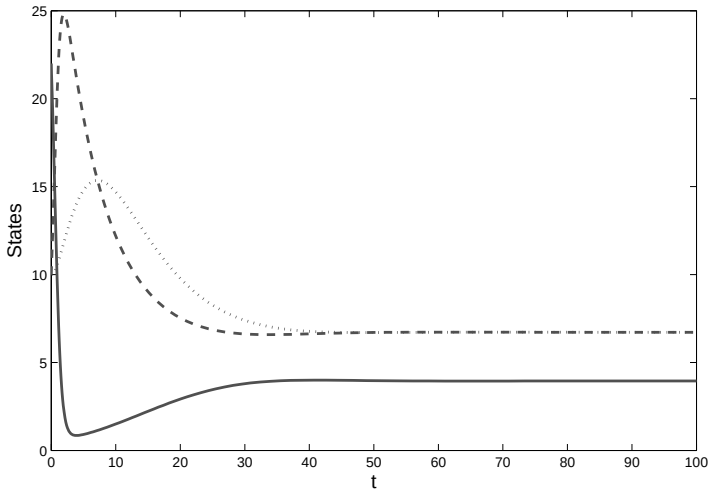
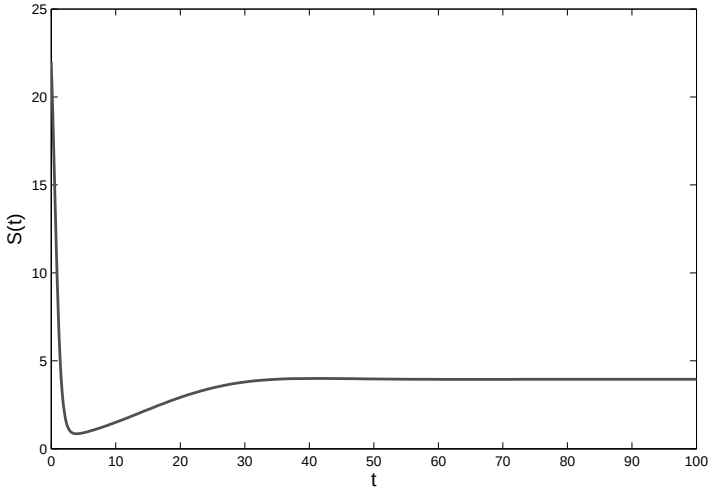
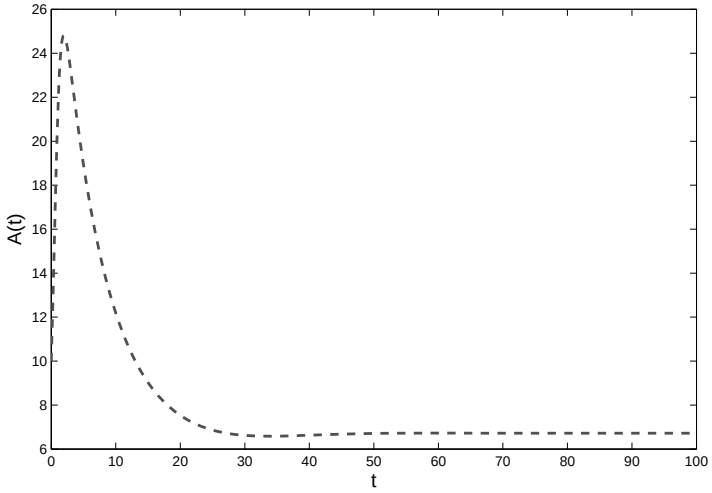


Figure 2.5: Time paths relating to the initial values $(S(0), A(0), C(0)) = (6, 3, 3)$ for the U.S. parameterization.


States
t


S(t)
t


A(t)
t

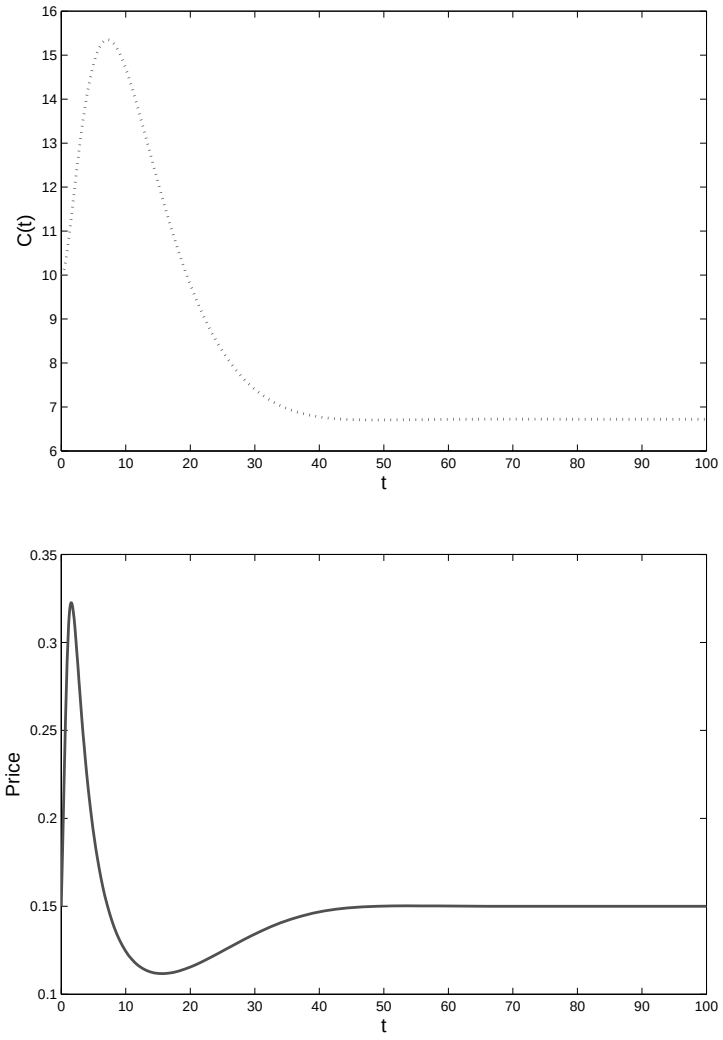
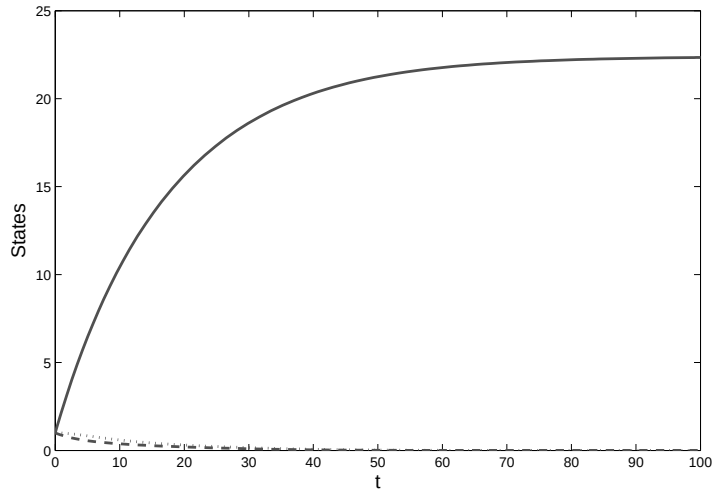
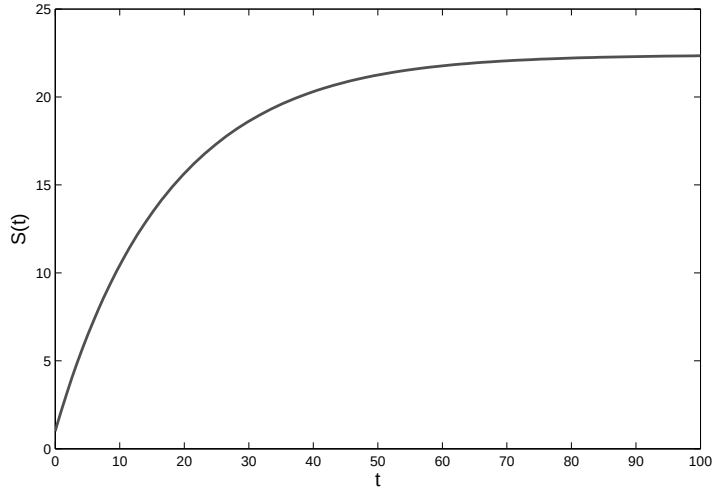
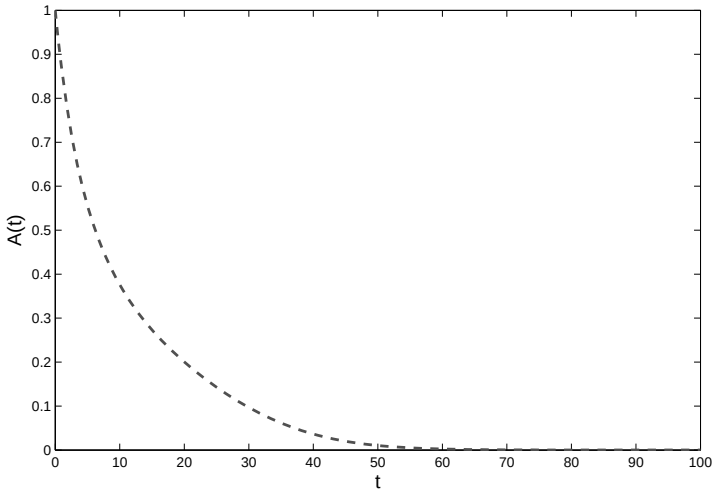


Figure 2.6: Time paths for the U.S. cocaine epidemic starting with the initial values $(S(0), A(0), C(0)) = (22, 10, 10)$.

In the last Figure 2.7, $S(t)$ is very low at the beginning of the epidemic but increases over the time while $A(t)$ and $C(t)$ are monotonously converging towards zero. It is interesting to see that despite the monotonicity of S, A, and C, the price shows a non-monotonic behaviour.


States
t
0
5
10
15
20
25
0
10
20
30
40
50
60
70
80
90
100


S(t)
t
0
5
10
15
20
25
0
10
20
30
40
50
60
70
80
90
100


A(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50
60
70
80
90
100

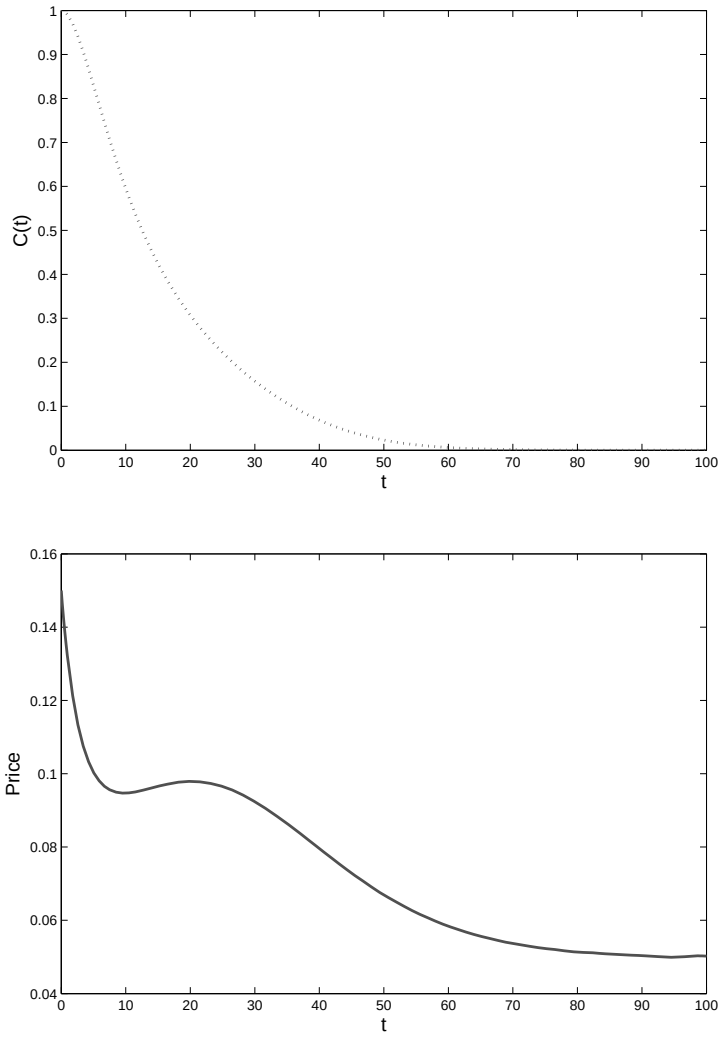
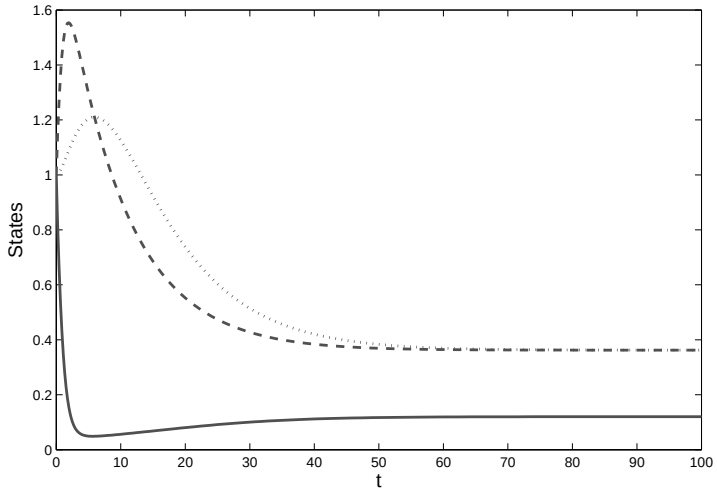


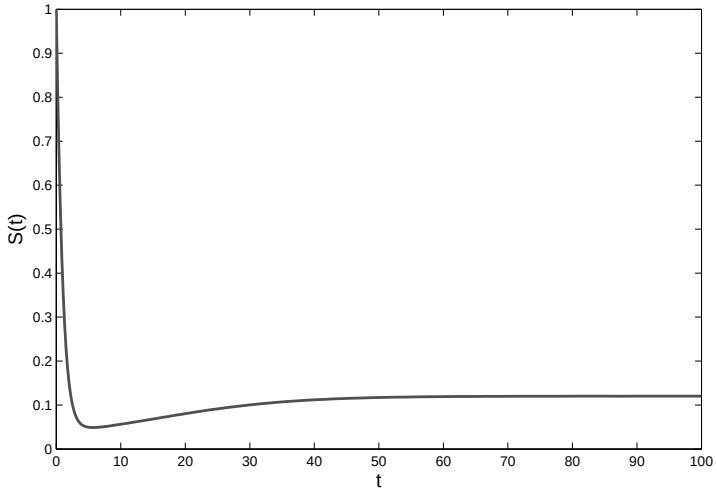
Figure 2.7: Time paths relating to the initial conditions $(S(0), A(0), C(0)) = (1, 1, 1)$ for the U.S. base parameter set.

2.3.2 Australia

For the Australian base case parameter set we look at two time paths to see the different dynamics of the model for altered initial states. The epidemic is described with the help of trajectories. The first trajectories emanate from high initial values $(S(0), A(0), C(0)) = (1, 1, 1)$, the second occur from rather low initial values $(S(0), A(0), C(0)) = (0.5, 0.25, 0.25)$.

Figure 2.8 and 2.9 show the paths over the first 100 years. At first, the number of susceptibles is declining. But then S slightly increases towards the steady state while A and C are growing during the first few years and then fall down rather rapidly.





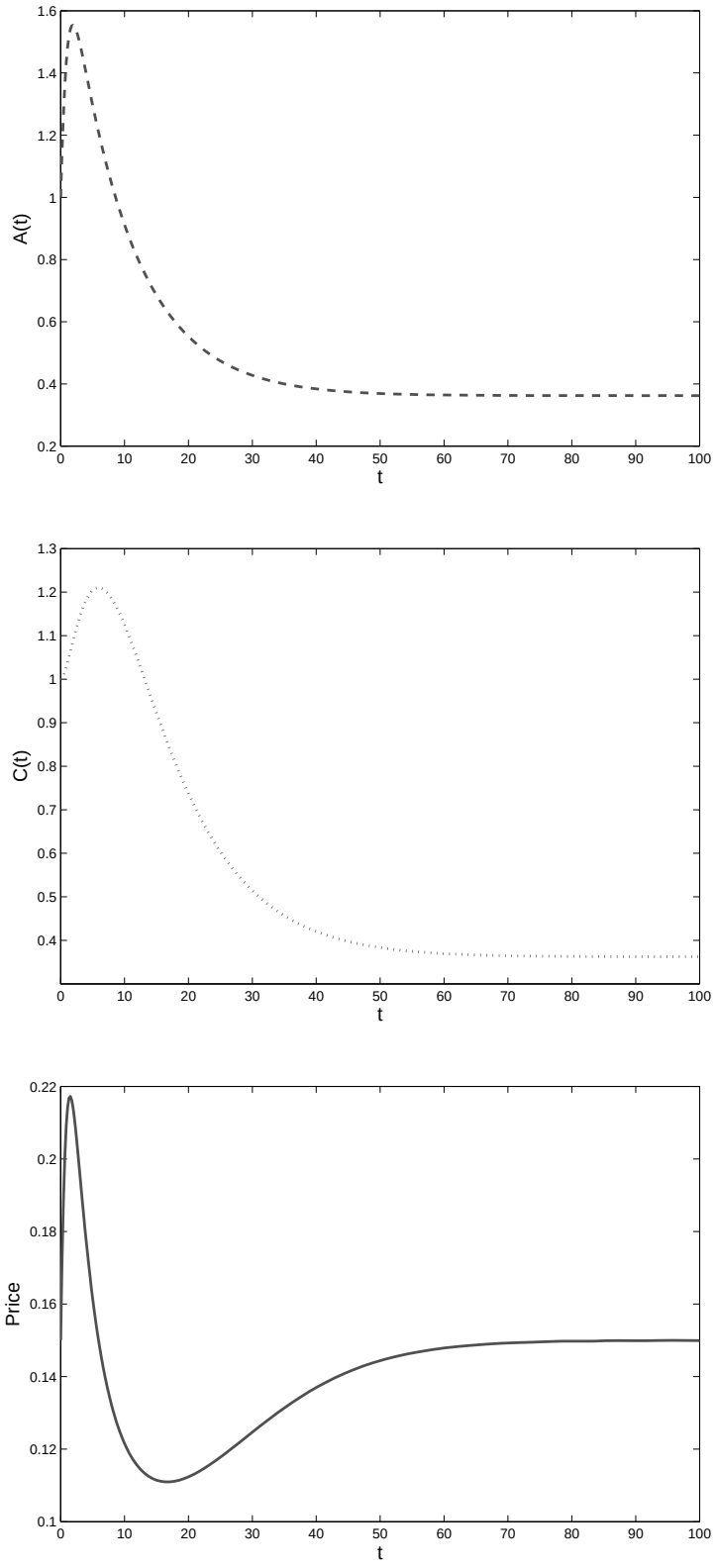
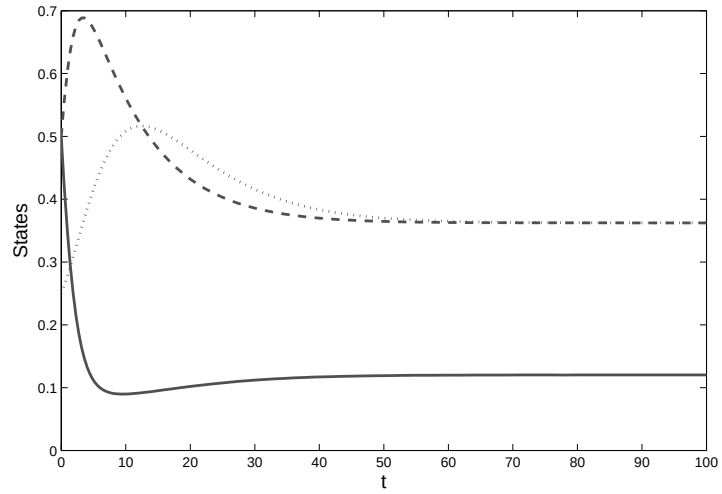
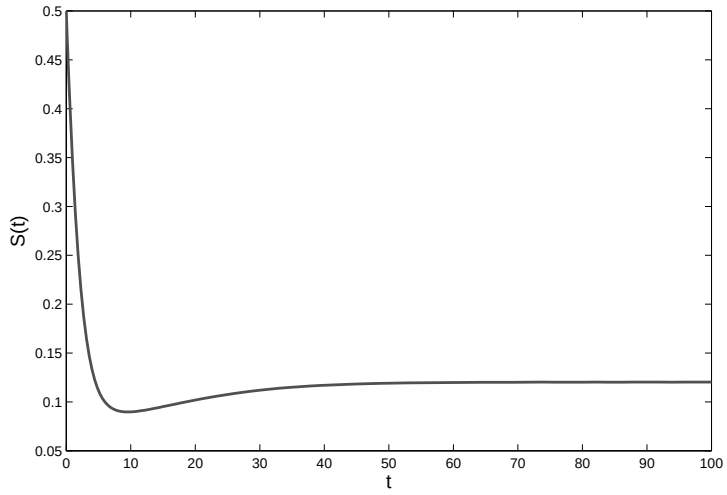
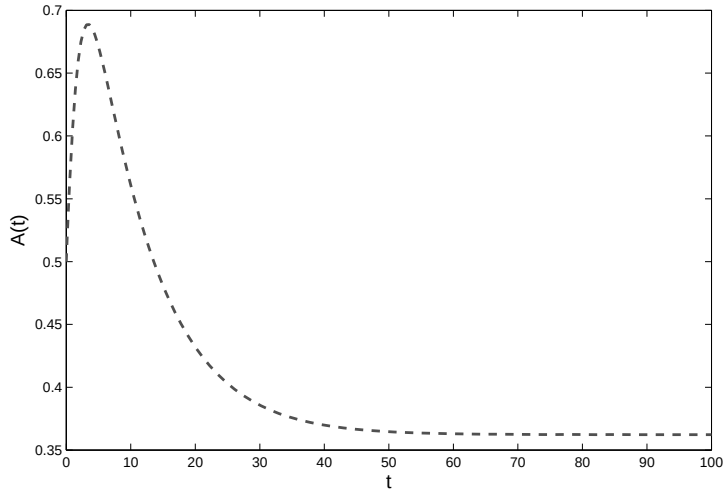


Figure 2.8: Time paths for the Australian IDU epidemic for the initial values $(S(0), A(0), C(0)) = (1, 1, 1)$.


States
t


S(t)
t


A(t)
t

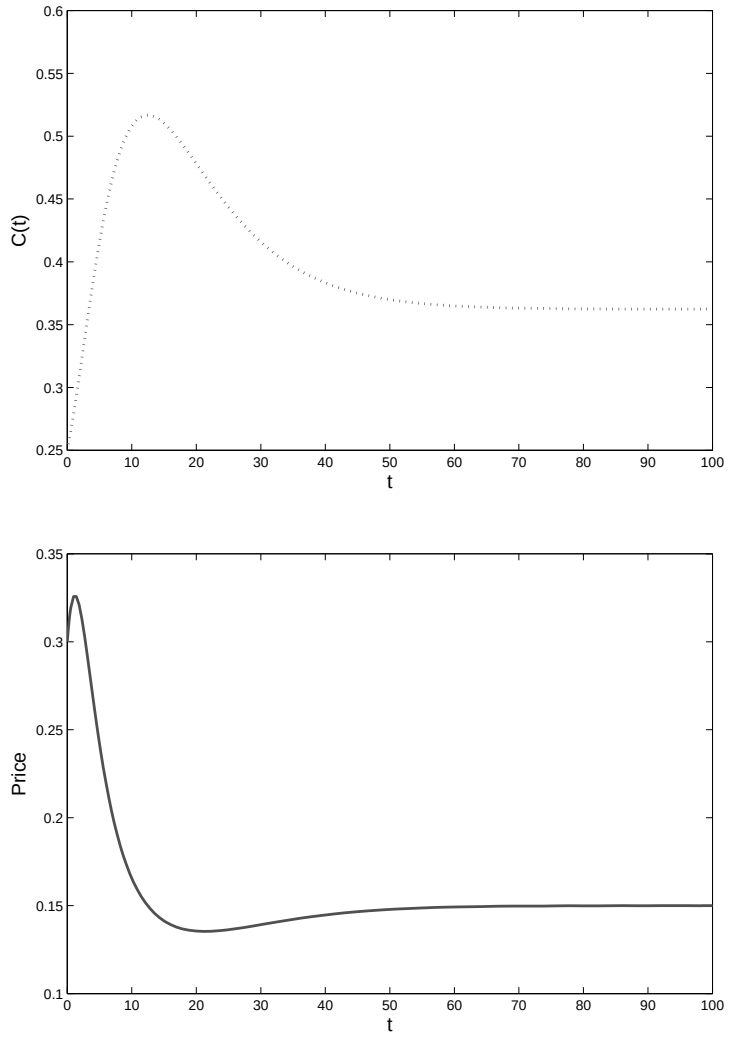


Figure 2.9: Time paths relating to the initial conditions $(S(0), A(0), C(0)) = (0.5, 0.25, 0.25)$ for the Australian parameterization.

Chapter 3

Sensitivity Analysis

In this chapter we will investigate the robustness of our model. We systematically change parameters contained in our model to determine the effects on the model output. First of all, we want to examine a 1% variation of the parameter values. That means that we are increasing one parameter while all the others are kept at their original level. Then, the values of the steady state will be recalculated, and we consider the percentage-wise effects. Afterwards, we will take a closer look at the parameter β to answer the question, how the exponent of the initiation function affects the steady states.

3.1 Sensitivity Analysis in the United States

Table 3.1 summarizes the percentage effects on the high steady state if a parameter is increased by 1%. It shows that the increases of k, β, and μ have the strongest impact on the steady state.

Parameter	$\hat{S}$	$\hat{A} = \hat{C}$
k	-0.7689	1.3868
δ	0.1402	-0.2497
α	-1.1265	0.2464
β	-3.3107	0.7240
a	-0.5388	0.1178
μ	1.7880	-1.3773
c	0.2840	-0.0621

Table 3.1: Effects on the high steady state of the U.S. base parameterization, if one parameter is increased by 1%, while the others are kept unchanged.

Some parameters, ηs and ηd, do not have any influence on the values, because in the steady state point $\hat{A} = \hat{C}$ must be achieved. Therefore, the price effect drops out of the model. These parameters will be left out in the tables.

Furthermore, Table 3.2 lists the percentage diversification of a 1% increase of a parameter on the saddle point steady state with the U.S. base parameterization.

Parameter	$\hat{S}$	$\hat{A} = \hat{C}$
k	1.1283	-1.9813
δ	-1.0729	1.9436
α	0.0819	-1.9024
β	-0.1409	3.2735
a	0.0397	-0.9122
μ	-0.1312	2.0295
c	-0.0211	0.4903

Table 3.2: Percentage alteration of the saddle point steady state of the U.S. cocaine epidemic, if a parameter is increased by 1%, while keeping the others at their original level.

Here, an increase of β has the greatest consequence on the steady state. The effects on the steady state values are mostly as one would expect knowing the dynamics of the model.

Table 3.3 shows how the artificial steady state $\hat{S} = \frac{k}{\delta}$ alters if the two parameters are increased by 1%.

Parameter	$\hat{S}$
k	1
δ	-0.9901

Table 3.3: Effects on the artificial steady state, if one parameter is increased by 1% for the United States.

Now, we will focus on a sensitivity analysis of the parameter β, the exponent in the initiation function.

A hypothetical parameter scenario is considered, where we decrease and increase β by a certain percentage while keeping the other parameters unchanged.

First, we look at the high steady state of the U.S. base parameterization, Table 3.4.

Change of β	new β	$\hat{S}$	$\hat{A} = \hat{C}$
-2%	1.5292	6.9424	-1.5183
-1.75%	1.5331	6.0508	-1.3233
-1.5%	1.5370	5.1660	-1.1298
-1.25%	1.5409	4.2880	-0.9378
-1%	1.5448	3.4169	-0.7473
-0.75%	1.5487	2.5526	-0.5582
-0.5%	1.5526	1.6950	-0.3707
-0.25%	1.5565	0.8442	-0.1846
$+0.25\%$	1.5643	-0.8376	0.1832
$+0.5\%$	1.5682	-1.6685	0.3649
$+0.75\%$	1.5721	-2.4929	0.5452
$+1\%$	1.5760	-3.3107	0.7240
$+1.25\%$	1.5799	-4.1221	0.9015
$+1.5\%$	1.5838	-4.9270	1.0775
$+1.75\%$	1.5877	-5.7255	1.2521
$+2\%$	1.5916	-6.5176	1.4254

Table 3.4: Sensitivity analysis for the exponent in the initiation function, β, on the high steady state of the U.S. base parameter set.

The impact on the high stable steady state $\hat{E}_2$ of the U.S. base case parameter set is not surprising. β has a strong oppositional influence on $\hat{S}$, the number of susceptible non-users at the equilibrium. If we increase the exponent in the initiation function, β, the number of active drug users, A, is growing somewhat so S is decreasing, which leads to a decline of $\hat{S}$.

Table 3.5 lists the effects on the saddle point equilibrium $\hat{E}_1$ of an alteration of β.

In this case, the increase of β has the same influence on the steady state as in the case of the stable high steady state $\hat{E}_2$. It has nevertheless a stronger impact on $\hat{A}$ and a weaker one on $\hat{S}$ than one would expect.

Change of β	new β	$\hat{S}$	$\hat{A} = \hat{C}$
-2%	1.5292	0.2921	-6.7886
-1.75%	1.5331	0.2548	-5.9223
-1.5%	1.5370	0.2178	-5.0611
-1.25%	1.5409	0.1809	-4.2049
-1%	1.5448	0.1443	-3.3539
-0.75%	1.5487	0.1079	-2.5078
-0.5%	1.5526	0.0717	-1.6668
-0.25%	1.5565	0.0358	-0.8309
$+0.25\%$	1.5643	-0.0355	0.8259
$+0.5\%$	1.5682	-0.0709	1.6467
$+0.75\%$	1.5721	-0.1060	2.4626
$+1\%$	1.5760	-0.1409	3.2735
$+1.25\%$	1.5799	-0.1755	4.0793
$+1.5\%$	1.5838	-0.2100	4.8803
$+1.75\%$	1.5877	-0.2443	5.6762
$+2\%$	1.5916	-0.2783	6.4673

Table 3.5: Sensitivity analysis for β on the saddle point steady state for the U.S. base parameterization.

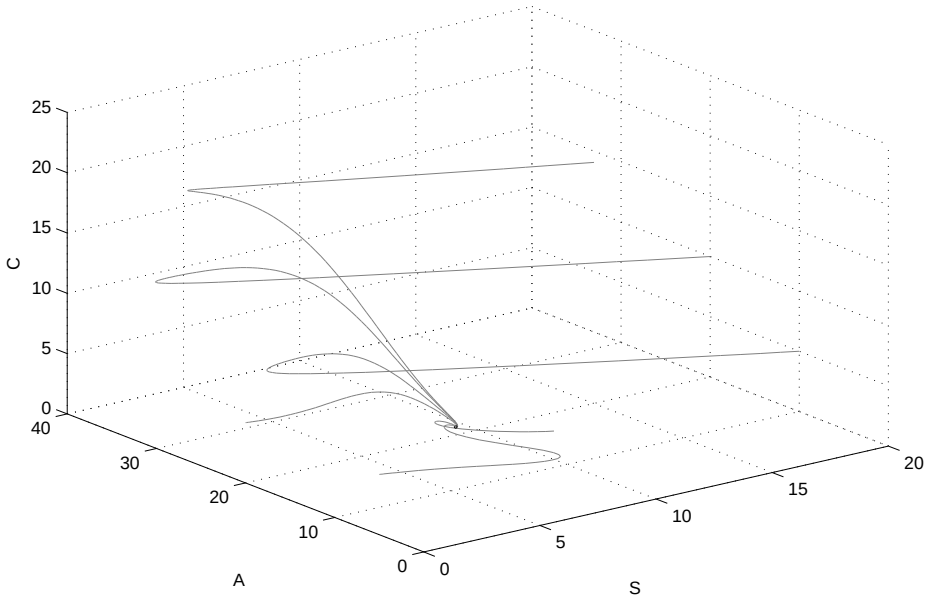


Figure 3.1: Phase portrait around the high steady state of the U.S. cocaine epidemic.

For a better overview we include the phase diagrams around the steady states in Figures 3.1 and 3.2 to see how the different trajectories behave.

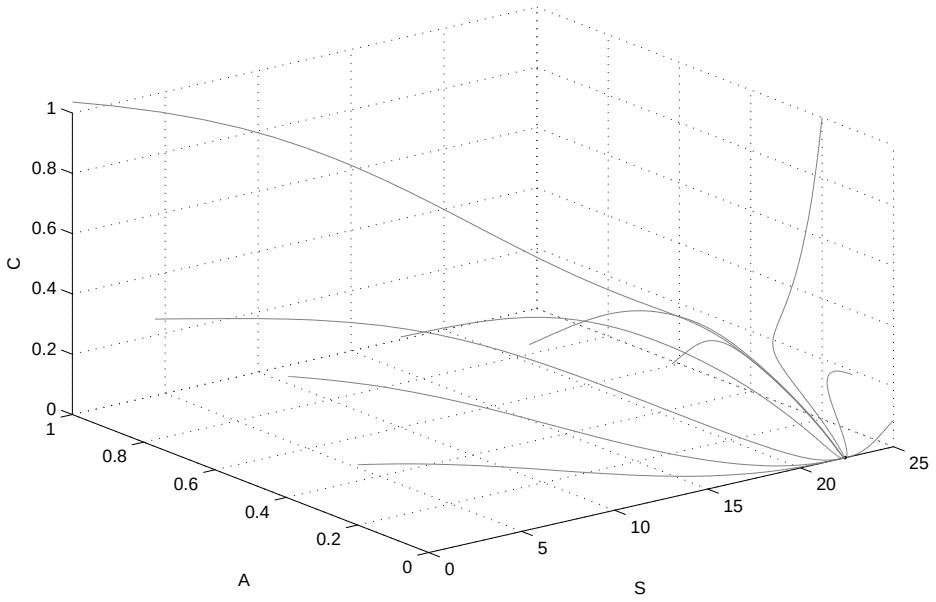


Figure 3.2: Phase portrait around the small steady state for the United States.

3.2 Sensitivity Analysis for the Australian Base Parameter Set

We conclude this chapter with studying how the output alters if different values of the input parameters are used in the Australian case.

Parameter	$\hat{S}$	$\hat{A} = \hat{C}$
k	0.1687	1.2312
δ	-0.0370	-0.2678
α	-0.9539	0.2653
β	0.8485	-0.2360
a	-0.4558	0.1268
μ	0.8295	-1.2185
c	0.2398	-0.0667

Table 3.6: Percentage alteration of the steady state for the Australian parameter base case if a parameter is increased by 1%.

Table 3.6 shows the percentage effects on the unique Australian steady state if a parameter is increased by 1%. As in the case of the U.S. base parameterization, the increases of k, β, and μ have the strongest impact on the stable steady state of the Australian IDU epidemic.

As for the cocaine epidemic in the United States, we will focus on a sensitivity analysis, with respect to β, the exponent in the initiation function.

Change of β	new β	$\hat{S}$	$\hat{A} = \hat{C}$
-2%	0.8450	-1.6653	0.4632
-1.75%	0.8471	-1.4594	0.4059
-1.5%	0.8493	-1.2529	0.3485
-1.25%	0.8514	-1.0457	0.2909
-1%	0.8536	-0.8379	0.2331
-0.75%	0.8557	-0.6294	0.1751
-0.5%	0.8579	-0.4203	0.1169
-0.25%	0.8600	-0.2105	0.0585
$+0.25\%$	0.8644	0.2111	-0.0587
$+0.5\%$	0.8665	0.4229	-0.1176
$+0.75\%$	0.8687	0.6354	-0.1767
$+1\%$	0.8708	0.8485	-0.2360
$+1.25\%$	0.8730	1.0623	-0.2955
$+1.5\%$	0.8751	1.2768	-0.3551
$+1.75\%$	0.8773	1.4920	-0.4150
$+2\%$	0.8794	1.7079	-0.4750

Table 3.7: Effects on the steady state values $(\hat{S}, \hat{A}, \hat{C})$ of a change in parameter β for the Australian IDU epidemic.

We decrease and increase β by a different percentage while fixing all other parameters. The results are listed in Table 3.7.

It is easy to see that $\hat{S}$ is shrinking with a smaller β while $\hat{A}$ is increasing, and vice versa $\hat{A}$ is decreasing with a higher value of β.

The effects are not as strong as in the case of the parameterization for the United States. Presumably this is due to the fact that the parameter values of the Australian problem are smaller and the steady state is, too.

To better illustrate the results we finally display the phase portrait around the steady state of the injection drug use epidemic in Australia in Figure 3.3. This allows for a better understanding of the dynamical system.

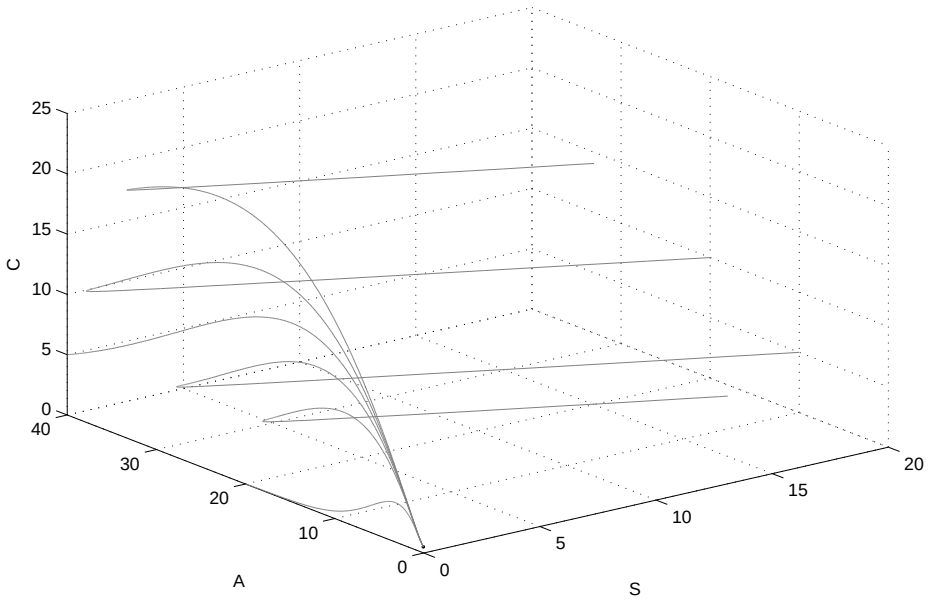


Figure 3.3: Phase portrait around the steady state for the Australian parameter set.

Chapter 4

Bifurcation Analysis

Bifurcation theory is used for the study of the change in the qualitative structure of a dynamical system. Bifurcations occur if a slight change of a parameter value causes a qualitative structural change in the model's behaviour. Examples of such changes are the gain or loss of steady states or the altered stability of equilibria. The parameter value where a bifurcation occurs is called the critical value of the system and will be denoted by $parameter_c$.

A *saddle-node bifurcation* is a local bifurcation where two fixed points of a dynamical system collide and annihilate. *Blue sky bifurcation* is another name in reference to the sudden accumulation of two fixed points. This bifurcation will occur in our model.

As mentioned in Chapter 3, the steady states $(\hat{S}, \hat{A}, \hat{C})$ are given by the solution of the SAC model (2.1) when $\dot{S}$, $\dot{A}$, and $\dot{C}$ are set equal to zero simultaneously. $\dot{C}$ is zero, if $\ln A = \ln C$, i.e., if A is equal to C. Thus, we get the following system of equations to solve:

$$\begin{aligned} 0 &= k - \delta S - \alpha A^{\beta} S c^{a}, \\ 0 &= \alpha A^{\beta} S c^{a} - \mu A. \end{aligned}$$

Adding $\dot{S} = 0$ and $\dot{A} = 0$ we obtain $k - \delta S - \mu A = 0$, and solving for S yields

$$\hat{S} = \frac{k - \mu \hat{A}}{\delta}.$$

Inserting $\hat{S}$ into the equation $\dot{A} = 0$ produces the simplified equation

$$0 = \hat{A}\left(\alpha k c^a \hat{A}^{\beta-1} - \alpha\mu c^a \hat{A}^{\beta} - \mu\delta\right).$$

The first solution is $\hat{A} = 0$.

To get other solutions we have to solve the equation given by

$$r\hat{A}^{\beta} - p\hat{A}^{\beta-1} + q = 0$$

with

$$\begin{aligned} r &= \alpha\mu c^a, \\ p &= \alpha k c^a, \\ q &= \mu\delta. \end{aligned}$$

We want to make a bifurcation analysis for two different values of β.

First, we set β equal to 2 therewith we have a convex initiation function. Second, we will simulate a concave initiation function by changing β to 1/2.

If $\beta = 2$, we get a quadratic equation, which can be solved analytically:

$$\begin{aligned} \hat{A}_1 &= \frac{p + \sqrt{p^2 - 4rq}}{2r}, \\ \hat{A}_2 &= \frac{p - \sqrt{p^2 - 4rq}}{2r}. \end{aligned}$$

The square root is greater than 0 if

$$\begin{aligned} k &> \sqrt{\frac{4\mu^2\delta}{\alpha c^a}}, \\ \delta &< \frac{\alpha k^2 c^a}{4\mu^2}, \\ \alpha &> \frac{4\mu^2\delta}{k^2 c^a}, \\ \mu &< \sqrt{\frac{\alpha k^2 c^a}{4\delta}}, \end{aligned}$$

or

$$c \;<\; \sqrt[a]{\frac{\alpha k^2}{4^2\delta}}.$$

For $\beta = 1/2$ we get just one steady state besides the artificial one $\hat{A} = 0$.

$$r\sqrt{\hat{A}} - p\frac{1}{\sqrt{\hat{A}}} + q = 0$$

After multiplying with $\sqrt{\hat{A}}$ and transforming $C = \sqrt{\hat{A}}$, $C^2 = \hat{A}$ we get the quadratic equation

$$rC^2 - p + qC = 0$$

with the solutions

$$\begin{aligned} \hat{A}_1 = C_1^2 &= \frac{-q + \sqrt{q^2 - 4rp}}{2r}, \\ \hat{A}_2 = C_2^2 &= \frac{-q - \sqrt{q^2 - 4rp}}{2r}. \end{aligned}$$

Regarding that C_2 is smaller than 0 and $C_2 = \sqrt{A_2}$, it is not a solution of our problem. Hence, no bifurcations emerge if β is equal to 1/2 .

4.1 Bifurcation Analysis for the United States

For the base case parameterization of the U.S. cocaine epidemic, Table 4.1 shows the critical values for $\beta = 2$.

parameter	critical value
k_c	0.6795
δ_c	0.2359
α_c	0.0023
μ_c	0.3280
c_c	34.6733

Table 4.1: Critical values for the base parameterization of the cocaine epidemic in the United States and $\beta = 2$.

These are exactly the values where the square root $\sqrt{p^2 - 4rq}$ is equal to zero.

We will also include the original parameter values, the base case parameters $parameter_{bc}$ in our figures.

4.1.1 Inflow into the State S, k

If $\beta = 2$, a change of the inflow rate to a lower value leads to a saddle-node bifurcation.

Figure 4.1 shows the bifurcation plots for A, the number of active users, and S, the group of susceptible non-users. The critical parameter value is located at $k_c = 0.6795$ with the corresponding bifurcation point $(\hat{S}_c, \hat{A}_c, \hat{C}_c) = (5.6154, 2.0454, 2.0454)$.

Note that the diagram of C would be the same as the one for A and therefore can be omitted.

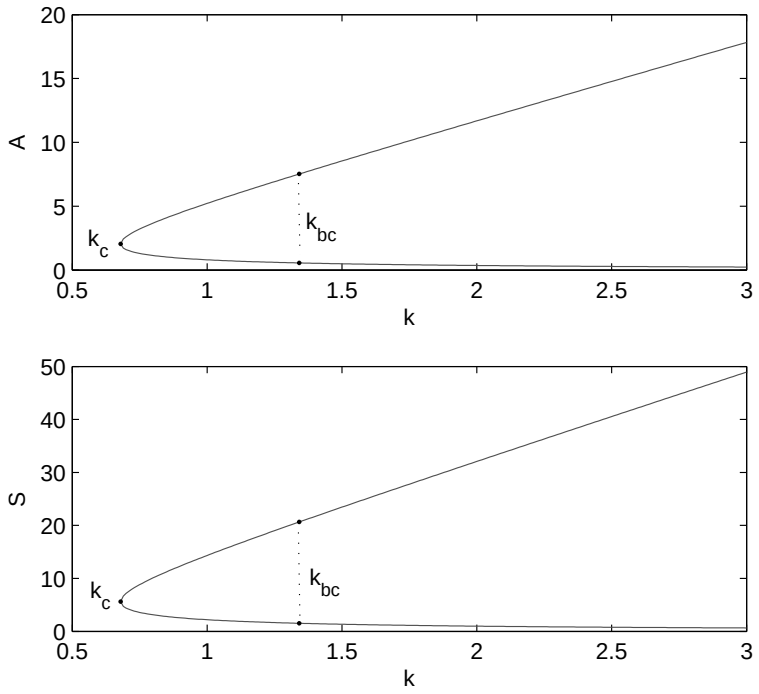


Figure 4.1: Bifurcation diagrams with respect to the inflow into state S, k, for the U.S. cocaine epidemic and $\beta = 2$.

For an example of the bifurcation analysis with a concave initiation function we refer to Figure 4.2.
It shows that a smaller exponent of the initiation function than 1, $\beta = 1/2$, does not lead to a modification in the dynamical behaviour. The parameters k, δ, α, μ, and c have no impact on the numbers of steady states or the stability if $\beta = 1/2$.

Please note that we omit the figures for these cases in this thesis.

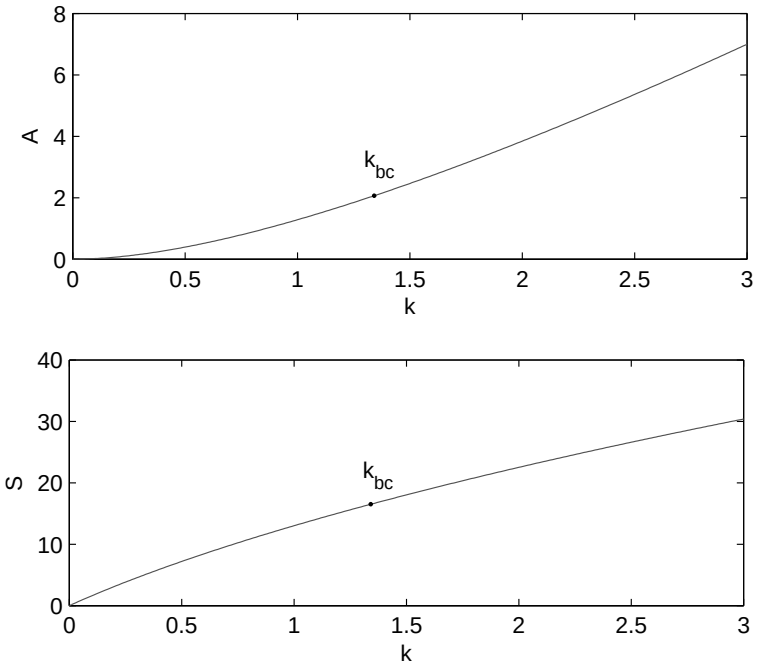


Figure 4.2: Bifurcation diagrams with respect to k, the inflow into state S, for the base parameter set of the cocaine epidemic in the United States of America and $\beta = 1/2$.

4.1.2 Exit from the State S, δ

Again, a saddle-node bifurcation occurs at the critical value $\delta_c = 0.2359$ if $\beta = 2$ as shown in Figure 4.3.
The critical value is almost four times as high as the base case parameter $\delta_{bc} = 0.0605$.

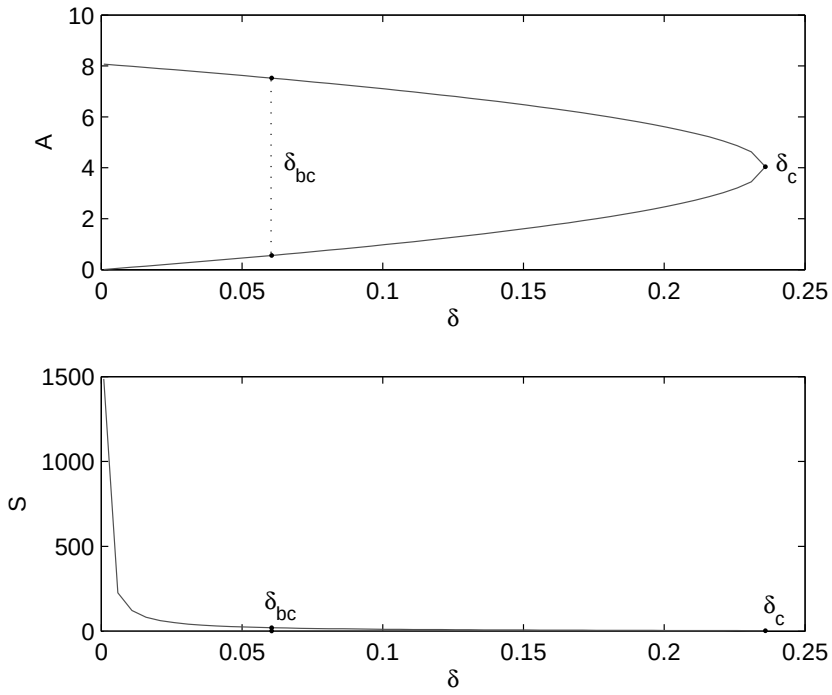


Figure 4.3: Bifurcation diagrams with respect to the exit from state S, δ, for the U.S. cocaine epidemic and $\beta = 2$.

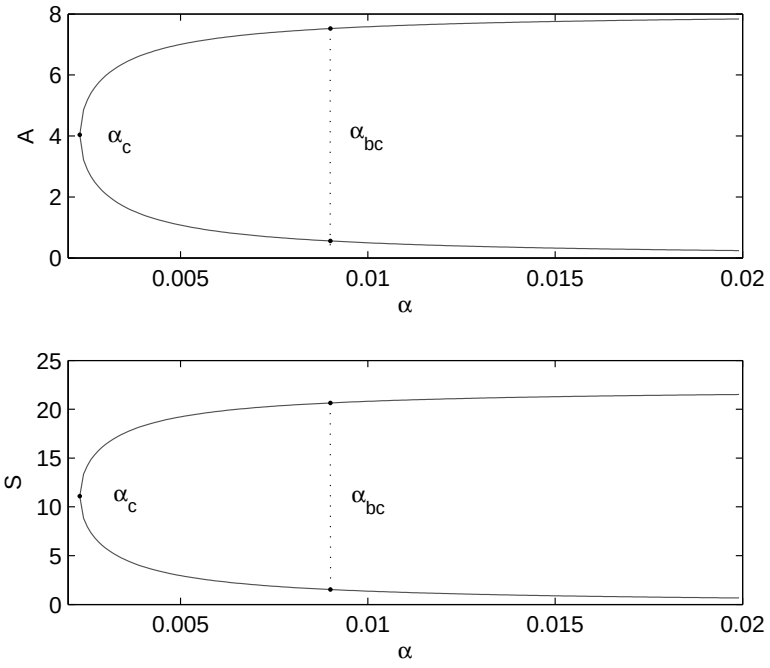


Figure 4.4: Bifurcation diagrams with respect to the coefficient in the initiation function, α, for the U.S. base parameterization and $\beta = 2$.

4.1.3 Initiation Function Coefficient, α

Figure 4.4 displays that a blue sky bifurcation occurs at α_c if β is set equal to 2. At the point $(\hat{S}_c, \hat{A}_c, \hat{C}_c) = (11.0884, 4.0388, 4.0388)$, the bifurcation point, the stable focus $\hat{E}_2$ and the saddle point steady state $\hat{E}_1$ collide.

4.1.4 Exit from the Active Use, μ

If $\beta = 2$, the critical parameter is smaller than the base case parameter $\mu_{bc} = 0.1661$. A saddle-node bifurcation occurs at the point $(\hat{S}_c, \hat{A}_c, \hat{C}_c) = (11.0884, 2.0454, 2.0454)$.

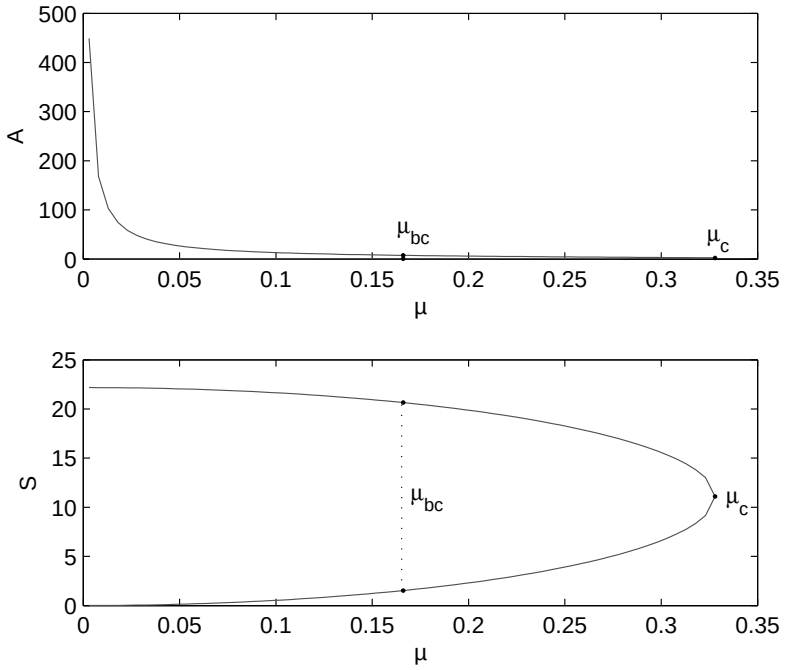


Figure 4.5: Bifurcation diagrams with respect to the exit of active use, μ, for the U.S. base parameter set and $\beta = 2$.

4.1.5 Coefficient in the Growth Rate, c

Figure 4.6 illustrates that the parameter value c has to be enhanced to nearly 35 in order to cause a saddle-node bifurcation. As discussed in [Ranner, 2009] and [Silbermayr, 2009], the adaptation of the SAC model to the original SA model is the higher the higher the parameter value c is.

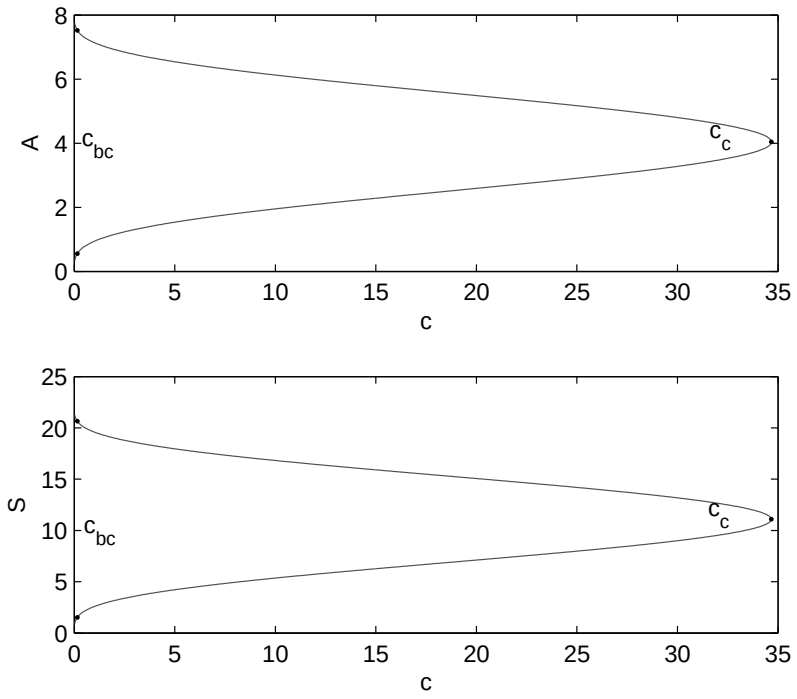


Figure 4.6: Bifurcation diagrams with respect to the coefficient in the growth rate, c, for the United States and $\beta = 2$.

4.2 Bifurcation Analysis for the Australian IDU Epidemic

By changing the exponent in the initiation function β to 2, a second and a third steady state appear for the Australian IDU epidemic. One is the artificial steady state where $\hat{A} = \hat{C} = 0$.

Keeping β equal to 1/2 represents the case of the injection drug use in Australia where the base case parameter $\beta_{bc} = 0.8622$ and we have a concave initiation function. Once more, no bifurcation occurs if the exponent in the initiation function equals 1/2.

For the Australian base parameters the critical values for $\beta = 2$ are listed in the Table 4.2.

parameter	critical value
k_c	0.0773
δ_c	0.0440
α_c	1.1054
μ_c	0.0773
c_c	0.0069

Table 4.2: Critical values for the Australian IDU base parameterization.

4.2.1 Inflow into the State S, k

Increasing the parameter k upon k_c evokes a blue sky bifurcation illustrated in Figure 4.7. That means there are values of k where we have just one steady state and some where we have 3. Exactly two steady states are reached if we equal parameter k to the critical value k_c in order to reach a congruence of $\sqrt{p^2 - 4rq} = 0$. Then, the two equilibria coincide leaving only the artificial steady state.

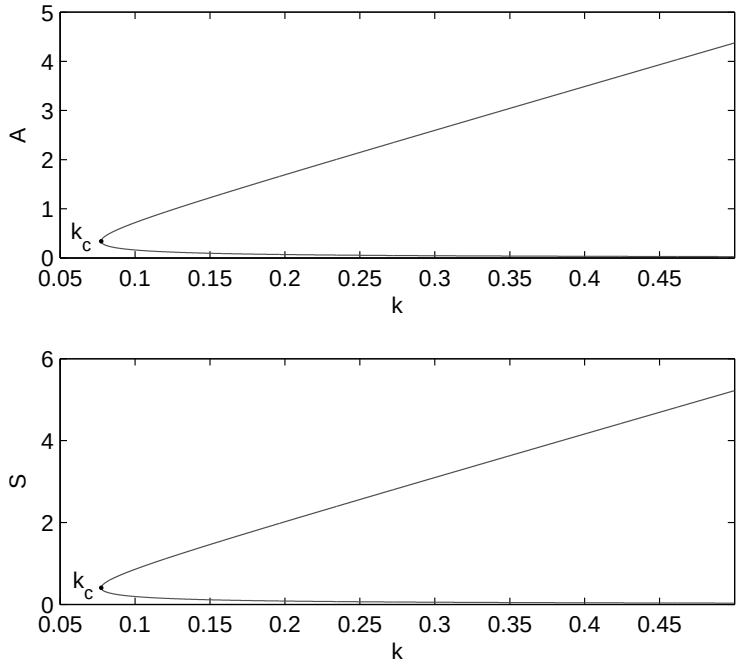


Figure 4.7: Bifurcation diagrams with respect to the inflow into state S, k, for the Australian IDU epidemic and $\beta = 2$.

4.2.2 Exit from the State S, δ

Figure 4.8 shows that a blue sky bifurcation emerges if we roughly halve the original value of δ. The bifurcation point is given by $(\hat{S}_c, \hat{A}_c, \hat{C}_c) = (0.5974, 0.2315, 0.2315)$.

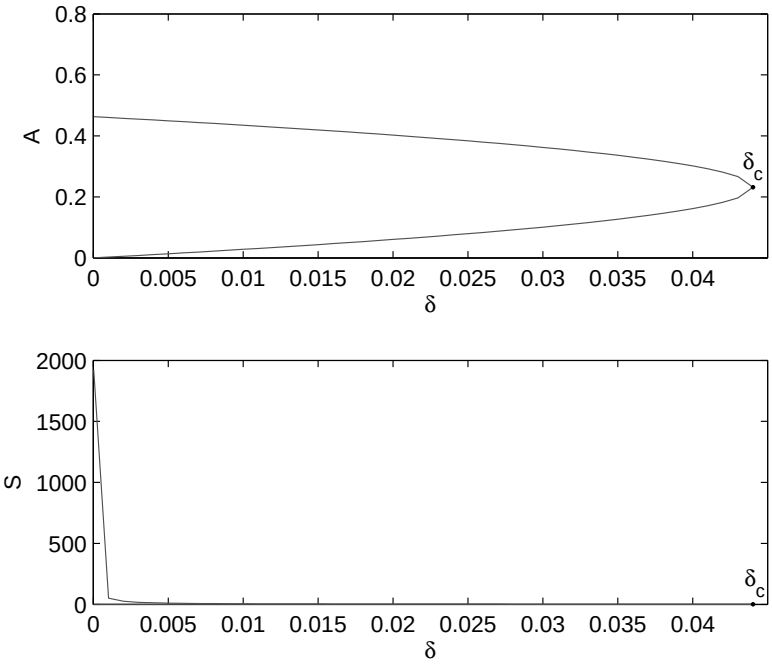


Figure 4.8: Bifurcation diagrams with respect to the exit from state S, δ, for the Australian base parameterization and $\beta = 2$.

4.2.3 Coefficient in the Initiation Function, α

Two equilibria occur if we increase α by more than 50% compared to the original parameter value $\alpha_{bc} = 0.5112$. As listed in the Table 4.2, the critical value of α is given by $\alpha_c = 1.1054$. The saddle-node bifurcation is shown in Figure 4.9.

4.2.4 Exit from the Active Use, μ

Again, two additional steady states occur if we abate the parameter for the exit from active use if $\beta = 2$. Figure 4.10 displays the blue sky bifurcation. $(\hat{S}_c, \hat{A}_c, \hat{C}_c) = (0.2763, 0.3404, 0.3404)$ denotes the bifurcation point, the point where the number of equilibria changes.

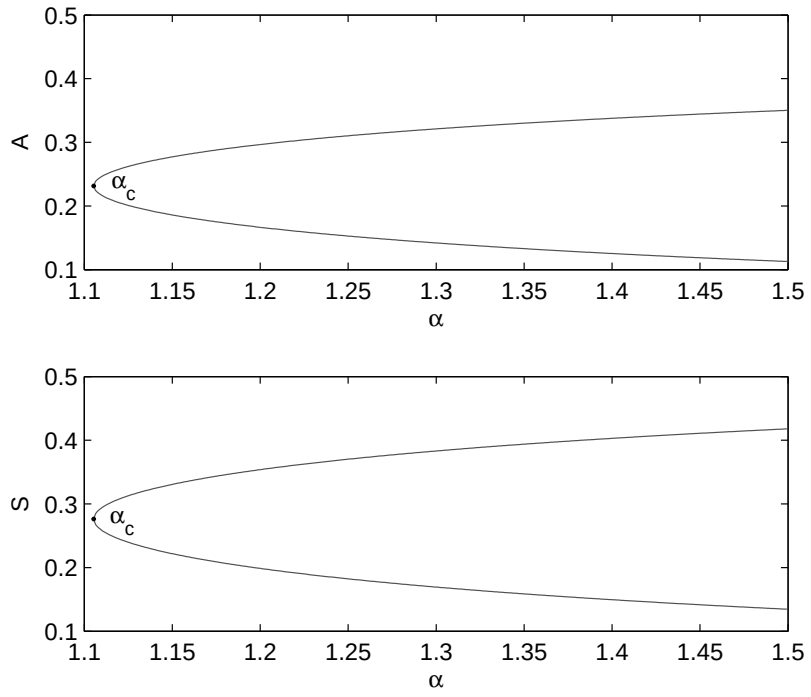


Figure 4.9: Bifurcation diagrams with respect to α for the Australian base parameter set and $\beta = 2$.

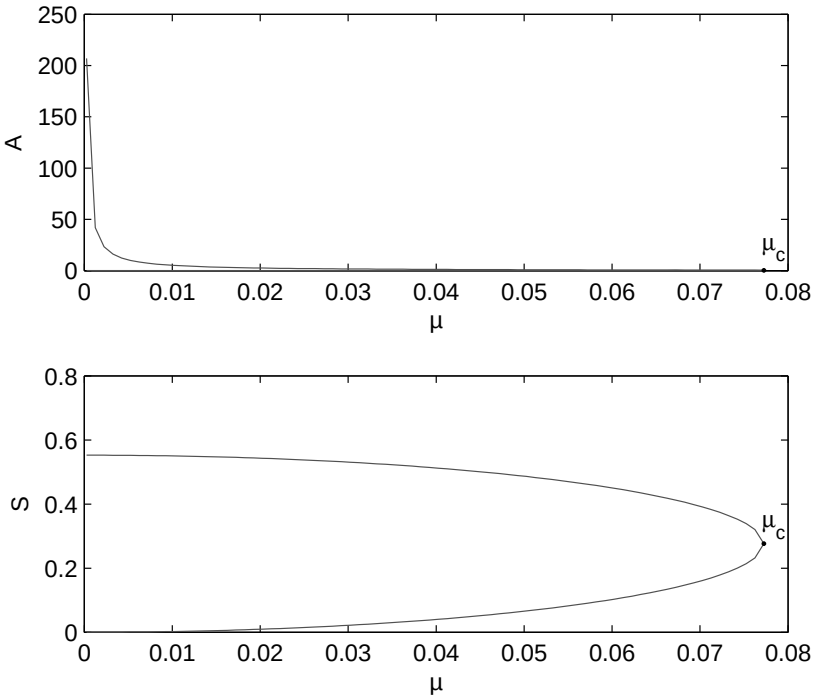


Figure 4.10: Bifurcation diagrams with respect to the exit from active use, μ, for Australia and $\beta = 2$.

4.2.5 Coefficient in the Growth Rate, c

Finally, we have a look at the bifurcation plots associated with the coefficient in the growth rate in Figure 4.11.

If $\beta = 2$, a blue sky bifurcation appears at the critical parameter value $c_c = 0.0069$, which is lower than the original value.

This is a big difference to the U.S. parameterization, where the coefficient in the growth rate c has to be enlarged to 34.6733 to get to the bifurcation point.

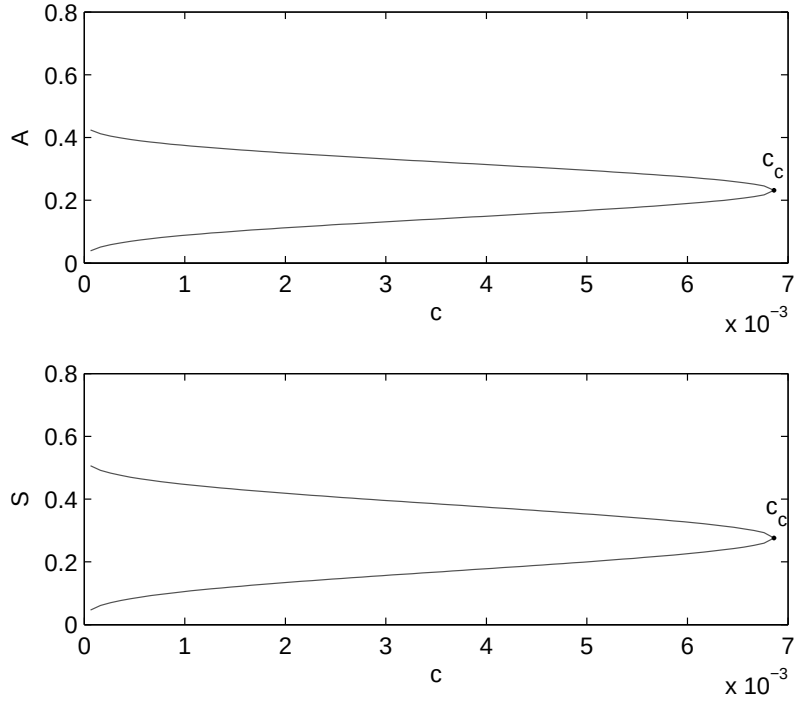


Figure 4.11: Bifurcation diagrams with respect to the coefficient in the growth rate, c, for the Australian IDU epidemic and $\beta = 2$.

4.3 Two-Dimensional Bifurcation Analysis

This part of the thesis is investigating the interplay of two parameters. We change two parameter values simultaneously and look at the number of steady

states.

We will see that there are parameter values where the number of equilibria is varying from one to three.

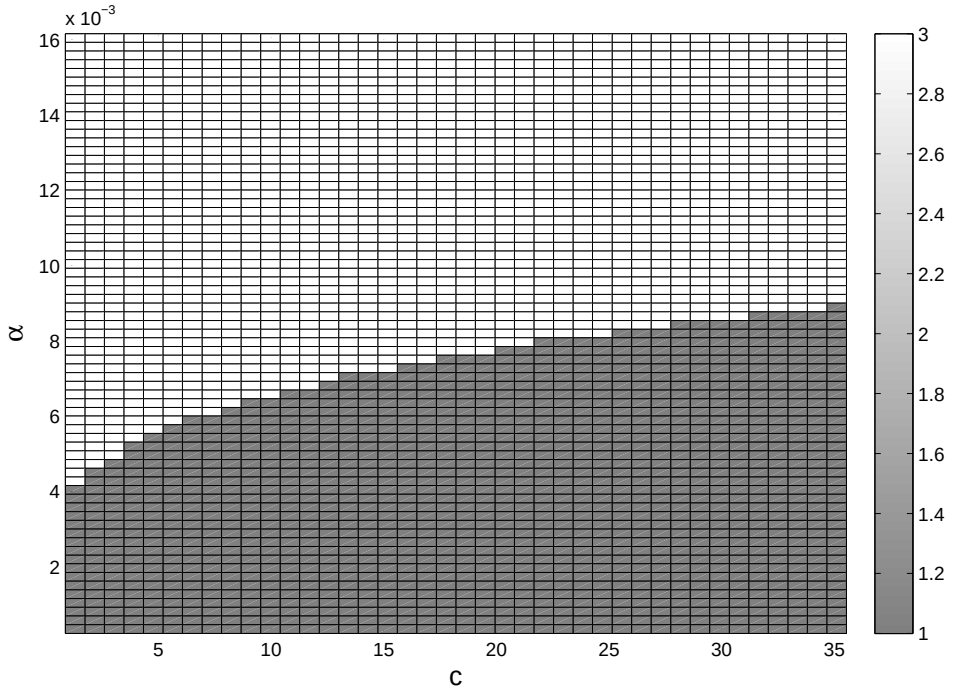


Figure 4.12: Two-dimensional bifurcation diagram with respect to the coefficient in the growth rate, c, and the coefficient of the initiation function, α, if $\beta = 2$ for the U.S. base parameterization.

Theoretically there can be exactly two steady states. The artificial one if $\hat{A} = 0$ and the steady state you get if you solve the equation with the two critical parameter values therewith the root is equal to 0 and $\hat{E}_1$ and $\hat{E}_2$ collide.

Figure 4.12 displays the interplay between c and α. These two parameters both have an impact on the initiation function $\alpha A^{\beta} S \left(c\frac{A}{C}\right)^{a}$.

If c is larged and α is diminished the initiation function is decreasing, meaning that S is increasing and A is decreasing. If we increase the coefficient of the initiation function above the original value, the function itself increases and consequently the number of drug users rises.

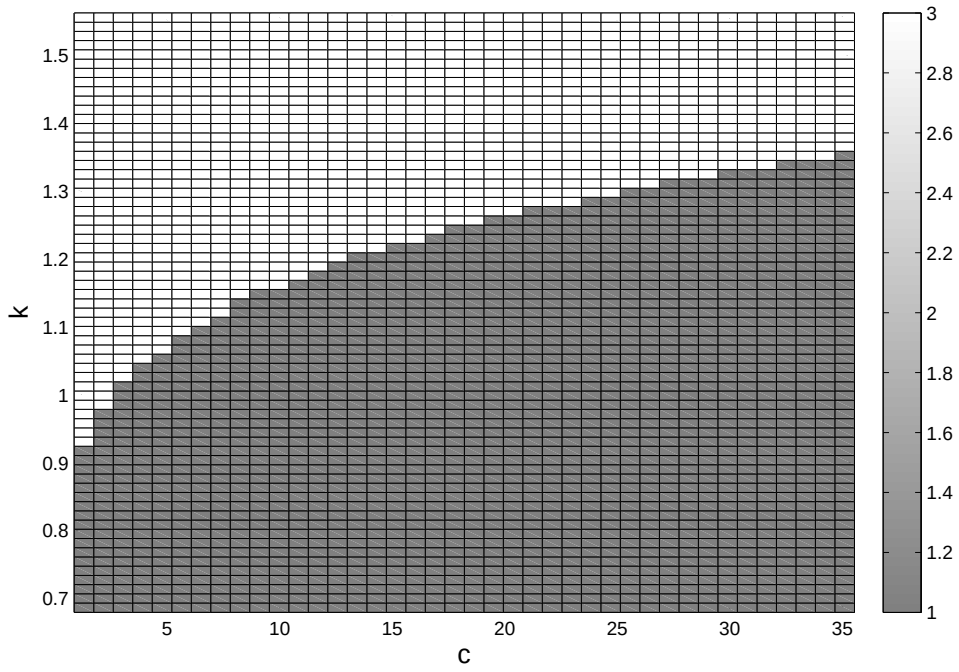


Figure 4.13: Two-dimensional bifurcation diagram with respect to c and the inflow into state S, k, if $\beta = 2$ for the U.S. cocaine epidemic.

Increasing the parameters k and c leads to a higher number of people who are susceptible to initiating into drug use. The parameter k, representing the inflow into the state S has an indirect effect on the number of active drug users A. Due to the great number of susceptible non-users caused by a higher inflow rate into state S, k, more people become drug users, so the state A rises, too.

Figure 4.13 displays the two-dimensional bifurcation analysis for these two parameters.

Figure 4.14 reproduces the two-dimensional bifurcation analysis for the exit of state S and the coefficient in the growth rate. The boost of δ reduces the numbers of S and A. The difference to the other two-dimensional bifurcation plots is that a small value of δ is required for two additional steady states.

Finally, we want to look at a two-dimensional bifurcation analysis for the Australian IDU parameterization.

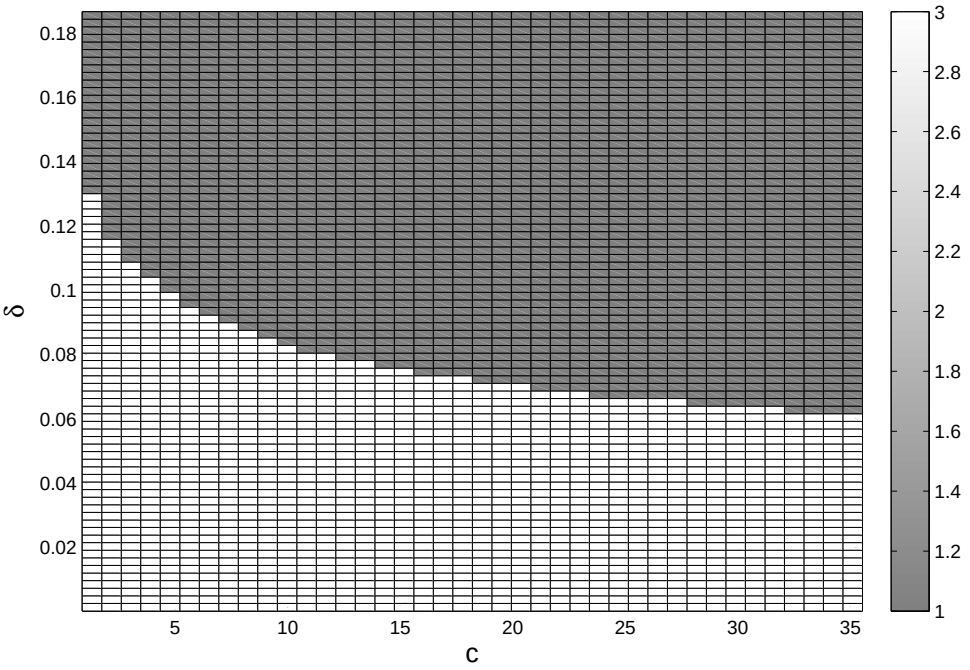


Figure 4.14: Two-dimensional bifurcation diagram with respect to c and the exit from state S, δ, if $\beta = 2$ for the U.S. base case parameters.

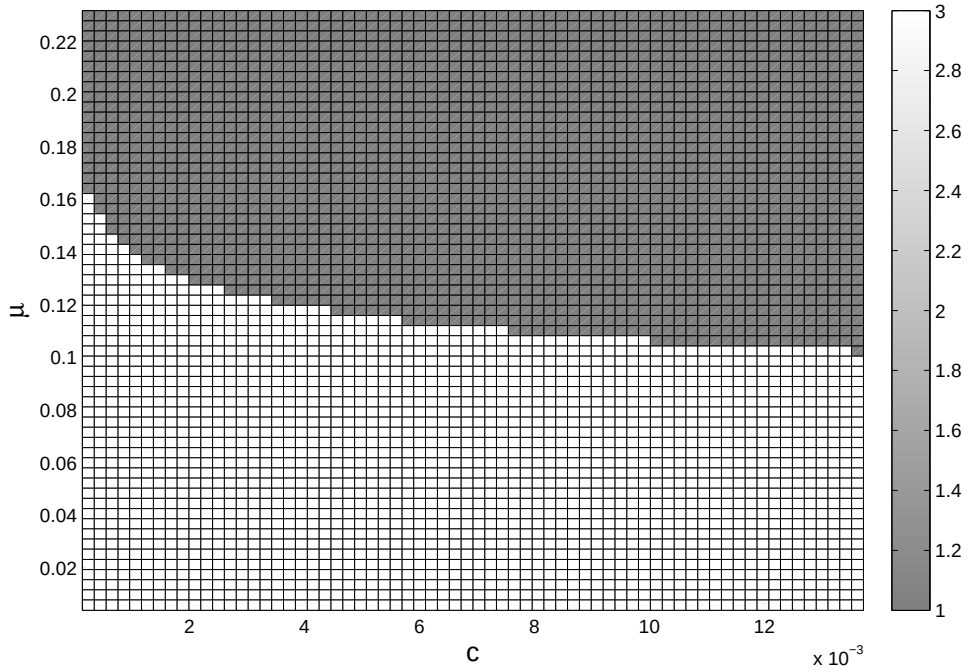


Figure 4.15: Two-dimensional bifurcation diagram with respect to c and the exit from active use A, μ, if $\beta = 2$ for the Australian IDU epidemic.

Again, the parameter β is set equal to 2, so we have a convex initiation function contrary to the initial situation of the Australian base case parameter set.

Figure 4.15 depicts the interaction between c and the parameter μ. Increasing μ has a negative impact on the number of drug users in contrast to a higher c.

Chapter 5

Enforcement

In this chapter we will model a control instrument denoted as enforcement, v. We let v be the amount of throughput capacity removed, which appears linearly in the third state equation $\dot{C}$.

$$\begin{aligned} \dot{S} &= k - \delta S - f(A)Sp(A,C)^a, \\ \dot{A} &= f(A)Sp(A,C)^a - \mu A, \\ \dot{C} &= g(p)C - vC. \end{aligned} \tag{5.1}$$

Throughput capacity measures how much smugglers can bring in on a current basis, which now is reduced by v. Enforcement can be seen as police work having an effect on the price.

However, the third state C will not be decreased for the whole epidemic. This will only be the case for the first year in this chapter.

After some preliminary analyses we conclude that the effectiveness of reducing C is most significant at the beginning of the epidemic.

5.1 Enforcement and its Effects in the U.S.A.

By varying v we get new equilibrium points for the high steady state and the saddle point.

The number of active users is smaller with a higher level of enforcement v

compared to the original value. In this case, the impact of enforcement is as one would expect it to be. The number of drug users decreases because of a higher price and the emanating smaller initiation, so the number of people who are susceptible to initiating into drug use increases.

$$\begin{aligned}(\hat{S}_{v=0\%}, \hat{A}_{v=0\%}, \hat{C}_{v=0\%}) &= (3.9490, 6.7201, 6.7201),\\ (\hat{S}_{v=10\%}, \hat{A}_{v=10\%}, \hat{C}_{v=10\%}) &= (4.8290, 6.3188, 3.2442),\\ (\hat{S}_{v=20\%}, \hat{A}_{v=20\%}, \hat{C}_{v=20\%}) &= (5.9153, 5.9231, 1.5613).\end{aligned}$$

We get the eigenvalues

$$\begin{aligned}\lambda_1 &= -0.1802,\\ \lambda_{2,3} &= -0.0980 \pm 0.1255i,\end{aligned}$$

if we have an enforcement policy with $v = 10\%$. This means we still have a stable focus as is the case for $v = 20\%$, in which the eigenvalues are given by

$$\begin{aligned}\lambda_1 &= -0.1793,\\ \lambda_{2,3} &= -0.0730 \pm 0.1143i.\end{aligned}$$

With a higher value of enforcement v the saddle point steady state is higher with respect to the state A. This means that we have more active users if we use enforcement to minimize the drug capital stock. On account of a smaller value of C we get a higher price p but consequently a smaller initiation if A and C are close to 0 and smaller than 1 and therefore a higher number of active users A.

$$\begin{aligned}(\hat{S}_{v=0\%}, \hat{A}_{v=0\%}, \hat{C}_{v=0\%}) &= (21.2619, 0.3332, 0.3332),\\ (\hat{S}_{v=10\%}, \hat{A}_{v=10\%}, \hat{C}_{v=10\%}) &= (20.9077, 0.4623, 0.2374),\\ (\hat{S}_{v=20\%}, \hat{A}_{v=20\%}, \hat{C}_{v=20\%}) &= (20.3898, 0.6509, 0.1716).\end{aligned}$$

Again, the control enforcement does not change the stability behaviour of the equilibrium and we still have a saddle point:

$$\begin{aligned}\lambda_1 &= 0.0729,\\ \lambda_2 &= -0.0583,\\ \lambda_3 &= -0.1773\end{aligned}$$

are the eigenvalues for the model with $v = 10\%$. For $v = 20\%$, the eigenvalues are given by

$$\lambda_1 = 0.0703,$$
$$\lambda_2 = -0.0572,$$
$$\lambda_3 = -0.1773.$$

If $C = 0$, enforcement has no impact on the equilibrium, so the artificial steady state stays the same.

In addition, we are looking closer at the dynamic behaviour of the model with different enforcement rates v.

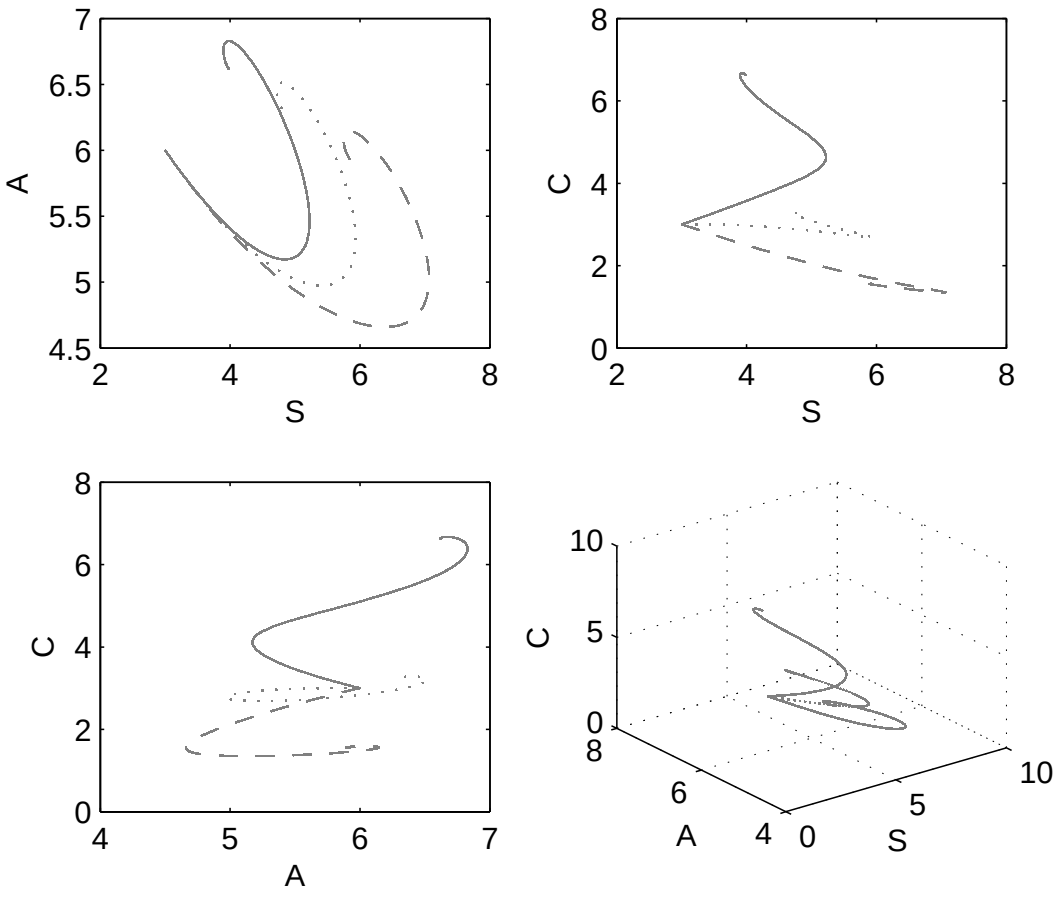


Figure 5.1: Trajectories with $v = 0\%$ (solid line), $v = 10\%$ (dotted line), and $v = 20\%$ (dashed line) enforcement rate per year and $(S(0), A(0), C(0)) = (3, 6, 3)$ for the U.S. cocaine epidemic.

In the Figures 5.1 and 5.2 we compare the behaviour of the trajectories with $v = 0\%$, $v = 10\%$, and $v = 20\%$ removal rate per year for the initial values $(S(0), A(0), C(0)) = (3, 6, 3)$ and $(S(0), A(0), C(0)) = (4.5, 4, 2)$, respectively.

It is plain to see that the trajectories converge to three different steady states. Looking at the (S, A)-plane in these figures reveals that the control v has a strong effect on the number of drug users A, as the performance of the trajectories for $v = 0\%$, $v = 10\%$, and $v = 20\%$ is very different.

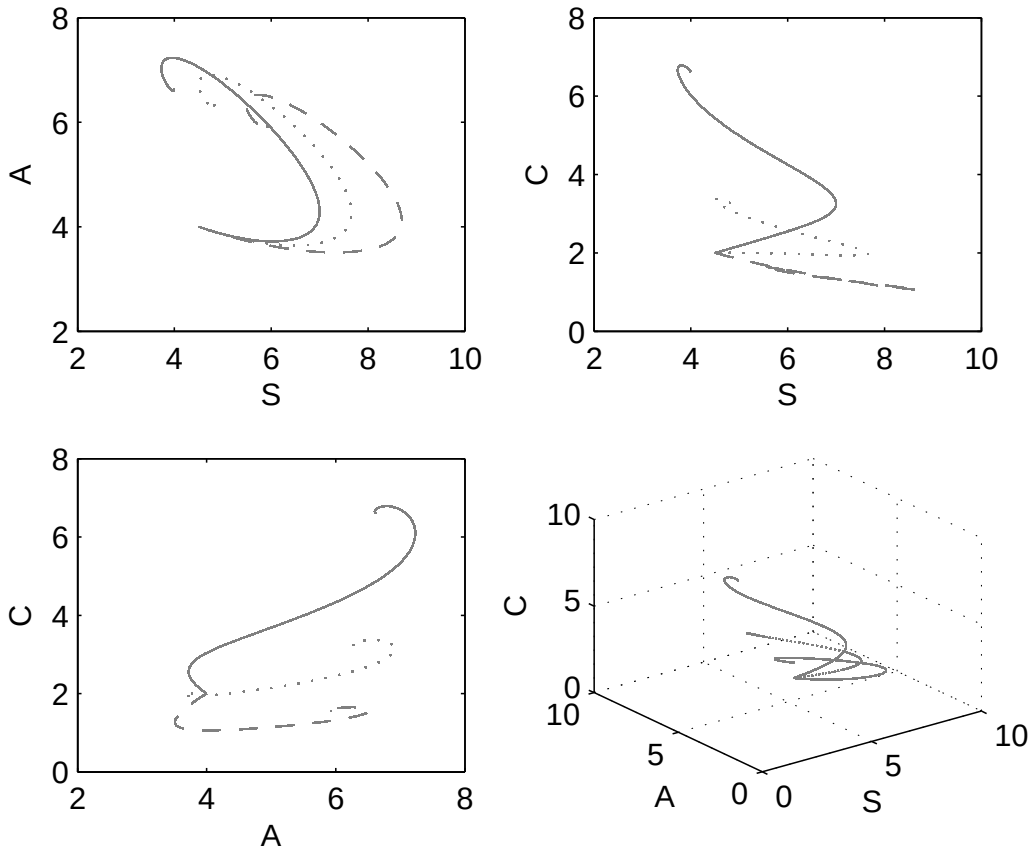


Figure 5.2: Trajectories with $v = 0\%$ (solid line), $v = 10\%$ (dotted line), and $v = 20\%$ (dashed line) enforcement rate per year and $(S(0), A(0), C(0)) = (4.5, 4, 2)$ for the U.S.A. base parameterization.

Next, we have a look at the social costs which occur during a drug epidemic.

$$J = \int_0^\infty e^{-rt} A(t)\, dt$$

describes the discounted amount of drug users over an infinite time horizon. We will use a finite horizon ($T = 500$) approximation to the infinite integral with an infinite error term:

$$J = \int_0^T e^{-rt} A(t)\ dt + \hat{A} \int_T^\infty e^{-rt}\ dt.$$

$\hat{J}$ calculates the social costs which arise from applying $v = 10\%$ or $v = 20\%$ in the first year:

$$\hat{J} = \int_0^1 e^{-rt} A_v(t)\ dt + \int_1^T e^{-rt} A(t)\ dt + \hat{A} \int_T^\infty e^{-rt}\ dt.$$

Figure 5.3 demonstrates the difference $(\hat{J} - J) * 100$ by applying the control $v = 10\%$. One may conclude from this that enforcement leads to smaller or equal costs because we have less users A. We final that the social costs can be reduced by up to nearly 80% if we have enforcement at the level $v = 10\%$. Starting at initial values with a high value of S and a small value of A it is cheaper not to use enforcement.

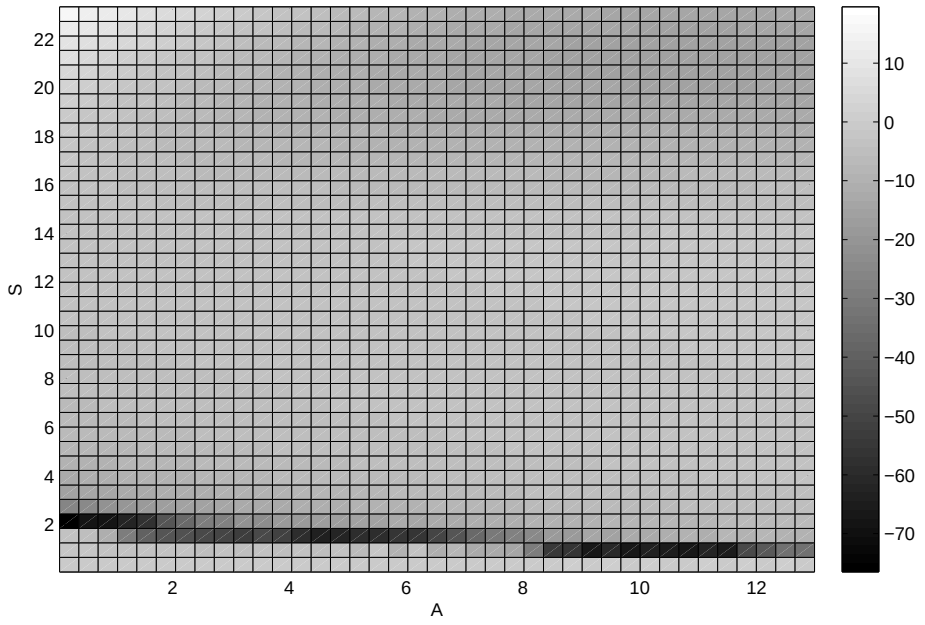


Figure 5.3: Reduction in J by applying control $v = 10\%$ for the cocaine epidemic in the United States.

The reduction of social costs in percentage terms, if we apply the control instrument enforcement in amount of $v = 20\%$ is shown in Figure 5.4. We can see that a higher reduction of the state C is more effective. There are some initial conditions of an epidemic where we can save more than 80% if we use enforcement to abate the drug stock C. The black area of the graphics shows those values of initial condition where enforcement has the strongest effect on the social costs.

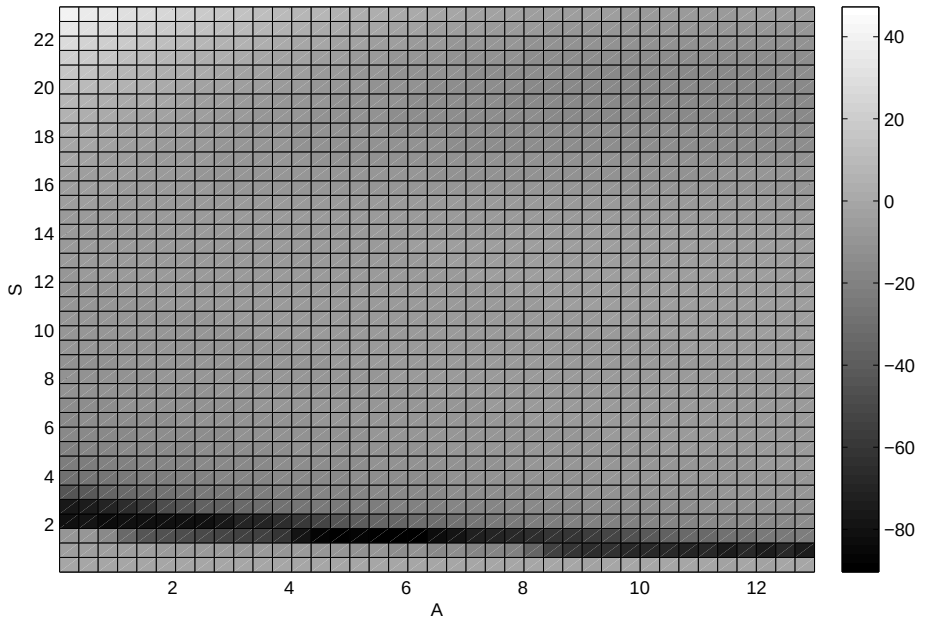


Figure 5.4: Reduction in J by applying control $v = 20\%$ for the U.S. base parameterization.

5.2 Enforcement and its Effects in Australia

If we modify the base case model SAC by adding a new parameter v we obtain new equilibria depending on the amount of the control enforcement v.

$$\begin{aligned}(\hat{S}_{v=0\%}, \hat{A}_{v=0\%}, \hat{C}_{v=0\%}) &= (0.1202, 0.3623, 0.3623),\\ (\hat{S}_{v=10\%}, \hat{A}_{v=10\%}, \hat{C}_{v=10\%}) &= (0.1411, 0.3448, 0.1770),\\ (\hat{S}_{v=20\%}, \hat{A}_{v=20\%}, \hat{C}_{v=20\%}) &= (0.1653, 0.3245, 0.0855).\end{aligned}$$

Once more, enforcement does not change the stability behaviour of our model.

The eigenvalues imply that the equilibria are stable foci:

$$\lambda_1 = -0.4540,$$
$$\lambda_{2,3} = -0.1132 \pm 0.0271i$$

derive from $v = 10\%$. The eigenvalues

$$\lambda_1 = -0.35303,$$
$$\lambda_{2,3} = -0.1076 \pm 0.0292i$$

come from setting $v = 20\%$.

In Figures 5.5 and 5.6 we compare the base case scenario with trajectories with $v = 10\%$ and $v = 20\%$ removal rate.

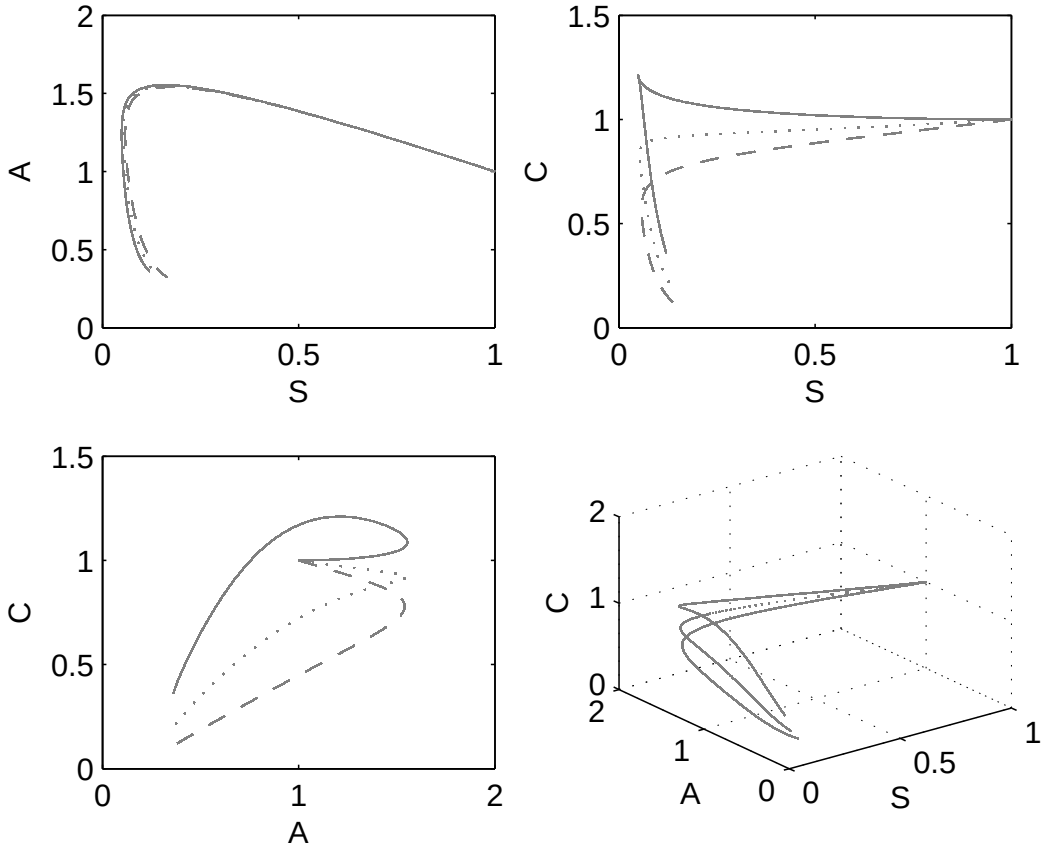


Figure 5.5: Trajectories with $v = 0\%$ (solid line), $v = 10\%$ (dotted line), and $v = 20\%$ (dashed line) enforcement rate per year and $(S(0), A(0), C(0)) = (1, 1, 1)$ for the Australian parameterization.

We start with the initial conditions $(S(0), A(0), C(0)) = (1, 1, 1)$ in Figure 5.5 and $(S(0), A(0), C(0)) = (0.5, 1, 0.5)$ in Figure 5.6. Looking at the first subplot of the figures, the (S, A)-plane, illustrates that enforcement v hardly affects A in the Australian IDU parameterization, since the trajectories for different values of v are very similar. This probably is due to the relatively small steady state in the Australian injection drug use epidemic.

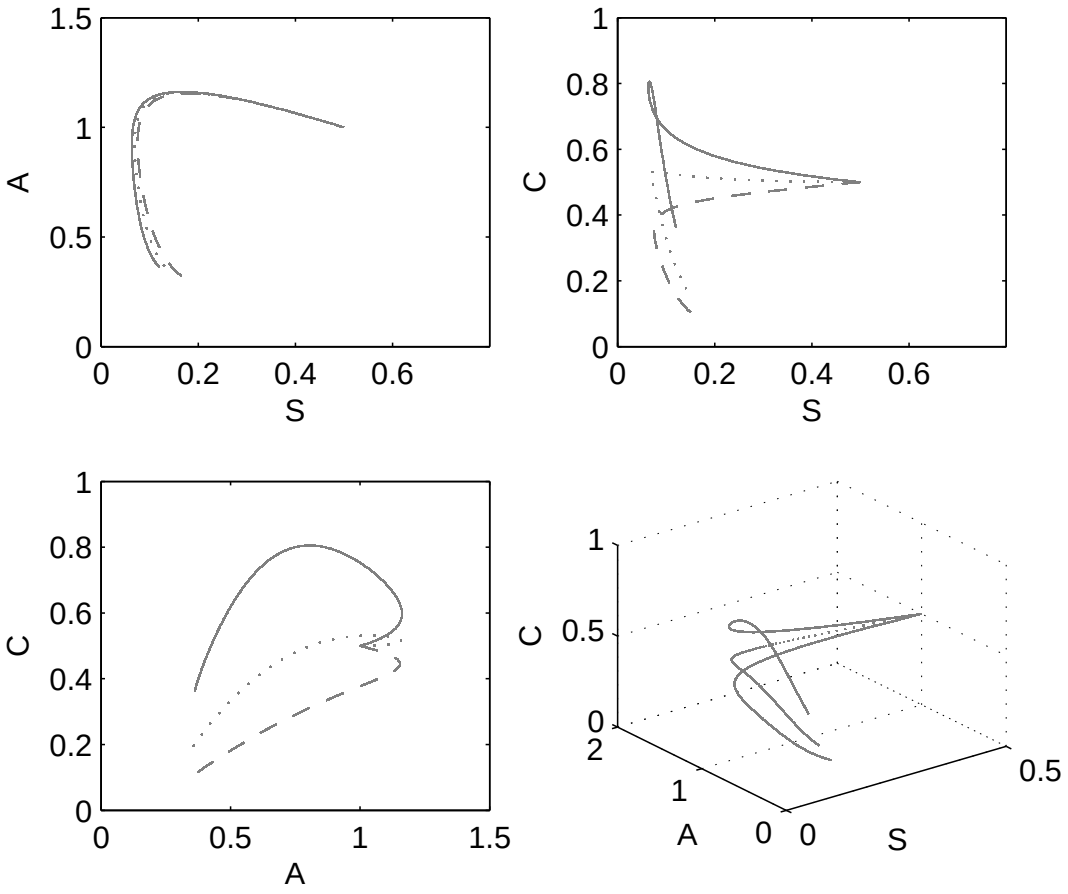


Figure 5.6: Trajectories with $v = 0\%$ (solid line), $v = 10\%$ (dotted line), and $v = 20\%$ (dashed line) enforcement rate per year and $(S(0), A(0), C(0)) = (0.5, 1, 0.5)$ for the injection drug use in Australia.

In the following analyses we will picture the reduction in social costs when applying the control $v = 10\%$ and $v = 20\%$ versus the base case model without enforcement. We are looking at the percental difference derived from initial values where $S(0)$ varies from $0.01\hat{S}$ to $2\hat{S}$, $A(0)$ varies from $0.01\hat{A}$ to $2\hat{A}$, and $C(0) = A(0)$.

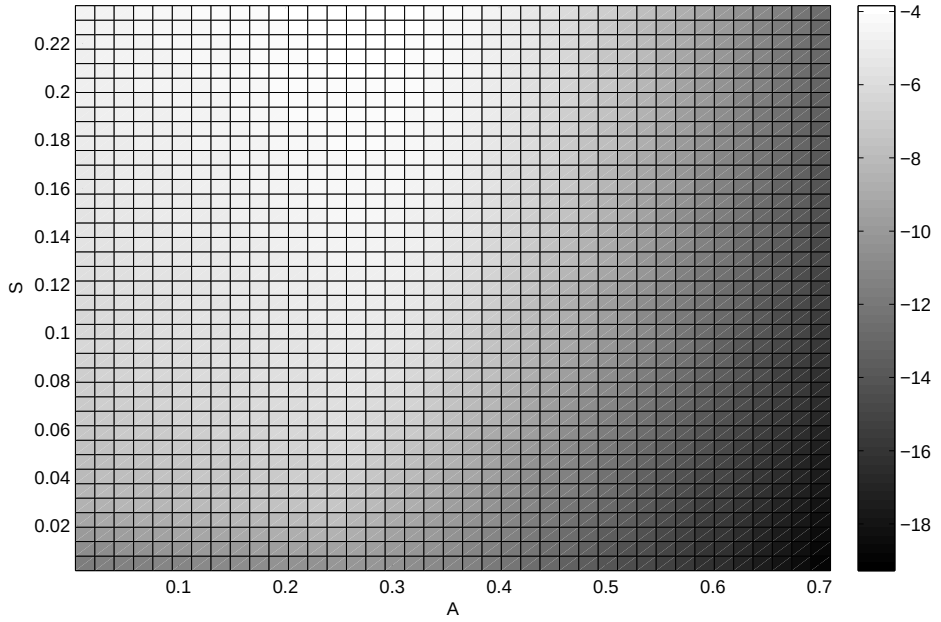


Figure 5.7: Reduction in J by applying control $v = 10\%$ for the Australian IDU.

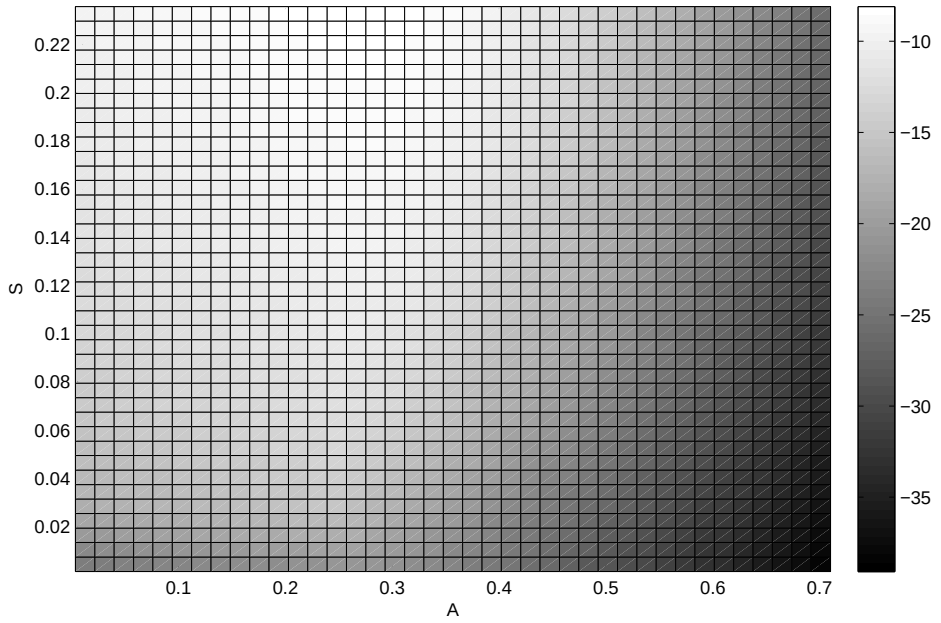


Figure 5.8: Reduction in J by applying control $v = 20\%$ for the Australian parameterization.

The (A, S)-plots in Figures 5.7 and 5.8 illustrate that we get the highest effects on the right hand side of the graphics for those initial conditions, where we have a high number of users A. Note that the current throughput capacity of the supply network of drugs equals the number of active users, $C(0) = A(0)$, in the beginning of the epidemic.

Again, we see that a higher value of v breeds a larger reduction in social costs. As mentioned before, enforcement in the Australian IDU epidemic is not as successful as in the cocaine epidemic in the United States.

Chapter 6

Supply Shocks

In this chapter, a supply shock and its consequences on the optimal dynamic drug policy are being analysed. Many illicit drug epidemics experienced such a supply reduction, for example the Australian "heroin drought"; see, e.g., [Bultmann et al., 2008a], [Bultmann et al., 2008b], [Degenhardt et al., 2005], [Weatherburn et al., 2002]. As introduced in Chapter 5, we reduce the current throughput capacity of the supply network of drugs. One can think of such a reduction as confiscation of a certain amount of the current drug stock or as more investment in the control of the supply side (dealers, etc.), so the drug price rises which consequently influences the consumers.

Here, we will simulate a supply shock to the extent of 50% by reducing the state $C(t)$ at the time t. Performing such a reduction of C at different stages of the epidemic and comparing the results will give us information about how expedient a supply reduction is at a certain stage in the epidemic. We will present the results in the (A, S)- and (A, C)-planes for the two different parameter sets for the United States and Australia.

6.1 Supply Shocks and their Effects in the U.S.A.

We simulate the effects of a supply reduction based on different initial scenarios. The outcome of a supply shock is examined for the first 50 years to detect which years affords the highest effects on the social costs.

In addition, we look at the benefit-cost ratio of a supply shock, which provides insight into the efficiency of this kind of control.

6.1.1 (A, S)-Plane

In the (A, S)-plane we distinguish between three different initial conditions for the throughput capacity of the supply network of drugs. $C(0)$ can either be twice as high as, equal to, or one half of $A(0)$ in the beginning of the epidemic. First, we want to look at the most efficient years for a supply shock of C by 50%. The results for the different initial conditions for $C(0)$ are very similar so we will just present the bottom line for the case $C(0) = 2A(0)$.

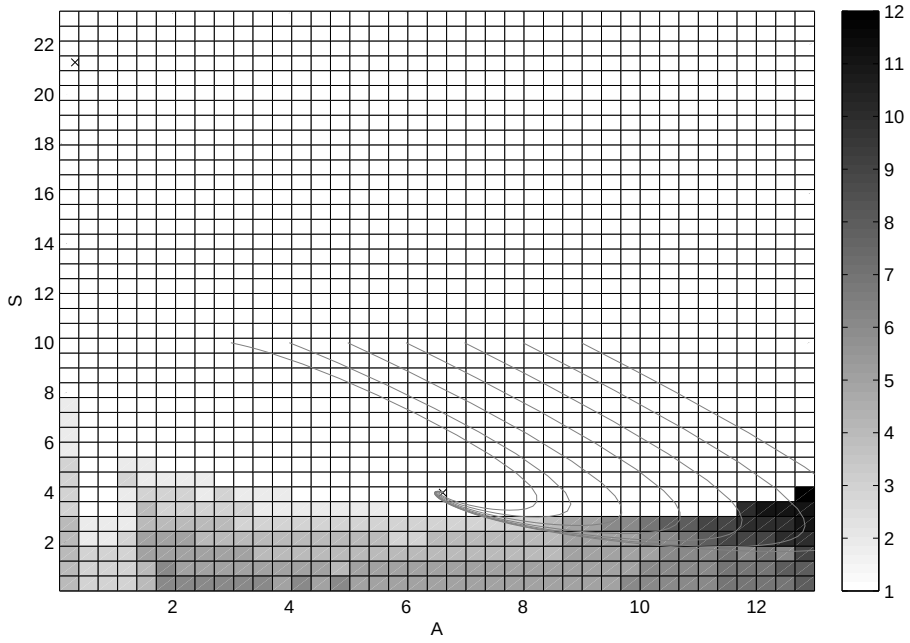


Figure 6.1: Most efficient years for a supply reduction by 50% for several initial values in the (A, S)-plane with a few trajectories for the U.S. base parameter set, $(C(0) = 2A(0))$.

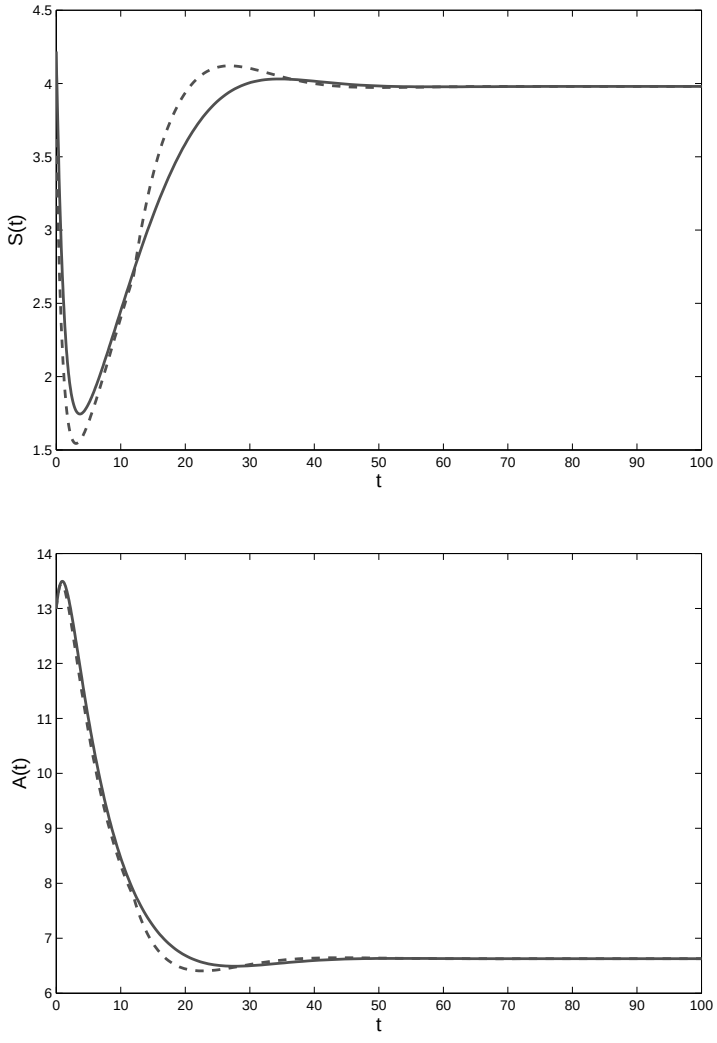
Figure 6.1 shows the most effective years, where a supply reduction for different initial values has the greatest impact on social costs. We include some trajectories to see if there is a connection between the trajectories and the most effective years for a reduction of C.

It appears that as long as the number of users is growing the first year is best suited for a supply reduction and then a later year becomes more effective. Please note that this is just an assumption but it was not proved or examined further.

As mentioned before, the qualitative behaviour of the solutions for $C(0) = A(0)$ and $C(0) = A(0)/2$ resemble $C(0) = 2A(0)$. For $C(0) = A(0)$ the latest year for a supply shock is the eleventh, for a drug stock half as large as the number of active users the latest most effective year for a supply shock is the 9th.

Next, we take a closer look at two different initial values. One within the white area, where it is optimal to have a supply reduction in the first year, and the second within the black area, where the maximum effect of a reduction of C by 50% is after 12 years.

Figure 6.2 illustrates the trajectories over the years for the number of susceptibles, the number of current users, the capital stock of the drug smuggling industry, the price function, and the initiation.



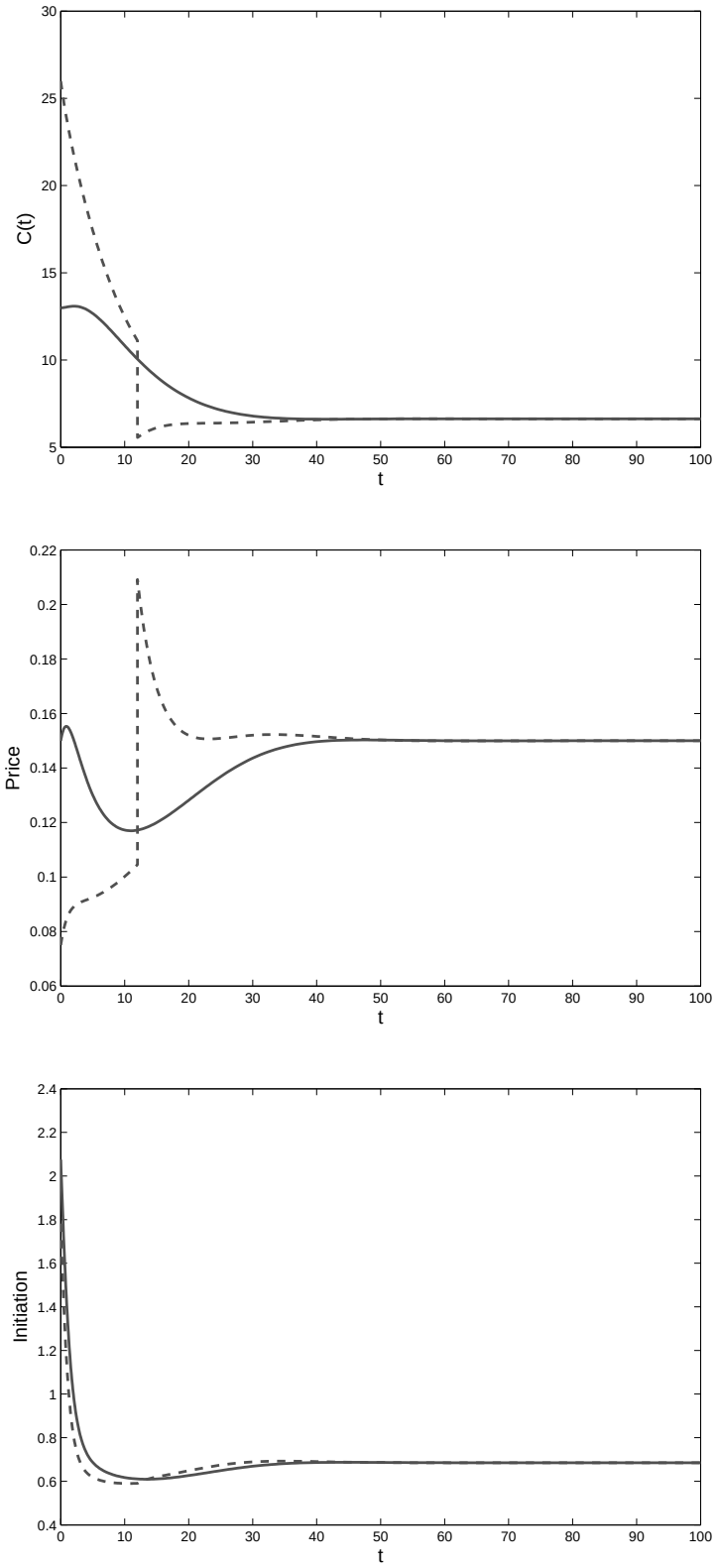


Figure 6.2: Comparison of the states, the price, and the initiation over the time for $(S(0), A(0), C(0)) = (4.2184, 12.9911, 25.9822)$ with a supply reduction of 50% in the first year (solid line) and for $(S(0), A(0), C(0)) = (3.6214, 12.9911, 25.9822)$ with a 50% supply reduction after 12 years (dashed line) of the U.S. cocaine epidemic.

If C drops, the number of users decreases as well but not as significantly. Accordingly, the price increases, which is plausible, because due to the stronger controls on the part of the police the dealers have to take higher hazards. This in turn has an impact on the initiation.

By comparing the respective state paths we can see that the numbers of susceptibles and users are very similar over the time but the recommended control is completely different. It is reasonable to assume that in an optimal control formulation of this model we would find so-called DNSS thresholds (named after Dechert, Nishimura, Sethi und Skiba), see [Grass et al., 2008].

Until now, the social costs have always been measured by the discounted accumulation of users over an infinite time horizon. Now we want to use another way to measure the costs.

$$J_2 = \int_0^\infty e^{-rt} C(t)\, dt$$

measures the social costs by looking at the number of drug sellers, which is proportional to $C(t)$. Later, we will recur to this subject.

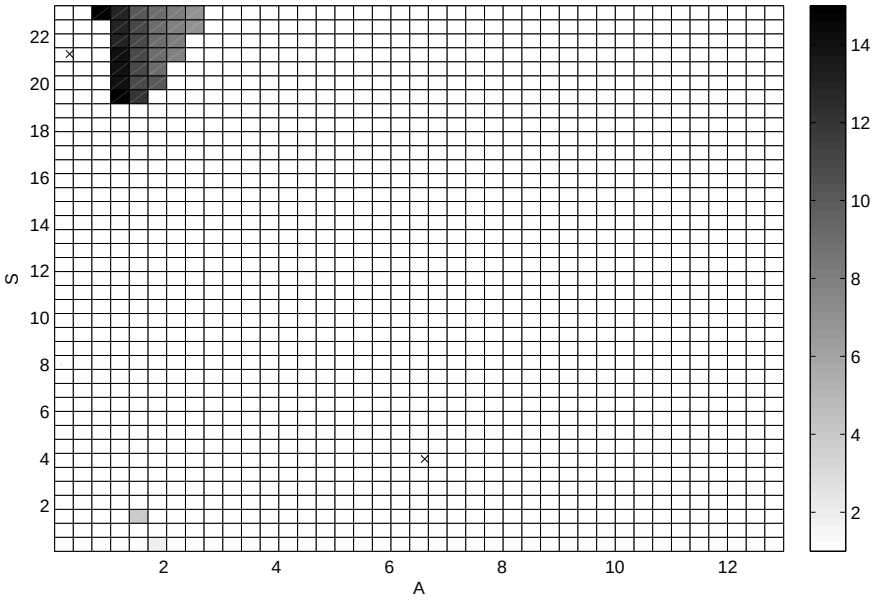


Figure 6.3: Most efficient years for a supply shock by 50% for several initial values in the (A, S)-plane measured with J_2 for United States, $(C(0) = 2A(0))$.

The most efficient years of a supply shock by 50% is shown in Figure 6.3 if we compare the costs which emerge from the number of drug sellers computed by using the state C instead of A.

The bottom line here is completely different to the results we get using the number of drug users to estimate the social costs.

Thus, we have to be careful with the decision which year is the most effective for a supply shock.

Moreover, we want to know how efficient the reduction of C by 50% in the first year is. For that purpose, we compute

$$\frac{J^i - \hat{J}^i}{\delta},$$

which can be interpreted as a benefit-cost ratio and gives an insight into the efficiency of a supply shock reducing C by 50% in the i-th year. The amount of the supply shock of C is called δ.

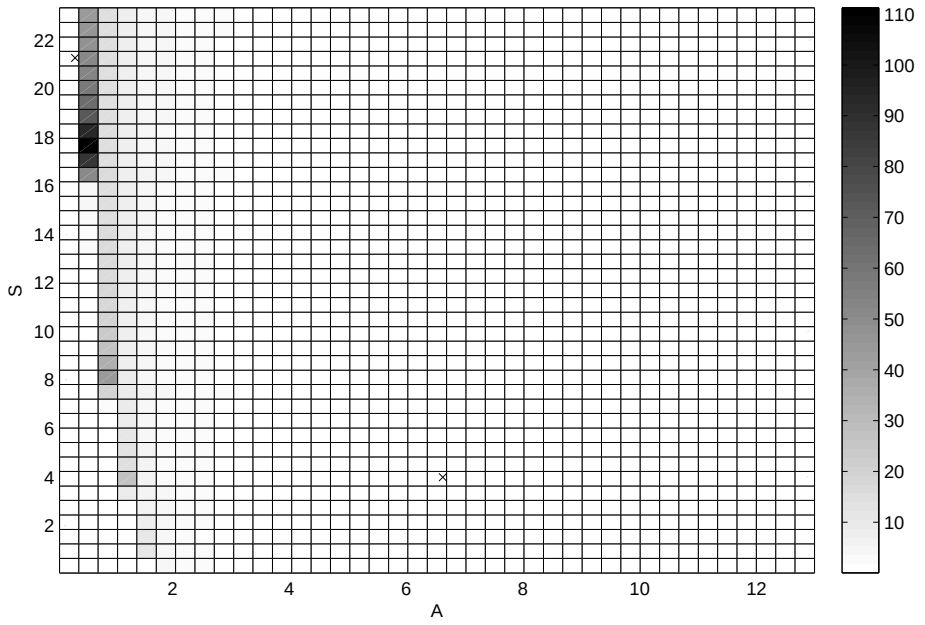


Figure 6.4: $\frac{J-\hat{J}}{\delta}$ for different initial values and $C(0) = 2A(0)$ in the (A, S)-plane for the U.S. cocaine epidemic.

Again, we investigate three different scenarios for $C(0)$, but because of the striking resemblance we will just look at $C(0) = 2A(0)$.

Figure 6.4 shows the results of a supply reduction in the first year where we have an initial drug stock twice as high as the number of current users. Note that the benefit-cost ratio is higher if we start with a smaller value of $C(0)$ because the denominator δ is smaller.

Once again, the effects of a supply reduction in the year i are measured by the discounted number of users over an infinite planning horizon:

$$\widehat{J_1^i} = \int_0^{i-1} e^{-rt} A(t)\ dt + \int_{i-1}^{i} e^{-rt} A_v(t)\ dt +$$

$$+ \int_i^T e^{-rt} A(t)\ dt + \hat{A} \int_T^{\infty} e^{-rt}\ dt.$$

Next, we want to add other ways to measure the social costs. First, we consider the number of drug sellers, which can be modeled as being proportional to $C(t)$:

$$J_2 = \int_0^{\infty} e^{-rt} C(t)\ dt.$$

Secondly, we let social costs be represented by the amount of drugs consumed, which is proportional to $A(t)$ times price raised to the short-run elasticity of demand ω:

$$J_3 = \int_0^{\infty} e^{-rt} p^{\omega} A(t)\ dt.$$

Finally, we may measure the social costs by looking at the amount of money spent for drugs:

$$J_4 = \int_0^{\infty} e^{-rt} p^{1+\omega} A(t)\ dt.$$

Note that $\omega = -0.5$ is assumed for the elasticity of demand in the objective function.

During our investigations we noticed that the qualitative behaviour of the solutions for J_1, J_2, J_3, and J_4 are very similar with changes only in the amount of the ratio. Therefore, we omit the figures for the different ways to measure the social costs. J_2 and J_4 provoke smaller benefit-cost ratios, but valuing with J_3 causes higher ones. For J_2 the highest ratios are 83.9786, for J_3 we have ratios up to 243.3547, and estimating the social cost with J_4 produces the highest benefit-cost ratios 53.7439.

To compare the results of a supply shock in the first year with the outcome of a 50% reduction of state C in the most efficient year we refer to Figure 6.5.

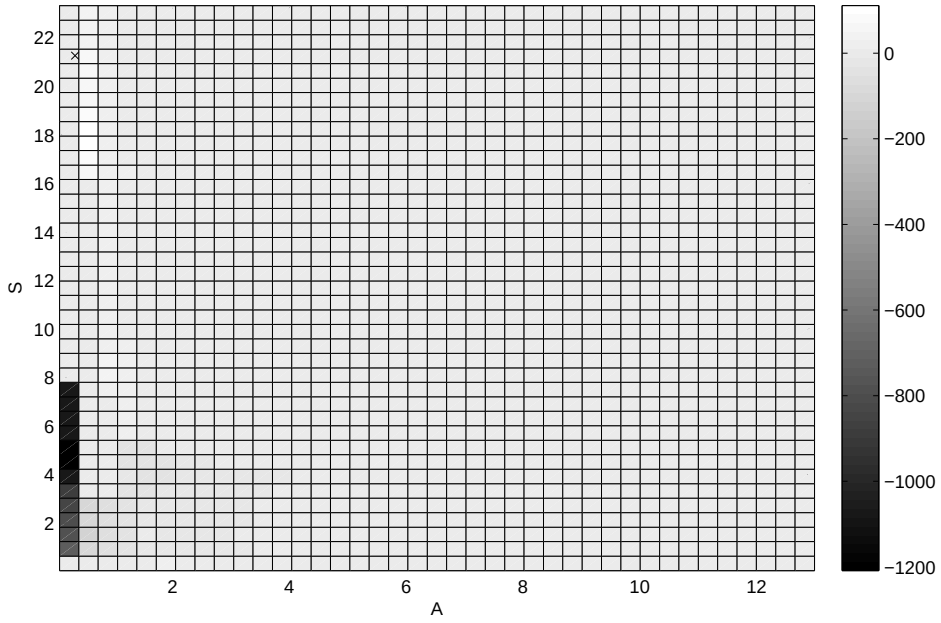


Figure 6.5: $\frac{J^i - \hat{J}^i}{\delta}$ for different initial values and $C(0) = 2A(0)$ in the (A, S)-plane with a supply reduction in the most efficient year for the cocaine epidemic in the United States.

The initial values where we got the highest benefit-cost ratio before when we had the supply shock at the very beginning of the epidemic are now afresh the ones where we have the best benefit-cost ratio.

Overall, we got the impression that the saddle point steady state has a high influence on the benefit-cost analysis. For this purpose, we take a closer look at this equilibrium in Figure 6.6.

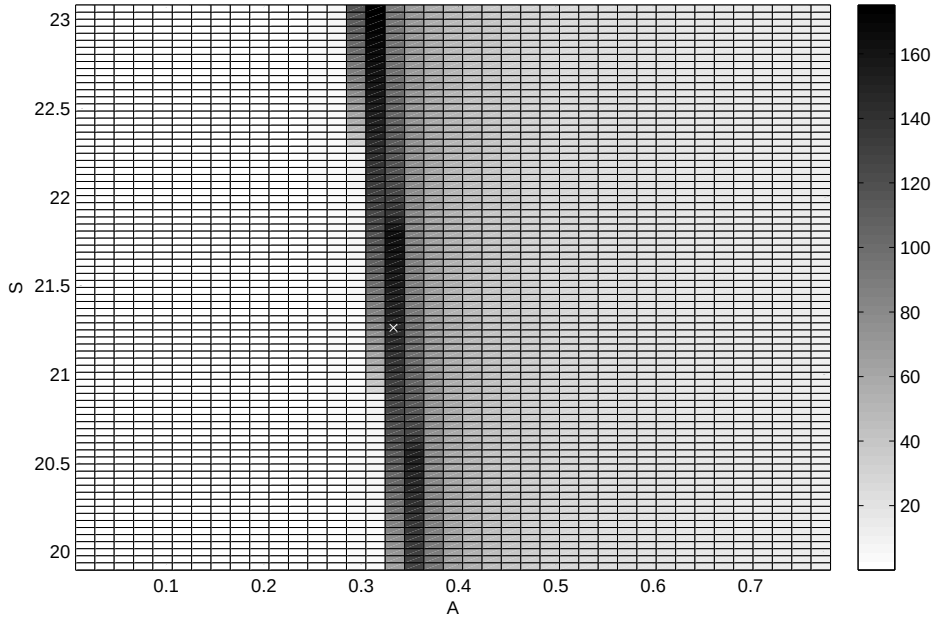
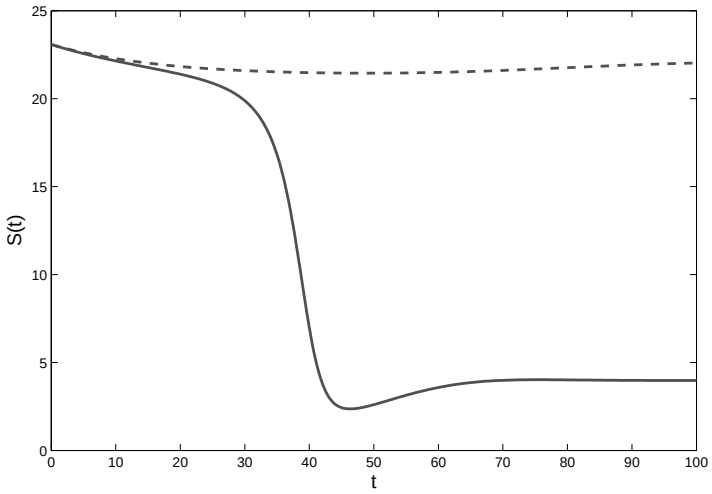


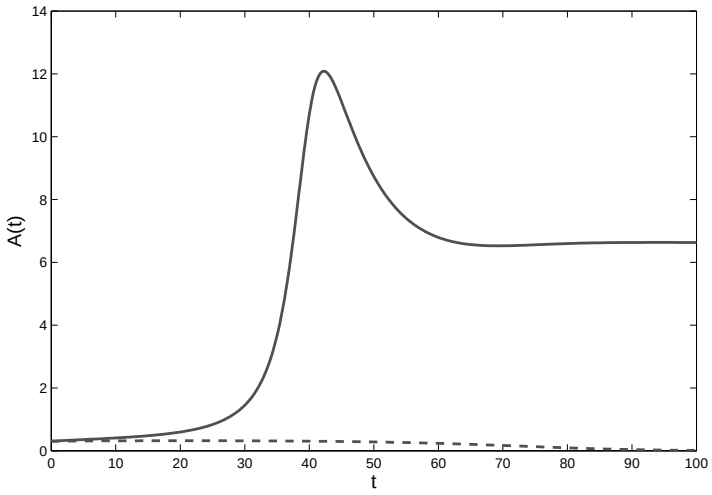
Figure 6.6: Zoom in on the saddle point steady state for the U.S. parameterization. $\frac{J-\hat{J}}{\delta}$ for different initial values and $C(0) = 2A(0)$ in the (A, S)-plane.

Needless to say that the quotients arisen with initial values along the instable manifold are the highest. It should be stressed that the qualitative behaviour of the solution is the same if the drug stock is not halved in the beginning but in the most efficient year.

We find that there is a band of initial values where a supply shock in the first year leads to a convergence towards another steady state. The U.S. cocaine parameters generate a tipping point curve separating the high-level stable focus from the low-level equilibrium related to the number of current users A. These tipping points alter with a reduction of C in the first year of the epidemic, e.g. [Caulkins et al., 2009a], [Caulkins et al., 2009b].

Figure 6.7 contrasts the original trajectories and the trajectories we get with a supply shock in the first year. We can see that the trajectories converge towards different steady states.


S(t)
t
0
5
10
15
20
25
0
10
20
30
40
50
60
70
80
90
100


A(t)
t
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2
4
6
8
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12
14
0
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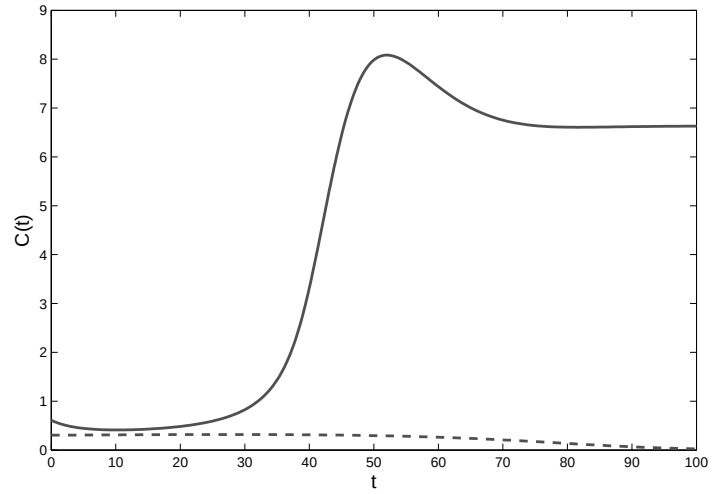
100


C(t)
t
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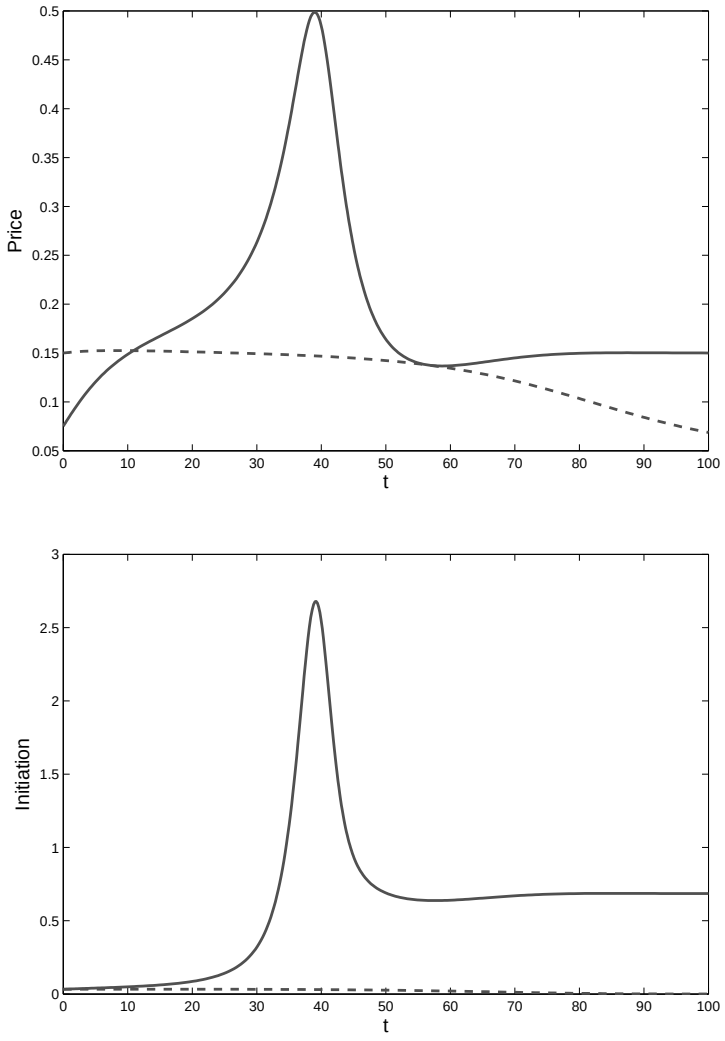


Figure 6.7: Drug use trajectories for the U.S. cocaine parameterization starting at $(S(0), A(0), C(0)) = (23.08168, 0.3048926, 0.6098)$ without (solid line) and with supply reduction (dashed line).

6.1.2 (A, C)-Plane

We continue with looking at the (A, C)-plane. Here, we will determine the results for five different but fixed values of $S(0)$ while $C(0)$ varies from $0.01\hat{C}$ to $2\hat{C}$. Either the number of susceptible non-users at the beginning of the epidemic is small, namely $S(0) = 2$, $S(0) = \hat{E}_2$, or $S(0) = 12$ or we start with a relatively high number of people who are susceptible to initiating into drug use, namely $S(0) = \hat{E}_1$ or $S(0) = 30$.

First, we want to investigate the most efficient years for a supply shock. Figure 6.8 illustrates the best years for a 50% reduction of C if we start with the initial condition $S(0) = 2$.

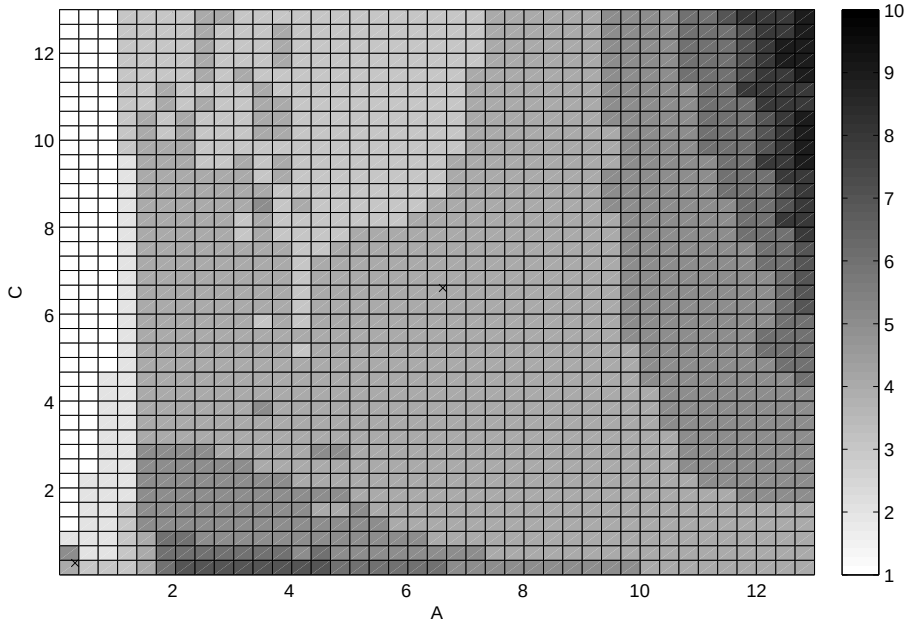


Figure 6.8: Most efficient years for a supply reduction of C by 50% if $S(0) = 2$ for several initial values in the (A, C)-plane for the U.S. parameterization.

The larger the number of people who are susceptible to initiating into drug use is at the beginning of the drug epidemic, the earlier a supply reduction is optimal.

This is shown in the next Figure 6.9 and by the fact that for the initial conditions $S(0) = 12$, $S(0) = \hat{E}_1$, and $S(0) = 30$ the first year is the most efficient year for a supply reduction of current throughput capacity of the supply net-

work of drugs, C, by 50%.

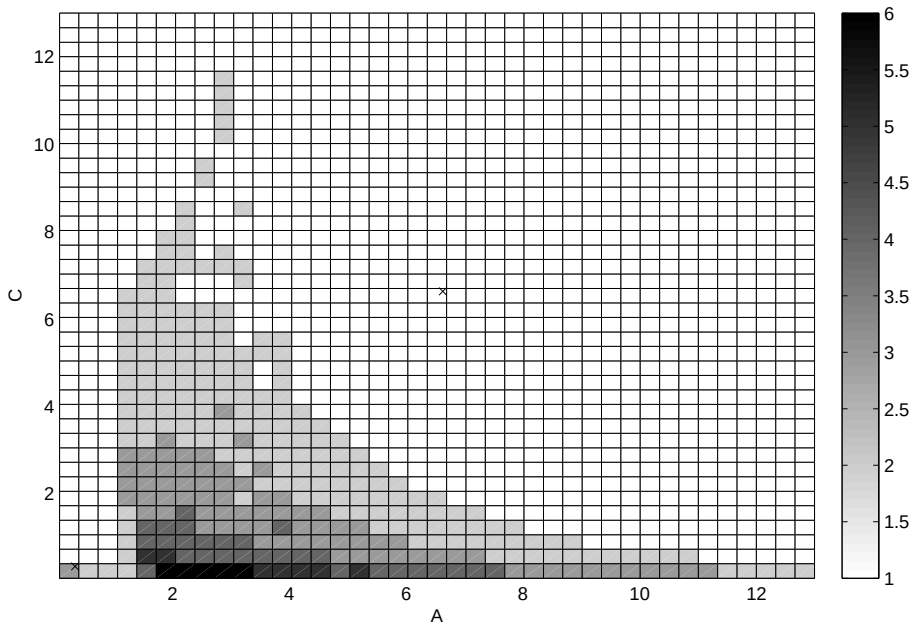


Figure 6.9: Most efficient years for a supply reduction of C in the U.S. cocaine epidemic if $S(0) = \hat{E}_2$ for several initial values in the (A, C)-plane.

Next, we want to look at the benefit-cost ratio $\frac{J-\hat{J}}{\delta}$ in the (A, C)-plane for different values of S. Once again, we just post the results for $S(0) = 2$, because the other ones are very similar.

Figure 6.10 shows the benefit-cost ratio for $S(0) = 2$ and for social costs measured with the discounted accumulation of users over an infinite time horizon. For $S(0) = \hat{E}_2$ we reach the value 249.8187. The benefit-cost ratio is higher for a higher initial value of S. For the highest considered value of $S(0) = 30$ we get the result 582.7231. The only exception here is the initial condition $S(0) = 2$, the smallest value of S we examined.

In turn, we want to investigate the difference between the benefit-cost ratio of a supply reduction in the first year and one in the most effective years.

Figure 6.11 shows that the quotient is smaller if there is a reduction of C in the most effective years because the costs do not get that high if there is an intervention at the right time.

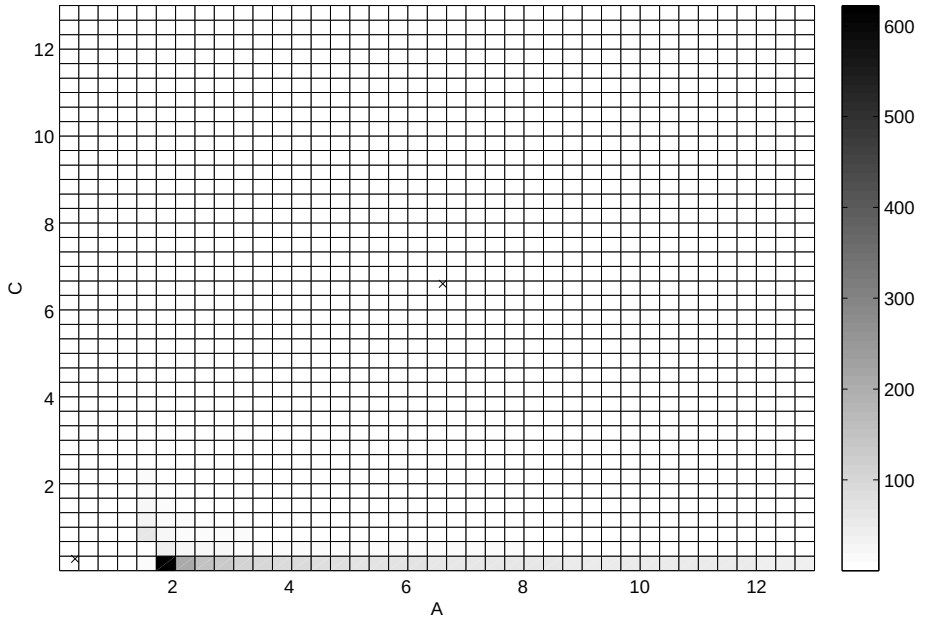


Figure 6.10: $\frac{J-\hat{J}}{\delta}$ for different initial values and $S(0) = 2$ in the (A, C)-plane for the U.S. base parameter set.

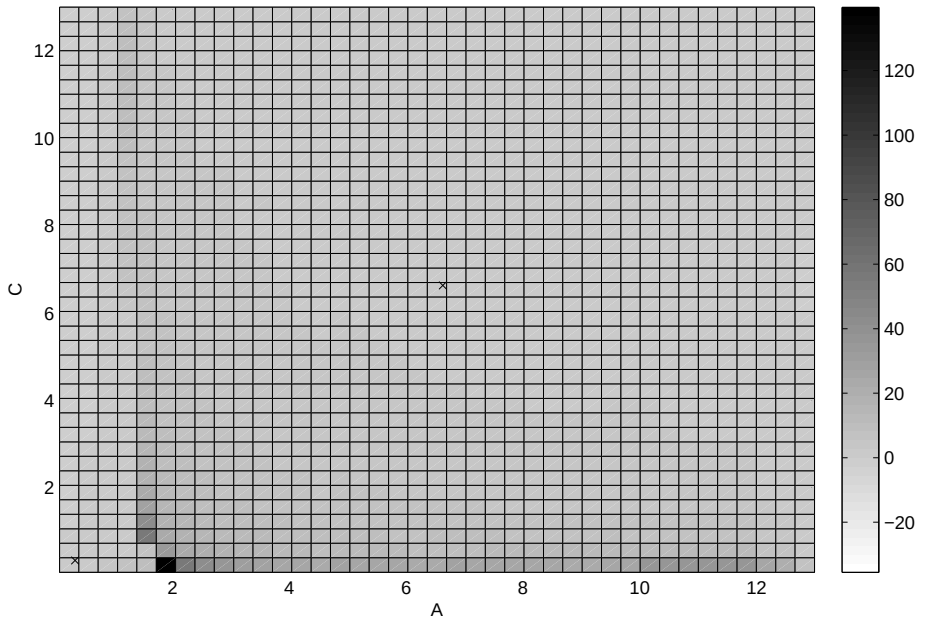


Figure 6.11: $\frac{J-\hat{J}}{\delta}$ for different initial values and $S(0) = 2$ in the (A, C)-plane with a supply shock in the most efficient years for the cocaine epidemic in the United States.

As mentioned before in the case of the (A, S)-plane, the different ways to measure the social costs lead to different benefit-costs ratios, but the qualitative behaviour of the solution stays the same. J_2 and J_4 determine a smaller benefit-cost ratio, 469.7652 and 228.2689, and J_3 breeds a higher ratio 1393.6846. Figure 6.12 exemplifies this.

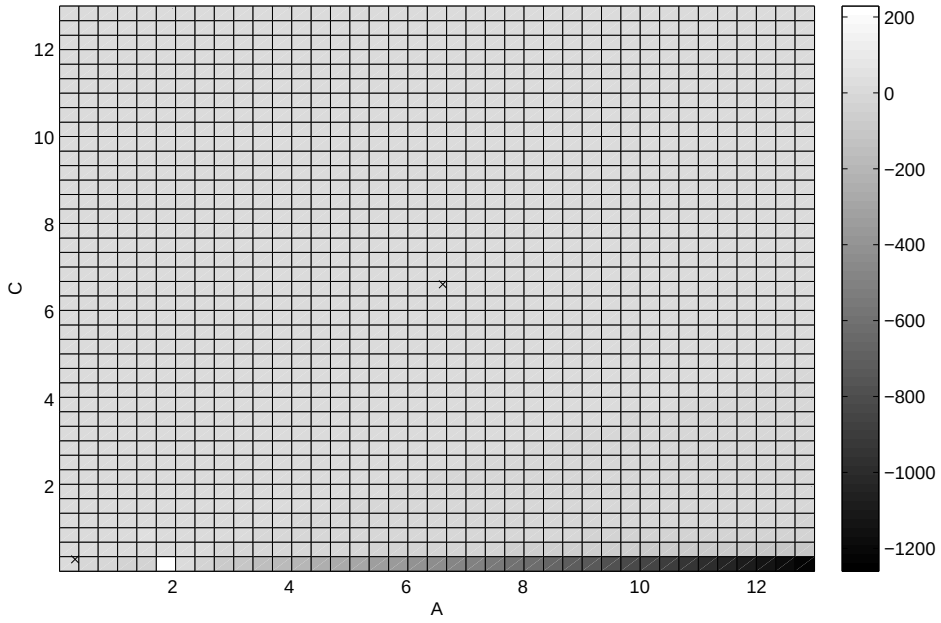


Figure 6.12: $\frac{J_4 - \hat{J}_4}{\delta}$ for different initial values and $S(0) = 2$ in the (A, C)-plane for the U.S. base parameterization.

6.2 Supply Shocks and their Effects in Australia

Until now, the analyses of this chapter only featured the U.S. parameter values. Hence, we will now look at the effects of a supply shock in the Australian injection drug use epidemic. Reducing $C(t)$ at time t and comparing the different results will give us an indication about how valuable a supply reduction at a certain stage in the epidemic is.

6.2.1 (A, S)-Plane

In this section we will analyse the results of a reduction of C by 50% for the Australian parameterization when we vary the number of people who are susceptible to initiating into drug use, $S(0)$, and the current users, $A(0)$.

In the beginning we quest for the best year for a supply reduction in terms of the year with the greatest effects on the social costs.

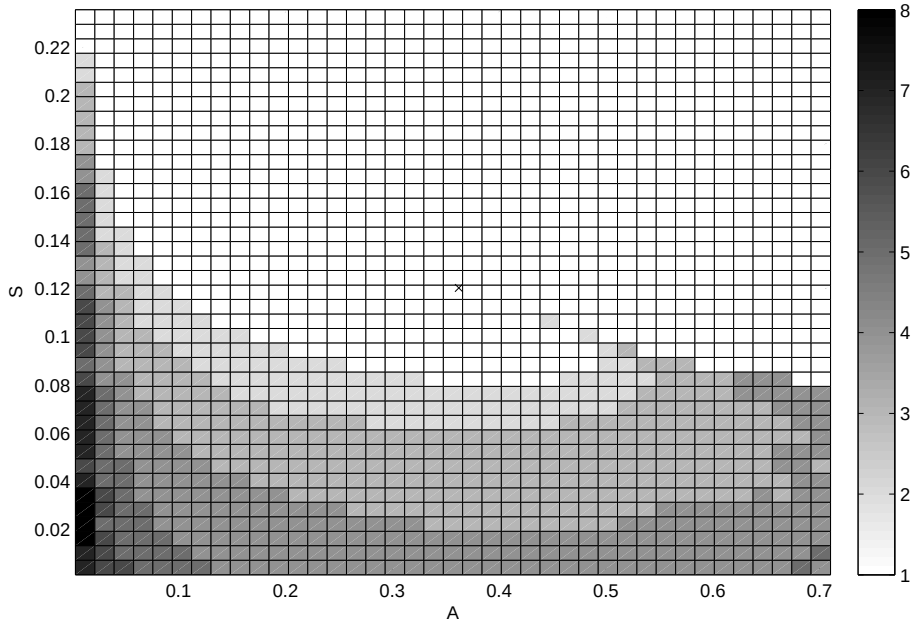


Figure 6.13: Years where the reduction of C by 50% have the highest consequences with the Australian base parameter set and $C(0) = 2A(0)$ in the (A, S)-plane.

Figure 6.13 shows the most efficient years for a sudden reduction of the through-

put capacity of the supply network of drugs with the initial condition $C(0) = 2A(0)$ and different initial values for $A(0)$ and $S(0)$. For $C(0) = A(0)$ year 9 is the latest most efficient, and for $C(0) = A(0)/2$ year 10 is the latest we found to be optimal.

In the next part of this section we want to inquire $\frac{J-\hat{J}}{\delta}$ with different initial values. Once again, we look at the three different initial conditions for $C(0)$ and the four different ways to measure the social costs.

Figure 6.14 illustrates the benefit-cost ratios for initial values where $S(0)$ varies from $0.01\hat{S}$ to $2\hat{S}$ and $A(0)$ from $0.01\hat{A}$ to $2\hat{A}$.

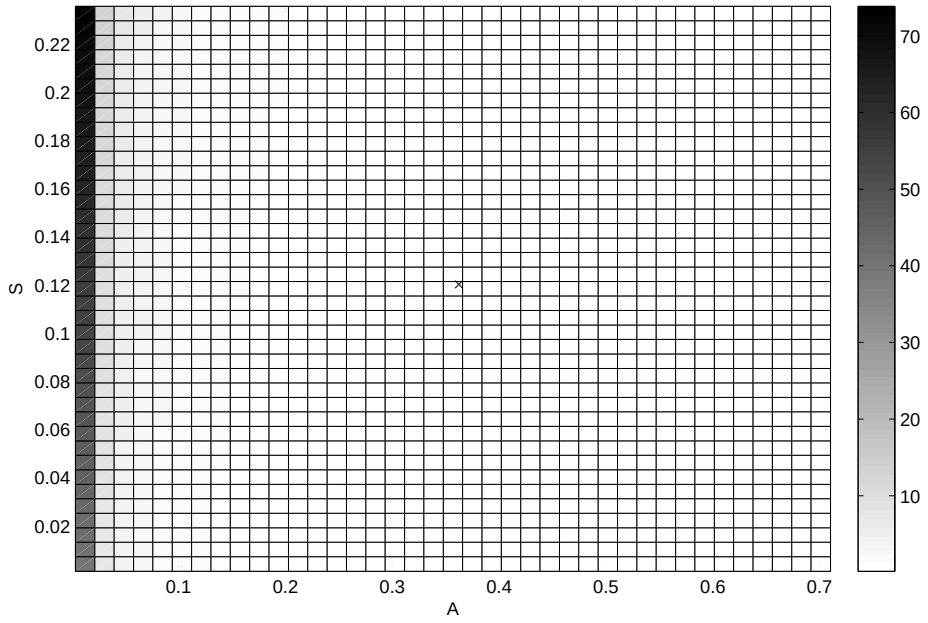


Figure 6.14: $\frac{J-\hat{J}}{\delta}$ for different initial values and $C(0) = 2A(0)$ in the (A, S)-plane with the Australian IDU parameters.

Due to the fact that the qualitative behaviour of the solution is the same for $C(0) = A(0)$ and $C(0) = A(0)/2$ we just include the previous figure. The benefit-cost ratio is higher for smaller initial values of C. The only exception is J_1 with the initial condition $C(0) = A(0)$. The benefit-cost ratio is a little smaller here compared to C being twice as high as A in the beginning of the epidemic. Again, we have the same pattern as in the U.S. parameter case. J_4 leads to the smallest ratio 8.4395 for $C(0) = 2A(0)$, J_2 to the second smallest.

We get 39.5511 for the smallest value of A and the highest value of S we contemplated. If we measure the social costs by the amount of drugs consumed we get 104.4562 as the highest result.

6.2.2 (A, C)-Plane

Finally, let us look at the consequences of a supply shock for the injection drug use in Australia by keeping S constant and varying C in the (A, C)-plane.

Figure 6.15 depicts the results for a high initial number of susceptible non-users, $S(0) = 2\hat{S}$.

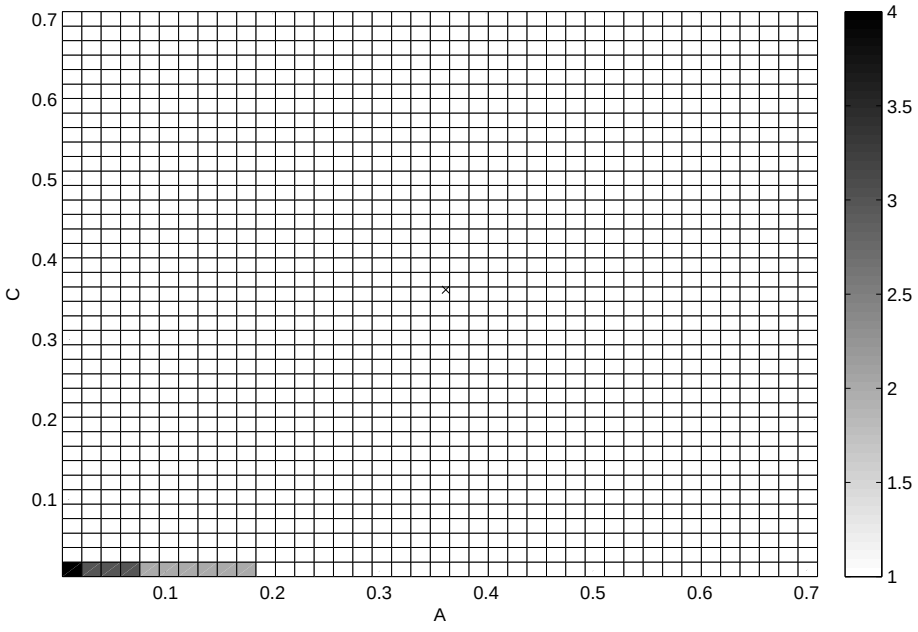
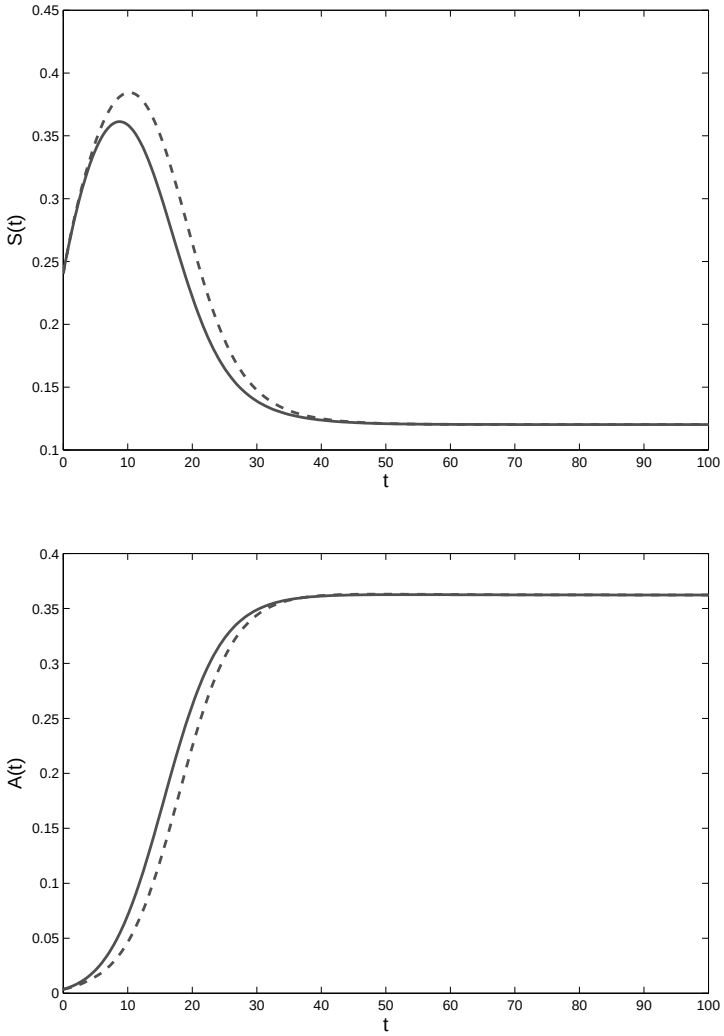


Figure 6.15: Years where a reduction of C by 50% has the most severe consequences for the Australian base parameter set and $S(0) = 2\hat{S}$ in the (A, C)-plane.

Figure 6.16 displays the time paths for the number of people who are susceptible to initiating into drug use, the active users, the current throughput capacity of the supply network of drugs, the price function, and the initiation function.

If we compare the state paths of $(S(0), A(0), C(0)) = (0.2404, 0.0036, 0.0217)$, where a supply shock in the first year is optimal, with those of $(S(0), A(0), C(0)) = (0.2404, 0.0036, 0.0036)$, where the greatest effect of a reduction of C by 50% is later in the 4th year, we see that the trajectories for $S(t)$ and $A(t)$ are very similar, although the applied control is completely different.

A reduction of C by 50% with the Australian parameter set is not as effective as for the cocaine epidemic in the United States of America. Here, we have a smaller steady state level and accordingly a smaller value of throughput capacity. Even the time paths for the drug stock $C(t)$ look quite similar. Interestingly, the price functions over time are significantly different with a supply reduction in the first versus the fourth year.



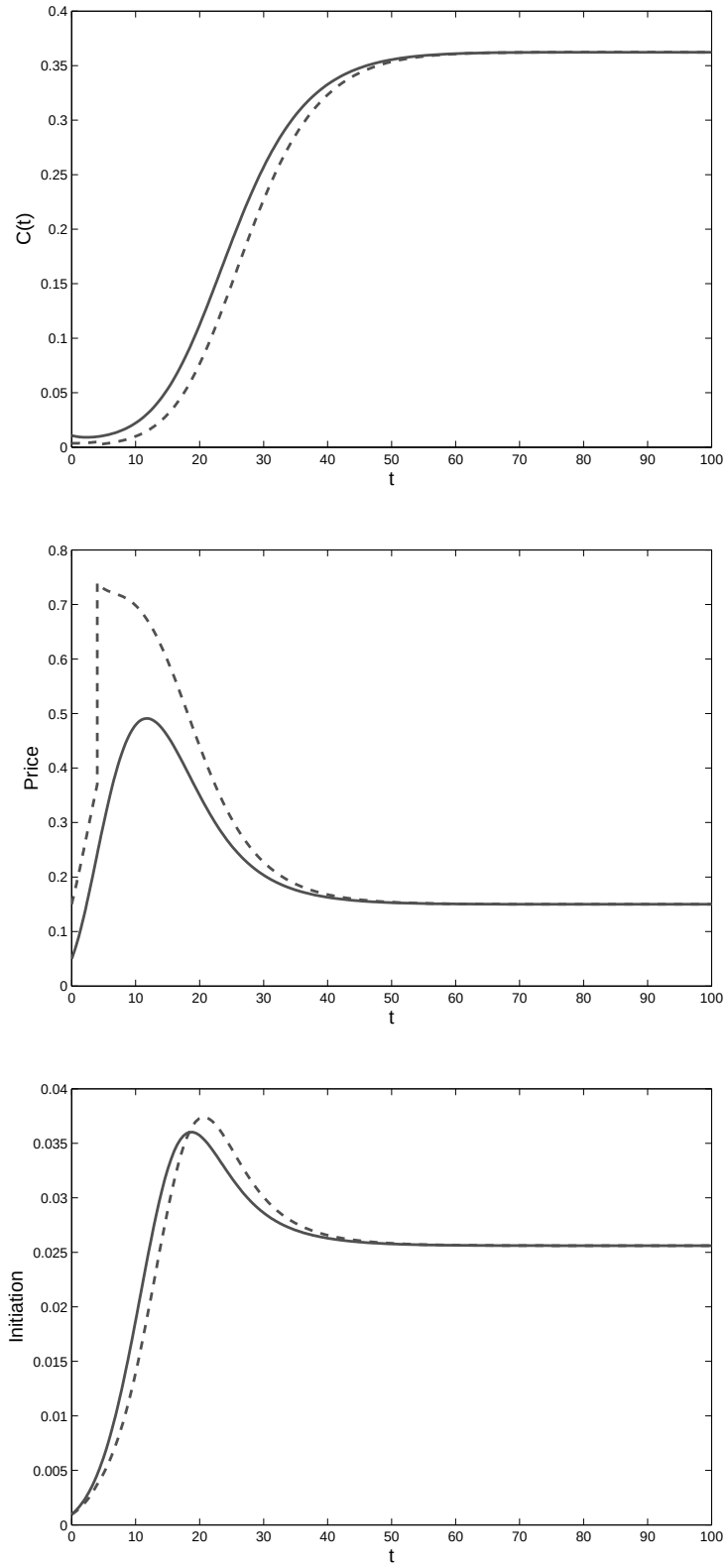


Figure 6.16: Comparison of the states S, A, and C, the price, and the initiation over time for $(S(0), A(0), C(0)) = (0.2404, 0.0036, 0.0217)$ with a supply reduction in the beginning (solid line) and for $(S(0), A(0), C(0)) = (0.2404, 0.0036, 0.0036)$ with a supply shock of 50% after 4 years (dashed line) for injection drug use in Australia.

It is easy to see that suddenly the price is nearly twice as high as before the shock, so a smaller drug stock C is leading to an increased price and therefore to a higher value of $S(t)$ combined with a lower number of current users A. Initiation is delayed; first it is suppressed, but later on it is even somewhat higher than in the case with the shock right at the beginning.

In contrast to the Australian case in the (A, S)-plane considering different initial values of $S(0)$ in the (A, C)-plane leads to a change in the recommended years for a supply reduction by 50% (Figures 6.17 - 6.20).

Figure 6.17 illustrates the outcome for the initial condition $S(0) = \hat{S}$, if we vary $A(0)$ from $0.01\hat{A}$ to $2\hat{A}$ and $C(0)$ from $0.01\hat{C}$ to $2\hat{C}$. Here we find more initial values where a later supply reduction is better than one in the beginning of the epidemic.

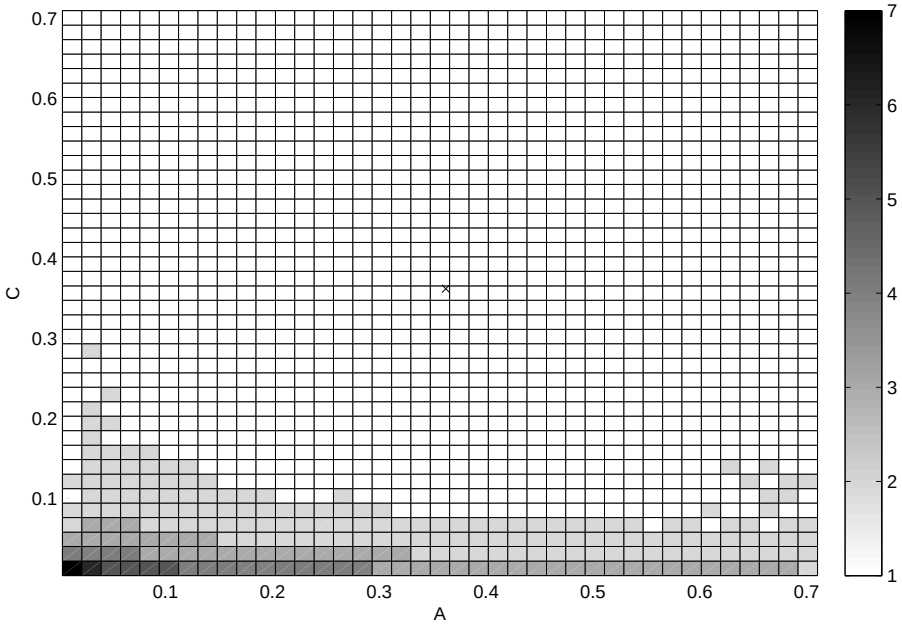


Figure 6.17: Years where the reduction of the current throughput capacity of the supply network of drugs by 50% has the strongest impact using the Australian base parameter set and $S(0) = \hat{S}$ in the (A, C)-plane.

Figure 6.18 illustrates that the best years for a reduction of the throughput capacity depends pretty much on the particular choice of the initial values.

For most of the examined initial conditions the first year for a supply reduction of C by 50% has the greatest effects on the social costs for a higher number of people who are susceptible to initiating into drug use (Figure 6.17). A supply shock later in the epidemic is more effective for smaller initial values of $S(0)$ (Figure 6.18).

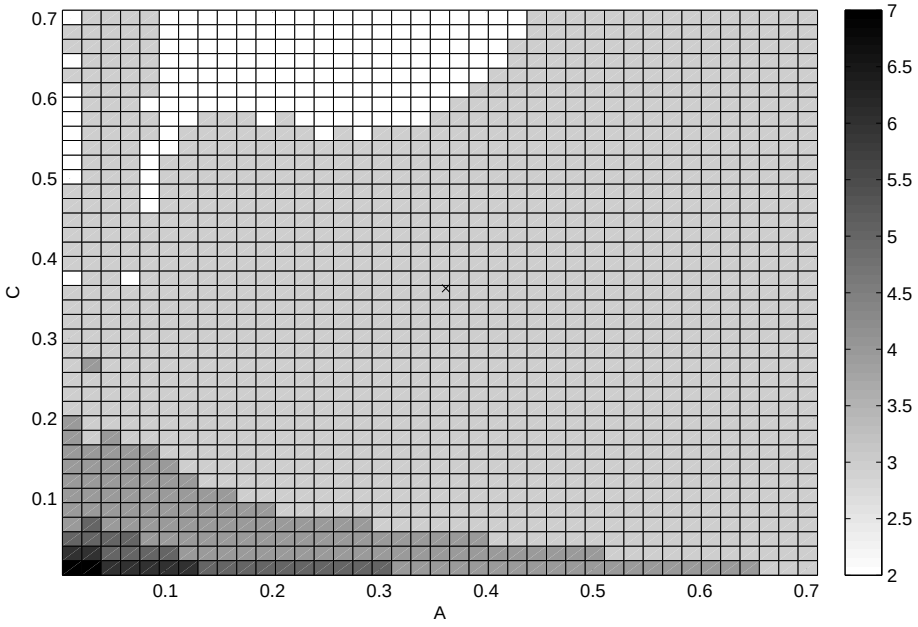


Figure 6.18: Years where a supply reduction of C by 50% has the strongest impact for the Australian injection drug use epidemic with $S(0) = \hat{S}/2$ in the (A, C)-plane.

Finally, we will have a look at the benefit-cost ratio (Figure 6.19 - 6.20). Once again, we display only one figure, (Figure 6.19) as an example for all initial scenarios, because the qualitative behaviour of the solutions are the same in the other corresponding exercises.

As mentioned before in the case of the U.S. cocaine epidemic, the benefit-cost ratio is smaller for smaller values of $S(0)$ if we vary $C(0)$ and $A(0)$ and keep the number of people who are susceptible to initiating into drug use, $S(0)$, fixed.

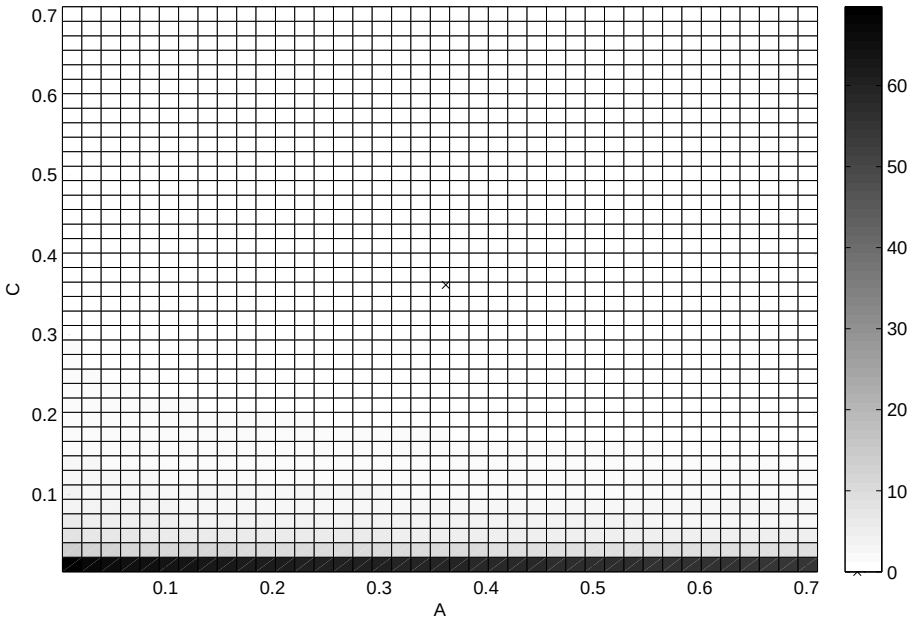


Figure 6.19: $\frac{J-\hat{J}}{\delta}$ for different initial values and $S(0) = 2\hat{S}$ in the (A, C)-plane with the Australian IDU parameters.

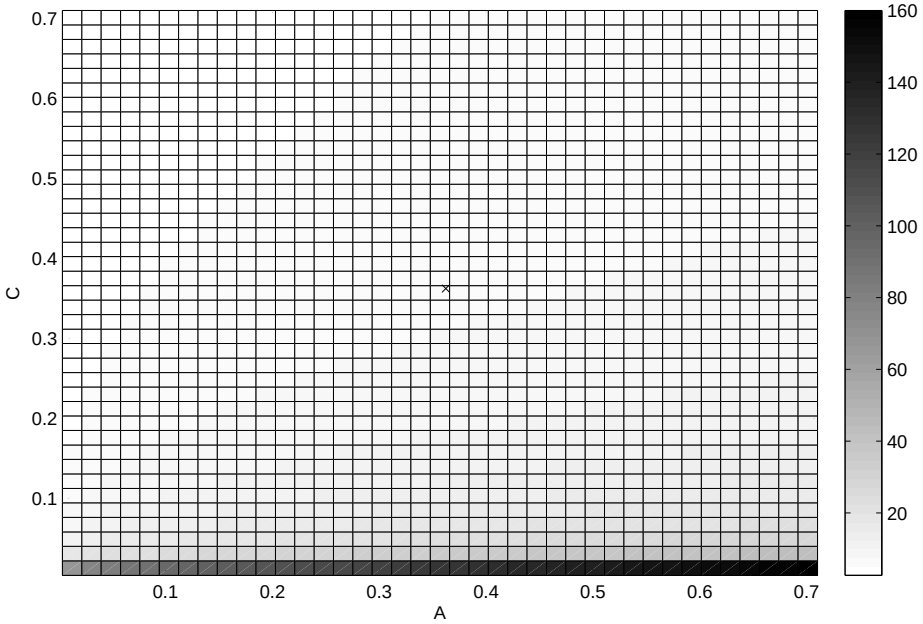


Figure 6.20: $\frac{J_2-\hat{J}_2}{\delta}$ for different initial values and $S(0) = 2\hat{S}$ in the (A, C)-plane for the Australian IDU parameterization.

As before, we notice a certain pattern in the benefit-cost ratios. J_4 provokes the smallest ratios, and J_3 the highest. We will look more deeply at this phenomenon in the following chapter.

The only exception here is J_2. For the first time we get a better benefit-cost ratio if we measure the costs by the number of drug dealers, which is illustrated in Figure 6.20.

Chapter 7

Social Costs

As mentioned in Chapter 6, working with different functional forms for the social costs resulting from a drug epidemic may generate pretty different results. In this thesis we consider four different integrands describing the social costs over an infinite planning horizon. In this chapter we will investigate whether or not they lead to consistent policy recommendations.

Recall that

$$J_1 = \int_0^\infty e^{-rt} A(t)\ dt$$

describes the discounted accumulation of users,

$$J_2 = \int_0^\infty e^{-rt} C(t)\ dt$$

represents the number of drug sellers,

$$J_3 = \int_0^\infty e^{-rt} p^\omega A(t)\ dt$$

stands for the quantity of drugs consumed, and

$$J_4 = \int_0^\infty e^{-rt} p^{1+\omega} A(t)\ dt$$

is the amount which is spent on drugs over an infinite planning horizon.

7.1 United States

For ease of comparison we first normalize the different social cost values. For that purpose, we weight all values to get the value represented by J_1. If we start with the high steady state of the U.S. base parameterization as initial value we get

	Social costs	$\frac{1}{\text{Weight}}$
J_1	165.7029	
J_2	165.7028	1
J_3	427.8429	2.5820
J_4	64.1764	0.3873

Starting with $(S(0), A(0), C(0)) = (21.2619, 0.3332, 0.3332)$, i.e. the saddle point steady state, we get nearly the same weights:

	Social costs	$\frac{1}{\text{Weight}}$
J_1	8.2236	
J_2	8.2445	1.0025
J_3	21.2592	2.5851
J_4	3.1813	0.3869

This suggests that the weighting is stable when we start in an equilibrium.

Furthermore, we look at a whole bench of different initial values and determine the corresponding weights. In particular, we vary $S(0)$ from 1 to 25 and let $A(0)$ vary from 1 to 10. We depict Figure 7.1 as one example to show how the weights alter for varied initial values. For J_2 the weights vary from 0.7594 up to 1.8947 if we start with the initial condition $(S(0), A(0), C(0) = 2A(0))$.

This is similar for J_3 and J_4 and for $C(0) = A(0)$ and $C(0) = A(0)/2$. We can conclude that the weights for J_2, J_3, and J_4 alter between minimum and maximum levels as depicted in the table below:

	Minimum	Maximum
J_2	0.6903	1.8947
J_3	2.0665	3.5489
J_4	0.2826	0.5312

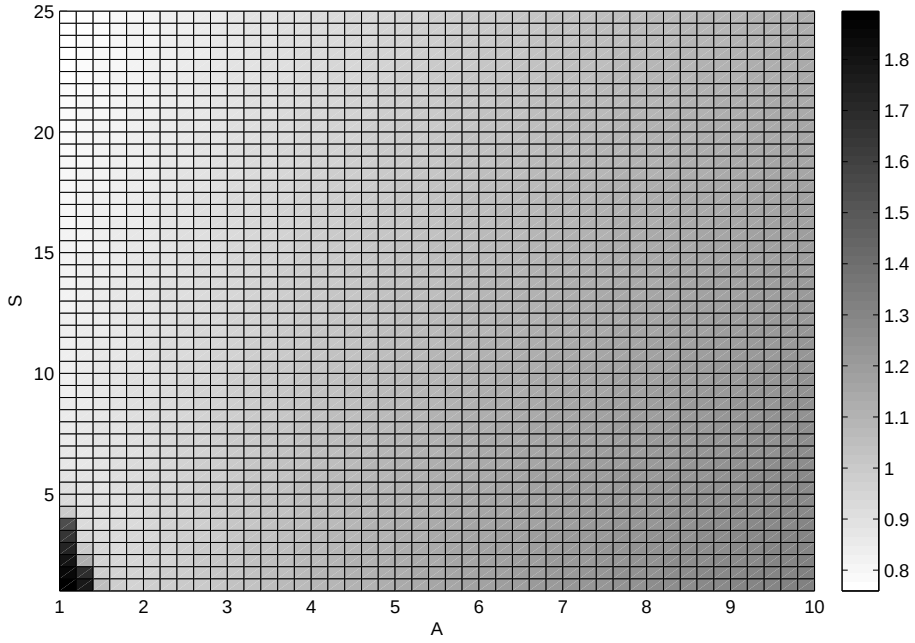


Figure 7.1: Weights for J_2 for different initial values with $C(0) = 2A(0)$ for the U.S. base parameter set.

Compared to the rather generous variation of initial values, the weights differ pretty moderately.
Table 7.1 lists the reference weights, computed as the arithmetic mean, for the U.S. cocaine epidemic and the initial conditions $C(0) = 2A(0)$, $C(0) = A(0)$, and $C(0) = A(0)/2$.

	J_2	J_3	J_4
$RW_{C=2A}$	1.0541	2.6287	0.3876
$RW_{C=A}$	0.9086	2.4376	0.4201
$RW_{C=A/2}$	0.8097	2.2847	0.4564

Table 7.1: Reference weights for the U.S. cocaine epidemic for $C(0) = 2A(0)$, $C(0) = A(0)$, and $C(0) = A(0)/2$.

In the next step, we will compare the results we got so far with the weighted sum of all four integrands used above to measure the social costs:

$$J_w = J_1 + RW_{J_2} J_2 + RW_{J_3} J_3 + RW_{J_4} J_4,$$

where J_w denotes the weighted sum of the integrals.

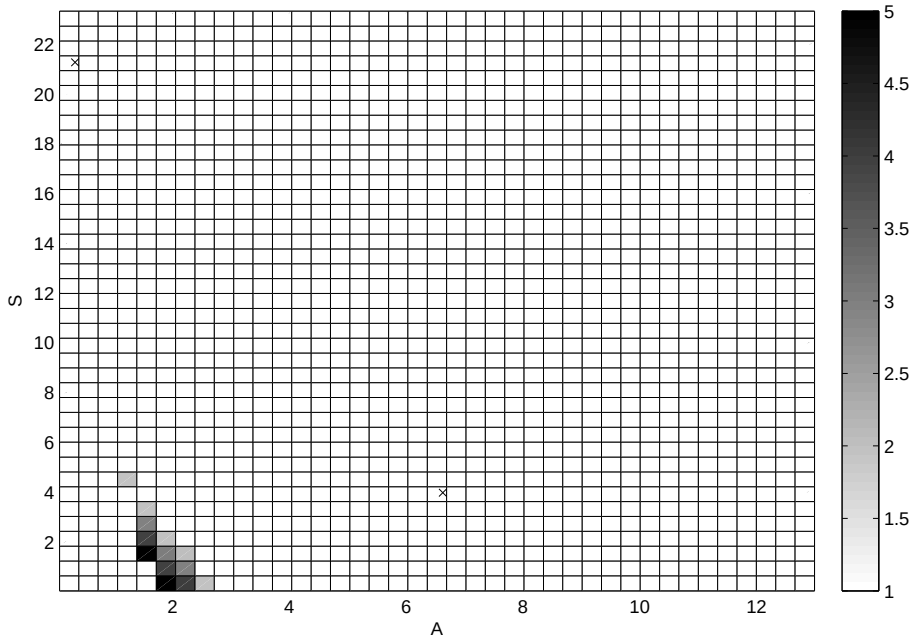


Figure 7.2: Most efficient years for a supply reduction for the U.S. cocaine epidemic with the initial condition $C(0) = 2A(0)$ based on J_w.

Figure 7.2 depicts the years where a supply shock has the strongest effect on the social costs which are measured by the weighted sum of J_1, J_2, J_3, and J_4. We start with the initial condition where the amount of drugs is twice as high as the number of users. As stated before, the recommended policy in which year to induce a supply shock strongly depends on the way social costs are measured.

In Chapter 6 we noticed a pattern how the different ways to calculate the benefit-cost ratio influences the results. Now, we want to investigate this ratio measured with a combination of the four distinct integrals.

Figure 7.3 shows the weighted benefit-cost ratios for several initial values where $C(0)$ is twice as high as the number of users at the beginning of the epidemic. The quotients are not four times as high as the ones measured with $\frac{J_1 - \hat{J}_1}{\delta}$ although we adapted the integrals to J_1. This most likely is due to the fact that we use one reference weight over a grand field of initial values.

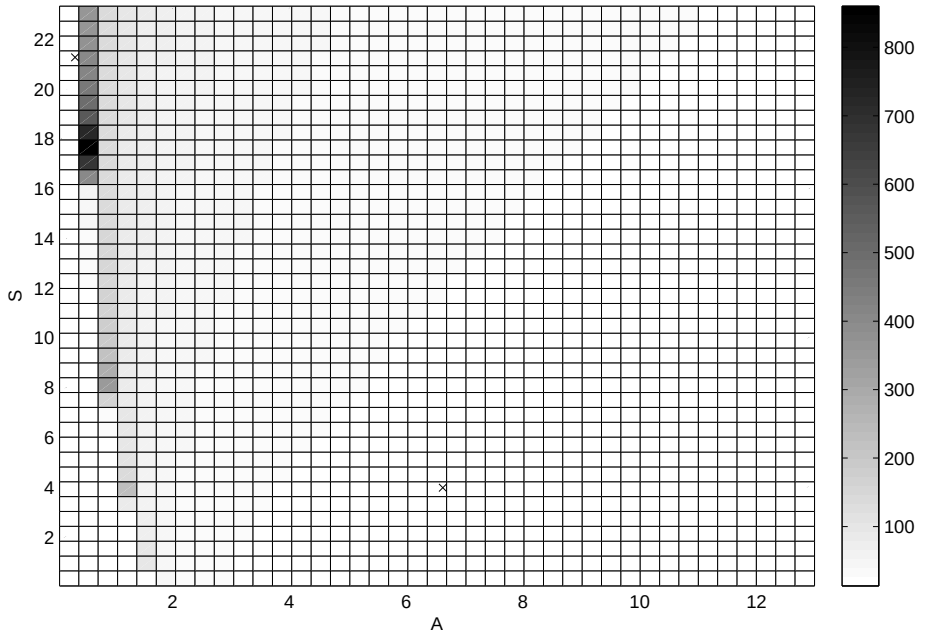


Figure 7.3: $\frac{J_w - \hat{J_w}}{\delta}$ for different initial values and $C(0) = 2A(0)$ for the U.S. parameter set.

In addition, we will now look at scenarios where we attach more or less importance to one method to calculate social costs and benefit-cost ratios. The core question of the analysis is whether leaping one component causes the strategy to change, or not.

First, we completely omit the supply state C to measure the costs. In Figure 7.4 the costs are calculated with a weighted sum of discounted accumulation of users, quantity of drugs consumed, and the amount spent on drugs. The qualitative behaviour of the solution, where the benefit-cost ratio is the highest, stays the same, but the amount changes, albeit not too much.

Second, we double the weight for J_2 and half the one for J_1. We lay more interest on the costs caused by drug dealers and less on the costs breed by users. In Figure 7.5 the benefit-cost ratios for $1/2J_1 + 2RW_{J_2}J_2 + RW_{J_3}J_3 + RW_{J_4}J_4$ are illustrated. We see that the calculation of the weighted costs is consistent, i.e. we get almost the same ratio with another weighted sum.

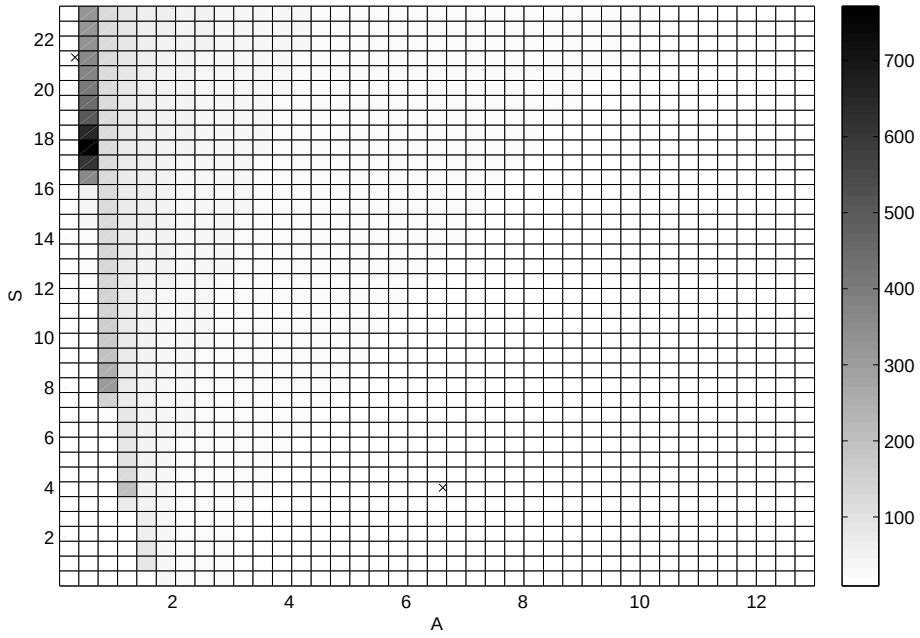


Figure 7.4: Benefit-cost ratio for the weighted sum of J_1, J_3, and J_4 for different initial values and $C(0) = 2A(0)$ for the United States.

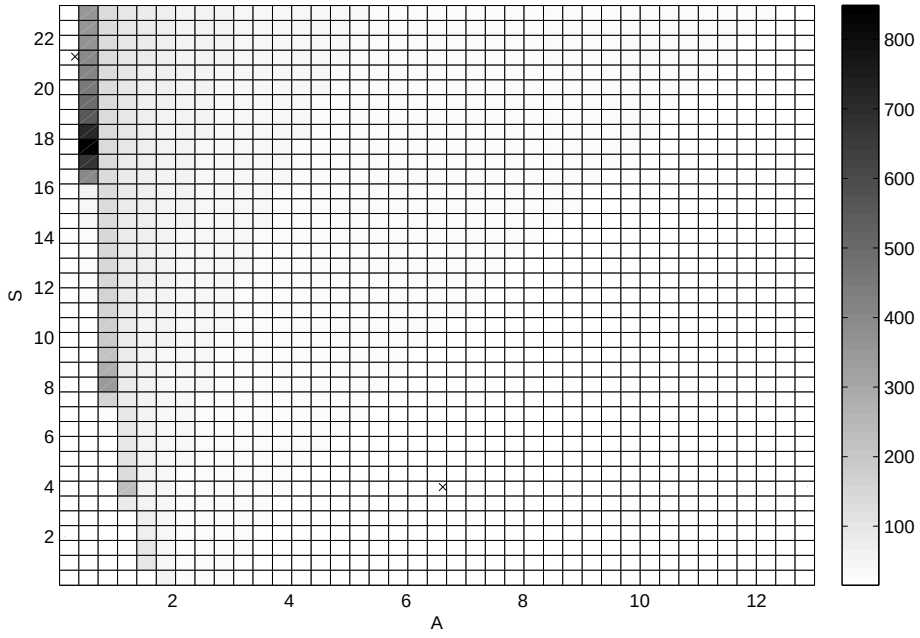


Figure 7.5: Benefit-cost ratio for the weighted sum of J_1, J_2, J_3, and J_4, if we lay more focus on J_2 and less on J_1 for different initial values and $C(0) = 2A(0)$ for the U.S. cocaine epidemic.

Accordingly, we have to be careful naming the amount of the social costs as they strongly depend on the way they are measured. However, looking at the weighted sum of the four different integrals produces similar results.

7.2 Australia

The original assumptions for the model parameters imply that we do get the same weighting as we did for the cocaine epidemic in the United States of America when we start with the stable steady state $(S(0), A(0), C(0)) = (0.1202, 0.3623, 0.3623)$ for the Australian base parameterization.

	Social costs	$\frac{1}{\text{Weight}}$
J_1	9.0567	
J_2	9.0569	1
J_3	23.3845	2.5820
J_4	3.5076	0.3873

Again, we calculate the weights for many different initial values, where $S(0)$ varies from 0.0001 to 0.3, $A(0)$ from 0.001 to 0.54, and $C(0) = 2A(0)$, $C(0) = A(0)$, or $C(0) = A(0)/2$.

Figure 7.6 pictures the different weights for J_3 when we start with the initial condition that $C(0)$ is a half of $A(0)$. As seen previously in the U.S. parameter case, we notice that the weighting is fairly stable for different initial values and functional forms we use to measure the social costs resulting from a drug epidemic.

The following table shows us that the variation of the weights for the different functional forms for the social costs resulting from a drug epidemic is closely bounded on the considered part of the state space.

	Minimum	Maximum
J_2	0.6934	1.303
J_3	2.0906	2.9251
J_4	0.3467	0.5153

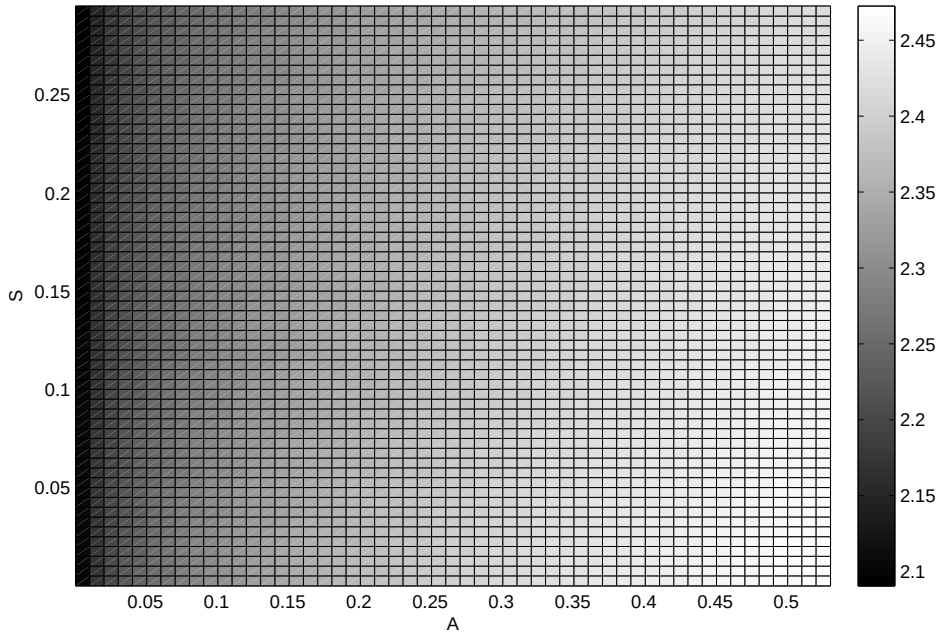


Figure 7.6: Weights for J_3 for different initial values with $C(0) = A(0)/2$ for the Australian base parameterization.

One more time, we calculate the reference weights to rerun different analyses with a weighted sum of all four social cost measurements and compare the obtained results. The average weights are listed in Table 7.2.

	J_2	J_3	J_4
$RW_{C=2A}$	1.0752	2.6571	0.3823
$RW_{C=A}$	0.9359	2.4898	0.4050
$RW_{C=A/2}$	0.8413	2.3534	0.4313

Table 7.2: Reference weights for the Australian injection drug use for $C(0) = 2A(0)$, $C(0) = A(0)$ and $C(0) = A(0)/2$.

We start the analysis with computing the most efficient years for a supply reduction by 50% when considering the weighted sum of the functional forms for the social costs resulting from a drug epidemic as we did for the cocaine base parameterization for the United States.

In turn, we come to the conclusion that we have to be prudent with the recommended control because the most efficient years for a reduction of C by 50% strongly depend on the functional form we use to measure the social costs resulting from a drug epidemic.

The results for the initial condition $C(0) = 2A(0)$ are depicted in Figure 7.7. A very big difference appears if we calculate the years where a supply shock has the strongest effect on the costs with other initial conditions or another weighting.

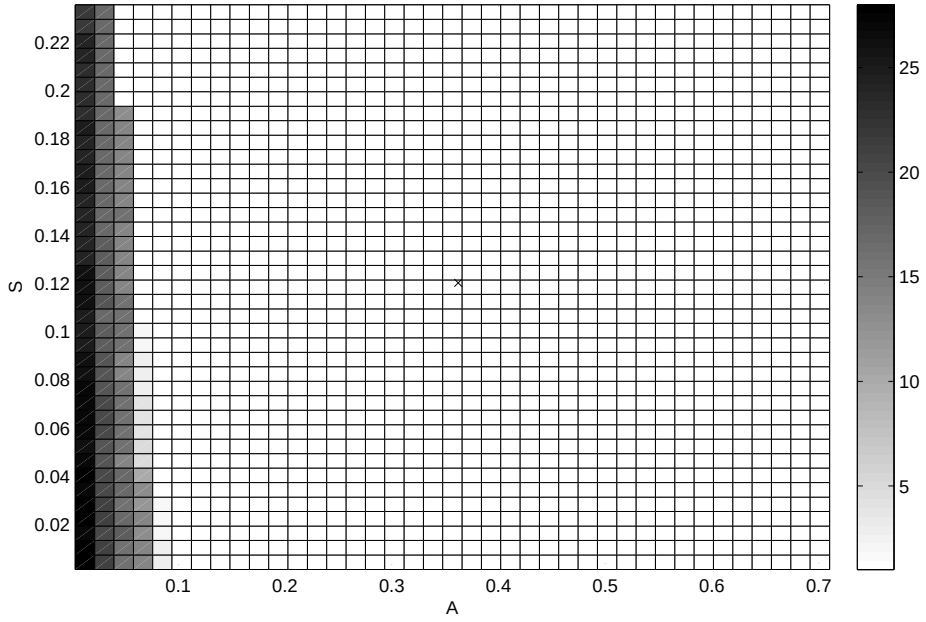


Figure 7.7: Most efficient years for a supply reduction of the state C by 50% calculated with J_w for the Australian injection drug use epidemic and the initial condition $C(0) = 2A(0)$.

Figure 7.8 illustrates the benefit-cost ratios for the weighted sum of all four different ways to measure the social costs we examined. We get completely different results compared to the benefit-cost ratio we got with the discounted accumulation of users over an infinite planning horizon, illustrated in Figure 6.4.

Next, we want to investigate how sensitive the results are with respect to the weights used. We first omit the third state C to calculate the social costs.

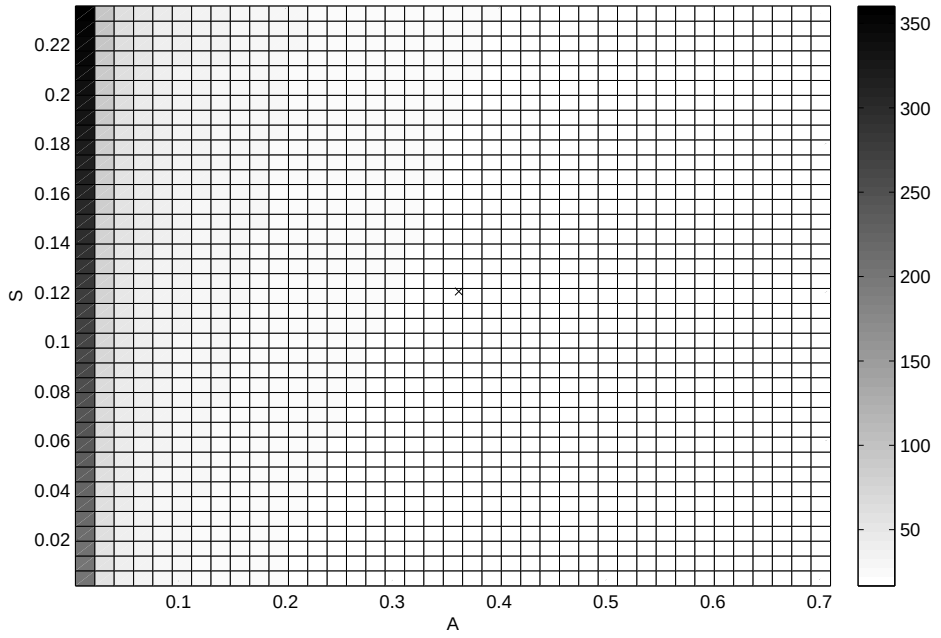


Figure 7.8: $\frac{J_w - \hat{J}_w}{\delta}$ for different initial values and $C(0) = 2A(0)$ for the Australian base parameterization.

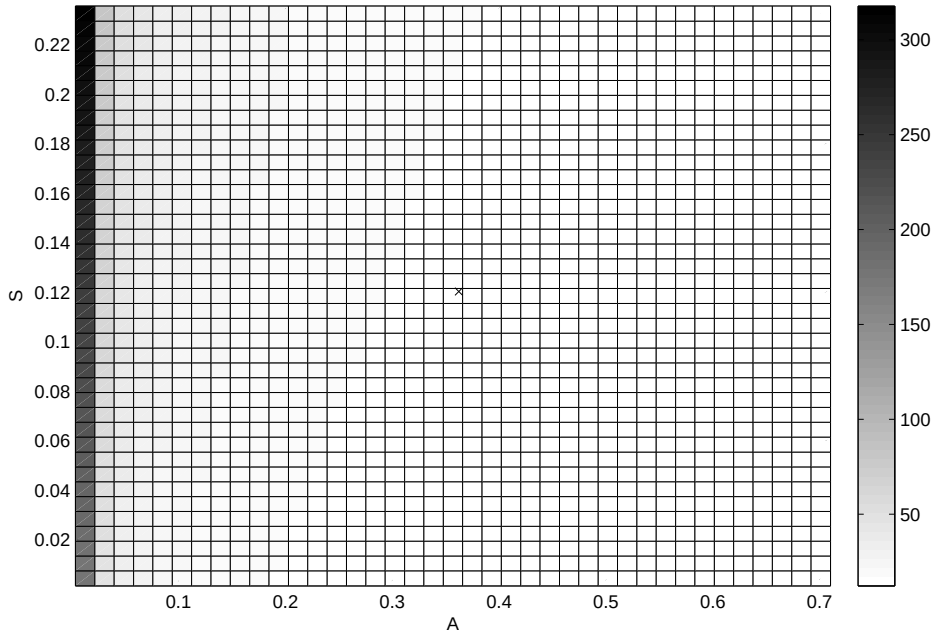


Figure 7.9: Benefit-cost ratio for the weighted sum of J_1, J_3, and J_4 for different initial values and $C(0) = 2A(0)$ for Australia.

From Figure 7.9 we conclude that we do not get a benefit-cost ratio three-fourths as high as for $\frac{J_w - \hat{J_w}}{\delta}$. As in the U.S. case, the reference weight is not the best one on the hole considered space.

Furthermore, we finally lay more focus on the costs caused by dealers and less on the costs caused by active users. Figure 7.10 shows that the benefit-cost ratio is consistent in the sense that we get similar benefit-cost ratios and the identical qualitative behaviour of the solutions as for $J_1 + RW_{J_2} J_2 + RW_{J_3} J_3 + RW_{J_4} J_4$.

Finally, it must be stated that the costs alter with the functional forms we use to measure them. However, the qualitative behaviour of the solution does not change too much, i.e., the initial values where the costs reach their highest level or the best benefit-cost ratio can be found are more or less the same, but not to the same amount.

In summary, the qualitative behaviour of the policy recommendation resembles, but the extent of the costs varies.

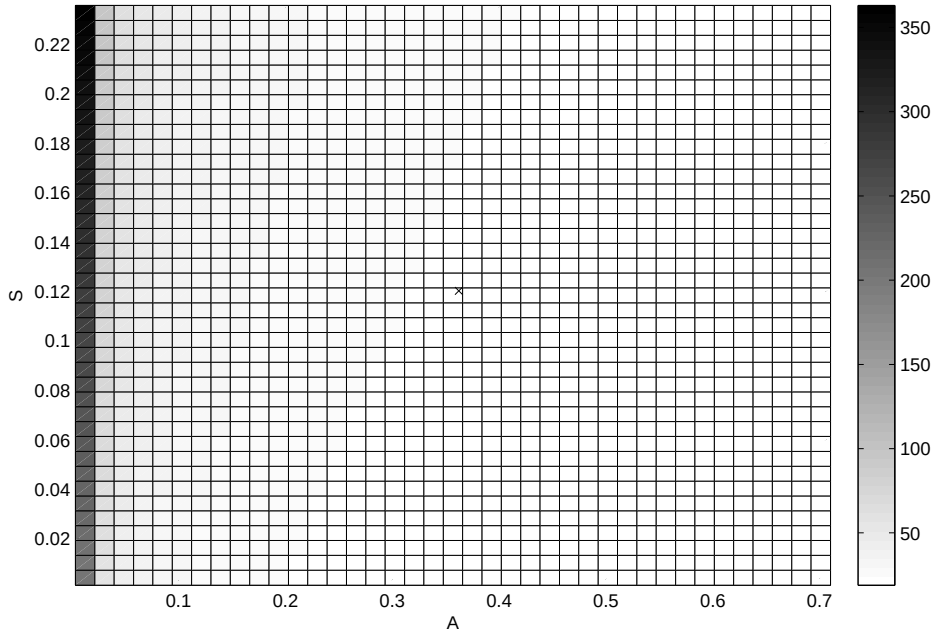


Figure 7.10: Benefit-cost ratio for the weighted sum of J_1, J_2, J_3, and J_4, if we lay more focus on J_2 and less on J_1 for different initial values and $C(0) = 2A(0)$ for Australia.

Chapter 8

Elasticity of Initiation

The basic motivation for this chapter is that we want to investigate how the dynamic SAC model behaves when the overall elasticity level is increased to -1. [Caulkins et al., 2000] find that the solution of a dynamical model is strongly dependent on the elasticity levels. In other dynamic models of drug use, the price occurs on three locations: the objective function, the initiation function, and the exit term from active use. A prominent example is provided by [Tragler et al., 2001]. One half of the absolute value of the overall elasticity can be found in the objective function, the elasticity of demand, and the other half at the initiation and the exit function. In our model, the exit rate is not price-dependent. In this chapter we either enhance the elasticity of initiation or we change our exit of active use to be price-dependent.

8.1 U.S. Parameterization

First, we change the parameter a, the elasticity of initiation, from $a = -0.25$ to $a = -0.5$ to make the initiation into cocaine use more dependent on price. The modification does not change the artificial stable node, $\hat{E} = (\hat{S}, \hat{A}, \hat{C}) = (22.1796, 0, 0)$, because the parameter a does not impact this equilibrium. However, we get two different new steady states

$$\begin{aligned} \hat{E}_1 &= (\hat{S}_1, \hat{A}_1, \hat{C}_1) = (21.8015, 0.1367, 0.1367), \\ \hat{E}_2 &= (\hat{S}_2, \hat{A}_2, \hat{C}_2) = (2.3612, 7.2176, 7.2176). \end{aligned}$$

It is obvious that the initiation and the number of users A in total are shrinking with an increasing absolute value of a, which is illustrated in Figure 8.1.

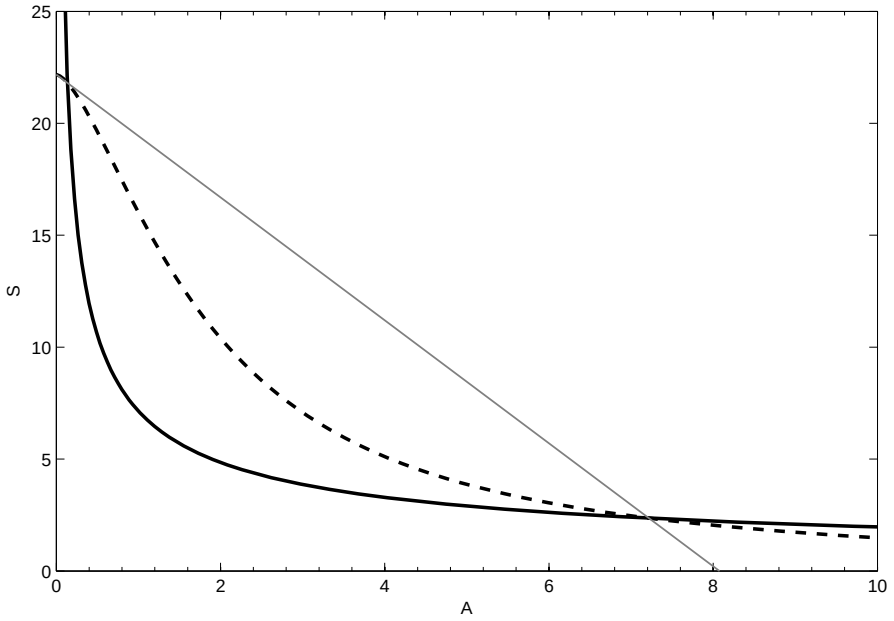


Figure 8.1: Isoclines and the linear relation between the steady state values for the cocaine epidemic in the United States if the elasticity of initiation is changed to $a = -0.5$.

The eigenvalues for $\hat{E}_1$ are given by

$$\begin{aligned}\lambda_1 &= 0.1729,\\ \lambda_2 &= -0.0594,\\ \lambda_3 &= -0.2112,\end{aligned}$$

implying that this equilibrium is a saddle point. $\hat{E}_2$ turns out to be a stable focus with the eigenvalues

$$\begin{aligned}\lambda_1 &= -0.2703,\\ \lambda_{2,3} &= -0.1339 \pm 0.1210i.\end{aligned}$$

Alternatively, we may add a new parameter $b = 0.25$ and consequently get a new system dynamic

$$\dot{S} = k - \delta S - f(A)Sp(A,C)^a,$$
$$\dot{A} = f(A)Sp(A,C)^a - \tilde{\mu}Ap^b,$$
$$\dot{C} = g(p)C.$$

At the equilibrium where $\dot{A} = \dot{C}$ has to be fulfilled we have $p = c$. In order that the two system dynamics have the same slope we set $\tilde{\mu} = \frac{\mu}{c}$. For this modified model there is just one equilibrium, the artificial steady state $\hat{E} = (\hat{S}, \hat{A}, \hat{C}) = (22.1796, 0, 0)$, the stable node which is visualized in Figure 8.2.

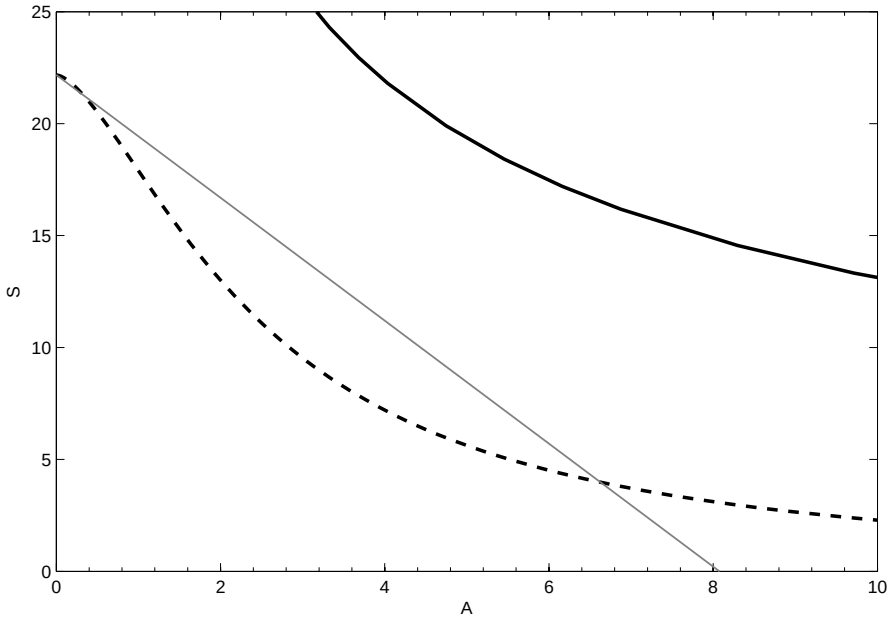


Figure 8.2: Isoclines and the linear relation between the steady state values when the outflow of $\dot{A}$ is changed in the cocaine epidemic in the United States.

By looking at the time paths of the three different system dynamics we are able to compare the behaviour of the different trajectories. Deliberately we choose $C(0) = A(0)$ to better explore the different evolutions of the states A and C. Firstly, we start with the initial values $(S(0), A(0), C(0)) = (3, 3, 3)$, illustraded in Figure 8.3. Secondly, we look at Figure 8.4 with $(S(0), A(0), C(0)) = (1, 1, 1)$. Finally, we will investigate the time paths for rather small initial values, namely $(S(0), A(0), C(0)) = (0.5, 0.5, 0.5)$ in Figure 8.5.

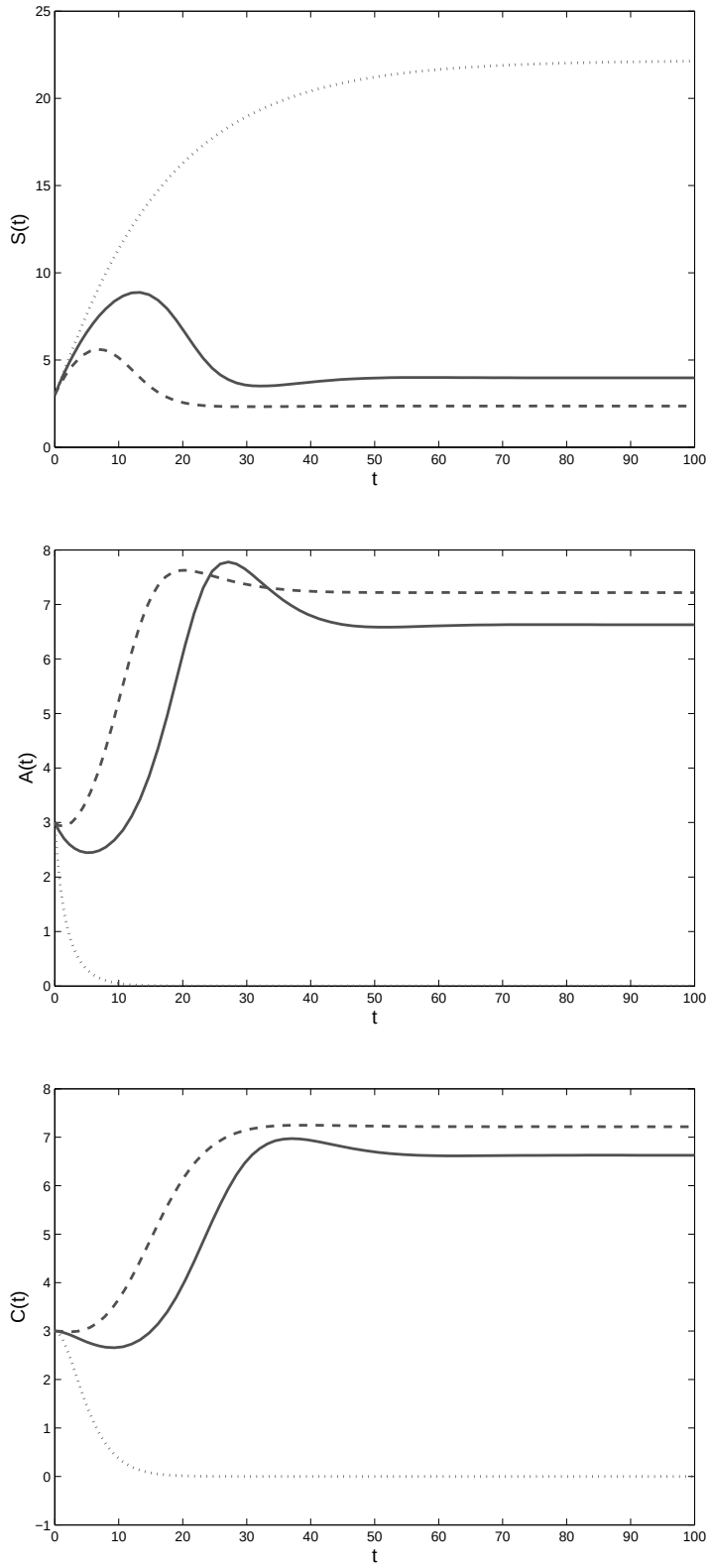
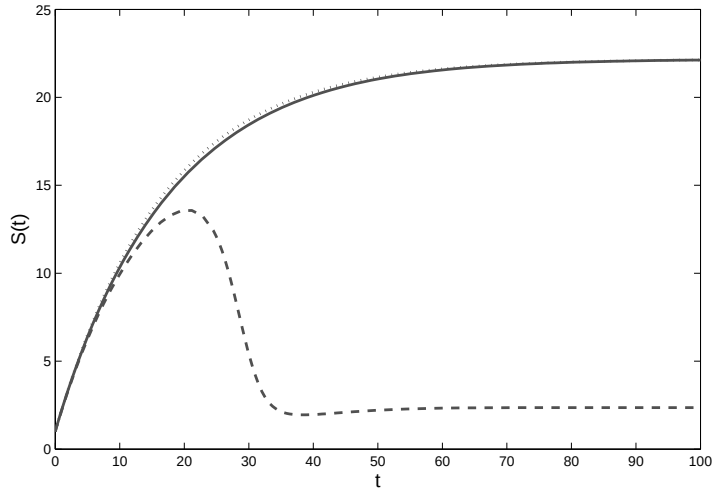


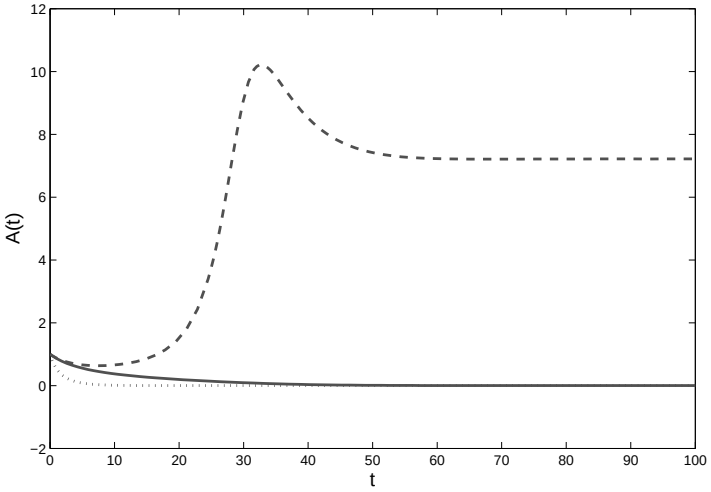
Figure 8.3: Time paths for the initial values $(S(0), A(0), C(0)) = (3, 3, 3)$ with the basic parameterization (solid line), $a = -0.5$ (dashed line), and $b = 0.25$ (dotted line) for the United States.

In those three figures we notice that the trajectories of the model with the new variable b converge to the artificial high equilibrium. In Figure 8.3 the trajectories of the two other system dynamics converge to the high steady state.

Figure 8.4 illustrates that although we start with the same initial values, the basic model and the model with the increased elasticity converge towards different steady states.

In Figure 8.5 all trajectories converge towards the small steady state, the artificial equilibrium. Interestingly, S looks almost the same in all three scenarios, while the evolutions of A and C are significantly different.





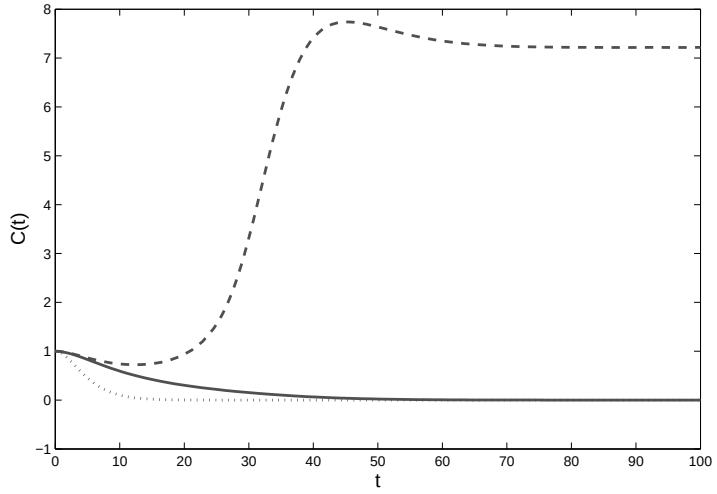
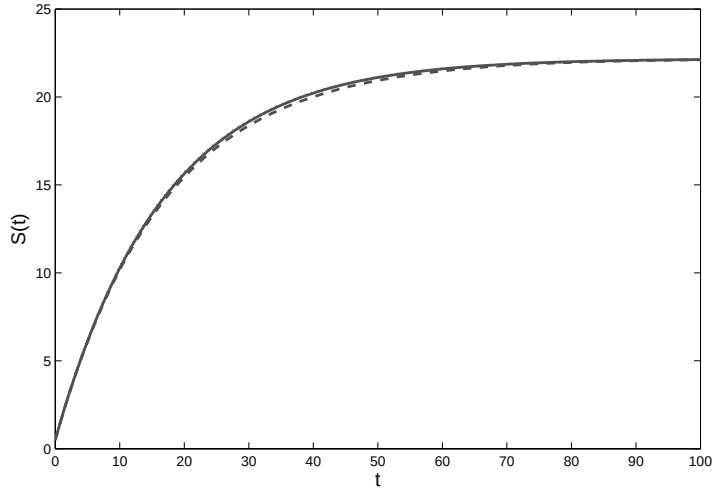
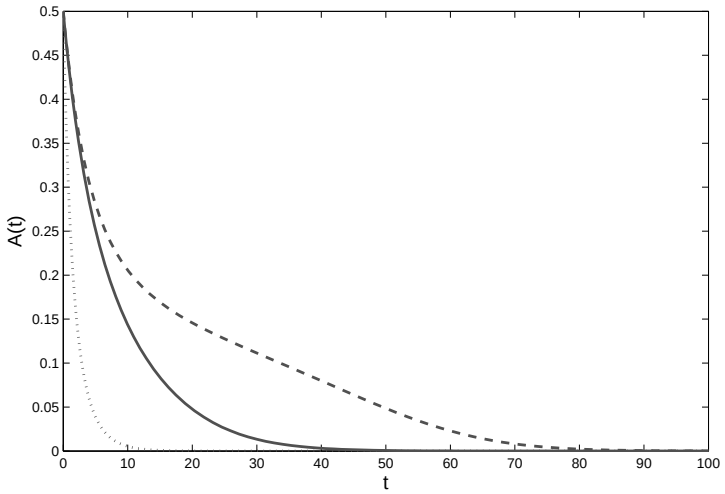


Figure 8.4: Time paths for the initial value $(S(0), A(0), C(0)) = (1, 1, 1)$ with the basic parameter set (solid line), $a = -0.5$ (dashed line), and $b = 0.25$ (dotted line) for the United States.





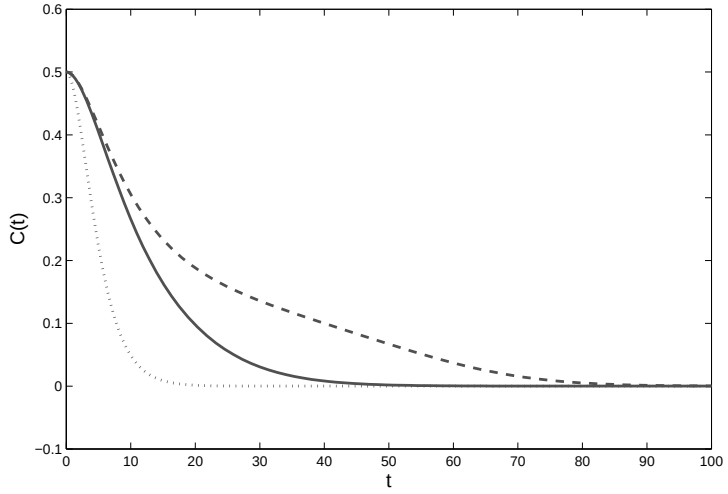


Figure 8.5: Time paths for the initial value $(S(0), A(0), C(0)) = (0.5, 0.5, 0.5)$ with the basic parameterization (solid line), $a = -0.5$ (dashed line), and $b = 0.25$ (dotted line) for the United States.

8.2 Australian Parameterization

Finally, we want to investigate the Australian IDU epidemic with a higher value for the elasticity of initiation. For the Australian base parameter set we got one steady state, which is not changing with an altered elasticity.

When we increase $a = -0.25$ to $a = -0.5$ we get a stable focus

$$\hat{E} \quad = \quad (\hat{S}, \hat{A}, \hat{C}) \quad = \quad (0.0758, 0.3995, 0.3995)$$

with the eigenvalues

$$\lambda_1 = -0.6076,$$
$$\lambda_{2,3} = -0.1175 \pm 0.0242i.$$

Due to the higher absolute value of the elasticity of initiation, the influence of the price function on initiation is increased in the considered scenarios. This implies that the number of current users, A, grows and the number of susceptible non-user, S, shrinks. Figure 8.6 depicts the new equilibrium, a stable focus.

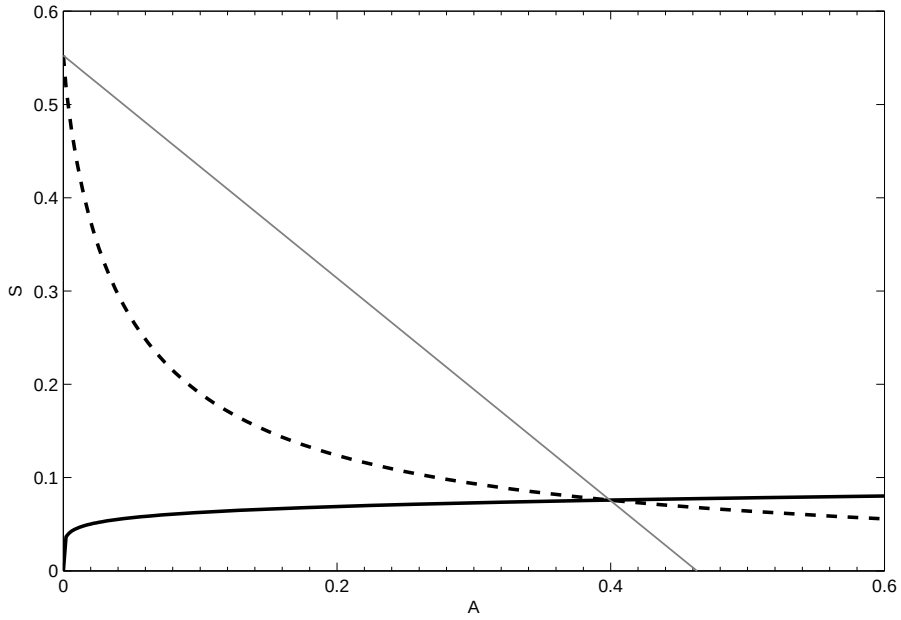


Figure 8.6: Isoclines and the linear relation between the steady state values if a is changed to -0.5 for the Australian IDU epidemic.

Note also that we do not lose steady states by adding the new term $\frac{\tilde{\mu}}{c}Ap^b$ to the Australian case. However, the stability behaviour changes, because

$$\hat{E} \quad = \quad (\hat{S}, \hat{A}, \hat{C}) \quad = \quad (0.3654, 0.0378, 0.0378)$$

is a saddle point with the eigenvalues

$$\begin{aligned} \lambda_1 &= 0.0334, \\ \lambda_2 &= -0.3688, \\ \lambda_3 &= -0.2708. \end{aligned}$$

Making the exit from active use price-dependent leads to an increased outflow of state A. Thus, the number of people who are susceptible to drug use rises. Figure 8.7 illustrates the linear relation between the steady state values and the isoclines.

To compare the different system dynamics we are looking at the time paths with the initial values $(S(0), A(0), C(0)) = (0.5, 0.5, 0.5)$, depicted in Figure 8.8.

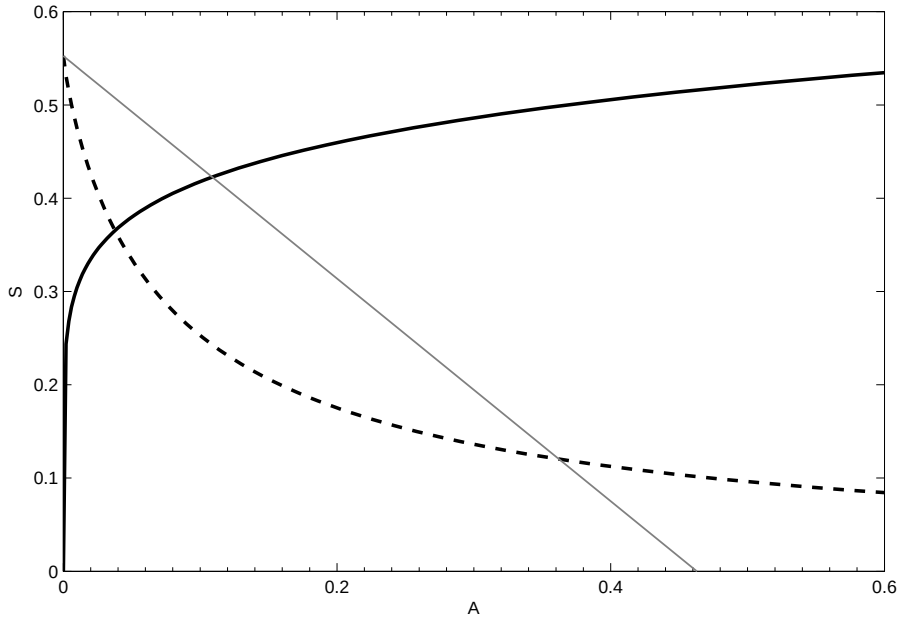
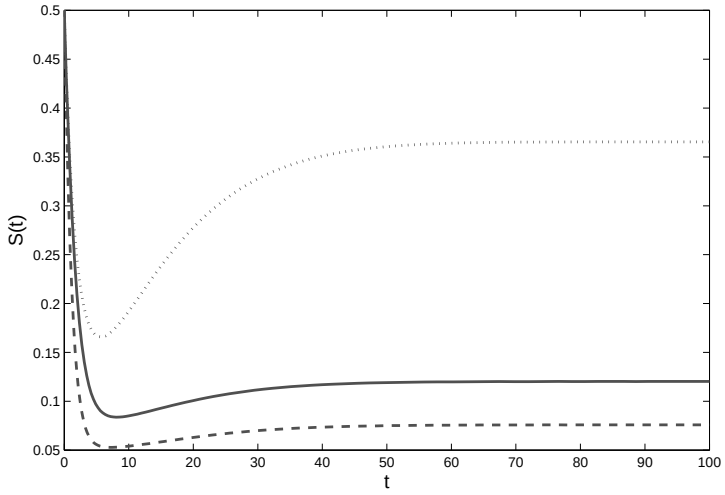


Figure 8.7: Isoclines and the linear relation between the steady state values when the outflow of $\dot{A}$ is changed in the Australian IDU model.



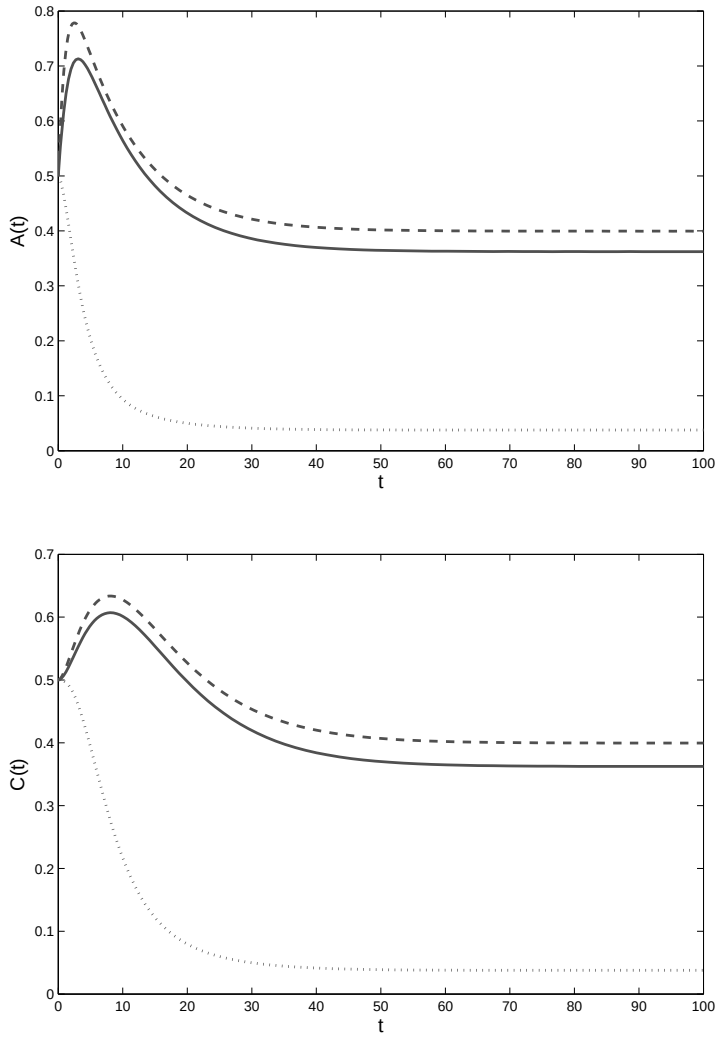


Figure 8.8: Time paths for the initial value $(S(0), A(0), C(0)) = (0.5, 0.5, 0.5)$ with the base case model (solid line), $a = -0.5$ (dashed line), and $b = 0.25$ (dotted line) for the injection drug use in Australia.

Note that the model with the increased absolute value of the elasticity of initiation leads to a higher number of active users, which is surprising at first sight. However, in our parameterization the price basically always is a value between 0 and 1 so that $p^{-0.5}$ is larger than $p^{-0.25}$. To obtain the expected results, the model would have to be rescaled.

Adding the new parameter b causes a significant change in $\hat{S}$. It is three times as high as the number of susceptible non-users with the base case parameterization in the steady state.

Chapter 9

Conclusions and Possible Extensions

This thesis was devoted to the dynamic three-state SAC model of illicit drug consumption derived from the well-known SA model discussed in [Caulkins et al., 2009a], [Caulkins et al., 2009b], and [Wallner, 2008]. The main differences are that a third state was added, i.e. the current throughput capacity of the supply network of drugs, and that the initiation function became price dependent. The analyses were performed for two different drug epidemics in two different countries, i.e. the cocaine use in the United States of America and the Australian injection drug use.

The intention of the first part was to analyze the uncontrolled model. Steady states and their stability properties were investigated. Then sensitivity and bifurcation analyses were carried out in order to deal with the inaccuracy of the model parameterizations. We detected a strong sensitivity of the solution with respect to the inflow into state S, k, the exit from state S, β, and the exit from active use, μ. In addition, we took a closer look at the interaction of two different parameter values and their influence on the number of equilibria.

Having completed this, we focused on the control instrument "enforcement". We presented the results in the (A, S)- and (A, C)-planes. It appears that the most efficient year for a supply reduction by 50% not only depends on the initial values and the relationship between A and C but also on the functional form we use to measure the social costs resulting from a drug epidemic. There is also a difference in the amount of the benefit-cost ratio if we consider the

aforementioned influences but not so much in the qualitative behaviour of the solutions. In summary, we could conclude that the benefit-cost ratio is in general higher for small values of active users.

The recommendation which year is the most effective one for a supply reduction has to be well considered preconceiving the dependence on the functional form for the examined costs resulting from a drug epidemic.

Furthermore, we investigated a weighted sum of up to four different functional forms to measure social costs. Again, we detected that the most efficient year for a supply shock strongly depends on the way the social costs are measured and weighted. It is hence a tough decision for policy makers to figure out what to focus on because the recommended policy and the measured costs strongly depend on the assumed functional forms. With different initial values, different objective functionals are more suitable, which emphasizes the dynamical character of the model.

Another important part of this thesis was to increase the overall elasticity level to -1 and look at the significant changes of the steady states. The motivation for this analysis was the discovery in [Caulkins et al., 2000] that the results are strongly dependent on the elasticity level.

Altogether, the U.S. base parameterization turned out to be the more interesting case. This perhaps can be explained by $\beta > 1$, the exponent in the initiation function, which leads to a convex initiation function and therefore to multiple steady states.

Finally, we want to conclude the thesis by pointing out some possible extensions that may be taken into consideration in future projects.

- In this thesis only enforcement as a control policy was considered. In future work, an interesting extension could be to investigate how the relative cost-effectiveness of treatment vs. prevention vs. enforcement varies during an epidemic. This could be done twice: first, let the removal of, say, users be in percentage terms and second, let it be a fixed number of people removed.
- Probably the most important extension that should be taken into consideration is to model treatment. It is commonly observed that incar-

cerating drug dealers may not be effective because they can be easily replaced. This replacement effect could be estimated by comparing the actual change in the present value of future drug use years with the no-replacement estimate.

- We chose the feedback effect of the initiation function to be logarithmic. Other functional forms might be considered.

- An extension of the cost effectiveness-analysis of Chapter 6 seems worthwhile. Instead of illustrating the results by the benefit-cost ratio, the outcome could be shown in terms of impact per person removed by the control policy. In addition, the number of active users, A, could be decreased by an absolute value λ, contrary to the indirect effect of the reduction of C, the current throughput capacity of the supply network of drugs, on A.

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Acknowledgements

This thesis is the concluding part of my studies at the Vienna University of Technology. Without the support and advise of several people it could not have been accomplished.

First off, I would like to thank my advisor Gernot Tragler for his guidance, for sharing his wisdom and having patience with me from the start to the final level.

Part of the work for this thesis was supported financially by the Austrian Science Fund (FWF).

Sincerely, I thank my parents and family for believing in me and making this possible in the first place.

Finally, I want to thank my friends and fellow students who supported me while I was working on this thesis and for making my years of study so special.