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DIPLOMARBEIT

Masses of Anti-de Sitter Spacetimes

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Kurzfassung

Die Erforschung von Schwarzen Löchern in anti-de Sitter Raumzeiten hat in den letzen Jahren durch die Aufstellung der Maldacena-Vermutung einiges an Aufmerksamkeit auf sich gezogen. Der Maldacena-Vermutung zufolge besteht eine Verbindung zwischen anti-de Sitter Raumzeiten mit Schwarzen Löchern und konformen Feldtheorien, die am Rand der Raumzeit definiert sind.
Das Hauptaugenmerk dieser Diplomarbeit liegt auf der Berechnung der Masse von verschiedenen anti-de Sitter Raumzeiten mithilfe eines quasi-lokalen Energie-Impuls-Tensors, der durch das Hinzufügen von Countertermen renormiert wird. Dabei wird die Entwicklung des Counterterm-Formalismus im Detail betrachtet und anschließend praktisch angewendet.

Abstract

The study of black holes in anti-de Sitter spacetimes has drawn a lot of attention in the last years since the Maldacena conjecture was proposed. According to the Maldacena conjecture, there exists a duality between anti-de Sitter spacetimes with black holes and conformal field theories defined on their boundaries.
The focus of this work lies on the calculation of the mass of various anti-de Sitter spacetimes with the use of a quasilocal stress tensor, which is renormalized by the addition of counterterms. The development of the counterterm formalism is reviewed in detail and subsequently illustrated on practical examples.

Contents

Notation and conventions. All metrics are defined with the signature $(-, +, +, ..., +)$, e.g., the time coordinate has a negative sign in the line element, whereas the space coordinates have a positive sign. For the determinant of the metric, the short notation $\det g_{\alpha\beta} \equiv g$ is used. Greek indices are used for the full $(n+1)$-dimensional spacetime, lower case latin indices are used for the n-dimensional boundary, and upper case latin indices are used for $(n-1)$-dimensional hypersurfaces on the boundary. Covariant derivatives are denoted by ∇_α or a semicolon in the lower indices, normal derivatives are denoted by ∂_α or a comma in the lower indices. The definitions of commonly known variables are given in Appendix A.

Chapter 1

Introduction

The study of black holes in anti-de Sitter (AdS) spacetimes has gained in importance in the last years. At the first glance, AdS does not seem to be a good physical spacetime, because its closed timelike curves violate the principle of causality. But due to the Maldacena conjecture [1], AdS is related to a conformal field theory that lives on the boundary of the AdS spacetime (this is also called the AdS/CFT correspondence). This correspondence is of great importance because the calculations in AdS (based on gravity) are much easier than in field theory. In this Introduction, we want to give only a brief insight into how this conjecture is motivated. For more details and further references see [1], [2], [3].

The Maldacena conjecture suggests a duality between superstring theory and a conformal field theory. String theory is a quantum theory that includes gravity. Particles are described as one-dimensional extended objects (the strings) instead of being pointlike. The strings can oscillate and different oscillation modes define different particles. Every string theory produces a massless particle with spin 2, which has only gravity as a consistent interaction, so gravity is naturally included. String theories do not work in every dimension, for flat space only 10 dimensions are possible. 10-dimensional string theory also includes fermions and gives rise to a supersymmetric theory (superstring theory).

Originally, string theory was introduced in the 1960's to uniquely describe the growing number of hadrons and mesons. It could indeed make some useful predictions. Anyway, it turned out later that hadrons and mesons are built out of quarks and obey the field theory of QCD (quantum chromo dynamics). QCD is a gauge theory with the symmetry group $SU(3)$ (there are 3 colors). It is asymptotically free, which means that the coupling constant is high at low energies (strong interaction at great distances) and small at high energies (weak interaction at small distances). Because of the energy dependence of the constant, one speaks of a running coupling constant. For the case of weak interactions, calculations can be done by the use of perturbation theory, but it is quite hard to handle the case of strong interaction for which perturbation theory breaks down. Today, mostly numerical lattice simulations are used for the calculations.

In 1974, t'Hooft made the suggestion that increasing N (the number of colors) could simplify the theory, and that the large N limit is a free string theory with coupling constant $1/N$. It turned out that this large N limit string theory is the same as the one describing quantum gravity, so the theories are dual. This duality holds for basically any gauge theory, so one has the freedom to choose a conformally invariant gauge theory, i.e., the coupling constant does not depend on the energy. There is not a large assortment of conformally invariant gauge theories. One possibility is the four-dimensional supersymmetric $SU(N)$ gauge theory with $\mathcal{N} = 4$ (the number of spinor supercharges), which has the conformal group $SO(4,2)$. The dual string theory should have the same symmetries as the field theory. Strings are not consistent in four flat dimensions, but they can be formulated in five dimensions. It turns out that five-dimensional anti-de Sitter space (AdS_5) is the only space with an $SO(4,2)$ isometry. Since the field theory is supersymmetric, this is also required for the string theory. As mentioned above, superstring theory requires ten dimensions. This can be achieved by adding a five sphere S^5 to get the 10-dimensional spacetime $\mathrm{AdS}_5 \times S^5$.

The great advantage now is that the couplings of the theories are inverse to each other. In the regions where the field theory is strongly coupled, the string theory is weakly coupled, whereas the string theory has strong coupling when the field theory is weakly coupled. This is quantified by the parameter $g_{YM}^2 N \sim g_s N \sim l/l_s$, where l is the radius of AdS and l_s is the intrinsic size of the graviton. If the radius of the spacetime becomes large compared to the size of the string length ($l/l_s \gg 1$) then the classical gravity description becomes reliable. On the other hand if $g_{YM}^2 N \ll 1$ then the field theory is valid. So one can make calculations on a great range. But it has also the disadvantage that it makes the conjecture hard to prove or disprove, because most of the calculations can only be done in one of the theories. But there are some quantities which are independent of the coupling constant and can be calculated in both theories (see [2] for details).

To search for a relationship between gauge theories and string theories was motivated by studies of D-branes and black holes in string theory. D-branes are solitons in string theory and occur in various dimensionalities, which is indicated by writing D-p-brane. p is the number of extended spatial dimensions. For $p = 0$, the D-brane behaves like a particle. In string perturbation theory, D-branes are defined as the surfaces on which open strings can end. The D-branes have solutions called black branes that are very similar to extremal charged black holes. The near horizon geometry of D-3-branes is $\mathrm{AdS}_5 \times S^5$. If we consider a black hole in AdS_5 with a Schwarzschild radius greater than the curvature radius of AdS, then it behaves like a black brane with R^3 substituted by S^3.

In the 1970's it was shown that black holes have thermodynamical properties (see [4]) and also a temperature (see [5]). To keep the black hole in equilibrium it is therefore necessary to assign the surrounding spacetime the same temperature. Unlike in de Sitter spacetime, anti-de Sitter spacetime and flat spacetime have no natural temperature associated with and hence can be assigned any temperature [6]. So the gravitational calculations done at finite temperature correspond to finite temperature field theories. An interesting application of the AdS/CFT correspondence at finite temperature was given in [7]. Experiments from RHIC (Relativistic Heavy Ion Collider at Brookhaven) revealed that Quark Gluon Plasma (QGP) appears to be strongly coupled and AdS/CFT provides a powerful tool to gain insight behind the physics. For example the QGP appears to have a very low entropy over shear viscosity and with the correspondence one could get way closer to the measurement than with other approaches. In [7] a lower bound for this quantity was conjectured.

In this work however we focus on calculating the mass and momentum of various AdS spacetimes with the help of the quasilocal stress tensor. Defining a mass for a closed region of spacetime is not a trivial problem. In the Newtonian case, the mass is given by the potential $\phi(r) \propto -M/r$. This definition can be applied to asymptotically flat spacetimes by the use of the ADM prescription (see Section 5.1.1), where the potential is derived from the time component of the metric $g_{tt} \propto -(1 + 2\phi(r))$. It is possible to apply this formalism also to spacetimes with non-flat asymptotic regions, such as asymptotic AdS, but then one needs to introduce a reference background. In the case of asymptotic AdS, one would introduce vacuum AdS as reference spacetime and assign it zero mass, so the mass difference between the two spacetimes is the mass of the asymptotic AdS spacetime. Another approach was given by Brown and York in [8], where they introduced a quasilocal stress tensor from which the mass and momentum of a spacetime can be derived. The quasilocal stress tensor is defined locally on the boundary metric and usually diverges as the boundary is taken to infinity. To get a finite result, one can again try to introduce a reference spacetime in which a boundary with the same metric is embedded, but it is not always possible to find a suitable background in which the boundary can be embedded. A solution for this problem was found by Balasubramanian and Kraus in [9]. In the light of the AdS/CFT correspondence, the divergences occurring for $r \to \infty$ can be interpreted as the standard ultraviolet divergences of quantum field theory, which can be removed by renormalization. So they introduced divergent counterterms defined from the boundary metric to keep the stress tensor finite. It turned out that only a few counterterms are needed and that there is no freedom in how to choose them. A formal description of deriving the counterterms was given by Batrachenko, Liu, McNees, Sabra and Wu in [10] by the use of the Hamilton-Jacobi formalism.

We will start by recalling some basics of the Einstein equations, and we will review in detail how they can be obtained through a variation principle. In Chapter 3, we will state some general properties of AdS spacetimes and show how to find suitable coordinate systems. In Chapter 4, we will draw our attention to Schwarzschild black holes in five-dimensional AdS. We will present the Kruskal extension and the Penrose diagram and discuss some properties of black holes. In Chapter 5, we will introduce the quasilocal stress tensor and the counterterm formalism, and in Chapter 6, we will explicitly calculate the stress tensor and the mass for some AdS spacetimes.

Chapter 2

The Einstein Equations

The Einstein equations are the field equations of general relativity. They cannot be deduced in a strict sense, but we can give some plausibility arguments by comparison with the field equations for electro-magnetism. The source of the gravitational field is the stress-energy-momentum tensor $T_{\alpha\beta}$ (referred to as stress tensor) that describes the matter and energy content of the spacetime. We require the properties that it is symmetric $T_{\alpha\beta} = T_{\beta\alpha}$ and covariant conserved $\nabla^\alpha T_{\alpha\beta} = 0$. $T_{\alpha\beta}$ is a d-dimensional symmetric tensor, so it has $d(d+1)/2$ independent components. As in electrodynamics, we want to express the stress tensor in terms of the field variable, which for the gravitational field is the metric tensor $g_{\alpha\beta}$. $T_{\alpha\beta}$ depends on $g_{\alpha\beta}$ and its first and second derivative. It can be shown that there is only one tensorial expression that fulfills all these requirements, and that is the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \tag{2.1}$$

$R_{\alpha\beta}$ is the Ricci tensor and R is the Ricci scalar (see Appendix A for the definitions). Additionally, one is free to add a constant term proportional to $g_{\alpha\beta}$, since $g_{\alpha\beta}$ fulfills the properties of $T_{\alpha\beta}$. This term is denoted by Λ and called the cosmological constant. The Einstein equations are then given by

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{2.2}$$

The left-hand side of the Einstein equations (the Einstein tensor) describes the structure of the spacetime, the right-hand side (the stress tensor) the energy and matter content. In contrast to electrodynamics, it is a quite non-trivial problem to find solutions for the field equations, because the equations are non-linear. Whereas the electromagnetic field variable A_μ only depends on the electrical and magnetic fields, the gravitational field variable $g_{\alpha\beta}$ depends on the structure of the spacetime itself. The problem is that one cannot simply calculate the metric from the stress tensor because one already needs to know the metric to verify that the stress tensor is indeed covariant conserved. To avoid this problem, one could simply define the stress tensor to be the result of the left-hand side of the equation, but this will lead in general to unreasonable physical properties of the stress tensor. Therefore, one requires that $T_{\alpha\beta}$ has to fulfill additional conditions, which are local causality and at least the weak energy condition. The weak energy condition can be related with the Einstein equation as follows

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} =& 8\pi T_{\alpha\beta} \qquad \mid \cdot\, g^{\alpha\beta} \\ R - \frac{1}{2}\delta^{\alpha}_{\alpha} R =& 8\pi T \\ \left(1 - \frac{d}{2}\right) R =& 8\pi T \end{aligned}$$

Now we substitute the Ricci scalar into the Einstein equations

$$R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{d-2} T g_{\alpha\beta} \right) \qquad \mid \cdot\, u^{\alpha} u^{\beta}$$

$$R_{\alpha\beta} u^{\alpha} u^{\beta} = 8\pi \left(T_{\alpha\beta} u^{\alpha} u^{\beta} - \frac{1}{d-2} T \right) \tag{2.3}$$

$T_{\alpha\beta} u^{\alpha} u^{\beta}$ is the energy density as measured by an observer moving with the velocity u^{α} ($u^{\alpha} u_{\alpha} = c^2 \equiv 1$). For a physically reasonable system, the energy density should be non-negative.

$$T_{\alpha\beta} u^{\alpha} u^{\beta} \geqslant 0 \tag{2.4}$$

This is called the weak energy condition. Furthermore, there exists the strong energy condition, which states that the right-hand side of eqn. (2.3) should be non-negative, so

$$T_{\alpha\beta} u^{\alpha} u^{\beta} \geqslant \frac{1}{d-2} T \tag{2.5}$$

With the above relation eqn. (2.3) this is related to the Ricci tensor, so the energy condition can be expressed through the metric as $R_{\alpha\beta} u^{\alpha} u^{\beta} \geqslant 0$. The strong energy condition does not imply the weak energy condition, it is just a stronger requirement on the physical properties. Finally, there is the dominant energy condition, which requires that if u^{α} is a future directed, timelike vector, then $-T^{\alpha}_{\beta} u^{\beta}$ should be a future directed timelike vector. $-T^{\alpha}_{\beta} u^{\beta}$ is the energy-momentum current density of matter, so the dominant energy condition states that energy flow of matter must always be smaller than the speed of light. The dominant energy condition does imply the weak energy condition.

The simplest solutions of the Einstein equations are the vacuum solutions (with $T_{\alpha\beta} = 0$) of spaces with high symmetry. The spaces with constant curvature and maximum symmetry are Minkowski space (flat), de Sitter space (positive curvature) and anti-de Sitter space (negative curvature), which is the spacetime we are interested in. We will discuss its properties in Chapter 3.

2.1 Variation Principle

We now want to show that the Einstein equations fulfill a variation principle. Therefore, we need an action S whose variation with respect to the field variable is required to be zero. From this condition one can then obtain the field equations. Notice that this is not a derivation of the Einstein equations, since the action is simply chosen to match the already known result. The variation must furthermore fulfill the boundary condition that S is held fixed on the boundary, i.e., the variation of the field variable is identical zero at the boundary.

Generally, the action S is defined as the integral over the manifold $\mathscr{M}$ of a Lagrange density $\mathscr{L}$, which is a scalar function of the field variables and their derivatives.

$$S = \int_{\mathscr{M}} dx^d \sqrt{-g}\, \mathscr{L} \tag{2.6}$$

The gravitational Lagrange density is simply given by $\mathscr{L} \propto R$, and the according action functional is called the Hilbert term $S_H[g]$.

$$S_H[g] = -\frac{1}{16\pi G} \int_{\mathscr{M}} d^{n+1}x \sqrt{-g}\, R \tag{2.7}$$

If one has also a cosmological constant Λ the Lagrange density is defined as $\mathscr{L} \propto R - 2\Lambda$, and the action is then called the bulk term S_{bulk}.

$$S_{bulk}[g] = -\frac{1}{16\pi G} \int_{\mathscr{M}} d^{n+1}x \sqrt{-g}\, (R - 2\Lambda) \tag{2.8}$$

Λ is a constant and therefore remains unaffected by the variation. So for conciseness we will concentrate on the Hilbert term for the following calculations. As we will see later, the action furthermore requires an additional term that is integrated over the boundary $\partial\mathscr{M}$ of $\mathscr{M}$ to guarantee a well-behaved variation principle. This term is called the boundary term or Gibbons-Hawking term S_{GH}.

$$S_{GH}[g] = \frac{1}{8\pi G}\epsilon \oint_{\partial\mathscr{M}} d^n x \sqrt{|h|}\, \mathcal{K} \tag{2.9}$$

$\mathcal{K}$ and h are the extrinsic curvature and the determinant of the induced metric on the boundary, respectively. $\epsilon = n^\alpha n_\alpha$ is the absolute value of the normal vector of the boundary and equals $+1$ on time-like surfaces and -1 on space-like surfaces. Together, these terms form the gravitational action $S_G = S_H + S_{GH}$.

2.1.1 Variation of the Hilbert Term

For convenience, we will vary the action with respect to $g^{\alpha\beta}$ instead of $g_{\alpha\beta}$. These variations are not independent from each other. If we consider the variation of the identity $g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$ (see Appendix A), we find the relation

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu} \tag{2.10}$$

For the Hilbert term we get for the variation $\delta g^{\alpha\beta}$

$$\delta S_H = -\frac{1}{16\pi G}\int_{\mathscr{M}} d^{n+1}x[\sqrt{-g}\; g^{\alpha\beta}\delta R_{\alpha\beta} + \sqrt{-g}\; R_{\alpha\beta}\delta g^{\alpha\beta} + R\delta\sqrt{-g}] \tag{2.11}$$

where we have used $R = R_{\alpha\beta}g^{\alpha\beta}$. The variation of $\sqrt{-g}$ (see Appendix A) gives

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\; g_{\alpha\beta}\delta g^{\alpha\beta} \tag{2.12}$$

Inserting this into the variation of the action gives

$$\delta S_H = -\frac{1}{16\pi G}\int_{\mathscr{M}} d^{n+1}x[\sqrt{-g}(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta})\delta g^{\alpha\beta} + \sqrt{-g}\; g^{\alpha\beta}\delta R_{\alpha\beta}] \tag{2.13}$$

We recognize the expression in round brackets as the Einstein tensor. So we can already leave this part as it is, but the last term needs further attention. The Ricci tensor is defined as the contraction of the Riemann tensor

$$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta} = \Gamma^\mu_{\alpha\beta,\mu} - \Gamma^\mu_{\alpha\mu,\beta} + \Gamma^\mu_{\nu\mu}\Gamma^\nu_{\alpha\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\alpha\mu} \tag{2.14}$$

If we calculate $R_{\alpha\beta}$ in a local Lorentz frame, the Christoffel symbols are zero. Notice that the derivations of the Christoffel symbols need not necessarily to be zero.

$$\delta R_{\alpha\beta} \mathrel{\hat{=}} \delta(\Gamma^\mu_{\alpha\beta,\mu} - \Gamma^\mu_{\alpha\mu,\beta}) \mathrel{\hat{=}} (\delta\Gamma^\mu_{\alpha\beta})_{;\mu} - (\delta\Gamma^\mu_{\alpha\mu})_{;\beta} \tag{2.15}$$

We have introduced the symbol $\stackrel{\circ}{=}$ for relations that hold only true in a local Lorentz frame. First we have used the fact that in a local Lorentz frame only the derivations of the Christoffel symbols contribute. Then we have exchanged the derivation with the variation (this only works for normal derivatives, because for covariant derivatives, the variation would also act on the Christoffel symbols), and finally, we have substituted the normal derivative by a covariant derivative, which makes no difference in a Lorentz frame since the Christoffel symbols are zero. But now the equation is tensorial and therefore holds in all coordinate systems. So we can use it to define the new variable

$$g^{\alpha\beta}\delta R_{\alpha\beta} = (g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^{\beta}_{\alpha\beta})_{;\mu} =: \tilde{\delta}v^{\mu}_{;\mu} \tag{2.16}$$

where we have used that the covariant derivative was defined to be metric compatible, i.e., $g_{\alpha\beta;\mu} = 0$, so the derivative can be written to act on the whole expression. The $\tilde{\delta}$-symbol was used because it has just a symbolic meaning and should not be understood as the variation of a quantity v^{μ}. We can insert this into the integral and perform the following transformations by the use of the Gauss theorem

$$\begin{aligned}\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\; g^{\alpha\beta}\delta R_{\alpha\beta} &= \int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\;\tilde{\delta}v^{\mu}_{;\mu} = \\ &= \oint_{\partial\mathscr{M}} d\Sigma_{\mu}\tilde{\delta}v^{\mu} = \\ &= \oint_{\partial\mathscr{M}} d^{n}x\sqrt{|h|}\epsilon n_{\mu}\tilde{\delta}v^{\mu}\end{aligned} \tag{2.17}$$

For the further calculation of $\tilde{\delta}v^{\mu}$, we take the definition of the Christoffel symbols and use the boundary condition of the variation, which states that the variation of the metric vanishes at the boundary $\delta g^{\alpha\beta}|_{\partial\mathscr{M}} = 0$. With this, the variation of the Christoffel symbols simplifies to

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}) \tag{2.18}$$

Inserting this into the definition of $\tilde{\delta}v^{\mu}$ gives

$$\begin{aligned}\tilde{\delta}v^{\mu}|_{\partial\mathscr{M}} =& g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^{\beta}_{\alpha\beta} = \\ =& \frac{1}{2}(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu})(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}) = \\ =& \frac{1}{2}g^{\mu\nu}(\delta g_{\nu\ ,\beta}^{\ \beta} + \delta g_{\nu\ ,\alpha}^{\ \alpha} - \delta g_{\alpha\ ,\nu}^{\ \alpha}) - \frac{1}{2}g^{\alpha\mu}(\delta g^{\beta}_{\ \alpha,\beta} + \delta g^{\beta}_{\ \beta,\alpha} - \delta g_{\alpha\ ,\nu}^{\ \nu}) = \\ =& g^{\mu\nu}(\delta g_{\nu\ ,\alpha}^{\ \alpha} - \delta g_{\alpha\ ,\nu}^{\ \alpha}) = \\ =& g^{\mu\nu}g^{\alpha\beta}(\delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu})\end{aligned} \tag{2.19}$$

and

$$\tilde{\delta} v_\mu|_{\partial\mathscr{M}} = g^{\alpha\beta}(\delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}) \tag{2.20}$$

To calculate the product $n^\mu \tilde{\delta} v_\mu|_{\partial\mathscr{M}}$, we use the completeness relation

$$g^{\alpha\beta} = \epsilon n^\alpha n^\beta + h^{\alpha\beta} \tag{2.21}$$

where the metric is separated into a part that is orthogonal to the boundary and one that is tangential. We then get

$$\begin{aligned} n^\mu \tilde{\delta} v_\mu|_{\partial\mathscr{M}} =& n^\mu(\epsilon n^\alpha n^\beta + h^{\alpha\beta})(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) = \\ =& n^\mu h^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \end{aligned} \tag{2.22}$$

In the first line we have used that $n^\mu n^\alpha$ is a symmetric quantity, whereas $(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu})$ is antisymmetric in the indices μ and α, so the first term must vanish. If we further consider the boundary condition $\delta g_{\alpha\beta}|_{\partial\mathscr{M}} = 0$, which tells us that the metric is held fixed on the boundary, we can conclude that the derivatives tangential to the boundary must be zero. So the term $h^{\alpha\beta}\delta g_{\mu\beta,\alpha}$ in the second line also drops out and we get

$$n^\mu \tilde{\delta} v_\mu|_{\partial\mathscr{M}} = -n^\mu h^{\alpha\beta} \delta g_{\alpha\beta,\mu} \tag{2.23}$$

Inserting this into the variation of the action functional leads us to the final result

$$\begin{aligned} 16\pi G\, \delta S_H =& - \int_{\mathscr{M}} d^{n+1}x \sqrt{-g}\; G_{\alpha\beta}\; \delta g^{\alpha\beta} + \\ & + \epsilon \oint_{\partial\mathscr{M}} d^n x \sqrt{|h|}\; h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^\mu \end{aligned} \tag{2.24}$$

The aim was to reproduce the Einstein equations, so we must get rid of the boundary term. Now it becomes clear that the Gibbons-Hawking term was introduced with the purpose to cancel this term.

2.1.2 Variation of the Gibbons-Hawking Term

We will show now that the variation of the Gibbons-Hawking term indeed cancels the boundary term that arose in the variation of the Hilbert term. From the boundary condition we know that the metric is constant at the boundary, and therefore, the boundary metric $h_{\alpha\beta}$ is also constant. So the only thing left that can be affected by the variation is the extrinsic curvature of the boundary $\mathcal{K}$.

$$\delta S_{GH}[g] = \frac{1}{8\pi G} \epsilon \oint_{\partial\mathscr{M}} d^n x \sqrt{|h|}\; \delta\mathcal{K} \tag{2.25}$$

The extrinsic curvature is defined as

$$\begin{aligned}\mathcal{K} =& - n^{\alpha}_{;\alpha} = -g^{\alpha\beta}n_{\alpha;\beta} = \\ =& - (\epsilon n^{\alpha}n^{\beta} + h^{\alpha\beta})n_{\alpha;\beta} = \\ =& - h^{\alpha\beta}n_{\alpha;\beta} = -h^{\alpha\beta}(n_{\alpha,\beta} - \Gamma^{\nu}_{\alpha\beta}n_{\nu})\end{aligned} \tag{2.26}$$

where we have used $n^{\alpha}n_{\alpha;\beta} = \frac{1}{2}(n^{\alpha}n_{\alpha})_{;\beta} = 0$ in the second line. The variation $\delta g_{\alpha\beta}$ only affects the Christoffel symbols

$$\begin{aligned}\delta\mathcal{K} =& h^{\alpha\beta}\delta\Gamma^{\nu}_{\alpha\beta}n_{\nu} = \\ =& \frac{1}{2}h^{\alpha\beta}n_{\nu}g^{\mu\nu}(\delta g_{\mu\alpha,\beta} + \delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) = \\ =& - \frac{1}{2}h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n_{\mu}\end{aligned} \tag{2.27}$$

For the variation of the Christoffel symbols we have first used the boundary condition $\delta g_{\alpha\beta}|_{\partial\mathscr{M}} = 0$, and then the resulting fact that the tangential derivatives of the variation of the metric must vanish as well. Inserting the above relation into the variation of the action integral gives

$$\delta S_{GH}[g] = -\frac{1}{16\pi G}\epsilon \oint_{\partial\mathscr{M}} d^n x\sqrt{|h|}\; h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n_{\mu} \tag{2.28}$$

which exactly cancels the boundary term in δS_H.

If we combine our results, we find that we have successfully reproduced the left-hand side of the Einstein equations

$$\delta(S_H + S_{GH}) = -\frac{1}{16\pi G}\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\; G_{\alpha\beta}\;\delta g^{\alpha\beta} \tag{2.29}$$

2.1.3 Variation of the Matter Terms

What is still left is the right-hand side of the Einstein equations, the stress tensor. The stress tensor describes the contribution of matter and fields (e.g., energy or pressure) to gravity. So we will define an action functional with a Lagrangian depending on some fields ϕ and their derivatives $\phi_{;\alpha}$

$$S_M[\phi, g] = -\frac{1}{G}\int d^{n+1}x\sqrt{-g}\mathscr{L}(\phi, \phi_{;\alpha}; g_{\alpha\beta}) \tag{2.30}$$

Without specifying an explicit expression for $\mathscr{L}$, we get for the variation

$$\begin{aligned}\delta S_M =& - \frac{1}{G}\int d^{n+1}x\left(\mathscr{L}\delta\sqrt{-g} + \frac{\partial\mathscr{L}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}\sqrt{-g}\right) = \\ =& - \frac{1}{G}\int d^{n+1}x\sqrt{-g}\;\delta g^{\alpha\beta}\left(\frac{\partial\mathscr{L}}{\partial g^{\alpha\beta}} - \frac{1}{2}\mathscr{L}g_{\alpha\beta}\right)\end{aligned} \tag{2.31}$$

The stress tensor is defined by

$$T_{\alpha\beta} := \mathscr{L} g_{\alpha\beta} - 2\frac{\partial \mathscr{L}}{\partial g^{\alpha\beta}} \tag{2.32}$$

so that

$$\delta S_M = \frac{1}{2G} \int d^{n+1}x \sqrt{-g}\, T_{\alpha\beta}\, \delta g^{\alpha\beta} \tag{2.33}$$

If we put all the parts of the action together and consider the requirement of the variation principle that the variation of the action must be zero

$$0 = \delta(S_H + S_{GH} + S_M) = -\frac{1}{16\pi G} \int_{\mathscr{M}} d^{n+1}x \sqrt{-g}\, (G_{\alpha\beta} - 8\pi T_{\alpha\beta})\, \delta g^{\alpha\beta} \tag{2.34}$$

we obtain the Einstein equations (requiring that the above expression should be valid for all variations $\delta g^{\alpha\beta}$)

$$G_{\alpha\beta} = 8\pi\, T_{\alpha\beta} \tag{2.35}$$

Chapter 3

Anti-de Sitter Spacetime

AdS is the maximally symmetric solution of the vacuum Einstein equations with constant negative curvature. For vacuum solutions of the Einstein equations in $d = n + 1$ dimensions, the cosmological constant is given by

$$\Lambda = \frac{n-1}{2(n+1)} R \tag{3.1}$$

(see Appendix A). Multiplying this equation with $g_{\alpha\beta}$ gives the general expression for the Ricci tensor

$$R_{\alpha\beta} = \frac{2}{n-1} \Lambda g_{\alpha\beta} \tag{3.2}$$

so the Ricci tensor is proportional to the metric.

AdS$_{n+1}$ has the topology $S^1 \times \mathbb{R}^n$, where the 1-sphere S^1 is time-like and $\mathbb{R}^n$ is space-like. To find a metric for AdS$_{n+1}$, we introduce a $(n+2)$-dimensional embedding space

$$ds^2 = -du^2 - dv^2 + \sum_{i=1}^{n} dx_i^2 \tag{3.3}$$

with two time-like coordinates u and v, and n space-like coordinates x_i. The defining equation for AdS$_{n+1}$ is then

$$-u^2 - v^2 + \sum_{i=1}^{n} x_i^2 = -l^2 \tag{3.4}$$

with l the curvature radius of AdS$_{n+1}$. For comparison, a d-sphere in $\mathbb{R}^{d+1}$ is defined by $\sum_{i=1}^{d+1} x_i^2 = l^2$. Equivalently, as one introduces spherical coordinates on the sphere for easier description, we now introduce new coordinates that are adapted to the symmetry of AdS.

$$\begin{aligned} u =& l \cosh \mu \sin \lambda \qquad 0 \leq \lambda < 2\pi \\ v =& l \cosh \mu \cos \lambda \qquad 0 \leq \mu < \infty \end{aligned} \tag{3.5}$$

These coordinates cover the whole manifold, so we get a global coordinate system. Furthermore, we require that

$$l \sinh \mu = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \tag{3.6}$$

This can be fulfilled by introducing polar coordinates in the x_i hypersurface. Substituting u and v by μ and λ in the line element eqn. (3.3) gives

$$ds^2 = -l^2(\sinh^2\mu \, d\mu^2 + \cosh^2\mu \, d\lambda^2) + \sum_{i=1}^{n} dx_i^2 \tag{3.7}$$

with

$$\begin{aligned} du &= l(\sinh \mu \, d\mu \sin \lambda + \cosh \mu \cos \lambda \, d\lambda) \\ dv &= l(\sinh \mu \, d\mu \cos \lambda - \cosh \mu \sin \lambda \, d\lambda) \end{aligned} \tag{3.8}$$

As yet we did not make any specification on the dimension. For further use, we will now focus on AdS_3 and AdS_5.

3.1 AdS_3

For AdS_3 we have to introduce polar coordinates for $n = 2$ dimensions that fulfill $l \sinh \mu = \sqrt{x_1^2 + x_2^2}$, so we define

$$\begin{aligned} x_1 =& l \sinh \mu \cos \varphi \\ x_2 =& l \sinh \mu \sin \varphi \end{aligned} \tag{3.9}$$

With the differentials

$$\begin{aligned} dx_1 =& l \cosh \mu \, d\mu \cos \varphi - l \sinh \mu \sin \varphi d\varphi \\ dx_2 =& l \cosh \mu \, d\mu \sin \varphi + l \sinh \mu \cos \varphi d\varphi \end{aligned} \tag{3.10}$$

we can calculate the line element

$$\sum_{i=1}^{2} dx_i = l^2 \cosh^2\mu \, d\mu^2 + l^2 \sinh^2\mu \, d\varphi^2 \tag{3.11}$$

where $d\varphi^2 = d\Omega_1^2$ is the line element of the 1-sphere S^1 (which is a circle). Inserting this into eqn. (3.7) leads to

$$ds^2 = l^2(d\mu^2 - \cosh^2\mu\, d\lambda^2 + \sinh^2\mu\, d\varphi^2) \tag{3.12}$$

Notice that the time-like coordinate λ is an angle, which means it is periodic: $\lambda = \lambda + 2\pi$, so we have closed time-like curves (which is consistent with the statement at the beginning of this Chapter that the topology of AdS includes a time-like S^1). A spacetime with closed time-like curves bears the problem that is not causal, so we want to get rid of this. Therefore, one has to unwrap the λ coordinate by not identifying λ with $\lambda+2\pi$, and go over to the universal covering space. We will denote the non-periodic λ as t/l and furthermore, we will set $l \sinh\mu = r$. With

$$d\lambda^2 = \frac{dt^2}{l^2} \quad \text{and} \quad dr^2 = l^2 \cosh^2\mu\, d\mu^2 = l^2(1+\sinh^2\mu)d\mu^2 = (l^2+r^2)d\mu^2$$

the line element evaluates to

$$ds^2 = -\left(1+\frac{r^2}{l^2}\right)dt^2 + \left(1+\frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\Omega_1^2 \tag{3.13}$$

These coordinates are called global coordinates, because they cover the whole spacetime.

Beside these global coordinates another possibility is to introduce Poincaré coordinates, which do not cover the whole spacetime. For AdS_3 we define

$$\begin{aligned} u &= (l^2 - t^2 + t^2 + x^2)\frac{1}{2r} \\ v &= l\frac{t}{r} \\ x_1 &= (l^2 + t^2 - t^2 - x^2)\frac{1}{2r} \\ x_2 &= l\frac{x}{r} \end{aligned} \tag{3.14}$$

with $r \geqslant 0$. With the defining equation of the line element eqn. (3.3) we get

$$ds^2 = \frac{l^2}{r^2}(-dt^2 + dr^2 + dx^2) \tag{3.15}$$

3.2 AdS_5

We introduce polar coordinates for $n = 4$ that fulfill

$$l \sinh\mu = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

as follows

$$\begin{aligned} x_1 &= l \sinh\mu \sin\chi \sin\vartheta \cos\varphi \\ x_2 &= l \sinh\mu \sin\chi \sin\vartheta \sin\varphi \\ x_3 &= l \sinh\mu \sin\chi \cos\vartheta \\ x_4 &= l \sinh\mu \cos\chi \end{aligned} \tag{3.16}$$

and calculate the line element the same way we did for AdS_3, resulting in

$$\begin{aligned} \sum_{i=1}^{4} dx_i^2 =& l\cosh^2\mu\, d\mu^2 + l^2\sinh^2\mu(d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) = \\ =& l\cosh^2\mu\, d\mu^2 + l^2\sinh^2\mu\, d\Omega_3^2 \end{aligned} \tag{3.17}$$

where $d\Omega_3^2$ is the line element of the 3-sphere S^3. Comparing this with the result we got for AdS_3 shows that the expressions almost match, except that $d\Omega_1^2$ is substituted by $d\Omega_3^2$. Inserting this into eqn. (3.7) and again introducing the variables t and r leads to an equation with the same structure as we had for AdS_3

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \tag{3.18}$$

3.3 $\mathbf{AdS}_{n+1}$

The general form of AdS_{n+1} in global coordinates is given by

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2 \tag{3.19}$$

with $d\Omega_{n-1}^2$ the metric of the $(n-1)$-sphere S^{n-1}. Notice that this is not the only possible expression for the metric of AdS. We want to mention here two other locally equivalent solutions of the same structure as eqn. (3.19). We can write all three combined in the following equation [11]

$$ds^2 = -\left(k + \frac{r^2}{l^2}\right) dt^2 + \left(k + \frac{r^2}{l^2}\right)^{-1} dr^2 + \frac{r^2}{l^2} d\Sigma_{k,n-1}^2 \tag{3.20}$$

$$d\Sigma_{k,n-1}^2 = \begin{cases} l^2 d\Omega_{n-1}^2 & \text{for } k = +1 \\ \sum_{i=1}^{n-1} dx_i^2 & \text{for } k = 0 \\ l^2 d\Xi_{n-1}^2 & \text{for } k = -1 \end{cases}$$

For $k = 1$, we get the metric of eqn. (3.19), for $k = 0$, we have flat space instead of the sphere, and for $k = -1$, $d\Xi_{n-1}^2$ denotes the metric of an

$(n-1)$-dimensional hyperbolic space H^{n-1} (notice that the lowest dimensional hyperbolic space is H^2.)

Anti-de Sitter spacetimes have the special property that massless particles can reach the boundary within finite time. To show this, we use the line element from eqn. (3.19). Massless particles travel on light-like paths, so $ds^2 = 0$. We have the freedom to choose the path in the direction where all angles have the value 0. For simplicity, we further let l be equal 1.

$$ds^2 = 0 = -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 \tag{3.21}$$

from which we deduce

$$dt^2 = (1+r^2)^{-2}dr^2$$

If we take the square root of this expression, we can integrate it and get

$$t = \int_0^R \frac{1}{1+r^2}dr = \arctan R \tag{3.22}$$

If we let $R \to \infty$, then we have $\arctan \infty = \pi/2$, and t remains finite. On the other hand, if we look on a massive particle, we have a time-like path with $ds^2 < 0$. Therefore we pick up an additional constant, which gives after integration a contribution proportional to R and therefore t is infinite when $R \to \infty$. So a massive particle can never reach the boundary.

Chapter 4

Schwarzschild Anti-de Sitter Spacetime

The Schwarzschild solution is an exact solution of the Einstein equations that describes the exterior field of a static, spherically symmetric body. We are interested in a vacuum solution of a spacetime with a Schwarzschild black hole. Schwarzschild black holes are the simplest black holes: they are non-rotating and uncharged (mass, angular momentum and charge are the only properties that are necessary to characterize a black hole).

The metric for Schwarzschild-AdS$_{n+1}$ is given by

$$ds^2 = -fdt^2 + \frac{1}{f}dr^2 + r^2 d\Omega_{n-1}^2, \qquad f = 1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^{n-2}} \tag{4.1}$$

To shorten the expression, we have introduced the constant r_0^2

$$r_0^2 = \frac{16\pi G_d M}{(n-1)Vol(S^{n-1})} \quad \text{and} \quad Vol(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

M is the mass of the black hole. In the limit $M \to 0$, the metric transforms to AdS without a black hole. G_d is the d-dimensional Newton's constant and $Vol(S^{n-1})$ the volume of a unit $(n-1)$-sphere (notice that the volume of an $(n-1)$-sphere is the surface area of an n-ball, not the enclosed area). The coordinate t is the Schwarzschild time, which is the time measured by a clock at rest in infinite spatial distance from the black hole. The coordinate r is the Schwarzschild radial coordinate. It does not measure proper spatial distance from the origin, but fulfills the condition that the volume of a 3-sphere centered at the origin with radius r is $2\pi^2 r^2$.

We want to examine the function f now a bit more in detail. If we rewrite it as

$$f = \frac{l^2 r^{n-2} + r^n - r_0^2 l^2}{l^2 r^{n-2}}$$

we find that for $n > 2$, f becomes singular at $r = 0$, which means that the component of the metric tensor containing f blows up to infinity, and on the other hand, $1/f = 0$. For such values, the metric is said to be singular, because the metric tensor becomes singular (i.e., not invertible). For the numerator we find that in five dimensions ($n = 4$) it has four zeros where $f = 0$ and $1/f$ goes to infinity

$$r = \pm\sqrt{-\frac{l^2}{2} \pm \sqrt{\frac{l^4}{4} + r_0^2 l^2}} \tag{4.2}$$

If we choose the sign under the square root to be negative, we get two imaginary zeros, otherwise we get two real zeros. r is a real coordinate, so the imaginary zeros do not bother us. We may also require that r is a radial coordinate, and therefore, we are only interested in positive values of r. Then there is just the value of r with two positive signs left (the most positive root). We will call this value r_+.

So we found that the metric becomes singular at $r = 0$ and $r = r_+$, but what does this mean physically? For $r = 0$, one finds that the spacetime has a true singularity where the spacetime curvature goes to infinity. For $r = r_+$, the metric seems to be singular here as well, but we will show in the next section that in this case it is just because of a bad choice of coordinates. Therefore, it is called a coordinate singularity. By introducing new coordinates, we will find an expression for the metric that is regular at $r = r_+$ and thus, the spacetime curvature remains finite.

4.1 Kruskal Coordinates

We will now introduce a new coordinate system for Schwarzschild AdS_5 in order to eliminate the singularity at r_+ and find the so-called complete analytical extension of the metric, i.e., the metric is free of coordinate singularities (see also [12]).

We start by introducing Eddington-Finkelstein coordinates. Eddington-Finkelstein coordinates are constant on null geodesics, so they are light-like coordinates. Only the r and t coordinates are transformed, the 3-sphere will not be affected. From r and t two light-like coordinates can be build. We define the first one as

$$v = t + \int \frac{1}{f} dr \tag{4.3}$$

where v is constant on radially ingoing null geodesics. So lines of constant v describe the paths of radial light rays starting at a finite time t from $r = \infty$

and ending at $t = \infty$ at $r = r_+$. With

$$dv = dt + \frac{1}{f}dr \quad \Longrightarrow \quad dt^2 = dv^2 - \frac{2}{f}drdv + \frac{1}{f^2}dr^2 \tag{4.4}$$

we can eliminate the t component in the line element (3.18), giving the ingoing extension

$$ds^2 = -f dv^2 + 2drdv + r^2 d\Omega^2 \tag{4.5}$$

The second light-like coordinate is defined as

$$u = t - \int \frac{1}{f}dr \tag{4.6}$$

where u is constant on radially outgoing null geodesics. So lines of constant u describe the paths of radial light rays starting at $t = -\infty$ at a finite radial distance, ending at a finite time t at $r = \infty$. With

$$du = dt - \frac{1}{f}dr \quad \Longrightarrow \quad dt^2 = du^2 + \frac{2}{f}drdv + \frac{1}{f^2}dr^2 \tag{4.7}$$

we can again substitute dt^2, which leads to the outgoing extension

$$ds^2 = -f du^2 + 2drdu + r^2 d\Omega^2 \tag{4.8}$$

We want to eliminate the r component also, so we use

$$\begin{aligned} dt =& \frac{1}{2}(dv + du) \\ dr =& \frac{1}{2}f(dv - du) \end{aligned} \tag{4.9}$$

and substitute this into the line element (4.1). The result is called the double null form

$$ds^2 = -f dudv + r^2 d\Omega^2 \tag{4.10}$$

r and t can be expressed as $r(u, v)$ and $t(u, v)$

$$\begin{aligned} t =& \frac{1}{2}(v + u) \\ \int \frac{1}{f}dr =& \frac{1}{2}(v - u) \end{aligned} \tag{4.11}$$

where r is only given implicitly. At the first glance, it seems as if we had already found what we were searching for, as the coefficients of the line element are only singular for $r = 0$ (which is no coordinate singularity). But we also have to examine if the new coordinates are everywhere well-defined. Evaluation of the integral from the definitions eqns. (4.3) and (4.6) gives

$$F(r) := \int \frac{1}{f} dr = \frac{\sqrt{l^3}\sqrt{2(\sqrt{4r_0^2+l^2}-l)}\ln\left(\frac{\sqrt{2}r-\sqrt{l}\sqrt{\sqrt{4r_0^2+l^2}-l}}{\sqrt{2}r+\sqrt{l}\sqrt{\sqrt{4r_0^2+l^2}-l}}\right)}{4\sqrt{4r_0^2+l^2}} + \\ + \frac{2\sqrt{l^3}\sqrt{2(\sqrt{4r_0^2+l^2}+l)}\arctan\left(\frac{\sqrt{2}r}{\sqrt{l}\sqrt{\sqrt{4r_0^2+l^2}+l}}\right)}{4\sqrt{4r_0^2+l^2}} \tag{4.12}$$

We recognize the expression $\sqrt{\frac{l}{2}}\sqrt{(\sqrt{4r_0^2+l^2}-l)}$ as the zero of $f(r)$ we called r_+. Using this, we can shorten the notation of the expression to

$$F(r) = \frac{lr_+ \ln\left(\frac{(r-r_+)}{(r+r_+)}\right) + 2lr_+ \arctan\left(\frac{r}{r_+}\right)}{2\sqrt{4r_0^2+l^2}} \tag{4.13}$$

The logarithm is singular for $r = r_+$, and therefore, u and v are still not well-defined at r_+.

The problems caused by the logarithm can be avoided by redefining the null coordinates in the following way

$$v' = \exp\left(\frac{v\sqrt{4r_0^2+l^2}}{lr_+}\right) \tag{4.14}$$

$$u' = -\exp\left(\frac{-u\sqrt{4r_0^2+l^2}}{lr_+}\right) \tag{4.15}$$

The factor in the exponent was chosen to cancel the factor of the logarithm in $F(r)$, up to a remaining factor 1/2, so the logarithm drops out and the expression remains well-defined at $r = r_+$. But due to the remaining factor 1/2 we now have the root of $r - r_+$, which is imaginary if $r < r_+$. So the coordinates v' and u' do not cover the whole spacetime. One could argue that we could have removed the factor 1/2 as well to avoid this problem. But since we have to make one further coordinate transformation anyway to return to non-light-like coordinates, we will find it convenient to leave this factor.

Before turning to new coordinates, we still have to calculate the line element for these coordinates. To substitute them into the double null form eqn. (4.10), we need

$$dv' = \frac{\sqrt{4r_0^2 + l^2}}{lr_+} \exp\left(\frac{v\sqrt{4r_0^2 + l^2}}{lr_+}\right) dv \tag{4.16}$$

$$du' = \frac{\sqrt{4r_0^2 + l^2}}{lr_+} \exp\left(\frac{-u\sqrt{4r_0^2 + l^2}}{lr_+}\right) du \tag{4.17}$$

with the result

$$ds^2 = f\frac{l^2 r_+^2}{v'u'(4r_0^2 + l^2)} dv'du' + r^2 d\Omega_3^2 \tag{4.18}$$

Now we do the final transformation to Kruskal coordinates by defining

$$r' = \frac{1}{2}(v' - u') \tag{4.19}$$

$$t' = \frac{1}{2}(v' + u') \tag{4.20}$$

If we substitute all u' and v' in the above line element, using the relation $(dt')^2 - (dr')^2 = du'dv'$, we receive the line element in Kruskal coordinates

$$ds^2 = f\frac{l^2 r_+^2}{4r_0^2 + l^2} \exp\left(-\frac{2F(r)\sqrt{4r_0^2 + l^2}}{lr_+}\right)(-(dt')^2 + (dr')^2) + r^2 d\Omega_3^2 \tag{4.21}$$

During the transformation we picked up a factor 2 in the exponent that now cancels the 1/2 that made the problems before. So we have finally found a set of global coordinates that is well-defined for all r.

For better visualization of our spacetime, we want to plot a diagram showing lines of constant t and r in our new coordinates t' and r'. This is called the Kruskal diagram. To plot the diagram, we have to find the expressions for $t'(r', t = const)$ and $t'(r', r = const)$. From the definitions of t' and r', (eqn. (4.20) and (4.19)) we find by eliminating either u' or v' and inserting the definitions for u', v', u and v (eqns. (4.15), (4.14), (4.6) and (4.3))

$$t' + r' = v' = \exp\left(\frac{(t + F(r))\sqrt{4r_0^2 + l^2}}{lr_+}\right) \tag{4.22}$$

$$t' - r' = u' = -\exp\left(\frac{-(t - F(r))\sqrt{4r_0^2 + l^2}}{lr_+}\right) \tag{4.23}$$

For lines of constant r we have to eliminate t in the above expression, which is done by

$$\begin{aligned} u'v' = (t')^2 - (r')^2 &= -\exp\left(\frac{2F(r)\sqrt{4r_0^2+l^2}}{lr_+}\right) = \\ &= -\frac{r-r_+}{r+r_+}\exp\left(2\arctan\left(\frac{r}{r_+}\right)\right) \end{aligned} \tag{4.24}$$

By eliminating $F(r)$, we can construct lines of constant t

$$\frac{v'}{u'} = \frac{t'+r'}{t'-r'} = -\exp\left(\frac{2t\sqrt{4r_0^2+l^2}}{lr_+}\right) \tag{4.25}$$

$$t' = r'\frac{\exp\left(\frac{2t\sqrt{4r_0^2+l^2}}{lr_+}\right)-1}{\exp\left(\frac{2t\sqrt{4r_0^2+l^2}}{lr_+}\right)+1} = r'\tanh\left(\frac{2t\sqrt{4r_0^2+l^2}}{lr_+}\right) \tag{4.26}$$

Plotting the function $t'(r')$ for constant r leads to the Kruskal diagram as shown in Fig. 4.1. Lines of constant t are not plotted in the diagram. They would be represented by straight lines through the origin where the value of t determines the gradient. The lines for $r = r_+$ are light-like (null-surfaces) and represent the event horizon of the black hole. They divide the diagram into four parts. The event horizon in the upper half plane is called the future event horizon, the one in the lower half is called the past event horizon. Notice that apart from the existence of a coordinate singularity, nothing extraordinary happens to the spacetime at the event horizon. Outside the black hole (region I and III), the lines of constant r are time-like (as in Minkowski space), but inside the black hole (region II and IV), the lines of constant r are space-like. This means, that one cannot remain on a position of constant r without exceeding the speed of light. So in region II one has to move towards the singularity at $r = 0$ (which is also a space-like surface). On the other hand, in region IV one is repulsed from the singularity. So one cannot stay in region IV but has to move into region I or III. Because this behavior is opposite to the black hole in region II, region IV is also called a white hole. The singularity in region IV lies in the past of every point, whereas the singularity in region II lies in the future of every point. The two regions outside the black hole are causally separated from each other, since there is no possibility to send signals from one region to the other. (Any signal must travel on a time-like or at least light-like curve, so there is no possibility to travel between the regions I and III)

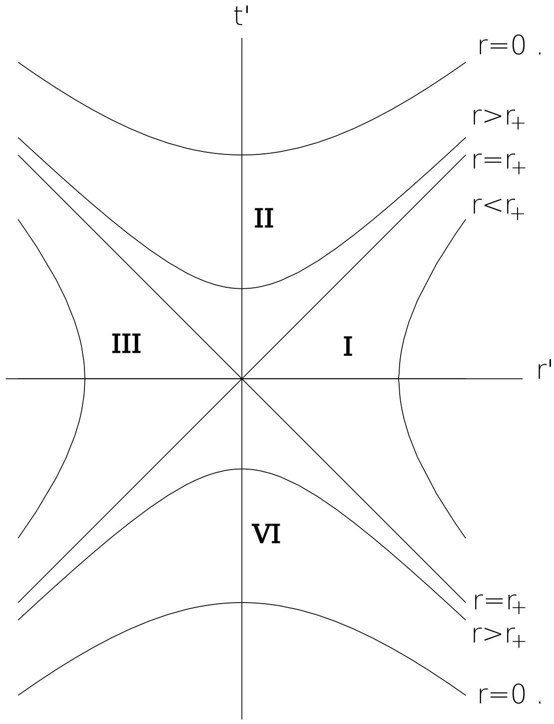

Figure 4.1: Kruskal diagram for Schwarzschild AdS. Every point represents a three-sphere. Light cones are at 45 degrees to the vertical.

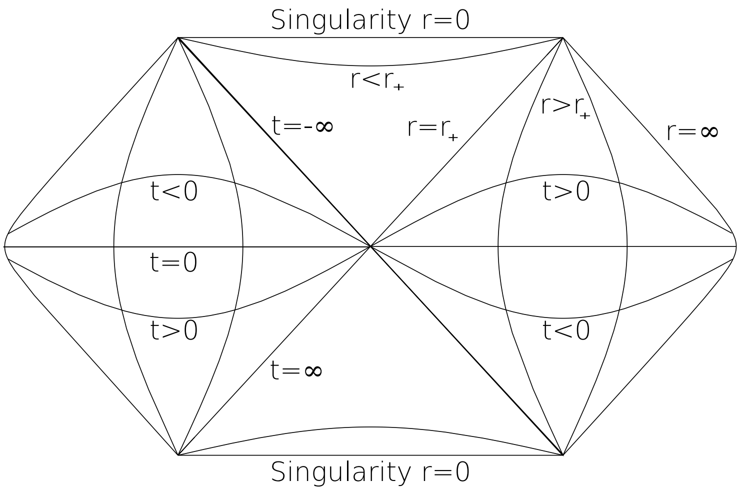

Figure 4.2: Penrose diagram for Schwarzschild AdS. Every point represents a three-sphere. Light cones are at 45 degrees to the vertical.

4.2 Penrose Diagram

The Kruskal diagram has the disadvantage that it is not suitable for mapping the whole infinite spacetime since $r', t' \in [-\infty, \infty]$. But through a further change of variables, we can construct the Penrose conformal diagram, which maps the infinite spacetime to a finite region, so we can plot the whole spacetime. With the function arctan, we can map the infinite interval $[-\infty, \infty]$ to the finite one $[-\pi/2, \pi/2]$. So we define new variables

$$v'' = \arctan(v'), \qquad -\frac{\pi}{2} < v'' < \frac{\pi}{2} \tag{4.27}$$

$$u'' = \arctan(u'), \qquad -\frac{\pi}{2} < u'' < \frac{\pi}{2} \tag{4.28}$$

with $-\frac{\pi}{2} < u'' + v'' < \frac{\pi}{2}$. Again, we want to plot lines of constant r and t, so we need to calculate the functions $u''(v'', t = const)$ and $u''(v'', r = const)$. By inserting the definitions for v', u', v and u (eqn. (4.14), (4.15), (4.3) and (4.6)), we get the inverse functions

$$\tan v'' = \exp\left(\frac{(t + F(r))\sqrt{4r_0^2 + l^2}}{lr_+}\right) \tag{4.29}$$

$$\tan u'' = -\exp\left(\frac{-(t - F(r))\sqrt{4r_0^2 + l^2}}{lr_+}\right) \tag{4.30}$$

Now we resolve this for $t + F(r)$ and $t - F(r)$

$$t + F(r) = \ln(\tan v'')\frac{lr_+}{\sqrt{4r_0^2 + l^2}} \tag{4.31}$$

$$t - F(r) = -\ln(-\tan u'')\frac{lr_+}{\sqrt{4r_0^2 + l^2}} \tag{4.32}$$

and get by addition and subtraction, respectively

$$t = \frac{1}{2}\ln\left(-\frac{\tan v''}{\tan u''}\right)\frac{lr_+}{\sqrt{4r_0^2 + l^2}} \tag{4.33}$$

$$F(r) = \frac{1}{2}\ln(-\tan v'' \tan u'')\frac{lr_+}{\sqrt{4r_0^2 + l^2}} \tag{4.34}$$

We stated above that t and r should be constant, so we take t and $F(r)$ as parameters and resolve the equations to get functions

$$u''(v'', t = const) = \arctan\left(-\tan(v'')\exp\left(-\frac{2t\sqrt{4r_0^2 + l^2}}{lr_+}\right)\right) \tag{4.35}$$

$$u''(v'', r = const) = \arctan\left(-\frac{1}{\tan(v'')}\exp\left(\frac{2F(r)\sqrt{4r_0^2 + l^2}}{lr_+}\right)\right) \tag{4.36}$$

In Fig. 4.2, the Penrose conformal diagram is plotted for lines of const r and t. It should be noticed that the line for $r = \infty$ is still spacelike, although it is very close to a lightlike curve. One could perform one more transformation that gives the diagram a rectangular shape, where $r = \infty$ is the vertical and $r = 0$ is the horizontal boundary of the rectangle. Then it becomes clear that $r = \infty$ indeed is a spacelike curve.

Chapter 5

The Quasilocal Stress Tensor

The quasilocal stress-energy-momentum tensor describes the energy and momentum of gravitational and matter fields in a spatially bounded region. A useful definition was first given in 1992 by Brown and York [8] by the use of the action principle and the Hamilton-Jacobi formalism.

5.1 Derivation of the Quasilocal Stress Tensor

In Chapter 2, we have demonstrated that the Einstein equations can be expressed as a variation principle. The stress tensor was defined as the variation of the Lagrangian of the matter action S_M, and by the requirement $\delta(S_M + S_G) = 0$ connected to the gravitational action S_G. Therefore, the Lagrangian in the definition of $T_{\alpha\beta}$ is equal to the gravitational Lagrangian. The quasilocal stress tensor T_{ij} can be defined equivalently. Remembering the steps from Section 2.1.3, we can define

$$T^{ij} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_G}{\delta \gamma_{ij}}. \tag{5.1}$$

where we have substituted the metric $g_{\alpha\beta}$ by the boundary metric γ_{ij}. $\mathscr{L}$ must also be expressed in terms of boundary variables. But instead of explicitly doing the calculation of the variation of S_G with respect to $\delta\gamma_{ij}$, we will use Hamilton-Jacobi theory to obtain the result. Hamilton-Jacobi theory tells us that the variation of the action S over the generalized coordinates q is the conjugate momentum $p = \frac{\partial S}{\partial q}$. Furthermore, in the Hamiltonian formalism the conjugate momentum is defined as

$$p = \frac{\partial}{\partial \dot{q}} (\sqrt{-g}\mathscr{L}) \tag{5.2}$$

In our case, the generalized coordinate is the boundary metric γ_{ij}. Eliminating p gives

$$\delta S = \frac{\partial}{\partial \dot{\gamma_{ij}}}(\sqrt{-g}\mathscr{L})\delta\gamma_{ij}. \tag{5.3}$$

Inserting the boundary metric for the generalized coordinate has lead to the formal expression $\dot{\gamma_{ij}}$, but it is still unclear how to interpret this quantity. This question will be answered in the next section. Afterwards we will deal with the problem to find an expression for $\mathscr{L}$ that depends only on boundary variables.

5.1.1 ADM Decomposition

In the ADM (Arnowitt, Deser and Misner) decomposition, the manifold $\mathscr{M}$ is sliced into non-intersecting space-like hypersurfaces. Therefore, we introduce an arbitrary scalar field $t(x^\alpha)$, where each $t = const$ describes a hypersurface Σ_t. The field t must only fulfill the conditions that it is a single-valued function of x^α, and that the unit normal to the hypersurfaces $n_\alpha \propto \partial_\alpha t$ is a future-directed time-like vector field. Now we introduce a new coordinate system (t, y^a), where y^a are the coordinates on the hypersurfaces. The coordinates on different hypersurfaces need not necessarily be connected with each other, but to construct a full coordinate system, we will have to introduce a relation between the former. To link the coordinates, we consider a congruence of curves γ that intersect the hypersurfaces. The intersection need not be orthogonal, nor need the curves be geodesics. t should be the parameter of the curves and t^α the tangent vector. If we follow a particular curve γ_P that has the intersection points P_i with the hypersurfaces Σ_{t_i}, we can identify the $y^a(P_i)$ so that y^a is held constant along the curves of the congruence. Expressed in the old coordinates $x^\alpha = x^\alpha(t, y^a)$, we have

$$t^\alpha = \left.\frac{\partial x^\alpha}{\partial t}\right|_{y^a} \tag{5.4}$$

for the tangential vectors to the congruence. The tangential vectors to the hypersurfaces Σ_t are defined as

$$e^\alpha_a = \left.\frac{\partial x^\alpha}{\partial y^a}\right|_t \tag{5.5}$$

and fulfill the relation $\pounds_t e^\alpha_a = 0$. Finally, we introduce the unit normal to the hypersurfaces

$$n_\alpha = -N\partial_\alpha t \tag{5.6}$$

	Σ_t (space-like)	$\mathscr{B}_r$ (time-like)	S_t (space-like)
Coordinates	y^a	z^i	ω^A
Tangential vector	e^α_a	e^α_i	e^α_A
Normal vector	$n_\alpha e^\alpha_a = 0$	$r_\alpha e^\alpha_i = 0$	$n_\alpha e^\alpha_A = 0$
Induced metric	$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$	$\gamma_{ij} = g_{\alpha\beta} e^\alpha_i e^\beta_j$	$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$
Completeness	$g^{\alpha\beta} = -n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b$	$g^{\alpha\beta} = r^\alpha r^\beta + \gamma_{ij} e^\alpha_i e^\beta_j$	$g^{\alpha\beta} = -n^\alpha n^\beta + r^\alpha r^\beta + \sigma_{AB} e^\alpha_A e^\beta_B$
Extrinsic curvature	$K_{ab} = n_{\alpha;\beta} e^\alpha_a e^\beta_b$	$\Theta_{ij} = r_{\alpha;\beta} e^\alpha_i e^\beta_j$	$\vartheta_{AB} = n_{\alpha;\beta} e^\alpha_A e^\beta_B$
Curvature scalar	$K = K_{ab} h^{ab} = n^\alpha_{;\alpha}$	$\Theta = \Theta_{ij} \gamma^{ij} = r^\alpha_{;\alpha}$	$\vartheta = \vartheta_{AB} \sigma^{AB}$
Metric determinant	$\sqrt{-g} = \sqrt{-g_{tt}}\sqrt{h}$	$\sqrt{-g} = \sqrt{g_{rr}}\sqrt{-\gamma}$	$\sqrt{-g} = \sqrt{-g_{tt} g_{rr}}\sqrt{\sigma}$

Table 5.1: Definitions of variables on different surfaces. Greek indices are used for the $(n+1)$-dimensional space, whereas for n-dimensional objects lower case Latin indices, and for $(n-1)$-dimensional ones upper case Latin indices are used.

The scalar function N is called the lapse and ensures proper normalization. n_α and e^α_a provide an orthogonal basis $n_\alpha e^\alpha_a = 0$. t^α is not orthogonal to the hypersurface and can be decomposed into basis vectors

$$t^\alpha = Nn^a + N^a e^\alpha_a \tag{5.7}$$

N^a is a vector on the hypersurface called the shift. To get the metric in the new coordinates, we start from the differential form of the coordinate transformation $x^\alpha = x^\alpha(t, y^a)$

$$\begin{aligned} dx^\alpha =& \frac{\partial x^\alpha}{\partial t}dt + \frac{\partial x^\alpha}{\partial y^a}dy^a = \\ =& t^\alpha dt + e^\alpha_a dy^a = \\ =& (Ndt)n^\alpha + (N^a dt + dy^a)e^\alpha_a \end{aligned} \tag{5.8}$$

and get with the definition $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ of the line element

$$ds^2 = -N^2dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt) \tag{5.9}$$

For our purpose, we are interested in decomposing the spacetime into time-like hypersurfaces $\mathscr{B}_r$ with $r = const$ rather than into space-like hypersurfaces Σ_t. By adapting the ADM formalism, one can write the metric in an ADM-like decomposition [9],

$$ds^2 = N^2dr^2 + \gamma_{ij}(dz^i + N^i dr)(dz^j + N^j dr) \tag{5.10}$$

where t was exchanged by r, and Σ_t by $\mathscr{B}_r$ with the coordinates z^i instead of y^a.

In the ADM decomposition the flow vector is t^α and the quantity $\dot{q}$ is defined as the Lie derivative along the flow vector [13]. For our ADM-like decomposition, the flow vector corresponds to r^α, so we will use a Lie derivative along a radial vector and define

$$\dot{\gamma_{ij}} = \pounds_r \gamma_{ij} \tag{5.11}$$

For the explicit calculation, we use that $\pounds_r e^\alpha_i = 0$, so the Lie derivative only acts on the metric tensor $g_{\alpha\beta}$

$$\pounds_r \gamma_{ij} = \pounds_r(g_{\alpha\beta}e^\alpha_i e^\beta_j) = \pounds_r g_{\alpha\beta} e^\alpha_i e^\beta_j \tag{5.12}$$

$$\begin{aligned} \pounds_r g_{\alpha\beta} =& g_{\alpha\beta;\mu}r^\mu + r^\mu_{;\alpha}g_{\mu\beta} + r^\mu_{;\beta}g_{\alpha\mu} = \\ =& \sqrt{g_{rr}}(\hat{r}_{\beta;\alpha} + \hat{r}_{\alpha;\beta}) = \\ =& -2\sqrt{g_{rr}}\Theta_{\alpha\beta} \end{aligned} \tag{5.13}$$

Here we have used $g_{\alpha\beta;\mu} = 0$ and introduced the symmetric tensor

$$\Theta_{\alpha\beta} = -\frac{1}{2}(\nabla_\alpha \hat{r}_\beta + \nabla_\beta \hat{r}_\alpha) \tag{5.14}$$

which is the extrinsic curvature tensor of the boundary metric. So the result is

$$\dot{\gamma_{ij}} = \pounds_r \gamma_{ij} = -2\sqrt{g_{rr}}\Theta_{\alpha\beta} \tag{5.15}$$

5.1.2 Foliation of the Spacetime

Now we will turn to the question how to express $\mathscr{L}$ in terms of boundary variables. Therefore, we need to distinguish between the different parts of the boundary. The boundary can be split up into two space-like surfaces Σ_1 and Σ_2 at constant times t_1 and t_2, and a time-like surface $\mathscr{B}_r$ at radius r

$$\partial\mathscr{M} = \Sigma_2 \cup (-\Sigma_1) \cup \mathscr{B}_r \tag{5.16}$$

Notice that the normal vector has to point outward, so we have to change the orientation of Σ_1, which is indicated by the minus sign in front of Σ_1 (we will assume $t_1 < t_2$). To foliate the $(n+1)$-dimensional spacetime with n-dimensional hypersurfaces, we will use surfaces of constant radius $\mathscr{B}_r$ in suitable coordinates. Table 5.1 gives an overview of the used symbols on different surfaces and some basic relations of important variables.

We want to rewrite the gravitational action

$$S_G[g] = \frac{1}{16\pi G}\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\; R + \frac{1}{8\pi G}\epsilon \oint_{\partial\mathscr{M}} d^n x\sqrt{|h|}\;\mathcal{K} \tag{5.17}$$

in terms of boundary variables and decompose it according to the above stated structure of the boundary. Therefore we start by splitting up the boundary term

$$\begin{aligned} 16\pi G S_G = &\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\; R + 2\int_{\mathscr{B}} d^n z\sqrt{-\gamma}\;\Theta + \\ &+ 2\int_{\Sigma_1} d^n y\sqrt{h}\; K - 2\int_{\Sigma_2} d^n y\sqrt{h}\; K \end{aligned} \tag{5.18}$$

With $\sqrt{h}$ we denote the metric determinant of surfaces of constant t. The $(n+1)$-dimensional Ricci scalar can be evaluated on an n-dimensional hypersurface with the Gauss-Codazzi equations (see [13] for details). Here, R was evaluated for the time-like boundary $\mathscr{B}_r$, with normal vector r^α and $\mathcal{R}$ the Ricci scalar with respect to the boundary metric γ_{ij}

$$R = \mathcal{R} - \Theta^{ij}\Theta_{ij} + \Theta^2 + 2(r^\alpha{}_{;\beta}r^\beta - r^\alpha r^\beta{}_{;\beta})_{;\alpha} \tag{5.19}$$

Substituting this into S_H gives with the use of the Gauss theorem

$$\begin{aligned}\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\; R = &\int dr \int_{\mathscr{B}_r} d^n z\sqrt{g_{rr}}\sqrt{-\gamma}\;(\mathcal{R} - \Theta^{ij}\Theta_{ij} + \Theta^2)\\ &+ 2\oint_{\partial\mathscr{M}} d\Sigma_\alpha (r^\alpha{}_{;\beta} r^\beta - r^\alpha r^\beta{}_{;\beta})\end{aligned} \tag{5.20}$$

where

$$\Sigma_\alpha = \begin{cases} +n_\alpha\sqrt{h}\; d^n y & \text{on } \Sigma_1 \\ -n_\alpha\sqrt{h}\; d^n y & \text{on } \Sigma_2 \\ +r_\alpha\sqrt{-\gamma}\; d^n z & \text{on } \mathscr{B}_r \end{cases}$$

For the evaluation of the surface integral, we use that the normal vectors are orthogonal $r^\alpha n_\alpha = 0$ and normalized $r^\alpha r_\alpha = +1$, $n^\alpha n_\alpha = -1$, that $r^\alpha{}_{;\beta} r_\alpha = \frac{1}{2}(r^\alpha r_\alpha)_{;\beta} = 0$ (this holds true also for n), and the definition of the extrinsic curvature $\Theta = r^\alpha{}_{;\alpha}$.

$$\begin{aligned}\oint_{\partial\mathscr{M}} d\Sigma_\alpha (r^\alpha{}_{;\beta} r^\beta - r^\alpha r^\beta{}_{;\beta}) = &- 2\int_{\Sigma_1} d^n y\sqrt{h}\; r^\alpha r^\beta n_{\alpha;\beta} + \\ &+ 2\int_{\Sigma_2} d^n y\sqrt{h}\; r^\alpha r^\beta n_{\alpha;\beta} - \\ &- 2\int_{\mathscr{B}_r} d^n z\sqrt{-\gamma}\;\Theta\end{aligned} \tag{5.21}$$

By inserting everything into eqn. (5.18), we see that the boundary term over $\mathscr{B}_r$ cancels out and we get

$$\begin{aligned}16\pi G S_G = &\int dr \int_{\mathscr{B}_r} d^n z\sqrt{g_{rr}}\sqrt{-\gamma}\;(\mathcal{R} - \Theta^{ij}\Theta_{ij} + \Theta^2) + \\ &+ 2\int_{\Sigma_1} d^n y\sqrt{h}\;(K - r^\alpha r^\beta n_{\alpha;\beta}) - \\ &- 2\int_{\Sigma_2} d^n y\sqrt{h}\;(K - r^\alpha r^\beta n_{\alpha;\beta})\end{aligned} \tag{5.22}$$

The boundary terms over Σ can be rewritten by the use of the definition of the extrinsic curvature $K = n_{\alpha;\beta}\; g^{\alpha\beta} = n_{\alpha;\beta}\; h^{\alpha\beta}$ (see eqn. (2.26)), and the relation $h^{\alpha\beta} = g^{\alpha\beta} + n^\alpha n^\beta$.

$$\begin{aligned}K - n_{\alpha;\beta} r^\alpha r^\beta = &n_{\alpha;\beta}(g^{\alpha\beta} + n^\alpha n^\beta) - n_{\alpha;\beta} r^\alpha r^\beta = \\ = &n_{\alpha;\beta}\sigma^{AB} e^\alpha_A e^\beta_B = \vartheta\end{aligned} \tag{5.23}$$

so that we finally get

$$
\begin{aligned}
16\pi S_G = & \int dr \int_{\mathscr{B}_r} d^n z \sqrt{g_{rr}} \sqrt{-\gamma}\ (\mathcal{R} - \Theta^{ij}\Theta_{ij} + \Theta^2) + \\
& + \int dr \int_{S_t} d^{n-1}\omega \sqrt{g_{rr}} \sqrt{\sigma}\ 2\vartheta \Big|_{t_1} - \\
& - \int dr \int_{S_t} d^{n-1}\omega \sqrt{g_{rr}} \sqrt{\sigma}\ 2\vartheta \Big|_{t_2}
\end{aligned}
\tag{5.24}
$$

Supposing a time-independent metric, the boundary terms over S_t cancel out. The gravitational Lagrangian written in boundary variables is then

$$
\mathscr{L} = \sqrt{g_{rr}} \sqrt{-\gamma}\ (\mathcal{R} - \Theta^{ij}\Theta_{ij} + \Theta^2) \tag{5.25}
$$

Now we have everything to calculate the conjugate momentum from eqn. (5.2), which we will denote with π_{ij}.

$$
\begin{aligned}
\pi^{ij} = & \frac{\partial}{\partial \pounds_r \gamma_{ij}} (\sqrt{-g}\mathscr{L}_G) = \\
= & \frac{\partial \Theta_{mn}}{\partial \pounds_r \gamma_{ij}} \frac{\partial}{\partial \Theta_{mn}} (\sqrt{-g}\mathscr{L}_G) = \\
= & -\frac{1}{16\pi G} \frac{-1}{2} \frac{1}{\sqrt{g_{rr}}} \delta_{im}\delta_{jn} \frac{\partial}{\partial \Theta_{mn}} [\mathcal{R} - (\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})\Theta_{ij}\Theta_{kl}] \sqrt{g_{rr}} \sqrt{-\gamma} = \\
= & -\frac{1}{16\pi G} \sqrt{-\gamma} (\Theta^{ij} - \gamma^{ij}\Theta)
\end{aligned}
\tag{5.26}
$$

With $\delta S = \pi^{ij}\delta\gamma_{ij}$, we get for the variation of the action functional

$$
\delta S_G = \frac{1}{16\pi G} \int_{\partial M} d^n x \sqrt{-\gamma}\ (\Theta^{ij} - \Theta\gamma^{ij})\delta\gamma_{ij} \tag{5.27}
$$

Knowing that $\delta S_G + \delta S_M = 0$, we get with the use of eqn. (2.33) the stress tensor for the boundary metric

$$
T_{ij} = \frac{1}{8\pi G} (\Theta_{ij} - \Theta\gamma_{ij}) \tag{5.28}
$$

5.2 Addition of Counterterms

The stress tensor T_{ij} was defined with respect to a boundary metric γ_{ij}, which was just the metric of a time-like hypersurface $\mathscr{B}_r$ where we made no specifications about r. If we want γ_{ij} to indeed describe the boundary of the spacetime, we have to take $r \to \infty$. The stress tensor will in general diverge in this case. Brown and York tried to solve the problem in [8] by introducing a reference spacetime, such as flat space, where they embedded a boundary with the same intrinsic metric γ_{ij}. Then one can subtract the reference spacetime from the spacetime of interest and get a finite result. The drawback of this method is that it is not always possible to find a suitable reference spacetime where the boundary can be embedded. For asymptotically anti-de Sitter spacetimes this problem was solved by Balasubramanian and Kraus in [9] with the counterterm formalism. Here the stress tensor is renormalized by the introduction of a finite series of boundary curvature invariants such as the metric, the Ricci tensor, or contractions of the Riemann tensor. For the definition of the counterterms we start from the gravitational action.

$$S_G = \frac{1}{16\pi G}\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\,(R-2\Lambda) - \frac{1}{8\pi G}\oint_{\partial\mathscr{M}} d^n x\sqrt{-\gamma}\,\Theta + S_{ct} \quad (5.29)$$

The action functional itself is divergent for $r \to \infty$, so the counterterm has to be chosen to cancel this divergence and must only depend on invariants of the boundary metric, such as the determinant of the metric tensor, the Ricci scalar, or contractions of the Ricci or the Riemann tensor. The number of required counterterms depends on the dimension of the spacetime. It turns out that there is no freedom in how to choose the counterterms. Through dimensional analysis one can show that in three dimensions only one counterterm is required, which is

$$S_1 = \int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma} \quad (5.30)$$

In four and five dimensions, the additional term

$$S_2 = \int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma}\,\mathcal{R} \quad (5.31)$$

is required. In six and seven dimensions, one needs two additional terms

$$S_3 = \int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma}\,\mathcal{R}^2 \quad (5.32)$$

$$S_4 = \int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma}\,\mathcal{R}_{ij}\mathcal{R}^{ij} \quad (5.33)$$

Higher dimensions require even more counterterms.

To get the general structure of a local counterterm for the stress tensor we start in analogy to S_M (see 2.1.3) with a general Lagrange function for S_{ct} and get

$$T^{ct}_{ij} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S^{ct}}{\delta\gamma^{ij}} \quad (5.34)$$

For the first two counterterms, this gives

$$T_{ij}^{(1)} = \frac{1}{8\pi G}\gamma_{ij} \tag{5.35}$$

$$T_{ij}^{(2)} = \frac{1}{8\pi G}(\mathcal{R}_{ij} - \frac{1}{2}\mathcal{R}) = \frac{1}{8\pi G}\mathcal{G}_{ij} \tag{5.36}$$

By doing the explicit calculations (see Chapter 6) and through the requirement that the counterterms must cancel the divergences, one finds the renormalized expressions

$$S_{reg} = S_{bulk} + S_{GH} - \frac{3}{l}S_1 \tag{5.37}$$

$$\begin{aligned} T_{ij}^{reg} =& T_{ij} - \frac{1}{l}T_{ij}^{(1)} = \\ =& \frac{1}{8\pi G}(\Theta_{ij} - \Theta\gamma_{ij} - \frac{1}{l}\gamma_{ij}) \end{aligned} \tag{5.38}$$

for AdS_3, and

$$S_{reg} = S_{bulk} + S_{GH} + \frac{3}{l}S_1 + \frac{l}{4}S_2 \tag{5.39}$$

$$\begin{aligned} T_{ij}^{reg} =& T_{ij} - \frac{3}{l}T_{ij}^{(1)} + \frac{l}{2}T_{ij}^{(2)} = \\ =& \frac{1}{8\pi G}(\Theta_{ij} - \Theta\gamma_{ij} - \frac{3}{l}\gamma_{ij} + \frac{l}{2}\mathcal{G}_{ij}) \end{aligned} \tag{5.40}$$

for AdS_5.

5.3 Hamilton-Jacobi Formalism

Balasubramanian and Kraus did not give a strict formalism on how to calculate the factors of the counterterms. They were simply chosen in the right way to get a finite result. It was shown by Batrachenko, Liu, McNees, Sabra and Wen in [10] that all the counterterms and factors can be derived from a Hamilton-Jacobi approach.

We start from a gravitational action

$$\begin{aligned} S[g_{\alpha\beta}, \phi^a, A^I_\alpha] =& -\frac{1}{16\pi G}\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\,(R - \frac{1}{2}g^{(\phi)}_{ab}\partial_\alpha\phi^a\partial^\alpha\phi^b - \\ & - \frac{1}{4}G_{IJ}(\phi)F^I_{\alpha\beta}F^{\alpha\beta\,J} - V(\phi)) \end{aligned} \tag{5.41}$$

which describes the bosonic part of a general matter-coupled system (see also [14]). The ϕ^a are scalar fields, $V(\phi)$ is a scalar potential, and the $F^I_{\alpha\beta}$ are the field strenght tensors, defined as

$$F^I_{\alpha\beta} = \partial_\alpha A^I_\beta - \partial_\beta A^I_\alpha$$

The indices a, b, I and J are field indices and not coordinate indices. We can again use the Gauss-Codacci equations to express the curvature scalar R in terms of boundary variables. In Section 5.1.2, we have found the relation

$$\int_{\mathcal{M}} \sqrt{-g}\, R = \int_{\mathcal{M}} \sqrt{-g}\, (\mathcal{R} - \Theta_{ij}\Theta^{ij} + \Theta^2) \tag{5.42}$$

which we substitute into eqn. (5.41). With the use of

$$g^{\alpha\beta} = r^\alpha r^\beta + \gamma^{\alpha\beta} e_i^\alpha e_j^\beta \qquad \text{and} \qquad \partial^\alpha = g^{\alpha\beta}\partial_\beta$$

we can rewrite the remaining terms so that we can express everything in boundary variables

$$\begin{aligned} S = &-\frac{1}{16\pi G}\int_{\mathscr{M}} d^{n+1}x\sqrt{-g}\, (\mathcal{R} - \Theta_{ij}\Theta^{ij} + \Theta^2 - \\ &-\frac{1}{2}g^{(\phi)}_{ab} r^\alpha \partial_\alpha \phi^a r^\beta \partial_\beta \phi^b - \frac{1}{2}g^{(\phi)}_{ab}\gamma^{ij}\partial_i\phi^a\partial_j\phi^b - \\ &-\frac{1}{4}G_{IJ}(\phi)F^{ij\,I}F^J_{ij} - \frac{1}{2}G_{IJ}(\phi)\gamma^{ij}r^\alpha F^I_{\alpha i} r^\beta F^J_{\beta j} - V(\phi)) \end{aligned} \tag{5.43}$$

In Hamilton theory, the Hamiltonian density $\mathscr{H}$ is

$$\mathscr{H} = \sum_i \pi_i \dot{q}_i - \mathscr{L} \tag{5.44}$$

and the Hamilton equations are

$$\dot{q}_i = \frac{\partial \mathscr{H}}{\partial \pi_i} \qquad \text{and} \qquad \dot{\pi}_i = \frac{\partial \mathscr{H}}{\partial q_i} \tag{5.45}$$

The conjugate momentum is defined as

$$\pi_i := \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \tag{5.46}$$

Usually, the Hamiltonian density and the conjugate momenta are defined with respect to t, but the holographic principle of flows in radial direction tells us to define them with respect to r instead. So the dotted variables are derivatives with respect to r. For the generalized coordinates q_i we have the field variables ϕ^a, A^I_α and γ_{ij}. The $\dot{q}_i$ are the Lie derivatives $\pounds_r$ of the field variables

$$\begin{aligned} \dot{\phi}^a &= \pounds_r \phi_a = r^\alpha \partial_\alpha \phi^a \\ \dot{A}^I_\alpha &= \pounds_r A^I_\alpha = r^\alpha \partial_\alpha A^I_\alpha \\ \dot{\gamma}_{ij} &= \pounds_r \gamma_{ij} = -2\sqrt{-\gamma}\,\Theta_{ij} \end{aligned} \tag{5.47}$$

For the last relation we have used eqn. (5.15). Now we can calculate the conjugate momenta.

$$\begin{aligned}\pi_a =&\frac{\partial \mathscr{L}}{\partial \dot{\phi}_a} = \\ =&\frac{1}{16\pi G}\frac{\partial}{\partial(r^\alpha \partial_\alpha \phi^a)}(\frac{1}{2}g^{(\phi)}_{ab} r^\alpha \partial_\alpha \phi^a r^\beta \partial_\beta \phi^b) = \\ =&\frac{1}{16\pi G} g^{(\phi)}_{ab}\hat{r}^\beta \partial_\beta \phi^b\end{aligned} \tag{5.48}$$

$g^{(\phi)}_{ab}$ is symmetric in a and b, so we can exchange ϕ^a and ϕ^b, and get an additional factor 2 from the derivation. For the calculation of the conjugate momentum of the field variable A^I_α, we use the definition of the field strength tensor $F^I_{\alpha\beta} = \partial_\alpha A^I_\beta - \partial_\beta A^I_\alpha$ and get

$$\begin{aligned}\pi^i_I =&\frac{\partial \mathscr{L}}{\partial \dot{A}^I_i} = \\ =&\frac{1}{16\pi G}\frac{\partial}{\partial(r^\alpha \partial_\alpha A^I_i)}(\frac{1}{2}G_{IJ}\gamma^{ij} r^\alpha(\partial_\alpha A^I_i - \partial_i A^I_\alpha) r^\beta F^J_{\beta j}) = \\ =&\frac{1}{16\pi G}\frac{1}{2}G_{IJ}\gamma^{ij} n^\beta F^J_{\beta j}\end{aligned} \tag{5.49}$$

The derivation of $\partial_i A^I_\alpha$ is zero, so this term vanishes. We have already calculated the momentum conjugate to the induced metric in eqn. (5.26), with the result

$$\begin{aligned}\pi^{ij} =&\frac{\partial \mathscr{L}}{\partial \dot{\gamma}_{ij}} = \\ =&\frac{1}{16\pi G}\frac{\partial}{-2\sqrt{g_{rr}}\,\partial\Theta_{ij}}(\Theta_{ij}\Theta^{ij} - \Theta^2) = \\ =&\frac{1}{16\pi G}\frac{1}{\sqrt{g_{rr}}}(\Theta\gamma^{ij} - \Theta^{ij})\end{aligned} \tag{5.50}$$

Now we have everything we need to calculate the Hamiltonian density. For the products $\pi_i \dot{q}_i$ we get

$$\begin{aligned}
\pi_a \dot{\phi}^a =& \pi^b g^{(\phi)}_{ab} r^\alpha \partial_\alpha \phi^a = \\
=& 16\pi G \pi^a \pi_b = 16\pi G \pi^a \pi^b g^{(\phi)}_{ab} \\
\pi^i_I \dot{A}^I_i =& G_{IJ} \gamma^{ij} \pi^J_j r^\alpha \partial_\alpha A^I_i = \\
=& G_{IJ} \gamma^{ij} \pi^J_j r^\alpha (F^I_{\alpha i} + \partial_i A^I_\alpha) = \\
=& G_{IJ} \gamma^{ij} \pi^J_j (16\pi G \pi^I_i + \partial_i A^I_\alpha) \\
\pi^{ij} \dot{\gamma}_{ij} =& -\frac{2}{16\pi G} (\gamma^{ij} \Theta_{ij} \Theta - \Theta^{ij} \Theta_{ij}) = \\
=& -\frac{2}{16\pi G} (\Theta^2 - \Theta^{ij} \Theta_{ij}) = \\
=& 2 \cdot 16\pi G (\pi^{ij} \pi_{ij} - \frac{1}{n-1} \pi^i_i \pi^j_j)
\end{aligned} \tag{5.51}$$

(see Appendix A for the details of the last step). If we further consider the substitution (see Appendix A)

$$\frac{1}{16\pi G} \frac{1}{2} G_{IJ} \gamma^{ij} r^\alpha F^I_{\alpha i} r^\beta F^J_{\beta j} = 16\pi G \frac{1}{2} G_{IJ} \gamma^{ij} \pi^I_i \pi^J_j \tag{5.52}$$

we can write down the Hamiltonian density $\mathscr{H}[\pi_a, \phi^a, \pi^i_I, A^I_i, \pi^{ij}, \gamma_{ij}]$ as

$$\begin{aligned}
\mathscr{H} =& 16\pi G \left(\frac{1}{2} G_{ab} \pi^a \pi^b + \frac{1}{2} G_{IJ} \gamma^{ij} \pi^I_i \pi^J_j + \pi^{ij} \pi_{ij} - \frac{1}{n-1} \pi^i_i \pi^j_j \right) + \\
&+ \frac{1}{16\pi G} \left(\mathcal{R} - \frac{1}{2} g^{(\phi)}_{ab} \gamma^{ij} \partial_i \phi^a \partial_j \phi^b - \frac{1}{4} GIJF^{ij\,I} F^J_{ij} - V \right) + \\
&+ G_{IJ} \gamma^{ij} \pi^J_j \partial_i A^I_\alpha
\end{aligned} \tag{5.53}$$

Now we turn to Hamilton-Jacobi theory. Here the conjugate momenta are written as functional derivatives of the on-shell action

$$\begin{aligned}
\pi_a &= \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \phi^a} \\
\pi^i_I &= \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta A^I_i} \\
\pi^{ij} &= \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ij}}
\end{aligned} \tag{5.54}$$

and we have the Hamilton-Jacobi equation

$$\mathscr{H} \left[\frac{\delta S}{\delta \phi^a}, \phi^a, \frac{\delta S}{\delta A^I_i}, A^I_i, \frac{\delta S}{\delta \gamma_{ij}}, \gamma_{ij} \right] = 0 \tag{5.55}$$

To determine the counterterms, we make the following ansatz for the action

$$S_{ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma}\Big(W(\phi) + C(\phi)\mathcal{R}\Big) \tag{5.56}$$

which contains the counterterms we need for up to five dimensions. In three dimensions we only need the first term, whereas in four and five dimensions, we need both terms. The conjugate momenta of the counterterm action are

$$\begin{aligned}
P_a =& \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\phi^a} = \\
=& \frac{1}{8\pi G}\left(\frac{\partial W}{\partial\phi^a} + \frac{\partial C}{\partial\phi^a}\mathcal{R}\right) \\
P^{ij} =& \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{ij}} = \\
=& \frac{1}{8\pi G}\left(\frac{1}{2}\gamma^{ij}W - C\mathcal{G}^{ij}\right) \\
P_i^I =& \frac{1}{\sqrt{-\gamma}}\frac{\delta S}{\delta A_i^I} = \\
=& 0
\end{aligned} \tag{5.57}$$

We substitute them into the Hamilton density eqn. (5.53)

$$\begin{aligned}
\mathscr{H} =& \frac{1}{16\pi G}\left(\mathcal{R} - \frac{1}{2}g^{(\phi)}_{ab}\gamma^{ij}\partial_i\phi^a\partial_j\phi^b - V - \frac{1}{4}G_{IJ}F^I_{ij}F^{ij\,J}\right) + \\
&+ \frac{2}{8\pi G}\Bigg[\frac{1}{2}g^{(\phi)}_{ab}\left(\frac{\partial W}{\partial\phi^a}\frac{\partial W}{\partial\phi^b} + \frac{\partial C}{\partial\phi^a}\frac{\partial C}{\partial\phi^b}\mathcal{R}^2 + \frac{\partial W}{\partial\phi^a}\frac{\partial C}{\partial\phi^b}\mathcal{R} + \frac{\partial W}{\partial\phi^b}\frac{\partial C}{\partial\phi^a}\mathcal{R}\right) + \\
&+ \frac{1}{4}\gamma^{ij}\gamma_{ij}W^2 + C^2\mathcal{G}^{ij}\mathcal{G}_{ij} - \gamma^{ij}\mathcal{G}_{ij}CW - \\
&- \frac{1}{n-1}\left(\frac{1}{4}\gamma^i_i\gamma^j_jW^2 + C^2\mathcal{G}^i_i\mathcal{G}^j_j - \gamma^i_i\mathcal{G}^j_jCW\right)\Bigg]
\end{aligned} \tag{5.58}$$

and use the Hamilton-Jacobi equation $\mathscr{H} = 0$. Solving this equation order-by-order in the metric gives three equations $\mathscr{H}_i = 0$, $i = 0, 1, 2$, where the index denotes the order of the metric. In zeroth order we have

$$\mathscr{H}_0 = \frac{1}{16\pi G}\left(2g^{(\phi)}_{ab}\frac{\partial W}{\partial\phi^a}\frac{\partial W}{\partial\phi^b} - \frac{n}{n-1}W^2 - V - \frac{1}{4}G_{IJ}F^I_{ij}F^{ij\,I}\right) = 0 \tag{5.59}$$

We can solve this for the scalar potential $V(\phi)$ in terms of the superpotential $W(\phi)$ and get

$$V = 2g^{(\phi)}_{ab}\frac{\partial W}{\partial\phi^a}\frac{\partial W}{\partial\phi^b} - \frac{n}{n-1}W^2 \tag{5.60}$$

The constant term $V_0 = -\frac{n}{n-1}W^2$ is related to the AdS length scale [10] by

$$l = \sqrt{-\frac{n(n-1)}{V_0}} \tag{5.61}$$

so we have found the relation

$$W = \frac{n-1}{l} \tag{5.62}$$

In first order, we have all terms containing the metric itself or the Ricci scalar. Both contributions are separately zero.

$$\begin{aligned}\mathscr{H}_1 =& \frac{1}{16\pi G}\left[2g^{(\phi)}_{ab}\left(\frac{\partial W}{\partial\phi^a}\frac{\partial C}{\partial\phi^b} + \frac{\partial W}{\partial\phi^b}\frac{\partial C}{\partial\phi^a}\right)\mathcal{R}+\right] \\ &+ \left[\left(\frac{n}{n-1} - 1\right)4\frac{2-n}{2}\mathcal{R}CW + \mathcal{R} - \frac{1}{2}g^{(\phi)}_{ab}\gamma^{ij}\partial_i\phi^a\partial_j\phi^b\right] = 0\end{aligned} \tag{5.63}$$

The last term has to vanish identically, and with $g^{(\phi)}_{ab}$ symmetric in a and b, we get for the $\mathcal{R}$ dependent terms

$$4g^{(\phi)}_{ab}\frac{\partial W}{\partial\phi^a}\frac{\partial C}{\partial\phi^b} + 2\frac{2-n}{n-1}CW + 1 = 0 \tag{5.64}$$

We can solve this equation for C by inserting the expression we found for W (eqn. (5.62)), and get

$$C = \frac{l}{2(n-2)} \tag{5.65}$$

Finally, the quadratic terms are left.

$$\begin{aligned}\mathscr{H}_2 =& \frac{1}{16\pi G}\left[2g^{(\phi)}_{ab}\frac{\partial C}{\partial\phi^a}\frac{\partial C}{\partial\phi^b}\mathcal{R}^2 + 4C^2\left(\mathcal{R}^{ij}\mathcal{R}_{ij} + \frac{n-4}{4}\mathcal{R}^2\right) - \right] \\ &- \left[\frac{(2-n)^2}{n-1}C^2\mathcal{R}^2\right]\end{aligned} \tag{5.66}$$

We had three equations for two unknown variables, so the last equation is already completely determined. Obviously, the terms do not cancel each other and the expression seems to be non-zero. Indeed, if the dual field theory contains Weyl anomalies, this term will not vanish. For the solutions we are interested in, one can argue that it goes to zero sufficiently fast if r is taken to infinity. The Ricci tensor of induced AdS is given by

$$\mathcal{R}_{tt} = 0, \qquad \mathcal{R}_{ij} \propto \gamma_{ij}\frac{1}{r^2} \quad \text{for } i,j \neq t \tag{5.67}$$

so $\mathcal{R}^2$ and $\mathcal{R}_{ij}\mathcal{R}^{ij}$ are proportional to r^{-4}.

Inserting for W eqn. (5.62) and for C eqn. (5.65) into the ansatz eqn. (5.56) gives the result

$$S_{ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma}\,\left(\frac{n-1}{l} + \frac{l}{2(n-2)}\mathcal{R}\right) \tag{5.68}$$

Comparing this for $n = 2$ (here we have only the first term) with eqn. (5.37) and for $n = 4$ with eqn. (5.39) shows that the derived results match the empirically found relations. Variation leads to the regulated stress tensor as defined in eqn. (5.38) and eqn. (5.40), respectively.

5.4 Mass and Momentum

From the stress tensor we can determine the mass and angular momentum of the entire matter and gravity system. Therefore, an ADM decomposition of the boundary metric γ_{ij} is used

$$ds^2 = \gamma_{ij}dz^i dz^j = -N_{S_t}^2 dt^2 + \sigma_{AB}(N^A dt + d\omega^A)(N^B dt + d\omega^B) \tag{5.69}$$

with N^A representing the shift vector embedded in the hypersurface S_t, and N_{S_t} being the lapse function that ensures normalization of the normal vector n_i to this hypersurface.

$$n_i e^i_A = 0 \qquad n^i = \frac{1}{N_{S_t}}\delta^{it} \qquad N_{S_t} = \sqrt{-g_{tt}}$$

According to [9], the mass is defined as

$$M = \int_{S_t} d^{n-1}x\sqrt{\sigma}\; n^i T_{ij}\xi^j \tag{5.70}$$

ξ^i is a time-like Killing vector generating an isometry of the boundary geometry

$$\pounds_\xi \sigma_{AB} = 0 \qquad \xi^i = \delta^{i0} = N_{S_t} n^i$$

The momentum is defined as

$$P_A = \int_{S_t} d^{n-1}x\sqrt{\sigma}\;\sigma_{AB}T^{Bi}n_i \tag{5.71}$$

Chapter 6

Calculation of the Quasilocal Stress Tensor

In the previous chapters, we have provided the theoretical framework for the quasilocal stress tensor and discussed some properties of anti-de Sitter spacetimes. Now we want to explicitly calculate the quasilocal stress tensor and the mass and momentum for various spacetimes. As already mentioned, for the AdS/CFT correspondence, $\mathrm{AdS}_5 \times S^5$ is of special interest. However, we will start with spacetimes of lower dimensions to get a better insight into the calculations. The first spacetime we will deal with is AdS_3. In three dimensions, we have a manageable number of components for the various tensors, so the calculations can easily be done "by hand". Then we will have a look at AdS_5 and finally, we will turn to $\mathrm{AdS}_5 \times S_5$ and generalizations of it, where the 10-dimensional metric has product form only asymptotically.
All metric constants used during the calculations, such as the Christoffel symbols, the Ricci scalar, or the Einstein tensor, are listed in Appendix B.

6.1 AdS_3

The metric of global AdS_3 was already given in eqn. (3.13) as

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_1^2 \tag{6.1}$$

Further we had the non-global Poincaré patch eqn. (3.15)

$$ds^2 = -\frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2} dx^2 \tag{6.2}$$

where $x = l\varphi$. The unit normal vector to surfaces of constant r is given by $\hat{r}_\alpha = \sqrt{g_{rr}}\ \delta_{\alpha r}$. We can see from the above definitions of the metrics that

Poincaré coordinates have the advantage that one does not have to make a series expansion when taking the square root of the metric component g_{rr}. We will start with the easier case of Poincaré coordinates, and deal with the global metric afterwards.

6.1.1 Poincaré AdS$_3$

For Poincaré coordinates, we have $\hat{r}_\alpha = \frac{l}{r}\delta_{\alpha r}$. The extrinsic curvature tensor was defined in eqn. (5.14) as

$$\Theta_{\alpha\beta} = -\frac{1}{2}(\nabla_\alpha \hat{r}_\beta + \nabla_\beta \hat{r}_\alpha)$$

If we explicitly write down the expression for the covariant derivative

$$\nabla_\alpha \hat{r}_\beta = \delta_{\alpha r}\partial_r \frac{l}{r}\delta_{\beta r} - \Gamma^\rho_{\alpha\beta}\delta_{\rho r}\frac{l}{r} \tag{6.3}$$

we see that this expression is already symmetric in α and β, so the extrinsic curvature tensor simplifies to

$$\Theta_{\alpha\beta} = \Theta_{\beta\alpha} = -\nabla_\alpha \hat{r}_\beta \tag{6.4}$$

From the Kronecker deltas in eqn. (6.3) we can deduce that the first term only contributes to Θ_{rr}, and that only Christoffel symbols $\Gamma^r_{\alpha\beta}$ occur in the second term (a list of the Christoffel symbols is given in Appendix B). Explicit calculation gives for the components of the extrinsic curvature tensor with non-vanishing Christoffel symbols

$$\begin{aligned} \Theta_{rr} &= \partial_r \frac{l}{r} - \Gamma^r_{rr}\frac{l}{r} = 0 \\ \Theta_{tt} &= -\Gamma^r_{tt}\frac{l}{r} = +\frac{r^2}{l^3} \\ \Theta_{xx} &= -\Gamma^r_{xx}\frac{l}{r} = -\frac{r^2}{l^3} \end{aligned} \tag{6.5}$$

The extrinsic curvature tensor is supposed to describe the curvature of the boundary, so it fits to our expectations that it has no r components. To make this explicit in the notation, we will from now on write Θ_{ij} instead of $\Theta_{\alpha\beta}$. The curvature scalar is defined as

$$\Theta = \Theta_{ij}\gamma^{ij} \tag{6.6}$$

and evaluates with $\gamma^{xx} = -\gamma^{tt} = \frac{l^2}{r^2}$ to

$$\Theta = -\frac{2}{l} \tag{6.7}$$

The stress tensor was given in eqn. (5.28) as $T_{ij} = \frac{1}{8\pi G}(\Theta_{ij} - \Theta\gamma_{ij})$. Inserting our results for Θ_{ij} and Θ gives

$$\begin{aligned} T_{tt} =& -\frac{1}{8\pi G}\frac{r^2}{l^3} \\ T_{xx} =& \frac{1}{8\pi G}\frac{r^2}{l^3} \\ T_{tx} =& T_{xt} = 0 \end{aligned} \tag{6.8}$$

The boundary of AdS lies at $r = \infty$, where the stress tensor diverges with r^2. The divergence can be eliminated by adding a suitable counterterm. Although we have already given the complete expressions for the regulated action and the regulated stress tensor in eqn. (5.37) and eqn. (5.38), respectively, we will go through the calculation step by step to make transparent what happens. We will first calculate the action according to eqn. (2.8) and eqn. (2.9)

$$\begin{aligned} S_{bulk} =& -\frac{1}{16\pi G}\int d^3x\sqrt{-g}\,(R - 2\Lambda) = \\ =& -\frac{1}{16\pi G}\int_0^\beta dt \int_0^{2\pi} l\, d\varphi \int_0^r dr'\frac{r'}{l}\left(-\frac{4}{l^2}\right) = \\ =& \frac{\beta\omega_1}{8\pi G}\frac{r^2}{l^2} \\ S_{GH} =& \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,\Theta = \\ =& \frac{1}{8\pi G}\int_0^\beta dt \int_0^{2\pi} l\, d\varphi\, \frac{r^2}{l^2}\left(-\frac{2}{l}\right) = \\ =& -\frac{\beta\omega_1}{8\pi G}\frac{2r^2}{l^2} \end{aligned} \tag{6.9}$$

where we have used $R = -6/l^2$ and $\Lambda = -2/l^2$ (see Appendix B). $\omega_1 = 2\pi$ is the volume of the unit 1-sphere S^1, and $\beta = 2\pi/T_H$ is the periodicity interval of time, with T_H the Hawking temperature (see e.g. [15] for further details). The complete action

$$S_G = S_{bulk} + S_{GH} = -\frac{\beta\omega_1}{8\pi G}\frac{r^2}{l^2} \tag{6.10}$$

diverges like r^2 for $r \to \infty$. We want to find a counterterm with the same r dependence that is a local, covariant function of the intrinsic geometry of the boundary. The simplest possible term is

$$S_1 = \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma} = \frac{\beta\omega_1}{8\pi G}\frac{r^2}{l} \tag{6.11}$$

which indeed has the right r dependence to cancel the divergence. In fact, it is readily shown that this is the only possible term to cancel the divergence. So the regulated action is given by

$$S_{reg} = S_{bulk} + S_{GH} + \frac{1}{l} S_1 \tag{6.12}$$

and has the result

$$S_{reg} = 0 \tag{6.13}$$

Variation of S_1 with respect to γ_{ij} (see Appendix A)

$$\delta S_1 = \frac{1}{8\pi G} \int d^2x \delta\sqrt{-\gamma} = \frac{1}{8\pi G} \int d^2x \frac{1}{2} \sqrt{-\gamma}\, \gamma^{ij} \delta\gamma_{ij} \tag{6.14}$$

in combination with eqn. (5.1) leads to the counterterm of the stress tensor

$$T^{(1)}_{ij} = \frac{1}{8\pi G} \gamma_{ij} \tag{6.15}$$

To eliminate the divergences in eqn. (6.8) we define

$$T^{reg}_{ij} = T_{ij} - \frac{1}{l} T^{(1)}_{ij} = \frac{1}{8\pi G}(\Theta_{ij} - \Theta\gamma_{ij} - \frac{1}{l}\gamma_{ij}). \tag{6.16}$$

This gives the result

$$T^{reg}_{ij} = 0 \qquad \forall\, i, j \tag{6.17}$$

which is clearly free of divergences. It follows immediately that the mass and the momentum are also zero since they are extracted from T^{reg}.

6.1.2 Global $\mathbf{AdS_3}$

With the metric from eqn. (6.1), we have the normal vector

$$\hat{r}_\alpha = \sqrt{g_{rr}} \delta_{r\alpha} = \frac{1}{\sqrt{1 + \frac{r^2}{l^2}}} \delta_{r\alpha} \tag{6.18}$$

We are interested in calculating objects on the boundary, so we can make a series expansion for $r \to \infty$

$$\lim_{r\to\infty} \hat{r} = \left(\frac{l}{r} - \frac{l^3}{2r^3} + \mathcal{O}\left(\frac{1}{r^5}\right) \right) \delta_{r\alpha} \tag{6.19}$$

The extrinsic curvature tensor can now be calculated from eqn. (6.4).

$$\begin{aligned}\Theta_{tt} &= -\Gamma^r_{tt}\hat{r} = \frac{r^2}{l^3} + \frac{1}{2l} - \frac{l}{8r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)\\ \Theta_{\varphi\varphi} &= -\Gamma^r_{\varphi\varphi}\hat{r} = -\frac{r^2}{l} - \frac{l}{2} + \frac{l^3}{8r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)\end{aligned} \tag{6.20}$$

and the curvature scalar $\Theta = \Theta_{ij}\gamma^{ij}$ evaluates to

$$\Theta = -\frac{2}{l} - \frac{l^3}{4r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{6.21}$$

Inserting Θ_{ij} and Θ into the definition of the stress tensor eqn. (5.28) gives

$$\begin{aligned}T_{tt} =& -\frac{1}{8\pi G}\left(\frac{r^2}{l^3} + \frac{3}{2l} + \frac{3l}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right)\\ T_{\varphi\varphi} =& -\frac{1}{8\pi G}\left(-\frac{r^2}{l} + \frac{l}{2} - \frac{3l^3}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right)\\ T_{t\varphi} =& T_{\varphi t} = 0\end{aligned} \tag{6.22}$$

One can see that the stress tensor again diverges with r^2, just as for Poincaré coordinates. For the action we have

$$\begin{aligned}S_{bulk} =& -\frac{1}{16\pi G}\int d^3x\sqrt{-g}\,(R - 2\Lambda) =\\ =& -\frac{1}{16\pi G}\int_0^\beta dt\int_0^{2\pi} d\varphi\int_0^r dr' r'\left(-\frac{4}{l^2}\right) =\\ =& \frac{\beta\omega_1}{8\pi G}\frac{r^2}{l^2}\\ S_{GH} =& \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,\Theta =\\ =& \frac{1}{8\pi G}\int_0^\beta dt\int_0^{2\pi} d\varphi\, r\sqrt{1+\frac{r^2}{l^2}}\left(-\frac{2}{l} - \frac{l^3}{4r^4}\right) =\\ =& -\frac{\beta\omega_1}{8\pi G}\left(\frac{2r^2}{l^2} + 1 + \mathcal{O}\left(\frac{1}{r^2}\right)\right)\end{aligned} \tag{6.23}$$

where we have used a series expansion to evaluate the square root in S_{GH}. The gravitational action is then

$$S_G = S_{bulk} + S_{GH} = \frac{\beta\omega_1}{8\pi G}\left(-\frac{r^2}{l^2} - 1 + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \tag{6.24}$$

The counterterm for the action was defined in eqn. (5.30) and gives

$$S_1 = \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma} = \frac{\beta\omega_1}{8\pi G}\left(\frac{r^2}{l} + \frac{l}{2} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \tag{6.25}$$

which we insert into the expression for the regulated action eqn. (5.37)

$$S_{reg} = S_{bulk} + S_{GH} + \frac{1}{l}S_1 = \frac{\beta\omega_1}{8\pi G}\frac{-1}{2} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{6.26}$$

For the stress tensor, we get with eqn. (5.38)

$$\begin{aligned} T^{reg}_{tt} =& -\frac{1}{8\pi G}\left(\frac{1}{2l} + \frac{3l}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T^{reg}_{\varphi\varphi} =& -\frac{1}{8\pi G}\left(\frac{l}{2} - \frac{3l^3}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T^{reg}_{t\varphi} =& T^{reg}_{\varphi t} = 0 \end{aligned} \tag{6.27}$$

The mass was defined in eqn. (5.70) and gives

$$\begin{aligned} M =& \int_{S_t} dx\sqrt{\sigma}\frac{1}{\sqrt{-g_{tt}}}\delta^{it}T^{reg}_{ij}\delta^{jt} = \\ =& \int_0^{2\pi} d\varphi\, r\left(\frac{l}{r} - \frac{l^3}{2r^3}\right)\frac{-1}{8\pi G}\left(\frac{1}{2l} + \frac{3l}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) = \\ =& -\frac{1}{8G} - \frac{l^2}{32Gr^2} \end{aligned} \tag{6.28}$$

The surprising result that the mass of Poincaré AdS_3 and global AdS_3 are not the same occurs because the time coordinates are different and therefore the definition of energy differs. The momentum is zero because T^{reg} has no off-diagonal components.

6.1.3 Perturbed Poincaré AdS_3

Now we want to study a spacetime with small deviations from AdS_3. The metric is given by

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}dx^2 + \delta g_{\alpha\beta}dx^\alpha dx^\beta \tag{6.29}$$

where $\delta g_{\alpha\beta}$ represents the perturbation. Working to first order in $\delta g_{\alpha\beta}$, the normal vector is $\hat{r}_\mu = (\frac{l}{r} + \frac{r}{2l}\delta g_{rr})\delta_{r\mu}$, and for the extrinsic curvature tensor we get with eqn. (6.4)

$$\begin{aligned}\Theta_{tt} &= \frac{r^2}{l^3} - \frac{r^4}{2l^5}\delta g_{rr} - \frac{r}{2l}\partial_r \delta g_{tt} \\ \Theta_{xx} &= -\frac{r^2}{l^3} + \frac{r^4}{2l^5}\delta g_{rr} - \frac{r}{2l}\partial_r \delta g_{xx} \\ \Theta_{tx} &= -\frac{r}{2l}\partial_r \delta g_{tx}\end{aligned} \tag{6.30}$$

The curvature scalar evaluates to

$$\Theta = \frac{r^2}{l^3}\delta g_{rr} - \frac{2}{l} + \frac{l}{r}(\partial_r \delta g_{tt} - \partial_r \delta g_{xx}) + \frac{1}{r^2}(\delta g_{xx} - \delta g_{tt}) \tag{6.31}$$

(the covariant components of the metric are given in Appendix B). Inserting in eqn. (5.38) yields the regulated stress tensor

$$\begin{aligned}T^{reg}_{tt} &= \frac{1}{8\pi G}\left(\frac{r^4}{2l^5}\delta g_{rr} + \frac{1}{l}\delta g_{xx} - \frac{r}{2l}\partial_r \delta g_{xx}\right) \\ T^{reg}_{xx} &= \frac{1}{8\pi G}\left(-\frac{r^4}{2l^5}\delta g_{rr} + \frac{1}{l}\delta g_{tt} - \frac{r}{2l}\partial_r \delta g_{tt}\right) \\ T^{reg}_{tx} &= \frac{1}{8\pi G}\left(\frac{1}{l}\delta g_{tx} - \frac{r}{2l}\partial_r \delta g_{tx}\right)\end{aligned} \tag{6.32}$$

With eqn. (5.70) and eqn. (5.71), we can calculate the mass and momentum, respectively.

$$\begin{aligned}M =& \int_0^{2\pi} dx \sqrt{g_{xx}} \frac{1}{\sqrt{-g_{tt}}} \delta^{it} T_{ij} \delta^{jt} = \\ =& \frac{1}{8\pi G}\int dx \left(\frac{r^4}{2l^5}\delta g_{rr} + \frac{1}{l}\delta g_{xx} - \frac{r}{2l}\partial_r \delta g_{xx}\right)\end{aligned} \tag{6.33}$$

$$\begin{aligned}P_x =& \int_0^{2\pi} dx \sqrt{g_{xx}}\, \sigma^{xA} \frac{1}{\sqrt{-g_{tt}}} \delta^{it} T_{iA} g_{xx} = \\ =& \frac{1}{8\pi G}\int dx \left(\frac{1}{l}\delta g_{tx} - \frac{r}{2l}\partial_r \delta g_{tx}\right)\end{aligned} \tag{6.34}$$

We want to compare these results with the spinning BTZ (Bañados, Teitelboim, Zanelli) solution (see [16], [17], [18]). The metric of the BTZ black hole is given by ([9])

$$ds^2 = -N^2 dt^2 + \rho^2 (d\psi + N^\psi dt)^2 + \frac{r^2}{N^2 \rho^2} dr^2 \tag{6.35}$$

with

$$N^2 = \frac{r^2(r^2 - r_+^2)}{l^2\rho^2} \qquad \rho^2 = r^2 + 4GMl^2 - \frac{r_+^2}{2}$$

$$r_+^2 = 8Gl\sqrt{M^2l^2 - J^2} \qquad N^\psi = -\frac{4GJ}{\rho^2}$$

Evaluating the metric components for $r \rightarrow \infty$ and assuming that $J^2 << M^2l^2$, we can identify the results with the metric components we had for perturbed Poincaré AdS_3

$$-N^2 + \rho^2(N^\psi)^2 \approx -\frac{r^2}{l^2} + 8GM = g_{tt} + \delta g_{tt}$$

$$\frac{r^2}{N^2\rho^2} \approx \frac{l^2}{r^2} + \frac{8Gl^4M}{r^4} = g_{rr} + \delta g_{rr}$$

$$\rho^2 N^\psi = -4GJ = \delta g_{t\psi}$$

$$\rho^2 \approx r^2 = g_{\psi\psi}$$

With the substitution $x \rightarrow l\psi$ and $\int dx \rightarrow l\int_0^{2\pi} d\psi$, we can insert these results into the equations for the mass (6.33) and the momentum (6.34), and get the identities $P_x = J$ and $M = M$, which shows that the equations received from the counterterm formalism reproduce the results of conventional techniques.

If we insert the values we have calculated for the mass and momentum we find that for $M = -\frac{1}{8G}$ and $J = 0$ the BTZ metric reproduces the metric of global AdS_3 (eqn. (6.1)), whereas for $M = 0$ and $J = 0$, it reproduces Poincaré AdS_3 (eqn. (6.2)).

6.1.4 Schwarzschild AdS_3

The metric of Schwarzschild AdS_3 is given by

$$ds^2 = -\left(1 + \frac{r^2}{l^2} - r_0^2\right)dt^2 + \left(1 + \frac{r^2}{l^2} - r_0^2\right)^{-1} dr^2 + r^2 d\Omega_1^2 \tag{6.36}$$

and has the normal vector

$$\hat{r}_\alpha = \sqrt{g_{rr}}\delta_{r\alpha} = \frac{1}{\sqrt{1 + \frac{r^2}{l^2} - r_0^2}}\delta_{r\alpha} \tag{6.37}$$

which we again expand for $r \rightarrow \infty$

$$\lim_{r\rightarrow\infty} \hat{r} = \left(\frac{l}{r} - \frac{l^3(1 - r_0)}{2r^3} + \mathcal{O}\left(\frac{1}{r^5}\right)\right)\delta_{r\alpha} \tag{6.38}$$

The extrinsic curvature tensor follows from eqn. (6.4)

$$\begin{aligned}\Theta_{tt} &= -\Gamma^r_{tt}\hat{r} = \frac{r^2}{l^3} + \frac{1-r_0^2}{2l} - \frac{l(1-r_0^2)^2}{8r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \\ \Theta_{\varphi\varphi} &= -\Gamma^r_{\varphi\varphi}\hat{r} = -\frac{r^2}{l} - \frac{l(1-r_0^2)}{2} + \frac{l^3(1-r_0^2)^2}{8r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)\end{aligned} \tag{6.39}$$

and the curvature scalar evaluates to

$$\Theta = -\frac{2}{l} + \frac{l^3(1-r_0^2)^2}{4r^4} + \mathcal{O}\left(\frac{1}{r^6}\right) \tag{6.40}$$

With the definition of the stress tensor eqn. (5.28), we get

$$\begin{aligned}T_{tt} &= -\frac{1}{8\pi G}\left(\frac{r^2}{l^3} + \frac{3(1-r_0^2)}{2l} + \frac{3l(1-r_0^2)^2}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T_{\varphi\varphi} &= -\frac{1}{8\pi G}\left(-\frac{r^2}{l} + \frac{l(1-r_0^2)}{2} - \frac{3l^3(1-r_0^2)^2}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T_{t\varphi} &= T_{\varphi t} = 0\end{aligned} \tag{6.41}$$

If we compare these results with the results for global AdS_3, we see that they are equivalent up to the factors $(1-r_0^2)$, which is not surprising since the constant terms in the metric are 1 for global AdS_3 and $(1-r_0^2)$ for Schwarzschild AdS_3. The divergent term proportional to r^2 is the same in both cases. The action is

$$\begin{aligned}S_{bulk} &= -\frac{1}{16\pi G}\int d^3x\sqrt{-g}\,(R-2\Lambda) = \\ &= -\frac{1}{16\pi G}\int_0^\beta dt\int_0^{2\pi} d\varphi\int_{r_+}^{r} dr' r'\left(-\frac{4}{l^2}\right) \\ &= \frac{\beta\omega_1}{8\pi G}\left(\frac{r^2}{l^2} - \frac{r_+^2}{l^2}\right) \\ S_{GH} &= \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma}\,\Theta = \\ &= \frac{1}{8\pi G}\int_0^\beta dt\int_0^{2\pi} d\varphi\, r\sqrt{1+\frac{r^2}{l^2}-r_0^2}\left(-\frac{2}{l} - \frac{l^3(1-r_0^2)^2}{4r^4}\right) = \\ &= -\frac{\beta\omega_1}{8\pi G}\left(\frac{2r^2}{l^2} + (1-r_0^2) + \frac{l^2}{4r^2}(1-r_0^2)^2 + \mathcal{O}\left(\frac{1}{r^4}\right)\right)\end{aligned} \tag{6.42}$$

where we have again used a series expansion to evaluate the square root in S_{GH}. Notice that the integration bounds of the r-integral now start at the horizon of the black hole and not at $r = 0$. If we use the defining equation of the horizon, $f(r_+) = 0$, we get the identity $(1 - r_0^2) = -\frac{r_+^2}{l^2}$. Substituting this into S_{bulk} leads to the following expression for the gravitational action

$$S_G = S_{bulk} + S_{GH} = -\frac{\beta\omega_1}{8\pi G}\left(\frac{r^2}{l^2} + \frac{l^2}{4r^2}(1 - r_0^2)^2\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{6.43}$$

The counterterm for the action eqn. (5.30) gives

$$\begin{aligned} S_1 =& \frac{1}{8\pi G}\int d^2x\sqrt{-\gamma} = \\ =& \frac{\beta\omega_1}{8\pi G}\left(\frac{r^2}{l} + \frac{l}{2}(1 - r_0^2) - \frac{l^3}{r^2}(1 - r_0^2)^2\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{6.44}$$

and for the regulated action eqn. (5.37), we get

$$\begin{aligned} S_{reg} =& S_{bulk} + S_{GH} + \frac{1}{l}S_1 = \\ =& \frac{\beta\omega_1}{8\pi G}\left(\frac{1}{2}(1 - r_0^2) - \frac{5l^2}{4r^2}(1 - r_0^2)^2\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{6.45}$$

The regulated stress tensor follows from eqn. (5.38) and gives

$$\begin{aligned} T_{tt}^{reg} =& -\frac{1}{8\pi G}\left(\frac{1 - r_0^2}{2l} + \frac{3l(1 - r_0^2)^2}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T_{\varphi\varphi}^{reg} =& -\frac{1}{8\pi G}\left(\frac{l(1 - r_0^2)}{2} - \frac{3l^3(1 - r_0^2)^2}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T_{t\varphi}^{reg} =& T_{\varphi t}^{reg} = 0 \end{aligned} \tag{6.46}$$

6.2 AdS$_5$

6.2.1 Schwarzschild AdS$_5$

The metric of Schwarzschild AdS$_5$ can be written as

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + r^2(d\psi^2 + \sin^2\psi\, d\vartheta^2 + \cos^2\psi\, d\varphi^2) \tag{6.47}$$

$$\text{with } f = 1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2}.$$

For the sphere we have used an alternative line element (see [9]) instead of the usual $d\psi^2 + \sin^2\psi(d\vartheta^2 + \sin^2\psi d\varphi^2)$. The used line element has the advantage that only angular functions of ψ occur, and therefore, some of the expressions take a simpler form. The unit normal vector is given by $\hat{r}_\alpha = \sqrt{1/f}\delta_{r\alpha}$ and has the large r expansion

$$\lim_{r\to\infty} \hat{r}_\alpha = \frac{l}{r} - \frac{l^3}{2r^3} + \frac{3l^5}{8r^5} + \frac{l^3 r_0^2}{2r^5} + \mathcal{O}\left(\frac{1}{r^6}\right) \tag{6.48}$$

For the calculation of the extrinsic curvature, we can use eqn. (6.4), with the result

$$\begin{aligned} \Theta_{tt} &= \frac{r^4 + l^2 r_0^2}{l^2 r^3}\sqrt{f} \\ \Theta_{ii} &= -\frac{1}{r}\sqrt{f}\gamma_{ii} \qquad \text{for } i = \psi, \vartheta, \varphi \end{aligned} \tag{6.49}$$

In the limit of $r \to \infty$ this evaluates to

$$\begin{aligned} \Theta_{tt} &= \frac{r^2}{l^3} + \frac{1}{2l} - \frac{l}{8r^2} + \frac{r_0^2}{2lr^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \\ \Theta_{\psi\psi} &= -\frac{r^2}{l} - \frac{l}{2} + \frac{l^3}{8r^2} + \frac{lr_0^2}{2r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \\ \Theta_{\vartheta\vartheta} &= \left(-\frac{r^2}{l} - \frac{l}{2} + \frac{l^3}{8r^2} + \frac{lr_0^2}{2r^2}\right)\sin^2\psi + \mathcal{O}\left(\frac{1}{r^4}\right) \\ \Theta_{\varphi\varphi} &= \left(-\frac{r^2}{l} - \frac{l}{2} + \frac{l^3}{8r^2} + \frac{lr_0^2}{2r^2}\right)\cos^2\psi + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{6.50}$$

and for the curvature scalar we find

$$\Theta = -\frac{4}{l} - \frac{l}{r^2} + \mathcal{O}\left(\frac{1}{r^6}\right) \tag{6.51}$$

For the stress tensor (eqn. (5.1), we get

$$\begin{aligned} T_{tt} &= \frac{1}{8\pi G}\left(-\frac{3r^2}{l^3} - \frac{9}{2l} - \frac{9(l^2 - 4r_0^2)}{8lr^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T_{\psi\psi} &= \frac{1}{8\pi G}\left(\frac{3r^2}{l} + \frac{l}{2} + \frac{l(l^2 - 4r_0^2)}{8r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T_{\vartheta\vartheta} &= \frac{1}{8\pi G}\left(\frac{3r^2}{l} + \frac{l}{2} + \frac{l(l^2 - 4r_0^2)}{8r^2}\right)\sin^2\psi + \mathcal{O}\left(\frac{1}{r^4}\right) \\ T_{\varphi\varphi} &= \frac{1}{8\pi G}\left(\frac{3r^2}{l} + \frac{l}{2} + \frac{l(l^2 - 4r_0^2)}{8r^2}\right)\cos^2\psi + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{6.52}$$

Again, we want to explicitly evaluate the action integral to find the counter-terms. By substituting $R = -\frac{20}{l^2}$ and $\lambda = -\frac{12}{l^2}$ (see Appendix B) into S_{bulk} (eqn. (2.8)), we get

$$\begin{aligned} S_{bulk} =& -\frac{1}{16\pi G}\int_0^\beta dt \int_{r_+}^r dr' r'^3 \int_{S^3} d\Omega_3 \,\frac{-8}{l^2} = \\ =& \frac{\beta\omega_3}{8\pi G}\frac{r^4 - r_+^4}{l^2} \end{aligned} \tag{6.53}$$

with $\omega_3 = 2\pi^2$ the volume of the unit 3-sphere S^3. The boundary term evaluates to

$$\begin{aligned} S_{GH} =& \frac{1}{8\pi G}\int_0^\beta dt \int_{S^3} d\Omega_3 \sqrt{f} r^3 \Theta = \\ =& -\frac{\beta\omega_3}{8\pi G}\left(\frac{4r^4}{l^2} + 3r^2 - 2r_0^2\right) + \mathcal{O}\left(\frac{1}{r^2}\right) \end{aligned} \tag{6.54}$$

where we have used the large r expansion

$$\lim_{r\to\infty} \sqrt{f} = \frac{r}{l} + \frac{l}{2r} - \frac{l^3}{8r^3} - \frac{lr_0^2}{2r^3} + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{6.55}$$

If we further consider that $f(r_+) = 0$, we can substitute $\frac{r_+^2}{l^2} = \frac{r_0^2 - r^2}{r^2}$ in S_{bulk} and get by addition of S_{GH} for the gravitational action

$$S_G = \frac{\beta\omega_3}{8\pi G}\left(-\frac{3r^4}{l^2} - 3r^2 + 2r_0^2 + l^2 - \frac{2l^2 r_0^2}{r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right). \tag{6.56}$$

To cancel the divergences for $r \to \infty$, we find that the counterterm introduced for AdS_3

$$S_1 = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^4x \sqrt{-\gamma} = \frac{\beta\omega_3}{8\pi G}\left(\frac{r^4}{l} + \frac{r^2 l}{2} - \frac{l^3}{8} - \frac{r_0^2 l}{2}\right) \tag{6.57}$$

removes the r^4 divergence, but there still remains an r^2 divergence. We have already stated in Section 5.2 that we need an additional counterterm, which was defined in eqn. (5.31)

$$S_2 = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^4x \sqrt{-\gamma}\,\mathcal{R} = \frac{1}{8\pi G}\left(\frac{6r^2}{l} + 3l - \frac{3l^3}{4r^2} - \frac{3r_0^2 l}{r^2}\right) \tag{6.58}$$

This counterterm removes the r^2 divergence, and for the regulated action we find in accordance with eqn. (5.39)

$$\begin{aligned} S_{reg} =& S_{bulk} + S_{GH} + \frac{3}{l}S_1 + \frac{l}{4}S_2 = \\ =& \frac{1}{8\pi G}\left(r_+^2 - \frac{r_o^2}{2} + \frac{3l^2}{8}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) \end{aligned} \tag{6.59}$$

The regulated stress tensor was defined in eqn. (5.40) as

$$T_{ij}^{reg} = \frac{1}{8\pi G}(\Theta_{ij} - \Theta\gamma_{ij} - \frac{3}{l}\gamma_{ij} + \frac{l}{2}\mathcal{G}_{ij}) \tag{6.60}$$

and evaluates to

$$\begin{aligned}
T_{tt}^{reg} &= \frac{1}{8\pi G}\left(\frac{3l}{8r^2} + \frac{3r_0^2}{2lr^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\
T_{\psi\psi}^{reg} &= \frac{1}{8\pi G}\left(\frac{l^3}{8r^2} + \frac{lr_0^2}{2r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \\
T_{\vartheta\vartheta}^{reg} &= \frac{1}{8\pi G}\left(\frac{l^3}{8r^2} + \frac{lr_0^2}{2r^2}\right)\sin^2\psi + \mathcal{O}\left(\frac{1}{r^4}\right) \\
T_{\varphi\varphi}^{reg} &= \frac{1}{8\pi G}\left(\frac{l^3}{8r^2} + \frac{lr_0^2}{2r^2}\right)\cos^2\psi + \mathcal{O}\left(\frac{1}{r^4}\right)
\end{aligned} \tag{6.61}$$

For the mass we get from eqn. (5.70)

$$\begin{aligned}
M = &\int_{S_t} d^3x\sqrt{\sigma}\frac{1}{\sqrt{-g_{tt}}}T_{tt}^{reg} = \\
=&\frac{3l^2}{32G} + \frac{3\pi r_0^2}{8G} + \mathcal{O}\left(\frac{1}{r^2}\right)
\end{aligned} \tag{6.62}$$

where the second term corresponds to the standard solution for the mass of Schwarzschild AdS_5. The first term is the mass of pure global AdS_5 when $r_0 = 0$ (i.e., when the mass of the black hole is zero). From the gravitational point of view, pure AdS_5 is a vacuum and should have zero mass, but from the view of the AdS/CFT correspondence it turns out that this mass comes from the Casimir energy (see [9]). The momentum is zero.

6.2.2 Electrical Charged Black Hole in AdS_5

We take the following metric as an ansatz for static electrically charged black hole solutions in AdS_{n+1}

$$ds^2 = -e^{-2(n-2)B(r)}f(r)dt^2 + e^{2B(r)}\left(\frac{1}{f(r)}dr^2 + r^2 d\Omega_{n-1}^2\right) \tag{6.63}$$

and set $n = 4$. $B(r)$ is some function depending on the charges which we will specify later. With the unit normal vector $\hat{r}_\alpha = e^B f^{-\frac{1}{2}}\delta_{r\alpha}$ we have for the extrinsic curvature tensor according to eqn. (6.4)

$$\begin{aligned}
\Theta_{tt} &= -\left(-2B' + \frac{f'}{2f}\right)\gamma_{tt}e^{-B}\sqrt{f} \\
\Theta_{ii} &= -\left(B' + \frac{1}{r}\right)\gamma_{ii}e^{-B}\sqrt{f} \qquad \text{for } i = \vartheta, \varphi, \psi
\end{aligned} \tag{6.64}$$

(primed variables are derived with respect to r), and the curvature scalar is

$$\Theta = -e^{-B}\sqrt{f}\left(B' + \frac{f'}{2f} + \frac{3}{r}\right) \tag{6.65}$$

So we get for the stress tensor (eqn. (5.1))

$$\begin{aligned} T_{tt} =& \frac{1}{8\pi G}\left(B' + \frac{1}{r}\right)3e^{-B}\gamma_{tt}\sqrt{f} \\ T_{ii} =& \frac{1}{8\pi G}\left(\frac{f'}{2f} + \frac{2}{r}\right)e^{-B}\gamma_{ii}\sqrt{f} \qquad \text{for } i = \vartheta, \varphi, \psi \end{aligned} \tag{6.66}$$

For the gravitational action we take

$$\begin{aligned} S[g_{\alpha\beta}, \phi^a, A^I_\alpha] =& -\frac{1}{16\pi G}\int_{\mathscr{M}} d^5x\sqrt{-g}\,(R - \frac{1}{2}g^{(\phi)}_{ab}\partial_\alpha\phi^a\partial^\alpha\phi^b - V(\phi) - \\ & -\frac{1}{4}G_{IJ}(\phi)F^I_{\alpha\beta}F^{\alpha\beta\ J}) + \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^4x\sqrt{-\gamma}\ \Theta \end{aligned} \tag{6.67}$$

which describes the bosonic part of a general matter coupled system ([14],[19]). The Einstein equation written in Ricci form is ([14],[19])

$$R_{\alpha\beta} = \frac{1}{2}g^{(\phi)}_{ab}\partial_\alpha\phi^a\partial_\beta\phi^b + \frac{1}{2}G_{IJ}(F^I_{\alpha\lambda}F_\beta{}^{\lambda\ J} - \frac{1}{6}g_{\alpha\beta}F^I_{\rho\sigma}F^{\rho\sigma\ J}) + \frac{1}{3}g_{\alpha\beta}V \tag{6.68}$$

Its trace gives the Ricci scalar

$$R = \frac{1}{2}g^{(\phi)}_{ab}\partial_\alpha\phi^a\partial^\alpha\phi^b + \frac{1}{2}G_{IJ}(1 - \frac{5}{6})F^I_{\rho\sigma}F^{\rho\sigma\ J} + \frac{5}{3}V \tag{6.69}$$

which we substitute into the action eqn. (5.41) and get the expression

$$S_{bulk} = -\frac{1}{16\pi G}\int_{\mathscr{M}} d^5x\sqrt{-g}\,\left(-\frac{1}{6}G_{IJ}F^I_{\alpha\beta}F^{\alpha\beta\ J} + \frac{2}{3}V\right) \tag{6.70}$$

For spherically symmetric black holes, the fields are functions only of the radial coordinate

$$\phi^a = \phi^a(r) \qquad \text{and} \qquad A^I_t = A^I_t(r)$$

The other components of A^I_α are zero. If we write down the $R_{\psi\psi}$ component of the Ricci tensor from eqn. (6.68), we find

$$R_{\psi\psi} = \frac{1}{2}g^{(\phi)}_{ab}\partial_\phi\phi^a\partial^b_\phi + \frac{1}{2}G_{IJ}(F^I_{\psi\lambda}F_\psi{}^{\lambda\ J} - \frac{1}{6}g_{\psi\psi}F^I_{\rho\sigma}F^{\rho\sigma\ J}) + \frac{1}{3}g_{\psi\psi}V \tag{6.71}$$

The first term is zero because ϕ^a is independent of ψ. The second term is also zero because A^I_α does neither depend on ψ, nor it has a ψ-component. Multiplication with $g_{\psi\psi}$ gives

$$2R^\psi_\psi = \left(-\frac{1}{6}G_{IJ}F^I_{\alpha\beta}F^{\alpha\beta\, J} + \frac{2}{3}V\right) \tag{6.72}$$

which is exactly the same expression we had in the action integral eqn. (6.70), so we can write

$$S_{bulk} = -\frac{1}{8\pi G}\int_{\mathscr{M}} d^5x\sqrt{-g}\; R^\psi_\psi \tag{6.73}$$

Now we calculate the explicit form of R^ψ_ψ with the Christoffel symbols (see Appendix B) and get

$$R^\psi_\psi = -\frac{1}{r^2}e^{-2B}(3fB'r + f'B'r^2 + f'r + 2f + fr^2B'' - 2) \tag{6.74}$$

which we insert into the action integral. Integration gives

$$S_{bulk} = \frac{\beta\omega_3}{8\pi G}(r^3fB' + r^2(f-1) + r_+^2) \tag{6.75}$$

For the boundary term we get with the curvature scalar from eqn. (6.65)

$$S_{GH} = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^4x\sqrt{\gamma}\;\Theta = -\frac{\beta\omega_3}{8\pi G}\left(r^3fB' + \frac{1}{2}r^3f' + 3r^2f\right) \tag{6.76}$$

Combining these terms, we find that the B' dependent terms cancel out and we get the divergent expression

$$S_{bulk} + S_{GH} = \frac{\beta\omega_3}{8\pi G}\left(-2r^2f - \frac{1}{2}r^3f' - r^2 + r_+^2\right) \tag{6.77}$$

This can be regulated by adding the counterterms defined in eqns. (5.30) and (5.31), which evaluate with the given metric to

$$S_1 = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^4x\sqrt{\gamma} = \frac{\beta\omega_3}{8\pi G}r^3\sqrt{f}e^B \tag{6.78}$$

$$S_2 = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} d^4x\sqrt{\gamma}\;\mathcal{R} = \frac{\beta\omega_3}{8\pi G}6r\sqrt{f}e^{-B} \tag{6.79}$$

The regulated action $S_{reg} = S_{bulk} + S_{GH} + \frac{3}{l}S_1 + \frac{l}{4}S_2$ is then

$$S_{reg} = \frac{\beta\omega_3}{8\pi G}\left(-2r^2f - \frac{1}{2}r^3f' - r^2 + r_H^2 + 3\frac{1}{l}r^3\sqrt{f}e^B + \frac{3}{2}lr\sqrt{f}e^{-B}\right) \tag{6.80}$$

In general, this result is not manifestly finite, but we will see that it is indeed finite for the case of R-charged black holes.

R-Charged Black Holes in AdS_5

For R-charged black holes, the function B is expressed through the harmonic function $\mathcal{H}$: $B = \frac{1}{6}\ln\mathcal{H}$. In the STU model, the harmonic function $\mathcal{H}$ is given by the product of three harmonic functions ([14], [19])

$$\mathcal{H} = H_1 H_2 H_3 = \prod_{i=1}^{3}\left(1 + \frac{q_i}{r^{n-2}}\right) \tag{6.81}$$

where the q_i are the charges. The metric for R-charged black holes in AdS_5 is then given by

$$ds^2 = -\mathcal{H}^{-\frac{2}{3}} f dt^2 + \mathcal{H}^{\frac{1}{3}}\left(\frac{1}{f}dr^2 + r^2 d\Omega_3^2\right) \tag{6.82}$$

with

$$f = 1 - \frac{r_0^2}{r^2} + \frac{r^2}{l^2}\mathcal{H}$$

Inserting f and $\mathcal{H}$ into the action integral (6.80) and making an expansion for $r \to \infty$ leads to

$$S_{reg} = \frac{\beta\omega_3}{8\pi G}\left(\frac{3}{8}l^2 + r_+^2 - \frac{1}{2}r_0^2 - \frac{1}{3l^2}Q^{(1)2} - \frac{1}{3l^4}Q^{(2)}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{6.83}$$

with the abbreviations

$$Q^{(1)} = q_1 + q_2 + q_3 \qquad \text{and} \qquad Q^{(2)} = q_1 q_2 + q_2 q_3 + q_3 q_1$$

So the action is indeed free of divergences. But we find that it is nonlinear in the charges, which is not inherently wrong, but unexpected since it does not correspond to a definition of mass for which BPS bounds could be formulated [19]. This problem can be solved by adding a finite counterterm, which corresponds to a renormalization procedure in field theory. Finite counterterms can be constructed from the matter fields ϕ^i and A^i_α. Since these fields are only functions of the radial coordinate r, we only consider the following counterterm, which does not have two-derivatives

$$S_{\phi^2} = \frac{\beta\omega_3}{8\pi G}\sqrt{-\gamma}\; g^{(\phi)}_{ab}\phi^a\phi^b \tag{6.84}$$

In the STU model, there are three $U(1)$ gauge fields X^a and two scalars ϕ_a, which are defined by

$$\begin{aligned} X^1 &= e^{-\frac{1}{\sqrt{6}}\phi_1 - \frac{1}{\sqrt{2}}\phi_2} = H_1^{-1}\mathcal{H}^{\frac{1}{3}} \\ X^2 &= e^{-\frac{1}{\sqrt{6}}\phi_1 + \frac{1}{\sqrt{2}}\phi_2} = H_2^{-1}\mathcal{H}^{\frac{1}{3}} \\ X^3 &= e^{-\frac{2}{\sqrt{6}}\phi_1} = H_3^{-1}\mathcal{H}^{\frac{1}{3}} \end{aligned} \tag{6.85}$$

so that $X^1 X^2 X^3 = 1$. The two independent scalars can then be expressed as

$$\begin{aligned}\phi_1 &= \frac{1}{\sqrt{6}}(\ln H_1 + \ln H_2 - 2\ln H_3)\\ \phi_2 &= \frac{1}{\sqrt{2}}(\ln H_1 - \ln H_2)\end{aligned} \tag{6.86}$$

With $g^{(\phi)}_{ab}\phi^a\phi^b = \phi^a\phi_a = \phi^2$ we get

$$\phi^2 = \phi_1^2 + \phi_2^2 = \frac{1}{r^4}\left(\frac{2}{3}Q^{(1)2} - 2Q^{(2)}\right) + \mathcal{O}\left(\frac{1}{r^6}\right) \tag{6.87}$$

Inserting this into eqn. (6.84) gives the finite expression

$$S_{\phi^2} = \frac{\beta\omega_3}{8\pi G}\left(\frac{2}{3}Q^{(1)2} - 2Q^{(2)}\right) \tag{6.88}$$

which can be used to cancel the charge dependence of the regulated action (6.83)

$$S_{reg} + \frac{1}{2l}S_{\phi^2} = \frac{\beta\omega_3}{8\pi G}\left(r_+^2 - \frac{1}{2}r_0^2 + \frac{3}{8}l^2\right) \tag{6.89}$$

This is now identical to the Schwarzschild AdS$_5$ solution (6.59). We can define a general expression for the action of black holes in AdS$_5$ with or without charge as

$$S_{complete} = S_{bulk} + S_{GH} + \frac{3}{l}S_1 + \frac{l}{4}S_2 + \frac{1}{2l}S_{\phi^2} \tag{6.90}$$

If the black hole has no charge, then the last term vanishes.

We have already calculated the general form of the extrinsic curvature in eqn. (6.64). Inserting the explicit values for the metric defined in eqn. (6.82) gives after series expansion for the unregulated stress tensor (eqn. (5.1))

$$T_{tt} = \frac{1}{8\pi G}\left(-\frac{3r^2}{l^3} - \frac{9}{2l} - \frac{Q^{(1)}}{l^3} + \frac{1}{r^2}\left(-\frac{9l}{8} + \frac{3Q^{(1)}}{l} + \frac{9r_0^2}{2l}\right)\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{6.91}$$

For the regulated stress tensor we had eqn. (5.40)

$$\begin{aligned}T_{tt}^{reg} =& T_{tt} - \frac{3}{l}T_{tt}^{(1)} + \frac{l}{4}T_{tt}^{(2)} =\\ =&\frac{1}{8\pi G}\frac{1}{lr^2}\left(\frac{3l^2}{8} + \frac{3r_0^2}{2} + Q^{(1)} + \frac{1}{3l^2}\left(Q^{(2)} - Q^{(1)2}\right)\right)\end{aligned} \tag{6.92}$$

Again, we have a nonlinear charge behavior, which can be removed by adding the finite counterterm

$$T_{ij}^{\phi^2} = \frac{1}{8\pi G}\gamma_{ij} g_{ab}^{(\phi)} \phi^a \phi^b \tag{6.93}$$

which follows from variation of S_{ϕ^2} (eqn. (6.84)). So we finally end with the renormalized stress tensor

$$T_{tt}^{ren} = T_{tt}^{reg} + \frac{1}{2l} T_{ij}^{\phi^2} \tag{6.94}$$

$$T_{tt}^{ren} = \frac{1}{8\pi G}\frac{1}{lr^2}\left(\frac{3l^2}{8} + \frac{3r_0^2}{2} + Q^{(1)}\right) \tag{6.95}$$

For the mass we get from eqn. (5.70)

$$\begin{aligned} M = & \int \sqrt{\sigma}\, d\Omega_3^2 \frac{1}{\sqrt{-g_{tt}}} T_{tt}^{ren} = \\ = & \frac{\pi}{4G}\left(\frac{3l^2}{8} + \frac{3r_0^2}{2} + Q^{(1)}\right) \end{aligned} \tag{6.96}$$

6.3 AdS$_5 \times S_5$

For spacetimes that can be (at least asymptotically) decomposed into AdS$_5 \times$ S_5 we can use the counterterm formalism of AdS$_5$ and do not need to introduce additional counterterms.

6.3.1 Rotating Three-Charged Black Hole in 10 Dimensions

The line element of a rotating three charged black hole in 10 dimensions is given by ([20],[21])

$$\begin{aligned} ds^2 = & \sqrt{\Delta}\left(-\frac{f}{\mathcal{H}}dt^2 + \frac{1}{f}dr^2 + r^2 d\mathbf{x}^2\right) + \\ & + \frac{1}{\sqrt{\Delta}} \sum_{i=1}^{3} l^2 H_i (d\mu_i^2 + \mu_i^2 (d\phi_i + A^i dt)^2) \end{aligned} \tag{6.97}$$

with

$$f = \frac{r^2}{l^2}\mathcal{H} - \frac{r_0^2}{r^2} \qquad A^i = \frac{\sqrt{q_i r_0^2}}{l q_i}(H_i^{-1} - 1) \qquad \sqrt{\Delta} = \mathcal{H}\sum_{i=1}^{3}\frac{\mu_i^2}{H_i} \tag{6.98}$$

The harmonic functions $\mathcal{H}$ and H_i are the same as defined in eqn. (6.81) with $n = 4$. The q_i are the charge parameters. One might wonder that the line element seems to have 11 instead of 10 variables, but the μ_i are not independent of each other. They parametrize a 2-sphere and have to fulfill the condition $\sum \mu_i^2 = 1$.

$$\mu_1 = \cos\vartheta \qquad \mu_2 = \sin\vartheta\cos\varphi \qquad \mu_3 = \sin\vartheta\sin\varphi \tag{6.99}$$

So we have in fact only two independent variables, ϑ and φ, with the familiar line element of the 2-sphere $\sum d\mu_i^2 = d\vartheta^2 + \sin^2 d\varphi^2$. $\mathbf{x}$ is a 3-vector and represents flat space. This corresponds to a spacetime with $k = 0$ (see Section 3.3).

For simplification we will choose all three q_i to be equal and get for the line element

$$ds^2 = -\frac{f}{H^2}dt^2 + \frac{H}{f}dr^2 + r^2 H d\mathbf{x}^2 + l^2 \sum_{i=1}^{3}(d\mu_i^2 + \mu_i^2(d\phi_i + Adt)^2) \tag{6.100}$$

Notice that the metric has off-diagonal components, and therefore, a decomposition into $AdS_5 \times S^5$ is not possible. But in the limit $r \to \infty$, the off-diagonal components vanish and the metric can be decomposed, where the variables t, r and $\mathbf{x}$ belong to AdS$_5$, and the variables μ_i and ϕ_i belong to the five-sphere. For later use we will introduce for the metric determinants the names γ_4 for the induced AdS$_5$, and S^5 for the S^5 (see Appendix B).

With the normal vector $\hat{r}_\alpha = \sqrt{g_{rr}}\,\delta_{r\alpha}$ of the simplified metric eqn. (6.100) we calculate the extrinsic curvature tensor (eqn. (6.4)),

$$\begin{aligned}
\Theta_{tt} &= \frac{l^2 r_0^2 + (q + r^2)}{l^3 (q + r^2)^2 a} \\
\Theta_{t\phi_i} &= -\frac{\sqrt{r_0^2 q}\mu_i}{(q + r^2)^2 a} \\
\Theta_{x_i x_i} &= -\frac{1}{la}
\end{aligned} \tag{6.101}$$

where we introduced for a short notation

$$a := \sqrt{\frac{q + r^2}{(q + r^2)^3 - l^2 r_0^2 r^2}} \tag{6.102}$$

For the curvature scalar, we get

$$\Theta = \frac{(l^2 r_0^2 (q + 2r^2) - 4(q + r^2)^3)a}{l(q + r^2)^2} \tag{6.103}$$

To calculate the stress tensor, we start from the action integral and use the Type IIB superstring theory low-energy effective action where we neglect the string corrections [21]

$$S_{bulk} = -\frac{1}{16\pi G_{10}} \int_{\mathcal{M}} d^{10}x\sqrt{-g}\left(R - \frac{1}{4.5!}(F_5)^2\right) \tag{6.104}$$

The Ricci scalar vanishes for the given metric, and the five form is given by

$$\frac{1}{4.5!}(F_5)^2 = \frac{4qr_0^2}{r^6H^3l^2} + \frac{8}{l^2} \tag{6.105}$$

Integration gives

$$S_{bulk} = \frac{\beta\omega_5 V_3}{8\pi G_{10}} l^3 \left(r^4 + 2qr^2 - r_+^4 - 2qr_+^2 + \frac{qr_0^2}{q+r_+^2} - \frac{qr_0^2}{q+r^2}\right) \tag{6.106}$$

where ω_5 (the volume of a 5-sphere) is the result of the integration over the μ_i and ϕ_i, and V_3 is the volume of the 3-dimensional space defined by the x_i. For the boundary term

$$S_{GH} = \frac{1}{8\pi G_{10}} \int_{\partial\mathcal{M}} d^9x\sqrt{-\gamma}\,\Theta \tag{6.107}$$

we get after integration

$$S_{GH} = \frac{\beta\omega_5 V_3}{8\pi G_{10}}\, l^3 \left(-4r^4 - 8qr^2 + 2r_0^2l^2 - 4q^2 + \frac{r_0^2 q}{r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{6.108}$$

As mentioned above, if we work in the limit $r \to \infty$, we can use $\sqrt{-\gamma_4} \cdot \sqrt{S^5}$ instead of $\sqrt{-\gamma}$ and get the same result. This is crucial for the counterterms, because therefore we can use the counterterms and prefactors of AdS$_5$, and do not need to bother about additional counterterms for higher dimensions. So for the first counterterm, we can use eqn. (5.30) and consider the five sphere as an additional factor

$$S_1 = \frac{1}{8\pi G_{10}} \int_{\partial AdS_5} d^4x\sqrt{-\gamma_4} \int_{S^5} d^5x\sqrt{S^5} \tag{6.109}$$

with the result

$$S_1 = \frac{\beta\omega_5 V_3}{8\pi G_{10}}\, l^4 \left(r^4 + 2qr^2 + q^2 - \frac{l^2 r_0^2}{2}\right) \tag{6.110}$$

which turns out to be already sufficient to cancel the divergences. The second counterterm

$$S_2 = \frac{1}{8\pi G_{10}} \int_{\partial AdS_5} d^4x\sqrt{-\gamma_4}\mathcal{R}_4 \int_{S^5} d^5x\sqrt{S^5} \tag{6.111}$$

is zero, because the Ricci scalar $\mathcal{R}_4$ vanishes. So the regulated action eqn. (5.39) reduces to

$$S_{reg} = S_{bulk} + S_{GH} + \frac{3}{l} S_1 \tag{6.112}$$

and has the result

$$\begin{aligned} S_{reg} = \frac{\beta \omega_5 V_3}{8\pi G_{10}}\, l^3 \Bigg(& \frac{r_0^2 l^2}{2} - q^2 - \frac{r_+^4}{2} - q r_+^2 + \\ & + q r_0^2 \left(\frac{1}{2(q + r_+^2)} - \frac{1}{2(q + r^2)} + \frac{l^2}{r^2} \right) \Bigg) + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{6.113}$$

It should be noticed that we only get a well defined result for S_{reg} if we use the decomposed metric. For the counterterm S_1 it makes no difference because it only contains the metric determinant, and for $r \to \infty$ we already stated that $\sqrt{-\gamma} \to \sqrt{-\gamma_4} \cdot \sqrt{S^5}$. But in the counterterm S_2, the Ricci scalar occurs, and $\mathcal{R}_4 \neq \mathcal{R}_9$. Whereas $\mathcal{R}_4 = 0$ and the divergences are clearly canceled by S_1, $\mathcal{R}_9 = 20/l^2$ would produce a counterterm S_2 that is proportional to S_1, and therefore, one could arbitrarily combine S_1 and S_2 to cancel the divergences.

From the regulated action one can calculate the thermodynamic potential [10]. The relation between them is given by

$$S_{reg} = \beta \Omega \tag{6.114}$$

To express everything in field variable quantities we use further the relation

$$\frac{1}{G_5} = \frac{\omega_5}{G_{10}} = \frac{2N^2}{\pi l^3} \tag{6.115}$$

where N is the number of colours in the field theory. So we get for the thermodynamic potential

$$\begin{aligned} \Omega = \frac{N^2 V_3}{4\pi} \Bigg(& \frac{r_0^2 l^2}{2} - q^2 - \frac{r_+^4}{2} - q r_+^2 + \\ & + q r_0^2 \left(\frac{1}{2(q + r_+^2)} - \frac{1}{2(q + r^2)} + \frac{l^2}{r^2} \right) \Bigg) + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{6.116}$$

Now we will calculate the stress tensor. From the definition in eqn. (5.28) we get

$$
\begin{aligned}
T_{tt} &= -\frac{1}{8\pi G_{10}}\ a\left(\frac{3}{l^3}(q+r^2)^2 - \frac{6r_0^2}{l} + lr_0^4\frac{(q+3r^2)}{(q+r^2)^3}\right) \\
T_{t\phi_i} &= \frac{1}{8\pi G_{10}}\ a\sqrt{r_0^2 q}\left(-5 + l^2 r_0^2 \mu_i^2 \frac{(q+3r^2)}{(q+r^2)^3}\right) \\
T_{x_i x_i} &= \frac{1}{8\pi G_{10}}\ a\left(\frac{3}{l}(q+r^2)^2 - lr_0^2\right) \\
T_{\vartheta\vartheta} &= \frac{1}{8\pi G_{10}}\ a\left(4l(q+r^2) - l^3 r_0^2 \frac{(q+2r^2)}{(q+r^2)^2}\right) \\
T_{\varphi\varphi} &= T_{\vartheta\vartheta}\ \sin^2\vartheta \\
T_{\phi_i\phi_i} &= T_{\vartheta\vartheta}\ \mu_i^2
\end{aligned}
\tag{6.117}
$$

For the regulated stress tensor, we can use the formula we had for AdS_5 (eqn. (5.40)). We already know from the action that we only need the first counterterm and that the second counterterm vanishes. So we get for the regulated stress tensor

$$
T_{ij}^{reg} = \frac{1}{8\pi G}(\Theta_{ij} - \Theta\gamma_{ij} - \frac{3}{l}\gamma_{ij}) \tag{6.118}
$$

which evaluates to

$$
\begin{aligned}
T_{tt}^{reg} &= \frac{1}{8\pi G_{10}}\frac{3r_0^2}{2r^2 l} + \mathcal{O}\left(\frac{1}{r^3}\right) \\
T_{t\phi_i}^{reg} &= -\frac{1}{8\pi G_{10}}\sqrt{r_0^2 q}\ \mu_i^2 \frac{2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \\
T_{x_i x_i}^{reg} &= \frac{1}{8\pi G_{10}}\frac{lr_0^2}{2r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \\
T_{\vartheta\vartheta}^{reg} &= \frac{1}{8\pi G_{10}} l + \mathcal{O}\left(\frac{1}{r^3}\right) \\
T_{\varphi\varphi}^{reg} &= \frac{1}{8\pi G_{10}} l\sin^2\vartheta + \mathcal{O}\left(\frac{1}{r^3}\right) \\
T_{\phi_i\phi_i}^{reg} &= \frac{1}{8\pi G_{10}} l\mu_i^2 + \mathcal{O}\left(\frac{1}{r^3}\right)
\end{aligned}
\tag{6.119}
$$

From the stress tensor we can calculate with eqn. (5.70) the mass

$$
M = \frac{\omega_5}{16\pi G_{10}} 3l^5 r_0^2\left(1 + \frac{q}{r^2}\right) + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{6.120}
$$

and with eqn. (5.71) the momenta

$$P_{\phi_i} = \frac{\omega_5}{8\pi G_{10}} l^6 \mu_i^2 \sqrt{q r_0^2} \left(1 + \frac{3l^2 r_0^2}{2r^4}\right) + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{6.121}$$

Chapter 7

Conclusions

With the use of the quasilocal stress tensor and the counterterm formalism, we were able to calculate the masses of several anti-de Sitter spacetimes in a consistent way. These results can be used for thermodynamical calculations. The ADM-mass we calculated is identified with the energy E, and the action S_{reg} is related with the thermodynamic potential by $S_{reg} = \beta\Omega$ (see [10]). For black hole thermodynamics, it has already been shown that the first law of thermodynamics, $dE = TdS$, holds rather generally. For these spacetimes, which exist for pure gravity (with a cosmological constant), the thermodynamic potential Ω is equivalent to the Helmholtz free energy F, therefore we have also satisfied $F = E - TS$. However, for AdS/CFT one also wants to consider more general matter coupled systems, such as the R-charged black hole discussed in Section 6.2.2. It turned out that in addition to the regularization with infinite counterterms it was necessary to introduce a finite counterterm for renormalization. After this renormalization we were able to recover the classical result, where the action and the stress tensor are linear in the charges. The thermodynamic potential is then given by $\Omega = E - TS - \phi^I Q_I$, where the Q_I are the conserved R-charges and the ϕ^I are the corresponding electric potentials. In classical thermodynamics, one would have the chemical potential instead of the ϕ^I. So black hole thermodynamics is very similar to ordinary thermodynamics, which can be used in the AdS/CFT correspondence to get results in finite temperature field theory.

In recent work, the counterterm formalism was also successfully applied to asymptotically flat spacetimes. This generalization is not trivial, because flat spacetimes do not have a dimensionful parameter such as the curvature scalar l to make the counterterms of the action dimensionless. For further work on this subject, see e.g. [22] and [23].

Appendix A

Definitions and Derivations

A.1 Definitions

Christoffel symbol

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \tag{A.1}$$

The Christoffel symbols are symmetric in the lower indices $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$ and metric compatible, i.e., the covariant derivative of the metric vanishes $g_{\mu\nu;\rho} = 0$.

Covariant derivative

$$\begin{aligned} \nabla_\alpha v_\beta =& \partial_\alpha v_\beta - \Gamma^{\mu}_{\alpha\beta} v_\mu \\ \nabla_\alpha v^\beta =& \partial_\alpha v_\beta + \Gamma^{\beta}_{\alpha\mu} v^\mu \\ \nabla_\alpha T^{\beta}{}_{\gamma} =& \partial_\alpha T^{\beta}{}_{\gamma} + \Gamma^{\beta}_{\alpha\mu} T^{\mu}{}_{\gamma} - \Gamma^{\mu}_{\alpha\gamma} T^{\beta}{}_{\mu} \end{aligned} \tag{A.2}$$

Lie derivative

$$\begin{aligned} \pounds_\xi v^\alpha =& \xi^\beta \partial_\beta v^\alpha - v^\beta \partial_\beta \xi^\alpha \\ \pounds_\xi v_\alpha =& \xi^\beta \partial_\beta v_\alpha + v_\beta \partial_\alpha \xi^\beta \\ \pounds_\xi T^{\alpha}{}_{\beta} =& \xi^\gamma \partial_\gamma T^{\alpha}{}_{\beta} - T^{\gamma}{}_{\beta} \partial_\gamma \xi^\alpha + T^{\alpha}{}_{\gamma} \partial_\beta \xi^\gamma \end{aligned} \tag{A.3}$$

Riemann tensor

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta} \Gamma^{\mu}_{\beta\gamma}. \tag{A.4}$$

Ricci tensor

$$R_{\beta\delta} = R^{\alpha}_{\ \beta\alpha\delta} \tag{A.5}$$

Ricci scalar

$$R = R^{\alpha}_{\ \alpha} \tag{A.6}$$

Cosmological constant for vacuum spacetimes in $(n+1)$ dimensions

$$\begin{aligned} R_{\alpha\beta} + \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} =& 0 \qquad | \cdot g^{\alpha\beta} \\ R - \frac{1}{2}R(n+1) + \Lambda(n+1) =& 0 \\ \frac{n-1}{2(n+1)}R =& \Lambda \end{aligned} \tag{A.7}$$

Volume of the $(n-1)$-sphere

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{A.8}$$

A.2 Derivations of Used Relations

A.2.1 Variation of the Metric

To derive the relation between $\delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta}$ we use the variation of the identity

$$\begin{aligned} 0 =& \delta(\delta^{\gamma}_{\alpha}) = \delta(g_{\alpha\beta}g^{\beta\gamma}) \\ 0 =& \delta g_{\alpha\beta}g^{\beta\gamma} + g_{\alpha\beta}\delta g^{\beta\gamma} \qquad | \cdot g_{\gamma\mu} \\ 0 =& \delta g_{\alpha\beta}\delta^{\beta}_{\mu} + g_{\alpha\beta}g_{\gamma\mu}\delta g^{\beta\gamma} \\ \delta g_{\alpha\beta} =& -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu} \end{aligned} \tag{A.9}$$

For the calculation of $\delta\sqrt{-g}$ we use the relation ([24])

$$tr\left[\frac{dA}{d\tau}A^{-1}\right] = \frac{1}{\det A}\frac{d}{d\tau}\det A \tag{A.10}$$

Notice that g denotes the determinant of $g_{\alpha\beta}$, so we have to insert $g_{\alpha\beta}$ for the matrix A. By substituting the derivation with the variation, we get

$$tr\left[\delta g_{\alpha\beta}\ g^{\beta\gamma}\right] = \frac{1}{g}\delta g$$

$$\delta g_{\alpha\beta}\ g^{\beta\alpha}\ g = \delta g$$
$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g$$
$$\delta\sqrt{-g} = \frac{1}{2}\frac{-g}{\sqrt{-g}}\ g^{\alpha\beta}\delta g_{\alpha\beta}$$

With the use of the relation in eqn. (A.9), we can express this in terms of $\delta g^{\alpha\beta}$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\ g_{\alpha\beta}\delta g^{\alpha\beta} \tag{A.11}$$

The variation of $\sqrt{-\gamma}$ is done exactly the same way.

A.2.2 Hamilton-Jacobi Formalism

$$\begin{aligned}\frac{1}{16\pi G}\frac{1}{2}G_{IJ}\gamma^{ij}r^{\alpha}F^{I}_{\alpha i}r^{\beta}F^{J}_{\beta j} =&\frac{1}{2}r^{\alpha}F^{I}_{\alpha i}\pi^{i}_{I} = \\ =&\frac{1}{2}G_{IJ}\gamma^{ij}\pi^{J}_{j}r^{\alpha}F^{I}_{\alpha i} = \\ =&16\pi G\frac{1}{2}\pi^{j}_{J}\pi^{J}_{j} = \\ =&16\pi G\frac{1}{2}G_{IJ}\gamma^{ij}\pi^{I}_{i}\pi^{J}_{j}\end{aligned} \tag{A.12}$$

$$\begin{aligned}\pi^{ij}\pi_{ij} =&\frac{1}{(16\pi G)^2}(h^{ij}h_{ij}\Theta^2 - 2h^{ij}\Theta_{ij}\Theta + \Theta^{ij}\Theta_{ij}) = \\ =&\frac{1}{(16\pi G)^2}((n-2)\Theta^2 + \Theta^{ij}\Theta_{ij}\end{aligned} \tag{A.13}$$

$$\begin{aligned}\pi^{i}{}_{i}\pi^{j}{}_{j} =&\frac{1}{(16\pi G)^2}(h^{i}{}_{i}h^{j}{}_{j}\Theta^2 - 2h^{i}{}_{i}\Theta^{j}{}_{j}\Theta + \Theta^{i}{}_{i}\Theta^{j}{}_{j}) = \\ =&\frac{1}{(16\pi G)^2}(n^2 - 2n + 1)\Theta^2 = \\ =&\frac{1}{(16\pi G)^2}(n-1)^2\Theta^2\end{aligned} \tag{A.14}$$

Appendix B

List of Metric Constants

B.1 AdS$_3$

General form of the line element

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\varphi^2 \tag{B.1}$$

Metric determinant

$$\sqrt{-g} = r \tag{B.2}$$

Christoffel symbols for AdS$_3$

$$\Gamma^r_{tt} = \frac{r}{l^2} f \qquad \Gamma^t_{rt} = -\Gamma^r_{rr} = \frac{r}{l^2}\frac{1}{f} \qquad \Gamma^r_{\varphi\varphi} = -rf \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \tag{B.3}$$

Ricci tensor

$$R_{\alpha\beta} = \frac{n}{l^2} g_{\alpha\beta} = \frac{2}{l^2} g_{\alpha\beta} \tag{B.4}$$

Ricci scalar

$$R = -\frac{n(n+1)}{l^2} = -\frac{6}{l^2} \tag{B.5}$$

Cosmological constant

$$\Lambda = -\frac{n(n-1)}{2l^2} = -\frac{1}{l^2} \tag{B.6}$$

B.1.1 Global AdS$_3$

$$f = 1 + \frac{r^2}{l^2} \tag{B.7}$$

Christoffel symbols

$$\begin{aligned} \Gamma^r_{tt} &= \frac{r(l^2+r^2)}{l^4} \qquad \Gamma^r_{rr} = -\Gamma^t_{rt} = -\frac{r}{l^2+r^2} \\ \Gamma^r_{\varphi\varphi} &= -\frac{r(l^2+r^2)}{l^2} \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \end{aligned} \tag{B.8}$$

B.1.2 Schwarzschild AdS_3

$$f = 1 + \frac{r^2}{l^2} - r_0^2 \tag{B.9}$$

Christoffel symbols

$$\begin{aligned} \Gamma^r_{tt} &= \frac{r(l^2+r^2-l^2r_0^2)}{l^4} \qquad \Gamma^t_{rt} = -\Gamma^r_{rr} = -\frac{r}{l^2+r^2-l^2r_0^2} \\ \Gamma^r_{\varphi\varphi} &= -\frac{r(l^2+r^2-l^2r_0^2)}{l^2} \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \end{aligned} \tag{B.10}$$

B.1.3 Poincaré AdS_3

$$f = \frac{r^2}{l^2} \qquad \varphi = \frac{x}{l} \tag{B.11}$$

Metric determinant

$$\sqrt{-g} = \frac{r}{l} \tag{B.12}$$

Christoffel symbols

$$\Gamma^r_{tt} = \frac{r^3}{l^4} \qquad \Gamma^r_{xx} = -\frac{r^3}{l^4} \qquad \Gamma^r_{rr} = \Gamma^t_{rt} = \Gamma^x_{rx} = \frac{1}{r} \tag{B.13}$$

B.1.4 Perturbed Poincaré AdS_3

$$f = \frac{r^2}{l^2} \qquad \varphi = \frac{x}{l} \tag{B.14}$$

Contravariant metric components (in the limit $r \to \infty$)

$$\begin{aligned} g^{tt} &= \left(-\frac{r^2}{l^2} + \delta g_{tt}\right)^{-1} \approx -\frac{l^2}{r^2} - \frac{l^4}{r^4}\delta g_{tt} \\ g^{rr} &= \left(\frac{l^2}{r^2} + \delta g_{rr}\right)^{-1} \approx \frac{r^2}{l^2} - \frac{r^4}{l^4}\delta g_{rr} \\ g^{xx} &= \left(\frac{r^2}{l^2} + \delta g_{xx}\right)^{-1} \approx \frac{l^2}{r^2} - \frac{l^4}{r^4}\delta g_{xx} \end{aligned} \tag{B.15}$$

Christoffel symbols

$$\begin{aligned} \Gamma^r_{tt} &= \frac{r^3}{l^4} - \frac{r^5}{l^6}\delta g_{rr} - \frac{1}{2}\frac{r^2}{l^2}\partial_r \delta g_{tt} \qquad \Gamma^r_{xt} = -\frac{1}{2}\frac{r^2}{l^2}\partial_r \delta g_{xt} \\ \Gamma^r_{xx} &= -\frac{r^3}{l^4} + \frac{r^5}{l^6}\delta g_{rr} - \frac{1}{2}\frac{r^2}{l^2}\partial_r \delta g_{xx} \end{aligned} \tag{B.16}$$

B.2 AdS$_5$

General form of the line element

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + r^2(d\psi^2 + \sin^2\psi d\vartheta^2 + \cos^2\psi d\varphi^2) \tag{B.17}$$

Metric determinant

$$\sqrt{-g} = r^3 \sin\psi \cos\psi \tag{B.18}$$

Christoffel symbols

$$\begin{aligned} \Gamma^r_{tt} &= \frac{1}{2} f \partial_r f & \Gamma^r_{\psi\psi} &= -rf \\ \Gamma^r_{\vartheta\vartheta} &= -rf\sin^2\psi & \Gamma^r_{\varphi\varphi} &= -rf\cos^2\psi \\ \Gamma^t_{rt} &= -\Gamma^r_{rr} = \frac{\partial_r f}{2f} & \Gamma^\psi_{r\psi} &= \Gamma^\vartheta_{r\vartheta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r} \\ \Gamma^\psi_{\vartheta\vartheta} &= -\cos\psi\sin\psi & \Gamma^\psi_{\varphi\varphi} &= \cos\psi\sin\psi \\ \Gamma^\varphi_{\psi\varphi} &= -\tan\psi & \Gamma^\vartheta_{\psi\vartheta} &= \cot\psi \end{aligned} \tag{B.19}$$

Ricci tensor

$$R_{\alpha\beta} = \frac{n}{l^2} g_{\alpha\beta} = \frac{4}{l^2} g_{\alpha\beta} \tag{B.20}$$

Ricci scalar

$$R = -\frac{n(n+1)}{l^2} = -\frac{20}{l^2} \tag{B.21}$$

Cosmological constant

$$\Lambda = -\frac{n(n-1)}{2l^2} = -\frac{6}{l^2} \tag{B.22}$$

B.2.1 Induced Metric on AdS$_5$

Einstein tensor

$$\mathcal{G}_{tt} = \frac{3f}{r^2} \qquad \mathcal{G}_{\psi\psi} = -1 \qquad \mathcal{G}_{\vartheta\vartheta} = -\sin^2\psi \qquad \mathcal{G}_{\varphi\varphi} = -\cos^2\psi \tag{B.23}$$

Ricci tensor

$$\mathcal{R}_{\psi\psi} = 2 \qquad \mathcal{R}_{\vartheta\vartheta} = 2\sin^2\psi \qquad \mathcal{R}_{\varphi\varphi} = 2\cos^2\psi \tag{B.24}$$

Ricci scalar

$$\mathcal{R} = \frac{6}{r^2} \tag{B.25}$$

B.2.2 Schwarzschild AdS$_5$

$$f = 1 + \frac{r^2}{l^2} - \frac{r_0^2}{r^2} \tag{B.26}$$

Christoffel symbols

$$\Gamma^r_{tt} = \frac{(l^2r^2 + r^4 - r_0^2 l^2)(r^4 + r_0^2 l^2)}{l^4 r^5} \qquad \Gamma^r_{\vartheta\vartheta} = -\frac{l^2r^2 + r^4 - r_0^2 l^2}{l^2 r}$$

$$\Gamma^r_{\varphi\varphi} = -\frac{(l^2r^2 + r^4 - r_0^2 l^2)\sin^2\vartheta}{l^2 r} \qquad \Gamma^r_{\psi\psi} = -\frac{(l^2r^2 + r^4 - r_0^2 l^2)\cos^2\vartheta}{l^2 r}$$

B.2.3 R-Charged Black Hole in AdS$_5$

Line element

$$ds^2 = -fe^{-4B}dt^2 + \frac{1}{f}e^{2B}dr^2 + r^2e^{2B}(d\psi^2 + \sin^2\psi d\vartheta^2 + \cos^2\psi d\psi^2) \tag{B.27}$$

$$f = 1 + H\frac{r^2}{l^2} - \frac{r_0^2}{r^2} \tag{B.28}$$

Metric determinant

$$\sqrt{-g} = r^3 e^{2B}\sin\psi\cos\psi \tag{B.29}$$

Christoffel symbols

$$\begin{aligned}
&\Gamma^r_{tt} = -\frac{1}{2}e^{-6B}f(4f\partial_r B - \partial_r f) && \Gamma^r_{\psi\psi} = -rf(1 + r\partial_r B) \\
&\Gamma^r_{\vartheta\vartheta} = -rf(1 + r\partial_r B)\sin^2\psi && \Gamma^r_{\varphi\varphi} = -rf(1 + r\partial_r B)\cos^2\psi \\
&\Gamma^r_{rr} = \frac{2f\partial_r B - \partial_r f}{2f} && \Gamma^t_{rt} = \frac{-4f\partial_r B + \partial_r f}{2f} \\
&\Gamma^\psi_{r\psi} = \Gamma^\vartheta_{r\vartheta} = \Gamma^\varphi_{r\varphi} = \frac{1 + r\partial_r B}{r} \\
&\Gamma^\psi_{\vartheta\vartheta} = -\cos\psi\sin\psi && \Gamma^\psi_{\varphi\varphi} = \cos\psi\sin\psi \\
&\Gamma^\varphi_{\psi\varphi} = -\tan\psi && \Gamma^\vartheta_{\psi\vartheta} = \cot\psi
\end{aligned} \tag{B.30}$$

B.3 Rotating Three-Charged Black Hole in 10 Dimensions

Line element

$$ds^2 = -\frac{f}{H^2}dt^2 + \frac{H}{f}dr^2 + r^2 H d\mathbf{x}^2 + l^2 \sum_{i=1}^{3}(d\mu_i^2 + \mu_i^2(d\phi_i + Adt)^2) \quad \text{(B.31)}$$

Metric determinant

$$\sqrt{-g} = l^5 r(q + r^2)\cos\vartheta\cos\varphi\sin^3\vartheta\sin\varphi \quad \text{(B.32)}$$

Ricci scalar

$$R = 0 \quad \text{(B.33)}$$

Determinant of the induced metric

$$\sqrt{-\gamma} = l^4\sqrt{(q+r^2)((q+r^2)^3 - l^2 m r^2)}\cos\vartheta\cos\varphi\sin^3\vartheta\sin\varphi \quad \text{(B.34)}$$

Ricci scalar of the induced metric

$$\mathcal{R} = \frac{20}{l^2} \quad \text{(B.35)}$$

Einstein tensor of the induced metric

$$\begin{aligned}
\mathcal{G}_{tt} &= \frac{10(q+r^2)}{l^4} - \frac{2m(3q+5r^2)}{l^2(q+r^2)^2} \\
\mathcal{G}_{t\phi_i} &= -\frac{6\mu_i^2\sqrt{mq}}{l(q+r^2)} \qquad \mathcal{G}_{x_i x_i} = -\frac{10(q+r^2)}{l^2} \\
\mathcal{G}_{\vartheta\vartheta} &= -6 \qquad \mathcal{G}_{\varphi\varphi} = -6\sin^2\vartheta \qquad \mathcal{G}_{\phi_i\phi_i} = -6\mu_i^2
\end{aligned} \quad \text{(B.36)}$$

B.3.1 AdS$_5 \times S_5$

$$ds^2 = \underbrace{\left(-\frac{f}{H^2} + l^2A^2\right)dt^2 + \frac{H}{f}dr^2 + r^2 H d\mathbf{x}^2}_{\mathrm{AdS}_5} + \underbrace{l^2\sum_{i=1}^{3}(d\mu_i^2 + \mu_i^2 d\phi_i^2)}_{S_5} \quad \text{(B.37)}$$

Metric determinant of S^5

$$\sqrt{S^5} = l^5\cos\vartheta\cos\varphi\sin^3\vartheta\sin\varphi \quad \text{(B.38)}$$

Metric determinant of induced AdS_5

$$\sqrt{-\gamma_4} = \frac{1}{l}\sqrt{(q+r^2)((q+r^2)^3 - l^2 r_0^2 r^2)} \tag{B.39}$$

Ricci scalar of induced AdS_5

$$\mathcal{R} = 0 \tag{B.40}$$

Einstein tensor of induced AdS_5

$$\mathcal{G}_{ij} = 0 \qquad \forall i, j \tag{B.41}$$

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