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Dynamic Evolution and Control of Terrorist Organisations

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durch

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Abstract

This thesis is targeting the dynamics and the stability behaviour of ordinary differential equation models describing a terrorist group. It deals with several models based on Hausken et al.'s considerations in their work *Composition and Dynamic Evolution of Terrorist Organizations*, [12, Hausken et al. 2012]. One main task is to investigate the stability behaviour with respect to changes of parameters in the model representing such a group.

Hausken et al. assumed that terrorist organisations consist most often of three groups: ideologues, criminal mercenaries and captive participants. Ideologues provide political purpose and direction and have a strong group commitment. Like every group a terrorist organisation often needs money to survive. The ideologues acquire money through capital support by sponsors or criminal activities. To assure permanent financial supporting, ideologues have to recruit mercenaries. These criminal mercenaries support the organisation by providing money, but they have a weak group commitment and may corrupt the ideological purity of the group. Captive participants have neither strong commitments nor financial interests but cannot leave without repercussions. The developement of the terrorist organisation is influenced by the strength of every single group and may turn into a criminal organisation or may break completely.

The following models, based on Hausken et al.'s work, analyse the development of such a group. Hausken et al. first present a model only with ideologues, which is then expanded by criminal mercenaries and finally by captive participants. Thereafter Hausken et al. also included government intervention to their model. Those models of different dimensions will be investigated accurately for the issue of an existing solution and the stability. In addition, an analysis for measuring the influence of a parameter change will be carried out to minimise the damage costs caused by the terrorist organisation.

Contents

Chapter 1

Introduction

1.1 Definition of Terrorism

Everybody has some association to the word *terrorism*, usually a bad one. But what is terrorism? In order to respond to terrorism, a clear definition is necessary. In history there were many attempts to give a legal definition of terrorism but none of them is widely accepted as the universal one.

1.1.1 The Difficulty of Defining Terrorism

The problem facing a global definition is the difficulty in taking account of special circumstances according to the type of action committed, the nature of victims or the type of method of the terrorist action, [23, Sorel 2003]. The enormity of existing definitions of terrorism is given by an enumerative, descriptive and also confusing mix of papers. According to J.M. Sorel it is much better to refer to the approaches of terrorism. One could focus on how the act was undertaken, by whom the act was perpetrated and the reason why the act was committed.

Anyway, trying to define terrorism runs the risk of getting into deeper and deeper water. For a satisfying definition the attitude of terrorism has to be distinguished from its acts or methods. Only the impact of the action has to be taken into account to a common law crime. This effect can also be characterised by a terror act that inculcates and causes a feeling of uncontrolled fear in the mind of society.

1.1.2 Some Aspects of Terrorism

For many people terrorism seems to be a random and senseless form of violence perpetrated by madmen. The idea to be susceptible to such apparently uncontrolable attacks certainly leads to an increasing sense of anxiety. An important psychological aspect of terrorism is the terrorist's ideological or political motivation making terrorism aking to war, [22, Ruby 2002]. That may engender a feeling of powerlessness in potential victims. This psychological aspect is likely to make people more sensitive to terrorism than they are to much greater risks such as traffic accidents.

As mentioned above the most difficult aspect of dealing with terrorism is probably the issue of defining it. The word *terrorism* has been used to describe a bunch of violent acts from domestic altercations to gang violence. But the popular view of terrorism does not include these acts.

Since 1983, the U.S. Department of State [6, Fox and Zawitz 2000] has used Title 22 of the United State Code, Section 2656f(d), to define terrorism. In the introduction to the Departement's *Patterns of Global Terrorism* it is defined as politically motivated violence perpetrated against noncombatant targets by subnational groups or clandestine agents, usually intended to influence an audience.

This definition has three key criteria to distinguish terrorism from crime, [22, Ruby 2002]. First of all terrorism has to be *politically motivated*. Terrorism is targeting political goals. Those acts are intended to influence the government policy. This criterion emphasizes that the social and psychological antecedents of personally or criminally motivated violence are different from the antecedents of terrorist violence.

The second one argues that terror acts are directed at *noncombatants*. It identifies terrorism as violence directed towards civilian populations or groups who are not prepared to defend against political violence.

The third criterion of the State Department's definition is that *subnational groups or clandestic agents* commit terror acts. From this point of view political violence is not terrorism, even if there is some probability to harm noncombatants (e.g. Pearl Harbor, 1941).

Additionally to the political motivation of the acts, targeting noncombatants and the clandestine perpetrators are other two important definition criteria. Kaplan [15, Kaplan 1978] argued that terrorism is intended to create an extremely fearful state of mind. Furthermore this fearful state is not intended only or primarily for the terrorist victims but rather for an audience who may have no relationship to the victims. Oots [19, Oots 1990] similarly said that terrorism was intended to "create fear and/or anxiety-inducing effects in a target audience larger than the immediate victims". Ruby notes also that the definition in the U.S. Army's textbook on militarly medicine echoes that terrorism is partly defined by its creation of fear in an audience beyond the immediate victim, [14, Jones and Fong 1994].

Taylor [24, Taylor 1998] discussed three perspectives that society uses in determining wether an act is terrorism or not. He presents these different perspectives to point out that even with a huge set of definition criteria people see terrorism as a legal issue. With this perspective, an act is considered as terrorism only if it is illegal. Governments are likely to use this perspective to interpret terrorism. However, the determination that an act is terrorism under this perspective depends on the government's interpretation. Obviously not all nations have the same definition of what is legal. Thus two governments may view a same incident differently.

A second perspective is *moral* in nature and would consider an act to be terrorism only if it had no moral justification. Some groups are willing to commit politically motivated illegal violence but do so by believing it is a necessary and morally justified act. As with a legal perspective, the use of a moral one in interpreting terrorism can result in different conclusions concerning the same act.

Taylor's third and final perspective is *behavorial*. With this perspective terrorism is defined purely by the behaviors involved, regardless of the laws or morality of those doing the defining. In reasoning from this perspective, different interpreters will necessarily reach the same conclusions as wether a particular act is terrorism or not.

Of course, philosophing about terrorism is a neverending story and neither the issue of this thesis. Further information on this topic can be read in lots of different papers such as Susan Tiefenbrun's *A Semiotic Approach to a Legal*

Definition of Terrorism Part III [25, Tiefenbrun 2002] in the *ILSA Journal of International & Comparative Law* where several definitions of terrorism (U.S., English, French, European Nations, Canadian) are listed.

1.2 Background and Motivation

Most terrorist groups do not exist very long. Their lifespan is strictly correlated to money, [21, Rapoport 1992]. They need capital fundings to carry out their ideologies and political ideas and waging war against organised societies. Terrorist groups can raise their money from outside fundings or through criminal activities. Without permanent sponsoring a group is not able to retain their ideology. It may disband or turn into a criminal organisation. Thus the maintenance of a terrorist group is perhaps closely related to its internal composition. The issue of the internal dynamics of such a group is not yet explored very accurately. And this is the inducement to Kjell Hausken's research and his models, [12, Hausken et al. 2012], which will be investigated in detail in this thesis.

Terror acts are not clearly distinguishable from criminal ones. Terrorists and criminals both work outside the law and their acts look pretty similar at the first sight. But terrorists go for a maximum of attention when exerting their acts whereas criminals organisations try to stay latent in the background. Too much attention on them would be bad for their business. Not all groups have a similar, political or religious, orientation. Gupta [8, Gupta 2008] argues that in time some terrorist groups transform automatically into criminal groups. One could follow up the transformation of some well-known terrorist groups into criminal ones. For instance, the colombian FARC was founded as a Marxist revolutionary group but ended up as a drug cartel due to enticement of money, [1, Betancourt 2011]. Also some splits of the IRA turned criminal, [5, English 2003].

Obviously there is only a thin line between terrorist and criminal organisations and both are known for acting in a violent way. However, there is an essential difference between them, namely their motivations. Criminals want to intimidate society. Terrorists on the other hand want to change the political or religious orientation of society and that is the point. This comes also through the definition of terrorism given in Section 1.1.1 and the definition of crime which is according to the Federal Bureau of Investigation a "criminal conspiracy ... motivated by greed".

Gupta [8, 9, Gupta 1990,2008] was convinced that some individuals of a terrorist group are not just motivated by their own selfish interests but also by the welfare of the entire group. Hausken argues similarly. According to him [10, 11, Hausken 1996] individuals try to find a perfect mix between selfishness and selflessness they can live with. That is the basic concept for understanding terrorism. Based on this utility function depending on two arguments Gupta filters out three types of actors within a terrorist group. These actors distinguish themselves in their primary motivation of joining such a group.

- Ideologues or the "true believers" have a strong group commitment. These actors devote their effort for the welfare of the group and would even sacrifice their lives for it, [13, Hoffer 2002].
- Criminal mercenaries or the "egoists" spend their lives in pursuit of their own selfish desires and are mostly interested in making money.
- Captive participants have neither a strong group commitment nor strong personal financial interests. They can not leave the group due to the fact that their cost of noncompliance of several things is too high. Their efforts are logistical supports such as providing safe houses, acting as look-outs and so on.

To keep up a group, ideologues to build up the foundation of an ideological framework are needed. Thereafter it needs money to operate and finance their activities. Those who are able to assure permanent financial support can keep their ideology, e.g. the group Laskar-e-Taibe receives fundings from the Pakistani military establishment, [4, Constable 2011]. Al-Qaeda can count on individual contributors from Saudi Arabia and some Gulf countries, [18, Napoleoni 2005].

Due to lack of fundings some groups are forced to turn towards criminal acts such as drug dealing, human trafficking or money laundering. When terrorist groups turn to criminal activities they need the support of those people who are just interested in making money, the mercenaries. The captive participants come under the influence of a group when it becomes sufficiently strong to impose its will on parts of the community which do not fully subscribe to their ideological or strategic goals, [12, Hausken et al. 2012].

Hausken's analysis starts with the assumption that every terrorist group consists of these three types of actors, [12, Hausken et al. 2012]. A group consisting only of ideologues needs fundings from outside to survive. If the ideologues recruit criminal mercenaries to the group the latter can provide money additionally to the capital fundings. If there is no more sponsoring from outside and a strong government has the ability to impose higher costs on the ideologues than on the mercenaries, the terrorist group tends to transform itself into a criminal organisation. But there are also other negative inducements for a terrorist group to change its orientation, i.e. the group's political base. A group consisting only of captive participants will fail in commitment and effectiveness and thus will cease to exist.

Hausken et al.'s assumptions suggest that every terrorist group has an optimal mix consisting of these three actors. It is a logistical problem of the group to keep up itself. Hence, terrorist groups evolve in time. All of them are founded as ideological groups trying to maintain their goals. Some of them can keep up their political or religious orientation and have a long lifespan (e.g. IRA, Hamas), some fail their goals and turn into a criminal organisation (i.e. FARC, Abu Sayaaf) and most of them disband (e.g. Japanese Red Army). It depends on the internal mix within the terrorist group, which way of development it takes.

1.3 Thesis Organisation

Chapter 2 presents Hausken et al.'s two-, three- and four-dimensional models without government intervention. These are all given by homogenous linear first ordered differential equations. The two-dimensional model includes only ideologues and capital funding. This one will be solved analytically and every possible case of stability will be investigated accurately. Finally, some plots of this model are displayed for better understanding. The three-dimensional model additionally includes criminal mercenaries and will be treated the same way. But the analysis claims much more pages because of the higher dimension and the associated numerous cases of stability. Also a solution of the four-dimensional model including captive participants is given in this chapter. But due to lack of space this solution will be just a theoretical one.

Chapter 3 also deals with the extended four-dimensional model but this time with government intervention. This leads to an inhomogenous system of differential equations, which can again be solved analytically.

The further parts of this thesis are restricted to the solution of the two-dimensional model. Chapter 4 concentrates on the examination of this solution.

Chapter 5 introduces some strategic investigations on Hausken et al.'s two-dimensional model. In order to minimise an objective functional measuring the damage caused by the terrorist group these examinations are first carried out analytically in Chapter 5 and then performed numerically with a chosen set of parameter values in Chapter 6.

Chapter 7 concludes, while some of the mathematical details can be found in the Appendix.

Chapter 2

The Model without Government Intervention

2.1 Ideological Purity

Hausken et al. [12, Hausken et al. 2012] presented their first model with $I(t)$ defined as the amount of labour exerted by ideologues to run a terrorist organisation and $K(t)$ defined as the amount of capital provided by sponsors with $t \geq 0$ representing the time argument. Over time the ideological effort increases with the injection of capital and is constrained by itself from unbounded growth. The willingness of sponsors to insert capital increases with the ideological effort as well and is also constrained by itself from unbounded growth. This results in the following system of two linear first ordered differential equations:

$$\begin{aligned} \dot{I}(t) &= aK(t) - bI(t) \\ \dot{K}(t) &= cI(t) - dK(t) \end{aligned}$$

where $a, b, c, d > 0$ are parameters. This can also be written as

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -b & a \\ c & -d \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} I(t) \\ K(t) \end{pmatrix}$$

with the initial conditions $I(0),\ K(0)$ and $\mathbf{A}$ being a matrix with constant coefficients. The solution of this system is given by

$$\begin{pmatrix} I(t) \\ K(t) \end{pmatrix} = e^{\mathbf{A}t} \begin{pmatrix} I(0) \\ K(0) \end{pmatrix}.$$

Therefore the eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{A}$ have to be calculated, which will be done in what follows.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -b-\lambda & a \\ c & -d-\lambda \end{vmatrix} = \lambda^2 + \lambda(b+d) + (bd - ac)$$

Determining the zeros of this polynomial results in the eigenvalues

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}\left(b + d - \sqrt{(b-d)^2 + 4ac}\right) \\ \lambda_2 &= -\frac{1}{2}\left(b + d + \sqrt{(b-d)^2 + 4ac}\right) \end{aligned}$$

which are both real and where the relation $\lambda_1 > \lambda_2$ is always true. Solving the linear equation system $(\mathbf{A} - \lambda\mathbf{I}) \cdot \mathbf{v} = \mathbf{0}$ leads to the eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} \frac{a}{a+b+\lambda_1} \\ \frac{c}{c+d+\lambda_1} \end{pmatrix} \qquad \text{for } \lambda_1$$

$$\mathbf{v}_2 = \begin{pmatrix} \frac{a}{a+b+\lambda_2} \\ \frac{c}{c+d+\lambda_2} \end{pmatrix} \qquad \text{for } \lambda_2.$$

Now the matrix $e^{\mathbf{A}t}$ can be diagonalised by the Jordan canonical form and the solution of the initial system is given by

$$\begin{pmatrix} I(t) \\ K(t) \end{pmatrix} = \mathbf{T} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \mathbf{T}^{-1} \begin{pmatrix} I(0) \\ K(0) \end{pmatrix}$$

where $\mathbf{T}$ is the transformation matrix which consists of the two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. For further information on this solution and the functions $I(t)$ and $K(t)$ see Chapter 4.

2.1.1 Stability Analysis

Next we want to address on the issue of the stability of the system. For that purpose it is useful to know what the phase portrait of the system looks like. We need to take a look at the signatures of the two eigenvalues λ_1 and λ_2. It is easy to see that the signature of λ_2 is negative for all $a, b, c, d > 0$, so we just have to figure out the signature of the first eigenvalue.

First of all we assume $\lambda_1 > 0$, which implies that $b + d < \sqrt{(b-d)^2 + 4ac}$ has to hold. Simple transformations give us the condition for $\lambda_1 > 0$, namely $bd < ac$. The analogous assumption and calculation for $\lambda_1 < 0$ gives us the condition $bd > ac$. The case of $bd = ac$ is still missing and at this constellation we get $\lambda_1 = 0$. Hence we have to consider the following three cases of our system accurately:

1. $bd > ac \quad \Leftrightarrow \quad \lambda_1 < 0,\ \lambda_2 < 0$
2. $bd < ac \quad \Leftrightarrow \quad \lambda_1 > 0,\ \lambda_2 < 0$
3. $bd = ac \quad \Leftrightarrow \quad \lambda_1 = 0,\ \lambda_2 < 0$

which will be discussed in what follows, respectively.

1. $bd > ac$

Here we have two real and distinct eigenvalues of the same sign, $\lambda_1, \lambda_2 < 0$ with the relation $\lambda_1 > \lambda_2$. In this case the matrix $\mathbf{A}$ has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as calculated above. The origin $(0, 0)$ is called an attractive node and the solution approaches the origin as t increases. Consequently the origin is said to be an asymptotically stable **attractive node**, [16, Mlitz 2008].

2. $bd < ac$

Under this condition both eigenvalues are real and of opposite sign, $\lambda_1 > 0$ and $\lambda_2 < 0$. In this case the matrix $\mathbf{A}$ has still two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The solution would approach the origin if it started on a point along $\mathbf{v}_2$, which belongs to λ_2, and move away from the origin if it started on a point along $\mathbf{v}_1$. In this case the origin $(0, 0)$ is called an unstable **saddle point**, [16, Mlitz 2008].

3. $bd = ac$

If $bd = ac$ holds, the matrix $\mathbf{A}$ is singular, which means that $\det(\mathbf{A}) = 0$ and our second eigenvalue is reduced to $\lambda_2 = -(b+d)$. In fact this happens if and only if we have more than one equilibrium point which is usually $(0,0)$. In this case we will have a line of equilibrium points and the characteristical trajectory for this line is the eigenvector associated to the eigenvalue zero. The corresponding characteristical trajectories are calculated by substituting $\lambda_1 = 0$ and $\lambda_2 = -b-d$ to the formula of the eigenvectors and we get the simple forms $\mathbf{v}_1 = (a,b)^T$ and $\mathbf{v}_2 = (-a,d)^T$. Therefore the general solution can be formed easily by

$$\begin{pmatrix} I(t) \\ K(t) \end{pmatrix} = \frac{1}{a(b+d)} \begin{pmatrix} a & -a \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-(b+d)t} \end{pmatrix} \begin{pmatrix} b & a \\ -d & a \end{pmatrix} \begin{pmatrix} I(0) \\ K(0) \end{pmatrix}.$$

Note that all the solutions are line parallel to the vector $\mathbf{v}_2$. For $t \to \infty$ the trajectory converges to the equilibrium point on the line of equilibrium points having $\mathbf{v}_1$ as a direction vector. The system is also stable but not asymptotically stable, [16, Mlitz 2008].

To get an idea of the dynamics of the system and what the phase portrait actually looks like the system is plotted on the next pages for certain randomly chosen cases.

2.1.2 Equilibrium Solution

An equilibrium solution is a constant solution of the system, which is usually called a *critical point*, [16, Mlitz 2008]. For a linear system of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ an equilibrium solution occurs at each solution of the (homogeneous) system $\mathbf{A}\mathbf{x} = \mathbf{0}$. So we have to solve the linear system of equations

$$\begin{aligned} aK(t) - bI(t) &= 0 \\ cI(t) - dK(t) &= 0. \end{aligned}$$

Beside the trivial equilibrium solution $I(t) = K(t) = 0$ there exist other equilibria under the condition $bd = ac$ as shown in Figure 2.5 and Figure 2.6. For further information see section 2.1.1, where the case $bd = ac$ is discussed in detail.

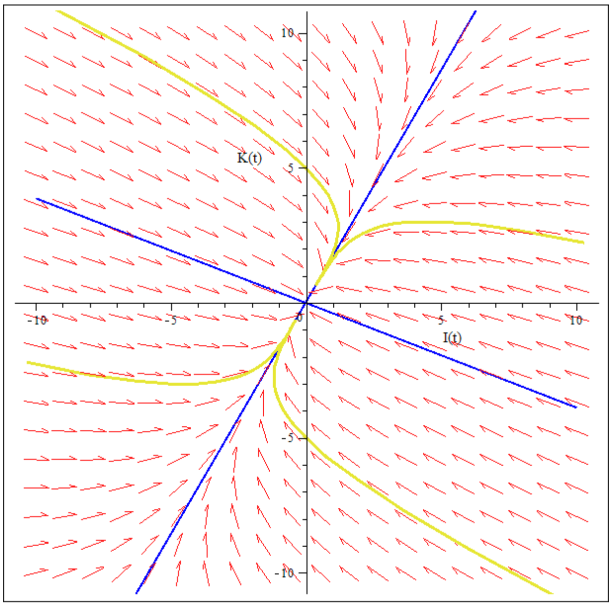

Figure 2.1: Plot of the phase portrait of the system with the parameters $a = 3,\ b = 7,\ c = 2,\ d = 3$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The origin is an attractive node.

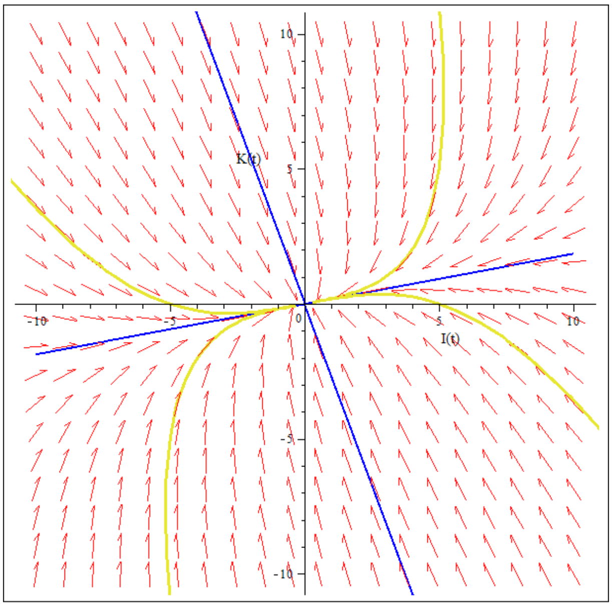

Figure 2.2: Plot of the phase portrait of the system with the parameters $a = 2,\ b = 3,\ c = 1,\ d = 8$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The origin is an attractive node.

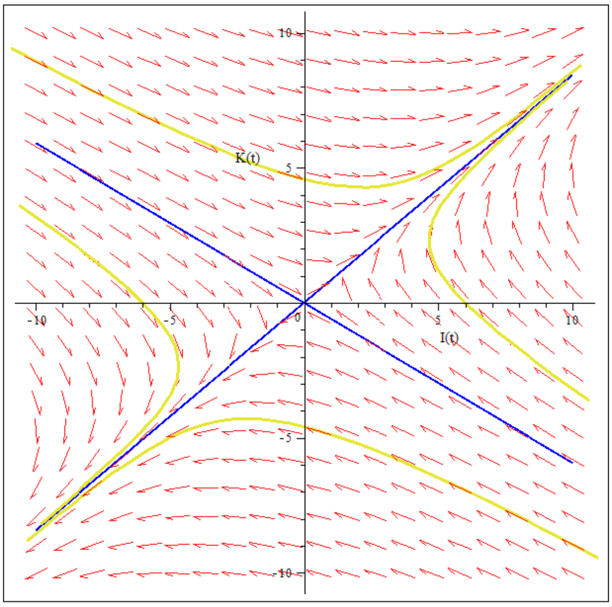

Figure 2.3: Plot of the phase portrait of the system with the parameters $a = 4,\ b = 2,\ c = 2,\ d = 1$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The origin is a saddle point.

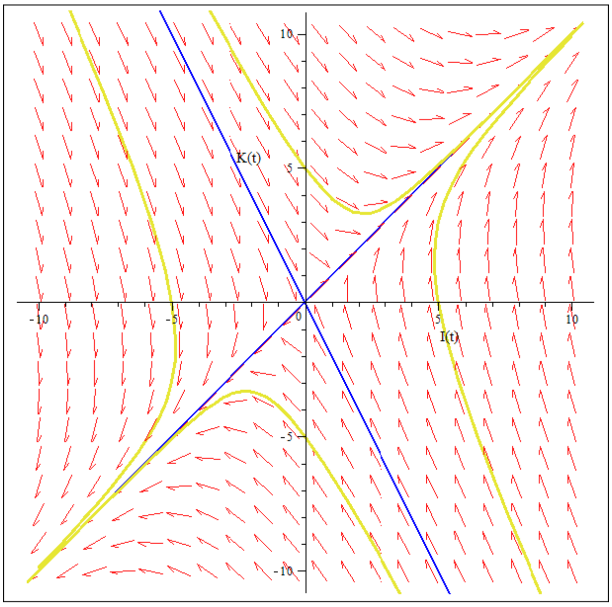

Figure 2.4: Plot of the phase portrait of the system with the parameters $a = 3,\ b = 1,\ c = 6,\ d = 4$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The origin is a saddle point.

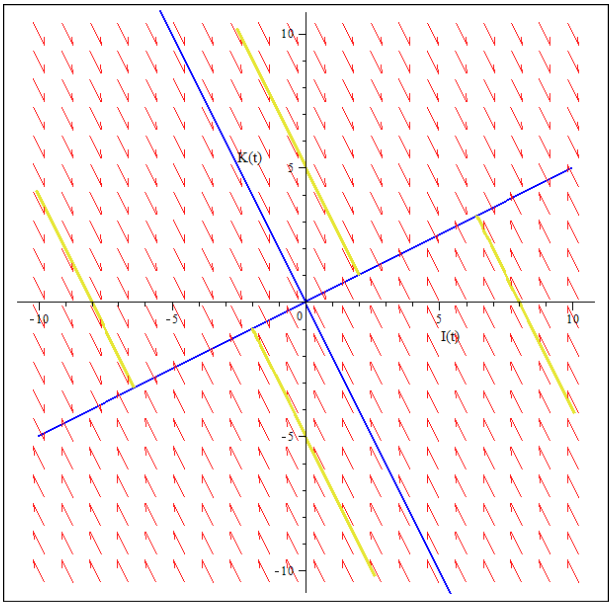

Figure 2.5: Plot of the phase portrait of the system with the parameters $a = 4,\ b = 2,\ c = 4,\ d = 8$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. All points on $\mathbf{v}_1$ are equilibrium points.

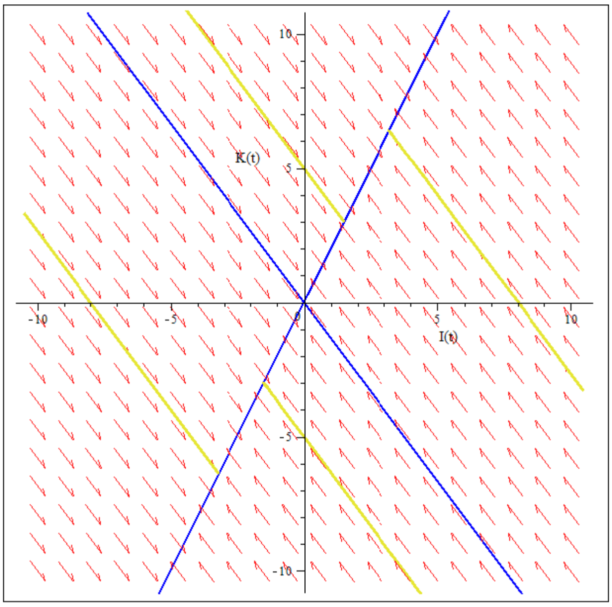

Figure 2.6: Plot of the phase portrait of the system with the parameters $a = 3,\ b = 6,\ c = 8,\ d = 4$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. All points on $\mathbf{v}_1$ are equilibrium points.

2.2 Including Criminal Mercenaries

To ensure financial support ideologues may recruit criminal mercenaries, [12, Hausken et al. 2012]. Thus the first model is generalised to

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -b & a & e \\ c & -d & -f \\ g & -h & -m \end{pmatrix}}_{\mathbf{B}} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix},$$

where $a, b, c, d, e, f, g, h, m > 0$ are parameters with the initial conditions $I(0)$, $K(0)$ and $M(0)$. $M(t)$ is defined as the amount of labour exerted by mercenaries. This system can be solved as shown above by calculating the eigenvalues with $\det(\mathbf{B} - \nu\mathbf{I}) = 0$ and the corresponding eigenvectors and diagonalising through the Jordan canonical form. The characteristic polynomial of the matrix $\mathbf{B}$ is given by

$$\begin{aligned} p(\nu) &= \nu^3 + \nu^2 \underbrace{(b + d + m)}_{A} + \nu \underbrace{(bd + bm + dm - ge - hf - ca)}_{B} + \\ &+ \underbrace{(bdm + afg + ech - dge - hfb - cam)}_{C}. \end{aligned}$$

The zeros of this polynomial can be determined by *Cardano's method* [3, Buechner 1857] by solving

$$z^3 + pz + q = 0$$

with

$$p = B - \frac{A^2}{3}, \qquad q = \frac{2A^3}{27} - \frac{AB}{3} + C,$$

which comes from the original characteristic polynomial reduced by substituting $\nu = z - \frac{A}{3}$. At this point, two variables u and v have to be introduced linked by the condition $z = u + v$. Substituting this into the reduced polynomial gives us

$$u^3 + v^3 + (3uv + p) + q = 0.$$

Comparing coefficients gives us another two conditions for u and v:

- $-p = 3uv \quad \Rightarrow \quad u^3v^3 = -\frac{p^3}{27}$
- $-q = u^3 + v^3$.

According to *Vieta* u^3 and v^3 are the two roots of the equation $t^2 + qt - \frac{p^3}{27} = 0$ and we finally get

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{D}}, \qquad v = \sqrt[3]{-\frac{q}{2} - \sqrt{D}} \qquad \text{with} \quad D := \frac{q^2}{4} + \frac{p^3}{27}.$$

According to the condition $z = u + v$ we achieve our first solution

$$z_1 = u + v = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}}.$$

Additionally we have to consider the other two solutions, which might be conjugate-complex. The two complex roots are obtained by considering the complex cubic roots. The fact that uv is real implies that they are obtained by multiplying one of the above cubic roots by $\epsilon_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and the other by $\epsilon_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Now the other two solutions can be written as $z_2 = u\epsilon_1 + v\epsilon_2$ and $z_3 = u\epsilon_2 + v\epsilon_1$:

$$z_2 = -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}}\right) + i \cdot \frac{\sqrt{3}}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{D}} - \sqrt[3]{-\frac{q}{2} - \sqrt{D}}\right)$$

$$z_3 = -\frac{1}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}}\right) - i \cdot \frac{\sqrt{3}}{2}\left(\sqrt[3]{-\frac{q}{2} + \sqrt{D}} - \sqrt[3]{-\frac{q}{2} - \sqrt{D}}\right).$$

To determine whether the zeros of the reduced polynomial are real or complex we have to consider the following three cases:

- $D > 0$
  The present condition $D > 0$ implies that $4p^3 + 27q^2 > 0$ has to hold:

$$\begin{aligned} 0 &< 4\left(B - \frac{A^2}{3}\right)^3 + 27\left(\frac{2A^3}{27} - \frac{AB}{3} + C\right)^2 \\ 0 &< 4B^3 - A^2B^2 + 4A^3C + 27C^2 - 18ABC. \end{aligned}$$

  This is the final condition for $D > 0$. If this is satisfied by the parameters

$a, b, c, d, e, f, g, h, m > 0$, the first solution of the reduced polynomial is real and the other two are conjugate-complex; $z_1 \in \mathbb{R}$ and $z_2, z_3 \in \mathbb{C}$.

- $D < 0$
  The same calculations as shown above give us the condition

$$4B^3 - A^2B^2 + 4A^3C + 27C^2 - 18ABC \quad < \quad 0,$$

  which has to be fulfilled for a negative D. We then receive three real and distinct solutions; $z_j \in \mathbb{R},\ j = \{1, 2, 3\}$.

- $D = 0$
  This case delivers

$$4B^3 - A^2B^2 + 4A^3C + 27C^2 - 18ABC \quad = \quad 0.$$

  Under this condition we also get three real solutions, $z_j \in \mathbb{R},\ j = \{1, 2, 3\}$, but they are not distinct as in the previous case. We get either one triple zero or one single and one double zero.

The terms containing the parameters $a, b, c, d, e, f, g, h, m > 0$, which occur in the condition for the signature of D, will not be given because they would cover more than one A4 paper.

Resubstituting by $\nu = z - \frac{A}{3}$ leads to the three zeros of the original characteristic polynomial, ν_1, ν_2 and ν_3, which are also classified by the sign of D. Here we finally have the three eigenvalues of the matrix $\mathbf{B}$. Solving the linear system of equations $(\mathbf{B} - \nu\mathbf{I}) \cdot \mathbf{w} = \mathbf{0}$ gives us the three corresponding eigenvectors

$$\mathbf{w}_1 \;=\; \begin{pmatrix} \frac{a(m+\nu_1-h)-h(e-a)}{(b+\nu_1+a)(m+\nu_1-h)-(g+h)(e-a)} \\ \frac{f}{d+\nu_1}\left(\frac{c}{f} - \frac{g+h}{m+\nu_1-h}\right)\left(\frac{a(m+\nu_1-h)-h(e-a)}{(b+\nu_1+a)(m+\nu_1-h)-(g+h)(e-a)}\right) + \frac{hf}{(m+\nu_1-h)(d+\nu_1)} \\ \left(\frac{g+h}{m+\nu_1-h}\right)\left(\frac{a(m+\nu_1-h)-h(e-a)}{(b+\nu_1+a)(m+\nu_1-h)-(g+h)(e-a)}\right) - \frac{h}{m+\nu_1-h} \end{pmatrix},$$

$$\mathbf{w}_2 \;=\; \begin{pmatrix} \frac{a(m+\nu_2-h)-h(e-a)}{(b+\nu_2+a)(m+\nu_2-h)-(g+h)(e-a)} \\ \frac{f}{d+\nu_2}\left(\frac{c}{f} - \frac{g+h}{m+\nu_2-h}\right)\left(\frac{a(m+\nu_2-h)-h(e-a)}{(b+\nu_2+a)(m+\nu_2-h)-(g+h)(e-a)}\right) + \frac{hf}{(m+\nu_2-h)(d+\nu_2)} \\ \left(\frac{g+h}{m+\nu_2-h}\right)\left(\frac{a(m+\nu_2-h)-h(e-a)}{(b+\nu_2+a)(m+\nu_2-h)-(g+h)(e-a)}\right) - \frac{h}{m+\nu_2-h} \end{pmatrix},$$

$$\mathbf{w}_3 = \begin{pmatrix} \frac{a(m+\nu_3-h)-h(e-a)}{(b+\nu_3+a)(m+\nu_3-h)-(g+h)(e-a)} \\ \frac{f}{d+\nu_3}\left(\frac{c}{f}-\frac{g+h}{m+\nu_3-h}\right)\left(\frac{a(m+\nu_3-h)-h(e-a)}{(b+\nu_3+a)(m+\nu_3-h)-(g+h)(e-a)}\right)+\frac{hf}{(m+\nu_3-h)(d+\nu_3)} \\ \left(\frac{g+h}{m+\nu_3-h}\right)\left(\frac{a(m+\nu_3-h)-h(e-a)}{(b+\nu_3+a)(m+\nu_3-h)-(g+h)(e-a)}\right)-\frac{h}{m+\nu_3-h} \end{pmatrix}.$$

Again the matrix $e^{\mathbf{B}t}$ can be diagonalised and the solution of the system of differential equations can be written as

$$\begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix} = \mathbf{T} \begin{pmatrix} e^{\nu_1 t} & 0 & 0 \\ 0 & e^{\nu_2 t} & 0 \\ 0 & 0 & e^{\nu_3 t} \end{pmatrix} \mathbf{T}^{-1} \begin{pmatrix} I(0) \\ K(0) \\ M(0) \end{pmatrix}$$

where $\mathbf{T}$ is once more the transformation matrix containing the three eigenvectors, $\mathbf{T} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$.

2.2.1 Stability Analysis

Generally it is easier to check the (un)stability of a system by taking a look at the complex variable domain of its characteristic polynomial than using any other methods, [7, Frey et al. 2008]. A system is

- *asymptotically stable* if every zero of its polynomial is in the open left half-plane, i.e. there is no zero on the imaginary axis or in the right half-plane,
- *(quasi-)stable* if every zero is in the left half-plane and there are just single zeros on the imaginary axis,
- *unstable* as soon as one single zero is in the open right half-plane or a multiple zero is on the imaginary axis.

Thus it is necessary to know the zeros of the characteristic polynomial to figure out the stability of the system. However, as we have seen above, this venture is not always easy or sometimes even impossible. Since it is obviously not practical to determine those zeros analytically we have to give consideration to some other methods to evaluate the dynamics of our three-dimensional system. The most common methods in stability analysis are *Hurwitz' stability criterion* [7,17, Frey et al. 2008, Mlitz 2008] and the classification of the zeros in a qualitative anaylsis by the *Routh-Hurwitz Theorem* and the *Cauchy Index*, [2, Bitmead and Anderson

1997]. For further information on Hurwitz' criterion and the qualitative analysis take a look at the Appendix. The following illustrative examples will be investigated by both methods. Since we will not skip anything we will give an example of every single case that may occur.

Example of an asymptotically stable system with $D < 0$

The first example we examine is

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -5 & 1 & 5 \\ 2 & -4 & -1 \\ 4 & -1 & -6 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 15\nu^2 + 51\nu + 37$.

- By using Hurwitz' stability criterion we have to test whether $p(\nu)$ is a Hurwitz polynomial or not. Considering $p(\nu)$ yields $p_3 = 1,\ p_2 = 15,\ p_1 = 51$ and $p_0 = 37$. Since all coefficients of $p(\nu)$ are non-zero and of the same sign we just need to check the condition

  $$p_1 p_2 - p_0 p_3 > 0,$$

  which is actually fulfilled by the present coefficients $p_i,\ i = 0, \ldots, 3$. Thus $p(\nu)$ is a Hurwitz polynomial and the present system is asymptotically stable.

- The qualitative analysis is a bit more detailed and needs further information. The first thing we need to know is the classification of the eigenvalues of the present system. Therefore we take a look at the term D from Cardano's method. By looking at $p(\nu)$ we denote that $A = 15, B = 51$ and $C = 37$. Applying these three values in the condition for the signature of D presented above yields a negative D. Hence we know that the three zeros of $p(\nu)$ are real and distinct; $\nu_i \in \mathbb{R},\ i = 1, 2, 3$.

  Designing the Hurwitz matrix of the characteristic polynomial and determining its minors gives us the number of zeros with a positive real part. (Since we know that $\nu_i \in \mathbb{R},\ \forall i = 1, 2, 3$ the Hurwitz matrix gives us the number of positive zeros.)

$$H(p) = \begin{pmatrix} 15 & 37 & 0 \\ 1 & 51 & 0 \\ 0 & 15 & 37 \end{pmatrix} \qquad \begin{aligned} M_1 &= 15 && > 0 \\ M_2 &= 728 && > 0 \\ M_3 &= 37 \cdot M_2 && > 0. \end{aligned}$$

We are now considering the sequence $p_3, M_1, \frac{M_2}{M_1}, \frac{M_3}{M_2}$ and its signatures, which is

$$p_3,\ M_1,\ \frac{M_2}{M_1},\ \frac{M_3}{M_2}: \qquad +, +, +, +.$$

Since we do not have a change of signatures there exists no positive zero of $p(\nu)$:

$$\nexists\ \nu_i \in \mathbb{R}: \qquad p(\nu_i) = 0\ \wedge\ \nu_i > 0, \qquad \forall i = 1, 2, 3.$$

In addition we may examine the Cauchy index by determining a Sturm chain of $p(\nu)$. The first member of the Sturm chain is the characteristic polynomial itself and the second one is a simplified form of its derivative. The others are calculated by Euclid's algorithm:

$$\begin{aligned}
&\Big(\ \nu^3 + 15\nu^2 + 51\nu + 37\Big) : \Big(\nu^2 + 10\nu + 17\Big) = \nu + 5 + \frac{-16\nu - 48}{\nu^2 + 10\nu + 17} \\
&\underline{-\nu^3 - 10\nu^2 - 17\nu} \\
&\qquad\quad 5\nu^2 + 34\nu + 37 \\
&\qquad\ \underline{-5\nu^2 - 50\nu - 85} \\
&\qquad\qquad\ -16\nu - 48
\end{aligned}$$

$R_2 := -16\nu - 48$ and thus

$$f_2 \cong -R_2 \qquad \Rightarrow \qquad f_2 = \nu + 3.$$

The last member of the Sturm chain is calculated in the same way and we get $f_3 = 1$. As a matter of lucidity all elements of the Sturm chain are listed below:

$$\begin{aligned} f_0 &= \nu^3 + 15\nu^2 + 51\nu + 37 \\ f_1 &= \nu^2 + 10\nu + 17 \\ f_2 &= \nu + 3 \\ f_3 &= 1. \end{aligned}$$

To calculate the Cauchy index we have to evaluate the sign of every single member of the Sturm chain at the limits of the considered interval which is of course $(-\infty, +\infty)$. Since 0 is not a zero of $p(\nu)$ we additionally evaluate the Sturm chain at the point 0 to distinguish between positive and negative zeros.

$$\begin{array}{llllll} -\infty: & f_0 = - & 0: & + & \infty: & + \\ & f_1 = + & & + & & + \\ & f_2 = - & & + & & + \\ & f_3 = + & & + & & + \end{array}$$

$$\Rightarrow \quad V(-\infty) = 3 \qquad\qquad V(0) = 0 \qquad\qquad V(\infty) = 0$$

where $V(.)$ describes the number of changes of the signature. Finally we are able to specify the Cauchy index:

$$\begin{aligned} I_{-\infty}^{0}(p(\nu)) &= V(-\infty) - V(0) \;=\; 3 \\ I_{0}^{+\infty}(p(\nu)) &= V(0) - V(+\infty) \;=\; 0. \end{aligned}$$

Thus we know that all three real zeros are negative. This result corresponds with the analysis of the Hurwitz matrix:

$$\nu_i \in \mathbb{R} \;\wedge\; \nu_i < 0 \qquad \forall\; i = 1, 2, 3.$$

Remembering the stability condition above yields that the present system is asymptotically stable, which equals the statement of Hurwitz' stability criterion.

Matrix of the System	$\begin{pmatrix} -5 & 1 & 5 \\ 2 & -4 & -1 \\ 4 & -1 & -6 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 15\nu^2 + 51\nu + 37$
D (from Cardano's Method)	< 0
Roots of $p(\nu)$	$\nu_i \in \mathbb{R} \ \wedge \ \nu_i < 0 \qquad \forall\, i = 1, 2, 3$
Result of Hurwitz' Stability Criterion	asymptotically stable
Result of the Qualitative Analysis	asymptotically stable

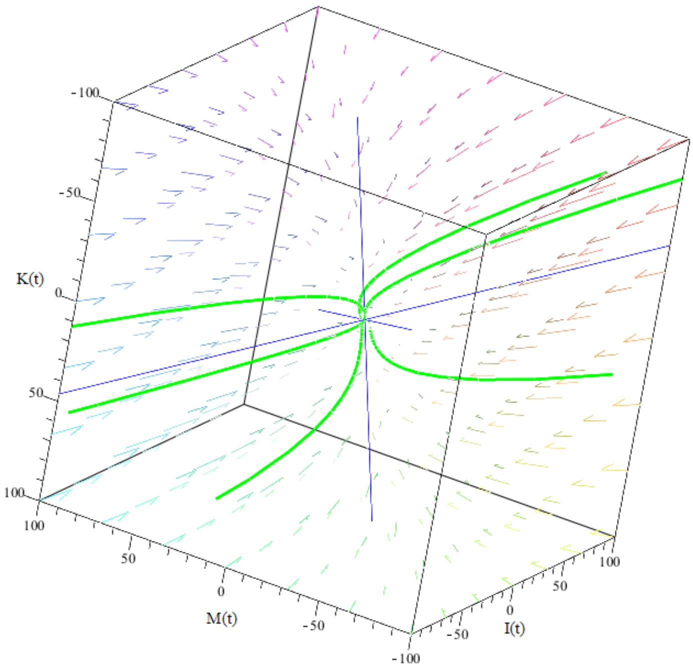

Figure 2.7: Plot of the phase portrait of the system with the parameters $a = 1,\ b = 5,\ c = 2,\ d = 4,\ e = 5,\ f = 1,\ g = 4,\ h = 1,\ m = 6$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

Example of an unstable system with *D < 0*

The second example is of the form

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ 5 & -1 & -3 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 10\nu^2 + 12\nu - 8$.

- The coefficients of $p(\nu)$ are $p_3 = 1,\ p_2 = 10,\ p_1 = 12$ and $p_0 = -8$. The necessary condition of Hurwitz' stability criterion is not fulfilled because not all p_i are of the same sign. Therefore we already know that $p(\nu)$ is no Hurwitz polynomial and the present system is unstable.
- For a qualitative analysis we need to check the Hurwitz matrix and its minors again.

$$H(p) = \begin{pmatrix} 10 & -8 & 0 \\ 1 & 12 & 0 \\ 0 & 10 & -8 \end{pmatrix} \qquad \begin{aligned} M_1 &= 10 && > 0 \\ M_2 &= 128 && > 0 \\ M_3 &= -8 \cdot M_2 && < 0. \end{aligned}$$

$$p_3,\ M_1,\ \frac{M_2}{M_1},\ \frac{M_3}{M_2}: \qquad +, +, +, -.$$

Here we get one change in this sequence so there has to be exactly one positive zero. By considering a Sturm chain of $p(\nu)$ and the Cauchy index we obtain a complete classification of the eigenvalues.

$$I_{-\infty}^{0}(p(\nu)) = 2, \quad I_{0}^{+\infty}(p(\nu)) = 1.$$

This result equals the analysis of the Hurwitz matrix and now we have an accurate classification of the three zeros:

$$\nu_i \in \mathbb{R},\ \forall i = 1,2,3 \qquad \wedge \qquad \nu_{1,2} < 0,\ \nu_3 > 0.$$

Matrix of the System	$\begin{pmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ 5 & -1 & -3 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 10\nu^2 + 12\nu - 8$
D (from Cardano's Method)	< 0
Roots of $p(\nu)$	$\nu_i \in \mathbb{R} \quad \wedge \quad \nu_{1,2} < 0,\ \nu_3 > 0$
Result of Hurwitz' Stability Criterion	unstable
Result of the Qualitative Analysis	unstable

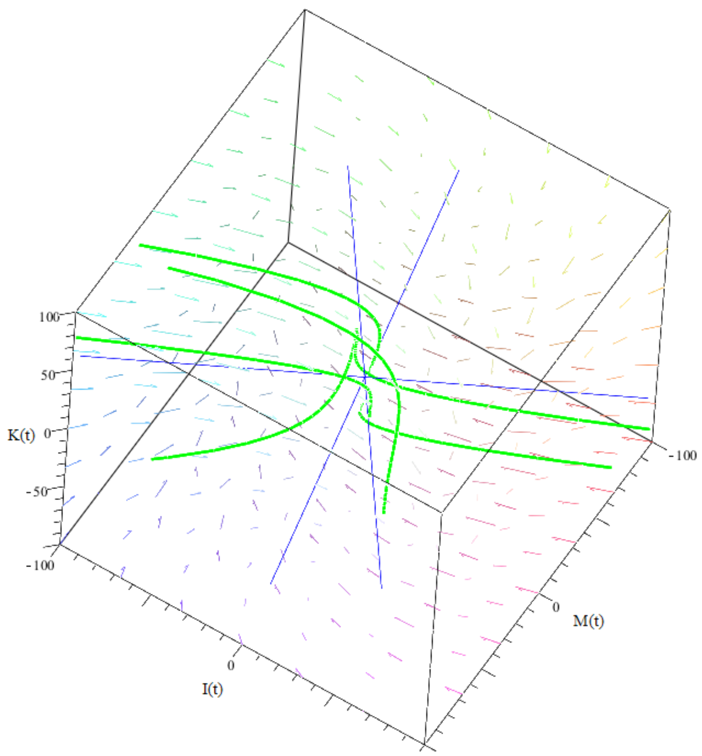

Figure 2.8: Plot of the phase portrait of the system with the parameters $a = 5,\ b = 3,\ c = 2,\ d = 4,\ e = 2,\ f = 1,\ g = 5,\ h = 1,\ m = 3$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

Example of a stable system with $D < 0$

The third system we examine has the following structure

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & -3 \\ 1 & -3 & -3 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 9\nu^2 + 16\nu = \nu(\nu^2 + 9\nu + 16)$.

Obviously the given matrix is singular. Considering $p(\nu)$ we know that one eigenvalue has to be zero. Hence we can split this zero from $p(\nu)$ and a polynomial of second degree remains. So we are able to determine the other two eigenvalues explicitly and we do not need any complicated analysis at all. The two zeros from the polynomial of second degree $\nu^2 + 9\nu + 16$ are

$$\begin{aligned} \nu_1 &= \frac{-9 - \sqrt{17}}{2} && < 0 \\ \nu_2 &= \frac{-9 + \sqrt{17}}{2} && < 0, \end{aligned}$$

which are both negative. In the following we seperately specify the classification of the three zeros again:

$$\nu_i \in \mathbb{R},\ \forall i = 1, 2, 3 \qquad \wedge \qquad \nu_{1,2} < 0,\ \nu_3 = 0.$$

In this situation of the zeros we have a stable case, which is not asymptotically stable, however.

Of course one could also use the stability criterion or additionally the qualitative analysis, which would both lead to the same result. But it is way more difficult and not necessary at all.

Matrix of the System	$\begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & -3 \\ 1 & -3 & -3 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 9\nu^2 + 16\nu$
D (from Cardano's Method)	< 0
Roots of $p(\nu)$	$\nu_i \in \mathbb{R} \quad \wedge \quad \nu_{1,2} < 0,\ \nu_3 = 0$
Result of Hurwitz' Stability Criterion	-
Result of the Qualitative Analysis	-

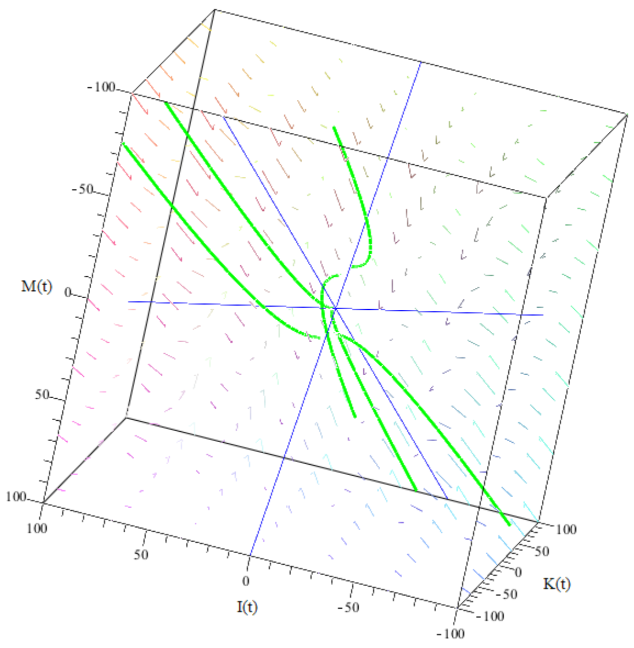

Figure 2.9: Plot of the phase portrait of the system with the parameters $a = 1,\ b = 3,\ c = 1,\ d = 3,\ e = 1,\ f = 3,\ g = 1,\ h = 3,\ m = 3$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

Example of an asymptotically stable system with $D > 0$

The next case we are interested in is of the form

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -8 & 27 & 1 \\ 1 & -3 & -4 \\ 2 & -2 & -6 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 17\nu^2 + 53\nu + 130$.

- Every p_i is non-zero and they all have the same signature. Thus the necessary condition is fulfilled. The sufficient condition $p_2 p_1 - p_0 p_3 > 0$ is also fulfilled, meaning we have a Hurwitz polynomial. Hence we know that the present system is asymptotically stable.

- To exert the other analysis we need to check the Hurwitz matrix again.

$$H(p) = \begin{pmatrix} 17 & 130 & 0 \\ 1 & 53 & 0 \\ 0 & 17 & 130 \end{pmatrix} \qquad \begin{aligned} M_1 &= 17 && > 0 \\ M_2 &= 771 && > 0 \\ M_3 &= 130 \cdot M_2 && > 0. \end{aligned}$$

$$p_3,\ M_1,\ \frac{M_2}{M_1},\ \frac{M_3}{M_2}: \qquad +,+,+,+.$$

Here we have to consider complex zeros, too, but as we can see there is no change of the signature in this sequence. Thus we do not have a zero with a positive real part. Calculating the Cauchy-Index yields

$$I_{-\infty}^{0}(p(\nu)) = 3, \quad I_{0}^{+\infty}(p(\nu)) = 0.$$

Obviously every zero is in the open left half-plane and thus the present system is asymptotically stable, which again equals the statement of Hurwitz' stability criterion.

$$\nu_1 \in \mathbb{R},\ \nu_{2,3} \in \mathbb{C} \qquad \wedge \qquad \nu_1 < 0,\ \mathrm{Re}(\nu_{2,3}) < 0.$$

Matrix of the System	$\begin{pmatrix} -8 & 27 & 1 \\ 1 & -3 & -4 \\ 2 & -2 & -6 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 17\nu^2 + 53\nu + 130$
D (from Cardano's Method)	> 0
Roots of $p(\nu)$	$\nu_1 \in \mathbb{R},\ \nu_{2,3} \in \mathbb{C}\ \wedge\ \nu_1 < 0,\ \mathrm{Re}(\nu_{2,3}) < 0$
Result of Hurwitz' Stability Criterion	asymptotically stable
Result of the Qualitative Analysis	asymptotically stable

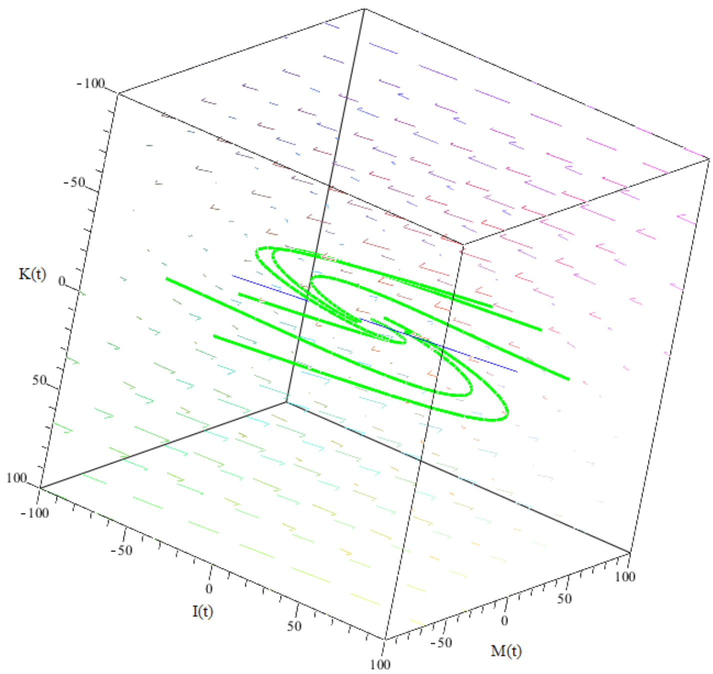

Figure 2.10: Plot of the phase portrait of the system with the parameters $a = 27,\ b = 8,\ c = 1,\ d = 3,\ e = 1,\ f = 4,\ g = 2,\ h = 2,\ m = 6$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

Example of an unstable system with $D > 0$

The next system looks as follows:

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -1 & 5 & 1 \\ 5 & -3 & -4 \\ 7 & -7 & -1 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 5\nu^2 - 53\nu + 104$.

- By looking at the signature of the coefficients we already know that the necessary condition required by Hurwitz' stability criterion is not fulfilled and thus the system is unstable.
- Analysing the Hurwitz matrix yields:

$$H(p) = \begin{pmatrix} 5 & 104 & 0 \\ 1 & -53 & 0 \\ 0 & 5 & 104 \end{pmatrix} \qquad \begin{array}{ll} M_1 = 5 & > 0 \\ M_2 = -369 & < 0 \\ M_3 = 104 \cdot M_2 & < 0. \end{array}$$

$$p_3, \; M_1, \; \frac{M_2}{M_1}, \; \frac{M_3}{M_2} : \qquad +, +, -, +.$$

Once again we have to consider complex zeros and we can see that there are two changes of the signature. Consequently we have two zeros with a positive real part. The Sturm chain and the Cauchy-Index give us

$$I_{-\infty}^{0}(p(\nu)) = 1, \;\; I_0^{+\infty}(p(\nu)) = 2.$$

Taking care of the result from the Hurwitz matrix we know that the two conjugate-complex zeros have a positive real part. The situation of the eigenvalues is as follows:

$$\nu_1 \in \mathbb{R}, \; \nu_{2,3} \in \mathbb{C} \qquad \wedge \qquad \nu_1 < 0, \; \mathrm{Re}(\nu_{2,3}) > 0.$$

We have two zeros in the right half plane, hence our present system is unstable.

Matrix of the System	$\begin{pmatrix} -1 & 5 & 1 \\ 5 & -3 & -4 \\ 7 & -7 & -1 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 5\nu^2 - 53\nu + 104$
D (from Cardano's Method)	> 0
Roots of $p(\nu)$	$\nu_1 \in \mathbb{R},\ \nu_{2,3} \in \mathbb{C}\ \wedge\ \nu_1 < 0,\ \mathrm{Re}(\nu_{2,3}) > 0$
Result of Hurwitz' Stability Criterion	unstable
Result of the Qualitative Analysis	unstable

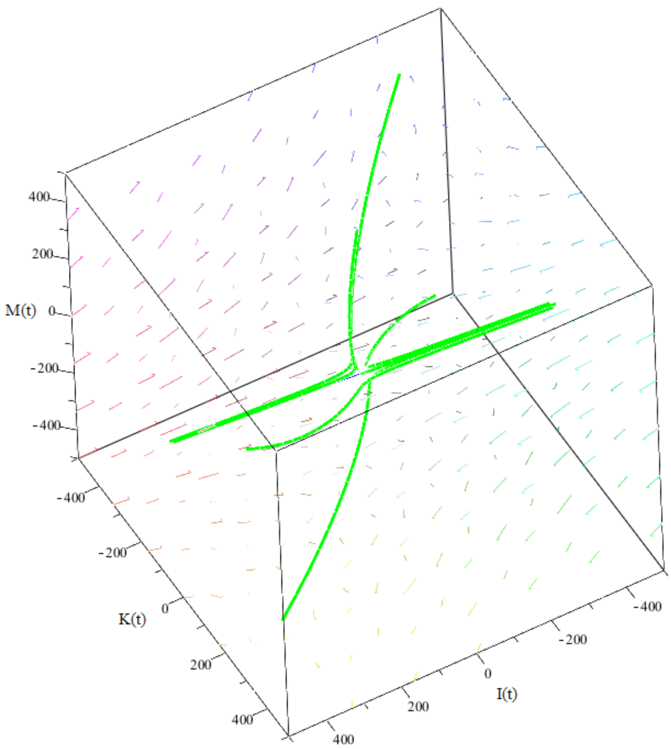

Figure 2.11: Plot of the phase portrait of the system with the parameters $a = 5,\ b = 1,\ c = 5,\ d = 3,\ e = 1,\ f = 4,\ g = 7,\ h = 7,\ m = 1$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

Example of an asymptotically stable system with $D = 0$

Note that we now have just two zeros of $p(\nu)$, whereas one of them is doubled since $D = 0$. In this case we have a simple structured matrix like the following

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 9\nu^2 + 24\nu + 20$.

- Checking the necessary and sufficient condition of Hurwitz' stability criterion brings out quickly that this system has to be asymptotically stable.
- For a qualitative analysis let us check the Hurwitz matrix again.

$$H(p) = \begin{pmatrix} 9 & 20 & 0 \\ 1 & 24 & 0 \\ 0 & 9 & 20 \end{pmatrix} \qquad \begin{array}{ll} M_1 = 9 & > 0 \\ M_2 = 196 & > 0 \\ M_3 = 20 \cdot M_2 & > 0. \end{array}$$

$$p_3,\ M_1,\ \frac{M_2}{M_1},\ \frac{M_3}{M_2}: \qquad +, +, +, +.$$

Since we do not have a change of signatures there is no positive zero of $p(\nu)$. Investigating the Sturm chain and the Cauchy-Index gives us

$$I_{-\infty}^{0}(p(\nu)) = 2, \quad I_0^{+\infty}(p(\nu)) = 0.$$

Now we know that both zeros are real and negative,

$$\nu_i \in \mathbb{R} \ \wedge \ \nu_i < 0 \qquad \forall\ i = 1, 2, 3$$

and therefore the present system is asymptotically stable.

Matrix of the System	$\begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -3 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 9\nu^2 + 24\nu + 20$
D (from Cardano's Method)	$= 0$
Roots of $p(\nu)$	$\nu_i \in \mathbb{R} \ \wedge \ \nu_i < 0 \qquad \forall\, i = 1, 2, 3$
Result of Hurwitz' Stability Criterion	asymptotically stable
Result of the Qualitative Analysis	asymptotically stable

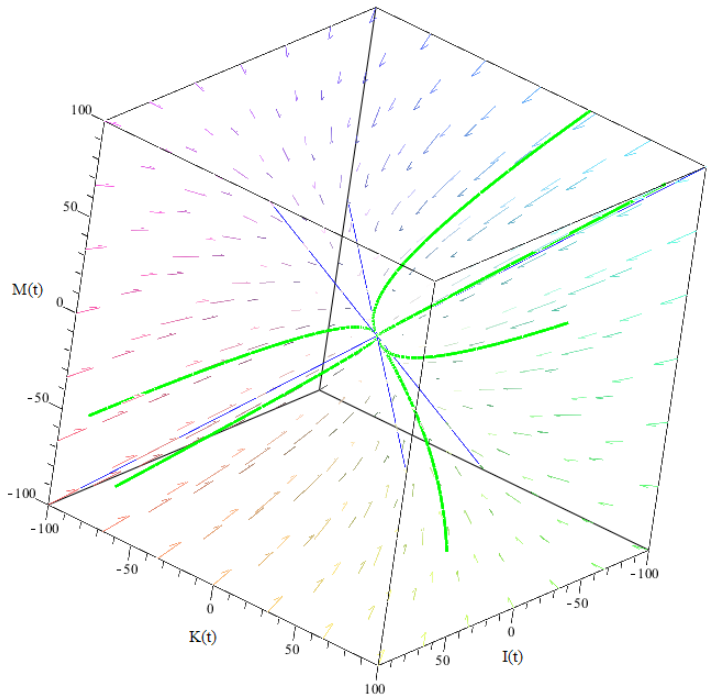

Figure 2.12: Plot of the phase portrait of the system with the parameters $a = 1,\ b = 3,\ c = 1,\ d = 3,\ e = 1,\ f = 1,\ g = 1,\ h = 1,\ m = 3$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

Example of an unstable system with $D = 0$

Now we consider the following system

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -2 & 5 & 5 \\ 5 & -2 & -5 \\ 5 & -5 & -2 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 6\nu^2 - 63\nu + 108$.

- One single look at the characteristical polynomial and the singatures within brings out that the system has to be unstable.
- The Hurwitz matrix and its minors looks as follows

$$H(p) = \begin{pmatrix} 6 & 108 & 0 \\ 1 & -63 & 0 \\ 0 & 6 & 108 \end{pmatrix} \qquad \begin{array}{ll} M_1 = 6 & > 0 \\ M_2 = -486 & < 0 \\ M_3 = 108 \cdot M_2 & < 0. \end{array}$$

$$p_3,\ M_1,\ \frac{M_2}{M_1},\ \frac{M_3}{M_2}: \qquad +, +, -, +.$$

Two changes in this sequence again imply two positive zeros which are of course the double ones. Again the Sturm chain and the Cauchy-Index yield

$$I_{-\infty}^{0}(p(\nu)) = 1, \quad I_{0}^{+\infty}(p(\nu)) = 1.$$

Knowing this we could also rewrite the characteristic polynomial $p(\nu) = \nu^3 + 6\nu^2 - 63\nu + 108 = (\nu + \nu_1)(\nu - \nu_2)^2$ and therefore

$$\nu_i \in \mathbb{R},\ \forall i = 1, 2, 3 \qquad \wedge \qquad \nu_1 < 0,\ \nu_{2,3} > 0$$

detect an unstable case. $(\nu_1 = 12,\ \nu_{2,3} = 3)$

Matrix of the System	$\begin{pmatrix} -2 & 5 & 5 \\ 5 & -2 & -5 \\ 5 & -5 & -2 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 6\nu^2 - 63\nu + 108$
D (from Cardano's Method)	$= 0$
Roots of $p(\nu)$	$\nu_i \in \mathbb{R},\ \forall i = 1, 2, 3 \quad \wedge \quad \nu_1 < 0,\ \nu_{2,3} > 0$
Result of Hurwitz' Stability Criterion	unstable
Result of the Qualitative Analysis	unstable

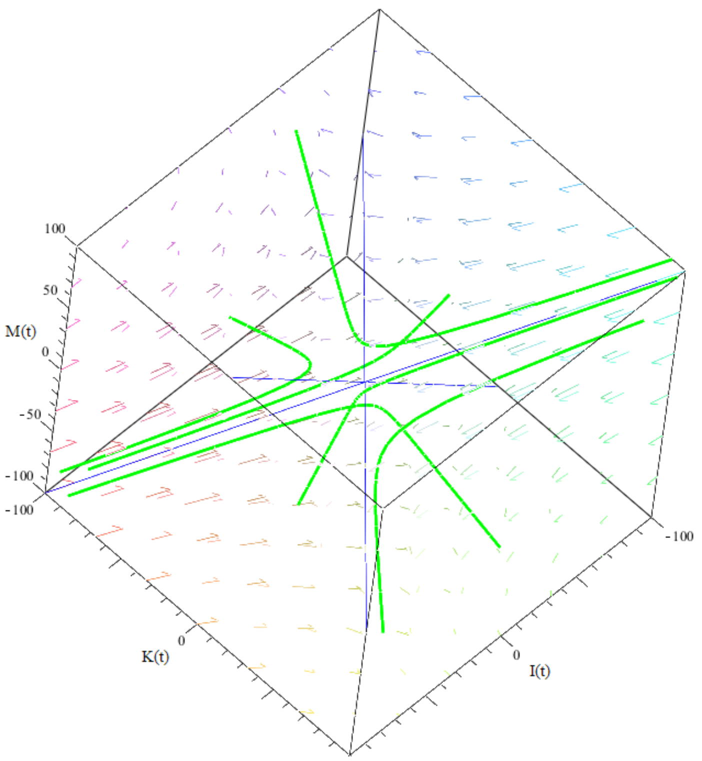

Figure 2.13: Plot of the phase portrait of the system with the parameters $a = 5,\ b = 2,\ c = 5,\ d = 2,\ e = 5,\ f = 5,\ g = 5,\ h = 5,\ m = 2$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

Example of a stable system with *D = 0*

The last presented example is a special case of the form

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \end{pmatrix} = \begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \end{pmatrix}$$

with the characteristic polynomial $p(\nu) = \nu^3 + 6\nu^2 = \nu^2(\nu + 6)$.

This is a special case since the matrix of the system is singular and our term D from Cardano's method is zero. We can read off the eigenvalues directly from the polynomial:

$$\nu_1 = -6 \qquad \nu_{2,3} = 0.$$

Regarding the minors of the Hurwitz matrix instantly shows that this case is stable and further calculations are not necessary.

Matrix of the System	$\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ 2 & -2 & -2 \end{pmatrix}$
Characteristic Polynomial	$p(\nu) = \nu^3 + 6\nu^2$
D (from Cardano's Method)	$= 0$
Roots of $p(\nu)$	$\nu_i \in \mathbb{R},\ \forall i = 1,2,3 \quad \wedge \quad \nu_1 < 0,\ \nu_{2,3} = 0$
Result of Hurwitz' Stability Criterion	-
Result of the Qualitative Analysis	-

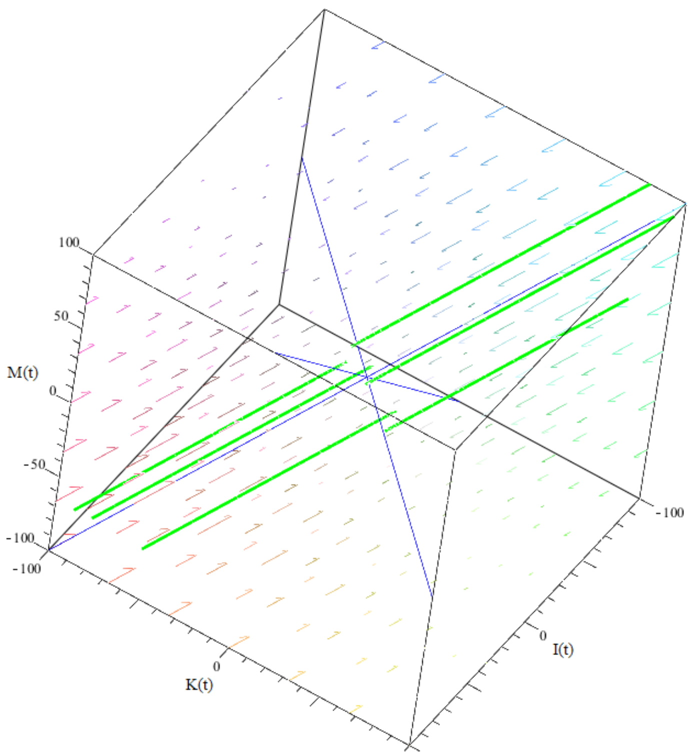

Figure 2.14: Plot of the phase portrait of the system with the parameters $a = 2,\ b = 2,\ c = 2,\ d = 2,\ e = 2,\ f = 2,\ g = 2,\ h = 2,\ m = 2$. The initial values of the plotted trajectories are chosen randomly. The blue lines represent the eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

2.2.2 Equilibrium Solution

In Hausken et al.'s three-dimensional model [12, Hausken et al. 2012] we now want to address the issue of the equilibrium points. Therefore we have to solve the following system of linear equations:

$$\begin{aligned} -bI + aK + eM &= 0 \\ cI - dK - fM &= 0 \\ gI - hK - mM &= 0. \end{aligned}$$

Besides the trivial equilibrium $I(t) = K(t) = M(t) = 0$ we get other equilibrium points under the condition

$$g(de - af) + m(ac - bd) + h(bf - ec) = 0,$$

which means that the matrix of the system is singular. The only situations with non-trivial equilibrium points are those with at least one eigenvalue equal to zero. When looking at the examples given above we have to consider the (not asymptotically) stable cases to get further informations.

2.3 Including Captive Participants

Ideologues and criminal mercenaries need captive participants to operate efficiently and it is assumed that sponsors are unaffected by them. Thus the model is generalised to

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \\ \dot{C}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -b & a & e & n \\ c & -d & -f & 0 \\ g & -h & -m & o \\ p & 0 & q & -r \end{pmatrix}}_{\mathbf{C}} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \\ C(t) \end{pmatrix}$$

where $n, o, p, q, r > 0$ are parameters. $C(t)$ is the amount of labour exerted by captive participants and is constrained by itself from unbounded growth. This system can be solved as previously. The zeros of the charactaristic polynomial of $\mathbf{C}$ lead to the four eigenvalues τ_1, τ_2, τ_3 and τ_4 and their corresponding eigenvectors $\mathbf{s}_1$, $\mathbf{s}_2, \mathbf{s}_3$ and $\mathbf{s}_4$. Again the solution of the differential equation system can be written as

$$\begin{pmatrix} I(t) \\ K(t) \\ M(t) \\ C(t) \end{pmatrix} = \mathbf{T} \begin{pmatrix} e^{\tau_1 t} & 0 & 0 & 0 \\ 0 & e^{\tau_2 t} & 0 & 0 \\ 0 & 0 & e^{\tau_3 t} & 0 \\ 0 & 0 & 0 & e^{\tau_4 t} \end{pmatrix} \mathbf{T}^{-1} \begin{pmatrix} I(0) \\ K(0) \\ M(0) \\ C(0) \end{pmatrix}$$

where $\mathbf{T}$ is once more the transformation matrix containing the four eigenvectors, $\mathbf{T} = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4)$.

2.3.1 Stability Analysis

In fact it is possible to investigate this four-dimensional model, too. For descriptive purpose we will show up the theoretical way for such an analysis. Actually we need the same methods as applied above but with some alterations.

Hurwitz' stability criterion [7, Frey et al. 2008] works in the same way as in the previous case but needs one more sufficient condition. The necessary condition on the characteristic polynomial is that the coefficients p_i are non-zero and of the same sign as above. The two sufficient conditions are as follows:

- $p_1 p_2 - p_0 p_3 > 0$ (as above)

- $p_1p_2p_3 - p_1^2p_4 - p_0p_3^2 > 0.$ (see Appendix)

If these two conditions are fulfilled, the characteristic polynomial is a Hurwitz polynomial and the present system is asymptotically stable.

The qualitative analysis works exactly the same way but is actually more difficult to examine since we always need to determine one more step because of the higher dimension. Unfortunately we do not have any means to find out whether we have real or complex eigenvalues. But indeed this fact will come out during the further analysis.

2.3.2 Equilibrium Solution

For the sake of completeness we will also give the condition, which gives us non-trivial equilibrium points. Besides the trivial equilibrium point $I(t) = K(t) = M(t) = C(t) = 0$ we receive other ones in case of

$$np(hf - dm) + nq(ch - gd) + (af - de)(op + rg) + (bd - ac)(rm - oq) + rh(ec - bf) = 0.$$

Chapter 3

The Model with Government Intervention

Hausken et al. did not only present the previously described models, which are all homogenous systems of differential equations. They also presented a model with government intervention, an inhomogenous one, see also [12, Hausken et al. 2012].

The government may target a terrorist organisation by attacking the four forces $I(t), K(t), M(t)$ and $C(t)$ of the previous section, i.e. the ideologues, the capital flow, the criminal mercenaries and the captive participants. Thus the last presented model can be extended by a residual so that the generalised model looks as follows:

$$\begin{pmatrix} \dot{I}(t) \\ \dot{K}(t) \\ \dot{M}(t) \\ \dot{C}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} -b & a & e & n \\ c & -d & -f & 0 \\ g & -h & -m & o \\ p & 0 & q & -r \end{pmatrix}}_{\mathbf{C}} \begin{pmatrix} I(t) \\ K(t) \\ M(t) \\ C(t) \end{pmatrix} + \begin{pmatrix} -sG_I \\ -uG_K \\ -vG_M \\ -wG_C \end{pmatrix}.$$

Again we have the initial conditions $I(0), K(0), M(0), C(0) > 0$. Hausken defined G_I, G_K, G_M and G_C to be the labour efforts exerted by the government to combat the four forces, respectively. $s, u, v, w > 0$ are parameters. More information on the coefficients G_I, G_K, G_M and G_C can be reviewed in [12, Hausken et al. 2012].

3.1 Solving the Inhomogenous Model

The general solution of an inhomogenous system consists of the general solution of the homogenous system and a specific solution of the inhomogenous one, [17, Mlitz 2008]. The model with government intervention whereof we are searching for a general solution is of the form

$$\dot{\mathbf{y}}(t) = \mathbf{C} \cdot \mathbf{y}(t) + \mathbf{l} \quad \in \mathbb{R}^4 \qquad\qquad \mathbf{y}(0) \in \mathbb{R}^4,$$

which we will use in what follows. Since we have already formed the solution of the homogenous system in the last chapter we are able to cut short the determination of the solution of the inhomogenous one. The solution given in Section 2.3 consists of the four functions $I(t), K(t), M(t), C(t)$. Expanding the matrices yields the concrete forms

$$\begin{pmatrix} I(t) \\ K(t) \\ M(t) \\ C(t) \end{pmatrix} = e^{\tau_1 t} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} + e^{\tau_2 t} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} + e^{\tau_3 t} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} + e^{\tau_4 t} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix}.$$

Since we already have the fundamental system we also know the corresponding *Wronski Matrix*

$$W(t) = \begin{pmatrix} \alpha_1 e^{\tau_1 t} & \beta_1 e^{\tau_2 t} & \gamma_1 e^{\tau_3 t} & \delta_1 e^{\tau_4 t} \\ \alpha_2 e^{\tau_1 t} & \beta_2 e^{\tau_2 t} & \gamma_2 e^{\tau_3 t} & \delta_2 e^{\tau_4 t} \\ \alpha_3 e^{\tau_1 t} & \beta_3 e^{\tau_2 t} & \gamma_3 e^{\tau_3 t} & \delta_3 e^{\tau_4 t} \\ \alpha_4 e^{\tau_1 t} & \beta_4 e^{\tau_2 t} & \gamma_4 e^{\tau_3 t} & \delta_4 e^{\tau_4 t} \end{pmatrix}.$$

The Wronski Matrix is always regular, $\det W(t) \neq 0$. Consequently its inverse $W(t)^{-1}$ also exists. The equation $\dot{W}(t) = \mathbf{C} \cdot W(t)$ is also true for the Wronski Matrix. With the approach $\mathbf{y}(t) = W(t) \cdot \mathbf{x}(t)$ we get

$$\begin{aligned} \dot{\mathbf{y}}(t) &= \dot{W}(t) \cdot \mathbf{x}(t) + W(t) \cdot \dot{\mathbf{x}}(t) \\ &= \mathbf{C} \cdot \mathbf{y}(t) + W(t) \cdot \dot{\mathbf{x}}(t). \end{aligned}$$

Substituting this into the original system yields $\mathbf{l} = W(t)\cdot\dot{\mathbf{x}}(t)$. As a consequence we get $\dot{\mathbf{x}}(t) = W(t)^{-1}\cdot\mathbf{l}$. Integrating $\dot{\mathbf{x}}(t)$ yields

$$\mathbf{x}(t) \;=\; \int W(j)^{-1}\cdot\mathbf{l}\,\mathrm{d}j.$$

Note that $\dot{\mathbf{x}}(t)$ is a vector and has to be integrated component wise. Now we have a specific solution of the inhomogenous system:

$$\mathbf{y}_s(t) \;=\; \int W(j)^{-1}\cdot\mathbf{l}\,\mathrm{d}j.$$

After all the general solution of the inhomogenous system is of the form

$$\mathbf{y}(t) \;=\; W(t)\cdot\left(\int_0^t W(j)^{-1}\cdot\mathbf{l}\,\mathrm{d}j + \mathbf{z}\right)$$

where $\mathbf{z}$ contains the initial values and $\mathbf{z} = W(0)^{-1}\cdot\mathbf{y}(0)$.

3.2 Stability Analysis

The stability analysis of the four-dimensional system is not the issue of this thesis and will not be given because it would take too much space. Theoretically it works exactly as described in the appendix.

Chapter 4

Parameter Constraints Assuring Feasibility of the State Variables

In our further investigations we will address only the case of asymptotically stable two-dimensional systems. Therefore we will focus on the functions $I(t)$ and $K(t)$ given in Section 2.1. Before looking at some particular two-dimensional systems we have to consider those solutions more accurately. For the purpose of realism we want these two solution functions to be positive, for $t > 0$, because negative values do not make sense. Therefore we eventually have to impose some constraints on the parameters $a, b, c, d > 0$ and the initial conditions $I(0), K(0) > 0$, which the functions depend on. That means, these constraints shell force the functions to be positive. Hence we have to examine the solution of the two-dimensional model, as given in Section 2.1, in more detail. Expanding the matrices yields the functions of the form

$$\begin{aligned} I(t) &= \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} \\ K(t) &= \gamma e^{\lambda_1 t} + \delta e^{\lambda_2 t} \end{aligned}$$

with the four terms

$$\alpha = \frac{aK(0)(c + d + \lambda_2) - cI(0)(a + b + \lambda_2)}{(c + d + \lambda_2)\sqrt{(b - d)^2 + 4ac}}$$

$$\begin{aligned}
\beta &= \frac{cI(0)(c+d+\lambda_1) - aK(0)(c+d+\lambda_1)}{(c+d+\lambda_1)\sqrt{(b-d)^2+4ac}} \\
\gamma &= \frac{cI(0)(a+b+\lambda_2) - aK(0)(c+d+\lambda_2)}{(a+b+\lambda_2)\sqrt{(b-d)^2+4ac}} \\
\delta &= \frac{aK(0)(a+b+\lambda_1) - cI(0)(a+b+\lambda_1)}{(c+d+\lambda_1)\sqrt{(b-d)^2+4ac}}.
\end{aligned}$$

Taking a closer look at the special case when $a+b = c+d$ implies $\lambda_2 = -(a+b) = -(c+d)$, which further implies that the terms α and γ are of the form $\frac{0}{0}$ and thus not defined, as is the eigenvector $\mathbf{v}_2 = (\frac{a}{a+b+\lambda_2}, \frac{c}{c+d+\lambda_2})^T$ of λ_2. This issue will be investigated accurately at the end of this chapter. So we have to examine the three seperate constellations of the parameters

1. $a+b < c+d$
2. $a+b > c+d$
3. $a+b = c+d$.

4.1 $a+b < c+d$

Before we address the positivity of $I(t)$ and $K(t)$ it might be useful to know the signs of the four terms $a+b+\lambda_{1,2}$ and $c+d+\lambda_{1,2}$, which occur very often. Because of $a+b < c+d$ and $\lambda_2 < \lambda_1$ we know that

$$a+b+\lambda_2 < \left\{ \begin{array}{c} a+b+\lambda_1 \\ c+d+\lambda_2 \end{array} \right\} < c+d+\lambda_1.$$

We now assume $c+d+\lambda_2$ to be positive. If this assumption holds true we know that $c+d+\lambda_1$ has to be positive, too.

$$\begin{aligned}
c+d+\lambda_2 &= c+d-\frac{1}{2}\left(b+d+\sqrt{(b-d)^2+4ac}\right) &> 0 \\
\Leftrightarrow \qquad & \sqrt{(b-d)^2+4ac} &< 2c+d-b.
\end{aligned}$$

To continue with the inequality it is necessary to take the square of it, which leaves equivalence only if both sides are non-negative. To guarantee this we need another constraint, which is $b < 2c+d$. In fact this is already implied by $a+b < c+d$ so we do not have to worry about this.

$$\begin{aligned} & (b-d)^2+4ac & < & \quad 4c^2+4c(d-b)+(d-b)^2 \\ \Leftrightarrow \quad & a+b & < & \quad c+d. \end{aligned}$$

This is a true statement and therefore we know that $c+d+\lambda_2 > 0$ and $c+d+\lambda_1 > 0$ without any further conditions. The analogous investigation for the assumption $a+b+\lambda_2 < 0$ leads to the condition $d < 2a+b$, which is no direct contradiction to $a+b < c+d$. However, with this information we are not able to figure out the sign of the corresponding term $a+b+\lambda_1$. We get the analogous result for the assumption $a+b+\lambda_1 > 0$ with the condition $d > 2a+b$. To keep an overview the whole situation is listed in the following table.

condition	$a+b+\lambda_1$	$a+b+\lambda_2$
$d < 2a+b$	?	< 0
$d > 2a+b$	> 0	?
$d = 2a+b$	> 0	< 0

Therefore we try to get the missing signs of the certain cases in another way. In case of $d < 2a+b$ we know that $a+b+\lambda_2 < 0$. To get the sign of $a+b+\lambda_1$ we consider the product of both terms.

$$\begin{aligned} & \underbrace{(a+b+\lambda_1)}_{?}\underbrace{(a+b+\lambda_2)}_{<0} & = & \quad a\cdot\underbrace{(a+b-c-d)}_{<0} \; < \; 0 \\ \Rightarrow \quad & (a+b+\lambda_1) & > & \quad 0 \end{aligned}$$

Hence, $a+b+\lambda_1$ is always positive, no matter if $d < 2a+b$ or $d > 2a+b$. In the same way we figure out that $a+b+\lambda_2$ is negative in both cases. Actually we do not have any additional conditions for these four terms and we know every sign of them. Any other assumption leads to a direct contradiction of the present condition $a+b < c+d$.

Now let us focus on the quest for $I(t) > 0$. Trandforming this inequality leads to the following constraint:

$$e^{\lambda_2 t}\frac{aK(0)(c+d+\lambda_1)-cI(0)(a+b+\lambda_1)}{(c+d+\lambda_1)\sqrt{(b-d)^2+4ac}} \quad < \quad e^{\lambda_1 t}\frac{aK(0)(c+d+\lambda_2)-cI(0)(a+b+\lambda_2)}{(c+d+\lambda_2)\sqrt{(b-d)^2+4ac}}$$

$$\frac{aK(0)(c+d+\lambda_1)-cI(0)(a+b+\lambda_1)}{aK(0)(c+d+\lambda_2)-cI(0)(a+b+\lambda_2)} \quad < \quad e^{(\lambda_1-\lambda_2)t}\frac{c+d+\lambda_1}{c+d+\lambda_2}.$$

Of course we have to do the analogous calculations to assure $K(t) > 0$, which leads us to the similar result

$$e^{(\lambda_1-\lambda_2)t}\frac{a+b+\lambda_1}{a+b+\lambda_2} \quad < \quad \frac{aK(0)(c+d+\lambda_1)-cI(0)(a+b+\lambda_1)}{aK(0)(c+d+\lambda_2)-cI(0)(a+b+\lambda_2)}.$$

Combining these inequalities results in the joint condition for $I(t)$ and $K(t)$ in case of $a+b < c+d$. Actually this condition is not very easy to deal with. Nevertheless we take a look at it in order to eventually replace it by some restrictions which are easier to handle. In this case the lower and upper bounds are exponential functions converging to $-\infty$ and ∞.

$$\underbrace{e^{(\lambda_1-\lambda_2)t}\frac{a+b+\lambda_1}{a+b+\lambda_2}}_{\to -\infty} < \frac{aK(0)(c+d+\lambda_1)-cI(0)(a+b+\lambda_1)}{aK(0)(c+d+\lambda_2)-cI(0)(a+b+\lambda_2)} < \underbrace{e^{(\lambda_1-\lambda_2)t}\frac{c+d+\lambda_1}{c+d+\lambda_2}}_{\to \infty}$$

To get an idea of what this condition actually looks like we take a look at Figure 4.1.

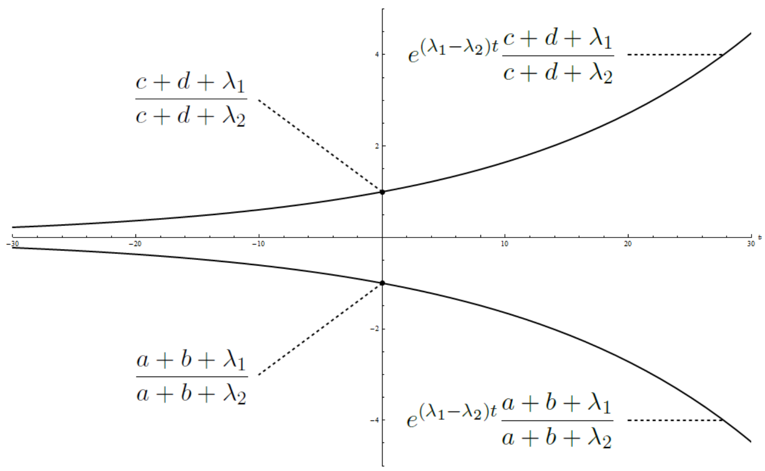

Figure 4.1: Randomly chosen plot of the upper and lower bound of the condition for positive functions in case $a+b < c+d$.

The value of the lower bound at $t = 0$ is equal to $\frac{a+b+\lambda_1}{a+b+\lambda_2}$. The value of the upper bound is equal to $\frac{c+d+\lambda_1}{c+d+\lambda_2}$. Since we are only interested in values for $t > 0$ we can claim for the middle term to be in the interval

$$\frac{aK(0)(c+d+\lambda_1) - cI(0)(a+b+\lambda_1)}{aK(0)(c+d+\lambda_2) - cI(0)(a+b+\lambda_2)} \in \left(\frac{a+b+\lambda_1}{a+b+\lambda_2}, \frac{c+d+\lambda_1}{c+d+\lambda_2}\right)$$

heading to the next obvious inequalities

$$\frac{a+b+\lambda_1}{a+b+\lambda_2} < \frac{aK(0)(c+d+\lambda_1) - cI(0)(a+b+\lambda_1)}{aK(0)(c+d+\lambda_2) - cI(0)(a+b+\lambda_2)} < \frac{c+d+\lambda_1}{c+d+\lambda_2},$$

which will be investigated, respectively. To handle those inequalities correctly note that now there is the condition $a + b < c + d$ and the signs $a + b + \lambda_2 < 0$, $a+b+\lambda_1 > 0$, $c+d+\lambda_2 > 0$ and $c+d+\lambda_1 > 0$. Starting with the first inequality yields:

$$\begin{aligned}
\frac{a+b+\lambda_1}{a+b+\lambda_2} &< \frac{aK(0)(c+d+\lambda_1) - cI(0)(a+b+\lambda_1)}{aK(0)(c+d+\lambda_2) - cI(0)(a+b+\lambda_2)} \\
(c+d+\lambda_1)(a+b+\lambda_2) &< (c+d+\lambda_2)(a+b+\lambda_1) \\
\lambda_1(a+b-c-d) &< \lambda_2(a+b-c-d) \\
\lambda_1 &> \lambda_2.
\end{aligned}$$

This is a true statement always. Handling the second inequality analogously implies that no further constraints are necessary to get positive functions $I(t), K(t)$ in case of $a + b < c + d$.

$$\forall\ a, b, c, d > 0,\ \forall\ t > 0 \text{ and } \forall\ I(0), K(0) > 0 : \qquad I(t), K(t) > 0$$

4.2 $a + b > c + d$

The analogous assumptions and calculations are performed for this case, too, and lead to a similar result, which is no real surprise. The mutual inequality for both functions looks as follows and has to be treated as in the previous case:

$$\underbrace{e^{(\lambda_1-\lambda_2)t}\frac{c+d+\lambda_1}{c+d+\lambda_2}}_{\to-\infty} < \frac{aK(0)(c+d+\lambda_1)-cI(0)(a+b+\lambda_1)}{aK(0)(c+d+\lambda_2)-cI(0)(a+b+\lambda_2)} < \underbrace{e^{(\lambda_1-\lambda_2)t}\frac{a+b+\lambda_1}{a+b+\lambda_2}}_{\to\infty}$$

Since this inequality turns out to hold true always we can also renounce any further constraints to obtain positive functions $I(t), K(t)$ in case of $a+b > c+d$.

$$\forall\ a,b,c,d > 0,\ \forall\ t > 0 \text{ and } \forall\ I(0), K(0) > 0 : \qquad I(t), K(t) > 0$$

4.3 $a + b = c + d$

Let us focus on this constellation. As mentioned at the beginning of this chapter, the problem starts with the eigenvector $\mathbf{v}_2$, which is actually not defined in this case. Hence there is no transformation matrix $\mathbf{T}$ and we can not even represent a solution of the system. Shortcutting the considerations, assume the second eigenvector to be $\mathbf{v}_2 = (-1, 1)^T$. The proof of this fact will be given at the end of this section.

Remember the second eigenvalue $\lambda_2 = -(a+b) = -(c+d)$. We can also reduce the first eigenvalue to $\lambda_1 = c - b = a - d$. The first eigenvector then reads as $\mathbf{v}_1 = (a, c)^T$. Thus the transformation matrix and its inverse are of the facile form

$$\mathbf{T} = \begin{pmatrix} a & -1 \\ c & 1 \end{pmatrix} \qquad\qquad \mathbf{T}^{-1} = \frac{1}{a+c}\begin{pmatrix} 1 & 1 \\ -c & a \end{pmatrix}.$$

Now we are able to determine the solution of the differential equation system. The functions $I(t)$ and $K(t)$ are of the same form as in the two previous cases but with modified coefficients, which are

$$\alpha = \frac{a(K(0)+I(0))}{a+c}$$

$$\beta = \frac{cI(0) - aK(0)}{a+c}$$

$$\begin{aligned} \gamma &= \frac{c(K(0)+I(0))}{a+c} \\ \delta &= \frac{aK(0)-cI(0)}{a+c}. \end{aligned}$$

Since $a+c \neq 0$ always applies, these modified coefficients are well defined and perfectly fit to the original ones. The claim of both functions to be positive leads to the simplified inequalities

$$I(t), K(t) > 0 \qquad \Leftrightarrow \qquad \underbrace{-e^{(\lambda_1-\lambda_2)t}c}_{\to -\infty} < \frac{aK(0)-cI(0)}{I(0)+K(0)} < \underbrace{e^{(\lambda_1-\lambda_2)t}a}_{\to\infty}$$

which are again always fulfilled for $t > 0$. Again it holds true that no further constraints are required to get positive functions $I(t), K(t)$ in case of $a+b = c+d$:

$$\forall\ a, b, c, d > 0,\ \forall\ t > 0 \text{ and } \forall\ I(0), K(0) > 0 : \qquad I(t), K(t) > 0$$

Nevertheless, the proof for the second eigenvector $\mathbf{v}_2 = (-1,1)^T$ is still missing. Let us consider the range of $(-1,1)^T$ under the linear transformation $\mathbf{A}$,

$$\begin{pmatrix} -b & a \\ c & -d \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} b+a \\ -c-d \end{pmatrix} = \begin{pmatrix} b+a \\ -b-a \end{pmatrix} = (-a-b)\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus, by definition, $(-1,1)^T$ is an eigenvector of the linear transformation $\mathbf{A}$ with the eigenvalue $-a-b = \lambda_2$.

■

4.4 Examples

In order to understand the previous calculations a little bit more easily, this section illustrates some examples for all three investigated cases with a plot of the phase portrait, representative examples for the corresponding functions $I(t)$ & $K(t)$ and the lower and upper bounds derived in the previous sections.

$a + b < c + d$

Starting with the first case $a+b<c+d$, a system with the following values

a	b	c	d	$I(0)$	$K(0)$	λ_1	λ_2
2	3	3	4	1	9	-1	-6

has the solution

$$\begin{aligned} I(t) &= \frac{21}{5}e^{-t} - \frac{16}{5}e^{-6t} \\ K(t) &= \frac{21}{5}e^{-t} + \frac{24}{5}e^{-6t}. \end{aligned}$$

The phase portrait and the solution of the system are plotted below.

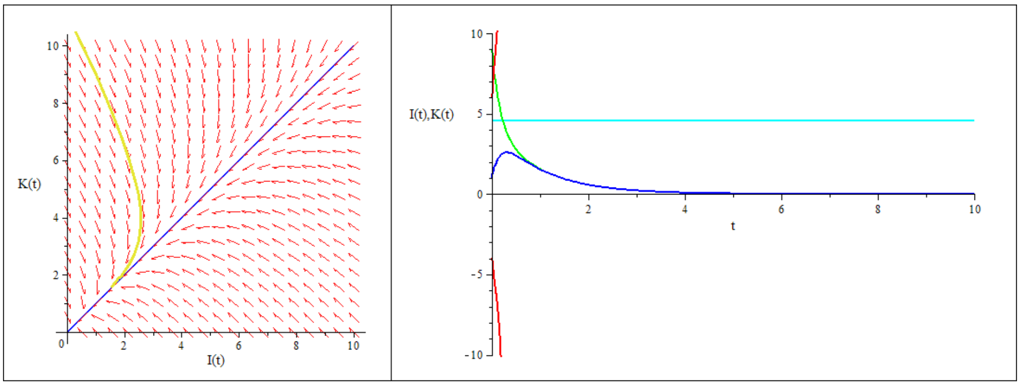

Figure 4.2: The left plot shows the phase portrait of the present system and the right one the corresponding functions. $I(t)$ is represented by the blue line and $K(t)$ by the green one. The red lines represent the upper and lower bounds assuring positivity as long as the cyan line (representing the middle term of the inequality) lies in between. As one can see the inequality is fulfilled as generally shown above.

$a+b>c+d$

An example of a system in the second case $a+b>c+d$ is given h the values

a	b	c	d	$I(0)$	$K(0)$	λ_1	λ_2
2	5	1	2	3	2	$-\frac{7-\sqrt{17}}{2}$	$-\frac{7+\sqrt{17}}{2}$

and the solution

$$\begin{aligned} I(t) &= \frac{25+\sqrt{17}}{17+\sqrt{17}}e^{-\frac{7-\sqrt{17}}{2}t}+\frac{25-\sqrt{17}}{17-\sqrt{17}}e^{-\frac{7+\sqrt{17}}{2}t} \\ K(t) &= \frac{25+\sqrt{17}}{-17+7\sqrt{17}}e^{-\frac{7-\sqrt{17}}{2}t}+\frac{25+\sqrt{17}}{17-7\sqrt{17}}e^{-\frac{7+\sqrt{17}}{2}t} \end{aligned}$$

The phase portrait and the corresponding functions look as follows.

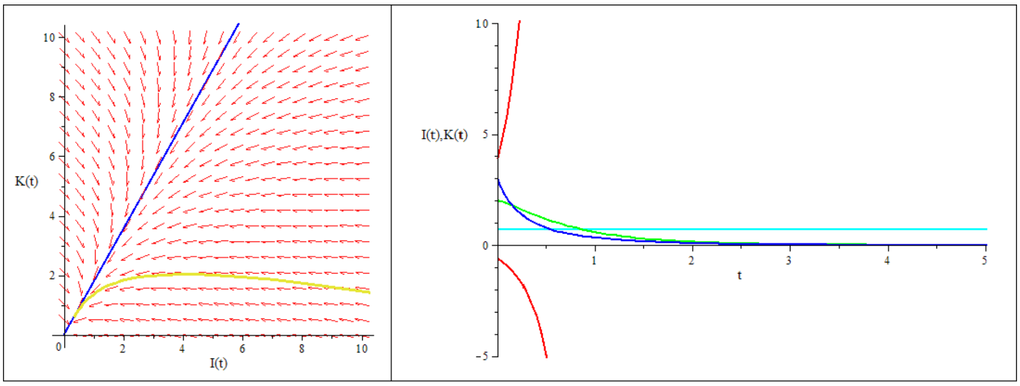

Figure 4.3: The left plot shows the phase portrait of the present system and the right one the corresponding functions. $I(t)$ is represented by the blue line and $K(t)$ by the green one. The red lines represent the upper and lower bounds assuring positivity as long as the cyan line (representing the middle term of the inequality) lies in between. As one can see the inequality is fulfilled as generally shown above.

$a + b = c + d$

The last and special case of $a + b = c + d$ is represented by the values

a	b	c	d	$I(0)$	$K(0)$	λ_1	λ_2
3	5	4	4	8	2	-1	-8

and the solution

$$\begin{aligned} I(t) &= \frac{30}{7}e^{-t} + \frac{26}{7}e^{-8t} \\ K(t) &= \frac{40}{7}e^{-t} - \frac{26}{7}e^{-8t}. \end{aligned}$$

The phase portrait and the corresponding functions look as follows.

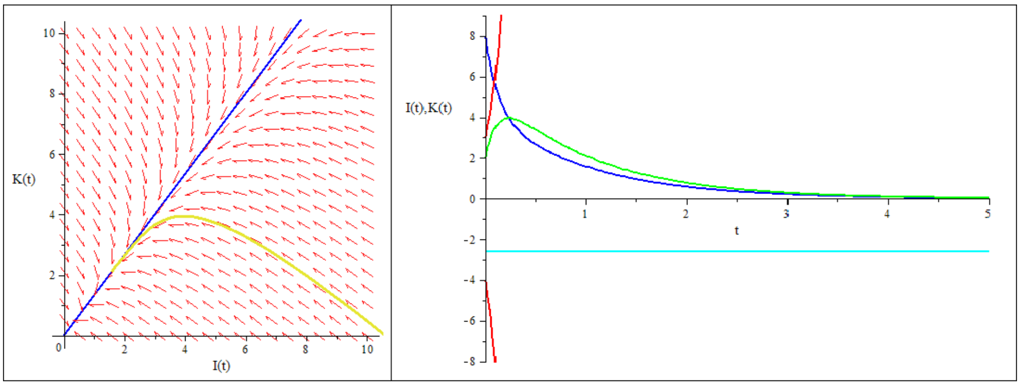

Figure 4.4: The left plot shows the phase portrait of the present system and the right one the corresponding functions. $I(t)$ is represented by the blue line and $K(t)$ by the green one. The red lines represent the upper and lower bounds assuring positivity as long as the cyan line (representing the middle term of the inequality) lies in between. As one can see the inequality is fulfilled as generally shown above.

Summarising this chapter we may conclude that no further conditions on the parameters or the initial values are required. In every case we get positive functions $I(t)$ and $K(t)$, $\quad \forall a, b, c, d > 0$, $\forall t > 0$ and $\forall I(0), K(0) > 0$.

Chapter 5

Dynamic Control of a Terrorist Organisation

In this chapter we want to discuss some strategic investigations in order to give some sort of pre-step to an optimal control model formulation. Our purpose is to determine the relative efficiency of some control at different stages of the terrorist organisation. For that purpose we consider some control intervention by changing the parameters a, b, c, d in the system representing the organisation. These changes will concern only one parameter per analysis, but we will carry out several examinations with every parameter. Furthermore those changes will not be made arbitrarily but rather for one year and at different stages of the terrorist organisation similar to the examinations pertaining to treatment of a drug epidemic in [20, Ranner 2009]. Note that - for ease of exposition - we call one time unit simply "one year". In particular one parameter will be increased or decreased in the first year, then in the second and so on. We will carry out these analyses for the first ten years. Finally we figure out the one year when the effectiveness of the control is the biggest. We will carry out the analysis on the objective functional

$$J = \int_0^\infty e^{-rt}(\psi I(t) + \varphi K(t))\mathrm{d}t,$$

which we determine to describe the damage caused by terrorists over an infinite planning horizon. r represents the annual discounting rate and ψ and φ are some weighting coefficients on the two functions $I(t)$ and $K(t)$. For these weightenings we claim the condition $\psi, \varphi \in [0, 1]$ with $\psi + \varphi = 1$. To carry out the analysis described above we have to require certain systems. On one hand we need the functions of the system with the original parameters but with different initial values. On the other hand we need those where one of the four parameters is changed for one year. The corresponding trajectories will be denoted by T_0 and T_1. But first of all we have to describe the computation of the objective damage functional. Therefore we replace the infinite planning horizon by some finite value T in order to eventually choose a finite planning horizon, which we will actually make use of later on. The value of J_0, the objective functional without any change of parameters, is calculated as follows.

$$\begin{aligned} J_0 &= \int_0^T e^{-rt}\,(\psi I_{T_0}(t) + \varphi K_{T_0}(t))\mathrm{d}t \\ &= \int_0^T e^{-rt}\left(\psi(\alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}) + \varphi(\gamma e^{\lambda_1 t} + \delta e^{\lambda_2 t})\right)\mathrm{d}t \\ &= \int_0^T e^{-rt}\left(e^{\lambda_1 t}(\psi\alpha + \varphi\gamma) + e^{\lambda_2 t}(\psi\beta + \varphi\delta)\right)\mathrm{d}t \\ &= (\psi\alpha + \varphi\gamma)\int_0^T e^{(\lambda_1 - r)t}\mathrm{d}t + (\psi\beta + \varphi\delta)\int_0^T e^{(\lambda_2 - r)t}\mathrm{d}t. \end{aligned}$$

Now we just have two integrals of a simple exponential function, which can be solved easily. This leads to the following result for the value of J_0:

$$J_0 = (\psi\alpha + \varphi\gamma)\frac{e^{(\lambda_1 - r)T} - 1}{\lambda_1 - r} + (\psi\beta + \varphi\delta)\frac{e^{(\lambda_2 - r)T} - 1}{\lambda_2 - r}.$$

If we are now able to switch on some control intervention in the very first year, the objective functional will be of the form

$$J_1 = \int_0^1 e^{-rt}\,(\psi I_{T_1}(t) + \varphi K_{T_1}(t))\mathrm{d}t + \int_0^T e^{-r(t+1)}\left(\psi \bar{I}_{T_0}(t) + \varphi \bar{K}_{T_0}(t)\right)\mathrm{d}t.$$

$\bar{I}_{T_0}$ and $\bar{K}_{T_0}$ are from the same system as I_{T_0} and K_{T_0}, but with different initial values. More precisely, we have the initial conditions

$$\begin{aligned} I_{T_1}(0) &= I(0) & \qquad \bar{I}_{T_0}(0) &= I_{T_1}(1) \\ K_{T_1}(0) &= K(0) & \qquad \bar{K}_{T_0}(0) &= K_{T_1}(1). \end{aligned}$$

The value of J_1 is calculated analogously to J_0, so we get

$$
\begin{aligned}
J_1 &= (\psi\tilde{\alpha} + \varphi\tilde{\gamma})\frac{e^{(\tilde{\lambda}_1 - r)} - 1}{\tilde{\lambda}_1 - r} + (\psi\tilde{\beta} + \varphi\tilde{\delta})\frac{e^{(\tilde{\lambda}_2 - r)} - 1}{\tilde{\lambda}_2 - r} \\
&+ e^{-r}\left((\psi\bar{\alpha} + \varphi\bar{\gamma})\frac{e^{(\lambda_1 - r)T} - 1}{\lambda_1 - r} + (\psi\bar{\beta} + \varphi\bar{\delta})\frac{e^{(\lambda_2 - r)T} - 1}{\lambda_2 - r}\right),
\end{aligned}
$$

where those parameters accentuated by $\tilde{}$ belong to the functions $I_{T_1}(t),\ K_{T_1}(t)$ and those with $\bar{}$ belong to $\bar{I}_{T_0}(t)$ and $\bar{K}_{T_0}(t)$, respectively. For the remaining years $i > 1$ we can also consider a continous function $J(i) =: J_i,\ i \in \mathbb{R},\ i > 0$ and not just discrete points at every year. This function is composed by the three integrals

$$
\begin{aligned}
J_i &= \int_0^i e^{-rt}\,(\psi I_{T_0}(t) + \varphi K_{T_0}(t))\mathrm{d}t + \int_0^1 e^{-r(t+i)}\,(\psi I_{T_1}(t) + \varphi K_{T_1}(t))\mathrm{d}t \\
&+ \int_0^T e^{-r(t+i+1)}\left(\psi \bar{I}_{T_0}(t) + \varphi \bar{K}_{T_0}(t)\right)\mathrm{d}t.
\end{aligned}
$$

Again we have to pay a special attention to several initial values which are

$$
\begin{aligned}
I_{T_0}(0) &= I(0) & I_{T_1}(0) &= I_{T_0}(i) & \bar{I}_{T_0}(0) &= I_{T_1}(1) \\
K_{T_0}(0) &= K(0) & K_{T_1}(0) &= K_{T_0}(i) & \bar{K}_{T_0}(0) &= K_{T_1}(1).
\end{aligned}
$$

After adjusting the initial values correctly the value of J_i is given by

$$
\begin{aligned}
J_i &= (\psi\alpha + \varphi\gamma)\frac{e^{(\lambda_1 - r)i} - 1}{\lambda_1 - r} + (\psi\beta + \varphi\delta)\frac{e^{(\lambda_2 - r)i} - 1}{\lambda_2 - r} \\
&+ e^{-ri}\left((\psi\tilde{\alpha} + \varphi\tilde{\gamma})\frac{e^{(\tilde{\lambda}_1 - r)} - 1}{\tilde{\lambda}_1 - r} + (\psi\tilde{\beta} + \varphi\tilde{\delta})\frac{e^{(\tilde{\lambda}_2 - r)} - 1}{\tilde{\lambda}_2 - r}\right) \\
&+ e^{-r(i+1)}\left((\psi\bar{\alpha} + \varphi\bar{\gamma})\frac{e^{(\lambda_1 - r)T} - 1}{\lambda_1 - r} + (\psi\bar{\beta} + \varphi\bar{\delta})\frac{e^{(\lambda_2 - r)T} - 1}{\lambda_2 - r}\right).
\end{aligned}
$$

After all these calculations we want to compare the value of J_0 with those of $J_i,\ i \in \mathbb{R},\ i > 0$ in order to find out which year is the best for the control intervention to be switched on.

To investigate the corresponding effects we need an indicator function which we choose of the form

$$E(i) \;=\; \frac{J_0 - J_i}{J_0} \cdot 100 \qquad\qquad i \in \mathbb{R},\; i > 0.$$

This indicator allows us to observe the effects of some control intervention over time. Considering J_i brings out that this function converges asymptotically to J_0. Thus the effect of the control intervention loses influence in later years and the effect converges to zero for $i \to \infty$. As a consequence the effect has to be the greatest in one of the first years. Due to the linearity of the investigated model we actually expect our control intervention to have the biggest effect very early.

Chapter 6

The Relative Efficiency of the Controls at Different Stages of a Terrorist Group

6.1 The Effects of an Intervention from Outside

Now we want to apply our considerations from above to some practical examples. To avoid unnecessary computing time we use a finite approximation T to the infinite planning horizon by considering

$$J = \int_0^T e^{-rt}(\psi I(t) + \varphi K(t))\mathrm{d}t + \int_T^\infty e^{-rt}(\psi \hat{I} + \varphi \hat{K})\mathrm{d}t,$$

where $\hat{I}$ and $\hat{K}$ represent the equilibrium state of the considered system. Since the equilibrium is actually $(\hat{I}, \hat{K}) = (0, 0)$, we can omit the second integral so that the objective functional consists only of the first one. Again it fits into the recently presented objective damage functional. In what follows we consider a mix of parameters which imply an asymptotically stable system. This means that the assumed terrorist group gradually approaches an equilibrium state. The purpose of the examinations in Chapter 5 is to observe the path to the equilibrium and the effects of the control intervention from outside on this path. The analysis will be carried out with the parameters

a	b	c	d
1	2	1	2

and for two different trajectories, namely those with the initial values $(I(0), K(0)) = (3, 6)$ and $(8, 1)$. In the first case the amount of labour exerted by ideologues is half the capital fundings from outside. The second trajectory describes a lack of money. Taking a look at the two-dimensional model in its original form and the signs of the parameters implies that not every parameter change, increase or decrease, does make sense. Therefore we will just increase b and d and decrease a and c, respectively, by the fixed proportion of 10%. Furthermore we will use a finite approximation of time, $T = 30$, and discount at an annual rate of $r = 0.04$. Moreover we will use a symmetric weightening of $I(t)$ and $K(t)$, $\psi = \varphi = 0.5$. Nothe that other values do not have a significant influence on the results, presumably due to the linearity of the model. The following figures show the results of our examinations. Note that the respective effects are expressed as percentage.

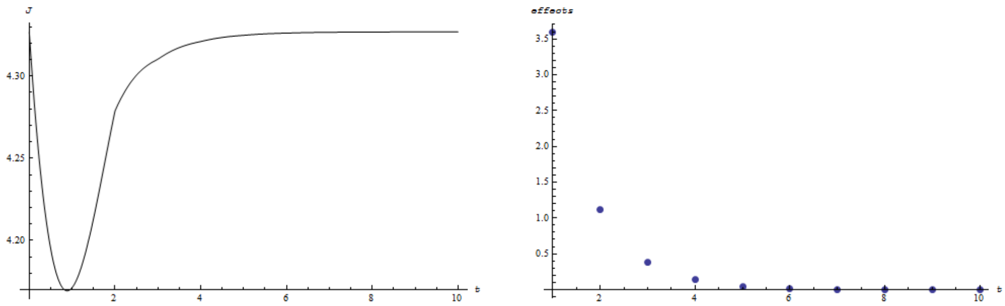

Figure 6.1: Plots of J_i and the effects per year when a is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

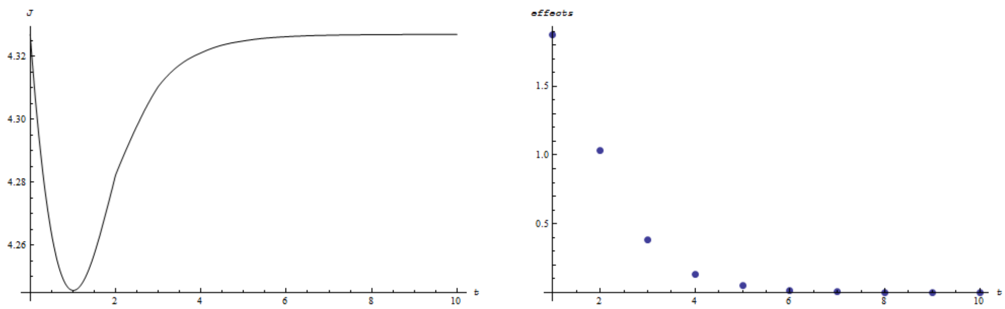

Figure 6.2: Plots of J_i and the effects per year when a is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

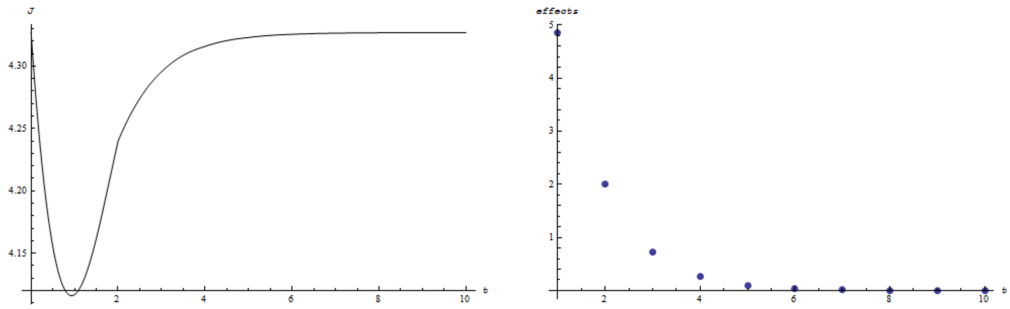

Figure 6.3: Plots of J_i and the effects per year when b is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

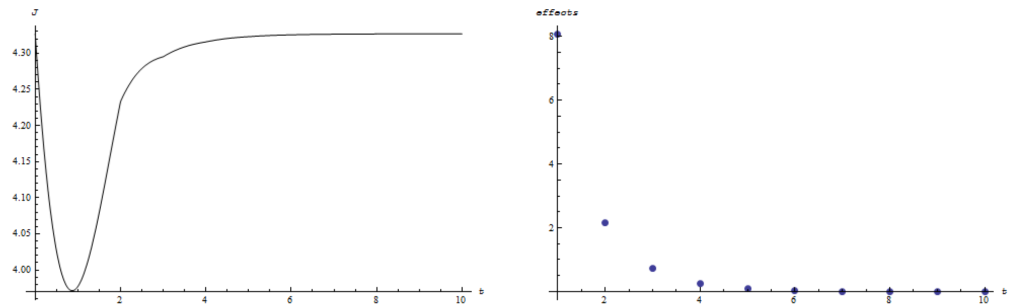

Figure 6.4: Plots of J_i and the effects per year when b is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

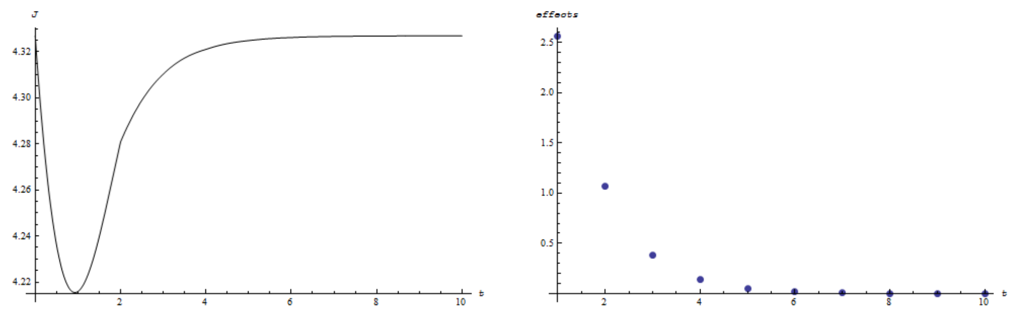

Figure 6.5: Plots of J_i and the effects per year when c is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

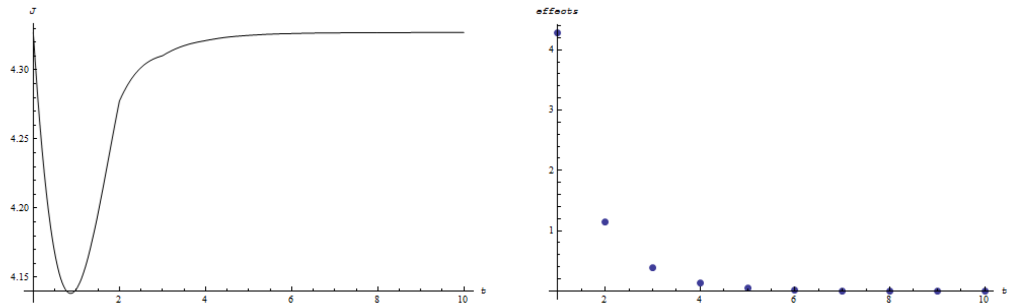

Figure 6.6: Plots of J_i and the effects per year when c is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

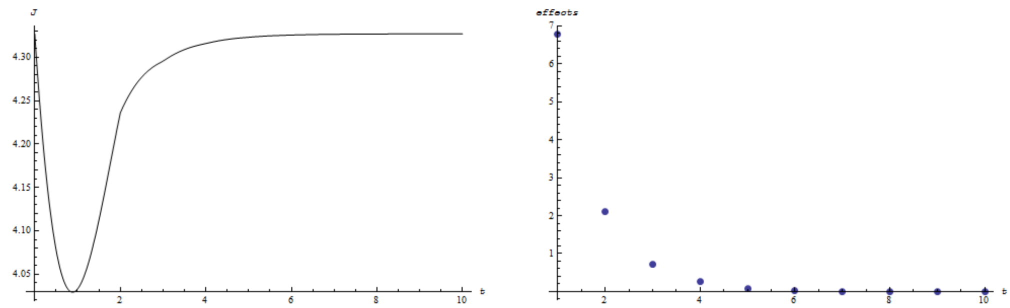

Figure 6.7: Plots of J_i and the effects per year when d is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

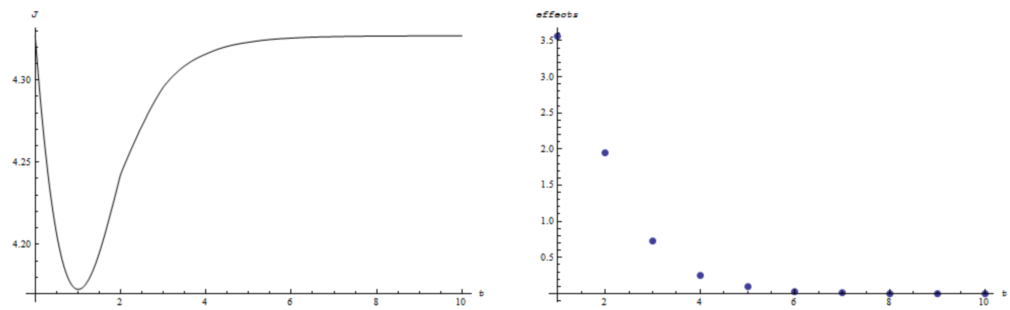

Figure 6.8: Plots of J_i and the effects per year when d is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

Obviously the results are pretty similar and not very surprising. We see clearly that the first year always fetches the biggest effect. This is no real surprise due to the linearity of the model as mentioned already in Chapter 5. This means that the control intervention in best switched on in the first year, at the very beginning of the option to apply the control. Please note that so far we just considered one point in every year. So this period is maybe too long for significant results. Taking an accurate look at the effects of Figure 6.2 or 6.8 gives us a reason to believe that the decrease of the effects is not always monotone within the first year. Perhaps there is even a little increase of the effects which can not be seen in those results. So we probably should not observe the trajectories year by year but rather scan them more accurately and consider periods which are much smaller than one entire year. In what fallows we consider periods with length 0.1 years. Again we consider the two trajectories with $(I(0), K(0)) = (3, 6)$ and $(8, 1)$.

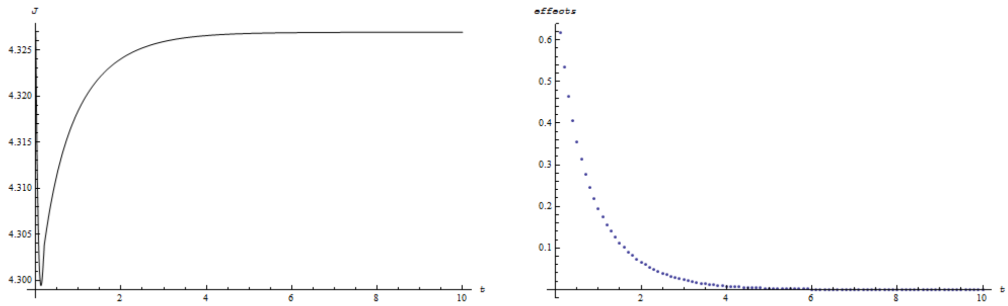

Figure 6.9: Plots of J_i and the effects per 0.1 years when a is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

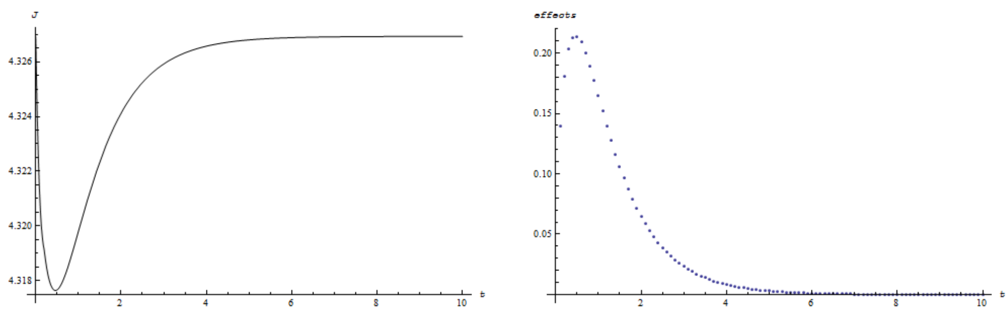

Figure 6.10: Plots of J_i and the effects per 0.1 years when a is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

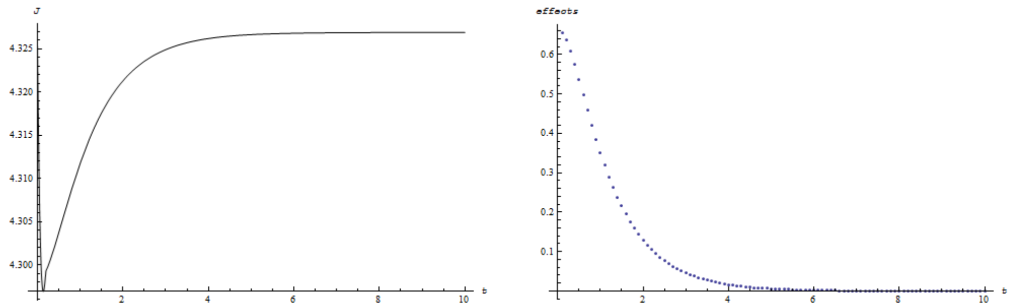

Figure 6.11: Plots of J_i and the effects per 0.1 years when b is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

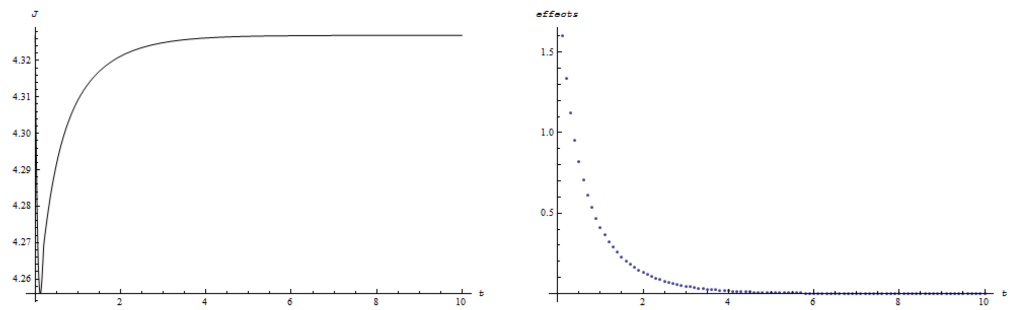

Figure 6.12: Plots of J_i and the effects per 0.1 years when b is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

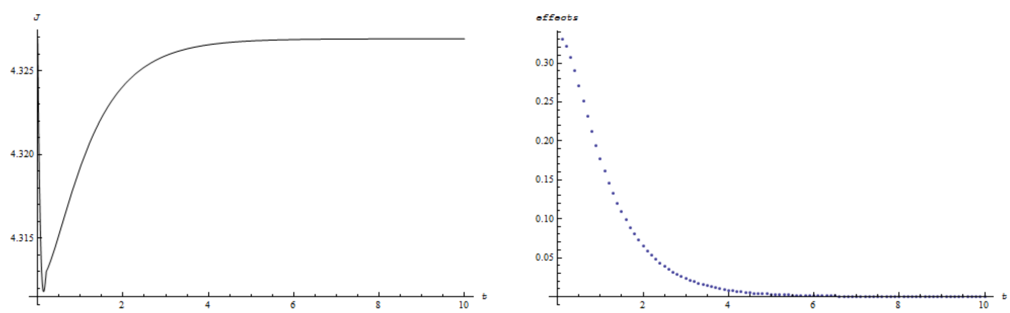

Figure 6.13: Plots of J_i and the effects per 0.1 years when c is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

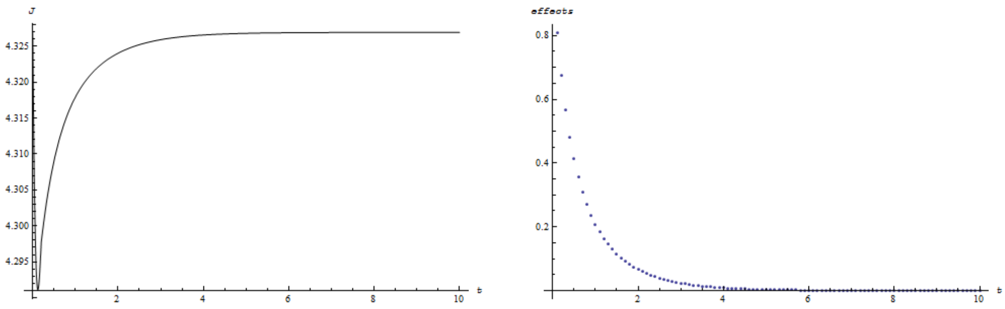

Figure 6.14: Plots of J_i and the effects per 0.1 years when c is decreased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

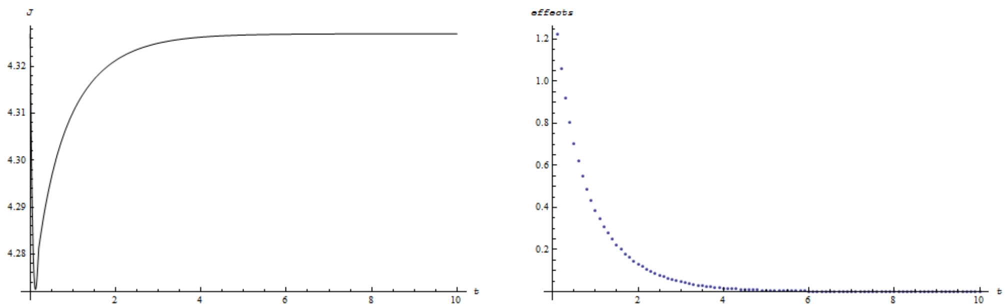

Figure 6.15: Plots of J_i and the effects per 0.1 years when d is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (3, 6)$.

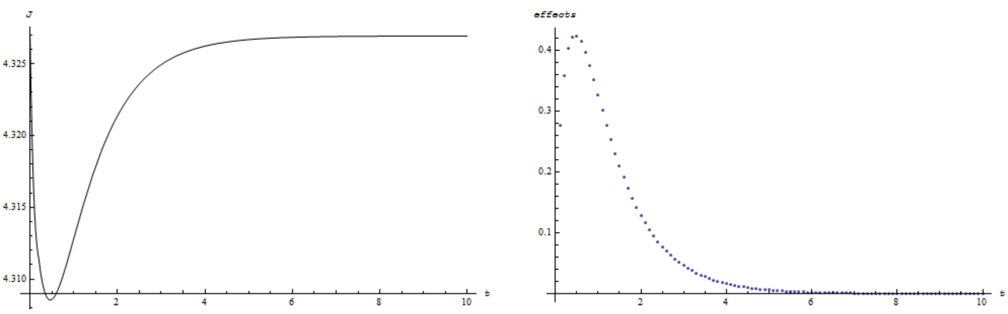

Figure 6.16: Plots of J_i and the effects per 0.1 years when d is increased by 10%. The initial conditions of this system are $(I(0), K(0)) = (8, 1)$.

The results above of a change in a and d (also b and c) are qualitatively pretty similar since they are connected to the same state variable. Therefore the next two figures are just given for a decrease of a and an increase of b. Apparently it is better for some initial values to switch on the control intervention at diferent stages as just at the beginning of the first year.

To figure this out in more detail we take a look at the normalised effects and the normalised state variables within one graphic. The first one shows a decrease of a by 10%, where the thick line describes the capital fundings from outside. Obviously there is a connection between the effects (dotted line) and this line. This is because a is directly influencing $K(t)$ according to Hausken et al.'s model. The more $K(t)$ is increasing the bigger are the effects of the intervention on the parameter influencing this state variable. The effect is the biggest shortly after $K(t)$ has reached its peak and starts to decrease afterwards. If the fundings are decreasing from the very beginning the effects also do. For comparative purpose we also display the solution for symmetric initial conditions, $I(0) = K(0)$, which is quite interesting because this state starts directly on the caracteristical trajectory.

The second graphic shows the analogous situation for an increase of b, where the thick line represents $I(t)$, the amount of labour exerted by the ideologues, with the corresponding initial values.

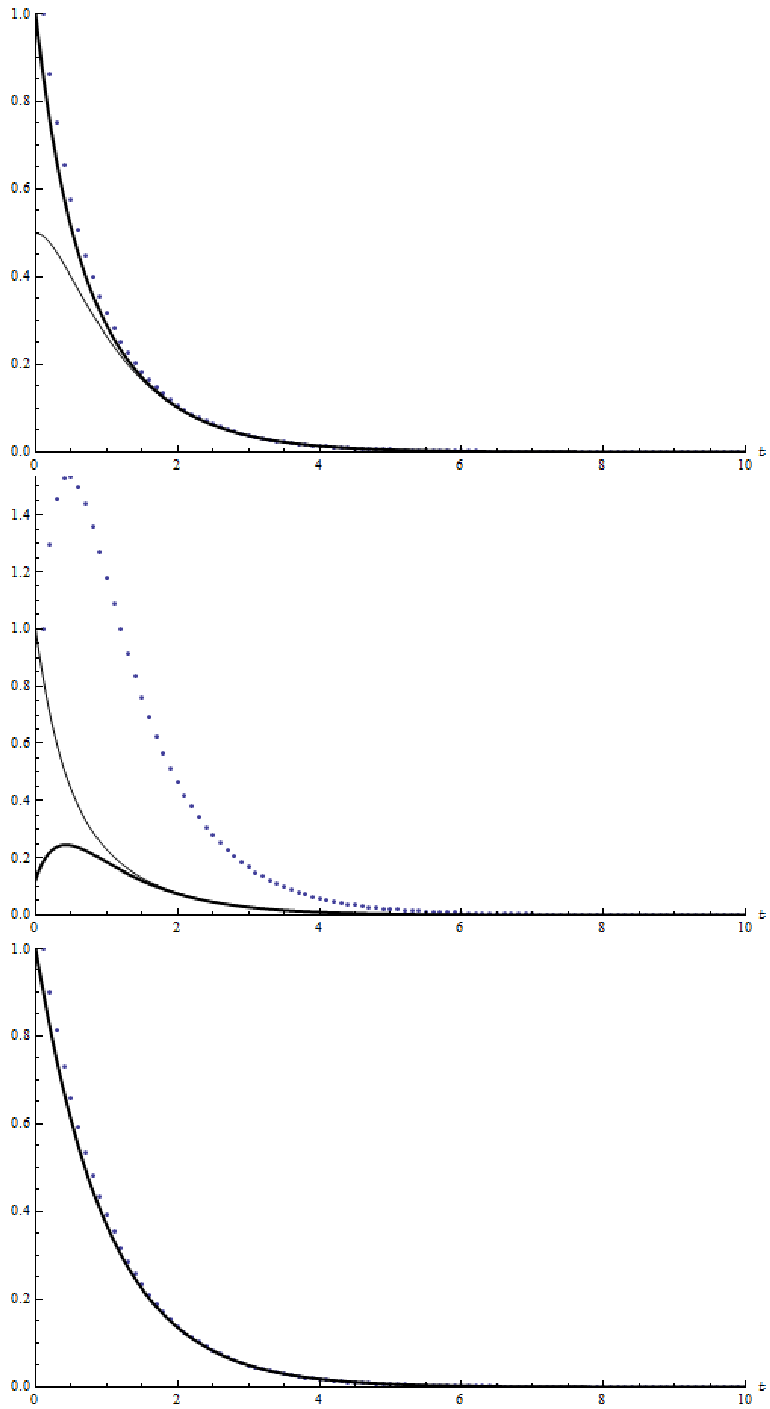

Figure 6.17: Normalised effects and state variables for $(I(0), K(0)) = (3, 6)$, $(8, 1)$ and $(5, 5)$ when a is decreased by 10%.

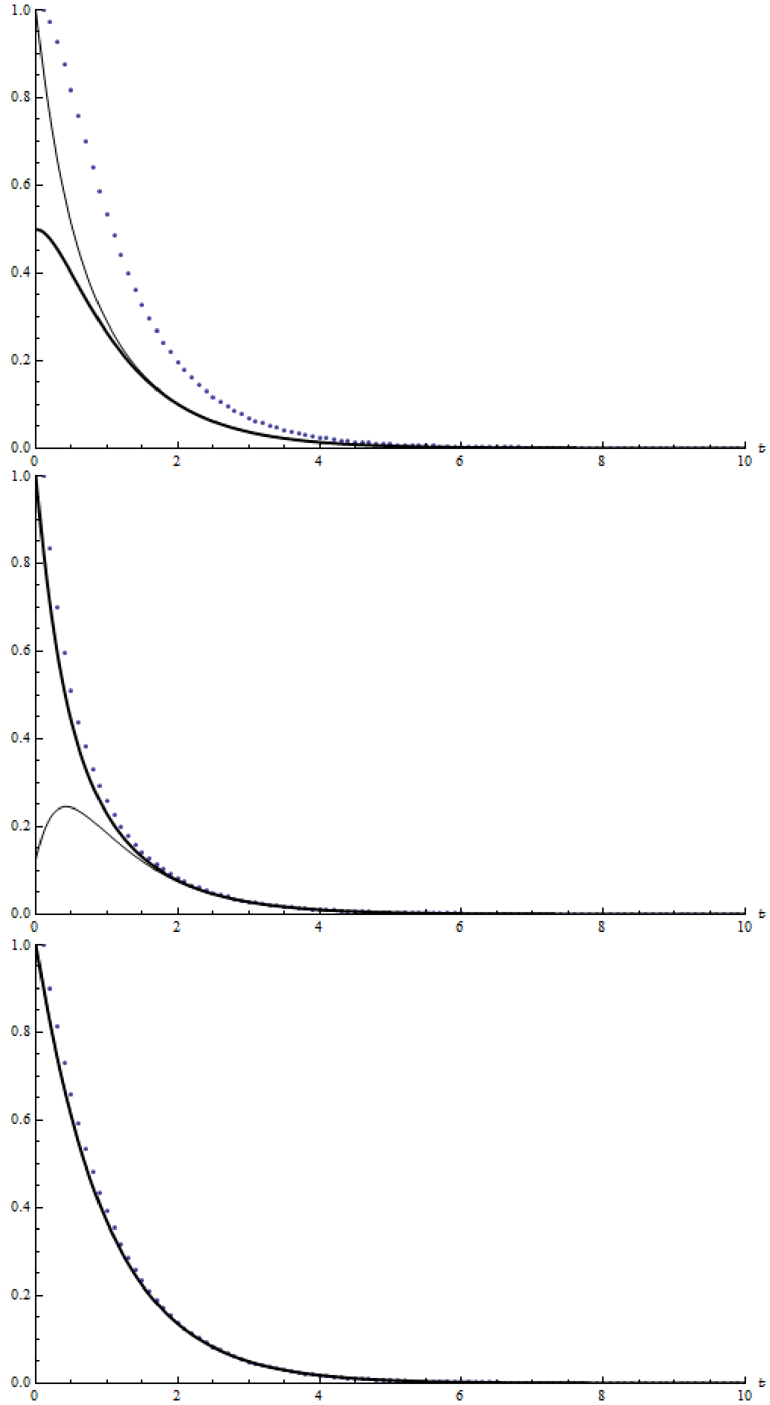

Figure 6.18: Normalised effects and state variables for $(I(0), K(0)) = (3, 6),\ (8, 1)$ and $(5, 5)$ when b is increased by 10%.

6.2 The Effects at the Beginning

Since the analyses presented above only took into account two initial values, we next want to expand our examinations. Therefore we consider a large area in the (I, K)-plane. We are constructing a grid of 40×40 points over the plane, where every point is representing one initial value.

If the control intervention is switched on we know that for most initial values the biggest effect is achieved in the very first period. The control intervention implies a change of the parameters by 10%. To compare the results we will also change the certain parameters by only 1%. The following figures display the (I, K)-plane coloured according to the respective efficiency of the control interventions when switched on at the very beginning of the planning horizon.

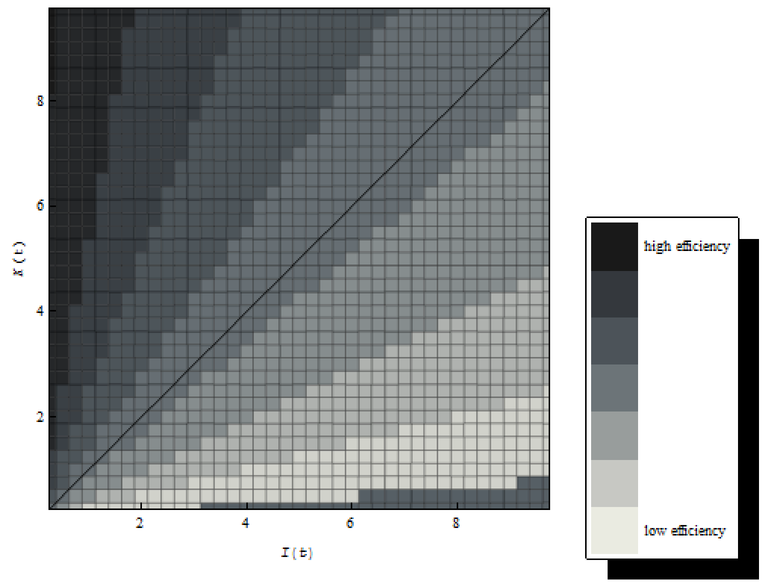

Figure 6.19: Efficiency zones when a is decreased by 1% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

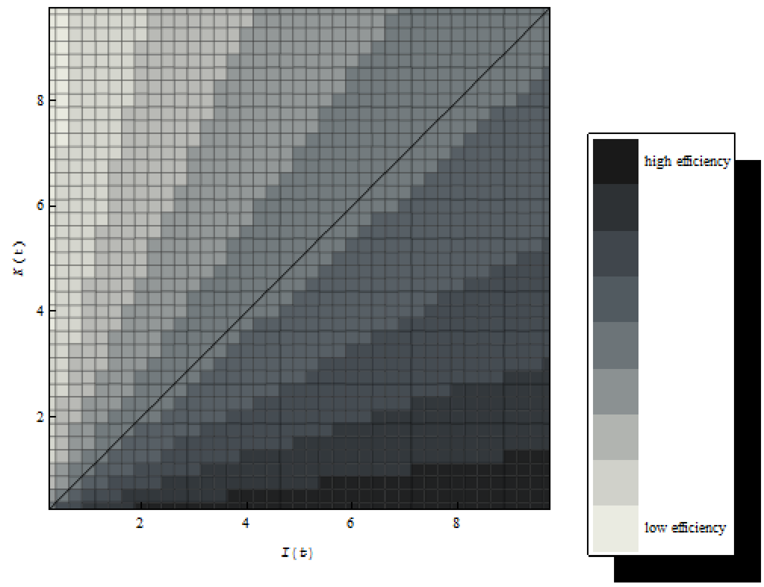

Figure 6.20: Efficiency zones when b is increased by 1% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

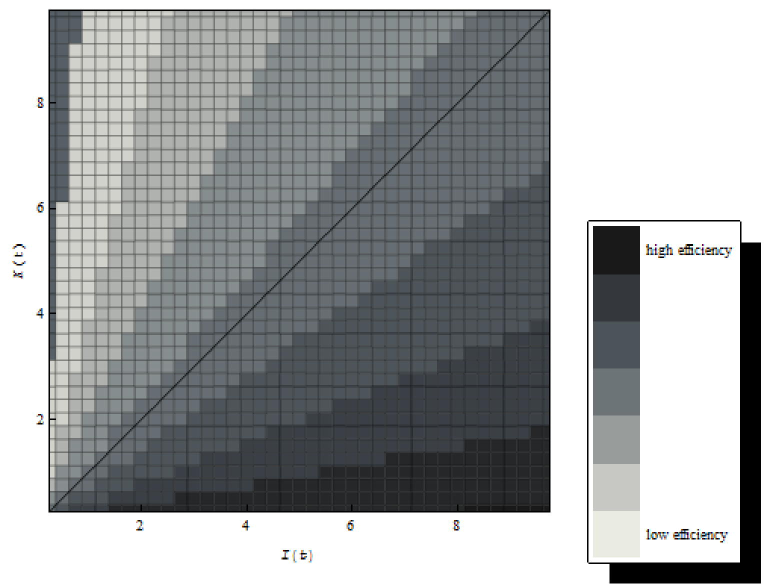

Figure 6.21: Efficiency zones when c is decreased by 1% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

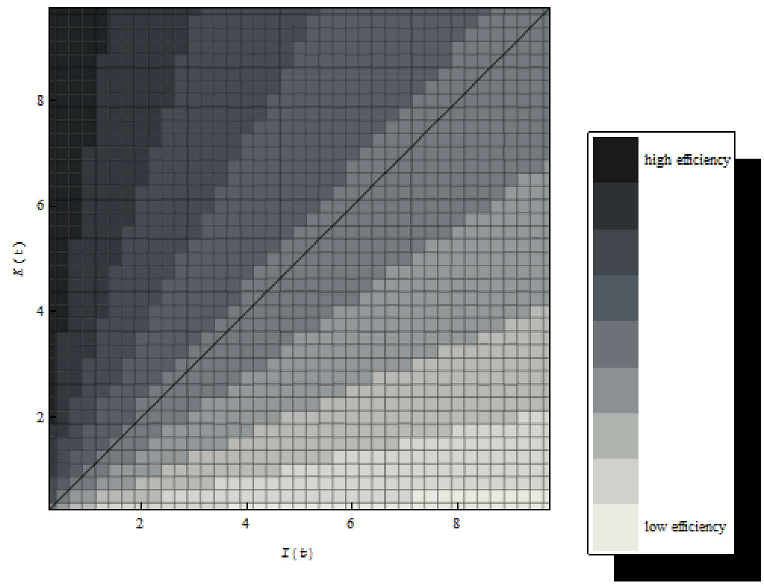

Figure 6.22: Efficiency zones when d is increased by 1% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

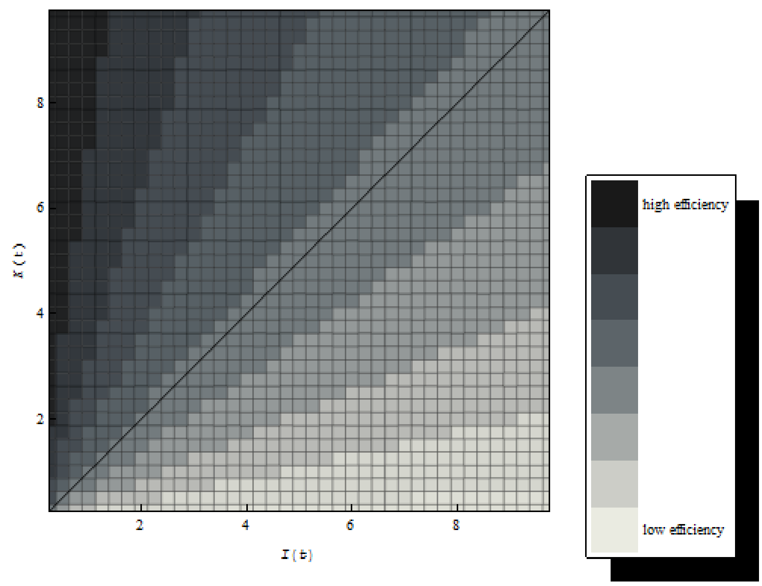

Figure 6.23: Efficiency zones when a is decreased by 10% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

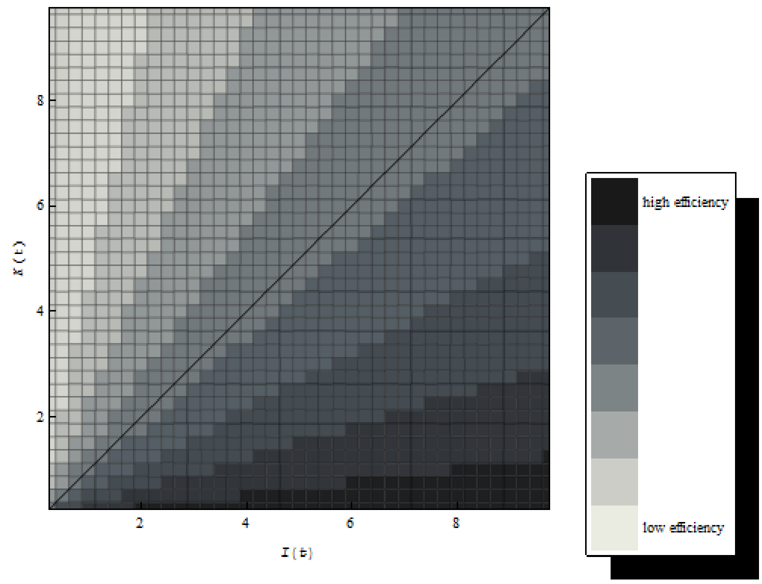

Figure 6.24: Efficiency zones when b is increased by 10% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

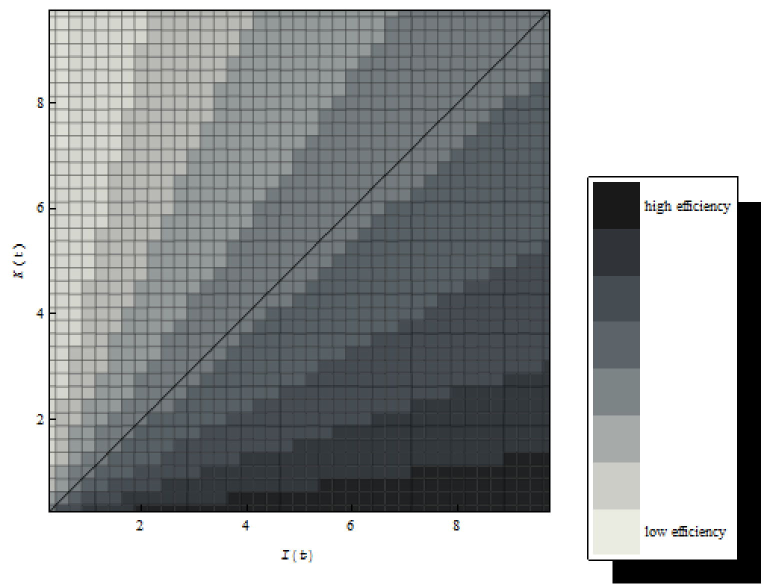

Figure 6.25: Efficiency zones when c is decreased by 10% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

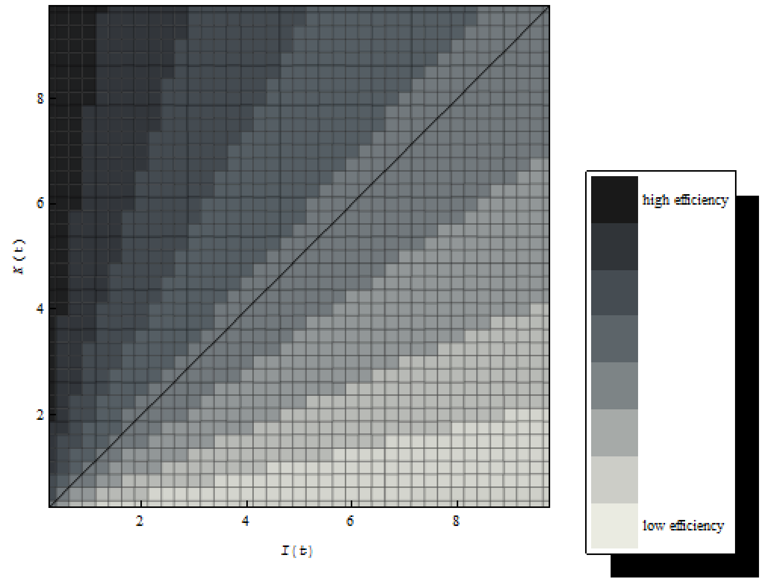

Figure 6.26: Efficiency zones when d is increased by 10% at the very beginning of the planning horizon. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

6.3 The Effects at Different Stages of the Terrorist Group

We only considered the control in the first period so far. But we know that there are some initial values achieving the biggest effect in later periods. Again we consider the (I, K)-plane with 40×40 points, and for each of those values the control intervention is exerted. Therefore we will increase or reduce the parameters by 10% as done so far and for comparative purpose by 1%, too. To avoid computing time we are only considering the first two years. The years afterwards are not relevant for our examination as we can see in the recently given pictures. Again these two years are scanned by a period length of one-tenth. After calculating the objective functionals the year with the strongest effect is assigned to every initial value. Hence the following graphics give some (I, K)-planes to show up which period is the best for the control intervention to switch on.

Obviously the results are all nearly the same. Hence the period for the control intervention is basically always the same no matter if the parameters are changed by 1% or 10%.

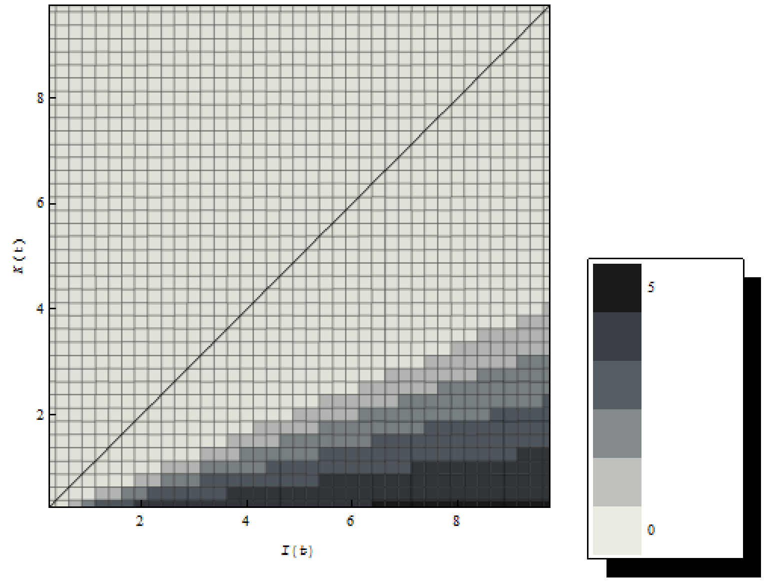

Figure 6.27: (I, K)-plane to discover which is the best period to change a by 1%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

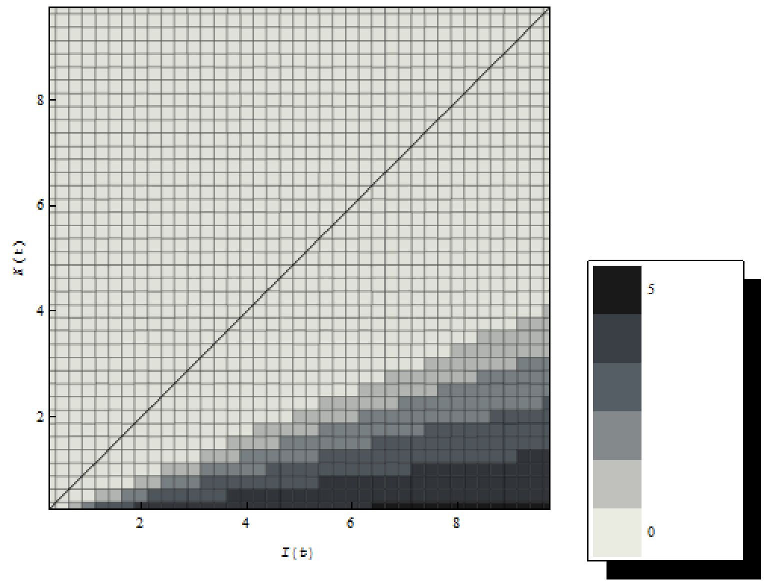

Figure 6.28: (I, K)-plane to discover which is the best period to change a by 10%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

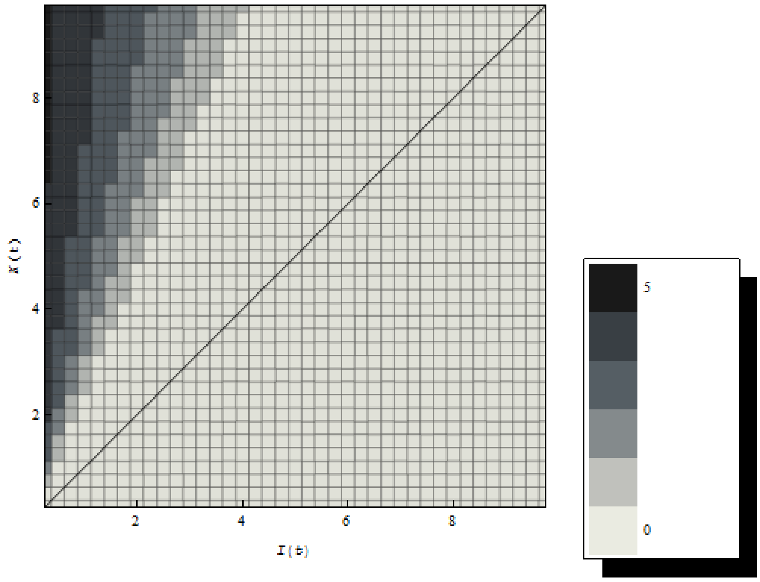

Figure 6.29: (I, K)-plane to discover which is the best period to change b by 1%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

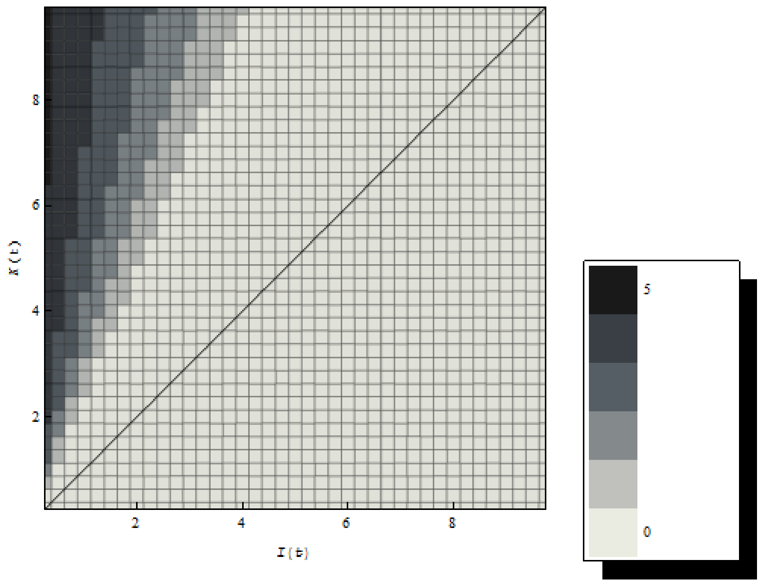

Figure 6.30: (I, K)-plane to discover which is the best period to change b by 10%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

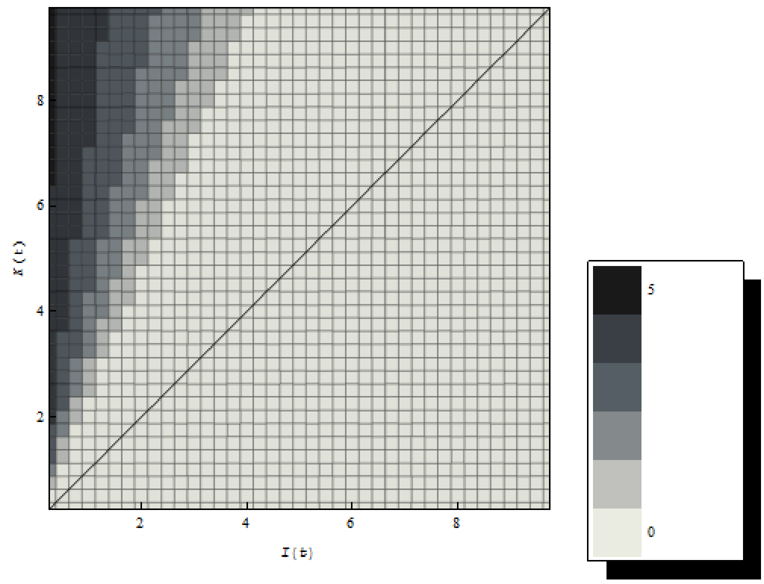

Figure 6.31: (I, K)-plane to discover which is the best period to change c by 1%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

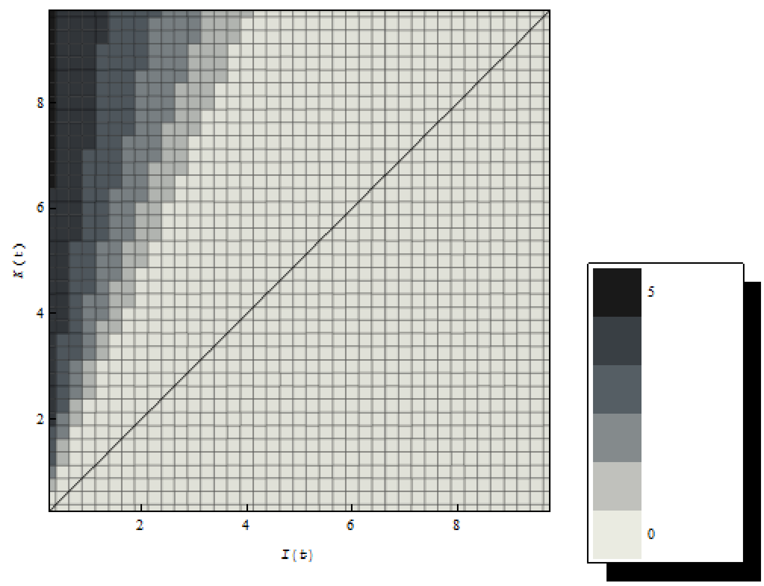

Figure 6.32: (I, K)-plane to discover which is the best period to change c by 10%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

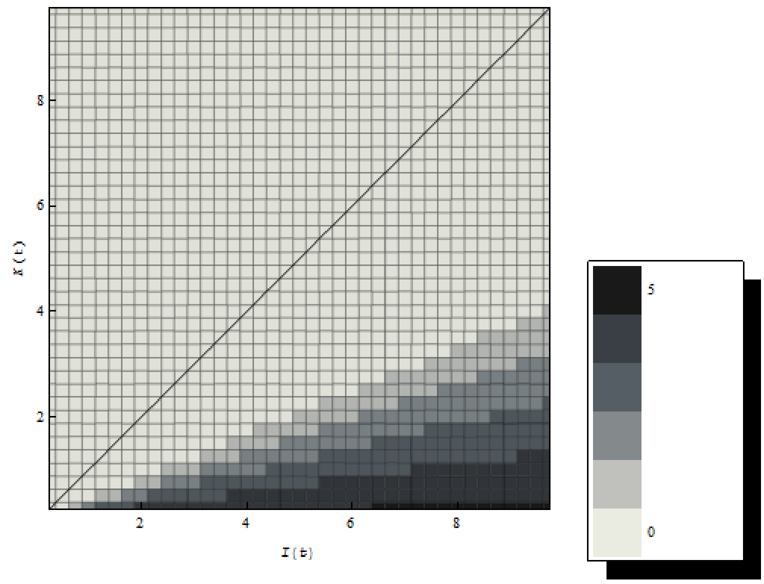

Figure 6.33: (I, K)-plane to discover which is the best period to change d by 1%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

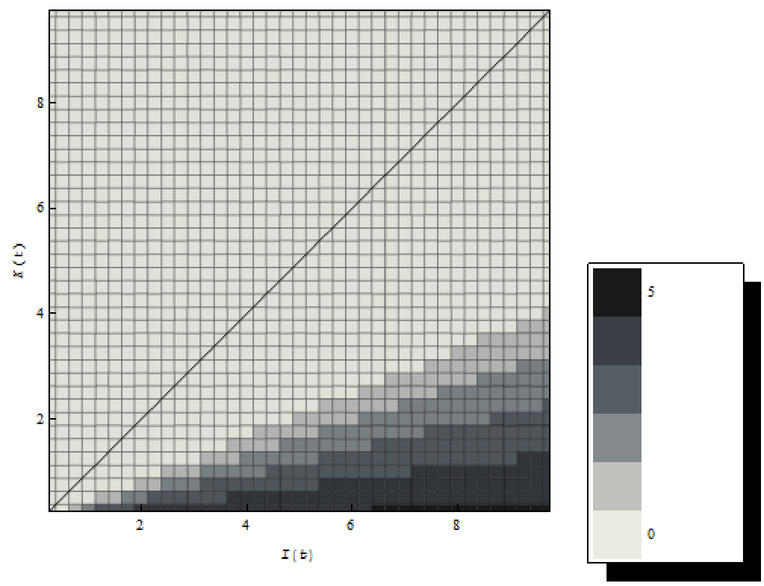

Figure 6.34: (I, K)-plane to discover which is the best period to change d by 10%. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

6.4 Approach of an Optimal Policy

What we do have so far is the information of the best point in time when the control should be switch on for one period. We have discovered the best period within the first year for every single parameter to change it by a certain percentage. Now we want to combine these informations. In the next examination we want to find out which parameter is the best one for a certain period to change by 10%. We know the effects of every parameter influencing the terrorist group so we already have all information that is required. We just need to compare those effects and assign the corresponding parameters to their initial values. We use again the 40×40 grid over the (I, K)-plane and assume the length of a period is a tenth year. First of all we will check the situation with the symmetric weightenings $\psi = \varphi = 0.5$ given by the Figure 6.35. This case is pretty unspectacular because the optimal policy is the same in every period. On one side of the characteristical trajectory it is optimal to change d and on the other side b. So we will assign different weightenings to the state variables to discover some more interesting results.

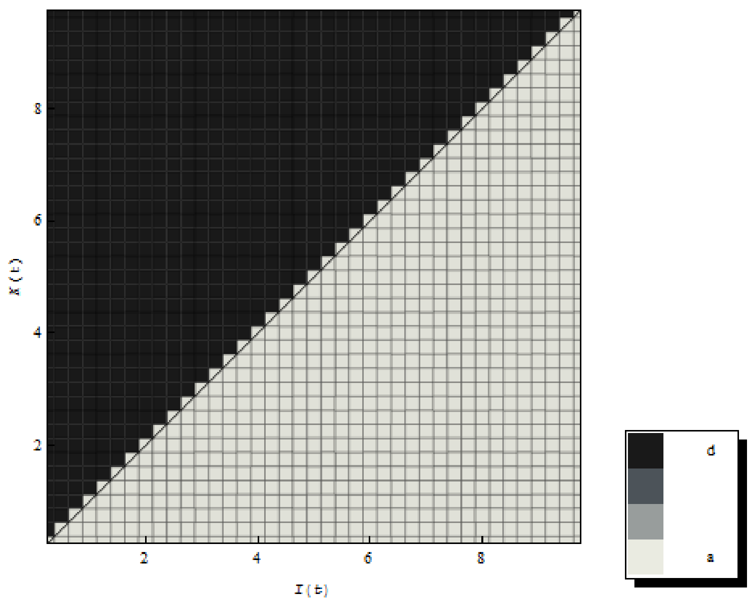

Figure 6.35: (I, K)-plane to discover which is the most efficient parameter to change by 10% in the 0^{th} period of the planning horizon, where $\psi = \varphi = 0.5$. The black line represents the characteristical trajectory of the system leading to the equilibrium point $(0, 0)$.

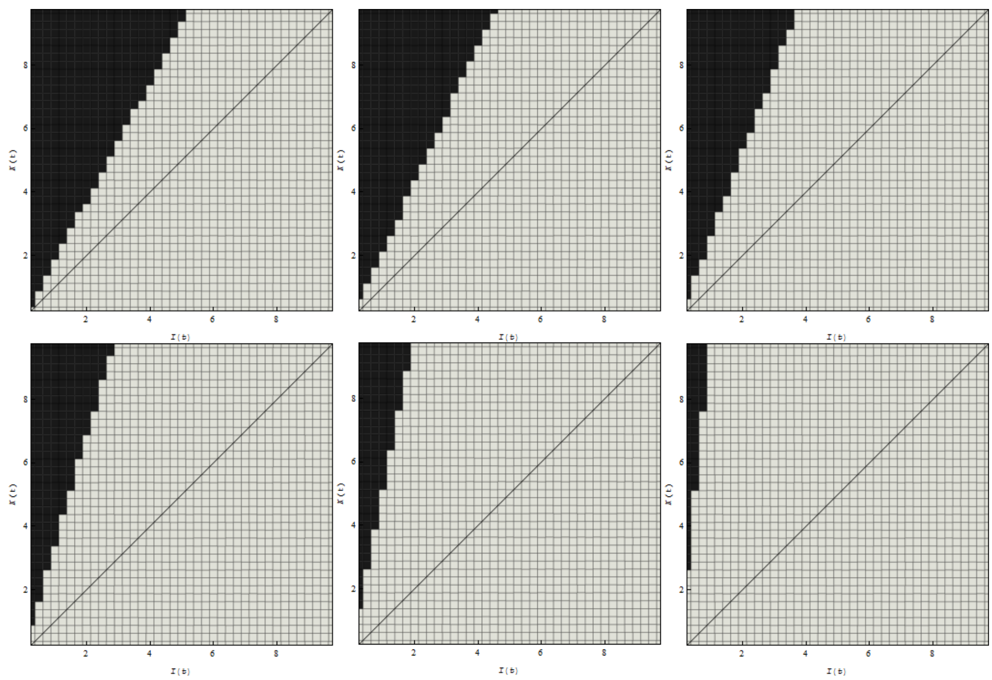

Figure 6.36: The first 6 periods to discover which is the most efficient parameter to change by 10%, where $\psi = 0.9$ and $\varphi = 0.1$.

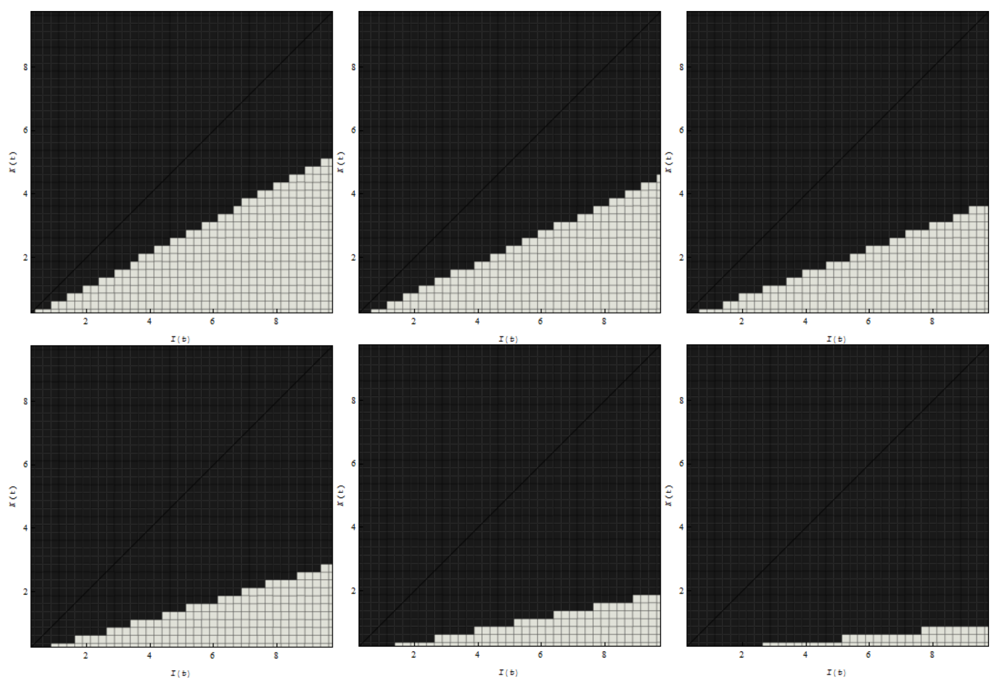

Figure 6.37: The first 6 periods to discover which is the most efficient parameter to change by 10%, where $\psi = 0.1$ and $\varphi = 0.9$.

As mentioned above the situation in Figure 6.35 is pretty unspectacular inasmuch as we have a symmetric constellation of the parameters and a symmetric weightening. Thus the optimal policy contains the same decision in every period. But if the functions receive different weightenings, we see the edge of our two decisions, b or d, shifting over time. Considering a trajectory moving towards this edge brings out the point of time when we have to make a change in our decisions. For a better view of this discovery we will display the next figures for the parameters $(a, b, c, d) = (1, 2, 1, 1)$, the trajectory starting at the point $(I(0), K(0)) = (9, 1)$ and the weightenings $\psi = \varphi = 0.5$.

Obviously there are two significant parameters, namely b and d. The other two parameters do not have an influence that strong and therefore they drop out of an optimal policy. We can see clearly that d is the most important parameter. The longer the terrorist group exists, the more important d gets for the control intervention. Since it is possible to consider only one period per figure we will show up the first thirteen periods because afterwards there are no more changes in the optimal policy. After the thirteenth period, d is the only parameter for the biggest effect on the terrorist group.

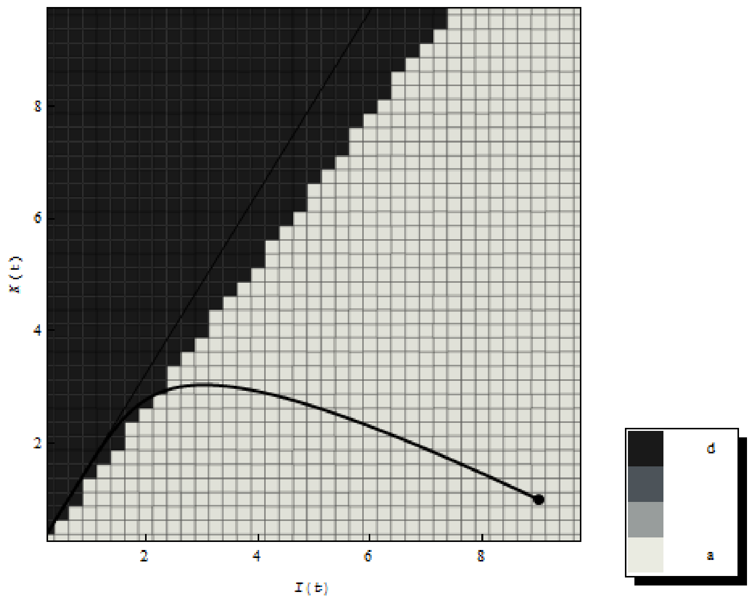

Figure 6.38: The 0^{th} period for the optimal policy. The black spot on the trajectory represents the actual point of time.

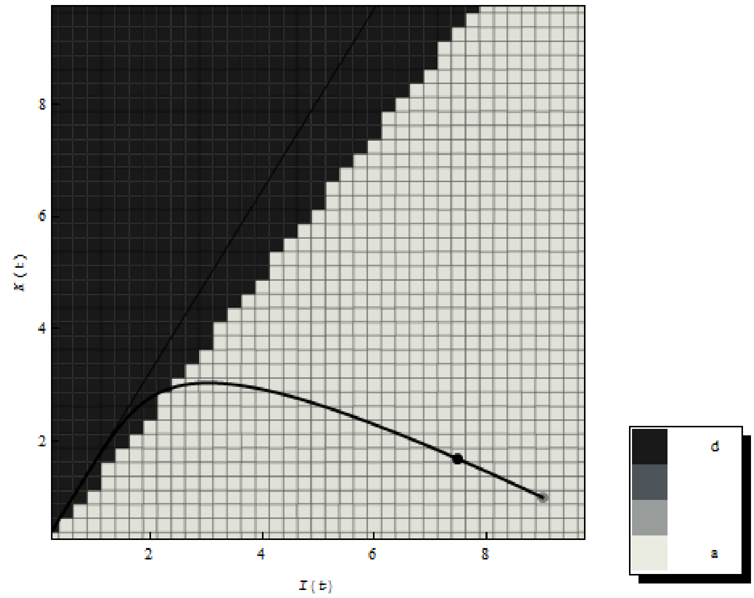

Figure 6.39: The 1^{st} period for the optimal policy. The black spot on the trajectory represents the actual point of time and the gray one the preceding period.

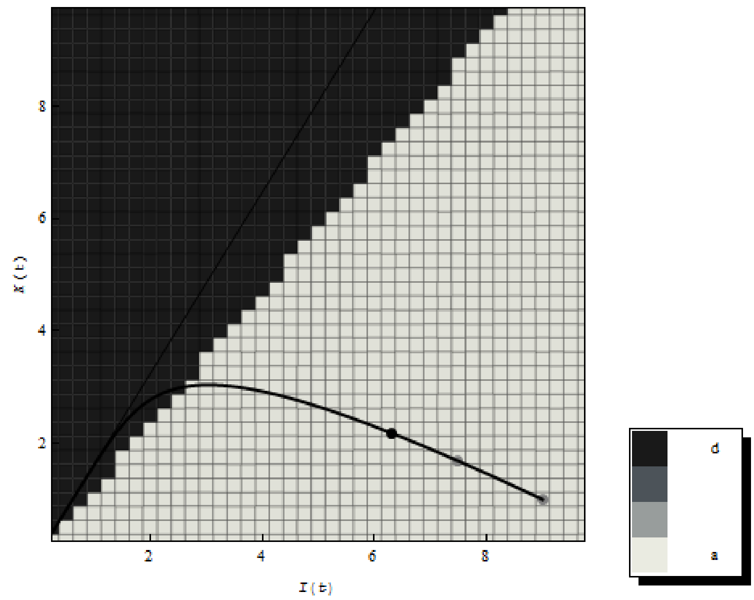

Figure 6.40: The 2^{nd} period for the optimal policy. The black spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

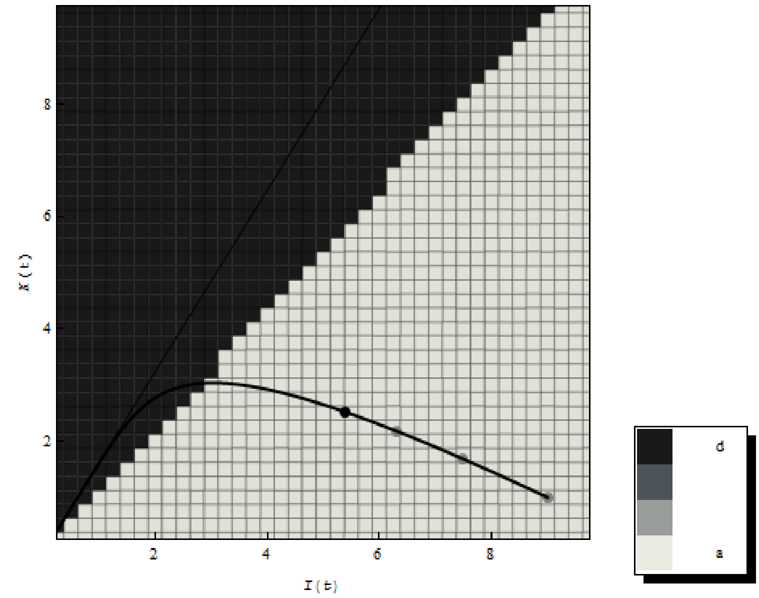

Figure 6.41: The 3^{rd} period for the optimal policy. The black spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

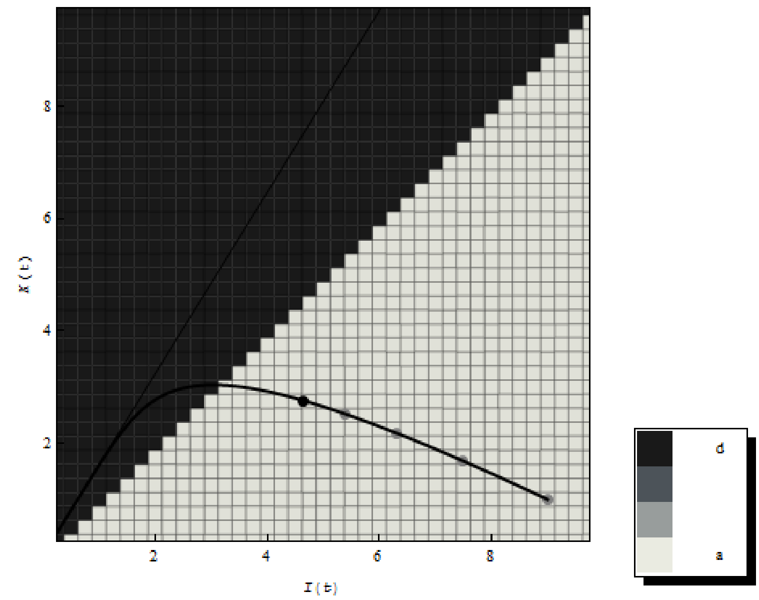

Figure 6.42: The 4^{th} period for the optimal policy. The black spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

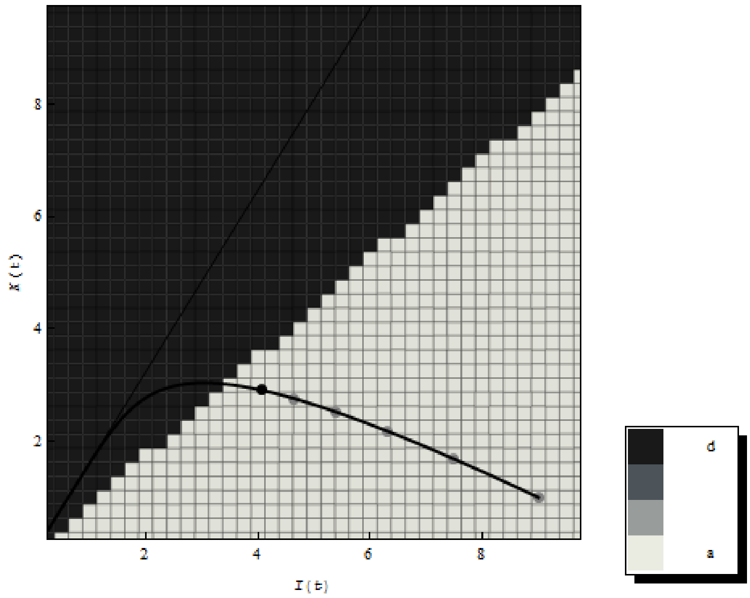

Figure 6.43: The 5^{th} period for the optimal policy. The black spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

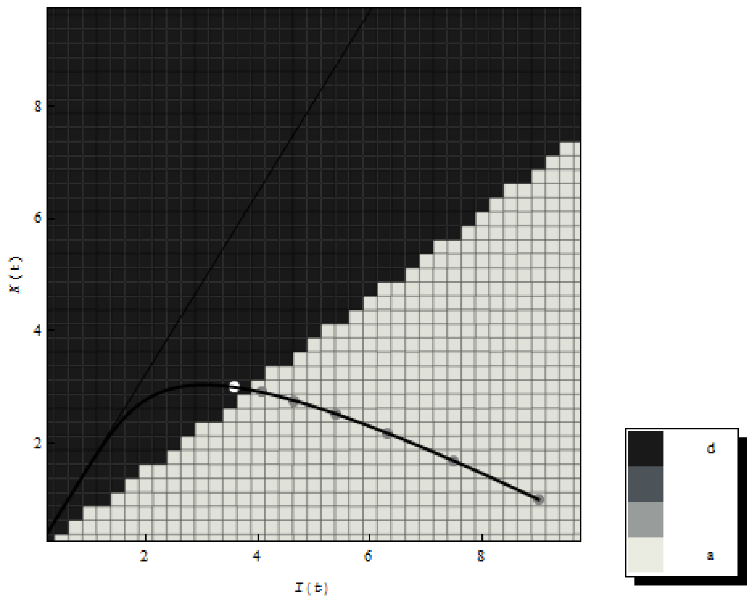

Figure 6.44: The 6^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

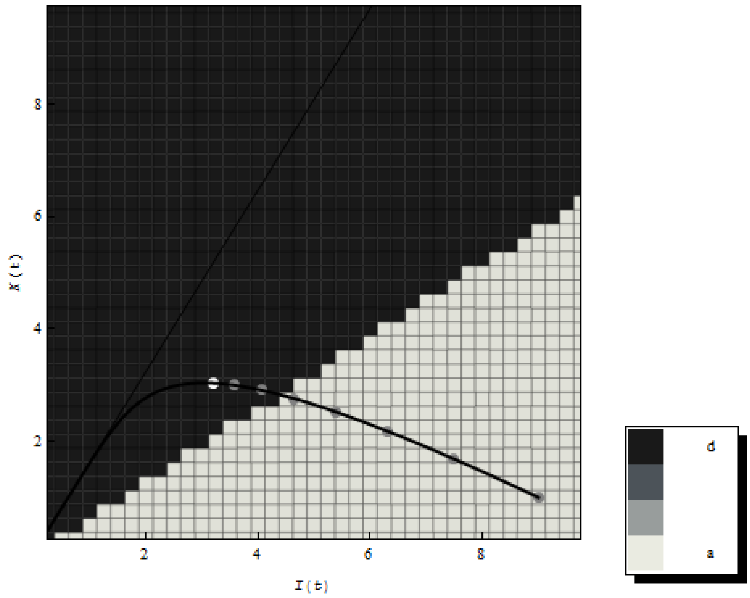

Figure 6.45: The 7^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

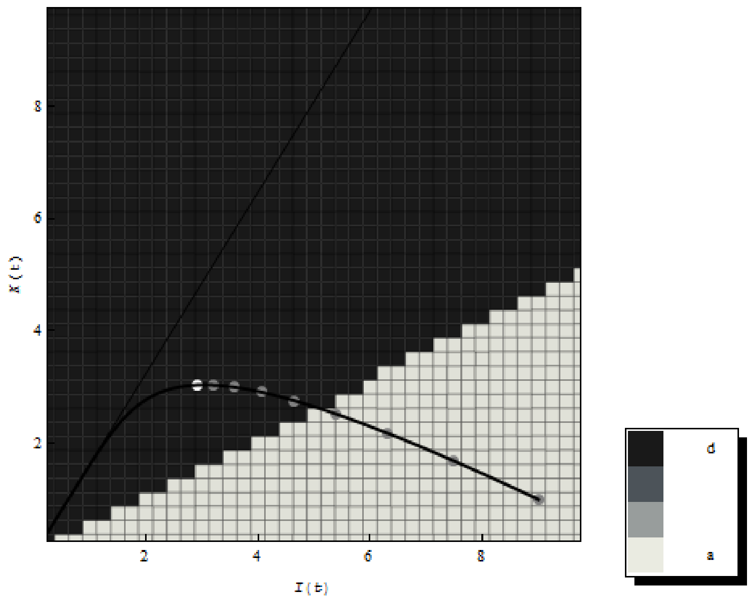

Figure 6.46: The 8^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

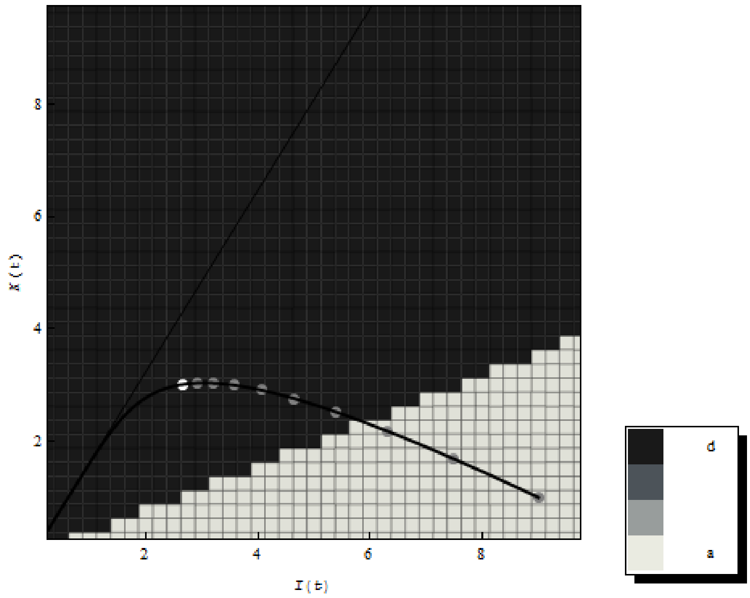

Figure 6.47: The 9^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

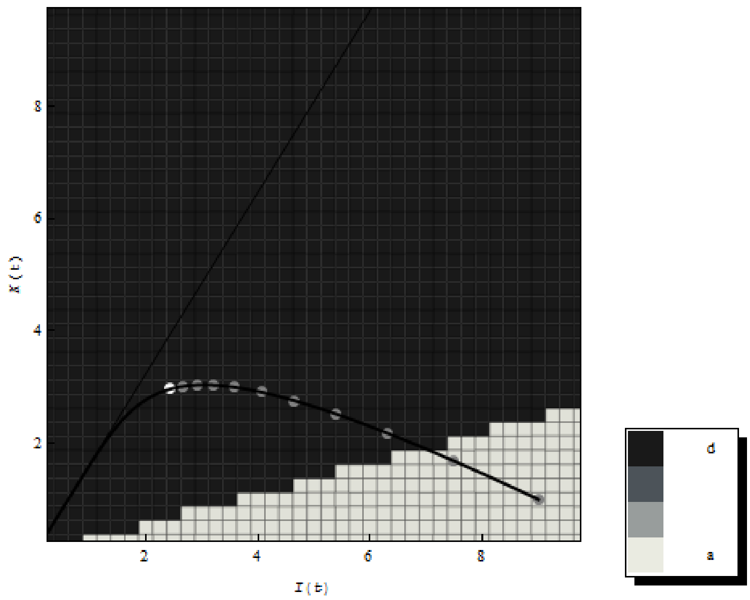

Figure 6.48: The 10^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

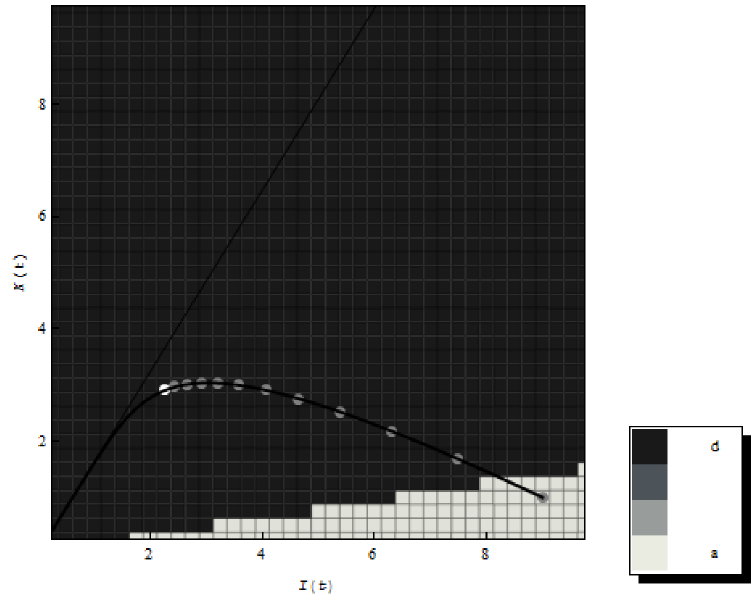

Figure 6.49: The 11^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

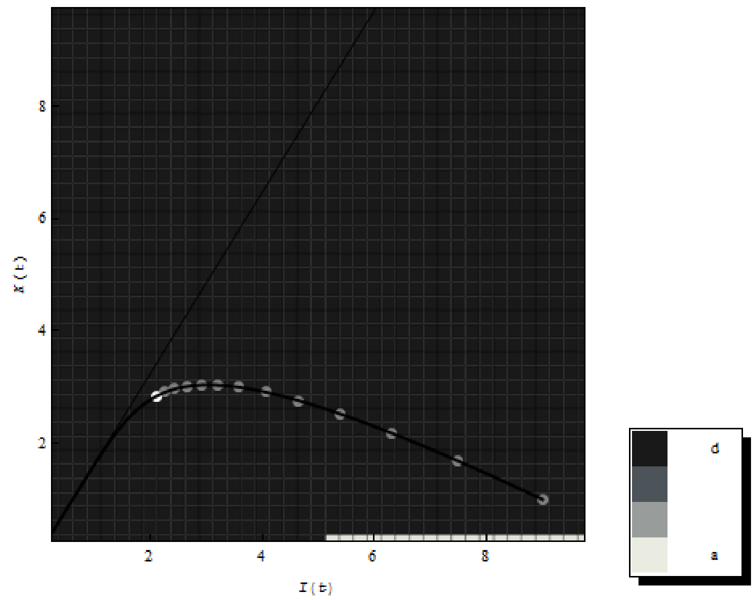

Figure 6.50: The 12^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

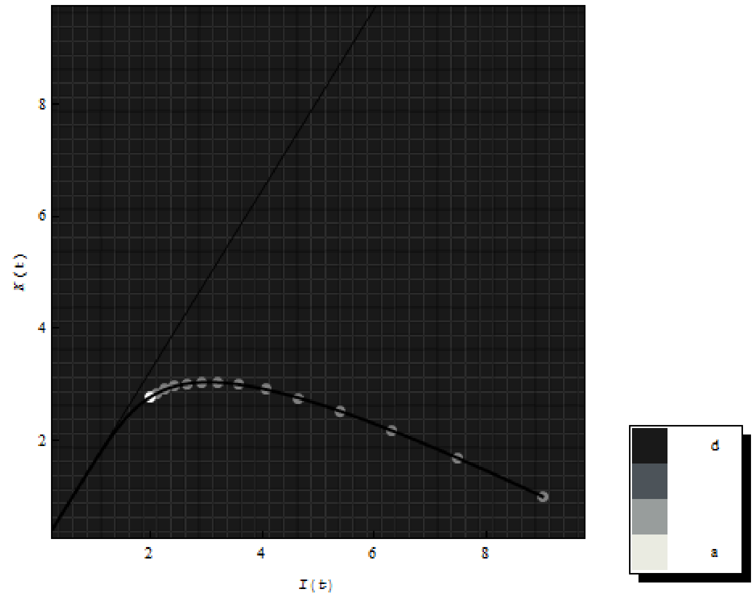

Figure 6.51: The 13^{th} period for the optimal policy. The white spot on the trajectory represents the actual point of time and the gray ones the preceding periods.

6.5 Simulating a 300% Shock of the Parameters

Let us expand our examinations one last time. We are now interested in the impact of a big shock in every parameter, respectively. Note that we still have to fulfill a stability condition, namely $bd > ac$. In order not to offend this condition we have carefully chosen the initial set of parameters $(a, b, c, d) = (1, 2, 1, 2)$ at the beginning of this chapter. Again this is the set we are going to deal with, where $\psi = \varphi = 0.5$. These parameters allow us to raise every single parameter by 300% and the system still rests asymptotically stable. Afterwards we will do the same examination as the last one of the previous section with several adjusted trajectories.

The results are not given seperately but rather within one graphic. Obviously a shock in a and c needs much more time to dominate the entire (I, K)-plane. A shock in b and especially d needs maximum half the time to do the same. This is due to the higher values of b and d from the the very beginning. A shock in a or c does not have an influence that strong and therefore much more periods are required.

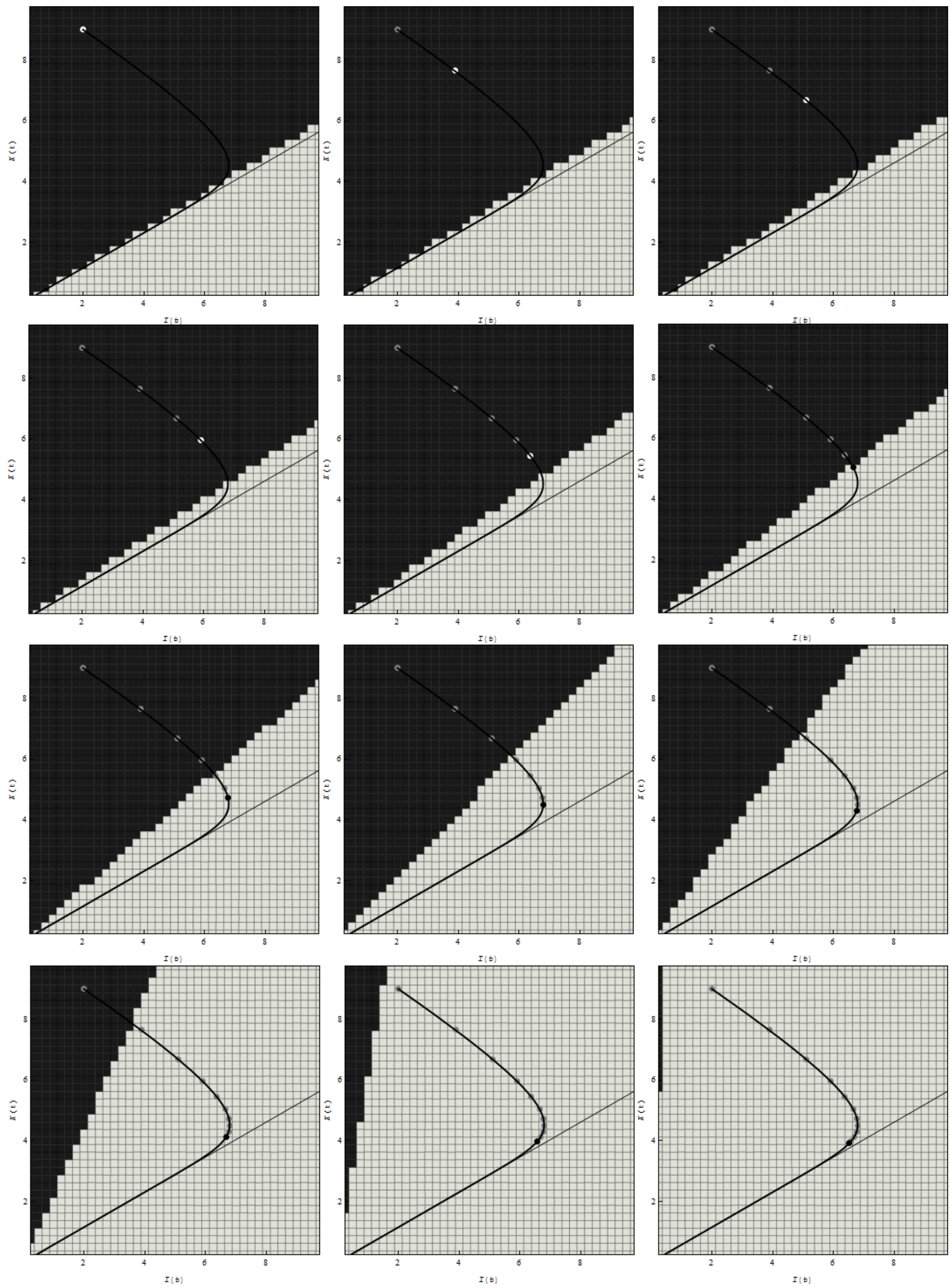

Figure 6.52: The first 12 periods with the optimal policy and the trajectory for the parameters $(a, b, c, d) = (3, 2, 1, 2)$

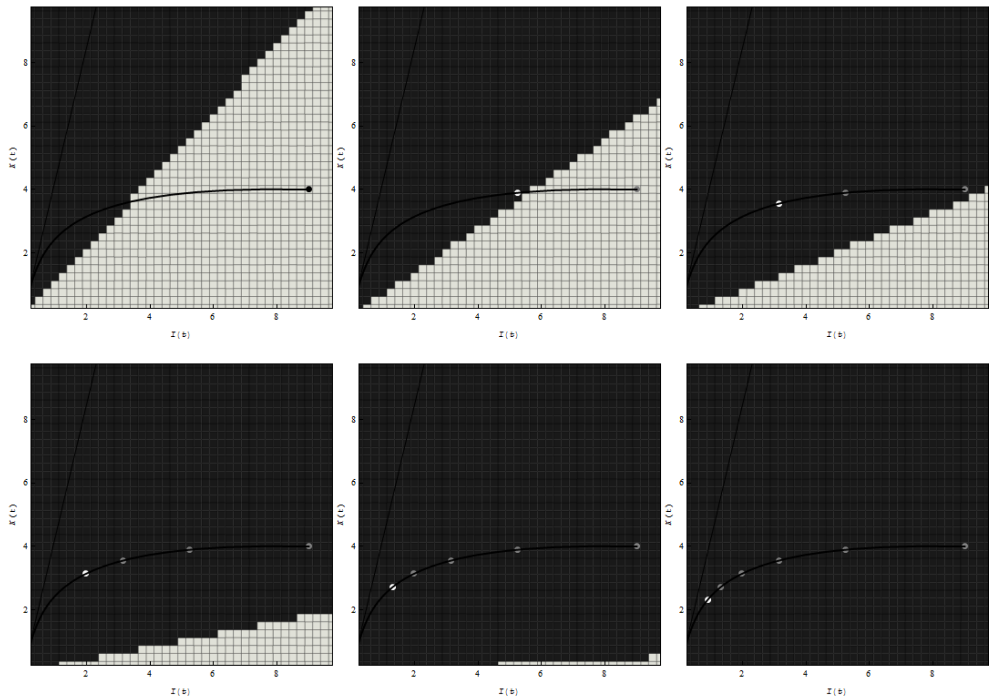

Figure 6.53: The first 6 periods with the optimal policy and the trajectory for the parameters $(a, b, c, d) = (1, 6, 1, 2)$

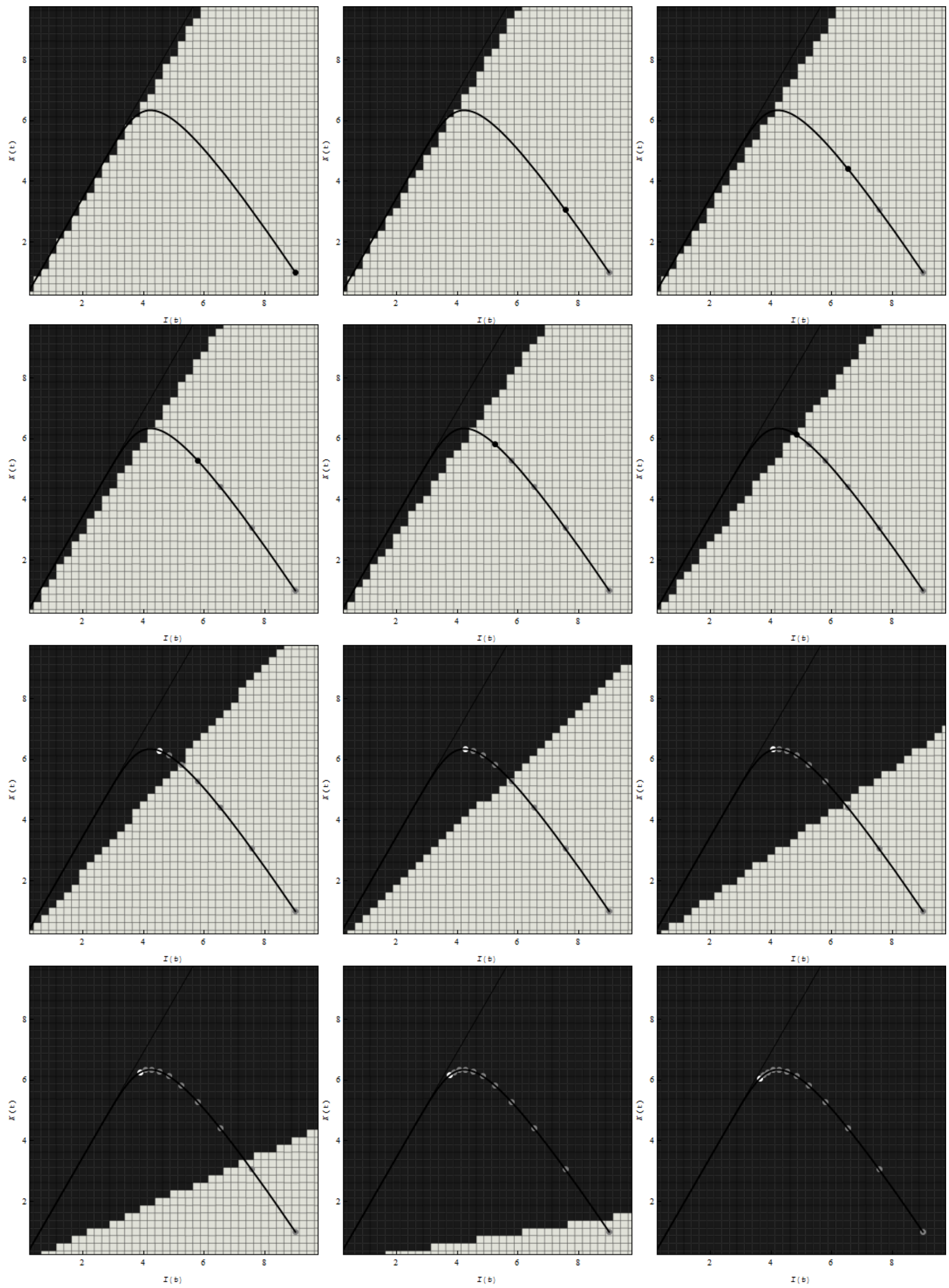

Figure 6.54: The first 12 periods with the optimal policy and the trajectory for the parameters $(a, b, c, d) = (1, 2, 3, 2)$

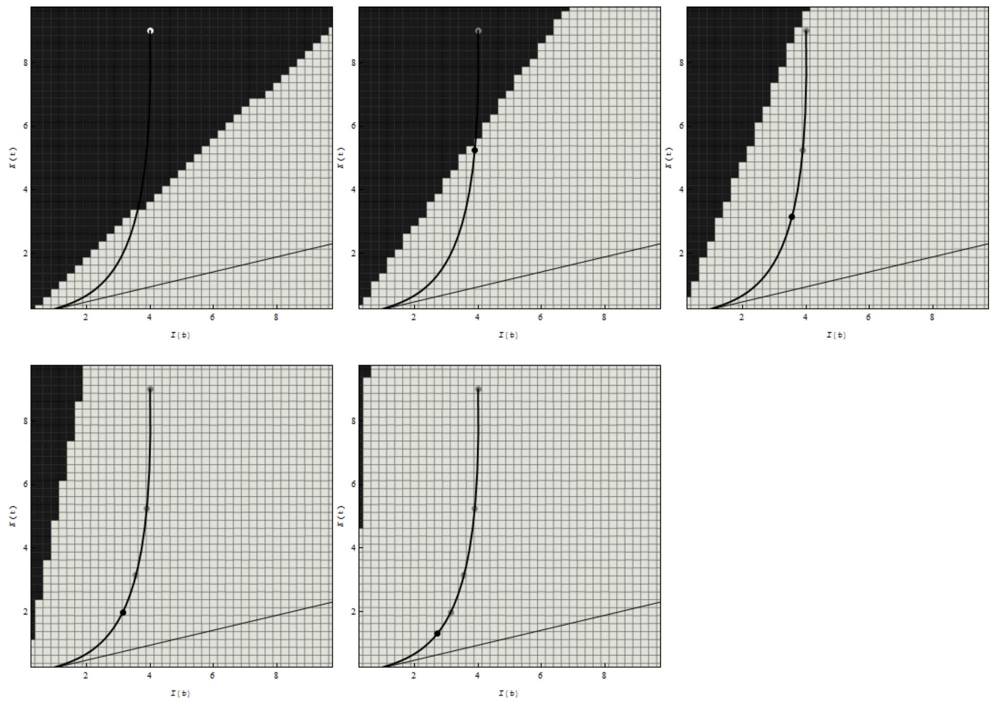

Figure 6.55: The first 5 periods with the optimal policy and the trajectory for the parameters $(a, b, c, d) = (1, 2, 1, 6)$

Chapter 7

Summary and Conclusions

We have discussed different aspects of terrorism and a variety of motives that prompt participants to join dissident movements. This thesis contains also an explanation of ideological migrations of terrorist organisations. The calculations are getting more complicated when they are faced with adversaries such as the government or competing dissident groups. Terrorist organisations with three types of actors are considered, as in Hausken et al. [12, Hausken et al. 2012]. Ideologues provide ideological purpose and have a strong group commitment. They are even willing to sacrifice their own interests for the good of the group. Criminal mercenaries have a weak group commitment but rather a personal financial interest. The third type of actors are captive participants who have neither a strong group commitment mor strong personal financial interests. They cannot leave the group due to the fact that their cost of defection is larger than their benefits.

We first considered a two-dimensional model without government intervention consisting of the labor exerted by altruists and capital fundings from outside. If a terrorist organisation consists only of ideologues, an external sponsor is needed. This model is represented by a system of two linear first order coupled differential equations. We solved this system generally and made some accurate remarks on this solution later in the thesis. We also discussed the stability of the two-dimensional model and every possibly occuring case. The main focus of this thesis was on the two-dimensional asymptotically stable case, which means that the terrorist organisation approaches an equilibrium over time. As mentioned above we did some research on the state variables of the solution an discovered a discontinouity at a specific constellation of the parameters occuring in the system. Therefore we adjusted the given solution for further examinations. To avoid such problems it is possible to choose another transformation matrix $\mathbf{T}$ while calculating the solution, e.g.

$$\mathbf{T} = \begin{pmatrix} -2a & b-d+\sqrt{bd-ac} \\ -(b-d+\sqrt{bd-ac}) & -2c \end{pmatrix},$$

with other eigenvectors of the system. The determinant of this matrix does not vanish for any constellation of the parameters a, b, c, d. Nevertheless the eigenvectors have to be of the same eigenspace as those of Section 2.1.

If a terrorist organisation consists of ideologues and mercenaries, the latter can provide money in addition to the capital fundings from outside if they do not corrupt the ideological purity. This extends the first model to a three-dimensional one. We also gave an accurate solution of this model and discussed the stability. We thereafter gave several plots of the system to visualise the dynamics.

We afterwards included captive participants and thereby extended the model to four dimensions. We also gave a general solution but due to the high dimension no further examinations were done. Finally we generalised this model by government interventions. This results in an inhomogenous system of differential equations. We also gave a direction to solve the inhomogenous system.

A main focus of this thesis is on the last two chapters. Chapter 5 discussed an presented the theory to control a terrorist organisation. In our case, the control intervention from outside was to change a parameter of the model by a certain percentage. The question of how this control intervention was carried out could be interpreted in many ways and was not done in this thesis. By $J_i,\ i \in \mathbb{R},\ i > 0$ we came up with a tool that interferes the development of the terrorist organisation on its way to the equilibrium. We also formed a tool that allows us to visualise the effects of the intervention over time. Chapter six applied this theory on a chosen set of parameters. First of all we tried to figure out when the effect is the biggest for two specific initial values. We observed the organisation year by year, which was actually not very practical. Thus we scanned the development of a group by periods of one tenth of a year. It was not very surprising that the biggest effect almost always appeared at the very beginning of our observation. This is due to the simplicity of the considered model, which gives us a reason to think twice about the assumptions. Perhaps it is possible to extend the model by nonlinear coefficients in order to approach a more realistic model. But actually it is the simplicity of the model that allowed us to do a variety of examinations. Nevertheless there are some initial values that show up the biggest effect in a later period. To discover those initial values we considered a grid over the (I, K)-plane and every point of the grid represented one initial value. We found out, which period is the best for the control intervention to switch on at every point. For that purpose we also used different weightenings on the considered state variables. Since the biggest effect appeared usually in the 0^{th} period, we plotted the "efficiency zones" on the (I, K)-plane. Actually there are some areas that show up bigger effects than others. Those areas are seperated by a clear edge. One could observe a trajectory of the terrorist group moving over these regions. Hence one could ask, e.g., for the costs while waiting for the terrorist group to move into a higher efficiency zone. We were also pursuing the issue of an optimal policy. Actually we did not just find out, which period is the best to change a specific parameter. We also found out, which parameter is the best to change in a certain period. Afterwards we had the information of when to change which parameter.

As mentioned above the considered model is pretty simple. As a consequence of this simplicity the results were pretty predictable. On the other hand, it allowed us to carry out a variety of investigations and would allow way more examinations than were done in this thesis. An elevating point would be to extend the model by nonlinear components. But in order to do that extension it is probably necessary to observe and study the phenomenon "terrorism" more accurately.

Appendix A

Routh-Hurwitz

A.1 Hurwitz' Stability Criterion

By means of this criterion it is possible to give a statement on the stability of a system of differnetial equations without knowing the zeros of its characteristic polynomial exactly. Therefore Hurwitz defines the so-called *Hurwitz-polynomial* which is a polynomial $p(\lambda)$ having all its zeros in the open left half-plane. If there are additionally some single zeros on the imaginary axis, $p(\lambda)$ is called a *modified Hurwitz-polynomial*. Hence a system is asymptotically stable if its polynomial $p(\lambda)$ is a Hurwitz-polynomial and it is stable if $p(\lambda)$ is a modified one.

A necessary but not sufficient condition for a Hurwitz-polynomial of n-th degree

$$p(\lambda) = p_n\lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_1\lambda + p_0, \qquad p_i \in \mathbb{R},\ i = 0, \ldots, n$$

is that all coefficients $p_i \in \mathbb{R}$ are available, which means that they are non-zero, and have the same signature. (In case of $n = 2$ this necessary condition is also sufficient.) For $n \geq 3$ we have to check further sufficient conditions such as the Hurwitz-determinants. Therefore we design the so called $n \times n$ *Hurwitz-matrix* [7, 17, Frey et al. 2008, Mlitz 2008] of the present polynomial given by

$$H(p) := \begin{pmatrix} p_{n-1} & p_{n-3} & p_{n-5} & \cdots & \cdots & 0 \\ p_n & p_{n-2} & p_{n-4} & \cdots & \cdots & 0 \\ 0 & p_{n-1} & p_{n-3} & \cdots & \cdots & 0 \\ 0 & p_n & p_{n-2} & \cdots & \cdots & 0 \\ 0 & 0 & p_{n-1} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_2 & p_0 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

and determine the main minors $M_1, \ldots, M_n$. Now the sufficient conditions can be written as:

- $p(\lambda)$ is a Hurwitz-polynomial if $M_i > 0,\ \forall i = 1, \ldots, n$.
- $p(\lambda)$ is a modified Hurwitz-polynomial if $M_1, \ldots, M_{n-2} > 0$ and $M_{n-1} = M_n = 0$.

Note that $M_n = p_0 \cdot M_{n-1}$. According to these informations the stability criterion of the three-dimensional system can be summarised as follows:

$p(\lambda) = p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0$ turns out to be a (modified) Hurwitz-polynomial if the coefficients $p_i \in \mathbb{R},\ i = 0, \ldots, 3$ are available and of the same signature and satisfy $p_1p_2 - p_0p_3 > 0\ (= 0)$.

And furthermore in case of a four-dimensional system we get:

$p(\lambda) = p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0$ turns out to be a (modified) Hurwitz-polynomial if the coefficients $p_i \in \mathbb{R},\ i = 0, \ldots, 4$ are available and of the same signature and satisfy $p_1p_2p_3 - p_1^2p_4 - p_0p_3^2 > 0\ (= 0)$.

A.2 Qualitative Analysis

This analysis uses the Hurwitz-matrix and the Cauchy Index [2,17, Bitmead and Anderson 1997, Mlitz 2008] as well to classify the zeros and therefore the eigenvalues of the investigated system. To obtain a first impression of the zeros of our characteristic polynomial $p(\lambda) = p_n\lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_1\lambda + p_0$ we have to take a look at the $n \times n$ Hurwitz matrix. Again we have to check the main minors of the Hurwitz matrix, $M_1, \ldots, M_n$, and figure out their signatures. According to Routh-Hurwitz there are no zeros on the imaginary axis if all three minors are unequal to zero. We now have to look at the sequence $p_n, M_1, \frac{M_2}{M_1}, \ldots, \frac{M_n}{M_{n-1}}$ and check for their specific signatures. The number of signature changes in this sequence equals the number of zeros with a positive real part. Note that we still do not know whether these zeros are real or complex.

For further analysis we may take a look at the Cauchy Index. The Cauchy Index $I_a^b\left(\frac{r}{q}\right)$ over an interval (a,b) of a real rational polynomial $p(\lambda) = \frac{r(\lambda)}{q(\lambda)}$ is defined as the number of jumps of $p(\lambda)$ from $-\infty$ to $+\infty$ minus the number from $+\infty$ to $-\infty$ as ν moves from a to b. Of course the interval (a,b) can be extended to $(-\infty, +\infty)$ to cover the entire domain of definition.

To calculate the Cauchy Index we have to consult *Sturm chains*, [17, Mlitz 2008]. A Sturm chain of a polynomial p is a sequence of polynomials associated to p and its derivative by a variant of *Euclid's algorithm for polynomials*. Sturm's theorem expresses the number of distinct real roots of p located in an interval. Applied to the interval of all the real numbers it gives the total number of real zeros of p.

A Sturm chain or Sturm sequence is a finite sequence of polynomials $f_0, \ldots, f_n$ of decreasing degree with these following properties:

- $f_0 = p(\nu)$ has only single zeroes;
- if $f_0(\xi) = 0$, then $\operatorname{sgn}(f_1(\xi)) = \operatorname{sgn}(p'(\xi))$;
- if $f_i(\xi) = 0$ for $0 < i < n$, then $\operatorname{sgn}(f_{i-1}(\xi)) = -\operatorname{sgn}(f_{i+1}(\xi))$;
- f_n does not change its signature.

This Sturm chain makes it easy to calculate the Cauchy Index. If $f_0, \ldots, f_n$ is a Sturm chain over (a,b), then

$$I_a^b\left(\frac{f_{i+1}}{f_i}\right) = V(a) - V(b),$$

where $V(.)$ denotes the number of signature changes in $f_0, \ldots, f_n$. Combining this analysis with the results from the Hurwitz matrix yields a clear classification of the zeros of the polynomial $p(\lambda)$, which allows us some remarks on the stability of the system.

List of Figures

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