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Non-classical approaches to Judgment Aggregation

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Michael Reitgruber, BSc.

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Kurzfassung

Urteilsaggregation befasst sich mit der Bildung eines gemeinsamen Urteils einer Gruppe aus den individuellen Urteilen der Gruppenmitglieder. Es kann als Generalisierung der klassischen Präferenzaggregation aus der Social Choice Theory gesehen werden. Daher ist es keineswegs überraschend, dass viele der bekannten Resultate, wie eine Variante von Arrows Theorem, auch für die Urteilsaggregation gelten. Das Umgehen jener Unmöglichkeiten und Paradoxa stellt daher auch eines der zentralen Forschungsgebiete der Urteilsaggregation dar. In letzter Zeit haben sich auch nicht-klassische Logiken als nützliches Werzeug zu diesem Zweck bemerkbar gemacht. Das Ziel dieser Arbeit ist es eine umfassende, einheitliche Übersicht über existierende Forschungsergebnise im Bereich der nicht-klassischen Urteilsaggregation zu erstellen, welche als Einstiegspunkt für weiterführende Forschung in dem Gebiet dienen soll.

Abstract

Judgment aggregation deals with the problem of collecting judgments on a predefined set of issues from a group of individuals into a single consistent judgment set. It can be seen as a generalization of classical preference aggregation from social choice theory. As such it runs into many of the same impossibilities and paradoxes, like a judgment aggregation variant of Arrow's Theorem. Similar to social choice theory, circumventing the impossibilities lies at the heart of the study of judgment aggregation. In recent years non-classical logics have shown to be a viable tool in that regard. With this work we want to give a comprehensive, uniform overview over existing research employing non-classical logics of any kind to study judgment aggregation, so that it can be used as a starting point for pursuing further research in the field.

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CHAPTER 1

Introduction

When faced with the task of having to decide on important matters one usually tends to prefer decisions made by a larger group of people (for example by some sort of majority vote), which is a commonplace occurrence in a democratic society. However, as will be illustrated by a number of examples, this is not always as straightforward as it might seem at first. One of the earliest examples of a case where such a simple democratic process falls short is the so-called 'Doctrinal Paradox', which we will briefly outline here, following Kornhauser and Sager [KS93]:
Suppose three judges have to come to a conclusion in a lawsuit over a breached contract. There are three propositions the court has to decide over:

- p: The defendant was contractually obliged to refrain from an action.
- q: The defendant did that action.
- r: The defendant is liable for a breach of contract.

According to contract law, whenever propositions p and q are the case, then also r has to be the case and vice versa. One might write this as a formal constraint of the form $(p \wedge q) \leftrightarrow r$. Next, suppose the judges cast their votes as follows:

	p	q	r
Judge 1	true	true	true
Judge 2	true	false	false
Judge 3	false	true	false
Majority	true	true	false

Table 1.1: An example of the doctrinal paradox

The question that now arises is whether the final decision should be based only on the majority vote on the conclusion r, or the vote on the individual premises p and q together with the prescriptions of contract law. Either case gives rise to an inconsistency. In the first case, often called conclusion based aggregation, the verdict will be 'not liable', while contract law clearly dictates that the defendant should be liable. In the second case, the premise based aggregation, the verdict will be 'liable', while the majority of judges believes the defendant to be not liable. Dietrich [DL08a] discusses judgment aggregation under this formalism.

Another way to look at this problem is by collecting the legal constraints together with the majority judgment into a set. This idea, called the *Discursive Dilemma*, was first presented by Pettit [Pet01] and Brennan [Bre01] as a more general approach to the topic. It is shown to be equivalent to the constraint based formalism in Endriss et al. [EGDHL16]. In case of the above example the set would be $\{p, q, \neg r, (p \wedge q) \leftrightarrow r\}$. This is clearly an inconsistent set. The advantage of this approach is that it treats each proposition equally and as such moves away from differentiating between premise and conclusion based aggregation, which allows the problem to be studied more easily, since it might not always be straightforward to partition propositions into to two groups. List and Pettit [LP02] were the first to study the above in an entirely formal setting.

Characterizing what exactly causes such inconsistencies to arise as well as finding ways to avoid them lies at the heart of Judgment Aggregation Theory.

There is a large amount of work available on the topic of classical judgment aggregation theory. Mongin and Dietrich [MD10], Mongin [Mon12] [Mon18] and List and Polak [LP10] provide a comprehensive overview, with Grossi and Pigozzi [GP14], List and Puppe [LP09] and Endriss [$\mathrm{E}^{+}16$] presenting the topic on a more introductory level. List [Lis07] presents an expansive bibliography concerning the doctrinal paradox, discursive dilemma and classical judgment aggregation in general. In contrast, use of non-classical logics as an approach has only started fairly recently. We aim to provide a systematic, unified view of the state-of-the-art of applications of non-classical logics to the field of judgment aggregation. A comprehensive review of the existing literature has shown that many of the existing works differ in their approach and as a result we will adopt a uniform terminology which we will use to present the results. We furthermore aim to recall connections to related topics, such as belief binarization. And finally we point out open questions remaining within the research area. As such this work can then be used as a starting point for future research in the field. While some overviews and surveys are available, most of them focus only on a specific topic and to our knowledge no comprehensive work has been published as of yet.

The rest of this work will be structured as follows. Chapter 2 introduces a general framework for judgment aggregation and recalls some results from the classical variant. In chapter 3 we will look at what happens when judgments are allowed to be fuzzy. Chapter 4 presents ways various other non-classical logics are used to analyze the problem of judgment aggregation. This includes substructural logics as well as conditional logic. A step away from the internal workings of JA, chapter 5 considers different approaches to reason about Judgment Aggregation. Finally, chapter 6 will provide the reader with a short outlook over what is still to come with regards to non-classical judgment aggregation.

CHAPTER 2

Basics of Judgment Aggregation

As a generalization of the above example we will now introduce a formal framework which will be used throughout this work when presenting various results in JA.

2.1 Formalization

In the following we will recall a general model for Judgment Aggregation first introduced by Dietrich [Die07] and subsequently updated in Dietrich and Mongin [DM10], starting with the definition of a general logic.

This kind of an abstract view on logic is convenient for our purposes, since it allows us to move away from any kind of concrete logical formalism and state properties of judgment aggregation theory in a general sense. An example of an important property of the theory are the so called rationality notions. They are consistency of judgment sets on the one hand and deductive closure on the other. One can define these properties both in a semantic (using entailment) or syntactic way (using provability/derivability).

For our purposes, it does not matter whether we define these rationality notions in a syntactic or semantic way. During the definitions a single relation will be used and it can be interpreted as either an entailment relation $\models$ or a derivability relation $\vdash$.

A general logic (with negation $\neg$) is now defined as a pair $(\mathbf{L}, \models)$ that consists of:

- A non-empty set $\mathbf{L}$ of propositions, such that whenever some $p \in \mathbf{L}$, then also $\neg p \in \mathbf{L}$.
- A binary relation $\models$ between sets of propositions $A \subseteq \mathbf{L}$ and propositions $p \in \mathbf{L}$. This relation is called the entailment relation and whenever $A \models p$, we read A entails p.

Using the entailment relation defined above, it is now possible to define the rationality notions as follows:

(1) *Consistency*: A set of propositions A is called inconsistent, whenever there is some $p \in A$, such that $A \models p$ and $A \models \neg p$. If no such p can be found A is called consistent.

(2) *Deductive Closure*: A set of propositions A is called deductively closed, whenever for any proposition p, $A \models p$ implies $p \in A$.

In order to obtain the well known impossibility results, one has to impose a number of (rather sensible) conditions on this general logic. They are:

(1) *self-entailment*: For every $p \in \mathbf{L}$, $p \models p$

(2) *monotonicity*: For every $p \in \mathbf{L}$ and every $A \subseteq B \subseteq \mathbf{L}$, if $A \models p$, then $B \models p$.

(3) *completability*: The empty set is consistent, and if $A \subseteq \mathbf{L}$ is consistent, then it has a consistent superset $B \subseteq \mathbf{L}$ that contains either p or $\neg p$ for each $p \in \mathbf{L}$

(4) *non-paraconsistency*: For every $A \subseteq \mathbf{L}$ and every $p \in \mathbf{L}$, if $A \cup \neg p$ is inconsistent, then $A \models p$

(5) *compactness*: For every $A \subseteq \mathbf{L}$ and $p \in \mathbf{L}$, if $A \models p$, then $B \models p$ for some finite $B \subseteq A$.

A general logic satisfying $(1) - (3)$ is sufficient for many problems of judgment aggregation. Some of them also require (4) and (5), however even then these properties are still satisfied by a lot of the logics used in these aggregation problems. An extension of this general logic that also encompasses non-monotonic logics can be found in Wen [Wen18].

In the following we will assume that every logic satisfies at least $(1) - (3)$ if not stated otherwise.

With the notion of a general logic firmly defined, we are now ready to formalize the typical components of judgment aggregation. Let $N = \{1, 2, ...n\}$, $(n \geqslant 2)$ be the group of individuals making judgments over some propositions.

The **agenda** is the set of propositions over which judgments are being made. Formally it is a set $X \subseteq \mathbf{L}$, where for each non-negated proposition $p \in X$ we also have $\neg p \in X$.

A **judgment set** is a subset $A \subseteq X$ of the agenda. $D(X)$ is the set of consistent and complete judgment sets over X. For any proposition $p \in A$ we say that p is accepted. A judgment set can have a number of properties, including:

- *Completeness*: For every $p \in X$, either $p \in A$ or $\neg p \in A$
- *Weak consistency*: For every $p \in X$, it is not the case that both $p \in A$ and $\neg p \in A$
- *Consistency*: There is no $p \in L$, for which both $A \models p$ an $A \models \neg p$
- *Deductive closure*: For every $p \in X$, if $A \models p$, then $p \in A$
- *Full rationality*: A is both consistent and complete.

Dietrich shows that, depending on the properties of the underlying language, some of the above rationality conditions are interrelated:

Whenever the underlying language satisfies *self-entailment*, *monotonicity* and *completability* one gets that:

- consistency implies weak consistency
- full rationality implies completeness, weak consistency and deductive closure

When the language also satisfies *non-paraconsistency*, the above implications become equivalences.

An **aggregation rule** is some function F that takes a profile $(A_1, ..., A_n)$ of individual judgment sets (taken from a given set of admissible profiles, called the domain of F or dom(F)), to a collective judgment set A. A profile is a n-tuple of individual judgment sets, with each A_i corresponding to the judgments made by the i-th individual. We now define a number of sensible requirements such an aggregation rule should ideally fulfill:

Universal Domain: The domain of F is the set of all profiles $(A_1, ..., A_n)$ of fully rational judgment sets.

Collective Rationality: The collective judgment set $F(A_1, ..., A_n)$ is fully rational for every profile $(A_1, ..., A_n) \in dom(F)$.

Anonymity: For any two profiles $(A_1, ..., A_n)$ and $(A'_1, ..., A'_n)$ ($\in dom(F)$), that are permutations of each other, $F(A_1, ..., A_n) = F(A'_1, ..., A'_n)$.

Neutrality: For every $p, q \in X$ and profile $(A_1, ..., A_n)$, if $p \in A_i \iff q \in A_i$ for every individual i, then $p \in F(A_1, ..., A_n) \iff q \in F(A_1, ..., A_n)$.

Systematicity: For every $p, q \in X$ and profiles $(A_1, ..., A_n)$,$(A'_1, ..., A'_n) \in dom(F)$, if for all i, $p \in A_i \iff q \in A'_i$, then $p \in F(A_1, ..., A_n) \iff q \in F(A'_1, ..., A'_n)$.

Independence: For every $p \in X$ and profiles $(A_1, ..., A_n)$,$(A'_1, ..., A'_n) \in dom(F)$, if for all i, $p \in A_i \iff p \in A'_i$, then $p \in F(A_1, ..., A_n) \iff p \in F(A'_1, ..., A'_n)$.
Informally this condition states that the collective judgment for any proposition p should only depend on the individuals judgments on this exact proposition.

Unanimity Preservation: For every proposition $p \in X$ and every profile $(A_1, ..., A_n)$, if $p \in A_i$ for every individual i, then $p \in F(A_1, ..., A_n)$.

Monotonicity: For every proposition $p \in X$ and every pair of profiles $(A_1, ..., A_n)$, $(A'_1, ..., A'_n)$, if $p \in A_i \implies p \in A'_i$ for every individual i and $p \notin A_j$, $p \in A'_j$ for some j, then $p \in F(A_1, ..., A_n) \implies p \in F(A'_1, ..., A'_n)$.

Acceptance-rejection neutrality: For every $p, q \in X$ and profile $(A_1, ...A_n)$, if $p \in A_i \iff q \notin A_i$ for every individual i, then $p \in F(A_1, ..., A_n) \iff q \notin F(A_1, ..., A_n)$.

Veto-dictatorial: For some $i \in N$, $p \in X$ and every judgment profile $(A_1, ..., A_n)$, $p \notin A_i \implies p \notin F(A_1, ..., A_n)$.

Dietrich and List [DL07d] show that the conditions of Independence together with Monotonicity prevent the alteration of the outcome of the aggregation by manipulating the agenda. However even with these conditions in place there is still some possibility of manipulation via different approaches, as was shown by Cariani et al. [CPS08]

A few examples of aggregation rules are:

- *Majority rule*: $F(A_1, ..., A_n) = \{p \in X : |\{i \in N : p \in A_i\}| > n/2\}$.
- *Quota rule with threshold t*: $F(A_1, ..., A_n) = \{p \in X : |\{i \in N : p \in A_i\}| >= t\}$. A quota rule is called strict if the inequality is strict. It is also possible to specify an acceptance threshold for each proposition individually.
- *Dictatorship*: $F(A_1, ..., A_n) = A_i$ for some fixed individual i
- *Premise based rule*: Let Y be the set of premises and G be a majority aggregation rule. Then a premised based aggregation rule is: $F(A_1, ..., A_n) = \{p \in X : G(A_1, ..., A_n) \cap Y \models p\}$.

The majority rule satisfies independence, but violates collective rationality even for very simple agendas (which can be seen in the example below). Nehring and Puppe [NP02] and [NP08] as well as Dietrich [DL07b] provide a detailed discussion on majority voting in particular. A Dictatorship satisfies both independence and collective rationality, but does so in a trivial sense. The premise based rule violates independence, because of the way the conclusions are chosen, but it satisfies collective rationality and as such produces consistent judgment sets.

As an example of a judgment aggregation problem formalized in this framework, we will use a modified version of the court example in the beginning.

The agenda is $X = \{p, \neg p, q, \neg q, r, \neg r, (p \wedge q) \leftrightarrow r, \neg((p \wedge q) \iff r))\}$. The individual judgment sets are as follows:

- $A_1 = \{p, q, r, (p \wedge q) \leftrightarrow r\}$
- $A_2 = \{p \neg q, \neg r, (p \wedge q) \leftrightarrow r\}$
- $A_3 = \{\neg p, q, \neg r, (p \wedge q) \leftrightarrow r\}$

It can easily be seen that each of the individual judgment sets are fully rational. Now let F be a simple majority rule, then $A = F(A_1, A_2, A_3) = \{p, q, \neg r, (p \wedge q) \leftrightarrow r\}$. The collective judgment set A is now inconsistent according to our definition because both $A \models r$ (since $\{p, q, (p \wedge q) \leftrightarrow r\} \models r$) and $A \models \neg r$ (since $\neg r \in A$).

2.2 The central impossibility theorem

We are now almost ready to present a version of the impossibility that lies at the center of judgment aggregation theory. But before we are able to do that it is necessary to introduce a condition that, in a sense, requires the agenda to contain a minimum amount of complexity (the impossibility theorems fall through for sufficiently simple agendas). For this we introduce the notion of **path-connectedness**.

Call a set of propositions $Y \subseteq \mathbf{L}$ minimally inconsistent, iff it is inconsistent and every proper subset of Y is consistent. An example of a minimally inconsistent set would be $\{p, p \rightarrow q, \neg q\}$, but not $\{p, q, \neg q\}$.

A proposition $q \in X$ **conditionally follows** from another proposition $p \in X$, written $p \models^* q$, if $p \neq \neg q$ and there is some minimally inconsistent $Y \subseteq X$ such that $\{p, \neg q\} \subseteq Y$. This can also be stated as $p \models^* q$, whenever $\{p\} \cup Y' \models q$, for some minimal set of additional premises Y' that are neither inconsistent together with p, nor with $\neg q$.

Now an agenda X is called **path-connected**, if for any $p, q \in X$, there are auxiliary formulas $p_1, ..., p_m \in X$, such that $p = p_1 \models^* p_2 \models^* ... \models^* p_m = q$.

This might at first glance look like a complicated condition. However even very simple Agendas already satisfy it. An example would be the courtroom agenda from above.

Using this definition, we can now state the following theorem:

Theorem 1 ([Mon18]) *If the agenda X is path-connected, then any aggregation rule F satisfying Universal Domain and Collective Rationality is either a Dictatorship or violates one or more of Unanimity Preservation, Monotonicity and Independence.*

This version of the canonical theorem originates from Nehring and Puppe [NP10], with slightly different versions appearing in Dokow and Holzman [DH05] and Dietrich and List [DM10]. Further impossibility results for classical judgment aggregation have been stated in Pauly and van Hees [PV06], Mongin [Mon08], Nehring and Puppe [NP08] as well as Dokow and Holzman [DH09] and [DH10]. List and Pettit [LP04] provide a comparison between this impossibility theorem and Arrow's [Arr63] famous impossibility theorem from social choice theory (see Gaertner [Gae09] for an introduction).

Imposing additional conditions onto the agenda allows some of the conditions (like Monotonicity) to be dropped. However similar impossibility theorems can always be obtained.

2.3 Avoiding the impossibility

Significant work has been put into finding ways to circumvent the impossibility dictated by the above mentioned theorem. The main focus of this work lies in exploring the ways non-classical logics can be applied to resolve the impossibilities. Other approaches, which we will briefly outline here, deal with the relaxation of the conditions placed upon the aggregation rules. Dietrich and List [DL07b], Nehring and Puppe [NP07] and Nehring et al. [NPP14] are only some examples that deal with this. A more comprehensive overview over these approaches can be found in [Lis12].

While relaxing Universal Domain and Collective Rationality is a strategy that is being explored in the literature, it might not be entirely desirable, since both are rather natural conditions to be expected of an aggregation rule. A more promising approach can be found in dropping the condition of Independence. A kind of rule that gives up Independence would be the premise based rule mentioned above. It was first introduced under the name *issue-by-issue voting* in Kornhauser and Sager [KS86] and Kornhauser [Kor92]. Further work on the premise based rule can be found in Petit [Pet01], List and Petit [LP02], Chapman [Cha02], Bowens and Rabinowicz [BR06] and Dietrich [Die06]. Another instance would be the so-called distance-based rules that try to minimize the distance between individual judgment sets and the collective judgment set. See Pigozzi [Pig06], Miller and Osherson [MO09], Lang et al. [LPSvdT11] and Endriss et al. [EGP10] for discussion of distance based aggregation.

CHAPTER 3

Many Valued Judgment Aggregation

Pauly and van Hees [PV06] and van Hees [VH07] first considered the application of many valued logics for judgment aggregation. They show that switching to a many-valued framework is usually not sufficient to avoid the impossibilities. In this chapter we present a few different approaches to many valued judgment aggregation that open up the way to possibilities, with the main focus on two abstract characterization results that help narrow down exactly when impossibilities occur. Finally we relate many-valued judgment aggregation to other fields such as belief-binarization.

3.1 Three Valued Logics

Perhaps the most natural way to come across the need for many-valuedness in judgment aggregation is when one considers the case where individuals have the choice of abstaining from judgment on some propositions. In [SJ11] Slavkovik et al. present an adaption of the so-called distance-based aggregation rules for the three-valued case. They then go on to extend this approach to allow assigning weights to agents (an approach already considered by Revesz [Rev97] in the context of information merging) as well as individual judgments, modeling a case in which certain individuals can have higher authority on some topics.

They give three examples of ternary logics that could be used, each of which assigns a different meaning to the third truth value:

- Łukasiewicz Logic: Here the third truth value is usually called $\frac{1}{2}$ and can be thought of to be a middle ground between true and false, i.e. signifying that the truth of the statement will be determined at a later state.

- Kleene Logic: In Kleene Logic, the three truth values are $\{T, I, F\}$, with I corresponding to an undefined truth value.
- Bochvar Logic: In this logic the third truth value is interpreted to mean "meaningless".

In the following Łukasiewicz Logic will be considered the underlying logic; however the definitions are analogous for the other logics. An interesting thing to note here is that Kleene Logic, together with a suitable inference relation $\models_K$, is not a member of the general logics defined in the introduction, while Łukasiewicz Logic (with $\models_L$) is. As such the classic impossibility results for *propositionwise aggregation* hold for Łukasiewicz Logic.

The underlying framework of judgment aggregation considered in their work is similar to the standard one defined in the introduction, where instead of judgment sets one considers judgment *sequences* $A_i \subseteq \{0, \frac{1}{2}, 1\}^{|X|}$. $A_i(a_j)$ denotes the judgment on the j-th element in judgment sequence A_i. It is always possible to go between judgment sequences and judgment sets by constructing a judgment set J_i as follows: For all $a \in A_i$: $a \in J_i$ iff $v(a) = 1$; $\neg a \in J_i$ iff $v(a) = 0$ and $a \notin J_i$ iff $v(a) = \frac{1}{2}$.

A distance based aggregation rule for ternary logics can now be defined as a function $\Delta^{d,\odot}$: $D(X)^n \to 2^{D(X)}$, such that $\Delta^{d,\odot}(A_1, ..., A_n) = argmin_{A \in D(X)} \odot (d(A, A_1), ..., d(A, A_n))$. Here $\odot$ is any function that satisfies non-decreasingness, minimality and identity, called an *aggregation function*. Examples of such aggregation functions are max and Σ. d is a distance function which, for all judgment sequences A, A', A'' in its domain, satisfies that $d(A, A') = 0$ iff $A = A'$; $d(A, A') = d(A', A)$ and $d(A, A') + d(A', A'') \geqslant d(A, A'')$ (triangle inequality). Examples are:

- Drastic Distance: $d_D(A, A') = 0$ iff $A = A'$, else $d_D(A, A') = 1$, i.e. it indicates whether the two judgment sequences differ.
- Hamming Distance: $d_H(A, A') = \sum_{i=1}^{|X|} \delta_H(A(a_i), A'(a_i))$, where $\delta_H(a, b) = 0$ iff $a = b$ and 1 otherwise, i.e. it counts the number of differences in the two judgment sequences.
- Taxicab Distance: $d_T = \sum_{i=1}^{|X|} |A(A_i) - A'(a_i)|$. The taxicab metric gives less weight to the difference between judgments of value $\frac{1}{2}$ and $\{0, 1\}$. However should all judgments in the sequence be binary it coincides with the Hamming distance.

The choice of distance function in the aggregation rule decides how the third truth value will affect the collective judgment. By choosing either the Hamming or Drastic distance, the third value will be interpreted as an alternative to accepting or rejecting a proposition, while the Taxicab Distance gives the third value an intermediate value between the two. By assigning a distance of zero from the third value to any of the others one causes the third value judgments (i.e. abstentions) to be ignored for the collective judgment set.

Of course it is also desirable that such a distance based aggregation rule satisfies a set of properties similar to propositionwise aggregation rules. The aggregation rule defined above satisfies *Universal Domain* by definition and *Anonymity* whenever the chosen aggregation function is symmetric. An aggregation function $\odot$ is symmetric when for all permutations π and argument vector x, $\odot(x) = \odot(\pi(x))$. The property of *Independence* however does not hold for distance based aggregation rules.

The above definition of a distance based aggregation rule can now easily be extended to allow for the specification of weights for each (individual, agenda item) pair. It has to be noted that in the following any distance function is assumed to be *granular*, where a granular distance function d^g is of the form $d^g(A, A') = \circledast_{i=1}^{|X|} \delta(A(a_i), A'(a_i))$, where $\circledast$ is a symmetric aggregation function with a unique minimum in 0. Intuitively a granular distance function is a distance function for which the distance between two judgment sequences can be expressed as some aggregation of the distances between the individual elements of the sequences. From the given distance functions the Hamming and Taxicab distances are granular, while the drastic distance is not. Now define a *weight matrix* $W = [w_{i,j}]_{n \times |X|}$, where $w_{i,j}$ signifies the weight of individual i's judgment on the j-item of the agenda. A distance based aggregation rule with weights then is a function $\Delta_W^{d^g,\odot} : D(X)^n \times (\mathbb{R}^+)^{(n \times |X|)} \rightarrow 2^D(X)$, such that $\Delta_W^{d^g,\odot}(A_1, ..., A_n, W) = argmin_{A \in D(X)} \odot_{i=1}^{n} \circledast_{j=1}^{|X|} w_{i,j} * \delta(A(a_j), A_i(a_j))$.

The weights can either be defined by the individual giving the judgment or by the one that aggregates the individual judgments, depending on the context. When the individual itself gives the weight it can be interpreted as the relevance that particular individual gives to that proposition or his confidence in the judgment made. When the aggregator specifies the weights they signify the reputation an individual has regarding certain items of the agenda.

When only considering weights for each individual it is enough to specify a weight vector $w = [w_i]$ with $w_i \geqslant 1$. An aggregation rule can then be defined as $\Delta_w^{d,\odot}(A_1, ..., A_n, w) = argmin_{A \in D(X)} \odot_{i=1}^{|X|} w_i * d(A, A_i)$, where d is an ordinary distance function.

Slavkovik et al. conclude by stating that it may be fruitful to study the behavior of different weighted distance-based aggregation rules $\Delta_W^{d,\odot}$ under various choices of d and $\odot$.

Additional considerations on distance based aggregation in a many-valued setting as well as a connection between many-valued judgment aggregation and belief merging can be found in Beg et al. [BB12].

3.2 Deductive closure of many-valued judgments via t-norms

For many real-world instances of judgment aggregation problems it is often not realistic for individuals to make binary choices on some issues. Consider for example the sentence: "The economy will grow". It is possible to simply allow Acceptance or Rejection, but the nuanced nature of the statement also lends itself to be judged to various degrees of "truthfulness". Another way many-valuedness might occur naturally is when new aggregation rules are considered. Table 3.1 shows an example of the simple averaging rule.

	p	q	$p \wedge q$
Individual 1	1	0	0
Individual 2	0	1	0
Individual 3	1	1	1
Average	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

Table 3.1: An example of the averaging rule, due to Duddy and Piggins [DP13]

In [DP13] Duddy and Piggins analyze the theory of judgment aggregation from a point of view in which each judgment, be it individual or collective, is allowed to be many valued. They employ the abstract logical model as described in the introduction, with the only difference being that the agenda X is now allowed to be any non-empty subset of the logical language. A judgment in this setting is any function $f : X \to [0, 1]$, that maps elements of the agenda to any value in the unit interval. As such individual and collective judgments are then expressed using judgments of that form, rather than judgment sets. An aggregation rule in this setting is any function $F : V^n \to V$, where V is the set of all deductively closed judgments. The main question they deal with is how to move this property of *Deductive Closure* of the individual and collective judgment sets from classical logic over to a many-valued setting. Consider again the example in Table 3.1. Clearly $\{p, q\} \models p \wedge q$. Let f be any judgment over the agenda of the example and let $f(p)$ and $f(q)$ denote the acceptance of p and q respectively. One might then want the conclusion to be accepted to a reasonable degree $f(p \wedge q)$ depending on the acceptance of the premises. A way to achieve this is by using a function $T : [0, 1]^2 \to [0, 1]$ to determine a lower bound on the degree to which the conclusion has to be accepted. In the example above $T(f(p), f(q))$ then has to be a lower bound for the value of $f(p \wedge q)$.

Such a function T is required to have a few natural properties:

- *Commutativity:* $T(f(p), f(q)) = T(f(q), f(p))$
- *Neutral Element:* $T(f(p), 1) = f(p)$

- *Monotonicity:* $T(f(p), f^1(q)) \geqslant T(f(p), f^2(q))$ if $f^1(q) \geqslant f^2(q)$
- *Associativity:* $T(x, T(y, z)) = T(T(x, y), z)$ $\forall x, y, z \in [0, 1]$

Neutral Element ensures compatibility with the classical case by enforcing $T(0, 1) = 1$ and $T(1, 1) = 1$. *Monotonicity* states that a weak increase in the judgment of a premise leads to a weak increase in the judgment of the conclusion. And finally, *Associativity* allows the function T to be extended to arbitrary many arguments.
Functions that satisfy the above conditions are called triangular norms (t-norms). For a standard reference on triangular norms and their applications see Klement et al. [KMP13]. A few examples of such t-norms are:

- Minimum t-norm $T_M(x, y) = min(x, y)$
- Product t-norm $T_P(x, y) = x * y$
- Łukasiewicz t-norm $T_L(x, y) = max(x + y - 1, 0)$

Using the above definitions it is now possible to define the notion of deductive closure in a many-valued setting. Let f be any judgment and $A \in P(X)$ be a set of propositions, then f_A denotes the tuple $(f(a_1), ..., f(a_{|A|}))$. Using this definition and a fixed t-norm T we can now state the following: For all non-empty sets A and any proposition $p \in X$,

$$A \models p \text{ implies } T(f_A) \leqslant f(p).$$

A judgment is deductively closed if and only if it satisfies this property.

Applying this definition of deductive closure to the example from above using each of the three different t-norms now yields an interesting result:

For the minimum t-norm we get that $T_M(f(p), f(q)) = \frac{2}{3}$, so $T_M(f(p), f(q)) > f(p \wedge q)$.

For the product t-norm we get something similar: $T_P(f(p), f(q)) = \frac{4}{9}$, which is again greater than $\frac{1}{3}$.

For the Łukasiewicz t-norm however we obtain that $T_L(f(p), f(q)) = \frac{1}{3}$, so $T_L(f(p), f(q)) \leqslant f(p \wedge q)$.

The reason for this behavior is that the Łukasiewicz t-norm possesses a certain property that the other two t-norms lack. Namely T_L has what is called a zero divisor. A zero divisor is any pair of non-zero numbers x, y, such that $T_L(x, y) = 0$. For example consider $T_L(\frac{1}{3}, \frac{2}{3}) = max(\frac{1}{3} + \frac{2}{3} - 1, 0) = 0$.

Using all of the above it is now possible to state the first result, concerning the averaging rule:

Theorem 2 ([DP13]) *If T is the Łukasiewicz t-norm then the averaging rule produces deductively closed collective judgments when applied to deductively closed individual judgments.*

Their central theorem however goes even further and establishes a general possibility depending on the properties of the underying t-norm:

Theorem 3 ([DP13]) *If T has a zero divisor then there exists an aggregation rule $F : V^n \rightarrow V$ that is not veto-dictatorial and satisfies Unanimity Preservation and Independence. If T has no zero divisor and X is non-trivially path-connected then no such aggregation rule exists.*

Two things that are of note here is that firstly, the possibility does not depend on the agenda at all. And secondly when considering judgments that can only take the value 0 or 1 (i.e. the classical case) the well known impossibility is again obtained.

Additional work on judgment aggregation in a many-valued setting involving t-norms has been done by Syed et al. in [SBK16].

3.3 Generalizing with MV-Algebras

In [Her13] Herzberg generalizes the above ideas for systematic propositional many-valued aggregation rules (in the following called attitude aggregators). Systematicity is a stronger variant of *Independence* that has been proven to be equivalent to it under certain conditions in Dietrich and List [DL10]. They build upon Dietrich and List [DL08c] and [DL10], in which they attempt to unify judgment aggregation and other related topics such as preference aggregation and probability aggregation.

The formal framework employed is similar to the general framework of judgment aggregation, but with a few key differences. Instead of judgment sets, each individual now represents his attitude towards propositions of the agenda via an attitude function $X \rightarrow M$, where M is the set of truth values. M will be assumed to have the structure of an MV-algebra. A profile is then nothing more than a sequence of n such attitude functions and an aggregator is simply a map from the set of profiles to the set of attitude functions.

The language of the agenda is that of many-valued propositional logic, containing countably many propositional variables, a constant '0', a symbol for strong disjunction $\oplus$ and negation $\neg$. Call the set of well formed formulas of that language $\mathbf{L}$. The standard operations of $\wedge$, $\vee$, $\rightarrow$ and $\otimes$ (strong conjunction) can be expressed using $\neg$ and $\oplus$. Concrete logics are obtained by further endowing the language with a provability relation $\vdash$. Under this relation one can define an equivalence relation $\equiv$ on $\mathbf{L}$ called *provable equivalence*. Two formulas $p, q \in \mathbf{L}$ are provably equivalent $p \equiv q$ iff both $\vdash p \rightarrow q$ and

$\vdash q \rightarrow p$. Call the set of equivalence classes $\mathbf{L}/\equiv$. It is now also possible to define the canoncial operations $\oplus$, $\neg$ representative-wise on the set of equivalence classes.

The semantics of the language are obtained by considering so-called MV-algebras. An MV-algebra M is a quadruple $(M, \oplus, \neg, 0)$, where $\oplus$ is associative and commutative over M with 0 as the neutral element. Furthermore it has to satisfy the following properties (where $1 = \neg 0$):

- $\neg\neg x = x$
- $x \oplus 1 = 1$
- $x \vee y = y \vee x$

An important observation to make is that the set of equivalence classes $\mathbf{L}/\equiv$ together with the operations $\oplus$ and $\neg$ also forms an MV-algebra. This allows one to treat the semantics of the defined logic algebraically, in accordance with Chang [Cha58] [Cha59]. From now on an arbitrary MV-algebra M referred to as the set of truth values will be fixed. An M-valuation will be defined as an MV-algebra homomorphism from $\mathbf{L}/\equiv$ to M. As a shorthand we can write $I(p)$ instead of $I([p]_\equiv)$, where I is such an M-valuation.

Examples of MV-algebras are the following:

- Any Boolean algebra (with $\vee$ and $\wedge$ corresponding to $\oplus$ and $\otimes$)
- The standard MV-algebra, with $M = [0,1]$, $\neg x = 1 - x$ and $x \oplus y = min(x + y, 1)$. This is also the set of truth values for the infinite valued logic $L_{\beth_1}$.
- The set $M = [0,1] \cap \mathbb{Q}$, with the operations defined as above. It forms the set of truth values for Łukasiewicz's $L_{\aleph_0}$.
- The set $M = \{0, 1/m, ..., (m-1)/m, 1\}$, again with the operations defined as above, forming the truth values for Łukasiewicz's $(m+1)$ valued logic L_{m+1}.

An attitude function A is called *rational* if it can be extended to an M-valuation, meaning there exists some M-valuation I, such that $A(p) = I(p)$ for all $p \in X$. This entails that any rational attitude function is not only well-defined on X but also on its closure under $\neg$ and $\oplus$. Therefore all rational attitude functions will be considered to be defined on the closure of the agenda in what follows. A profile is rational if all of the contained attitude functions are rational.

As was the case for judgment aggregation rules, it is also important to state some responsiveness conditions for attitude aggregators. In fact many of them can be directly adopted from the JA case. Beforehand two definitions are necessary. Firstly a formula $p \in \mathbf{L}$ is called *strictly contingent* iff there exists for all $x \in M$ some M-valuation I such that $I(p) = x$. And secondly we call an agenda X complex iff there exists some strictly contingent $p_0 \in X$ and strictly contingent propositions p_1, p_2, p_3, q_1, q_2 in the closure of X, such that for all M-valuaitons I it is the case that $I(p_1) \oplus I(p_2) = I(p_3)$ and $\neg I(q_1) = I(q_2)$. If p_1, p_2, p_3, q_1, q_2 are even in X then it is called *rich*. Using these definitions and the abbreviation $\underline{A}(p) = (A_i(p))_{i \in N}$ we can now state the responsiveness conditions:

- Rationality: An attitude aggregator F is rational iff for all rational profiles $\underline{A} \in Dom(F)$, $F(\underline{A})$ is also a rational attitude function.
- Universality: An attitude aggregator F is universal iff it's domain consists of all rational profiles.
- Independence: An attitude aggregator F is independent iff there exists a function $G : M^N \times X \to M$, such that for all profiles $\underline{A} \in Dom(F)$ and for all $p \in X$, $F(\underline{A})(p) = G(\underline{A}(p), p)$.
- Systematicity: An attitude aggregator F is systematic iff there exists a map $f : M^N \to M$, such that for all profiles $\underline{A} \in Dom(F)$ and all $p \in X$, it holds that $F(\underline{A})(p) = f(\underline{A}(p))$. Intuitively this means that the aggregate attitude towards a propositions p only depends on the individuals attitude on that very proposition. f is called a decision criterion and it is unique whenever X contains some strictly contingent formula p_0.
- Strong Systematicity: An attitude aggregator F is called strongly systematic iff it is systematic over the closure of X.
- Pareto Principle: An attitude aggregator F is Paretian iff for all profiles $\underline{A} \in Dom(F)$ and all $p \in X$, $A_i(p) = 0 \ \forall i \in N \implies F(\underline{A})(p) = 0$.
- Strong systemisability: Call a systematic attitude aggregator F for a complex agenda X strongly systematisable iff either F is strongly systematic or X is rich.

Herzbergs main result now notes that M^N, i.e. the product of $|N|$ copies of M is again an MV-algebra, with $\oplus_N$ and $\neg_N$ defined componentwise and the zero element 0_N being an N-sequence of 0s. It details a one-to-one correspondence between structure preserving maps from M^N to M and rational, universal, Paretian and strongly systematisable attitude aggregators. The theorem reads as follows:

Theorem 4 ([Her13]) *If F is a rational, universal, Paretian and strongly systematisable attitude aggregator, then the decision criterion of F is an MV-homomorphism.*

Conversely, if f is an MV-homomorphism and F is defined by the equation $F(\underline{A})(p) = f(\underline{A}(p))$ for all profiles $\underline{A} \in Dom(F)$ and all $p \in X$, then F is rational, universal, Paretian and systematic.

When considering the Boolean algebra $M = \{0, 1\}$ with the usual boolean operations, then M^N is again a Boolean algebra isomorphic to the powerset Boolean algebra of N. Via an argument on ultrafilters one now obtains a version of the central impossibility theorem of binary judgment aggregation, previously proven in Herzberg [Her10], as a simple corollary:

Theorem 5 ([Her13]) *Suppose F is a rational, universal, Paretian and strongly systematisable attitude aggregator. If the algebra of truth values is Boolean, then the decision criterion of F is a Boolean homomorphism.*

If the algebra of truth values is Boolean and and the set of individuals N is finite, then F is a dictatorship.

Similar arguments over filters and ultrafilters to obtain impossibilities have also been employed by Daniëls [Dan11] and Klamler and Eckert [KE+09]. A general correspondence between ultrafilters and judgment aggregation rules is presented in Herzberg [Her08].

A possibility result can be obtained for the case in which the underlying logic is the infinite valued $L_{\beth_1}$, since for the corresponding MV-algebra M, there exist continuum-many structure preserving maps from M^N to M and as such also uncountably many systematic attitude aggregators that aren't dictatorships. Herzberg concludes by stating that the question of whether it is possible to obtain possibility results for Łukasiewicz's $L_{\aleph_0}$ and finite sets of individuals N is still open and might be answered by checking whether there are any MV-homomorphisms other than projections from M^N to M when $M = [0, 1] \cap \mathbb{Q}$.

Related work on the application of algebraic methods in the field of social choice theory can be found in Kim and Roush [KR80], Brown [Bro74], Aleskerov [Ale13] and Daniëls and Pacuit [DP09].

Herzberg also mentions the model theoretic approach by Lauwers and Van Liedekerke [LVL95] as an alternative to the algebraic approach, although it has as of yet not been applied to many-valued aggregation problems. Further reading on this topic can be found in Herzberg et al. [HLVLF10] and Herzberg and Eckert [HE12a] [HE12b].

3.4 Abstract Algebraic Logic

In [EPZ15] Esteban et al. further generalize the abstract generalization result for MV-algebras of Herzberg to encompass agendas formulated in any self-extensional logic. These are all logics admitting a general version of many-world semantics called referential semantics. They include, but are not limited to, classical, intuitionistic, modal, many-valued and relevance logics. For a detailed overview over abstract algebraic logic refer to Font et al. [FJP03]. Characterizations of self-extensional logics can be found in Wójcicki [Wój79] and Jansana and Palmigiano [JP06].

Before we can present the obtained results it is necessary to briefly describe the formal framework employed. A consequence operation, in the following also called closure operator, over a set A is a function $C : 2^A \to 2^A$ for which it holds that for every $X, Y \subseteq A$:

- $X \subseteq C(X)$
- if $X \subseteq Y$, then $C(X) \subseteq C(Y)$
- $C(C(X)) = C(X)$

The operator is finitary, if $C(X) = \cup\{C(Z) \mid Z \subseteq X,\ Z\ finite\}$. A closure system $\mathcal{C}$ is any collection $\mathcal{C} \subseteq 2^A$, such that $\mathcal{C}$ is closed under intersection and $A \in \mathcal{C}$. A closure system is algebraic if it is closed under unions of up-directed families; where for any poset $< P, \leqslant >$ a set $U \subseteq P$ is called up-directed if, for any $a, b \in U$, there exists some $c \in U$, such that $a, b \leqslant c$. An example of an algebraic closure system on A would be the set of all C-closed subsets of A, where C is a finitary closure operator.

A logic is now a pair $S =< \mathbf{Fm}_L, \vdash_S >$ where $\mathbf{Fm}_L$ is the algebra of formulas over the propositional language L and $\vdash_S$ is a consequence relation for which it holds that the corresponding consequence operation is invariant under substitution.

For any algebra A, a S-filter of A is a subset $F \subseteq A$, for which it holds that for every $\Gamma \cup \{p\} \subseteq \mathbf{Fm}$ and every $h \in Hom(\mathbf{Fm}, A)$: If $\Gamma \vdash_S p$ and $h[\Gamma] \subseteq F$, then $h(p) \in F$. The collection of all S-filters of A is a closure system, denoted by $Fi_S(A)$.

The algebraic counterpart of a logic S are now the so-called S-algebras in the class $\mathbb{A}lgS$. An S-Algebra is defined via the concept of *Tarski-congruence*. For any algebra A and closure system $\mathcal{C}$ on A, the *Tarski-congruence* of $\mathcal{C}$ relative to A is denoted by $\tilde{\Omega}_A(\mathcal{C})$. It is the greatest congruence for which it holds that for any $F \in \mathcal{C}$ it does not relate elements in F to elements that do not belong to F. An S-algebra is now any algebra A, such that the *Tarski-congruence* of $Fi_S(A)$ relative to A is the identity. From now on let S be any self-extensional logic and $\mathbf{B} \in \mathbb{A}lgS$.

Let $X \subseteq \mathbf{Fm}$ be the agenda as usual, with $\bar{X}$ denoting the closure of the agenda over the logical connectives of the considered language. Such an agenda is called n-pseudo-rich, if it contains at least n formulas $\{p_1, ..., p_n\}$, such that each p_i is provably equivalent to some x_i for some set $\{x_1, ..., x_n\}$ of pairwise different propositional variables. Informally this property states that the agenda has to contain at least n formulas that behave like like propositional variables.

Attitude functions, profiles and aggregators can be defined in a similar manner to Herzberg, with the only difference that attitude aggregators are now *partial* functions: $F : (\mathbf{B}^{|X|})^n \nrightarrow \mathbf{B}^{|X|}$. The properties of rationality, universality, independence, systematicity, strong systematicity as well as the notion of a decision criterion can again be defined as before.

Esteban et al. now obtain the following general characterization results:

Theorem 6 ([EPZ15]) *Let F be a rational, universal and strongly systematic attitude aggregator. Then the decision criterion of F is a homomorphism of S-algebras.*

and

Theorem 7 ([EPZ15]) *Let $f : \boldsymbol{B}^n \nrightarrow \boldsymbol{B}$ be a homomorphism of S-algebras. Then the function $F : (\boldsymbol{B}^{|X|})^n \nrightarrow \boldsymbol{B}^{|X|}$, defined for any rational profile $(A_1, ..., A_n)$ and any $p \in X$ by the following assignment:*

$$F(A_1, ..., A_n)(p) = f(A_1(p), ..., A_n(p))$$

is a rational, universal and strongly systematic attitude aggregator.

The well-known impossibility in classical judgment aggregation can again be obtained from the above results by letting **B** be the standard two-valued Boolean Algebra, using the result that there exists a bijection between boolean homomorphisms $f : \{0,1\}^n \to \{0,1\}$ and ultrafilters on $\{0,1\}^n$ and the observation that whenever n is finite such an ultrafilter is principal (since principal ultrafilters correspond to dictatorial decision criteria).

3.5 Unifying many-valued JA and Preference Aggregation

Grossi [Gro09] describes an interesting correspondence between a certain subset of many-valued judgment aggregation and preference aggregation. They begin by considering the well-known result that for any total preorder $\leqslant_P$ there exists a ranking function $u : X \to [0,1]$, such that for every $x, y \in X$: $x \leqslant_P y$ iff $u(x) \leqslant u(y)$. Treating this ranking function as a many-valued interpretation of propositions now gives a bridge between preferences and logical implications:

$$x \leqslant_P y \text{ iff } u(x) \leqslant u(y) \text{ iff } u \models x \to y$$

The translation from a Preference aggregation setting to a JA setting is now given by the following function, which informally maps preference relations to fuzzy implication agendas:

$$J(\leqslant_p) := \{x \to y | (x,y) \in \leqslant_P\} \cup \{\neg(x \to y) | (x,y) \notin \leqslant_P\}$$

The above translation is a bijection between PA structures and JA structures with fuzzy implication agendas and ensures that each PA structure corresponds to exactly one such JA structure. Using this translation it is now possible to import various impossibilities from preference aggregation theory into the world of judgment aggregation. An example would be Arrows Theorem:

Theorem 8 ([Gro09]) *For any translation of a preference aggregation structure into judgment aggregation on fuzzy implication agendas, there exists no aggregation function which satisfies Unanimity, Independence and Non-Dictatorship.*

Another, perhaps more interesting, result is the translation of the so-called impossibility of a Paretian liberal, cf. Sen [Sen70]. For this consider the property called *Minimal Liberalism*, which informally states that there are at least two agents who always dictate the ordering of at least one pair of issues each. Using a suitable translation (via the above translation function) of the property to Judgment aggregation one obtains:

Theorem 9 ([Gro09]) *For any translation of a preference aggregation structure into judgment aggregation on fuzzy implication agendas, there exists no aggergation function that satisfies Unanimity and Minimal Liberalism.*

Informally the result states that there is no way to aggregate judgments while preserving unanimity if there are at least two individuals that can impose acceptance/rejection of at least one implication by themselves.

Grossi then goes on to extend the above correspondence to more complex agendas, by considering preference relations over more complex, logically interconnected issues formulated in Gödel-Dummet Logic, called Gödel-Dummet Preferences. In this more general setting they obtain a characterization of Dictatorships:

Theorem 10 ([Gro09]) *Let the set of issues of the Gödel-Dummet preference aggregation problem contain a subset of the form $\{p, q, p \wedge q\}$. Then an aggregation function for the JA-translation of the problem satisfies Systematicity if and only if it is Dictatorial.*

They conclude by stating that it should be possible to obtain other impossibility (or possibility) results in a similar fashion to the one above.

3.6 Belief Binarization

In [DL18] Dietrich and List present a connection between the fields of Belief Binarization and Judgment aggregation and use known impossibility results from JA to derive similar theorems for Belief Binarization. The main question of Belief Binarization is whether it is possible to express binary beliefs as a function of degrees of beliefs without running into problems such as the well known Lottery Paradox.
Formally, we have a non-empty set X of propositions, that is closed under negation, on which beliefs are held. X is called the *proposition set*. A *degree-of-belief function* Cr is a function that assigns to any proposition $p \in X$ a certain numerical value $Cr(p) \in [0, 1]$. A *belief set* is a subset $B \subseteq X$. It is

- *consistent*, if B is a consistent set.

- *complete*, if it contains one of each pair of propositions and negations in X.
- *implication-closed*, if it contains every proposition $p \in X$ that is entailed by B.

A function f, that maps a degree of belief function Cr to a belief set $B = f(Cr)$ is called a belief binarization rule. An example of a whole class of these rules are the so-called threshold rules. A threshold rule accepts a proposition p into the generated belief set, if the value of $Cr(p)$ exceeds some threshold value t. I.e. $B = \{p \in X \mid Cr(p) \textit{ exceeds } t\}$, where *exceeds* can either mean $>$ or $\geqslant$ depending on whether the threshold rule employs a strict or weak threshold. A threshold rule for which the threshold varies for each proposition is called *non-uniform*.

Similar to the properties of judgment aggregation rules, we might want a belief binarization rule to have the following four properties:

- **Universal Domain:** The domain of f is the set of all degree-of-belief functions over X.
- **Belief consistency and completeness:** $f(Cr)$ is consistent and complete for every $Cr \in Dom(f)$.
- **Propositionwise independence:** For all $Cr, Cr' \in Dom(f)$ and all $p \in X$, if $Cr(p) = Cr'(p)$ then $p \in B \iff p \in B'$, where $B = f(Cr)$ and $B' = f(Cr')$. Intuitively this says that an agents belief of p is not influenced by his degrees of beliefs in propositions other than p.
- **Certainty preservation:** For all $Cr \in Dom(f)$ and $p \in X$, if $Cr(p) = 1$ then $p \in B$ and if $Cr(p) = 0$, then $p \notin B$, meaning that if an agent already believes fully (or not at all) in all proposition then this should be reflected in the binary beliefs as well.

An intuition on the similarities between judgment aggregation and belief binarization can easily be gained by considering an example. Compare the following table to Table 3.1:

	p	q	$p \wedge q$
World 1	1	0	0
World 2	0	1	0
World 3	1	1	1
Degree of Belief	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

Table 3.2: Correspondence between judgments and beliefs

When swapping voting agents with possible worlds in which a belief is held, the common structure of judgment aggregation and belief binarization becomes apparent. An important difference however is that in belief binarization one usually does not have information about the belief in a proposition in a certain world, i.e. one only has access to the last row of the table, the overall degree of belief, while in a judgment aggregation problem the whole set of individual judgments is available. As such belief binarization problems are more closely related to judgment aggregation problems in which the aggregation rule satisfies *Anonymity*.

This correspondence between the two can be made formal. Let f be a belief binarization function over X. A judgment aggregation rule F over X can then be constructed in the following two steps:
Firstly, convert the individual judgment sets into an anonymous profile (i.e. the proportion of agents accepting some proposition $p \in X$): For each judgment profile $(A_1, ..., A_n)$ a function $Cr_{(A_1,...,A_n)}$ is constructed that assigns to each proposition p the proportion of individuals accepting it.

$$Cr_{(A_1,...,A_n)}(p) = \frac{|\{i \in N | p \in A_i\}|}{n}$$

Second, the belief binarization function f is applied to the newly constructed function $Cr_{(A_1,...,A_n)}$, which yields a belief set that can also be interpreted as a collective judgment set. This yields the following judgment aggregation rule:

$$F(A_1, ..., A_n) = f(Cr_{(A_1,...,A_n)})$$

It is important to note that this aggregation function F is *Anonymous* and that, whenever the underlying binarization function f satisfies universal domain, belief consistency and completeness, propositionwise independence and certainty preservation, then, for any number of individuals n, F satisfies universal domain, collective rationality, independence and unanimity preservation.

Using this formalization an important impossibility theorem for belief binarization can now be directly imported from judgment aggregation:

Theorem 11 ([DL18]) *For any non-trivial proposition set X, there exists no belief binarization rule satisfying universal domain, belief consistency and completeness, propositionwise independence and certainty preservation.*

This can be proven by simply assuming that such a rule f exists and constructing the corresponding aggregation rule F. By the central impossibility theorem of judgment aggregation however, such a rule can not exist.

Dietrich and List then analyze various ways one can attempt to sidestep the impossibility by relaxing one of the properties imposed upon belief binarization rules. In particular the following relaxations are discussed:

- Certainty preservation: It is a fairly simple requirement that is also very intuitive and as such would not make much sense to be removed.

- Universal Domain: An important requirement when talking about universal belief binarization rules. However, as is sometimes employed in judgment aggregation, it can be dropped when discussing certain domain restrictions.
- Consistency: It makes sense to assume that in a real-world setting people do not always harbor fully consistent beliefs.
- Implication-closure: For large and complex proposition sets this requirement might represent unrealistic demands on the individuals reasoning abilities.
- Propositionwise independence: Some proposition might be related to others in certain ways and as such it also makes sense to consider the case where an individuals belief on this proposition depends on their belief on the ones it is related to.
- Completeness: Contrary to judgment aggregation theory it is not strictly necessary for an agent to have a belief for every element of the proposition set.

In the following subsections we recall the effect each of these relaxations has on the underlying belief binarization rule and whether it is a suitable way to sidestep the impossibility.

3.6.1 Relaxing Completeness

Relaxing the condition that generated Belief Sets have to be complete has been identified as the most obvious way to try and circumvent impossibilities, since it is quite natural for an agent to not have beliefs about every item in the proposition set. A straightforward relaxation of this condition would require the generated belief sets to only be implication-closed instead of complete. This however does not do much in regards to the impossibility and only yield the following triviality result:

Theorem 12 ([DL18]) *For any non-trivial proposition set X, any belief binarization rule satisfying universal domain, belief consistency and implication closure, propositionwise independence and certainty preservation is a threshold rule with a uniform threshold of 1.*

This result stems from a well known result in Judgment Aggregation Theory, namely that for every path-connected agenda X, any aggregation rule satisfying universal domain, collective consistency and implication closure, independence and unanimity preservation is oligarchic, cf. Dokow and Holzman [DH10]. However, since all aggregation rules corresponding to belief binarization functions are also anonymous, we get that the aggregation rule has to be the unanimity rule, i.e. a quota rule with threshold 1.

Further discussion on the relaxation of completeness in judgment aggregation can be found in List and Pettit [LP02], Gärdenfors [Gär06], Dietrich and List [DL07b] [DL07c] [DL08b] as well as Dokow and Holzman [DH10].

3.6.2 Relaxing Propositionwise Independence

Another plausible way to try and obtain possibility is to relax the condition on propositional independence and as such allow relations between propositions to affect the outcome of the binarization function. This can be made formal by introducing a so-called *relevance relation* R over the propositions in X, where pRq is read as "p is relevant to q". The set of propositions relevant to a fixed proposition p is denoted by $R(p) = \{q \in X \mid qRp\}$. Using this definition, one can now state the relaxed version of propositionwise independence:

Independence of irrelevant propositions: For any $Cr, Cr' \in dom(f)$ and any $p \in X$, if $Cr(q) = Cr'(q)$ for all $q \in R(p)$, then $p \in B \iff p \in B'$, where again $B = f(Cr)$ and $B' = f(Cr')$.

Given suitable interpretations for the relevance relation R one then obtains various variants of aggregation rules:

Premise-based rules: For these rules, a subset $Y \subseteq X$ of premises is identified. Then for all propositions $p \in X \backslash Y$ outside of the premises, we take the whole set of premises to be relevant to p: $R(p) = Y$, while for a premise $q \in Y$ only itself is relevant: $R(q) = q$. A binarization rule for such a premise based approach can be obtained by first generating the binary belief set for the premises using some binarization rule g and then derive the beliefs on the remaining propositions by logical inference. Formally:

$$f(Cr) = \{p \in X \mid g(Cr|Y) \models p\}$$

As long as the set of premises Y and the binarization function g are chosen in such a way that the outcome of $g(Cr|Y)$ is always a consistent set, it is guaranteed that f will always yield a consistent and implication closed belief set.

Sequential-priority rules: This is a generalization of the premise-based approach where a linear ordering is imposed on the propositions in X. The belief set B is then built in stages by considering the propositions $p \in X$ in order by first checking whether p is already entailed by B. If it is, p will be put into B; if p is not entailed then a simple binarization function (such as uniform threshold) is applied and p is included if and only if this binarization function includes p and $\{p\} \cup B$ is not an inconsistent belief set. Rules of this kind may produce implication-closed belief sets depending on which binarization function is applied and on the order of priority imposed.

Generalized-priority rules: This is again a generalization of the previous approach, by allowing the relevance relation to be a priority graph instead of a linear order. One then starts the binarization process with a proposition for which no other propositions are relevant and then recursively applies the procedure from before to every adjacent node in the priority graph. This procedure yields a consistent belief set as long as the priority graph is transitive, negation invariant (if pRq then also $\neg pRq$ and $pR\neg q$) and there are no relevance relations between ancestors of two mutually irrelevant propositions.

All three of the above rules have relevant counterparts in the world of judgment aggregation, such as premise-based aggregation for which similar results have been obtained. Sequential-priority rules for judgment aggregations are presented in List [Lis04] and Li [Li10] and the corresponding generalized-priority rules in Dietrich [Die15].

Another possible type of rule are so-called **distance-based rules**. These kind of rules try to form a belief set that is minimally distant from any chosen degree-of-belief function. Different distance measures can be employed. A well-known metric from judgment aggregation theory is the so called *Hamming Distance*, which is defined to be the number of propositions over which two judgment sets disagree. A belief binarization rule that employs the hamming distance will yield consistent and complete belief sets and it furthermore satisfies the requirement of independence of irrelevant propositions, when all premises in X are considered to be relevant to each other. A discussion of distance based rules from the perspective of belief merging can be found in Konieczny and Pérez [KP02].

Other kinds of non-independent belief-binarization rules for which results may carry over to judgment aggregation are Leitgeb's *P-stability-based rules* [Lei13] as well as Lin and Kelly's *camera-shutter rules* [LK12b] [LK12a].

3.6.3 Relaxing Implication Closure

Above we discussed a relaxation of completeness by replacing it with the condition of implication closure. One might go even further and even relax this property by replacing it with something called *closure under implication by singletons*. A belief set B is closed under implication by singletons if it contains any $p \in X$, for which there is some $q \in B$, such that q entails p. Using this weakened form of implication-closure allows one to obtain the following result:

Theorem 13 ([DL18]) *Let k be the size of the largest minimally inconsistent subset of X. Any threshold rule with a strict threshold of $\frac{k-1}{k}$ (or higher) for each proposition satisfies universal domain, belief consistency and closure under implication by singletons, propositionwise independence and certainty preservation.*

An important thing to note here is that the size of the minimally inconsistent subset k grows with complexity of the proposition set X. So for large and complex sets X, the threshold will be very close to 1.

The corresponding result in judgment aggregation was shown by Dietrich and List [DL07a]:
If k is the size of the largest minimally inconsistent subset of the agenda, then any quota rule with a strict threshold of at least $\frac{k-1}{k}$ satisfies each of universal domain, collective consistency, independence and unanimity preservation.

3.6.4 Relaxing Consistency

The condition of consistency of belief sets can be relaxed by replacing it with a condition that limits the size of inconsistent subsets that are allowed to be in B. This condition is called *Belief k-consistency*, where a belief set is called k-consistent if it does not contain any inconsistent subsets of size up to k. Using this a possibility result can be obtained:

Theorem 14 ([DL18]) *Any threshold rule with a strict threshold of at least $\frac{k-1}{k}$ for each proposition satisfies universal domain, belief k-consistency, proposition wise independence and certainty preservation.*

In [Lis14] List presents the corresponding result from judgment aggregation theory, which is a statement about quota rules similar to the above theorem.

3.6.5 Relaxing Universal Domain

By suitably restricting the domain of the belief binarization function it is possible to find functions that satisfy all other requirements. Consider for example a function f, whose domain is restricted to degree-of-belief functions Cr with the following property: For every minimally inconsistent subset $Y \subseteq X$, there is at least one proposition $p \in Y$, such that $Cr(p) \leqslant t$. From that it follows that any threshold binarization function with a strict threshold of t will never produce an inconsistent belief set.

This domain restriction corresponds to the restrictions on judgment profiles that ensure the consistency of the majority rule and supermajority rules with threshold t.

3.6.6 Relaxing Certainty Preservation

Relaxing Certainty Preservation, while possible, doesn't do much to sidestep the impossibility. Define an atom to be any proposition $p \in X$, such that p entails either q or $\neg q$ for every $q \in X$. Call a proposition set atom-closed, if it contains a maximal number of atoms. For these atom-closed proposition sets the following result can be stated:

Theorem 15 ([DL18]) *Any atom-closed proposition set that contains more than one proposition, negation pair, any belief binarization rule satisfying universal domain, belief consistency and completeness and propositionwise independance is constant.*

The corresponding result in judgment aggregation theory states that for any atom-closed agenda that contains more than one proposition, negation pair, any aggregation rule satisfying universal domain, collective rationality and independence is either dictatorial or constant, cf. Dietrich [Die06].

CHAPTER 4

Other non-classical logics

This chapter deals with non-classical logics that are not many-valued in nature. For some of them (such as Relevant Logic) it is the case that a general impossibility result has already been obtained via the abstract characterization result in Section 3.4. However new insight can still be gained by considering the details of the judgment aggregation procedures involved.

4.1 Substructural Logics

In this section we are going to present an analysis of various substructural logics in accordance to Porello [Por17]. Beforehand we are going to give a short recollection of sequent calculus and how it is used to obtain substructural logics, as well as how these procedures affect the underlying logical language. For a more complete introduction to substructural logics refer to Paoli [Pao13] and Restall [Res02].

For our discussion it is sufficient to recall that in a sequent calculus formulas, or rather sequences of formulas called sequents, are derived from axioms by applying certain rules. These rules consist of so-called structural rules, logical rules and the cut rule. The logical rules are used to introduce the connectives of the language. An example is the rule for introducing $\wedge$ in classical logic:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge_a B, \Delta} \wedge_a, r$$

Structural rules on the other hand are used to modify the structure of the sequents themselves. These are the rules that are important for the discussion of substructural logics. We are only going to introduce the left sided variant of these rules, the right variants are analogous:

$$\frac{\Gamma, B, A, \Sigma \;\vdash\; \Delta}{\Gamma, A, B, \Sigma \;\vdash\; \Delta}\text{exchange, l}$$

$$\frac{\Gamma \;\vdash\; \Delta}{\Gamma, A \;\vdash\; \Delta}\text{weakening, l}$$

$$\frac{\Gamma, A, A \;\vdash\; \Delta}{\Gamma, A \;\vdash\; \Delta}\text{contraction, l}$$

A substructural logic is now obtained by removing any combination of structural rules from the calculus. To get an idea of the effects this can have on the underlying language, consider this alternative formulation of the $\wedge$ rule from above:

$$\frac{\Gamma \;\vdash\; A, \Delta \qquad \Sigma \;\vdash\; A, \Pi}{\Gamma, \Sigma \;\vdash\; A \wedge_m B, \Delta, \Pi}\wedge_m, r$$

In classical sequent calculus the formulations for $\wedge_a$ and $\wedge_m$ are provably equivalent using multiple applications of exchange and weakening. However, if these two rules are absent, the equivalence disappears and we obtain two versions of the connective, namely an additive ($\wedge_a$) and a multiplicative ($\wedge_m$) version of $\wedge$. This has significance in the context of judgment aggregation because it introduces two different ways one can look at the conjunction of propositions.

Another straightforward effect of restricting the structural rules is that, depending on which rules are no longer available, the order as well as the multiplicity of formulas in a sequent matter. In classical sequent calculus one can freely permute formulas, as well as collapse multiple occurrences of the same formula into a single occurrence. This allows the sequent to be represented as a pair of sets of formulas. When now any of the structural rules are removed from the calculus this is no longer possible and one has to consider the sequent as either a multiset or even an ordered list.

We will go into details on how this change of the underlying language affects the judgment aggregation procedure in the following sections on specific substructural logics.

Since substructural logics are obtained by restricting the sequent calculus of classical logic it is only natural to approach this analysis from a proof-theoretical point of view as well. We are now dealing with logics lacking a few of the features present in classical logic and as such some of the terms defined in the introduction need to be adapted accordingly.

Firstly, since some of the logics considered are lacking some, or all, of either Exchange, Weakening and Contraction we shall from now on consider the **Agenda** X to be either a multiset or a list of propositions from the language. This also entails that a **Judgment Set** A is now a multiset or a list of elements from X. In the following we will apply the usual set theoretic notions of $\in$ and powerset also to multisets and lists. These applications can be interpreted with their respective intuitive meanings, i.e. for a list A, $a \in A$ iff there is an occurrence of a somewhere in A.

As we are working with a variety of different logics we will from now on indicate which logic we are currently working in by indexing the agenda. So X_L will be defined as an agenda X in which all contained propositions are taken from the language of the logic L.

Let $\sim p$ be the **complement** of a formula p, then: $\sim p = \neg p$ if p is not a negated formula and $\sim p = q$ if p is negated and $p = \neg q$. This definition is necessary since in some of the considered logics the semantics of the negation differ from the one in classical logic. For example, already in intuitionistic logic $\neg\neg p \not\Leftrightarrow p$. We can now use the defined complement to update the conditions on judgment sets we outlined in Chapter 2 where necessary:

- *Weak consistency*: For every $p \in X$, it is not the case that both $p \in A$ and $\sim p \in A$.
- *Completeness*: For every $p \in X$, either $p \in A$, or $\sim p \in A$.

Consistency and *Deductive Closure* remain unchanged as they only depend on the notion of derivability in the logic, which is clearly defined by way of a sequent calculus in all the logics considered in this chapter.

Another deviation from classical logic that has significant impact on the study of judgment aggregation in substructural logics is that for some of the logics considered here consistency does no longer imply weak consistency. This can be seen when considering an inconsistent set Y. Then by definition $Y \vdash$ and furthermore by Weakening, for every $Y' \supset Y$: $Y' \vdash$. Now in a logic lacking Weakening, it is possible for a consistent set to have inconsistent subsets, which may violate weak consistency. Because of this a condition is defined which is quite a bit stronger than just consistency alone, namely **Robust Consistency**. A set (multiset, list) of propositions Y is called robustly consistent, if Y is consistent and every proper subset (submultiset, sublist) $Y' \subset Y$ is as well. Robust consistency now clearly implies both consistency and weak consistency. In the previous chapters we considered all individual judgment sets to be fully rational (that is complete and consistent), however here we will only assume them to be robustly consistent unless they are explicitly mentioned to be complete as well. That is due to the fact that in logics lacking classical negation it is not appropriate to model acceptance and rejection of a proposition via negation.
$J(X_L)$ is defined to be the set of all robustly consistent judgment sets and $J^*(X_L)$ to be the set of all judgment sets that are both robustly consistent and complete.

Another interesting generalization considered by Porello is when one allows the aggregation functions to depend on two separate agendas. One agenda X_L on which the individuals make their judgments and a second agenda $X_{L'}$ over which the rationality conditions of the collective judgment set are evaluated. It is important to note here, that X_L and $X_{L'}$ are the same syntactically and only differ in the underlying logics L and L'. For most of what is to follow however, both L and L' will refer to the same logic. And as such it is necessary to slightly alter the definition of aggregation functions: An aggregation function is a function $F : J(X_L) \to 2^{X_{L'}}$ where 2^A refers to the powerset of A. Responsiveness conditions for such an aggregation function F can be defined in a way similar to how we have defined them in the introduction.

Let $[X_L, X_{L'}]$ be the class of aggregation functions from judgment sets defined over X_L to judgment sets defined over $X_{L'}$. Now for any $F \in [X_L, X_{L'}]$ the usual rationality conditions with respect to the considered logic are defined as:

- F is *weakly consistent* iff for every agenda A, $F(A)$ is weakly consistent w.r.t. L'
- F is *consistent* iff for every agenda A, $F(A)$ is consistent w.r.t. L'
- F is *robustly consistent* iff for every agenda A, $F(A)$ is robustly consistent w.r.t. L'
- F is *deductively closed* iff for every agenda A, $F(A)$ is deductively closed w.r.t. L'
- F is *complete* iff for every agenda A, $F(A)$ is complete w.r.t. L'
- F is *weakly rational* iff for every agenda A, $F(A)$ is complete and weakly consistent w.r.t L'

Define AX to be a set of axioms then $[X_L, X_{L'}](AX)$ defines the set of aggregation functions that satisfy all axioms in AX. The axioms considered in the following are: Weak Rationality (WR), Anonymity (A), Independence (I), Neutrality (N), Monotonicity (M) ans well as Acceptance-rejection neutrality (arN). It is interesting to note that the only rule satisfying the set of axioms $\{WR, A, I, N, M\}$ is the majority rule; this can be shown for the logics considered here by slightly adapting the proof from [EGP12]. Call that set of axioms MAJ. Dropping weak rationality from this set characterizes the uniform quota rules.

Adapting the concept of **safety** as presented by Endriss et al. in [EGP12] is fairly straightforward: For a set of axioms AX, a pair of agendas $[X_L, X_{L'}]$ is safe for axioms AX iff every $F \in [X_L, X_{L'}](AX)$ is robustly consistent. This definition can be extended to entire logics as follows: A pair of logics (L, L') is safe for axioms AX iff every pair of agendas (X_L, X'_L) is safe for axioms AX.

And finally a few properties that define classes of agendas are generalized:

- *Median Property (MP)*: An agenda X_L has the median property iff every minimally inconsistent subset of X_L has cardinality at most 2.
- *Simplified Median Property (SMP)*: An agenda X_L has the simplified median property iff every (non-trivially) inconsistent subset of X_L contains a subset $\{\phi, \psi\}$, such that $\phi \vdash_L \neg\psi$ and $\neg\psi \vdash_L \phi$.
- *Syntactic Simplified Median Property (SSMP)*: An agenda X_L has the syntactic simplified median property iff every (non-trivially) inconsistent subset of X_L has a subset $\{\phi, \neg phi\}$.
- *k-Median Property (kMP)*: An agenda X_L has the k-median property iff every minimally inconsistent subset of X_L has cardinality at most k.

Having all of the above definitions under our belt, we are now ready to present the main results for a variety of different substructural logics.

4.1.1 Intuitionistic Logic (IL)

The first logic considered by Porello is intuitionistic logic. It is usually not considered to be a substructural logic but since it is an important logic and can be obtained by modifying a sequent calculus it still fits into this discussion nicely. A sequent calculus for intuitionistic logic can be constructed from the sequent calculus of classical logic by imposing the condition that only a single formula may occur on the right hand side of a sequent. It is easy to see, and not very surpising, that one runs into much of the same problems with regard to judgment aggregation in intuitionistic logic as in classical logic. This leads to the following theorem:

Theorem 16 ([Por17]) *Intuitionistic logic is not safe for MAJ.*

To give an example of where judgment aggregation fails to produce a consistent outcome for intuitionistic logic consider the example from the introduction. Recall the collective judgment set obtained was $\{p, q, \neg r, (p \wedge q) \leftrightarrow r\}$. This set can be shown to be inconsistent with the following sequent proof (representing the biconditional as a conjunction of implications):

$$
\wedge,l\,\dfrac{\rightarrow,l\,\dfrac{WL\,\dfrac{WL\,\dfrac{\wedge,r\,\dfrac{WL\,\dfrac{p \vdash p}{p,q \vdash p} \qquad WL\,\dfrac{q \vdash q}{p,q \vdash q}}{p,q \vdash p \wedge q}}{p,q,r \rightarrow (p \wedge q) \vdash p \wedge q}}{p,q,\neg r, r \rightarrow (p \wedge q) \vdash p \wedge q} \qquad WL\,\dfrac{WL\,\dfrac{WL\,\dfrac{WL\,\dfrac{r \vdash r}{\neg r, r \vdash}}{\neg r, r \rightarrow (p \wedge q), r \vdash}}{q, \neg r, r \rightarrow (p \wedge q), r \vdash}}{p,q,\neg r, r \rightarrow (p \wedge q), r \vdash}}{p,q,\neg r,(p \wedge q) \rightarrow r, r \rightarrow (p \wedge q) \vdash}}{p,q,\neg r,((p \wedge q) \rightarrow r) \wedge (r \rightarrow (p \wedge q)) \vdash}
$$

As is the case with classical logic the median property characterizes safe agendas:

Theorem 17 ([Por17]) *An agenda X_{IL} is safe for MAJ iff it satisfies the median property.*

And with regards to the larger class of uniform quota aggregation functions defined by the set of axioms $\{A, I, N, M\}$ we get again behavior similar to classical logic. Namely, any agenda X_{IL} is safe for $\{A, I, N, M\}$ iff it satisfies the kMP for an aggregation function F_m, with threshold $m > n - \frac{n}{k}$.

4.1.2 Lambek Calculus L

Lambek Calculus was originally developed by Lambek to model the syntax of natural languages, so it is not far fetched to assume that it might be a suitable choice for modeling the problem of judgment aggregation. It is the most restrictive substructural logic and is obtained by dropping all of Exchange, Weakening and Contraction from the sequent calculus of intuitionistic logic. This has multiple effects on the underlying logical language. First of all, the order, as well as the multiplicity of formulas occurring now matters. This leads to the inception of two order-sensitive implications $A\backslash B$ and A/B. In the first case, we conclude B whenever it is preceded by A, while in the second case B is concluded when it is succeeded by A. Lastly, since Exchange is rejected, the commutativity of conjunction is lost, meaning that $A \wedge B$ is no longer equivalent to $B \wedge A$. In a small variation of the original definition of Lambek Calculus Porello adds the constant false $\bot$ to the language. This allows the definition of a kind of negation using the order sensitive implication in the form of $A\backslash\bot$.

To accommodate the properties of Lambek Calculus it is now necessary to consider judgment sets, as well as the agenda to be ordered lists. For the majority rule this means that one has to count not only how many individuals judge the same propositions to be true, but also which occurrence of each proposition they judge to be true (in case the same proposition occurs multiple times in the agenda). For the Lambek Calculus one again obtains an impossibility result when considering the majority rule:

Theorem 18 ([Por17]) *The Lambek Calculus is not safe for MAJ.*

For example, consider any agenda X that includes $[p, p\backslash\bot, p\backslash q, (p\backslash q)\backslash\bot, q, q\backslash\bot]$. Then a vote on this partial agenda could look as follows:

	p	$p\backslash\bot$	$p\backslash q$	$(p\backslash q)\backslash\bot$	q	$q\backslash\bot$
Individual 1	1	0	0	1	0	1
Individual 2	0	1	1	0	0	1
Individual 3	1	0	1	0	1	0
Majority	1	0	1	0	0	1

Table 4.1: An agenda in lambek calculus where majority voting fails

Now the collective judgment list is $[p, p\backslash q, q\backslash\bot]$, which can be shown to be inconsistent with the following sequent proof:

$$\backslash,l\ \dfrac{\backslash,l\ \dfrac{p \vdash p \qquad q \vdash q}{p, p\backslash q \vdash q} \qquad \bot \vdash \bot}{p, p\backslash q, q\backslash\bot \vdash \bot}$$

Furthermore, since in the Lambek Calculus consistency and robust consistency are not equivalent, problems can already arise when considering a very simple agenda. As an example consider the agenda $X = [p, p\backslash\bot, q, q\backslash\bot]$, which satisfies the median property, and the following vote:

	p	$p\backslash\bot$	q	$q\backslash\bot$
Individual 1	1	1	1	0
Individual 2	1	0	0	0
Individual 3	0	1	0	0
Majority	1	1	0	0

Table 4.2: A simple agenda for which majority voting fails

Each individual judgment list is consistent, the collective judgment list $[p, p\backslash\bot]$ however is not.

So if the individual judgment lists are assumed to be only consistent, the median property is no longer enough to ensure that the collective judgments are consistent. To obtain the following result, one has to assume that the individual judgment lists are **robustly consistent**:

Theorem 19 ([Por17]) *An agenda X_L is safe for MAJ iff it satisfies the median property.*

For the class of uniform quota rules the same result as before is again obtained, but again with the condition of assuming robust consistency.

4.1.3 (Exponential Free) Linear Logic LL

In this section we deal with the fragment of Linear Logic called Exponential Free Linear Logic or Multiplicative Additive Linear Logic. It can be obtained by adding the Exchange rule to the sequent calculus for the Lambek calculus. We are now dealing with a logic that contains additive and multiplicative versions of each logical connective. This entails that when formalizing a judgment aggregation problem in LL the choice of connective can be ambiguous. By adding Exchange one again gains commutativity for both the multiplicative and additive variants of conjunction and disjunction, as well as the ability to freely reorder formulas within Judgment Sets and Agendas. However multiple occurrences of the same formula need to be accounted for still, since Weakening and Contraction are not available. In this sense LL provides a model for a certain kind of resource-awareness. This allows one to model, for example, judgments depending on multiple instances of the same formula. An, admittedly contrived, real-life example could be a long voting session in which the same proposal is voted over multiple times. Because of this we now represent the judgment sets as multisets. In this section the judgment sets are assumed to be complete. One again obtains the same impossibility result as before:

Theorem 20 ([Por17]) *Linear logic is not safe fore MAJ.*

As an example, consider the following agenda: $X = \{p, q, p \multimap q, \neg p, \neg q, \neg(p \multimap q)\}$, where $\multimap$ is the multiplicative implication in LL. Then three individuals might vote as follows:

	p	q	$p \multimap q$	$\neg p$	$\neg q$	$\neg(p \multimap q)$
Individual 1	1	0	0	0	1	1
Individual 2	0	0	1	1	1	0
Individual 3	1	1	1	0	0	0
Majority	1	0	1	0	1	0

Table 4.3: An agenda in linear logic where majority voting fails

It is easy to see that the individual judgment sets above are all consistent. The collective judgment set $\{p, \neg q, p \multimap q\}$ however is not, as can be seen by the following sequent proof:

$$\dfrac{p \vdash p \qquad \dfrac{q \vdash q}{q, \neg q \vdash}\,\neg, l}{p, p \multimap q, \neg q \vdash}\,\multimap, l$$

For characterizing safe agendas using the median property one has to assume robust consistency for the individual judgment sets and then obtains again the following result:

Theorem 21 ([Por17]) *An agenda X_{LL} is safe for MAJ iff it satisfies the median property.*

Since the individual judgment sets are assumed to be complete and weakly consistent it might be interesting to see what happens in classes of functions that extend the majority rule. This can be done by taking the set of axioms MAJ, keeping weak rationality and dropping one or more of the others. Then the following holds (again assuming robust consistency):

- X_{LL} is safe for $\{WR, A, N, I\}$ iff it satsifies the SMP
- X_{LL} is safe for $\{WR, A, N\}$ iff it satisfies the SMP
- X_{LL} is safe for $\{WR, A, N\}$ iff it satisfies the SSMP

As was the case with the Lambek Calculus, Linear Logic behaves the same as above with regards to uniform quota rules when assuming robust consistency.

Restricting oneself to the multiplicative fragment of Linear logic (MLL) one obtains the same results:

- Multiplicative linear logic is not safe for MAJ
- X_{MLL} is safe for MAJ iff it satisfies the median property.

All the above can be obtained for the intuitionistic fragments of Linear Logic and Multiplicative Linear Logic as well.

4.1.4 Additive Linear Logic (ALL)

The additive fragment of Linear Logic deserves it's own section because it allows to state an interesting possibility result (Judgment sets are again assumed to be complete):

Theorem 22 ([Por17]) *Additive Linear Logic is safe for MAJ*

This result is mainly due to the fact that in additive linear logic every provable sequent can contain at most two formulas. This entails that there can be no minimally inconsistent sets with cardinality > 2 and as such every agenda stated in ALL satisfies the median property. As an example consider the agenda $\{p, q, p\&q, \neg p, \neg q, \neg(p\&q)\}$. This agenda satisfies the median property in additive linear logic since $p, q \vdash p\&q$ is not derivable in ALL and as a consequence the subset $\{p, q, \neg(p\&q)\}$ cannot be proven to be inconsistent.

This possibility result still holds even if we add contraction.

However, when weakening is added to the sequent calculus, the resulting logic ALL+W is no longer safe for MAJ. This is demonstrated by considering the above agenda and the following vote:

	p	q	$p\&q$	$\neg p$	$\neg q$	$\neg(p\&q)$
Individual 1	1	1	1	0	0	0
Individual 2	1	0	0	0	1	1
Individual 3	0	1	0	1	0	1
Majority	1	1	0	0	0	1

Table 4.4: An agenda for which majority voting fails in ALL+W

It can easily be seen that all individual judgment sets are consistent. The collective judgment set $\{p, q, \neg(p\&q)\}$ however is proven inconsistent as follows:

$$
\begin{prooftree}
\AxiomC{$p \vdash p$}
\AxiomC{$q \vdash q$}
\LeftLabel{$\otimes, R$}
\BinaryInfC{$p, q \vdash p \otimes q$}
\AxiomC{$p \vdash p$}
\LeftLabel{WL}
\UnaryInfC{$p, q \vdash p$}
\AxiomC{$q \vdash q$}
\RightLabel{WL}
\UnaryInfC{$p, q \vdash q$}
\RightLabel{$\&, R$}
\BinaryInfC{$p, q \vdash p\&q$}
\RightLabel{$\otimes, L$}
\UnaryInfC{$p \otimes q \vdash p\&q$}
\RightLabel{cut}
\BinaryInfC{$p, q \vdash p\&q$}
\RightLabel{$\neg, L$}
\UnaryInfC{$p, q, \neg(p\&q) \vdash$}
\end{prooftree}
$$

If extensions of the majority rule are again allowed, one runs into impossibilities even for the simple case of ALL:

- ALL is not safe for $\{WR, A, N\}$
- ALL is not safe for $\{WR, A, I\}$
- ALL is not safe for $\{A, N, I, M\}$ (quota rules)

By restricting the quota rules to require a super-majority another possiblity result can be obtained:

Theorem 23 ([Por17]) *ALL + Contraction is safe for quota rules F_m, where $m > \frac{n}{2}$.*

And again the extensions of the majority rule can be made safe by requiring something stronger than the median property (as above).

4.1.5 Relevant Logic R

Next we consider an extension of the previously defined Linear Logic LL, called Relevant Logic. The idea behind Relevant Logic is to more closely capture the intended meaning of conditionals, namely by adding the notion of relevance between antecedent and succeedent of an implication. This is achieved by disallowing the introduction of arbitrary formulas on either side of a sequent. As such Relevant Logic differs from LL mainly in the fact that Contraction is available and that the language is distributive over the additive connectives. It is not possible to directly enforce distributivity by modifying the underlying sequent calculus. Rather, one considers the equivalent Hilbert system and adds axioms for Contraction and Distributivity of the additives. Since Weakening does not hold it will again be necessary to represent judgment sets as multisets. We also have to assume robust consistency when characterizing safe agendas. The full language of R behaves similar to LL and as such one gets the same impossibility:

Theorem 24 ([Por17]) *Relevant logic is not safe for MAJ.*

For an example of an agenda where the impossibility can be obtained, consider the example used shown in Table 4.3. It works analogous for R.

And again the median property is enought to characterize safe agendas.

By restricting to the additive fragment of relevant logic (AR) it is possible to obtain the same possibility results as for additive linear logic. Namely:

Theorem 25 ([Por17]) *Additive Relevant Logic is safe for MAJ*

The proof makes again use of the fact that in AR every minimally inconsistent set has cardinality 2.

The other safety results can similarly be obtained for relevant logic.

4.1.6 Judgment Aggregation over two Logics

Finally Porello analyzes aggregation rules which allow the collective judgment set to be formulated in a different logic than the individual judgment sets, in particular the case in which the individuals reason in classical logic. Then a possibility result can be obtained for the two pairs of logics (CL, ALL) and (CL, AR):

Theorem 26 ([Por17]) *Both (CL, ALL) and (CL, AR) are safe for MAJ*

The converse does however not hold. I.e. if the individuals reason in ALL and rationality conditions of the collective judgment set are evaluated w.r.t. classical logic. A simple counter example to this is the following set $\{A, B, \neg(A \wedge B)\}$, which is assumed to be the output of a majority aggregation rule. This set is not inconsistent in ALL, but it is inconsistent in classical logic.

4.2 Conditional Logic via Subjunctive Implications

In [Die10] Dietrich analyzes different classes of agendas over a logic in which implications are interpreted subjunctively. A subjunctive interpretation of the statement "if a then b" considers the truth of b in case a is hypothetically true, whereas a classical interpretation considers the actual truth value of a. In that sense a subjunctive interpretation is supposed to more closely resemble the way conditionals are used in day-to-day speech. As an example consider the following nonsense sentence:

If the moon is blue, then the moon is red.

When the conditional is taken as a classical material implication this statement is true, since the antecedent is false. However if we now interpret the above subjunctively and assume the moon to be blue then it is obvious that it cannot also be red, thus making the statement false. And indeed; This interpretation seems to follow the intuitive meaning of the statement more closely.

Dietrich now shows that some possibility results can be obtained when considering agendas over a logic with subjunctive implications. We will provide an overview of their results here, but before that it is again necessary to state a few definitions.

Dietrich employs the abstract model of a logical language defined in the introduction, however for most of the work only languages containing $\wedge$, $\neg$, and $\rightarrow$ are considered. In the following we will also use the defined connective $\leftrightarrow$, where $a \leftrightarrow b$ is defined as $(a \rightarrow b) \wedge (b \rightarrow a)$. The question is then how to properly give semantics to these symbols to avoid some of the impossibilities of judgment aggregation.

Let p and q each be a conjunction of one or more atomic propositions, then

- Statements of the form $p \rightarrow q$ are called uni-directional connection rules. Call them non-degenerate if $p \rightarrow q$ is not a tautology.
- Statements of the form $p \leftrightarrow q$ are called bi-directional connection rules. Call them non-degenerate if neither $p \rightarrow q$ nor $q \rightarrow p$ is a tautology.

An agenda is called an **implication agenda** if it only consists of non-degenerate connection rules as well as the atomic propositions contained within them. Such an implication agenda is called **simple** if all connection rules are uni-directional. By X^+ we denote all non-negated propositions in an agenda X.

Dietrich gives semantics to the symbols in the language by examining two desirable properties a logic should have in connection with the aggregation of judgments:

a If a connection rule r is accepted, it places a restriction on atomic propositions that can also be accepted. For example, $p \to q$ is inconsistent with $\{p, \neg q\}$, but is consistent with each of $\{p, q\}$, $\{\neg p, q\}$ and $\{\neg p, \neg q\}$. As $p \to q$ can be accepted together with the latter 3 sets, but never the first one.

b The acceptance of a negated connection rule $\neg r$ (or the non-acceptance of a positive connection rule r) should not restrict the atomic propositions that can be accepted. This means that, for example, $\neg(p \to q)$ should be consistent with all 4 of the sets from above and as such also be able to be accepted together with them.

It is quite easy to see that the material implication of classical logic satisfies property a but not property b. One can for example consider $\neg(a \to b)$, which is inconsistent with every set that contains either $\neg a$ or b. (Since in classical logic it is possible to write the above implication as $\neg(\neg a \lor b)$, which is further equivalent to $a \land \neg b$.

An interpretation of $\to$ that satisfies both properties can be found in the Conditional Logic C^+ introduced by Stalnaker [Sta68] and D. Lewis [Lew13]. We will now quickly recall the semantics of C^+.

A C^+ interpretation consists of the following:

- A non-empty set of possible worlds denoted W
- For every proposition $p \in L$ a function f_p which maps a world w to all worlds that are similar to w and where p is true.
- For every world $w \in W$ a function v_w that states whether a proposition holds in w.

It furthermore has to satisfy the following properties, for all worlds $w \in W$ and all propositions $p, q \in L$:

- $v_w(\neg p) = T$ iff $v_w(p) = F$
- $v_w(p \land q) = T$ iff $v_w(p) = T$ and $v_w(q) = T$
- $v_w(p \to q) = T$ iff $v_{w'}(q) = T$ for all worlds $w' \in f_p(w)$
- if $w' \in f_p(w)$ then $v_{w'}(p) = T$ (this is ensured by the definition of f_p

- if $v_w(p) = T$ then $w \in f_p(w)$

With this we are now ready to define entailment in C^+ and in turn consistency. Some set of formulas $A \subseteq L$ entails $p \in L$ ($A \models p$ if, for all C^+ interpretations $(W, (f_p), (v_w))$ and all worlds $w \in W$, whenever all $q \in A$ hold at w then also p holds. Using the definition of consistency from the introduction we can now state:

- A set $A \subset L$ is consistent iff there is some C^+ interpretation $(W, (f_p), (v_w))$ and a world $w \in W$, such that at w all formulas $q \in A$ hold.

4.2.1 Results on simple implication agendas

For simple implication agendas Dietrich is able to give the following result:

Theorem 27 ([Die10]) *Any quota rule $F_{(m_p)_{p \in X^+}}$ for a simple implication agenda X is consistent if and only if*

$$m_b \leqslant m_a + m_{a \to b} - n \text{ for all } a \to b \in X$$

This theorem states that it is possible to obtain a consistent quota rule for judgment aggregation over conditional logic. An extreme example can found by setting $m_p = n$ for all $p \in X^+$. However it is also possible to find more forgiving quota rules, in which one is only constricted by the length of implication chains $a \to b, b \to c, \ldots$ and the occurrence of cycles $(a \to b, b \to a)$. The threshold monotonically increases along implication chains and stays constant along cycles. However cycles in the agenda are not desirable since all of the connection rules contained in the cycle have to have an acceptance threshold of n. This can be seen by setting $m_a = m_b$ in the above theorem.

The theorem is proven by first identifying possible sources of inconsistencies and then showing that the derived inequality is necessary and sufficient to prevent the types of inconsistencies found.

This is done in particular by first noticing that any collective judgment set generated by a quota rule has to contain exactly one of either p or $\neg p$ for all $p \in X$. Then the theorem follows from the following two lemmas:

- For a simple implication agenda X, a set $A \subset X$ satisfying the above condition is consistent if and only if it doesn't contain a triple $a, a \to b, \neg b \in X$.
- For a simple implication agenda X, a quota rule $F_{(m_p)_{p \in X^+}}$ never accepts any triple $a, a \to b, \neg b \in X$ if and only if the inequality from theorem 27 holds.

4.2.2 Other special cases of implication agendas

The next step is to consider what happens when the possible range of implication agendas is expanded to include other special types that occur commonly in the wild, namely:

- *semi-simple* implication agendas, in which all connection rules are implications of the form $p \to b$, where b is atomic and p is a conjunct of atoms.
- *bi-simple* implication agendas, in which all connection rules are bi-conditionals $a \leftrightarrow b$, where both a and b are atomic.

For semi-simple agendas one can adapt the two lemmas from above to again obtain an inequality that characterizes consistent quota rules (again using the fact that any subset obtained by a quota rule contains exactly one of the pair $\{p, \neg p\}$ for any $p \in X$):

- For a semi-simple implication agenda X, a set $A \subset X$ is consistent, if and only if it doesn't contain any subset of the form $C(p) \cup \{p \to b, \neg b\}$, where $C(p)$ is the set of conjuncts in p.
- For a semi-simple implication agenda X, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent, if and only if the following inequality holds:

$$\sum_{a \in C(p)} (n - m_a) + m_b \leqslant m_{p \to b} \text{ for all } p \to b \in X$$

In a similar vein one can obtain similar results when considering bi-simple implication agendas:

- For bi-simple implication agendas the excluded-subsets are of the form $\{a, \neg b, a \leftrightarrow b\}$, $\{\neg a, b, a \leftrightarrow b\}$ and $\{a \leftrightarrow b, \neg(b \leftrightarrow a)\}$.
- For bi-simple implication agendas X a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent, if and only if the following inequality holds:

$$m_{a \leftrightarrow b} = n \text{ and } m_a = m_b \text{ for all } a \leftrightarrow b \in X$$

4.2.3 General implication agendas

For general implication agendas Dietrich obtained the following theorem for characterizing consistent quota rules:

Theorem 28 ([Die10]) *A quota rule $F_{(m_p)_{p \in X^+}}$ for an implication agenda X is consistent if and only if the thresholds satisfy the following:*

- *for every $p \rightarrow q \in X$*

$$\sum_{a \in C(p)} (n - m_a) + \max_{b \in C(q) \setminus C(p)} m_b \leqslant m_{p \rightarrow q} \leqslant n - \max_{S \in X_{p \rightarrow q}} \sum_{s \in S : p \rightarrow s \in X} (n - m_{p \rightarrow s})$$

- *for every $p \leftrightarrow q \in X$,*

  (i) $m_{p \leftrightarrow q} = n$

  (ii) $m_a = n$ *for all* $a \in C(p) \cap C(q)$

  (iii) m_a *is the same for all* $a \in C(p) \triangle C(q)$ *and equals* n *if* $|C(p) \triangle C(q)| \geqslant 3$

The way towards obtaining the above conditions for consistent quota rules is again the same as it has been for the previous cases. The only difference is that due to the relatively unconstrained nature of general implication agendas it is quite hard to identify possible sources of inequality. Dietrich distinguishes between two types of inconsistencies that can occur in a general implication agenda:

- Sets representing an inconsistency between a non-negated connection rule and atomic or negated atomic propositions, for example $\{a, \neg b, a \leftrightarrow b\}$.
- Sets representing an inconsistency between a negated connection rule and non-negated connection rules, for example $\{\neg(a \rightarrow (b \wedge c)), a \rightarrow b, a \rightarrow c\}$

Formulating inequalities corresponding to the identified inconsistent subsets and further simplifying them yields the conditions in the above theorem.

The obtained conditions might seem quite complex at first glance, as they are, but in practice the conditions usually simplify. For example in case one deals with simple, semi-simple or bi-simple agendas the conditions reduce to the previously obtained result. For agendas only containing uni-directional rules the second condition can be dropped and vice versa for agendas containing only bi-directional connection rules.

As a corollary the following possibility result for quota rules over implication agendas defined in C^+ can be obtained:

Theorem 29 ([Die10]) *For an implication agenda X, there exists:*

- *a consistent quota rule* $F_{(m_p)_{p \in X^+}}$ *that satisfies the usual responsiveness conditions*
- *a single consistent uniform quota rule* F_m*, namely the unanimity rule with* $m = n$

4.2.4 Abstract characterization result

In this final section we will give a quick recapitulation of Dietrich's result characterizing consistent quota rules for arbitrary agendas X. This result is general and independent of the choice of logic, which is why we are only going to briefly mention it here.

Theorem 30 ([Die10]) *For any simplicity relation $<$, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent if and only if*

$$\sum_{p \in Y} (n - m_p) < n \textit{ for all } Y \in IR_< \; (\textit{where } m_{\neg p} := n - m_p + 1 \; \forall p \in X^+)$$

$<$ is a so-called simplicity relation, which can be any relation between two sets that satisfies the following two conditions:

- Let Y and Z be two inconsistent sets, then $Y \subset Z \implies Y < Z$.
- $<$ is well founded.

Given such a simplicity relation $<$, the set $IR_<$ is the set of irreducible sets, i.e. sets that cannot be simplified any further, generalizing the irreducibility notion in Dietrich and List [DL06]. Since this set is used to obtain the conditions in theorem 30 it is desirable to keep it as small as possible. This can be done by choosing fine grained notions of simplicity for $<$ that allow for many reductions. For example, $IR_\subset$ is usually quite large, while choosing $<$ to be a comparison of size generates a smaller set of irreducibles.

CHAPTER 5

Logics for Reasoning about JA

In this final chapter we will take a step away from the internals of judgment aggregation in non-classical logics and instead present two different approaches of using non-classical logics to model the problem of judgment aggregation itself. Any such logic should be expressive enough to model the common aggregation rules and their properties as well as model each individuals judgment sets and allow quantification of some sorts over both individuals and their judgment sets. The goal of these logics is to provide a formal framework for reasoning over various aspects of judgment aggregation such as the properties of aggregation rules. This in turn also allows one to rigorously prove several impossibility results in both judgment aggregation and social choice theory. An early example of a logic designed to characterize the relationship between different aggregation rules is Pauly [Pau07].

5.1 Judgment Aggregation Logic

The first logic we are going to present is the so called Judgment Aggregation Logic (JAL) as introduced in [ÅvdHW11] by Ågotnes et al. This logic is able to express all concepts occurring in usual Judgment aggregation problems. It can express all common aggregation rules (i.e. majority voting and quota rules) as well as individual properties such rules can have (Independence, Monotonicity, etc.). Furthermore one can formulate and prove various paradoxes of judgment aggregation, such as the discursive dilemma in JAL.

5.1.1 Syntax and Semantics

The language of Judgment Aggregation Logic is $L(N, X)$, where N and X are parameters corresponding to the set of agents and the agenda respectively. It consists of the following atoms:

$$\Pi = N \cup \{\mathbf{h}_p | p \in A\} \cup \{\sigma\}$$

where σ is a constant denoting that the current agenda item is being selected by the current aggregation rule.

The formulas of the language are defined by the grammar

$$\phi = \alpha \mid \Box\phi \mid \blacksquare\phi \mid \phi \wedge \phi \mid \neg\phi$$

where $\alpha \in \Pi$. Similarly to other modal logics a dual to $\Box$ and $\blacksquare$ can be defined as follows: $\Diamond\phi = \neg\Box\neg\phi$ and $\blacklozenge\phi = \neg\blacksquare\neg\phi$. Implicaton $\rightarrow$ and disjunction $\vee$ are obtainable as defined connectives in the usual way. Informally formulas of this language can be interpreted as follows:

Symbol	**Intended Meaning**
i ($i \in N$)	Agent i judges the current agenda item to be true in the current profile
σ	The current agenda item is in the collective judgment set
$\mathbf{h}_p$	The current agenda item is p
$\Box\phi$ ($\Diamond\phi$)	ϕ is true in every (resp. some) judgment profile
$\blacksquare\phi$ ($\blacklozenge\phi$)	ϕ is true in every (resp. some) agenda item

Table 5.1: Symbols of Judgment Aggregation Logic

Ågotnes et al. now give the formal semantics for this language:
A model w.r.t. $L(N, X)$ and some underlying logic $\mathbf{L}$ is defined to be an aggregation rule F over X. A *table* is a tuple $T = (F, A, p)$, where $A \in D(X)^n$ is some complete and consistent profile of judgment sets $(A_1, ..., A_n)$ and $p \in X$. A formula in JLA is now interpreted on a table as follows:

$$F, A, p \models_{\mathbf{L}} \mathbf{h}_q \iff p = q$$

$$F, A, p \models_{\mathbf{L}} i \iff p \in A_i$$

$$F, A, p \models_{\mathbf{L}} \sigma \iff p \in F(A)$$

$$F, A, p \models_{\mathbf{L}} \Box\psi \iff \forall A' \in D(X)^n : \ F, A', p \models_{\mathbf{L}} \psi$$

$$F, A, p \models_{\mathbf{L}} \blacksquare\psi \iff \forall p' \in X : \ F, A, p' \models_t extbfL\psi$$

$$F, A, p \models_{\mathbf{L}} \phi \wedge \psi \iff F, A, p \models_{\mathbf{L}} \phi \text{ and } F, A, p \models_{\mathbf{L}} \psi$$

$$F, A, p \models_{\mathbf{L}} \neg\phi \iff F, A, p \not\models_{\mathbf{L}} \phi$$

One can write $F \models_{\mathbf{L}} \phi$, if and only if $F, A, p \models_{\mathbf{L}} \phi$ for every profile A and $p \in X$. A formula ϕ is valid $\models_{\mathbf{L}} \phi$, if and only if $F \models_{\mathbf{L}}$ for all aggregation rules F. A formula is said to express a property of an aggregation rule, if the formula is true for some rule F, if and only if F has that property.

5.1.2 Expressing properties of aggregation rules

It is now possible to model various properties of aggregation rules using JAL. A few examples are:

- Non-dictatorship: $ND = \bigwedge_{i \in N} \Diamond\blacklozenge\neg(\phi \rightarrow i)$
- Unanimity: $UN = \Box\blacksquare((1 \wedge ... \wedge n) \rightarrow \sigma)$

For these formulas it can be shown that they hold for some aggregation rule F, if and only if F has the respective property they are expressing.

As a final example Ågotnes et al. model the so called discursive paradox using JAL. Firstly, the property of being a majority rule can be expressed as:

$$MV = \sigma \leftrightarrow \bigvee_{G \subseteq N, |G| \geqslant \frac{n}{2}} \bigwedge_{i \in G} i$$

Assuming now that $|N| \geqslant 3$ and the agenda X contains some subset of the form $\{p, q, p \rightarrow q\}$, the following can be shown:

Theorem 31 ([ÅvdHW11]) *Let* $\perp = \sigma \wedge \neg\sigma$*, then*

$$\models_{\boldsymbol{L}} \Diamond((\blacksquare MV) \rightarrow \perp),\ \textit{or equivalently}\ \models_{\boldsymbol{L}} \Diamond\blacklozenge\neg MV$$

5.1.3 Axiomatisation

Ågotnes et al. also provide a Hilbert style axiomatisation of the logic JAL($\mathbf{L}$) with the following axioms, a few of which we will present here (x ranges over $\{\sigma, i : i \in N\}$, $\mathbf{h'}_p$ means $\mathbf{h}_q$ when $p = \neg q$ and $\mathbf{h}_{\neg p}$ otherwise):

$$\begin{aligned}
\text{Atmost: } & \neg(\mathbf{h}_p \wedge \mathbf{h}_q) \\
\text{Atleast: } & \bigvee_{p \in X} \mathbf{h}_p \\
\text{Agenda: } & \blacklozenge\mathbf{h}_p \\
\text{Once: } & \blacklozenge(\mathbf{h}_p \wedge \phi) \rightarrow \blacksquare(\mathbf{h}_p \wedge \phi) \\
\text{CpJS: } & \blacklozenge(\mathbf{h}_p \wedge x) \vee \blacklozenge(\mathbf{h'}_p) \wedge x)
\end{aligned}$$

Atmost states that there can only be one agenda item on the table at the time, while Atleast states that there is always an item on the table. Agenda says that every agenda item will appear on the table and Once states that each item will only appear once.

Finally CpJS encodes the requirement that judgment sets have to be complete.

For the derivability relation $\vdash_{JAL(\mathbf{L})}$ both soundness and completeness can be obtained:

Theorem 32 (Soundness and completeness ([ÅvdHW11])) *If the agenda is finite, we have that for any formula ψ, $\vdash_{JAL(\mathbf{L})} \psi$, if and only if $\models_{\mathbf{L}} \psi$.*

One thing to note about JAL is that, while it has all axioms of the modal logic S5 (that is K, T, 4 and 5) as well as the rules Modus Ponens and Necessitation, the principle of uniform substitution does not hold. A simple counter example is the following:

$$\Box\blacklozenge\sigma$$

which states that any judgment aggregation rule will always make a judgment, however substituting σ leads to the following formula which is not valid:

$$\Box\blacklozenge(\sigma \wedge i)$$

which states that the judgment aggregation rule will always make the same judgments as some agent i, which clearly only holds for rules that are dictatorships for i.

Perkov [Per16] presents a natural deduction system for judgment aggregation logic.

5.1.4 Model Checking

When working with a formal system such as JAL, a question one might ask rather often is the following:
Given some aggregation function F, a profile A and a formula ϕ, is it the case that $F, A, p \models_{\mathbf{L}} \phi$.

This sort of model checking can be useful to verify various properties of judgment aggregation rules, once an appropriate formalization has been found. With that the question arises of what the computational complexity of such a model checking procedure might be. Ågotnes et al. show that for a reasonable representation of judgment aggregation rules the following holds:

Theorem 33 ([ÅvdHW11]) *Assuming a reasonable representation of judgment aggregation rules, the model checking problem for JAL is $\triangle_2^p$-hard and it is NP-hard even if the formula to be checked is of the form $\Diamond\psi$, where ψ contains no further $\Box$ or $\Diamond$.*

Reasonable representations of judgment aggregation rules F are any representations that satisfy the following conditions:

(i) The size of the representation is polynomial in the size of the agenda

(ii) There is a polynomial time algorithm which takes as an input the representation of the rule and some profile A and produces as output $F(A)$

There can be many such representations, however Ågotnes et al. present a very general one, namely where a judgment aggregation rule is represented by a polynomially bounded two-tape Turing machine. The first tape carries the judgment profile and the resulting collective judgment set is written on the second tape.

Finally a normal form is presented which shows that (in case of a finite agenda) any property that can be expressed in JAL can also be expressed as a formula in which no modal operator occurs in its own scope (that is, the modal depth of formulas in JAL can be restricted to two). We will not repeat the transformation $d(\phi)$ into the normal form here and only present the following theorem:

Theorem 34 ([ÅvdHW11]) *For any F, A, p and formula ϕ,*

$$F, A, p \models_{\boldsymbol{L}} \phi \text{ , if and only if } F, A, p \models_{\boldsymbol{L}} d(\phi)$$

For practical purposes however it might still be necessary to search for a succinct representation of a property rather than relying on the normal form since the transformed formula $d(\phi)$ can often be exponential in the number of elements of the agenda, which for most real-world agendas is an unacceptable overhead.

5.2 Dynamic Logic of Propositional Assignments

This next formalism differs from the previously defined JAL by the fact that it is not a newly designed formalism explicitly developed for the purpose of modeling judgment aggregation. Rather it is an embedding into an already existing formalism, namely the so-called Dynamic Logic of Propositional Assignments (DLPA), cf. Balbiani et al. [BHT13]. This embedding was first presented in [NGH18] by Novaro et al. Employing DLPA for the purpose of modeling JA is interesting because there exists a translation between programs of DLPA and ordinary propositional logic. That allows SAT-solvers and other already existing automated reasoning tools to be used to verify properties of judgment aggregation rules modeled in DLPA and in some cases even discover new properties and impossibility theorems of JA, as has been done by Geist et al. for the field of Social Choice Theory in [GE11].

For the sake of easier modeling, Novaro et al. consider a slight variation of the formalism described in the introduction. In their case agents only vote on atomic propositions, which are then linked together by a set of complex formulas called Integrity Constraints (IC). Both formulations are however equivalent, and a translation of a JA problem from one into the other is easily possible. Acceptance and rejection are modeled by an m-tuple $B_i = (b_1, ...b_m)$ called ballot where each $b_j \in \{0, 1\}$, where 0 and 1 signify rejection and acceptance of item j of the agenda respectively. A profile $B = (B_1, ..., B_n)$ is then the collection of ballots from all individuals. Here again a straightforward translation into the formalism defined in the introduction is possible with ballots being translated into individual judgment sets. The final definition is that of a model $Mod(IC)$ corresponding to the set of all ballots satisfying the integrity constraints. This is the set of all consistent judgment profiles. An aggregation rule is now any function F that maps profiles consisting of ballots taken from $Mod(IC)$ to a collective ballot $F(B)$.

5.2.1 Syntax and Semantics of DLPA

The language of DLPA can be given by the following grammar:

$$\begin{aligned} \phi &::= p \mid \top \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \langle\pi\rangle\phi \\ \pi &::= +p \mid -p \mid \pi;\pi \mid \pi \cup \pi \mid \phi? \end{aligned}$$

where $p \in \mathbb{P}$ is a propositional variable from a countable set $\mathbb{P} = \{p, q, ...\}$. Other connectives, as well as the dual modality $[\pi]\phi$ can be defined as usual. Intuitively $+p$ assigns the value true to the variable p, while $-p$ assigns false. $\pi;\pi'$ means that π and π' are executed in sequence, while $\pi \cup \pi'$ nondeterministically executes either π or π'. $\phi?$ checks wheter ϕ holds and fails otherwise.

Formally, the semantics of DLPA are given by valuations $v \subset \mathbb{P}$. $\mathbb{V} = 2^{\mathbb{P}}$ is the set of all valuations. A variable p is said to be true, if $p \in v$. A program is then interpreted as follows:

$$\begin{aligned} ||p|| &= \{v \in \mathbb{V} \mid p \in v\} \\ ||\top|| &= 2^{\mathbb{P}} \\ ||\bot|| &= \{\} \\ ||\neg\phi|| &= 2^{\mathbb{P}} \backslash ||\phi|| \\ ||\phi \vee \psi|| &= ||\phi|| \cup ||\psi|| \\ ||\langle\pi\rangle\phi|| &= \{v \in \mathbb{V} \mid \exists v_1 \ s.t. \ (v, v_1) \in ||\pi|| \ and \ v_1 \in ||\phi||\} \\ ||+p|| &= \{(v_1, v_2) \mid v_2 = v_1 \cup \{p\}\} \\ ||-p|| &= \{(v_1, v_2) \mid v_2 = v_1 \backslash \{p\}\} \\ ||\pi;\pi'|| &= ||\pi|| \circ ||\pi'|| \\ ||\pi \cup \pi'|| &= ||\pi|| \cup ||\pi'|| \\ ||\phi?|| &= \{(v, v) \mid v \in ||\phi||\} \end{aligned}$$

It is possible to express most standard programming constructs, such as $skip$ and $if-else$ in DLPA. Assignements of the form $p \leftarrow q$ are obtainable via $if-else$ constructs.

Natural numbers can be expressed in their binary representations by a conjunction of variables. These numbers can be compared to each other, incremented and set to zero. Repeated execution of a particular program can be achieved as follows: $\pi^n := \pi; \pi^{n-1}$ and $\pi^{\leqslant n} := (skip \cup \pi); \pi^{\leqslant n-1}$.

5.2.2 Judgment Aggregation in DLPA

Let $\mathbb{B}^{n,m} := \{p_{ij} \mid i \in N, 1 \leqslant j \leqslant |X|\} \subset \mathbb{P}$, where p_{ij} encodes the decision of agent i on agenda item j. Furthermore $\mathbb{O}^m := \{p_j \mid 1 \leqslant j \leqslant |X|\}$ is the set of variables encoding the possible outcomes for item j. A final set $\mathbb{U} = \{q_i \mid i \in \mathbb{N}\}$ is defined, which will be used to encode finitely many counters in the program.

A valuation v_B is said to translate a profile $B = (B_1, ..., B_n)$ if

- $v_B \subseteq \mathbb{B}^{n,m}$
- $p_{ij} \in v_B$ if and only if $b_{ij} = 1$

A program $f(\mathbb{B}^{n,m})$ is said to translate an aggregation rule F, if for all possible profiles B and valuations v_B encoding B it is the case that $V^f_{v_B} = F(B)$, where $V^f_{v_B}$ is the set of valuations reachable from v_B in $f(\mathbb{B}^{n,m})$ restricted to the set of outcomes $\mathbb{O}^m$.

Novaro et al. now give the following result:

Theorem 35 (Novaro) *For every set of agents N, agenda X and set of integrity constraint IC, all aggregation rules $F: \ Mod(IC)^n \to 2^{\{0,1\}^{|X|}}$ are expressible as DLPA-Programs.*

A few examples of common aggregation rules translated into $DLPA$ are

- $dictatorship_i(\mathbb{B}^{n,m}) \ := ;_{j \in X} (p_j \leftarrow p_{ij})$
- $maj(\mathbb{B}^{n,m}) \ := ;_{j \in X} (zero(pro \cup con); ;_{i \in N}$ (if p_{ij} then $incr(pro)$ else $incr(con)$); if $pro > con$ do $+p_j)$

where pro and con are two disjoint subsets of $\mathbb{U}$ which count the number of agents that accept and reject a proposition.

It is also possible to express (uniform) quota rules as well as different kinds of minimization/maximization rules, such as rules based on Hammond Distance between ballots.

Another use for DLPA is to express various axiomatic properties of aggregation rules, such as Independence, Monotonicity, etc. As an example we repeat the translation of the property of unanimity of a rule:

$$U \;:=\; \bigwedge_{j \in X}(((\bigwedge_{i \in N} p_{ij}) \rightarrow p_j) \wedge ((\bigwedge_{i \in N} \neg p_{ij}) \rightarrow \neg p_j)).$$

Then it holds that F satisfies U if and only if $\models Prof_{IC}(\mathbb{B}^{n,m}, \mathbb{O}^m) \rightarrow [f(\mathbb{B}^{n,m}]U$, where $Prof_{IC}(\mathbb{B}^{n,m}, \mathbb{O}^m)$ is a formula that is true if and only if we are in a valuation that corresponds to a valid encoding of a profile.

Finally we will present the translation of DLPA into propositional logic. It first simplifies programs into atomic programs, then distributes the atomic programs over the logical connectives inside formulas and at the end eliminates atomic programs. It is defined as follows:

$$\begin{aligned}
[\phi?]\psi &\iff \phi \rightarrow \psi \\
[\pi_1; \pi_2]\phi &\iff [\pi_1][\pi_2]\phi \\
[\pi_1 \cup \pi_2]\phi &\iff [\pi_1]\phi \wedge [\pi_2]\phi \\
[\pi]\neg\phi &\iff \neg[\pi]\phi \\
[\pi](\phi_1 \wedge \phi_2) &\iff [\pi]\phi_1 \wedge [\pi]\phi_2 \\
[+p]q &\iff \begin{cases} \top \text{ if } p = q \\ q \text{ otherwise} \end{cases} \\
[-p]q &\iff \begin{cases} \bot \text{ if } p = q \\ q \text{ otherwise} \end{cases}
\end{aligned}$$

Something to be aware of when working with this translation is again the possibility of exponential blowup for certain formulas.

CHAPTER 6

Topics for further research

In this concluding chapter we will present an overview over a few different avenues of further research on the topic of non-classical judgment aggregation.

6.1 Different meanings of degrees

One of the things to consider when working with many-valued judgment aggregation is what meaning one gives to the degrees assigned by the individuals. The most common formulation, which is also captured by a wide array of many-valued logics, is that individuals assign a degree of **truth** to the propositions.

A straightforward alternative would be the case in which individuals have to state how much they agree with certain propositions, i.e. give a degree of **assent**. The difference to the truth-based approach mainly lies in how logical connectives are treated for statements of assent. As an example, it might not be the case that "I assent to p AND I assent to q" is equivalent to "I assent to $p \wedge q$". This means that, depending on the interpretation, some connectives in a logic of assent may not be truth-functional, which might prove to be a possible avenue to avoid impossibilities in judgment aggregation.

Another approach is to consider the degrees assigned by the individuals not as truth directly but rather as the **probability** that the chosen proposition is true. In this case the underlying connectives are known to not be truth-functional. Treating degrees in this way might allow results from probability theory as well as other fields, such as probabilistic opinion pooling (see McConway [McC81]) and probability logics (Hájek [Háj01]) to carry over. A discussion on how probability theory can be related to fuzzy logic can be found in Gaines [Gai78].

Considering the assigned degrees to be degrees of **belief** is another option that allows results from belief-binarization (see Section 3.6) as well as belief-merging (Konieczny and Pérez [KP02]) and related fields to be applied to judgment aggregation problems. Huber et al. [HSP08] provide a comprehensive overview over prevailing theories of degrees of belief. Connectives over beliefs are again not truth-functional, making this approach another interesting candidate for sidestepping the impossibilities of judgment aggregation.

As a generalization of the above we can also consider the case in which judgments are **two-dimensional**, i.e. an individual assigns a pair of degrees (x, y) to each proposition. Section 3.1 presents an instance of such a two-dimensional judgment, in which agents can give their judgment weighted by their expertise on the topic. Other combinations are still left to be explored, for example when individuals give a confidence score in addition to their judgment.

6.2 Relaxing consistency

The property of consistency of individual and collective judgment sets lies at the heart of classical judgment aggregation theory. Even in classical logic it is already highly demanding of the individuals to provide fully consistent judgment sets. In the case of many-valued logics this requirement becomes even more demanding and as such it might make sense to consider relaxations of this requirement. Section 3.6 presents such a relaxation from the point of view of belief-binarization in which consistency is replaced by a weaker criterion called belief-k-consistency.

Another approach would be to let individuals give their judgment in a range (for example a confidence interval). This not only softens the demand on each individual to give consistent judgments, but it might also allow the import of results from interval-based fuzzy logic.

6.3 Generalizing valuation based approaches

In this section we introduce a generalization of existing valuation based approaches to judgment aggregation, as seen for example in Section 3.1. By dropping the requirement that individual judgments J_i have to be atomically closed we obtain that a judgment now corresponds to an entire set of valuations, namely all judgments that extend J_i to a valuation over the entire agenda. As an example consider the following simple classical agenda $X = \{p, q, p \vee q\}$ and the individual judgment $J_i = \{p \vee q\}$. The set of valuations corresponding to this individual judgment is now $\mathbb{J}_i = \{\{p, q, p \vee q\}, \{p, p \vee q\}, \{q, p \vee q\}\}$.

The above can easily be extended to a many-valued setting by considering the individual judgments to be *valuation functions* $j_i : Y \to [0, 1]$, where $Y \subseteq X$. Now, instead of only considering the valuation given by the individual themselves, we consider the set $\mathbb{J}_i$ of all valuations that extend j_i to a consistent valuation of the entire agenda. A fact one has to be careful about is that in contrast to the classical case this set can be (uncountably) infinite, depending on the underlying logic of the aggregation problem and the structure of the agenda.

Aggregation in this setting now means picking a set of valuations that is closest to all individual sets of valuations. For this it is first necessary to define the notion of distance between sets of valuations. One possibility would be to consider the valuations in terms of the set of values they produce when applied to the agenda. It is then possible to employ measures of distance between sets of sets of real numbers (for example a generalized Hamming Distance [BK96]). Similarly to standard valuation based approaches, the behavior of the underlying aggregation procedure depends on what measure of distance between sets of valuations is employed and how closeness to a set of valuations is defined.

Once the collective set of valuations $\mathbb{J}$ has been obtained one can (if necessary) pick a valuation $j \in \mathbb{J}$ to serve as the single collective judgment of the group of individuals. There are multiple possible ways to achieve this, one possibility being that j is the 'average' of the valuations in $\mathbb{J}$.

Again one of the main benefits of this new formulation is the ability to exploit existing literature, for example literature on fuzzy measures (Sugeno [Sug93]), fuzzy aggregation (Vaníček et al. [VVA09]) and distance-based fuzzy implication (Ruspini [Rus91]).

6.4 Connection to (semi-)fuzzy quantifiers

There is an interesting connection between judgment aggregation and fuzzy quantification that, to our knowledge, has not been explored as of yet. As an example, consider a majority vote on a proposition p. In informal language, one might say that p is accepted whenever **More than half** of individuals agree with p. This kind of phrasing is often encountered when dealing with fuzzy quantification. To be able to employ fuzzy quantification as a way to deal with judgment aggregation problems, the following translation is necessary:

1 For every $p \in X$ define a new monadic predicate symbol $\bar{p}$.

2 Interpret $\bar{p}(i)$ to mean 'Individual i agrees with p'.

3 Add postulates of the form $\overline{p \circ q}(i) \leftrightarrow \bar{p}(i) \circ' \bar{q}(i)$ and simplify accordingly. This allows for different MV-interpretations of the propositional connectives and makes it possible to limit the introduction of new predicates to atomic propositions only.

A monadic MV-quantifier is now a truth function $\tilde{Q} : V^D \to V$, where V is the set of truth values and D the domain. Let the domain D be the set of individuals N, then any truth function $\tilde{Q}$ *is* already an aggregation function F_Q over a singleton agenda $X = \{p\}$ and vice versa. Proposition-wise aggregation over general agendas can be achieved by applying the fuzzy quantifier to each proposition in the agenda individually and collecting the results: $\langle \tilde{Q}_F(\bar{p}) \rangle_{p \in X}$. This vector of truth values can then be easily translated back into a set of propositions. The properties of an aggregation rule defined in such a way directly depend on the properties of the used quantifier. By definition, such a rule satisfies

Systematicity, and in turn also *Independence*, as well as *Universal Domain* and *Neutrality*. It is *monotonous* whenever the underlying quantifier is and *anonymous* whenever the quantifier is **logical**.

Many of the common aggregation rules can now be expressed in this new formalism. Consider for example the semi-fuzzy quantifier **At least k** ($\tilde{Q}_k$). Informally, a sentence $\tilde{Q}_k x(p(x))$ is true, whenever $p(x)$ is true for at least k distinct domain elements. An aggregation rule constructed from $\tilde{Q}_k$ in the above way now corresponds to a **quota rule** with threshold $t = \frac{k}{|N|}$. By letting k be equal to $\frac{|N|+1}{2}$ one in turn obtains the **majority rule**. The **Unanimity rule** can be achieved by either letting $k = |N|$ or by constructing the aggregation rule from the **All** quantifier ($\forall$).

For a domain element $d \in D$ the quantifier $\tilde{Q}_d$ is defined in such a way that the truth value of a sentence $\tilde{Q}_d x(p(x))$ corresponds to the truth of $p(d)$. In our setting, for an individual $i \in N$, an aggregation rule constructed from $\tilde{Q}_i$ is a **Dictatorship** by individual i.

To illustrate the procedures defined above, consider the following simple example of a majority vote:

	p	q	$p \wedge q$
Individual 1	1	0	0
Individual 2	0	1	0
Individual 3	1	1	1
Majority	1	1	0

Table 6.1: A simple majority vote

Following our translation from above, we obtain two predicate symbols $\bar{p}$ and $\bar{q}$, with the following interpretations:

$$\bar{p} = \{i_1, i_3\}$$
$$\bar{q} = \{i_2, i_3\}$$

The domain of the translated example is $D = \{i_1, i_2, i_3\}$. Since we are dealing with a majority vote in a setting with 3 individuals, the corresponding quantifier is **At least 2** ($\tilde{Q}_2$). Applying this quantifier to each individual proposition yields the following results:

$$\tilde{Q}_2 x(\bar{p}(x)) = 1$$
$$\tilde{Q}_2 x(\bar{q}(x)) = 1$$
$$\tilde{Q}_2 x(\bar{p}(x) \wedge \bar{q}(x)) = 0$$

Translating back into a judgment set now yields $\{p, q, \neg(p \wedge q)\}$, as expected.

Under this new formalism many of the impossibilities of judgment aggregation reduce to well-known properties of fuzzy quantifiers, such as the fact that quantifiers of the form **At least k** ($\tilde{Q}_k$), where $k < |D|$ does not distribute over conjunction: $\tilde{Q}_k x(p(x) \wedge q(x)) \not\equiv \tilde{Q}_k x(p(x)) \wedge \tilde{Q}_k x(q(x))$, as shown by the example above. We believe that by exploiting such connections to the existing literature on fuzzy quantification (such as Glöckner [Glö08]) new results for the field of judgment aggregation can be obtained. An overview over various fuzzy quantifiers and their interpretations can also be found in Liu et al. [LK98].

6.5 Computational aspects

A topic that is very sparsely discussed in the literature is that of the computational aspects of the proposed aggregation procedures. Endriss et al. present some considerations on the complexity of classical judgment aggregation in [EGP12]. Slavkovik et al. also briefly mention the complexity of their weight-based aggregation of three-valued judgments in [SJ11]. Such complexity considerations as well as concrete implementations of aggregation procedures are however still missing for a majority of published results in non-classical judgment aggregation.

CHAPTER 7

Summary and Conclusion

The above chapters provide a comprehensive overview over the current state of affairs concerning applications of non-classical logics to judgment aggregation, as well as some pointers to related topics and open research questions. We covered the following approaches:

- A novel weight-based aggregation procedure by Slavkovik et al. [SJ11] that allows for fine-tuning of individual judgments
- An extension of the concept of deductive closure of judgment sets to the many-valued case employing triangular norms, first presented by Duddy and Piggins in [DP13].
- Two abstract characterization results, namely Herzberg's approach involving MV-Algebras presented in [Her13] and an extension of the former employing Abstract Algebraic Logic due to Esteban et al. [EPZ15], that extend the core impossibility theorem of judgment aggregation to a wide range of non-classical logics
- Grossi's embedding of preference aggregation into many-valued judgment aggregation presented in [Gro09]
- A connection to the field of Belief Binarization, first presented by Dietrich and List in [DL18].
- Porello's [Por17] examination of various substructural logics, in which possibility results for certain restricted classes of logics are obtained.
- An analysis of a conditional logic in which implications are interpreted in a more natural, subjunctive, manner, in which Dietrich [Die10] obtains possibility results for certain classes of agendas
- Two different formalisms that allow reasoning over judgment aggregation due to Ågotnes et al. [ÅvdHW11] and Novaro et al. [NGH18].

Concludingly, we would like to put some focus on topics we think have been somewhat neglected in the available research literature. One of these topics is the search for actual applications of the obtained results. An example that immediately springs to mind, but has to our knowledge not been investigated yet, is the connection between non-classical aggregation rules and the search for alternative voting procedures. For this purpose it might also prove fruitful to examine a possible connection of judgment aggregation to the active research field of deliberation and investigate how various choices of aggregation procedures can influence the process of individual deliberation and vice-versa.

We also noticed the fact that, for many of the proposed non-classical models of judgment aggregation it is not entirely clear how the real-life counterpart of the employed judgments looks like. For some of them, such as the three valued judgments in Section 3.1 this is quite straightforward. However, as an example, in case of the substructural logics presented by Porello in [Por17] it is not so clear. In a similar vein it might be interesting to put more focus on the reversed case, in the sense that one starts by considering a real-world judgment aggregation problem and then works backwards to find a suitable logical model, similar to how Dietrich [Die10] arrived at their Conditional Logic with subjunctive implications.

Finally, concerning future research in the field, we believe that there are two equally promising approaches to take when trying to obtain new results. The first, and most straightforward is to extend on existing abstract-characterization results, such as those by Herzberg and Esteban et al. By identifying classes of logics that are not covered by these characterizations a starting point is reached from which one can either obtain a possibility for the concerned class of logics, or a newer, more general variant of the characterization result. The other approach concerns the cross-application of existing results from a variety of related fields. In recent years many, sometimes surprising, connections of judgment aggregation to other research fields have been discovered. For some of them, such as the relations to Belief Binarization and Preference Aggregation, work has already been done on formally establishing the connection and applying existing results. However, we believe there to be many relations that are as of yet undiscovered, an example being the connection of judgment aggregation to fuzzy quantification. The main focus in this area of research should nonetheless be two-fold: On the one hand, research on the cross-application of results from fields with a known connection to judgment aggregation is far from complete, while on the other hand it is crucial that more connections are uncovered.

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