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On duality relations for Asian options

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Klaus Haider, BSc
Matrikelnummer: 01411889

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der Fakultät für Mahematik und Geoinformation
der Technischen Universität Wien

unter der Anleitung von

Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Friedrich Hubalek

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Kurzfassung

Das Ziel dieser Arbeit ist es die Theorie über Dualitätsrelationen für Optionspreise in einem Binomialmodell mit Dividendenzahlungen mit einem Fokus auf asiatische Optionen bezüglich des arithmetischen Mittels verständlich und umfassend zu präsentieren. Im Allgemeinen sind Dualitätsrelationen Aussagen darüber, dass der Preis einer Option im ursprünglichen Modell dem Preis einer bestimmten, anderen Option im dualen Modell entspricht. Im Spezialfall eines Binomialmodells mit Dividendenzahlungen handelt es sich beim dualen Modell wiederum um ein Binomialmodell mit Dividendenzahlungen bezüglich anderer Parameter. Im ersten Teil der Arbeit diskutieren wir das Modell und betrachten einige Beispiele, wohingegen wir im zweiten Teil die Theorie über Dualitätsrelationen für europäische Vanilla-Optionen, Forward-start-Optionen, Lookback-Optionen und asiatische Optionen und amerikanische Vanilla-Optionen behandeln. Da die Dualitätsrelationen bezüglich europäischer, asiatischer Optionen nicht auf ihre amerikanischen Gegenstücke erweitert werden können, versuchen wir im letzten Teil dieser Arbeit eine Menge von Stoppzeiten, welche eine spezielle Zeitumkehr-Eigenschaft aufweisen, zu charakterisieren und die Relationen für asiatische Optionen bezüglich des arithmetischen Mittels für Stoppzeiten zu erweitern.

Abstract

The purpose of this thesis is to comprehensively state the theory on duality relations for option prices in a binomial model with dividend yield with a focus on Asian options with respect to the arithmetic mean. In general, duality relations state that the calculation of prices of specific options for the original model is equivalent to the calculation of prices for specific other options under a dual model. In the special case of a binomial model with dividend yield the corresponding dual model is again a binomial model with dividend yield with respect to other parameters. In the first part of this thesis we discuss the model and give examples, whereas we treat the theory on duality relations for vanilla options, forward start options, lookback options and Asian options of European type and vanilla options of American type in the second part. As the duality relations for Asian options break down when passing over to the American case, finally, we characterize a set of stopping times which satisfy a specific time reversal property and try to extend the relations for Asian options with respect to the arithmetic mean to stopping times.

Eidesstattliche Erklärung

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Wien, am 13. Mai 2020

Klaus Haider

Inhaltsverzeichnis

1. Introduction

Option pricing is one of the fundamental fields of work of financial mathematics. Unfortunately, there often is not a closed-form solution for the price of an option in a specific market model available. Therefore, it is very important to obtain relations between different types of options, because it can reduce the complexity of calculations immensely, if the price of a more complicated option can be related to the price of a less complicated one. In particular, this might be a great deal for path-dependent options such as forward start options, lookback options or Asian options. For example in the case of a path-dependent forward start vanilla options one is able to relate the option price to a non-path-dependent usual vanilla option with different parameters and in the case of Asian options one can relate the prices of floating and fixed strike options, which reduces the problem of computing the expected value of two highly dependent random variables, the stock price and the average price, to the calculation of the expected value of one single random variable, the average price.

To achieve this, one uses a change of measure to the dual measure and looks at the dual process and symmetry relations of the original process and the dual process. The dual process usually is equal to a random factor times the reciprocal of the original price process and the dual measure is characterised through having the normalized original price process as density process with respect to the original martingale measure. A comprehensive, thorough and mathematically demanding theory about this approach in a general and abstract semimartingale setting can be found in the papers [2] and [3] of Eberlein, Papapantoleon and Shiryaev. The purpose of this thesis is to present this theory for the special case of a binomial model with dividend yield, which is still often used as an approximation of the popular model by Black and Scholes. In this special case, the dual model again can be identified with a binomial model with dividend yield, hence, it can be beautifully illustrated how the general approach works and which symmetries are used or appear. Furthermore, we will try to extend the theory about relations for fixed and floating strike Asian options to stopped processes and investigate which sets of stopping times are stable under time reversal.

The next chapter starts with a very short reminder of some very basic definitions and results about stochastic processes and stopping times used in discrete time finance. Then we continue with the definition and discussion of a binomial model with dividend yield. A very comprehensive treatment of the notions and theory used in discrete time finance can be found in the book of Föllmer and Schied [4] and theory about binomial models in particular is treated in the books of Shreve [8] and van der Hoek and Elliott [9].

In Chapter 3 we introduce some standard options and the notations we use throughout this thesis. Further, we illustrate duality relations for vanilla options, forward start options, forward start Asian options and forward start lookback options of European type and vanilla options of American type with examples with just few time steps. We also show that contrary to vanilla options the duality relations for Asian options with fixed and floating strike cannot be extended to their American counterparts.

Chapter 4 deals with the theory of duality relations presented by Eberlein, Papapantoleon and Shiryaev in a general semimartingale setting for the special case of binomial models with dividend yield. We prove several distributional relations for the underlying random walks between a binomial model and its dual model and use them to obtain the duality relations for options of European type observed in the previous chapter. We continue with identifying stopping times with respect to the generated filtration of the underlying random walk with antichains with respect to the partial order of extension on the set of finite $\{-1, 1\}$-sequences. Using this identification, we define a bijection from the set of stopping times of the original binomial model to the set of stopping times for the dual binomial model with changed parameters. Applying this bijection, we are able to prove the duality relations for vanilla options of American type.

Throughout Chapter 5 we try to extend the theory on duality relations for Asian fixed and floating strike options with respect to the arithmetic mean for binomial models with dividend yield. We characterise a set of stopping times which satisfy a distributional time reversal property for the underlying random walk which can be understood as a generalisation of the time reversal property used for the proof of the duality relations for Asian options of European type. We discuss restrictions on how such a stopping time might look like and finally, we use this property to give proof of duality relations for Asian options with respect to the stopped processes.

2. Preliminaries and model

2.1. Martingales and stopping times

In this section we want to recall few basic definitions and results about martingales and stopping times. The idea behind this section is to give a reader of this thesis who is unfamiliar with martingales and stopping times, but has some basic knowledge about probability theory, a very short introduction to the topic, so that he or she is able to follow the proofs in the next section and especially Chapter 4 without other literature. Therefore, the definitions, theorems and proofs presented in this section are formulated just in such a generality as we need it for this thesis, but of course can be easily generalised. A reader familiar with these topics can skip this section without problems and start reading at the beginning of the next section.

We start with the definition of a discrete-time martingale with finite time horizon.

Definition 2.1. Let $N \in \mathbb{N}_0$. A sequence of random variables $M = (M_n)_{0 \leq n \leq N}$ is said to be a martingale with respect to a filtration $\mathbb{G} = (\mathcal{G}_n)_{0 \leq n \leq N}$ and a probability measure $\mathbb{Q}$ if

- M is $\mathbb{G}$-adapted: M_n is $\mathcal{G}_n$-measurable for all $n \in \{0, \dots, N\}$,
- M is $\mathbb{Q}$-integrable: $\mathbb{E}_{\mathbb{Q}}\left[|M_n|\right] < \infty$ for all $n \in \{0, \dots, N\}$ and
- M has the martingale property: $\mathbb{E}_{\mathbb{Q}}\left[M_n|\mathcal{G}_k\right] = M_k$ for all $k \leq n$, $k, n \in \{0, \dots, N\}$.

The concept of martingales corresponds to the idea of a fair game, where the expected amount of money of the gambler after n rounds of the game, given the information up to round k, is equal to the gambler's amount of money after round k, which means that the gambler is neither expected to win nor to lose money.

We continue with the definition of a discrete-time stopping time.

Definition 2.2. Let $N \in \mathbb{N}_0$ and $\mathbb{G} = (\mathcal{G}_n)_{0 \leq n \leq N}$ be a filtration. A random variable τ with values in $\{0, \dots, N\}$ is called a $\mathbb{G}$-stopping time if $\{\tau \leq n\} \in \mathcal{G}_n$ for all $n \in \{0, \dots, N\}$.

The property $\{\tau \leq n\} \in \mathcal{G}_n$ for all $n \in \{0, \dots, N\}$ can be interpreted as that the information whether one should stop or not before or at time n should be contained in the information available up to time n. In this sense stopping times can be used to model the exercise strategy of a buyer of an option of American type.

The next lemma deals with the σ-algebra of the information up to the stopping time.

Lemma 2.3. *Let $N \in \mathbb{N}_0$, $\mathbb{G} = (\mathcal{G}_n)_{0 \leq n \leq N}$ be a filtration and τ be a $\mathbb{G}$-stopping time bounded by N. The set $\mathcal{G}_\tau := \{A \in \mathcal{G}_N | A \cap \{\tau \leq n\} \in \mathcal{G}_n \text{ for all } n \in \{0, \dots, N\}\}$ is a σ-algebra. We call it the σ-algebra associated with τ.*

Proof. We take $n \in \{0, \dots, N\}$ arbitrarily. First we have to show $\Omega \in \mathcal{G}_\tau$. We have

$$\Omega \cap \{\tau \leq n\} = \{\tau \leq n\} \in \mathcal{G}_n,$$

because τ is a $\mathbb{G}$-stopping time. Hence we get $\Omega \in \mathcal{G}_\tau$.
Second, we take $A \in \mathcal{G}_\tau$ and have to show $A^{\mathrm{C}} \in \mathcal{G}_\tau$. As a consequence of $A \in \mathcal{G}_\tau$, we have $A \in \mathcal{G}_N$ and, therefore, $A^{\mathrm{C}} \in \mathcal{G}_N$. So we have verified the first condition in the definition of $\mathcal{G}_\tau$. Further, we have

$$A^{\mathrm{C}} \cap \{\tau \leq n\} = (A^{\mathrm{C}} \cup \{\tau > n\}) \cap \{\tau \leq n\} = (A \cap \{\tau \leq n\})^{\mathrm{C}} \cap \{\tau \leq n\}.$$

Since $A \in \mathcal{G}_\tau$, we have $A \cap \{\tau \leq n\} \in \mathcal{G}_n$ and, therefore, $(A \cap \{\tau \leq n\})^{\mathrm{C}} \in \mathcal{G}_n$, because $\mathcal{G}_n$ is a σ-algebra. As already mentioned above, we also have $\{\tau \leq n\} \in \mathcal{G}_n$ because τ is a stopping time. Hence, we get $A^{\mathrm{C}} \cap \{\tau \leq n\} \in \mathcal{G}_n$ because σ-algebras are intersection stable. Since $n \in \{0, \dots, N\}$ was chosen arbitrarily, this shows $A^{\mathrm{C}} \in \mathcal{G}_\tau$.
The third property we have to show is that $\mathcal{G}_\tau$ is stable under taking countable unions. Therefore, we take a sequence $(A_k)_{k\geq 1}$ with $A_k \in \mathcal{G}_\tau$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ $A_k \in \mathcal{G}_\tau$ implies $A_k \in \mathcal{G}_N$ and further $\bigcup_{k\geq 1} A_k \in \mathcal{G}_N$ because $\mathcal{G}_N$ is a σ-algebra. We also have

$$\bigcup_{k\geq 1} A_k \cap \{\tau \leq n\} = \bigcup_{k\geq 1} (A_k \cap \{\tau \leq n\})$$

and, therefore, get $\bigcup_{k\geq 1} A_k \cap \{\tau \leq n\} \in \mathcal{G}_n$ because $\mathcal{G}_n$ is a σ-algebra and $A_k \cap \{\tau \leq n\} \in \mathcal{G}_n$ as a consequence of $A_k \in \mathcal{G}_\tau$ for every $k \in \mathbb{N}$. As $n \in \{0, \dots, N\}$ was chosen arbitrarily we get $\bigcup_{k\geq 1} A_k \in \mathcal{G}_\tau$, which finishes the proof. □

Now we are able to formulate a statement about the measurability of the random variable of an adapted process at a stopping time.

Lemma 2.4. *Let $N \in \mathbb{N}_0$, $\mathbb{G} = (\mathcal{G}_n)_{0\leq n\leq N}$ be a filtration and $(Z_n)_{0\leq n\leq N}$ be a $\mathbb{G}$-adapted process. Then Z_τ define via $Z_\tau(\omega) = Z_{\tau(\omega)}(\omega)$ for all $\omega \in \Omega$ is $\mathcal{G}_\tau$-measurable for a $\mathbb{G}$-stopping time τ bounded by N.*

Proof. Let B an element of the Borel-σ-algebra of $\mathbb{R}$ and $n \in \{0, \dots, N\}$. We have

$$\begin{aligned}
\{Z_\tau \in B\} \cap \{\tau \leq n\} &= \bigcup_{0\leq k\leq n} \{Z_\tau \in B, \tau = k\} \\
&= \bigcup_{0\leq k\leq n} \{Z_k \in B, \tau = k\} \\
&= \bigcup_{0\leq k\leq n} (\{Z_k \in B\} \cap \{\tau = k\}).
\end{aligned}$$

For all $k \in \{0, \dots, n\}$ Z_k is $\mathcal{G}_n$- measurable because Z is $\mathbb{G}$-adapted and $\mathcal{G}_k \subset \mathcal{G}_n$. Hence, we have $\{Z_k \in B\} \in \mathcal{G}_n$. As a consequence of $\{\tau = k\} = \{\tau \leq k\} \cap \{\tau \leq k-1\}^{\mathrm{C}}$ and $\{\tau \leq k\} \in \mathcal{G}_k$, $\{\tau \leq k-1\}^{\mathrm{C}} \in \mathcal{G}_{k-1}$, because τ is a $\mathbb{G}$-stopping time, we get $\{\tau = k\} \in \mathcal{G}_k \subset \mathcal{G}_n$ for all $k \in \{0, \dots, n\}$. Since $\mathcal{G}_n$ is a σ-algebra and, therefore, stable under taking finite intersections and unions and the equation above, we obtain $\{Z_\tau \in B\} \cap \{\tau \leq n\} \in \mathcal{G}_n$ for all $n \in \{0, \dots, N\}$ and as a special case $\{Z_\tau \in B\} \in \mathcal{G}_N$, because $\{\tau \leq N\} = \Omega$. This implies $\{Z_\tau \in B\} \in \mathcal{G}_\tau$, which shows that Z_τ is $\mathcal{G}_\tau$-measurable, since B was taken arbitrarily. □

The following theorem is a simplified version of Doob's optional sampling theorem which tells us that the martingale property can be extended to stopping times in some sense.

Theorem 2.5. *Let $N \in \mathbb{N}_0$, $\mathbb{G} = (\mathcal{G}_n)_{0 \leq n \leq N}$ be a filtration, $(M_n)_{0 \leq n \leq N}$ be a $\mathbb{G}$-martingale with respect to a probability measure $\mathbb{Q}$ and τ be a $\mathbb{G}$-stopping time bounded by $n \in \{0, \dots, N\}$. Then we have*

$$\mathbb{E}_{\mathbb{Q}}\left[M_n | \mathcal{G}_\tau\right] = M_\tau.$$

Proof. We take $A \in \mathcal{G}_\tau$. For $k \in \{0, \dots, n\}$ we have $A \cap \{\tau = k\} = (A \cap \{\tau \leq k\}) \setminus (A \cap \{\tau \leq k-1\}) \in \mathcal{G}_k$, because of the definition of $\mathcal{G}_\tau$, $\mathcal{G}_{k-1} \subset \mathcal{G}_k$ and the fact that σ-algebras are closed with respect to taking differences of sets. Hence, we get using the law of total expectation for conditional expectations and taking out what is known of the conditional expectation

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[M_n \mathbb{1}_A\right] &= \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[M_n \mathbb{1}_{A \cap \{\tau = k\}}\right] \\
&= \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[M_n \mathbb{1}_{A \cap \{\tau = k\}} | \mathcal{G}_k\right]\right] \\
&= \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[M_n | \mathcal{G}_k\right] \mathbb{1}_{A \cap \{\tau = k\}}\right].
\end{aligned}$$

Now we can use the martingale property of M and again the linearity of the expected value to further obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[M_n \mathbb{1}_A\right] &= \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[M_k \mathbb{1}_{A \cap \{\tau = k\}}\right] \\
&= \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[M_\tau \mathbb{1}_{A \cap \{\tau = k\}}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[M_\tau \mathbb{1}_A\right],
\end{aligned}$$

for all $A \in \mathcal{G}_\tau$, which together with Lemma 2.4 and the definition of the conditional expectation implies that $\mathbb{E}_{\mathbb{Q}}\left[M_n | \mathcal{G}_\tau\right] = M_\tau$. □

In the next section we will introduce binomial models with dividend yield.

2.2. Binomial model with dividend yield

In the remaining part of this thesis we will deal with binomial models with dividend yield. A binomial model with dividend yield is a discrete-time market model which consists of one riskless bond with price process B and one risky, dividend paying stock with price process S. We usually view everything up to a finite time horizon $T \in \mathbb{N}_0$, which most of the time corresponds to the maturity of an option.
We assume that the price of the riskless bond changes by a constant factor $b \in \mathbb{R}_+$ in every time step from time t to time $t+1$ for $t \in \{0, \dots, T-1\}$ and that the price of the stock, dividend payments included, goes "up" by a factor $u \in \mathbb{R}_+$ or goes "down" by a factor $d \in \mathbb{R}_+$ in each of these time steps. Using this interpretation we can assume $d < u$ without loss of generality. Hence, we model the behaviour of the stock price in every time step with a modified Bernoulli random variable $X_t \in \{-1, 1\}$ for every $t \in \{1, \dots, T\}$ on a probability space with sample space Ω, σ-algebra $\mathcal{F} := \sigma(X_1, \dots, X_T)$ and a probability measure $\mathbb{P}$ with positive probabilities for going "up" or "down" in each time step and, therefore, get for the stock price process starting with a deterministic, known price $S_0 \in \mathbb{R}_+$

$$S_t = S_0 \, u^{\frac{t+\sum_{k=1}^{t} X_k}{2}} \, d^{\frac{t-\sum_{k=1}^{t} X_k}{2}} \text{ for } t \in \{0, \dots, T\}, \tag{2.1}$$

whereas we get for the price process of the riskless, hence, deterministic bond, which corresponds to the time value of money and hence starts with price $B_0 = 1$ at time $t = 0$,

$$B_t = b^t \text{ for } t \in \{0, \dots, T\}. \tag{2.2}$$

Further, we introduce the filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$ with

$$\mathcal{F}_0 := \{\emptyset, \Omega\} \qquad \text{and} \qquad \mathcal{F}_t := \sigma(X_1, \dots, X_t) \qquad \text{for } t \in \{1, \dots, T\} \tag{2.3}$$

and for the reason of simplicity the random variables

$$Y_t := \sum_{1 \le k \le t} X_k \text{ for } t \in \{0, \dots, T\} \tag{2.4}$$

and rewrite the price process of the stock as

$$S_t = S_0 \, \exp\left(\frac{t+Y_t}{2} \log(u) + \frac{t-Y_t}{2} \log(d) \right). \tag{2.5}$$

We should note that in general we assume no additional information about the probability measure $\mathbb{P}$ except

$$\mathbb{P}[X_1 = i_1, \dots, X_T = i_T] > 0 \quad \text{for} \quad (i_1, \dots, i_T) \in \{-1, 1\}^T, \tag{2.6}$$

which even corresponds to a situation of "Knightian uncertainty" in contrast to just a situation of "risk", where the exact distributions or at least confidence intervals for the parameters of the distributions are known. But as we will see in the next theorem we do not need to know more information about $\mathbb{P}$ to find a "fair" price for options in our model. We continue with the description of our assumptions about the dividend payments in the model. We assume that we hold a portfolio of shares with value process V which starts with one share and in which we reinvest the dividends paid at each time $t \in \{1, \dots, T\}$, so that no money vanishes on the level of the portfolio. Further, we assume that there exists a dividend factor $a \in \mathbb{R}_+$ so that the dividend payment at time $t \in \{1, \dots, T\}$ is equal to $(a^t - a^{t-1})S_t$, therefore, we get

$$V_0 = S_0, \quad V_1 = S_1 + (a-1)S_1 = aS_1 \quad V_2 = aS_2 + (a^2 - a)S_2 = a^2 S_2,$$

which leads us per induction to

$$V_t = a^t S_t \text{ for } t \in \{0, \dots, T\}. \tag{2.7}$$

We should note that for a dividend factor $a > 1$ the dividend payments would be increasing over time for constant stock prices and that for $a = 1$ there are no dividend payments, we have $V = S$ and end up with a usual binomial model without dividends. Hence, we should further note that using this interpretation only $a \geq 1$ makes sense.
A second possible and maybe easier to understand and accept interpretation for Equation 2.7 is that there are no dividend payments, but that the stock is traded in another currency and we assume that the conversion rate between the two currencies changes by the deterministic, constant conversion factor $a \in \mathbb{R}_+$ in each time step.

The next chapters of this thesis deal with calculations and relations for the prices of options on S. Therefore, the next theorem deals with martingale measures for the discounted value process V/B, which is necessary for dealing with prices of options on S.

Theorem 2.6. *In our binomial model with dividend yield there exists a unique probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which is equivalent to $\mathbb{P}$ so that the discounted price process $(V_t/B_t)_{0 \leq t \leq T}$ is a martingale with respect to the filtration $\mathbb{F}$ and the measure $\mathbb{Q}$ if and only if $d < b/a < u$. In this case, the random variables $(X_t)_{0 \leq t \leq T}$ are independent and identical distributed under $\mathbb{Q}$ with*

$$\mathbb{Q}[X_1 = 1] = \frac{\frac{b}{a} - d}{u - d} \qquad \text{and} \qquad \mathbb{Q}[X_1 = -1] = \frac{u - \frac{b}{a}}{u - d}$$

and we call $\mathbb{Q}$ the martingale measure for our model.

Proof. Clearly, the process V/B is adapted to $\mathbb{F}$ as a consequence of $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$, $B_t = b^t$ and

$$V_t = a^t S_0 \exp\left(\frac{t + \sum_{1 \leq k \leq t} X_k}{2} \log(u) + \frac{t - \sum_{1 \leq k \leq t} X_k}{2} \log(d)\right)$$

for all $t \in \{0, \dots, T\}$. Further, V/B is integrable with respect to every probability measure on $(\Omega, \mathcal{F})$, because $|V_t/B_t|$ is bounded by $a^t/b^t S_0 u^t$ for every $t \in \{0, \dots, T\}$. Therefore, we just need to find a probability measure $\mathbb{Q}$ so that V/B satisfies the martingale property in Definition 2.1 with respect to the filtration $\mathbb{F}$. As a consequence of the tower property of conditional expectation and $\mathcal{F}_t \subset \mathcal{F}_{t+1}$, the martingale property is equivalent to

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{V_{t+1}}{B_{t+1}} \middle| \mathcal{F}_t\right] = \frac{V_t}{B_t} \qquad \forall t \in \{0, \dots, T-1\}.$$

So we fix $t \in \{0, \dots, T-1\}$ and get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\frac{V_{t+1}}{B_{t+1}}\middle|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{a^{t+1}S_0 \exp\left(\frac{t+1+\sum_{k=1}^{t+1} X_k}{2}\log(u) + \frac{t+1-\sum_{k=1}^{t+1} X_k}{2}\log(d)\right)}{b^{t+1}}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{V_t}{B_t}\frac{a}{b}\exp\left(\frac{1+X_{t+1}}{2}\log(u) + \frac{1-X_{t+1}}{2}\log(d)\right)\middle|\mathcal{F}_t\right] \\ &\overset{!}{=} \frac{V_t}{B_t}.\end{aligned}$$

Now we can use the fact that V_t/B_t is $\mathcal{F}_t$-measurable to take it out of the conditional expectation and then divide the last equation by $V_t/B_t > 0$ to obtain

$$\frac{a}{b}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\frac{1+X_{t+1}}{2}\log(u) + \frac{1-X_{t+1}}{2}\log(d)\right)\middle|\mathcal{F}_t\right] \overset{!}{=} 1.$$

In the next step we divide the equation by a/b and use that X_{t+1} takes only two values to see

$$\begin{aligned}\frac{b}{a} &\overset{!}{=} \mathbb{Q}\left[X_{t+1} = 1|\mathcal{F}_t\right]u + \left(1 - \mathbb{Q}\left[X_{t+1} = 1|\mathcal{F}_t\right]\right)d \\ &= d + (u-d)\,\mathbb{Q}\left[X_{t+1} = 1|\mathcal{F}_t\right].\end{aligned}$$

Hence our desired equation holds if and only if

$$\mathbb{Q}\left[X_{t+1} = 1|\mathcal{F}_{t+1}\right](\omega) = \frac{\frac{b}{a} - d}{u-d} \qquad \mathbb{Q}\text{-a.s.},$$

which is equivalent to

$$\mathbb{Q}\left[X_{t+1} = 1\right] = \frac{\frac{b}{a} - d}{u-d} \qquad \text{and} \qquad X_{t+1} \perp\!\!\!\perp \mathcal{F}_t.$$

This defines a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, which corresponds to positive probabilities for up and down, if and only if, $d < b/a < u$.
As a consequence of $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$ and $t \in \{0, \dots, T-1\}$ chosen arbitrarily we further get that the random variables $(X_t)_{1 \le t \le T}$ are independent and identically distributed with

$$\mathbb{Q}\left[X_1 = 1\right] = \frac{\frac{b}{a} - d}{u-d} \qquad \text{and} \qquad \mathbb{Q}\left[X_1 = -1\right] = 1 - \mathbb{Q}\left[X_1 = 1\right] = \frac{u - \frac{b}{a}}{u-d},$$

which finishes the proof. □

The name binomial model is a consequence of the fact that the price of the stock at a given time can be understood as being determined by the outcome of a sum of i.i.d. Bernoulli random variables under the martingale measure $\mathbb{Q}$ and hence is connected in some sense to a binomial distribution and binomial coefficients.

In the remaining part of this thesis we will always assume that the conditions for a, b, u and d in the theorem above are satisfied and denote the unique martingale measure for our binomial model with $\mathbb{Q}$.

The next corollary deals with the "change of numéraire"-technique, where one changes the numéraire from the bond B to the value process V.

Corollary 2.7. *The process $M_t = B_0/V_0 \cdot V_t/B_t$ for $t \in \{0, \dots, T\}$ is a martingale with respect to the filtration $\mathbb{F}$ and the martingale measure $\mathbb{Q}$ and $d\mathbb{Q}'/d\mathbb{Q} := M_T$ defines a probability measure $\mathbb{Q}'$ on $(\Omega, \mathcal{F})$ which is equivalent to $\mathbb{Q}$. We call this measure the dual measure for our model.*

Proof. By Theorem 2.6, the process V_t/B_t is a martingale with respect to $\mathbb{F}$ and $\mathbb{Q}$ and still remains integrable and adapted after multiplication with the constant factor $B_0/V_0 = 1/S_0$. We fix $s < t$, $s, t \in \{0, \dots, T\}$. By taking the constant, hence, $\mathcal{F}_s$-measurable factor B_0/V_0 out of the conditional expectation and using the martingale property of V_t/B_t, we get

$$\mathbb{E}_{\mathbb{Q}}\left[M_t|\mathcal{F}_s\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{V_t}{B_t}\frac{B_0}{V_0}\middle|\mathcal{F}_s\right] = \frac{B_0}{V_0}\mathbb{E}_{\mathbb{Q}}\left[\frac{V_t}{B_t}\middle|\mathcal{F}_s\right] = \frac{B_0}{V_0}\frac{V_s}{B_s} = M_s,$$

which shows that M is a martingale.
Further, we have $V_T > 0, B_T > 0$, therefore, $M_T > 0$ $\mathbb{Q}$-a.s. and get by using the martingale property of M and the law of total expectation for the conditional expectation

$$\mathbb{E}_{\mathbb{Q}}\left[M_T\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[M_T|\mathcal{F}_0\right]\right] = \mathbb{E}_{\mathbb{Q}}\left[M_0\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{V_0}{B_0}\frac{B_0}{V_0}\right] = 1,$$

which implies that $d\mathbb{Q}'/d\mathbb{Q} = M_T$ defines a probability measure $\mathbb{Q}'$ which is equivalent to $\mathbb{Q}$. □

In the next chapter we continue with examples of binomial models and calculations of option prices.

3. Options and examples

In this chapter we introduce some types of options and the notation for their prices which will be used throughout the rest of this thesis. We also look at examples where we calculate option prices and then notice relations for option prices. Proofs that these relations hold in general can be found in Chapter 4.

3.1. Examples for vanilla options of European type

We start with one of the simplest and most popular types of options.

Definition 3.1. We denote the prices of vanilla call and put options of European type with maturity $T \in \mathbb{N}_0$ and strike price $K \in \mathbb{R}_+$ in a binomial model with dividend yield with parameters (S_0, a, b, u, d), as introduced in the previous chapter, and equivalent martingale measure $\mathbb{Q}$ the following way:

$$\mathbb{C}(S_0, K, a, b, u, d, T) := \frac{1}{b^T}\ \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+],$$
$$\mathbb{P}(S_0, K, a, b, u, d, T) := \frac{1}{b^T}\ \mathbb{E}_{\mathbb{Q}}[(K - S_T)^+].$$

Now let us look at an example.

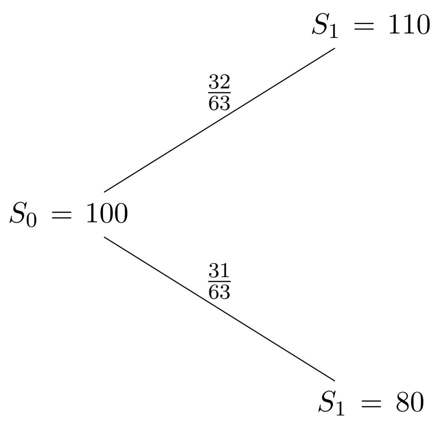
Example 3.2. At time zero our stock should be worth $S_0 = 100$. The stock price either increases by the factor $u = 11/10$ or decreases by the factor $d = 4/5$ and the riskless bond increases by the factor $b = 16/15$ in each time step. The factor which corresponds to reinvesting dividends (or changing currencies) should be $a = 28/25$. We want to determine the price of vanilla call and put options with strike price $K = 90$ and maturity $T = 1$. We start with the calculation of the equivalent martingale measure $\mathbb{Q}$. By Theorem 2.6 we get

$$\mathbb{Q}[X_1 = 1] = \frac{\frac{b}{a} - d}{u - d} = \frac{\frac{16}{15}\frac{25}{28} - \frac{4}{5}}{\frac{11}{10} - \frac{4}{5}} = \frac{\frac{20}{21} - \frac{4}{5}}{\frac{3}{10}} = \frac{32}{63}$$

and

$$\mathbb{Q}[X_1 = -1] = 1 - \mathbb{Q}[X_1 = 1] = \frac{31}{63}.$$

Hence, we get the following binomial tree for our stock price S.



We calculate the prices of the options and get

$$\begin{aligned}\mathbb{C}\left(100, 90, \frac{28}{25}, \frac{16}{15}, \frac{11}{10}, \frac{4}{5}, 1\right) &= \frac{15}{16}\ \mathbb{E}_{\mathbb{Q}}[(S_1 - 90)^+] \\ &= \frac{15}{16}\left(\frac{32}{63}(110-90)^+ + \frac{31}{63}(80-90)^+\right) \\ &= \frac{200}{21}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\left(100, 90, \frac{28}{25}, \frac{16}{15}, \frac{11}{10}, \frac{4}{5}, 1\right) &= \frac{15}{16}\ \mathbb{E}_{\mathbb{Q}}[(90 - S_1)^+] \\ &= \frac{15}{16}\left(\frac{32}{63}(90-110)^+ + \frac{31}{63}(90-80)^+\right) \\ &= \frac{775}{168}.\end{aligned}$$

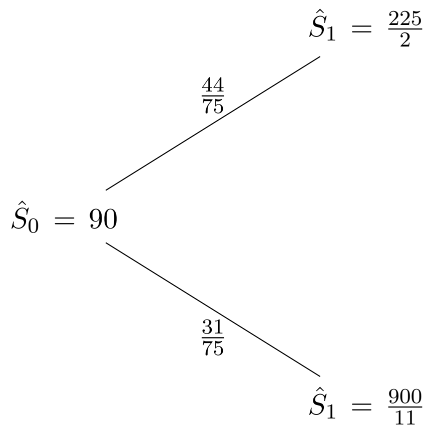
We also take a look at a second binomial model with parameters

$$(\hat{S}_0, \hat{a}, \hat{b}, \hat{u}, \hat{d}) = \left(K, b, a, \frac{1}{d}, \frac{1}{u}\right) = \left(90, \frac{16}{15}, \frac{28}{25}, \frac{5}{4}, \frac{10}{11}\right)$$

for the stock price $\hat{S}$ and calculate the equivalent martingale measure $\hat{\mathbb{Q}}$ for this model using Theorem 2.6

$$\begin{aligned}\hat{\mathbb{Q}}[\hat{X}_1 = 1] &= \frac{\frac{\hat{b}}{\hat{a}} - \hat{d}}{\hat{u} - \hat{d}} = \frac{\frac{28}{25}\frac{15}{16} - \frac{10}{11}}{\frac{5}{4} - \frac{10}{11}} = \frac{\frac{21}{20} - \frac{10}{11}}{\frac{15}{44}} = \frac{31}{75}, \\ \hat{\mathbb{Q}}[\hat{X}_1 = -1] &= 1 - \hat{\mathbb{Q}}[\hat{X}_1 = 1] = \frac{44}{75}.\end{aligned}$$

So we get the following tree for the stock price $\hat{S}$.



Again we calculate the prices for vanilla options of European type, but this time with strike price $\hat{K} = S_0 = 100$ and maturity $\hat{T} = T = 1$:

$$
\begin{aligned}
\mathbb{C}\left(90, 100, \frac{16}{15}, \frac{28}{25}, \frac{5}{4}, \frac{10}{11}, 1\right) &= \frac{25}{28}\,\mathbb{E}_{\hat{\mathbb{Q}}}[(\hat{S}_1 - 100)^+] \\
&= \frac{25}{28}\left(\frac{31}{75}\left(\frac{225}{2} - 100\right)^+ + \frac{44}{75}\left(\frac{900}{11} - 100\right)^+\right) \\
&= \frac{775}{168}, \\
\mathbb{P}\left(90, 100, \frac{16}{15}, \frac{28}{25}, \frac{5}{4}, \frac{10}{11}, 1\right) &= \frac{25}{28}\,\mathbb{E}_{\hat{\mathbb{Q}}}[(100 - \hat{S}_1)^+] \\
&= \frac{25}{28}\left(\frac{31}{75}\left(100 - \frac{225}{2}\right)^+ + \frac{44}{75}\left(100 - \frac{900}{11}\right)^+\right) \\
&= \frac{200}{21}.
\end{aligned}
$$

Hence, we obtain the duality relations

$$
\begin{aligned}
\mathbb{C}\left(100, 90, \frac{28}{25}, \frac{16}{15}, \frac{11}{10}, \frac{4}{5}, 1\right) &= \mathbb{P}\left(90, 100, \frac{16}{15}, \frac{28}{25}, \frac{5}{4}, \frac{10}{11}, 1\right), \\
\mathbb{P}\left(100, 90, \frac{28}{25}, \frac{16}{15}, \frac{11}{10}, \frac{4}{5}, 1\right) &= \mathbb{C}\left(90, 100, \frac{16}{15}, \frac{28}{25}, \frac{5}{4}, \frac{10}{11}, 1\right).
\end{aligned}
$$

The general result which corresponds to the relations in the example is Theorem 4.3.

3.2. Examples for forward start options of European type

In this section we want to give examples for forward start call and put options. Contrary to vanilla options the strike price of a forward start call or put option is the stock price at a fixed point in time before maturity and hence a random variable itself.

Definition 3.3. The payoff of a forward start call option with maturity $T \in \mathbb{N}_0$ and forward start time $t \in \{0, \ldots, T\}$ of European type is $(S_T - S_t)^+$ and the payoff of a forward start put option with maturity $T \in \mathbb{N}$ and forward start time $t \in \{0, \ldots, T\}$ of European type is $(S_t - S_T)^+$. In a binomial model with dividend yield with parameters (S_0, a, b, u, d) and equivalent martingale measure $\mathbb{Q}$ we denote the prices of these options by

$$\mathbb{C}_{fwd}(S_0, a, b, u, d, T, t) := \frac{1}{b^T} \, \mathbb{E}_{\mathbb{Q}}[(S_T - S_t)^+]$$

and

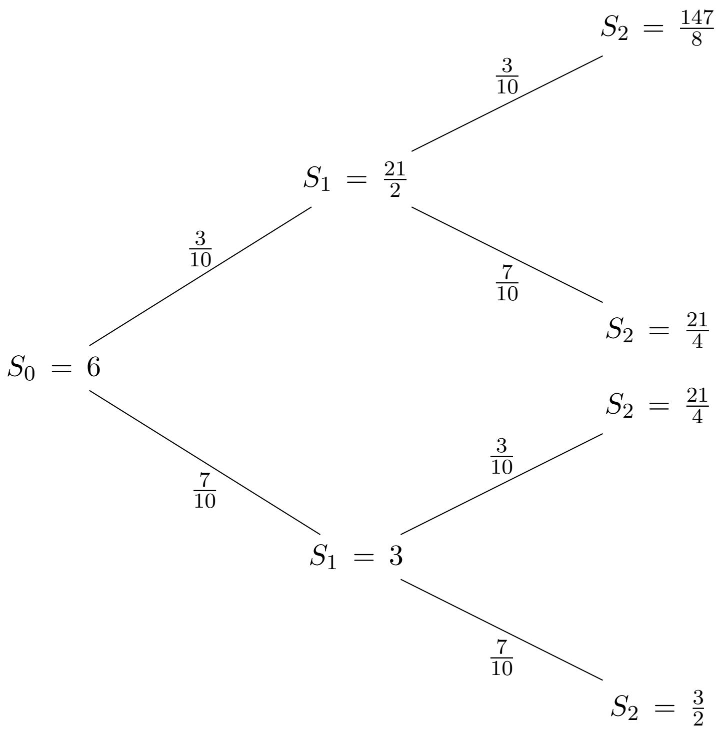
$$\mathbb{P}_{fwd}(S_0, a, b, u, d, T, t) := \frac{1}{b^T} \, \mathbb{E}_{\mathbb{Q}}[(S_t - S_T)^+].$$

Let us look at an example.

Example 3.4. We choose $S_0 = 6, a = 4/3, b = 7/6, u = 7/4, d = 1/2$ as parameters for our binomial model and we want to calculate the prices of forward start call and put options with maturity $T = 2$ and forward start time $t = 1$. We start with the calculation of the equivalent martingale measure $\mathbb{Q}$ and as a consequence of Theorem 2.6 obtain

$$\mathbb{Q}[X_1 = 1] = \frac{\frac{b}{a} - d}{u - d} = \frac{\frac{3}{4}\frac{7}{6} - \frac{1}{2}}{\frac{7}{4} - \frac{1}{2}} = \frac{\frac{7}{8} - \frac{1}{2}}{\frac{5}{4}} = \frac{3}{10},$$
$$\mathbb{Q}[X_1 = -1] = 1 - \mathbb{Q}[X_1 = 1] = \frac{7}{10}.$$

Hence, we get the following tree for the stock prices S.



Now we are able to calculate the prices of the options:

$$\begin{aligned}
\mathbb{C}_{fwd}\left(6, \frac{4}{3}, \frac{7}{6}, \frac{7}{4}, \frac{1}{2}, 2, 1\right) &= \left(\frac{6}{7}\right)^2 \mathbb{E}_{\mathbb{Q}}[(S_2 - S_1)^+] \\
&= \left(\frac{6}{7}\right)^2 \left[\left(\frac{3}{10}\right)^2 \left(\frac{147}{8} - \frac{21}{2}\right) + \frac{7}{10}\frac{3}{10}\left(\frac{21}{4} - 3\right)\right] \\
&= \left(\frac{3}{7}\right)^2 \left[\left(\frac{3}{5}\right)^2 \frac{63}{8} + \frac{7}{5}\frac{3}{5}\frac{9}{4}\right] \\
&= \frac{243}{280}, \\
\mathbb{P}_{fwd}\left(6, \frac{4}{3}, \frac{7}{6}, \frac{7}{4}, \frac{1}{2}, 2, 1\right) &= \left(\frac{6}{7}\right)^2 \mathbb{E}_{\mathbb{Q}}[(S_1 - S_2)^+] \\
&= \left(\frac{6}{7}\right)^2 \left[\frac{3}{10}\frac{7}{10}\left(\frac{21}{2} - \frac{21}{4}\right) + \left(\frac{7}{10}\right)^2 \left(3 - \frac{3}{2}\right)\right] \\
&= \left(\frac{3}{7}\right)^2 \left[\frac{3}{5}\frac{7}{5}\frac{21}{4} + \left(\frac{7}{5}\right)^2 \frac{3}{2}\right] \\
&= \frac{27}{20}.
\end{aligned}$$

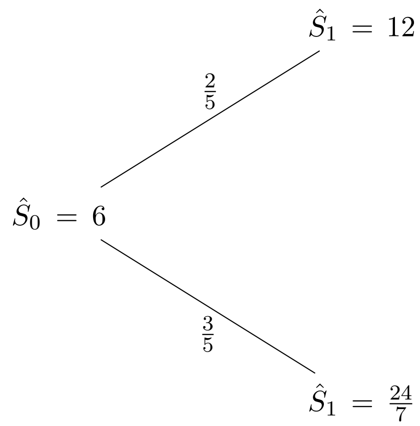
Let us take a quick look at a second binomial model with parameters

$$(\hat{S}_0, \hat{a}, \hat{b}, \hat{u}, \hat{d}) = \left(S_0, b, a, \frac{1}{d}, \frac{1}{u}\right) = \left(6, \frac{7}{6}, \frac{4}{3}, 2, \frac{4}{7}\right)$$

for the stock price $\hat{S}$. We want to calculate the prices for vanilla call and put options with strike price $\hat{K} = S_0 = 6$ and maturity $\hat{T} = T - t = 1$. Again we calculate the equivalent martingale measure $\hat{\mathbb{Q}}$ for this model by Theorem 2.6 and get

$$\begin{aligned}
\hat{\mathbb{Q}}[\hat{X}_1 = 1] &= \frac{\frac{\hat{b}}{\hat{a}} - \hat{d}}{\hat{u} - \hat{d}} = \frac{\frac{6}{7}\frac{4}{3} - \frac{4}{7}}{2 - \frac{4}{7}} = \frac{\frac{8}{7} - \frac{4}{7}}{\frac{10}{7}} = \frac{2}{5}, \\
\hat{\mathbb{Q}}[\hat{X}_1 = -1] &= 1 - \hat{\mathbb{Q}}[\hat{X}_1 = 1] = \frac{3}{5},
\end{aligned}$$

hence, obtain the following tree for the stock prices $\hat{S}$.



The prices of the options are

$$\begin{aligned}\mathbb{C}\left(6, 6, \frac{7}{6}, \frac{4}{3}, 2, \frac{4}{7}, 1\right) &= \frac{3}{4}\ \mathbb{E}_{\hat{\mathbb{Q}}}[(\hat{S}_1 - 6)^+] \\ &= \frac{3}{4}\frac{2}{5}(12 - 6) \\ &= \frac{9}{5}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\left(6, 6, \frac{7}{6}, \frac{4}{3}, 2, \frac{4}{7}, 1\right) &= \frac{3}{4}\ \mathbb{E}_{\hat{\mathbb{Q}}}[(6 - \hat{S}_1)^+] \\ &= \frac{3}{4}\frac{3}{5}\left(6 - \frac{24}{7}\right) \\ &= \frac{81}{70}.\end{aligned}$$

Therefore, we obtain the duality relations

$$\mathbb{C}_{fwd}\left(6, \frac{4}{3}, \frac{7}{6}, \frac{7}{4}, \frac{1}{2}, 2, 1\right) = \left(\frac{3}{4}\right)^1 \mathbb{P}\left(6, 6, \frac{7}{6}, \frac{4}{3}, 2, \frac{4}{7}, 2 - 1\right)$$

and

$$\mathbb{P}_{fwd}\left(6, \frac{4}{3}, \frac{7}{6}, \frac{7}{4}, \frac{1}{2}, 2, 1\right) = \left(\frac{3}{4}\right)^1 \mathbb{C}\left(6, 6, \frac{7}{6}, \frac{4}{3}, 2, \frac{4}{7}, 2 - 1\right).$$

The generalisation of this result is Theorem 4.5.

3.3. Examples for Asian and lookback options of European type

In this section we want to give examples for forward start Asian and lookback call and put options of European type. Both types of options have in common that there exist floating strike and fixed strike versions of them. We will start with Asian options.

Asian options are options which depend on the arithmetic mean of the stock price, which is a random variable itself.

Definition 3.5. Let $T \in \mathbb{N}_0$ and $t \in \{0, \dots, T\}$. Let $A(t,T) := 1/(T-t+1)\sum_{k=t}^{T} S_k$ denote the arithmetic mean of the stock prices from time t up to time T. The payoffs of floating strike, forward start Asian call and put options with maturity T and forward start time t of European type are $(S_T - A(t,T))^+$, respectively $(A(t,T) - S_T)^+$, and the payoffs of fixed strike, forward start Asian call and put options with maturity T, forward start time t and strike price $K \in \mathbb{R}_+$ of European type are $(A(t,T) - K)^+$, respectively $(K - A(t,T))^+$. In a binomial model with dividend yield with parameters (S_0, a, b, u, d) and equivalent martingale measure $\mathbb{Q}$ we denote the prices of these options as

$$\begin{aligned}
\mathbb{AC}_{float}(S_0, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(S_T - A(t,T)^+],\\
\mathbb{AP}_{float}(S_0, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(A(t,T) - S_T)^+],\\
\mathbb{AC}_{fix}(S_0, K, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(A(t,T) - K)^+],\\
\mathbb{AP}_{fix}(S_0, K, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(K - A(t,T))^+].
\end{aligned}$$

As a special case we denote the non-forward start versions of these options as

$$\begin{aligned}
\mathbb{AC}_{float}(S_0, a, b, u, d, T) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(S_T - A(0,T)^+],\\
\mathbb{AP}_{float}(S_0, a, b, u, d, T) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(A(0,T) - S_T)^+],\\
\mathbb{AC}_{fix}(S_0, K, a, b, u, d, T) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(A(0,T) - K)^+],\\
\mathbb{AP}_{fix}(S_0, K, a, b, u, d, T) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}[(K - A(0,T))^+].
\end{aligned}$$

There also exist definitions of Asian options which use the geometric or harmonic mean instead of the arithmetic mean, but we will focus on the options above, as they are the most popular.

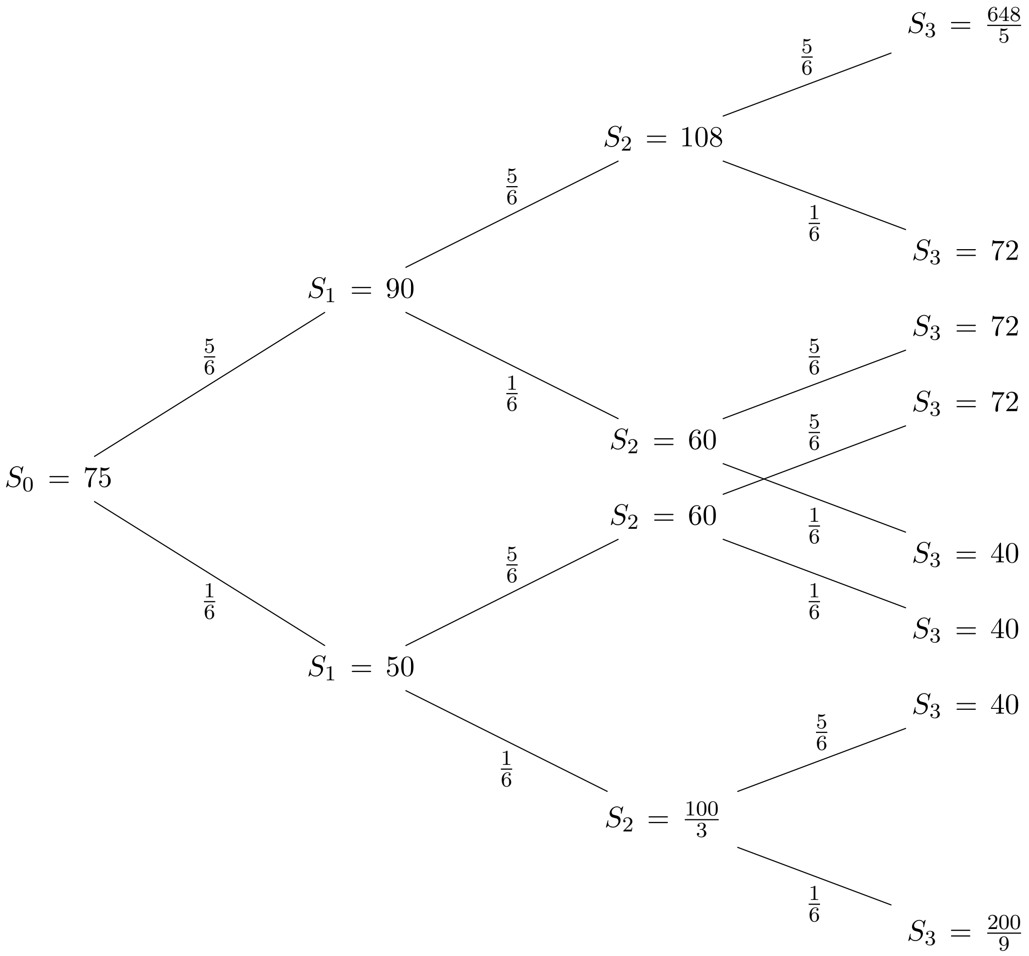
In general the prices of fixed strike Asian options are simpler to calculate, as they depend only on the arithmetic mean. Now let us look at an example.

Example 3.6. For the stock prices S we look at a binomial model with dividend yield with parameters $S_0 = 75, a = 9/8, b = 5/4, u = 6/5, d = 2/3$ and our aim is to calculate the option prices of floating strike, forward start Asian call and put options with maturity

$T = 3$ and forward start time $t = 1$. Therefore, first we have to calculate the equivalent martingale measure $\mathbb{Q}$ for this model using Theorem 2.6:

$$\mathbb{Q}[X_1 = 1] = \frac{\frac{b}{a} - d}{u - d} = \frac{\frac{8}{9}\frac{5}{4} - \frac{2}{3}}{\frac{6}{5} - \frac{2}{3}} = \frac{\frac{10}{9} - \frac{2}{3}}{\frac{8}{15}} = \frac{5}{6},$$
$$\mathbb{Q}[X_1 = -1] = 1 - \mathbb{Q}[X_1 = 1] = \frac{1}{6}.$$

Hence, we obtain the tree below.



Second, we have to calculate the values of the arithmetic mean of the stock prices from forward start time $t = 1$ to the time of maturity $T = 3$ and get

$$A(1,3) = \begin{cases} \frac{1}{3}\left(90+108+\frac{648}{5}\right) = \frac{546}{5} & \text{if } (X_1,X_2,X_3) = (1,1,1) \\ \frac{1}{3}\left(90+108+72\right) = 90 & \text{if } (X_1,X_2,X_3) = (1,1,-1) \\ \frac{1}{3}\left(90+60+72\right) = 74 & \text{if } (X_1,X_2,X_3) = (1,-1,1) \\ \frac{1}{3}\left(90+60+40\right) = \frac{190}{3} & \text{if } (X_1,X_2,X_3) = (1,-1,-1) \\ \frac{1}{3}\left(50+60+72\right) = \frac{182}{3} & \text{if } (X_1,X_2,X_3) = (-1,1,1) \\ \frac{1}{3}\left(50+60+40\right) = 50 & \text{if } (X_1,X_2,X_3) = (-1,1,-1) \\ \frac{1}{3}\left(50+\frac{100}{3}+40\right) = \frac{370}{9} & \text{if } (X_1,X_2,X_3) = (-1,-1,1) \\ \frac{1}{3}\left(50+\frac{100}{3}+\frac{200}{9}\right) = \frac{950}{27} & \text{if } (X_1,X_2,X_3) = (-1,-1,-1) \end{cases}.$$

Now we are able to calculate the options prices:

$$\begin{aligned}
&\mathbb{AC}_{float}\left(75,\frac{9}{8},\frac{5}{4},\frac{6}{5},\frac{2}{3},3,1\right) = \left(\frac{4}{5}\right)^3 \mathbb{E}_{\mathbb{Q}}[(S_3 - A(1,3))^+] \\
&= \left(\frac{4}{5}\right)^3 \left[\left(\frac{5}{6}\right)^3\left(\frac{648}{5}-\frac{546}{5}\right) + \frac{1}{6}\left(\frac{5}{6}\right)^2\left(72-\frac{182}{3}\right)\right] \\
&= \left(\frac{4}{5}\right)^3 \left[\left(\frac{5}{6}\right)^3\frac{102}{5} + \frac{1}{6}\left(\frac{5}{6}\right)^2\frac{34}{3}\right] \\
&= \frac{544}{81}, \\
&\mathbb{AP}_{float}\left(75,\frac{9}{8},\frac{5}{4},\frac{6}{5},\frac{2}{3},3,1\right) = \left(\frac{4}{5}\right)^3 \mathbb{E}_{\mathbb{Q}}[(A(1,3) - S_3)^+] \\
&= \left(\frac{4}{5}\right)^3 \left[\left(\frac{5}{6}\right)^2\frac{1}{6}(90-72) + \left(\frac{5}{6}\right)^2\frac{1}{6}(74-72) + \frac{5}{6}\left(\frac{1}{6}\right)^2\left(\frac{190}{3}-40\right)\right. \\
&\qquad\left. + \left(\frac{1}{6}\right)^2\frac{5}{6}(50-40) + \left(\frac{1}{6}\right)^2\frac{5}{6}\left(\frac{370}{9}-40\right) + \left(\frac{1}{6}\right)^3\left(\frac{950}{27}-\frac{200}{9}\right)\right] \\
&= \left(\frac{4}{5}\right)^3 \left[\left(\frac{5}{6}\right)^2 3 + \left(\frac{5}{6}\right)^2\frac{1}{3} + \frac{5}{6}\left(\frac{1}{6}\right)^2\frac{70}{3} + \left(\frac{1}{6}\right)^2\frac{25}{3} + \left(\frac{1}{6}\right)^2\frac{5}{6}\frac{10}{9} + \left(\frac{1}{6}\right)^3\frac{350}{27}\right] \\
&= \frac{1184}{27}.
\end{aligned}$$

If we look at a second binomial model for the stock prices $\hat{S}$ with parameters

$$(\hat{S}_0, \hat{a}, \hat{b}, \hat{u}, \hat{d}) = \left(S_0, b, a, \frac{1}{d}, \frac{1}{u}\right) = \left(75, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}\right)$$

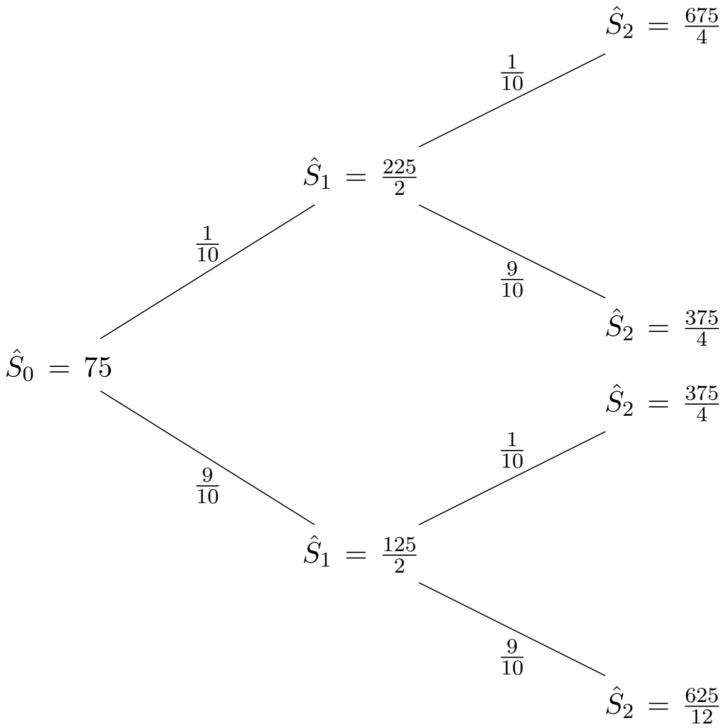
and equivalent martingale measure $\hat{\mathbb{Q}}$, then by Theorem 2.6 we get

$$\hat{\mathbb{Q}}[\hat{X}_1 = 1] = \frac{\frac{\hat{b}}{\hat{a}} - \hat{d}}{\hat{u} - \hat{d}} = \frac{\frac{4}{5}\frac{9}{8} - \frac{5}{6}}{\frac{3}{2} - \frac{5}{6}} = \frac{\frac{9}{10} - \frac{5}{6}}{\frac{2}{3}} = \frac{1}{10}$$

and

$$\hat{\mathbb{Q}}[\hat{X}_1 = -1] = 1 - \hat{\mathbb{Q}}[\hat{X}_1 = 1] = \frac{9}{10}.$$

As a consequence we obtain the following tree.



We want to know the prices for fixed strike, Asian call and put options of European type with strike price $\hat{K} = S_0 = 75$, maturity $\hat{T} = T - t = 2$ and, therefore, have to calculate the arithmetic mean of the stock prices $\hat{S}$ up to time $\hat{T} = 2$:

$$\hat{A}(0,2) = \begin{cases} \frac{1}{3}\left(75 + \frac{225}{2} + \frac{675}{4}\right) = \frac{475}{4} & \text{if } (\hat{X}_1, \hat{X}_2) = (1,1) \\ \frac{1}{3}\left(75 + \frac{225}{2} + \frac{375}{4}\right) = \frac{375}{4} & \text{if } (\hat{X}_1, \hat{X}_2) = (1,-1) \\ \frac{1}{3}\left(75 + \frac{125}{2} + \frac{375}{4}\right) = \frac{925}{12} & \text{if } (\hat{X}_1, \hat{X}_2) = (-1,1) \\ \frac{1}{3}\left(75 + \frac{125}{2} + \frac{675}{12}\right) = \frac{2275}{36} & \text{if } (\hat{X}_1, \hat{X}_2) = (-1,-1) \end{cases}.$$

Now we are able to compute the option prices:

$$
\begin{aligned}
&\mathbb{AC}_{fix}\left(75, 75, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}, 2\right) = \left(\frac{8}{9}\right)^2 \mathbb{E}_{\hat{\mathbb{Q}}}[(\hat{A}(0,2) - 75)^+] \\
&= \left(\frac{8}{9}\right)^2 \left[\left(\frac{1}{10}\right)^2 \left(\frac{475}{4} - 75\right) + \frac{1}{10}\frac{9}{10}\left(\frac{375}{9} - 75\right) + \frac{9}{10}\frac{1}{10}\left(\frac{925}{12} - 75\right)\right] \\
&= \left(\frac{8}{9}\right)^2 \left[\left(\frac{1}{10}\right)^2 \frac{175}{4} + \frac{1}{10}\frac{9}{10}\frac{75}{4} + \frac{9}{10}\frac{1}{10}\frac{25}{12})\right] \\
&= \frac{148}{81}, \\
&\mathbb{AP}_{fix}\left(75, 75, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}, 2\right) = \left(\frac{8}{9}\right)^2 \mathbb{E}_{\hat{\mathbb{Q}}}[(75 - \hat{A}(0,2))^+] \\
&= \left(\frac{8}{9}\right)^2 \left(\frac{9}{10}\right)^2 \left(75 - \frac{2275}{36}\right) \\
&= \left(\frac{8}{10}\right)^2 \frac{425}{36} \\
&= \frac{68}{9}.
\end{aligned}
$$

Hence, we obtain the duality relations

$$
\mathbb{AC}_{float}\left(75, \frac{9}{8}, \frac{5}{4}, \frac{6}{5}, \frac{2}{3}, 3, 1\right) = \left(\frac{8}{9}\right)^1 \mathbb{AP}_{fix}\left(75, 75, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}, 3 - 1\right)
$$

and

$$
\mathbb{AP}_{float}\left(75, \frac{9}{8}, \frac{5}{4}, \frac{6}{5}, \frac{2}{3}, 3, 1\right) = \left(\frac{8}{9}\right)^1 \mathbb{AC}_{fix}\left(75, 75, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}, 3 - 1\right).
$$

The general result which corresponds to this example is Theorem 4.7.

Now we will treat lookback options. These options depend on the supremum or infimum of the stock prices.

Definition 3.7. Let $T \in \mathbb{N}_0$ and $t \in \{0, \dots, T\}$. The payoffs of floating strike, forward start, supremum lookback call and put options with maturity T, forward start time t and scaling factor $\alpha \in \mathbb{R}_+$ of European type are $(S_T - \alpha \max_{t \le k \le T} S_k)^+$, respectively $(\alpha \max_{t \le k \le T} S_k - S_T)^+$, and the payoffs of fixed strike, forward start supremum lookback call and put options with maturity T, forward start time t and strike price $K \in \mathbb{R}_+$ of European type are $(\max_{t \le k \le T} S_k - K)^+$, respectively $(K - \max_{t \le k \le T} S_k)^+$. The payoffs of infimum lookback options are defined analogously. In a binomial model with dividend yield with parameters (S_0, a, b, u, d) and equivalent martingale measure $\mathbb{Q}$ we denote the prices of these options as

$$
\begin{aligned}
\overline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(S_T - \alpha \max_{t \le k \le T} S_k\right)^+\right], \\
\overline{\mathbb{LP}}_{float}(S_0, a, b, u, d, T, t, \alpha) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(\alpha \max_{t \le k \le T} S_k - S_T\right)^+\right], \\
\underline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(S_T - \alpha \min_{t \le k \le T} S_k\right)^+\right], \\
\underline{\mathbb{LP}}_{float}(S_0, a, b, u, d, T, t, \alpha) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(\alpha \min_{t \le k \le T} S_k - S_T\right)^+\right], \\
\overline{\mathbb{LC}}_{fix}(S_0, K, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(\max_{t \le k \le T} S_k - K\right)^+\right], \\
\overline{\mathbb{LP}}_{fix}(S_0, K, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(K - \max_{t \le k \le T} S_k\right)^+\right], \\
\underline{\mathbb{LC}}_{fix}(S_0, K, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(\min_{t \le k \le T} S_k - K\right)^+\right], \\
\underline{\mathbb{LP}}_{fix}(S_0, K, a, b, u, d, T, t) &:= \frac{1}{b^T}\, \mathbb{E}_{\mathbb{Q}}\left[\left(K - \min_{t \le k \le T} S_k\right)^+\right].
\end{aligned}
$$

As a special case we denote the non-forward start versions of these options without the parameter t, analogously to the notation of Asian options of European type (see Definition 3.5).

Let us again look at an example.

Example 3.8. We continue using the models from Example 3.6. This means that we have the parameters

$$
\begin{aligned}
(S_0, a, b, u, d) &= \left(75, \frac{9}{8}, \frac{5}{4}, \frac{6}{5}, \frac{2}{3}\right) \\
(\hat{S}_0, \hat{a}, \hat{b}, \hat{u}, \hat{d}) &= \left(S_0, b, a, \frac{1}{d}, \frac{1}{u}\right) = \left(75, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}\right)
\end{aligned}
$$

for the stock prices S and $\hat{S}$ and the corresponding equivalent martingale measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ with

$$
\begin{aligned}
&\mathbb{Q}[X_1 = 1] = \frac{5}{6},\ \mathbb{Q}[X_1 = -1] = \frac{1}{6}, \\
&\hat{\mathbb{Q}}[\hat{X}_1 = 1] = \frac{1}{10},\ \hat{\mathbb{Q}}[\hat{X}_1 = -1] = \frac{9}{10}.
\end{aligned}
$$

First, we want to calculate the price of a floating strike, forward start, supremum lookback call option with forward start time $t = 2$, maturity $T = 3$ and scaling factor $\alpha = 5/7$ on the stock prices S. We get

$$\begin{aligned}\overline{\mathbb{LC}}_{float}\left(75, \frac{9}{8}, \frac{5}{4}, \frac{6}{5}, \frac{2}{3}, 3, 2, \frac{5}{7}\right) &= \left(\frac{4}{5}\right)^3 \mathbb{E}_{\mathbb{Q}}\left[\left(S_3 - \frac{5}{7}\max_{2\le k\le 3} S_k\right)^+\right] \\ &= \left(\frac{4}{5}\right)^3 \left[\left(\frac{5}{6}\right)^3 \left(\frac{648}{5} - \frac{5}{7}\frac{648}{5}\right) + \left(\frac{5}{6}\right)^2 \frac{1}{6}\left(72 - \frac{5}{7}72\right)\right. \\ &\qquad \left. + \frac{1}{6}\left(\frac{5}{6}\right)^2\left(72 - \frac{5}{7}72\right) + \left(\frac{1}{6}\right)^2 \frac{5}{6}\left(40 - \frac{5}{7}40\right)\right] \\ &= \left(\frac{4}{5}\right)^3 \left[\left(\frac{5}{6}\right)^3 \frac{1296}{35} + 2\left(\frac{5}{6}\right)^2 \frac{1}{6}\frac{144}{7} + \left(\frac{1}{6}\right)^2 \frac{5}{6}\frac{80}{7}\right] \\ &= \frac{2560}{189}.\end{aligned}$$

Second, we compute the price of a fixed strike, supremum lookback call option with strike price $\hat{K} = S_0/\alpha = 105$ and maturity $\hat{T} = T - t = 1$ on the stock prices $\hat{S}$ and get

$$\begin{aligned}\overline{\mathbb{LP}}_{fix}\left(75, 105, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}, 1\right) &= \frac{8}{9}\,\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(105 - \max_{0\le k\le 1} \hat{S}_k\right)^+\right] \\ &= \frac{8}{9}\frac{9}{10}(105 - 75) \\ &= 24.\end{aligned}$$

Hence, we obtain the following relation

$$\frac{7}{5}\,\overline{\mathbb{LC}}_{float}\left(75, \frac{9}{8}, \frac{5}{4}, \frac{6}{5}, \frac{2}{3}, 3, 2, \frac{5}{7}\right) = \left(\frac{8}{9}\right)^2 \overline{\mathbb{LP}}_{fix}\left(75, 105, \frac{5}{4}, \frac{9}{8}, \frac{3}{2}, \frac{5}{6}, 3-2\right).$$

The general result which corresponds to the relation in the example is Theorem 4.9.

3.4. Examples for options of American type

In this section we want to look at examples of options of American type. Contrary to options of European type, options of American type can be exercised at any point in time up to the time of maturity. Hence, they are worth more than their European counterparts as one can exercise them at time of maturity too, but also at other points in time. An valid exercising strategy has to depend only on the information available up to the time of exercising, therefore, can be modelled with a stopping time.

We start with vanilla options of American type.

Definition 3.9. Let $T \in \mathbb{N}_0$. Let $\mathcal{T}_T$ denote the set of stopping times with respect to the filtration $\mathbb{F}$ that are bounded by time T. We denote the prices of vanilla call and put options of American type with maturity T and strike price $K \in \mathbb{R}_+$ in a binomial model with dividend yield with parameters (S_0, a, b, u, d) and equivalent martingale measure $\mathbb{Q}$ the following way:

$$\mathbb{c}(S_0, K, a, b, u, d, T) := \sup_{\tau \in \mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}(S_\tau - K)^+\right],$$
$$\mathbb{p}(S_0, K, a, b, u, d, T) := \sup_{\tau \in \mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}(K - S_T)^+\right].$$

Let us look at an example.

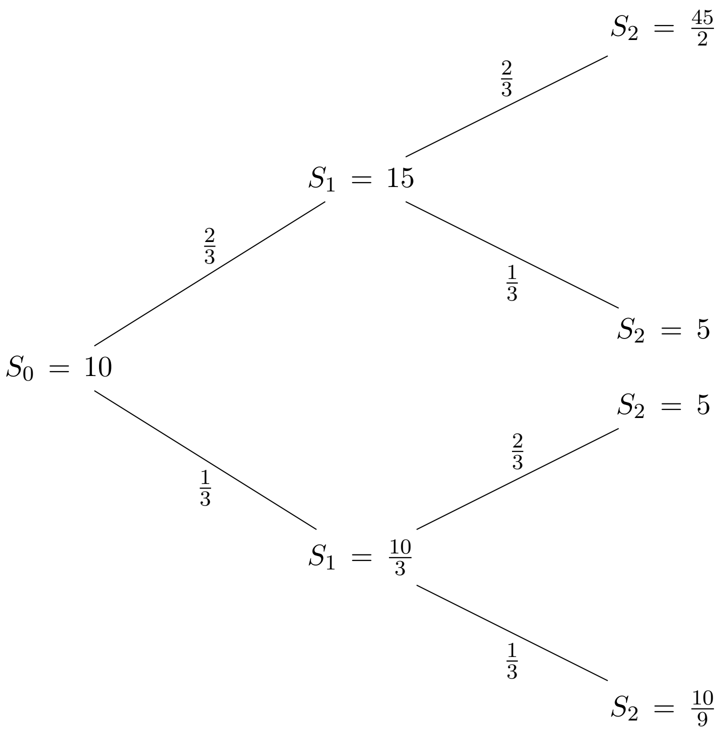
Example 3.10. We choose $S_0 = 10, a = 6/5, b = 4/3, u = 3/2, d = 1/3$ as parameters for the binomial model of the stock prices S and we want to calculate the prices of a vanilla put option of American type with strike price $K = 12$ and maturity $T = 2$. Therefore, we need to compute the equivalent martingale measure and using Theorem 2.6 get

$$\mathbb{Q}[X_1 = 1] = \frac{\frac{b}{a} - d}{u - d} = \frac{\frac{5}{6}\frac{4}{3} - \frac{1}{3}}{\frac{3}{2} - \frac{1}{3}} = \frac{\frac{10}{9} - \frac{1}{3}}{\frac{7}{6}} = \frac{2}{3}$$

and

$$\mathbb{Q}[X_1 = -1] = 1 - \mathbb{Q}[X_1 = 1] = \frac{1}{3}.$$

Hence, we obtain the following tree for the stock prices S.



It is easy to see (for a proof see Lemma 4.13) that there are only five stopping times,

$$\tau_1 \equiv 0, \quad \tau_2 \equiv 1, \quad \tau_3 \equiv 2,$$

which can be identified with exercising at the maturities $T = 0, T = 1$, or $T = 2$ (and, hence, options of European type), and

$$\tau_4 = \begin{cases} 1 & \text{if } X_1 = 1 \\ 2 & \text{else} \end{cases}, \quad \tau_5 = \begin{cases} 1 & \text{if } X_1 = -1 \\ 2 & \text{else} \end{cases}$$

in $\mathcal{T}_2$. We calculate

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{3}{4}\right)^{\tau_1}(12 - S_{\tau_1})^+\right] &= (12 - 10) = 2, \\
\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{3}{4}\right)^{\tau_2}(12 - S_{\tau_2})^+\right] &= \frac{1}{3}\frac{3}{4}\left(12 - \frac{10}{3}\right) = \frac{13}{6}, \\
\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{3}{4}\right)^{\tau_3}(12 - S_{\tau_3})^+\right] &= \frac{2}{3}\frac{1}{3}\left(\frac{3}{4}\right)^2(12 - 5) + \frac{1}{3}\frac{2}{3}\left(\frac{3}{4}\right)^2(12 - 5) \\
&\quad + \left(\frac{1}{3}\right)^2\left(\frac{3}{4}\right)^2\left(12 - \frac{10}{9}\right) = \frac{175}{72}, \\
\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{3}{4}\right)^{\tau_4}(12 - S_{\tau_4})^+\right] &= \frac{1}{3}\frac{2}{3}\left(\frac{3}{4}\right)^2(12 - 5) + \left(\frac{1}{3}\right)^2\left(\frac{3}{4}\right)^2\left(12 - \frac{10}{9}\right) = \frac{14}{9}, \\
\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{3}{4}\right)^{\tau_5}(12 - S_{\tau_5})^+\right] &= \frac{2}{3}\frac{1}{3}\left(\frac{3}{4}\right)^2(12 - 5) + \frac{1}{3}\frac{3}{4}\left(12 - \frac{10}{3}\right) = \frac{73}{24},
\end{aligned}$$

and obtain

$$\begin{aligned}
\mathbb{p}\left(10, 12, \frac{6}{5}, \frac{4}{3}, \frac{3}{2}, \frac{1}{3}, 2\right) &= \sup_{\tau \in \mathcal{T}_2} \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{3}{4}\right)^{\tau}(12 - S_\tau)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{3}{4}\right)^{\tau_5}(12 - S_{\tau_5})^+\right] \\
&= \frac{73}{24}.
\end{aligned}$$

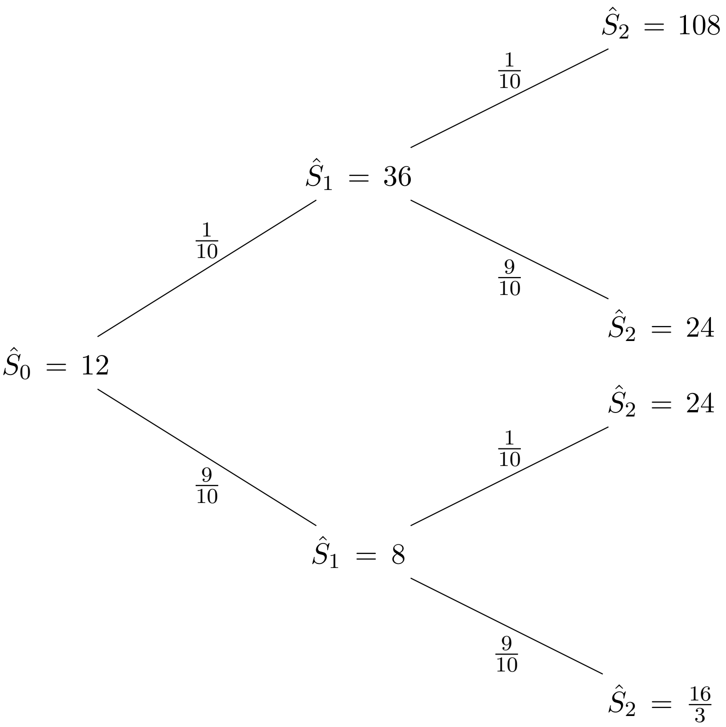
We also take a look at a second binomial model with parameters

$$(\hat{S}_0, \hat{a}, \hat{b}, \hat{u}, \hat{d}) = \left(K, b, a, \frac{1}{d}, \frac{1}{u}\right) = \left(12, \frac{4}{3}, \frac{6}{5}, 3, \frac{2}{3}\right)$$

for the stock price $\hat{S}$ and calculate the equivalent martingale measure $\hat{\mathbb{Q}}$ for this model using Theorem 2.6:

$$\hat{\mathbb{Q}}[\hat{X}_1 = 1] = \frac{\frac{\hat{b}}{\hat{a}} - \hat{d}}{\hat{u} - \hat{d}} = \frac{\frac{3}{4}\frac{6}{5} - \frac{2}{3}}{3 - \frac{2}{3}} = \frac{\frac{9}{10} - \frac{2}{3}}{\frac{7}{3}} = \frac{1}{10},$$
$$\hat{\mathbb{Q}}[\hat{X}_1 = -1] = 1 - \hat{\mathbb{Q}}[\hat{X}_1 = 1] = \frac{9}{10}.$$

So we get the following tree for the stock prices $\hat{S}$.



We want to calculate the price of a vanilla call option of American type with maturity $\hat{T} = T = 2$ and strike price $\hat{K} = S_0 = 10$. Analogously to above, there are only the stopping times

$$\hat{\tau}_1 \equiv 0, \quad \hat{\tau}_2 \equiv 1, \quad \hat{\tau}_3 \equiv 2,$$
$$\hat{\tau}_4 = \begin{cases} 1 & \text{if } \hat{X}_1 = 1 \\ 2 & \text{else} \end{cases}, \quad \hat{\tau}_5 = \begin{cases} 1 & \text{if } \hat{X}_1 = -1 \\ 2 & \text{else} \end{cases}$$

in $\hat{\mathcal{T}}_2$. We get

$$\begin{aligned}
\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\frac{5}{6}\right)^{\hat{\tau}_1}\left(\hat{S}_{\hat{\tau}_1}-10\right)^+\right] &= (12-10) = 2,\\
\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\frac{5}{6}\right)^{\hat{\tau}_2}\left(\hat{S}_{\hat{\tau}_2}-10\right)^+\right] &= \frac{1}{10}\frac{5}{6}(36-10) = \frac{13}{6},\\
\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\frac{5}{6}\right)^{\hat{\tau}_3}\left(\hat{S}_{\hat{\tau}_3}-10\right)^+\right] &= \left(\frac{1}{10}\right)^2\left(\frac{5}{6}\right)^2(108-10) + \frac{1}{10}\frac{9}{10}\left(\frac{5}{6}\right)^2(24-10)\\
&\quad + \frac{9}{10}\frac{1}{10}\left(\frac{5}{6}\right)^2(24-10) = \frac{175}{72},\\
\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\frac{5}{6}\right)^{\hat{\tau}_4}\left(\hat{S}_{\hat{\tau}_4}-10\right)^+\right] &= \frac{1}{10}\frac{5}{6}(36-10) + \frac{9}{10}\frac{1}{10}\left(\frac{5}{6}\right)^2(24-10) = \frac{73}{24},\\
\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\frac{5}{6}\right)^{\hat{\tau}_5}\left(\hat{S}_{\hat{\tau}_5}-10\right)^+\right] &= \left(\frac{1}{10}\right)^2\left(\frac{5}{6}\right)^2(108-10) + \frac{1}{10}\frac{9}{10}\left(\frac{5}{6}\right)^2(24-10) = \frac{14}{9}.
\end{aligned}$$

We should note that for every stopping time $\tau \in \mathcal{T}_2$ there is exactly one stopping time $\hat{\tau} \in \hat{\mathcal{T}}_2$ so that the expected payoffs of the corresponding exercise strategies of the options coincide. This is a consequence of Lemma 4.15. We further obtain

$$\begin{aligned}
\mathbb{c}\left(12, 10, \frac{4}{3}, \frac{6}{5}, 3, \frac{2}{3}, 2\right) &= \sup_{\hat{\tau}\in\hat{\mathcal{T}}_2} \mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\frac{5}{6}\right)^{\hat{\tau}}(\hat{S}_{\hat{\tau}})^+\right]\\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\frac{5}{6}\right)^{\hat{\tau}_4}\left(\hat{S}_{\hat{\tau}_4}-10\right)^+\right]\\
&= \frac{73}{24}.
\end{aligned}$$

and

$$\mathbb{p}\left(10, 12, \frac{6}{5}, \frac{4}{3}, \frac{3}{2}, \frac{1}{3}, 2\right) = \mathbb{c}\left(12, 10, \frac{4}{3}, \frac{6}{5}, 3, \frac{2}{3}, 2\right).$$

The general result corresponding to the relation above is Theorem 4.16.

The second type of options of American type with which we will deal are Asian Options of American type.

Definition 3.11. Let $T \in \mathbb{N}_0$ and $K \in \mathbb{R}_+$. In a binomial model with dividend yield with parameters (S_0, a, b, u, d) and equivalent martingale measure $\mathbb{Q}$ we denote the prices of floating strike Asian call and put options of American type with maturity T as

$$\mathbb{ac}_{float}(S_0,a,b,u,d,T) := \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}(S_\tau - A(0,\tau)^+\right],$$
$$\mathbb{ap}_{float}(S_0,a,b,u,d,T) := \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}(A(0,\tau) - S_T)^+\right]$$

and the prices of fixed strike Asian call and put options of American type with maturity T and strike price K as

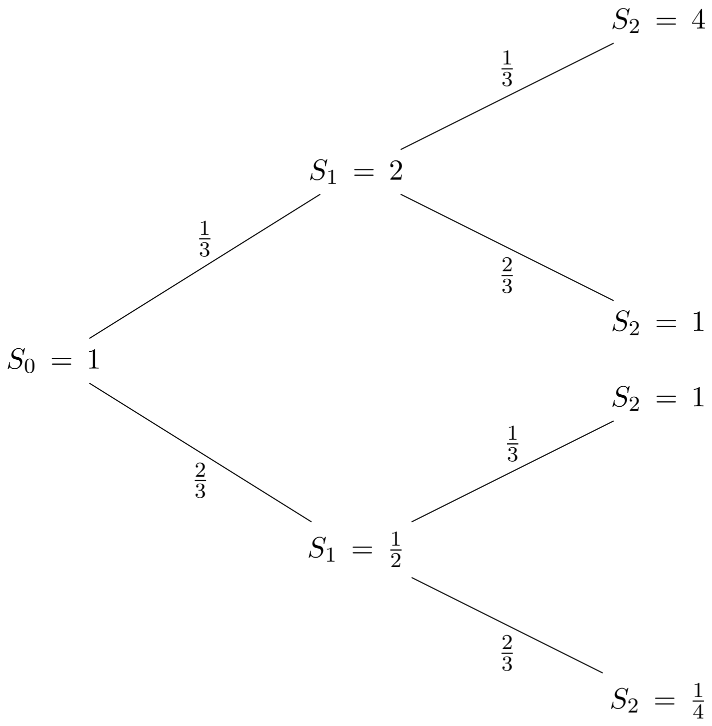
$$\mathbb{ac}_{fix}(S_0,K,a,b,u,d,T) := \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}(A(0,\tau) - K)^+\right],$$
$$\mathbb{ap}_{fix}(S_0,K,a,b,u,d,T) := \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}(K - A(0,\tau))^+\right].$$

Let us illustrate these options with the following example.

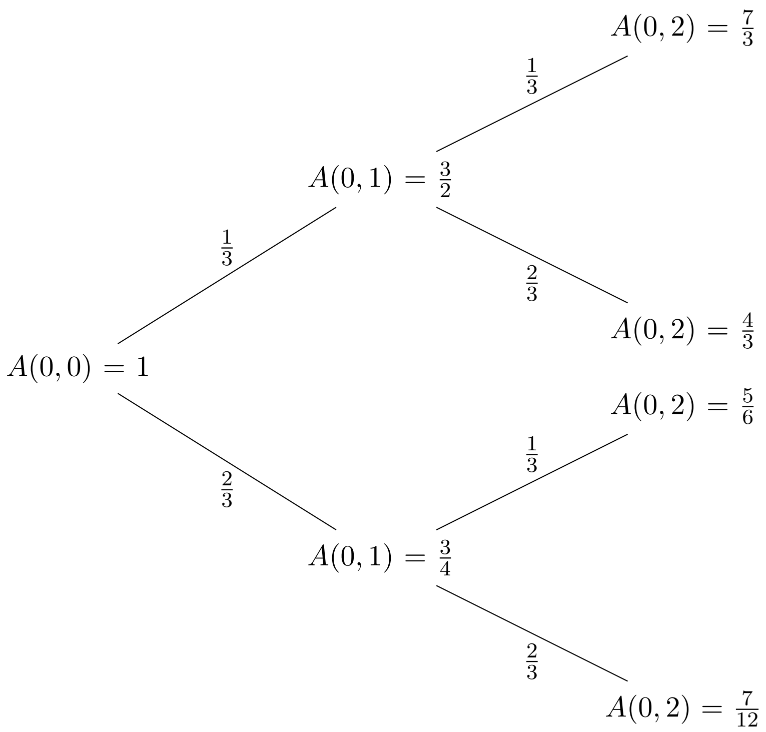
Example 3.12. For the stock prices S we look at a binomial model with parameters $S_0 = a = b = 1, u = 2, d = 1/2$ and hence by Theorem 2.6 get

$$\mathbb{Q}[X_1 = 1] = \frac{\frac{b}{a} - d}{u - d} = \frac{1 - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3},$$
$$\mathbb{Q}[X_1 = -1] = 1 - \mathbb{Q}[X_1 = 1] = \frac{2}{3}$$

for the equivalent martingale measure $\mathbb{Q}$. As a consequence, we have the following tree for the stock prices.



We want to calculate the price of a floating strike Asian put option of American type with maturity $T = 2$. Therefore, we have to calculate the arithmetic means and obtain the following tree.



Again, there are only the five stopping times, $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 of Example 3.10, in $\mathcal{T}_2$, because of Lemma 4.13. Hence, we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_1)-S_{\tau_1})^+\right] &= 0,\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_2)-S_{\tau_2})^+\right] &= \frac{2}{3}\left(\frac{3}{4}-\frac{1}{2}\right)=\frac{1}{6},\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_3)-S_{\tau_3})^+\right] &= \frac{1}{3}\frac{2}{3}\left(\frac{4}{3}-1\right)+\left(\frac{2}{3}\right)^2\left(\frac{7}{12}-\frac{1}{4}\right)=\frac{2}{9},\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_4)-S_{\tau_4})^+\right] &= \left(\frac{2}{3}\right)^2\left(\frac{7}{12}-\frac{1}{4}\right)=\frac{4}{27},\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_5)-S_{\tau_5})^+\right] &= \frac{1}{3}\frac{2}{3}\left(\frac{4}{3}-1\right)+\frac{2}{3}\left(\frac{3}{4}-\frac{1}{2}\right)=\frac{13}{54},
\end{aligned}$$

and obtain

$$\begin{aligned}
\mathrm{ap}_{float}\left(1,1,1,2,\frac{1}{2},2\right) &= \sup_{\tau\in\mathcal{T}_2}\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau)-S_\tau)^+\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_5)-S_{\tau_5})^+\right]\\
&= \frac{13}{54}.
\end{aligned}$$

Let us note that we have

$$(\hat{S}_0,\hat{a},\hat{b},\hat{u},\hat{d}) = \left(S_0,b,a,\frac{1}{d},\frac{1}{u}\right)=\left(1,1,1,2,\frac{1}{2}\right)=(S_0,a,b,u,d)$$

and hence the dual model can be identified with the original model. Now we look at the price of a fixed strike Asian call option with strike price $\hat{K}=S_0=1$ and maturity $\hat{T}=T=2$. We calculate

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_1)-1)^+\right] &= 0,\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_2)-1)^+\right] &= \frac{1}{3}\left(\frac{3}{2}-1\right)=\frac{1}{6},\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_3)-1)^+\right] &= \left(\frac{1}{3}\right)^2\left(\frac{7}{3}-1\right)+\frac{1}{3}\frac{2}{3}\left(\frac{4}{3}-1\right)=\frac{2}{9},\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_4)-1)^+\right] &= \frac{1}{3}\left(\frac{3}{2}-1\right)=\frac{1}{6},\\
\mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_5)-1)^+\right] &= \left(\frac{1}{3}\right)^2+\left(\frac{7}{3}-1\right)+\frac{1}{3}\frac{2}{3}\left(\frac{4}{3}-1\right)=\frac{2}{9}.
\end{aligned}$$

and hence get

$$\mathbb{ac}_{fix}\left(1,1,1,1,2,\frac{1}{2},2\right) = \sup_{\tau\in\mathcal{T}_2} \mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau)-1)^+\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_5)-1)^+\right]$$
$$= \frac{2}{9}.$$

We further obtain

$$\mathbb{AP}_{float}\left(1,1,1,2,\frac{1}{2},t\right) = \mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_t)-S_{\tau_t})^+\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[(A(0,\tau_t)-1)^+\right] = \mathbb{AC}_{fix}\left(1,1,1,1,2,\frac{1}{2},t\right),$$

for $t \in \{0,1,2\}$, which corresponds to Corollary 4.8, but also

$$\mathbb{ap}_{float}\left(1,1,1,2,\frac{1}{2},2\right) \neq \mathbb{ac}_{fix}\left(1,1,1,1,2,\frac{1}{2},2\right),$$

which shows that this result in general can not be extended to the case of Asian options of American type.

In Chapter 5 we try to find subsets of $\mathcal{T}_T$ so that the duality relations for Asian options of American type restricted to one of these subsets hold again.

4. Theoretical results for the binomial model with dividend yield

In this chapter we will show that the relations for vanilla options of European and American type, forward start options, Asian options of European type and lookback options illustrated in the examples of the previous chapter hold in general for binomial models with dividend yield.

In the whole chapter we look at an arbitrage free binomial model with dividend yield and parameters $(a, b, u, d) \in (\mathbb{R}_+ \setminus \{0\})^4$ and $T \in \mathbb{N}_0$ as described in Section 2.2. We denote the stock price for $t \in \{0, \ldots, T\}$ by $S_t = S_0 \exp((t + Y_t)/2 \cdot \log(u) + (t - Y_t)/2 \cdot \log(d))$ with $S_0 > 0$ and $Y_t = \sum_{k=1}^{t} X_k$, where the random variables X_k take values in $\{-1, 1\}$ for $1 \le k \le T$. Further, we look for $t \in \{0, \ldots, T\}$ at a riskless bond $B_t = b^t$ and at the value process $V_t = a^t S_t$ of a portfolio which reinvests the paid dividends in the stock. By $\mathbb{Q}$ we denote the unique martingale measure for our model with respect to the filtration $\mathbb{F}$ consisting of the sigma algebras $\mathcal{F}_t = \sigma(X_1, \ldots, X_t)$ for $0 \le t \le T$ and by $\mathbb{Q}'$ the dual measure.

We also should have a second binomial model with parameters $(b, a, 1/d, 1/u)$ in our minds. We denote the corresponding processes by $\hat{S}, \hat{Y}, \hat{X}, \hat{B}$ and $\hat{V}$ and the unique martingale measure for this model by $\hat{\mathbb{Q}}$. The starting value $\hat{S}_0$ depends on which relation we want to show but either will be equal to S_0 or $K > 0$.

4.1. Vanilla options of European type

Before we treat explicit relations we have to show distributional properties of the random variables $(X_t)_{1 \le t \le T}$ under the dual measure $\mathbb{Q}'$ with the density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T$.

Lemma 4.1. *Define the dual measure $\mathbb{Q}'$ via $d\mathbb{Q}'/d\mathbb{Q} := B_0/V_0 \cdot V_T/B_T$. Then $(X_t)_{1 \le t \le T}$ are i.i.d. under $\mathbb{Q}'$ and hence especially exchangeable under $\mathbb{Q}'$, i.e. for every permutation $\pi : \{1, \ldots, T\} \to \{1, \ldots, T\}$ we have*

$$Law(X_t; 1 \le t \le T \mid \mathbb{Q}') = Law(X_{\pi(t)}; 1 \le t \le T \mid \mathbb{Q}').$$

Proof. First, we note that $\mathbb{Q}'$ is a well defined measure as shown in Corollary 2.7. Second we fix $r_t \in \mathbb{R}$ for $1 \le t \le T$ and begin to calculate the characteristic function of $(X_t)_{1 \le t \le T}$ under $\mathbb{Q}'$:

$$\begin{aligned}\varphi_{X_1,\dots,X_T}(r_1,\dots,r_T) &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(i\sum_{t=1}^{T} r_t X_t\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(i\sum_{t=1}^{T} r_t X_t\right)\frac{B_0}{V_0}\frac{V_T}{B_T}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(i\sum_{t=1}^{T} r_t X_t\right)\frac{a^T S_T}{b^T S_0}\right].\end{aligned}$$

Now we use the identities $S_T = S_0 \exp((T+Y_T)/2\cdot\log(u)+(T-Y_T)/2\cdot\log(d))$, $Y_T = \sum_{t=1}^{T} X_t$ to further get

$$\begin{aligned}\varphi_{X_1,\dots,X_T}(r_1,\dots,r_T) &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(i\sum_{t=1}^{T} r_t X_t\right)\frac{a^T\exp\left(\frac{T+Y_T}{2}\log(u)+\frac{T-Y_T}{2}\log(d)\right)}{b^T}\right] \\ = \mathbb{E}_{\mathbb{Q}}&\left[\exp\left(i\sum_{t=1}^{T} r_t X_t\right)\frac{a^T}{b^T}\exp\left(\frac{T+\sum_{t=1}^{T} X_t}{2}\log(u)+\frac{T-\sum_{t=1}^{T} X_t}{2}\log(d)\right)\right] \\ = \mathbb{E}_{\mathbb{Q}}&\left[\prod_{t=1}^{T}\left(\mathrm{e}^{ir_tX_t}\frac{a}{b}\exp\left(\frac{1+X_t}{2}\log(u)+\frac{1-X_t}{2}\log(d)\right)\right)\right].\end{aligned}$$

We continue with using the fact that $(X_t)_{1\le t\le T}$ are i.i.d. under $\mathbb{Q}$ by Theorem 2.6 and the martingale property of $B_0/V_0\cdot V_T/B_T$ under $\mathbb{Q}$ (see Corollary 2.7) to obtain

$$\begin{aligned}\varphi_{X_1,\dots,X_T}(r_1,\dots,r_T) &= \prod_{t=1}^{T}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{ir_tX_t}\frac{a}{b}\exp\left(\frac{1+X_t}{2}\log(u)+\frac{1-X_t}{2}\log(d)\right)\right] \\ &= \prod_{t=1}^{T}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{ir_tX_1}\frac{a}{b}\exp\left(\frac{1+X_1}{2}\log(u)+\frac{1-X_1}{2}\log(d)\right)\right] \\ &= \prod_{t=1}^{T}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{ir_tX_1}\frac{B_0}{V_0}\frac{V_1}{B_1}\right] \\ &= \prod_{t=1}^{T}\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{ir_tX_1}\mathbb{E}_{\mathbb{Q}}\left[\frac{B_0}{V_0}\frac{V_T}{B_T}\middle|\mathcal{F}_1\right]\right].\end{aligned}$$

Because X_1 is $\mathcal{F}_1 = \sigma(X_1)$ measurable per definition we can pull the first factor into the conditional expectation and then as a consequence of the law of total expectation for conditional expectation conclude

$$\begin{aligned}
\varphi_{X_1,\ldots,X_T}(r_1,\ldots,r_T) &= \prod_{t=1}^{T} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{ir_tX_1}\frac{B_0}{V_0}\frac{V_T}{B_T}\middle|\mathcal{F}_1\right]\right] \\
&= \prod_{t=1}^{T} \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{ir_tX_1}\frac{B_0}{V_0}\frac{V_T}{B_T}\right] \\
&= \prod_{t=1}^{T} \mathbb{E}_{\mathbb{Q}'}\left[\mathrm{e}^{ir_tX_1}\right] \\
&= \prod_{t=1}^{T} \varphi_{X_1}(r_t),
\end{aligned}$$

which proofs our first claim.

In general, i.i.d. random variables are exchangeable, which can be easily seen by fixing a permutation $\pi : \{1,\ldots,T\} \to \{1,\ldots,T\}$ to observe

$$\varphi_{X_1,\ldots,X_T}(r_1,\ldots,r_T) = \prod_{t=1}^{T} \varphi_{X_1}(r_t) = \varphi_{X_{\pi(1)},\ldots,X_{\pi(T)}}(r_1,\ldots,r_T),$$

which implies exchangeability. □

With the next lemma we take a closer look at the relations between our two binomial models.

Lemma 4.2. *The processes $X, Y, \hat{X}$ and $\hat{Y}$ satisfy the following two relations:*

1. *$Law(X_t; 1 \le t \le T \mid \mathbb{Q}') = Law(-\hat{X}_t; 1 \le t \le T \mid \hat{\mathbb{Q}})$,*
2. *$Law(Y_t; 0 \le t \le T \mid \mathbb{Q}') = Law(-\hat{Y}_t; 0 \le t \le T \mid \hat{\mathbb{Q}})$.*

Proof. We know from Lemma 4.1 and Theorem 2.6 that $(X_t)_{1\le t\le T}$ are i.i.d. under $\mathbb{Q}'$ and $(\hat{X}_t)_{1\le t\le T}$ are i.i.d. under $\hat{\mathbb{Q}}$. Therefore, it suffices to focus on the marginal distributions of X_T and $\hat{X}_T$. We start with using the density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T$, the definitions of V_T, B_T, S_T and Y_T and the fact that $(X_t)_{1\le t\le T}$ are i.i.d. under $\mathbb{Q}$ by Theorem 2.6 to get

$$
\begin{aligned}
\mathbb{Q}'[X_T = 1] &= \mathbb{E}_{\mathbb{Q}'}\left[\mathbb{1}_{\{X_T=1\}}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_T=1\}} \frac{B_0}{V_0}\frac{V_T}{B_T}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_T=1\}} \frac{a^T S_T}{b^T S_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_T=1\}} \frac{a^T S_0 \exp\left(\frac{T+Y_T}{2}\log(u) + \frac{T-Y_T}{2}\log(d)\right)}{b^T S_0}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_T=1\}} \frac{a^T}{b^T} \exp\left(\frac{T+\sum_{t=1}^T X_t}{2}\log(u) + \frac{T-\sum_{t=1}^T X_t}{2}\log(d)\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_T=1\}} \frac{a}{b} \exp\left(\frac{1+X_T}{2}\log(u) + \frac{1-X_T}{2}\log(d)\right)\right] \\
&\quad \cdot \mathbb{E}_{\mathbb{Q}}\left[\frac{a^{T-1}}{b^{T-1}} \exp\left(\frac{T-1+\sum_{t=1}^{T-1} X_t}{2}\log(u) + \frac{T-1-\sum_{t=1}^{T-1} X_t}{2}\log(d)\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{X_T=1\}} \frac{a}{b}u\right] \cdot \mathbb{E}_{\mathbb{Q}}\left[\frac{B_0}{V_0}\frac{V_{T-1}}{B_{T-1}}\right] \\
&= \frac{a}{b}u \cdot \mathbb{Q}\left[X_T = 1\right] \cdot \mathbb{E}_{\mathbb{Q}}\left[\frac{B_0}{V_0}\frac{V_{T-1}}{B_{T-1}}\right].
\end{aligned}
$$

It follows from Corollary 2.7 that $B_0/V_0 \cdot V_T/B_T$ is a martingale with constant expectation equal to one under the martingale measure $\mathbb{Q}$ and by Theorem 2.6 we have

$$
\begin{aligned}
\mathbb{Q}'[X_T = 1] &= \frac{a}{b}u\, \mathbb{Q}\left[X_T = 1\right] \\
&= \frac{a}{b}u\, \frac{\frac{b}{a} - d}{u - d} \\
&= u\frac{1 - \frac{a}{b}d}{u-d} \\
&= \frac{ud}{ud}\, \frac{\frac{1}{d} - \frac{a}{b}}{\frac{1}{d} - \frac{1}{u}} \\
&= \hat{\mathbb{Q}}[\hat{X}_T = -1] \\
&= \hat{\mathbb{Q}}[-\hat{X}_T = 1].
\end{aligned}
$$

Since X_T and $\hat{X}_T$ only take values in $\{-1, 1\}$ this equation proofs the first relation.

To show the second relation, we fix $r_t \in \mathbb{R}$ for $0 \leq t \leq T$ and look at the characteristic function of $(Y_0, ..., Y_T)$:

$$\begin{aligned}\varphi_{Y_0,\dots,Y_T}(r_0,\dots,r_T) &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(i\sum_{t=0}^{T} r_t Y_t\right)\right]\\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(i\sum_{t=0}^{T} r_t \sum_{k=1}^{t} X_k\right)\right].\end{aligned}$$

In the next step we can use the first relation to get

$$\begin{aligned}\varphi_{Y_0,\dots,Y_T}(r_0,\dots,r_T) &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(i\sum_{t=0}^{T} r_t \sum_{k=1}^{t} -\hat{X}_k\right)\right]\\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(i\sum_{t=0}^{T} r_t(-\hat{Y}_t)\right)\right]\\ &= \varphi_{-\hat{Y}_0,\dots,-\hat{Y}_T}(r_0\dots,r_T),\end{aligned}$$

which finishes the proof. □

Now we have all prerequisites to start with the proof of the duality relation between vanilla put and call options of European type demonstrated in Example 3.2.

Theorem 4.3. *For vanilla call and put options of European type the option prices in a binomial model with dividend yield satisfy the relations*

$$\mathbb{C}(S_0,K,a,b,u,d,T) = \mathbb{P}(K,S_0,b,a,1/d,1/u,T) \tag{4.1}$$

and

$$\mathbb{P}(S_0,K,a,b,u,d,T) = \mathbb{C}(K,S_0,b,a,1/d,1/u,T). \tag{4.2}$$

Proof. We start with the price of a vanilla call option of European type and get

$$\begin{aligned}\mathbb{C}(S_0,K,a,b,u,d,T) &= \frac{1}{b^T}\,\mathbb{E}_{\mathbb{Q}}\left[(S_T-K)^+\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{S_0}\frac{a^T S_T}{b^T}\frac{1}{a^T}(S_0-K\frac{S_0}{S_T})^+\right].\end{aligned}$$

Now we use the identities $V_t = a^t S_t$ and $B_t = b^t$ in a first step, perform a change of numeraire from B to V, which corresponds to a change of measure from our original pricing measure $\mathbb{Q}$ to the dual measure $\mathbb{Q}'$ with density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T$ in a second step and then rewrite everything in terms of Y_T to obtain

$$\begin{aligned}\mathbb{C}(S_0,K,a,b,u,d,T) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{B_0}{V_0}\frac{V_T}{B_T}\frac{1}{a^T}(S_0-K\frac{S_0}{S_T})^+\right]\\ &= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}(S_0-K\frac{S_0}{S_T})^+\right]\\ &= \frac{1}{a^T}\,\mathbb{E}_{\mathbb{Q}'}\left[\left(S_0-K\,\exp\left(-\frac{T+Y_T}{2}\log(u)-\frac{T-Y_T}{2}\log(d)\right)\right)^+\right].\end{aligned}$$

Further, we use the relation $-\log(x) = \log(1/x)$, apply Lemma 4.2 to replace $\mathbb{Q}'$ with $\hat{\mathbb{Q}}$ and Y_T with $-\hat{Y}_T$ and rewrite the formula in terms of $\hat{S}_T$ to get Equation 4.1:

$$\begin{aligned}
&\mathbb{C}(S_0, K, a, b, u, d, T) = \\
&= \frac{1}{a^T}\,\mathbb{E}_{\mathbb{Q}'}\left[\left(S_0 - K\ \exp\left(\frac{T+Y_T}{2}\log\left(\frac{1}{u}\right) + \frac{T-Y_T}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right] \\
&= \frac{1}{a^T}\,\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(S_0 - K\ \exp\left(\frac{T-\hat{Y}_T}{2}\log\left(\frac{1}{u}\right) + \frac{T+\hat{Y}_T}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right] \\
&= \frac{1}{a^T}\,\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(S_0 - \hat{S}_T\right)^+\right] = \mathbb{P}(K, S_0, b, a, 1/d, 1/u, T).
\end{aligned}$$

The proof of Equation 4.2 follows analogously. □

4.2. Forward start options of European type

In this section we will treat the duality relations for forward start options illustrated in Example 3.4. To prove the relations we first have to show a "time reversal property" for the random walk $(Y_t)_{0\leq t\leq T}$.

Lemma 4.4. *The random walk $(Y_t)_{0\leq t\leq T}$ satisfies the following "time reversal" property*

$$\forall\ 0 \leq t \leq T : Law(Y_T - Y_t \mid \mathbb{Q}') = Law(Y_{T-t} \mid \mathbb{Q}'). \tag{4.3}$$

Proof. First we fix a $t \in \{0, \dots, T\}$ and a $r \in \mathbb{R}$ and look at the characteristic function of $Y_T - Y_t$ under $\mathbb{Q}'$

$$\begin{aligned}
\varphi_{Y_T - Y_t}(r) &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(ir(Y_T - Y_t)\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(ir\left(\sum_{k=1}^{T} X_k - \sum_{k=1}^{t} X_k\right)\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(ir\sum_{k=t+1}^{T} X_k\right)\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{k=t+1}^{T} irX_k\right)\right] \\
&= \varphi_{X_{t+1},\dots,X_T}(r, \dots, r).
\end{aligned}$$

From Lemma 4.1 we know that the random variables $(X_i)_{1\leq i\leq T}$ are exchangeable under the dual measure $\mathbb{Q}'$ and, therefore, the distributions of $(X_{t+1}, \dots, X_T)$ and $(X_1, \dots, X_{T-t})$

coincide. Hence, by using this relation we get

$$\begin{aligned}\varphi_{Y_T-Y_t}(r) &= \varphi_{X_1,\dots,X_{T-t}}(r,\dots,r)\\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(ir\sum_{k=1}^{T-t}X_k\right)\right]\\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(irY_{T-t}\right)\right]\\ &= \varphi_{Y_{T-t}}(r),\end{aligned}$$

which finishes the proof.

□

Now we are able to give the proof of the following theorem, which relates the prices of forward start call and put options to the prices of vanilla call and put options of European type.

Theorem 4.5. *The prices of forward start call and put options with forward start time $t \leq T$ in a binomial model with dividend yield satisfy the following two relations:*

$$\mathbb{C}_{fwd}(S_0,a,b,u,d,T,t) = \frac{1}{a^t}\ \mathbb{P}(S_0,S_0,b,a,1/d,1/u,T-t) \tag{4.4}$$

$$\mathbb{P}_{fwd}(S_0,a,b,u,d,T,t) = \frac{1}{a^t}\ \mathbb{C}(S_0,S_0,b,a,1/d,1/u,T-t). \tag{4.5}$$

Proof. We fix $t \in \{0,\dots,T\}$ and start with the price of a forward start call option to get

$$\begin{aligned}\mathbb{C}_{fwd}(S_0,a,b,u,d,T,t) &= \frac{1}{b^T}\ \mathbb{E}_{\mathbb{Q}}[(S_T-S_t)^+]\\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^T}\frac{a^TS_T}{S_0}\frac{S_0}{a^TS_T}(S_T-S_t)^+\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{a^TS_T}{b^TS_0}\frac{1}{a^T}\left(S_0-S_0\frac{S_t}{S_T}\right)^+\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{B_0}{V_0}\frac{V_T}{B_T}\frac{1}{a^T}\left(S_0-S_0\frac{S_t}{S_T}\right)^+\right]\end{aligned}$$

Now we can perform a change to the dual measure $\mathbb{Q}'$ with density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T$ and use the definition $S_t = S_0\exp((t+Y_t)/2\cdot\log(u) + (t-Y_t)/2\cdot\log(d))$ to obtain

$$\mathbb{C}_{fwd}(S_0,a,b,u,d,T,t) = \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{S_t}{S_T}\right)^+\right]$$
$$= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{S_0\exp\left(\frac{t+Y_t}{2}\log(u)+\frac{t-Y_t}{2}\log(d)\right)}{S_0\exp\left(\frac{T+Y_T}{2}\log(u)+\frac{T-Y_T}{2}\log(d)\right)}\right)^+\right]$$
$$= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\exp\left(-\frac{T-t+(Y_t-Y_t)}{2}\log(u) - \frac{T-t-(Y_T-Y_t)}{2}\log(d)\right)\right)^+\right].$$

We continue with using the identity $-\log(x) = \log(1/x)$ and substituting Y_{T-t} for $Y_T - Y_t$ as a consequence of Lemma 4.4:

$$\mathbb{C}_{fwd}(S_0,a,b,u,d,T,t) =$$
$$= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\exp\left(\frac{T-t+Y_{T-t}}{2}\log\left(\frac{1}{u}\right) - \frac{T-t-Y_{T-t}}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right].$$

Due to Lemma 4.2 we can replace $\mathbb{Q}'$ with $\hat{\mathbb{Q}}$ and Y_{T-t} with $-\hat{Y}_{T-t}$ in the first step, rewrite the formula in terms of $\hat{S}_{T-t}$ in the second step and conclude

$$\mathbb{C}_{fwd}(S_0,a,b,u,d,T,t) =$$
$$= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^T}\left(S_0 - S_0\exp\left(\frac{T-t-\hat{Y}_{T-t}}{2}\log\left(\frac{1}{u}\right) - \frac{T-t+\hat{Y}_{T-t}}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right]$$
$$= \frac{1}{a^t}\,\mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{T-t}}\left(S_0 - S_{T-t}\right)^+\right] = \frac{1}{a^t}\,\mathbb{P}(S_0,S_0,b,a,1/d,1/u,T-t).$$

Again Formula 4.5 follows analogously. □

These relations are interesting, because they tell us that instead of calculating the expected value of two dependent random variables in a binomial model with T time steps it suffices to calculate the expected value of a single random variable in a smaller binomial model with just $T-t$ time steps.

4.3. Asian options of European type

As already mentioned in Section 3.3 there are two types of Asian options we are dealing with, namely floating strike call and put options and fixed strike call and put options. The aim of this section is to give a proof for the relations between this types which were pictured in Example 3.6. A proof for corresponding relations in a Black-Scholes setting can be found in the paper by Vanmaele, Deelstra, Liinev and Dhaene [10] for a discrete average and in the paper by Henderson and Wojakowski [6] for a continuous average.

Before we can start to prove the relations, we again have to show a distributional property of our random walk first.

Lemma 4.6. *For every $t \in \{0, \dots, T\}$ the random walk $(Y_k)_{t \le k \le T}$ satisfies the following "time reversal" property*

$$Law(Y_T - Y_k; t \le k \le T \mid \mathbb{Q}') = Law(Y_{T-k}; t \le k \le T \mid \mathbb{Q}'). \tag{4.6}$$

Proof. First, we fix $t \in \{0, \dots, T\}$ and $(r_0, \dots, r_{T-t}) \in \mathbb{R}^{T-t}$. Then we look at the characteristic function of $(Y_T - Y_t, \dots, Y_T - Y_T)$ and transform it to

$$\begin{aligned} \varphi_{Y_T - Y_t, \dots, Y_T - Y_T}(r_0, \dots, r_{T-t}) &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{k=t}^{T} i r_{k-t} \left(Y_T - Y_k\right)\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{k=t}^{T} i r_{k-t} \left(\sum_{j=1}^{T} X_j - \sum_{j=1}^{k} X_j\right)\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{k=t}^{T} i r_{k-t} \sum_{j=k+1}^{T} X_j\right)\right]. \end{aligned}$$

Now we change the order of summation and get

$$\begin{aligned} \varphi_{Y_T - Y_t, \dots, Y_T - Y_T}(r_0, \dots, r_{T-t}) &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{j=t+1}^{T} i \left(\sum_{k=t}^{j-1} r_{k-t}\right) X_j\right)\right] \\ &= \varphi_{X_{t+1}, \dots, X_T}(r_0, r_0 + r_1, \dots, \sum_{k=0}^{T-t-1} r_k). \end{aligned}$$

We continue with the application of Lemma 4.1, which states that the distribution of $(X_{t+1}, \dots, X_T)$ is the same as the distribution of $(X_{T-t}, \dots, X_1)$ under the dual measure $\mathbb{Q}'$. Hence, their characteristic functions are identical and we conclude

$$\begin{aligned} \varphi_{Y_T - Y_t, \dots, Y_T - Y_T}(r_0, \dots, r_{T-t}) &= \varphi_{X_{T-t}, \dots, X_1}(r_0, r_0 + r_1, \dots, \sum_{k=0}^{T-t-1} r_k) \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{j=0}^{T-t-1} i \left(\sum_{k=0}^{j} r_k\right) X_{T-t-j}\right)\right]. \end{aligned}$$

Because we want to rewrite everything in terms of Y_{T-k} for $t \le k \le T$, we have to change the order of summation again and obtain

$$\begin{aligned}\varphi_{Y_T-Y_t,\dots,Y_T-Y_T}(r_0,\dots,r_{T-t}) &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{k=0}^{T-t-1} ir_k \sum_{j=k}^{T-t-1} X_{T-t-j}\right)\right]\\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{k=0}^{T-t-1} ir_k Y_{T-t-k}\right)\right]\\ &= \mathbb{E}_{\mathbb{Q}'}\left[\exp\left(\sum_{k=0}^{T-t} ir_k Y_{T-t-k}\right)\right]\\ &= \varphi_{Y_{T-t},\dots,Y_0}(r_0,\dots,r_{T-t}).\end{aligned}$$

Since the characteristic functions of $(Y_T - Y_t,\dots,Y_T - Y_T)$ and $(Y_{T-t},\dots,Y_0)$ agree on whole $\mathbb{R}^{T-t+1}$, the distributions have to coincide. □

The previous lemma again states kind of a "time reversal" property like the one already shown in Lemma 4.4. Although they look nearly identical at first sight, it has to be made clear that the property stated in the previous lemma is stronger in the sense that it is a statement about multidimensional distributions, therefore, implies the statement of Lemma 4.4 as a special case.

Now we can state the relations we want to show and give their proofs.

Theorem 4.7. *In a binomial model with dividend yield the calculation of the prices of Asian call and put options with forward start time $t \in \{0,\dots,T\}$ and floating strike $A(t,T) = 1/(T-t+1)\cdot\sum_{k=t}^{T} S_t$ can be reduced to the calculation of the prices of Asian put and call options with fixed strike:*

$$\mathbb{AC}_{float}(S_0,a,b,u,d,T,t) = \frac{1}{a^t}\,\mathbb{AP}_{fix}(S_0,S_0,b,a,1/d,1/u,T-t) \tag{4.7}$$

$$\mathbb{AP}_{float}(S_0,a,b,u,d,T,t) = \frac{1}{a^t}\,\mathbb{AC}_{fix}(S_0,S_0,b,a,1/d,1/u,T-t). \tag{4.8}$$

Proof. We restrict ourselves to the proof of Relation 4.7 as the proof of Relation 4.8 works analogously. At first we fix $t \in \{0,\dots,T\}$. Then we start with the price of an Asian call option with floating strike $A(t,T)$ and change the measure from the original martingale measure to the dual measure $\mathbb{Q}'$ using the density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T$ to get

$$\begin{aligned}
\mathbb{AC}_{float}(S_0,a,b,u,d,T,t) &= \frac{1}{b^T}\,\mathbb{E}_{\mathbb{Q}}\left[(S_T - A(t,T))^+\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^T}\frac{a^T S_T}{S_0}\frac{S_0}{a^T S_T}(S_T - A(t,T))^+\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{a^T S_T}{b^T S_0}\frac{1}{a^T}\left(S_0 - S_0\frac{A(t,T)}{S_T}\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{B_0}{V_0}\frac{V_T}{B_T}\frac{1}{a^T}\left(S_0 - S_0\frac{A(t,T)}{S_T}\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{A(t,T)}{S_T}\right)^+\right].
\end{aligned}$$

Now we use the definitions $A(t,T) = 1/(T-t+1)\cdot\sum_{k=t}^{T} S_k$ and $S_t = S_0\exp((t+Y_t)/2\cdot\log(u) + (t-Y_t)/2\cdot\log(d))$ and rewrite the expression in terms of Y_t to obtain

$$\begin{aligned}
&\mathbb{AC}_{float}(S_0,a,b,u,d,T,t) = \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{\frac{1}{T-t+1}\sum_{k=t}^{T} S_k}{S_T}\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{\frac{1}{T-t+1}\sum_{k=t}^{T} S_0\exp\left(\frac{k+Y_k}{2}\log(u) + \frac{k-Y_k}{2}\log(d)\right)}{S_0\exp\left(\frac{T+Y_T}{2}\log(u) + \frac{T-Y_T}{2}\log(d)\right)}\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{1}{T-t+1}\sum_{k=t}^{T}\exp\left(-\frac{T-k+(Y_T-Y_k)}{2}\log(u)\right.\right.\right. \\
&\qquad\qquad\left.\left.\left. -\frac{T-k-(Y_T-Y_k)}{2}\log(d)\right)\right)^+\right].
\end{aligned}$$

Due to Lemma 4.6 we can use the identity $-\log(x) = \log(1/x)$, replace $Y_T - Y_k$ with Y_{T-k} for all $k \in \{t,\dots,T\}$ and change the summation index from k to $j = T-k$:

$$
\begin{aligned}
&\mathbb{AC}_{float}(S_0, a, b, u, d, T, t) = \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{1}{T-t+1}\sum_{k=t}^{T}\exp\left(\frac{T-k+(Y_T-Y_k)}{2}\log\left(\frac{1}{u}\right)\right.\right.\right. \\
&\qquad\qquad\left.\left.\left.+\frac{T-k-(Y_T-Y_k)}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{1}{T-t+1}\sum_{k=t}^{T}\exp\left(\frac{T-k+Y_{T-k}}{2}\log\left(\frac{1}{u}\right)\right.\right.\right. \\
&\qquad\qquad\left.\left.\left.+\frac{T-k-Y_{T-k}}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{1}{T-t+1}\sum_{j=0}^{T-t}\exp\left(\frac{j+Y_j}{2}\log\left(\frac{1}{u}\right)+\frac{j-Y_j}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right].
\end{aligned}
$$

By applying Lemma 4.2 we are able to interchange $\mathbb{Q}'$ with $\hat{\mathbb{Q}}$ and Y_j with $-\hat{Y}_j$ for all $j \in \{0, \dots, T-t\}$ to get

$$
\begin{aligned}
&\mathbb{AC}_{float}(S_0, a, b, u, d, T, t) = \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^T}\left(S_0 - S_0\frac{1}{T-t+1}\sum_{j=0}^{T-t}\exp\left(\frac{j-\hat{Y}_j}{2}\log\left(\frac{1}{u}\right)+\frac{j+\hat{Y}_j}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right].
\end{aligned}
$$

Now rewriting the formula in terms of $\hat{S}$ leads to

$$
\begin{aligned}
\mathbb{AC}_{float}(S_0, a, b, u, d, T, t) &= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^T}\left(S_0 - \frac{1}{T-t+1}\sum_{j=0}^{T-t}\hat{S}_j\right)^+\right] \\
&= \frac{1}{a^t}\,\mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{T-t}}\left(S_0 - \hat{A}(0, T-t)\right)^+\right] \\
&= \frac{1}{a^t}\,\mathbb{AP}_{fix}\left(S_0, b, a, 1/d, 1/u, T-t\right),
\end{aligned}
$$

which finishes the proof. □

Those relations are very useful, because instead of calculating the expected value of a function of two highly dependent random variables, namely S_T and $A(t,T)$, to get the price of an Asian option with forward start time $t \in \{0, \dots, T\}$ and floating strike one just has to calculate the price of an Asian option with fixed strike, which just corresponds to the calculation of the expected value of the single random variable $A(0, T-t)$. Hence, this is at least theoretically, a great simplification as instead of the two dimensional distribution of the pair $(S_T, A(t,T))$ one just needs to know the one dimensional distribution of $A(0, T-t)$.

It is very interesting to note that, because of the "time reversal" property of Y_t, the right hand side of the relations does not involve the stock prices after time $T-t$, whereas the left hand side does heavily depend on them .

One also should mention that with slight modifications of the poof above similar relations can be shown for Asian options on the geometric average $G(t,T) := \sqrt[T-t+1]{\prod_{k=t}^{T} S_k}$ in place of the arithmetic average $A(t,T)$.

Of course we get relations for usual Asian options on the arithmetic mean $A(0,T)$ as a corollary of the previous theorem for the special case $t = 0$.

Corollary 4.8. *In a binomial model with dividend yield the prices of Asian call and put options with floating strike $A(0,T) = 1/(T+1) \cdot \sum_{t=0}^{T} S_t$ satisfy the following relations to the prices of Asian put and call options with fixed strike:*

$$\mathbb{AC}_{float}(S_0, a, b, u, d, T) = \mathbb{AP}_{fix}(S_0, S_0, b, a, 1/d, 1/u, T) \tag{4.9}$$

$$\mathbb{AP}_{float}(S_0, a, b, u, d, T) = \mathbb{AC}_{fix}(S_0, S_0, b, a, 1/d, 1/u, T). \tag{4.10}$$

4.4. Lookback options of European type

In this section we want to proof relations between lookback options with floating strike and lookback options with fixed strike such as the one we have seen in Example 3.8. One important ingredient of the proof is again Lemma 4.6, which should not surprise us as the relations for lookback options look very similar to the relations for Asian options shown in the section before. Analogously to the relations for Asian options we will proof everything for forward start lookback options.

Theorem 4.9. *The prices of lookback call and put options with forward start time $t \in \{0, \ldots, T\}$ and $\alpha > 0$ in a binomial model with dividend yield satisfy the following relations:*

$$\frac{1}{\alpha}\, \overline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) = \frac{1}{a^t}\, \overline{\mathbb{LP}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T-t) \tag{4.11}$$

$$\frac{1}{\alpha}\, \overline{\mathbb{LP}}_{float}(S_0, a, b, u, d, T, t, \alpha) = \frac{1}{a^t}\, \overline{\mathbb{LC}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T-t) \tag{4.12}$$

$$\frac{1}{\alpha}\, \underline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) = \frac{1}{a^t}\, \underline{\mathbb{LP}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T-t) \tag{4.13}$$

$$\frac{1}{\alpha}\, \underline{\mathbb{LP}}_{float}(S_0, a, b, u, d, T, t, \alpha) = \frac{1}{a^t}\, \underline{\mathbb{LC}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T-t). \tag{4.14}$$

Proof. At first we fix $t \in \{0, \ldots, T\}$ and $\alpha > 0$. We start with the price of a lookback call option with floating strike $\max_{t \le k \le T} S_k$ and change $\mathbb{Q}$ to the dual measure $\mathbb{Q}'$ with density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T$ to get

$$\begin{aligned}
\overline{\mathbb{LC}}_{float}(S_0,a,b,u,d,T,t,\alpha) &= \frac{1}{b^T}\,\mathbb{E}_{\mathbb{Q}}\left[\left(S_T-\alpha\max_{t\le k\le T}S_k\right)^+\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^T}\frac{a^TS_T}{S_0}\frac{S_0}{a^TS_T}\left(S_T-\alpha\max_{t\le k\le T}S_k\right)^+\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{a^TS_T}{b^TS_0}\frac{1}{a^T}\left(S_0-\alpha S_0\frac{\max_{t\le k\le T}S_k}{S_T}\right)^+\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{B_0}{V_0}\frac{V_T}{B_T}\frac{1}{a^T}\left(S_0-\alpha S_0\frac{\max_{t\le k\le T}S_k}{S_T}\right)^+\right]\\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0-\alpha S_0\frac{\max_{t\le k\le T}S_k}{S_T}\right)^+\right].
\end{aligned}$$

Further, we use the relation $S_t = S_0\exp((t+Y_t)/2\cdot\log(u)+(t-Y_t)/2\cdot\log(d))$ to rewrite the expression in terms of Y_t, pull $S_0>0$ out of the maximum and pull the positive denominator into the maximum:

$$\begin{aligned}
&\overline{\mathbb{LC}}_{float}(S_0,a,b,u,d,T,t,\alpha) =\\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0-\alpha S_0\frac{\max_{t\le k\le T}\left(S_0\exp\left(\frac{k+Y_k}{2}\log(u)+\frac{k-Y_k}{2}\log(d)\right)\right)}{S_0\exp\left(\frac{T+Y_T}{2}\log(u)+\frac{T-Y_T}{2}\log(d)\right)}\right)^+\right]\\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0-\alpha S_0\max_{t\le k\le T}\frac{\exp\left(\frac{k+Y_k}{2}\log(u)+\frac{k-Y_k}{2}\log(d)\right)}{\exp\left(\frac{T+Y_T}{2}\log(u)+\frac{T-Y_T}{2}\log(d)\right)}\right)^+\right].
\end{aligned}$$

In the next step we use the defining functional equation of the exponential function, the identity $\log(x)=\log(1/x)$ and Lemma 4.6 to substitute Y_T-Y_k with Y_{T-k} for $t\le k\le T$ to obtain

$$\begin{aligned}
&\overline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) = \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - \alpha S_0 \max_{t\le k\le T}\left(\exp\left(-\frac{T-k+(Y_T-Y_k)}{2}\log(u)\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.-\frac{T-k-(Y_T-Y_k)}{2}\log(d)\right)\right)\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - \alpha S_0 \max_{t\le k\le T}\left(\exp\left(\frac{T-k+(Y_T-Y_k)}{2}\log\left(\frac{1}{u}\right)\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.+\frac{T-k-(Y_T-Y_k)}{2}\log\left(\frac{1}{d}\right)\right)\right)\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^T}\left(S_0 - \alpha S_0 \max_{t\le k\le T}\left(\exp\left(\frac{T-k+Y_{T-k}}{2}\log\left(\frac{1}{u}\right)\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.+\frac{T-k-Y_{T-k}}{2}\log\left(\frac{1}{d}\right)\right)\right)\right)^+\right].
\end{aligned}$$

As a consequence of Lemma 4.2 we change the measure from $\mathbb{Q}'$ to $\hat{\mathbb{Q}}$, replace Y_{T-k} by $-\hat{Y}_{T-k}$ for $t \le k \le T$, pull $S_0 > 0$ into the maximum and rewrite everything in terms of $\hat{S}$ to get

$$\begin{aligned}
&\overline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) = \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^T}\left(S_0 - \alpha S_0 \max_{t\le k\le T}\left(\exp\left(\frac{T-k+\hat{Y}_{T-k}}{2}\log\left(\frac{1}{u}\right)\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.+\frac{T-k-\hat{Y}_{T-k}}{2}\log\left(\frac{1}{d}\right)\right)\right)\right)^+\right] \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^T}\left(S_0 - \alpha \max_{t\le k\le T}\left(S_0\exp\left(\frac{T-k+\hat{Y}_{T-k}}{2}\log\left(\frac{1}{u}\right)\right.\right.\right.\right. \\
&\qquad\left.\left.\left.\left.+\frac{T-k-\hat{Y}_{T-k}}{2}\log\left(\frac{1}{d}\right)\right)\right)\right)^+\right] \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^T}\left(S_0 - \alpha \max_{t\le k\le T}\hat{S}_{T-k}\right)^+\right].
\end{aligned}$$

Now we change the index and pull the constant α out to conclude

$$\begin{aligned}
\overline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) &= \frac{\alpha}{a^t}\,\mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{T-t}}\left(\frac{S_0}{\alpha} - \max_{0\le j\le T-t}\hat{S}_j\right)^+\right] \\
&= \frac{\alpha}{a^t}\,\overline{\mathbb{LP}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T-t).
\end{aligned}$$

The other three relations can be shown analogously. □

We should note that not every choice of $\alpha > 0$ does make sense for every type of lookback option. For example, if we choose $\alpha \geq 1$ we get $\overline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, t, \alpha) = 0$ independent of the other parameters chosen, because $(S_T - \alpha \max_{t \leq k \leq T} S_k)^+ = 0$, therefore, also the expected value vanishes. Restricting to $0 < \alpha < 1$ does not make sense either for every type of lookback option, as we then have $\underline{\mathbb{LP}}_{float}(S_0, a, b, u, d, T, t, \alpha) = 0$ with a similar argument.

For the choice of $t = 0$ in the theorem above we get simpler relations for non forward start lookback options as a corollary.

Corollary 4.10. *In a binomial model with dividend yield the prices of lookback call and put options with $\alpha > 0$ satisfy*

$$\frac{1}{\alpha}\,\overline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, \alpha) = \overline{\mathbb{LP}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T) \tag{4.15}$$

$$\frac{1}{\alpha}\,\overline{\mathbb{LP}}_{float}(S_0, a, b, u, d, T, \alpha) = \overline{\mathbb{LC}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T). \tag{4.16}$$

$$\frac{1}{\alpha}\,\underline{\mathbb{LC}}_{float}(S_0, a, b, u, d, T, \alpha) = \underline{\mathbb{LP}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T) \tag{4.17}$$

$$\frac{1}{\alpha}\,\underline{\mathbb{LP}}_{float}(S_0, a, b, u, d, T, \alpha) = \underline{\mathbb{LC}}_{fix}(S_0, S_0/\alpha, b, a, 1/d, 1/u, T). \tag{4.18}$$

4.5. Vanilla options of American type

In this section we want to proof relations as in Theorem 4.3 for vanilla options of American type. Therefore, we have to take a closer look at the set of stopping times $\mathcal{T}_T = \{\tau \mid \tau \leq T, \tau\ \mathbb{F}\text{-stopping time}\}$. Our aim is to obtain a bijection between the set $\mathcal{T}_T$ and the analogously defined set $\hat{\mathcal{T}}_T$ of $\hat{\mathbb{F}}$- stopping times. As the notion of stopping times is a very probabilistic concept, we want to go over to something non-probabilistic to obtain such a bijection. This is of course very easily possible as we work with binomial models.
We start with the following definition.

Definition 4.11. Let $n \in \mathbb{N}$. We define $\mathcal{A}_n$ to be the set of $\{-1, 1\}$-sequences of at most length n, i.e.

$$\mathcal{A}_n := \{(i_1, \ldots, i_k) \mid k \in \mathbb{N}_0, k \leq n, \forall j \in \{1, \ldots, k\} : i_j \in \{-1, 1\}\}.$$

We define the binary relation $R_{\leq,n} \subseteq \mathcal{A}_n^2$ as the set of all pairs $(x, y) \in \mathcal{A}_n^2$ so that y is an extension of x and denote $(x, y) \in \mathcal{R}_{\leq,n}$ as $x \leq y$, that means for $x = (i_1, \ldots, i_k), y = (j_1, \ldots, j_l)$

$$x \leq y \qquad :\Longleftrightarrow \qquad ((k \leq l) \wedge ((i_1, \ldots, i_k) = (j_1, \ldots, j_k))).$$

This order is also known as prefix-order and is also treated in the theory of formal languages (see for example the paper by Kundu [7]). The use of the symbol "≤" is justified by the next lemma.

Lemma 4.12. *The relation of extension $\mathcal{R}_{\le,n}$ is a partial order on $\mathcal{A}_n$.*

Proof. We have to show reflexivity, antisymmetry and transitivity. Let $x = (i_1, \dots, i_k), y = (j_1, \dots, j_l), z = (s_1, \dots, s_m) \in \mathcal{A}_n$. Then clearly $x \le x$, hence $\mathcal{R}_{\le,n}$ is reflexive.

If we suppose $x \le y$ and $y \le x$, we get $k \le l$ and $l \le k$, which shows that x and y are of equal length. As a consequence of $x \le y$ we further get $x = (i_1, \dots, i_k) = (j_1, \dots, j_k) = (j_1, \dots, j_l) = y$, which shows that $\mathcal{R}_{\le,n}$ is antisymmetric.

We assume now $x \le y$ and $y \le z$. Then x also has to be shorter than z and we have $x = (i_1, \dots, i_k) = (j_1, \dots, j_k) = (s_1, \dots, s_k)$. Thus z is an extension of x and $\mathcal{R}_{\le,n}$ is transitive, which finishes the proof. □

Now we are able to state a bijection between our set of stopping times $\mathcal{T}_T$ and a set of specific subsets of $\mathcal{A}_{T-1}$.

Lemma 4.13. *The function*

$$\begin{aligned} p_T : \mathcal{T}_T &\to \mathcal{B}_{T-1} := \{A \mid A \in \mathcal{P}(\mathcal{A}_{T-1}), \nexists (a,b) \in A^2 : a < b\} \\ \tau &\mapsto A_\tau := \{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) < T\} \end{aligned}$$

is a bijection.

Proof. Step 1): Given $\tau \in \mathcal{T}_T$ we want to show $A_\tau \in \mathcal{B}_{T-1}$. It is clear that A_τ is a subset of $\mathcal{A}_{T-1}$, so it only remains to show that A_τ satisfies the second condition in the definition of $\mathcal{B}_{T-1}$. Let $\omega_1, \omega_2 \in \Omega$ so that $\tau(\omega_1) < T$, $\tau(\omega_2) < T$. Without loss of generality we can assume $\tau(\omega_1) \le \tau(\omega_2)$.
Suppose we have $(X_1(\omega_1), \dots, X_{\tau(\omega_1)}(\omega_1)) = (X_1(\omega_2), \dots, X_{\tau(\omega_1)}(\omega_2))$, which is equivalent to $(X_1(\omega_1), \dots, X_{\tau(\omega_1)}(\omega_1)) \le (X_1(\omega_2), \dots, X_{\tau(\omega_2)}(\omega_2))$ in the sense of Definition 4.11. Our aim is to show that equality must hold, which is under our assumption equivalent to $\tau(\omega_1) = \tau(\omega_2)$.
Because τ is a stopping time we have $\{\omega \in \Omega \mid \tau(\omega) = \tau(\omega_1)\} \in \mathcal{F}_{\tau(\omega_1)} = \sigma(X_1, \dots, X_{\tau(\omega_1)})$. This implies that the indicator function $\mathbb{1}_{\{\tau(\omega)=\tau(\omega_1)\}}$ has to be constant on the atoms of $\mathcal{F}_{\tau(\omega_1)}$. Therefore, $\mathbb{1}_{\{\tau(\omega)=\tau(\omega_1)\}}$ must be constant on the set $\{\omega \in \Omega \mid X_1(\omega) = X_1(\omega_1), \dots, X_{\tau(\omega_1)}(\omega) = X_{\tau(\omega_1)}(\omega_1)\}$. As a consequence of our assumption we have $\omega_1, \omega_2 \in \{\omega \in \Omega \mid X_1(\omega) = X_1(\omega_1), \dots, X_{\tau(\omega_1)}(\omega) = X_{\tau(\omega_1)}(\omega_1)\}$ and hence we get $\mathbb{1}_{\{\tau(\omega_2)=\tau(\omega_1)\}} = \mathbb{1}_{\{\tau(\omega_1)=\tau(\omega_1)\}} = 1$. This shows our claim.

Step 2): In this step we want to show that the mapping p_T is injective. We start with two stopping times $\tau_1, \tau_2 \in \mathcal{T}_T$ so that $\tau_1 \ne \tau_2$. Our aim is to show $A_{\tau_1} \ne A_{\tau_2}$.
There must exist an $\omega_1 \in \Omega$ with $\tau_1(\omega_1) \ne \tau_2(\omega_1)$. Without loss of generality we assume $\tau_1(\omega_1) < \tau_2(\omega_2)$. Because of $\tau_2 \le T$ we have $\tau_1(\omega_1) < T$ and, therefore, get $(X_1(\omega_1), \dots, X_{\tau_1(\omega_1)}(\omega_1)) \in A_{\tau_1}$.
We assume there would exist an $\omega_2 \in \Omega$ with $(X_1(\omega_2), \dots, X_{\tau_2(\omega_2)}(\omega_2)) = (X_1(\omega_1), \dots, X_{\tau_1(\omega_1)}(\omega_1))$, which is equivalent to $(X_1(\omega_1), \dots, X_{\tau_1(\omega_1)}(\omega_1)) \in A_{\tau_2}$. Then $(X_1(\omega_2), \dots, X_{\tau_2(\omega_2)}(\omega_2)) = (X_1(\omega_1), \dots, X_{\tau_2(\omega_2)}(\omega_1))$ would hold and because of $\{\omega \in \Omega \mid \tau(\omega) = \tau(\omega_1)\} \in \mathcal{F}_{\tau(\omega_1)} = \sigma(X_1, \dots, X_{\tau(\omega_1)})$ we would get $\tau_2(\omega_1) = \tau_2(\omega_2) =$

$\tau_1(\omega_1)$, which is a contradiction. Hence, we have $(X_1(\omega_1), \ldots, X_{\tau_1(\omega_1)}(\omega_1)) \notin A_{\tau_2}$ and, therefore, $A_{\tau_1} \neq A_{\tau_2}$.

Step 3): It only remains to show that p_T is surjective. Given $A \in \mathcal{B}_{T-1}$ we define

$$\tau_A(\omega) := \begin{cases} k & \text{if } \exists (i_1, \ldots, i_k) \in A \text{ s.t.} (X_1(\omega), \ldots, X_k(\omega)) = (i_1, \ldots, i_k) \\ T & \text{else} \end{cases}.$$

It is obvious that τ_A is a stopping time bounded by T and that $p_T(\tau_A) = A$, which finishes the proof. □

The previous lemma tells us that every $\tau \in \mathcal{T}_T$ can be identified with an antichain of $(\mathcal{A}_{T-1}, \leq)$, which means a subset of incomparable elements of $\mathcal{A}_{T-1}$.
In some sense the mapping p_T can be understood as a translation of stopping times from our original binomial model and the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ to a version of our binomial model, where the new sample space just consists of $\{-1, 1\}$-sequences of length up to T and the generated sets $A_\tau \in \mathcal{B}_{T-1}$ can be viewed as subtrees of a full, complete, rooted and directed binary tree of height T, where the level of the leaf of each path starting at the root gives us the information at which time we should stop, if the sequence realized by the random variables $(X_t)_{1 \leq t \leq T}$ up to this time coincides with the path. If the random variables do not follow any path existing in the subtree corresponding to A_τ to its end, then we stop at time T. The case that we stop at time 0 almost surely corresponds to $A_\tau = \{()\}$, which can be identified with the subtree only consisting of the root, whereas the case that we stop at time T almost surely corresponds to $A_\tau = \emptyset$, which can be identified with the empty subtree.
We can see that it is very important that the sets in $\mathcal{B}_{T-1}$ are antichains as otherwise we would not get a bijection using this procedure. Let us take a look at a short example. We look at the two sets $A_1 = \{(1, -1)\}$ and $A_2 = \{(1, -1), (1, -1, -1)\}$. The first set tells us that we should stop at time 2 if we start with "up" in the first step and "down" in the second step of our random walk and else stop at time T. The second set gives us the same information plus the piece of information that we should stop at time 3 if our random walk goes "up", "down" and "down" at the beginning. It is obvious that this extra information we get from our second set is irrelevant, because we have to stop already at time 2 in this case. Therefore, the relevant information encoded in our two sets using this algorithm would be the same and hence we would not get a bijection if we allowed for sets which are not antichains, like A_2.
We can also look at this problem from a different point of view. If we go to step 3 of the proof above and look at τ_{A_2}, then we will notice that this mapping is not well defined because of the extra information compared to A_1 and, therefore, not a stopping time, which will be problematic in the next section about Asian options of American type.

Before we deal with Asian options of American type we still want to proof duality relations for vanilla options of American type in this section. In the next corollary we want to use the bijection we obtained in the previous lemma to get a quick estimate for a lower bound for the number of stopping times in a binomial model.

Corollary 4.14. *The number of stopping times up to time T in a binomial model is at least double exponentially growing in T and we get as a lower bound*

$$\sum_{k=0}^{T-1}(2^{(2^k)}-1) \leq |\mathcal{T}_T|.$$

Proof. For $T \in \mathbb{N}$ we have $|\mathcal{T}_T| = |\mathcal{B}_{T-1}|$ because of Lemma 4.13. Now we define for $k \in \{0,\dots,T-1\}$ subsets of $\mathcal{B}_{T-1}$ which only consist of antichains consisting of sequences of length k:

$$\mathcal{B}_{T-1,k} := \{\{x_1,\dots,x_l\} \mid l \in \mathbb{N}, \forall i \in \{1,\dots,l\} : (x_i \in A_{T-1} \wedge \ |x_i| = k), \nexists\, i,j : x_i < x_j\}.$$

These subsets are clearly disjoint, therefore, we get $\sum_{k=0}^{T-1} |\mathcal{B}_{T-1,k}| \leq |\mathcal{B}_{T-1}|$. Further, we define for $k \in \{0,\dots,T-1\}$

$$\begin{aligned} f_{T-1,k} : \mathcal{P}(\mathcal{P}(\{1,\dots,k\})) \setminus \{\emptyset\} &\to \mathcal{B}_{T-1,k} \\ \{C_1,\dots,C_l\} &\mapsto \bigcup_{j=1}^{l} \{((-1)^{1+\mathbb{1}_{\{1\in C_j\}}},\dots,(-1)^{1+\mathbb{1}_{\{k\in C_j\}}})\}. \end{aligned}$$

As a consequence of the fact that $f_{T-1,k}$ maps to sets of sequences which all have the same length, these sets are antichains, therefore, elements of $\mathcal{B}_{T-1,k}$.
Let $C_1, C_2 \in \mathcal{P}(\{1,\dots,k\})$ with $C_1 \neq C_2$. Then there must exist $r \in \{1,\dots,k\}$ with $r \in C_1, r \notin C_2$ and hence we have

$$((-1)^{1+\mathbb{1}_{\{1\in C_1\}}},\dots,(-1)^{1+\mathbb{1}_{\{k\in C_1\}}}) \neq ((-1)^{1+\mathbb{1}_{\{1\in C_2\}}},\dots,(-1)^{1+\mathbb{1}_{\{k\in C_2\}}}).$$

This implies that $f_{T-1,k}$ is injective. Therefore, we get

$$\sum_{k=0}^{T-1}(2^{(2^k)}-1) = \sum_{k=0}^{T-1} |\mathcal{P}(\mathcal{P}(\{1,\dots,k\})) \setminus \{\emptyset\}| \leq \sum_{k=0}^{T-1} |\mathcal{B}_{T-1,k}| \leq |\mathcal{B}_{T-1}| = |\mathcal{T}_T|,$$

which is what we wanted to show. ☐

The previous lemma tells us that it does not make any sense to deal with every single stopping time on its own as the number of stopping times is growing extremely fast if we increase the number of time steps. The sets $\mathcal{B}_{T-1,k}$ correspond to stopping times which either stop at time k or at time T.

As an important next step on our way to show relations for vanilla options of American type we give a bijection between the sets $\mathcal{T}_T$ and $\hat{\mathcal{T}}_T$ of stopping times and show a distributional property of this bijection and the random walks Y and $\hat{Y}$.

Lemma 4.15. *Define the function*

$$\begin{aligned} g_T : \mathcal{B}_{T-1} &\to \mathcal{B}_{T-1} \\ \{(i_{1,1},\dots,i_{1,k_1}),\dots,(i_{n,1},\dots,i_{n,k_n})\} &\mapsto \{(-i_{1,1},\dots,-i_{1,k_1}),\dots,(-i_{n,1},\dots,-i_{n,k_n})\}. \end{aligned}$$

Then g_T is a bijection. Given the bijections p_T from Lemma 4.13 and the analogously defined $\hat{p}_T : \hat{\mathcal{T}}_T \to \mathcal{B}_{T-1}$ further define the function $h_T := \hat{p}_T^{-1} \circ g_T \circ p_T$. Then $h_T : \mathcal{T}_T \to \hat{\mathcal{T}}_T$ is bijective and for every $\tau \in \mathcal{T}_T$ the random walks Y and $\hat{Y}$ satisfy

$$Law(\tau, Y_\tau \mid \mathbb{Q}') = Law(h_T(\tau), -\hat{Y}_{h_T(\tau)} \mid \hat{\mathbb{Q}}). \tag{4.19}$$

Proof. It is clear that g_T is bijective and well defined, because it is self-inverse and changing the signs of all sequences does not change anything regarding the relation of extensions and, therefore, the antichain property still holds and so g_T really maps to $\mathcal{B}_{T-1}$. As p_T, g_T and $\hat{p}_T^{-1}$ are bijective also h_T, the composition of them, is bijective.
Hence, it just remains to show that Equation 4.19 holds for all stopping times of our original binomial model. Let $\tau \in \mathcal{T}_T, t \in \{0, \ldots, T-1\}$ and $k \in \mathbb{Z}$. Using the definition of $p_T(\tau) = A_\tau$ we then get

$$\begin{aligned}
&\mathbb{Q}'[\tau = t, Y_\tau = k] = \mathbb{Q}'[\tau = t, Y_t = k] \\
&= \mathbb{Q}'\left(\dot{\bigcup_{\substack{(i_1,\ldots,i_t)\in A_\tau \\ i_1+\cdots+i_t=k}}} \{\omega \in \Omega \mid X_1(\omega) = i_1, \ldots, X_t(\omega) = i_t\} \right) \\
&= \sum_{\substack{(i_1,\ldots,i_t)\in A_\tau \\ i_1+\cdots+i_t=k}} \mathbb{Q}'[X_1 = i_1, \ldots, X_t = i_t].
\end{aligned}$$

Now we use Lemma 4.1, which tells us that $(X_i)_{1\le i\le T}$ are i.i.d under $\mathbb{Q}'$ and the definition of g_T to obtain

$$\begin{aligned}
\mathbb{Q}'[\tau = t, Y_\tau = k] &= \sum_{\substack{(i_1,\ldots,i_t)\in A_\tau \\ i_1+\cdots+i_t=k}} \prod_{j=1}^{t} \mathbb{Q}'[X_j = i_j] \\
&= \sum_{\substack{(i_1,\ldots,i_t)\in A_\tau \\ i_1+\cdots+i_t=k}} \mathbb{Q}'[X_1 = 1]^{\frac{t+k}{2}} \mathbb{Q}'[X_1 = -1]^{\frac{t-k}{2}} \\
&= \sum_{\substack{(\hat{i}_1,\ldots,\hat{i}_t)\in g_T(A_\tau) \\ \hat{i}_1+\cdots+\hat{i}_t=-k}} \mathbb{Q}'[X_1 = 1]^{\frac{t+k}{2}} \mathbb{Q}'[X_1 = -1]^{\frac{t-k}{2}}.
\end{aligned}$$

In the next two steps we first apply Lemma 4.2 1. and then use the identities $g_T(A_\tau) = g_T \circ p_T(\tau) = \hat{p}_T \circ \hat{p}_T^{-1} \circ g_T \circ p_T(\tau) = \hat{p}_T \circ h_T(\tau) = \hat{A}_{h_T(\tau)}$ to conclude

$$\mathbb{Q}'[\tau = t, Y_\tau = k] = \sum_{\substack{(\hat{i}_1,\dots,\hat{i}_t)\in g_T(A_\tau)\\ \hat{i}_1+\dots+\hat{i}_t=-k}} \hat{\mathbb{Q}}[\hat{X}_1 = -1]^{\frac{t+k}{2}}\hat{\mathbb{Q}}[\hat{X}_1 = 1]^{\frac{t-k}{2}}$$
$$= \sum_{\substack{(\hat{i}_1,\dots,\hat{i}_t)\in \hat{A}_{h_T(\tau)}\\ \hat{i}_1+\dots+\hat{i}_t=-k}} \hat{\mathbb{Q}}[\hat{X}_1 = -1]^{\frac{t+k}{2}}\hat{\mathbb{Q}}[\hat{X}_1 = 1]^{\frac{t-k}{2}}.$$

Because the random variables $(\hat{X}_i)_{1\le i\le T}$ are i.i.d under $\hat{\mathbb{Q}}$ we continue with

$$\mathbb{Q}'[\tau = t, Y_\tau = k] = \sum_{\substack{(\hat{i}_1,\dots,\hat{i}_t)\in \hat{A}_{h_T(\tau)}\\ \hat{i}_1+\dots+\hat{i}_t=-k}} \prod_{j=1}^{t} \hat{\mathbb{Q}}[\hat{X}_j = \hat{i}_j]$$
$$= \sum_{\substack{(\hat{i}_1,\dots,\hat{i}_t)\in \hat{A}_{h_T(\tau)}\\ \hat{i}_1+\dots+\hat{i}_t=-k}} \hat{\mathbb{Q}}[\hat{X}_1 = \hat{i}_1, \dots, \hat{X}_t = \hat{i}_t].$$

In the last steps of our proof we use that the events in our sum are disjoint and the definition of $\hat{A}_{h_T(\tau)} = \hat{p}_T(h_T(\tau))$ to get

$$\mathbb{Q}'[\tau = t, Y_\tau = k] = \hat{\mathbb{Q}}\left(\bigcup_{\substack{(\hat{i}_1,\dots,\hat{i}_t)\in \hat{A}_{h_T(\tau)}\\ \hat{i}_1+\dots+\hat{i}_t=-k}} \{\omega \in \Omega \mid \hat{X}_1(\omega) = \hat{i}_1, \dots, \hat{X}_t(\omega) = \hat{i}_t\} \right)$$
$$= \hat{\mathbb{Q}}[h_T(\tau) = t, \hat{Y}_t = -k]$$
$$= \hat{\mathbb{Q}}[h_T(\tau) = t, -\hat{Y}_{h_T(\tau)} = k],$$

which finishes the proof for the case $0 \le t \le T-1$.

For the case $t = T$ we define the sets $A_{\tau,T} := \{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) = T\}$ and $\hat{A}_{h_T(\tau),T} := \{(\hat{X}_1(\omega), \dots, \hat{X}_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : h_T(\tau)(\omega) = T\}$. We note for fixed $(i_1, \dots, i_T)$ that

$$\begin{aligned}(i_1, \dots, i_T) \in A_{\tau,T} &\iff \nexists j \in \{0, \dots, T-1\} : (i_1, \dots, i_j) \in A_\tau\\ &\iff \nexists j \in \{0, \dots, T-1\} : (-i_1, \dots, -i_j) \in g_T(A_\tau)\\ &\iff \nexists j \in \{0, \dots, T-1\} : (-i_1, \dots, -i_j) \in \hat{A}_{h_T(\tau)}\\ &\iff (-i_1, \dots, -i_T) \in \hat{A}_{h_T(\tau),T}.\end{aligned}$$

Using these equivalences the case $t = T$ can be shown analogously to the case $t \in \{0, \dots, T-1\}$ above, which finishes the proof.

□

Now we are able to proof the relations for vanilla options of American type.

Theorem 4.16. *For vanilla call and put options of American type the option prices in a binomial model with dividend yield satisfy the relations*

$$\mathbb{c}(S_0, K, a, b, u, d, T) = \mathbb{p}(K, S_0, b, a, 1/d, 1/u, T) \tag{4.20}$$

and

$$\mathbb{p}(S_0, K, a, b, u, d, T) = \mathbb{c}(K, S_0, b, a, 1/d, 1/u, T). \tag{4.21}$$

Proof. We start with the price of a vanilla call option of American type and get

$$\begin{aligned}\mathbb{c}(S_0, K, a, b, u, d, 0, T) &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}\,(S_\tau - K)^+\right] \\ &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}\frac{a^T S_T}{b^T S_0}\frac{b^T S_0}{a^T S_T}(S_\tau - K)^+\right].\end{aligned}$$

Now we use the identities $V_t = a^t S_t$ and $B_t = b^t$ and perform a change of measure from our original pricing measure $\mathbb{Q}$ to the dual measure $\mathbb{Q}'$ with density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T$ in the first step and then use the law of total expectation for conditional expectations to obtain

$$\begin{aligned}\mathbb{c}(S_0, K, a, b, u, d, 0, T) &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{b^\tau}\frac{b^T S_0}{a^T S_T}(S_\tau - K)^+\right] \\ &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}'}\left[\mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{b^\tau}\frac{b^T S_0}{a^T S_T}(S_\tau - K)^+\middle|\mathcal{F}_\tau\right]\right].\end{aligned}$$

We continue with using the fact that $1/b^\tau(S_\tau - K)^+$ is $\mathcal{F}_\tau$ measurable as a consequence of Lemma 2.4, which implies that we can take it out of the conditional expectation and then use the optional sampling theorem (see Theorem 2.5) for the bounded stopping times $\tau \in \mathcal{T}_T$ and the $\mathbb{Q}'$-martingale $(B_t V_0)/(V_t B_0) = (b^t S_0)/(a^t S_t)$:

$$\begin{aligned}\mathbb{c}(S_0, K, a, b, u, d, 0, T) &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}'}\left[\mathbb{E}_{\mathbb{Q}'}\left[\frac{b^T S_0}{a^T S_T}\middle|\mathcal{F}_\tau\right]\frac{1}{b^\tau}(S_\tau - K)^+\right] \\ &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}'}\left[\frac{b^\tau S_0}{a^\tau S_\tau}\frac{1}{b^\tau}(S_\tau - K)^+\right] \\ &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^\tau}\left(S_0 - K\frac{S_0}{S_\tau}\right)^+\right].\end{aligned}$$

Further, we rewrite everything in terms of Y, use the relation $-\log(x) = \log(1/x)$ and apply Lemma 4.15 to replace $\mathbb{Q}'$ with $\hat{\mathbb{Q}}$, τ with $h_T(\tau)$ and Y_τ with $-\hat{Y}_{h_T(\tau)}$ to get

$$\begin{aligned}
&\mathbb{c}(S_0, K, a, b, u, d, 0, T) = \\
&= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^\tau}\left(S_0 - K\exp\left(-\frac{\tau+Y_\tau}{2}\log(u) - \frac{\tau-Y_\tau}{2}\log(d)\right)\right)^+\right] \\
&= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^\tau}\left(S_0 - K\exp\left(\frac{\tau+Y_\tau}{2}\log\left(\frac{1}{u}\right) + \frac{\tau-Y_\tau}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right] \\
&= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{h_T(\tau)}}\left(S_0 - K\exp\left(\frac{h_T(\tau)-\hat{Y}_{h_T(\tau)}}{2}\log\left(\frac{1}{u}\right)\right.\right.\right. \\
&\qquad\qquad\qquad\qquad\left.\left.\left. + \frac{h_T(\tau)+\hat{Y}_{h_T(\tau)}}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right].
\end{aligned}$$

Now we rewrite everything in terms of $\hat{Y}$ and use the fact that $h_T : \mathcal{T}_T \to \hat{\mathcal{T}}_T$ is a bijection to obtain

$$\begin{aligned}
\mathbb{c}(S_0, K, a, b, u, d, 0, T) &= \sup_{\tau\in\mathcal{T}_T} \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{h_T(\tau)}}\left(S_0 - \hat{S}_{h_T(\tau)}\right)^+\right] \\
&= \sup_{\hat{\tau}\in\hat{\mathcal{T}}_T} \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{\hat{\tau}}}\left(S_0 - \hat{S}_{\hat{\tau}}\right)^+\right] \\
&= \mathbb{p}(K, S_0, b, a, 1/d, 1/u, 0, T),
\end{aligned}$$

which finishes the proof of the first relation. The second relation can be shown analogously. □

5. Possible extensions for Asian options

Our aim in this chapter is to find a subset $\mathcal{T}_T^*$ of $\mathcal{T}_T$ on which we can extend the relations for Asian options shown in Theorem 4.7. As Example 3.12 shows, this is not possible on the whole set $\mathcal{T}_T$, because then we would have the relations for Asian options of American type. Hence, we have to find a proper subset of stopping times so that we can prove a time reversal property with respect to these stopping times.

5.1. Time reversal with respect to stopping times

We would like to have a set $\mathcal{T}_T^* \subseteq \mathcal{T}_T$ of stopping times and a function (bijection) $r_T : \mathcal{T}_T^* \to \mathcal{T}_T^*$ so that for every $\tau \in \mathcal{T}_T^*$

$$Law(\tau, Y_\tau - Y_\tau, \ldots, Y_\tau - Y_0 \mid \mathbb{Q}') = Law(r_T(\tau), Y_0, \ldots, Y_{r_T(\tau)} \mid \mathbb{Q}'), \tag{5.1}$$

which is equivalent to

$$Law(\tau, X_\tau, \ldots, X_1 \mid \mathbb{Q}') = Law(r_T(\tau), X_1, \ldots, X_{r_T(\tau)} \mid \mathbb{Q}') \tag{5.2}$$

would hold, because then we could extend the proof of Theorem 4.7 to a proof of relations for Asian options of American type restricted to the set $\mathcal{T}_T^*$ like we extended the proof of Theorem 4.3 to a proof of Theorem 4.16. We start our quest with the following obviously useful definition.

Definition 5.1. Let $n \in \mathbb{N}$. We define

$$f_n : \mathcal{B}_{n-1} \to \mathcal{P}(\mathcal{A}_{n-1})$$
$$\{(i_{1,1}, \ldots, i_{1,k_1}), \ldots, (i_{n,1}, \ldots, i_{n,k_n})\} \mapsto \{(i_{1,k_1}, \ldots, i_{1,1}), \ldots, (i_{n,k_n}, \ldots, i_{n,1})\}$$

and for $A \in \mathcal{B}_{n-1}$ we call $f_n(A)$ the time reversal of A.

We should note that the time reversal of $A \in \mathcal{B}_{T-1}$ has not to be necessarily an element of $\mathcal{B}_{T-1}$ again. If we take for example $T = 3$ and $A = \{(1), (-1, 1)\}$, which corresponds to the stopping time which stops at the first time at which the random walk Y goes up, then we observe that $f_T(A) = \{(1), (1, -1)\}$, which is not an antichain with respect to the relation of extension defined in 4.11, therefore, does not correspond to any stopping time in $\mathcal{T}_3$. Hence, we will take now a closer look at the question for which subsets of $\mathcal{B}_{T-1}$ the time reversal is again an element of $\mathcal{B}_{T-1}$. For this purpose, we give the following definition.

Definition 5.2. Let $n \in \mathbb{N}$. We define the binary relation $R_{\preccurlyeq,n} \subseteq \mathcal{A}_n^2$ as the set of all pairs $(x, y) \in \mathcal{A}_n^2$ so that y is a backward extension of x and denote $(x, y) \in \mathcal{R}_{\preccurlyeq,n}$ as $x \preccurlyeq y$, that means for $x = (i_1, \ldots, i_k), y = (j_1, \ldots, j_l)$

$$x \preccurlyeq y \qquad :\Longleftrightarrow \qquad ((k \le l) \wedge ((i_1, \ldots, i_k) = (j_{l-k+1}, \ldots, j_l))).$$

This relation is also known as suffix-order in formal language theory. The notation used for this relation is justified by the next lemma, which can be proven analogously to Lemma 4.12.

Lemma 5.3. *The relation of backward extension $R_{\preccurlyeq,n}$ is a partial order on $\mathcal{A}_n$.*

Now we are able to formulate the following lemma.

Lemma 5.4. *For $A \in \mathcal{B}_{T-1}$ the time reversal $f_T(A)$ is an element of $\mathcal{B}_{T-1}$ if and only if A is an antichain with respect to the relation of backward extension. If we further define the function $r_T := p_T^{-1} \circ f_T \circ p_T$ on $\mathcal{T}_T$, then for $\tau \in \{\tau' \in \mathcal{T}_T \mid f_T(p_T(\tau')) \in \mathcal{B}_{T-1}\}$ $r_T(\tau)$ is again an element of $\mathcal{T}_T$. We call $\tau \in \{\tau' \in \mathcal{T}_T \mid f_T(p_T(\tau')) \in \mathcal{B}_{T-1}\}$ a forward-backward stopping time and $r_T(\tau)$ the time reversal of τ.*

Proof. Let $A \in \mathcal{B}_{T-1}$. For $(i_1, \dots, i_k), (j_1, \dots j_l) \in \mathcal{A}_{T-1}$ we have

$$(i_1, \dots, i_k), (j_1, \dots, j_l) \in A \iff (i_k, \dots, i_1), (j_l, \dots, j_1) \in f_T(A)$$

and

$$\begin{aligned}(i_1, \dots, i_k) \prec (j_1, \dots, j_l) &\iff ((k < l) \wedge ((i_1, \dots, i_k) = (j_{l-k+1}, \dots, j_l)) \\ &\iff ((k < l) \wedge ((i_k, \dots, i_1) = (j_l, \dots, j_{l-k+1})) \\ &\iff (i_k, \dots, i_1) < (j_l, \dots, j_1),\end{aligned}$$

which shows the first claim. The second claim is just a consequence of the definitions of the functions and the set $\{\tau \in \mathcal{T}_T \mid f_T(p_T(\tau)) \in \mathcal{B}_{T-1}\}$. □

We should note that our function r_T is not a bijection from $\mathcal{T}_T$ to itself and that even for forward-backward stopping times the distributional property 5.2 does not necessarily hold, because the proof for the case $t = T$ in an analogous proof to that of Lemma 4.15 fails. If we take for example the forward-backward stopping time $\tau \in \mathcal{T}_3$ which corresponds to the set $A_\tau = \{(1, -1), (1, 1)\}$, then we have that the forward-backward stopping time $r_3(\tau)$ corresponds to the set $A_{r_3(\tau)} = \{(-1, 1), (1, 1)\}$. Then we get for positive probabilities for "up" and "down"

$$\mathbb{Q}'[\tau = 3, X_3 = 1, X_2 = 1, X_1 = -1] = \mathbb{Q}'[X_3 = 1, X_2 = 1, X_1 = -1] > 0,$$

whereas we have

$$\mathbb{Q}'[r_3(\tau) = 3, X_1 = 1, X_2 = 1, X_3 = -1] = 0,$$

because $r_3(\tau) = 2$ on the set $\{\omega \in \Omega \mid X_1(\omega) = 1, X_2(\omega) = 1\}$ as a consequence of $(1, 1) \in A_{r_3(\tau)}$.
Hence, we have to restrict the set of stopping times we are looking at even more.

Definition 5.5. We say that a stopping time τ is a strong forward-backward stopping time if and only if the set $\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) \leq T\}$ is an antichain with respect to the relation of backward extension.

The last step we have to take before being able to prove the distributional property 5.1 we want, is the next lemma, which shows how exactly the time reversal of a strong forward-backward stopping time looks like.

Lemma 5.6. *Every strong forward-backward stopping time $\tau \in \mathcal{T}_T$ is a forward-backward stopping time and satisfies*

$$f_{T+1}(\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\}) = \{(X_1(\omega), \dots, X_{r_T(\tau)(\omega)}(\omega)) \mid \omega \in \Omega\}.$$

Every almost surely constant stopping time $\tau \in \mathcal{T}_T$ is a strong forward-backward stopping time and satisfies $r_T(\tau) = \tau$.

Proof. Let $\tau \in \mathcal{T}_T$ be a strong forward-backward stopping time. Then by definition $\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) \leq T\}$ is an antichain regarding the order of backward extension. Because every subset of an antichain is an antichain too and

$$p_T(\tau) = \{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) < T\},$$

we get that $p_T(\tau)$ is an antichain regarding the relation of backward extension. Then Lemma 5.4 implies $f_T(p_T(\tau)) \in \mathcal{B}_{T-1}$, which means that τ is a forward-backward stopping time by definition.

Now let us prove the stated equation. By definition of f_T, f_{T+1} and r_T it is clear that we have

$$\begin{aligned} & f_{T+1}(\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) < T\} = p_T(r_T(\tau)) \\ & = \{(X_1(\omega), \dots, X_{r_T(\tau)(\omega)}(\omega)) \mid \omega \in \Omega : r_T(\tau)(\omega) < T\}). \end{aligned}$$

The stopping time τ must be defined on every atom of $\mathcal{F}_T$ and every sequence in the set $\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) < T\}$ can be identified with the number of atoms on which this sequence is realized by the starting sequence of the random variables $(X_i)_{1 \leq i \leq T}$. Hence, we know the number of sequences that have to be in $f_{T+1}(\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) = T\}$, because every sequence in this set counts for exactly one atom and we further know, using the same argument for $r_T(\tau)$ again, that this number has to be equal to the number of sequences in the set $\{(X_1(\omega), \dots, X_{r_T(\tau)(\omega)}(\omega)) \mid \omega \in \Omega : r_T(\tau)(\omega) = T\}$ as a consequence of the equation above. Because τ is a strong forward-backward stopping time it must be possible to add every sequence in the first set to the second set or equivalently to define $r_T(\tau)$ to take the value T on the corresponding atoms without getting a contradiction to $r_T(\tau)$ being a stopping time. On the other hand this has to be the only possible choice, because p_T is a bijection (see Lemma 4.13) and we have

$$(p_T)^{-1}(f_{T+1}(\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega : \tau(\omega) < T\}) = r_T(\tau).$$

Hence, the sets have to be equal.

For the proof of the last statement let $\tau \in \mathcal{T}_T$ be an almost surely constant stopping time which stops at time t. That implies that the set $p_{T+1}(\tau)$ has to contain all $\{1,-1\}$-sequences of length t. Hence, we have $f_{T+1}(p_{T+1}(\tau)) = p_{T+1}(\tau)$ and, therefore, $r_T(\tau) = \tau$, because reversing them does not change anything and it is clear that the set is an antichain with respect to backward extension as all sequences have the same length. □

Now we are finally able to prove Property 5.1 for the set of strong forward-backward stopping times and a slightly generalised version of the distributional property in Lemma 4.15, which we will both use to prove relations for Asian options.

Lemma 5.7. *Let $\tau \in \mathcal{T}_T$ be a stopping time.*

1. *Then τ is a strong forward-backward stopping time if and only if τ satisfies*
$$Law(\tau, Y_\tau - Y_\tau, \dots, Y_\tau - Y_0 \mid \mathbb{Q}') = Law(r_T(\tau), Y_0, \dots, Y_{r_T(\tau)} \mid \mathbb{Q}').$$
Further $r_T(\tau) \in \mathcal{T}_T$ is the only stopping time with this property.

2. *Then τ satisfies*
$$Law(\tau, Y_0, \dots, Y_\tau \mid \mathbb{Q}') = Law(h_T(\tau), -\hat{Y}_0, \dots, \hat{Y}_{h_T(\tau)} \mid \hat{\mathbb{Q}}).$$

Proof. We start to prove the first statement. First, we note that

$$Law(\tau, Y_\tau - Y_\tau, \dots, Y_\tau - Y_0 \mid \mathbb{Q}') = Law(r_T(\tau), Y_0, \dots, Y_{r_T(\tau)} \mid \mathbb{Q}')$$

is equivalent to

$$Law(\tau, X_\tau, \dots, X_1 \mid \mathbb{Q}') = Law(r_T(\tau), X_1, \dots, X_{r_T(\tau)} \mid \mathbb{Q}')$$

as a consequence of $Y_t = \sum_{k=1}^{t} X_k$ for $t \in \{0, \dots, T\}$.

We start with the assumption that $\tau \in \mathcal{T}_T$ is a strong forward-backward stopping time and fix $t \in \{0, \dots, T\}$ and $i_l \in \{-1, 1\}$ for $l \in \{1, \dots, T\}$. We obtain

$$\begin{aligned}
&\mathbb{Q}'[\tau = t, X_\tau = i_t, \dots, X_1 = i_1] = \mathbb{Q}'[\tau = t, X_t = i_t, \dots, X_1 = i_1] \\
&= \mathbb{Q}'\left(\dot{\bigcup_{\substack{(j_1,\dots,j_t) \in \{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\} \\ (j_1,\dots,j_t) = (i_1,\dots,i_t)}}} \{\omega \in \Omega \mid X_t(\omega) = i_t, \dots, X_1(\omega) = i_1\} \right) \\
&= \sum_{\substack{(j_1,\dots,j_t) \in \{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\} \\ (j_1,\dots,j_t) = (i_1,\dots,i_t)}} \mathbb{Q}'[X_t = i_t, \dots, X_1 = i_1].
\end{aligned}$$

Now we use the exchangeability of the random variables $(X_i)_{1 \le i \le T}$ (see Lemma 4.1), Definition 5.1 and then Lemma 5.6 to get

$$\begin{aligned}
\mathbb{Q}'[\tau = t, X_\tau = i_t, \dots, X_1 = i_1] &= \sum_{\substack{(j_1,\dots,j_t)\in\{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega))\mid\omega\in\Omega\}\\(j_1,\dots,j_t)=(i_1,\dots,i_t)}} \mathbb{Q}'[X_1 = i_t, \dots, X_t = i_1]\\
&= \sum_{\substack{(j_t,\dots,j_1)\in f_{T+1}(\{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega))\mid\omega\in\Omega\})\\(j_t,\dots,j_1)=(i_t,\dots,i_1)}} \mathbb{Q}'[X_1 = i_t, \dots, X_t = i_1]\\
&= \sum_{\substack{(j_t,\dots,j_1)\in\{(X_1(\omega),\dots,X_{r_T(\tau)(\omega)}(\omega))\mid\omega\in\Omega\}\\(j_t,\dots,j_1)=(i_t,\dots,i_1)}} \mathbb{Q}'[X_1 = i_t, \dots, X_t = i_1]\\
&= \mathbb{Q}'\left(\dot{\bigcup_{\substack{(j_t,\dots,j_1)\in\{(X_1(\omega),\dots,X_{r_T(\tau)(\omega)}(\omega))\mid\omega\in\Omega\}\\(j_t,\dots,j_1)=(i_t,\dots,i_1)}}} \{\omega\in\Omega \mid X_1(\omega) = i_t, \dots, X_t(\omega) = i_1\}\right)\\
&= \mathbb{Q}'[r_T(\tau) = t, X_1 = i_t, \dots, X_t = i_1] = \mathbb{Q}'[r_T(\tau) = t, X_1 = i_t, \dots, X_{r_T(\tau)} = i_1],
\end{aligned}$$

which finishes the proof of the first direction of the equivalence.

For the other implication, we assume that $\tau \in \mathcal{T}_T$ is not a strong forward-backward stopping time. So by Definition 5.5 $\{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega) \mid \omega \in \Omega\}$ is not a backward antichain, which by Definition 5.2 means that there exist $k, l \in \{0, \dots, T\}$ so that $k < l \leq T$ and $(i_l, \dots, i_1), (i_k, \dots, i_1) \in \{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega) \mid \omega \in \Omega\}$. Then we have

$$\mathbb{Q}'[\tau = l, X_l = i_1, \dots, X_1 = i_l] = \mathbb{Q}'[X_l = i_1, \dots, X_1 = i_l] > 0$$

as a consequence of $(i_l, \dots, i_1) \in \{(X_1(\omega), \dots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\}$ and hence $\{\omega \in \Omega \mid X_1(\omega) = i_l, \dots, X_l(\omega) = i_1\} \subseteq \{\omega \in \Omega \mid \tau(\omega) = l\}$ and the fact that $(X_i)_{1\leq i\leq T}$ are i.i.d. under $\mathbb{Q}'$ (see Lemma 4.1), we have assumed $0 < \mathbb{P}[X_1 = 1] < 1$ (see Section 2.2) and know that $\mathbb{Q}'$ is equivalent to $\mathbb{Q}$ by definition (see Lemma 4.1) and $\mathbb{Q}$ is equivalent to $\mathbb{P}$ by definition (see Theorem 2.6). Because we know $k < T$ we get $(i_k, \dots, i_1) \in A_\tau$, which implies $(i_1, \dots, i_k) \in f_T(A_\tau)$. As a consequence of Lemma 5.4 we have $f_T(A_\tau) \in \mathcal{B}_{T-1}$ and hence $(i_1, \dots, i_l) \notin f_T(A_\tau)$. This yields

$$\begin{aligned}
&\mathbb{Q}'[r_T(\tau) = l, X_1 = i_1, \dots, X_l = i_l] =\\
&= \mathbb{Q}'\left(\dot{\bigcup_{\substack{(j_1,\dots,j_l)\in f_T(A_\tau)\\(j_1,\dots,j_l)=(i_1,\dots,i_l)}}} \{\omega\in\Omega \mid X_1(\omega) = i_1, \dots, X_l(\omega) = i_l\}\right)\\
&= \mathbb{Q}'(\emptyset) = 0,
\end{aligned}$$

which shows that the desired property does not hold and hence finishes the proof of the equivalence.

Now we want to show uniqueness. Let $\tilde{\tau} \in \mathcal{T}_T$ with $\tilde{\tau} \neq r_T(\tau)$. Without loss of generality there exists $\tilde{\omega} \in \Omega$ with $\tilde{\tau}(\tilde{\omega}) < r_T(\tau)(\tilde{\omega})$. Because $\tilde{\tau}$ and $r_T(\tau)$ are stopping times we have $\{\omega \in \Omega \mid \tilde{\tau}(\omega) = \tilde{\tau}(\tilde{\omega})\}, \{\omega \in \Omega \mid r_T(\tau)(\omega) = \tilde{\tau}(\tilde{\omega})\} \in \mathcal{F}_{\tilde{\tau}(\tilde{\omega})}$. Hence, the indicator functions $\mathbb{1}_{\{\tilde{\tau}=\tilde{\tau}(\tilde{\omega})\}}$ and $\mathbb{1}_{\{r_T(\tau)=\tilde{\tau}(\tilde{\omega})\}}$ have to be constant on the atoms of $\mathcal{F}_{\tilde{\tau}(\tilde{\omega})}$. Therefore, we get by the same arguments as above

$$\mathbb{Q}'[\tilde{\tau} = \tilde{\tau}(\tilde{\omega}), X_1 = X_1(\tilde{\omega}), \dots, X_{\tilde{\tau}} = X_{\tilde{\tau}}(\tilde{\omega})] = \mathbb{Q}'[X_1 = X_1(\tilde{\omega}), \dots, X_{\tilde{\tau}} = X_{\tilde{\tau}(\tilde{\omega})}(\tilde{\omega})] > 0$$

and

$$\mathbb{Q}'[r_T(\tau) = \tilde{\tau}(\tilde{\omega}), X_1 = X_1(\tilde{\omega}), \dots, X_{\tilde{\tau}} = X_{r_T(\tau)}(\tilde{\omega})] = \mathbb{Q}'(\emptyset) = 0,$$

which implies that the desired property cannot hold for $\tilde{\tau}$ and as a consequence shows uniqueness.

Now we want to give a proof for the second statement. For this purpose we only assume that τ is a stopping time. Again we note that

$$Law(\tau, Y_0, \dots, Y_\tau \mid \mathbb{Q}') = Law(h_T(\tau), -\hat{Y}_0, \dots, -\hat{Y}_{h_T(\tau)} \mid \hat{\mathbb{Q}})$$

is equivalent to

$$Law(\tau, X_1, \dots, X_\tau \mid \mathbb{Q}') = Law(h_T(\tau), -\hat{X}_1, \dots, -\hat{X}_{h_T(\tau)} \mid \hat{\mathbb{Q}})$$

as a consequence of $Y_t = \sum_{k=1}^t X_k$ and $\hat{Y}_t = \sum_{k=1}^t \hat{X}_k$ for $t \in \{0, \dots, T\}$. Again we fix $t \in \{0, \dots, T\}$ and $i_l \in \{-1, 1\}$ for $l \in \{1, \dots, T\}$ and obtain

$$\begin{aligned}
&\mathbb{Q}'[\tau = t, X_1 = i_1, \dots, X_\tau = i_t] = \mathbb{Q}'[\tau = t, X_1 = i_1, \dots, X_t = i_t] = \\
&= \mathbb{Q}'\left(\dot{\bigcup_{\substack{(j_1,\dots,j_t) \in \{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\} \\ (j_1,\dots,j_t) = (i_1,\dots,i_t)}}} \{\omega \in \Omega \mid X_t(\omega) = i_t, \dots, X_1(\omega) = i_1\} \right) \\
&= \sum_{\substack{(j_1,\dots,j_t) \in \{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\} \\ (j_1,\dots,j_t) = (i_1,\dots,i_t)}} \mathbb{Q}'[X_1 = i_1, \dots, X_t = i_t].
\end{aligned}$$

Now we use the first statement of Lemma 4.2, then view τ as stopping time in $\mathcal{T}_{T+1}$ and use Lemma 4.15 to get

$$\begin{aligned}
&\mathbb{Q}'[\tau = t, X_1 = i_1, \dots, X_\tau = i_t] = \\
&= \sum_{\substack{(j_1,\dots,j_t) \in \{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\} \\ (j_1,\dots,j_t) = (i_1,\dots,i_t)}} \hat{\mathbb{Q}}[-\hat{X}_1 = i_1, \dots, -\hat{X}_t = i_t] \\
&= \sum_{\substack{(j_1,\dots,j_t) \in g_{T+1}(\{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\}) \\ (j_1,\dots,j_t) = (-i_1,\dots,-i_t)}} \hat{\mathbb{Q}}[-\hat{X}_1 = i_1, \dots, -\hat{X}_t = i_t].
\end{aligned}$$

We know $g_{T+1}(\{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega\in\Omega : \tau(\omega)<T\}) = g_T(\{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega\in\Omega : \tau(\omega)<T\}) = \hat{p}_T(h_T(\tau)) = \{(\hat{X}_1(\omega),\dots,\hat{X}_{h_T(\tau)(\omega)}(\omega)) \mid \omega\in\Omega : h_T(\tau)(\omega)<T\}$. We further use the fact that τ and $h_T(\tau)$ are stopping times and note

$$
\begin{aligned}
&(j_1,\dots,j_T)\in g_{T+1}(\{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega\in\Omega\}) \iff\\
&\iff \nexists\, 0\le k\le T-1 : (j_1,\dots,j_k)\in g_T(\{(X_1(\omega),\dots,X_{\tau(\omega)}(\omega)) \mid \omega\in\Omega : \tau(\omega)<T\})\\
&\iff \nexists\, 0\le k\le T-1 : (j_1,\dots,j_k)\in \{(\hat{X}_1(\omega),\dots,\hat{X}_{h_T(\tau)(\omega)}(\omega)) \mid \omega\in\Omega : h_T(\tau)(\omega)<T\}\\
&\iff (j_1,\dots,j_T)\in\{(\hat{X}_1(\omega),\dots,\hat{X}_{h_T(\tau)(\omega)}(\omega)) \mid \omega\in\Omega\}.
\end{aligned}
$$

Hence, we continue

$$
\begin{aligned}
&\mathbb{Q}'[\tau=t, X_1=i_1,\dots,X_\tau=i_t] =\\
&= \sum_{\substack{(j_1,\dots,j_t)\in\{(\hat{X}_1(\omega),\dots,\hat{X}_{h_T(\tau)(\omega)}(\omega))\mid\omega\in\Omega\}\\ (j_1,\dots,j_t)=(-i_1,\dots,-i_t)}} \hat{\mathbb{Q}}[-\hat{X}_1=i_1,\dots,-\hat{X}_t=i_t]\\
&= \hat{\mathbb{Q}}\left(\bigcup_{\substack{(j_1,\dots,j_t)\in\{(\hat{X}_1(\omega),\dots,\hat{X}_{h_T(\tau)(\omega)}(\omega))\mid\omega\in\Omega\}\\ (j_1,\dots,j_t)=(-i_1,\dots,-i_t)}} \{\omega\in\Omega \mid -\hat{X}_1(\omega)=i_1,\dots,-\hat{X}_t(\omega)=i_t\}\right)\\
&= \hat{\mathbb{Q}}[h_T(\tau)=t,-\hat{X}_1=i_1,\dots,-\hat{X}_t=i_t] = \hat{\mathbb{Q}}[h_T(\tau)=t,-\hat{X}_1=i_1,\dots,-\hat{X}_{h_T(\tau)}=i_t],
\end{aligned}
$$

which finishes the proof. □

5.2. Duality relations for strong forward-backward stopping times

In this section we prove duality relations for Asian options with respect to strong forward-backward stopping times and discuss in which sense they extend or relate to the relations shown in Corollary 4.8.

The following theorem gives us relations for Asian options with fixed and floating strike prices for single strong forward-backward stopping times.

Theorem 5.8. *Define $w_T := h_T\circ r_T$. Let $\tau\in\mathcal{T}_T$ be a strong forward-backward stopping time for our binomial model with parameters (S_0,a,b,u,d). Then $w_T(\tau)$ is a stopping time for the binomial model with parameters $(S_0,b,a,1/d,1/u)$ and we have the relations*

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}\ (S_\tau - A(0,\tau))^+\right] = \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{w_T(\tau)}}\ \left(\hat{S}_0 - \hat{A}(0,w_T(\tau))\right)^+\right]$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^\tau}\ (A(0,\tau) - S_\tau)^+\right] = \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{w_T(\tau)}}\ \left(\hat{A}(0,w_T(\tau)) - \hat{S}_0\right)^+\right].$$

Proof. We restrict ourselves to the proof of the first relation as the proof of the second relation works analogously. We start with the left-hand side and change the measure from the original martingale measure to the dual measure $\mathbb{Q}'$ using the density $d\mathbb{Q}'/d\mathbb{Q} = B_0/V_0 \cdot V_T/B_T = (a^T S_T)/(b^T S_0)$ to get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\frac{a^T S_T}{b^T S_0}\frac{b^T S_0}{a^T S_T}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{b^{\tau_{call}}}\frac{b^T S_0}{a^T S_T}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right].\end{aligned}$$

Now we use the law of total expectation for conditional expectations and the fact that $1/b^{\tau_{call}}(S_{\tau_{call}} - A(0,\tau_{call}))^+$ is $F_{\tau_{call}}$-measurable as a consequence of Lemma 2.4 and the property of taking out what is known of the conditional expectation to obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] &= \mathbb{E}_{\mathbb{Q}'}\left[\mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{b^{\tau_{call}}}\frac{b^T S_0}{a^T S_T}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+ \mid F_{\tau_{call}}\right]\right] \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\mathbb{E}_{\mathbb{Q}'}\left[\frac{b^T S_0}{a^T S_T} \mid F_{\tau_{call}}\right]\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right].\end{aligned}$$

We continue with applying the optional sampling theorem (see Theorem 2.5) for the bounded stopping time $\tau_{call} \in \mathcal{T}_T$ and the $\mathbb{Q}'$-martingale $(B_t V_0)/(V_t B_0) = (b^t S_0)/(a^t S_t)$ and the definition $A(0,t) = 1/(t+1) \cdot \sum_{k=0}^{t} S_k$:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] &= \mathbb{E}_{\mathbb{Q}'}\left[\frac{b^{\tau_{call}} S_0}{a^{\tau_{call}} S_{\tau_{call}}}\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^{\tau_{call}}}\left(S_0 - S_0\frac{\frac{1}{\tau_{call}+1}\sum_{k=0}^{\tau_{call}} S_k}{S_{\tau_{call}}}\right)^+\right].\end{aligned}$$

Further, we use the definition $S_t = S_0 \exp((t+Y_t)/2 \cdot \log(u) + (t-Y_t)/2 \cdot \log(d))$ and rewrite the expression in terms of Y_t to obtain

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] = \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^{\tau_{call}}}\left(S_0 - S_0\frac{\frac{1}{\tau_{call}+1}\sum_{k=0}^{\tau_{call}} S_0\exp\left(\frac{k+Y_k}{2}\log(u) + \frac{k-Y_k}{2}\log(d)\right)}{S_0\exp\left(\frac{\tau_{call}+Y_{\tau_{call}}}{2}\log(u) + \frac{\tau_{call}-Y_{\tau_{call}}}{2}\log(d)\right)}\right)^+\right] \\ &= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^{\tau_{call}}}\left(S_0 - S_0\frac{1}{\tau_{call}+1}\sum_{k=0}^{\tau_{call}}\exp\left(-\frac{\tau_{call}-k+(Y_{\tau_{call}}-Y_k)}{2}\log(u)\right.\right.\right. \\ &\qquad \left.\left.\left.- \frac{\tau_{call}-k-(Y_{\tau_{call}}-Y_k)}{2}\log(d)\right)\right)^+\right].\end{aligned}$$

Due to Lemma 5.7 we can replace τ_{call} with $r_T(\tau_{call})$ and $Y_{\tau_{call}} - Y_k$ with $Y_{r_T(\tau_{call})-k}$ for all $k \in \{0,\dots,r_T(\tau_{call})\}$, use the identity $-\log(x) = \log(1/x)$ and change the summation index to get

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] =$$
$$= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^{r_T(\tau_{call})}}\left(S_0 - S_0\frac{1}{r_T(\tau_{call})+1}\sum_{k=0}^{r_T(\tau_{call})}\exp\left(\frac{r_T(\tau_{call}) - k + Y_{r_T(\tau_{call})-k}}{2}\log\left(\frac{1}{u}\right)\right.\right.\right.$$
$$\left.\left.\left.+\frac{r_T(\tau_{call}) - k - Y_{r_T(\tau_{call})-k}}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right]$$
$$= \mathbb{E}_{\mathbb{Q}'}\left[\frac{1}{a^{r_T(\tau_{call})}}\left(S_0 - S_0\frac{1}{r_T(\tau_{call})+1}\sum_{j=0}^{r_T(\tau_{call})}\exp\left(\frac{j+Y_j}{2}\log\left(\frac{1}{u}\right)\right.\right.\right.$$
$$\left.\left.\left.+\frac{j-Y_j}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right].$$

By applying the second statement of Lemma 5.7 we are able to interchange $\mathbb{Q}'$ with $\hat{\mathbb{Q}}$, $r_T(\tau_{call})$ with $w_T(\tau_{call}) = h_T(r_T(\tau_{call}))$ and Y_j with $-\hat{Y}_j$ for all $j \in \{0,\dots,w(\tau_{call})\}$ to get

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] =$$
$$= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{w_T(\tau_{call})}}\left(S_0 - S_0\frac{1}{w_T(\tau_{call})+1}\sum_{j=0}^{w_T(\tau_{call})}\exp\left(\frac{j-\hat{Y}_j}{2}\log\left(\frac{1}{u}\right)\right.\right.\right.$$
$$\left.\left.\left.+\frac{j+\hat{Y}_j}{2}\log\left(\frac{1}{d}\right)\right)\right)^+\right].$$

Now rewriting the formula in terms of $\hat{S}$ and $\hat{A}$ leads to

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{b^{\tau_{call}}}\left(S_{\tau_{call}} - A(0,\tau_{call})\right)^+\right] = \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{w_T(\tau_{call})}}\left(\hat{S}_0 - \frac{1}{w_T(\tau_{call})+1}\sum_{j=0}^{w_T(\tau_{call})}\hat{S}_j\right)^+\right]$$
$$= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{1}{a^{w_T(\tau_{call})}}\left(\hat{S}_0 - \hat{A}(0, w_T(\tau_{call}))\right)^+\right],$$

which finishes the proof. □

The next lemma shows that w_T is a bijection and maps (strong) forward-backward stopping times to (strong) forward-backward stopping times.

Lemma 5.9. *The function $w_T := h_T \circ r_T$ is a bijection from $\{\tau \in \mathcal{T}_T | \tau$ is a forward-backward stopping time$\}$ to $\{\hat{\tau} \in \hat{\mathcal{T}} | \hat{\tau}$ is a forward-backward stopping time$\}$. If $\tau \in \mathcal{T}_T$ is a strong forward-backward stopping time, then $w_T(\tau) \in \hat{\mathcal{T}}_T$ is also a strong-forward backward stopping time.*

Proof. Let $\tau \in \mathcal{T}_T$. As a consequence of the definition of forward-backward stopping times (see Lemma 5.4) $r_T(\tau)$ is a stopping time. Because τ is a stopping time, $p_T(\tau)$ is an antichain with respect to forward extension (see Lemma 4.13) and hence it is easy to see that $f_T(p_T(\tau)) = p_T(r_T(\tau))$ is an antichain with respect to backward extension, which is obviously equivalent to $f_T(p_T(r_T(\tau)))$ being an antichain with respect to forward extension. This shows that $r_T(\tau)$ is a forward-backward antichain. Obviously, f_T is injective and hence r_T is injective on the set of forward-backward stopping times to itself, which shows that it is bijective on this set.

For τ being a strong forward-backward stopping time, we get using Lemma 4.13 for $n = T+1$ and τ viewed as an element of $\mathcal{T}_{T+1}$ that the set $(\{(X_1(\omega), \ldots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\})$ is an antichain with respect to forward extension. Hence, $f_{T+1}(\{(X_1(\omega), \ldots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\})$ is an antichain with respect to backward extension. Combined with Lemma 5.6, which tells us

$$f_{T+1}(\{(X_1(\omega), \ldots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\}) = \{(X_1(\omega), \ldots, X_{r_T(\tau)(\omega)}(\omega)) \mid \omega \in \Omega\},$$

this shows that $r_T(\tau)$ is a strong forward-backward stopping time.

Lemma 4.15 tells us that h_T is a bijection too. Therefore, $w_T = h_T \circ r_T$ is a bijection. Clearly g_T and g_{T+1} leave the antichain properties with respect to forward and backward extension unchanged and hence h_T maps (strong) forward backward stopping times to (strong) forward backward stopping times, which finishes the proof. □

The next corollary is a statement about duality relations for Asian options of American type restricted to the smaller set of strong forward-backward stopping times, which was the result we intended to show in this section.

Corollary 5.10. *Let $\mathcal{T} \subseteq \mathcal{T}_T$ be a set of strong forward-backward stopping times for our binomial model with parameters (S_0, a, b, u, d). Then we have*

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{b^{\tau}} \; (S_\tau - A(0,\tau))^+ \right] = \sup_{\hat{\tau} \in w_T(\mathcal{T})} \mathbb{E}_{\hat{\mathbb{Q}}} \left[\frac{1}{a^{\hat{\tau}}} \; \left(\hat{S}_0 - \hat{A}(0,\hat{\tau}) \right)^+ \right]$$

and

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{b^{\tau}} \; (0, A(\tau) - S_\tau)^+ \right] = \sup_{\hat{\tau} \in w_T(\mathcal{T})} \mathbb{E}_{\hat{\mathbb{Q}}} \left[\frac{1}{a^{\hat{\tau}}} \; \left(\hat{A}(0,\hat{\tau}) - \hat{S}_0 \right)^+ \right].$$

Proof. This is a direct consequence of Theorem 5.8. □

There is still the open question, whether there exist strong forward-backward stopping times that are not constant. The next lemma shows that strong forward-backward stopping times cannot only take exactly two consecutive points in time as values, which at least constrains the set of possible strong forward-backward stopping times a little bit.

Lemma 5.11. *Let $\tau \in \mathcal{T}_T$ be a strong forward-backward stopping time with*

$$\max_{\omega\in\Omega} \tau(\omega) - \min_{\omega\in\Omega} \tau(\omega) \leq 1.$$

Then τ is constant.

Proof. Let τ be such a stopping time. First, we denote $n := \min_{\omega\in\Omega} \tau(\omega)$,

$$A := \{(X_1(\omega), \ldots, X_{\tau(\omega)}(\omega)) \mid \omega \in \Omega\}$$

and $(i_1, \ldots, i_n) := (X_1(\omega), \ldots, X_{\tau(\omega)}(\omega))$ for an $\omega \in \Omega$ with $\tau(\omega) = n$. Our aim is to show that A contains only sequences of length n. For this purpose we will reconstruct A using an iterative procedure.

We start with $A^0 := \{(i_1, \ldots, i_n)\}$. Because τ is a strong forward-backward stopping time the set A is a backward antichain. Hence, $(i_1, \ldots, i_n) \in A$ implies $(1, i_1, \ldots, i_n) \notin A$ and $(-1, i_1, \ldots, i_n) \notin A$. Using this result, the fact that there exist $\omega_1, \omega_2 \in \Omega$ with $(1, i_1, \ldots, i_n) = ((X_1(\omega_1), \ldots, X_{n+1}(\omega_1))$ and $(-1, i_1, \ldots, i_n) = ((X_1(\omega_2), \ldots, X_{n+1}(\omega_2))$ and $\tau(\omega_1), \tau(\omega_2) \in \{n, n+1\}$ by our assumption, we obtain

$$(1, i_1, \ldots, i_{n-1}), (-1, i_1, \ldots, i_{n-1}) \in A.$$

We define

$$A^0 := A^1 \cup \{(j_1, i_1, \ldots, i_{n-1}) \mid j_1 \in \{-1, 1\}\}$$

and get

$$A^1 \subseteq A.$$

Now we can iterate the arguments above for every element of $\{(j_1, i_1, \ldots, i_{n-1}) \mid j_1 \in \{-1, 1\}\}$ and define for every $k \in \{2, \ldots, n\}$

$$A^k := A^{k-1} \cup \{(j_1, \ldots, j_k, i_1, \ldots, i_{n-k}) \mid j_1, \ldots, j_k \in \{-1, 1\}\} \quad \text{with} \quad A_k \subseteq A.$$

We should note that the sequence $(A^k)_{0\leq k\leq n}$ is strictly increasing because we have $|A^0| = 1$ and $|A^k| \leq |A^{k-1}| + 2^k$, therefore, by induction

$$|A^{k-1}| \leq \sum_{r=0}^{k-1} 2^r = 2^k - 1 < 2^k = |\{(j_1, \ldots, j_k, i_1, \ldots, i_{n-k}) \mid j_1, \ldots, j_k \in \{-1, 1\}\}|.$$

We further note that this iterative procedure stops in step n and we have

$$A^n = \{(j_1, \ldots, j_n) \mid j_1, \ldots, j_n \in \{-1, 1\}\} \subseteq A.$$

We see that A^n contains every $\{-1, 1\}$-sequence of length n and nothing else. As adding any other sequence would violate the forward (or backward) antichain property, we must conclude $A^n = A$, which finishes the proof. □

Because we know (see Lemma 5.6) that constant stopping times are strong forward-backward stopping times, Corollary 4.8 can be derived as special case of Corollary 5.10. If it turned out that there exist strong forward-backward stopping times which are not constant, then Corollary 5.10 would be a strictly stronger result than Corollary 4.8.

Otherwise, if it turns out that the set of strong forward-backward stopping times coincides with the set of almost surely constant stopping times, then Theorem 5.8 and Corollary 5.10 will be direct consequences of Corollary 4.8, hence both corollaries and Theorem 5.8 will be equivalent and we will further get that constant stopping times are the only stopping times in a binomial model with properties 5.1 and 5.2, which in my opinion will reduce the chance to extend the relations for Asian options to stopping times drastically. So both outcomes of the open question whether strong forward-backward stopping times that are not constant exist imply interesting consequences for the theory presented in this section.

A. Appendix

In this appendix we just want to shortly show how the valuation of Asian options of American type with respect to the arithmetic mean in a binomial model with dividend yield can be implemented. For this purpose, we use the free software Octave [1].

A.1. Code for Asian options of American type

The theory about pricing American options in discrete time models using the Snell envelope can be found in chapter 6 of the book of Föllmer and Schied [4]. The following code is a basic implementation of the defining backward recursion of the Snell envelope to get the prices of all four types of Asian options of American type discussed in this thesis. The parameters are denoted as in the theory presented in the other chapters.

```
function [fix_call, fix_put, float_call, float_put] = american_asian(S_0,K,a,b,u,d,T)
 #calculate equivalent martingale measure
 p=(b/a-d)/(u-d);
 q=1-p;

 #create matrix with paths of the stock price process as rows
 paths=ones(2^T,1)*S_0; #column for time t=0
 for t=1:T
   factor=mod(linspace(0,2^t-1,2^t),2)+1;
   count=ones(1,2^t)*2^(T-t);
   paths(:,t+1)=paths(:,t).*(repelems([u,d],[factor;count]))'; #add column for time t
 endfor

 #create matrix with paths of the arithmetic mean process as rows
 temp=triu(ones(T+1,T+1))./linspace(1,T+1,T+1);
 average=paths*temp;


 ##calculate option prices

 #calculate value of the options at each time step if they are exercised
 fix_call=max(average-K,0);
 fix_put=max(K-average,0);
 float_call=max(paths-average,0);
 float_put=max(average-paths,0);

 #backward recursions for option values
 for t=T:-1:1
  for k=1:2^(T+1-t):2^T
```

```
30      #calculate maximum of value if exercised and discounted expected value at time t+1
31      fix_call(k,t)=max(fix_call(k,t),(p*fix_call(k,t+1)+q*fix_call(k+2^(T-t),t+1))/b);
32      fix_put(k,t)=max(fix_put(k,t),(p*fix_put(k,t+1)+q*fix_put(k+2^(T-t),t+1))/b);
33      float_call(k,t)=max(float_call(k,t),(p*float_call(k,t+1)+q*float_call(k+2^(T-t),t+1))/b);
34      float_put(k,t)=max(float_put(k,t),(p*float_put(k,t+1)+q*float_put(k+2^(T-t),t+1))/b);
35    endfor
36  endfor
37
38  #return only prices at time t=0
39  fix_call=fix_call(1,1);
40  fix_put=fix_put(1,1);
41  float_call=float_call(1,1);
42  float_put=float_put(1,1);
43 endfunction
```

The problem with the algorithm above is that it takes $\Theta(2^T)$ time and hence is only useful to get the option prices for Asian options of American type up to $T = 25$ time steps. A faster algorithm for lower and upper bounds for option prices of Asian options of American type with respect to the arithmetic mean is for example discussed in Gaudenzi, Zanette and Lepellere [5] and another algorithm for approximate pricing devised by John Hull and Alan White can be found in Section 8.5 of the book by van der Hoek and Elliott [9].

Sometimes it might be useful to get exact prices for small examples like in Chapter 3. We should note that if the input parameters S_0, K, a, b, u, d for the algorithm above are rational numbers, then the option prices will be rational numbers too. Hence, we can use the symbolic package for Octave to calculate exact prices for rational parameters. Unfortunately, we cannot use the code above to do this, because it is not compatible with symbolic variables, but we only have to replace the lines 6 to 12 of the code above with the following lines.

```
#create matrix with paths of the stock price process as rows and matrix with paths of the arithmetic mean process as rows
paths=ones(2^T,1)*S; #column for time t=0
average=paths; #column for time t=0
for t=1:T
  factor=1;
  for k=1:2^(T-t):2^T
    paths(k:k+2^(T-t)-1,t+1)=paths(k:k+2^(T-t)-1,t)*(factor*u+(1-factor)*d);
    if (factor==1)
      factor=0;
    else
      factor=1;
    endif
  endfor
  average(:,t+1)=(average(:,t)*t+paths(:,t+1))/(t+1);
endfor
```

As these changes and using symbolic variables make the code take even longer it really should be used only for very small examples with just few time steps. To calculate with symbolic rational numbers in Octave, we first have to load the symbolic package and then initialise the rational input parameters as in the following example.

```
pkg load symbolic;

syms S_0 u d a b K;
S_0=sym(3);
u=sym(5)/2;
d=sym(1)/3;
a=sym(3)/2;
b=sym(5)/4;
K=sym(4)/3;
T=5;

[fix_call,fix_put,float_call,float_put]=american_asian_sym(S_0,K,a,b,u,d,T);
```

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