utility based asset pricing under high risk aversion

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December 2007

Abstract

This work aims to present three methods of pricing an asset.
A price of a derivative is the amount of money a buyer agrees to pay to the seller at time 0 in order to receive the derivative at maturity time T. If the market is complete, this price is uniquely given by the initial wealth of the portfolio in stocks and bonds that recreates the terminal payoff (replication). But in reality transaction costs or non-traded assets cause that the financial market is not complete. Then different prices consistent with the No Arbitrage Condition exist as each corresponds to a different martingale measure. The *superreplication price* is defined as the supremum of these martingale measures and therefore unrealistically high, but all risks and uncertainty is removed. Hence we want to find another way of pricing in an incomplete market but remain risk averse.
For these purposes we introduce the *utility indifference price* after explainig the concept of *utility maximization* in chapter 3 and giving a short definition on *risk aversion* in chapter 4. This price considers the risk aversion and can also depend on the agents initial wealth. Unlike the superreplication price the utility indifference price is not linear in the number of units of the claim, but converges to the superreplication price if the risk aversion tends to infinity. This statement is proved in chapter A.2. The utility indifference price can be considered as an interpolation between the totally risk averse superreplication price and the *marginal utility price*, which we introduce as the third price in chapter 6.
By means of two examples all these properties will be verified in the last chapter.

Contents

1 Prearrangements

This section cites the mathematical requirement which will be needed for this work and are obtained from the lecture notes of Prof. Teichmann [3].

1.1 Basics from Convex Analysis

This section concentrates on the methods from convex analysis, which will be needed in the present work.
For a real vector space V with norm and $\dim V < \infty$ we define the pairing as

$$\langle ., . \rangle : V \times V' \to \mathbb{R}$$
$$\langle v, l \rangle \mapsto l(v)$$

where V' denotes the dual vector space, i.e. the space of linear functionals $l : V \to \mathbb{R}$; $V' := \{l : V \to \mathbb{R} | l \text{ linear}\}$. We have the dual space with a natural dual norm

$$\| l \| := \sup_{\|v\| \leq 1} |l(v)|$$

The following dual relations hold:

- If for some $v \in V$ $\langle v, l \rangle = 0$ holds for all $l \in V'$, then $v = 0$.
- If for some $l \in V'$ $\langle v, l \rangle = 0$ holds for all $v \in V$, then $l = 0$.

In the present work we deal with an euclidean vector space, i.e. we have a scalar product $\langle ., . \rangle : V \times V' \to \mathbb{R}$ which is symmetric and positive definite. Thus we can identify V' with V and every linear functional $l \in V'$ can uniquely be represented as $l = \langle ., x \rangle$ for some $x \in V$.

Definition (convex)
Let C be a finite dimensional vector space. $C \subset V$ is called convex, if for all $x, y \in C$

$$tx + (1-t)y \in C, \quad t \in [0,1]$$

Definition (convex hull)
The convex hull of a subset $M \subset V$ is defined as

$$\langle M \rangle_{\text{conv}} := \cup_{\substack{M \subset C \subset V \\ C \text{ convex}}} C$$

The closed convex hull $\overline{\langle M \rangle_{\text{conv}}}$ is the smallest closed convex subset that contains M. If M is compact, the convex hull $\langle M \rangle_{\text{conv}}$ is already closed and therefore compact.

Definition (extreme point)
Let C be a closed, convex set. $x \in C$ is called extreme point if for all $y, z \in C$ with $x = ty + (1-t)z$, $t \in [0,1]$ either $t = 0$ or $t = 1$ holds, i.e. there do not exist $x_1 \neq x_2 \in C$ such that $x = \frac{1}{2}(x_1 + x_2)$.

Lemma 1.1. ***(Hahn-Banach)*** *Let V be a finite-dimensional, euclidean vector space and C a closed, convex set that does not contain the origin, i.e. $0 \notin C$.*
Then a linear functional $l \in V'$ exists such that $l(x) \geq \alpha$ holds for all $x \in C$, i.e. the linear functional l seperates 0 and C.

Lemma 1.2. ***(Krein-Milman)*** *Let C be a compact and convex set.*
Then C is the convex hull of all its extreme points, i.e

$$\langle ext(C) \rangle_{conv,\ closed} = C$$

Definition (convex cone)
A subset $C \subset V$ is called convex cone, if for all $v_1, v_2 \in C$

- $v_1 + v_2 \in C$ and
- $\lambda v_1 \in C$ for $\lambda \geq 0$

The polar C^0 of a given cone C is defined by

$$C^0 := \{l \in V \text{ such that } \langle l, v \rangle \leq 0 \text{ for all } v \in C\}$$

Lemma 1.3. ***(Bipolar Theorem)*** *Let $C \in V$ be a convex cone, then the bipolar $C^{00} := (C^0)^0 \subset V$ is the closure of C, i.e. $C^{00} = \overline{C}$.*

Proof First take $v \in \overline{C}$. Then $\langle l, v \rangle \leq 0$ for all $l \in C^0$ by definition of the polar C^0 and therefore $v \in C^{00}$. If we could find $v \in C^{00} \setminus \overline{C}$, then for all $l \in C^0$ we have $\langle l, v \rangle \leq 0$ by definition.
On the other hand it follows from the seperation theorem that we can find $l \in V$ such that $\langle l, x \rangle \leq 0$ for all $x \in \overline{C}$ and $\langle l, v \rangle > 0$. Therefore take l and α such that $\langle l, x \rangle \leq \alpha$ and $\langle l, v \rangle > \alpha$. As $0 \in \overline{C}$, we have $\alpha \geq 0$ and if there were $x \in \overline{C}$ satisfying $\langle l, x \rangle > 0$ then we would have for all $\lambda \geq 0$ that $\langle l, \lambda x \rangle = \lambda \langle l, x \rangle \leq \alpha$, which is a contradiction; so $\langle l, x \rangle \leq 0$. By assumption we have $l \in C^0$, however this yields a contradiction, since $\langle l, v \rangle > 0$ and $v \in C^{00}$.

The following lemma is by Rásonyi and Stettner [1]:

Lemma 1.4. *Define $\mathcal{H} \subset \mathcal{F}$ as a σ-algebra that contains all $\mathbb{P}$-nullsets and denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel sets of $\mathbb{R}^d$. Let $\eta_n : \mathbb{R} \times \Omega \to \mathbb{R}^d$ be a sequence of $\mathcal{B}(\mathbb{R}) \otimes \mathcal{H}$-measurable functions such that for nearly all ω*

$$\forall x \qquad \liminf_{n \to \infty} |\eta_n(x, \omega)| < \infty$$

Then there exists a sequence n_k of $\mathcal{B}(\mathbb{R}) \otimes \mathcal{H}$-measurable $\mathbb{N}$-valued functions $n_k < n_{k+1}, k \in \mathbb{N}$ such that $\widetilde{\eta}_k(x, \omega) := \eta_{n_k}(x, \omega)$ converges for all x to some $\widetilde{\eta}(x, \omega)$ as $k \to \infty$, in brief: there is a convergent random subsequence.

1.2 Basics from Optimization Theory

Definition (local maximum, local minimum)
Let $U \subset \mathbb{R}^m$ be a subset of V, where V is open in $\mathbb{R}^m$ and $F : V \to \mathbb{R}$ a C^2-function.
A point $x \in U$ is called a local maximum of F on U, if a neighbourhodd W_x of x exists in V such that for $y \in U \cap W_x$

$$F(y) \leq F(x)$$

holds. For a local minimum we have $F(y) \geq F(x)$ respectively.

Lemma 1.5. *Given the above settings and a C^2-curve $c :]-1,1[\to V$ with $c(0) = x$ and $c(t) \in U$ for $t \in]-1,1[$, i.e. the curve always lies in U and intersects x, then the following condition holds true*

$$\frac{d}{dt}|_{t=0} F(c(t)) = \left\langle grad\ F(x), c'(0) \right\rangle = 0$$

We can now prove a version of the Lagrangian multiplier theorem for affine subspaces $U \subset \mathbb{R}^m$. We take an affine subspace $U \subset \mathbb{R}^m$ and an open neighbourhood $V \subset \mathbb{R}^m$ such that $U \cap V \neq \emptyset$, where a C^2-function $F : V \to \mathbb{R}$ is given.

Lemma 1.6. *Let x be a local maximum (local minimum) of F on $U \cap V$ and assume that there are $k := m - dim\, U$ vectors $l_1, \ldots, l_k \in \mathbb{R}^m$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ such that the subset*

$$U = \{x \in V \mid \langle l_i, x \rangle = \alpha_i,\ i = 1, \ldots, k\}$$

Then

$$grad\ F(x) \in \langle l_1, \ldots, l_k \rangle$$

hence there exist real numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$grad\ F(x) = \lambda_1 l_1 + \cdots + \lambda_k l_k.$$

Hence we can formulate a recipe; a necessary condition for an extremal point of $F : V \to \mathbb{R}$ under the condition $\langle l_i, x \rangle = \alpha_i$ for $i = 1, \ldots, k$, is to solve the unconstraint extended problem with the Lagrangian L

$$L(x, \lambda_1, \ldots, \lambda_k) = F(x) - \sum_{i=1}^{k} \lambda_i (\langle l_i, x \rangle - \alpha_i)$$

Indeed, as we want to find the extrema of L we have to calculate the partial derivatives to x and λ and equal them to zero:

$$\text{grad } F(x) - \sum_{i=1}^{k} \lambda_i l_i = 0$$

$$\langle l_i, x \rangle = \alpha_i$$

If this can be solved, i.e. if we find an $\widehat{x}$ which solves the equations, then $\widehat{x} \in U$ is a local minimum and there exist $\widehat{\lambda}_1, \dots, \widehat{\lambda}_k \in \mathbb{R}$, such that grad $F(\widehat{x}) = \sum_{i=1}^{k} \widehat{\lambda}_i l_i$ and $\langle l_i, \widehat{x} \rangle = \alpha_i$

Remark 1.7. The gradient of a C^1-function $F : V \to \mathbb{R}$ on a finite dimensional vector space V is defined through

$$\langle \text{grad } F(x), z \rangle = \frac{d}{ds}|_{s=0} F(x + sz)$$

for $x \in V$ and $z \in \mathbb{R}^n$ and a scalar product $\langle ., . \rangle$. This can be calculated to any basis and gives a coodrdinate representation. The derivative of F is understood as element of the dual space

$$dF(x)(z) := \frac{d}{ds}|_{s=0} F(x + sz)$$

for $x \in V$ and $z \in \mathbb{R}^n$. If we have a Euclidean vector space and a orthonormal basis $e_1, \dots, e_N$, we can calculate

$$(\text{grad}_{e_i} F(x))_{i=1,\dots,N} := (dF(x)(e_i))_{i=1,\dots,N}$$

1.3 Basics from Probability Theory

Definition (σ-algebra) A σ- algebra is a collection of subsets of a given set (here Ω) which contains the empty set(and therefore is a nonempty set) and is closed under complementation and countable union of its members. For example a σ-algebra of the set $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ could be $\mathcal{F}_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}\}$. In this work the subset $\mathcal{F} \subset 2^{\Omega}$ of the power set will denote a σ-Algebra.

Definition (probability measure) A *probability measure* is a map

$$\mathbb{P} : \mathcal{F} \to \mathbb{R}$$

such that

$$\mathbb{P}(\cup_{n\geq 0} A_n) = \sum_{n\geq 0} \mathbb{P}(A_n)$$

for all mutually disjoint sequences $(A_n)_{n\geq 0} \in \mathcal{F}$ and

$$\mathbb{P}(\Omega) = 1$$

In case of finite probability spaces a measure is given by its values on the *atoms* of the σ-algebra.

Definition (atom) An *atom* is a measurable set which does not contain a smaller non-empty e So given a measure space $(\Omega, \mathcal{F})$ and a finite measure $\mathbb{P}$ on that space, then a set $A \in \mathcal{F}$, $A \neq \emptyset$ is called atom of the σ-Algebra

$\mathcal{F}$, if for any subset $B \subset A$ with $B \in \mathcal{F}$, we have either $B = \emptyset$ or $B = A$. Any set $C \in \mathcal{F}$ can be decomposed uniquely into atoms, i.e.

$$C = \bigcup_{\substack{A \text{ is atom} \\ A \subset C}} A$$

Lemma 1.8. *Let $(\Omega, \mathcal{F})$ be a finite space.*

(i) $A \cap B = \emptyset$ holds for two different atoms A and B of $\mathcal{F}$

(ii) The union of all atoms of $\mathcal{F}$ is finite, disjoint and forms Ω

(iii) To each $\omega \in \Omega$ exists exactly one atom $A \in \mathcal{F}$ with $\omega \in A$

Proof

(i) Assume $A \cap B = C \neq \emptyset$. If $C = A$, then $B \supset A$, since A and B are different. Therefor B cannot be an atom, which contradicts the assumption. If $C \subset A$, then A cannot be an atom and this contradicts the assumption as well.

(ii) Finiteness follows from the fact that $\mathcal{F}$ is finite, disjointness from part (i). Assume that the union B of all atoms of $\mathcal{F}$ does not equal Ω. Then B^C can neither be an atom nor contain atoms. Since the σ-algebra $\mathcal{F}$ is finite, it has to contain either the smallest, nonempty, strict subset of B^C, which then is an atom per definition, or B^C is an atom himself. Both cases lead to a contradiction.

(iii) Due to part (ii) the union of all atoms of $\mathcal{F}$ is disjoint, hence each $\omega \in \Omega$ can only be contained in one atom. Since the same union forms Ω, each $\omega \in \Omega$ has to be in at least one atom.

We denote the set of atoms by $\mathcal{A}(\mathcal{F})$ and the set of all probability measures on $(\Omega, \mathcal{F})$ by $\mathbb{P}(\Omega)$. These measures can be characterised as maps from the atoms of $\mathcal{F}$ to the realvalued nonnegative numbers such that the sum over all atoms equals 1.
We always assume that $\mathcal{F}$ is complete with respect to $\mathbb{P}$, i.e. for every set $B \subset \Omega$, such that $B \subset A$ with $A \in \mathcal{F}$ and $P(A) = 0$, we have $B \in \mathcal{F}$. Such sets are called $\mathbb{P}$-nullsets. The assumption on $\mathbb{P}$ being complete allows us to deal with maps, which are defined up to sets of probability 0.
A random variable $X : (\Omega, \mathcal{F}) \to \mathbb{R}^d$ is a measurable map, i.e. the inverse image of Borel measurable sets is measurable in $\mathcal{F}$. A measurable map takes constant values on each atom of the measurable space and we denote these values by $X(A)$ for A an atom in $\mathcal{F}$, i.e.

$$X \text{ is } \mathcal{F}\text{-measurable} \iff \exists\, \alpha_1, \ldots, \alpha_m \in \mathbb{R} \text{ with } X = \sum_{i=1}^{m} \alpha_i 1_{A_i}$$

We denote by $\sigma(M)$ the smallest σ-algebra that contains the set $M \subset \Omega$.
If the set M is given as inverse image of Borel subsets from $\mathbb{R}^d$ via a map $X : \Omega \to \mathbb{R}^d$, then we write for the generated σ-algebra $\sigma(X)$. This is the smallest σ-algebra such that X is measurable $X : (\Omega, \sigma(X)) \to \mathbb{R}^d$.
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we can define the expectation $\mathbb{E}(X)$ of a random variable via

$$\mathbb{E}(X) := \sum_{A \text{ is atom}} \mathbb{P}(A) X(A)$$

for finitely valued X.

Definition Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a minimal probability space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$, $A_1, \ldots, \mathcal{A}_m$ denote the atoms of $\mathcal{G}$ with $\mathbb{P}(A_j) > 0$ and $X : \Omega \to \mathbb{R}$ a random variable. Then

$$\mathbb{E}(X|\mathcal{G}) : \Omega \to \mathbb{R}, \qquad \mathbb{E}(X|\mathcal{G}) : \omega \mapsto \sum_{j=1}^{m} \frac{\mathbb{E}(X 1_{A_j})}{\mathbb{P}(A_j)} 1_{A_j}(\omega)$$

is the conditional expectation of X under σ-algebra $\mathcal{G}$

Lemma 1.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be complete sub-σ-algebras, then the conditional expectation satisfies following properties:*

- $\mathbb{E}(X|\mathcal{G}) = X$ *for all $\mathcal{G}$-measurable X, i.e.* $X \in L^1(\Omega, \mathcal{G}, \mathbb{P})$
- $\mathbb{E}(X|\mathcal{G}) \geq 0$ *if* $X \geq 0$
- *This property is known as the " tower law ":*
  $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ *for all* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.
- *This property is known as the " Jensen's inequality":*
  $\phi(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(\phi(X)|\mathcal{G})$ *for convex* $\phi : \mathbb{R} \to \mathbb{R}$ *and* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.
- $\mathbb{E}(ZX|\mathcal{G}) = Z\,\mathbb{E}(X|\mathcal{G})$ *for all* $Z \in L^1(\Omega, \mathcal{G}, \mathbb{P})$
- $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ *if X is independent of $\mathcal{G}$*
- *Let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ be given and take σ-algebras $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$. Assume $A \in \mathcal{G}_1 \cap \mathcal{G}_2$ such that $X = Y$ on A and $A \cap \mathcal{G}_1 = A \cap \mathcal{G}_2$. Then $\mathbb{E}(X|\mathcal{G}_1) = \mathbb{E}(Y|\mathcal{G}_2)$ on A.*

1.3.1 Martingale Theory

The term is of great importance when it comes to the notation of No Arbitrage.

Definition (Filtration) Let $(\Omega, \mathcal{F})$ be a finite probability space.
A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a finite sequence $\mathcal{F} = \{\mathcal{F}_n\}_{0,\dots,T}$ of σ-algebras

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_T \subset 2^\Omega$$

where $\mathcal{F} = \mathcal{F}_T$ for $T \geq 1$. Filtrations reflect the idea that the number of informations increases as time goes by.

Definition Denote by $\mathcal{F} = \{\mathcal{F}_n\}_{0,\dots,T}$ a filtration and by $(S_n)_{n=0,\dots,T}$ a sequence of real-valued random variables on $(\Omega, \mathbb{P})$. We introduce the following terms:

stochastic process A stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence of $\mathbb{R}^d$-valued random variables $(S_n)_{n=0,\dots,T}$

adapted The sequence (stochastic process) $(S_n)_{n=0,\dots,T}$ is said to be *adapted to the filtration* $(\mathcal{F}_n)_{n=0,\dots,T}$, if S_n is $\mathcal{F}_n$-measurable for $n = 0, \dots, T$. It suffices to say that the process is *adapted*, if there is no doubt about the filtration.

predictable A stochastic process $(\phi_n)_{n=0,\dots,T}$ is called *predictable*, if ϕ_0 is constant and ϕ_n is is $\mathcal{F}_{n-1}$-measurable for $n = 1, \dots, T$. So any predictable process is also adapted.

stochastic integral Let S be an $\mathbb{R}^d$-valued Martingal and ϕ and $\mathbb{R}^d$-valued predictable (at least adapted) process. Then the 'stochastic integral' is defined as

$$(\phi \cdot S)_n := \sum_{t=1}^{n} \phi_t \left(S_t - S_{t-1}\right) = \sum_{t=1}^{n} \langle \phi_t, \Delta S_t \rangle \,, \; t = 0, \dots T$$

where $\langle ., . \rangle$ denotes the inner product in $\mathbb{R}^d$ and $\Delta S_t := S_t - S_{t-1}$. The random variable $(\phi \cdot S)_n$ will model the gain or loss occured up to time to time n. We have following basic partial integration relation

$$(\phi \cdot S)_n = \phi_T S_T - \phi_0 S_0 - (S_{*-1} \cdot \phi)_n$$

where $(S_{*-1})_n := S_{n-1}$ for $1 \leq n \leq T$ and $(S_{*-1})_0 := S_0$

With these informations the definition of a martingale can be given.

Definition (Martingale) A sequence $(S_n)_{n=0,\dots,T}$ of $\mathbb{R}^d$-valued random variables is called a martingale, if $(S_n)_{n=0,\dots,T}$ is adapted to the filtration $(\mathcal{F}_n)_{n=0,\dots,T}$ with

$$\mathbb{E}(S_n|\mathcal{F}_m) = S_m \quad \text{for } 0 \leq m \leq n \leq T.$$

A sequence $(S_n)_{n=0,\dots,T}$ of random variables $S_n = (S_n^0, \dots S_n^d)$ is a martingale, if each component $(S_n^j)_{n=0,\dots,T}$ for $j = 0, \dots, d$ is a martingale.
To show the martingale property of a sequence of random variables $(S_n)_{n=0,\dots,T}$, it is also common to verify the equivalent condition $\mathbb{E}(S_{n+1} - S_n|\mathcal{F}_n) = 0$ instead of $\mathbb{E}(S_{n+1}|\mathcal{F}_n) = S_n$ for $n = 0, \dots, T-1$.

Definition (stopping times) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n=0,\dots,T}$ a filtration. A random variable $\tau : \Omega \to \mathbb{N}_{\geq 0}$ is said to be a stopping time, if

$$\{\tau \leq n\} \in \mathcal{F}_n$$

for $n = 0, \dots, T$. For an adapted process S and a stopping tim τ with $\tau \leq T$ we can define for $\omega \in \Omega$

$$S_\tau(\omega) := S_{\tau(\omega)}(\omega),$$

and the stopped process S^τ is defined for any stopping time τ and $n = 0, \dots, T$

$$S_n^\tau := S_{\tau \wedge n}.$$

The stopped σ-algebra

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \text{ such that } A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for } n = 0, \dots, T\}$$

contains all informations up to the stopping time τ.

Lemma 1.10. *Let $\tau, \eta, \eta_1, \eta_2, \dots$ be stopping times, then*

- $\sum_{i=1}^k \eta_k$, $\inf \eta_i$, $\sup \eta_i$, $\limsup \eta_i$, $\liminf \eta_i$ *are stopping times, too.*
- *If $\tau \leq \eta$ bounded by T, then $\mathcal{F}_\tau \subset \mathcal{F}_\eta$ and $\{\tau \leq \eta\}$ and $\{\eta \leq \tau\}$ lie in $\mathcal{F}_{\tau\wedge\eta} = \mathcal{F}_\tau \cap \mathcal{F}_\eta$.*
- *If τ, η bounded by T, then $\{\tau \leq \eta\} \cap \mathcal{F}_\tau \subset \mathcal{F}_{\tau\wedge\eta}$.*
- *If τ bounded by T, then $\mathcal{F}_\tau = \mathcal{F}_n$ on $\{\tau = n\}$, i.e. $\{\tau = n\} \cap \mathcal{F}_\tau = \{\tau = n\} \cap \mathcal{F}_n$.*
- *If $A \in \mathcal{F}_\tau$ and τ is bounded by T, then $\tau_A = \tau 1_A + T 1_{A^C}$ is a stopping time.*
- *For an adapted sequence of random variables S and stopping times τ, η which are bounded by T, it follows that S_τ is $\mathcal{F}_\tau$-measurable and $\mathbb{E}(S_\tau|\mathcal{F}_\eta)$ is $\mathcal{F}_{\tau\wedge\eta}$-measurable.*

Lemma 1.11. ***(Doob's optional sampling)*** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space, $(\mathcal{F}_n)_{n=0,\dots,T}$ a filtration and $(S_n)_{n=0,\dots,T}$ an adapted process.*

(i) If S is a martingale, then for every predictable process $(\phi_n)_{n=0,\dots,T}$ the stochastic integral $(\phi \cdot S)$ is a martingale. In particular $\mathbb{E}((\phi \cdot S)_T) = 0$ and $\mathbb{E}(S_\tau) = \mathbb{E}(S_0)$ for all stopping times $\tau \leq T$.

(ii) If the stochastic integral $(\phi \cdot S)$ satisfies for every predictable process ϕ

$$\mathbb{E}((\phi \cdot S)_T) = 0,$$

then it follows that S is a martingale

(iii) If for all stopping times $\tau \leq T$

$$\mathbb{E}(S_\tau) = \mathbb{E}(S_0)$$

holds, it follows that S is a martingale.

(iv) If S is a martingale, then for all stopping times $\eta \leq \tau \leq T$ almost surely

$$\mathbb{E}(S_\tau | \mathcal{F}_\eta) = S_\eta$$

or generally formulated for all stopping times $\eta, \tau \leq T$

$$\mathbb{E}(S_\tau | \mathcal{F}_\eta) = S_{\tau \wedge \eta}$$

Proof The four assertions will be proved step by step.

- If S is a martingale, we can use the martingale property, the predictability of ϕ and Lemma 1.9, such that the following equation holds for $m \leq n$

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} \phi_i \, (S_i - S_{i-1}) | \mathcal{F}_m\right) &= \mathbb{E}\left(\sum_{i=1}^{m} \phi_i \, (S_i - S_{i-1}) | \mathcal{F}_m\right) \\
&\quad + \mathbb{E}\left(\sum_{i=m+1}^{n} \phi_i \, \mathbb{E}\left(S_i - S_{i-1} | \mathcal{F}_{i-1}\right) | \mathcal{F}_m\right) = \\
&= 0 + (\phi \cdot S)_m
\end{aligned}$$

As $(\phi \cdot S)_0 = 0$ it follows that $\mathbb{E}((\phi \cdot S)_T) = 0$.
Define $\phi_0 = 0$ and for $n = 1, \dots, T$ the predictable process

$$\phi_n := 1_{\{\tau > n-1\}} = 1 - 1_{\{\tau \leq n-1\}}$$

in what follows

$$(\phi \cdot S)_T = S_\tau - S_0.$$

- We construct a predictable process ϕ for fixed $j = 1, \dots, T$ and $A \in \mathcal{F}_j$ by

$$\phi_n = 0 \text{ for } n \neq j+1$$
$$\phi_{n+1} = 1_A,$$

i.e. we only invest at time $j+1$ if state A occured. then $\mathbb{E}(1_A(S_{j+1} - S_j)) = 0$ which leads to $\mathbb{E}(S_{j+1}|\mathcal{F}_j) = S_j$.

- For the constant stopping time $n = 1, \dots, T$ we know that

$$\tau = 1_A n + T 1_{A^C}$$

is a stopping time for $A \in F_n$ and $\mathbb{E}(S_n) = \mathbb{E}(S_0)$. But then

$$\begin{aligned} \mathbb{E}(S_T 1_{A^C} + S_n 1_A) &= \mathbb{E}(S_0), \\ \mathbb{E}((S_T - S_n)1_{A^C} + S_n) &= \mathbb{E}(S_0), \\ \mathbb{E}((S_T - S_n)1_{A^C}) &= 0. \end{aligned}$$

Consequently $\mathbb{E}(S_T|\mathcal{F}_\tau) = S_n$ which yields to the martingale property.

- Assume S is a martingale, then for $n \leq T$

$$\mathbb{E}(S_T|\mathcal{F}_\tau) = \mathbb{E}(S_T|\mathcal{F}_n) = S_n = S_\tau$$

holds on $\{\tau = n\}$ by Lemma 1.10. Therefor $\mathbb{E}(S_T|\mathcal{F}_\tau) = S_\tau$ on $\{\tau \leq T\}$. For $\eta \leq \tau \leq T$ we have

$$\begin{aligned} \mathbb{E}(S_\tau|\mathcal{F}_\eta) &= \mathbb{E}(\mathbb{E}(S_T|\mathcal{F}_\tau)|\mathcal{F}_\eta) \\ &= \mathbb{E}(S_T|\mathcal{F}_\eta) = S_\eta \end{aligned}$$

by the tower law. Now the general case for two stopping times η, τ

$$\mathbb{E}(S_\tau|\mathcal{F}_\eta) = \mathbb{E}(S_\tau|\mathcal{F}_{\tau\wedge\eta}) = S_{\tau\wedge\eta} \text{ on } \{\eta \leq \tau\}$$

since the σ-algebras $\mathcal{F}_\eta$ and $\mathcal{F}_{\tau\wedge\eta}$ agree on $\{\eta \leq \tau\}$. Further

$$\mathbb{E}(S_\tau|\mathcal{F}_\eta) = \mathbb{E}(S_{\tau\wedge\eta}|\mathcal{F}_\eta) = S_{\tau\wedge\eta} \text{ on } \{\eta \geq \tau\}$$

since the random variables S_τ and $S_{\tau\wedge\eta}$ agree on $\{\eta \geq \tau\}$.

Definition (equivalent probability measure)
If two probability measures Q and $\mathbb{P}$ share the same null sets, they are said to be equivalent. So the set of events that occur with probability 0 under one measure is the same as the set of events that occur with probability 0 under the other measure, i.e. $\mathbb{P}(A) = 0$ iff $Q(A) = 0$. We will write $Q \sim \mathbb{P}$.

Definition (absolutely continuous probability measure)
If we have for all $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ that $Q(A) = 0$ too, then the measure Q is called absolutely continuous with respect to $\mathbb{P}$.

In the present setting we consider a finite probability space Ω with $\mathbb{P}(\omega) > 0$ for each $\omega \in \Omega$, therefor we have $Q \sim \mathbb{P}$ iff $Q(\omega) > 0$ for each $\omega \in \Omega$. We shall often identify the measure Q with its Radon-Nikodym derivative $\frac{dQ}{d\mathbb{P}}$, which, as Ω is finite, simply means

$$\frac{dQ}{d\mathbb{P}}(\omega) = \frac{Q(\omega)}{\mathbb{P}(\omega)}$$

Lemma 1.12. ***(change of measure)***
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_n)_{n=0,\dots,T}$ and Q be an equivalent probability measure such that

$$\frac{dQ}{d\mathbb{P}}(\omega) = X$$

for $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $Q|_{\mathcal{F}_n}$ are equivalent probability measures on $(\Omega, \mathcal{F}_n, \mathbb{P}|_{\mathcal{F}_n})$ for $n = 0, \dots, T$ and

$$\frac{dQ_n}{d\mathbb{P}_n}(\omega) =: X_n$$

is a $\mathbb{P}$-martingale, where $\mathbb{P}_n$ denotes the restriction of $\mathbb{P}$ to $\mathcal{F}_n$. Additionally the following formulas

$$\mathbb{E}_P(X|\mathcal{F}_n) = X_n$$

and

$$\mathbb{E}_Q(Y|\mathcal{F}_n) = \frac{1}{X_n}\mathbb{E}_P(YX|\mathcal{F}_n)$$

for all $Y \in L^1(\Omega, \mathcal{F}, Q)$ hold.

Proof By Radon-Nikodym we know $\mathbb{E}_{\mathbb{P}}(X) = 1$. It is obvious that Q_n are equivalent probability measures and we have

$$\begin{aligned} \mathbb{E}_{Q_n}(Y) &= \mathbb{E}_{P_n}(YX_n) \\ &= \mathbb{E}_P(YX_n) \end{aligned}$$

for all $Y \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$, but also

$$\begin{aligned} \mathbb{E}_{Q_n}(Y) &= \mathbb{E}_Q(Y) \\ &= \mathbb{E}_P(YX) \end{aligned}$$

where from it follows due to the definition of conditional expectation that $\mathbb{E}_P(X|\mathcal{F}_n) = X_n$, hence X_n is a martingale. By calculating the conditional expectation with respect to Q, we get for $Y \in L^1(\Omega, \mathcal{F}, Q)$ and $Z \in L^1(\Omega, \mathcal{F}_n, Q)$

$$\begin{aligned}
\mathbb{E}_Q(YZ) &= \mathbb{E}_P(YZX) \\
&= \mathbb{E}_P(\mathbb{E}_P(YX|\mathcal{F}_n)Z) \\
&= \mathbb{E}_Q(\frac{1}{X_n}(\mathbb{E}_P(YX|\mathcal{F}_n)Z))
\end{aligned}$$

2 Introduction

2.1 The Model

In this section the mathematical model of a discrete financial market will be introduced.

2.1.1 The market

A financial market is considered to be a place where assets can be traded at fixed prices at fixed (trading) times. We will postulate following basic requirements on the model:

- There are only finitely many trading times $n = 0, \dots, T \in \mathbb{N}$ that allow to trade assets.

- The market consists of finitely many tradable assets $j = 0, \dots, d \in \mathbb{N}$

- Each asset has a price at any time. This price is well known as soon as the trading time has arrived (but not necessarily before). Hence the price of an asset $j \in (0, \dots d)$ at time $n \in (0, \dots, T)$ can be modeled by a nonegative, real-valued random variable S_n^j on an appropriate probability space Ω.

- The market is defined on a minimal finite probability space $(\Omega, \mathbb{P})$. This means that $\mathcal{F} = 2^\Omega$ and $\mathbb{P}(A) > 0$ for all $A \in \mathcal{F}$ and that the sample space Ω consists of finitely many elements, i.e. $\Omega = (\omega_1, \dots, \omega_m)$, where m denotes the number of possible states, at which the market and the price of each asset respectively can possibly be at each point in time $n \in (0, \dots, T)$. The event ω_i occurs with probability $\mathbb{P}(\omega_i)$.

- If it is well known, which state $\omega \in \Omega$ will occur, then the price of each asset are well known too. We have now just postulated that only the present prices of the assets have to be known, but not the future prices. Now consider that for each time $n < T$ we do not know which exact state occured, but which event $A_n \subseteq \Omega$ with $\omega \in A_n$.

We additionally require that the information about the state ω, which occurs at time $n = T$, increases within time. This can only happen, if the number of possible states in $A_n \subseteq \Omega$ decreases if time goes by, i.e. we require $A_n \subseteq A_{n-1}$ for $n = 1, \dots, T$. Obviously we can model the information and its behaviour by a filtration $\mathcal{F} = \{\mathcal{F}_0, \dots, \mathcal{F}_T\}$ with $\mathcal{F}_n := \sigma(\mathcal{A}_n)$, where $\mathcal{A}_n$ denotes the set of all possible events A_n that represent a state of the market at time $n \in \{0, \dots, T\}$. It is reasonable that $\mathcal{A}_n$ is a disjoint decomposition of Ω and therefore a system of atoms of $\mathcal{F}_n$, which means that the states of the market at time $n \in \{0, \dots, T\}$ are represented by the atoms of the σ-Algebra $\mathcal{F}_n$. Due to Lemma 1.8 (ii) such a decompostition of atoms does always exist. The prices S_n^j of each asset $j = 0, \dots d$ are now said to be $\mathcal{F}_n$-measurable. Therewith our requirement that for each asset $j = 0, \dots, d$ the price is well defined at every time, if the point in time has already arrived - but not necessarily in an earlier point of time, since S_n^j is constant on the well known event $A_n \in \mathcal{F}_n$ at time $n \in \{0, \dots, T\}$ - is satisfied. Moreover define $\mathcal{F}_0$ as the trivial σ-algebra, i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$, which means that the market is deterministic at time $n = 0$

The term "information" has to be understood as the knowledge of how the market has been evolving so far. The more trading times have passed by, the more accurate the market can be divided into possible events and the set of possible outcomes at the end of time, i.e. at time $n = T$ can be constrained. Consequently one says that no information is available at time $n = 0$ and "full" information is available at time $n = T$.
Further we require a "reasonable" market. At time n it should not be possible to predict which state will take place in the next moment $n + 1$. This means that the atoms of σ-algebra $\mathcal{F}_{n+1}$ do not change compared to the atoms of $\mathcal{F}_n$. Consider that the stock price of this state changes between time n and $n+1$. Then it would be possible to make profit without risk by buying and selling securities. This possibility will be eliminated by the *No Arbitrage Theory*, which will be introduced in the next section.
We will now formulate a few more postulations on the market:
The asset $j = 0$ is assumed to be a so called *riskless asset* or bond in every market, i.e. the price process $(S_n^0)_{n=0,\dots,T}$ is assumed to be strictly positive and $\mathcal{F}_0$-measurable for all $n \in \{0, \dots, T\}$. We stipulate that the price of the bond starts at $S_0^0 := 1$ and increases at interest rate i. Hence $S_n^0 = (1+i)S_{n-1}^0 = (1+i)^n$. The coefficients $v_n := \frac{1}{S_n^0}$ for $n \in \{0, \dots, T\}$ are called *discount factors*. Discounting allows to compare prices at time 0 to prices at time $n > 0$, i.e. $\tilde{S}_n^j := v^n S_n^j$ is the price of asset j at time n discounted on time 0. The assets $S_n^1, \dots, S_n^d$ are called *risky assets*, because we do not know their future values.
In order that trading is possible from the beginning to the end, we further assume that $S_n^j \neq 0$, $j = 0, \dots, d$, $n = 0, \dots T - 1$.

We will make an additional request on the bond by defining $S_n^0 \equiv 1$ constant - doing so does not interfere the model, since we may always express the values of the other assets in units of the bond. As a final remark it can be said that the probability measure $\mathbb{P}$ can arise from different sources, like historical observations, or it is only a result of market analysis. However, it will be shown that the knowledgement of the underlying probability measure will not be of great importance for pricing of claims.

2.1.2 The agent

The agent interferes the market development by buying and selling assets at the present price using a trading strategy. We assume that the agents activity will not change the market.
A *trading strategy* is a real-valued random variable ϕ_n^j, defined on Ω, and indicates how many units of asset j the agent owns at time n. A positive valued ϕ_n^j means that the agent has bought an asset and respectively a negative value ϕ_n^j announces that the agent sold the asset. We have to assume that both - buying and selling - is allowed. Note that the value ϕ_n^j does not have to be an integer, i.e. each asset is divisible.
Unlike the prices of assets, which are adapted to the filtration $\mathcal{F}$ that underlies the market, the trading strategies are predictable. This makes sense, since the value ϕ_n^j already indicates how much the agent holds at time n, and therefore the agent has made decisions at time $n-1$ at the latest. Consequently the trading strategy ϕ_n^j has to be $\mathcal{F}_{n-1}$ measurable. The $\mathcal{F}_0$-measurable strategy ϕ_0 indicates how much the agent contributes to the market.
The set of all trading strategies is described by Φ and equals the set of predictable families $\phi = (\phi_0, \ldots, \phi_T)$ with $\phi_n = (\phi_n^0, \ldots, \phi_n^d)$, $n = 0, \ldots, T$, where $\phi_n^j : \Omega \to \mathbb{R}$ for $n = 0, \ldots, T$, $j = 0, \ldots, d$ are $\mathcal{F}_{n-1}$-measurable random variables.
An agents entire holding at every time $n \in \{1, \ldots, T\}$ is called *portfolio.* So a portfolio at time n is formed by an amount of ϕ_n^0 in the bank account and ϕ_n^j units of risky assets. So the value V_n of such a portfolio at time n is

$$V_n(\phi) = \phi_n S_n := \sum_{i=0}^{d} \phi_n^i S_n^i$$

for $n = 0, \ldots, T$. Remark that ϕ_0 is always constant.
The *discounted value process* is given through

$$\tilde{V}_n(\phi) = v_n(\phi_n S_n) = \phi_n \tilde{S}_n$$

for $n = 0, \ldots, T$, where $\tilde{S}_n = v_n S_n$ denotes the *discounted price process.* Recalling the definition of the stochastic integral we can now give it following interpretation: If S denotes the development of the stock and ϕ_n the number of assets hold at time n, then the capital changes by $\phi_n(X_n - X_{n-1})\phi_n \Delta S_m$ through changes in the stock between time $n-1$ and n. Hence the stochastic integral $\phi \cdot S_n$ reflects the cumulative gains and losses that derive from changes in the stock and not from selling or buying of securities. So if the agent owns a portfolio that consists of several assets, S and ϕ become vectors. Then ϕ_n^i denotes the number of assets of type i hold at time n and S_n^i its value. Consequently the gains between time $n-1$ and n result in the scalar product $\phi_n^T(S_n - S_{n-1})$.

If we denote by $\phi_n(S_n - S_{n-1})$ the stock price gain at time n, it means that the portfolio ϕ_n was bought before the stock changed from S_{n-1} to S_n, hence at the the end of time $n-1$ after the value S_{n-1} settled. Therefore only the information until time $n-1$ can be available for the choice of ϕ_n, especially the value S_n has to remain unknown.

We will always assume that the agents wants to maximize profit. Let $G_n(\phi)$ denote the agents cumulative gains or losses for $n = 0, \ldots, T$, $\tilde{G}_n(\phi)$ the discounted. Hence for $n = 0, \ldots, T$,

$$G_n(\phi) := \sum_{j=1}^{n} \langle \phi_j, \Delta S_j \rangle = (\phi \cdot S)_n, \quad \tilde{G}_n(\phi) := \sum_{j=1}^{n} \left\langle \phi_j, \Delta \tilde{S}_j \right\rangle = (\phi \cdot \widetilde{S})_n,$$

where $\Delta S_j := S_j - S_{j-1}$ and $\Delta \tilde{S}_j := \tilde{S}_j - \tilde{S}_{j-1}$ for $j = 0, \ldots, d$.

In the following, it will make sense to constrain the set of trading strategies. It shall not be possible to bring in or take out wealth of the market. This means that the portfolios value changes only because the stock prices change during time. The trading strategy is then said to be *self - financing*.

Definition (self - financing) A trading strategy $\phi \in \Phi$ is said to be *self - financing*, if

$$\phi_n S_n = \phi_{n+1} S_n$$

for $n = 0, \ldots, T-1$. Obviously this condition is equivalent to

$$\phi_{n+1}(S_{n+1} - S_n) = \phi_{n+1} S_{n+1} - \phi_n S_n$$

for $n = 0, \ldots, T-1$, and therefore

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1}(S_{n+1} - S_n)$$

for $n = 0, \ldots, T-1$, which makes clear that changes of the value processes are due to changes in the stock prices, i.e. by changing the portfolio from ϕ_{n-1} to ϕ_n, no money is put in or out.

Lemma 2.1. *Let $S = (S^0, \ldots, S^d)$ be a discrete model of a financial market and ϕ a trading strategy. Then the following assertions are equivalent:*

(i) The strategy ϕ is self-financing

(ii) $V_n(\phi) - V_{n-1}(\phi) = (\phi \cdot S)_n,\ n = 1, \ldots, T$

(iii) $\widetilde{V}_n(\phi) - \widetilde{V}_{n-1}(\phi) = (\phi \cdot \widetilde{S})_n,\ n = 0, \ldots, T-1$

(iv) $V_n(\phi) = V_0(\phi) + (\phi \cdot S)_n,\ n = 0, \ldots, T$

(v) $\widetilde{V}_n(\phi) = V_0(\phi) + (\phi \cdot \widetilde{S})_n,\ n = 0, \ldots, T$

Proof First we will show $(i). \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (i)$. By replacing $V_n(\phi)$ and S_n by $\widetilde{V}_n(\phi)$ and $\widetilde{S}_n$ $(i) \Rightarrow (iii) \Rightarrow 5. \Rightarrow (i)$ can be shown in the same way.

$(i) \Rightarrow (ii)$ Using the definition of self-financing, i.e. $\phi_n S_n = \phi_{n+1} S_n$

$$\begin{aligned} V_n(\phi) - V_{n-1}(\phi) &= \langle \phi_n, S_n \rangle - \langle \phi_{n-1}, S_{n-1} \rangle = \langle \phi_n, S_n \rangle - \langle \phi_n, S_{n-1} \rangle \\ &= (\phi \cdot S)_n \end{aligned}$$

$(ii) \Rightarrow (iv)$ For $n = 0, \ldots, T$:

$$\begin{aligned} V_n(\phi) &= V_0(\phi) + \textstyle\sum_{k=1}^n (V_k(\phi) - V_{k-1}(\phi)) = V_0(\phi) + \sum_{k=1}^n \langle \phi_k, \Delta S_k \rangle \\ &= V_0(\phi) + G_n(\phi) = V_0(\phi) + (\phi \cdot S)_n \end{aligned}$$

$(iv) \Rightarrow (i)$

$$\begin{aligned} V_{n+1}(\phi) - V_n(\phi) &= V_0(\phi) + \sum_{k=1}^{n+1} \langle \phi_k, \Delta S_k \rangle - V_0(\phi) - \sum_{k=1}^{n} \langle \phi_k, \Delta S_k \rangle \\ &= \langle \phi_{n+1}, \Delta S_{n+1} \rangle \\ &= \langle \phi_{n+1}, S_{n+1} \rangle - \langle \phi_{n+1}, S_n \rangle \end{aligned}$$

$$\begin{aligned} \Longleftrightarrow \langle \phi_{n+1}, S_n \rangle &= V_n(\phi) + \langle \phi_{n+1}, S_{n+1} \rangle - V_{n+1}(\phi) = V_n(\phi) \\ &= \langle \phi_n, S_n \rangle \end{aligned}$$

Lemma 2.2. *For any $\mathbb{R}^d$-valued predictable process $(\phi^1, \ldots, \phi^d)$ and for any value V_0 there exists a unique predictable process ϕ^0 such that the $\mathbb{R}^{d+1}$-valued predictable process $(\phi^0, \ldots, \phi^d)$ is a self-financing trading strategy with $V_0(\phi) = V_0$ and $\widetilde{V}_n(\phi) = V_0 + (\phi \cdot \widetilde{S})_n$ for $n = 0, \ldots, T$*

Proof For a self-financing trading strategy the formula

$$\begin{aligned} \widetilde{V}_n(\phi) &= \phi_n^0 + \phi_n^1 \widetilde{S}_n^1 + \cdots + \phi_n^d \widetilde{S}_n^d \\ &= V_0 + (\phi \cdot \widetilde{S})_n \end{aligned}$$

holds, which now allows to calculate the trading strategy ϕ^0. The predictability of the process is easily seen:

$$\phi_n^0 = V_0 + (\phi \cdot \widetilde{S})_{n-1} - \phi_n^1 \widetilde{S}_{n-1}^1 - \cdots - \phi_n^d \widetilde{S}_{n-1}^d$$

So this Lemma states that instead of taking any trading strategy at time 0, one can trade with a self-financing trading strategy without making severe restrictions, since the self-financing property can easily be achieved by changing the bond of the trading strategy.
To sum up: The $\mathcal{F}_n$-measurable random variable $\sum_{j=0}^{d} \phi_n^j \widetilde{S}_n^j = \sum_{j=0}^{d} \phi_{n+1}^j \widetilde{S}_n^j$ is interpreted as the value $\widetilde{V}_n$ of the portfolio at time n defined by the trading strategy ϕ. It is much easier to describe the movements of the value $\phi_n S_n$ if we use discounted prices and take the asset $\widetilde{S}^0$ as numéraire. We can then conclude that $\widetilde{S}_n^0$ units of money at time n is the same as 1 unit of money at time 0. We will replace prices $\widetilde{S}$ by discounted prices $\frac{\widetilde{S}}{\widetilde{S}^0} = (\frac{\widetilde{S}_0^j}{\widetilde{S}_n^0}, \frac{\widetilde{S}_1^j}{\widetilde{S}_n^0}, \ldots, \frac{\widetilde{S}_d^j}{\widetilde{S}_n^0})$ and use

$$S_n^j := \frac{\widetilde{S}_n^j}{\widetilde{S}_n^0}, \quad \text{for } j = 1, \ldots, d \text{ and } n = 0, \ldots, T$$

realising that it is not necessary to include the 0-th coordinate as $S_n^0 = 1$. For a self-financing trading strategy $(\phi_n)_{n=0,\ldots,T} = (\phi_n^0, \phi_n^1, \ldots, \phi_n^d)$ we then have for the initial investment $\widetilde{V}_0$

$$\widetilde{V}_0 = \sum_{j=0}^{d} \phi_1^j \; \widetilde{S}_0^j = \phi_1^0 + \sum_{j=1}^{d} \phi_1^j \; \widetilde{S}_0^j = \phi_1^0 + \sum_{j=1}^{d} \phi_1^j \; S_0^j$$

since we assumed that $\widetilde{S}_0^0 = 0$.
Let $(\overline{\phi}_n)_{n=1,\ldots,T} = (\overline{\phi}_n^0, \overline{\phi}_n^1, \ldots, \overline{\phi}_n^d)_{n=1,\ldots,T}$ be an $\mathbb{R}^{d+1}$-dimensional predictable self-financing process. By dropping the first coordinate $\overline{\phi}_n^0$ one obtains the $\mathbb{R}^d$-valued process $(\phi_n)_{n=1,\ldots,T} = (\phi_n^1, \ldots, \phi_n^d)_{n=1,\ldots,T}$. The observation is that for every $\mathbb{R}^d$-valued predictable process (ϕ_n) exists *exactly one self-financing* $\mathbb{R}^{d+1}$ valued predictable process $(\overline{\phi}_n)$ such that $\phi_n^1, \ldots, \phi_n^d = \overline{\phi}_n^1, \ldots, \overline{\phi}_n^d$ and $\overline{\phi}_1^0 = 0$. The values $\overline{\phi}_{n+1}^0$ for $n = 1, \ldots, T-1$ can inductively be determined by $\sum_{j=0}^{d} \phi_n^j \widetilde{S}_n^j = \sum_{j=0}^{d} \phi_{n+1}^j \widetilde{S}_n^j$. As we required $S_n^0 > 0$ for all n it follows that there is exactly one function ϕ_{n+1}^0 such that $\sum_{j=0}^{d} \overline{\phi}_n^j \widetilde{S}_n^j = \sum_{j=0}^{d} \overline{\phi}_{n+1}^j \widetilde{S}_n^j$ holds. Of course the ϕ_{n+1}^0 is $\mathcal{F}_n$-measurable. Economically speaking the above argument is obvious: for any given trading strategy $(\phi_n)_{n=1,\ldots,T} = (\phi_n^1, \ldots, \phi_n^d)_{n=1,\ldots,T}$ in the d risky assets one may always add a trading strategy $(\overline{\phi}_n^0)_{n=1,\ldots,T}$ in the numéraire asset 0 such that the whole strategy becomes self-financing. This trading strategy becomes unique if we normalize $\overline{\phi}_1^0 = 0$. This also allows us to interpret the asset 0 as a cash account where at all times $n = 0, \ldots, T-1$ the gains and losses that arise from the investments in the d risky assets are collected and from which the investments in the risky assets are financed. The requirement $\overline{\phi}_1^0 = 0$ illustrates an empty cash account at the beginning. The following development of the holdings in the cash account is determined uniquely since it

derives from the holdings in the risky assets $1, \dots, d$.
Additionally we observe that the investment $(\overline{\phi}_n^0)_{n=1,\dots,T}$ in the num/'eraire asset does not change the discounted value of the portfolio. Hence the discounted value $V_n(\phi)$ of the portfolio

$$V_n(\phi) = \frac{\widetilde{V}_n(\phi)}{\widetilde{S}_n^0}, \quad n = 0, \dots, T$$

depends only on the $\mathbb{R}^d$-valued process $(\phi_n)_{n=1,\dots,T} = (\phi_n^1, \dots, \phi_n^d)_{n=1,\dots,T}$. By the normalisation requirements $\widetilde{S}_0^0 = 1$ and $\overline{\phi}_1^0 = 0$ it follows

$$\widetilde{V}_0(\phi) = V_0(\phi) = \sum_{j=1}^{d} \phi_1^j \, S_0^j.$$

and

$$V_T(\phi) = V_0(\phi) + (\phi \cdot S)_T$$

2.2 Asset Pricing

Pricing an asset is the main goal of financial market theory.

Definition (derivative or contingent claim) A good whose future value derives from other goods (so called *underlyings*) is called *derivative* or *contingent claim*. This good usually exists in form of a contract, where obligation and rights that derive from the future value of the underlying, are fixed.

Typical examples for derivatives are options.

Definition (European Call Option & European Put Option)
In finance options are types of derivative contracts, including *call options* and *put options*, where the future payoffs to the buyer and seller of the contract are determined by the price of another security, such as a common stock. More specifically, a call option is an agreement in which the buyer (holder) has the right (but not the obligation) to exercise by buying an asset at a *strike price* K on a future date (at *maturity* T); and the seller (writer) has the obligation to honor the terms of the contract.
A put option is an agreement in which the buyer has the right (but not the obligation) to exercise by selling an asset at the strike price on date; and the seller has the obligation to honor the terms of the contract.

From a mathematical point of view, the options exercising depends only on the fact of how the price of the underlying good develops. In case of a Call Option, it makes sense to exercise only if the price of the underlying good is greater than the strike price, such that the holder of the option makes

profit from the price difference. If the price of the underlying good turns out to be lower than the strike price, the option would not be exercised by the rational holder, since it would be cheaper to buy this good in a direct way. The same applies to the Put Option.
So derivatives itselves form goods in the usual sense and therefore have a price vector. In the following the price vector of a Call Option is denoted by $C = (C_0, \ldots, C_T)$ and in case of a Put Option by $P = (P_0, \ldots, P_T)$.

2.2.1 Example of a call option

The buyer of a call expects that the price may go up. He pays a premium (which he will never get back) and has then the right to exercise the option at the strike price K.
Now consider an investor, who buys a call on a stock with a strike price of $K = 50$ and pays a premium of 5. The current price is 40.

- Assume the price rises and is 60 on the strike date. The investor would then exercise the option and could then share, or sell it in the open market for 60. So the profit would be 10 minus the fee paid for the option, 5, for a net profit of 5. The investor has thus doubled his money, having paid 5 and ending up with 10.
- If however the share price never rises to 50 (that is, it stays below the strike price) up through the exercise date, then the option would expire as worthless. The investor loses the premium of 5.

Thus, in any case, the loss is limited to the fee (premium) initially paid to purchase the stock, while the potential gain is theoretically unlimited (consider if the share price rose to 100).
From the viewpoint of the seller, if the seller thinks the stock is a good one, (s)he is better (in this case) by selling the call option, should the stock in fact rise. However, the strike price (in this case, 50) limits the seller's profit. In this case, the seller does realize the profit up to the strike price (that is, the 10 rise in price, from 40 to 50, belongs entirely to the seller of the call option), but the increase in the stock price thereafter goes entirely to the buyer of the call option.
From the above, it is clear that a call option has positive monetary value when the underlying instrument has a price S above the strike price K. Since the option will not be exercised unless it is "in-the-money", the payoff for a Call Option is

$$C_T = max(S_T - K, 0) = (S_T - K)^+$$

Obviously the payoff of the Put Option is given by

$$P_T = max(K - S_T, 0) = (S_T - K)^-$$

Since derivatives derive from the future value of the underlying, this value is firstly the only one known at this future time. Generally let the future date, on which the value of a derivative is certain, be at time $n = T$. Further considerations are based upon the question if and how the knowledge of the price at time $n = T$ can help to find an appropriate price at time $n = 0, \ldots, T-1$. This question leads to the No Arbitrage Theory.

2.3 No Arbitrage Theory for discrete models & the Fundamental Theorem of Asset Pricing

An arbitrage opportunity is the possibility to make a profit in a financial market without risk and without net investment of capital. Consequently a market with arbitrage opportunity could not exist in the long run. So the principle of no arbitrage (NA) states that a mathematical model of a financial market should not allow arbitrage possibilities.

Definition A self-financing trading strategy ϕ with $V_n(\phi) \geq 0$ is called *admissible* or in other words:
a trading strategy ϕ is admissible if the gains process $(\phi \cdot S)$ is uniformly bounded from below by a non random constant, i.e. if there exists a constant $C \geq 0$ such that for all $n = 0, \ldots, T$

$$G_n(\phi) := (\phi \cdot S)_n \geq -C \qquad \mathbb{P} - a.s.$$

If only admissible strategies ϕ are allowed, it is ensured that investors have a fixed credit limit.
An admissible trading strategy ϕ with $V_0(\phi) = 0$, $V_T(\phi) \gneq 0$ is called *arbitrage - strategy.*
The market is said to be *arbitrage - free*, if no arbitrage - strategy in Φ exists.

A claim at time T is an $\mathcal{F}_T$-measurable random variable $X \geq 0$. (The previously introduced C_T and P_T describe claims). The price $q_0(X)$ of a contingent claim X should be chosen in a way such that the risks are optimally covered. The idea of replication (or hedging) helps to find such a price.

Definition (Replication)
A replicating strategy is a self-financing strategy satisfying $V_T(\phi) = X$. The claim X is then said to be *replicable* or *attainable.*

This idea gives the clue that the price of the claim should develop in the same way as the value of the replicating strategy. Otherwise one can make profit by selling the more expensive and buying the cheaper portfolio, where both positions even up at time T- and this would violate the (NA)-condition. Consequently we require an arbitrage-free market.

Definition (complete market) If every claim can be replicated, the market is said to be *complete.*

Definition (incomplete market) If at least one claim can not be replicated, the market is said to be *incomplete.*

Now a claim represents the definable part of a price vector of a derivative, since the derivative derives from its underlying at this time. The postulation of (NA) has now enourmous effects on pricing a claim, because it restricts the selection of possible prices for the claim. Theorem 2.4 states how (NA) helps to price claims such that the market remains arbitrage-free. But at first some facts about portfolios and gains are introduced.
Consider that $G_n(\phi)$ and $V_n(\phi)$ (resp. $\tilde{G}_n(\phi)$ and $\tilde{V}_n(\phi)$) are functions of $\mathcal{F}_n$-measurable random variables and therefore they are $\mathcal{F}_n$-measurable as well, for all $n = 1, \ldots, T$ and $\phi \in \Phi$.

Lemma 2.3. *Q is a measure under which the discounted stock price process $(\tilde{S}_0, \ldots, \tilde{S}_T)$ form a martingale iff the expected gain equals 0 in expectation:*

$$\mathbb{E}_Q[\tilde{G}_T(\phi)] = 0, \quad \forall \phi \in \Phi$$

This means that under the measure Q no gains or losses are expected.

Proof Since the trading strategy $\phi \in \Phi$ is predictable, we can use 1.11 to obtain that the discounted prices $(\tilde{S}_0, \ldots, \tilde{S}_T)$ form a martingale under measure Q iff

$$\mathbb{E}_Q[\tilde{G}_T(\phi)] = \mathbb{E}_Q\left[\sum_{n=1}^{T} \langle \phi_n, S_n \rangle\right] = 0, \quad \forall \phi \in \Phi$$

Definition We call the subspace $\mathcal{K}$ of $L^0(\Omega, \mathcal{F}, \mathbb{P})$, defined by

$$\begin{aligned} \mathcal{K} &:= \{\tilde{V}_T(\phi) | \phi \text{ self-financing strategy, } \tilde{V}_0(\phi) = 0\} \\ &= \{(\phi \cdot \tilde{S})_T | \phi \text{ predictable}\} \end{aligned}$$

the set of contingent claims attainable at price 0.

The economic interpretation is the following: the random variables $X = (\phi \cdot S)_T$, for some $\phi \in \Phi$ are precisely those contingent claims, i.e., the pay-off functions at time T depending on $\omega \in \Omega$ in an $\mathcal{F}_T$ -measurable way that can be replicated with zero initial investment, by pursuing some predictable trading strategy ϕ.
The affine space $\mathcal{K}(x) = x + \mathcal{K}$ is then called the set of contingent claims attainable at price x and is obtained by shifting $\mathcal{K}$ by the constant function $x \in \mathbb{R}$, so it is the space that contains all random variables of the form $x + (\phi \cdot S)_T$. Its economic interpretation therefore translates into that these are those contingent claims an agent may replicate by investing an initial capital x by pursuing some predictable trading strategy.

Definition

$$\begin{aligned} C &:= \{Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \mid \text{there is } X \in \mathcal{K} \text{ s.t. } X \geq Y\} \\ &= \mathcal{K} - L^2_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P}) \end{aligned}$$

denotes the convex cone of contingent claims super-replicable at price 0.
A contingent claim X is replicable at price x at time T if a self-financing strategy ϕ exists s.t.

$$X = x + (\phi \cdot S)_T \ \in x + \mathcal{K}$$

and superreplicable at price x at time T if a self-financing strategy ϕ exists s.t.

$$X \leq x + (\phi \cdot S)_T \ \in x + \mathcal{K} \quad \text{i.e. } X \in C$$

So the contingent claim Y is superreplicable at price 0 if we can attain it with zero investment by pursuing some predictable strategy. As a consequence we arrive at some contingent claim X and would "throw away" money if it is necessary to arrive at claim Y.

Definition (No Arbitrage Condition) Since $\mathcal{K}$ is a subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and the positive cone $L^2_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P})$ is polyhedral, it follows that C is closed.
A financial market satisfies the No-Arbitrage Condition (NA), if

$$\mathcal{K} \cap L^2_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$$

where 0 denotes the function equal to zero. The interpretation reads as follows: it should not be possible to find a contingent claim sold at price zero, i.e. an element $X \in \mathcal{K}$ such that $X \geq 0$ a.s. and $\mathbb{P}(X > 0) > 0$, since this results in an arbitrage opportunity. So in case of arbitrage one can find a trading strategy ϕ such that he starts with initial investment zero and ends up with a contingent claim which is nonnegative and not identically equal to zero, i.e. $X = (\phi \cdot S) > 0$.
Note that the (NA) - condition $\mathcal{K} \cap L^2_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$ is equivalent to

$$C \cap L^2_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}.$$

as agents are allowed to "throw away" money.

Definition (equivalent and absolutely continuous martingale measure)
If $\mathbb{P}$ represents the 'real world' probability measure, then Q is called the equivalent martingale measure for S with respect to the numéraire S^0, if $\mathbb{P}$ and Q are equivalent and the discounted price process is a martingale under Q.

We denote the set of equivalent martingale measures with respect to numéraire S^0 by $\mathcal{M}^e(S, S^0) = \mathcal{M}^e(\widetilde{S})$,
the set of absolutely continuous martingale measures with respect to the numéraire S^0 by $\mathcal{M}^a(S, S^0) = \mathcal{M}^a(\widetilde{S})$. As we assume $S^0 \equiv 1$ we shall write $\mathcal{M}^e(S)$ and $\mathcal{M}^a(S)$ respectively.
Now we can formulate the "fundamental Theorem of Asset Pricing":

2.3.1 The Fundamental Theorem of Asset Pricing

The fundamental theorem of asset pricing states that the (NA)-condition is equivalent to the existence of a linear functional $q > 0$, such that $q|_{\mathcal{K}} \leq 0$ which - because $\mathcal{K}$ is a vector space - is equivalent to $q|_{\mathcal{K}} = 0$. This linear functional q can be interpreted as the density $\frac{dQ}{d\mathbb{P}}$ of a probability measure Q equivalent to P under which the process S is a martingale. The basic message of this theorem is that a model of a financial market is free of arbitrage if and only if there is a probability measure Q, equivalent to the original $\mathbb{P}$ (i.e., $\mathbb{P}(A) = 0$ iff $Q(A) = 0$), such that the stock price process is a martingale under Q.

Lemma 2.4. ***(Fundamental Theorem of Asset Pricing)*** *Let S be a discrete model for a financial market, then the following two assertions are equivalent:*

(i) The model is arbitrage-free

(ii) The set of equivalent martingale measures is non-empty, $\mathcal{M}^e(\widetilde{S}) \neq \emptyset$.

Proof

$\Leftarrow$ So let Q be such an equivalent martingale measure to $\mathbb{P}$, $Q \sim P$, and ϕ a self-financing trading strategy satisfying $V_0(\phi) = 0$. Then, due to Lemma 2.3, it follows that

$$\mathbb{E}_Q(\tilde{V}_T(\phi)) = \mathbb{E}_Q(V_0 + \tilde{G}_T(\phi)) = \mathbb{E}_Q(\tilde{G}_T(\phi)) = 0$$

So $V_T(\phi) \geq 0$, since Q does not contain any nullsets. Consequently no arbitrage strategy exists, which means that the market is arbitrage-free.

$\Rightarrow$ Now we assume that the market is arbitrage-free. Then

$$\mathcal{K} \cap L_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}$$

It is the goal to seperate the disjoint convex sets $\mathcal{K}$ and

$$\{Y \in L^2_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P}) \mid \mathbb{E}_{\mathbb{P}}(Y) = 1\}$$

by a hyperplane induced by a linear functional q. Then $q(X) = 0$ for all $X \in \mathcal{K}$ and $q(Y) > 0$ for all $Y \in L^2_{\geq 0}(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}_{\mathbb{P}}(Y) = 1$. For all measurable sets $A \in \mathcal{F}$ define

$$Q(A) = \frac{q(1_A)}{q(1_\Omega)}$$

and obtain an equivalent probability measure $Q \sim \mathbb{P}$, since $q(1_A) > 0$ for sets with $\mathbb{P}(A) > 0$. Since each $X \in \mathcal{K}$ is described by a stochastic integral with ϕ self-financing $X = (\phi \cdot S)_T$, we have $\mathbb{E}_Q[(\phi \cdot S)_T] = 0$ which implies that S is a martingale due to Doobs Optional Sampling 1.11.

As already mentioned in the definition of the term "replication", we have to require an arbitrage-free market and as we now know, this is equivalent to the requirement of the existence of a martingale measure.

Definition (pricing rule) A pricing rule for contingent claims $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ at time T is a map

$$X \mapsto \pi(X)$$

where $(\pi(X)_n)_{n=0,\ldots,T}$ is an adapted stochastic process which determines the price of the claim at time T at previous time $n \leq T$. Using the idea of replication we want this stochastic process to end up such that $\pi(X)_T = X$. We will now add contingent claims $X_1, \ldots, X_k$ sold at price $\pi(X_1), \ldots, \pi(X_k)$ to the financial market S. If the introduction of these claims does not create an arbitrage opportunity, the prices $\pi(X_1), \ldots, \pi(X_k)$ are called arbitrage-free prices. Mathematically speaking, a pricing rule is arbitrage-free if for any finite set of claims $X_1, \ldots, X_k$ the expanded model of a financial market

$$(S^0_n, \ldots, S^d_n, \pi(X_1), \ldots, \pi(X_k))$$

satisfies (NA).

Lemma 2.5. ***(arbitrage-free prices)*** *From a mathematic point of view we want to find a self financing trading strategy ϕ that satisfies $q_0(X) = V_0(\phi)$ and $V_T(\phi) = X$.*
In order to avoid vagueness we will now not assume a constant bond.
Let π be an arbitrage-free pricing rule for a set of contingent claims $\mathcal{X}$, then the discrete model $(S^0, \ldots, S^d)$ is arbitrage-free and there is $Q \in \mathcal{M}^e(\widetilde{S})$ such that for all $X \in \mathcal{X}$

$$\pi(X)_n = \mathbb{E}_Q\left[\frac{S^0_n}{S^0_T} X | \mathcal{F}_n\right].$$

Vice versa, if the model S is arbitrage-free, then

$$\pi(X)_n = \mathbb{E}_Q\left[\frac{S_n^0}{S_T^0} X | \mathcal{F}_n\right]$$

defines an arbitrage-free pricing rule for all contingent claims $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. This leads to the conclusion that the only arbitrage-free prices are conditional expectation of the discounted claims with respect to Q and pricing rules are always linear.

Proof If the market $(S^0, \dots, S^d, \pi(X))$ is arbitrage-free we know that an equivalent martingale measure Q exists such that the discounted prices are martingales. Particularly

$$\frac{\pi(X)_n}{S_n^0}$$

is a Q-martingale, therefor

$$\mathbb{E}\left[\frac{\pi(X)_T}{S_T^0} | \mathcal{F}_n\right] = \mathbb{E}\left[\frac{X}{S_T^0} | \mathcal{F}_n\right] = \frac{\pi(X)_n}{S_n^0}$$

Given an arbitrage-free discrete model S and define the pricing rules by the above relation for one equivalent martingale measure $Q \in \mathcal{M}^e(\widetilde{S})$, then the whole market is arbitrage-free, since at least one equivalent martingale measure exists which is Q.

Remark 2.6. The reason why we take an equivalent martingale measure and not an absolutely continuous martingale measure $Q \in \mathcal{M}^a(\widetilde{S})$ is that we could then find at least one measurable set A such that $Q(A) = 0$ and $P(A) > 0$. But the claim 1_A with $\mathbb{P}(A) > 0$ would have price 0, which implies arbitrage if we enter this contract $X = 1_A$. Hence only equivalent martingale measures are possible for pricing.
The set $\mathcal{M}^e(\widetilde{S})$ of equivalent martingale measures is convex whereas the set of absolutely continuous martingale measures $\mathcal{M}^a(\widetilde{S})$ is convex and compact. Therefore the analysis of extreme points of $\mathcal{M}^a(\widetilde{S})$ becomes important.

Lemma 2.7. *Let S be a discrete model for a financial market, $\mathcal{M}^e(\widetilde{S}) \neq \emptyset$ and $\mathcal{M}^a(\widetilde{S}) = \langle Q_1, \dots, Q_m \rangle$.*
Then the following assertions are equivalent for all $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$:

(i) $X \in K$ $(X \in C)$

(ii) For all $Q \in \mathcal{M}^e(\widetilde{S})$ we have $\mathbb{E}_Q(X) = 0$ ($\mathbb{E}_Q(X) \leq 0$)

(iii) For all $Q \in \mathcal{M}^a(\widetilde{S})$ we have $\mathbb{E}_Q(X) = 0$ ($\mathbb{E}_Q(X) \leq 0$)

(iv) For all $i = 1, \dots, m$ we have $\mathbb{E}_{Q_i}(X) = 0$ ($\mathbb{E}_{Q_i}(X) \leq 0$)

Proof The polar of the cone C is given by

$$C^0 = \{Z \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \text{ s.t. } \mathbb{E}_P(ZX) \leq 0\}$$

If we calculate the Radon-Nikodym-derivative $\frac{dQ}{d\mathbb{P}}$ for $Q \in \mathcal{M}^a(\widetilde{S})$ we see

$$\mathbb{E}_P(\frac{dQ}{d\mathbb{P}}X) = \mathbb{E}_Q(X) = \mathbb{E}_Q((\phi \cdot \widetilde{S})_T + Y)$$

for $Y \leq 0$, and as Q is a martingale measure the expectation of the stochastic integral disappears, it follows that

$$\mathbb{E}_P(\frac{dQ}{d\mathbb{P}}X) = \mathbb{E}_Q(Y) \leq 0$$

Hence $\frac{dQ}{d\mathbb{P}} \in C^0$. Given $Z \in C^0$, we have

$$\mathbb{E}_P(ZX) = \mathbb{E}_Q(Y) \leq 0$$

for all $X \in C$. The model is arbitrage-free, so $Z \geq 0$. Assuming $Z \neq 0$, so

$$\mathbb{E}_P\left(\frac{Z}{\mathbb{E}_P(Z)}(\phi \cdot \widetilde{S})_T\right) \leq 0$$

for all self-financing strategies ϕ. Replacing ϕ by $-\phi$

$$\mathbb{E}_P(\frac{Z}{\mathbb{E}_P(Z)}(\phi \cdot \widetilde{S})_T) = 0$$

which means that $\frac{Z}{\mathbb{E}_P(Z)} \in \mathcal{M}^a(\widetilde{S})$. As the polar cone C is exactly given by the cone generated by $\frac{dQ}{d\mathbb{P}} \in \mathcal{M}^a(\widetilde{S})$ all the assertion hold by the bipolar theorem.

$$\begin{aligned}
C^0 &= \left\langle \frac{dQ_1}{d\mathbb{P}}, \ldots, \frac{dQ_m}{d\mathbb{P}} \right\rangle_{\text{cone}}, \\
C^{00} &= C = \{X \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \text{ such that } \mathbb{E}_{Q_i}(X) \leq 0 \text{ for } i = 1, \ldots, m\}. \\
\mathcal{K}^0 &= \left\langle \frac{dQ_1}{d\mathbb{P}}, \ldots, \frac{dQ_m}{d\mathbb{P}} \right\rangle_{\text{vector}}, \\
\mathcal{K}^{00} &= \mathcal{K} = \{X \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \text{ such that } \mathbb{E}_{Q_i}(X) = 0 \text{ for } i = 1, \ldots, m\}.
\end{aligned}$$

We can now reformulate the definition of a complete and an incomplete market.
Let S be a discrete model for a financial market and $\mathcal{M}^e(\widetilde{S}) \neq \emptyset$.

Definition (complete market) We call a financial market *complete*, if $\mathcal{M}^e(\widetilde{S}) = \{Q\}$, i.e. the set of equivalent martingale measures consists of a single element.

Definition (incomplete market) We call a financial market *incomplete*, if $\mathcal{M}^e(\widetilde{S})$ consists of more than one element. Then $\mathcal{M}^a(\widetilde{S}) = \langle Q_1, \dots, Q_m \rangle_{\text{convex}}$ for linearly independent measures $Q_i, i = 1, \dots, m$ and $m \geq 2$.

Lemma 2.8. ***(complete markets)***
The following assertions are equivalent:

(i) *S is a complete financial market, i.e.* $\mathcal{M}^e(\widetilde{S}) = \{Q\}$

(ii) *For every claim X exists a self-financing trading strategy ϕ such that the discounted claim can be replicated, i.e.*

$$V_T(\phi) = X$$

(iii) *For every claim X exists a predictable process ϕ and a unique number x such that the discounted claim can be replicated, i.e.*

$$\widetilde{X} = \frac{X}{S_T^0} = x + (\phi \cdot \widetilde{S})_T$$

(iv) *There is a unique fair pricing rule for every claim X.*

Proof The equivalence of (ii) and (iii) is obcious since it is the same by discounting.
$(i) \Rightarrow (ii), (iii)$: If S is complete, it follows that a unique martingale measure Q exists such that the discounted stock prices are Q-martingales. By Lemma 2.5 we know that

$$\pi(X)_n = \frac{S_n^0}{S_T^0} \mathbb{E}_Q \left[X | \mathcal{F}_n \right]$$

is the only arbitrage-free price for a claim X at time n, as the set of equivalent martingale measures consists only of the one element Q. Taking a look at the final value of the martingale $(\frac{\pi(X)_n}{S_n^0})_{n=0,\dots,T}$

$$\frac{\pi(X)_T}{S_T^0} = x + (\phi \cdot \widetilde{S})_T$$

Building the expectation with respect to Q on both sides leads to
$\mathbb{E}_Q \left[\frac{\pi(X)_T}{S_T^0} - x \right] = 0$ and therefor $\frac{\pi(X)_T}{S_T^0} - x \in \mathcal{K}$ by 1.11 which proves (iii) and consequently (ii).
$(ii) \Rightarrow (iv)$: Assume we have a portfolio ϕ that replicates the claim X. By definition of replication this means that

$$\pi(X)_n = V_n(\phi), \qquad \forall n = 0, \dots, T$$

defines a pricing rule. Therefor the pricing rule is uniquely given through the portfolios values as the values of the portfolio are unique du to (NA).
$(iv) \Rightarrow (i)$: If we have a unique pricing rule $\pi(X)$ for any claim X, then we know by Lemma 2.5 that we have an equivalent martingale measure.

Lemma 2.9. ***(incomplete markets)***
The following assertions are equivalent:

(i) *S is an incomplete financial market*

(ii) *For every claim X exists a self-financing trading strategy ϕ such that the discounted claim can be super-replicated, i.e.*

$$V_T(\phi) \geq X$$

and at least once $V_T(\phi) > X$ holds true, so there is at least one claim that cannot be replicated.

(iii) *For every claim X exists a predictable process ϕ and a unique number x such that the discounted claim can be super-replicated, i.e.*

$$\widetilde{X} = \frac{X}{S_T^0} \leq x + (\phi \cdot \widetilde{S})_T$$

and at least once $\widetilde{X} < x + (\phi \cdot \widetilde{S})_T$ holds true, so there is at least one claim that cannot be replicated.

Define

$$\Pi_{\downarrow}(X) = \inf\{\mathbb{E}_Q(\frac{X}{S_T^0}) \text{ for } Q \in \mathcal{M}^e(\widetilde{S})\}$$
$$\Pi_{\uparrow}(X) = \sup\{\mathbb{E}_Q(\frac{X}{S_T^0}) \text{ for } Q \in \mathcal{M}^e(\widetilde{S})\}$$

then we have either $\Pi_{\uparrow}(X) = \Pi_{\downarrow}(X)$ or $\Pi_{\downarrow}(X) < \Pi_{\uparrow}(X)$.
If $\Pi_{\uparrow}(X) = \Pi_{\downarrow}(X)$, X is attainable at price $\pi(X) := \Pi_{\uparrow}(X) = \Pi_{\downarrow}(X)$, i.e. $X = \pi(X) + (\phi \cdot S)_T$ for some ϕ and therefor $\pi(X)$ is the only arbitrage-free price for X.
If $\Pi_{\downarrow}(X) < \Pi_{\uparrow}(X)$, we have that the arbitrage-free prices at time 0 form an open interval

$$]\ \Pi_{\downarrow}(X), \Pi_{\uparrow}(X)\ [= \{E_Q(\frac{X}{S_T^0}) | Q \in \mathcal{M}^e(\widetilde{S})\}$$

Proof Assume that the market satisfies (NA). Then $\mathcal{M}^a(\widetilde{S}) = \langle Q_1, \ldots, Q_m \rangle_{\text{convex}}$, $m \geq 2$ because the market is incomplete. The polar cone of C is generated by $\frac{dQ_1}{d\mathbb{P}}, \ldots, \frac{dQ_m}{d\mathbb{P}}$,

$$C^0 = \left\langle \frac{dQ_1}{d\mathbb{P}}, \ldots, \frac{dQ_m}{d\mathbb{P}} \right\rangle_{\text{cone}}$$

Hence we can find numbers x for each claim X such that $\mathbb{E}_{Q_i}[\widetilde{X} - x] \leq 0$ for $i = 1, \ldots, m$, so by the bipolar theorem

$\widetilde{X} = \frac{X}{S_T^0} = x+(\phi\cdot\widetilde{S})_T - Y \leq x+(\phi\cdot\widetilde{S}_T)$, where Y denotes some non-negative random variable. If every claim could be replicated, it would mean that we have only one martingale measure due to the previous lemma for complete markets.
Under the assumption of replication

$$\widetilde{X} = x + (\phi \cdot \widetilde{S})_T$$

only one price exists. This follows immediately from

$$\mathbb{E}_Q(\frac{X}{S_T^0}) = x$$

for all $Q \in \mathcal{M}^a(\widetilde{S})$ by Doob's Theorem (see also Remark below). Let $\Pi_\downarrow(X) < \Pi_\uparrow(X)$ and $\pi(X)$ define an arbitrage-free pricing rule such that $\pi(X)_0 \geq \Pi_\downarrow(X)$. Then an equivalent martingale measure $Q \in \mathcal{M}^e(\widetilde{S})$ exists satisfying $\pi(X)_0 = \mathbb{E}_Q(\frac{X}{S_T^0})$. By the (NA)-assumption $\pi(X)_0 = \Pi_\downarrow(X)$. Therefor $\mathbb{E}_Q(\frac{X}{S_T^0} - \Pi_\downarrow(X)) \leq 0$, i.e. $\frac{X}{S_T^0} - \Pi_\downarrow(X) \in C$. So there is a predictable strategy ϕ such that $(\phi \cdot \widetilde{S})_T \geq \frac{X}{S_T^0} - \Pi_\downarrow(X)$, but building the expectation with respect to the martingale measures yields to

$$0 = \mathbb{E}_Q((\phi \cdot \widetilde{S})_T) \geq \mathbb{E}_Q(\frac{X}{S_T^0} - \Pi_\downarrow(X)) = 0$$

This would mean that $(\frac{X}{S_T^0} - \Pi_\downarrow(X)) \equiv (\phi \cdot \widetilde{S})_T$, thus $\Pi_\downarrow(X) = \Pi_\uparrow(X)$, which is a contradiction. Therefore $\pi(X)_0 < \Pi_\downarrow(X)$. For $\pi(X)_0 > \Pi_\downarrow(X)$ we simply have to pass from X to the negative claim $-X$. If the pricing interval is reduced to

$$\{x\} = \{\mathbb{E}_Q(\frac{X}{S_T^0}) \text{ for } Q \in \mathcal{M}^e(\widetilde{S})\}$$

then $E_Q(\frac{X}{S_T^0} - x) = 0$ for all $Q \in \mathcal{M}^e(\widetilde{S})$ and therefore a predictable strategy ϕ exists such that

$$\frac{X}{S_T^0} - x = (\phi \cdot \widetilde{S})_T$$

Remark 2.10. Let S be an arbitrage-free market and X an attainable contingent claim. Then X is of the form

$$\frac{X}{S_T^0} = x + (\phi \cdot \widetilde{S})_T$$

for $x \in \mathbb{R}$ and some trading strategy ϕ. Then the constant x and the process $(\phi \cdot \widetilde{S})_n$ are uniquely determined by the equation above and satisfy for every $Q \in \mathcal{M}^e(\widetilde{S})$

$$x = E_Q(\frac{X}{S_T^0}) \quad \text{and } x + (\phi \cdot \widetilde{S})_T = \mathbb{E}_Q[\frac{S_n^0 X}{S_T^0}|\mathcal{F}_n] \text{ for } n = 0, \dots, T.$$

Assume that X can be represented in two ways: $X = x_1 + (\phi_1 \cdot S)_T$ and $X = x_2 + (\phi_2 \cdot S)_T$ with $x_1 \neq x_2$. W.l.o.g. let $x_1 > x_2$. Then we can find an arbitrage opportunity by using the trading strategy $\phi_1 - \phi_2$ which produces a strictly positive relust at time T, since $x_1 - x_2 = ((\phi_1 - \phi_2) \cdot S)_T > 0$.
The uniqueness of the process $(\phi \cdot S)$ can be shown in the same way. Assume that X can be represented as $X = x + (\phi_1 \cdot S)_T$ and $X = x + (\phi_2 \cdot S)_T$ where the processes $\phi_1 \cdot S$ and $\phi_2 \cdot S$ are not identical, i.e. for an $n = 0, \dots, T$ holds $(\phi_1 \cdot S)_n \neq (\phi_2 \cdot S)_n$. W.l.o.g. suppose that $A := \{(\phi_1 \cdot S)_n > (\phi_2 \cdot S)_n\}$ is a non-empty event which obviously lies in $\mathcal{F}_n$. By definition we have the fact $(\phi_1 \cdot S)_T = (\phi_2 \cdot S)_T$, so we can find the trading strategy $\phi := (\phi_1 - \phi_2)1_A \cdot 1_{]n,T]}$ which is a predictable process that creates arbitrage as $(\phi \cdot S)_T = 0$ outside A while it contradicts (NA) if $(\phi \cdot S)_T = (\phi_1 \cdot S)_n - (\phi_2 \cdot S)_n > 0$ on A.
Finally the equations introduced above result freom the fact that for every predictable process ϕ and every $Q \in \mathcal{M}^a(S)$, we have that the process $\phi \cdot S$ is a Q-martingale.

Lemma 2.11. ***(superreplication)*** *Let S satisfy (NA). Then*

$$\begin{aligned}
\Pi_\uparrow(X) &= \sup\{E_Q[X] \mid Q \in \mathcal{M}^e(S)\} \\
&= \max\{E_Q[X] \mid Q \in \mathcal{M}^e(S)\} \\
&= \min\{x \mid \text{ there exists } Y \in \mathcal{K},\ x + Y \geq X\}
\end{aligned}$$

Proof As already shown, $X - \Pi_\uparrow(X) \in C$, consequently

$$\begin{aligned}
X &= \Pi_\uparrow(X) + G \qquad \text{for some } G \in C \\
&= \Pi_\uparrow(X) + Y - L \qquad \text{for some } Y \in \mathcal{K},\ L \in L_{\geq 0} \\
&\leq \Pi_\uparrow(X) + Y \qquad \text{for some } Y \in \mathcal{K}
\end{aligned}$$

Therefor $\Pi_\uparrow(X) \geq \inf\{x \mid \text{ there exists } Y \in \mathcal{K},\ x + Y \geq X\}$.
Assume $x < \Pi_\uparrow(X)$. Then it will follow that no element $Y \in \mathcal{K}$ exists such that $x + Y \geq X$, which shows that
$\Pi_\uparrow(X) = \inf\{x \mid \text{ there exists } Y \in \mathcal{K}, x + Y \geq X\}$ and that the infimum is a minimum. As $x < \Pi_\uparrow(X)$ holds, there exists a $Q \in \mathcal{M}^e(S)$ such that $x < \mathbb{E}_Q(X)$ which implies that for all $Y \in \mathcal{K}$ we have $\mathbb{E}_Q(x + Y) = x < \mathbb{E}_Q(X)$ contradicting the relation $x + Y \geq X$.

Taking the results together, we can give a precise definition of the superreplication price and will then illustrate its character on the basis of a little example.

Definition (superreplication price)
If we are in the situation of an incomplete market, more than one equivalent martingale measure exists and we cannot find a replicating strategy. A price for a contingent claim X is fair (i.e. prevent from arbitrage opportunities), if it equals the the expectation of the discounted payoff under an equivalent martingale measure. As the set of equivalent martingale measures $\mathcal{M}^e(S)$ is convex, the set $\{\mathbb{E}_Q(X), Q \in \mathcal{M}^e(S)\}$ is an interval and any choice of initial price outside this interval would lead to arbitrage.

For a claim X and $x > 0$ we denote by $\Pi(X)$ the super-replication price which is given by

$$\begin{aligned}\Pi(X) &= \sup\{Q \in \mathcal{M}^e(S) \mid \mathbb{E}_Q[X] \leq x\} \\ &= \inf\{x \mid X \leq x + (\phi \cdot S)_T\}\end{aligned}$$

Putting these equations into words words leads to following interpretation: A claim is superreplicated if the hedging portfolio is guaranteed to produce at least the payoff of the claim. The superreplication price is the supremum of possible prices consistent with (NA). It is the smallest intial value an agent needs in order to superreplicate the payoff of the claim, i.e. to eliminate the possibility of any losses from claim X by dynamical trading.
example
Consider a one-step model for a financial market with constant bond $B \equiv 1$ and stock S moving according to following tree:

$$\begin{tikzcd} & S_1 = \frac{3}{2} \\ S_0 = 1 \arrow[ur] \arrow[r] \arrow[dr] & S_1 = 1 \\ & S_1 = \frac{1}{2} \end{tikzcd}$$

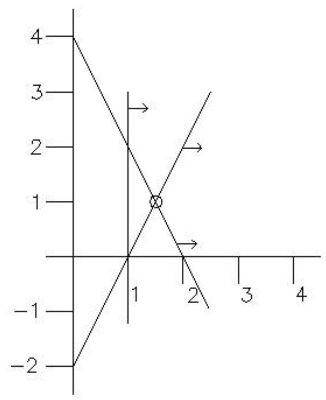
and claim

$$X = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

In order to determine $\Pi = \inf\{x | x + (\phi \cdot S) \geq X\}$ we can draw following figure which represents the necessary equations

$$\begin{aligned} x + \frac{\phi}{2} &\geq 2 \\ x \quad &\geq 1 \\ x - \frac{\phi}{2} &\geq 1 \end{aligned}$$

and conclude that $\Pi = 1.5$.



We come to the same result if we calculate the set of martingale measures and then compute $\sup_Q E_Q(X)$. Solving the system

$$\begin{aligned} \frac{3}{2}q_1 + q_2 + \frac{1}{2}q_3 &= 1 \\ q_1 + q_2 + q_3 &= 1 \end{aligned}$$

leads to the set of martingale measures

$$Q = \left\{ t \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} + (1-t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, t \in]0, 1[\right\}$$

which allows to calculate

$$
\begin{aligned}
\sup_Q E_Q \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} &= \sup_{t\in(0,1)} 2 \cdot \frac{1}{2}t + 1 \cdot (1-t) + 1 \cdot \frac{1}{2}t \\
&= \sup_{t\in(0,1)} 1 + \frac{t}{2} \\
&= 1 + \frac{1}{2} \\
&= 1.5
\end{aligned}
$$

3 Utility Maximization Problem

The following Lemmas and their proofs are mainly based on lecture notes by J. Teichmann [3] and W. Schachermayer [4]. We still consider a discrete market model, which is - as defined above - an adapted $(d+1)$-dimensional stochastic process S on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$. The price process $S = (S_n^j)_{n=0,\dots,T}^{j=0,\dots,d}$ consists of $d+1$ assets, at which the price process $S = (S_n^0)_{n=0,\dots,T}$ is strictly positive and called the *riskless asset* or *bond*.
For the present, it is sufficient to take the following version of No Arbitrage condition:

Assumption 1 The set $\mathcal{M}^e(S)$ is not empty.

Utility theory is based on the belief that each individual agent possesses a utility function and if the agent has to make a decision, she always acts in such a way that expected utility is maximized.
Thus, in addition to the model S of a financial market, we have to introduce the function $U(x)$, which is supposed to model the utility of an agents wealth x at terminal time T.

Definition (utility function) A utility function U assigns each initial wealth $x \in dom(U)$ a number:

$$U : \mathrm{dom}(U) \to \mathbb{R} \cup \{-\infty\}$$

Furthermore a utility function U is assumed to be increasing and strictly concave. These properties reflect the general situation that more money has "more utility" and the agents character of being risk averse - so the agent prefers a certain payment of x instead of taking X with $\mathbb{E}[X] = x$, i.e. the agent would rather choose a guaranteed payment of 100 Euro than to run the risk of receiving 200 Euro or nothing with a fifty-fifty chance (see section on Risk Aversion).

From an economic point of view it is also natural to require that the marginal utility tends to zero when wealth x tends to infinity, and to infinity when the wealth x tends to its lower bound:

$$U'(\infty) = \lim_{x \to \infty} U'(x) = 0, \quad U'(0) = \lim_{x \to 0} U'(x) = \infty$$

We have to distinguish between two cases if we want to take a look at the behaviour of the marginal utility at the other end of the wealth scale.

case 1 (negative wealth not allowed) in case of $dom(U) = (0, \infty)$ we assume that U fulfills the conditions that
$U(x) = -\infty$ for $x < 0$ while
$U(x) > -\infty$ for $x > 0$ and

$$U'(0) \quad := \quad \lim_{x \searrow 0} U'(x) = \infty$$

The last property is also known as the INADA condition.

case 2 (negative wealth allowed) In the situation where $dom(U) = \mathbb{R}$ we assume that
$U(x) > -\infty$ for all $x \in R$ and that

$$U'(-\infty) \quad := \quad \lim_{x \searrow -\infty} U'(x) = \infty$$

Now we can formulate the utility maximization problem as the problem of finding the optimal trading strategy ϕ, such that the expected utility of terminal wealth is maximized.

Definition (Utility maximization problem) The utility maximization problem is defined as the determination of $u(x)$ for $x \in dom(U)$,

$$u(x) := \sup_{\phi} \mathbb{E}_P \left[U(x + (\phi \cdot S)_T) \right]$$

So the value function $u(x)$ indicates the expected utility of an agent at time T for a given initial endowment x - considering that the agent invests optimally in the financial market S.
We will now study the problem of finding the optimal strategy $\widehat{\phi}$ which depends on the initial wealth x. If we would not have required an arbitrage free market, one could find a portfolio with initial value 0 such that

$0 + (\phi \cdot S) > 0$, which would imply that no optimizer exists. We will now reformulate the optimization problem as

$$\begin{aligned} u(x) &= \sup_{Y \in \mathcal{K}} \mathbb{E}\left[U(x+Y)\right] \\ \mathcal{K} &\subset L^2(\Omega, \mathcal{F}, P) \end{aligned}$$

where for the affine subspace $x + \mathcal{K}$ and a random variable X

$$\begin{aligned} X \in x + \mathcal{K} \Leftrightarrow \quad & \mathbb{E}_Q[X] = x & \text{for all } Q \in \mathcal{M}^a(S) \\ \Leftrightarrow \quad & \mathbb{E}_P\left[\tfrac{dQ}{dP} X\right] = x & \text{for all } Q \in \mathcal{M}^a(S) \end{aligned}$$

holds true. Then the problem can be translated into a problem of Lagrangian multipliers, i.e.

$$L(X, \lambda_1, \ldots, \lambda_m) \quad = \quad \mathbb{E}_P\left(U(X)\right) - \sum_{i=1}^{m} \lambda_i \left(\mathbb{E}_P(\frac{dQ^i}{dP} X) - x\right)$$

If $\widehat{X} = x + \widehat{Y} = x + (\widehat{\phi \cdot S})_T$ solves the optimization problem for $x \in dom(U)$, then there exist $\widehat{\lambda}_1, \ldots, \widehat{\lambda}_m$ such that

$$U'(\widehat{X}) = U'(x + \widehat{Y}) = \sum_{i=1}^{m} \widehat{\lambda}_i \frac{dQ^i}{dP}$$

and

$$(\widehat{X}, \widehat{\lambda}_1, \ldots, \widehat{\lambda}_m) \text{ satisfies } L(\widehat{X}, \widehat{\lambda}_1, \ldots, \widehat{\lambda}_m) = 0$$

So far the problem of utility maximization translated into a Lagrange problem, i.e. L has an extremum which is a saddle point (neither a minimum nor a maximum).
Writing

$$L(X, y, Q) = \mathbb{E}_P\left[U(X)\right] - y\left(\mathbb{E}_Q(X) - x\right)$$

where $X \in L^2(\Omega, \mathcal{F}, P)$. By defining $y := \sum_{i=1}^{m} \widehat{\lambda}_i > 0$ and using the fact that Q, as an absolutely continuous martingale measure, $Q \in \mathcal{M}^a(S)$, can be written as a convex combination $yQ = \sum_{i=1}^{m} y\mu_i Q_i = \sum_{i=1}^{m} \lambda_i Q_i$, the new Lagrange function is justified. Defining

$$\begin{aligned} \Phi(X) &:= \inf_{y,Q} L(X, y, Q) \quad & \text{for } X \in L^2(\Omega, \mathcal{F}, P) \\ \Psi(y, Q) &:= \sup_{X} L(X, y, Q) \quad & \text{for } y > 0,\ Q \in \mathcal{M}^a(S) \end{aligned}$$

then it follows due to the saddle point property that

$$\begin{aligned} u(x) = \sup_{X \in L^2(\Omega,\mathcal{F},P)} \inf_{\substack{y>0 \\ Q \in \mathcal{M}^a(S)}} L(X, y, Q) &= \inf_{\substack{y>0 \\ Q \in \mathcal{M}^a(S)}} \sup_{X \in L^2} L(X, y, Q) \quad (1) \\ = \sup_{X \in L^2} \Phi(X) &= \inf_{y,Q} \Psi(y, Q) \quad (2) \end{aligned}$$

Lemma 3.1.

$$u(x) = \sup_{X \in L^2} \Phi(X) = \sup_{Y \in \mathcal{K}} \mathbb{E}(U(x+Y))$$

Proof Taking a closer look at the definition of $\Phi(X)$

$$\Phi(X) = \inf_{y,Q} \left(\mathbb{E}[U(X)] - y(\mathbb{E}_Q[X] - x) \right),$$

we recognize that we have to distinguish between the two possible cases for $(\mathbb{E}_Q[X] - x)$:

- $\mathbb{E}_Q[X] - x > 0$: In this case it follows that there is at least one Q such that $\Phi(X) = -\infty$
- $\mathbb{E}_Q[X] - x \leq 0$: Here $\Phi(X) = \mathbb{E}[U(X)]$ for all $Q \in \mathcal{M}^a(S)$. Remark that $(\mathbb{E}_Q[X] - x) \leq 0$ implies that $X - x \in C$, i.e. X can be superreplicated.

Consequently we get

$$\begin{aligned} \sup_{X \in L^2(\Omega,\mathcal{F},P)} \Phi(X) &= \sup_{\mathbb{E}_Q[X] \leq x} \mathbb{E}[U(X)] \\ &= \sup_{\substack{X \text{ superreplicable} \\ \text{at price } x}} \mathbb{E}[U(X)] = \sup_{\substack{X \text{ replicable} \\ \text{at price } x}} \mathbb{E}[U(X)] \\ &= \sup_{Y \in \mathcal{K}} \mathbb{E}[U(x+Y)] = u(x) \end{aligned}$$

which proves the first $'' ='' $ of (2). The step

$$\sup_{\substack{X \text{ superreplicable} \\ \text{at price } x}} \mathbb{E}[U(X)] = \sup_{\substack{X \text{ replicable} \\ \text{at price } x}} \mathbb{E}[U(X)]$$

is justified as "X superreplicable" means that $X = x + (\phi \cdot S)_T - Z$, where $Z \geq 0$ stands for the consumption. So leaving out Z only enlarges the value of $E\left[U(x)\right]$ as U is an increasing function.

Now it remains to look at $\Psi(y, Q)$, which leads to the idea of duality:

Lemma 3.2. *Given an arbitrage-free financial market* $(S^0, \ldots, S^d)$*, the function*

$$\Psi(y, Q) = \sup_{X \in L^2(\Omega,\mathcal{F},P)} L(X, Q, y)$$

can be expressed by the conjugate function V *of* U*,*

$$\Psi(y, Q) = \mathbb{E}_P \left[V(y \frac{dQ}{dP}) \right] + yx$$

Proof

$$\begin{aligned}\Psi(y,Q) &= \sup_{X\in L^2} \mathbb{E}_P\left[U(X)\right] - y\left(\mathbb{E}_Q(X) - x\right)\\ &= \mathbb{E}_P\left[\sup_{X\in L^2} U(X) - y\frac{dQ}{dP}X\right] + yx\\ &\overset{*}{=} \mathbb{E}_P\left[V(\frac{dQ}{dP}y)\right] + yx\end{aligned}$$

By definition of the convex conjugate the last step $\overset{*}{=}$ holds:

Definition (convex conjugate) The convex conjugate function V of the agents concave utility function U is defined by

$$V(y) = \sup_{x\in dom(U)} (U(x) - xy), \quad y > 0$$

which is just the Legendre-transform of $x \mapsto -U(-x)$

So if we know the conjugate function, we can calculate the supremum, where we only need the conjugate at $y\frac{dQ}{dP}$.

3.1 The duality approach

Calling

$$u(x) = \sup_{Y\in\mathcal{K}} \mathbb{E}_P(U(x+Y)) \quad (P)$$

where $x \in dom(U)$ is given, the *primal problem* (P), we can now formulate its dual problem (P*) as

$$v(y) = \inf_{Q\in\mathcal{M}^a(S)} \mathbb{E}[V(y\,\frac{dQ}{dP})] \quad (P^*)$$

which we want to calculate for all $y > 0$. The next Lemma proves that a solution to the dual optimization problem exists.

Lemma 3.3. *If U is a utility function and if $\mathcal{M}^e(S) \neq \emptyset$, i.e. under the abscence of arbitrage, then there exists a unique optimizer $\widehat{Q}(y)$ such that*

$$v(y) = \inf_{Q\in\mathcal{M}^a(S)} \mathbb{E}_P\left[V(y\,\frac{dQ}{dP})\right] = \mathbb{E}_P\left[V(y\,\frac{d\widehat{Q}}{dP})\right].$$

Furthermore

$$\inf_{y>0}(v(y) + xy) = \inf_{\substack{y>0\\ Q\in\mathcal{M}^a(S)}} \mathbb{E}_P\left[V(y\,\frac{dQ}{dP})\right] + xy = \inf_{\substack{y>0\\ Q\in\mathcal{M}^a(S)}} \Psi(y,Q)$$

Proof Since $V : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is strictly convex, C^2 on $(0,\infty)$ and $V'(0) = -\infty$ we obtain by compactness the existence of an optimizer $\widehat{Q}(y)$ and by $V'(0) = -\infty$ that the optimizer is an equivalent martingale measure. By strict convexity the optimizer is also unique. The gradient condition for $\widehat{Q}(y)$ reads as follows

$$E_P\left[V'(\widehat{Q}(y))\,(\frac{d\widehat{Q}(y)}{dP} - \frac{dQ}{dP})\right] = 0$$

for all $Q \in \mathcal{M}^a(S)$. The value function v shares the same qualitive properties as V.
Fixing $x \in dom(U)$ and taking the optimizer $\widehat{y} = \widehat{y}(x) > 0$, it follows

$$\begin{aligned}\inf_{y>0}(v(y) + xy) &= v(\widehat{y}) + x\widehat{y} \leq \inf_{Q\in\mathcal{M}^a(S)} \mathbb{E}_P\left[V(y\,\frac{dQ}{dP})\right] + xy \\ &\leq \mathbb{E}_P\left[V(y\,\frac{dQ}{dP})\right] + xy\end{aligned}$$

for all $Q \in \mathcal{M}^a(S)$ and $y > 0$, so

$$\inf_{y>0}(v(y) + xy) \leq \inf_{\substack{y>0\\ Q\in\mathcal{M}^a(S)}} \mathbb{E}_P\left[V(y\,\frac{dQ}{dP})\right] + xy$$

Take $y_1 > 0$ and $Q_1 \in \mathcal{M}^e(S)$ for some $\epsilon > 0$ and assume that

$$\begin{aligned}\inf_{y>0}(v(y) + xy) + 2\epsilon &\geq v(y_1) + xy_1 + \epsilon \\ &\geq \mathbb{E}_P\left[V(y_1\,\frac{dQ_1}{dP})\right] + xy_1 \\ &\geq \inf_{\substack{y>0\\ Q\in\mathcal{M}^a(S)}} \mathbb{E}_P\left[V(y\,\frac{dQ}{dP})\right] + xy\end{aligned}$$

Since this holds for every $\epsilon > 0$, we are finished.

Now pulling the results from the primal and dual problem together, we see what remains to prove:

$$\inf_{y>0} \underbrace{\overbrace{\inf_{Q\in\mathcal{M}^a(S)} \mathbb{E}_P\left[V(y\,\frac{dQ}{dP})\right]}^{v(y)} + xy}_{\Psi(y,Q)} = u(x) \left(= \sup_{X\in L^2} \underbrace{\inf_{\substack{y>0\\ Q\in\mathcal{M}^a(S)}} L(X,y,Q)}_{\Phi(X)} \right)$$

Thus it is only left to show the following lemma:

Lemma 3.4. *Let U be a utility function and $(S^0, \ldots, S^d)$ an arbitrage-free market. Then*

$$\inf_{y>0}(v(y) + xy) \quad = \quad u(x),$$

and the mini-max assertion holds.

Proof For fixed $x \in dom(U)$ exists an optimal portfolio $\widehat{X}(x)$ and there exist Lagrangian multipliers $\widehat{\eta}_1, \ldots, \widehat{\eta}_m \geq 0$ such that

$$\text{grad} L(\widehat{X}(x), \widehat{\eta}_1, \ldots, \widehat{\eta}_m) = 0.$$

Since the optimal portfolio satisfies the condition $\mathbb{E}_{Q_i}(\widehat{X}(x)) = x$ for $i = 1, \ldots, m$ it follows that

$$L(\widehat{X}(x), \widehat{\eta}_1, \ldots, \widehat{\eta}_m) = u(x).$$

We now define a measure $\widehat{Q}$ such that

$$U'(\widehat{X}) = \widehat{y}\,\frac{d\widehat{Q}}{dP},$$

where $\widehat{y} := \sum_{i=1}^m \widehat{\eta}_i > 0$. This assertion follows from the Lagrangian multiplier method as

$$U'(\widehat{X}) - \widehat{y}\sum_{i=1}^m \frac{\widehat{\eta}_i}{\widehat{y}}\frac{dQ_i}{dP} = 0,$$

where we can now see that

$$\widehat{y}\frac{d\widehat{Q}}{dP} = \widehat{y}\sum_{i=1}^m \frac{\widehat{\eta}_i}{\widehat{y}}\frac{dQ_i}{dP},$$

therefore $\widehat{Q} \in \mathcal{M}^e(S)$. So $\widehat{Q}$ is recognized as the optimal measure for the dual problem. Moreover

$$\mathbb{E}_P\left[V(\widehat{y}\,\frac{d\widehat{Q}}{dP})\right] + x\widehat{y} = \inf_{Q\in\mathcal{M}^a(S)} \mathbb{E}_P\left[V(\widehat{y}\,\frac{dQ}{dP})\right] + x\widehat{y}$$

since $V'(y) = -(U')^{-1}(y)$ and $Q_* \in \mathcal{M}^e(S)$ is a minimum if and only if

$$\mathbb{E}_P\left[V'(y\,\frac{dQ_*}{dP})(\frac{dQ_*}{dP} - \frac{dQ}{dP})\right] = 0$$

for all $Q \in \mathcal{M}^a(S)$. This property is satisfied by $\widehat{Q}$. By definition of V we also obtain that

$$\begin{aligned}
\mathbb{E}_P\left[V(\widehat{y}\,\frac{d\widehat{Q}}{dP})\right] + x\widehat{y} \quad &= \quad \sup_{X\in L^2(\Omega,\mathcal{F},P)} L(X, \widehat{y}, \widehat{Q}) \\
&= \quad L(\widehat{X}, \widehat{y}, \widehat{Q}),
\end{aligned}$$

since $U'(\widehat{X}) = \widehat{y}\frac{d\widehat{Q}}{dP}$ and the following holds:
By definition

$$\begin{aligned} V(y) &= \sup_{x \in dom(U)} (U(x) - yx) \\ &= U(\widehat{x}_y) - y\widehat{x}_y \end{aligned}$$

it follows that

$$\begin{aligned} 0 &= U'(\widehat{x}_y) - y \Leftrightarrow \\ U'(\widehat{x}_y) &= y \Leftrightarrow \\ \widehat{x}_y &= (U')^{-1}(y) \end{aligned}$$

which leads to

$$V(y) = U((U')^{-1}(y) - y(U')^{-1}(y))$$

and this allows

$$V(\widehat{y}\frac{d\widehat{Q}}{dP}) = U(\widehat{X}) - \frac{d\widehat{Q}}{dP}\widehat{y}\widehat{X}.$$

However $L(\widehat{X}, \widehat{y}, \widehat{Q}) = u(x)$ by assumption on optimality of $\widehat{X}$. Therefore

$$\mathbb{E}_P\left[V(\widehat{y}\frac{d\widehat{Q}}{dP})\right] + x\widehat{y} = u(x),$$

and $\widehat{y}$ is the minimizer since

$$\mathbb{E}_P\left[V'(\widehat{y}\frac{d\widehat{Q}}{dP})\frac{d\widehat{Q}}{dP}\right] = -x$$

by assumption. Hence by using the formulas for V and the definitions it follows

$$\begin{aligned} \inf_{\substack{y>0 \\ Q \in \mathcal{M}^a(S)}} \mathbb{E}_P\left[V(y\frac{dQ}{dP})\right] + xy &= \inf_{y>0} \mathbb{E}_P\left[V(y\frac{d\widehat{Q}}{dP})\right] + xy \\ &= u(x) \\ &= \mathbb{E}_P\left[U(\widehat{X})\right]. \end{aligned}$$

So by this theorem we can formulate the following dual relation: Given a utility maximization problem for $x \in dom(U)$,

$$u(x) = \sup_{Y \in \mathcal{K}} \mathbb{E}_P\left[U(x+Y)\right],$$

then we can associate a dual problem for $y > 0$

$$v(y) = \inf_{Q \in \mathcal{M}^a(S)} \mathbb{E}_P \left[V(y \frac{dQ}{dP}) \right]$$

From the minimax condsiderations we know that

$$u(x) = \inf_{y>0} v(y) + xy.$$

So we can solve the dual problem first, where from we obtain $y \mapsto \widehat{Q}(y)$. Then we can calculate $\widehat{y}(x)$ for given $x \in dom(U)$ and receive

$$\begin{aligned} v(\widehat{y}(x)) + x\widehat{y}(x) &= u(x) \\ u'(\widehat{X}(x)) &= \widehat{y}(x) \frac{d\widehat{Q}(\widehat{y}(x))}{dP} \end{aligned}$$

The results are now resumed in the following Theorem:

Theorem 3.5. *Let S be a financial market, $\mathcal{M}^e(S) \neq \emptyset$ and U a utility function. The value functions are represented by $u(x)$ and $v(y)$*

$$u(x) = \sup_{Y \in C} \mathbb{E}_P \left[U(x+Y) \right] \tag{3}$$

$$v(y) = \inf_{Q \in \mathcal{M}^a(S)} \mathbb{E}_P \left[V(y \frac{dQ}{dP}) \right] \tag{4}$$

since $\mathcal{K} = \{V_T(\phi) \mid \phi \text{ self-financing, } V_0(\phi) = 0\}$, $\mathcal{K} \subset L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{K} - L^2_{\geq 0}(\Omega, \mathcal{F}, P) = C$ the convex cone with $C^0 = \left\{ y \frac{dQ}{dP} \mid Q \in \mathcal{M}^a(S),\ y \geq 0 \right\}$, it is clear that

$$u(x) = \sup_{\substack{\phi \text{ self-} \\ \text{financing}}} \mathbb{E}_P \left[U(V_T(\phi)) \right] = \sup_{Y \in \mathcal{K}} \mathbb{E}_P \left[U(x+Y) \right] = \sup_{Y \in C} \mathbb{E}_P \left[U(x+Y) \right]$$

Then we have

(i) The value functions $u(x)$ and $v(y)$ are conjugate and u inherits the qualitative properties of U.

(ii) The optimizers $\widehat{X}(x) = x + \widehat{Y}$ in (3) and $\widehat{Q}(y) \in \mathcal{M}^e(S)$ in (4) exist, are unique and satisfy

$$\widehat{X}(x) = -V'\left(y \frac{d\widehat{Q}(y)}{dP} \right), \text{ or equivalently } \ y \frac{d\widehat{Q}(y)}{dP} = U'(\widehat{X}(x)) \tag{5}$$

where $x \in dom(U)$ and $y > 0$ are linked by $u'(x) = y$ or equivalently $x = -v'(y)$

(iii) The following formulas hold true:

$$u'(x) = \mathbb{E}_P\left[U'(\widehat{X}(x))\right], v'(y) = \mathbb{E}_{\widehat{Q}}\left[V'\left(y\,\frac{d\widehat{Q}(y)}{dP}\right)\right] \tag{6}$$

$$xu'(x) = \mathbb{E}_P\left[\widehat{X}(x)U'(\widehat{X}(x))\right], yv'(y) = \mathbb{E}_P\left[y\frac{d\widehat{Q}(y)}{dP}V'\left(y\frac{d\widehat{Q}(y)}{dP}\right)\right] \tag{7}$$

In case of a complete market the set of equivalent martingal measures consists of only one element Q and therefore $\widehat{Q}(y)$ can be replaced by Q in the formulas above.

Proof It is only left to show the formulas for $u'(x)$ and $v'(y)$, whereat the formulas for $v'(y)$ in (6) and (7) are obvious since differentiating

$$v(y) = \mathbb{E}_P\left[V\left(y\,\frac{dQ}{dP}\right)\right] = \sum_{n=1}^{N} p_n V\left(y\,\frac{q_n}{p_n}\right) \tag{8}$$

immediately leads to the results - under consideration that $\mathbb{E}_P[\frac{dQ}{dP}] = 1$ by Radon-Nikodym.
Now the formula for $u'(x)$ in (6) is exactly the relation property that has been shown in 2: $u'(x) = y = \mathbb{E}_P[y\frac{dQ}{dP}] = \mathbb{E}_P\left[U'(\widehat{X}(x))\right]$ and (7) can be shown by using both relations $x = -v'(y)$ and $y = u'(x)$, as $xu'(x)$ becomes $-v'(y)y$ whose formula has just been proved:

$$\begin{aligned} xu'(x) &= -v'(y)y \\ &\stackrel{*}{=} -\mathbb{E}_P\left[y\,\frac{dQ}{dP}V'(y\,\frac{dQ}{dP})\right] \\ &= \mathbb{E}_P\left[y\,\frac{dQ}{dP}(-V'(y\,\frac{dQ}{dP}))\right] \\ &= \mathbb{E}_P\left[U'(\widehat{X}(x))\,\widehat{X}(x)\right] \end{aligned}$$

where $\stackrel{*}{=}$ uses equation (7) which has just been verified to hold true.

Now the economic interpretation of the formulas above is of special interest. Regarding the *Arrow-Debreu assets* $\mathbf{1}_{\{\omega_n\}}$, we have the following relation for the price at time $t = 0$:

$$\mathbb{E}_Q[\mathbf{1}_{\{\omega_n\}}] = Q[\omega_n] =: q_n$$

and each such asset $\mathbf{1}_{\{\omega_n\}}$ can be represented as $\mathbf{1}_{\{\omega_n\}} = Q(\omega_n) + (\phi \cdot S)_T$, where ϕ is some predictable trading strategy.
So equation (5)

$$U'\left(\widehat{X}(x)(\omega_n)\right) = \widehat{y}(x)\,\frac{\widehat{q}_n(\widehat{y}(x))}{p_n}, \;\; n = 1, \ldots, N$$

means that in every possible state of the world ω_n, the marginal utility of the wealth of an optimal investing agent at time T, i.e. $U'\left(\widehat{X}(x)(\omega_n)\right)$ is proportional to the ratio of q_n, which is the price of the corresponding Arrow security $\mathbf{1}_{\{\omega_n\}}$, and the probability of its success $p_n = \mathbb{P}[\omega_n]$. (It will later be elaborated why this interpretation is also clear for the case of an incomplete market).
This proportionality relation must hold true for an optimal investment, otherwise an agent could invest a little more in the more favorable asset and a little less in the less favorable one and and so strictly increase her expected utility under the same budget constraint. Moreover we derive that $y = u'(x)$ is the proportionality factor for the investors initial endowment x.
Now this lemma reveals an easy way to solve the treated utility maximization problem. First calculate $v(y)$ by (4), which leads to a simple one-dimensional computation; as soon as $v(y)$ is known, all the other quantities like $\widehat{X}(x), u(x), u'(x), \ldots$ can be calculated by using the formulas introduced by this lemma.
Additionally one comes to the conclusion that the value function $x \mapsto u(x)$ can be regarded as a utility function itself since it shares all the qualitative attributes of the original utility function U. This also makes sense from an economic point of view since the 'indirect utility' function $u(x)$ denotes the expected utility at time T of an agent with initial endowment x, after having optimally invested in the financial market S.
The formula for $u'(x)$ has an interesting economic interpretation as well: Assume that the initial is now $x+h$ instead of x and let h be a small real number. Then the economic agent could use the additional endowment h to fincance h units of the cash account and invest x to the optimal pay-off function $\widehat{X}(x)$. Hence the agent finds himself with the pay-off function $\widehat{X}(x) + h$ at time T. This investment strategy compared to the optimal one corresponding to the initial endowment $x+h$ which is $\widehat{X}(x+h)$ and under the consideration that u is differentiable, it follows that

$$\begin{aligned}
\lim_{h\to 0} \frac{u(x+h) - u(x)}{h} &= \lim_{h\to 0} \frac{\mathbb{E}[U(\widehat{X}(x+h)) - U(\widehat{X}(x))]}{h} && (9)\\
&= \mathbb{E}[U'(\widehat{X}(x))] && (10)\\
& && (11)
\end{aligned}$$

Since h can be positive as well as negative, we have equality and therefore discovered another proof of formula (6) for $u'(x)$. From an economic point of view the proof shows that an optimally investing agent is indifferent of first order towards a (small) investment into the cash account.

An analogous calculation as in (9) leads to the formula for $u'(x)$ in (7), if the additional endowment $h \in \mathbb{R}$ is used to finance an additional investment into the optimal portfolio $\widehat{X}(x)$, where we then get to the pay-off function $\frac{x+h}{x}\widehat{X}(x)$ and compare this investment to $\widehat{X}(x+h)$. So the economic optimally investing agent is indifferent of first order towards marginal variation of the investment into the portfolio $\widehat{X}(x)$.
So now it becomes obvious that formula (6)

$$u'(x) = \mathbb{E}_P\left[U'(\widehat{X}(x))\right]$$

and (7)

$$xu'(x) = \mathbb{E}_P\left[\widehat{X}(x)U'(\widehat{X}(x))\right]$$

are simply special cases of a more general principle. In a complete market for each $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathbb{E}_Q[f]\, u'(x) = \lim_{h\to 0} \frac{\mathbb{E}_{\mathbb{P}}[U(\widehat{X}(x) + hf) - U(\widehat{X}(x))]}{h} \tag{12}$$

If an additional endowment of $h\mathbb{E}_Q[f]$ is invested in order to finance the contingent claim hf, then the increase in expected utility is of first order equal to $h\mathbb{E}_Q[f]\, u'(x)$. Again the agent is of first order indifferent towards an additional investment into the claim f.
For the incomplete case a variation of the optimal pay-off function $\widehat{X}(x)$ by a small unit of an arbitraray pay-off function $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the formula (12) becomes

$$\mathbb{E}_{\widehat{Q}(y)}[f]\, u'(x) = \lim_{h\to 0} \frac{\mathbb{E}_{\mathbb{P}}[U(\widehat{X}(x) + hf) - U(\widehat{X}(x))]}{h} \tag{13}$$

The only difference to the one of the complete case is that Q has to be replaced by $\widehat{Q}(y)$.
Now we emphasize this formula is not only valid for contingent claims attainable at price x, i.e. for claims of the form $f = x + (\phi \cdot S)_T$, but also for arbitrary claims f for which we cannot calculate the price from no arbitrage considerations.
Taking a closer look at this formula (13) leads to following interpretation: the pricing rule $f \mapsto \mathbb{E}_{\widehat{Q}(y)}[f]$ gives all those prices at which an optimally investing agent with initial endowment x and utility function U is indifferent of first order towards adding a (small) unit of the contingent claim f to her optimal portfolio $\widehat{X}(x)$.

Changing the point of view in the way of M. Davis, a different way of pricing can be discovered: one may define $\widehat{Q}(y)$ by (13) and then check that this is an equivalent martingale measure for S. This pricing rule can be interpreted as 'pricing by marginal utility'.
In order to prove formula (13) we will now introduce 'fictitious securities'. Let $(f^1, \ldots, f^k)$ be finitely many elements of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $(f^1, \ldots, f^k)$, the space $\mathcal{K} = \{(\phi \cdot S)_T : \phi \in \Phi\}$ and the constant function $\mathbf{1}$ linearly span $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$S_t^{d+j} \quad = \quad \mathbb{E}_{\widehat{Q}(y)}[f^j|\mathcal{F}_t], \quad j = 1, \ldots, k, \;\; t = 0, \ldots, T \tag{14}$$

as k processes for fixed $x \in \mathrm{dom}(U)$ and $y = u'(x)$.
The $\mathbb{R}^{d+1}$-valued process $S = (S^0, \ldots, S^d)$ can now be expanded to an $\mathbb{R}^{d+1+k}$-valued process $\bar{S} = (S^0, \ldots, S^d, S^{d+1}, \ldots, S^{d+k})$ by adding these new coordinates. By (14) $\bar{S}$ is a martingale under $\widehat{Q}(y)$. As $\widehat{Q}(y)$ is now the unique probability measure by the choice of $(f^1, \ldots, f^k)$ and lemma 2.8 we are in the situation of Theorem (3.5) for a complete market. Comparing (5)

$$\widehat{X}(x) = -V'\left(y\,\frac{dQ}{dP}\right) \tag{15}$$

for a complete market and

$$\widehat{X}(x) = -V'\left(y\,\frac{d\widehat{Q}(y)}{dP}\right) \tag{16}$$

we see that the optimal pay-off function $\widehat{X}(x)$ has not changed. So in the 'completed' market $\bar{S}$ the optimal investment can be achieved by trading in the first $d+1$ assets only - without using the 'fictitious securities' $S^{d+1}, \ldots, S^{d+k}$.
Actually we can now use formula (12) of the complete market to receive formula (13) for incomplete markets by applying Q to $\widehat{Q}(y)$.
So the pricing rule induced by $\widehat{Q}(y)$ is such that the interpretation of the optimal investment $\widehat{X}(x)$ defined in formula (5)

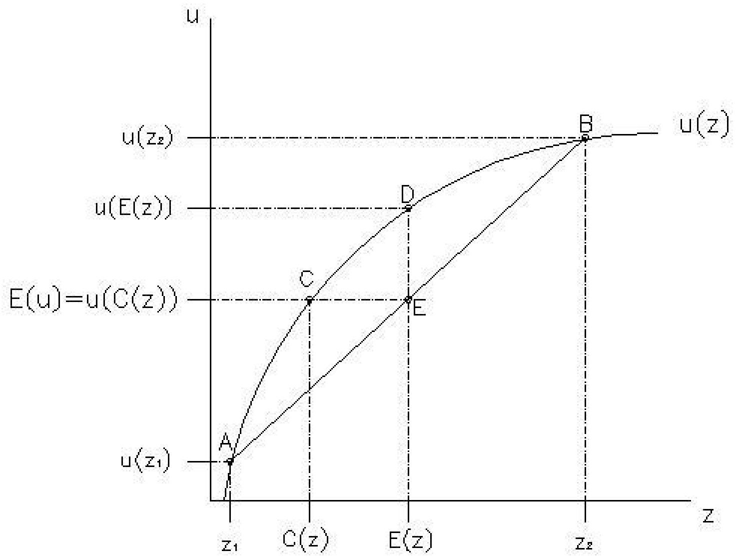
$$U'(\widehat{X}(x)) = y\,\frac{d\widehat{Q}(y)}{dP}$$

remains the same for the case of an incomplete market.
Note that by defining the measure $\widehat{Q}$ via $y\frac{dQ}{dP}$ we always obtain a measure with mass less than one.

4 Risk aversion

It is reported that Bernoulli was the first who stated that the determination of the value of an item must not be based on the price, but rather on the

utility it yields. There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount. Now we introduce the idea of risk aversion in a mathematic way. Intuitively a risk averse agent who has the choice between two comparable returns prefers the less risky one.
The facts of risk aversion will now be visualized in order to draw conclusions about the concavity requirement of utility functions.
Assume that a random variable Z has only two possible outcomes z_1 and z_2,



where z_1 occures with probability p and z_2 with probability $(1-p)$. Then the expected outcome $\mathbb{E}[Z] = pz_1 + (1-p)z_2$ is on the horizontal axis between z_1 and z_2 since it is a convex combination of them. As already known a utility function U is concave - therefore expected utility $\mathbb{E}(U) = pU(z_1) + (1-p)U(z_2)$ (denoted by point $'E'$) lies on the chord that connects the points $A = (z_1, U(z_1))$ and $B = (z_2, U(z_2))$. As the utility function U is concave it follows that the point E lays below the point $D = (\ \mathbb{E}[Z], U(\mathbb{E}[Z])\ )$, i.e. the utility of the expected outcome $U(\mathbb{E}[Z])$ is greater than expected utility $\mathbb{E}[U]$; $U(pz_1 + (1-p)z_2) > pU(z_1) + (1-p)U(z_2)$.
So imagine that there are two lotteries: one pays $\mathbb{E}[Z]$ with certainty and the other z_1 with probability p or z_2 with probability $1-p$. The utility of the first lottery is then $U(\mathbb{E}[Z])$ whereas the utility of the second lottery is $pU(z_1) + (1-p)U(z_2)$. The expected income is the same for both lotteries, but the risk averse agent would prefer $\mathbb{E}[Z]$ with certainty to $\mathbb{E}[Z]$ with uncertainty - and this effect is illustrated in the figure as $U(\mathbb{E}[Z]) > \mathbb{E}[U]$.

Another way to capture this effect is to find the so called certainty equivalent. Therefore consider a third lottery which pays $C(Z)$ with certainty and is such that it yields the same utility as the random lottery, i.e. $U(C(Z)) = \mathbb{E}[U]$. As seen in the figure, $C(Z)$ is less than the expected income $\mathbb{E}[Z]$; $C(Z) < \mathbb{E}[Z]$. However we know that the agent is indifferent between receiving $C(Z)$ with certainty and $\mathbb{E}[Z]$ with uncertainty. The difference $\mathbb{E}[Z]-C[Z] = \pi(Z)$ defines the risk-premium which is the maximum amount of income that an agent would forego in order to obtain a portfolio without risk. (Pratt) Hence the agent is risk-averse if $\pi(Z) \geq 0$ or equivalently $C(Z) \leq \mathbb{E}[Z]$ holds and we can easily prove that a utility function U is concave if and only if preferences are risk-averse:

Proof Let U be concave, i.e. $U(\lambda x + (1-\lambda)y) \geq \lambda U(x) + (1-\lambda)U(y)$ for all $\lambda \in (0,1)$, $x, y \in \mathbb{R}$. We have $\mathbb{E}[Z] = \lambda x + (1-\lambda)y$ and $\mathbb{E}[U] = \lambda U(x) + (1-\lambda)U(y)$ and therefor $U(\mathbb{E}[Z]) \geq \mathbb{E}[U]$. By definition of the certainty equivalent $E[U] = U(C(Z))$ - consequently $U(\mathbb{E}[Z]) \geq U(C(Z))$. As U is an increasing function it follows that $\mathbb{E}[Z] \geq C(Z)$, which is just the definition of risk-aversion.

Summing up: If x has the same utility as the expected utility of a random payoff X, i.e. if

$$U(x) = \mathbb{E}[U(X)],$$

then x is called the certainty equivalent of the random payoff. The utility function is called *risk averse*, if the certainty equivalent is less than the expected value of a random payment, i.e. $x < \mathbb{E}[X]$, which is the case for all concave utility functions. In order to quantify the strength of risk aversion, consider a random payoff X with $\mathbb{E}[X] = 0$ and $var(X) = 1$. Then one can see by how much the certainty equivalent reduces, i.e.

$$U(x+\delta) = \mathbb{E}[U(x+\epsilon X)].$$

The bigger the ratio $-\delta/\epsilon$ the more risk averse the agent is. Using taylor series expansion, it can be seen that

$$U(x) + \delta U^{'}(x) \approx U(x) + U^{'}(x)\mathbb{E}[X] + \frac{1}{2}\epsilon^2 U^{''}(x)\mathbb{E}[X^2].$$

and that is why the risk aversion of U is defined by

$$-U^{''}(x)/U^{'}(x)$$

So for risk averse investors, the pain from losing 1EUR is greater than the pleasure of winning 1EUR. Thus, such investors have to be compensated with additional return to induce them to hold risky assets. The following example helps to understand the concept of risk aversion: Consider an agent with exponential utility function, i.e.

$$U(x) = -\gamma \exp(-\gamma x)$$

and suppose $\gamma = 1/10$. To illustrate the risk aversion assume there is a choice of receiving either a guaranteed cash amount of $x = 100$ or $x = d$ (down) with probability q and $x = u(\text{up})$ with probability $(1 - q)$. For $d = 99$ and $u = 101$ the agent would be indifferent only if $q \sim 0.48$, for $d = 90$ and $u = 110$ it will have to be $q \sim 0.27$ and for $d = 0$ and $u = 200$ even $q \sim 0.000045$. Equation:

$$qU(\text{down}) + (1 - q)U(\text{up}) = U(100)$$

Hence, the agent would only choose the uncertain way, if the chance to lose tends to zero.

5 The Utility Indifference Price

Now we formulate the utility indifference approach for the pricing and hedging in incomplete markets. Therefore we need these following notations:

Definition The *utility maximization problem* for $x \in \text{dom}(U)$ is defined as

$$u(G, x) := \sup_{\phi \in \Phi} \mathbb{E}_P[U(x + (\phi \cdot S)_N - G)]$$

So the expression that has previously been introduced in the last chapter is now simply expanded for contingent claims G.

The *utility indifference price* is defined as the unique price p at which the agent is indifferent (in the sense that her expected utility under optimal trading remains unchanged) between receiving a premium p and selling the claim G or doing nothing.

$$p(G, x) := \inf\{w \in \mathbb{R} : u(G, x + w) \geq u(0, x)\}$$

Hence the agent only accepts the offer if it increases her expected utility or will be indifferent if expected utility remains the same. Equivalently the utility indifference price can be formulated from the perspective of a buyer, where $p(G, x) := \inf\{w \in \mathbb{R} : u(-G, x - w) \geq u(0, x)\}$ in our notation, which is then also known als the utility indifferent bid price. For a risk manager who possesses a risky asset G, the buyer's indifference price is the largest amount another investor would be willing to pay in order to take the risk that goes with G.
Anyway we need to solve the agents utility maximization problem with and without the claim.

The utility indifference price has following properties:

(i) *non-linear pricing*
utility indifference prices are non-linear in the number of options ϵ. The investor is not willing to pay twice as much for twice as many options, but requires a reduction in this price to take on the additional risk. Alternatively, a seller requires more than twice the price for taking on twice the risk. This property can be seen from the value function

$$u(\epsilon G, x) := \sup_{\phi \in \Phi} \mathbb{E}_P[U(x + (\phi \cdot S)_N - \epsilon G)]$$

since U is a concave function.

(ii) *recovery of complete market price*
If the market is complete or if the claim G can be replicated, then the utility indifference price $p(\epsilon G, x)$ equals the price of the complete market for ϵ units. Let $X_T = x + (\phi \cdot S) = x + \widetilde{X}_T$ for some $\widetilde{X}_T \in \mathcal{K}(0)$ ($\mathcal{K}(0)$ denotes the set of claims that can be replicated with zero initial endowment iff $x + \widetilde{X} \in \mathcal{K}(x)$). Let G be replicable from an initial value g, i.e. $G = g + \widetilde{X}_T^G$ with $\widetilde{X}_T^G \in \mathcal{K}(0)$,

$$X_T - \epsilon G = x + \widetilde{X}_T - \epsilon g - \epsilon \widetilde{X}_T^G = (x - \epsilon g) + \widetilde{X}'_T$$

where $\widetilde{X}'_T \in \mathcal{K}(0)$ and therefore $X_T - \epsilon G \in \mathcal{K}(x - \epsilon g)$. Consequently

$$u(\epsilon G, x) = \sup_{X_T \in \mathcal{K}(x)} \mathbb{E}[X_T - \epsilon G] = \sup_{X_T \in \mathcal{K}(x - \epsilon g)} \mathbb{E}[X_T] = u(0, x - \epsilon g)$$

and therefore the utility indifference price $p(\epsilon G, x)$ is just ϵ units of the complete market price g, i.e. $p(\epsilon G, x) = \epsilon g$.
(Or equivalently: If claim $G = (\phi \cdot S)_T$ then $p(x, G) = 0$)

(iii) *monotonicity*
If $G \leq C$ then $p(x, G) \leq p(x, C)$, which follows from the monotonicity of $u(x)$.

(iv) *convexity*
The utility indifference sellers price for contingent claims G_1, G_2 and $\lambda \in [0, 1]$

$$p(x, \lambda G_1 + (1 - \lambda) G_2) \leq \lambda p(x, G_1) + (1 - \lambda) p(x, G_2)$$

Intuitively the utility indifference price converges to the superreplication price if the risk aversion tends to infinity. This is also the main result of M. Rásonyi and L. Carassus [2].

Theorem 5.1. *Suppose that $x \in (0,\infty)$ and S is bounded. Furthermore assume that the risk aversion tends to infinity and that (a strengthened 17) No Arbitrage- condition hold.*
Then the utility indifference-prices are well-defined and converge to the superreplication price.

See Appendix below for the proof of this theorem.

6 The Marginal Utility Price

As already noted in section 3 another concept of pricing has been studied by M. Davis. Before we give the definition of the so called 'marginal utility price', we note that

$$\begin{aligned} u(G,x) &:= \sup_{X\in\mathcal{K}(x)} \mathbb{E}_{\mathbb{P}}(U(X-G)) \text{ for } x \in dom(u) \\ v(G,y) &:= \inf_{Q\in\mathcal{M}^a} \mathbb{E}_{\mathbb{P}}(V(y\,\frac{dQ}{dP}) - y\,\frac{dQ}{dP}G) \text{ for } y>0 \end{aligned}$$

and as we still have a dual relation the following statements hold:

$$\begin{aligned} u(G,x) &= \inf_{y>0} v(G,y) + yx \\ &= v(G,\widehat{y}(x)) + \widehat{y}(x)x \end{aligned}$$

$$\begin{aligned} v(G,y) &= \sup_{x\in dom(U)} u(G,x) - yx \\ &= u(G,\widehat{x}(y)) - y\widehat{x}(y) \end{aligned}$$

with the optimizers $x \mapsto \widehat{X}(G,x) \in \mathcal{K}(x)$ and $y \mapsto \widehat{Q}(G,y) \in \mathcal{M}^e$.

Definition (**marginal utility price**) The marginal utility price at G for a claim H is given by

$$q(G,x;H) = \mathbb{E}_{\widehat{Q}(G,\widehat{y}(x+p(G,x)))}(H),$$

where the dual optimizer is taken from the problem of selling G and receiving $p(G,x)$ - which is is the utility indifference price of G referring to utility function U and initial endowment $x \in dom(U)$, i.e.

$$p(G,x) = \inf\{w \text{ such that } u(G,x+w) = u(0,x)\}.$$

We have following relation:

$$\lim_{\delta\to 0} \frac{p(x,\delta G)}{\delta} = q(0,x;G),$$

or more generally

$$\frac{d}{d\delta}p(x,\delta G) = q(\delta G, x; G).$$

The reason why the marginal utility price is defined in this way is motivated by following question: which small units of a contingent claim f can be added to the agents optimal portfolio such that she is indifferent up to first order in expectation wheter to invest the optimal portfolio and add the claim or only invest the optimal portfolio.

$$\begin{aligned}
\frac{d}{d\epsilon}|_{\epsilon=0}\ \mathbb{E}[U(\widehat{X}(0,x)+\epsilon f)] &= \mathbb{E}[U'(\widehat{X}(0,x)+0)\ f] \\
&\overset{(5)}{=} \mathbb{E}[\widehat{y}(x)\frac{d\widehat{Q}(0,\widehat{y}(x))}{dP}\ f] \\
&= \widehat{y}(x)\mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}\,(f) \\
&= u'(0,x)\ \mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}\,(f).
\end{aligned}$$

Therefor the utility of first order in ϵ will not be changed by adding a contingent claim f, if f satisfies $\mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}(f) = 0$; thus the marginal utility price is given by $q(0,x;G)$, hence the appropiate pricing measure is the minimal distance martingale measure, where the distance is induced by the convex conjugate of the utility function.

The connection between the utility indifference price $p(G,x)$ and the marginal utility price allows to formulate the following equations. By definition of the utility indifference price , we know that

$$u(G, x + p(G,x)) = u(0,x)$$

so

$$\frac{d}{d\epsilon}u(\epsilon G, x + p(\epsilon G, x)) = 0$$

and therefore

$$\frac{d}{d\epsilon}\ |_{\epsilon=0}\ \ u(\epsilon G, x) + u'(0,x)\frac{d}{d\epsilon}|_{\epsilon=0}\ p(\epsilon G, x) = 0$$

Since

$$u(\epsilon G, x) = \sup_{X\in\mathcal{K}(x)} \mathbb{E}_P[U(X-\epsilon G)] = \mathbb{E}_P[U(\widehat{X}-\epsilon G)]$$

$$\text{so } \frac{d}{d\epsilon}u(\epsilon G, x) = \mathbb{E}_P[U'(\widehat{X}-\epsilon G)\ (-G)]$$

Considering this and $U'(\widehat{X}) = \widehat{y}\frac{d\widehat{Q}}{dP}$, it follows

$$\begin{aligned}
\frac{d}{d\epsilon}|_{\epsilon=0}\, u(\epsilon G, x) &= \mathbb{E}_P[U'(\widehat{X} - G)\ (-G)] \\
&= \mathbb{E}_P[\widehat{y}\frac{d\widehat{Q}}{dP}\ (-G)] \\
&= \widehat{y}(x)\mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}[(-G)] \\
&= u'(0,x)\ \mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}[(-G)] \\
&= -u'(0,x)\ \mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}[G]
\end{aligned}$$

Consequently

$$\begin{aligned}
-u'(0,x)\ \mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}[G] + u'(0,x)\ \frac{d}{d\epsilon}|_{\epsilon=0}\ p(\epsilon G, x) &= 0, \\
\text{so} \qquad u'(0,x)\ \left(\frac{d}{d\epsilon}|_{\epsilon=0}\ p(\epsilon G, x) - \mathbb{E}_{\widehat{Q}(0,\widehat{y}(x))}[G]\right) &= 0,
\end{aligned}$$

what clears up that

$$\frac{d}{d\epsilon}|_{\epsilon=0}\ p(\epsilon G, x) = \mathbb{E}_{\widehat{Q}(0,\hat{y}(x))}[G] = q(0, x; G)$$

Furthermore we can prove

$$\frac{d}{d\epsilon}\ p(\epsilon G, x) = q(\epsilon G, x; G)$$

since by same consideration

$$\frac{d}{d\epsilon_1}|_{\epsilon_1=\epsilon}\ u(\epsilon_1 G, x + p(\epsilon G, x)) + u'(\epsilon G, x + p(\epsilon G, x))\frac{d}{d\epsilon}\ p(\epsilon G, x) = 0$$

leads to

$$\begin{aligned}
&\frac{d}{d\epsilon_1}|_{\epsilon_1=\epsilon}\ u(\epsilon_1 G, x + p(\epsilon G, x)) = \\
&= u'(\epsilon G, x + p(\epsilon G, x))\mathbb{E}_{\widehat{Q}(\hat{y}(\epsilon G, x + p(\epsilon G, x)))}(\frac{d}{d\epsilon}\widehat{X}(\epsilon G, x + p(\epsilon G, x)) - G) \\
&= -u'(\epsilon G, x + p(\epsilon G, x))\mathbb{E}_{\widehat{Q}(\hat{y}(\epsilon G, x + p(\epsilon G, x)))}[G]
\end{aligned}$$

Theorem 6.1. *Let U_n be a sequence of utility functions satisfying the conditions of A.2. Finally we have following relations for the indifference prices and the marginal utility prices at G:*

(i) The utility indifference price $\epsilon \to p_n(\epsilon G, x)$ is convex and increasing.

(ii) The derivative of the sequence of indifference prices with respect to ϵ is given by

$$\frac{d}{d\epsilon} p_n(\epsilon G, x) = q_n(x, \epsilon G; G)$$

for $n \geq 1$ and $\epsilon \geq 0$

(iii) The limit of the utility indifference price is the superreplication price. (see 5 and Appendix)

Therefore we can conclude that

$$\epsilon q(0, x; G) \leq p(\epsilon G, x) \leq \epsilon \Pi(G)$$

as the first $\leq$ follows from the fact that the utility indifference price is convex function and therefore its derivative (which is the marginal utility price) always lies below and the second $\leq$ is the main result of M. Rásonyi and L. Carassus in [2]. Since the marginal utility price is defined as the derivative of the convex utility indifference price, where from we have already stated that it converges to the superreplication price, it follows that the marginal utility price has to converge, too.

The next chapter shows two examples which also illustrate this property.

7 Examples

TASK 1

We consider a one-step model for a financial market with bond B and stock S. The bond is assumed to be constant, i.e. $B \equiv 1$, the stock S is assumed to move according to following tree:

$$\begin{tikzcd}
 & S_1 = 4 \\
S_0 = 2 \arrow[ur] \arrow[r] \arrow[dr] & S_1 = 2 \\
 & S_1 = 1
\end{tikzcd}$$

Assume furthermore that

$$\mathbb{P}(S_1 = 4) = \frac{1}{2} \text{ and } \mathbb{P}(S_1 = 2) = \mathbb{P}(S_1 = 1) = \frac{1}{4}$$

We define the exponential utility utility function U by

$$U_n(x) := -e^{-\gamma_n x}, x \in \mathbb{R}.$$

Exponential utility has the feature that the wealth or initial endowment of the investor has no impact on the problem which makes the mathematics tractable but is also a strong assumption as different investors with varying initial wealths are unlikely to assign the same value to a claim. Consider the utility maximization problem, where G denotes a european call with strike price $K = 2$.

$$u(\epsilon G, x+p) := \sup_{Y \in K} \mathbb{E}[U(x + Y - \epsilon G)],$$

Obviously the risk aversion R of the utility function U is given by γ.
The set K is given by:

$$K = \left\{ \begin{pmatrix} 2a \\ 0 \\ -a \end{pmatrix}, \ a \in \mathbb{R} \right\}$$

Now we can compute the value function

$$\begin{aligned}
u(\epsilon G, x+p) &= \sup_{Y \in K} \mathbb{E}[U(x + Y - G)] \\
&= \sup_{a \in \mathbb{R}} \mathbb{E}\left(U \begin{pmatrix} x+p+2a-2\epsilon \\ x+p \\ x+p-a \end{pmatrix} \right) \\
&= \sup_{a \in \mathbb{R}} \mathbb{E} \begin{pmatrix} e^{-\gamma(x+p+2a-2\epsilon)} \\ e^{-\gamma(x+p)} \\ e^{-\gamma(x+p-a)} \end{pmatrix} \\
&= -e^{-\gamma(x+p)} \inf_{a \in \mathbb{R}} \left(\underbrace{\frac{1}{2} e^{-2\gamma a + 2\gamma\epsilon} + \frac{1}{4} + \frac{1}{4} e^{\gamma a}}_{=: f(a)} \right) \\
&= -e^{-\gamma(x+p)} \left(\frac{1}{2} e^{-2\gamma \widehat{a} + 2\gamma\epsilon} + \frac{1}{4} + \frac{1}{4} e^{\gamma \widehat{a}} \right).
\end{aligned}$$

The optimizer $\widehat{a}$ is the solution of $f'(a) \overset{!}{=} 0$

$$\begin{aligned}
f'(\widehat{a}) &= \frac{1}{2}(-2\gamma) e^{-2\gamma\widehat{a} + 2\gamma\epsilon} + \frac{1}{4}\gamma e^{\gamma\widehat{a}} = 0 \\
&\Leftrightarrow \ \widehat{a} = \frac{\ln(4) + 2\gamma\epsilon}{3\gamma}.
\end{aligned}$$

This leads to

$$u(\epsilon G, x+p) \quad = \quad -e^{-\gamma(x+p)} \left(\frac{1}{2} e^{-\frac{2\ \ln(4)}{3} + \frac{2\gamma\epsilon}{3}} + \frac{1}{4} + \frac{1}{4} e^{\frac{\ln(4)}{3}} \right).$$

Hence we have that

$$\begin{aligned} u(0,x) &= \sup_{Y \in K} \mathbb{E}[u(x+Y)] \\ &= \sup_{a \in \mathbb{R}} \mathbb{E} \left(u \begin{pmatrix} x+2a \\ x \\ x-a \end{pmatrix} \right) \\ &= \sup_{a \in \mathbb{R}} \mathbb{E} \begin{pmatrix} -e^{-\gamma(x+2a)} \\ -e^{-\gamma x} \\ -e^{-\gamma(x-a)} \end{pmatrix} \\ &= -e^{-\gamma x} \inf_{a \in \mathbb{R}} \left(\underbrace{\frac{1}{2} e^{-2\gamma a} + \frac{1}{4} + \frac{1}{4} e^{\gamma a}}_{=: f(a)} \right). \end{aligned}$$

The optimizer is $\widehat{a} = \frac{\ln(4)}{3\gamma}$ and leads to:

$$u(0,x) \quad = \quad -e^{-\gamma x} \left(\frac{1}{2} e^{-\frac{2\ \ln(4)}{3}} + \frac{1}{4} + \frac{1}{4} e^{\frac{\ln(4)}{3}} \right)$$

Equating $u(0,x) = u(\epsilon G, x+p)$ following equation will be obtained:

$$\underbrace{2e^{-\frac{2\ \ln(4)}{3}} + 1 + e^{\frac{\ln(4)}{3}}}_{A} = e^{-\gamma p} \ \underbrace{(2e^{-\frac{2\ \ln(4)}{3}} + 1 + e^{\frac{\ln(4)}{3}} e^{\frac{2\gamma\epsilon}{3}} \ )}_{B}$$

$$\Rightarrow p = \frac{\ln(A) - \ \ln(B)}{-\gamma}.$$

For the risk aversion $\gamma \to \infty$ the utility price $p = p(G,x)$ converges to the superreplication price:

$$-\frac{\ln(A)}{\gamma} \longrightarrow 0 \text{ for } \gamma \to \infty$$

$$\begin{aligned} \frac{\ln(B)}{\gamma} &\geq \frac{1}{\gamma} \ln \Big(e^{\frac{2\gamma\epsilon}{3}} \ [2e^{\frac{-2\ \ln(4)}{3}} + e^{\frac{\ln(4)}{3}}] \Big) \\ &= \frac{1}{\gamma} \ln(e^{\frac{2\gamma}{3}}) + \underbrace{\ln(2e^{\frac{-2\ \ln(4)}{3}} + e^{\frac{\ln(4)}{3}})}_{\longrightarrow 0 \text{ for } \gamma \to \infty} \to \frac{2}{3} \end{aligned}$$

$$\frac{\ln(B)}{\gamma} \quad \leq \quad \frac{1}{\gamma}\ \ln\Big(e^{\frac{2\gamma\epsilon}{3}}\ [1 + 2e^{\frac{-2\ \ln(4)}{3}} + e^{\frac{\ln(4)}{3}}]\ \Big) \to \frac{2}{3}$$

So p converges to the superreplication price $\frac{2}{3} = 2\frac{1}{3} + 0 + 0\frac{2}{3}$, since the set of equivalent martingale measures is given by

$$Q = \left\{ t \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix} + (1-t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ t \in]0,1[\right\}$$

and the superreplication price is defined as

$$\Pi(X) = \sup_{Q} \mathbb{E}_Q(X)$$

We can now illustrate the fact that the utility indifference price converges to the superreplication as risk aversion tends to infinity. It is also shown that the utility indifference price is close to the marginal utility price for small ϵ, since the marginal utility price is defined as the derivative of the utility indifference at $\epsilon = 0$

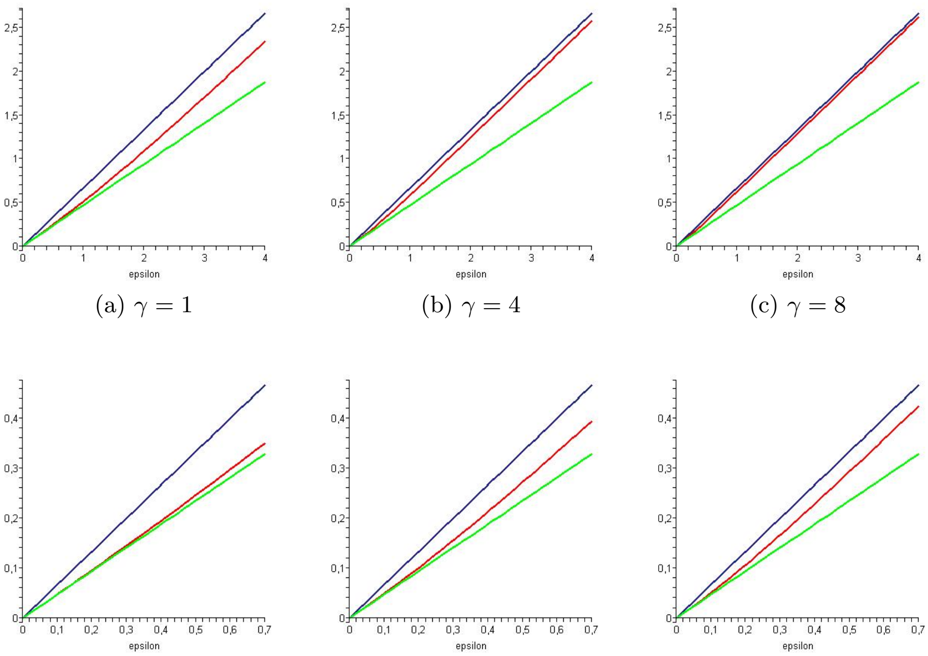


Figure 1: superreplication, utility indifference and marginal utility price

TASK 2

We still consider a one-step model for a financial market with constant bond B and stock S which is assumed to move according to following tree

$$\begin{tikzcd}[row sep=small] & S_1 = 3 \\ S_0 = 2 \arrow[ur] \arrow[r] \arrow[dr] & S_1 = 2 \\ & S_1 = 1 \end{tikzcd}$$

and assume furthermore that

$$\mathbb{P}(S_1 = 3) = \frac{1}{5}\ ,\ \mathbb{P}(S_1 = 2) = \frac{11}{20} \text{ and } \mathbb{P}(S_1 = 1) = \frac{1}{4}$$

The set K is given by

$$K = \left\{ \left(\begin{array}{r} a \\ 0 \\ -a \end{array} \right),\ a \in \mathbb{R} \right\}$$

and we will take a look at a european call option

$$G = \left\{ \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \right\}$$

The utility function U is no longer exponential

$$U(x) := -\frac{1}{2}\, e^{-\gamma x} - \frac{1}{2}\, e^{-2\,\gamma x}$$

, i.e. we can not factor out the inital wealth anymore which makes the calculation of $u(\epsilon G, x+p)$ and $u(0,x)$ difficult.
First we need the optimizer of the value function

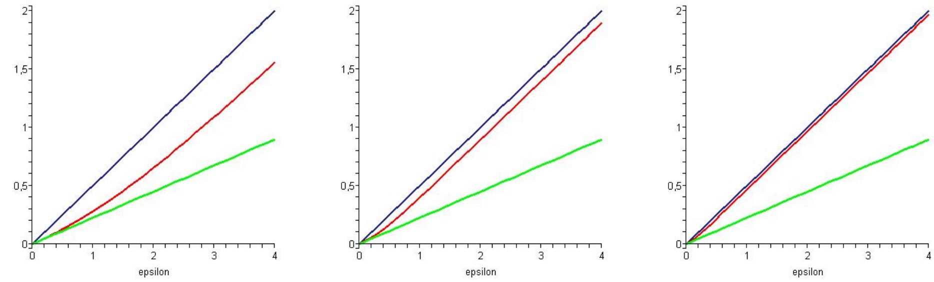
$$\begin{aligned} u(\epsilon G, x+p) \quad = \sup\nolimits_{a \in \mathbb{R}} \quad & -1/10\, e^{-\gamma(x+p+a-\epsilon G\,\gamma)} - 1/10\, e^{-2\,\gamma(x+p+a-\epsilon G\,\gamma)} \\ & -\frac{11}{40}\, e^{-\gamma(x+p)} - \frac{11}{40}\, e^{-2\,\gamma(x+p)} \\ & -1/8\, e^{-\gamma(x+p-a)} - 1/8\, e^{-2\,\gamma(x+p-a)} \end{aligned}$$

Hence we want to find an a which satisfies

$$
\begin{aligned}
\frac{d}{da}u(\epsilon G, x+p) &= 1/10\,\gamma e^{-\gamma(x+p+a-\epsilon\, G)} + 1/5\,\gamma e^{-2\,\gamma(x+p+a-\epsilon\, G)} \\
&\quad -1/8\,\gamma e^{-\gamma(x+p-a)} - 1/4\,\gamma e^{-2\,\gamma(x+p-a)} = 0
\end{aligned}
$$

With the help of maple we get a result which does not only depend on ϵG and γ but also on x and p. Solving the same problem for $p = 0$ and $\epsilon = 0$ leads to the optimizer for the value function $u(0, x)$. Approximatively $\widehat{a} = \widehat{a}_0 + \frac{\epsilon}{2}$ holds, where $\widehat{a}_0$ denotes the optimizer of $u(0, x)$ and $\widehat{a}$ the optimizer of $u(\epsilon G, x + p)$ respectively. By this fact we can calculate the utility indifference price. In order to get the marginal utility price, we only have to build the first derivative with respect to ϵ and then equal ϵ to zero.

Similar to the first example the following figures reveal the convexity of the utility indifference price, whereas the marginal utility price and superreplication price are linear in ϵ and show the behaviour of the utility indifference price for small ϵ.



(a) $\gamma = 1$ (b) $\gamma = 4$ (c) $\gamma = 12$

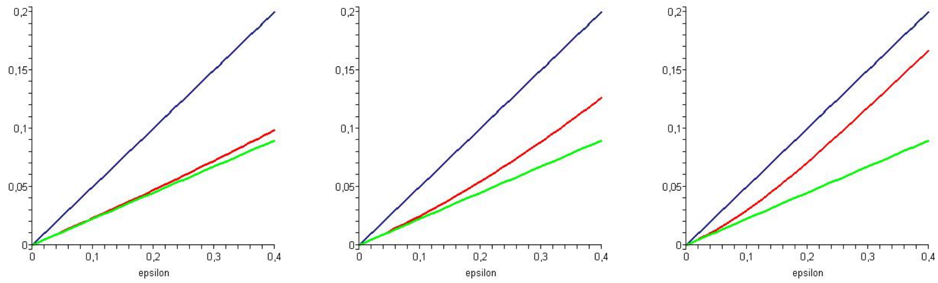


Figure 2: superreplication, utility indifference and marginal utility price

We receive following values for $x = -0.5$ and $\gamma = 12$:

ϵ	$p(\epsilon G, x)$	$\epsilon\ \Pi_{\uparrow}$
0.2	0.0709801424	0.1
0.4	0.1669791934	0.2
0.6	0.2665928022	0.3
0.8	0.3665564405	0.4
1	0.4665527989	0.5
2	0.9665522742	1.
3	1.466552274	1.5
4	1.966552273	2.
5	2.466552273	2.5

In order to get some more information about this value function, we also take a look at the dual problem.
First we determine the convex conjugate function of $U(x)$ which is given by

$$V(y) = \sup_{x \in \mathrm{dom}(u)} U(x) - xy$$

For our utility function $-1/2\, e^{-\gamma x} - 1/2\, e^{-2\,\gamma x}$ we get

$$\begin{aligned} V(y) &= \sup_{x \in \mathrm{dom}(u)} U(x) - xy \\ &= U(\widehat{x}) - y\widehat{x} \end{aligned}$$

where $\widehat{x}$ is the solution to

$$U'(x) = 1/2\,\gamma e^{-\gamma x} + \gamma e^{-2\,\gamma x} = y$$

Hence we get

$$V(y) = -\frac{-\gamma + \sqrt{\gamma\,(\gamma + 16\,y)} + 8\,y + 32\,y\ln(2) - 16\,y\ln\left(\frac{-\gamma+\sqrt{\gamma(\gamma+16\,y)}}{\gamma}\right)}{16\gamma}$$

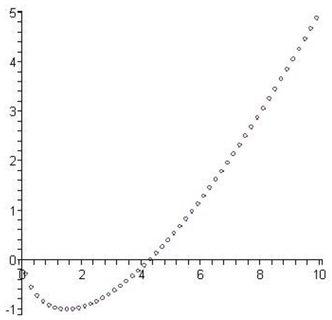
Its value function $v(y)$ is given by

$$\begin{aligned} v(\epsilon G, y) &= \inf_{Q \in \mathcal{M}^a(S)} \mathbb{E}_P \left[V(y\frac{dQ}{dP}) - y\frac{dQ}{dP}\epsilon G \right] \\ &= \inf_{t \in [0,1]} \mathbb{E}_P \left[V(y\frac{dQ}{dP}) - y\frac{dQ}{dP}\epsilon G \right] \\ &= \inf_{t \in [0,1]} \frac{1}{5}\, V\left(\frac{5}{2}\,yt\right) - \frac{1}{2}\,\epsilon\, 1yt + \frac{11}{20}\, V\left(\frac{20}{11}\,y\,(1-t)\right) + \frac{1}{4}\, V\,(2\,yt) \end{aligned}$$

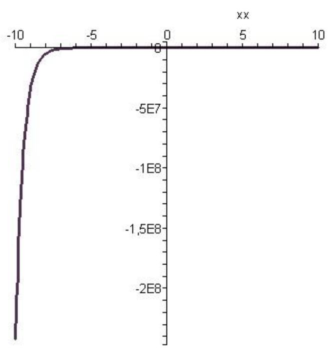
Unfortunately maple is not able to evaluate t, which depends on y. Therefore we determine the needed values by defining an iteration which solves the problem for several y whereby γ and ϵ are given. As an example let $\gamma = 1$ and $\epsilon = 0$. Then we calculate the value function $v(0, y)$for several y and can draw the function pointwise:

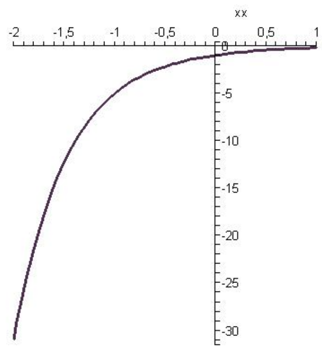
```
> restart; with(plots):
Warning, the name changecoords has been redefined
> V:=y-> -1/16*(-g+(g*(g+16*y))^(1/2)+8*y+32*y*ln(2)
 -16*y*ln((-g+(g*(g+16*y))^(1/2))/g))/g:
> q1:=1/2: q3:=1-q1: p1:=1/5: p3:=1/4: p2:=1-p1-p3: G:=1:
> A:=p1*V(y*q1*t/p1)-epsilon*G*y*q1*t:
> B:=p2*V( y*(1-t)/p2):
> C:=p3*V(y*q3* t/p3):
> vy:=y->simplify(A+B+C):
> dvyt:=diff(vy(y),t):
> tglg:=dvyt=0:
> dvyy:=diff(vy(y),y):
> yglg:=dvyy=-x:
> epsilon:=0: g:=1: yBegin:=0.1: yEnd:=10: step:=0.2:
> for j from yBegin to yEnd by step do
> y:=j;
> t[y]:=fsolve(tglg,t);
> t:=t[y];
> v[j]:=vy(y);
> x[y]:=solve(yglg,x);
> eq:=y=k*exp(-g*x[y]) + 2*k*exp(-2*g*x[y]);
> c:=fsolve(eq,k);
> yhut:=c*exp(-g*xx) + 2*c*exp(-2*g*xx);
> y:=yhut;
> ueps[j]:=vy(yhut)+yhut*xx;
> t:='t';
> print("****");
> end do:
> eq:=yEnd=yBegin + step*r: sol:=solve(eq,r):
iEnd:=floor(sol):
> yValues:=[seq(yBegin + step*i, i=0..iEnd)]:
> vyValues:=[seq(v[yBegin+step*i],i=0..iEnd)]:
> Approximation[g,epsilon]:=zip((yps,vyps)->[yps,vyps], yValues,vyValues):
> colorlist:=[violet, red, gold, green, navy];
> col:=g+epsilon;
> vyPoints[g,epsilon]:=plot(Approximation[g,epsilon],style = point,
color=colorlist[col]):
> display(vyPoints[g,epsilon],text[g,epsilon]);

> m:=min(seq(v[yBegin+step*i],i=0..iEnd));
                       m := -0.997489024
> m:=min(seq(v[yBegin+step*i],i=0..iEnd)):
> for h from 1 to iEnd do
> k:=h:
```

```
> if (vyValues[k]=m) then yx:=yValues[k]; print(‘Minimum‘); print(m);
> print(‘achieved at y‘);print(yx); print(yValues[k]) else h:=h end if;
> end do:
>
                              Minimum
                            -0.997489024
                           achieved at y
                                1.5
                                1.5

> u[g, epsilon]:=ueps[yx];
> uf[g,epsilon]:=plot(u[g,epsilon],xx =-2..1, color=colorlist[col],
     thickness=2):
> display(uf[g,epsilon]);
```



(a) $v(\epsilon G, y)$





(b) $u(\epsilon G, x)$ (c) detail of $u(\epsilon G, x)$

Figure 3: $\gamma = 1$, $\epsilon = 0$

We see and have calculated that the minimum of $v(\epsilon G, y)$ is obtained at $y = 1.5$. As $v'(y) = -x$ holds, we can calculate $\widehat{y}(x)$ which is needed to determine the value function $u(\epsilon G, x)$:

$$
\begin{aligned}
u(\epsilon G, x) &= \inf_{y>0} v(\epsilon G, y) + xy \\
&= v(\epsilon G, \widehat{y}(x)) + x\widehat{y}(x)
\end{aligned}
$$

Now we can apply the same procedure for $\epsilon = 1$ and compare these results with those of $\epsilon = 0$:

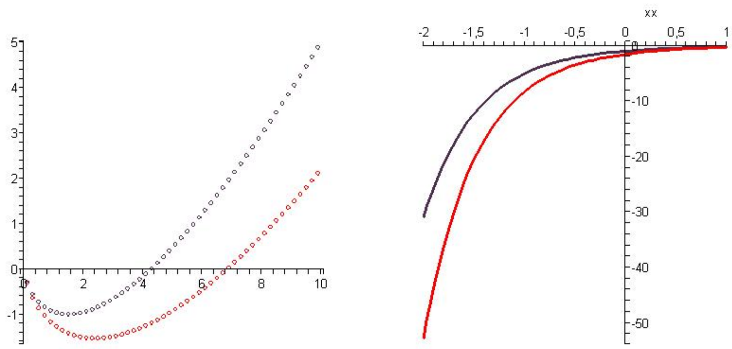
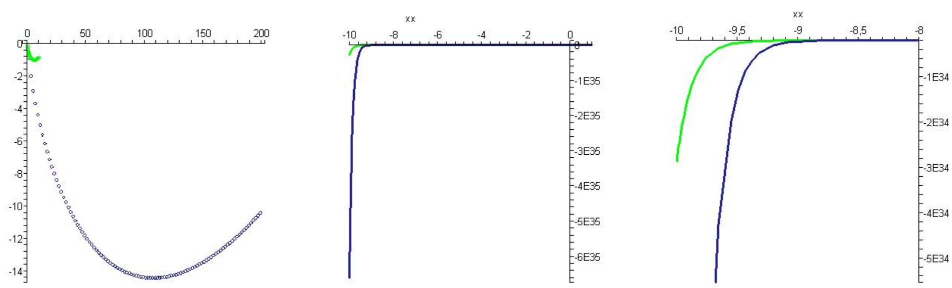


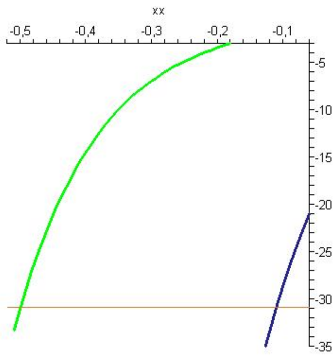
Figure 4: value functions with $\epsilon = 0$ and $\epsilon = 1$ for $\gamma = 1$

For $\gamma = 1$ and $\epsilon = 1$ the minimum is achieved at $y = 2.5$ whereas for larger γ, for example $\gamma = 4$, we see that the number ϵ of claims has enormous effects on the functions:



value functions $v(\epsilon G, y)$ and $u(\epsilon G, x)$ for $\gamma = 4$

To be more precisely, we also view a more detailed picture of the last figure, which allows us to see that the utility indifference price $p(\epsilon G, x)$ for $\epsilon = 1$, $\gamma = 4$ and inital wealth $x = -0.5$ is $p(1, -0.5) = -.1010576242$



which is the result we can get from solving the primal problem. Consequently we have also been able to price an asset, where the price depends on the initial wealth, which is not the case if the utility function is exponential.

A APPENDIX

Before we show the convergence of the utility indifference price to the superreplication price $\Pi(G)$, we have to formulate some definitions and four lemmas which are denoted by (A) - (D).

A.1 Definitions

Following sets have to be introduced in order to formulate a strenghtened (NA) condition which we will need now.
Denote the set of $\mathcal{F}_t$-measurable d-dimensional random variables by Ξ_t and let $D_t(\omega)$ characterise the smallest affine hyperplane that contains the support of the conditional distribution of ΔS_t with respect to $\mathcal{F}_{t-1}$. There are no redundant assets, if $D_t = \mathbb{R}^d$. Elsewise one can always replace $\phi_t \in \Xi_{t-1}$ (regarding that ϕ is a predictable process, i.e. ϕ_t is $\mathcal{F}_{t-1}$-measurable) by its orthogonal projection $\hat{\phi}$ on D_t without changing the portfolios value, since

$$\langle \phi_t, \Delta S_t \rangle = \langle \hat{\phi}_t, \Delta S_t \rangle \text{ a.s.}$$

Define

$$\tilde{\Xi}_t := \{\xi \in \Xi_t : \xi \in D_{t+1} \text{ a.s., } |\xi| = 1 \text{ on } \{D_{t+1} \neq \{0\}\}\}$$

Lemma A.1. *(NA) holds iff there exist $\mathcal{F}_t$-measurable random variables $\beta_t > 0$, $0 \leq t \leq T-1$ such that*

$$ess. \inf_{\xi \in \tilde{\Xi}_t} \mathbb{P}(\langle \xi, \Delta S_{t+1} \rangle < -\beta_t | \mathcal{F}_t) > 0 \ a.s. \ on \ \{D_{t+1} \neq \{0\}\}$$

Proof Assume $D_{t+1} \neq \{0\}$ a.s. For t fixed and a sequence $\delta_n \searrow 0$ define the set

$$A_n := \left\{ \omega : ess \inf_{\xi \in \tilde{\Xi}_t} \mathbb{P}(\langle \xi, \Delta S_{t+1} \rangle < -\delta_n | \mathcal{F}_t) = 0 \right\}$$

Let the essential infimum be achieved by some $\xi_n^* \in \widetilde{\Xi}_t$. Indeed, take $\xi_n^k \in \widetilde{\Xi}_t$ such that

$$\lim_k \mathbb{P}(\left\langle \xi_n^k, \Delta S_{t+1} \right\rangle < -\delta_n | \mathcal{F}_t) = ess \inf_{\xi \in \widetilde{\Xi}_t} \mathbb{P}(\langle \xi, \Delta S_{t+1} \rangle < -\delta_n | \mathcal{F}_t),$$

applying Lemma 1.4 one gets a random subsequence $\widetilde{\xi}_n^k$ that converges to some ξ_n^*. Now define

$$G_k := \{\left\langle \widetilde{\xi}_n^k, \Delta S_{t+1} \right\rangle < -\delta_n\}, \qquad G := \{\langle \xi_n^*, \Delta S_{t+1} \rangle < -\delta_n\},$$

and ensure that $G \subset \liminf_k G_k$ and therefore $\liminf_k \mathbf{1}_{G_k}(\omega) = \mathbf{1}_{\liminf_k G_k}(\omega)$.
The Fatou Lemma guarantees that

$$\mathbb{P}(\langle \xi_n^*, \Delta S_{t+1} \rangle < -\delta_n | \mathcal{F}_t) \leq \lim_k \mathbb{P}(\left\langle \widetilde{\xi}_n^k, \Delta S_{t+1} \right\rangle < -\delta_n | \mathcal{F}_t),$$

consequently ξ_n^* attains the essential infimum.
Since $A_{n+1} \subset A_n$, define

$$A := \bigcap_{n=1}^{\infty} A_n$$

and we intend to show that $\mathbb{P}(A) = 0$, otherwise one would have a random subsequence $\widetilde{\xi}_n^*$ of ξ_n^* converging to some $\widetilde{\xi}$. A Fatou lemma as above shows

$$\mathbb{P}(\left\langle \widetilde{\xi}, \Delta S_{t+1} \right\rangle < 0 | \mathcal{F}_t) \leq \liminf_{n \to \infty} \mathbb{P}(\left\langle \widetilde{\xi}_n^*, \Delta S_{t+1} \right\rangle < -\delta_n | \mathcal{F}_t) = 0$$

on A, so, necessarily

$$\mathbb{P}(\left\langle \widetilde{\xi} 1_A, \Delta S_{t+1} \right\rangle \geq 0 | \mathcal{F}_t) = 1$$

hence, (NA) implies that

$$\mathbb{P}(\left\langle \widetilde{\xi} 1_A, \Delta S_{t+1} \right\rangle = 0 | \mathcal{F}_t) = 1$$

which contradicts $\widetilde{\xi} \in D_{t+1}$ and therefore $\mathbb{P}(A) = 0$ must hold.
Define

$$\beta_t := \sum_{n=1}^{\infty} \delta_n 1_{A_n^C / A_{n-1}^C} \qquad \text{with } A_0^C := \emptyset$$

Since $\mathbb{P}(A) = 0$ this is an almost everywhere positive function, hence

$$\mathbb{P}(\langle \xi, \Delta S_{t+1} \rangle < -\beta_t | \mathcal{F}_t) > 0 \; a.s. \qquad \forall p \in \widetilde{\Xi}_t$$

The following assumption is needed in order to derive bounds on trading strategies - otherwise it may happen that the supremum of expected utility is ∞.

Assumption (strengthened (NA)-condition)
There exists a constant $\beta > 0$ such that for $0 \leq t \leq T-1$

$$\operatorname{ess.}\inf_{\xi \in \tilde{\Xi}_t} \mathbb{P}(\langle \xi, \Delta S_{t+1} \rangle < -\beta_t | \mathcal{F}_t) > 0 \text{ a.s. on } \{D_{t+1} \neq \{0\}\} \tag{17}$$

Remark A.2. Assume $D_t = \mathbb{R}^d$ for simplicity and let $C_t(\omega)$ denote the closed convex hull of the support of the conditional distribution of ΔS_t with respect to $\mathcal{F}_{t-1}$. The "uniform no arbitrage" condition in Schäl(2000) asserts that there should exist a ball of fixed, deterministic radius around the origin which is contained in C_t a.s. Clearly, this holds iff each halfspace whose bordering hyperplane is closer to the origin than some fixed constant contains some point of $C_t(\omega)$, which is the previous Assumption.

Lemma A.3. ***(A)*** *Let $x > \Pi(G)$. Suppose that S is bounded and the strengthened (NA)-Condition 17 holds. Take any strategy $\phi \in \mathcal{A}(G,x)$ that satisfies $\phi_t \in D_t$, $1 \le t \le T$*
Then increasing functions $M_t(x) \ge 0$- note that they do not depend on the particular choice of the strategy ϕ - exist, such that

$$V_t^{x,\phi} \le M_t(x)$$

Proof For $t = 0$ take $M_0(x) := x$. Now define the sets

$$A := \left\{ |\phi_t| > \frac{V_{t-1}^{x,\phi}}{\beta} \right\} \in \mathcal{F}_{t-1}, \qquad B := \left\{ \left(\frac{\phi_t}{|\phi_t|} \cdot S_t \right) > -\beta \right\}$$

Then $A \cap B \subset \{V_t^{x,\phi} < 0\}$ and

$$\mathbb{P}(A \cap B) = \mathbb{E}\left[\mathbb{E}\left[I_{A\cap B}|\mathcal{F}_{t-1}\right]\right] = \mathbb{E}\left[I_A\left[\mathbb{E}(I_B|\mathcal{F}_{t-1})\right]\right]$$

Since we assumed that (NA) holds, $\mathbb{P}(B|\mathcal{F}_{t-1}) > 0$. If $\mathbb{P}(A) > 0$, it would follow that $\mathbb{P}(V_t^{x,\phi} < 0) > 0$. But as $V_T^{x,\phi} \ge G \ge 0$ a.s., the (NA)-condition implies $V_t^{x,\phi} \ge 0$ a.s. for all t. This contradiction shows that

$$|\phi_t| \le \frac{V_{t-1}^{x,\phi}}{\beta}$$

Thus by induction hypothesis

$$V_t^{x,\phi} \le M_{t-1}(x) + \|\Delta S_t\|_\infty M_{t-1}(x)/\beta =: M_t(x)$$

which defines a suitable $M_t(x)$

Lemma A.4. ***(B)*** *Let $x > \Pi(G)$. Suppose that S is bounded and Assumption 1, the strengthened (NA)-Condition, holds. Then $u_n(G,x)$ is well-defined, finite and*

$$u_n(G,x) = \sup_{\phi\in\mathcal{A}(G,x),\phi_t\in D_t} \mathbb{E}U_n(V_T^{x,\phi} - G)$$

Proof Take some strategy $\tilde{\phi} \in \mathcal{A}(G,x)$ which fullfills $V_T^{x,\tilde{\phi}} \geq G + \epsilon$ with $\epsilon > 0$. Then $V_T^{x,\tilde{\phi}} - G \geq \epsilon$ and therefore by definition of $u_n(G,x)$

$$u_n(G,x) \geq U_n(\epsilon) > -\infty$$

With the help of Lemma A we can show $u_n(G,x) < \infty$: $V_T^{x,\phi} \geq G$ holds, since $\phi \in \mathcal{A}(G,x)$. Therefore we can state that $U_n(V_T^{x,\phi} - G) \leq U_n(M_T(x))$.

Lemma A.5. ***(C)*** *Let $B \in L^0$ such that $B \notin \mathcal{K}(z) - L^0_+$. Then there exists $\epsilon > 0$ such that*

$$\inf_{\theta \in \mathcal{K}(z)} \mathbb{P}(\theta \leq B - \epsilon) \geq \epsilon$$

Proof At first it is to mention that under (NA) the set $\mathcal{K}(z) - L^0_+$ is closed in probability. Suppose that there exist $\theta_n \in \mathcal{K}(z)$ such that the statement is false, i.e.

$$\mathbb{P}(\theta_n \leq B - 1/n) \leq 1/n$$

Then it would follow for the set $\kappa_n := [\theta_n - (B - 1/n)] I_{\theta_n \geq (B-1/n)} \in L^0_+$ that

$$\mathbb{P}(\theta_n - \kappa_n = B - 1/n) \geq 1 - 1/n$$

This would mean that $\theta_n - \kappa_n \to B$ in probability, hence $B \in \overline{\mathcal{K}(z) - L^0_+} = \mathcal{K}(z) - L^0_+$ is a contradiction to our assumption.

Lemma A.6. ***(D)*** *Suppose that U_n, $n \in \mathbb{N}$ satisfy the assumption of risk aversion.*

$$r_n(x) = -\frac{U_n''(x)}{U_n'(x)} \to \infty, n \to \infty$$

as well as

$$\forall n \in \mathbb{N} \qquad U_n(x) = 0, \quad U_n^{'}(x) = 1.$$

Then, for $n \to \infty$

$$\begin{aligned} U_n(y) &\to -\infty \quad \forall 0 < y < x \\ U_n(y) &\to 0 \qquad \forall y \geq x \end{aligned}$$

Proof At first we treat the case $y < x$: Since $U_n^{'}(x)$ is a decreasing function and $U_n^{'}(x) = 1$,

$$U_n^{'}(u) \geq U_n^{'}(x) = 1 \qquad \text{for } u \leq x$$

$U_n^{'}(u) \geq 1$ effects the risk aversion

$$r_n(u) = -\frac{U_n''(u)}{U_n'(u)} \leq -U_n''(u)$$

Then

$$\begin{aligned} U_n^{'}(y) &= U_n^{'}(x) - \int_y^x U_n^{''}(u)du \\ &\geq 1 + \int_y^x r_n(u)du \to \infty \end{aligned}$$

Making use of the fact that $U_n^{'}(y) \to \infty$ and $U_n(x) = 0$, it follows

$$U_n^{'}(y) = U_n^{'}(x) - \int_y^x U_n^{'}(u)du \to -\infty \quad \text{q.e.d.}$$

Now take any $y > x$. We claim that $U_n^{'}(y) \to 0$. Suppose this statement was wrong for a subsequence n_k, i.e.

$$U_{n_k}^{'}(y) \geq \alpha > 0$$

Again we can make use of the monotonicity of U: Because $U_n^{'}(y)$ is a decreasing function, it follows that $U_n^{'}(u) > U_n^{'}(y)$ holds for all $u \leq y$. Now the assumption of risk aversion $r_n(u) = -\frac{U_n^{''}(u)}{U_n^{'}(u)} \to \infty$ implies that $U_{n_k}^{''}(u) \to \infty$ for $k \to \infty$, $u \leq y$. Then

$$0 \leq U_{n_k}^{'}(y) = U_{n_k}^{'}(x) + \int_y^x U_{n_k}^{''}(u)du = 1 + \int_y^x U_{n_k}^{''}(u)du \to -\infty$$

is a contradiction. Combining the facts $U_n^{'}(y) \to 0$ and $U_n(x) = 0$

A.2 Convergence of utility indifference prices to the superreplication price

We will now make use of the Lemmas (A) - (D). Lemma (A) and (B) ensure that under the assumptions made before, the utility maximization problem is well-defined and that the value processes $V_t(\phi)$ are uniformly bounded. Since the superreplication price can be considered as the utility indifference price for the function
$U_\infty := -\infty$, $y < x$,
$U_\infty := 0$, $y \geq x$ Lemma (D) reduces the problem to the case where $U_n \to U_\infty$ and U_∞ is given as just now introduced. With the help of Lemma (C) and the uniform bounds on strategies (Lemma B) it is possible to show that $u_n(G, y) \to -\infty$ for $\Pi(G) < y < \Pi(G) + x$. Proof of Theorem: Fix $x > 0$. It will neither have an effect on the assumption of risk aversion, nor change the utility indifference price , if each U_n is replaced by $\alpha_n U_n + \beta_n$ for $\alpha_n > 0$, $\beta_n \in \mathbb{R}$. Now, by choosing $\alpha_n := \frac{1}{U_n^{'}(x)}$ and $\beta_n := -\frac{U_n(x)}{U_n^{'}(x)}$, it can be assumed that

$$U_n(x) = 0, \quad U_n^{'}(x) = 1$$

for all $n \in \mathbb{N}$. Now fix $\Pi(G) < y < x + \Pi(G)$. Then - by definition of the superreplication price -

$$x + G \notin \mathcal{K}(y) - L^0_+$$

This gives the opportunity to use Lemma C with $B := x + G$ and $y = z$. Considering that the increasing function $M_T(.) \geq 0$ as defined in Lemma A, is independent of the choice of the strategy ϕ, it is possible to choose the strategy ϕ uniformly for all $\phi \in \mathcal{A}(G, y)$ such that $\phi_t \in D_t$ for all t. For such a ϕ, define the set

$$A_\phi := \{\omega \in \Omega : V_T^{y,\phi}(\omega) \leq x + G(\omega) - \epsilon\}$$

and by Lemma C it follows that $\mathbb{P}(A_\phi) \geq \epsilon$. Finally we get

$$\begin{aligned}
u_n(G, y) &= \mathbb{E}U_n(V_T^{y,\phi} - G) \\
&\leq \mathbb{E}I_{A_\phi} U_n(x - \epsilon) + \mathbb{E}I_{A_\phi^C} U_n(M_T(y)) \\
&\leq \mathbb{P}(A_\phi) U_n(x - \epsilon) + \mathbb{P}(A_\phi^C) U_n(M_T(y) + x) \\
&\leq \epsilon U_n(x - \epsilon) + U_n(M_T(y) + x).
\end{aligned}$$

So $u_n(G, y) \leq \epsilon U_n(x - \epsilon) + U_n(M_T(y) + x) \to -\infty$
And by definition of $u_n(0, x) = \mathbb{E}U_n(x + (\phi^* \cdot S)$

$$\liminf_{n\to\infty} u_n(0, x) \geq \liminf_{n\to\infty} U_n(x) = 0$$

Now it will be shown that

$$p_n(G, x) \leq \Pi(G)$$

Taking a strategy $\hat{\phi}(\delta) \in \mathcal{A}(G, x + \Pi(G) + \delta$ such that

$$\Pi(G) + \delta + (\hat{\phi} \cdot S)_T \geq G$$

and considering the fact that U_n is a non-decreasing function, it can be shown that $u_n(0, x) \leq u_n(G, x + \Pi(G) + \delta)$:

$$\begin{aligned}
u_n(0, x) &= \sup_{\phi \in \mathcal{A}(0,x)} \mathbb{E}U_n(x + (\phi \cdot S)_T) \\
&\leq \sup_{\phi \in \mathcal{A}(0,x)} \mathbb{E}U_n(x + (\phi \cdot S)_T + \Pi(G) + \delta + (\hat{\phi} \cdot S)_T - G) \\
&\leq \sup_{\phi \in \mathcal{A}(G, x+\Pi(G)+\delta)} \mathbb{E}U_n(x + (\phi \cdot S)_T + \Pi(G) + \delta - G) \\
&= u_n(G, x + \Pi(G) + \delta)
\end{aligned}$$

By definition of the utility indifference price as

$$p_n(G, x) = \inf_{z \in \mathbb{R}} \{u_n(0, x) \leq u_n(G, x + z)\}$$

it follows that $p_n(G,x) \leq \Pi(G) + \delta$. Thus, letting $\delta \to 0$, $p_n(G,x) \leq \Pi(G)$ Now it has to be shown that

$$\liminf_{n\to\infty} p_n(G,x) \geq \Pi(G)$$

Suppose that for some $x > \eta > 0$ and a subsequence n_k

$$p_{n_k}(G,x) \leq \Pi(G) - \eta$$

holds, for all $k \in \mathbb{N}$. By definition of the utility indifference price $p_n(G,x)$ and remembering that $u_n(G,y) \leq \epsilon U_n(x-\epsilon) + U_n(M_T(y)+x) \to -\infty$ for $\Pi(G) < y < x + \Pi(G)$ it is easy to see that the left hand side tends to $-\infty$ since y corresponds to $x + \Pi(G) - \eta$. But the right hand side has just been proved to be nonnegative. Therefore this is a contradiction.

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