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DISSERTATION

A van Kampen theorem for profinite 2-complexes

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Diese Dissertation haben begutachtet:

1. Begutachter

2. Begutachter

Contents

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Summary/Abstract

A theory of profinite 2-complexes is presented aiming at a 2-dimensional analog of a theory of profinite fundamental groups of profinite graphs of groups as introduced by O.V.Mel'nikov and P.A.Zalesskii in [32] and [33]. We tried to "simulate" elementary homotopies but avoided introducing profinite homotopy categories. In a sense, our approach is more flexible than using something like profinite simplicial complexes. For example we allow 2-cells with infinite boundaries and loops.

Chapter 1 reviews basic facts from profinite graph theory as presented in the book by L.Ribes and P.A.Zalesskii [25]. Some remarks on the Vietoris topology follow and we also introduce unoriented profinite graphs – they turn out to be useful for defining 1-skeletons. "Big lines" and "circles" are introduced – the latter are candidates for boundaries of 2-cells. A big line is a directed profinite graph with its edges linearly ordered.

In Chapter 2 we introduce a category of "profinite 2-complexes" with morphisms allowed to collapse 2-cells to edges or vertices and edges to vertices. These conditions "simulate" homotopies which one can – in a finite complex – interpret as elementary homotopies of geometric models. The 1-skeleton of a 2-complex is an unoriented profinite graph. At the end of the chapter we prove the existence of (co)equalizers.

Chapter 3 reviews the theory of profinite groupoids and draws mainly from work of J.Almeida & P.Weil [1] and P.R.Jones [12]. We show that certain subcategories are closed under forming projective limits and that every object is the projective limit of finite objects. Profinite groupoids and 2-complexes lead to defining a category of "continuous actions". Of particular interest to us are actions with complexes admitting a continuous section for the projection onto the space of its connected components.

Chapter 4 is devoted to the study of continuous disc free actions (any disc intersects trivially with its image). This category turns out to be closed under taking projective limits.

Chapter 5 deals with "based Galois actions", i.e., disc free actions such that the underlying complex admits a continuous section for the canonical projection onto its component space. Constructions introduced in Chapter 4 are specialized.

Chapter 6 introduces the concepts of "universal based Galois action" and "fundamental groupoid". It is proved that such actions always exist. They play a role similar to that of the universal covering space on which the fundamental group acts. For a connected profinite graph interpreted as a 1-complex the universal Galois cover as introduced in [32] appears.

Chapter 7 provides a van Kampen theorem for gluing schemes of based Ga-

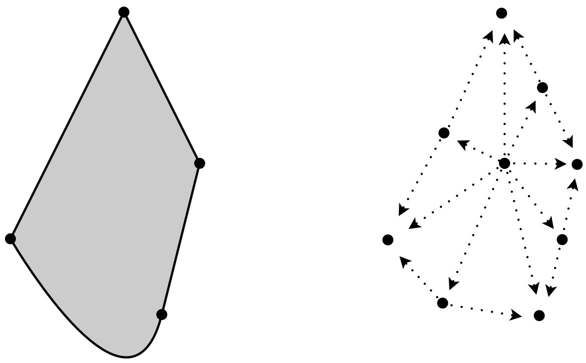


Figure 0.0.1: The figure has two parts. On the left a geometric visualization of the abstract data structure indicated on the right is pictured. The abstract data structure consists of a space of points, the nodes in the right part of the picture, and a continuous boundary map, indicated by the dashed arrows.

lois actions. Essentially it says that a profinite graph of actions (carrying the data of complexes and their fundamental group(oid)s) gives rise to a graph of profinite groups such that gluing the complexes yields a complex whose fundamental group(oid) is the fundamental group(oid) of the profinite graph of group(oid)s. In describing the fundamental groupoid of a graph of groupoids we use free and universal groupoids.
Chapter 8 shows how to use the van Kampen theorem for defining and studying Cayley complexes. We finish the work with a construction that yields for a given profinite groupoid $\mathcal{G}$ a 2-complex with $\mathcal{G}$ as fundamental groupoid.

Hints for viewing figures

All figures that appear within the text realizing a graph or a 2 dimensional complex are drawn such that vertices are points, edges are curves in $\mathbb{R}^2$, and 2-cells are homeomorphic to D^2. This style of drawing is a convenient visualization of abstract data structures in $\mathbb{R}^2$. If the data structures are finite the drawings correspond to geometric realizations. To avoid confusion, especially when considering infinite structures, Figure 0.0.1 states the relation between the "pictures" and the abstract graphs and complexes.

Chapter 1

Profinite graphs

1.1 Basic concepts

Concepts of profinite graph theory and hyperspace topologies are reviewed. For a profinite space X let $\mathrm{F}X$ denote the set of closed non empty subsets of X. Equip it with the Vietoris topology to become a profinite space. Recall that the set of all $W(U_1,\dots,U_n)=\{C\in \mathrm{F}X:\ i=1,\dots,n\ U_i\cap C\neq\emptyset,\ C\subseteq \bigcup_{i=1}^n U_i\}$ with $U_1,\dots,U_n$ open in X and $n\in\mathbb{N}$, is a basis of the Vietoris topology ([20], [10]). The Vietories functor F is defined on the category of profinite spaces sending each X to $\mathrm{F}X$ and $\varphi:\ X\to Y$ to $\mathrm{F}\varphi:\ C\mapsto\varphi(C)$. Since X is compact T_2, $\mathrm{F}X$ is compact T_2 and the finite subsets of X, with at most n elements are a compact subspace of $\mathrm{F}X$. We denote it by $\binom{X}{n}$.
An **oriented graph** (X,d_0,d_1) is a profinite space X together with maps $d_0,d_1:\ X\to X$ satisfying $d_id_j=d_j$ for $i,j\in\{0,1\}$. An **unoriented graph** (X,δ) is a profinite space X together with a map $\delta:\ X\to\binom{X}{2}$ such that $\delta x=\{x\}$ whenever $x\in\delta y$ for some y in X.
A **vertex** in (X,δ) (in (X,d_0,d_1)) satisfies $\delta x=\{x\}$ (respectively $d_ix=x$ for some $i\in\{0,1\}$). Set VX the set of vertices and $EX:=X\setminus VX$ the set of **edges**. Note that VX is closed. Using these definitions one has $\delta:\ X\to\binom{VX}{2}$. A **loop** is an edge x with $\delta x=\{u\}$ (with $d_0x=d_1x=u$). A **loop-free graph** is a (un)oriented graph without loops.
Every oriented graph can be viewed to be unoriented with the same set of vertices and edges. The converse holds if X is second countable but not in general [4].
A **graph morphism** f between oriented graphs (G,d_0,d_1) and (H,d_0',d_1') is a map from G to H such that $d_i'f=fd_i$, $i\in\{0,1\}$ (see [25]).
A **graph morphism** f between unoriented graphs (G,δ) and (H,δ') is a map from G to H such that $\delta'fx=\{fy:\ y\in\delta x\}$.

If H carries the final topology with respect to f (i.e., $U \subseteq H$ is open if and only if $f^{-1}U$ open in G) then (H, d_0', d_1') is a **quotient graph** of G and f is the **quotient map**. (Un)oriented graphs together with their morphisms form a category.

$\mathcal{C}$ will be a class of finite groups closed under taking subgroups, quotients, and extensions. A projective limit of groups in $\mathcal{C}$ is a **pro-$\mathcal{C}$ group**. We refer to [32] for the concept of **pro-$\mathcal{C}$ fundamental group** of a profinite graph.

During this work let <u>Set</u>, <u>Top</u>, <u>p-OGraph</u>, <u>p-UGraph</u>, <u>Group</u>, <u>Groupoid</u>, <u>PGroupoid</u>, <u>ContAct</u>, <u>GalAct</u>, <u>BasedAct</u>, <u>BGalAct</u>, denote the respective categories of sets, topological spaces, oriented graphs, unoriented graphs, groups, groupoids, profinite groupoids, continuous actions, Galois actions, based actions and based Galois actions. Our source for category theory are [29] and [15]. Concepts of profinite groupoids are discussed in [23].

1.2 Oriented graphs

Oriented graphs have been treated in [19], [26], [32], [33] and [34].

In <u>p-OGraph</u> the universal constructions of (co)products, (co)equalizer and thus pushout and pullbacks are directly inhereted from <u>Top</u>.

Definition 1.2.1: *The pushout $(\Gamma_1 \sqcup_A \Gamma_2, d_0, d_1)$ of the diagram*

$$\begin{tikzcd} (A, d_0^A, d_1^A) \arrow[r, "\iota_1"] \arrow[d, "\iota_2"] & (\Gamma_1, d_0^{\Gamma_1}, d_1^{\Gamma_1}) \\ (\Gamma_2, d_0^{\Gamma_2}, d_1^{\Gamma_2}) & \end{tikzcd}$$

with injections ι_1, ι_2 exists. We called it **oriented adjunction graph**.

Next we introduce the barycentric refinement of an oriented graph. Refinement does not change the fundamental group(oid) of the oriented graph.

Construction 1.2.2: *Let (Γ, d_0, d_1) be an oriented graph and make $\Gamma_V, \Gamma_0, \Gamma_1$ different copies of Γ. Form the adjunction space $\mathcal{B}\Gamma := \Gamma_V \sqcup_{V\Gamma} \Gamma_0 \sqcup_{V\Gamma} \Gamma_1$ in* <u>TOP</u> *and let $\iota_V, \iota_0, \iota_1$ be the canonical embeddings of $\Gamma_V, \Gamma_0, \Gamma_1$ into $\mathcal{B}\Gamma$. The boundary maps $d_0^{\mathcal{B}}, d_1^{\mathcal{B}} : \mathcal{B}\Gamma \to \mathcal{B}\Gamma$*

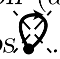
$$d_0^{\mathcal{B}}(x) := \begin{cases} x & : x \in \Gamma_V \\ \iota_V d_0 \iota_0^{-1} x & : x \in \Gamma_0 \\ \iota_V \iota_1^{-1} x & : x \in \Gamma_1 \end{cases}$$

and

$$d_1^{\mathcal{B}}(x) := \begin{cases} x & : x \in \Gamma_V \\ \iota_V \iota_0^{-1} x & : x \in \Gamma_0 \\ \iota_V d_1 \iota_1^{-1} x & : x \in \Gamma_1 \end{cases}$$

turn out to be well defined.

Proposition 1.2.3: *$(\mathcal{B}\Gamma, d_0^{\mathcal{B}}, d_1^{\mathcal{B}})$ is an oriented graph with the following properties:*

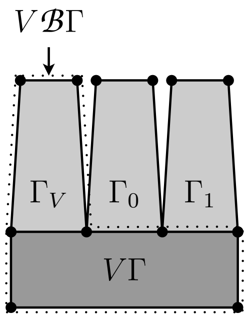
- *$V\mathcal{B}\Gamma = \Gamma_V$.*
- *For all $v \in V\mathcal{B}\Gamma$ the intersection $(d_0^{\mathcal{B}})^{-1}v \cap (d_1^{\mathcal{B}})^{-1}v = \{v\}$. In other words $\mathcal{B}\Gamma$ does not contain loops*  *.*
- *For every $v \in V\mathcal{B}\Gamma \setminus V\Gamma$ there exist exactly two disctinct edges e_1, e_2 in $\mathcal{B}\Gamma$ such that $d_0^{\mathcal{B}} e_1 = v$ and $d_1^{\mathcal{B}} e_2 = v$.*

Proof : For proving the continuity of $d_0^{\mathcal{B}}, d_1^{\mathcal{B}}$ a symmetry argument shows that it is enough to consider $d_0^{\mathcal{B}}$. It is obvious that there is a basis of open sets U in $\mathcal{B}\Gamma$ such that either $U \subseteq \iota_v E\Gamma \cup \iota_0 E\Gamma \cup \iota_1 E\Gamma$ or $\iota_v^{-1} U = \iota_0^{-1} U = \iota_1^{-1} U$. In the first case $(d_0^{\mathcal{B}})^{-1} U$ is open because it is either U or empty. In the second case we set $U' = \iota_V^{-1} U$ and observe that $(d_0^{\mathcal{B}})^{-1} U = (\iota_V U') \cup (\iota_1 U') \cup (\iota_0 d_0^{-1} U') \cup (\iota_1 d_1^{-1} U')$ is open.

Since Γ_V is closed in $\mathcal{B}\Gamma$ and $d_i^{\mathcal{B}} x = x \Leftrightarrow x \in \Gamma_V$ one has $V\mathcal{B}\Gamma = \Gamma_V$. Thus $(\mathcal{B}\Gamma, d_0^{\mathcal{B}}, d_1^{\mathcal{B}})$ is an oriented graph.

By the definition of $d_0^{\mathcal{B}}$ for $v \in \iota_V E\Gamma$ we have $(d_0^{\mathcal{B}})^{-1} v = \{\iota_1 \iota_V^{-1} v\}$ and $(d_1^{\mathcal{B}})^{-1} v = \{\iota_0 \iota_V^{-1} v\}$. Thus there are exactly two edges adjacent to each $v \in \iota_V E\Gamma$. On the other hand if $v \in \iota_V V\Gamma$ then $(d_0^{\mathcal{B}})^{-1} v = \iota_0 d_0^{-1} \iota_V^{-1} v \subseteq \iota_0 E\Gamma \cup \{v\}$ and $(d_1^{\mathcal{B}})^{-1} v = \iota_1 d_1^{-1} \iota_V^{-1} v \subseteq \iota_1 E\Gamma \cup \{v\}$. Thus e_1 and e_2 intersect in $\{v\}$ only. $\square$

On the right the amalgamation $\mathcal{B}\Gamma = \Gamma_V \sqcup_{V\Gamma} \Gamma_0 \sqcup_{V\Gamma} \Gamma_1$ and $V\mathcal{B}\Gamma$ are visualized. Below the realization of a barycentric refinement is sketched.



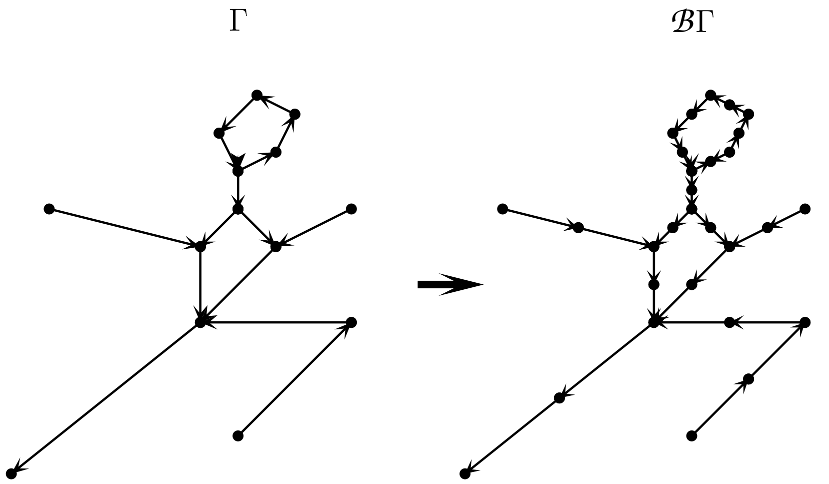


Figure 1.2.1: Visualization and structure of $\mathcal{B}\Gamma$

Observe that there is a canonical graph morphism $\varphi : (\Gamma, d_0^{\mathcal{B}}, d_1^{\mathcal{B}}) \to (\Gamma, d_0, d_1)$ collapsing the connected components of the closed subgraph $\Delta := \iota_V\Gamma \cup \iota_0\Gamma$. For every $v \in \iota_V E\Gamma$ one has $(d_0^{\mathcal{B}})^{-1}v \subseteq i_v E\Gamma \cup i_0 E\Gamma$ and $(d_1^{\mathcal{B}})^{-1}v \subseteq i_v E\Gamma \cup i_1 E\Gamma$ respectively. Moreover $d_0^{\mathcal{B}}\iota_0 E\Gamma \cup d_1^{\mathcal{B}}\iota_1 E\Gamma \subseteq V\Gamma$. Thus the connected components of Δ consist of single edges together with their endpoints. Hence the quotient is isomorphic to Γ. We refer to φ as the **canonical morphism** from $\mathcal{B}\Gamma$ to Γ.

Definition 1.2.4: *The oriented graph* $(\mathcal{B}\Gamma, d_0^{\mathcal{B}}, d_1^{\mathcal{B}})$ *is the* **barycentric refinement** *of* (Γ, d_0, d_1).

Theorem 1.2.5: *There is an isomorphism of pro-$\mathcal{C}$ fundamental groups* $\pi_1^{\mathcal{C}}(\mathcal{B}\Gamma, d_0^{\mathcal{B}}, d_1^{\mathcal{B}}) \cong \pi_1^{\mathcal{C}}(\Gamma, d_0, d_1)$.

Proof : The canonical morphism ϕ collapses the trees of the forest Δ. So the first Proposition on page 486 in [35] shows that the fundamental pro-$\mathcal{C}$ groups of Γ and $\mathcal{B}\Gamma$ coincide. □

1.3 Unoriented graphs

This paragraph is devoted to the study of universal constructions in p-UGraph. The following is an analog of 1.7 in [33].

Theorem 1.3.1:

1. *For any inverse system of graphs* $((G_\alpha, \delta_\alpha), p_{\alpha\beta})$ *the projective limit* $(G, \delta) = \varprojlim_\alpha (G_\alpha, \delta_\alpha)$ *exists.*
2. *Every* (G, δ) *is the projective limit of an inverse system* $((G_\alpha, \delta_\alpha), p_{\alpha\beta})$ *of finite unoriented graphs.*

Proof :

1. Let $G := \varprojlim_\alpha G_\alpha$ be the projective limit of profinite spaces. Since the Vietoris functor F is right exact, the system $(G_\alpha, p_{\alpha\beta})$ gives rise to an inverse system $(\mathrm{F}G_\alpha, \mathrm{F}p_{\alpha\beta})$ with projective limit $\varprojlim_\alpha \mathrm{F}G_\alpha$ equal to $\mathrm{F}G$. The universal property of $\varprojlim_\alpha \mathrm{F}G_\alpha$ yields $\delta : G \to \mathrm{F}G$ turning (G, δ) into an unoriented graph. It remains to show that the canonical projections $p_\alpha : G \to G_\alpha$ are

graph morphisms and that for any (Y, δ') with compatible graph morphisms $p'_\alpha : (Y, \delta') \to (G_\alpha, \delta_\alpha)$ the universal map $u : Y \to G$ is a graph morphism. The commutativity of the following diagram shows the validity of these assertions.

$$\begin{tikzcd}
 & G \arrow[d, "p_\alpha"] \arrow[dr, "p_\beta"] \arrow[ddd, dotted, bend right, "\delta"'] & \\
Y \arrow[r, "p'_\alpha"] \arrow[d, "\delta'"] \arrow[ur, dotted, bend left, "u"] \arrow[rr, bend left=60, "p'_\beta"] & G_\alpha \arrow[r, "p_{\alpha\beta}"] \arrow[d, "\delta_\alpha"] & G_\beta \arrow[d, "\delta_\beta"] \\
\mathrm{F}Y \arrow[r, "\mathrm{F}p'_\alpha"] \arrow[dr, dotted, bend right, "\mathrm{F}u"'] & \mathrm{F}G_\alpha \arrow[r, "\mathrm{F}p_{\alpha\beta}"] & \mathrm{F}G_\beta \\
 & \mathrm{F}G \arrow[u, "\mathrm{F}p_\alpha"] \arrow[ur, "\mathrm{F}p_\beta"'] &
\end{tikzcd}$$

2. It suffices for any clopen partition $\mathbb{P}$ of G to find a refined clopen partition $\mathbb{P}'$, such that the induced quotient map is a graph morphism. Let $\mathbb{M}$ be the set of all non empty clopen sets of the form $\delta^{-1}W(P) \cap P = \{x \in P : \delta x \subseteq P\}$ with $P \in \mathbb{P}$. Every $P \in \mathbb{P}$ which intersects all sets in $\mathbb{M}$ trivially, is contained in EG. Therefore, letting $\mathbb{M}'$ be the set of all non empty sets of the form $\delta^{-1}W(P', Q') \cap EG$ with $P' \neq Q' \in \mathbb{M}$, one finds $\mathbb{P}' := \mathbb{M}' \cup \mathbb{M}$ to be a clopen partition of the desired sort. □

Corollary 1.3.2: *When (G, δ) is loop-free, then the inverse system can be chosen to be loop-free.*

Proof : Let $((G_\alpha, \delta_\alpha), p_{\alpha\beta})$ be an inverse system as in Theorem 1.3.1 1. For each α set G'_α the graph obtained from G_α by collapsing all loops to their endpoints. Then $((G'_\alpha, \delta'_\alpha), p'_{\alpha\beta})$ is a cofinal inverse system because for any two elements g, g' in G there is α such that $p_\alpha g \neq p_\alpha g'$ and $p_\alpha a \neq p_\alpha b$ for $\{a, b\}$ denoting either δg or $\delta g'$. □

Let F be an arbitrary functor from a small category to p-UGraph. We will show that colim F exists. Since a directed set can be viewed as a small arrow

category, we will refer to a diagram from a directed set to $\underline{\text{p-UGraph}}$, as a directed $\underline{\text{p-UGraph}}$-diagram. Therefore projective limits in $\underline{\text{p-UGraph}}$ are colimits of directed $\underline{\text{p-UGraph}}$-diagrams. Note that the existence of arbitrary colimits in $\underline{\text{p-UGraph}}$ implies Theorem 1.3.1.
As shown in [29], the existence of arbitrary colimits is equivalent to the existence of arbitrary products and equalizers.
Using this equivalence we will prove our assertion in two parts. First we show that $\underline{\text{p-UGraph}}$ admits arbitrary products, and second, we construct equalizers.
Products in $\underline{\text{p-OGraph}}$ are products in $\underline{\text{Set}}$ and a suitable definition of boundary maps. For defining products in $\underline{\text{p-UGraph}}$ we avail ourselves of a more complex set theoretical construction.

Construction 1.3.3: *Let I be an arbitrary index set and $(G_i, \delta_i)_{i\in I}$ be a collection of unoriented graphs. In $\underline{\text{Top}}$ we define $X := \prod_{i\in I} VG_i$.*
Take $P := \left(\prod_{i\in I} G_i\right) \times \binom{X}{2}$ and note that it is compact. Denote the elements of P by $(m, \{u, v\}) = ((m_i)_{i\in I}, \{(u_i)_{i\in I}, (v_i)_{i\in I}\})$.
The subset

$$\Gamma := \{(m, \{u, v\}) \in P : \ \forall i \in I \ \delta_i m_i = \{u_i, v_i\}\}$$

is compact. Define $\delta : \ \Gamma \to \binom{\Gamma}{2}$ by sending $(m, \{u, v\})$ to $\{(u, \{u\}), (v, \{v\})\}$. Then (Γ, δ) is an unoriented graph, the product, denoted by $\prod_{i\in I}(G_i, \delta_i)$. There are canonical projections $p_i : \prod_{i\in I}(G_i, \delta_i) \to (G_i, \delta_i)$, defined as $p_i(m, \{u, v\}) = m_i$.

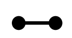
Proof : It is easy to see that the assignment $(u, \{u\}) \mapsto u$ is a homeomorphism from $\Delta \to X$, where $\Delta = \left\{(u, \{u\}) \in X \times \binom{X}{2}\right\}$, and therefore we can identify δ with the projection of P to $\binom{X}{2}$. Hence δ is continuous. To see that Γ is a closed subset of P, take a point $p_0 \in P$ and observe: If every neighbourhood of $p_0 = (m, \{u, v\})$ contains some $g = (m', \{u', v'\}) \in \Gamma$ then for fixed $i \in I$ in every neighbourhood of m_i, u_i and v_i there is some m'_i, u'_i and v'_i with $\delta_i m'_i = \{u'_i, v'_i\}$. Continuity of δ_i, for all $i \in I$, yields $\delta_i m_i = \{u_i, v_i\}$ and thus $p_0 \in \Gamma$, such that finally (Γ, δ) turns out to be an unoriented graph.

It remains to show that (Γ, δ) has the universal property of the product in $\underline{\text{p-UGraph}}$. Let (Γ', δ') be given and $p'_i : (\Gamma', \delta') \to (\Gamma_i, \delta_i)$ be projections. The universal morphism $\sigma : (\Gamma', \delta') \to (\Gamma, \delta)$ turns out to be $\sigma(m', \{u', v'\}) =$

$((p'_i m')_{i\in I}, \{(p'_i u')_{i\in I}, (p'_i v')_{i\in I}\})$. One observes that σ is well defined and that for every $i \in I$ the diagram

$$\begin{tikzcd}
(\Gamma, \delta) \arrow[r, "p_i"] & (\Gamma_i, \delta_i) \\
(\Gamma', \delta') \arrow[u, dotted, "\sigma"] \arrow[ur, "p'_i"'] &
\end{tikzcd}$$

is commutative. Therefore σ is unique as a consequence of its definition. $\square$

Now we turn to the existence of equalizers. It is obvious that equalizers in p-OGraph are obtained by using the forgetful functor p-OGraph $\rightarrow$ Set , but this method does not work for equalizers in p-UGraph: Consider the one edge graph  . The equalizer of the endomorphism, flipping the two endpoints, and the identity morphism is the empty set, because the equalizer in Set consists of the edge e only which is not a graph. So in this case the equalizer in p-UGraph will turn out to be empty.

Construction 1.3.4: *Given two graph morphisms:*

$$(G, \delta) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (H, \delta)$$

The equalizer of φ and ψ is the set difference $E \setminus S$, where $E := \{g \in G : \varphi g = \psi g\}$ is the set theoretic equalizer and $S := \{g \in EG : \varphi u \neq \psi u, \ \{u, v\} = \delta g\}$ the set of edges e, for which φ and ψ differ on the endpoints of e.
Set $\delta_\restriction := \delta\restriction(E \setminus S)$ and $\iota : E \setminus S \to G$ the natural embedding. The set S is open and $(E \setminus S, \delta_\restriction)$ is a subgraph of (G, δ), which together with ι is the equalizer of φ and ψ, represented by the diagram

$$(E \setminus S, \delta_\restriction) \xrightarrow{\iota} (G, \delta) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (H, \delta)$$

and often denoted as $E(\varphi, \psi)$.

Proof : To see that S is open we observe that δ is continuous, and therefore the complement $\{g \in G : \psi u = \varphi u, \ \forall u \in \delta g\}$ of S is closed. Since E is closed $E \setminus S$ is closed and thus compact. Moreover since all $e \in EG$ for which there is a vertex $u \in \delta e$ such that $\varphi u \neq \psi u$ lie in S, we conclude that $E \setminus S$ is a subgraph of (G, δ).

Obviously ι has the desired property $\varphi \circ \iota = \psi \circ \iota$.

It remains to show that $((E \setminus S, \delta_{\restriction}), \iota)$ is universal, i.e. for any $((F, \delta^*), \kappa)$, with $\kappa : (F, \delta^*) \to (G, \delta)$ and $\varphi \circ \kappa = \psi \circ \kappa$ there is a unique morphism $\sigma : (F, \delta^*) \to (E \setminus S, \delta_{\restriction})$ such that $\varphi \circ \kappa = \varphi \circ \iota \circ \sigma = \psi \circ \iota \circ \sigma = \psi \circ \kappa$.

Observe that the image of κ is contained in $E \setminus S$. Then $\sigma = \iota^{-1} \circ \kappa$. $\square$

Definition 1.3.5: *An (un)oriented graph is* **connected** *if it is not the disjoint union of two non empty closed subgraphs. The* **connected components** *of an (un)oriented graph are its maximal connected subgraphs.*

In [34] a graph is termed connected if it is the projective limit of finite connected graphs (in the sense that the "realization" is connected). In light of Lemma 1.6 [34] our definition is equivalent.

1.4 Big line

A **LOT space** (linear ordered topological, [10]) is a linearly ordered set $(X, \leq)$ with order topology (i.e. the open intervals $(a, b)_{\leq} := \{x : a < x < b\}$ form a basis). If no confusion arises we simply write (a, b) for $(a, b)_{\leq}$. An element $x \in X$ is a **successor** of a if $a < x$ and $(a, x) = \emptyset$. A successor of a is unique and will be denoted by a^+.

Definition 1.4.1: *On a quotient space Y of a LOT space $(X, \leq)$ define the* **strictly induced relation** $\preccurlyeq$ *as follows:*

$$q \preccurlyeq q' \text{ if and only if } \quad \forall x \in f^{-1}q, \forall x' \in f^{-1}q' : \; x < x' \text{ or } q = q'$$

The following is an immediate consequence of the definition.

Proposition 1.4.2: *"$\preccurlyeq$" is a partial order. It is linear if and only if $f^{-1}(y)$ is an interval for every y in Y.*

Remark that a continuous surjection f from a compact space X to a T_2 space Y is a quotient map. Therefore the induced map $g : X/f \to Y$ is a continuous bijection. Since X is compact and Y is a T_2 space, g is a homeomorphism.

Proposition 1.4.3: *Let Y be a space, $(X, \leq)$ a LOT space and $f : X \to Y$ a surjective map with "$\preccurlyeq$" the strictly induced relation on Y. Then the order topology induced by "$\preccurlyeq$" agrees with the quotient topology modulo f. If, in addition, X is profinite then the topology on Y equals the quotient topology as well.*

Proof : For proving the first statement it suffices to show the quotient topology on Y is coarser than the $\preccurlyeq$-topology. Since $f^{-1}y$ is an interval for every $y \in Y$, the preimage $f^{-1}V$ of any quotient open set $V \subseteq Y$, is the union of f-saturated intervals. It shows that every connected component of $f^{-1}V$ is f-saturated and maps onto a $\preccurlyeq$-open interval in V.

If X is profinite the quotient topology is induced by "$\preccurlyeq$". □

Construction 1.4.4: *Let $(X, \leq)$ be a profinite LOT space and $C := \{a \in X : \exists a^+\}$. Form $X^* := X \amalg C$. Define a map $\iota : X^* \to X$ by setting $\iota(x) = x$ for $x \in X$ and $\iota(c) := c$ for $c \in C$. Extend $\leq$ to a linear order $\leq^*$ on X^* by setting $x <^* y$ if either $\iota x < \iota y$ or $\iota x = \iota y$ and $x \in X$, $y \in C$. Then X^* is a LOT space.*
Writing $\chi : X \to X^*$ and $\zeta : C \to X^*$ for the natural identifications define maps $d_0, d_1 : X^* \to X^*$ by setting

$$d_0 x^* := \chi \iota x^*$$
$$d_1 x^* := \begin{cases} \chi\iota(x^*) & : x^* \in \chi(X) \\ \chi\iota(x^*)^+ & : x^* \in \zeta(C) \end{cases}$$

and observe $d_0\zeta(c) = \chi(c)$, $d_1\zeta(c) = \chi(c^+)$, $d_0\chi = d_1\chi = \chi$.

Proposition 1.4.5: *Every morphism $f: (X, \leq) \to (Y, \leq')$ of profinite LOT spaces lifts to a unique morphism f^* and gives rise to commutative diagrams:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, bend left, "\chi_X"] & Y \arrow[d, bend left, "\chi_Y"] \\ X^* \arrow[u, bend left, "\iota_X"] \arrow[r, dotted, "f^*"'] & Y^* \arrow[u, bend left, "\iota_Y"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, bend left, "\zeta_X"] & Y \arrow[d, bend left, "\zeta_Y"] \\ X^* \arrow[u, bend left, "\iota_X"] \arrow[r, dotted, "f^*"'] & Y^* \arrow[u, bend left, "\iota_Y"] \end{tikzcd}$$

Moreover, if f is injective (surjective), so is f^*. The assignment $X \to X^*$, $f \to f^*$ gives rise to an exact functor.

Proof : During the proof write $\chi, \zeta, \chi', \zeta'$ for $\chi_X, \zeta_X, \chi_Y, \zeta_Y$. Define

$$f^*(x^*) := \begin{cases} \chi'(f(\iota x^*)) : x^* \in \chi(X) \text{ or } f((\iota x^*)^+) = f(\iota x^*) \\ \zeta'(f(\iota x^*)) : \text{else} \end{cases}$$

f^* is a morphism of LOT spaces, injective (surjective) if and only f is a morphism of LOT spaces and that the above diagrams commute.

Uniqueness of f^*: Assume $g^*: X^* \to Y^*$ is a morphisms of LOT spaces such that the diagrams commute. Then the commutativity reads $g^* = \chi' \circ g \circ \iota$ so that for every x^* either $g^*(x^*) = \chi' \circ f \circ \iota x^*$ or $g^*(x^*) = \zeta' \circ f \circ \iota x^*$ holds. If x^* is in $\chi(X)$ the first of these equalities is satisfied. Otherwise x^* is in $\zeta(C)$. Since g^* is a morphism of LOT spaces, the first equality holds if g identifies $\iota(x^*)$ with its successor, and, the second otherwise. Hence $f^* = g^*$. □

Corollary 1.4.6: *Any $(X, \leq)$ is a projective limit $\varprojlim_\alpha (X_\alpha, \leq_\alpha)$ of finite LOT spaces X_α, and $(X^*, \leq^*) = \varprojlim_\alpha (X^*_\alpha, \leq^*_\alpha)$, where all X^*_α are finite.*

Proof : This follows from the right exactness of "*". □

Proposition 1.4.7: *The triple (X^*, d_0, d_1) is an oriented graph.*

We give two proofs, a direct one, and one using a projective limit argument and Corollary 1.4.6:

Proof direct : Continuity of d_i is proved by showing that the inverse image of an open interval (a, b) in X is an open interval:

$$d_0^{-1}(a,b) = \left(\min_{\leq^*}(\iota^{-1}a), \max_{\leq^*}(\iota^{-1}b)\right)$$

and

$$d_1^{-1}(a,b) = \left(c, \min_{\leq^*}(\iota^{-1}b)\right)$$

where, in case there is x with $a = x^+$, we put $c := x$ and $c := a$ else.

To prove compactness of X^* suppose $\{J_i : i \in I\}$ is an open cover of X^* by non-empty open intervals J_i. Since X is compact (in the induced topology as well) there is a finite subcover $\{J_i : i \in I_0\}$ of X. When $e \in C$ is not covered by the finite subcover there are 2 indices $i, j \in I_0$ with $J_i = (a, e), J_j = (e, b)$. Therefore almost all elements of X^* are covered and thus there is a finite subcover $\{J_i : i \in I_1\}$ of X^*.

X^* is T_2 since its topology is induced by an order.

Totally disconnectedness of X^* is shown if for any $x <^* y$ one can provide clopen sets A, B with $x \in A$, $y \in B$, $A \cap B = \emptyset$ and $A \cup B = X^*$. When $d_0 y = d_1 x$ then set $A = (-\infty, y)$, $B = (d_0 y, \infty)$ if x, y are edges and $A = (-\infty, y)$, $B = (d_0 y, \infty)$ else. Otherwise if $d_0 y \neq d_1 x$ separate $d_0 y$ and $d_1 x$ with two clopen intervals I, J in X such that $I \cup J = X$. Since X^* is a LOT space $A = d_0^{-1} I$, $B = d_0^{-1} J$ is a suitable choice. □

Proof with Prop. : For any finite quotient $(X_\alpha, \leq_\alpha)$ it is immediate that $(X_\alpha^*, d_{\alpha,0}, d_{\alpha,1})$ is an oriented graph. Proposition 1.4.6 implies that X^* is profinite. Denoting $p_{\alpha\beta} : X_\alpha \to Y_\beta$ the canonical projections of an inverse system of X, one observes:

- $d_{\beta,0} p_{\alpha\beta}^* = \chi_\beta \iota_\beta p_{\alpha\beta}^* = \chi_\beta p_{\alpha\beta} \iota_\alpha = p_{\alpha\beta}^* \chi_\alpha \iota_\alpha = p_{\alpha\beta}^* d_{\alpha,0}$
- For x^* is in $\chi_\alpha(X_\alpha)$ or $p_{\alpha\beta}\iota_\alpha x^* = p_{\alpha\beta}\iota_\alpha(x^*)^+$ one has $d_{\beta,1} p_{\alpha\beta}^*(x^*) = \chi_\beta \iota_\beta p_{\alpha\beta}^*(x^*) = \chi_\beta p_{\alpha\beta} \iota_\alpha(x^*) = p_{\alpha\beta}^* \chi_\alpha \iota_\alpha(x^*) = p_{\alpha\beta}^* d_{\alpha,1}(x^*)$.
- For x^* is in $\zeta_\alpha(X_\alpha)$ and $p_{\alpha\beta}\iota_\alpha x^* \neq p_{\alpha\beta}\iota_\alpha(x^*)^+$ one has $d_{\beta,1} p_{\alpha\beta}^*(x^*) = \zeta_\beta \iota_\beta p_{\alpha\beta}^*(x^*) = \zeta_\beta p_{\alpha\beta} \iota_\alpha(x^*) = p_{\alpha\beta}^* \zeta_\alpha \iota_\alpha(x^*) = p_{\alpha\beta}^* d_{\alpha,1}(x^*)$. Therefore each $p_{\alpha\beta}^*$ is a graph morphism and so $(X^*, \varprojlim_\alpha d_{\alpha,0}, \varprojlim_\alpha d_{\alpha,1}) = (X^*, d_0, d_1)$ is a profinite graph. □

Corollary 1.4.8: *The morphism f^* of Proposition 1.4.5 is a graph morphism.*

Proof : We have to prove that $f^* d_j = d'_j f^*$ holds for $j = 0, 1$. This follows from the properties of f^* (1.4.5). $\square$

Definition 1.4.9: *An (un)oriented graph as in Construction 1.4.4 is a(n unoriented)* **big line.** *The (unoriented)* **big circle obtained** *from it is the quotient graph with endpoints identified. Define the two classes* $\mathcal{I}$:= $[I : I \text{big line}]$, $\mathcal{S}$:= $[S : S \text{big circle}]$ *and a class function* φ: $\mathcal{I} \to \mathcal{S}$ *that assigns to each* I *the big circle* S *obtained from it. Denote by* π_I: $I \to S$ *the respective quotient map. In the unoriented situation we set* $\mathcal{S}_u := [S : S \text{ unoriented big circle}]$.

Note that the class of unoriented big circles $\mathcal{S}_u = \mathcal{S}/\zeta$ where $\zeta : \mathcal{S} \to \mathcal{S}$ operates by reversing orientation for each $S \in \mathcal{S}$. For any big line $(L, \leq)$ the following statements can be readily verified:

Corollary 1.4.10:

- *For different vertices* v, w *there is an edge* e *in* $(v, w) \cup (w, v)$.
- *For all* e *in* EL *one has* $(d_0 e, d_1 e) = \{e\}$.

Lemma 1.4.11: *The topology on a big circle* (S, d_0, d_1), *containing more than 1 element, is not induced by an order "*$\leq_S$*" on* S *such that for all edges* $e \in S$ *one has* $d_0 e \leq_S e \leq_S d_1 e$.

Proof : Assume on the contrary that an order "$\leq_S$" induces the topology. Since S is compact there are vertices m, the minimum, and m', the maximum. By definition there exists a big line $(L, \leq)$ and a morphism $f : L \to S$, identifying $l := \min_{\leq} L$ and $l' := \max_{\leq} L$. Observe that either $f^{-1}m \neq l$ or $f^{-1}m' \neq l'$. Assume w.l.o.g. $f^{-1}m \neq l$ and $f^{-1}m \leq f^{-1}m'$. Set $I := (f^{-1}m, l')$. By 1.4.10 for all $v \in VI$ there is an edge contained in $(f^{-1}m, v)$. Thus either there is an $e_0 \in I$ such that $f^{-1}m = d_0 e_0$ or for every neighbourhood U of $f^{-1}m$ there is an edge $e > f^{-1}m$. Similarly for $I' := (l, f^{-1}m)$ either there is an edge e'_0 with $f^{-1}m = d_1 e'_0$ or for every neighbourhood U of $f^{-1}m$ there is an edge $e' \in U$ with $e' < f^{-1}m$. Note that $fI \cup fI' \cup \{fl, m\} = S$. For any $v \in I$ the image $f[f^{-1}m, v)$ is an open set containing a non-empty interval $[m, x)_S$. Recall that f restricted to $L \setminus \{l, l'\}$ is a homeomorphism. Use 1.4.10 again and conclude for every $e \in EI$ the validity of $(d_0 fe, d_1 fe)_S \cup (d_1 fe, d_0 fe)_S = \{fe\}$. Thus because

$f^{-1}m \neq l$ and L is a big line the existence of e_0 implies those of e_0'. This leads to either $fe_0 >_S m$ or $fe_0' >_S m$, both a contradiction. Since $f \restriction L \setminus \{l, l'\}$ is a homeomorphism, every closed neighbourhood A of m contains an edge $e' \in fI'$ and an edge $e \in fI$. Because A is compact, $v := \min\{x' \in V(f(I') \cap A) : x' > \max\{d_0 e, d_1 e\}, i \in \{0,1\}\}$ exists in A and thus $v \neq m$. Since $fI \cup fI' \cup \{fl, m\} = S$, either v is in $\{d_0 e_0, d_1 e_0\}$ for some $e_0 \in fEI$, or else, every neighbourhood of v contains some $e \in fEI$, both a contradiction, because $(d_0 I \cup d_1 I) \cap (d_0 I' \cup d_1 I') = \emptyset$ and $\overline{I} \cap \overline{I'} \subset \{f^{-1}m, l, l'\}$. □

Proposition 1.4.12: *For a surjective graph morphism f from a big line (G, δ) to a graph (H, δ') the following conditions are equivalent:*

(1) H is a big line and the restriction of f to $f^{-1}EH$ is a bijection.

(2) For all $y \in H$, $f^{-1}y$ is a closed interval.

Proof :

(1) ⇒ (2) Since H is a LOT space it is T_2. Therefore $f^{-1}y$ is closed for every $y \in H$. Assume there is $y \in H$ and $f^{-1}y$ is not an interval. Then there is an interval (a, b) which intersects $f^{-1}y$ trivially but $(-\infty, a] \cap f^{-1}y \neq \emptyset$ and $[b, \infty) \cap f^{-1}y \neq \emptyset$. Now f restricted to $f^{-1}EH$ is a bijection and thus $f^{-1}y$ is in VH and (a, b) contains an edge e. Since $[a, b]$ is compact there exist $m := \max\{c' \in [a, b] : c' < e, \exists d' > e : fc' = fd'\}$ and $m' := \min\{c' \in [a, b] : c' > e, \exists d' < e : fc' = fd'\}$, and the bijectivity of f implies $m, m' \in VH$. Restricting f to the subgraph $G' := [m, m']$, and observing that G' is a big line, it turns out that $f[m, m']$ is a big circle with order topology, contradicting Lemma 1.4.11.

(2) ⇒ (1) If $f^{-1}y$ is an interval in $\leq$, by Proposition 1.4.2 the relation $\preccurlyeq$ induces the topology on H. Using Proposition 1.4.5 lift $f \restriction VG : VG \to VH$ to the unique graph morphism $f^* : G \to (VH)^*$. We show that H coincides with $(VH)^*$. Since f is a graph morphism $\{fz : z \in \delta x\}$ equals $\delta' fx$. By requisite if δx is mapped onto a single vertex y so is x. On the other hand if f is bijective on δx, x is mapped to an edge y, so $f^{-1}y$ has to be a single point, and hence f is bijective on $f^{-1}EH$. Thus there is a bijective graph morphism $g : (VH)^* \to H$ such that $f = g \circ f^*$. So g is a morphism of LOT spaces and thus a homeomorphism. The restriction of f^* to $f^{*-1}E((VH)^*)$ is a bijection thus the second statement in (1) holds. □

Remark that any morphism of unoriented big lines which respects condition (2) of Proposition 1.4.12 uniquely determines a morphism between the big circles, obtained from them. Moreover since any big line has a basis consisting of clopen intervals we conclude from Proposition 1.4.12:

Corollary 1.4.13: *A big line (big circle) is the projective limit of an inverse system of finite big lines (big circles).*

Proposition 1.4.14: *Given a big circle S and $x \in VS$. Then there exists a big line $(L, \leq)$ in $\varphi^{-1}\{S\}$ with x the image of* $\min_{\leq}$ *and* $\max_{\leq}$. *Removing an edge from S yields a big line.*

Proof : Let $f : L \to S$ be the quotient map and $v \in VL$ map to x. We can assume $v \notin \{\min, \max\}$ otherwise there is nothing to prove. Cover L with $[\min, v], [v, \max]$ and build $L' := [\min, v] \amalg_{\min=\max} [v, \max]$. Since $[\min, v]$ and $[v, \max]$ naturally carry the induced order topology we can define a relation $\leq'$ on L' setting $l \leq' m \Leftrightarrow (l \leq m$ whenever l, m both in $[\min, v]$ or both in $[v, \max]$, or $l \in [v, \max]$ and $m \in [\min, v])$. This relation is a well defined linear order and induces the topology on L'. Since both, $[v, \max]$ and $[\min, v]$, are big lines, L' is a big line. $(\min, v) \cup (v, \max)$ is homeomorphically embedded into both L and L' and thus S is homeomorphic to L' mod $\min_{\leq'} L'$ and $\max_{\leq'} L'$.

For every $e \in ES$: $f^{-1}ES \setminus \{e\}$ is the union of two big lines L and L'. Then $f(L \cup L')$ is a big line. □

For all $x \in S$ let "$\leq_x$" be the order induced on $S \setminus \{x\}$.

Corollary 1.4.15: *Let S', S'' be big circles and $g : S'' \to S$, $f : S \to S'$ morphisms of graphs injective on $g^{-1}ES$, $f^{-1}ES'$ respectively. Then there is a morphism of big lines $g' : I'' \to I$ and a unique morphism of big lines $f' : I \to I'$ giving rise to a commutative diagram:*

$$\begin{tikzcd} S'' \arrow[r, "g"] & S \arrow[r, "f"] & S' \\ & I \arrow[u, "\pi_I"'] & \end{tikzcd} \qquad \begin{tikzcd} S'' \arrow[r, "g"] & S \arrow[r, "f"] & S' \\ I'' \arrow[u, "\pi_{I''}"'] \arrow[r, dotted, "\exists g'"'] & I \arrow[u, "\pi_I"'] \arrow[r, dotted, "\exists! f'"'] & I' \arrow[u, "\pi_{I'}"'] \end{tikzcd}$$

,

Proof : Set $l := \min_{\leq} I$ and $m := \max_{\leq} I$. Since f is bijective on $f^{-1}ES'$ removing any edge $e' \in ES'$ turns $f \upharpoonright S \setminus \{f^{-1}e'\}$ into a morphism of big lines (Proposition 1.4.14). Thus for every $y \in S'$ with $y \neq f\pi_I l$, the preimage $\pi_I^{-1}f^{-1}y$ is a closed interval, and $\pi_I^{-1}f^{-1}f\pi_I(l) = [l, a] \cup [b, m]$ the union of two closed intervals with $a < b$. Hence $\mathbb{P} := \{[l, a]\} \cup \{[b, m]\} \cup \{\pi_I^{-1}f^{-1}y : \ y \in S' \setminus f\pi_I(l)\}$ partitions I into closed intervals and so $I' := I/\mathbb{P}$ serves the purpose. On the other hand choose a point in $g^{-1}\pi l$ and use the first part of Proposition 1.4.14 to find a big line I'', such that S'' is obtained from it by mapping $\min_{\leq''}$ and $\max_{\leq''}$ to the chosen point. Then a morphism of big lines $g' : \ I'' \to J$ is induced, such that S is obtained from J. Obviously I and J are homeomorphic. So g' has the desired properties. $\square$

Corollary 1.4.16: *Let S be a big circle and $f : \ S \to S'$ a surjective graph morphism. Suppose that for every y in S' there are a big line $(I_y, \leq)$, S is obtained from I_y, and, $\pi_I^{-1}f^{-1}y$ is an interval in I_y.Then S' is either a big circle or a point.*

Proof : Construct a partition $\mathbb{P}$ of I like in Corollary 1.4.15. Now $I' = I/\mathbb{P}$ enjoys $\varphi I' = S'$. $\square$

Corollary 1.4.17: *Let $((S_\alpha, \delta_\alpha), p_{\alpha\beta})$ be an inverse system of big circles, and the morphisms $p_{\alpha\beta}$ injective on $p_{\alpha\beta}^{-1}ES_j$. Then the projective limit $(S, \delta) = \varprojlim_A (S_\alpha, \delta_\alpha)$ is a big circle.*

Proof : By Corollary 1.4.15 there exist an inverse system of big lines $((I_\alpha, \leq_\alpha), p'_{\alpha\beta})$ such that $\pi_{I_\beta}p'_{\alpha\beta} = p_{\alpha\beta}\pi_{I_\alpha}$ holds for all $\alpha \geq \beta$. Since inverse images of intervals are intervals, the projective limit $\varprojlim_\alpha (I_\alpha, \leq_\alpha)$ is a big line $(I, \leq)$ which, by the universal property of projective limits, maps onto $(S, \delta) := \varprojlim_\alpha (S_\alpha, \delta_\alpha)$. Since each $\pi_{I_\alpha} \upharpoonright (\min, \max)_{\leq_\alpha}$ is a homeomorphism we can see that (S, δ) is obtained from $(I, \leq)$. $\square$

Chapter 2

Complexes

2.1 Pre-complexes

We introduce a category of profinite complexes and show that it is closed under taking projective limits. The notion of pre-complex is needed:

Definition 2.1.1: *A* **pre-complex** *is a triple* (X, o, δ) *where*

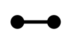
1. X *is a topological space.*
2. $o \in \mathbb{N}^X$ *is a map. When* $o(x) = n$ *then* x *is an* n-**cell**.
3. (a) $\delta : \; X \to FX$ *is a continuous map.*
   (b) *For each* $x \in X$ *all cells* y *in* $\delta x \setminus \{x\})$ *satisfy* $o(y) \leq o(x) - 1$. *If* $\delta x \setminus \{x\}$ *is not empty then it contains some* y *satisfying* $o(y) = o(x) - 1$.
   (c) *For all* x *in* X *one has* $\{x\} \cup \bigcup \{\delta y : \; y \in \delta x\} \subseteq \delta x$.

The **disc** *and the* **boundary** *of* x *are respectively* δx *and* $\beta xx := \delta x \setminus \{x\}$. *Note that* $\delta x = \{x\}$ *whenever* $o(x) = 0$. *A* **vertex** *is a* 0-*cell and an* **edge** *is a* 1-*cell.*
The **dimension** $O(A)$ *of* A *in* 2^X *is defined as* $O(A) := -1$ *if* $A = \emptyset$ *and* $O(A) := \sup_{x \in A} o(x)$ *else. We allow* $O(A) = +\infty$.

3.(b) can be rephrased:

3.$(b.')$: Forall $x \in X : \; O(\beta x) = o(x) - 1$.

A pre-complex $(\{v, e\}, o, \delta)$ with $o(v) = 0$ and $o(e) = 1$ is a **loop** . A pre-complex $(\{v_0, v_1, e\})$ with $o(v_0) = o(v_1) = 0$ and $\delta e = \{v_0, v_1\}$ is a **single**

edge  .

Definition 2.1.2: *A pre-complex of* **spherical type** S^j, $j = -1, 0, 1$ *or, for short, a* **sphere**, *is one of the following:*

- $S^{-1} := \emptyset$.
- $S^0 := \{\{x, y\}, o, \delta\}$ *with* $o(x) = o(y) = 0$ *and* $\delta x := \{x\},\ \delta y := \{y\}$.
- S^1 *is a big circle with at least one edge interpreted as a* 1*-pre-complex (Proposition 2.3.1).*

Definition 2.1.3: *A* **sub pre-complex** *of* (X, o, δ) *is a pre-complex* (A, o_a, δ_a) *with* $A \subseteq X$, $o \restriction A = o_a$, $\delta \restriction A = \delta_a$. *The subset* $X^{\leq n} := o^{-1}([-1, n])$ *together with the restriction of* o *and* δ *to* $X^{\leq n}$ *is a pre-complex, the* n**-skeleton** *of* X. *Set* $X^{=n} := o^{-1}(n)$ *and* $X^{>n} := o^{-1}([n + 1, \infty))$
To shorten notation for low dimensional complexes we set $VX = X^{=0}$ *the vertices of* X, $EX = X^{=1}$ *the edges of* X *and* $DX = X^{=2}$ *the* 2*-cells of* X.

Note that $X^{=n}$ need not be a subpre-complex.
A subset $Y \subseteq X$ is a subpre-complex if and only if $\beta y \subseteq Y$ holds for all $y \in Y$.

Proposition 2.1.4: *When* (A_i, o_i, δ_i) *are subpre-complexes of* (X, o, δ) *then their intersection* $\bigcap A_i$ *and their union* $\bigcup A_i$ *are both subpre-complexes.*

Proof : It suffices to define the functions $o_{\bigcap a_i}$, $\delta_{\bigcap A_i}$ by restriction. A similar argument works for the union. □

2.2 2-complexes

Definition 2.2.1: *A* 2-**complex** *is a pre-complex* (X, o, δ) *such that the following holds:*

*(a) (***Boundary condition)** *The boundary* βx *of every* 2*-cell* x *is a sphere* S^1*. For every* 1*-cell* x *its boundary* βx *is a sphere* S^0*.*

*(b) (***Base property)** **:** *For every edge* l *with a single endpoint there is a base of neighbourhoods* $\mathcal{B}(l)$ *such that for all* $B \in \mathcal{B}(l)$ *the closure of the set* $\{e \in EX : \exists d \in DX,\ e \in \beta d \cap B,\ |\beta d \cap B| \text{ is odd}\}$ *does not contain* l*.*

The base property is needed to prevent 2-cells to converge to loops in a non "homotopic" manner (see Figure 2.2.1).
Every subpre-complex of a complex is itself a complex.

Definition 2.2.2: *A subpre-complex of a complex is a* **subcomplex***.*

For any cell x its boundary βx, and its disk δx are subcomplexes. Remark that β need not be continuous! An example is the complex with triangles converging to the single edge as indicated in Figure 2.2.2:

Definition 2.2.3: *A* 1*-complex is* **connected** *if it is not the disjoint union of two closed subcomplexes. A complex is* **connected** *it its* 1*-skeleton* $X^{\leq 1}$ *is connected.*

Corollary 2.3.2 will relate this definition to the one for graphs (Definition 1.3.5).

Definition 2.2.4: *A* **morphism** *of* 2*-complexes* $\phi : (X, o, \delta) \to (Y, o', \delta')$ *is a continuous map* $\phi : X \to Y$ *such that* 0*-cells map to* 0*-cells and the following properties hold:*

(i) For every $x \in X$ *the equation* $\delta' \phi x = \{\phi y : y \in \delta x\}$ *hold.*

(ii) Let $x \in X$*,* $y = \phi x \in Y$ *be* 2*-cells. Then the restriction of* ϕ *to* $E(\beta x) \cap \phi^{-1} E(\beta y)$ *is a bijection onto* $E(\beta y)$*.*

(iii) *: For all* 2*-cells* $x \in X$ *and all edges* $y \in Y$ *with* $\phi x = y$ *the cardinality of the set* $\phi^{-1} y \cap \beta x$ *is finite and even.*

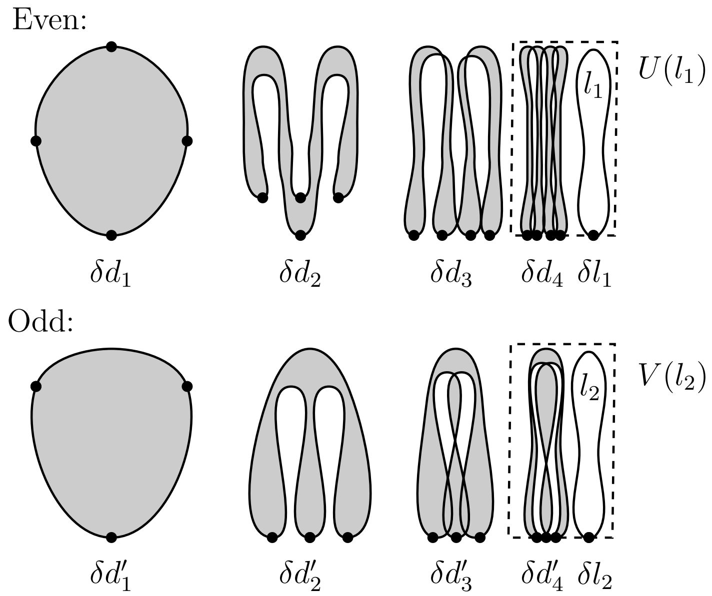


Figure 2.2.1: Two sequences of 2-cells $(d_n)_{n\in\mathbb{N}} \to l_1$ and $(d'_n)_{n\in\mathbb{N}} \to l_2$ together with their discs δd_n and $\delta d'_n$ are displayed where δl_1 and δl_2 are loops. In the even case the two "nooses" tend to become identified while converging to l_1. This is okay. In the odd case the boundary condition is violated. The three vertical "strips" tend to be identified along parts of the boundary, so that one rather would expect a "twisted 2-manifold" then the loop l_2 as the limit of the sequence.

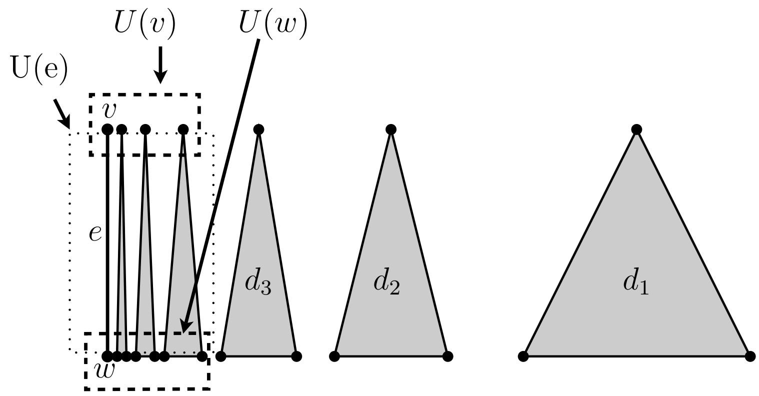


Figure 2.2.2: The sequence $(d_n)_{n\in\mathbb{N}}$ of 2-cells (the triangles) converges to the edge e on the left. The boundary βe consists of v and w. Looking at the neighbourhoods $U(e), U(v), U(w)$ makes clear that $\beta d_n \to \delta e$ and certainly $\delta e \neq \beta e$.

It can be shown that $o'(\phi x) \leq o(x)$ for every cell x. The conditions imply that ϕX is a subcomplex of Y.
If Y has no loops and is T_1 then (iii) follows from the properties of S^n, a counting argument and the continuity of δ.

Lemma 2.2.5: *When (X, o, δ) is connected and $\varphi : (X, o, \delta) \to (Y, o, \delta)$ a surjective morphism of complexes, then (Y, o, δ) is connected.*

Proof : Assume that (Y, o, δ) is not connected. Then Y is the disjoint union of two subcomplexes Y_1, Y_2. Therefore X is the disjoint union of the subcomplexes (here we use 2.2.4 (i)) $\varphi^{-1}Y_1$ and $\varphi^{-1}Y_2$ and therefore X is not connected, a contradiction. $\square$

Proposition 2.2.6: *Complexes and their morphisms (Definition 2.2.4) form a subcategory* <u>COMP</u> *of* <u>TOP</u>.

Proof : We check by induction that composition of morphisms $\phi : (X, o, \delta) \to (Y, o', \delta')$ and $\psi : (Y, o', \delta') \to (Z, o'', \delta'')$ is a morphism.

When $n = 0$ then $\psi\phi$ certainly maps 0-cells to 0-cells. Suppose $\psi\phi \restriction X^{\leq n-1}$ has been shown to be a morphism for some $n \geq 1$. Then the first condition holds for n since

$$\delta''\psi\phi x = \{\psi z :\ z \in \{\phi y :\ y \in \delta x\}\} = \{\psi\phi y :\ y \in \delta x\}.$$

Let $n > 1$, $x \in X$, $y := \phi x \in Y$, $z := \psi\phi x \in Z$ be n-cells then ψ is a bijection from $\beta x^{=n-1} \cap \psi^{-1} y^{=n-1}$ to $\beta y^{=n-1}$ and obviously $\psi\phi$ is a surjection onto $\beta z^{=n-1}$, the composition $\psi\phi$ is a bijection from $\beta x^{=n-1} \cap \phi^{-1}\psi^{-1} z^{=n-1}$ to $\beta z^{=n-1}$. Then the second condition holds.

Let $n > 1$, $x \in X$, $y := \phi x \in Y$, $z := \psi\phi x \in Z$ be cells where $o(x) = n$, $o(z) = n - 1$ and $o(y)$ is either n or $n - 1$. So

$$|(\phi^{-1}\psi^{-1} z \cap \beta x^{=n-1})| = \begin{cases} |(\psi^{-1} z \cap \beta' y^{=n-1})| : o'(y) = n \\ |(\phi^{-1} y \cap \beta x^{=n-1})| \ : o'(y) = n - 1 \end{cases}$$

and thus the third condition is valid for $\psi \circ \phi$. □

2.3 Profinite complexes

2.3.1 1-complexes and unoriented graphs

It is convenient to interpret unoriented graphs as 1-complexes and vice versa. More formally let <u>p-1-Comp</u> be the category of 1-complexes where the underlying space is profinite.

Proposition 2.3.1: *There is a fully faithful surjective functor*

$$F :\ \underline{\text{p-UGraph}} \rightarrow \underline{\text{p-1-Comp}}$$

sending an object (X, δ) to (X, o, δ'), where $\delta' x := \delta x \cup \{x\}$ and $o(x) = 1$ if $x \in EX$ and 0 else. $F(\varphi)$ is determined by φ as a map of underlying spaces and the properties of complexes.

Proof : Observe $\delta^{-1} W(U_1, U_2) = \delta'^{-1} W(X, U_1, U_2)$ and $\delta'^{-1} W(U_1, U_2) = \delta^{-1} W(U_1, U_2) \cap W(U_1, U_2)$, showing that δ' is continuous if and only if δ is. □

As a consequence of the definition of F we have:

Corollary 2.3.2:

- *The underlying spaces of (X, δ) and $F(X, \delta)$ is X.*
- *The subgraphs of (X, δ) are in 1-1 correspondence to the subcomplexes of $F(X, \delta)$. Thus (X, δ) is a connected graph if and only if $F(X, \delta)$ is a connected complex.*
- *F preserves products, equalizers, co-products, co-equalizers and projective limits.*

In the sequel we switch between the two categories whenever it is of advantage.

In particular a loop in p-1Comp is of the form $F(L)$ where L is a .

Recall that a map from X to Y is a quotient map if and only if it is surjective and it induces the topology on Y.

Definition 2.3.3: *A complex morphism $f : (X, o, \delta) \to (Q, o_q, \delta_q)$ with Q the quotient space modulo f is a* **quotient morphism** *and (Q, o_q, δ_q) is the* **quotient complex***.*

Any clopen partition $\mathbb{P}$ of a profinite space X induces a quotient map onto a finite set. For a complex, with underlying profinite space, the quotient in Top need not inherit the structure of a complex. The example indicated in Figure 2.3.1 is a 2-complex with underlying profinite space – it is not the projetive limit of finite complexes.

2.3.2 The quotient property and projective limits

For beeing able to describe a "profinite" complex as the projective limit of finite complexes with quotient maps complex morphisms, conditions on clopen partitions $\mathbb{P}$ of the complex need to be formulated. Let $\mathbb{P}$ be a clopen partition of the 2-complex (X, o, δ) where X is a profinite space.

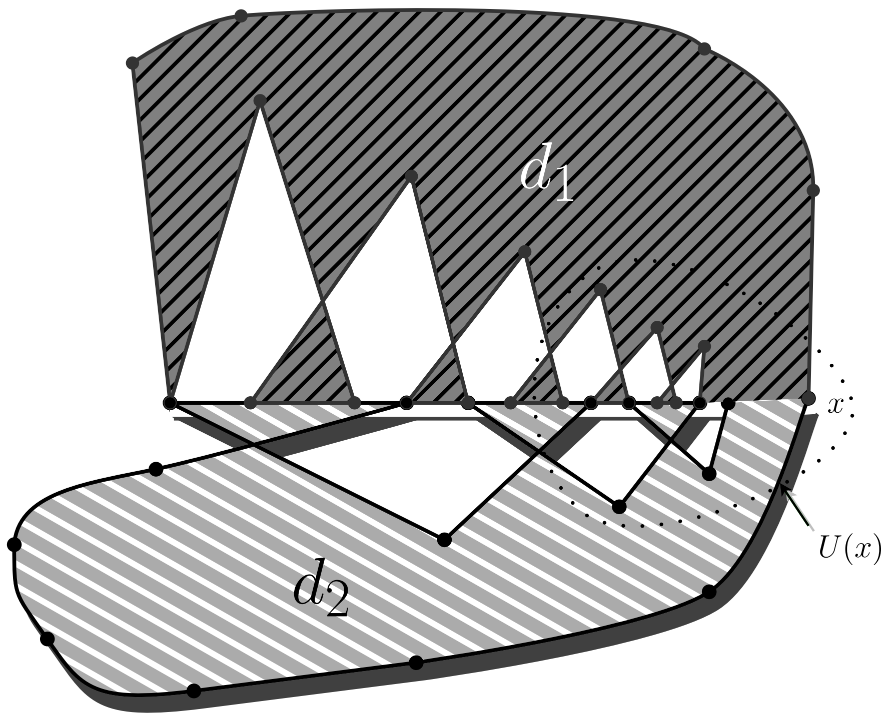


Figure 2.3.1: Each vertical "strip" of disc d_1 is twisted by 180° before glued towards βd_2. Similarly the "strips" of d_2 are treated. On the right the slots "converge" to x. Observe that a typical clopen neighbourhood $U(x)$ of x does not contain an open neighbourhood which satisfies (3) of Lemma 2.3.4, i.e. the intersection of $U(x)$ with the boundaries of either d_1 or with d_2 is not connected. There must be a clopen partition containing a subset of $U(x)$. Collapsing the partition results in a complex consisting of two discs sharing a vertex: Collapsing $U(x)$ itself creates a disc with a "hole", coming from a loop. Since a disc cannot have such a hole (see boundary condition Definition 2.2.1 (a)), the loop needs to be contracted. In this way all the holes in the picture have to be contracted to x.

Lemma 2.3.4: *Suppose that (X, o, δ) is a 2-complex with X profinite and $\mathbb{P}$ a clopen partition of X.*
Define on the quotient $X/\mathbb{P}$ the functions:

$$o'(P) := \min\{o(x) :\ x \in P\}$$

$$\textit{For any}\quad \mathbb{Q} \subseteq \mathbb{P} :\ O'(\mathbb{Q}) := \sup_{P\in\mathbb{Q}} o'(P)$$

$$\delta'(P) := \{Q \in \mathbb{P} :\ \exists x \in P,\ \delta x \cap Q \neq \emptyset\}.$$

Then $(X/\mathbb{P}, o', \delta')$ is a 2-complex if and only if the following conditions hold:

(1) *For all P and Q in $\mathbb{P}$, and for all elements p, p' in P the set $\delta p \cap Q$ is not empty if and only if $\delta p' \cap Q \neq \emptyset$.*

(2) *For $P \in \mathbb{P}$, $O'(\beta' P) = o'(P) - 1$.*

(3) *For every $P \in \mathbb{P}$ with $o'(P) = 2$, every $x \in P$ and every $Q \in \mathbb{P}$ the boundary βx is either a*  *or one can find a big line $(L, \leq)$, with at least three elements, such that βx, interpreted as an unoriented graph, is obtained from it, and the inverse image of the projection $\pi^{-1}(Q \cap \beta x)$ is a closed interval.*

(4) *For all P with $o'(P) = 1$ and all $x \in DX \cap P$ the intersection $E(\beta x) \cap P$ is of even cardinality.*

Proof : Set $Y = X/\mathbb{P}$ and f the canonical projection. Pick $y \in Y$ arbitrary:

$$\begin{array}{ll} & \forall x, x' \in f^{-1}y :\ \{Q \in \mathbb{P} :\ Q \cap \delta x \neq \emptyset\} = \{Q \in \mathbb{P} :\ Q \cap \delta x' \neq \emptyset\} \\ \Leftrightarrow & \forall x \in f^{-1}y :\ \delta' y = \{fz :\ z \in \delta x\} \\ \Leftrightarrow & \forall x \in f^{-1}y :\ \delta' fx = \{fz :\ z \in \delta x\} \end{array}$$

Thus (1) is equivalent to Definition 2.2.4-(i).

$\Rightarrow$ If f is a complex morphism, then (2) and (4) hold. For any $x \in X$ and $y \in Y$ with $o'(y) = 2$ and $fx = y$ one has f bijective on $f^{-1}E\beta' y \cap \beta x$. Suppose there is a big line I such that βx is obtained from it. Then by 1.4.15 there is a big line I', such that $\beta' fy$ is obtained from it, and an order preserving graph morphism $\tilde{f} :\ I \to I'$. Because f is a complex morphism $\beta' fy$ is either a loop or has at least three elements. In the latter case Proposition 1.4.12 renders a choice of I, with $\pi^{-1}(\mathbb{P} \cap \beta y)$ a closed interval, possible. Thus (3) holds.

$\Leftarrow$ To show the converse we check the relevant properties of Defintions 2.1.1, 2.2.1 and 2.2.4.

2.1.1-3.(a) δ' is a continuous map from $Y \to \mathrm{F}Y$.

2.1.1-3.(b) $\forall y \in Y :\ O'(\beta' y) = o'(y) - 1$.

2.1.1-3.(c) $\forall y \in Y :\ \bigcup\{\delta' z :\ z \in \delta' y\} \subseteq \delta y$.

2.2.4-(ii) For 2-cells $x \in X$, $y \in Y$ with $fx = y$ the restriction of f to $\beta x^{=n-1} \cap f^{-1}y^{=n-1}$ is a bijection onto $\beta' y^{=n-1}$.

2.2.1 (a) For all 2-cells $y \in Y$, $\beta' y$ is an S^1

2.2.1 (b) (Base Property) for .

2.2.4-(iii) For all 2-cells $x \in X$ and all edges $y \in Y$ with $\phi x = y$ the cardinality of the set $\phi^{-1}y \cap \beta x$ is finite and even.

2.1.1-3.(b) is equivalent to (2). 2.1.1-3.(c) follows obviously form (1) and the complex properties of X. 2.1.1-3.(b) shows that $f \restriction X^{\leq 1}$ is a graph morphism. Thus use (3) and Corollary 1.4.16 to obtain 2.2.1. Referring to Corollary 1.4.15 there is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] & S' \\ I \arrow[u, "\pi"] \arrow[r, "f'"'] & I' \arrow[u, "\pi'"'] \end{tikzcd}.$$

Thus Proposition 1.4.12 yields 2.2.4-(ii).

We show 2.1.1-3.(a). Since f is continuous and X is compact, f is closed. Thus f can be lifted to $Ff :\ \mathrm{F}X \to \mathrm{F}Y$ by letting $FfA := \{fa :\ a \in A\}$. To see that Ff is a continuous functor, pick an open set $U = W(O_1, \dots, O_k)$ in $\mathrm{F}Y$. Then $Ff^{-1}U = \{A \in \mathrm{F}X :\ fA \cap O_i \neq \emptyset,\ fA \subset \bigcup O_i\} = \{A \in \mathrm{F}X :\ A \cap f^{-1}[O_i] \neq \emptyset,\ A \subset \bigcup f^{-1}[O_i]\} = W(f^{-1}[O_1], \dots, f^{-1}[O_k])$ so that $\mathrm{F}^{-1}U$ is open. Since for all x in X, $Ff\delta x = \{fz :\ z \in \delta x\}$ we can conclude from (a) that $fx = fx' \Rightarrow \{fz :\ z \in \delta x\} = \{fz :\ z \in \delta x'\}$. Hence the map $\delta'' y := Ff \circ \delta[f^{-1}y]$ is well defined and $\delta'' = \delta'$.

2.2.1 (b) is satisfied since Y is finite.

Finally we show 2.2.4-(iii). For every $P \in \mathbb{P}$ with $o'(P) = 1$ and every 2-cell x in P the intersection $E(\beta x) \cap P$ is finite because βx is a big circle. We show that it is even by proving the following claim:

For every edge $e \in EX$ and every $P \in \mathbb{P}$ with $o'(P) = 1$ the edge e is not contained in the closure of $\{d \in DX :\ |\beta d \cap P| \text{ is odd}\}$.

If βe is a singleton set, item (4) of the quotient property implies the validity of the claim. Now let βe be not a singleton set. Then there are neighbourhoods

U and V, with $U \cap V = U \cap P = V \cap P = \emptyset$ such that $W(U,V,P)$ is a Vietoris neighbourhood of δe. Since U and V are disjoint, for every $d \in DX$, with $\delta d \in W(U,V,P)$, the intersection $\beta d \cap P$ is indeed finite and even. $\square$

Definition 2.3.5: *A clopen partition of a complex giving rise to a quotient complex has the* **quotient property***.*

Definition 2.3.6: *A* **profinite 2-complex** *is the projective limit of finite 2-complexes.*

Lemma 2.3.7: *The category of profinite 2-complexes is closed under forming projective limits.*

Proof : Let $((X_i, o_i, \delta_i), p_{ij})_{i \in I}$ be an inverse system of profinite 2-complexes. Set $X = \varprojlim_I X_i$. The projective limit $(X', o', \delta') := \varprojlim_I (X_i^{\leq 1}, o_i, \delta_i)$ exists in P-1-COMP by Corollary 2.3.2 and Theorem 1.3.1. X' can be viewed as a closed subspace of X. Set $o(x) = o'(x)$ if $x \in X'$ and $o(x) = 2$ else. The right exactness of the Vietoris functor (i.e., F in the diagram accompanying 1.3.1) shows that $\delta := \varprojlim \delta_i$ is a well-defined map.

Since FX is the projective limit of the inverse system (FX_i, p_{ij}^*) with $p_{ij}^* = F(p_{ij})$, one obtains a 1-complex $B := \varprojlim \beta_i x_i$. Corollary 1.4.17 shows that $B = \beta x$ is a big circle, as desired.

Definition 2.2.4 (iii) implies that Definition 2.2.1 (b) holds. Therefore (X, o, δ) is a complex. $\square$

Theorem 2.3.8: *A 2-complex (X, o, δ) is a profinite 2-complex if and only if:*

"space" *X is a profinite space.*

"base" *Every clopen partition of X has a clopen refinement with the quotient property.*

Proof : This follows from Lemma 2.3.4 and Lemma 2.3.7. $\square$

2.4 Universal constructions in <u>COMP</u>

The category of profinite 2-complexes is closed under taking projective limits. We focus our interest on the existence of (co)-products and (co)-equalizers. In <u>P-2-COMP</u> the coproduct of finitely many profinite 2-complexes is their disjoint union.

The existence of products we prove only under restrictions:

Definition 2.4.1: *A 2-complex (X, o, δ) is* **homogeneously bounded** *if the sets $VX,\ EX,\ DX$ and $B_\kappa(X) := \{x :\ |\beta x| = \kappa\}$ are closed for all cardinalities κ. It is* **finitely bounded** *if for every $x \in X$ the boundary βx is finite.*

For any homogeneously bounded profinite complex X the sets $VX,\ EX,\ DX$ and $B_\kappa(X)$ are clopen.

We now provide a similar construction as in 1.3.3.

Construction 1st part 2.4.2: *(X_1, o_1, δ_1) and (X_2, o_2, δ_2) be finitely and homogeneously bounded profinite 2-complexes. Make $M = X_1 \times X_2$, set p_i the i-th coordinate projection, define in <u>TOP</u> the profinite space $P := M \times F(M)$ and denote the canonical projections onto the factors by π_M and π_C respectively. We write, for short, $m_i := p_i m$ and $C_i := p_i C$.*

An element (m, C) in P is a vertex, an edge or a 2-cells according to the following rule:

$$(m, C) \in \begin{cases} VP & m_1 \in VX_1,\ m_2 \in VX_2 \\ EP & m_1 \in X_1^{\leq 1}, m_2 \in X_2^{\leq 1},\ m_1 \text{ or } m_2 \text{ is an edge} \\ DP & m_1 \text{ or } m_2 \text{ is a 2-cell} \end{cases}$$

One observes that $\Gamma := X_1^{\leq 1} \times X_2^{\leq 1} \subseteq P$. We let Γ be the 1- skeleton of the product. Define $\delta :\ P \to FP$ by $\delta(m, C) = \{(m', C') \in \Gamma,\ C' \subseteq C\} \cup (m, C)$ and $o :\ P \to \mathbb{N}$ by

$$o(x) = \begin{cases} 0 & x \in VP \\ 1 & x \in EP \\ 2 & x \in DP \end{cases}.$$

Construction 2$^{\text{nd}}$ part 2.4.3: *Make X the elements $(m, C) \in P$ which meet for all $i = 1, 2$ the following requirements:*

1. *$C_i = \delta_i m_i$ (compare Definition 2.2.4 i.).*
2. *$m \in C$.*
3. *If $(m, C) \in EP$ then $C \in \binom{P}{3}$.*
4. *$m_i \in DX_i \Rightarrow p_i$ is injective on $C \cap p_i^{-1} EC_i$ (compare Definition 2.2.4 ii.).*
5. *If $(m, C) \in DP$ then $\delta(m, C) \setminus \{(m, C)\}$ is a big circle.*

(X, o, δ) is a finite 2-complex – the product of the given complexes (X_i, o_i, δ_i). We denote it by $(X_1, o_1, \delta_1) \times (X_2, o_2, \delta_2)$.

Proof : We compile a list of items to be shown.

a.) All $x \in X$ have to satisfy the following complex properties:
   - δ is continuous
   - $o(\beta x) = o(x) - 1$
   - The boundary condition 2.2.1 (a)
   - The disc δx is a subcomplex
   - The base property 2.2.1 (b)

b.) X is a closed subset of P and thus X is profinite.

c.) The projections p_i are complex morphisms

d.) The universal property of the product

Let us process the list:

Continuity of δ: Pick $x = (m, C)$ and a Vietoris-open neighbourhood of δx of the form

$$V := W\left(V_1^1 \times W(V_1^1, \ldots, V_{n(1)}^1), \ldots, V_1^k \times W(V_1^k, \ldots, V_{n(k)}^k)\right)$$

(there is no loss of generality in such a choice of V). Since the elements of δx are of the form (m', C') with $C' \subseteq C$ we can assume $V_l^i \in \{V_1^1, \ldots, V_{n(1)}^1\}$ for all $l \geq 2$ and every $i \in \{1, 2\}$. Hence $\delta^{-1} V$ equals the open set $V_1^1 \times W(V_1^1, \ldots V_{n(1)}^1)$ and therefore δ is continuous.

$o(\beta x) = o(x) - 1$ and the boundary condition follow directly, for vertices from 1, for edges from 1 – 3 and for 2-cells from 5.

δx is a subcomplex because of 1, 3 and 5.

The base property is immediate because for every $e \in EX$ there is a neighbourhood which does not contain any disc.

The p_i are continuous and $\delta_i p_i = \{q \in \delta p\}$. Moreover property 4 yields 2.2.4 ii. and thus the p_i are complex morphisms.

Now we turn to the universal property of products. Assume given a complex (Z, u, γ) with projections $q_1 : \; Z \to X_1$ and $q_2 : \; Z \to X_2$ such that q_1, q_2 are complex morphism. Let $\eta(y) := ((q_1 y, q_2 y), \{(q_1 z, q_2 z) : \; z \in \delta y\})$ and observe that it is a complex morphism from $Z \to X$, commuting with p_i and q_i, as desired.

We still have to show that X is closed in P. For the 1-skeleton there is nothing to show and thus 3 holds. Take a net of discs $((m_\nu, C_\nu))_{\nu \in \Lambda} \to (m, C)$. Because $m_\nu \in C_\nu$ it follows by the properties of the Vietories topology that $m \in C$. Thus the argument showing continuity of δ in X holds for $\bar{X}$ and hence $p_i[\delta m] = \delta_i m_i = C_i = p_i C$. It remains to show 4 and 5. Since, by assumption, for $i = 1, 2$ the sets $\{x_i \in X_i : \; |\delta_i x_i| = n\}$ are clopen in X_i the boundaries $\beta_i m_i$ are isomorphic to $\beta_i p_i m_\nu$ for ν large enough. Therefore there exists a Vietoris open neighbourhood $W(V_1, \ldots, V_n)$ of C, such that $V_j \cap V_k \neq \emptyset$ implies for $1 \leq j \leq k \leq n$ that $j = k$. Moreover for all $\nu \in \Lambda$ large enough the intersections $V_j \cap C$ and $V_j \cap C_\nu$ both contain a single element. Hence 4 holds and because δ is continuous $\delta(m, C)$ is connected and 5 holds.

□

In general a product for two profinite complexes does not exist if one contains a 2-cell with infinite boundary and the other one a single edge  .

Example 2.4.4: Let (X_1, o_1, δ_1) be a 2-cell d with infinite boundary $\beta_1 d$, and (X_2, o_2, δ_2) a single edge with edge e and endpoints w_1, w_2. Now assume that their exists their product (X, o, δ). The universality of products yields canonical projections p_1, p_2 such that for every (Y, o', δ') with morphisms $q_1 : (Y, o', \delta') \to (X_1, o_1, \delta_1)$, $q_2 : (Y, o', \delta') \to (X_2, o_2, \delta_2)$ there exists a unique morphism σ and the following diagram commutes:

$$\begin{tikzcd}
(Y,o',\delta') \arrow[dr, dotted, "\sigma"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& (X,o,\delta) \arrow[r, "p_2"] \arrow[d, "p_1"] & (X_2,o_2,\delta_2) \\
& (X_1,o_1,\delta_1) &
\end{tikzcd}$$

For the following construction compare Figure 2.4. Let a sequence of edges $(e_n)_{n\in\mathbb{N}}$ in (X_1, o_1, δ_1) converge to a vertex v. Denote the order induced by v (Proposition 1.4.14) by $\leq$. We can assume $e_n \leq e_m \Leftrightarrow n \leq m$.
We construct inductively morphisms $q_2^n : (X_1, o_1, \delta_1) \to (X_2, o_2, \delta_2)$.
Set $q_2^1(x) = w_1$ for all $x \in X_1$. For $n > 0$

$$q_2^n(y) := \begin{cases} q_2^{n-1}(y) & n \text{ is odd or } y \in \beta \setminus [e_{n-1}, e_n]_{\leq_v} \\ e & n \text{ is even and } y \in \{e_{n-1}, e_n\} \\ w_2 & n \text{ is even and } y \in (e_{n-1}, e_n)_{\leq_v} \end{cases}$$

Let $q_1^n = 1_{X_1}$ for all $n \in \mathbb{N}$. By the universal property of X, for every n there is a unique morphism $\sigma_n : X_1 \to X$ with $p_1\sigma_n = 1_{X_1}$ and $p_2\sigma_n = q_2^n$.
Observe that for every $n \in \mathbb{N}$ and every $y \in [v, e_{n-1}]_{\leq}$ one has $\sigma_{n-1}y = \sigma_n y$. Therefore the sequence of images of $\sigma_n X_1$ converges to some δy because δ is continuous. Since $\delta y \supseteq [v, e_n]$ for a fixed $n \in \mathbb{N}$ the boundary βy has cardinality greater than 2 and thus y must be a 2-cell. On the other hand since $q_2^n e_n = \{e\}$ and $(q_1 e_n = q_1 e_m \Leftrightarrow n = m)$, it is clear that $(\sigma_n e_n)_{n\in\mathbb{N}}$ is an infinite sequence of edges in βy. Since y is a disc βy is a complex of type S^1 and therefore converges to the vertex $\sigma_1 v$. The set of vertices $Z = \{z \in \beta y : p_2 z = w_2\}$ is closed since p_2 is continuous and βy is closed. But as indicated $\sigma_n e_n \to \sigma_1 v$ and therefore the intersection $\{\sigma_n w_n \in \delta_1 e_n,\ n \in \mathbb{N}\} \cap Z$ is infinite and gives rise to a subsequence in Z converging to $\sigma_1 v$, a contradiction.

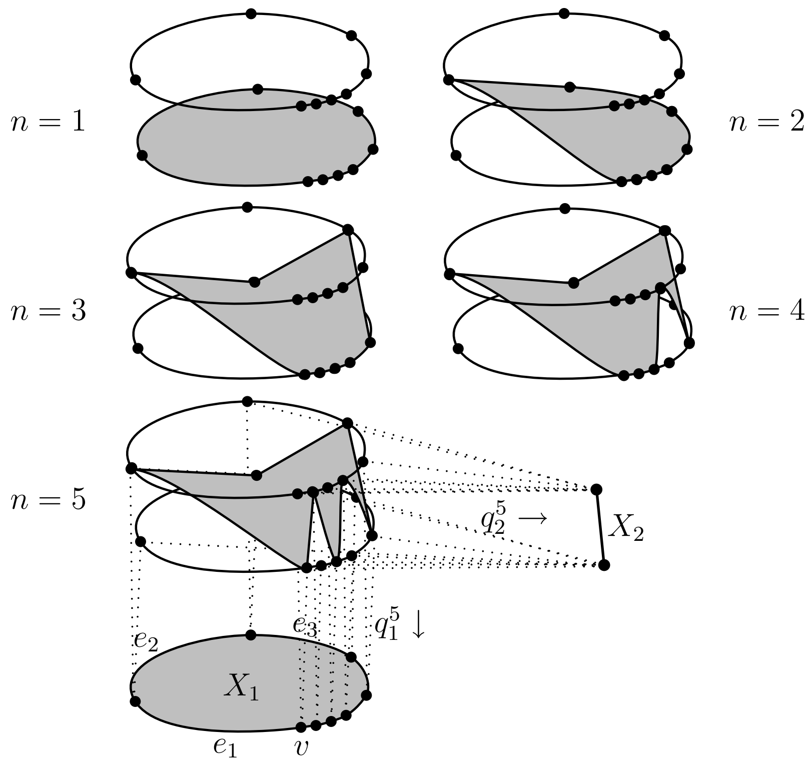


Figure 2.4.1: A finite model of Example 2.4.4 is sketched. The dotted lines symbolize natural projections, the "vertical" morphism $q_1^5 : X_1 \to X_1$, and the "horizontal" morphism $q_2^5 : X_1 \to X_2$. The projections q_1^5 and q_2^5 of the complex (the shaded area) are presented in space, and for better visibility, circles at the upper and lower level of X_2 help with orientation. As one can see with $n = 5$ increasing n adds more an more vertices in the "zig-zag" line.

Now we turn to (co)equalizers of 2-complexes, which we need for the Constructions 4.3.1, 4.3.2 and finally for Corollary 4.3.3.

Construction 2.4.5: *Let there be given morphisms of profinite 2-complexes*

$$(X, o, \delta) \underset{g}{\overset{f}{\rightrightarrows}} (Y, o', \delta').$$

Define $S := \{x \in X : \exists z \in X \ fz = gz, \ fx \neq gx, \ x \in \delta' z\}$. *Let* $o_{\restriction}$, $\delta_{\restriction}$ *denote the restrictions of* o, δ *to* $X \setminus S$. *Then* S *is open and* $(X \setminus S, o_{\restriction}, \delta_{\restriction})$ *is a complex which serves as an equalizer:*

$$(X \setminus S, o_{\restriction}, \delta_{\restriction}) \rightarrow (X, o, \delta) \underset{g}{\overset{f}{\rightrightarrows}} (Y, o', \delta').$$

Proof : The proof given in Construction 1.3.4 for graphs carries over to the present situation with minor changes. □

We next construct an adjunction complex. In Construction 2.4.8 co-equalizers will be established.

Definition 2.4.6: *The* **adjunction complex** *of two profinite complexes* (X, o, δ), (Y, o', δ') *along* A *is the pushout of the diagram*

$$\begin{tikzcd} (A, o_A, \delta_A) \arrow[r, "\iota_1"] \arrow[d, "\iota_2"] & (X, o, \delta) \\ (Y, o', \delta') & \end{tikzcd}$$

with embeddings ι_1, ι_2 .

Proposition 2.4.7: *Let* (X, o, δ), (Y, o', δ') *be profinite* 2*-complexes and* (A, o, δ) *be a subcomplex of* X *and of* Y *with embeddings* e_X, e_Y. *Then the co-equalizer* $C(e_X, e_Y)$ *of the diagram*

$$(A, o, \delta) \underset{e_Y}{\overset{e_X}{\rightrightarrows}} (X, o, \delta) \sqcup (Y, o', \delta')$$

is given by the co-equalizer in TOP. *Thus the adjunction complex along* A *exists.*

Proof : Since the images of the embeddings are disjoint one has a partition of $Z := (X, o, \delta) \sqcup (Y, o', \delta')$ given by $\{\{e_X a, e_Y a\} :\ a \in A\} \cup \{z \in Z \setminus e_X \sqcup e_Y A\}$. It is easily seen that this partition has the quotient property and thus the quotient space is a complex. The universal property of an adjunction complex can be readily verified. $\square$

Proposition 2.4.8: *For f and g morphisms of 2-complexes*

$$(X, o, \delta) \underset{g}{\overset{f}{\rightrightarrows}} (Y, o', \delta').$$

There exists the coequalizer $C(f, g)$.

Proof : Let Λ denote the set of all surjective morphisms $\lambda :\ (Y, o', \delta') \to (C_\lambda, o_\lambda, \delta_\lambda)$ to finite C_λ such that $\lambda f = \lambda g$. We devise an order "$\leq$" on Λ by setting $\lambda \leq \lambda'$ if and only if there is $\varphi_{\lambda\lambda'}$ with $\varphi_{\lambda\lambda'}\lambda' = \lambda$. The set Λ is not empty since our category has a terminal object (a single point). We claim that $(\Lambda, \leq)$ is a directed set. Pick λ, λ'. Using Construction 2.4.3 we can form the product complex $(C_\lambda, o_\lambda, \delta_\lambda) \times (C_{\lambda'}, o_{\lambda'}, \delta_{\lambda'})$ as indicated in the diagram below

$$\begin{tikzcd}
 & & (C_\lambda, o_\lambda, \delta_\lambda) \\
(X, o, \delta) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & (Y, o', \delta') \arrow[ur, "\lambda"] \arrow[r, dashed, "\sigma"] \arrow[dr, "\lambda'"'] & C_\lambda \times C_{\lambda'} \arrow[u] \arrow[d] \\
 & & (C_{\lambda'}, o_{\lambda'}, \delta_{\lambda'})
\end{tikzcd}$$

The universal property of the product yields the dashed arrow σ. Since $\sigma f = \sigma g$ the canonical image σY is a non empty complex and so σ belongs to Λ. Moreover $\sigma \geq \lambda$ and $\sigma \geq \lambda'$. The projective limit of the above inverse system provides us with a morphism ρ from Y to a 2-complex, say R.

We claim $R = C(f, g)$. Take a morphism $\kappa :\ Y \to R'$ with $\kappa f = \kappa g$. Since for every finite quotient of R' the composition of the canonical projection and κ belongs to λ a projective limit argument yields a morphism from $R \to R'$.

$\square$

Chapter 3

Profinite Groupoids

3.1 Basic definitions

Our concepts of (continuous) groupoids and profinite categories are taken from [14], [5], [23], [12] and [21]. Let $\mathcal{C}$ be a category. $\mathrm{Ob}(\mathcal{C})$ denotes the objects and $\mathrm{Hom}_{\mathcal{C}}$ the morphisms of $\mathcal{C}$. For objects $x \neq y$ of $\mathcal{C}$ we define $\mathcal{C}(x, y) := \mathrm{Hom}(x, y)$ and $\mathcal{C}(x) := \mathrm{Hom}(x, x)$.

Definition 3.1.1: *A* **groupoid** $\mathcal{G}$*, or sometimes* $(\mathcal{G}, \cdot)$*, is a small category with invertible arrows, i.e.* $\mathcal{G}^{\mathrm{opp}} \cong \mathcal{G}$*. Here "$\cdot$" denotes composition of arrows. For* $c \in \mathcal{G}(A, B)$*,* $d \in \mathcal{G}(B, C)$ *we shall denote composition by* $c \cdot d$*.*
A **subgroupoid** $\mathcal{H} < \mathcal{G}$ *is a subcategory such that* $\mathcal{H} \cong \mathcal{H}^{\mathrm{opp}}$*. It is* **wide** *if* $V\mathcal{G} = V\mathcal{H}$ *and* **full** *if* $\mathcal{H}(A, B) = \mathcal{G}(A, B)$ *for every pair of objects* $A, B \in V\mathcal{H}$*. A covariant functor* $F : \mathcal{G} \to \mathcal{H}$ *with* $\mathrm{im}(F)$ *a subcategory of* $\mathcal{H}$ *is a* **groupoid morphism***.*
F *is* **injective** *(***surjective***) if it is injective (surjective) on morphisms.*
A **normal subgroupoid** $\mathcal{H} \lhd \mathcal{G}$ *is a wide subgroupoid of* $\mathcal{G}$ *such that for every two objects* A, B *in* $\mathcal{H}$*, every morphisms* $h_0 \in \mathcal{H}(A)$ *and* $g \in \mathcal{G}(A, B)$*, there is a morphism* $h_1 \in \mathcal{H}(B)$ *with* $h_1 = g^{-1} \cdot h_0 \cdot g$*.*
A **quotient groupoid** $\mathcal{G}/\mathcal{H}$ *of a groupoid* $\mathcal{G}$ *by a normal subgroupoid* $\mathcal{H} \lhd \mathcal{G}$ *is the groupoid with object set the connected components of* $\mathcal{H}$*, and morphisms* $g\mathcal{H} := \{g' : \exists h, h' \in \mathcal{H},\ g' = h' \cdot g \cdot h\}$*.* $\mathcal{H}$ *has* **finite index** *in* $\mathcal{G}$ *or, for short,* $\mathcal{H} \lhd_{\mathrm{fin}} \mathcal{G}$*, if for all* $v \in \mathrm{Ob}(\mathcal{G}/\mathcal{H})$ *one has* $|\mathcal{G}/\mathcal{H}(v)| < \infty$*. For details see [14].*
A groupoid is **totally disconnected** *if for all* $x, y \in \mathcal{G}$ *the homset* $\mathcal{G}(x, y) = \emptyset$*.*

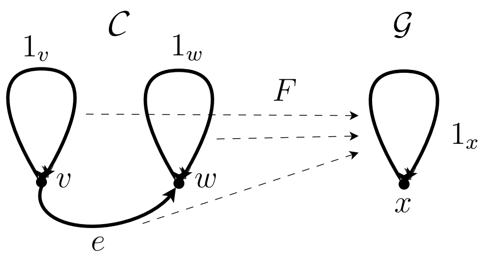


Figure 3.1.1: The category $\mathcal{C}$, on the left, consists of two objects v, w and three arrows $1_v, 1_w, e$. The dashed arrows indicate a functor F from $\mathcal{C}$ to the trivial groupoid $\mathcal{G}$ with one vertex x. F sends v, w to x and $1_v, 1_w, e$ to 1_x. Because F is injective on each homset of $\mathcal{C}$ it is faithful, but $\mathcal{C}$ is not a groupoid. Therefore $\underline{\text{PGroupoid}}$ is not a pseudovariety.

Definition 3.1.2: *A* **boolean groupoid** *consists of an oriented graph $(\mathcal{G}, d_0, d_1)$, a closed subset D of $E\mathcal{G} \times E\mathcal{G}$ and a map $\cdot : D \to E\mathcal{G}$ such that:*

1. *$(\mathcal{G}, \cdot)$ is a groupoid.*
2. *$(e, f) \in D \Leftrightarrow d_1 e = d_0 f$.*
3. *$d_0(e \cdot f) = d_0 e$, $d_1(e \cdot f) = d_1 f$*
4. *Inversion in $(\mathcal{G}, \cdot)$ is continuous.*
5. *d_0 restricted to the identities of the groupoid $(\mathcal{G}, \cdot)$ is a homeomorphism.*

A **profinite continuous groupoid** *or, for short, a* **profinite groupoid** *(pc-groupoid) is the projective limit of finite boolean groupoids.*
Our definition is coherent with the notions given in [23], [12] and [21].
The **quotient** *of a profinite continuous groupoid by a profinite normal subgroupoid is the quotient groupoid equipped with the quotient topology.*
Morphisms of profinite groupoids *are graph morphisms which at the same time are groupoid morphisms.*
The **kernel** *of a morphism F of groupoids is the wide normal subgroupoid whose edges map to identity elements. We denote it by $\ker F$.*

Remarks: For boolean groupoids $E\mathcal{G}$ is closed.
If a functor of groupoids is injective on objects it is a groupoid morphism. $\underline{\text{PGroupoid}}$ is not closed under divisors, i.e. the existence of a faithful functor from a category $\mathcal{C}$ to a groupoid $\mathcal{G}$ does not imply that $\mathcal{C}$ is a groupoid (Example 3.1.1).

Thus pc-groupoids do not form a pseudovariety of categories and thus results of Almeida, Weil [1] and Jones [12], do not carry over to our situation.
The canonical projection of a groupoid to a quotient groupoid is a groupoid morphism. Moreover for profinite groupoids this projection is continuous.
If $\mathcal{G}$ is connected, $\mathcal{G}/\mathcal{G}$ considered as an oriented graph is a  .
A profinite group G is a profinite groupoid $\mathcal{G}(G)$ if we define $E\mathcal{G} = G$ and $V\mathcal{G} = \{1_G\}$.

3.2 Universal constructions

Proposition 3.2.1: *The category* PGroupoid *allows forming arbitrary products, equalizers and projective limits.*

Proof : We define products and equalizers in PGroupoid explicitly. $\mathcal{G} \times \mathcal{H} = E\mathcal{G} \times E\mathcal{H} \cup V\mathcal{G} \times V\mathcal{H}$ and $d_0, d_1, \cdot, .^{-1}$ are defined coordinatewise. The topology on $\mathcal{G}\times\mathcal{H}$ is the usual product topology. Certainly $\mathcal{G}\times\mathcal{H}$ is a continuous groupoid and the product of $\mathcal{G}$ and $\mathcal{H}$ in PGroupoid. For the infinite case take the Tychonoff product $\prod E\mathcal{G}_i \cup \prod V\mathcal{G}_i$. Define the operations coordinatewise and observe that they are continuous.

Given two morphisms $\varphi, \psi : \mathcal{G} \to \mathcal{H}$. The set theoretic equalizer $\{g \in E\mathcal{G} : \varphi g = \psi g\} \cup \{g \in V\mathcal{G} : \varphi g = \psi g\}$ of φ and ψ is a subgroupoid of $\mathcal{G}$ – the equalizer of φ and ψ in PGroupoid.

The third assertion is a consequence of the previous ones. □

The next result is easy to prove.

Proposition 3.2.2: *The coproduct of a finite family of profinite groupoids $\mathcal{G}_i$, $i \in I$ can be constructed in* Top *by forming the coproducts $V(\coprod \mathcal{G}_i) = \coprod(V\mathcal{G}_i)$, $E(\coprod \mathcal{G}_i) = \coprod(E\mathcal{G}_i)$ and $D(\coprod \mathcal{G}_i) = \coprod D_i$.*

Proof : The coproduct in Top can be turned into a profinite groupoid by defining the operations componentwise. The universal property of the coproduct is a consequence of its universality in Top. □

3.2.1 Projective limits and "congruences"

The notion of normal subgroupoids connects to the concept congruence relation in universal algebra. We need a more general form. We need a more general form.

Definition 3.2.3: *A* **congruence** *relation on a profinite groupoid G is a closed equivalence relation $\sim$ on $\mathcal{G}$, such that the quotient $\mathcal{G}/_\sim$ is a pc-groupoid.*
The **quotient groupoid** *modulo $\sim$ has objects and arrows the respective equivalence classes. The quotient map is a morphism of profinite groupoids.*

In a connected boolean groupoid the normal subgroupoids determine the congruence relations and vice versa.
For boolean totally disconnected groupoids the next result will turn out useful.

Lemma 3.2.4: *In a compact totally disconnected boolean groupoid for each open subgroup $G \lhd \mathcal{G}(x)$ and each open neighbourhood of $U \in \mathfrak{U}(1_x)$ with $U \cap \mathcal{G}(x) = G$ one can find a neighbourhood $V \subseteq U$ such that $V \cap \mathcal{G}(y) \lhd \mathcal{G}(y)$ for all $y \in Ob(\mathcal{G})$.*

Proof : We first show that one can achieve $V \cap \mathcal{G}(y) \leq \mathcal{G}(y)$. Assume on the contrary that for each V there is a $y \in V\mathcal{G}$ and edges $u_V, n_V \in \mathcal{G}(y) \cap V$ with $u_V n_V = g_V \notin G(y) \cap V$. We are going to choose V in a specific manner. For each $O \in \mathfrak{U}(x)$ the intersections $O \cap V\mathcal{G}$ and $O' := d_0^{-1}(O \cap V\mathcal{G})$ are open. Thus $O' \cap U$ is open and $y \in O$ implies $V(O) = \mathcal{G}(y) \cap O' \cap U = \mathcal{G}(y) \cap U$. Therefore the open neighbourhoods $V(O)$ satisfy $G(x) \cap V(O) = G$. By assumption for each O there are $u_{V(O)}$ and $n_{V(O)}$ such that their product $g_{V(O)}$ is not contained in $V(O)$ and hence not in U. By the continuity of the composition the closure $\{g_{V(O)} : O \in \mathfrak{U}(x)\} \subseteq U^c$ intersects the open set U, a contradiction.

In a similar fashion with conjugation instead multiplication one shows that $V \cap \mathcal{G}(y)$ can be turned into a normal subgroup of $\mathcal{G}(y)$ for all $y \in V\mathcal{G}$. □

Corollary 3.2.5: *If $G < \mathcal{G}(x)$ has finite index then $\mathcal{G}(y) \cap V$ is a subgroup of finite index in $G(y)$. In addition one can achieve that for every $a \in \mathcal{G}(x)$ there is an open neighbourhood W_a with $aV = W_a \cap \mathcal{G}(x)$ and, whenever $W_a \cap \mathcal{G}(y) \neq \emptyset$ then $W_a \cap \mathcal{G}(y) = a_y V \cap \mathcal{G}(y)$ holds for a suitable $a_y \in \mathcal{G}(y)$. Moreover $d_0 W_a = d_0 V$.*

Proof : Assume for any V there is a $y \in \mathrm{Ob}(\mathcal{G})$ such that the subgroup $\mathcal{G}(y) \cap V$ has infinite index. Then there are infinitely many $a_\nu \in \mathcal{G}(y)$ representing the cosets of $V \cap \mathcal{G}(y)$ in $\mathcal{G}(y)$. Now for each y choose two different a_ν, a'_ν such that they converge to $a, a' \in \mathcal{G}(x)$ such that $aV \cap \mathcal{G}(x) = a'V \cap \mathcal{G}(x)$. Then $a_\nu a'^{-1}_\nu$ converges to an $v \in \mathcal{G}(x) \cap V$ which is a contradiction since they do not lie in V. Then $|G(x) : \mathcal{G}(x) \cap V| \geq |\mathcal{G}(y) : V \cap \mathcal{G}(y)|$, for ν big enough, shows that the first statement of the corollary holds. The same argument works for every $a \in \mathcal{G}(x)$ and W_a. For different a, a' use T_2 to see that $W_a \cap W_{a'}$ can be chosen empty. Because the index of G is finite in $\mathcal{G}(x)$ there are only finitely many W_a. The set $A := \{a_y \in \mathcal{G}(y) : \ a_y \notin \bigcup W_a\}$ is therefore closed and one observes that $d_0 A$ is closed. Hence the sets $W_a \cap d_0^{-1} d_0 A$ together with the given sets $V \cap d_0^{-1} d_0 A$ give rise to a finite quotient. □

Proposition 3.2.6: *Every boolean totally disconnected groupoid $\mathcal{G}$ is profinite.*

Proof : A proof can be found in [13]. For convenience we sketch a direct one. Let e_1, e_2 be in $E\mathcal{G}$. Each object group is the projective limit of finite quotients thus it suffices to consider a congruence relation on $\mathcal{G}$ such that e_1, e_2 belong to different congruence classes and the quotient modulo the relation is finite.

If $d_0 e_1 \neq d_0 e_2$ take a clopen partition $\mathbb{P}$ of $V\mathcal{G}$ separating e_1 and e_2, otherwise let $\mathbb{P}$ be the trivial partition. If $d_0 e_1 \neq d_0 e_2$ set $N_x = \mathcal{G}(x)$ for all $x \in \mathrm{Ob}(\mathcal{G})$. Otherwise take $N_x \lhd_{\mathrm{fin}} \mathcal{G}(x)$ for $x = d_0 e_1$ with $e_1 N_x \neq e_2 N_x$. By Corollary 3.2.5 N_x can be extended to a clopen congruence on some neigbhourhood of x. All such congruences yield an clopen cover of $\mathcal{G}$ such that compactness yields a finite subcover, w.l.o.g. containing the congruence generated by $N_{d_0 x}$. Then one finds a clopen congruence relation using the finite subcover and the partition $\mathbb{P}$ such that the quotient $\mathcal{G}/\mathbb{P}$ is a finite groupoid separating e_1 and e_2. □

The next propositions concern groupoids with finitely many objects.

Proposition 3.2.7: *Every boolean groupoid $\mathcal{G}$ with finitely many objects is profinite.*

Proof : We refer to [12] 4.1 where it has been shown that $\mathcal{G}$ is the projective limit of finite categories V. Since groupoids map to groupoids the inverse system can be taken to consist of groupoids only. □

Now we turn to the connected case

Proposition 3.2.8: *Every connected boolean groupoid $\mathcal{G}$ is profinite.*

Proof : It suffices to find, for given different elements $a, b \in E\mathcal{G}$, a groupoid morphism $\varphi : \mathcal{G} \to \mathcal{H}$ to a finite groupoid $\mathcal{H}$ with $\varphi a \neq \varphi b$. Fix vertices x, y in $V\mathcal{G}$ with $a \in \mathcal{G}(x)$ and $b \in \mathcal{G}(y)$. We choose a clopen partition $\mathbb{P}$ of $V\mathcal{G}$ and a normal subgroup $N \lhd \mathcal{G}(x)$ as follows:

If x is different from y then $N = \mathcal{G}(x)$ and $\mathbb{P}$ is any clopen partition such that x and y belong to different classes. If x is equal to y then N is any open normal subgroup of $\mathcal{G}(x)$ such that $ab^{-1} \notin N$ and $\mathbb{P}$ has a single class, $V\mathcal{G}$.

$\mathcal{G}(x)$ acts freely by left multiplication on $\bigcup_{y \in V\mathcal{G}} \mathcal{G}(x, y) = d_0^{-1}x$. By [25] Lemma 5.6.5 there is a continuous section $\sigma : d_0^{-1}x/\mathcal{G}(x) \to d_0^{-1}x$. Note that $\sigma(y) \in \mathcal{G}(x, y)$ and we can achieve that $\sigma(\mathcal{G}(x)/\mathcal{G}(x)) = 1_{\mathcal{G}(x)}$. For $P \in \mathbb{P}$ consider the subgroupoid $\mathcal{N}_P := \bigcup_{y, y' \in P} \sigma^{-1}(y) N \sigma(y')$. Then $\bigcup_{P \in \mathbb{P}} \mathcal{N}_P$ is a normal subgroupoid of $\mathcal{G}$ with finite quotient. When φ denotes the canonical projection, then $\varphi a \neq \varphi b$ by construction. □

The next example is a contribution of Karl Auinger [2].

Example 3.2.9: *Take the Cantor set C, represented by the dyadic expansion, i.e. binary sequences $(i_n)_{n \in \mathbb{N}}$ where each sequence is a label for a different point of C. There is an equivalence relation on C generated by*

$$(i_n)_{n \in \mathbb{N}} \sim (j_n)_{n \in \mathbb{N}} \Leftrightarrow \exists n \in \mathbb{N} \begin{cases} i_m = j_m : m < n \\ i_m < j_m : m = n \\ i_m > j_m : m > n \end{cases}$$

The relation $\sim$ induces a map $C \to [0, 1] \subset \mathbb{R}$. Observe that the equivalence classes of $\sim$ have either 1 or 2 elements. Defining $E\mathcal{G} :=\sim \subset C \times C$, $V\mathcal{G} := C$ and setting $d_0, d_1 : C \times C \to C$ the coordinate projections, turns $\mathcal{G} = V\mathcal{G} \sqcup E\mathcal{G}$ into a boolean groupoid. There is no (local) continuous section for the connected components of $\mathcal{G}$. More precisely every finite quotient of $\mathcal{G}$ is totally disconnected. Thus there does not exist an inverse system of finite groupoids $\mathcal{G}_i$ with $\mathcal{G} = \varprojlim_i \mathcal{G}_i$.

3.3 Sheaves of groups and free constructions

To unify notations of different sources we prove:

Proposition 3.3.1: *A profinite totally disconnected groupoid is a sheaf of profinite groups in the sense of [19] and conversely.*

Proof :

- $\Rightarrow$ Define $\gamma: \mathrm{Hom}_{\mathcal{G}} \to \mathrm{Ob}(\mathcal{G})$ in a canonical way. Then γ is continuous. $\mathcal{G}(x)$ is a profinite group for every $x \in \mathrm{Ob}(\mathcal{G})$ and since composition of arrows is continuous on $D(\cdot)$; therefore it is continuous in the sense of [32]. So $\{\{G(x): x \in \mathrm{Ob}(\mathcal{G})\}, \gamma, \mathrm{Ob}(\mathcal{G})\}$ is a sheaf of profinite groups.
- $\Leftarrow$ Given $(\mathcal{G}, \gamma, T)$ a sheaf of profinite groups. Define $\mathrm{Ob}(\mathcal{H}) := T$ and for all $t \in T$ set $\mathcal{H}(t) := \gamma^{-1}t$. Obviously this forms a groupoid with $D := \bigcup_{t \in T} \mathcal{H}(t) \times \mathcal{H}(t)$, when "$\cdot$" is the group multiplication then $(\mathcal{H}, \cdot)$ is a profinite groupoid.

  □

We now consider special universal objects in the category PGroupoid.

Definition 3.3.2: *Let (Γ, d_0, d_1) be an oriented graph. A profinite groupoid $F(\Gamma)$ is the* **free profinite groupoid** *on Γ if there is a graph morphism $\eta: (\Gamma, d_0, d_1) \to F(\Gamma)$ such that for every profinite groupoid $\mathcal{G}$ and morphism of graphs $\varphi: (\Gamma, d_0, d_1) \to \mathcal{G}$ there is a unique groupoid morphism $\sigma: F(\Gamma) \to \mathcal{G}$ with $\varphi = \sigma \circ \eta$.*
Let $\mathcal{G}$ be a profinite groupoid and σ a map from $V\mathcal{G}$ to a profinite space X. The profinite **universal groupoid** *$U_\sigma(\mathcal{G})$ is a profinite groupoid for which the following holds:*

1. *$X \cong VU_\sigma(\mathcal{G})$*
2. *σ extends to a unique groupoid morphism $\tilde{\sigma}: \mathcal{G} \to U_\sigma(\mathcal{G})$.*
3. *For every groupoid $\mathcal{H}$ and morphism $\tilde{\tau}: \mathcal{G} \to \mathcal{H}$ there is a unique $\tilde{\tau}': U_\sigma(\mathcal{G}) \to \mathcal{H}$ with $\tilde{\tau} = \tilde{\tau}' \circ \tilde{\sigma}$ if there is a morphism $\tau': VU_\sigma(\mathcal{G}) \to VH$ with $\tau' \circ \sigma = \tau \restriction VG$.*

The diagrams below display the universal properties of $F\Gamma$ and $U_\sigma(\mathcal{G})$.

$$\begin{tikzcd}
\Gamma \arrow[r, "\eta"'] \arrow[dr, "\varphi"] & F\Gamma \arrow[d, dotted, "\exists!\sigma"] \\
 & \mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
V\mathcal{G} \arrow[rr, "\sigma"] \arrow[dd] \arrow[dr, "\tau"] & & X \arrow[dd] \arrow[dl, dotted, "\tau'"'] \\
& V\mathcal{H} \arrow[dd] & \\
\mathcal{G} \arrow[rr, "\tilde{\sigma}"] \arrow[dr, "\tilde{\tau}"'] & & U_\sigma(\mathcal{G}) \arrow[dl, dotted, "\exists!\tilde{\tau}'"] \\
& \mathcal{H} &
\end{tikzcd}$$

A simple example is the following free groupoid. Take an oriented graph L which is a  . The free groupoid $F(L)$ in <u>Groupoid</u> computes to be $F(L) = \mathcal{G}(\mathbb{Z})$ while in <u>PGroupoid</u> one has $F(L) = \mathcal{G}(\mathbb{Z})$.

Proposition 3.3.3: *For every connected profinite graph* (Γ, d_0, d_1) *with closed set of edges* $F(\Gamma)$ *exists. For every connected groupoid and every morphism of profinite spaces* $\sigma : V\mathcal{G} \to X$ *there exists* $U_\sigma(\mathcal{G})$.

Proof : For proving the first statement we present $\Gamma = \varprojlim \Gamma_\alpha$ for finite graphs Γ_α. When $F_0(\Gamma_\alpha)$ denotes the (abstract) free groupoid in the sense of [5] chapter 8, then there is an induced inverse system of these abstract groupoids, which in turn, give rise to an inverse system of their profinite completions (compare [12] chapter 6). The completions are connected as well and their projective limit is our canditate for $F(\Gamma)$. We need to prove the universal property only for finite groupoids $\mathcal{H}$. Now one can fix α such that the map $\Gamma \to \mathcal{H}$ factors through $\Gamma \to \Gamma_\alpha$. Hence the induced groupoid morphism $F(\Gamma_\alpha) \to \mathcal{H}$ lifts to the desired groupoid morphism $F(\Gamma) \to \mathcal{H}$.

To prove the second statement factorize the equivalence relation induced by σ^{-1} on $V\mathcal{G}$. Then one obtains an oriented graph (Γ, d_0, d_1). Take $F(\Gamma)$ and the normal totally disconnected subgroupoid generated by the relation $\sim$ on Γ, where $e \sim f$ is equivalent to e, f share the same endpoints in Γ and there is an $h \in E\mathcal{G}$ such that within $\mathcal{G}$ the equality $e = h^{-1}fh$ holds. Then $F(\Gamma)/_\sim$ will serve as the universal object. $\square$

Chapter 4

Profinite Continuous Actions

We will introduce actions of profinite group(oid)s on a profinite 2-complex and Galois actions.

4.1 Definition and basic properties

Definition 4.1.1: *A* **profinite action** *is a triple* $(G,(X,o,\delta),\mu)$ *consisting of a profinite group* G*, a profinite 2-complex* (X,o,δ) *and a continuous map* $\mu:\ G\times X\to X$ *such that* $\mu(h,\mu(g,x))=\mu(hg,x)$ *and* $\delta\mu(x,g)=\{\mu(y,g):\ y\in\delta x\}$.
A **continuous groupoid action**, *or for short* **continuous action** $(\mathcal{G},(X,o,\delta),*,p)$ *is a profinite continuous groupoid* $(\mathcal{G},\cdot)$*, a profinite 2-complex* (X,o,δ)*, a map* $p:X\to V\mathcal{G}$*, a closed subset* $D(*)$ *of* $X\times E\mathcal{G}$ *and a map* $*:\ D(*)\to X$ *such that for arbitrary* $x\in X$ *and* $g\in\mathcal{G}$ *all of the following holds:*

- $D(*)=\{(x,g):\ p(x)=d_0g\}$
- $x*1_{p(x)}=x$
- $p(x*g)=d_1g$
- $(x*g)*h=x*(g\cdot h)$
- $\delta(x*g)=\{y*g:\ y\in\delta x\}$

We shall find it convenient to denote a continuous action by $(p,*)$.

Definition 4.1.2: *For a given action $(\mathcal{G},(X,o,\delta),p,*)$ we say $\mathcal{G}$ acts on X.*
$x\mathcal{G}:=\{x*g:\ g\in\mathcal{G}\}$ is the orbit and ${}_x\mathcal{G}:=\{g\in G: x*g=x\}$ the stabilizer of x.
An action is **free** *if ${}_x\mathcal{G}$ is trivial for every $x\in X$. It is* **disc free** *if it is free and $x\mathcal{G}\cap\delta y=\{x\}$ for all $y\in DX$ and $x\in\delta y$.*
A **Galois action** *is a continuous disc free action $(\mathcal{G},(X,o,\delta),p,*)$.*
The **quotient of an action** *$(p,*)$ is the quotient complex induced by the partition $\{x\mathcal{G}:x\in X\}$.*
A **connected** *or* **componentwise action** *is a continuous action such that for all $x\in X$ the orbit $x\mathcal{G}$ is contained in the connected component of x.*
A **morphism of continuous actions** *$(\mathcal{G},(X,o,\delta),p,*)\to(\mathcal{H},(Y,o',\delta'),p',*')$ is a pair (μ,η), with $\mu:\ \mathcal{G}\to\mathcal{H}$ a morphism of profinite groupoids and $\eta:\ (X,o,\delta)\to(Y,o',\delta')$ a complex morphism satisfying the following compatibility condition:*

$$\forall(x,g)\in D(*):\ \eta(x*g)=\eta x*'\mu g$$

A morphism is **surjective**, **injective**, *or* **bijective** *if both, μ and η, have these properties.*

Remarks: Any profinite action $(G,(X,o,\delta),\mu)$ can be regarded as a continuous action by setting $\mathcal{G}=\mathcal{G}(G)$, $p(x)=v$ for all $x\in X$ where $\{v\}=V\mathcal{G}$, the composition $*=\mu$ and $D(*)=X\times E\mathcal{G}$.
For each $g\in\mathcal{G}$ the map $x\mapsto x*g$ is injective because $(x*g)*g^{-1}=x*1_{p(x)}=x$.
If $x*g$ is defined than $y*g$ is defined for every $y\in\delta x$, because g is invertible and the set $\delta(x*g\cdot g^{-1})=\delta x$ is compact. Thus for every $g\in\mathcal{G}$ the preimage $p^{-1}(d_0g)$ is a closed subcomplex of (X,o,δ) and g induces a complex morphism from $p^{-1}(d_0g)$ to $p^{-1}(d_0g^{-1})$.
The equality $\delta(x*g)=\{y*g:\ y\in\delta x\}$ and the injectivity of the map $x\mapsto x*g$ imply $o(x)=o(x*g)$ for all $x\in X$.

Lemma 4.1.3: *For every connected action $(\mathcal{G},(X,o,\delta),p,*)$ the groupoid $\mathcal{G}$ is totally disconnected.*

Proof : Assume X is a connected complex. A clopen partition $\mathbb{P}$ of $\mathrm{Ob}(\mathcal{G})$ gives rise to a clopen partition $\{p^{-1}[P]:\ P\in\mathbb{P}\}$ of X. But for all $x\in X$ and $g\in p(x)$ one has $\delta(x*g)=\{y*g:\ y\in\delta x\}$ and thus $p(x)=p(y)$ for every $y\in\delta x$.

Therefore the inverse image $p^{-1}g$ of $g \in \mathrm{Ob}(\mathcal{G})$ is a subcomplex. X is thus a finite union of disjoint subcomplexes. It is not connected if $|\mathbb{P}| > 1$ since $\mathbb{P}$ has the quotient property and $X/\mathbb{P}$ is a finite set of vertices. Therefore $\mathcal{G}$ is groupoid with a single vertex.

Applying this observation to the connected components of X shows that $\mathcal{G}$ is indeed totally disconnected. $\square$

4.2 Galois actions

4.2.1 Existence of projective limits

Lemma 4.2.1: *For a Galois action $(\mathcal{G}, (X, o, \delta), p, *)$ the space $X/\mathcal{G}$ is a quotient complex.*
Moreover $\{(\mathcal{G}/\mathcal{H}, (X/\mathcal{H}, o_{\mathcal{H}}, \delta_{\mathcal{H}}), p_{\mathcal{H}}, *_{\mathcal{H}}) : \ \mathcal{H} \lhd_{\text{fin}} \mathcal{G}\}$ is an inverse system of (possibly infinite) Galois actions with projective limit $(\mathcal{G}, (X, o, \delta), p, *)$.

Proof : To prove that the quotient $X/\mathcal{G}$ is a complex set $P(x) := \{x * g : \ g \in d_0^{-1}p(x)\}$ and $\mathbb{P} := \{P(x) : \ x \in X\}$. Let o', O' and δ' be defined as in Lemma 2.3.4. $\mathbb{P}$ is a closed partition and for each $x \in X$ and each $g \in p(x)$ the discs δx and $\delta(x * g)$ are disjoint. We prove (a) and (b) of Definition 2.2.1 to hold. It suffices to show (a) for $x \in DX$. Take a big line $(L, \leq)$ such that βx is obtained from it. Since $\delta(x * g) = \{y * g : \ y \in \delta x\}$ the action of $\mathcal{G}$ on δx is completely determined by the action of $\mathcal{G}$ on x. Thus for each $y \in \delta x$ and each $g \in d_0^{-1}py$ the action $y \mapsto y * g$ induces an injective morphism $\delta x \to \delta x * g$. Therefore $\bigcup_{g \in d_0^{-1}p(x)} \delta_x * g/\mathbb{P}$ is homeomorphic to δx and thus a big circle. So (a) holds.

We prove (b). Let x be an edge with single endpoint and V a neighbourhood of x. The set $U(V) := \{y * g : \ g \in p(y), \ y \in V\}$ is open because $U(V) = \pi_X(*^{-1}(V))$ (here π_X is the projection onto X). For every given open $\mathcal{G}$-saturated set W, one can find V with $U(V) \subseteq W$. Moreover V can be chosen to belong to the base of neighbourhoods of x satisfying (b).

For $x \in D(X)$ and $|\beta x * g \cap V|$ even for all $g \in d_0^{-1}p(x)$ one concludes that $|\beta x \cap U(V)|$ is even. By assumption there does not exist a net of 2-cells $(x'_\nu)_{\nu \in \Lambda} \to x$ with odd $|\beta x'_\nu \cap V|$ for infinitely many ν. Therefore $U(V)$ is an open set containing $x\mathcal{G}$ satisfying property (b) for all $x' \in x\mathcal{G}$. Thus by (a) one concludes that (b) is true for the quotient $U(V)/\mathbb{P}$.

The canonical map $\varphi : \ X \to X/\mathbb{P}$ obviously fulfills (i),(ii),(iii) of Definition 2.2.4 and thus the first assertion of the lemma is proved.

Next let us show that every $\mathcal{H} \lhd \mathcal{G}$ induces a Galois action $\mathcal{G}/\mathcal{H}$ on $X/\mathcal{H}$. In light of Lemma 4.1.3 we can assume $\mathcal{G}$ to be a profinite group and $\mathcal{H}$ a normal subgroup. On the quotient $X/\mathcal{H}$ define $*'$:

$$x\mathcal{H} *' g\mathcal{H} := (x * g)\mathcal{H}$$

One observes that this is welldefined. Define

$$p'(x\mathcal{H}) := p(x)\mathcal{H}$$

Now routine calculation shows that $(*', p')$ is a continuous action.

Since $\mathcal{G}$ operates disc free on X so does $\mathcal{G}/\mathcal{H}$ on $X/\mathcal{H}$. Therefore $(*', p')$ is a Galois action.

Now the pair (μ, η) of quotient maps $\mu : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ and $\eta : X \to X/\mathcal{H}$ is a morphism of continuous actions, sending $(*, p)$ to $(*', p')$,

Therefore $\mathbb{P} := \{\{(\mathcal{G}/\mathcal{H}, (X/\mathcal{H}, o_{\mathcal{H}}, \delta_{\mathcal{H}}), p_{\mathcal{H}\mathcal{H}'}, *_{\mathcal{H}\mathcal{H}'}) : \mathcal{H} \lhd_{\text{fin}} \mathcal{G}\}, (\mu_{\mathcal{H}}, \eta_{\mathcal{H}})\}$ with $\mu_{\mathcal{H}\mathcal{H}'}$, $\eta_{\mathcal{H}\mathcal{H}'}$ are the morphisms, defined only for $\mathcal{H}' \lhd \mathcal{H}$, is a projective system. By Proposition 3.2.6 $\mathcal{G}$ is the projective limit of an inverse system of finite groupoids. For each of these finite groupoids $\mathcal{K}$ there is an $\mathcal{H} \lhd_{\text{fin}} \mathcal{G}$ and a surjective morphism $\mathcal{G}/\mathcal{H} \to \mathcal{K}$. Therefore $\mathcal{G} = \varprojlim_{\mathcal{H} \lhd_{\text{fin}} \mathcal{G}} \mathcal{G}/\mathcal{H}$ and consequently $X = \varprojlim_{\mathcal{H} \lhd_{\text{fin}} \mathcal{G}} X/\mathcal{H}$. It then turns out that$(*, p) = \varprojlim \mathbb{P}$.

□

Lemma 4.2.2: *Let $(\mathcal{G}, (X, o, \delta), *, p)$ be a Galois action with $\mathcal{G}$ finite then it is the projective limit of an inverse system of Galois actions $(\mathcal{G}, (Y_i, o_i, \delta_i), *_i, p_i)$ with each Y_i finite.*

Proof : Let $\mathbb{P}$ be a clopen partion of X. We want for every $g \in E\mathcal{G}$ and $P \in \mathbb{P}$ that $p * g$ is defined either for all or for no $p \in P$. The set $\{p^{-1}(g) : g \in V\mathcal{G}\}$ is a clopen partition of X as well and we replace $\mathbb{P}$ by the common refinement. Thus w.l.o.g. we assume that $V\mathcal{G}$ is a singleton set. We pass to a partition $\mathbb{P}'$ that arises by further refinement of $\mathbb{P}$ and $\{P * g : P \in \mathbb{P}, g \in E\mathcal{G}\}$.

There is a maximal subset M of $\mathbb{P}'$ such that $M * g \cap M = \emptyset$ for all $g \in E\mathcal{G}$. Therefore $g \neq h$ implies $M * g \cap M * h = \emptyset$ and $\bigcup_{g \in E\mathcal{G}, P \in M} P * g = X$. Set $Y = \bigcup\{P \in M\}$ and define $\tilde{\delta} : Y \to \mathrm{F}Y$ by $\tilde{\delta} y = \{x * g \in Y : x \in \delta y, g \in E\mathcal{G}\}$. Observe that $(X/\mathcal{G}, o/\mathcal{G}, \delta/\mathcal{G}) = (Y, o, \tilde{\delta})$ and let $p : (X, o, \delta) \to (Y, o, \tilde{\delta})$ be the canonical projection. By Lemma 4.2.1 $(Y, o, \tilde{\delta})$ is a profinite 2-complex. Therefore there is a refinement $\mathbb{Q}$ of $\{p(P) : P \in \mathbb{P}\}$ which has the quotient property with respect to $\tilde{\delta}$. Lifting $\mathbb{Q}$ to X yields a clopen partition $\mathbb{Q}' := \{p^{-1}Q : Q \in \mathbb{Q}\}$. Let $Q_1 \in \mathbb{Q}$. Observe that for $\tilde{\delta}' Q_1 = \{Q_1, \ldots, Q_n\}$ there are $g_2, \dots, g_n \in E\mathcal{G}$ such that $\delta' Q_1 = \{Q_1, Q_2 * g_2, \dots, Q_n * g_n\}$. Therefore it is immediate that $\mathbb{Q}'$ is a $\mathcal{G}$-invariant clopen partition with the quotient property. $\square$

Lemma 4.2.3: *A Galois action $(\mathcal{G}, (X, o, \delta), *, p)$ is the projective limit of finite Galois actions $(\mathcal{G}_i, (X_i, o_i, \delta_i), *_i, p_i)$*

Proof : By Lemma 4.2.1 $(\mathcal{G}, (X, o, \delta), *, p) = \varprojlim_{\mathcal{H} \lhd_{\text{fin}} \mathcal{G}} (\mathcal{G}/\mathcal{H}, (X/\mathcal{H}, o_{\mathcal{H}}, \delta_{\mathcal{H}}), *_{\mathcal{H}}, p_{\mathcal{H}})$.

By Lemma 4.2.2 each $(\mathcal{G}/\mathcal{H}, (X/\mathcal{H}, o_{\mathcal{H}}, \delta_{\mathcal{H}})$ is the projective limit of an inverse system of finite Galois actions $(\mathcal{G}/\mathcal{H}, (Y^{\mathcal{H}}_\alpha, o^{\mathcal{H}}_\alpha, \delta^{\mathcal{H}}_\alpha), *^{\mathcal{H}}_\alpha, p^{\mathcal{H}}_\alpha)$.

For any $\mathcal{H}' \lhd \mathcal{H}$ every $(\mathcal{G}/\mathcal{H}, (Y^{\mathcal{H}}_\alpha, o^{\mathcal{H}}_\alpha, \delta^{\mathcal{H}}_\alpha), *^{\mathcal{H}}_\alpha, p^{\mathcal{H}}_\alpha)$ is a finite quotient of $(\mathcal{G}/\mathcal{H}', (X/\mathcal{H}', o_{H'}, \delta_{H'}), *_{\mathcal{H}'}, p_{\mathcal{H}'})$. Therefore there exists α' and a surjective morphism of Galois actions $(\eta_{\alpha'\alpha}, \varphi_{\alpha'\alpha}) : (*^{\mathcal{H}'}_{\alpha'}, p^{\mathcal{H}'}_{\alpha'}) \to (*^{\mathcal{H}}_\alpha, p^{\mathcal{H}}_\alpha)$. Invoking the universal property of projective limits yields a natural isomorphism of the Galois actions $(\mathcal{G}', (Y, o', \delta'), *', p') = \varprojlim_{\mathcal{H} \lhd \mathcal{G},\ \alpha_{\mathcal{H}}} (*^{\mathcal{H}}_{\alpha_{\mathcal{H}}}, p^{\mathcal{H}}_{\alpha_{\mathcal{H}}})$ and $(\mathcal{G}, (X, o, \delta), *, p)$. The subsequent diagram visualizes the situation. The dotted arrows result from the universal properties of projective limits, the dashed ones are the surjective morphisms constructed above.

$$
\begin{tikzcd}
(*, p) \arrow[r, bend left] \arrow[rr, bend left] \arrow[d, dotted, shift right, "\cong"'] & (*_{\mathcal{H}'}, p_{\mathcal{H}'}) \arrow[r] \arrow[d] \arrow[dd, bend right] & (*_{\mathcal{H}}, p_{\mathcal{H}}) \arrow[d] \arrow[dd, bend left] \\
(*', p') \arrow[u, dotted, shift right] \arrow[ur, dotted] \arrow[urr, dotted, bend left] \arrow[r, bend left] \arrow[rr, bend left] \arrow[dr] \arrow[drr, bend right] & (*^{\mathcal{H}'}_{\alpha'}, p^{\mathcal{H}'}_{\alpha'}) \arrow[r, dashed] \arrow[d] & (*^{\mathcal{H}}_{\alpha}, p^{\mathcal{H}'}_{\alpha}) \arrow[d] \\
 & (*^{\mathcal{H}'}_{\beta'}, p^{\mathcal{H}'}_{\beta'}) \arrow[r, dashed] & (*^{\mathcal{H}}_{\beta}, p^{\mathcal{H}}_{\beta})
\end{tikzcd}
$$

$\square$

Proposition 4.2.4: <u>GalAct</u> *is closed under forming projective limits.*

Proof : Let $(\mathcal{G}_\alpha, (X_\alpha, o_\alpha, \delta_\alpha), *_\alpha, p_\alpha)$ be an inverse system of Galois actions.

Lemma 2.3.7 and Proposition 3.2.1 show that $(X, o, \delta) = \varprojlim(X_\alpha, o_\alpha, \delta_\alpha)$ and $\mathcal{G} = \varprojlim \mathcal{G}_\alpha$ exist. We define an action of $\mathcal{G}$ on (X, o, δ). The map $p := \varprojlim p_\alpha$, and the map $* : D(*) \to X$ are given by projective limits. It is not hard to verify the conditions in Definition 4.1.1. In particular the action is disc free because for every non trivial $g \in \mathrm{E}\mathcal{G}$ there is a finite quotient where it is disc free. □

4.3 Universal constructions

In the next section the existence of pullbacks of profinite actions is needed. Thus we construct the product of two profinite actions and the equalizer of two action morphisms.

Construction 4.3.1: *We define the product of continuous actions* $(\mathcal{G}, (X, o, \delta), *, p)$ *and* $(\mathcal{H}, (Y, o', \delta'), *', p')$ *provided that the product of the complexes* X *and* Y *exists (as shown in Construction 2.4.3 this is the case for* X *and* Y *homogeneously finitely bounded.).*
Form $\mathcal{G} \times \mathcal{H}$ *and let it act coordinatewise on* $(X, o, \delta) \times (Y, o', \delta')$.

Proof : The product $(X, o, \delta) \times (Y, o', \delta')$ is a subset of $X \times Y \times \mathrm{F}(X \times Y)$. Define the map $p \times p' : (X, o, \delta) \times (Y, o', \delta') \to \mathcal{G} \times \mathcal{H}$ by $p \times p' := (p \circ \pi_X, p' \circ \pi_Y)$ and a candidate "$\diamond$" for the action by

$$\begin{aligned} (m, C) \diamond (g, h) &= ((x, y), \{(x', y') \in C\}) \diamond (g, h) \\ &:= ((x * g, y *' h), \{(x' * g, y' *' h) : (x', y') \in C\}). \end{aligned}$$

Because $x' \in \delta x$ and $y' \in \delta' y$, the map $\diamond$ is well defined. Now we prove for $\mathcal{H}, (X, o, \delta) \times (Y, o', \delta'), \diamond, p \times p')$ the axioms of beeing a continuous action (Definition 4.1.1):

- By construction it follows that $(m, C) \diamond (g, h) = ((x, y), C) \diamond (g, h)$ is defined whenever $p \times p'(m, C) = p \times p'((x, y), C) = (p(x), p'(y)) = (d_0 g, d_0 h) = d_0(g, h)$.
- $(m, C) \diamond 1_{p \times p'(m,C)} = ((x, y), C) \diamond (1_{p(x)}, 1_{p(y)}) = ((x, y), C)$ as desired.
- $p \times p'((m, C) \diamond (g, h)) = p \times p'(((x, y), C) \diamond (g, h)) = (p(x * g), p'(y * h)) = (d_1 g, d_1 h) = d_1(g, h)$
- As before one checks $((m, C) \diamond (g, h)) \diamond (g', h') = (m, C) \diamond (gg', hh') = (m, C) \diamond ((g, h) \cdot (g', h'))$.

- Set $(Z, o_\pi, \delta_\pi) = (X, o, \delta) \times (Y, o', \delta')$. We want to show that $\delta_\pi((m, C) \diamond (g, h)) = \{(m', C') \diamond (g, h) : (m', C') \in \delta_\pi(m, C)\}$. For this purpose it suffices to show that the elements $(m', C') \diamond (g, h)$ belong to Z. For every disc x in the product we have to verify the properties of the list in Construction 1.3.3 for βx. Items 1–4 hold since they hold coordinatewise. Items 5–6 hold because (g, h) induces a continous bijection from C onto $\{(x' * g, y' * h) : (x', y') \in C\}$.

$\square$

Construction 4.3.2: *The equalizer*

$$(\mathcal{K}, (E, o_e, \delta_e), *_e, p_e) \overset{(\psi,\varphi)}{\rightarrow} (\mathcal{G}, (X, o, \delta), *, p) \underset{(\mu,\eta)}{\overset{(\mu',\eta')}{\rightrightarrows}} (\mathcal{H}, (Y, o', \delta'), *', p')$$

of two morphisms of actions (μ, η), (μ', η'), *denoted by* $E((\mu, \eta), (\mu', \eta'))$, *is given by the equalizer in* <u>PGroupoid</u>

$$\mathcal{K} \overset{\psi}{\rightarrow} \mathcal{G} \underset{\mu}{\overset{\mu'}{\rightrightarrows}} \mathcal{H}$$

together with the equalizer in <u>p-2-Comp</u>

$$(E, o_e, \delta_e) \overset{\varphi}{\rightarrow} (X, o, \delta) \underset{\eta}{\overset{\eta'}{\rightrightarrows}} (Y, o', \delta').$$

Proof : Observe that if $v \in V\mathcal{G} \setminus \psi V\mathcal{K}$ then one has $\eta y \neq \eta' y$ for all $y \in p^{-1}(v)$. Therefore $E \cap p^{-1}V(\varphi\mathcal{K}) = E$. Since φ and ψ are embeddings let us identify E and $\mathcal{K}$ with their respective images. Define $*_e := * \restriction D(*) \cap E \times E\mathcal{K}$ and p_e to be the respective restriction of p to E. Observe that E consists exactly of those elements z for which $\eta z = \eta' z$ and $p(z) \in \mathrm{Ob}(\mathcal{K})$. This yields $\eta(z * k) = \eta z *' \mu k = \eta' z *' \mu' k = \eta'(z * k)$ and therefore E is invariant under the action of $\mathcal{K}$. Thus $(\mathcal{K}, (E, o_e, \delta_e), *_e, p_e)$ is an action such that (μ, η) and (μ', η') agree on $(\mathcal{K}, (Z, o_e, \delta_e), *_e, p_e))$. Thus $(\mu', \eta') \circ (\psi, \varphi) = (\mu, \eta) \circ (\psi, \varphi)$.

We prove the universal property of equalizers for $E((\mu, \eta), (\mu', \eta'))$. Let $(\psi', \varphi') : (\mathcal{K}', (Z', o'_e, \delta'_e), *'_e, p'_e) \rightarrow (\mathcal{G}, (X, o, \delta), *, p)$ be a morphism of actions satisfying $(\mu \circ \psi', \eta \circ \varphi') = (\eta' \circ \varphi', \mu' \circ \psi')$. Then $\eta \circ \varphi' = \eta' \circ \varphi'$ and $\mu \circ \psi' = \mu' \circ \psi'$ hold and therefore conclude that $\mathrm{im}(\varphi') \subseteq E$ and $V\mathcal{K} = V\mathcal{K}'$. If there is a $k \in E\mathcal{K}'$ with $\mu k' = \mu' k'$ then $k' \in \mathcal{K}$. Thus $\mathcal{K}'$ is contained in $\mathcal{K}$. $\square$

Corollary 4.3.3: *Consider actions with homogeneously finitely bounded complexes. Then every diagram*

$$\begin{tikzcd} & (\tilde{*},\tilde{p}) \arrow[d, "(\tilde{\mu},\tilde{\eta})"] \\ (*',p') \arrow[r, "(\mu,\eta)"] & (*,p) \end{tikzcd} \quad \text{admits a pullback} \quad \begin{tikzcd} (*_e,p_e) \arrow[r, dotted, "(\tilde{\mu}_e,\tilde{\eta}_e)"] \arrow[d, dotted, "(\mu'_e,\eta'_e)"'] & (\tilde{*},\tilde{p}) \arrow[d, "(\tilde{\mu},\tilde{\eta})"] \\ (*',p') \arrow[r, "(\mu,\eta)"] & (*,p) \end{tikzcd}$$

In concrete terms, $(*_e,p_e)$ is the equalizer

$$(*_e,p_e) \overset{(\psi,\varphi)}{\to} (*',p') \times (\tilde{*},\tilde{p}) \underset{(\mu,\eta)\circ\pi_{(*',p')}}{\overset{(\tilde{\mu},\tilde{\eta})\circ\pi_{(\tilde{*},\tilde{p})}}{\rightrightarrows}} (*,p),$$

where $\pi_{(*',p')}$ and $\pi_{(\tilde{*},\tilde{p}')}$ are the canonical projections from the product of actions onto its factors.

Proposition 4.3.4: *Let (μ,η) and (μ',η') be morphisms of continuous (Galois) actions*

$$(\mathcal{G},(X,o,\delta),*,p) \underset{(\mu,\eta)}{\overset{(\mu',\eta')}{\rightrightarrows}} (\mathcal{H},(Y,o',\delta'),*',p').$$

There exists a coequalizer $C((\mu,\eta),(\mu',\eta'))$ in <u>ContAct</u> or <u>GalAct</u>. When considering continuous actions the groupoid $\mathcal{K}$ of the coequalizer $C((\mu,\eta),(\mu',\eta'))$ agrees with the coequalizer $C(\mu,\mu')$ in <u>PGroupoid</u>. This can fail to happen if the coequalizer is constructed in <u>GalAct</u>. (Example 4.3.6). Example 4.3.5 shows that the underlying 2-complex of $C((\mu,\eta),(\mu',\eta'))$ does not necessarily equals the coequalizer $C(\eta,\eta')$.

Proof : The proof is an extension of the proof of Proposition 2.4.8. Let Λ denote the set of all morphisms $(\gamma,\lambda): (*',p') \to (\mathcal{G}_l,(X_l,o_l,\delta_l),*_l,p_l)$ to finite (Galois) actions $(*_l,p_l)$ such that $(\gamma,\lambda)\circ(\mu,\eta) = (\gamma,\lambda)\circ(\mu',\eta')$. We devise an order "$\leq$" on Λ by setting $(\gamma,\lambda) \leq (\gamma',\lambda')$ if and only if there are $\psi_{\gamma\gamma'}$ and $\varphi_{\lambda\lambda'}$ with $\psi_{\gamma\gamma'}\gamma' = \gamma$ and $\varphi_{\lambda\lambda'}\lambda' = \lambda$. The set Λ is not empty since our category has a terminal object (a single point on which the groupoid acts). We claim that $(\Lambda,\leq)$ is a directed set. Pick (γ,λ), (γ',λ'). Using Construction 4.3.1 we can form the product action $(*_l,p_l)\times(*_{l'},p_{l'})$ as indicated in the diagram below

$$\begin{tikzcd}
& & (*_l, p_l) \\
(*, p) \arrow[r, bend left, "(\mu,\eta)"] \arrow[r, bend right, "(\mu',\eta')"'] & (*', p') \arrow[ur, "(\gamma,\lambda)"] \arrow[dr, "(\gamma',\lambda')"'] \arrow[r, dotted, "(\tau,\sigma)"] & (*_l, p_l) \times (*_{l'}, p_{l'}) \arrow[u] \arrow[d] \\
& & (*_{l'}, p_{l'})
\end{tikzcd}$$

The universal property of the product yields the dotted arrow (τ, σ). Take the action generated by the image of τ and σ. Proposition 4.3.7 will show that $(*_l, p_l) \times (*_{l'}, p_{l'})$ is Galois if the two factors are. One checks that $(\tau, \sigma) \circ (\mu, \eta) = (\tau, \sigma) \circ (\mu', \eta')$ and thus (τ, σ) belongs to Λ. Moreover $(\tau, \sigma) \geq (\gamma, \lambda)$ and $(\tau, \sigma) \geq (\gamma', \lambda')$. The projective limit of the above inverse system provides us with a morphism (κ, ρ) from $(*', p')$ to a continuous action, say $(\ast, q)$.

We claim that $(\ast, q) = C((\mu, \eta), (\mu', \eta'))$. Take a morphism to a finite quotient $(\gamma_\alpha, \lambda_\alpha) : C((\mu, \eta), (\mu', \eta')) \to (\ast_\alpha, \lambda_\alpha)$. Observe that $(\gamma_\alpha, \lambda_\alpha) \circ (\tau', \sigma') \in \Lambda$, where (τ, σ) is the universal morphism of the coequalizer. Therefore there is a morphism from $(\ast, q) \to C((\mu, \eta), (\mu', \eta'))$ which is obviously surjective and thus the claimed equality $(\ast, q) = C((\mu, \eta), (\mu', \eta'))$ holds.

We observe that the groupoid of $C((\mu, \eta), (\mu', \eta'))$ is gernerated by $\mathrm{im}(\tau)$, and thus cannot be bigger than $\mathcal{K}(\mu, \mu')$. To show that, if considering continuous actions in general, it cannot be smaller, we construct an action of $\mathcal{K}(\mu, \mu')$ on the equalizer $C(\eta, \eta')$, such that the action is the surjective image of $(*, p)$. The universal property of the coequalizer then shows that $\mathcal{K}(\mu, \mu')$ equals the groupoid of $C((\mu, \eta), (\mu', \eta'))$.

Let $\tau' : \mathcal{H} \to \mathcal{K}(\mu, \mu')$, $\sigma' : Y \to C(\eta, \eta')$ be the universal morphisms of groupoid- and complex coequalizers respectively.

Set

$$c \diamond k := \bigcup \sigma' \circ \eta \left[D(*) \cap (\eta^{-1} \sigma'^{-1} c \times \mu^{-1} \tau'^{-1} k) \right]$$

and $D(\diamond) = \{(c, k) : c \diamond k \neq \emptyset\}$ Note that $\diamond$ is a well defined map from $D(\diamond) \to C(\eta, \eta')$. Define

$$q'(c) := \tau' p' \sigma'^{-1} c.$$

Then a short calculation shows that $(\mathcal{K}(\mu,\mu'), C(\eta,\eta'), \diamond, q')$ is a continuous action. Let $y = \eta x$ and $h = \mu g$ then

$$\sigma'(y *' h) = \sigma'\eta\eta^{-1}(y *' h) = \sigma'\eta(x * g) = \sigma'\eta x \diamond \tau'\mu g = \sigma' y \diamond \tau' h$$

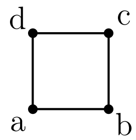
Thus (τ',σ') is a morphism of actions having the property $(\tau',\sigma')\circ(\mu,\eta) = (\tau',\sigma')\circ(\mu',\eta')$.

The pair τ',σ' turns out to be a surjective morphism of actions from $(\diamond, q') \to (\ast, q)$. $\square$

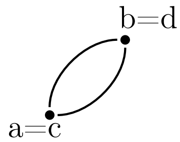
Example 4.3.5: Let $\Gamma(\hat{\mathbb{Z}}_p)$ be the oriented Cayley graph of $\hat{\mathbb{Z}}_p$ interpreted as a 1-complex. Take the complex $\Gamma(\hat{\mathbb{Z}}_p) \sqcup \{z\} \sqcup \bullet\!\!-\!\!\bullet$ where z is a vertex. Let $\mathcal{G}(d_0, d_1)$ be the groupoid consisting of three discrete vertices v, z', w such that $d_0^{-1}v = \hat{\mathbb{Z}}_p$ and $d_0^{-1}z' = 1_{z'}$, $d_0^{-1}w = 1_w$. There is a natural profinite action $(\mathcal{G}, \Gamma(\hat{\mathbb{Z}}_p), *, p)$ with $p\Gamma(\hat{\mathbb{Z}}_p) = \{v\}$, $pz = z'$ and $p(\bullet\!\!-\!\!\bullet) = \{w\}$.
Let $a \in V\Gamma(\hat{\mathbb{Z}}_p)$ and $b \in V(\bullet\!\!-\!\!\bullet)$. We define two morphisms of actions $\partial_0, \partial_1 : (*, p) \to (*, p)$ such that $\partial_0 = (\mu, \eta)$ and $\partial_1 = (\mu', \eta')$. The morphisms μ, η, μ', η' restricted to $d_0^{-1}\{v, w\}$ and $\Gamma(\hat{\mathbb{Z}}_p) \sqcup \bullet\!\!-\!\!\bullet$ are the identities while $\mu 1_{z'} = 1_v$, $\mu' 1_{z'} = 1_w$ and $\eta z = a$, $\eta' z = b$. The morphisms ∂_0, ∂_1 turn out to be morphisms of continuous actions.
Let us construct $C(\partial_0, \partial_1)$. It is immediate that $C(\mu, \mu') = \hat{\mathbb{Z}}_p$ and $C(\eta, \eta') = \Gamma(\hat{\mathbb{Z}}_p) \sqcup_{a=b} \bullet\!\!-\!\!\bullet$. Observe that every continuous action on $C(\eta, \eta')$ has to stabilize a. Therefore $C(\mu, \mu')$ only acts trivially on $C(\eta, \eta')$. Denote this action by $(\ast, q)$.
We show that this action cannot be the coequalizer $C(\partial_0, \partial_1)$ by constructing an action $(\ast', q')$ and a morphism of actions from $(*, p) \to (\ast', q')$ which does not factor through $(\ast, q)$.
Set $Y := G \sqcup E \sqcup V := \Gamma(\hat{\mathbb{Z}}_p) \sqcup V\Gamma(\hat{\mathbb{Z}}_p) \sqcup V\Gamma(\hat{\mathbb{Z}}_p)$. Let $o \upharpoonright G \sqcup V$ be inherited from the dimension maps on $\Gamma(\hat{\mathbb{Z}}_p)$, and set $o(E) = \{1\}$. For every e in E define $\delta e := \{g, e, v\}$ with $g \in G, v \in V$ all representing the same vertex in $\Gamma(\hat{\mathbb{Z}}_p)$. Obviously (Y, o, δ) is a profinite 2-complex and there is a natural continuous action $(\mathcal{G}(\hat{\mathbb{Z}}_p), (Y, o, \delta), \ast', q')$ coinciding with the inherited actions of $\hat{\mathbb{Z}}_p$ on G, E, V.
Observe that every morphism of actions $(\ast, q) \to (\ast', q')$ maps $\hat{\mathbb{Z}}_p$ to 1 but the morphism which embeds $C(\eta, \eta')$ into Y naturally extends to a morphism of actions (σ', τ') such that the image of σ' is not trivial.

Remark that it is possible to show that $C(\partial_0, \partial_1) = (\ast', q')$.

Example 4.3.6: *Let (X, o, δ) be the complex indicated by*



and let $\mathcal{G} = \mathcal{G}(C_2)$, where C_2 is the cyclic group with 2 elements. Define an operation of $\mathcal{G}$ on X rotating the square such that $a \to c$ and $b \to d$. This operation yields a disc free continuous action $(*, p)$ of $\mathcal{G}$ on X. Now let $r(x) := x * g$ where g is the not trivial element of C_2. Observe $(id_{\mathcal{G}}, r)$ and $(id_{\mathcal{G}}, id_X)$ are automorphisms of $(*, p)$. Their coequalizer consists of the complex (C, o', δ') indicated by



and the groupoid $\mathcal{G}(C_2)$ which does not operate disc free on C.
It turns out that the coequalizer $C((r, id_{\mathcal{G}}), (id_X, id_{\mathcal{G}}))$ in GalAct is the trivial groupoid $\mathcal{G}(1)$ acting on C.

In general forming the coequalizer for two morphisms in GalAct and in ContAct yields different underlying complexes.

Proposition 4.3.7: *Let $(*', p')$ and $(\tilde{*}, \tilde{p})$ be continuous actions with morphisms $(\mu, \eta), (\mu', \eta') : (*', p') \to (*, p)$.*
Let $(\ast, q)$ be any of either the equalizer or the product. When the given actions are (disc) free so is $(\ast, q)$.
If $(\ast, q)$ is the product then the universal morphisms are surjective.
Let $(\ast, q)$ be the pullback action of the diagram in Corollary 4.3.3. (μ'_e, η'_e) is surjective provided (μ', η') is and $(\tilde{\mu}_e, \tilde{\eta}_e)$ is surjective if $(\tilde{\mu}, \tilde{\eta})$ is. If $\tilde{\eta}$ maps different connected components to different connected components so does η'_e. If $(\tilde{\mu}, \tilde{\eta})$ is a quotient of actions so is $(\mu'_e, \tilde{\eta}'_e)$.

Proof : Let $(\ast, q)$ be the product of $(*, p)$ and $(*', p')$. Observe that the action $(\ast, q)$ is defined component-wise:

$$\begin{aligned}(m, C)\ast(g, h) &= ((x, y), \{(x', y') \in C\})\ast(g, h) \\ &:= ((x * g, y *' h), \{(x' * g, y' *' h) : (x', y') \in C\}).\end{aligned}$$

and thus it is (disc) free if both $(*, p)$ and $(*', p')$ are. The complex morphisms $p_X : ((x, y), C) \mapsto x$ and $p_Y : ((x, y), C) \mapsto Y$ are surjective and the coordinate projections from a product of profinite groupoids onto its components are surjective. Thus the composite morphisms are surjective.

If $(\ast, q)$ is the equalizer of (η, μ) and (η', μ') then it embeds into $(*', p')$ and is thus a (disc) free action if $(*', p')$ is. As a consequence the pullback of (disc) free actions is disc free.

Let finally $(\ast, q)$ be the pullback of the diagram in Corollary 4.3.3. Let $(\mathcal{K}, (Z, o_e, \delta_e), *_e, p_e)$, $(\mathcal{H}, (Y, o', \delta'), *', p')$, $(\tilde{\mathcal{G}}, (\tilde{X}, \tilde{o}, \tilde{\delta}), \tilde{*}, \tilde{p})$ and $(\mathcal{G}, (X, o, \delta), *, p)$ be the data of the diagram. Assume (μ', η') is surjective. We show that (μ'_e, η'_e) is surjective. Take $\tilde{g} \in E\tilde{\mathcal{G}}$ and $\tilde{x} \in \tilde{X}$. Let $g = \tilde{\mu}\tilde{g}$ and $x = \tilde{\eta}\tilde{x}$. Then there is some $h \in E\mathcal{H}$, $y \in Y$ with $\mu' h = g$ and $\eta' y = x$. Obviously $(g', \tilde{g}) \in \mathcal{K}$. It is not hard to construct at least one $C \in \mathrm{F}Z$ such that $((y, \tilde{x}), C) \in Z$. This shows surjectivity of (μ'_e, η'_e).

Similarly one concludes that the surjectivity of $(\tilde{\mu}, \tilde{\eta})$ implies the surjectivity of $(\tilde{\mu}_e, \tilde{\eta}_e)$.

We prove that if $\tilde{\eta}$ respects components so does $\tilde{\eta}_e$. Let $((a', \tilde{a}), C)$, $((b', \tilde{b}), D)$ be elements of Z. Assume $a' = b'$. By the property of the equalizer we have $\eta' a' = \tilde{\eta}\tilde{a}$ and $\eta b' = \tilde{\eta}\tilde{b}$. Therefore $\eta\tilde{a} = \eta\tilde{b}$ follows and thus $\tilde{a}$ and $\tilde{b}$ are in the same component of $\tilde{X}$. Moreover, if a', b' lie in the same component of Y, then their images in X belong to the same component. Let A', $\tilde{A}$ are sets of representatives of components of Y and $\tilde{X}$ such that their images in X coincide. For every $a' \in A'$ there is a unique $\tilde{a} \in \tilde{A}$ with $((a', \tilde{a}), C) \in Z$. Therefore we find η'_e to respect components.

We now prove that if $(\tilde{\mu}, \tilde{\eta})$ is a quotient morphism so is $(\tilde{\mu}_e, \tilde{\eta}_e)$. First we show that for given $\tilde{x} \in \tilde{X}$ and $y \in Y$ with $\eta' y = \tilde{\eta}\tilde{x}$ one has exactly one C in $\mathrm{F}Z$ such that $((y, \tilde{x}), C) \in Z$. If $\tilde{x}$ is a vertex the claim is true. If $\tilde{x}$ is an edge then $C = \{(y', \tilde{x}') : \tilde{x}' \in \tilde{\beta}\tilde{x},\ y' \in \eta'^{-1}\tilde{\eta}\tilde{x}'\} \cup \{(y', \tilde{x}) : y' \in \eta'^{-1}\tilde{\eta}\tilde{x}\}$. Because $\tilde{\eta}$ is injective on $\tilde{\delta}\tilde{x}$ one checks that C is cannot be smaller and thus the claim is true. If $\tilde{x}$ is a 2-cell the proof works similarly.

Therefore $\tilde{\eta}_e : ((y, \tilde{x}), C) \mapsto y$ is the same as factorizing Z modulo $\mathcal{G}$ because it respects components.

□

Chapter 5

Based (continuous) actions

Our definition of Galois actions requires to consider based actions so that an abstract analog of unique "homotopy lifting property" can be achieved for morphisms.

5.1 Definitions and basic properties

Definition 5.1.1: *Let A be a subset of vertices, representatives of the connected components of a complex (X, o, δ). Then (X, o, δ) is a* **based complex** *with* **base** *A. Denote it by $((X, A), o, \delta)$. A* **based continuous action**, *denoted by $(\mathcal{G}, ((X, A), o, \delta), *, p)$, is a continuous action $(\mathcal{G}, (X, o, \delta), *, p)$ such that A is a base for X and p restricted to A is injective.*
We use $(p, *, A)$ as shorthand.
A morphism $(\mu, \eta) : (\mathcal{G}, ((X, A), o, \delta), *, p) \to (\mathcal{H}, ((Y, A'), o', \delta'), *', p')$ is **based** *if $\eta A \subseteq A'$.*
(μ, η) is **A-injective** *if η is injective on A.*

Definition 5.1.2: *A* **based Galois action** *$(\mathcal{G}, ((X, A), o, \delta), *, p)$ is a disc free based continuous action with A is closed.*

There is a disc free based continuous action with A not closed.

Example 5.1.3: *Take the space X of [25] example 5.6.9 on which C_2 operates such that there is no section for the quotient morphism $x \mapsto \bar{x}$. Take 2 copies X_v and X_e of X and build $Y := X_v \sqcup_{x_0} X_e$ where x_0 is the fixed point of the action of C_2. Define $\delta : Y \to FY$ by setting*

$$\delta x := \begin{cases} \{x\} & : x \in X_v \\ \{x\} \cup \bar{x} & : x \in X_e \end{cases} .$$

Obviously δx is well defined and continuous. One sets $o : Y \to \mathbb{N}$ by $o(X_v) = 0$ and $o(X_e) = 1$. Thus (Y, o, δ) is a profinite 1-complex. Define the groupoid $\mathcal{G}$ by setting $V\mathcal{G} := X/C_2$, $\mathcal{G}(x_0)$ is trivial and $\mathcal{G}(x) = C_2$ else. Observe that there is a natural continuous action of $\mathcal{G}$ on X by setting $px = \bar{x}$ and the induced action "$*$" of C_2. The action $(\mathcal{G}, ((Y, A), o, \delta), *, p)$ is a disc free based action if one selects A appropriately. Since there is no continuous section for the action of C_2 the base A cannot be chosen to be closed.

We provide an analog of Proposition 1.4 in [34].

Proposition 5.1.4: *Suppose we are given a commutative diagram of Galois actions*

$$\begin{tikzcd}
(\ast, p, A) \arrow[r, shift left, "(\mu,\eta)"] \arrow[r, shift right, "(\mu',\eta')"'] \arrow[dr, "(\nu,\sigma)"'] & (\tilde{\ast}, \tilde{q}, \tilde{B}) \arrow[d, "(\tau,\pi)"', "A-\text{injective}"] \\
 & (\ast, q, B)
\end{tikzcd}$$

and the complex morphisms η and η' agree on A. Then $(\mu, \eta) = (\mu', \eta')$.

Proof : We introduce full notation for the continuous actions

$$(\mathcal{G}, ((X, A), o, \delta), \ast, p)$$
$$(\tilde{\mathcal{H}}, ((\tilde{Y}, \tilde{B}), \tilde{o}', \tilde{\delta}'), \tilde{\ast}, \tilde{q})$$

If the theorem holds for $(\tilde{\ast}, \tilde{q}, \tilde{B})$ finite a projective limit argument allows us to find an inverse system of A-injective morphisms $(\tau_\alpha, \pi_\alpha)$ such that (τ, π) is its projective limit. Let p_α be a shorthand for the projection from $(\tilde{\ast}, \tilde{q}, \tilde{B})$ onto a finite quotient. If the Proposition is shown for the finite case, one has $p_\alpha \circ (\mu, \eta) = p_\alpha \circ (\mu', \eta')$ and a projective limit argument yields that $(\mu, \eta) = (\mu', \eta')$.

Therefore let us assume Y is finite. Because of the A-injectivity of (τ, π) further assume that Y is connected.

First we show $\eta = \eta'$. For the proof of this statement it is enough to consider an arbitrary connected component C of X. Assume that $\eta \neq \eta'$ on C. The morphisms η and η' coincide on A and thus the equalizer $E(\eta, \eta')$ intersected with C is a subcomplex of C which is not empty. By the finiteness of Y the sets $\{\eta^{-1}y : \ y \in Y\}$ and $\{\eta'^{-1}y : \ y \in Y\}$ are clopen partitions of C and thus their common refinement is a clopen partition $\mathbb{P}$ of C. Each $P \in \mathbb{P}$ has the property that either $\eta P = \eta' P$ or $\eta P \cap \eta' P = \emptyset$ holds. Hence the equalizer is a finite union of clopen sets and thus clopen itself. Because C is connected and $E(\eta, \eta')$ is a subcomplex we can find an edge or a 2-cell $x \in C$ such that that $\eta x \neq \eta' x$ and $\eta v = \eta' v$ for some vertex v contained in $\beta x \cap E(\eta, \eta')$. Since $\pi\eta = \sigma = \pi\eta'$ the images ηx and $\eta' x$ lie in the same orbit of $\tilde{\mathcal{H}}$, i.e. there is a $h \in E\tilde{H}$ with $\eta x = \eta' x \tilde{*} h$. The equations $\eta v = \eta' v$ and $\tilde{\delta} x \tilde{*} h = \{y \tilde{*} h : \ y \in \delta x\}$ yield that $\eta v \tilde{*} h = \eta v$ contradicting the free action of $\tilde{\mathcal{H}}$. Thus $\eta x = \eta' x$, a contradiction to the choice of x.

Therefore η and η' agree on X.

We now show $\mu = \mu'$. Assume on the contrary $\mu \neq \mu'$, i.e. it exists some $g \in E\mathcal{G}$ such that $\mu g \neq \mu' g$. Observe that for every $x \in p^{-1}g$ the equality $\eta(x * g) = \eta x \tilde{*} \mu g = \eta' x \tilde{*} \mu' g = \eta'(x * g)$ holds and hence $\mu = \mu'$ and thus $(\mu, \eta) \neq (\mu', \eta')$ as desired. □

Note that the pair η, η' respects fibers in the sense of Proposition 1.4 in [34].

Lemma 5.1.5: *For a based continuous action $(\mathcal{G}((X, A), o, \delta)*, p)$ the map p is open.*

Proof : Assume that this is false. Then there is an open subset U of X with $p(U)$ not open. Hence there is a net $(g_\nu)_{\nu \in \Lambda}$ in the complement $p(U)^c \cap V\mathcal{G}$ which converges to $g \in p(U)$. Thus there is a corresponding net $1_{\mathcal{G}(g_\nu)}$ of the respective identities converging to $1_{\mathcal{G}(g)}$. Take a net $(x_\nu)_{\nu \in \Lambda}$ in X with $px_\nu = g_\nu$ and observe $x_\nu \notin U$ for all $\nu \in \Lambda$. Now $1_{\mathcal{G}(g_\nu)} \to 1_{\mathcal{G}(g)}$ but $g \neq px$, contradicting to $D(*)$ closed. □

Lemma 5.1.6: *Given a based continuous action $(\mathcal{G}, ((X, A), o, \delta), *, p)$. A is closed if and only if for every U open in A the inverse image $p^{-1}pU$ is open in X.*

Proof :

$\Rightarrow$ Take U open in A. Assume $V := p^{-1}pU$ is not open then there exists a net $(x_\nu)_{\nu\in\Lambda} \to x \in V$ where all x_ν are in the complement of V and w.l.o.g. lie in different connected components of X. Therefore there is a corresponding net of $(a_\nu)_{\nu\in\Lambda}$ such that each a_ν belongs to the connected component of x_ν for every ν in Λ. Since V is a union of connected components of X by construction $a_\nu \notin U$. But A is closed and thus $a_\nu \to a \in A$. Since X is profinite and δ continuous, connected components converge to subsets of connected components. Thus a and x are members of the same connected component of X. This yields a contradiction to the openness of U, because $a_\nu \notin U$ for all $\nu \in \Lambda$.

$\Leftarrow$ Assume A is not closed. Then there is a net $(a_\nu)_{\nu\in\Lambda}$ in A converging to $x \notin A$. Let $\{a\}$ be the intersection of A with the connected component of x. Because the set $B := \{a_\nu : \nu \in \Lambda\} \cup \{x\}$ is closed and X is profinite there are two open neighbourhoods U of B, and V of a with void intersection. Let $V' = V \cap A$ be an open neighbourhood of a in A. Then $p^{-1}pV'$ does not contain any a_ν but it contains x a contradiction. $\square$

Corollary 5.1.7: *Given a based continuous action $(\mathcal{G}((X,A),o,\delta)*,p)$ with closed A. Then p is a homeomorphism from A to $V\mathcal{G}$.*

Proof : Since p is continuous every open set $V \in V\mathcal{G}$ yields an open subset $p^{-1}V \cap A$ of A. On the other hand since A is closed the open subsets U in A give rise to open sets $p^{-1}pU$ in X (Lemma 5.1.6). Since A meets each connected component in one point we find $pp^{-1}pU = pU$ and Lemma 5.1.5 yields $pp^{-1}pU$ is open. $\square$

5.2 Universal constructions and projective limits

The existence of equalizers and products of based actions can be deduced from the constructions for continuous actions (4.3.1 and 4.3.2). It is an easy observation that these constructions are naturally based.

Let us consider pullbacks of based continuous actions:

Proposition 5.2.1: *For a given diagram of based actions*

$$\begin{tikzcd} & (\tilde{*},\tilde{q},\tilde{B}) \arrow[d, "(\tau,\pi)", "A-\text{injective}"'] \\ (\ast,p,A) \arrow[r, "(\nu,\sigma)"] & (*,q,B) \end{tikzcd} \quad \textit{the pull back} \quad \begin{tikzcd} (\tilde{\ast},\tilde{p},\tilde{A}) \arrow[r, dotted, "(\tilde{\nu},\tilde{\sigma})"] \arrow[d, dotted, "(\tau',\pi')", "A-\text{injective}"'] & (\tilde{*},\tilde{q},\tilde{B}) \arrow[d, "(\tau,\pi)", "A-\text{injective}"'] \\ (\ast,p,A) \arrow[r, "(\nu,\sigma)"] & (*,q,B) \end{tikzcd}$$

exists, if it exists in <u>ContAct</u>. *Moreover the morphism $(\tilde{\nu},\tilde{\sigma})$ is uniquely determined.*

Proof : Take a pull back given by the continuous actions forgetting base points. Then Proposition 4.3.7 allows us to lift $A, B, \tilde{B}$ uniquely to representatives of the connected components of $\tilde{A}$. In light of Proposition 5.1.4 any compatible morphism $(\tilde{\nu}',\tilde{\sigma}') : (\tilde{\ast},\tilde{p},\tilde{A}) \to (\tilde{*},\tilde{p},\tilde{B})$ that agrees with $(\tilde{\nu},\tilde{\sigma})$ on $\tilde{A}$ must be $(\tilde{\nu},\tilde{\sigma})$. □

Corollary 5.2.2: *Let $(\tilde{\ast}',\tilde{p}',\tilde{A}')$ be universal for $(\ast,p,A)$ then there is a unique morphism $(\tilde{\nu}',\tilde{\sigma}') : (\tilde{\ast}',\tilde{p}',\tilde{A}') \to (\tilde{*},\tilde{q},\tilde{B})$ lifting (ν,σ).*

Proof : By the universal property of the universal action there exists a unique morphism $(\tilde{\tau},\tilde{\pi}) : (\tilde{\ast}',\tilde{p}',\tilde{A}') \to (\tilde{\ast},\tilde{p},\tilde{A})$. □

Proposition 5.2.3: *Every based Galois action $(\mathcal{G},((X,A),o,\delta),*,p)$ is the projective limit of finite based Galois actions. Conversely the projective limit of finite based Galois actions is a based Galois action.*

Proof :

⇒ By 4.2.2 an inverse system of finite Galois actions $(*_\nu,p_\nu)$ has a projective limit $(*,p)$. The inverse system consists of based morphisms and based actions $(*_\nu,p_\nu,A_\nu)$ so that $A = \varprojlim A_\nu$ is closed. Observe that A meets each component of the projective limit in one element and thus $(*,p,A)$ is a based Galois action.

⇐ Given $(\mathcal{G},((X,A),o,\delta),*,p)$ with A closed. By Lemma 4.2.1 the action $(*,p)$ is the projective limit of finite continuous actions. Take a member $(\mathcal{H},((Y,o',\delta'),*',p')$ of this inverse system. Y lifts to a clopen partition $\mathbb{P}$ of X. We refine this partition to a partition such that the quotient morphism is a morphism of based continuous actions: Take a clopen partition $\mathbb{P}(A)$ of A finer than the partition induced by $\mathbb{P}$. By Corollary 5.1.7 for all $P \in \mathbb{P}$ the set $p(P)$ is clopen in $V\mathcal{G}$. Therefore setting $\mathbb{P}' := \{p^{-1}p(P) : \; P \in \mathbb{P}(A)\}$ yields a clopen partition which is coarser than that consisting of connected components. Thus the common refinement $\mathbb{Q}$ of $\mathbb{P}$ and $\mathbb{P}'$ inherits the quotient property from $\mathbb{P}$ and the image of A in the quotient $X/\mathbb{Q}$ meets each connected component once. Take $\mathcal{H}' = \coprod_{P \in \mathbb{P}(A)} \mathcal{H}(a(P))$, where $a(P)$ is a chosen representative of P. The action of $\mathcal{H}'$ on $X/\mathbb{Q}$ is naturally induced by the action of $\mathcal{H}$ on Y and thus is disc free as desired. $\square$

Proposition 5.2.4: *Let $(*, p, A)$, $(*', p', B)$ be based actions. If $(*, p) \times (*', p')$ exists so does $(*, p, A) \times (*', p', B)$.*

Proof : Consider the product of actions $(*, p) \times (*', p')$ and observe that $\{((a, b), \{(a, b)\}) : \; a \in A, b \in B\}$ is a base for $(*, p) \times (*', p')$ compatible with the coordinate projections. Thus one finds that $(*, p, A) \times (*', p', B)$ exists. $\square$

Proposition 5.2.5: *Let (μ, η) and (μ', η') be morphisms of based continuous actions*

$$(\mathcal{G}, ((X, A)o, \delta), *, p) \underset{(\mu,\eta)}{\overset{(\mu',\eta')}{\rightrightarrows}} (\mathcal{H}, ((Y, B), o', \delta'), *', p').$$

Then the coequalizer constructed in Proposition 4.3.4 serves as a coequalizer in <u>BasedAct</u>.

Proof : We rewrite the proof of Proposition 4.3.4. Let Λ denote the set of all morphisms $(\gamma, \lambda) : \; (*', p', B) \to (\mathcal{G}_l, (X_l, o_l, \delta_l), *_l, p_l, B_l)$ to finite based (Galois) actions $(*_l, p_l, B_l)$ such that $(\gamma, \lambda) \circ (\mu, \eta) = (\gamma, \lambda) \circ (\mu', \eta')$. We devise an order "$\leq$" on Λ by setting $(\gamma, \lambda) \leq (\gamma', \lambda')$ if and only if there are $\psi_{\gamma\gamma'}$ and $\varphi_{\lambda\lambda'}$ with $\psi_{\gamma\gamma'}\gamma' = \gamma$ and $\varphi_{\lambda\lambda'}\lambda' = \lambda$. The set Λ is not empty since our category has a terminal object (a single point on which the groupoid acts). We claim that $(\Lambda, \leq)$ is a directed set. Pick (γ, λ), (γ', λ'). Using Proposition 5.2.4 we can form the product action $(*_l, p_l, B_l) \times (*_{l'}, p_{l'}, B_{l'})$

as indicated in the diagram below

$$
\begin{tikzcd}
& & & (*_l, p_l, B_l) \\
(*, p, A) \arrow[r, bend left, "(\mu,\eta)"] \arrow[r, bend right, "(\mu',\eta')"'] & (*', p', B) \arrow[urr, "(\gamma,\lambda)"] \arrow[drr, "(\gamma',\lambda')"'] \arrow[rr, dotted, "(\tau,\sigma)"] & & (*_l, p_l, B_l) \times (*_{l'}, p_{l'}, B_{l'}) \arrow[u] \arrow[d] \\
& & & (*_{l'}, p_{l'}, B_{l'})
\end{tikzcd}
$$

The universal property of the product yields the dotted arrow (τ, σ). Take the action generated by the image of τ and σ. Proposition 4.3.7 shows that $(*_l, p_l, B_l) \times (*_{l'}, p_{l'}, B_{l'})$ is based Galois if the two factors are. One checks that $(\tau, \sigma) \circ (\mu, \eta) = (\tau, \sigma) \circ (\mu', \eta')$ and thus (τ, σ) belongs to Λ. Moreover $(\tau, \sigma) \geq (\gamma, \lambda)$ and $(\tau, \sigma) \geq (\gamma', \lambda')$. One checks that the above inverse system is cofinal for the inverse system constructed in Proposition 4.3.4. Thus the projective limit is equal to the coequalizer formed in the category ContAct. Because all morphisms and actions considered this coequalizer serves as the coequalizer in BasedAct. □

Proposition 5.2.6: *Proposition 4.3.7 is valid for based actions. In particular the pullback of based Galois action is a based Galois action.*

Proof : Since the construction of (co)products, (co)equalizers and pullbacks in ContAct can be performed with morphisms of based actions Proposition 4.3.7 stays valid. □

Chapter 6

Simply connected complexes and universal based Galois actions

In this section, for any given based Galois action, we will construct the universal based Galois action and later use it to define pro-$\mathcal{C}$ fundamental groupoids. We first carry out the basic constructions for finite 2-complexes and (pro)finite groups and then consider the general case. In this way we follow the strategy in [34] where the existence of a universal Galois cover has been proved for any connected oriented graph.

6.1 Basic definitions

Definition 6.1.1: *A based Galois action* $(\tilde{\mathcal{G}}, ((\tilde{X}, \tilde{A}), \tilde{o}, \tilde{\delta}), \tilde{*}, \tilde{p})$ *is* **universal** *for a based Galois action* $(\mathcal{G}, ((X, A), o, \delta), *, p)$ *if all of the following holds*

1. *There is a quotient morphism of actions* $(\mu, \eta) : (\tilde{*}, \tilde{p}, \tilde{A}) \to (*, p, A)$.
2. *Every quotient morphism* (μ', η') *from a based Galois action* $(\mathcal{H}, ((Y, A'), o', \delta'), *', p')$ *to* $(*, p)$ *lifts to a based morphism* $(\tilde{\mu}, \tilde{\eta})$ *yielding the commutative diagram*

$$\begin{tikzcd}
(\tilde{*}, \tilde{p}) \arrow[r, "(\tilde{\mu}, \tilde{\eta})"] \arrow[d, "(\mu, \eta)"'] & (*', p') \arrow[dl, "(\mu', \eta')"] \\
(*, p) &
\end{tikzcd}$$

A based Galois action is **universal** *if it is universal for itself.*

It is immediate that the lifted morphism is a quotient morphism.

It is convenient, to interpret any profinite 2-complex as a based action with trivial groupoid:

Definition 6.1.2: *Let $((X, A), o, \delta)$ be a based profinite 2-complex. Let 1_A be the groupoid defined by $Ob(1_A) = A$ and $Hom 1_A = A$ consisting of the identities only. Moreover let $p_A : X \to V1_A$ be the map sending $x \mapsto a$ if a and x lie in the same connected component of X and let $*_A : X \times E1_A \to X$ be defined as $(x, e) \mapsto x$ if and only if $d_0 e = p_A x$.*
Then $(1_A, ((X, A), o, \delta), *_A, p_A)$ is the **trivial based action** *for the complex $((X, A), o, \delta)$.*

6.2 Finite constructions

We need the "concept" of finite tree. A profinite tree is defined by homology properties [33]. For the discrete case the concepts of tree and simply connected graph are equivalent. The concept of realization of finite graphs can be shown to be unique up to homeomorphism.
For an infinite profinite graph beeing simply connected is a stronger property than that of beeing a profinite tree ([33]).

Definition 6.2.1: *A* **circuit** *is a finite big circle. A finite* **tree** *is a connected (un)oriented graph which does not contain a circuit. A* **finite tree complex** *is a complex which interpreted as an unoriented graph is a finite tree. A finite tree subcomplex T in (X, o, δ) is a* **spanning tree** *if $VT = VX$.*

Definition 6.2.2: *A based profinite 2-complex $((X, A), o, \delta)$ is* **simply connected** *if the trivial based action $(1_A, ((X, A), o, \delta), *_A, p_A)$ is universal.*

A finite connected 2-complex contains a maximal simply connected subcomplex.
Considering a finite tree (in the sense of [34]) as a 1-complex (2.3.1) it turns out immediately that it is simply connected.
Our next goal is showing that any finite connected based 2-complex $((X, a), o, \delta)$ possesses a maximal simply connected subcomplex which we denote by $K(X, a)$.
We employ a straight forward algorithm which implicitly uses combinatorial homotopies ([24]).

Algorithm 6.2.3:

➩ *Fix a spanning tree T_0 in X which contains a.*

➩ *Suppose T_{n-1} has been constructed for $n \geq 1$. If there is a 2-cell in $X \setminus T$ and $e \in EX$ with $\delta x \setminus T \subseteq \{x, e\}$ then set $T_n := T_{n-1} \cup \{x, e\}$, else terminate the algorithm.*

✔ *Since X is finite, there exists $n \in \mathbb{N}$ for which the algorithm terminates. In this case we let $K(X, a) := T_{n-1}$.*

Proposition 6.2.4: *$K(X, a)$ is a simply connected subcomplex of $((X, a), o, \delta)$.*

Proof : Let $(\mu, \eta) : (\mathcal{G}, ((Y, b), o', \delta'), *, p) \to (1_a, ((X, a), o, \delta), *_a, p_a)$ be a morphism of based actions. For all $g \in \mathcal{G}$ and all $y \in Y$ the equation $\eta(y * g) = \eta y$ holds. Therefore and because T_0 is simply connected $\eta^{-1}T_0$ has connected components each isomorphic to T_0. Suppose $\eta^{-1}T_0$ were not connected. Since $((Y, b), o', \delta')$ is connected and $VT_0 = VX$ there is an edge $y \in EY$ such that $\delta' y$ meets at least two connected components of $\eta^{-1}T_0$. Then y connects two different copies C, C' of T_0. Algorithm 6.2.3 yields a disc d in X with $\beta d \setminus T_0 = \eta y$. Hence there is an element d' of $\eta^{-1}d$ such that y is in the boundary of d'. But $C \cup C' \cup y$ does not contain a circuit and therefore η is not injective on the set of edges of $\beta d' \cap \eta^{-1}\beta d$. Thus η is not a morphism of complexes, a contradiction. □

6.3 Existence of universal based Galois actions

Let $(\mathcal{G}, ((X, \tilde{a}), o, \delta), *, p)$ be a finite based Galois action. We construct a based Galois action $(\tilde{\mathcal{G}}, ((\tilde{X}, b), \tilde{o}, \tilde{\delta}), \tilde{*}, \tilde{p})$ with $(*, p, a)$ a quotient.
Using Proposition 1.4.14 for every 2-cell x we select $v_x \in V\beta x$ and an order "$\leq_v$" on βx.
Similarly for every edge or vertex x we select a vertex $v_x \in \delta x$.

Construction 6.3.1: *Let $((X,a),o,\delta)$ be a finite 2-complex and $K(X,a)$ a maximal simply connected subcomplex. By Construction 6.2.3 $K(X,a)$ exists.*
Let $\mathcal{F}_{\mathcal{C}}(EX)$, be the free pro-$\mathcal{C}$ groups generated by all edges of X.
For every 2-cell x with $E\beta x = \{e_1,\dots,e_n\}$ there is a unique word $w(x) = e_1^{\epsilon_1} e_2^{\epsilon_2} \dots e_n^{\epsilon_n} \in \mathcal{F}_{\mathcal{C}}(EX)$ such that for $0 < i < j \leq n$ one has $e_i <_{v_x} e_j$ and $\epsilon_i = \begin{cases} +1 : v_{e_i} <_{v_x} e_i \\ -1 : \text{else} \end{cases}$.
Let D be the normal subgroup of $\mathcal{F}_{\mathcal{C}}(EX)$ generated by the set

$$EK \cup \{w(x) : \ x \in DX \setminus K\}$$

The following diagram arises:

$$EX \rightarrowtail \mathcal{F}_{\mathcal{C}}(EX) \xrightarrow{\tau} \mathcal{F}_{\mathcal{C}}(EX)/D$$

Set $G := im(\tau)$, $U := G \times X$ and $b := (1,a)$.
Now assume that $x \in X \setminus K$ is a 2-cell. The word $w(x)$ can be written in the form $w(x) = w_1(x)\dots w_k(x)$ such that for $i = 0,\dots,k$ all of the following holds:

1. *$w_i(x) = e_{i1}^{\epsilon_{i1}} \dots e_{im}^{\epsilon_{im}}$ and $\{e_{i1},\dots,e_{im}\} \setminus K \subseteq \{e_{i1}, e_{im}\}$.*
2. *If $e_{i1} \notin K$ then $\epsilon_{i1} = -1$*
3. *If $e_{im} \notin K$ then $\epsilon_{im} = 1$*

For every 2-cell x define $\kappa_x(y) : \ \beta x \to \mathcal{F}_{\mathcal{C}}(EX)$ by setting

$$\kappa_x(y) := w_1(x) w_2(x) \dots w_i(x)$$

where i is determined by the condition

$$\min\{e_{i1}, v_{e_{i1}}\} \leq_{v_x} y \leq_{v_x} \max\{e_{im}, v_{e_{im}}\}$$

For $(f,x) \in U$ set

$$o'(f,x) := o(x)$$

and

$$\delta'(f,x) := \begin{cases} \{(f,y) : y \in \delta x\} & : x \in K \\ \{(f,x),(f,x_f),(xf,y_f) : \ \delta x = \{x_f,y_f\}, x_f = v_x\} & : x \in EX \setminus K \\ \{(\tau\kappa_x(y)f, y) : \ y \in \beta x\} \cup \{(f,x)\} & : x \in DX \setminus K \end{cases}$$

We define an operation $* : \ U \times E\mathcal{G}(G) \to U$ setting $u*g = (x,h)*g := (x,hg)$ and $p : \ U \to V\mathcal{G}(G)$.

Our construction corresponds to the one for the universal Galois cover of an oriented graph [34].

Proposition 6.3.2: *$((U,b),o',\delta')$ is a based profinite 2-complex. A different choice of b, v_x's and $<_x$ results in an isomorphism of 2-complexes. $(\mathcal{G}(G),((U,b),o',\delta'),*,p)$ is a universal based Galois action for $(1_a,((X,a),o,\delta),*_a,p_a)$.*

Proof : For checking that (U,o',δ') is a 2-complex we turn to Definitions 2.1.2 and 2.2.1. The only part of proving which does not appear to us routine is checking the continuity of δ'. For instance 2.2.1 (b) holds since there are no loops in U. Since X is finite the Vietoris open sets $W(V_1,\dots,V_n)$ with $V_i = V_i' \times \{x_i\}$ and V_i' is an open in $E\mathcal{G}$ form a basis of the topology in FU. Set $Z := \{x_1,\dots,x_n\}$. If $\delta^{-1}Z \subseteq K$ then the set $\delta'^{-1}W(V_1,\dots,V_n) = \bigcap_{i=1}^n V_i' \times \delta^{-1}Z$.

Suppose that $\delta^{-1}Z \setminus K \neq \emptyset$. Then observing $\delta^{-1}Z \subseteq K$ if $|Z| \leq 2$ we have the cases:

(a) $|Z| = 2$,

(b) $|Z| = 3$, and

(c) $|Z| > 3$

In (a) and (b) one can w.l.o.g. assume that $n = |Z|$. Then we have

(a) $Z = \{e,v\}$ is a loop. Then $\delta'^{-1}W(V_1,V_2) = (V_1' \cap V_2' \cap eV_1') \times \{e\}$ is open.

(b) $Z = \{e,v,u\}$. Since we can assume that e is an edge not a loop (else Z contains a 2-cell with boundary a loop and would be contained in K) the set $\delta'^{-1}W(V_1,V_2,V_3) = (V_1' \cap V_2' \cap eV_3') \times \{e\}$ is open.

(c) We must have $Z = \delta x$ for some 2-cell x. Therefore w.l.o.g. we can assume that $x = x_1$, and βx can be ordered

$$x_2 \leq_{v_x} \cdots \leq_{v_x} x_n.$$

Setting $g_1 = 1$ and $g_i = (\tau\kappa_x(x_i))^{-1}$ for $i \geq 2$ the set $\delta'^{-1}W(V_1,\dots,V_n) = (\bigcap g_i V_i') \times \delta'^{-1}Z$ is open.

Thus δ' is continuous.

Let us show that U is connected and hence is based. It is not hard to see that $(*,p)$ induces a disc free continuous action on U. Let C be a connected component of U. Then by construction, for every generator e of G the union

$C \cup C * e$ is connected and hence C is $\mathcal{G}(G)$-invariant. Since $X = U/\mathcal{G}$ is connected and so is U.

Our proof shows that $(\mathcal{G}(G), ((U, b), o', \delta'), *, p)$ is a based Galois action.

The construction does evidently not depend upon the choice of b. Now we show that changing v_x determines a complex isomorphism. We observe that only $\delta' \restriction G \times X \setminus K$ depends on the choice of v_x. It suffices to consider (U_1, o', β') arising by changing v_x to v'_x for a single point $x \in X$ and provide a complex isomorphism from $U_1 \to U$.

Assume x is a 2-cell. We see that $\beta(f, x) = \{(\tau\kappa_x(y)f, y) : \ y \in \beta x\}$ and $\beta'(f, x) = \{(g\tau\kappa_x(y)f, y) : \ y \in \beta x\}$, where $g = \tau w_1(x) \dots w_k(x)$ such that v'_x is contained in δe for some e occuring in $w_k(x)$. Thus there is a complex isomorphism from $U \to U_1$ by mapping $(f, x) \in U$ to $(g^{-1}f, x)$ and (f, y) to (f, y) if $y \neq x$.

In a similar fashion one shows that inverting the order $\leq_x$ yields an isomorphism of the desired form. Finally, when x is an edge and $\delta x = \{x, u, v\}$ it does not matter when $v_x = u$ or $v_x = v$. We omit a formal proof.

It is left to prove that U is universal for $(1_a, ((X, a), o, \delta), *_a, p_a)$.

Let $(\mu, \eta) : \ (\mathcal{G}(H), ((Y, c), \tilde{o}, \tilde{\delta}), *, p) \to (1_a, ((X, a), o, \delta), *_a, p_a)$ be a quotient morphism of based Galois actions. We construct a quotient morphism $(\mu', \eta') : \ (*', p', b) \to (*, p, y)$.

Because $K(X, a)$ is finite and simply connected there is a unique embedding $\iota : \ K(X, a) \to (Y, c)$ sending a to c. The selection of elements v_x and orders $\leq_{v_x}$ extends in a unique fashion to the elements of (Y, b). Because $\mathcal{H}$ operates disc free one finds for each element $x \in X \setminus K$ a unique $y \in Y$ such that $v_y \in \mathrm{im}(\iota)$. Form the set C by adding all these y to $\mathrm{im}(\iota)$. One can apply the same procedure to construct a set C' with respect to the unique embedding $\iota' : \ K(X, a) \to U$ satisfying $\iota'(a) = b$. For $y \in C$ there are finitely many elements $h_i \in H$ such that $\tilde{\delta} y = \{y_1 * h_1, \dots, y_2 * h_2, \dots, y_n * h_n\}$. Observe that one can choose $h_1 = 1$ and thus there is a morphism μ' from $G \to H$ sending $\tau\kappa_x(y_i)$ to h_i. Moreover one checks that there is a morphism $\eta' : \ U \to Y$ defined by sending $\{g\} \times C' \mapsto \{\mu' g\} \times C$. Since the compatibility condition $\eta'(g*c) = \eta' g * \mu' c$ is valid (μ', η') is a based morphism of actions. Its image is a closed connected subcomplex of Y and because $\mathcal{G}(H)$ operates dics free and Y is connected one finds μ' and thus η' to be surjective. Therefore (μ', η') is a quotient morphism of based Galois actions as desired.

□

Corollary 6.3.3: *Let B be a finite set. The action $(\mathcal{G},((U,B),o',\delta'),*,p) := \coprod_{b\in B}(\mathcal{G},((U,b),o',\delta'),*,p)$ is universal for an action $(\mathcal{H},((Y,C),o'',\delta'').*',p')$ if and only if there is quotient morphism of actions $(*',p',B) \to (1_A,((X,A),o,\delta)*_A,p_A)$.*

Proof : Apply Proposition 6.3.2 componentwise. □

Theorem 6.3.4: *For every based Galois action $(\mathcal{G},((X,A),o,\delta),*,p)$ there exists a universal Galois action.*

Proof : By Proposition 5.2.3 there is an inverse system of finite actions with projective limit $(\mathcal{G},((X,A),o,\delta),*,p)$. By Corollary 6.3.3 each $(*_\alpha,p_\alpha,A_\alpha)$ in the inverse system has a universal action $(\tilde{*}_\alpha,\tilde{p}_\alpha,\tilde{A}_\alpha)$. By Corollary 4.3.3 the morphism $(\mu_{\alpha\beta},\eta_{\alpha\beta}) : (*_\alpha,p_\alpha,A_\alpha) \to (*_\beta,p_\beta,A_\beta)$ induces a morphism of actions $(\tilde{\mu}_{\alpha\beta},\tilde{\eta}_{\alpha\beta}) : (\tilde{*}_\alpha,\tilde{p}_\alpha,\tilde{A}_\alpha) \to (\tilde{*}_\beta,\tilde{p}_\beta,\tilde{A}_\beta)$. This morphism is unique by Proposition 5.1.4, because $(\tilde{\mu}_\alpha,\tilde{\eta}_\alpha) : (\tilde{*}_\alpha,\tilde{p}_\alpha,\tilde{A}_\alpha) \to (*_\alpha,p_\alpha,A_\alpha)$ is a quotient morphism and thus prescribes $(\tilde{*}_\alpha,\tilde{p}_\alpha)$ at $\tilde{A}_\alpha$. Thus we have an inverse system of universal actions which has an inverse limit the action $(\tilde{*},\tilde{p},\tilde{A})$ (Proposition 4.2.4).

Let us prove the universal property of $(\tilde{*},\tilde{p},\tilde{A})$. Conider a based Galois action $(+,q,B)$ with groupoid $\mathcal{K}$ and a quotient morphism $(\mu^+,\eta^+) : (+,q,B) \to (*,p,A)$. By Proposition 5.2.3 there is an inverse system $(+_\alpha,q_\alpha,B_\alpha)$ with projective limit $(+,q,B)$. Denote $(\tau_\alpha,\pi_\alpha) : (+,q,B) \to (+_\alpha,q_\alpha,B_\alpha)$. Take $\ker\mu^+$ and observe that for each α one has $\mathcal{G}_\alpha := \tau_\alpha \ker\mu^+$ a normal subgroupoid in $\tau_\alpha\mathcal{K}$. Moreover the $(+_\alpha,q_\alpha,B_\alpha)/\mathcal{G}_\alpha$ form an inverse system for $(*,p,A)$. One checks that they possess universal Galois actions which form an inverse system for $(\tilde{*},\tilde{p},\tilde{A})$. Thus there is a quotient morphism $(\tilde{*},\tilde{p},\tilde{A}) \to (+,q,B)$.

□

Corollary 6.3.5: *Let $(\mathcal{G},((X,A),o,\delta),*,p)$ then finding the universal Galois action for $(*,p,A)$ is equivalent to finding an action $(\tilde{*},\tilde{p},\tilde{A})$ and a quotient morphism $(\tilde{*},\tilde{p},\tilde{A}) \to (*,p,A)$ such that the complex of $(*,p,A)$ is simply connected.*

We list useful observations:

Corollary 6.3.6: *The projective limit of an inverse system of universal based actions is universal.*
A simply connected oriented graph in the sense of [34] is a simply connected 1*-complex.*

Proof : The first statement is an immediate consequence of the proof of Theorem 6.3.4. For the second statement one checks that the definition of $U_C(\Gamma, T)$ (see Lemma 2.3 in [34]) is coherent to Construction 6.3.1 if $T = K(X, a)$, the relations $D = \{1\}$ and $v_x = d_0 x$. Therefore the pro-$\mathcal{C}$ fundamental group in the sense of [34] coincides in case of orientable graphs. $\square$

Chapter 7

Fundamental groupoids of graphs of complexes

In this chapter we will prove a van Kampen theorem, namely the fundamental groupoid of graphs of complexes turns out to be the fundamental groupoid of a certain graph of groupoids.

7.1 Description of the fundamental groupoid of a graph of groupoids

In this section we give a brief description of the fundamental groupoid of a graph of groups. Graphs of groups first appeared in [30]. A combinatorial approach to graphs of groups and graphs of groupoids can be found in [22]. A profinite version of graphs of groups and their fundamental groups can be found in [32] and [33].

Definition 7.1.1: *A* **graph of groups** $(\mathcal{G}, \partial_0, \partial_1)$ *consists of a totally disconnected groupoid* $\mathcal{G}$ *and a pair of morphisms* $\partial_i : \mathcal{G} \to \mathcal{G}$ *for* $i = 0, 1$ *satisfying* $\partial_i \partial_j = \partial_j$*, so that* $\partial_i \mathcal{G}(v) \subseteq \mathcal{G}(v)$ *implies* $\partial_i \restriction \mathcal{G}(v) = id_{\mathcal{G}(v)}$ *for all* $v \in V\mathcal{G}$*. With a graph of groups we connect an oriented graph (***gluing graph***)* (Γ, d_0, d_1) *whose elements coincide with* $V\mathcal{G}$*. The* d_i *are defined according to the equality* $d_i p = p\partial_i$*.*

To define the fundamental group(oid) of a graph of groups we sketch two approaches.
The first is taken from [32]. Consider the universal covering $\tilde{\Gamma}$ of the gluing graph Γ and pick a 0-connected section J for Γ. Search for a J-specialization (β, β_1) into a profinite group H such that the standard graph $S(\mathcal{G}, \beta, H)$

is connected and simply connected. Then H is the fundamental group of $(\mathcal{G}, \partial_0, \partial_1)$ if β is injective on fibers.
The second approach is similar to that one found in [22]. Let $(\mathcal{G}, \partial_0, \partial_1)$ be a connected graph of groups and Γ its gluing graph. Let $\sigma : V\Gamma \sqcup V\Gamma \to V\Gamma$ be the morphism sending $v \mapsto v$, for all $v \in V\Gamma$ and let $\mathcal{G}'$ be the subgroupoid corresponding to the vertices of Γ. Set $p : V\mathcal{G} \to \Gamma$ to be the canonical map and set $\mathcal{G}' * F(\Gamma) := U_\sigma(\mathcal{G}' \sqcup F(\Gamma))$. There is a natural relation "$\sim$" on $\mathcal{G}' * F(\Gamma)$ generated by setting $f \cdot h \sim g \cdot f$, whenever all of the following holds

- $f \in F(\Gamma)$
- There is a $w \in p^{-1}f$ and a $e \in \mathcal{G}(w)$ such that either $\partial_0 e = h,\ \partial_1 e = g$ or $\partial_1 e = h,\ \partial_0 e = g$.

Then obtain the fundamental groupoid $\pi_1(\mathcal{G}, \partial_0, \partial_1)$ by forming $\mathcal{G}' * F(\Gamma)/_\sim$. To obtain the fundamental group of the graph of groups, one has to find a 0-connected section J for Γ and amalgamate $\pi_1(\mathcal{G}, \partial_0, \partial_1)$ along the embedding of J into $F(\Gamma)$.
In this work we focus on the category of based Galois actions and thus we do not consider graphs of groups on their own. This approach leads to a theory, which covers graphs of groups implicitly. Therefore constructing the fundamental groupoid of a graph of based Galois actions yields a fundamental group(oid) of the underlying graph of groups.

7.2 Gluing schemes and a van Kampen theorem

Definition 7.2.1: *A* **gluing scheme** $(*, p, A), \partial_0, \partial_1)$ *consists of a based action* $(*, p, A)$, *and a pair of morphisms* $\partial_i : (*, p, A) \to (*, p, A)$ *for* $i = 0, 1$ *satisfying* $\partial_i \partial_j = \partial_j$, *so that* $\partial_i C \subseteq C$ *implies* $\partial_i \upharpoonright C = id_C$ *for all connected components of the underlying profinite* 2*-complex. With a gluing scheme we connect an oriented graph (***gluing graph***)* (Γ, d_0, d_1) *whose elements are the connected components of the underlying complex of* $(*, p, A)$. *The* d_i *are defined according to the equality* $d_i p = p\partial_i$. *The co-equalizer* $C(\partial_0, \partial_1)$ *will be termed* **glued action**.

Theorem 7.2.2: *Let $((*, p, A), \partial_0, \partial_1)$ be a gluing scheme and $C(\partial_0, \partial_1)$ the result of gluing. Let the gluing graph Γ be connected.*
There is a gluing scheme $((\tilde{*}, \tilde{p}, \tilde{A}), \tilde{\partial}_0, \tilde{\partial}_1)$ giving rise to a commutative diagram

$$
\begin{tikzcd}
(\tilde{\Gamma}, \tilde{d}_0, \tilde{d}_1) \arrow[r, "\pi_\Gamma"] & (\Gamma, d_0, d_1) \\
(\tilde{*}, \tilde{p}, \tilde{A}) \arrow[u, "\tilde{p}"] \arrow[r, two heads, "\pi_A"] \arrow[d, "\tilde{\sigma}"'] & (*, p, A) \arrow[u, "p"'] \arrow[d, "\sigma"] \\
C(\tilde{\partial}_0, \tilde{\partial}_1) \arrow[r, dashed, two heads, "A\text{-injective}"] & C(\partial_0, \partial_1)
\end{tikzcd}
$$

Here

- $C(\tilde{\partial}_0, \tilde{\partial}_1)$ *is a universal based action for* $C(\partial_0, \partial_1)$*;*
- π_A *is a surjective morphism of based actions which respects the morphisms* ∂_i*.*

Proof : We first prove the theorem for simply connected Γ

By Theorem 6.3.4 there is a universal based action $(\tilde{*}, \tilde{p}, \tilde{A})$ for the based action $(*, p, A)$. Since $\Gamma = pA$ there is a natural graph morphism from $\tilde{\Gamma} := \tilde{p}\tilde{A}$ to Γ. By Proposition 5.2.1 there is for $i = 0, 1$ a unique morphism of based action s $\tilde{\partial}_i$ making the following diagram commutative:

$$
\begin{tikzcd}
(\tilde{*}, \tilde{p}, \tilde{A}) \arrow[r, "\tilde{\partial}_i"] \arrow[d, "A\text{-injective}"] & (\tilde{*}, \tilde{p}, \tilde{A}) \arrow[d, "A\text{-injective}"] \\
(*, p, A) \arrow[r, "\partial_i"] & (*, p, A)
\end{tikzcd}
$$

It is not hard to see that $((\tilde{*}, \tilde{p}, \tilde{A}), \tilde{\partial}_0, \tilde{\partial}_1)$ is a gluing scheme. Let us show that the connected components of Γ are in a 1-1 correspondence with those of $\tilde{\Gamma}$. Pick a connected component C of Γ and take its preimage $X_C := \gamma^{-1}C$ in the 2-complex X of the based action $(*, p, A)$. The restrictions of ∂_i to X_C have a coequalizer, say C_0. Since C is connected, so is C_0, as a result of gluing. It shows every connected component of C_0 to be contained in a connected component of $C(\partial_0, \partial_1)$.

We now claim that different connected components C_1, C_2 of Γ cannot be in the same connected component of $\tilde{\Gamma}$. To see this we first find a clopen graph partition of Γ so that the C_i are not in the same member of the partition. Now, in the inverse images of the members of the partition in X the bonding maps ∂_i map each member of the clopen partition of X into itself. Therefore, after collapsing components, C_1 and C_2 must belong to different connected component s of $\tilde{\Gamma}$, as claimed.

We shall successively show the existence of P, λ, κ, $\tilde{\sigma}$ and give more explanation about the arrows in the diagram

$$\begin{tikzcd}
(\tilde{*}, \tilde{p}, \tilde{A}) \arrow[dd] \arrow[dr, dashed, "\exists!"] \arrow[ddr, dashed, "\tilde{\sigma}"'] \arrow[drr, "A\text{-inj}"] & & \\
& P \arrow[r, dashed, "A\text{-inj}"'] \arrow[d, dashed] & (*, A, p) \arrow[d] \\
C(\tilde{\partial}_0, \tilde{\partial}_1) \arrow[r, dashed, bend right, "\lambda"'] & \tilde{C}(\partial_0, \partial_1) \arrow[l, bend right, "\kappa"'] \arrow[r, "A\text{-inj}"] & C(\partial_0, \partial_1)
\end{tikzcd}$$

We first construct $\tilde{\sigma}$. To this end we form the pullback P of the vertical arrow on the right of the diagram and the universal arrow from $\tilde{C}(\partial_0, \partial_1) \to C(\partial_0, \partial_1)$. The universal property of the pullback yields a unique arrow from $(\tilde{*}, \tilde{p}, \tilde{A})$ to P and $\tilde{\sigma}$ is defined by composition.

Since $\tilde{\sigma}\tilde{\partial}_1 = \tilde{\sigma}\tilde{\partial}_0$ the universal property of the co-equalizer $C(\tilde{\partial}_0, \tilde{\partial}_1)$ yields a morphism λ of based action s such that the triangle on the left commutes. Since $\tilde{\sigma}$ is an epimorphism, so is λ. Since the groupoid components of $C(\partial_0, \partial_1)$ and $C(\tilde{\partial}_0, \tilde{\partial}_1)$ are in 1-1 correspondence the universal property of the universal action $\tilde{C}(\partial_0, \partial_1)$ for $C(\partial_0, \partial_1)$ provides us with a morphism κ of based actions. Hence we may identify the result of gluing w.r.t. the scheme on the upper left corner of the diagram with the universal action of the result of gluing the scheme on the right upper corner.

So, when Γ is simply connected, our theorem is proved.

In our next step we devise a gluing scheme P with $\tilde{\Gamma}$ the universal covering of Γ and a morphism of based action s to the gluing scheme $((*, p, A), \partial_0, \partial_1)$.

The action of $\pi_1(\Gamma)$ on $\tilde{\Gamma}$ will lift to an action on P with quotient the gluing scheme $((*, p, A), \partial_0, \partial_1)$.

Our candidate for P is the pullback in the diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & ((*, A, p), \partial_0, \partial_1) \arrow[d] \\
(\tilde{\Gamma}, \tilde{d}_0, \tilde{d}_1) \arrow[r, "\pi_\Gamma"] & (\Gamma, d_0, d_1)
\end{tikzcd}$$

Let $\beta : \Gamma \to \pi_1(\Gamma)$ be canonically constructed from the 0-connected section $j : \Gamma \to \tilde{\Gamma}$ [32]. As sets we may form $\tilde{A} := A \times \pi_1(\Gamma)$, $\tilde{X} := X \times \pi_1(\Gamma)$, $\tilde{\mathcal{G}} := \mathcal{G} \times \pi_1(\Gamma)$ and, finally, $\tilde{\Gamma} := \Gamma \times \pi_1(\Gamma)$. As shown in [32] form $(\tilde{\Gamma}, \tilde{d}_0, \tilde{d}_1)$ and let $\pi_1(\Gamma)$ act canonically on it with quotient Γ. Now lift $\tilde{d}_i$ in the canonical fashion to morphisms of based action s $\tilde{\partial}_i$ on P.

It is not difficult to see that the connected components of $\tilde{\Gamma}$ are in a 1-1 correspondence with the connected components of Γ.

Our next task is to explain and establish the arrows in the following diagram and to show that it is commutative

$$\begin{tikzcd}
(\tilde{\Gamma}, \tilde{d}_0, \tilde{d}_1) \arrow[r, "\pi_\Gamma"] & (\Gamma, d_0, d_1) \\
P \arrow[u] \arrow[r, two heads, "\pi_{\Gamma P}"] \arrow[d, "\sigma_P"'] & (*, p, A) \arrow[u, "p"'] \arrow[d, "\sigma"] \\
C(\partial_{0P}, \partial_{1P}) \arrow[r, dashed, two heads, "\tilde{\pi}_{\Gamma P}"] & C(\partial_0, \partial_1)
\end{tikzcd}$$

For defining $\tilde{\pi}_{\Gamma P}$ pick x with $\partial_0 x \neq \partial_1 x$. Then both, $\partial_0 x$ and $\partial_1 x$ go to the same element in $C(\partial_0, \partial_1 P)$. Hence, if $\pi_{\Gamma P}(y) = x$ we set $\tilde{\pi}_{\Gamma P}(\sigma_P \partial_{0P}(y)) := \sigma x$. This shows that $\tilde{\pi}_{\Gamma P}$ is well-defined and surjective on the underlying spaces.

By Proposition 5.2.6 $C(\partial_{0P}, \partial_{1P})$ is a based Galois action. By construction of P one checks $\sigma \pi_{\Gamma P} \partial_{0P} = \sigma \pi_{\Gamma P} \partial_{1P}$ and thus there is a unique morphism $C(\partial_{0P}, \partial_{1P}) \to C(\partial_{0P}, \partial_{1P})$ which coincides with $\tilde{\pi}_{\Gamma P}$. Thus $\tilde{\pi}_{\Gamma P}$ is an A-injective morphism of based Galois actions and we have completed the construction of P.

In our final step we start from an arbitrary based action $(*, p, A)$ with connected Γ and prove the theorem.

To this end we shall explain step by step the arrows of the diagram

$$
\begin{tikzcd}
(\tilde{\Gamma}, \tilde{d}_0, \tilde{d}_1) \arrow[r, "1"] & (\tilde{\Gamma}, \tilde{d}_0, \tilde{d}_1) \\
(\tilde{*}, \tilde{p}, \tilde{A}) \arrow[u, "\tilde{p}"'] \arrow[r, "A\text{-inj}"] \arrow[d, "\tilde{\sigma}_P"'] & P \arrow[u] \arrow[d, "\sigma_P"] \\
C(\tilde{\partial}_{0P}, \tilde{\partial}_{1P}) \arrow[r, dashed, "A\text{-inj}"] & C(\partial_{0P}, \partial_{1P})
\end{tikzcd}
$$

As shown in the previous step form the pullback P and the let the right column of the present diagram be exactly the left hand side of the previous diagram (on page 77). Now $\tilde{\Gamma}$ is simply connected and we can construct the universal arrow for P and let it be the gluing scheme for $C(\tilde{\partial}_{0P}, \tilde{\partial}_{1P})$. Since $C(\tilde{\partial}_{0P}, \tilde{\partial}_{1P})$ is a universal based action for $C(\partial_{0P}, \partial_{1P})$ and the latter a universal based action for $C(\partial_0, \partial_1)$, the universality of $C(\tilde{\partial}_{0P}, \tilde{\partial}_{1P})$ for $C(\partial_{0P}, \partial_{1P})$ is established. $\square$

Chapter 8

Cayley complexes

Cayley graphs in discrete group theory are the standard models of spaces with fundamental group a given group G. In algebraic topology a more general concept of Cayley complex of a given group G arises. It consists of the Cayley graph of $G = \langle E|R \rangle$, where $\langle E|R \rangle$ is a presentation of G, and for each relation $r \in R$ and every vertex in the Cayley graph, the unique sequence of edges corresponding to r, is used as boundary for adjoining a disc. Since the relator words correspond to circles in the graph one gets a simply connected 2-CW complex such that factoring the action of the group G yields a complex with fundamental group G.

In the profinite situation this method is not applicable since one cannot transverse relations edgewise in the Cayley graph of a group.

8.1 Preliminaries and definition of the Cayley complex

To make this introductionary explanation more exact and generalize it to the profinite situation we cite the Cayley graph definition of [34]. Every profinite group G is the quotient of a free profinite group $\mathcal{F}(X)$, where X is a generating set converging to 1 or, more generally, X is a topological space (embedded into G), see free presentations [25]. The definition of the Cayley graph is given w.r.t. X.

Definition 8.1.1: *Let $(X, *)$ be a based profinite space ($*$ is the base point), and $\mu : X \to G$ a continuous map of X into a pro-$\mathcal{C}$-group G such that $\mu(*) = 1$. The* **Cayley graph** *$\Gamma(G, \mu, X)$ corresponding to the map μ is given by*

$$\Gamma(G, \mu, X) = G \times X,\ V(\Gamma(G, \mu, X)) = G \times \{*\}$$

$$d_0(g, x) = (g, *),\ d_1(g, x) = (g\mu(x), *).$$

Since $\mathcal{F}_{\mathcal{C}}(X)$ is the free pro-$\mathcal{C}$ group on X the map μ induces a unique homomorphism $\sigma : \mathcal{F}_{\mathcal{C}}(X) \to G$. Theorem 3.1. [34] states that X generates G if and only if $\Gamma(G, \mu, X)$ is connected and $\pi_1^{\mathcal{C}}(\Gamma(G, \mu, X)) \cong \ker \sigma$.
We want to come to our definition of the Cayley complex and need certain constructions involving free groups.

Construction 8.1.2: *Let (X, o, δ) be a profinite 2-complex without .
We construct a corresponding simply connected complex $(C(X), o', \delta')$, the* **cone complex** *over (X, o, δ).
Take the coproduct $C(X) = X \sqcup X^{\leq 1} \sqcup \{0\}$ where 0 is a single point, the* **peak** *of $C(X)$. Let ι and ι_1 be the canonical embeddings of X and $X^{\leq 1}$ into $C(X)$.
Define two maps $d_X(y) := \{\iota z :\ z \in \delta(\iota^{-1}y)\}$ and $d_{X^{\leq 1}}(y) := \{\iota_1 z :\ z \in \delta\iota_1^{-1}y\} \cup \{\iota z :\ z \in \delta\iota_1^{-1}y\} \cup \{0\}$. Now set*

$$\delta' y := \begin{cases} d_X(y) & y \in \iota X \\ d_{X^{\leq 1}}(y) & y \in \iota_1 X^{\leq 1} \\ \delta'(y) = \{0\} & y = 0 \end{cases}$$

and

$$o'(y) = \begin{cases} o(\iota^{-1}y) & y \in im(\iota) \\ o(\iota_1^{-1}y) + 1 & y \in im(\iota_1) \\ 0 & y = 0 \end{cases}$$

We extend the definition to based complexes $((X, A), o, \delta)$ by choosing a point $a \in A$ and construct $((C(X), a), o', \delta')$.

Proposition 8.1.3: *For every based profinite 2-complex $((X, A), o, \delta)$ without and every $a \in VX \cup \{0\}$ the cone $((C(X), a), o', \delta')$ is a connected and simply connected profinite 2-complex.*

Proof : The space $C(X)$ is profinite and δ', o' are well defined. We show that $(C(X), o', \delta')$ is a pre-complex (compare 3. (a) - (c) 2.1.1).

δ' restricted to ιX is continuous and symmetric with respect to $X, X^{\leq 1}$ otherwise. Thus for showing continuity of δ' it suffices to consider sets of the form $W(\{0\}, U_1, \ldots, U_n)$ with $\iota_1 \iota^{-1} U_1 = U_1 \cap \mathrm{im}(\iota_1)$. Observe

$$\begin{aligned}\delta'^{-1} W(\{0\}, U_1, \ldots, U_n) = &\\ &\{0\} \cup \iota^* \delta^{-1} \iota^{*-1} W(U_1, \ldots, U_n) \cup \\ &\big(\iota_1^* \delta^{-1} \iota_1^{*-1} W(U_1, \ldots, U_n) \cap \iota_1^* \iota_1^{*-1} W(U_1, \ldots, U_n)\big)\end{aligned}$$

which is open because ι^*, ι_1^* are the canonical liftings of ι, ι_1 to the Vietoris topology of $\mathrm{F}C(X)$ ([20]).

Since $\beta' y \cap \mathrm{im}(\iota_1) \neq \emptyset$ is equivalent to $y \in \mathrm{im}(\iota_1)$, and for $y \in \mathrm{im}(\iota_1)$ the boundary map β' is symmetric in X and $X^{\leq 1}$, one has $O'(\beta' y) = o'(y) - 1$.

In particular one has for $y \in VX$ that $\beta' \iota_1 y = \{\iota y, \iota_1 y, 0\}$ and for $y \in EX$ that $\beta' \iota_1 y = \iota^* \delta y \cup \iota_1^* \beta y \cup \{0\} = \{\iota v, \iota w, \iota y, \iota_1 v, \iota_1 w, 0\}$ with $\beta y = \{v, w\}$.

It turns out that $\bigcup_{z \in Y} \delta' z \subseteq \delta' y$. Thus (C, o', δ') is a pre-complex. To show it is a complex it is left to check (a) and (b) of Definition 2.2.1.

Because X has no loops for all $y \in EX$ the boundary has two elements. Thus for each $y \in EX$ the boundary can be ordered in the form $\iota v \leq \iota_1 v \leq 0 \leq \iota_1 w \leq \iota w \leq \iota y$ and so is a big circle. Hence condition (a) is valid.

To show (b) it suffices to take an open neighbourhood U of e where $e = \iota_1 v$ and $v \in VX$. Now take $U' = \delta^{-1} \iota_1^{-1} U \cap \iota_1^{-1} U$, observe that U' is open in X and if $x \in U'$ then $\delta x \in U'$. Hence $V = \{\iota_1 u : \ u \in U'\}$ and $V \subseteq U$ is an open neighbourhood of e such that for all 2-cells z the intersection $\beta' z \cap V$ is empty or contains exactly two elements.

Since for every $c \in C(X)$ there is an edge $e \in \mathrm{im}(\iota_1)$ with $\{c, 0\} \subset \delta' e$ we conclude that $(C(X), o', \delta')$ is connected.

We complete the proof by showing that for every $a \in VC(X)$ the complex $((C(X), a), o', \delta')$ is simply connected. Assume there is a based pc-action $(\mathcal{G}, ((Y, b), o', \delta'), *, p)$ and a morphism of actions (μ, η) from $(*, p, a)$ to the trivial action $(*_a, p_a, a)$ on $C(X)$. For every $g \in E\mathcal{G}$ with $0 * g \neq 0$ and hence for all $y \in \mathrm{im}(\iota_1)$ one has $y * g \neq y$. Since the action of $\mathcal{G}$ is disc free and every $c \in C(X)$ is in the boundary of an element in $\mathrm{im}(\iota_1)$ we have $C(X) * g \cap C(X) = \emptyset$. But $C(X)$ is connected and so Y is connected. Thus for every g the equation $0 * g = 0$ holds and therefore $(*_a, p_a, a)$ is universal.

□

Corollary 8.1.4: *Let $((X,A),o,\delta)$ be a based profinite 2-complex, $C(X)$ be the cone over X and S be any subcomplex of X. Then there is an injective morphism of complexes $\iota_S:\ C(S)\to C(X)$.*

Proof : The natural embedding $\iota_S:\ S\sqcup S^{\leq 1}\sqcup\{0\}\to x\sqcup X^{\leq 1}\sqcup\{0\}$ turns out to be the desired complex morphism. □

Lemma 8.1.5: *Let $((X,a),o,\delta)$ be a connected and simply connected complex containing a cone C with peak 0. Let $e_1,e_2\in EX$ and $v,w\in VX$ such that $\beta e_1=\{0,v\}$ and $\beta e_2=\{0,w\}$. Consider $\Gamma=$*  *and set $(Y,o',\delta'):=(X,o,\delta)\sqcup\bullet\!\!-\!\!\bullet$ where $\bullet\!\!-\!\!\bullet\ :=\{u_1,u_2,e\}$ and e is an edge. Let $(1_{\{a,u_1\}},((Y,\{a,u_1\}),o',\delta'),\ast,q)$ be a trivial based action. Denote the two components of $1_{\{a,u_1\}}$ by 1_a and 1_{u_1}. Define $\eta_0,\eta_1:\ Y\to Y$ by setting*

$$
\begin{array}{ll}
\eta_0\restriction X=id(X) & ,\ \eta_1\restriction X=id(X)\\
\eta_0(u_1)=0 & ,\ \eta_1(u_1)=0\\
\eta_0(u_2)=v & ,\ \eta_1(u_2)=w\\
\eta_0(e)=e_1 & ,\ \eta_1(e)=e_2
\end{array}
$$

and define $\mu_0,\ \mu_1:\ 1_{\{a,u_1\}}\to 1_{\{a,u_1\}}$ by setting

$$
\begin{array}{ll}
\mu_0\restriction 1_a=id(1_a) & ,\ \mu_1\restriction 1_a=id(1_a)\\
\mu_0[1_{u_1}]=1_a & ,\ \mu_1[1_{u_1}]=1_a
\end{array}.
$$

Let $\partial_i=(\mu_i,\eta_i)$ then $((\ast,q,A),\partial_0,\partial_1)$ is a gluing scheme such that $C(\partial_0,\partial_1)$ is a universal based Galois action.

Proof : Because $\bullet\!\!-\!\!\bullet$ is simply connected and $(\ast,q,A)$ turns out to be a based Galois action, $(\ast,q,A)$ is universal. η_1,η_2 are complex morphism and μ_1,μ_2 morphisms of profinite groupoids. The property $\eta_i(y\ast g)=\eta_i y\ast\mu_i g$ is satisfied for $i=1,2$ and thus ∂_0,∂_1 are morphisms of based Galois actions.

Therefore $((\ast,q,A),\partial_0,\partial_1)$ is a gluing scheme. The free pro-$\mathcal{C}$ group in one generator $\hat{\mathbb{Z}}_{\mathcal{C}}$, acts on the universal Galois cover $\tilde{\Gamma}$ of Γ. Theorem 7.2.2 (our version of the van Kampen Theorem) yields a lifting $(((\tilde{\ast},\tilde{q},\tilde{A})),\tilde{\partial}_0,\tilde{\partial}_1)$ of $((\ast,q,A),\partial_0,\partial_1)$ with underlying complex $(\tilde{Y},\tilde{o},\tilde{p})$. Observe that $\tilde{Y}=Y\times\hat{\mathbb{Z}}_{\mathcal{C}}$ as a set. It turns out that within $C(\tilde{\partial}_0,\tilde{\partial}_1)$ all peaks of the cones $C\times\hat{\mathbb{Z}}_{\mathcal{C}}$ are identified by $\tilde{\partial}_0,\ \tilde{\partial}_1$. Thus all the cones are identified and hence $C(\partial_0,\partial_1)=C(\tilde{\partial}_0,\tilde{\partial}_1)$. The van Kampen theorem 7.2.2 yields that$C(\partial_0,\partial_1)$ is universal. □

We now look at the Cayley graph of a profinite group G and identifiy relations.
Given a generating set X of G converging to 1 then the Cayley graph of $\Gamma(G, \mu, X)$ is connected and $\mathcal{C}$-simply connected by the proof of Theorem 3.1 [34]. Thus interpreted as a based 1-complex it is simply connected by Corollary 6.3.6.

Definition 8.1.6: *Let $\mathcal{F}_{\mathcal{C}}(X)$ be the free group on X and $\Gamma(\mathcal{F}_{\mathcal{C}}(X), \mu_F, X)$ the Cayley graph of $\mathcal{F}_{\mathcal{C}}(X)$. A geodesic $L(v, w)$ for two elements $v, w \in \Gamma$ is a minimal simply connected and connected subgraph of Γ containing v and w.*

Lemma 8.1.7: *$L(v, w)$ exists for every v, w in $\Gamma(\mathcal{F}_{\mathcal{C}}(X), \mu_F, X)$. The geodesic is unique.*

Proof : Since the Cayley graph is connected and simply connected for each v, w there is a connected simply connected subgraph containing them. Take the set of all such subgraphs and define an order relation by inclusion. Since such subgraphs are obviously closed the intersection over each chain is a closed minimal subgraph, therefore a geodesic.

Assume the geodesic is not unique then there are two geodesics L, L' with not empty symmetric difference $\delta(L, L')$. There are minimal closed connected subgraphs $l \subset L$, $l' \subset L'$ with intersection $l' \cap l \subset V\Gamma$ and cardinality at least 2. Since $V\Gamma$ is totally disconnected we can partition $l \cap l'$ into two clopen subgraphs V_1, V_2 of $l \cup l'$ and partition the complement into two clopen subsets U_1, U_2. Let C_2 be the cyclic group with order 2 and generator a. Then define $C := l \cup l' \times C_2$ and $d_0', d_1' : C \to C$ by setting $d_0'(x, g) = (d_0 x, g)$ and $d_1'(x, g) = (d_1 x, g)$ if x and $d_1 x$ are in the same clopen set, and $d_1'(x, g) = \{d_1 x, g \cdot a\}$ else. The result is a connected graph such that C_2 acts on it. The quotient C/C_2 is not $\mathcal{C}$-simply connected, a contradiction to Lemma A [32]. $\square$

It may be the case that $L(v, w) = \Gamma$, e.g. take the Cayley graph of $\hat{\mathbb{Z}}_2$ and two points $v = 2,\ w = 2^{-1}$.

Let us construct the Cayley complex for a given group G. There exists a free presentation $\langle E|N \rangle$ of G [25], i.e. a set of generators X and an epimorphism $\varphi : \mathcal{F}_{\mathcal{C}}(X) \to G$. Let $N = \ker \varphi$.
For a finite group, with presentation the set of generators A and the set of relations R, one expects the Cayley graph to contain circuits for each

relation $r \in R$. In combinatorial group theory (see [17]) constructing the Cayley complex means adjoining to each circuit a certain number of 2-cells. We want to imitate this construction and extend it to the infinite case. Since the Cayley graph of the free pro-$\mathcal{C}$ group $\mathcal{F}_{\mathcal{C}}(X)$ is not the projective limit of simply connected finite graphs one is forced to add some 2-cells to get a "Cayley" complex which is the projective limit of finite "Cayley" complexes. Since in our setting it seems quite unnatural to adjoin 2-cells to a simply connected complex, we will introduce two concepts: The Cayley complex and the reduced Cayley complex. To construct the Cayley complex we will adjoin 2-cells in a general but simple manner. In this case every Cayley complex will turn out to be the projective limit of finite ones. To construct the reduced Cayley complex we will adjoin 2-cell to some geodesics only. The advandage of reduced Cayley complexes is that they are much closer to "Cayley complexes" described in topology and discrete group theory (see next section for examples).

Construction 8.1.8: *Let G be a pro-$\mathcal{C}$ group with generating set X converging to 1 and $\langle X|N\rangle$ be a free presentation of G, i.e. there is an epimorphism $\varphi: \mathcal{F}_{\mathcal{C}}(X) \to G$ such that $\ker\varphi = N$.*
Take $\mathcal{B}\Gamma(\mathcal{F}_{\mathcal{C}}(X), \mu_F, X)$ the barycentric refinement of the Cayley graph of the free pro-$\mathcal{C}$ group on generators X and $\Delta := \mathcal{B}\Gamma(\mathcal{F}_{\mathcal{C}}(X)\mu_F, X)/N$. Then interpret Δ as a 1-complex (Δ, o, δ).
Form the cone $C(\mathcal{B}\Gamma)$ and define the space $C := C(\Delta)\times G \quad \sqcup \quad \Delta \times G$. Setting $\delta' := \delta \times id$ and $o' := o$ turns (C, o', δ') into a profinite 2-complex. Let $\iota_C: \Delta \to C(\Delta)$ be the natural embedding. Define complex morphisms $\iota_0, \iota_1: C \to C$:

$$\iota_0(g,f) := \begin{cases} (g,f) & : g \in C(\Delta)\times G \\ (\iota_C g, 1) & : \text{else} \end{cases}$$

$$\iota_1(g,f) := \begin{cases} (g,f) & : g \in C(\Delta)\times G \\ (\iota_C g, f) & : \text{else} \end{cases}$$

$K(G)$:= $C(\iota_0, \iota_1)$ is the **Cayley complex** *of the pro-C group G.*

Proof : We have to show that (C, o', δ') is a complex and that ι_0, ι_1 are complex morphisms. Observe $\delta'(g,f) = \delta g \times \{f\}$. Thus C is a profinite 2 complex. Because ι_C is an embedding of complexes ι_0, ι_1 turn out to be complex morphisms. □

Proposition 8.1.9: *$K(G)$ is a connected and simply connected complex. Moreover there is a disc free action of G on $K(G)$, which coincides with the natural action of G on Δ.*

Proof : By Proposition 8.1.3 each cone $C(\Delta) \times \{f\}$ is connected and simply connected. Set $\partial_i := (id, \iota_i)$, for $i = 0, 1$, and $(*, p, A)$ to be the trivial based action on C with $A := \{(1, g) : \ g \in G\}$. Then $((*, p, A), \partial_0, \partial_1)$ is a gluing scheme with all connected components, appart from $\mathcal{B}\Gamma/N$, simply connected and a simply connected underlying gluing graph. Theorem 7.2.2 yields that in the lifted gluing scheme $(\tilde{C}, \tilde{\partial}_0, \tilde{\partial}_1)$ the action of N on $\mathcal{B}\Gamma$ is anhilated in $C(\tilde{\partial}_0, \tilde{\partial}_1)$. Thus $C(\partial_0, \partial_1)$ is universal. We define an action of $\mathcal{F}_C(X)$ on $C(\eta_0, \eta_1)$.

Observe there is an action $(\mathcal{G}_A, C, \diamond, q)$ of the connected groupoid $\mathcal{G}_A$ with object set A and object groups trivial. To verify this we define q and $\diamond$. Let $q(c) := a \in A$ with $\pi_{\mathcal{F}_C(X)} a = \pi_{\mathcal{F}_C(X)} c$ and define for $(g, f) \in q^{-1} a$ and $h \in \mathcal{G}_A(a, a')$ with $a \neq a'$ the action $c \diamond h := (gh, fh)$. For $i = 0, 1$ let $\mu_i : \ \mathcal{G}_A \to \mathcal{G}_A$ be the unique maps such that $\eta_i(c \diamond f) = \eta_i c \diamond \mu_i f$ and let $\partial'_i := (\mu_i, \eta_i)$. Observe that the underlying complex of the coequalizer $C(\partial'_0, \partial'_1)$ is $C(\eta_0, \eta_1)$. The image of $\mathcal{G}_A$ acts disc free on $C(\eta_0, \eta_1)$ since it acts disc free on Δ which embeds into $C(\eta_0, \eta_1)$. It is a routine calculation that the image of $\mathcal{G}_A$ within $C(\partial'_0, \partial'_1)$ generates the group G. By construction the embedding $\Delta \to C(\eta_0, \eta_1)$ can be extended to an embedding of actions.

□

Because all peaks of the cones in C are embedded into $K(G)$, we call x a **peak in** $K(G)$ if it is the image of a peak of a cone in C.

Corollary 8.1.10: *The Cayley complex $K(\mathcal{F}_C(X))$ is the projective limit of finite Cayley complexes.*

Proof : By 4.2.1 G acts disc free on $K(\mathcal{F}_C(X))/N$. Define a relation on $K(\mathcal{F}_C(X))/N$ by setting

$$g \sim g' \Leftrightarrow \begin{cases} g = g' & : \\ \exists x & : x/N \in \beta g = \beta g', \ x \text{ a peak in } K(G) \\ \exists x, \ \exists e \in E\mathcal{B}\Gamma & : \{x/N, e\} \subset \beta g \cap \beta g', \ x \text{ a peak in } K(G) \end{cases}$$

The relation $\sim$ is closed since N is. By construction $g \sim g' \Rightarrow o(g) = o(g')$. Thus by Lemma 2.3.4 the quotient is a profinite connected 2-complex with disc free action of G. It is easily seen that $(K(\mathcal{F}_C(X))/N)/_\sim = K(G)$. Therefore we have a morphism $(\mu_N, \eta_N) : (*, p, 1_{\mathcal{F}_C(X)}) \to (*_N, p_N, 1_G)$ of based Galois actions $(\mathcal{G}(\mathcal{F}_C(X)), K(F_C(X)), *, p)$ and $(\mathcal{G}(G), K(G), *_N, p_N)$. Therefore an inverse system system for $\mathcal{F}_C(X)$ can be extended to an inverse system for $(*, p, 1_{\mathcal{F}_C(X)})$. $\square$

Corollary 8.1.11: *Every Cayley complex $K(G)$ is the projective limit of an inverse system of finite Cayley complexes $K(G/N)$.*

Proof : Restrict the inverse system of $(*, p, 1_{\mathcal{F}_C(X)})$ to all projections of $(*', p', 1_G)$ and observe that, $(*', p', 1_G)$ is the projective limit of the restricted system. $\square$

Next we will formally construct the reduced Cayley complex. Let $\mathcal{F}_C(X)$ be a free pro-$\mathcal{C}$ group and $\Gamma(\mathcal{F}_C(X), X, \mu_F, X)$ its Cayley graph. Let $\mathrm{Geo}(\Gamma, S)$ be the subspace of $\mathrm{F}\Gamma$ (= hyperspace of Γ) consisting of all the geodesics of the form $L(f, f \cdot s)$ with $s \in S \leq \mathcal{F}_C(X)$.

Construction 8.1.12: *Let G be a pro-$\mathcal{C}$ group and $\langle X|N\rangle$ be a free presentation of G. We modifiy Construction 8.1.8.*
Take $\mathcal{B}\Gamma(\mathcal{F}_{\mathcal{C}}(X), \mu_F, X)$ the barycentric refinement of the Cayley graph of the free pro-$\mathcal{C}$ group with generators X. Interpret $\mathcal{B}\Gamma$ as a 1-complex $(\mathcal{B}\Gamma, o, \delta)$.
With help of Proposition 1.2.3 each geodesic $L(f, f\cdot n)$, $f \in \mathcal{F}_{\mathcal{C}}(X)$, $n \in N$ can be uniquely lifted to a geodesic $L_{\mathcal{B}}(f, f\cdot n)$ within $\mathcal{B}(\Gamma(F_C(X), \mu_F, X))$. For each lifted geodesic $L_{\mathcal{B}}(f, f\cdot n)$ with $n \in N$ form the cone $C(L_{\mathcal{B}}(f, f\cdot n))$. Remember the definitions of ι and ι_S ($S = C(L_{\mathcal{B}}(f, f\cdot n))$) from Construction 8.1.2 and Corollary 8.1.4. Define $\iota(f,n) :\ C(L_{\mathcal{B}}(f, f\cdot n)) \sqcup_{L_{\mathcal{B}}(f, n\cdot f)} \mathcal{B}\Gamma \to$ Geo$(\mathcal{B}(\Gamma), N) \times C(\mathcal{B}(\Gamma))$ by sending c to $(L_{\mathcal{B}}(f, f\cdot n), \iota_{C(L_{\mathcal{B}}(f,f\cdot n))}(c))$ if $c \in C(L_{\mathcal{B}}(f, f\cdot n))$ and c to $(L_{\mathcal{B}}(f, f\cdot n), \iota(g))$ else.
The union $U := \bigcup_{f\in\mathcal{F}_{\mathcal{C}}(X),\ n\in N} \mathrm{im}(\iota(f,n))$ is a closed subcomplex of Geo$(\mathcal{B}(\Gamma), N) \times C(\mathcal{B}(\Gamma))$.
Set $C := U \sqcup (\{L_{\mathcal{B}}(f, f\cdot n) :\ f \in \mathcal{F}_{\mathcal{C}}(X),\ n \in N\} \times \mathcal{B}\Gamma)$. Let $\iota_C :\ \mathcal{B}\Gamma \to C(\mathcal{B}\Gamma)$ be the natural embedding. Define complex morphisms $\iota_0, \iota_1 :\ C \to C$:

$$\iota_0(L_{\mathcal{B}}(f, f\cdot n), g) := \begin{cases} (L_{\mathcal{B}}(f, f\cdot n), g) & : g \in U \\ (L_B(f, f\cdot n), \iota_C g) & : \text{else} \end{cases}$$

$$\iota_1(L_{\mathcal{B}}(f, f\cdot n), g) := \begin{cases} (L_{\mathcal{B}}(f, f\cdot n), g) & : g \in U \\ (L_B(1,1), \iota_C g) & : \text{else} \end{cases}$$

Remark that $L_{\mathcal{B}}(f, f)$ is a singelton and thus $C(L_{\mathcal{B}}(f, f))$ consists of 3 elements only.
There is a disc free action of $\mathcal{F}_{\mathcal{C}}(X)$ on $C(\iota_0, \iota_1)$. Form the quotient $C(\iota_0, \iota_1)/N$ and define an equivalence relation $\sim$ (compare proof of Corollary 8.1.10):

$$g \sim g' \Leftrightarrow \begin{cases} g = g' & : \\ \exists x & : x/N \in \beta g = \beta g',\ x \text{ a peak in } C(\iota_0, \iota_1) \\ \exists x,\ \exists e \in E\mathcal{B}\Gamma & : \{x/N, e\} \subset \beta g \cap \beta g',\ x \text{ a peak in } C(\iota_0, \iota_1) \\ \begin{array}{c} g = (L_{\mathcal{B}}(f,f), k)/N \\ g' = (L_{\mathcal{B}}(f,f), k')/N \end{array} & : k, k' \in C(L_{\mathcal{B}}(f,f))/N \end{cases}$$

Then $K_R(G) := (C(\iota_0, \iota_1)/N)/_\sim$ is the **reduced Cayley complex** *of $\mathcal{F}_{\mathcal{C}}(X)$.*

Proof : It is immediate that ι_0 and ι_1 are complex morphisms and that $\sim$ is a closed relation. Thus $K_R(G)$ exists. $\square$

Proposition 8.1.13: *$K_R(G)$ is a connected an simply connected profinite 2-complex with disc free action of G.*

Proof : The same argument (using Theorem 7.2.2) as in the proof of Proposition 8.1.9 turns $C(\iota_0, \iota_1)$ into a connected and simply connected complex with disc free action of $\mathcal{F}_C(X)$. We refere to this action as $(*, p, 1_{\mathcal{F}_C(X)})$. Compare the proof of Corollary 8.1.10 to deduce that $\sim$ is a G invariant relation. The only part to show is that $(C(\iota_0, \iota_1)/N)/_\sim$ is simply connected. We will sketch the argument. Observe that $(C(L(f, f \cdot n))/N)/_\sim$ is again a cone, i.e. the cone $C(L(f, f \cdot n)/N)$. Proposition 5.2.1 yields a surjective morphism of based actions from $(*, p, 1_{\mathcal{F}_C(X)}) \to (*', p', 1')$ where $(*', p', 1')$ is the universal action for the action of G on $(C(\iota_0, \iota_1)/N)/_\sim$. The relation $\sim$ lifts to a G invariant relation $\sim'$ on $(*, p, 1_{\mathcal{F}_C(X)})$ such that $(*, p, 1_{\mathcal{F}_C(X)})/_{\sim'} = (*', p', 1')$. Observe that there is no G invariant relation on $C(\iota_0, \iota_1)$ which is strictly contained in $\sim'$. Thus $(C(\iota_0, \iota_1)/N)/_\sim$ is simply connected. Alternatively one can define an appropriate gluing scheme and use Theorem 7.2.2 again.

□

8.2 Examples of Cayley complexes

Example 8.2.1: *Let $G := \mathcal{F}_C(X)$. Then $K_R(G)$ is the barycentric refined Cayley graph $\mathcal{B}\Gamma(G, \mu_G, X)$ and $K(G)$ is the $K_R(G)$ together with a cone $C_g(K_R(G))$ for each $g \in G$ (compare Construction 8.1.8).*

Example 8.2.2: *Let p be a natural number and C_p be the cyclic group with order p. Then the Cayley complex $K(C_p) = K_R(C_p)$ consists of a $2p$-gon P, i.e. a big circle with $2p$ edges, and p cones $C_0(P), \ldots, C_{p-1}(P)$ (see Figure 8.2.1).*

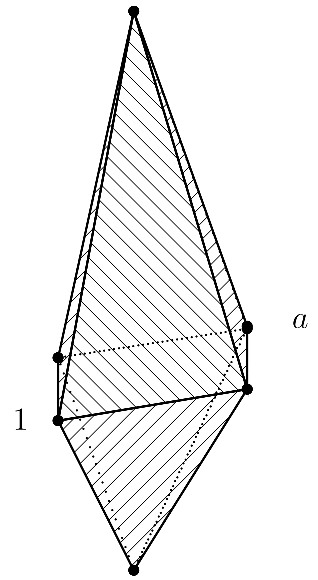


Figure 8.2.1: The Cayley complex of the cyclic group C_2 with generator a. There are 2 cones amalgamated to a rectangle visualizing the barycentric refinement of the Cayley graph of C_2. The two lables 1 and a are the image of the embedding of $V\Gamma$ into $\mathcal{B}\Gamma$. The realization of $K(C_2)$ is a sphere and the realization of the quotient $K(C_2)/C_2$ the projective space $\mathbb{P}_2$ as desired.

Example 8.2.3: *Let* $G := C_2 \times C_2 = \{(1,1),(a,1),(1,a),(a,a)\}$. *Then the Cayley complex* $K(G) \neq K_R(G)$. $K(G)$ *consist of the barycentric refined Cayley graph* $\mathcal{B}\Gamma(\{(1,a),(a,1)\},\mu,G)$ *(see Figure 8.2.2) and* 4 *cones* $C_g(\Gamma)$ *while the reduced Cayley complex is more difficult to describe.*
Let $\alpha := (a,1)$ *and* $\beta := (1,a)$ *and let* $R := \{\alpha^2,\beta^2,\alpha\beta\alpha\beta,\beta\alpha\beta\alpha\}$. *Each* r *in* R *gives rise to a geodesic* $L_{\mathcal{B}}(f,f\cdot r)$ *in* $\mathcal{B}\Gamma(\mathcal{F}_C(\{\alpha,\beta\}),\mu_F,\{\alpha,\beta\})$. *Observe that the pairs*

$$(f,r) \in \{(r,1),(r,\alpha),(r,\beta),(r,\beta\alpha):\ r \in R\}$$

are representatives for the geodesics $L_{\mathcal{B}}(f,f\cdot r)$ *which map to different subgraphs of* $K_R(G)$.
Thus $K_R(G)$ *consists of the barycentric refined Cayley graph of* G *and* 16 *adjoint cones (see Figure 8.2.3).*
Remark that $S^2 \subset \mathbb{R}^3$ *is the unit* 2*-sphere and* $[0,1]$ *the unit interval in* $\mathbb{R}$. *The geometric realization of the complex* $K(G)/G$ *is a space* $X := X_0 \sqcup_{[0,1]} X_1$, *where* $X_i = S^2/C_2$ *for* $i = 0,1$.
The complex $K_R(G)/G$ *turns out to resemble a finite version of a torus, i.e. it can be realized by forming* $Y = X_0 \sqcup_{\{0\}} X_1$ *and adjoining two disc along the path* $S_0S_1S_0S_1$ *where* 0 *is the starting point of simple closed paths* S_0 *and* S_1 *traversing a unit circle* $S^1 \subset \mathbb{R}^2 \cap X_i$ *for* $i = 0,1$ *respectively.*

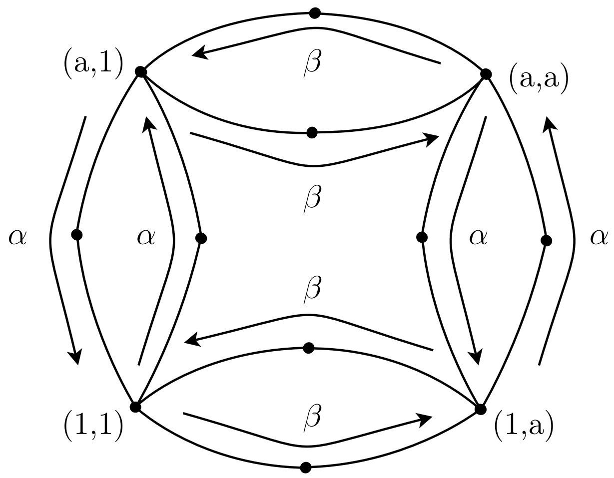


Figure 8.2.2: The barycentric refined Cayley graph $\mathcal{B}\Gamma$ of $C_2 \times C_2$ viewed as a 1-complex. Here $C_2 = \{1, a\}$ and thus $(1,1), (1,a), (a,1)$ and (a,a) are the vertices of the Cayley graph embedded into its barycentric refinement. The arrows symbolize the action of $C_2 \times C_2$ on $\mathcal{B}\Gamma$, e.g. $(1,1) * \alpha = (a,1)$, $(a,1) * \beta = (a,a)$.

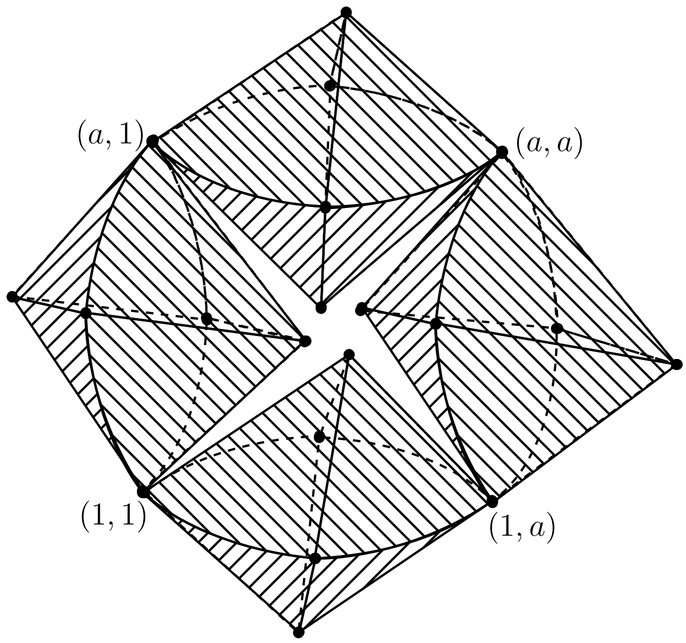


Figure 8.2.3: 8 cones of the reduced Cayley complex $K_R(C_2 \times C_2)$ are indicated. These cones correspond to the geodesics $L_{\mathcal{B}}(g, g \cdot \alpha^2)$ and $L_{\mathcal{B}}(g, g \cdot \beta^2)$ with $g \in C_2 \times C_2$. Look at the Cayley graph in figure 8.2.2. The boundaries of the cones can be determined by traversing the simple closed pathes within $\mathcal{B}\Gamma$ indicated by the relations of R (the oriented circuits labled by α and β, e.g. the "outer circle" of the Cayley graph starting at $(1, 1)$ corresponds to the relation $\beta\alpha\beta\alpha$).

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List of abbrevations

$(*, p)$	Shorthand for $(\mathcal{G}, (X, o, \delta), *, p)$.
$(*, p, A)$	Shorthand for $(\mathcal{G}, ((X, A), o, \delta), *, p)$.
<u>BGalAct</u>	The category of based Galois actions.
<u>BasedAct</u>	The category of based actions.
<u>ContAct</u>	The category of continuous actions.
<u>GalAct</u>	The category of Galois actions.
<u>Groupoid</u>	The category of (abstract) groupoids $(\mathcal{G}, \cdot)$.
<u>Group</u>	The category of groups.
<u>Pgroupoid</u>	The category of profinite groupoids $(\mathcal{G}, d_0, d_1)$ and continuous groupoid morphisms.
<u>Set</u>	The category of sets.
<u>Top</u>	The category of topological spaces X and continuous maps.
<u>p-OGraph</u>	The category of oriented graphs.
<u>p-UGraph</u>	The category of unoriented graphs.
LOT	Linearly ordered topological.
pc-groupoid	Profinite continuous groupoid.

List of symbols

$\mathcal{F}_{\mathcal{C}}(X)$	The free pro-$\mathcal{C}$ group with generating set X sort.
$L(v, w)$	The geodesic connecting v and w. Both v and w are vertices in the Cayley graph of a free pro-$\mathcal{C}$ group $\mathcal{F}_{\mathcal{C}}(X)$.
$(\mathcal{G}, ((X, A), o, \delta), *, p)$	A based action of a pc-groupoid on a profinite 2-complex.
$(\mathcal{G}, (X, o, \delta), *, p)$	A continuous action of a pc-groupoid on a profinite 2-complex.
(μ, η)	A morphism of actions or based actions.
$x\mathcal{G}$	The orbit of x under the action of $\mathcal{G}$ defined by $x\mathcal{G} := \{x * g : \ g \in \mathcal{G}\}$.
$(G, (X, o, \delta), \mu)$	A profinite action of a profinite group G on a profinite 2-complex (X, o, δ).

${}_x\mathcal{G}$	The stabilizer of x under the action of $\mathcal{G}$ defined by ${}_x\mathcal{G} := \{g \in G : x * g = x\}$.
$(\tilde{\mathcal{G}}, ((\tilde{X}, \tilde{A}), \tilde{o}, \tilde{\delta}), \tilde{*}, \tilde{p})$	This is often used for universal Galois actions.
$(\Gamma_1 \sqcup_A \Gamma_2, d_0, d_1)$	The oriented adjunction graph along the embedding of A into Γ_1 and Γ_2.
$(\mathcal{B}\Gamma, d_0^{\mathcal{B}}, d_1^{\mathcal{B}})$	The oriented barycentric refinement of the graph (Γ, d_0, d_1).
$\mathcal{S}$	The class of all oriented big circles. $\mathcal{S}_u$ is the class of unoriented big circles.
I	The class of all big lines.
φ	In the first chapter "φ" is designated for representing the class function $I \to \mathcal{S}$.
π_I	The quotient map sending a big line I to the big circle S obtained from I.
βx	The boundary of $x \in (X, o, \delta)$ defined by $\beta x = \delta x \setminus \{x\}$.
$\mathcal{C}^{\mathrm{opp}}$	The dual category, i.e. the category with object set $\mathrm{Ob}(\mathcal{C})$ and $\mathcal{C}^{\mathrm{opp}}(A, B) = \mathcal{C}(B, A)$ for all objects A, B.
$K(G)$	The Cayley complex of G.
$\Gamma(G, \mu, X)$	The Cayley graph of the pro-$\mathcal{C}$ group G and generators X.
$K_R(G)$	The reduced Cayley complex of G.
U^c	If U is a subset of a set X, which is clear from context, then U^c is the complement of U in X.
(X, o, δ)	A (pre-)complex with underlying space X, dimension map o and disc map δ.
$((X, A), o, \delta)$	A based profinite 2-complex, i.e. A is a set of representatives of the connected components of (X, o, δ).
$K(X, a)$	A maximal simply connected and connected subcomplex of X containing a.
$(C(X), o', \delta')$	The cone over a complex X. $C(X)$ is a connected and simply connected complex.
D^2	A homeomorphic copy of the open unit ball in $\mathbb{R}^2$.
$O(A)$	The dimension of a subset A of a (pre-)complex (X, o, δ) defined by $O(A) := \sup_{a \in A} o(a)$.
EX	Subspace of edges of an (un)oriented graph or a complex.

$\mathfrak{U}(x)$	The neighbourhood filter of x.
$E(\varphi, \psi)$	The equalizer of two morphisms $\varphi, \psi : X \to Y$.
$C(\varphi, \psi)$	The coequalizer of two morphisms $\varphi, \psi : X \to Y$.
$F(\Gamma)$	The free groupoid generated by an oriented graph Γ.
$\mathrm{F}X$	The hyperspace of closed subsets of X.
$\binom{X}{n}$	The space of subsets of X with cardinality at most n.
$\mathrm{Geo}(\Gamma, S)$	The hyperspace of geodesics in Γ of the form $L(f, f \cdot s)$, where Γ is the Cayley graph of a free pro-$\mathcal{C}$ group $\mathcal{F}_{\mathcal{C}}(X)$ and s is a subgroup of $\mathcal{F}_{\mathcal{C}}(X)$.
$L_{\mathcal{B}}(f, f \cdot n)$	The unique lifting of the geodesic connecting $L(f, f \cdot n)$ to the barycentric refinement of the Cayley graph $\mathcal{B}\Gamma(\mathcal{F}_{\mathcal{C}}(X), \mu_F, X)$.
$\mathcal{G}(G)$	The groupoid with one vertex and vertex group G.
$(*, p, A), \partial_0, \partial_1)$	A gluing scheme of a based action with gluing morphisms ∂_0, ∂_1.
$(\mathcal{G}, \partial_0, \partial_1)$	A graph of groups with gluing morphisms ∂_0 and ∂_1.
(X, δ)	An unoriented graph with boundary map $\delta : \Gamma \to \binom{X}{2}$.
$\mathcal{G}$	A groupoid $\mathcal{G}$. Here $\mathcal{G}$ is an acronym for the algebraic $(\mathcal{G}, \cdot)$ or the profinite $(\mathcal{G}, d_0, d_1)$.
$\langle E \vert R \rangle$	A presentation of a group by a set of generators E and a set of relations in these generators R.
$\langle E \vert N \rangle$	A free presentation of a free pro-$\mathcal{C}$ group G by a set of generators E and a a normal subgroup of $N \lhd \mathcal{F}_{\mathcal{C}}(E)$ such that $\mathcal{F}_{\mathcal{C}}(E)/N$ is G.
$\mathrm{Hom}_{\mathcal{C}}$	The morphisms of a category $\mathcal{C}$.
$\mathcal{C}(x, y)$	The set of morphisms $h : x \to y$, with $x, y \in \mathrm{Ob}(\mathcal{C})$. The shorthand for $\mathcal{C}(x, x)$ is $\mathcal{C}(x)$.

$1_x,\ 1_X$	This symbols stand for the identity morphism on X, or the identity element of the object group $\mathcal{G}(x)$, or the trivial groupoid with object set X.
$\mathrm{im}(F)$	Denotes the image of the morphism or functor F.
$(a,b)_\leq$	An open interval of a linearly ordered set or LOT space $(X,\leq)$, with infimum a and supremum b. If the order is clear from context $(a,b)_\leq$ is abbreviated to (a,b). Closed and half open intervals are $[a,b]_\leq,\ [a,b)_\leq$ and $(a,b]_\leq$.
$(X_\alpha, p_{\alpha\beta})$	Inverse system of objects X_α, i.e. $\alpha \in \Lambda$ is a directed set and for all $\alpha > \beta$ there is a morphism $p_{\alpha\beta} : X_\alpha \to X_\beta$ such that $p_{\alpha\gamma} = p_{\beta\gamma}p_{\alpha\beta}$ if $\alpha > \beta > \gamma$.
$\ker F$	The kernel of a morphism F, i.e. all elements mapped to identities.
$\mathrm{colim}\, F$	The colimit of a functor F.
$(X,\leq)$	A linearly ordered set or a LOT space.
(X^*, d_0, d_1)	The oriented graph generated by a LOT space $(X,\leq)$ with order relation "$\leq^*$".
$X/_\sim,\ X/N$	The quotient of X modulo a relation $\sim$ or a relation generated by N.
$\mathcal{H} \lhd \mathcal{G}$	$\mathcal{H}$ is a normal subgroupoid of $\mathcal{G}$. For groups write $H \lhd G$.
$\mathcal{H} \lhd_{\mathrm{fin}} \mathcal{G}$	$\mathcal{H}$ is a normal subgroupoid of $\mathcal{G}$ with finite index. For groups write $H \lhd_{\mathrm{fin}} G$.
$\mathrm{Ob}(\mathcal{C})$	The objects of a category $\mathcal{C}$.
$\mathbb{P}$	A partition of some topological space.
$\pi_1^{\mathcal{C}}(\Gamma, d_0, d_1)$	The pro-$\mathcal{C}$ fundamental group of the oriented graph Γ.
$\prod_{i\in I} X_i$	The product of objects X_i over the index set I, e.g. $\prod_{i\in I}(X_i,\delta_i)$ is a product of unoriented graphs. If I is finite $\prod_{i\in I} X_i = X_1 \times \cdots \times X_n$.
$\coprod_{i\in I} X_i$	The coproduct of objects X_i over the index set I, e.g. $\coprod_{i\in I}(X_i,\delta_i)$ is a coproduct of unoriented graphs. If I is finite $\coprod_{i\in I} X_i = X_1 \sqcup \cdots \sqcup X_n$.

$\varprojlim_{\lambda\in\Lambda} X_\lambda$	The projective limit of the inverse system of the X_α formed in the respective category, e.g. $\varprojlim_{\alpha\in\Lambda}(X_\alpha,\delta_\alpha)$ is considered in the category $\underline{\text{P-UGRAPH}}$ while $\varprojlim_{\alpha\in\Lambda} X_\alpha$ is formed in the category $\underline{\text{TOP}}$.
$\mathbb{N}$	The natural numbers (containing 0).
$\mathbb{R}$	The real numbers.
$\restriction$	The restriction of a relation, or commonly of a function, e.g for $f:\ A\to B$ and $C\subset A$ one has $f\restriction C:\ C\to B$.
S^j	(Pre-)complexes of spherical type. In this work $j=-1,0,1$ only.
$S(\mathcal{G},\beta,H)$	The standard graph of a graph of groups.
$<$	Symbolizes either an order relation or a substructure relation. In the latter case e.g. $H<G$, $\mathcal{H}<\mathcal{G}$ reads H is a subgroup of G and $\mathcal{H}$ is a subgroupoid of $\mathcal{G}$.
a^+	For $a\in(X,\leq)$, a^+ denotes the successor of a.
VX	Subspace of vertices of an (un)oriented graph or a complex.
2^X	The hyperspace of closed subsets of a topological space X.
$W(U_1,\ldots,U_n)$	A base set of the Vietoris topology, on 2^X, with respect to the open sets $U_1,\ldots,U_n\subseteq X$.
$X^{\leq n}$	The n-skeleton of X. Thus $X^{\leq n}$ contains cells x with dimension $o(x)\leq n$. The symbols $X^{<n}$, $X^{=n}$, $X^{\geq n}$ and $X^{>n}$ are defined analogously.
$\mathbb{Z}$	The integers.
$\hat{\mathbb{Z}}$	The profinite completion of the group $\mathbb{Z}$. $\hat{\mathbb{Z}}_C$ is the pro-C completion and $\hat{\mathbb{Z}}_p$ the pro-p completion of $\mathbb{Z}$.

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