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Optimality of Adaptive 2D Boundary Element Method

Ausgeführt am Institut für

Analysis und Scientific Computing

unter der Anleitung von

Ao. Univ.-Prof. Dipl.-Math. Dr. techn. Dirk Praetorius

durch

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Contents

CHAPTER 1

Introduction & Outline

1.1. What is adaptivity?

What is *adaptivity*? A question, which I heard very often during the process of writing this work. My parents, the non-mathematicians among my friends, even my grandparents were—up to a certain amount—interested in understanding what I am doing. Often, when I tried to reply to that question, I gave the following answer.

In the beginning, there is always the problem, one wants to solve. In most of the practical cases, this would be some kind of differential equation, because these mathematical objects are superbly suited to describe processes which take place in the real world. For this introduction, however, I want to consider a much simpler problem, we all know from school: Calculating the area of a geometric figure. Our task shall be to calculate the area of the blue domain under the red curve in Figure 1. Maybe the first attempt would be to find a formula which describes the curve, and perform integration by hand. What is expedient in school, happens to be impossible in most of the practical cases. Even if one knows the correct formula for the curve, one is hardly able to calculate the area except for very simple cases. Furthermore, the curve is often described only by a set of data points, and there is no chance to find a formula.

But one simple thing one can definitely do, is shown in the middle image in Figure 1. One chooses a point on the axes between a and b and draws the biggest possible rectangles which fit under the curve. The area of rectangles is easy to calculate, just *width* times *height*, and it only remains to sum up. Of course one may argue that this is only an approximation to the real area and even a pretty bad one. But an approximation to the exact solution of a problem is often the best result, we can expect. So, lets refine this approach and draw many rectangles under the curve as seen in the rightmost image in Figure 1. This is called *uniform refinement* in literature because each rectangle has the same width. Again, we calculate the area of the rectangles and sum up. Now, this seems like a good solution. We are pretty close to the real area, and as mentioned before, a good approximation is the best we can hope for. But although one could perform the calculation of the rectangle area with a computer, one can imagine that every additional rectangle increases the computational work. This might not be a problem for our simple example, but in real world applications, computational resources are limited and cost lots of money. Therefore, we are interested to deploy our resources as smart as possible. When we go back to the middle image in Figure 1, we see that the approximation in the left part of the image is not too bad. The problem is the right part, where the area of

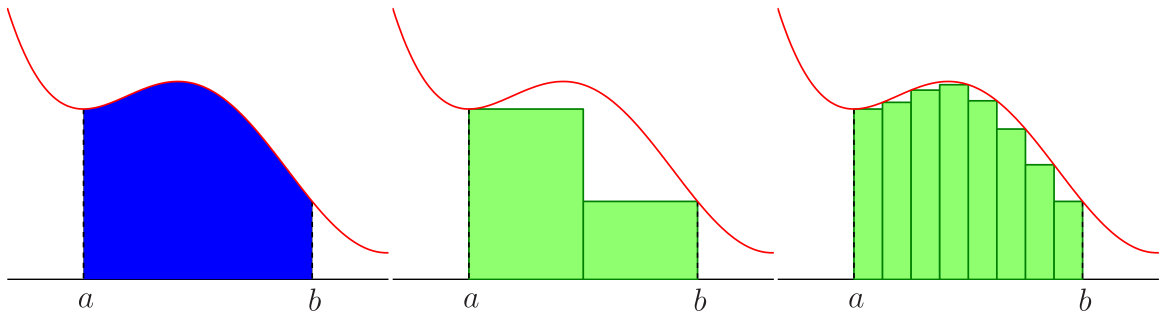

FIGURE 1. A simple quadrature problem.

the rectangle is nowhere near the exact area. Maybe it would be sufficient to refine only the right rectangle.

Here comes adaptivity into place. But how to quantify that some rectangles are *better* than others? One could consider a so called *uniform refinement* of the two rectangles into four rectangles as seen in the leftmost image in Figure 2. Now, we compare these two approximations. We see that the refinement of the right rectangle yields a huge advantage in accuracy, whereas the improvement for the left rectangle is moderate. This helps to decide which rectangle we have to refine and which not. In literature, this strategy is known as $(h-h/2)$-based error estimation, and it applies to a very wide field of interesting problems.

Another approach is to consider a simpler problem. As mentioned before, we are not able to calculate the area under the curve—which is not covered with the rectangles—exactly. But we are able to calculate the distance between the upper edge of the rectangle and the curve as depicted in the rightmost image in Figure 2. Again, we may use this quantity (perhaps weighted with the width of the rectangles) to decide, whether we should refine the rectangle or not. In both cases, we would end up with the approximation in the middle image in Figure 2. Of course, one refinement is often not enough to reach the

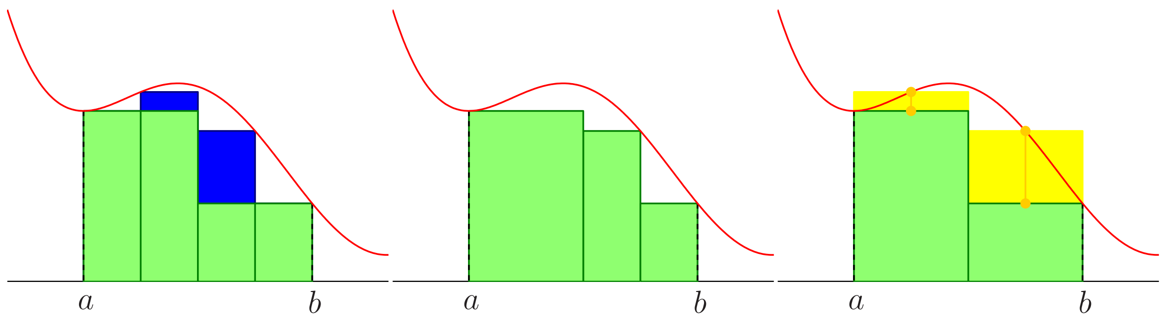

FIGURE 2. Smart improvement of the approximation.

desired quality of approximation. Thus, we iterate the described procedure until we are satisfied with the outcome in Figure 3. There, we see that the rectangles are adaptively distributed. In regions with steep slope, many refinements have been performed, whereas only a few refinements took place in regions, where the curve is flat. In contrast to this example, in many practical application it is not possible to *see* the error between approximation and the exact solution. Therefore, we are interested in *reliable* error estimators, which means that the quantity we use to decide which rectangle to refine gives also an upper estimate of the approximation error. In our particular case, the second strategy depicted in the rightmost image in Figure 2 yields a *reliable* error estimate. In order not to overestimate the real error—and consequently perform to many refinements for the desired accuracy—we talk of *efficient* error estimators. In many applications, $(h - h/2)$-based error estimators are efficient.

Now, as we thought of different error estimation strategies, two questions arise naturally.

First: Does the strategy produce arbitrarily accurate approximations if we only perform sufficiently many iterations? Note that the answer to this question seems obvious if we perform uniform refinements of the rectangles. But in context of adaptivity, it is not clear that each rectangle is refined arbitrarily often.

Second: Is the strategy the best possible for particular situations and is it even better than the naive approach of uniform refinement? This work tries to deal with these questions in case of adaptive algorithms for general Galerkin schemes and particularly for the boundary element method.

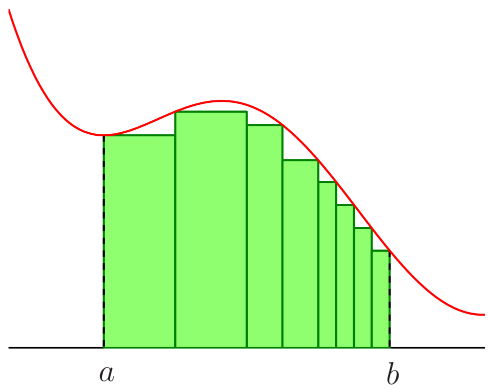

FIGURE 3. Adaptive approximation of the area under the curve.

1.2. Outline

Recently, there was a major breakthrough in the mathematical understanding of convergence and quasi-optimality of h-adaptive FEM for second-order elliptic PDEs. However, many of the ingredients which appear in the proofs in [**18, 23, 33**] were mathematically open for adaptive BEM. In Chapter 2 we consider a general adaptive algorithm (e.g. BEM or FEM) and work out these ingredients to develop a fully abstract framework for proving convergence and quasi-optimality of general adaptive algorithms of the type:

`solve` $\longrightarrow$ `estimate` $\longrightarrow$ `mark` $\longrightarrow$ `refine`

Moreover, we formulate a set of sufficient assumptions, which are used to prove the three main results of this work:

- convergence of the adaptive algorithm,
- optimal convergence rate of the estimator,
- characterization of the approximation class and therefore optimal convergence rate of the error.

In contrast to prior works on optimality of adaptive algorithms, we show that efficiency of the error estimator is only needed to characterize the approximation class, whereas convergence and optimality of the adaptive algorithm mainly depend on discrete local reliability of the estimator. Chapter 4 applies the abstract analysis to a concrete model problem, i.e. for a polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$, we analyze Symm's integral equation

$$V\phi = (K + \tfrac{1}{2})g \quad \text{on the boundary } \partial\Omega$$

for some given boundary data $g \in H^{1/2}(\partial\Omega)$. We use the weighted-residual error estimator from [**17**] to steer the mesh refinement. We prove discrete local reliability and a new inverse estimate, which allows us to prove convergence of the adaptive algorithm. To get optimality of the adaptive algorithm, we present two 1D mesh refining strategies in Section 4.4 which guarantee uniform shape regularity of the constructed meshes and satisfy several other properties which are needed to apply the abstract analysis of Chapter 2. Whereas, e.g. *newest vertex bisection* fulfills all required properties for meshes in 2D and 3D, the proof in 1D is inherently different, because we cannot rely on the angle condition to prevent the collapse of an element, but we need to bound the ratio of the diameters of two neighboring elements. Under some additional regularity assumptions on the boundary data g, we are able to prove efficiency of the weighted-residual error estimator on locally refined meshes, which was priorly only known on quasi uniform meshes under even slightly stronger regularity assumptions (see [**13**]). Chapter 5 incorporates the approximation of the Dirichlet data g by a discrete function G_ℓ in each step of the adaptive algorithm. Using the concept of modified Dörfler marking (cf. [**42, 3**]), we prove convergence and quasi-optimality of the corresponding adaptive algorithm. Finally, Chapter 6 provides some numerical experiments, which underline the results of this work and give a comparison to naive uniform mesh-refinement. We conclude the work with some remarks on the saturation assumption and give a slightly weaker result in the Appendix.

CHAPTER 2

Abstract Analysis of Adaptive Algorithms

The goal of this section is to analyze and understand the mathematical framework of quasi-optimal convergence rates for adaptive mesh-refining algorithms. Note carefully that this chapter does not rely on adaptive BEM, but applies to general adaptive algorithms. Emphasis is laid on the observation that quasi-optimality of the adaptive algorithm with respect to a certain abstract approximation class introduced below does *not* need efficiency of the error estimator. In particular, the marking parameter for the optimality of the Dörfler marking does essentially depend only on the reliability constant (see also e.g. [**18, 42, 33**], where also the efficiency constant is involved). Later-on, the efficiency is used to characterize the approximation class involved.

2.1. Abstract setting

Suppose that $\mathcal{H}$ is a separable Hilbert space with scalar product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ and corresponding norm $|\!|\!|\cdot|\!|\!|$, $f \in \mathcal{H}^*$ is a linear and continuous functional on $\mathcal{H}$, and $\phi \in \mathcal{H}$ is the unique solution of the variational formulation

$$\langle\!\langle \phi\,,\,\psi \rangle\!\rangle = f(\psi) \quad \text{for all } \psi \in \mathcal{H}. \tag{2.1}$$

Note that existence and uniqueness of ϕ follow from the Lax-Milgram lemma. Based on some triangulation $\mathcal{T}_\star$, we assume that $\mathcal{X}_\star = \mathcal{X}(\mathcal{T}_\star)$ is a finite dimensional subspace of $\mathcal{H}$ with corresponding Galerkin solution $\Phi_\star$, i.e.

$$\langle\!\langle \Phi_\star\,,\,\Psi_\star \rangle\!\rangle = f(\Psi_\star) \quad \text{for all } \Psi_\star \in \mathcal{X}_\star. \tag{2.2}$$

As in the continuous setting, existence and uniqueness of $\Phi_\star$ follow from the Lax-Milgram lemma. We stress the Galerkin orthogonality

$$\langle\!\langle \phi - \Phi_\star\,,\,\Psi_\star \rangle\!\rangle = 0 \quad \text{for all } \Psi_\star \in \mathcal{X}_\star \tag{2.3}$$

which, in fact, characterizes $\Phi_\star$. Moreover, (2.3) implies the Pythagoras theorem

$$|\!|\!|\phi - \Psi_\star|\!|\!|^2 = |\!|\!|\phi - \Phi_\star|\!|\!|^2 + |\!|\!|\Phi_\star - \Psi_\star|\!|\!|^2 \quad \text{for all } \Psi_\star \in \mathcal{X}_\star. \tag{2.4}$$

In particular, this yields the best approximation property

$$|\!|\!|\phi - \Phi_\star|\!|\!| = \min_{\Psi_\star \in \mathcal{X}_\star} |\!|\!|\phi - \Psi_\star|\!|\!| \tag{2.5}$$

of the Galerkin solution also known as *Céa's lemma*. Finally, we assume that for each element $T \in \mathcal{T}_\star$ we can compute a corresponding *refinement indicator* $\eta_\star(T)$ which —at least heuristically— measures the error between ϕ and $\Phi_\star$ on T. Under these hypotheses, the usual adaptive algorithm reads as follows:

Algorithm 2.1. INPUT: *Initial mesh* $\mathcal{T}_0$, *adaptivity parameter* $0 < \theta < 1$, *counter* $\ell := 0$

(i) *Compute discrete solution* Φ_ℓ *corresponding to* $\mathcal{T}_\ell$.
(ii) *Compute refinement indicators* $\eta_\ell(T)$ *for all* $T \in \mathcal{T}_\ell$.
(iii) *Determine set* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *of minimal cardinality such that Dörfler marking*

$$\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2. \tag{2.6}$$

is satisfied.
(iv) *Refine (at least) marked elements* $T \in \mathcal{T}_\ell$ *to obtain new mesh* $\mathcal{T}_{\ell+1}$.
(v) *Increase counter* $\ell \mapsto \ell + 1$ *and iterate.*

OUTPUT: *Discrete solutions* Φ_ℓ *and error estimators* $\eta_\ell := \Big(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2}$ *for* $\ell \ge 0$.

In the following, we want to work out the properties of the error estimator η_ℓ and the mesh-refinement which guarantee convergence $\Phi_\ell \to \phi$ in $\mathcal{H}$ as $\ell \to \infty$ even with quasi-optimal convergence rate.

2.2. Assumptions on discretization & mesh-refinement

We fix a mesh-refining strategy, e.g., red-green-blue refinement or newest vertex bisection for triangular meshes in $\mathbb{R}^2$, see e.g. [**43**, Chapter 4] or bisection of elements for partitions of a 1D manifold (see Section 4.4). For a given mesh $\mathcal{T}$ and $\mathcal{M} \subseteq \mathcal{T}$ the set of marked elements, we denote by

$$\mathcal{T}' := \texttt{refine}(\mathcal{T}, \mathcal{M}) \tag{2.7}$$

the coarsest mesh such that all marked elements $T \in \mathcal{M}$ have been refined. Moreover, we write

$$\mathcal{T}' = \texttt{refine}(\mathcal{T}) \tag{2.8}$$

if there exist finitely many meshes $\mathcal{T}^{(0)}, \dots, \mathcal{T}^{(n+1)}$ and $\mathcal{M}^{(j)} \subseteq \mathcal{T}^{(j)}$ such that $\mathcal{T} = \mathcal{T}^{(0)}$, $\mathcal{T}^{(j+1)} = \texttt{refine}(\mathcal{T}^{(j)}, \mathcal{M}^{(j)})$ for all $j = 0, \dots, n$, and $\mathcal{T}' = \mathcal{T}^{(n+1)}$. Put differently, $\mathcal{T}'$ is obtained by finitely many steps of refinement. For the fixed initial mesh $\mathcal{T}_0$ from the adaptive loop, we may now define

$$\mathbb{T} := \big\{\mathcal{T}' \,:\, \mathcal{T}' = \texttt{refine}(\mathcal{T}_0)\big\} \tag{2.9}$$

as the set of all triangulations which can be obtained from $\mathcal{T}_0$ by use of the fixed refinement strategy. Now, the assumptions on the discretization read as follows:

(D1) The discrete spaces are nested, i.e. $\mathcal{T}' = \texttt{refine}(\mathcal{T})$ implies $\mathcal{X}(\mathcal{T}) \subseteq \mathcal{X}(\mathcal{T}')$ for the corresponding spaces, as well as conforming, i.e. $\mathcal{X}(\mathcal{T}) \subset \mathcal{H}$ for all $\mathcal{T} \in \mathbb{T}$.
(D2) Uniform mesh-refinement, i.e. $\mathcal{T}^{(0)} = \mathcal{T}_0$ and $\mathcal{T}^{(\ell+1)} = \texttt{refine}(\mathcal{T}^{(\ell)}, \mathcal{T}^{(\ell)})$ for $\ell \ge 0$, yields convergence

$$|||\phi - \Phi^{(\ell)}||| \xrightarrow{\ell\to\infty} 0 \tag{2.10}$$

with $\Phi^{(\ell)} \in \mathcal{X}(\mathcal{T}^{(\ell)})$ being the corresponding Galerkin solution.

For the mesh-refinement, we state the following assumptions:

(R1) The mesh-refinement strategy guarantees that all constants involved in error estimates in Section 2.3 remain bounded.

(R2) For $\mathcal{T}' = \texttt{refine}(\mathcal{T})$, each refined element $T \in \mathcal{T}\backslash\mathcal{T}'$ is refined into at least two sons, i.e. $\#(\mathcal{T}\backslash\mathcal{T}') \le \#\mathcal{T}' - \#\mathcal{T}$.

(R3) The additional refinements which ensure (R1) do not lead to substantially more elements, i.e. for $\mathcal{T}_\ell = \texttt{refine}(\mathcal{T}_0)$

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\rm mesh} \sum_{j=0}^{\ell-1} \#\mathcal{M}^{(j)} \tag{2.11}$$

with some ℓ-independent constant $C_{\rm mesh} > 0$ which depends only on $\mathcal{T}_0$.

(R4) For two meshes $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$, there is a coarsest common refinement $\mathcal{T} \oplus \mathcal{T}' \in \mathbb{T}$, i.e. $\mathcal{X}(\mathcal{T}) \cup \mathcal{X}(\mathcal{T}') \subseteq \mathcal{X}(\mathcal{T} \oplus \mathcal{T}')$, which satisfies

$$\#(\mathcal{T} \oplus \mathcal{T}') \le \#\mathcal{T} + \#\mathcal{T}' - \#\mathcal{T}_0. \tag{2.12}$$

We stress that (D1),(D2), and (R1) are used to prove convergence of the adaptive loop, whereas (R2)–(R4) are only used to prove quasi-optimal convergence rates.

Remark. *In practice,* (R1) *is satisfied if all meshes $\mathcal{T} \in \mathbb{T}$ are, e.g., uniformly shape regular, and* (D1) *is obvious for piecewise polynomials.* (D2) *is usually guaranteed by Céa's lemma* (2.5) *and approximation properties of the discrete spaces $\mathcal{X}(\mathcal{T})$ since the mesh-size tends to zero uniformly. Moreover,* (D2) *is necessary to allow for convergence of the adaptive algorithm. Assumption* (R2) *is trivially satisfied in practice. Only* (R3)–(R4) *are mathematically demanding:* (R3) *has first been proven for newest vertex bisection of triangular meshes in 2D in* [**11**]*. The proof has been generalized to newest vertex bisection of simplicial meshes and arbitrary dimension in* [**42**]*. Finally,* (R4) *has first been observed for newest vertex bisection of triangular meshes in 2D in* [**42**] *and has been generalized in* [**18**] *to arbitrary dimension.* □

Remark. *All assumptions* (D1), (D2), (R1)–(R4) *are satisfied for newest vertex bisection in arbitrary dimension if the initial mesh $\mathcal{T}_0$ satisfies a certain distribution of the reference edges. For red-green-blue refinement, only* (R4) *fails to hold in general, see* [**37**]*. In Section 4.4, we provide a local 1D refinement for adaptive 2D BEM, which satisfies* (D1), (D2) *as well as* (R1)–(R4) *and additionally guarantees that the local mesh-ratio $\kappa(\mathcal{T}_\ell)$ remains bounded.* □

2.3. Assumptions on the error estimator

We assume that the error estimator η_ℓ satisfies the following properties with certain constants $C_{\rm stab}, C_{\rm red}, C_{\rm rel}, C_{\rm eff}, C_{\rm hot}, C_{\rm dlr}, C_{\rm ref} \ge 1$ and $0 < q_{\rm red} < 1$, which depend only on the given meshes. Recall, that Assumption (R1) on the mesh refinement strategy guarantees that the constants remain uniformly bounded for all meshes $\mathcal{T} \in \mathbb{T}$.

(E1) For $\mathcal{T}_\star, \mathcal{T}_\ell \in \mathbb{T}$, the error estimator is stable on joint elements

$$\Big|\Big(\sum_{T\in\mathcal{T}_\star\cap\mathcal{T}_\ell} \eta_\star(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\star\cap\mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2}\Big| \le C_{\rm stab}\,|||\Phi_\star - \Phi_\ell|||, \tag{2.13}$$

where $\Phi_\star$ and Φ_ℓ are the corresponding Galerkin solutions.

(E2) For $\mathcal{T}_\ell \in \mathbb{T}$ and $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\ell)$, the error estimator satisfies some reduction property on refined elements

$$\Big(\sum_{T\in\mathcal{T}_\star\backslash\mathcal{T}_\ell} \eta_\star(T)^2\Big)^{1/2} \le q_{\rm red}\Big(\sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_\star} \eta_\ell(T)^2\Big)^{1/2} + C_{\rm red}\,|||\Phi_\star - \Phi_\ell|||. \tag{2.14}$$

(E3) The error estimator is reliable, i.e. it provides an upper bound

$$C_{\rm rel}^{-1}\,|||\phi - \Phi_\ell|||^2 \le \eta_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2. \tag{2.15}$$

(E4) The error estimator satisfies the discrete local reliability

$$|||\Phi_\star - \Phi_\ell|||^2 \le C_{\rm dlr} \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2 \tag{2.16}$$

for all refinements $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\ell)$ with corresponding Galerkin solution $\Phi_\star$ and a certain subset $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ which contains the refined elements $\mathcal{T}_\ell\backslash\mathcal{T}_\star \subseteq \mathcal{R}_\ell$ and satisfies $\#\mathcal{R}_\ell \le C_{\rm ref}\#(\mathcal{T}_\ell\backslash\mathcal{T}_\star)$.

(E5) The error estimator is efficient up to terms of higher order, i.e. it provides a lower bound

$$C_{\rm eff}^{-1}\,\eta_\ell^2 \le |||\phi - \Phi_\ell|||^2 + {\rm hot}_\ell^2. \tag{2.17}$$

In addition, we assume that uniform mesh-refinement, i.e. $\mathcal{T}^{(0)} = \mathcal{T}_0$ and $\mathcal{T}^{(\ell+1)} = \texttt{refine}(\mathcal{T}^{(\ell)}, \mathcal{T}^{(\ell)})$ for $\ell \ge 0$, yields convergence of the higher-order term with a certain rate. This means that for $\mathcal{T}_\ell = \texttt{refine}(\mathcal{T}^{(\ell)})$, there holds

$${\rm hot}_\ell \le C_{\rm hot}(\#\mathcal{T}^{(\ell)} - \#\mathcal{T}_0)^{-s_\star} \tag{2.18}$$

for constants $C_{\rm hot} \ge 1$ and $s_\star > 0$ which depend only on $\mathcal{T}_0$ and (R1).

In the spirit of [**18**], the assumptions (E1)–(E3) are used to prove convergence and even contraction of the adaptive loop. Moreover, (E1) and (E4) are used to prove optimality of the marking strategy which generalizes the (concrete) analysis of [**42**] and [**18**], where also efficiency (E5) is involved. Together with the contraction result which follows from (E1)–(E3) and the Pythagoras theorem (2.4), the latter allows to prove that the adaptive loop asymptotically leads to the best possible convergence rate for the estimator. Finally, we only need the property of efficiency (E5) to characterize the approximation class and to conclude that the best possible convergence rate for the error coincides with the one of the estimator.

Remark. *Assumption* (E1) *usually follows from the triangle inequality and certain scaling arguments as e.g. inverse estimates and therefore explicitly exploits assumption* (R1).

The observation that the mesh-size decreases uniformly on refined elements, combined with the foregoing arguments also proves (E2)*. Reliability* (E3) *is usually satisfied for residual error estimators. Finally,* (E4) *has first been observed in* [**42**] *for lowest-order adaptive FEM for the 2D Poisson problem, where* $\mathcal{R}_\ell$ *is the set of refined elements plus an additional layer of elements, and* $C_{\rm ref} > 0$ *depends only on uniform shape regularity of the meshes under consideration. In* [**18**]*, one finds the improved result with* $\mathcal{R}_\ell = \mathcal{T}_\ell \backslash \mathcal{T}_\star$ *being the set of refined elements only.* □

Remark. *We stress that discrete local reliability* (E4) *and convergence* (D2) *for uniform mesh-refinement together with nestedness* (D1) *and shape regularity* (R1) *prove the reliability* (E3)*. To see this, we argue as follows: For fixed* $\ell \in \mathbb{N}$ *and arbitrary* $\varepsilon > 0$*, we may use* (D2) *to obtain that* k *uniform refinements of* $\mathcal{T}_0$ *provide a mesh* $\mathcal{T}_{0,k}$ *which satisfies*

$$|||\phi - \Phi_{0,k}||| \le \varepsilon\, |||\phi - \Phi_\ell|||$$

for the corresponding Galerkin solution $\Phi_{0,k}$*. The nestedness assumption* (D1) *implies that* k *uniform refinements of* $\mathcal{T}_\ell$ *provide a mesh* $\mathcal{T}_{\ell,k}$ *with* $\mathcal{X}(\mathcal{T}_{0,k}) \subseteq \mathcal{X}(\mathcal{T}_{\ell,k})$*. In particular, the Céa lemma* (2.5) *reveals*

$$|||\phi - \Phi_{\ell,k}||| \le |||\phi - \Phi_{0,k}||| \le \varepsilon\, |||\phi - \Phi_\ell|||.$$

Together with $\mathcal{X}_\ell = \mathcal{X}(\mathcal{T}_\ell) \subseteq \mathcal{X}(\mathcal{T}_{\ell,k})$*, the triangle inequality and the discrete local reliability* (E4) *for* $\mathcal{T}_\star = \mathcal{T}_{\ell,k}$ *yield*

$$|||\phi - \Phi_\ell||| \le |||\phi - \Phi_{\ell,k}||| + |||\Phi_{\ell,k} - \Phi_\ell||| \le \varepsilon |||\phi - \Phi_\ell||| + \Big(C_{\rm dlr} \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2}.$$

Hence, we see

$$|||\phi - \Phi_\ell|||^2 \le \frac{C_{\rm dlr}}{(1-\varepsilon)^2} \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 = \frac{C_{\rm dlr}}{(1-\varepsilon)^2}\, \eta_\ell^2,$$

which holds for all $\varepsilon > 0$*. Altogether, we thus end up with* (E3) *even with* $C_{\rm rel} = C_{\rm dlr}$*.* □

2.4. Convergence of adaptive algorithm

We start with the observation that reduction of the error estimator (E2) and stability (E1) lead to a perturbed contraction of the error estimator in each step of the adaptive loop. From now on, we assume that boundedness of the constants (R1) holds for the chosen discretization.

Lemma 2.2. *Under assumptions* (E1) *and* (E2)*, one obtains*

$$\eta_{\ell+1}^2 \le q_{\rm est}\, \eta_\ell^2 + C_{\rm est}\, |||\Phi_{\ell+1} - \Phi_\ell|||^2 \quad \text{for all } \ell \ge 0. \tag{2.19}$$

The constants $0 < q_{\rm est} < 1$ *and* $C_{\rm est} > 0$ *depend only on the constants in* (E1) *and* (E2) *and on the marking parameter* $0 < \theta < 1$ *of the Dörfler marking* (2.6)*.*

Proof. Recall the Young inequality $(a+b)^2 \le (1+\delta)a^2 + (1+\delta^{-1})b^2$ for all $a, b \in \mathbb{R}$ and $\delta > 0$. This and the application of (E1) and (E2) gives

$$\begin{aligned}\eta_{\ell+1}^2 &= \sum_{T\in\mathcal{T}_{\ell+1}\setminus\mathcal{T}_\ell} \eta_{\ell+1}(T)^2 + \sum_{T\in\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell} \eta_{\ell+1}(T)^2 \\ &\le (1+\delta)q_{\rm red}^2 \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}} \eta_\ell(T)^2 + (1+\delta) \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}} \eta_\ell(T)^2 + C_{\rm est}|||\Phi_{\ell+1}-\Phi_\ell|||^2,\end{aligned}$$

where $C_{\rm est} = (1+\delta^{-1})(C_{\rm red}^2 + C_{\rm stab}^2)$. Therefore, $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}$ and the Dörfler marking (2.6) give

$$\begin{aligned}\eta_{\ell+1}^2 &\le (1+\delta)\Big(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 - (1-q_{\rm red}^2) \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}} \eta_\ell(T)^2\Big) + C_{\rm est}|||\Phi_{\ell+1}-\Phi_\ell|||^2 \\ &\le (1+\delta)\big(1-(1-q_{\rm red}^2)\theta\big) \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 + C_{\rm est}|||\Phi_{\ell+1}-\Phi_\ell|||^2.\end{aligned}$$

Finally, we choose $\delta > 0$ sufficiently small so that $q_{\rm est} = (1+\delta)\big(1-(1-q_{\rm red}^2)\theta\big) < 1$. □

It has already been observed in the seminal work [**9**] that the Céa lemma (2.5) combined with nestedness $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ for all $\ell \ge 0$ implies a priori convergence

$$\lim_{\ell\to\infty} |||\Phi_\infty - \Phi_\ell||| = 0 \tag{2.20}$$

towards a certain (unknown) limit $\Phi_\infty \in \mathcal{H}$. Note that Φ_∞ is the Galerkin solution with respect to the subspace $\mathcal{X}_\infty$ which is the closure of $\bigcup_{\ell=0}^\infty \mathcal{X}_\ell$ in $\mathcal{H}$. In particular, this proves that $|||\Phi_{\ell+1} - \Phi_\ell||| \to 0$ as $\ell \to 0$, and hence the sequence of error estimators is contractive up to a non-negative zero sequence, see (2.19). It is therefore a consequence of elementary calculus that (2.19) leads to convergence, see [**4**, Lemma 2.3].

Corollary 2.3. *If the discrete spaces are nested* (D1)*, the estimator reduction* (2.19) *implies estimator convergence* $\lim_\ell \eta_\ell = 0$*. Under reliability* (E3)*, this proves convergence of the adaptive algorithm* $\lim_\ell |||\phi - \Phi_\ell||| = 0$. □

This *estimator reduction concept* is studied in [**4**] and applies to a quite general class of problems and estimators, e.g. non-symmetric or even nonlinear problems and residual as well as $(h-h/2)$-type error estimators. Note carefully that the Pythagoras theorem (2.4), i.e. symmetry of $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ has not been used so far. In our framework of symmetric problems, however, we may now use the Pythagoras theorem (2.4) to improve (2.19) and to obtain even linear convergence of the adaptive loop in the spirit of [**18**].

Theorem 2.4. *Assume that* (2.19) *holds true, since e.g.* (E1) *and* (E2) *are valid. Provided nestedness* (D1) *and reliability* (E3)*, there exist constants* $0 < \gamma, \kappa < 1$ *such that*

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \textit{for all } \ell \ge 0 \textit{ and } \Delta_\ell := |||\phi - \Phi_\ell|||^2 + \gamma\,\eta_\ell^2. \tag{2.21}$$

The constants $0 < \gamma, \kappa < 1$ *depend only on* $q_{\rm est}$ *and* $C_{\rm est}$ *from* (2.19).

Proof. By use of nestedness (D1) and the Pythagoras theorem $|||\phi - \Phi_\ell|||^2 = |||\phi - \Phi_{\ell+1}|||^2 + |||\Phi_{\ell+1} - \Phi_\ell|||^2$, we see

$$\begin{aligned}\Delta_{\ell+1} &= |||\phi - \Phi_\ell|||^2 + \gamma\,\eta_{\ell+1}^2 - |||\Phi_{\ell+1} - \Phi_\ell|||^2\\ &\le |||\phi - \Phi_\ell|||^2 + \gamma q_{\rm est}\,\eta_\ell^2 + (\gamma C_{\rm est} - 1)|||\Phi_{\ell+1} - \Phi_\ell|||^2.\end{aligned}$$

For sufficiently small $0 < \gamma < 1$, it holds that $\gamma C_{\rm est} \le 1$, and the last term can be dropped. With an additional parameter $\varepsilon > 0$ and reliability (E3), we obtain

$$\Delta_{\ell+1} \le |||\phi - \Phi_\ell|||^2 + \gamma q_{\rm est}\,\eta_\ell^2 = (1-\varepsilon)\,|||\phi - \Phi_\ell|||^2 + \gamma(q_{\rm est} + \varepsilon C_{\rm rel})\,\eta_\ell^2 \le \kappa\,\Delta_\ell,$$

where $\kappa := \max\big\{1-\varepsilon\,,\,q_{\rm est} + \varepsilon C_{\rm rel}\big\}$ satisfies $0 < \kappa < 1$ for $\varepsilon > 0$ sufficiently small. □

2.5. Optimality of Dörfler marking

So far, we have seen that Dörfler marking (2.6) guarantees the (perturbed) contraction properties (2.19), (2.21) and hence $\lim_\ell \eta_\ell = 0$, see Lemma 2.2 resp. Theorem 2.4. Now, we prove —in some sense— the converse.

Proposition 2.5. *Suppose that the assumptions* (E1) *and* (E4) *hold. Then, for all* $0 < \kappa_\star < 1$ *there exists a constant* $0 < \theta_\star < 1$ *such that for all* $0 < \theta \le \theta_\star$ *and all* $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\ell)$ *it holds that*

$$\eta_\star^2 \le \kappa_\star \eta_\ell^2 \quad\Longrightarrow\quad \theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2 \tag{2.22}$$

with $\mathcal{T}_\ell \backslash \mathcal{T}_\star \subseteq \mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ *from* (E4)*. The constant* $\theta_\star$ *depends only on the constants in* (E1) *and* (E4) *as well as on* $\kappa_\star$*.*

Proof. Note that the Young inequality, (E1), and (E4) give

$$\begin{aligned}\eta_\ell^2 &= \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_\star} \eta_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\ell(T)^2\\ &\le \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_\star} \eta_\ell(T)^2 + (1+\delta) \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\star^2 + (1+\delta^{-1})C_{\rm stab}^2|||\Phi_\star - \Phi_\ell|||^2\\ &\le (1+\delta)\kappa_\star\eta_\ell^2 + \big(1 + (1+\delta^{-1})C_{\rm stab}^2 C_{\rm dlr}\big) \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2.\end{aligned}$$

Rearranging these terms, we see

$$\frac{1-(1+\delta)\kappa_\star}{1+(1+\delta^{-1})C_{\rm stab}^2 C_{\rm dlr}}\,\eta_\ell^2 \le \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2.$$

For arbitrary $0 < \kappa_\star < 1$ and sufficiently small $\delta > 0$, we have

$$\theta_\star := \frac{1-(1+\delta)\kappa_\star}{1+(1+\delta^{-1})C_{\rm stab}^2 C_{\rm dlr}} > 0. \tag{2.23}$$

Then, $0 < \theta_\star < 1$ concludes the proof of (2.22). □

Remark. *Under the assumption* $\eta_\star^2 \le \kappa_\star \eta_\ell^2$, *the preceding proof shows that stability* (E1) *and discrete local reliability* (E4) *prove the optimality of the Dörfler marking* (2.22). *Provided the error estimator* η_ℓ *is reliable* (E3) *and that the discrete spaces are nested* (D1), *it is easy to show that Dörfler marking for* $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ *does also imply the discrete local reliability:*

$$\theta_\star \,|||\Phi_\star - \Phi_\ell|||^2 \le \theta_\star \,|||\phi - \Phi_\ell|||^2 \le \theta_\star C_{\rm rel}\, \eta_\ell^2 \le C_{\rm rel} \sum_{T \in \mathcal{R}_\ell} \eta_\ell(T)^2,$$

whence (E4) *holds with* $C_{\rm dlr} = \theta_\star^{-1} C_{\rm rel}$. □

Therefore, the discrete local reliability (E4) is not only sufficient but in case of $\eta_\star \le \kappa_\star \eta_\ell$ even necessary to prove optimality of the Dörfler marking.

2.6. Quasi-optimality of adaptive algorithm

Finally, we aim to analyze the best possible algebraic convergence rate which can be obtained by the adaptive algorithm. To that end, we define the set

$$\mathbb{T}_N := \big\{ \mathcal{T}_\star \in \mathbb{T} \,:\, \#\mathcal{T}_\star - \#\mathcal{T}_0 \le N \big\}$$

of all triangulations which have at most $N \in \mathbb{N}$ elements more than the initial mesh and can be generated from $\mathcal{T}_0$ by use of the fixed mesh refinement strategy. Since the adaptive algorithm is steered by the error estimator, it is natural to ask for the best algebraic convergence rate $\mathcal{O}(N^{-s})$ for the error estimator. This is mathematically characterized as follows: For given data $f \in \mathcal{H}^*$ and exact solution $\phi \in \mathcal{H}$ of (2.1), we write

$$(\phi, f) \in \mathbb{A}_s^\eta \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \|(\phi, f)\|_{\mathbb{A}_s^\eta} := \sup_{N \in \mathbb{N}} \inf_{\mathcal{T}_\star \in \mathbb{T}_N} (N^s \eta_\star) < \infty, \tag{2.24}$$

where $\eta_\star$ denotes the error estimator corresponding to the triangulation $\mathcal{T}_\star$. By definition, a convergence rate $\eta_\star = \mathcal{O}(N^{-s})$ is possible if the optimal meshes are chosen. Conceptually, we say that the adaptive algorithm is quasi-optimal if the sequence of adaptively generated meshes $\mathcal{T}_\ell$ leads to $\eta_\ell = \mathcal{O}(N^{-s})$, whenever $(\phi, f) \in \mathbb{A}_s^\eta$. The precise result is given in Theorem 2.7.

Lemma 2.6. *Suppose that the mesh-refinement satisfies* (R2) *and* (R4) *and that* (D1) *is satisfied by the discretization. Assume that the estimator fulfills the assumptions* (E1)–(E3). *Then, for* $(\phi, f) \in \mathbb{A}_s^\eta$, $\mathcal{T}_\ell \in \mathbb{T}$, *and* $\kappa_\star \in (0,1)$, *there is a certain refinement* $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\ell)$ *with*

$$\eta_\star^2 \le \kappa_\star \eta_\ell^2 \quad \textit{and} \quad \#\mathcal{T}_\star - \#\mathcal{T}_\ell \le C_1 \eta_\ell^{-1/s}, \tag{2.25}$$

where the set $\mathcal{R}_\ell \supseteq \mathcal{T}_\ell \setminus \mathcal{T}_\star$ *from Propostition 2.5 satisfies*

$$\#\mathcal{R}_\ell \le C_2 \eta_\ell^{-1/s}, \tag{2.26}$$

as well as the Dörfler marking (2.6) *for all* $0 < \theta \le \theta_\star$. *The constants* $C_1 > 0$ *and* $C_2 > 0$ *depend only on* $\|(\phi, f)\|_{\mathbb{A}_s^\eta}$.

Proof. Let $\lambda > 0$ be a free parameter which is fixed later-on. According to the definition of the approximation class $\mathbb{A}_s^\eta$, we find for given $\varepsilon^2 := \lambda\eta_\ell^2 > 0$ some triangulation $\mathcal{T}_\varepsilon \in \mathbb{T}$ such that

$$\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \lesssim \varepsilon^{-1/s} \quad \text{and} \quad \eta_\varepsilon \leq \varepsilon. \tag{2.27}$$

We now consider the coarsest common refinement $\mathcal{T}_\star := \mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell$ and first note that

$$\#\mathcal{T}_\star - \#\mathcal{T}_\ell \leq (\#\mathcal{T}_\varepsilon + \#\mathcal{T}_\ell - \#\mathcal{T}_0) - \#\mathcal{T}_\ell = \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \tag{2.28}$$

according to (R4). Due to $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\varepsilon)$, we may argue as in Lemma 2.2 to see, for arbitrary $\delta > 0$,

$$\eta_\star^2 \leq (1+\delta)\eta_\varepsilon^2 + (1+\delta^{-1})(C_{\rm red}^2 + C_{\rm stab}^2)|||\Phi_\star - \Phi_\varepsilon|||^2, \tag{2.29}$$

where we have roughly estimated $q_{\rm red} \leq 1$. By use of nestedness (D1), reliability (E3), and the Céa Lemma (2.5), one sees

$$|||\Phi_\star - \Phi_\varepsilon||| \leq |||\phi - \Phi_\star||| + |||\phi - \Phi_\varepsilon||| \lesssim |||\phi - \Phi_\varepsilon||| \lesssim \eta_\varepsilon.$$

Together with (2.29), this proves

$$\eta_\star^2 \lesssim \eta_\varepsilon^2 \leq \lambda\eta_\ell^2.$$

Choosing $\lambda > 0$ sufficiently small, we enforce $\eta_\star^2 \leq \kappa_\star\eta_\ell^2$. Next, we employ Proposition 2.5 to obtain that $\mathcal{R}_\ell$ from (E4) satisfies the Dörfler marking. For fixed $\lambda > 0$, we obtain $\varepsilon \simeq \eta_\ell$. Together with (2.27) and (2.28), this proves (2.25). To see (2.26), we use (R2) as well as (2.28) and end up with

$$\#\mathcal{R}_\ell \lesssim \#(\mathcal{T}_\ell \backslash \mathcal{T}_\star) \leq \#\mathcal{T}_\star - \#\mathcal{T}_\ell \leq \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \lesssim \varepsilon^{-1/s} \simeq \eta_\ell^{-1/s}.$$

This concludes the prove. □

Now, the quasi-optimality for a general adaptive algorithm which fits in the framework of this chapter is stated in the following theorem.

Theorem 2.7. *Let the discretization satisfy* (D1). *Suppose that the mesh-refinement satisfies* (R1)–(R4) *and that the Dörfler marking is optimal in the sense of Proposition 2.5. Assume that Δ_ℓ is equivalent to η_ℓ^2 and contractive, i.e. $\Delta_{\ell+1} \leq \kappa\,\Delta_\ell$ for some $0 < \kappa < 1$, cf. Theorem 2.4, and that the error estimator satisfies* (E1)–(E3). *For $0 < \theta \leq \theta_\star$, then there holds equivalence*

$$(\phi, f) \in \mathbb{A}_s^\eta \quad \Longleftrightarrow \quad \eta_\ell \leq C_{\rm opt}\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \geq 0 \tag{2.30}$$

for all $s > 0$. The constant $C_{\rm opt} > 0$ depends only on $\|(\phi, f)\|_{\mathbb{A}_s^\eta}$, and (R1).

Proof. First, we show that $\mathcal{M}_\ell \lesssim \eta_\ell^{-1/s}$. Therefore, we apply Lemma 2.6 to obain $\#\mathcal{R}_\ell \lesssim \eta_\ell^{-1/s}$ from (2.26) and that the set $\mathcal{R}_\ell$ satisfies Dörfler marking (2.6). In step (iii) of Algorithm 2.1, however, $\mathcal{M}_\ell$ was chosen to be the *minimal* set which satisfies the Dörfler marking (2.6). We thus obtain

$$\#\mathcal{M}_\ell \leq \#\mathcal{R}_\ell \lesssim \eta_\ell^{-1/s}. \tag{2.31}$$

Now suppose that $c\,\eta_\ell^2 \le \Delta_\ell \le C\,\eta_\ell^2$ and $\Delta_{\ell+1} \le \kappa\,\Delta_\ell$ for some $0 < \kappa < 1$, cf. Theorem 2.4. We employ (2.31) and optimality (R3) of the mesh closure to see

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \lesssim \sum_{j=0}^{\ell-1} \eta_j^{-1/s} \lesssim \sum_{j=0}^{\ell-1} \Delta_j^{-1/s}.$$

Inductively, we see $\Delta_\ell \le \kappa^{\ell-j}\Delta_j$ and hence

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \Delta_\ell^{-1/s} \sum_{j=0}^{\ell-1} \kappa^{(\ell-j)/s} \le \frac{1}{1-\kappa^{1/s}}\,\Delta_\ell^{-1/s} \lesssim \eta_\ell^{-1/s}$$

by use of the geometric series. For $(\phi, f) \in \mathbb{A}_s^\eta$, this proves the estimate

$$\eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-1/s} \quad \text{for all } \ell \ge 1.$$

The sufficiency of the condition (2.30) follows by definition of the approximation class $\mathbb{A}_s^\eta$. $\square$

Finally, we want to characterize the abstract approximation class $\mathbb{A}_s^\eta$ in terms of the Galerkin error.

Theorem 2.8. *Suppose reliability* (E3) *and efficiency* (E5) *of* η_ℓ. *Let* $0 < s \le s_\star$. *For given data* $f \in \mathcal{H}^*$ *and corresponding solution* $\phi \in \mathcal{H}$ *of* (2.1), *it holds*

$$(\phi, f) \in \mathbb{A}_s^\eta \quad \Longleftrightarrow \quad \phi \in \mathbb{A}_s,$$

where the approximation class $\mathbb{A}_s$ *is defined by*

$$\mathbb{A}_s := \big\{\psi \in \mathcal{H} \,:\, \|\psi\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}} \min_{\mathcal{T}_*\in\mathbb{T}_N} \inf_{\Psi_*\in\mathcal{X}(\mathcal{T}_*)} |||\psi - \Psi_*|||N^s < \infty\big\},$$

i.e. the approximation class $\mathbb{A}_s^\eta$ *is characterized by the optimal rate of convergence of the best approximation in the natural energy norm.*

Proof. First, we assume $(\phi, f) \in \mathbb{A}_s^\eta$. Then the reliability estimate (E3) together with Céa's lemma (2.5) prove

$$\|\phi\|_{\mathbb{A}_s} \le C_{\rm rel}\|(\phi, f)\|_{\mathbb{A}_s^\eta} < \infty,$$

i.e. $\phi \in \mathbb{A}_s$.

Second, we assume $\phi \in \mathbb{A}_s$ for some $s \in (0, s_\star]$. The definition of the approximation class $\mathbb{A}_s$ guarantees a mesh $\mathcal{T}_{N/2} \in \mathbb{T}$ with

$$\#\mathcal{T}_{N/2} - \#\mathcal{T}_0 \le N/2 \quad \text{and} \quad \inf_{\Psi_{N/2}\in\mathcal{X}(\mathcal{T}_{N/2})} |||\phi - \Psi_{N/2}|||(N/2)^s \le \|\phi\|_{\mathbb{A}_s}.$$

Because of the Céa Lemma (2.5), we get

$$|||\phi - \Phi_{N/2}|||(N/2)^s \le \|\phi\|_{\mathbb{A}_s}.$$

For $N > 4\#\mathcal{T}_0$, we construct a quasi-uniform mesh $\mathcal{T}_u \in \mathbb{T}$ by splitting each element $T \in \mathcal{T}_0$ uniformly in exactly $k = \lfloor N/(2\#\mathcal{T}_0)\rfloor$ parts. Then, it holds

$$\#\mathcal{T}_u - \#\mathcal{T}_0 = (k-1)\#\mathcal{T}_0 \leq \frac{N}{2\#\mathcal{T}_0}\#\mathcal{T}_0 = N/2.$$

We define the overlay $\mathcal{T}_+ := \mathcal{T}_{N/2} \oplus \mathcal{T}_u$. The mesh $\mathcal{T}_u$ has at least

$$k\#\mathcal{T}_0 \geq \Big(\frac{N}{2\#\mathcal{T}_0} - 1\Big)\#\mathcal{T}_0 = N/2 - \#\mathcal{T}_0 \geq N/4$$

elements. Therefore and by Equation (2.18), it holds that $\mathrm{hot}_+ \leq 4^{s_\star} C_{\mathrm{hot}} N^{-s_\star}$. With the Céa lemma (2.5)

$$|||\phi - \Phi_+||| \leq |||\phi - \Phi_{N/2}|||,$$

we then obtain

$$\begin{aligned}(|||\phi - \Phi_+|||^2 + \mathrm{hot}_+^2)N^{2s} &\leq |||\phi - \Phi_{N/2}|||^2 (\tfrac{N}{2})^{2s}\, 2^{2s} + 4^{s_\star} C_{\mathrm{hot}} N^{2s-2s_\star}\\ &\leq 4^s \|\phi\|_{\mathbb{A}_s}^2 + 4^{s_\star} C_{\mathrm{hot}} < \infty.\end{aligned}$$

Efficiency (E5) now gives

$$\eta_+^2 \lesssim |||\phi - \Phi_+|||^2 + \mathrm{hot}_+^2 \tag{2.32}$$

and therefore

$$\eta_+^2 N^{2s} \lesssim (|||\phi - \Phi_+|||^2 + \mathrm{hot}_+^2)N^{2s} \lesssim 4^s \|\phi\|_{\mathbb{A}_s}^2 + 4^{s_\star} C_{\mathrm{hot}}. \tag{2.33}$$

The overlay estimate (R4) finally yields $\#\mathcal{T}_+ - \#\mathcal{T}_0 \leq \#\mathcal{T}_{N/2} + \#\mathcal{T}_u - 2\#\mathcal{T}_0 \leq N$. This proves $(\phi, f) \in \mathbb{A}_s^\eta$. $\square$

CHAPTER 3

A Short Introduction to the Boundary Element Method

The purpose of this section is to give a quick dive into the boundary element method and to clarify notation for the following sections. Throughout, we consider the (formal) Laplace operator

$$\Delta u(x) := \sum_{j=1}^{d} \partial_{x_j}^2 u(x)$$

for a smooth function $u \in C^2(\overline{\Omega})$ on a bounded domain $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$. As a model problem serves the following elliptic partial differential equation: Find $u \in C^2(\Omega)$ such that

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{3.1}$$

and u fulfills some prescribed Dirichlet boundary conditions, i.e.

$$u|_\Gamma = g \quad \text{on } \Gamma := \partial\Omega \tag{3.2}$$

for a given function $g \in C(\Omega)$. We now aim to reformulate (3.1) and (3.2) equivalently. Therefore, we introduce the Newton kernel

$$G(z) := \begin{cases} -\frac{1}{2\pi} \log|z| & \text{for } d = 2, \\ +\frac{1}{4\pi}|z|^{-1} & \text{for } d = 3. \end{cases}$$

Obviously, the function $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is smooth. Now, the solution of (3.1) can be characterized with the *representation formula* (see e.g. [**39**, Theorem 3.1.6]).

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ. Let $u \in C^2(\Omega)$ be a solution of* (3.1)*. Then, there holds*

$$u(x) = \int_\Gamma G(x-y)\partial_{n(y)}u(y)\,dy - \int_\Gamma \partial_{n(y)}G(x-y)u(y)\,dy \tag{3.3}$$

for all $x \in \Omega$. Here, $n(y)$ denotes the outer normal unit field at $y \in \Gamma$.

This formula motivates the definition of the following *formal* integral operators:

- The simple-layer potential of $\psi : \Gamma \to \mathbb{R}$

$$\big(\widetilde{V}\psi\big)(x) := \int_\Gamma G(x-y)\psi(y)\,dy \tag{3.4}$$

for all $x \in \Omega$.

- The double-layer potential of $v : \Gamma \to \mathbb{R}$
$$\big(\widetilde{K}v\big)(x) := \int_\Gamma \partial_{n(y)} G(x-y)v(y)\, dy \tag{3.5}$$
for all $x \in \Omega$.

With the definitions above, the representation formula (3.3) reads
$$u(x) = \big(\widetilde{V}\partial_{n(y)}u\big)(x) - \big(\widetilde{K}u\big)(x).$$
for all $x \in \Omega$.

Now, consider the Hilbert space
$$H^1(\Omega) := \big\{v \in L^2(\Omega) \,:\, \nabla v \in L^2(\Omega)^d\big\}, \tag{3.6}$$
equipped with the norm $\|v\|^2_{H^1(\Omega)} := \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$. Here, the gradient ∇ is understood in the weak sense. It is possible to define the trace $v|_\Gamma$ of a function $v \in H^1(\Omega)$ by continuous extension of the classical trace for smooth functions. To that end, we use the famous result of Meyers-Serrin that $C^\infty(\overline{\Omega})$ is a dense subset of $H^1(\Omega)$. In a similar fashion, one may define a weak normal derivative $\partial_n v$ for functions $v \in H^1(\Omega)$, which additionally satisfy $\Delta v \in \widetilde{H}^{-1}(\Omega)$. The space $\widetilde{H}^{-1}(\Omega)$ denotes the dual space of $H^1(\Omega)$. This allows the definition of

- the trace space of $H^1(\Omega)$:
$$H^{1/2}(\Gamma) := \big\{v \in L^2(\Gamma) \,:\, \text{exists } w \in H^1(\Omega) \text{ with } v = w|_\Gamma\big\} \tag{3.7}$$
associated with the norm $\|v\|_{H^{1/2}(\Gamma)} := \inf\big\{\|w\|_{H^1(\Omega)} \,:\, v = w|_\Gamma \text{ for } w \in H^1(\Omega)\big\}$ as well as
- its dual space
$$H^{-1/2}(\Gamma) := H^{1/2}(\Gamma)^*.$$

We revisit the integral operators from (3.4) and (3.5) and continuously extend them to the following boundary integral operators:
$$V\psi := (\widetilde{V}\psi)|_\Gamma \,:\, H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$
$$(K - \tfrac{1}{2})v := \big(\widetilde{K}v\big)|_\Gamma \,:\, H^{1/2}(\Gamma) \to H^{1/2}(\Gamma).$$
One can explicitly represent the boundary integral operators as
$$(V\psi)(x) := \int_\Gamma G(x-y)\psi(y)\, dy \quad \text{and} \quad (Kv)(x) := \text{p.v.} \int_\Gamma \partial_{n(y)} G(x-y)v(y)\, dy$$
for all $x \in \Gamma$. Here, p.v. $\int_\Gamma$ denotes Cauchy's principal value. With this, we may reformulate the representation formula (3.3) for functions $u \in H^1(\Omega)$.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ. Let $u \in H^1(\Omega)$ be a weak solution of* (3.1)*. Then, there holds*
$$u = \widetilde{V}\phi - \widetilde{K}u|_\Gamma \quad \textit{in } H^1(\Omega), \tag{3.8}$$

where $\phi := \partial_n u \in H^{-1/2}(\Gamma)$ *denotes the normal derivative of* u.

Applying the trace to the equation above and rearranging the terms, we obtain Symm's integral equation: Given $g = u|_\Gamma \in H^{1/2}(\Gamma)$, find $\phi \in H^{-1/2}(\Gamma)$ such that

$$V\phi = (K + \tfrac{1}{2})g \quad \text{in } H^{1/2}(\Gamma). \tag{3.9}$$

The link between (3.9) and (3.1)–(3.2) is given via the representation formula (3.8), which provides the solution u of (3.1)–(3.2) by the solution ϕ of (3.9), i.e.

$$u = \widetilde{V}\phi - \widetilde{K}g \quad \text{in } H^1(\Omega). \tag{3.10}$$

Conversely, computing the normal derivative $\phi = \partial_n u$ of the solution of (3.1) and (3.2) yields the solution of (3.9).

Among others, the boundary element method denotes the approximation of the solution ϕ of (3.9) with a Galerkin scheme, i.e. for a finite dimensional subspace $\mathcal{X} \subset H^{-1/2}(\Gamma)$, we formulate the following problem: Find $\Phi \in \mathcal{X}$, such that

$$\langle V\Phi\,,\,\Psi\rangle_\Gamma = \langle (K + \tfrac{1}{2})g\,,\,\Psi\rangle_\Gamma \quad \text{for all } \Psi \in \mathcal{X}. \tag{3.11}$$

Here, $\langle\cdot\,,\,\cdot\rangle_\Gamma$ denotes the $L^2(\Gamma)$ scalar product, which can be extended to the duality pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ by continuity. Elementarily, we observe that (3.11) is equivalent to a finite dimensional linear system of equations if we expand the solution Φ in terms of basis functions. It is well-known (see e.g. [**41**, Theorem 6.23]) that $|||\cdot||| := \langle V\cdot\,,\,\cdot\rangle^{1/2}$ admits an equivalent norm on $H^{-1/2}(\Gamma)$, provided that $\operatorname{diam}(\Omega) < 1$. Therefore, $\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle := \langle V\cdot\,,\,\cdot\rangle$ is even a scalar product on $H^{-1/2}(\Omega)$, i.e. it holds

$$C_{\rm norm}^{-1}|||\psi||| \le \|\psi\|_{H^{-1/2}(\Gamma)} \le C_{\rm norm}|||\psi||| \quad \text{for all } \psi \in H^{-1/2}(\Gamma) \tag{3.12}$$

for some Γ dependent constant $C_{\rm norm} > 0$.

Thus, we may apply the Lax-Milgram lemma to prove that both, the continuous problem (3.9) as well as the discrete formulation (3.11) allow for a unique solution.

Theorem 3.3. *Let* ϕ *denote the solution of* (3.9) *and let* $\Phi \in \mathcal{X}$ *denote its Galerkin approximation* (3.11). *Then, there holds the Céa lemma*

$$|||\phi - \Phi||| = \inf_{\Psi \in \mathcal{X}} |||\phi - \Psi|||, \tag{3.13}$$

for all closed subspaces $\mathcal{X} \subseteq H^{-1/2}(\Gamma)$. *For a partition* $\mathcal{T}$ *of the boundary* Γ *with mesh-width* $h > 0$, *consider the test space* $\mathcal{X} := \mathcal{P}^0(\mathcal{T}_h)$ *of* $\mathcal{T}$*-piecewise constant functions. Under the additional regularity assumption* $\phi \in L^2(\Gamma) \cap H^\nu(\mathcal{T})$ *(see Section 4.5.1), there holds the* a priori *error estimate*

$$|||\phi - \Phi||| \le C_{\rm a\,priori}\, h^{1/2+\min\{\nu,1\}}. \tag{3.14}$$

The constant $C_{\rm a\,priori} > 0$ *depends only on* Γ.

PROOF. For the proof of (3.13), see e.g. [**41**, Theorem 8.1]. To see (3.14), let $\Pi_h : L^2(\Gamma) \to \mathcal{X}$ denote the L^2-orthogonal projection. With the approximation properties of Π_h (cf. [**14**]) and (3.13), we prove

$$|||\phi - \Phi||| \le |||(1 - \Pi_h)\phi||| \lesssim \|h^{1/2}(1 - \Pi_h)\phi\|_{L^2(\Gamma)}.$$

Applying the elementwise Poincaré inequality, we end up with

$$|||\phi - \Phi||| \lesssim h^{1/2+\nu}.$$

This concludes the proof. □

CHAPTER 4

Application of Abstract Analysis to Symm's Integral Equation in 2D

4.1. Model problem

The remainder of this paper deals with the following concrete model problem and applies the abstract results from Chapter 2: We consider Symm's Integral equation

$$V\phi = (K + \tfrac{1}{2})g \quad \text{on } \Gamma \tag{4.1}$$

where $\Gamma := \partial\Omega$ is the boundary of a polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ with $\mathrm{diam}(\Omega) < 1$. To abbreviate notation, we will denote the right-hand side of Equation (4.1) by $f := (K + 1/2)g$. Whereas $g \in H^{1/2}(\Gamma)$ is sufficient to guarantee the solvability of (4.1), the weighted-residual error estimator needs the given boundary data to satisfy $g \in H^1(\Gamma)$.

This section provides an overview on this work and its main results. We start with a discussion of the concrete realization of the modules which compose the adaptive algorithm (Algorithm 2.1).

4.1.1. Algorithm 2.1, Step (i): `solve`. Let $\mathcal{T}_\ell$ denote a partition of the boundary Γ generated from an initial partition $\mathcal{T}_0$ by local mesh–refinement with Algorithm 4.1. As usual, we denote the L^2-scalar product on the boundary Γ by $\langle \cdot\,,\,\cdot\rangle_{L^2(\Gamma)}$ and extend it to the duality brackets of $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ by continuity. The lowest-order Galerkin discretization of the continuous model problem (4.1) reads: Find $\Phi_\ell \in \mathcal{X}(\mathcal{T}_\ell) := \mathcal{P}^0(\mathcal{T}_\ell)$ such that

$$\langle\!\langle \Phi_\ell\,,\,\Psi_\ell \rangle\!\rangle = \langle (K + \tfrac{1}{2})g\,,\,\Psi_\ell\rangle_{L^2(\Gamma)} \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell), \tag{4.2}$$

where we use the polynomial spaces $\mathcal{P}^p([0,1]) := \big\{v \in C^\infty([0,1])\,:\,\frac{\partial^{p+1}}{\partial s^{p+1}}v = 0\big\}$ to define

$$\mathcal{P}^p(\mathcal{T}_\ell) := \{v \in L^2(\Gamma)\,:\, v \circ F_T \in \mathcal{P}^p([0,1]) \text{ for all } T \in \mathcal{T}_\ell\}.$$

Here, $F_T : [0,1] \to T$ is an affine transformation which maps the unit interval onto the element $T \in \mathcal{T}_\ell$. As in the continuous setting, it is well known that (4.2) allows for a unique solution. For simplicity, we assume that the module `solve` computes the exact discrete solution. However, it would be possible to include an approximate solver into our analysis.

4.1.2. Algorithm 2.1, Step (ii): `estimate`. We recall the definition of the residual-based error estimator η_ℓ which dates back to the seminal work [**19**] for 2D and has been extended in [**20**] to 3D. The local contributions of η_ℓ are defined by

$$\eta_\ell(T) := \operatorname{diam}(T)^{1/2}\|\tfrac{\partial}{\partial s}(V\Phi_\ell - f)\|_{L^2(T)} \quad \text{for all } T \in \mathcal{T}_\ell. \tag{4.3}$$

Here, $\frac{\partial}{\partial s}$ denotes the arclength derivative along Γ. We define the local mesh-width function $h_\ell \in L^\infty(\Gamma)$ by $h_\ell|_T := \operatorname{diam}(T)$, where $\operatorname{diam}(T)$ denotes the Euclidean length of an element $T \in \mathcal{T}_\ell$. Now, η_ℓ reads

$$\eta_\ell := \Big(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2} = \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(V\Phi_\ell - f)\|_{L^2(\Gamma)}. \tag{4.4}$$

Note that due to the assumption $g \in H^1(\Gamma)$, we obtain by use of the mapping properties of V and K that $f = (K + \frac{1}{2})g \in H^1(\Gamma)$ as well as $V\Phi_\ell \in H^1(\Gamma)$ (cf. [**34**]). Therefore, the estimator η_ℓ is well defined.

4.1.3. Algorithm 2.1, Step (iii): `mark`. As in the abstract framework of chapter 2, we use Dörfler markin (2.6).

4.1.4. Algorithm 2.1, Step (iv): `refine`. Locally refined meshes $\mathcal{T}_\ell$ are obtained from an initial partition $\mathcal{T}_0$. For a given set $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ of marked elements, we refine $\mathcal{T}_\ell$ such that at least all elements $T \in \mathcal{M}_\ell$ are refined and the K-mesh constant remains bounded, i.e. for

$$\kappa(\mathcal{T}_\ell) := \max\big\{h_\ell|_T/h_\ell|_{T'} \,:\, T, T' \in \mathcal{T}_\ell \text{ with } T\cap T' \neq \emptyset\big\} \tag{4.5}$$

holds

$$\kappa(\mathcal{T}_\ell) \le 2\kappa(\mathcal{T}_0) \tag{4.6}$$

for all meshes $\mathcal{T}_\ell$ generated from $\mathcal{T}_0$. In contrast to the abstract setting in Chapter 2, we fix the refinement routine and perform local mesh refinement with the following Algorithm, which guarantees (4.6):

Algorithm 4.1. Input: *Partition $\mathcal{T}_\ell$, marked elements $\mathcal{M}_\ell^{(0)} := \mathcal{M}_\ell$, counter $i := 0$.*

(i) *Define $\mathcal{U}^{(i)} := \bigcup_{T\in\mathcal{M}_\ell^{(i)}} \big\{T' \in \mathcal{T}_\ell\backslash\mathcal{M}_\ell^{(i)}$ neighbor of $T \,:\, h_\ell|_{T'} > \kappa(\mathcal{T}_0)\, h_\ell|_T\big\}$.*

(ii) *If $\mathcal{U}^{(i)} \neq \emptyset$, define $\mathcal{M}_\ell^{(i+1)} := \mathcal{M}_\ell^{(i)} \cup \mathcal{U}^{(i)}$, increase counter $i \mapsto i+1$, and goto* (i).

(iii) *Otherwise, bisect all marked elements $T \in \mathcal{M}_\ell^{(i)}$ to obtain $\mathcal{T}_{\ell+1}$.*

Output: *Refined boundary partition $\mathcal{T}_{\ell+1}$ as well as $\mathcal{M}_\ell^{(i)} = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$.*

A detailed analysis of this algorithm is given in Section 4.4.

4.2. Function spaces involved

The definition of $H^{1/2}(\Gamma)$ as the trace space of $H^1(\Omega)$ is given in (3.7). For $\nu \in (0, 5/2]$, we define $H^\nu(\Gamma)$ as a trace space, i.e.

$$H^\nu(\Gamma) := \big\{v|_\Gamma \,:\, v \in H^{\nu+1/2}(\Omega)\big\}$$

equipped with the norm

$$\|w\|_{H^\nu(\Gamma)} := \inf\big\{\|v\|_{H^{\nu+1/2}(\Omega)} \,:\, w = v|_\Gamma,\, v \in H^{\nu+1/2}(\Gamma)\big\}.$$

In the sense that the corresponding sets are equal and the norms are equivalent, this definition coincides with the classical definition of $H^\nu(\Gamma)$ as a Sobolev space on the 1D Lipschitz-manifold Γ for $\nu \in (0,1)$. Moreover, we may use the *Sobolev-Slobodeckij* norm

$$|w|_{H^\nu(\Gamma)} := \int_\Gamma\int_\Gamma \frac{|w(x)-w(y)|^2}{|x-y|^{1+2\nu}}\,dx\,dy$$

to define the equivalent norm $(\|\cdot\|^2_{L^2(\Gamma)} + |\cdot|^2_{H^\nu(\Gamma)})^{1/2}$ on $H^\nu(\Gamma)$, for $\nu \in (0,1)$. Furthermore for $\nu = 1$, $H^1(\Gamma)$ is equipped with the equivalent norm

$$\|w\|^2_{H^1(\Gamma)} := \|w\|^2_{L^2(\Gamma)} + \|\tfrac{\partial}{\partial s}w\|^2_{L^2(\Gamma)} \quad \text{for all } w \in H^1(\Gamma),$$

where $\frac{\partial}{\partial s}$ denotes the arclength derivative along Γ.

Finally, for $\nu \in (0,1)$, we may equivalently define $H^\nu(\Gamma)$ as the real interpolation space of $L^2(\Gamma)$ and $H^1(\Gamma)$ (cf. [**10**]). All mentioned definitions of $H^\nu(\Gamma)$ are—at least for $\nu \in (0,1)$—equivalent. The norm equivalence constants, however, depend on the boundary Γ.

4.3. Main results

First, we prove convergence of adaptive BEM (see Theorem 2.4 for a more general version of this result).

Theorem 4.2. *Let the sequence of meshes $\mathcal{T}_\ell$ and the corresponding solutions of Φ_ℓ of Equation* (4.2) *be obtained from Algorithm 2.1. Then, there exist constants* $0 < \gamma, \kappa < 1$ *such that*

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \textit{for all } \ell \ge 0 \qquad \textit{with} \qquad \Delta_\ell := |||\phi - \Phi_\ell|||^2 + \gamma\,\eta_\ell^2. \tag{4.7}$$

The constants $0 < \gamma, \kappa < 1$ *are independent of* $\ell \in \mathbb{N}_0$.

Now, for given boundary data $g \in H^{s_{\rm reg}}(\Gamma)$ for some $s_{\rm reg} > 2$, we are able to prove efficiency of η_ℓ.

Theorem 4.3. *Let the given boundary data satisfy $g \in H^{s_{\rm reg}}(\Gamma)$ for some $s_{\rm reg} > 2$. Let ϕ denote the solution of* (4.1)*. Then, for $\mathcal{T}_\ell \in \mathbb{T}$ the error estimator η_ℓ is efficient in the following sense*

$$C_{\rm eff}^{-1}\eta_\ell \le \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + {\rm hot}_\ell. \tag{4.8}$$

Here, $C_{\rm eff} > 0$ depends only on Γ and $\kappa(\mathcal{T}_\ell)$. The higher-order term ${\rm hot}_\ell$ *is given in detail in Definition 4.19. For all $\varepsilon > 0$, it satisfies*

$${\rm hot}_\ell = \Big(\sum_{T\in\mathcal{T}_\ell} {\rm hot}_\ell(T)^2\Big)^{1/2} \quad \textit{and} \quad {\rm hot}_\ell(T) \le C_{\rm hot}(h_\ell|_T)^{\min\{s_{\rm reg},5/2\}-1/2-\varepsilon}, \tag{4.9}$$

where $C_{\rm hot} > 0$ depends only on Γ, $\kappa(\mathcal{T}_\ell)$, $s_{\rm reg} > 2$, and $\varepsilon > 0$.

Following the steps of adaptive FEM, we prove discrete local reliability for η_ℓ, see Proposition 4.9. With this results, we are able to prove the optimal rate of convergence of the estimator. With fixed mesh refinement strategy, we recall the definition of

$$\mathbb{T}_N := \big\{\mathcal{T}_\star \in \mathbb{T} \,:\, \#\mathcal{T}_\star - \#\mathcal{T}_0 \le N\big\}$$

and

$$(\phi, f) \in \mathbb{A}_s^\eta \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \|(\phi, f)\|_{\mathbb{A}_s^\eta} := \sup_{N\in\mathbb{N}} \inf_{\mathcal{T}_\star\in\mathbb{T}_N} (N^s \eta_\star) < \infty, \tag{4.10}$$

from Chapter 2. Using the efficiency estimate in Theorem 4.3, we are able to characterize the approximation class $\mathbb{A}_s^\eta$ in terms of the Galerkin error only. Therefore, we introduce

$$\phi \in \mathbb{A}_s \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \|\psi\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}} \inf_{\mathcal{T}_*\in\mathbb{T}_N} \inf_{\Psi_*\in\mathcal{P}^0(\mathcal{T}_*)} |||\psi - \Psi_*|||N^s < \infty.$$

Precisely, this quasi-optimality is stated in the following theorem.

Theorem 4.4. *Let $(\Phi_\ell)_{\ell\in\mathbb{N}}$ denote the sequence of solutions generated by Algorithm 2.1 driven by the weighted residual-based error estimator η_ℓ. Assume that the corresponding sequence of meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ is created by local refinement as stated in Algorithm 4.1. Then, for sufficiently small adaptivity parameter $0 < \theta < 1$, Algorithm 2.1 is optimal in the following sense*

$$\phi \in \mathbb{A}_s^\eta \quad \Longleftrightarrow \quad \eta_\ell \le C_3\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \textit{for all } \ell \in \mathbb{N}.$$

The constant $C_3 > 0$ depends only on $\|(\phi, f)\|_{\mathbb{A}_s^\eta}$ and Γ. In addition, let the given boundary data additionally satisfy $g \in H^{s_{\rm reg}}(\Gamma)$ for some $s_{\rm reg} > 2$. Then, for $0 < s < \min\{s_{\rm reg}, 5/2\} - 1/2$ and sufficiently small $0 < \theta < 1$, Algorithm 2.1 is optimal in the following sense

$$\phi \in \mathbb{A}_s \quad \Longleftrightarrow \quad |||\phi - \Phi_\ell||| \le C_4\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \textit{for all } \ell \in \mathbb{N},$$

where the constant $C_4 > 0$ depends only on $\|\phi\|_{\mathbb{A}_s}$ and Γ.

4.4. Optimal local mesh-refinement for 2D BEM

In this section, we aim to develop a local mesh-refinement strategy for 2D BEM which mathematically guarantees the assumptions (D1), (D2) and (R1)–(R4). Clearly, emphasis is laid on the mesh closure (R3) and the overlay estimate (R4). We stress that the considered mesh-refining algorithms of Section 4.4.1 and Section 4.4.2 have first been analyzed in the technical report [**7**], where the focus is, however, only on (R3). For the sake of completeness, we elaborate the proofs of [**7**] and extend them by considering also (R4).

Suppose that $\mathcal{T}_0 = \{T_1, \dots, T_N\}$ is a given initial partition of Γ into affine boundary segments T_j and that a sequence of meshes $\mathcal{T}_\ell$ is obtained inductively by local refinement, where

$$\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell) \tag{4.11}$$

is generated from $\mathcal{T}_\ell$ by refinement of (at least) certain marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$. Here, refinement of an element $T \in \mathcal{M}_\ell$ means that T is bisected into two elements $T_1, T_2 \in \mathcal{T}_{\ell+1}$ of half length, i.e., there holds $h_{\ell+1}|_T = \frac{1}{2}\, h_\ell|_T$. In particular, assumption (R2) is already satisfied. Moreover, we aim to guarantee that the K-mesh constant (4.5) satisfies the uniform boundedness

$$\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0) \quad \text{for all } \ell \in \mathbb{N}. \tag{4.12}$$

We stress that, for our analysis below, the uniform boundedness (4.12) implies (R1).

Remark. *Clearly, the boundedness estimate* (4.12) *cannot be improved in general. For instance, let $\mathcal{T}_0$ be a uniform partition with $\#\mathcal{T}_0 > 1$ and $\#\mathcal{M}_0 = 1$. Provided that the obtained partition satisfies $\#\mathcal{T}_1 < 2\,\#\mathcal{T}_0$, i.e., the local refinement does not lead to a uniform refinement, there holds $\kappa(\mathcal{T}_0) = 1$, whereas $\kappa(\mathcal{T}_1) = 2$.* □

4.4.1. Level-Based Mesh-Refinement. To use the analytical techniques developed in [**11, 42**], we introduce the level of an element by induction: For $T \in \mathcal{T}_0$, let $\texttt{level}(T) := 0$. If $T \in \mathcal{T}_\ell$ is bisected into two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$, we define $\texttt{level}(T_1) := \texttt{level}(T_2) := \texttt{level}(T) + 1$.

Algorithm 4.5. INPUT: *Partition $\mathcal{T}_\ell$, marked elements $\mathcal{M}_\ell^{(0)} := \mathcal{M}_\ell$, counter $i := 0$.*

- (i) *Define $\mathcal{U}^{(i)} := \bigcup_{T\in\mathcal{M}_\ell^{(i)}} \big\{T' \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell^{(i)}$ neighbor of $T \,:\, \texttt{level}(T') < \texttt{level}(T)\big\}$.*
- (ii) *If $\mathcal{U}^{(i)} \neq \emptyset$, define $\mathcal{M}_\ell^{(i+1)} := \mathcal{M}_\ell^{(i)} \cup \mathcal{U}^{(i)}$, increase counter $i \mapsto i+1$, and goto* (i).
- (iii) *Otherwise, bisect all marked elements $T \in \mathcal{M}_\ell^{(i)}$ to obtain $\mathcal{T}_{\ell+1}$.*

OUTPUT: *Refined boundary partition $\mathcal{T}_{\ell+1}$ as well as $\mathcal{M}_\ell^{(i)} = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$.*

Note that Algorithm 4.5 is well-defined in the sense that it terminates for some counter $0 \le i \le \#\mathcal{T}_\ell - 1$. Moreover, the following lemma states that $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$.

Lemma 4.6 (Assumption (R1) for Algorithm 4.5)**.** *Assume that $\mathcal{T}_0$ is a given initial partition and that the partitions $\mathcal{T}_\ell$ are inductively generated by Algorithm 4.5, where the sets $\mathcal{M}_j \subseteq \mathcal{T}_j$ of marked elements are arbitrary. Then, neighboring elements satisfy*

$$|\,\texttt{level}(T) - \texttt{level}(T')| \le 1 \quad \textit{for all } T, T' \in \mathcal{T}_\ell \textit{ with } T \cap T' \neq \emptyset. \tag{4.13}$$

Moreover, there holds $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$ for all $\ell \in \mathbb{N}$.

PROOF. The estimate (4.13) easily follows from induction and the definition of the $\mathcal{U}^{(i)}$ in step (i) of Algorithm 4.5. Now, let $T, T' \in \mathcal{T}_\ell$ be neighbors, i.e., $T \neq T'$ and $T \cap T' \neq \emptyset$. Consequently, the unique ancestors $\widehat{T}, \widehat{T}' \in \mathcal{T}_0$ with $T \subseteq \widehat{T}$ and $T' \subseteq \widehat{T}'$ either coincide or are neighbors as well. Moreover, according to bisection, there hold $h_\ell|_T = 2^{-\texttt{level}(T)} h_0|_{\widehat{T}}$ and $h_\ell|_{T'} = 2^{-\texttt{level}(T')} h_0|_{\widehat{T}'}$. Together with (4.13), we obtain

$$\frac{h_\ell|_T}{h_\ell|_{T'}} = 2^{\texttt{level}(T') - \texttt{level}(T)} \frac{h_0|_{\widehat{T}}}{h_0|_{\widehat{T}'}} \le 2\kappa(\mathcal{T}_0).$$

Taking the supremum over all possible neighbors, we conclude $\kappa(\mathcal{T}_\ell) \le 2\kappa(\mathcal{T}_0)$. □

Theorem 4.7. *Algorithm 4.5 satisfies* (R2)–(R4) *as well as uniform boundedness* $\kappa(\mathcal{T}) \le 2\kappa(\mathcal{T}_0)$ *for all meshes* $\mathcal{T}$ *that can be generated from* $\mathcal{T}_0$.

Proof. The boundedness of $\kappa(\mathcal{T})$ was proved in Lemma 4.6. Moreover, the assumption (R2) is satisfied by definition of the refinement strategy. We aim to use the arguments from [**42**] to verify (R3). In the latter work, the focus is on newest vertex bisection for simplicial meshes in $\mathbb{R}^d$ with $d \ge 2$. To adopt the notation of [**42**], note that the sets $\mathcal{M}_j$ are pairwise disjoint. Therefore, there holds $\#\mathcal{M} = \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$ with $\mathcal{M} := \bigcup_{j=0}^{\ell-1} \mathcal{M}_j$. Finally, [**42**, Theorem 6.1] states the estimate

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le \#\mathcal{T}_\ell - \#(\mathcal{T}_\ell \cap \mathcal{T}_0) = \#(\mathcal{T}_\ell \backslash (\mathcal{T}_\ell \cap \mathcal{T}_0)) \lesssim \#\mathcal{M} = \sum_{j=0}^{\ell-1} \#\mathcal{M}_j,$$

where the notation $\lesssim$ suppresses the constant $C_{\rm mesh}$ from (R3). From now on, our proof only aims to point out the modifications to ensure that the proof of [**42**, Theorem 6.1] applies to our case as well. — In our context, we call a partition $\mathcal{T}$ *conforming* provided that the level property (4.13) holds. It is easily observed that Algorithm 4.5 provides the coarsest conforming refinement $\mathcal{T}_{\ell+1}$ of the partition $\mathcal{T}_\ell$ such that all elements $T \in \mathcal{M}_\ell$ are refined. Moreover, we note that our refinement rule, i.e. refinement of an element by bisection, leads to a binary refinement tree as does newest vertex bisection in $\mathbb{R}^d$. Therefore, we can even call the refinement routine elementwise: Suppose that $\mathcal{T}' = \texttt{refine}(\mathcal{T}, \mathcal{M})$ is a realization of Algorithm 4.5 which applies refinement for the set $\mathcal{M} \cap \mathcal{T}$, where we define $\mathcal{T}' := \mathcal{T}$ in case of $\mathcal{M} \cap \mathcal{T} = \emptyset$. Suppose that $\mathcal{M}_\ell = \{T_1, \dots, T_m\}$. By induction, we may define

$$\mathcal{T}_\ell^{(0)} = \mathcal{T}_\ell \quad \text{and} \quad \mathcal{T}_\ell^{(i)} := \texttt{refine}(\mathcal{T}_\ell^{(i-1)}, \{T_i\}) \quad \text{for } i = 1, \dots, m.$$

Then, there holds $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell) = \mathcal{T}_\ell^{(m)}$. These observations provide the framework for the analysis of [**42**].

- First, we note that the definition of

$$d := \min_{\widehat{T} \in \mathcal{T}_0} \operatorname{diam}(\widehat{T}) \quad \text{and} \quad D := \max_{\widehat{T} \in \mathcal{T}_0} \operatorname{diam}(\widehat{T}),$$

leads to

$$2^{-\texttt{level}(T)} d \le \operatorname{diam}(T) \le 2^{-\texttt{level}(T)} D \quad \text{for all } T \in \mathcal{T}_\ell \text{ and } \ell \in \mathbb{N}_0, \tag{4.14}$$

which follows from the fact that $\operatorname{diam}(T) = 2^{-\texttt{level}(T)} \operatorname{diam}(\widehat{T})$, where $\widehat{T} \in \mathcal{T}_0$ is the unique ancestor of $T \in \mathcal{T}_\ell$, i.e. $T \subseteq \widehat{T}$. This observation corresponds to [**42**, Equation (4.1)].

- Second, [**42**, Corollary 4.6] is satisfied due to Estimate (4.13).
- Third, suppose that $T' \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$ is generated by a call of $\texttt{refine}(\mathcal{T}_\ell, \{T\})$ for some $T \in \mathcal{M}_\ell$. By definition of Algorithm 4.5, there are some elements $T_0, \dots, T_r \in \mathcal{T}_\ell$ such that T_j is a neighbor of T_{j-1} with $\texttt{level}(T_j) < \texttt{level}(T_{j-1})$, $T_0 = T$, and $T' \subset T_r$. This

implies $\texttt{level}(T') = \texttt{level}(T_r) + 1 < \texttt{level}(T_0) + 1 = \texttt{level}(T) + 1$ for $r > 0$ and verifies the analogon of [**42**, Theorem 5.1].

- Fourth, [**42**, Theorem 5.2] is a consequence of [**42**, Equation (4.1)] and [**42**, Theorem 5.1] and therefore holds in our case as well.
- Finally, the proof of [**42**, Theorem 6.1] only relies on [**42**, Theorem 5.1–5.2] and [**42**, Equation (4.1)] and therefore applies to our mesh-refinement as well.
- It remains to prove the overlay estimate (R4). We aim to proof even a little bit more, i.e. for meshes $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$, there holds $\mathcal{T} \oplus \mathcal{T}' \in \mathbb{T}$ and

$$\begin{aligned} \mathcal{T} \oplus \mathcal{T}' = \mathcal{T}_\oplus := &\big\{T \in \mathcal{T} \,:\, \text{exists } T' \in \mathcal{T}' \text{ with } T \subseteq T'\big\} \\ &\cup \big\{T' \in \mathcal{T}' \,:\, \text{exists } T \in \mathcal{T} \text{ with } T' \subseteq T\big\}. \end{aligned} \tag{4.15}$$

If the characterization of $\mathcal{T} \oplus \mathcal{T}'$ above holds true, the estimate in (R4) is fulfilled trivially. First, we show that $\mathcal{T}_\oplus$ as defined in (4.15) is a refinement of $\mathcal{T}$ and $\mathcal{T}'$. Assume it exists $T \in \mathcal{T}$ with $T \notin \mathcal{T}_\oplus$. Then, for all $T' \in \mathcal{T}'$, it holds $T \nsubseteq T'$. Because, the refinement rule generates a binary refinement tree, this implicates $T' \subseteq T$ or $|T \cap T'| = 0$ for all $T' \in \mathcal{T}'$. Therefore, we have $T'_1, \dots, T'_k \in \mathcal{T}'$ with

$$T = \bigcup_{i=1}^{k} T'_i.$$

By definition of $\mathcal{T}_\oplus$, $T'_i \in \mathcal{T}_\oplus$ for all $i = 1, \dots, k$ and therefore $\mathcal{T}_\oplus$ is a refinement of $\mathcal{T}$. The same argumentation for $\mathcal{T}'$ yields that $\mathcal{T}_\oplus$ is a refinement of $\mathcal{T}'$. Obviously, $\mathcal{T}_\oplus$ is the coarsest common refinement of $\mathcal{T}$ and $\mathcal{T}'$. Next, we show by contradiction that

$$|\texttt{level}(T) - \texttt{level}(T')| \le 1 \quad \text{for all } T, T' \in \mathcal{T}_\oplus \text{ with } T \cap T' \neq \emptyset. \tag{4.16}$$

Therefore, assume neighbors $T, T' \in \mathcal{T}_\oplus$ with $\texttt{level}(T) > \texttt{level}(T') + 1$. Because $T, T' \in \mathcal{T}_\oplus \subseteq \mathcal{T} \cup \mathcal{T}'$ and (4.16) is guaranteed for $\mathcal{T}$ and $\mathcal{T}'$ by Lemma 4.6, we obtain immediately $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$. By definition of $\mathcal{T}_\oplus$, it exists $\widehat{T} \in \mathcal{T}$ with $T' \subseteq \widehat{T}$. Thus, we have

$$\texttt{level}(\widehat{T}) \le \texttt{level}(T') < 1 + \texttt{level}(T), \text{ i.e. } \quad |\texttt{level}(\widehat{T}) - \texttt{level}(T')| > 1.$$

This contradicts Lemma 4.6 because $\widehat{T}, T \in \mathcal{T}$ are neighbors or coincide. Therefore, we prove (4.16). Consequently, we may generate $\mathcal{T}_\oplus$ by iterative refinement of $\mathcal{T}_0 := \mathcal{T}$

$$\mathcal{T}_{i+1} := \texttt{refine}(\mathcal{T}_i, \mathcal{T}_i \setminus \mathcal{T}_\oplus)$$

for all $i \ge 0$ with $\mathcal{T}_i \setminus \mathcal{T}_\oplus \neq \emptyset$. This yields $\mathcal{T}_\oplus \in \mathbb{T}$ and therefore $\mathcal{T}_\oplus = \mathcal{T} \oplus \mathcal{T}'$, which concludes the proof. □

4.4.2. $\boldsymbol{\kappa}$-Based Mesh-Refinement. In this section, we use the level-based mesh-refinement to prove that the mesh-refinement proposed in [**25, 26, 30**] (see Algorithm 4.1) is also optimal. The advantage of this is that there is no need to compute or store the level function.

Remark. *For the implementation of Algorithm 4.1, it pays to sort the elements* $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ *by its diameter, i.e., one determines a permutation* π *with* $\mathrm{diam}(T_{\pi(j)}) \le \mathrm{diam}(T_{\pi(j+1)})$ *for* $j = 1, \dots, N-1$. *Up to* $\mathcal{O}(N \log N)$ *for sorting, Algorithm 4.1 can then be realized in linear complexity.* $\square$

We note that, by definition, Algorithm 4.1 provides the coarsest refinement $\mathcal{T}_{\ell+1}$ of a partition $\mathcal{T}_\ell$ with $\kappa(\mathcal{T}_\ell) \le 2\,\kappa(\mathcal{T}_0)$ such that all elements $T \in \mathcal{M}_\ell$ are refined and that there holds $\kappa(\mathcal{T}_{\ell+1}) \le 2\,\kappa(\mathcal{T}_0)$. The following theorem states optimality of the mesh-refinement from Algorithm 4.1. The proof will be achieved by comparison of Algorithm 4.1 with Algorithm 4.5.

Theorem 4.8. *Algorithm 4.1 satisfies* (R2)–(R4) *as well as uniform boundedness* $\kappa(\mathcal{T}) \le 2\kappa(\mathcal{T}_0)$ *for all* $\mathcal{T} \in \mathbb{T}$.

Proof. First, we prove the uniform boundedness (4.12) of the K-mesh constant. To that end, let $T, T' \in \mathcal{T}_{\ell+1}$ be neighbors, i.e., $T \ne T'$ and $T \cap T' \ne \emptyset$. Consequently, the fathers $\widehat{T}, \widehat{T}' \in \mathcal{T}_\ell$ of T and T' either coincide or are neighbors as well. We aim to provide an upper bound for the quotient $h_{\ell+1}|_{T'}/h_{\ell+1}|_T$. In case of $\widehat{T} = \widehat{T}'$, there holds $h_{\ell+1}|_T = h_{\ell+1}|_{T'}$. Therefore, we may assume that $\widehat{T} \ne \widehat{T}'$. We now consider four cases:

(a) If $\widehat{T}, \widehat{T}'$ are both not refined, there holds $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}$ and $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}$.
(b) If $\widehat{T}, \widehat{T}'$ are both refined, there holds $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}/2$ and $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}/2$.
(c) If $\widehat{T}'$ is refined and $\widehat{T}$ is not, there holds $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}/2$ and $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}$.
(d) If $\widehat{T}'$ is not refined and $\widehat{T}$ is refined, there holds $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}$ and $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}/2$. Moreover, Algorithm 4.1 implies $h_\ell|_{\widehat{T}'} \le \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}}$.

In the cases (a)–(c), we thus observe $h_{\ell+1}|_{T'}/h_{\ell+1}|_T \le h_\ell|_{\widehat{T}'}/h_\ell|_{\widehat{T}} \le \kappa(\mathcal{T}_\ell)$. In case (d), there holds $h_{\ell+1}|_{T'}/h_{\ell+1}|_T = 2\,h_\ell|_{\widehat{T}'}/h_\ell|_{\widehat{T}} \le 2\,\kappa(\mathcal{T}_0)$. Altogether, this proves

$$\frac{h_{\ell+1}|_{T'}}{h_{\ell+1}|_T} \le \max\{\kappa(\mathcal{T}_\ell), 2\,\kappa(\mathcal{T}_0)\} \quad \text{for all neighboring elements } T, T' \in \mathcal{T}_{\ell+1},$$

whence $\kappa(\mathcal{T}_{\ell+1}) \le \max\{\kappa(\mathcal{T}_\ell), 2\,\kappa(\mathcal{T}_0)\}$. By induction, we conclude $\kappa(\mathcal{T}_{\ell+1}) \le 2\,\kappa(\mathcal{T}_0)$.

Second, the optimality (R3) for the κ-based mesh-refinement is obtained via the estimate for the level-based mesh-refinement from the previous section. To that end, let $\widetilde{\texttt{refine}}$ denote the level-based mesh-refinement from Section 4.4.1. By induction, we now define an additional sequence of partitions by

$$\widetilde{\mathcal{T}}_{\ell+1} := \widetilde{\texttt{refine}}(\widetilde{\mathcal{T}}_\ell, \widetilde{\mathcal{M}}_\ell) \quad \text{with} \quad \widetilde{\mathcal{M}}_\ell := \mathcal{M}_\ell \cap \widetilde{\mathcal{T}}_\ell,$$

where $\widetilde{\mathcal{T}}_0 := \mathcal{T}_0$ and $\widetilde{\mathcal{M}}_0 := \mathcal{M}_0$. In the following, we prove that the partitions $\mathcal{T}_\ell$ generated by Algorithm 4.1 are coarser than the partitions $\widetilde{\mathcal{T}}_\ell$ generated by Algorithm 4.5 in the sense that each element $T \in \mathcal{T}_\ell$ is the union of elements from $\widetilde{\mathcal{T}}_\ell$, i.e.,

$$\forall \ell \in \mathbb{N}_0 \, \forall T \in \mathcal{T}_\ell \, \exists \mathcal{V}_\ell \subseteq \widetilde{\mathcal{T}}_\ell \qquad T = \bigcup_{\widetilde{T} \in \mathcal{V}_\ell} \widetilde{T}. \tag{4.17}$$

This implies $\#\mathcal{T}_\ell \le \#\widetilde{\mathcal{T}}_\ell$. Moreover, there holds $\#\widetilde{\mathcal{M}}_\ell \le \#\mathcal{M}_\ell$ by definition of the set $\widetilde{\mathcal{M}}_\ell$. Using the optimality (R3) of the level-based refinement, we therefore infer optimality of the κ-based refinement

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le \#\widetilde{\mathcal{T}}_\ell - \#\widetilde{\mathcal{T}}_0 \lesssim \sum_{j=0}^{\ell-1} \#\widetilde{\mathcal{M}}_j \le \sum_{j=0}^{\ell-1} \#\mathcal{M}_j.$$

Here, the symbol $\lesssim$ suppresses the constant $C_{\rm mesh}$ from (R3). Altogether, it thus only remains to verify (4.17).

This is done by induction on $\ell \in \mathbb{N}_0$: The case $\ell = 0$ follows by definition $\mathcal{T}_0 = \widetilde{\mathcal{T}}_0$. Now, suppose that (4.17) holds for $\mathcal{T}_\ell$ and $\widetilde{\mathcal{T}}_\ell$ and consider an arbitrary element $T \in \mathcal{T}_{\ell+1}$. We have to distinguish certain cases:

- First, let $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}$. By the induction hypothesis, there is some $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_\ell$ such that

$$T = \bigcup_{\widetilde{T} \in \mathcal{V}} \widetilde{T}.$$

For any $\widetilde{T} \in \mathcal{V}$, there holds either $\widetilde{T} \in \widetilde{\mathcal{T}}_{\ell+1}$ or $\widetilde{T} = \widetilde{T}' \cup \widetilde{T}''$ for some $\widetilde{T}', \widetilde{T}'' \in \widetilde{\mathcal{T}}_{\ell+1}$. Consequently, this implies

$$T = \bigcup_{\widetilde{T} \in \widetilde{\mathcal{V}}} \widetilde{T} \quad \text{with} \quad \widetilde{\mathcal{V}} := \big\{ \widetilde{T}' \in \widetilde{\mathcal{T}}_{\ell+1} \,:\, \exists \widetilde{T} \in \mathcal{V} \quad \widetilde{T}' \subseteq \widetilde{T} \big\}.$$

- Second, let $T \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$, fix the unique $\widehat{T} \in \mathcal{T}_\ell$ with $T \subsetneqq \widehat{T}$, and assume that $\widehat{T} \in \mathcal{T}_\ell \backslash \widetilde{\mathcal{T}}_\ell$. By the induction hypothesis, there is some $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_\ell$ such that

$$\widehat{T} = \bigcup_{\widetilde{T} \in \mathcal{V}} \widetilde{T}.$$

Moreover, $\widehat{T} \in \mathcal{T}_\ell \backslash \widetilde{\mathcal{T}}_\ell$ implies $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_{\ell+1}$. Now, recall that bisection leads to a binary refinement tree. Consequently, the two sons of $\widehat{T}$ have an analogous representation. In particular, this implies

$$T = \bigcup_{\widetilde{T} \in \widetilde{\mathcal{V}}} \widetilde{T} \quad \text{with} \quad \widetilde{\mathcal{V}} := \big\{ \widetilde{T} \in \mathcal{V} \,:\, \widetilde{T} \subseteq T \big\} \subseteq \widetilde{\mathcal{T}}_{\ell+1}.$$

- Finally, let $T \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$, fix the unique $\widehat{T} \in \mathcal{T}_\ell$ with $T \subsetneqq \widehat{T}$, and assume that $\widehat{T} \in \mathcal{T}_\ell \cap \widetilde{\mathcal{T}}_\ell$. In particular, $\widehat{T}$ is refined by the κ-based mesh-refinement from Algorithm 4.1. We now aim to show that $\widehat{T}$ will be marked for refinement by the level-based mesh-refinement from Algorithm 4.5 as well. To that end, we again consider all possible cases:

• First, we note that $\widehat{T} \in \mathcal{M}_\ell$ implies $\widehat{T} \in \widetilde{\mathcal{M}}_\ell$ due to $\widehat{T} \in \mathcal{T}_\ell \cap \widetilde{\mathcal{T}}_\ell$. Therefore, we obtain $T \in \widetilde{\mathcal{T}}_{\ell+1}$.

• Second, assume that $\widehat{T} \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell$ has a marked neighbor $\widehat{T}' \in \mathcal{M}_\ell$ which leads to the additional marking of $\widehat{T}$, i.e., $h_\ell|_{\widehat{T}} > \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}'}$. Let $\widehat{T}_0, \widehat{T}_0' \in \mathcal{T}_0$ be the —not necessarily distinct— unique elements with $\widehat{T} \subseteq \widehat{T}_0$ and $\widehat{T}' \subseteq \widehat{T}_0'$. By definition of $\kappa(\mathcal{T}_0)$, there holds $h_0|_{\widehat{T}_0} \le \kappa(\mathcal{T}_0) h_0|_{\widehat{T}_0'}$. From the definition of the level-function, we infer

$h_\ell|_{\widehat{T}} = 2^{-\texttt{level}(\widehat{T})} h_0|_{\widehat{T}_0}$ and $h_\ell|_{\widehat{T}'} = 2^{-\texttt{level}(\widehat{T}')} h_0|_{\widehat{T}_0'}$. Combining these relations, we obtain $2^{-\texttt{level}(\widehat{T})} h_0|_{\widehat{T}_0} = h_\ell|_{\widehat{T}} > \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}'} = \kappa(\mathcal{T}_0) 2^{-\texttt{level}(\widehat{T}')} h_0|_{\widehat{T}_0'}$ and end up with

$$\kappa(\mathcal{T}_0) \geq \frac{h_0|_{\widehat{T}_0}}{h_0|_{\widehat{T}_0'}} > \kappa(\mathcal{T}_0) 2^{\texttt{level}(\widehat{T}) - \texttt{level}(\widehat{T}')}$$

and hence $\texttt{level}(\widehat{T}') > \texttt{level}(\widehat{T})$. According to the induction hypothesis for $\widehat{T}' \in \mathcal{T}_\ell$ and the level-estimate (4.13), we infer that $\widehat{T}' \in \widetilde{\mathcal{T}}_\ell$. Consequently, $\widehat{T}' \in \mathcal{M}_\ell$ implies $\widehat{T}' \in \widetilde{\mathcal{M}}_\ell$ according to our first observation. Now, $\widehat{T}' \in \widetilde{\mathcal{M}}_\ell$ and $\texttt{level}(\widehat{T}') > \texttt{level}(\widehat{T})$ enforces refinement of $\widehat{T}$ by the level-based Algorithm 4.5. This and $\widehat{T} \in \widetilde{\mathcal{T}}_\ell$ imply $T \in \widetilde{\mathcal{T}}_{\ell+1}$.

• Finally, for any element $\widehat{T} \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell$ which is refined by Algorithm 4.1, we find a marked element $\widehat{T}^{(0)} \in \mathcal{M}_\ell$ and a chain of elements $\widehat{T}^{(1)}, \dots, \widehat{T}^{(i)} \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell$ such that

$$\kappa(\mathcal{T}_0)\, h_\ell|_{\widehat{T}^{(j-1)}} < h_\ell|_{\widehat{T}^{(j)}} \quad \text{for } j = 1, \dots, i \quad \text{and} \quad \widehat{T}^{(i)} = \widehat{T}.$$

In particular, all these elements will be refined by call of Algorithm 4.1. Proceeding as in the previous step, we see that there holds $\widehat{T}^{(j)} \in \widetilde{\mathcal{T}}_\ell$ for all $j = 0, \dots, i$ as well as $\widehat{T}^{(0)} \in \widetilde{\mathcal{M}}_\ell$ and that all these elements will be refined by the level-based mesh-refinement as well. As above, we thus obtain $T \in \widetilde{\mathcal{T}}_{\ell+1}$.

- To verify the overlay estimate (R4), we propose for $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$

$$\mathcal{T} \oplus \mathcal{T}' = \mathcal{T}_\oplus \tag{4.18}$$

with $\mathcal{T}_\oplus$ from (4.15). Analogously to the last step of the proof of Theorem 4.7, we see that $\mathcal{T}_\oplus$ is a refinement of $\mathcal{T}$ and $\mathcal{T}'$. It remains to show $\kappa(\mathcal{T}_\oplus) \leq 2\kappa(\mathcal{T}_0)$. We argue by contradiction. Therefore, assume neighbors $T, T' \in \mathcal{T}_\oplus$ with $\mathrm{diam}(T)/\mathrm{diam}(T') > \max\{\kappa(\mathcal{T}), \kappa(\mathcal{T}')\}$. By definition of the K-mesh constant κ, we obtain $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$. The definition of $\mathcal{T}_\oplus$ thus gives an element $\widehat{T}' \in \mathcal{T}'$ with $T \subset \widehat{T}'$. Now, we obtain the contradiction

$$\max\{\kappa(\mathcal{T}), \kappa(\mathcal{T}')\} < \frac{\mathrm{diam}(T)}{\mathrm{diam}(T')} \leq \frac{\mathrm{diam}(\widehat{T}')}{\mathrm{diam}(T')} \leq \kappa(\mathcal{T}'),$$

where we used that T and $\widehat{T}'$ are neighbors in $\mathcal{T}'$ or coincide. The remainder of the proof follows analogously to the last step of the proof of Theorem 4.7. □

4.5. Proof of main results

This section applies the abstract analysis from Chapter 2 to the concrete model problem from Section 4.1. Therefore, $\phi \in L^2(\Gamma)$ and $\Phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ denote the solutions from (4.1) and (4.2), respectively. The estimator η_ℓ was defined in Section 4.1.2.

4.5.1. Notation. As stated in Section 4.1, $\mathcal{T}_\ell$ denotes a partition of the boundary Γ into affine line segments $T \in \mathcal{T}_\ell$ generated by refinement of the initial mesh $\mathcal{T}_0$ with Algorithm 4.1 or Algorithm 4.5. By $\mathcal{N}_\ell$, we denote the set of nodes of $\mathcal{T}_\ell$. For any element

$T \in \mathcal{T}_\ell$, we define an affine transformation $F_T : \widehat{T} \to T$, where $\widehat{T} := [0,1]$ denotes the reference interval. We define the L^2-orthogonal projection $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$ for all $\ell \in \mathbb{N}$. For any subset $\Gamma' \subseteq \Gamma$, the patch is

$$\omega_\ell(\Gamma') := \big\{T \in \mathcal{T}_\ell \,:\, T \cap \Gamma' \neq \emptyset\big\}, \tag{4.19}$$

and consequently for any subset $\mathcal{E}_\ell \subseteq \mathcal{T}_\ell$, we define

$$\omega_\ell(\mathcal{E}_\ell) := \omega_\ell\Big(\bigcup \mathcal{E}_\ell\Big). \tag{4.20}$$

Furthermore, the first-order spline space with respect to $\mathcal{T}_\ell$ is defined as

$$\mathcal{S}^1(\mathcal{T}_\ell) := \{v \in H^1(\Gamma) \,:\, v|_T \circ F_T \in \mathcal{P}^1([0,1]) \text{ for all } T \in \mathcal{T}_\ell\},$$

i.e. $\mathcal{S}^1(\mathcal{T}_\ell)$ is the space of piecewise affine and global continuous functions on Γ. For $\nu > 0$, we define broken Sobolev spaces as

$$H^\nu(\mathcal{T}_\ell) := \{v \in L^2(\Gamma) \,:\, v|_T \in H^\nu(T) \text{ for all } T \in \mathcal{T}_\ell\}.$$

Finally,

$$\mathcal{T}_{\ell,k} := \mathrm{unif}^{(k)}(\mathcal{T}_\ell) \tag{4.21}$$

denotes a mesh which is generated by bisecting all elements $T \in \mathcal{T}_\ell$ k-times and $\mathrm{unif}^{(k)}(T)$ denotes the set of sons $T_i \in \mathrm{unif}^{(k)}(\mathcal{T}_\ell)$, $i = 1, \dots, 2^k$ of $T \in \mathcal{T}_\ell$.

4.5.2. Scott-Zhang quasi-interpolation. We recall the definition of the Scott-Zhang projection $J_\ell : L^2(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$ from [**40**] for our particular situation: For all nodes $z \in \mathcal{N}_\ell$, we choose an element $T_z \in \mathcal{T}_\ell$ with $z \in T_z$. For $v \in L^2(\Gamma)$, $J_\ell v \in \mathcal{S}^1(\mathcal{T}_\ell)$ is then defined nodewise

$$(J_\ell v)(z) := \langle \psi_z \,,\, v \rangle_{L^2(T_z)}$$

for all nodes $z \in \mathcal{N}_\ell$. Here, $\psi_z \in \mathcal{P}^1(T_z)$ denotes the L^2-dual basis function defined by $\langle \psi_z \,,\, \zeta_{z'} \rangle_{L^2(T_z)} = \delta_{zz'}$. The hat-function $\zeta_{z'}$ with respect to the node $z' \in \mathcal{N}_\ell$ is given by

$$\zeta_{z'} \in \mathcal{S}^1(\mathcal{T}_\ell) \quad \text{and} \quad \zeta_{z'}(z) := \begin{cases} 1 & z = z' \\ 0 & z \neq z' \end{cases} \tag{4.22}$$

for all $z \in \mathcal{N}_\ell$. By definition, J_ℓ is a projection, i.e.

$$J_\ell V_\ell = V_\ell \quad \text{for all } V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell).$$

Additionally, for $T \in \mathcal{T}_\ell$, we have the following local stability properties (cf. [**40**, Corollary 4.1])

$$\begin{aligned} \|(1 - J_\ell)v\|_{L^2(T)} &\le C_{\mathrm{sz}} \|v\|_{L^2(\bigcup \omega_\ell(T))} \quad \text{for all } v \in L^2(\Gamma), \\ \|(1 - J_\ell)v\|_{H^1(T)} &\le C_{\mathrm{sz}} \|\tfrac{\partial}{\partial s} v\|_{L^2(\bigcup \omega_\ell(T))} \quad \text{for all } v \in H^1(\Gamma), \end{aligned} \tag{4.23}$$

as well as the local approximation property

$$\|(1 - J_\ell)v\|_{L^2(T)} \le C_{\mathrm{sz}} \,\mathrm{diam}(T) \|\tfrac{\partial}{\partial s} v\|_{L^2(\bigcup \omega_\ell(T))} \quad \text{for all } v \in H^1(\Gamma), \tag{4.24}$$

where the constant $C_{\mathrm{sz}} > 0$ depends only on $\kappa(\mathcal{T}_\ell)$ and Γ. $H^{1/2}$-stability of J_ℓ follows from interpolation arguments.

4.5.3. Proof of estimator related assumptions (E1)–(E4)**.** The purpose of this section is to prove that the residual based error estimator η_ℓ satisfies the assumptions (E1)–(E4) given in Section 2.3, which are sufficient for the optimality of the adaptive algorithm. The discrete local reliability (E4) has first been proved for a more general model problem in [**28**, Proposition 4.3]. For our particular situation in 2D, we provide a simplified proof.

Proposition 4.9 (discrete local reliability (E4))**.** *Let $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\ell)$ denote an refinement of $\mathcal{T}_\ell \in \mathbb{T}$ with associated Galerkin solution $\Phi_\star \in \mathcal{P}^0(\mathcal{T}_\star)$. Let $\mathcal{R}_\ell := \omega_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star)$ denote the enriched set of refined elements. Then, it holds*

$$|||\Phi_\star - \Phi_\ell||| \le C_{\mathrm{dlr}} \Big(\sum_{T \in \mathcal{R}_\ell} \eta_\ell(T)^2 \Big)^{1/2} \tag{4.25}$$

with some constant $C_{\mathrm{dlr}} > 0$ which depends only on Γ and $\kappa(\mathcal{T}_\ell)$. Moreover, we have $\mathcal{T}_\ell \setminus \mathcal{T}_\star \subseteq \mathcal{R}_\ell$ with $\#\mathcal{R}_\ell \le C_5 \#(\mathcal{T}_\ell \setminus \mathcal{T}_\star)$ for some constant $C_5 > 0$, which depends only on $\kappa(\mathcal{T}_\ell)$ and therefore is uniformly bounded due to (4.6)*.*

Proof. For each node $z \in \mathcal{N}_\ell$, let $\zeta_z \in \mathcal{S}^1(\mathcal{T}_\ell)$ denote the nodal hat function from (4.22). To abbreviate notation, we define for a function $\xi : \Gamma \to \mathbb{R}$ and a subset $\omega \subseteq \Gamma$

$$\xi|_\omega := \begin{cases} \xi & \text{in } \omega, \\ 0 & \text{in } \Gamma \setminus \omega. \end{cases}$$

Recall $f = (K + 1/2)g$. For $\Psi_\star \in \mathcal{P}^0(\mathcal{T}_\star)$, the Galerkin orthogonality of Φ_ℓ yields

$$\langle f - V\Phi_\ell \,,\, \Psi_\star \rangle_{L^2(\Gamma)} = \langle f - V\Phi_\ell \,,\, (1 - \Pi_\ell)\Psi_\star \rangle_{L^2(\Gamma)}. \tag{4.26}$$

Note that $(1 - \Pi_\ell)\Psi_\star = 0$ in $\bigcup(\mathcal{T}_\ell \cap \mathcal{T}_\star) = \overline{\Gamma \setminus \bigcup(\mathcal{T}_\ell \setminus \mathcal{T}_\star)}$. With $\mathcal{N}_{\ell,\star} := \mathcal{N}_\ell \cap \bigcup(\mathcal{T}_\ell \setminus \mathcal{T}_\star)$, we may write $\sum_{z \in \mathcal{N}_{\ell,\star}} \zeta_z = 1$ on $\bigcup(\mathcal{T}_\ell \setminus \mathcal{T}_\star)$. Therefore, it holds

$$\begin{aligned} \langle f - V\Phi_\ell \,,\, \Psi_\star \rangle_{L^2(\Gamma)} &= \Big\langle \sum_{z \in \mathcal{N}_{\ell,\star}} \zeta_z (f - V\Phi_\ell) \,,\, (1 - \Pi_\ell)\Psi_\star \Big\rangle_{L^2(\Gamma)} \\ &= \Big\langle \sum_{z \in \mathcal{N}_{\ell,\star}} \zeta_z (f - V\Phi_\ell) \,,\, \Psi_\star \Big\rangle_{L^2(\Gamma)} \\ &\qquad - \Big\langle \sum_{z \in \mathcal{N}_{\ell,\star}} \zeta_z (f - V\Phi_\ell) \,,\, (\Pi_\ell \Psi_\star)|_{\bigcup S_\ell} \Big\rangle_{L^2(\Gamma)} \end{aligned} \tag{4.27}$$

where $S_\ell := \mathcal{R}_\ell \setminus (\mathcal{T}_\ell \setminus \mathcal{T}_\star) \subseteq \mathcal{T}_\ell \cap \mathcal{T}_\star$. In the last equality, we used

$$\Big(\sum_{z \in \mathcal{N}_{\ell,\star}} \zeta_z \Big)\Big|_{\bigcup(\mathcal{T}_\ell \setminus \mathcal{R}_\ell)} \equiv 0 \quad \text{as well as} \quad \langle f - V\Phi_\ell \,,\, (\Pi_\ell \Psi_\star)|_{\bigcup(\mathcal{T}_\ell \setminus \mathcal{T}_\star)} \rangle_{L^2(\Gamma)} = 0.$$

By use of $h_\ell = h_\star$ on S_ℓ and elementwise stability of Π_ℓ on $S_\ell \subset \mathcal{T}_\ell \cap \mathcal{T}_\star$, one derives

$$\begin{aligned}\langle f - V\Phi_\ell\,,\,\Psi_\star\rangle_{L^2(\Gamma)} &\le \Big\| \sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)} \|\Psi_\star\|_{H^{-1/2}(\Gamma)} \\ &\qquad + \Big\| h_\ell^{-1/2} \sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell)\Big\|_{L^2(\Gamma)} \|h_\star^{1/2}\Pi_\ell\Psi_\star\|_{L^2(\bigcup S_\ell)} \\ &\le \Big\| \sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)} \|\Psi_\star\|_{H^{-1/2}(\Gamma)} \\ &\qquad + \Big\| h_\ell^{-1/2} \sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell)\Big\|_{L^2(\Gamma)} \|h_\star^{1/2}\Psi_\star\|_{L^2(\Gamma)}.\end{aligned} \tag{4.28}$$

With the local inverse inequality from [**32**, Theorem 3.6], we finally prove

$$\begin{aligned}&\langle f - V\Phi_\ell\,,\,\Psi_\star\rangle_{L^2(\Gamma)} \\ &\lesssim \Big(\Big\| \sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell)\Big\|_{H^{1/2}(\Gamma)} + \Big\| h_\ell^{-1/2} \sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell)\Big\|_{L^2(\Gamma)}\Big) \|\Psi_\star\|_{H^{-1/2}(\Gamma)}.\end{aligned} \tag{4.29}$$

Now, we use that the function $\sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell) \in H^1(\Gamma)$ is continuous on Γ. Due to $\langle (f - V\Phi_\ell)\,,\,\Psi_\ell\rangle_{L^2(\Gamma)} = 0$ for all $\Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$, the function has a zero on each element. Analogously to [**20**, Theorem 3.3], Friedrich's inequality allows us to estimate the right-hand side of the above estimate by

$$\begin{aligned}\langle f - V\Phi_\ell\,,\,\Psi_\star\rangle_{L^2(\Gamma)} &\lesssim \Big\| h_\ell^{1/2} \tfrac{\partial}{\partial s} \sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z(f - V\Phi_\ell)\Big\|_{L^2(\bigcup \mathcal{R}_\ell)} \|\Psi_\star\|_{H^{-1/2}(\Gamma)} \\ &\le \Big(\Big\| h_\ell^{1/2}\Big(\sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z\Big) \tfrac{\partial}{\partial s}(f - V\Phi_\ell)\Big\|_{L^2(\bigcup \mathcal{R}_\ell)} \\ &\qquad + \|h_\ell^{-1/2}(f - V\Phi_\ell)\|_{L^2(\bigcup \mathcal{R}_\ell)}\Big) \|\Psi_\star\|_{H^{-1/2}(\Gamma)},\end{aligned}$$

where we used the pointwise estimate $\frac{\partial}{\partial s}\sum_{z\in\mathcal{N}_{\ell,\star}} \zeta_z \lesssim h_\ell^{-1}$. Again, we apply Friedrich's inequality and finally end up with

$$\langle f - V\Phi_\ell\,,\,\Psi_\star\rangle_{L^2(\Gamma)} \lesssim \|h_\ell^{1/2} \tfrac{\partial}{\partial s}(f - V\Phi_\ell)\|_{L^2(\bigcup \mathcal{R}_\ell)} \|\Psi_\star\|_{H^{-1/2}(\Gamma)}.$$

Plugging in $\Psi_\star = \Phi_\star - \Phi_\ell$, we get

$$|\!|\!|\Phi_\star - \Phi_\ell|\!|\!|^2 = \langle f - V\Phi_\ell\,,\,\Psi_\star\rangle_{L^2(\Gamma)} \lesssim \|h_\ell^{1/2} \tfrac{\partial}{\partial s}(f - V\Phi_\ell)\|_{L^2(\bigcup \mathcal{R}_\ell)} \|\Psi_\star\|_{H^{-1/2}(\Gamma)}.$$

Canceling the term $\|\Psi_\star\|_{H^{-1/2}(\Gamma)}$ and norm equivalence (3.12) prove the statement. □

As a next step, we prove a new inverse estimate which is the heart of the matter to verify (E1), (E2), and (E5). We need a slightly more general version than in [**28**]. The main difference to [**28**, Proposition 3.3] is that this result holds for $\psi \in L^2(\Gamma)$ instead of discrete $\psi = \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ only. Note that with adapted notation, the arguments of the proof also hold in the 3D case.

Lemma 4.10. *Let $\mathcal{T}_\ell \in \mathbb{T}$. For $\psi \in L^2(\Gamma)$, it holds*

$$C_{\rm V}^{-1}\|h_\ell^{1/2}\tfrac{\partial}{\partial s}V\psi\|_{L^2(\Gamma)} \le \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}, \tag{4.30}$$

where the constant $C_{\rm V} > 0$ depends only on Γ and the fixed mesh-refinement strategy of Algorithm 4.1 resp. Algorithm 4.5.

Proof. First, we define the reference element $\widehat{T} := [0,1]$. For $\delta > 0$ and for all $T \in \mathcal{T}_\ell$, let $B_T := \bigcup_{x\in T}\{z \in \mathbb{R}^2 \,:\, |z-x| \le \delta\} \subset \mathbb{R}^2$ denote a certain neighborhood of T. We choose $\delta > 0$ sufficiently small such that

$$B_T \cap \Gamma \subset \omega_\ell(T) \quad \text{for all } T \in \mathcal{T}_\ell.$$

Note that δ depends only on the shape of Γ. Let $\widetilde{V} : H^{-1/2}(\Gamma) \to H^1_{\ell oc}(\mathbb{R}^2)$ (cf. [**34**, Theorem 6.11]) denote the simple-layer potential corresponding to V. Furthermore, for all $T \in \mathcal{T}_\ell$ let $u_T^{\rm near} := \widetilde{V}(\psi\chi_{\bigcup\omega_\ell(T)})$ and $u_T^{\rm far} := \widetilde{V}(\psi\chi_{\Gamma\setminus\bigcup\omega_\ell(T)})$ denote the near-field resp. the far-field of $\widetilde{V}\psi$. Here χ_ω denotes the characteristic function with respect to the set $\omega \subset \mathbb{R}^2$. We observe with $h_T := \operatorname{diam}(T)$

$$\|h_\ell^{1/2}\tfrac{\partial}{\partial s}V\psi\|_{L^2(\Gamma)}^2 = \sum_{T\in\mathcal{T}_\ell} h_T\|\tfrac{\partial}{\partial s}V\psi\|_{L^2(T)}^2 \lesssim \sum_{T\in\mathcal{T}_\ell}\Big(h_T\|\tfrac{\partial}{\partial s}u_T^{\rm near}\|_{L^2(T)}^2 + h_T\|\tfrac{\partial}{\partial s}u_T^{\rm far}\|_{L^2(T)}^2\Big), \tag{4.31}$$

where we used $u_T^{\rm near}|_\Gamma = V(\psi\chi_{\bigcup\omega_\ell(T)})$ as well as $u_T^{\rm far}|_\Gamma = V(\psi\chi_{\Gamma\setminus\bigcup\omega_\ell(T)})$. First, we treat the near-field term in the estimate using the stability of $V : L^2(\Gamma) \to H^1(\Gamma)$

$$\sum_{T\in\mathcal{T}_\ell} h_T\|\tfrac{\partial}{\partial s}u_T^{\rm near}\|_{L^2(T)}^2 \lesssim \sum_{T\in\mathcal{T}_\ell} h_T\|\psi\|_{L^2(\omega_\ell(T))}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2. \tag{4.32}$$

Second, we treat the far-field term. With [**28**, Lemma 3.6], we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_\ell} h_T\|\tfrac{\partial}{\partial s}u_T^{\rm far}\|_{L^2(T)}^2 &\lesssim \sum_{T\in\mathcal{T}_\ell}|u_T^{\rm far}|_{H^1(B_T)}^2\\ &\lesssim \sum_{T\in\mathcal{T}_\ell}\Big(\|\widetilde{V}\psi\|_{H^1(B_T)}^2 + |u_T^{\rm near}|_{H^1(B_T)}^2\Big)\\ &\lesssim \|\widetilde{V}\psi\|_{H^1(S)}^2 + \sum_{T\in\mathcal{T}_\ell}|u_T^{\rm near}|_{H^1(B_T)}^2\\ &\lesssim \|\psi\|_{H^{-1/2}(\Gamma)}^2 + \sum_{T\in\mathcal{T}_\ell}|u_T^{\rm near}|_{H^1(B_T)}^2,\end{aligned} \tag{4.33}$$

with $S := \bigcup_{T\in\mathcal{T}_\ell} B_T$ and where we used the stability of $\widetilde{V} : H^{-1/2}(\Gamma) \to H^1(S)$. It remains to take care of the near field terms in the estimate above. For $T \in \mathcal{T}_\ell$, we consider the affine transformation

$$F_T : \mathbb{R}^2 \to \mathbb{R}^2,\ x \mapsto h_T A x + b$$

with $b \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$ being orthonormal such that $F_T(\widehat{T} \times \{0\}) = T$. We define $\widehat{B}_T := F_T^{-1}(B_T)$. Because F_T is affine, we may write $F_T^{-1}(\omega_\ell(T)) := \widehat{T}_0 \cup \widehat{T} \cup \widehat{T}_1$ as union of three affine line segments. The angles between $\widehat{T}_0, \widehat{T}$, and $\widehat{T}_1$ are determined by the finitely many corners of Γ. Additionally, for two neighboring elements T, T' holds

$$h_T/h_{T'} \in \big\{2^k h_{T_0}/h_{T_0'} \,:\, k \in \mathbb{Z},\, T_0, T_0' \in \mathcal{T}_0 \text{ such that } 2^k h_{T_0}/h_{T_0'} \in [\kappa(\mathcal{T}_\ell)^{-1}, \kappa(\mathcal{T}_\ell)]\big\}.$$

Since the mesh-refinement ensures $1 \le \kappa(\mathcal{T}_\ell) \le 2\kappa(\mathcal{T}_0)$, we obtain $\#\big\{\mathrm{diam}(\widehat{T}_i) \,:\, i = 0, 1,\ T \in \mathcal{T}_\ell,\ \ell \in \mathbb{N}\big\} < \infty$. Therefore $\widehat{\omega}_\ell(T) := F_T^{-1}(\omega_\ell(T))$ belongs to a finite set of reference patches. We compute

$$\begin{aligned}
-2\pi\left(u_T^{\mathrm{near}} \circ F_T\right)(x) &= \int_{\omega_\ell(T)} \log|F_T(x) - y| \psi(y)\, ds_y \\
&= h_T \int_{\widehat{\omega}_\ell(T)} \log|h_T A(x - \widehat{y})| \psi(F_T(\widehat{y}))\, ds_{\widehat{y}} \\
&= h_T\Big(\int_{\widehat{\omega}_\ell(T)} \log|h_T|\, \psi(F_T(\widehat{y}))\, ds_{\widehat{y}} + \int_{\widehat{\omega}_\ell(T)} \log|x - \widehat{y}|\, \psi(F_T(\widehat{y}))\, ds_{\widehat{y}}\Big) \\
&= h_T\Big(\int_{\widehat{\omega}_\ell(T)} \log|h_T|\, \psi(F_T(\widehat{y}))\, ds_{\widehat{y}} - 2\pi \widehat{V}_T(\psi \circ F_T)(x)\Big),
\end{aligned}$$

where $\widehat{V}_T : \widetilde{H}^{-1/2}\big(\widehat{\omega}_\ell(T)\big) \to H^1(\widehat{B}_T)$ denotes one of finitely many single-layer potentials on the reference patches. Note that the first term in the last line of the equation above doesn't depend on $x \in \mathbb{R}^2$. This and the stability of $\widehat{V}_T$ give

$$\begin{aligned}
|u_T^{\mathrm{near}}|^2_{H^1(B_T)} \lesssim |u_T^{\mathrm{near}} \circ F_T|^2_{H^1(\widehat{B}_T)} &\lesssim h_T^2 \|\widehat{\psi}\|^2_{\widetilde{H}^{-1/2}(\widehat{\omega}_\ell(T))} \\
&\lesssim h_T^2 \|\widehat{\psi}\|^2_{L^2(\widehat{\omega}_\ell(T))} \simeq h_T \|\psi\|^2_{L^2(\omega_\ell(T))}.
\end{aligned} \tag{4.34}$$

Plugging the estimates (4.32), (4.33), and (4.34) into (4.31), we end up with

$$\begin{aligned}
\|h_\ell^{1/2} \tfrac{\partial}{\partial s} V\psi\|^2_{L^2(\Gamma)} &\lesssim \|h_\ell^{1/2}\psi\|^2_{L^2(\Gamma)} + \|\psi\|^2_{H^{-1/2}(\Gamma)} + \sum_{T\in\mathcal{T}_\ell} |u_T^{\mathrm{near}}|^2_{H^1(B_T)} \\
&\lesssim \|h_\ell^{1/2}\psi\|^2_{L^2(\Gamma)} + \|\psi\|^2_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

This yields the assertion. □

The last result allows us to prove the assumptions (E1) and (E2).

Proposition 4.11 (stability assumption (E1))**.** *The weighted-residual error estimator satisfies* (E1) *and* $C_{\mathrm{stab}} \ge 1$ *depends only on* $\kappa(\mathcal{T}_\ell)$, Γ, *and the constant* $C_{\mathrm{V}} > 0$ *from Lemma 4.10.*

Proof. With the reverse triangle inequality and $h_\ell = h_\star$ on $\mathcal{T}_\ell \cap \mathcal{T}_\star$, we see

$$\Big|\Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\ell(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\star(T)^2\Big)^{1/2}\Big| \le \|h_\star^{1/2} \tfrac{\partial}{\partial s} V(\Phi_\ell - \Phi_\star)\|_{L^2(\bigcup(\mathcal{T}_\ell\cap\mathcal{T}_\star))}.$$

The next step is to apply the inverse inequality from Lemma 4.10 as well as the inverse inequality from [**32**, Theorem 3.6]

$$\begin{aligned}\|h_\star^{1/2}\tfrac{\partial}{\partial s}V(\Phi_\ell-\Phi_\star)\|_{L^2(\bigcup(\mathcal{T}_\ell\cap\mathcal{T}_\star))} &\le \|h_\star^{1/2}\tfrac{\partial}{\partial s}V(\Phi_\ell-\Phi_\star)\|_{L^2(\Gamma)}\\ &\lesssim \|\Phi_\ell-\Phi_\star\|_{H^{-1/2}(\Gamma)}+\|h_\star^{1/2}(\Phi_\ell-\Phi_\star)\|_{L^2(\Gamma)}\\ &\lesssim \|\Phi_\ell-\Phi_\star\|_{H^{-1/2}(\Gamma)} \simeq |||\Phi_\ell-\Phi_\star|||,\end{aligned}$$

where we used norm equivalence (3.12). The combination of the last two estimates proves the result. □

Proposition 4.12 (reduction assumption (E2))**.** *The weighted-residual error estimator satisfies* (E2) *with* $q_{\rm red}=\sqrt{1/2}$ *and* $C_{\rm red}>0$ *depends only on* $\kappa(\mathcal{T}_\ell)$, Γ *and the constant* $C_{\rm V}>0$ *from Lemma 4.10.*

Proof. The triangle inequality shows

$$\Big(\sum_{T\in\mathcal{T}_\star\setminus\mathcal{T}_\ell}\eta_\star(T)^2\Big)^{1/2} \le \|h_\star^{1/2}\tfrac{\partial}{\partial s}(V\Phi_\ell-f)\|_{L^2(\bigcup(\mathcal{T}_\star\setminus\mathcal{T}_\ell))}+\|h_\star^{1/2}\tfrac{\partial}{\partial s}V(\Phi_\star-\Phi_\ell)\|_{L^2(\Gamma)}. \tag{4.35}$$

To treat the first term on the right-hand side of the above estimate, we use $h_\star\le h_\ell/2$ on $\mathcal{T}_\star\setminus\mathcal{T}_\ell$ and obtain

$$\|h_\star^{1/2}\tfrac{\partial}{\partial s}(V\Phi_\ell-f)\|_{L^2(\bigcup(\mathcal{T}_\star\setminus\mathcal{T}_\ell))}\le\sqrt{1/2}\Big(\sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_\star}\eta_\ell(T)^2\Big)^{1/2}.$$

The second term on the right-hand side of (4.35) is estimated as in the previous proof, by help of Lemma 4.10

$$\begin{aligned}\|h_\star^{1/2}\tfrac{\partial}{\partial s}V(\Phi_\star-\Phi_\ell)\|_{L^2(\Gamma)} &\lesssim \|\Phi_\ell-\Phi_\star\|_{H^{-1/2}(\Gamma)}+\|h_\star^{1/2}(\Phi_\ell-\Phi_\star)\|_{L^2(\Gamma)}\\ &\lesssim \|\Phi_\ell-\Phi_\star\|_{H^{-1/2}(\Gamma)}\simeq|||\Phi_\ell-\Phi_\star|||.\end{aligned}$$

Plugging everything together, we conclude the proof. □

Now, as we have proved the assumptions (E1), (E2), and (E4), we see that convergence (Theorem 2.4) and quasi-optimality (Theorem 2.7) hold true for our particular situation. The next step is to characterize the approximation class $\mathbb{A}_s$ in terms of the energy norm error. Therefore, we need an efficiency estimate for η_ℓ.

4.5.4. Proof of Efficiency (E5)**.** The key to efficiency is again Lemma 4.10. We want to use the statement with $\psi:=\phi-\Phi_\ell$, which gives us

$$\eta_\ell=\|h_\ell^{1/2}\tfrac{\partial}{\partial s}V(\phi-\Phi_\ell)\|_{L^2(\Gamma)}\lesssim\|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}+\|h_\ell^{1/2}(\phi-\Phi_\ell)\|_{L^2(\Gamma)}. \tag{4.36}$$

With the L^2-orthogonal projection $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$, we see

$$\begin{aligned}
\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} &\le \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\Gamma)} + \|h_\ell^{1/2}(\Pi_\ell\phi - \Phi_\ell)\|_{L^2(\Gamma)} \\
&\lesssim \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\Gamma)} + \|\Pi_\ell\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \\
&\le \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\Gamma)} + \|(1-\Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} + \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \\
&\lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\Gamma)},
\end{aligned} \tag{4.37}$$

where we used the approximation properties of Π_ℓ (cf. [**14**, Theorem 4.1]). Note, that the last term on the right-hand side of the estimate above is a priori at least of order $\mathcal{O}(h^{1/2})$ on uniform meshes. This allows us to formulate the next proposition.

Proposition 4.13. *Let the solution of* (4.1) *additionally satisfy* $\phi \in L^2(\Gamma) \cap H^\nu(\mathcal{T}_0)$ *for some* $\nu > 0$. *Then, for all* $0 < s \le \min\{\nu, 1\} + 1/2$, *it holds* $(\phi, f) \in \mathbb{A}_s^\eta$. *Moreover, for* $0 < \theta < 1$ *sufficiently small, it also holds*

$$|||\phi - \Phi_\ell||| \le \widetilde{C}_{\mathrm{opt}}(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}$$

for all $\ell \ge 0$ *and hence* $\phi \in \mathbb{A}_s$. *The constant* $\widetilde{C}_{\mathrm{opt}} > 0$ *depends only on* $\|(\phi, f)\|_{\mathbb{A}_s^\eta}$. *In particular, this states that the adaptive algorithm yields at least the same rate of convergence as uniform mesh refinement (cf. Theorem 3.3).*

Proof. The estimate (4.37) gives an efficiency estimate (E5) with higher-order term $\|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\Gamma)}$. Let $\mathcal{T}_\ell = \texttt{refine}(\mathcal{T}^{(\ell)})$ for a uniform mesh $\mathcal{T}^{(\ell)}$ with mesh-width $h^{(\ell)} \simeq (\#\mathcal{T}^{(\ell)} - \mathcal{T}_0)^{-1}$. Without loss of generality, assume $\nu \le 1$. We aim to prove (2.18) by use of the elementwise Poincaré inequality

$$\begin{aligned}
\|h_\ell^{1/2}(1-\Pi_\ell)\phi\|^2_{L^2(\Gamma)} \le h^{(\ell)} \sum_{T\in\mathcal{T}_0} \|(1-\Pi^{(\ell)})\phi\|^2_{L^2(T)} &\lesssim \big(h^{(\ell)}\big)^{1+2\nu} \sum_{T\in\mathcal{T}_0} \|\phi\|^2_{H^\nu(T)} \\
&\simeq (\#\mathcal{T}^{(\ell)} - \mathcal{T}_0)^{-1-2\nu} \sum_{T\in\mathcal{T}_0} \|\phi\|^2_{H^\nu(T)}.
\end{aligned}$$

This proves $s_\star = \min\{\nu, 1\} + 1/2$ in (E5). Together with the a priori convergence result in Theorem 3.3, the estimate above yields $(\phi, f) \in \mathbb{A}_s^\eta$ for $0 < s \le s_\star$. Therefore, we may employ Theorem 2.7 to prove the assertion. $\square$

Because the regularity of the solution ϕ is influenced by the boundary Γ, this result is not fully satisfactory. Thus, we aim to estimate the term $\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}$ to get a better result. To formulate the next statement, recall the definition of the uniform refined mesh $\mathrm{unif}^{(k)}(\mathcal{T}_\ell)$ given in (4.21).

Proposition 4.14. *Let the given boundary data satisfy* $g \in H^{s_{\mathrm{reg}}}(\Gamma)$ *for some* $s_{\mathrm{reg}} > 2$. *We consider a mesh* $\mathcal{T}_\ell \in \mathbb{T}$. *Then, the unique solution of* (4.1) *can be decomposed as* $\phi = \phi_0 + \phi_{\mathrm{sing}}$. *The smooth part satisfies* $\phi_0 \in H^{\nu_{\mathrm{reg}}-1-\varepsilon}(\mathcal{T}_\ell)$ *for all* $\varepsilon > 0$, *where* $\nu_{\mathrm{reg}} := \min\{s_{\mathrm{reg}}, 5/2\}$. *The singular part fulfills* $\phi_{\mathrm{sing}} \in L^2(\Gamma)$. *Moreover, it exists* $h_0 > 0$

such that for all $\mathcal{T}_\ell$ with mesh-width $\|h_\ell\|_{L^\infty(\Gamma)} < h_0$ and for all $\kappa > 0$, there exists $k \in \mathbb{N}$ such that

$$\|h_\ell^{1/2}(1-\Pi_{\ell,k})\phi\|_{L^2(\Gamma)} \le \kappa\|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\Gamma)} + C_6\|h_\ell^{1/2}(1-\Pi_\ell^{(1)})\phi_0\|_{L^2(\Gamma)} \tag{4.38}$$

where $\Pi_{\ell,k} : L^2(\Gamma) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(\mathcal{T}_\ell))$ and $\Pi_\ell^{(1)} : L^2(\Gamma) \to \mathcal{P}^1(\mathcal{T}_\ell)$ denote the respective L^2-orthogonal projections. The constants $C_6 > 0, h_0 > 0$, and $k \in \mathbb{N}$ depend only on Γ and $\kappa(\mathcal{T}_\ell)$. The function ϕ_0 depends on $\mathcal{T}_\ell$ and $s_{\mathrm{reg}} > 2$, but for all $\varepsilon > 0$ the elementwise norm is bounded uniformly, i.e.

$$\sum_{T\in\mathcal{T}_\ell} \|\phi_0\|^2_{H^{\nu_{\mathrm{reg}}-1-\varepsilon}(T)} \le C_{\mathrm{hot}} < \infty \tag{4.39}$$

and $C_{\mathrm{hot}} > 0$ depends only on Γ, $\kappa(\mathcal{T}_\ell)$, $s_{\mathrm{reg}} > 2$, and $\varepsilon > 0$.

The proof of this proposition needs several preliminary lemmata and the definition of the space of singularity functions: Let $\beta_j \in (-1/2, 2]$, $j = 1, \dots, m$, with $\beta_j \ne \beta_i$ for $i \ne j$. Then, for an interval $T \subseteq \mathbb{R}$

$$\begin{aligned}\mathcal{H}_{\mathrm{sing}}(T, (\beta_j)_{j=1}^m) := \mathrm{span}\Big(&\big\{s \mapsto s^{\beta_j} \,:\, j = 1,\dots,m\big\} \\ &\cup \big\{s \mapsto s^{\beta_j}\log(s) \,:\, j = 1,\dots,m\big\}\Big) \oplus \mathcal{P}^1(T)\end{aligned} \tag{4.40}$$

is called the singularity space for $(\beta_j)_{j=1}^m$.

Lemma 4.15. *Assume $h > 0$ and $r \ge h/\nu$ for some $\nu > 0$. Let x_0, s_1, $s_2 \in [r, r+h]$. For $|s_2 - x_0| \ge h/4$ and $\beta \in (-1/2, 2]$, it holds*

$$\frac{|\int_{s_2}^{s_1} t^{\beta-2}dt|}{|\int_{x_0}^{s_2} t^{\beta-2}dt|} \le C_7. \tag{4.41}$$

where $C_7 = 4/(1+\nu)^{\beta-2} > 0$.

Proof. Because $\beta - 2 \le 0$, we may estimate

$$\max_{t\in[r,r+h]} t^{\beta-2} = r^{\beta-2} \quad \text{and} \quad \min_{t\in[r,r+h]} t^{\beta-2} = (r+h)^{\beta-2} \ge (1+\nu)^{\beta-2}r^{\beta-2}.$$

With $|s_2 - x_0| \ge h/4$, we see

$$\frac{|\int_{s_2}^{s_1} t^{\beta-2}dt|}{|\int_{x_0}^{s_2} t^{\beta-2}dt|} \le \frac{hr^{\beta-2}}{h/4(1+\nu)^{\beta-2}r^{\beta-2}} \le \frac{4}{(1+\nu)^{\beta-2}}.$$

This concludes the proof. □

In the following, we will write $(\cdot)' = \frac{\partial}{\partial s}$ to abbreviate the notation.

Lemma 4.16. *Assume $h > 0$ and $r \ge h/\nu$ for some $\nu > 0$. Consider the interval $T := [r, r+h]$ and $\psi \in \mathcal{H}_{\mathrm{sing}}(T, (\beta_j)_{j=1}^m)$. Then, there exists $r_0 > 0$ such that for $r < r_0$*

$$\max_T |\psi'| \le C_8 \min_{T'} |\psi'|, \tag{4.42}$$

where $T' := [r, r+h/4]$ or $T' := [r+3h/4, r+h]$. The constants $C_8 > 0$ and $r_0 > 0$ depend only on $\nu > 0$ and $(\beta_j)_{j=1}^m \in (-1/2, 2]$.

PROOF. The function ψ can be written as

$$\psi(s) = \sum_{j=1}^{m} a_j s^{\beta_j} + b_j s^{\beta_j} \log(s) + a_T, \tag{4.43}$$

with $a_T \in \mathcal{P}^1(T)$ and $a_j, b_j \in \mathbb{R}$. First, we observe that $(s \mapsto s^1) \in \mathcal{P}^1(T)$. Therefore, we assume $a_j = 0$ for $\beta_j = 1$. Note that the statement (4.42) is trivial if ψ' is constant. Due to the last observation, this happens only if all coefficients a_j, b_j are zero. Therefore, we may additionally assume that at least one coefficient a_j or b_j is non-zero. Let $|\psi'(x_0)| = \min_{x\in T} |\psi'(x)|$ for $x_0 \in T$. We use the minimality of $|\psi'(x_0)|$ to show that, for all $s \in T$, either one of the terms $|\psi'(x_0)|$ and $\int_{x_0}^s \psi''(t)dt$ is zero or that both terms must have the same sign. We argue by contradiction and assume $\psi'(x_0)\int_{x_0}^s \psi''(t)dt < 0$, i.e. both terms have opposite sign for some $s \in [r, r+h]$. We choose $x_1 \in [r, r+h]$ such that

$$\Big| \int_{x_0}^{x_1} \psi''(t)dt \Big| < |\psi'(x_0)| \quad \text{and} \quad \psi'(x_0) \int_{x_0}^{x_1} \psi''(t)dt < 0. \tag{4.44}$$

This is possible because $\int_{x_0}^{x_1} \psi''(t)dt \to 0$ for $x_1 \to x_0$. With (4.44), we obtain

$$|\psi'(x_1)| = \Big|\psi'(x_0) + \int_{x_0}^{x_1} \psi''(t)dt\Big| = |\psi'(x_0)| - \Big|\int_{x_0}^{x_1} \psi''(t)dt\Big| < |\psi'(x_0)|, \tag{4.45}$$

which is a contradiction to the minimality of $|\psi'(x_0)|$. We just proved

$$\psi'(x_0) \int_{x_0}^{s} \psi''(t)dt \geq 0 \quad \text{for all } s \in T$$

i.e. both terms have the same sign or at least one of them is zero. With this result, we may write

$$|\psi'(s)| = \Big|\psi'(x_0) + \int_{x_0}^{s} \psi''(t)dt\Big| = |\psi'(x_0)| + \Big|\int_{x_0}^{s} \psi''(t)dt\Big| \tag{4.46}$$

for all $s \in T$.

Now, we fix the index j_0 with the smallest exponent $\beta_{j_0} \in (-1/2, 2]$ and $a_{j_0} \neq 0$ or $b_{j_0} \neq 0$ in (4.43). Note that we can explicitly compute ψ'', i.e.

$$\psi''(s) = \sum_{j=1}^{m} a_j\beta_j(\beta_j - 1)s^{\beta_j - 2} + b_j s^{\beta_j-2}\big(\beta_j(\beta_j-1)\log(s) + 2\beta_j - 1\big).$$

Now, we have to distinguish two cases:

Case 1: It holds that $b_{j_0} = 0$. Due to our assumptions, we have $\beta_{j_0} \neq 1$, since $a_{j_0} \neq 0$. Then, we choose $r_0 < 1/(1+\nu)$ sufficiently small such that for all $0 < s < r_0(1+\nu)$ holds

$$0 < \tfrac{1}{2}|a_{j_0}\beta_{j_0}(\beta_{j_0}-1)s^{\beta_{j_0}-2}| \leq |\psi(s)''| \leq 2|a_{j_0}\beta_{j_0}(\beta_{j_0}-1)s^{\beta_{j_0}-2}|, \tag{4.47}$$

which is possible because $\beta_j - 2 \leq 0$ and the term with the smallest exponent dominates the function ψ''.

Case 2: It holds that $b_{j_0} \neq 0$. If $\beta_{j_0} \neq 1$, we choose $r_0 < 1/(1+\nu)$ sufficiently small such that for all $0 < s < r_0(1+\nu)$ holds

$$0 < \tfrac{1}{2}|b_{j_0}\beta_{j_0}(\beta_{j_0}-1)\log(s)s^{\beta_{j_0}-2}| \leq |\psi(s)''| \leq 2|b_{j_0}\beta_{j_0}(\beta_{j_0}-1)\log(s)s^{\beta_{j_0}-2}|, \tag{4.48}$$

which is possible because $\beta_j - 2 \leq 0$ and the term with the smallest exponent dominates the function ψ''. If $\beta_{j_0} = 1$, the log-term vanishes and we get

$$0 < \tfrac{1}{2}|b_{j_0}s^{-1}| \leq |\psi(s)''| \leq 2|b_{j_0}s^{-1}|, \tag{4.49}$$

i.e. case 1 with different constants. All arguments for case 1 in the proof below work analogously for this case.

In either case, we see that for $r < r_0$, we get $r + h \leq r(1+\nu) \leq r_0(1+\nu)$. Therefore, $s \in T$ satisfies $s \leq r_0(1+\nu)$, and we get with (4.47)–(4.48) that ψ'' has no zero on T. Using this and (4.46), we get for $s_1, s_2 \in T$, $s_2 \neq x_0$

$$\begin{aligned}\frac{|\psi'(s_1)|}{|\psi'(s_2)|} &= \frac{|\psi'(x_0)| + |\int_{x_0}^{s_1}\psi''(t)dt|}{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''(t)dt|} \leq \frac{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''(t)dt| + |\int_{s_2}^{s_1}\psi''(t)dt|}{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''(t)dt|} \\ &\leq 1 + \frac{|\int_{s_2}^{s_1}\psi''dt|}{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''dt|} \leq 1 + \frac{|\int_{s_2}^{s_1}\psi''dt|}{|\int_{x_0}^{s_2}\psi''dt|}.\end{aligned} \tag{4.50}$$

Again we use (4.47) and (4.48), to estimate

$$\frac{|\psi'(s_1)|}{|\psi'(s_2)|} \leq 1 + \frac{2|a_{j_0}\beta_{j_0}(\beta_{j_0}-1)\int_{s_2}^{s_1}t^{\beta_{j_0}-2}dt|}{\frac{1}{2}|a_{j_0}\beta_{j_0}(\beta_{j_0}-1)\int_{x_0}^{s_2}t^{\beta_{j_0}-2}dt|} = 1 + 4\frac{|\int_{s_2}^{s_1}t^{\beta_{j_0}-2}dt|}{|\int_{x_0}^{s_2}t^{\beta_{j_0}-2}dt|}, \tag{4.51}$$

for case 1, and by use of $r + h \leq r_0(1+\nu) < 1$

$$\frac{|\psi'(s_1)|}{|\psi'(s_2)|} \leq 1 + \frac{2|b_{j_0}\beta_{j_0}(1-\beta_{j_0})\int_{s_2}^{s_1}t^{\beta_{j_0}-2}\log(t)|dt}{\frac{1}{2}|b_{j_0}\beta_{j_0}(1-\beta_{j_0})\int_{x_0}^{s_2}t^{\beta_{j_0}-2}\log(t)dt|} \leq 1 + 4\frac{|\int_{s_2}^{s_1}t^{\beta_j-2}dt|}{|\int_{x_0}^{s_2}t^{\beta_j-2}dt|}\frac{|\log(r)|}{|\log(r+h)|} \tag{4.52}$$

for case 2. If we restrict ourselves to $|s_2 - x_0| \geq h/4$, all assumptions of Lemma 4.15 are satisfied, and we get for case 1

$$\frac{|\psi'(s_1)|}{|\psi'(s_2)|} \leq 1 + 4C_7. \tag{4.53}$$

by help of Equation (4.51). For case 2, we additionally have to bound $|\log(r)/\log(r+h)| \leq C_9$ in (4.52) by

$$C_9 = \sup_{0<r<r_0}\frac{|\log(r)|}{|\log(r+h)|} \leq \sup_{0<r<r_0}\frac{|\log(r)|}{|\log(r)+\log(1+\nu)|} < \infty,$$

where we used $r + h \leq r(1 + \nu) \leq r_0(1 + \nu) < 1$. With the definition

$$T' := \begin{cases} [r, r + h/4] & \text{for } x_0 \in [r + h/2, r + h] \\ [r + 3h/4, r + h] & \text{for } x_0 \in [r, r + h/2) \end{cases}$$

and $s_2 \in T'$, we ensure $|s_2 - x_0| \geq h/4$. Plugging everything together, we use (4.53) to prove the statement (4.42). □

Lemma 4.17. *Assume $h > 0$ and $r \geq h/\nu$ for some $\nu > 0$. Consider the interval $T := [r, r + h]$ and $\psi \in \mathcal{H}_{\rm sing}(T, (\beta_j)_{j=1}^m)$. Then, there exists $r_0 > 0$ such that for $r < r_0$, it holds*

$$\|(1 - \Pi_k)\psi\|_{L^2(T)}^2 \leq C_{10} 2^{-2k} \|(1 - \Pi)\psi\|_{L^2(T)}^2, \tag{4.54}$$

where the constants $C_{10} > 0$ and $r_0 > 0$ depend only $(\beta_j)_{j=1}^m \in (-1/2, 2]$ and $\nu > 0$. Here, $\Pi_k : L^2(T) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(T))$ and $\Pi : L^2(T) \to \mathcal{P}^0(T)$ denote the L^2-orthogonal projections.

Proof. The statement is trivial for constant ψ, i.e. we may assume $\psi'(s) \neq 0$ for at least one $s \in T$. For $r < r_0$, Lemma 4.16 proves

$$0 < \max_T |\psi'| \leq C_8 \min_{T'} |\psi'|. \tag{4.55}$$

Next, we use that $(1-\Pi_k)\psi$ has a zero s_{T_i} on each $T_i \in \mathrm{unif}^{(k)}(T), i = 1, \ldots, 2^k$. Therefore

$$|(1 - \Pi_k)\psi(s)| = \Big| \int_{s_{T_i}}^s ((1 - \Pi_k)\psi)' dt \Big| = \Big| \int_{s_{T_i}}^s \psi' dt \Big| \leq h_{T_i} \max_{T_i} |\psi'|$$

for $s \in T_i$. With (4.55) and $h_{T_i} = h_{T_1} = 2^{-k}h$, we conclude

$$\begin{aligned} \|(1 - \Pi_k)\psi\|_{L^2(T)}^2 &\lesssim \sum_{i=1}^{2^k} h_{T_i}^3 \max_{T_i} |\psi'|^2 \lesssim h_{T_1}^3 2^k \max_T |\psi'|^2 \lesssim h_{T_1}^3 2^k \min_{T'} |\psi'|^2 \\ &= 2^{-2k} h^3 \min_{T'} |\psi'|^2. \end{aligned} \tag{4.56}$$

Now, we calculate for $s_0, s \in T'$

$$h^3 \min_{T'} |\psi'| \simeq \int_{T'} \Big(\int_{s_0}^s \min_{T'} |\psi'| dt \Big)^2 ds \leq \int_{T'} \Big| \int_{s_0}^s \psi' dt \Big|^2 ds, \tag{4.57}$$

where we used $4|T'| = |T| = h$ and the fact that ψ' doesn't change sign on T' because of (4.55). To bound the last term in the estimate above, we introduce the L^2-orthogonal projection $\Pi' : L^2(T') \to \mathcal{P}^0(T')$. Let $s_0 \in T'$ denote the zero of $(1 - \Pi')\psi$ and note that

$((1-\Pi')\psi)' = \psi'$ on T'. With this and the estimates (4.56) and (4.57), we end up with

$$\begin{aligned}\|(1-\Pi_k)\psi\|^2_{L^2(T)} &\lesssim 2^{-2k}\int_{T'}\left|\int_{s_0}^{s}((1-\Pi')\psi)'dt\right|^2 ds \\ &= 2^{-2k}\|(1-\Pi')\psi\|^2_{L^2(T')} \leq 2^{-2k}\|(1-\Pi)\psi\|^2_{L^2(T')} \\ &\leq 2^{-2k}\|(1-\Pi)\psi\|^2_{L^2(T)},\end{aligned} \tag{4.58}$$

due to the best-approximation property of Π' on T'. This proves the assertion. □

Lemma 4.18. *Assume $h > 0$. Consider the interval $T := [0,h]$ as well as $\psi \in \mathcal{H}_{\rm sing}(T,(\beta_j)_{j=1}^m)$. Then, there holds*

$$\|(1-\Pi_k)\psi\|^2_{L^2(T)} \leq C_{11}2^{-2\varepsilon k}\|(1-\Pi)\psi\|^2_{L^2(T)}, \tag{4.59}$$

where the constant $C_{11} > 0$ and $\varepsilon > 0$ depend only on $(\beta_j)_{j=1}^m \in (-1/2, 2]$. Here, $\Pi_k : L^2(T) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(T))$ and $\Pi : L^2(T) \to \mathcal{P}^0(T)$ denote the L^2-orthogonal projections.

Proof. For $\varepsilon = (\min_{j=1,\ldots,m}\beta_j + 1/2)/2$, we consider $\mu \in \mathcal{H}_{\rm sing}(T,(\beta_j)_{j=1}^m) \subset H^\varepsilon(T)$ (cf. [**22**, Corollary 4.9]). We define the fractional Sobolev norms by interpolation. Recall that all definitions of the fractional Sobolev norms are equivalent on the whole space $H^\varepsilon(\Gamma)$. But as the constants depend on the domain, we get some elementwise properties like the Poincaré inequality (cf. [**10**])

$$\|(1-\Pi)v\|_{L^2(T)} \lesssim \|h^\varepsilon v\|_{H^\varepsilon(T)} \quad \text{for all } v \in H^\varepsilon(T)$$

more easily if we choose the definition by interpolation. Let $\widehat{\mu}(s) := \mu(hs)$. First we prove that $\widehat{\mu}$ belongs to a finite dimensional space:

$$\widehat{\mu}(s) = \sum_{j=1}^m a_j h^{\beta_j}s^{\beta_j} + b_j\big(h^{\beta_j}s^{\beta_j}\log(s) + h^{\beta_j}\log(h)s^{\beta_j}\big) + a_T(hs) \in \mathcal{H}_{\rm sing}([0,1],(\beta_j)_{j=1}^m),$$

where $\dim\mathcal{H}_{\rm sing}([0,1],(\beta_j)_{j=1}^m) \leq 2m+2$. With this and standard scaling arguments, one obtains

$$\|\mu\|^2_{H^\varepsilon(T)} \lesssim h^{1-2\varepsilon}\|\widehat{\mu}\|^2_{H^\varepsilon([0,1])} \lesssim h^{1-2\varepsilon}\|\widehat{\mu}\|^2_{L^2([0,1])} \lesssim h^{-2\varepsilon}\|\mu\|^2_{L^2(T)}, \tag{4.60}$$

where the second estimate holds because of norm equivalence on finite dimensional spaces. By use of (4.60) with $\mu = (1-\Pi)\psi$, we conclude

$$\begin{aligned}\|(1-\Pi_k)\psi\|^2_{L^2(T)} &= \|(1-\Pi_k)(1-\Pi)\psi\|^2_{L^2(T)} \\ &= \sum_{i=1}^{2^k} \|(1-\Pi_k)(1-\Pi)\psi\|^2_{L^2(T_i)} \\ &\lesssim h_{T_1}^{2\varepsilon} \sum_{i=1}^{2^k} \|(1-\Pi)\psi\|^2_{H^\varepsilon(T_i)} \\ &\lesssim h_{T_1}^{2\varepsilon} \|(1-\Pi)\psi\|^2_{H^\varepsilon(T)} \\ &\lesssim (h_{T_1}/h)^{2\varepsilon} \|(1-\Pi)\psi\|^2_{L^2(T)} \lesssim 2^{-2\varepsilon k} \|(1-\Pi)\psi\|^2_{L^2(T)},\end{aligned} \tag{4.61}$$

where we used the Poincaré inequality for fractional Sobolev norms and the fact that $\sum_{i=1}^{2^k} \|w\|^2_{H^\varepsilon(T_i)} \lesssim \|w\|^2_{H^\varepsilon(T)}$ for all $w \in H^\varepsilon(T)$ (see [**10**]). □

Now, we are ready to prove Proposition 4.14.

Proof of Proposition 4.14. According to [**22**, Theorem 4.8], the solution ϕ has the form

$$\phi(x) = \widetilde{\phi}_0(x) + \phi_{\text{sing}} := \widetilde{\phi}_0(x) + \sum_{j=1}^m \chi_j(x)\phi_j(|x-c_j|) \quad \text{for all } x \in \Gamma, \tag{4.62}$$

where $m \in \mathbb{N}$ is the number of corners c_j of Γ and $\widetilde{\phi}_0 \in H^{\nu_{\text{reg}}-1-\varepsilon}(\mathcal{T}_0)$ for all $\varepsilon > 0$. The singularity functions ϕ_j satisfy

$$\phi_j(s) = \sum_{i=1}^M a_{i,j} s^{\beta_{i,j}} + b_{i,j} s^{\beta_{i,j}} \log(s) \in \mathcal{H}_{\text{sing}}\Big([0,\infty], \big((\beta_{i,j})_{i=1}^M\big)_{j=1}^m\Big), \tag{4.63}$$

where the exponents $\beta_{i,j} > -1/2$ are determined by the inner angle α_j in c_j through $\beta_{i,j} + 1 = k_i\pi/\alpha_j$ for some non-negative integer $k_i \in \mathbb{N}$. χ_j is a smooth cutoff function with $c_i \notin \operatorname{supp}(\chi_j)$ for all $i \neq j$. For each χ_j, it exists a neighborhood $U_j \subset \Gamma$ of c_j such that $\chi_j \equiv 1$ in U_j. We choose $h_0 > 0$ sufficiently small so that the ball $B_{h_0}(c_j) \cap \Gamma \subset U_j$ for all $j = 1, \dots, m$. Additionally, we observe that for $\beta_{i,j} > 2$ the corresponding term in (4.63) is smoother than $\widetilde{\phi}_0$. Thus, it is sufficient to consider $\beta_{i,j} \in (-1/2, 2]$. Of course, we want to exploit Lemma 4.17 and Lemma 4.18. We prove estimate (4.38) elementwise, i.e.

$$\|h_\ell^{1/2}(1-\Pi_{\ell,k})\phi\|^2_{L^2(\Gamma)} = \sum_{T\in\mathcal{T}_\ell} h_\ell|_T \|(1-\Pi_{\ell,k})\phi\|^2_{L^2(T)}. \tag{4.64}$$

By use of an affine transformation, we can treat each element that appears in the sum as an interval on the real axis, i.e. we identify the corner c_j with zero and $T = [r, r+h]$

for some $r \geq 0$, $h = h_\ell|_T$. If $r > 0$, there exists at least one element T' with $T' \cap T \neq \emptyset$, which is located between the corner c_j and T. Mesh regularity thus gives

$$r \geq h_\ell|_{T'} \geq \frac{h_\ell|_T}{\kappa(\mathcal{T}_\ell)}. \tag{4.65}$$

Now, we consider equation (4.64) and distinguish three cases:

(i) If $T = [0, h]$, the assumption on the mesh-width shows $h < h_0$ and therefore

$$\|(1 - \Pi_{\ell,k})\phi\|^2_{L^2(T)} \lesssim \|(1 - \Pi_{\ell,k})(\phi_j + a_T)\|^2_{L^2(T)} + \|\widetilde{\phi}_0 - a_T\|^2_{L^2(T)}.$$

We choose $a_T = (\Pi^{(1)}_\ell \widetilde{\phi}_0)|_T \in \mathcal{P}^1(T)$ and apply Lemma 4.18 to estimate the first term

$$\begin{aligned}
&\|(1 - \Pi_{\ell,k})\phi\|^2_{L^2(T)} \\
&\lesssim 2^{-2\varepsilon k}\|(1 - \Pi_\ell)(\phi_j + a_T)\|^2_{L^2(T)} + \|(1 - \Pi^{(1)}_\ell)\widetilde{\phi}_0\|^2_{L^2(T)} \\
&\lesssim 2^{-2\varepsilon k}\|(1 - \Pi_\ell)\phi\|^2_{L^2(T)} + \|(1 - \Pi^{(1)}_\ell)\widetilde{\phi}_0\|^2_{L^2(T)}.
\end{aligned}$$

(ii) If $T = [r, r+h]$ with $r+h < h_0$ and additionally $r < r_0$ with the constant $r_0 > 0$ from Lemma 4.17, we obtain

$$\begin{aligned}
&\|(1 - \Pi_{\ell,k})\phi\|^2_{L^2(T)} \\
&\lesssim 2^{-2k}\|(1 - \Pi_\ell)(\phi_j + a_T)\|^2_{L^2(T)} + \|(1 - \Pi^{(1)}_\ell)\widetilde{\phi}_0\|^2_{L^2(T)} \\
&\lesssim 2^{-2k}\|(1 - \Pi_\ell)\phi\|^2_{L^2(T)} + \|(1 - \Pi^{(1)}_\ell)\widetilde{\phi}_0\|^2_{L^2(T)}
\end{aligned}$$

by use of Lemma 4.17.

(iii) If $T = [r, r+h]$ with $r \geq r_0$ or $r + h \geq h_0$, we obtain by use of mesh regularity $r\kappa(\mathcal{T}_\ell) \geq h$ that $r \geq \min\{r_0, h_0/(1 + \kappa(\mathcal{T}_\ell))\} > 0$. Therefore, $\phi_{\rm sing}|_T$ is smooth and $\phi|_T \in H^{\nu_{\rm reg}-1-\varepsilon}(T)$. We apply Lemma 4.17 with $\psi = a_T := (\Pi^{(1)}_\ell \phi)|_T$ to see

$$\begin{aligned}
&\|(1 - \Pi_{\ell,k})\phi\|^2_{L^2(T)} \\
&\lesssim 2^{-2k}\|(1 - \Pi_\ell)a_T\|^2_{L^2(T)} + \|(1 - \Pi^{(1)}_\ell)\phi\|^2_{L^2(T)} \\
&\lesssim 2^{-2k}\|(1 - \Pi_\ell)\phi\|^2_{L^2(T)} + \|(1 - \Pi^{(1)}_\ell)\phi\|^2_{L^2(T)}.
\end{aligned}$$

Finally, we define ϕ_0 elementwise by

$$\phi_0|_T := \begin{cases} \widetilde{\phi}_0|_T & \text{for cases (i),(ii)} \\ \phi|_T & \text{for case (iii)} \end{cases}$$

and obtain $\phi_0 \in H^{\nu_{\rm reg}-1-\varepsilon}(\mathcal{T}_\ell)$ for all $\varepsilon > 0$. Choosing $k \in \mathbb{N}$ sufficiently large in the estimates above, we insert in (4.64) to prove the assertion. □

With this result, we may prove the first estimate (4.8) of Theorem 4.3.

Proof of Theorem 4.3. Due to (4.36), it remains to estimate the term $\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}$. Let $\Pi_{\ell,k} : L^2(\Gamma) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(\mathcal{T}_\ell))$ denote the L^2-orthogonal projection. First, note that due to the approximation properties of $\Pi_{\ell,k}$ (cf. [**14**, Theorem 4.1]), it holds

$$\begin{aligned} \|\Pi_{\ell,k}\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} &\le \|(1-\Pi_{\ell,k})\phi\|_{H^{-1/2}(\Gamma)} + \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \\ &\le C\|h_{\ell,k}^{1/2}(1-\Pi_{\ell,k})\phi\|_{L^2(\Gamma)} + \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \\ &= C2^{-k/2}\|h_\ell^{1/2}(1-\Pi_{\ell,k})\phi\|_{L^2(\Gamma)} + \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \end{aligned} \tag{4.66}$$

for all $k \in \mathbb{N}$. Here, the constant $C > 0$ stems from the inverse estimate in [**32**, Theorem 3.6] and is independent of $\ell, k \in \mathbb{N}$. Consequently, we may estimate

$$\begin{aligned} \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} &\le \|h_\ell^{1/2}(\phi - \Pi_{\ell,k_1}\phi)\|_{L^2(\Gamma)} + \|h_\ell^{1/2}(\Pi_{\ell,k_1}\phi - \Phi_\ell)\|_{L^2(\Gamma)} \\ &\le \|h_\ell^{1/2}(\phi - \Pi_{\ell,k_1}\phi)\|_{L^2(\Gamma)} + C\|h_\ell/h_{\ell,k_1}\|_{L^\infty(\Gamma)}^{1/2}\|\Pi_{\ell,k_1}\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \\ &\le (1+C)\|h_\ell^{1/2}(\phi - \Pi_{\ell,k_1}\phi)\|_{L^2(\Gamma)} + C2^{k_1/2}\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)}, \end{aligned} \tag{4.67}$$

where $C > 0$ again stems from the inverse estimate in [**32**, Theorem 3.6]. With $h_0 > 0$ from Proposition 4.14, we choose k_1 sufficiently large such that $\|h_{\ell,k_1}\|_{L^\infty(\Gamma)} < h_0$ for all $\ell \in \mathbb{N}$. For $k_2 \in \mathbb{N}$, we get

$$\begin{aligned} &\|h_\ell^{1/2}(1-\Pi_{\ell,k_1})\phi\|_{L^2(\Gamma)} \\ &\quad\le \|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + \|h_\ell^{1/2}(1-\Pi_{\ell,k_1})\Pi_{\ell,k_1+k_2}\phi\|_{L^2(\Gamma)} \\ &\quad\le \|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + \|h_\ell^{1/2}\Pi_{\ell,k_1+k_2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} \\ &\quad\le \|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + C2^{(k_1+k_2)/2}\|\Pi_{\ell,k_1+k_2}(\phi - \Phi_\ell)\|_{H^{-1/2}(\Gamma)} \\ &\quad\le (1+C)\|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + C2^{(k_1+k_2)/2}\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)}, \end{aligned} \tag{4.68}$$

where we applied the inverse inequality from [**32**, Theorem 3.6] as well as (4.66). Given $\kappa > 0$, Proposition 4.14 now provides $k_2 \in \mathbb{N}$ such that

$$\begin{aligned} &\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} \\ &\quad\le \kappa\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1})\phi\|_{L^2(\Gamma)} + C_6\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1}^{(1)})\phi_0\|_{L^2(\Gamma)} \\ &\quad\le \kappa\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1})\phi\|_{L^2(\Gamma)} + C_6\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1}^{(1)})\phi_0\|_{L^2(\Gamma)}. \end{aligned} \tag{4.69}$$

Plugging (4.69) into (4.68) and rearranging the terms, we get

$$(1-(1+C)\kappa)\|h_\ell^{1/2}(1-\Pi_{\ell,k_1})\phi\|_{L^2(\Gamma)} \lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(1-\Pi_{\ell,k_1}^{(1)})\phi_0\|_{L^2(\Gamma)}.$$

For $\kappa > 0$ sufficiently small, combine the estimate above with (4.67) to prove the assertion. Note that $\kappa > 0$ determines $k_2 \in \mathbb{N}$ as well as h_0 determines $k_1 \in \mathbb{N}$. Therefore, the hidden constants in the estimate above are fixed uniformly. □

Definition 4.19. *Let the given boundary data satisfy* $g \in H^{s_{\mathrm{reg}}}(\Gamma)$ *for some* $s_{\mathrm{reg}} > 2$. *With* $\nu_{\mathrm{reg}} := \min\{s_{\mathrm{reg}}, 5/2\}$, *we define the higher-order term* hot_ℓ *by*

$$\mathrm{hot}_\ell := \|h_\ell^{1/2}(1-\Pi_{\ell,k}^{(1)})\phi_0\|_{L^2(\Gamma)} \quad \textit{with} \quad \mathrm{hot}_\ell(T) := \|h_\ell^{1/2}(1-\Pi_{\ell,k}^{(1)})\phi_0\|_{L^2(T)}$$

for all $T \in \mathcal{T}_\ell$. *Here,* $k = k_1 \in \mathbb{N}$ *as in the proof of Theorem 4.3 depends only on* Γ. *As stated in Proposition 4.14, the function* $\phi_0 \in H^{\nu_{\mathrm{reg}}-1-\varepsilon}(\mathcal{T}_{\ell,k})$ *for all* $\varepsilon > 0$ *depends on* $\mathcal{T}_\ell$ *and* $s_{\mathrm{reg}} > 2$, *but the piecewise norm is uniformly bounded, i.e.*

$$\sum_{T \in \mathcal{T}_{\ell,k}} \|\phi_0\|^2_{H^{\nu_{\mathrm{reg}}-1-\varepsilon}(T)} \le C_{\mathrm{hot}} < \infty, \tag{4.70}$$

where $C_{\mathrm{hot}} > 0$ *depends only on* Γ, $\kappa(\mathcal{T}_\ell)$, $s_{\mathrm{reg}} > 2$, *and* $\varepsilon > 0$. *Therefore, the Poincaré inequality for fractional Sobolev norms yields*

$$hot_\ell(T)^2 = \sum_{T_i \in \mathrm{unif}^{(k)}(T)} \|h_\ell^{1/2}(1-\Pi_{\ell,k}^{(1)})\phi_0\|^2_{L^2(T_i)} \lesssim (h_\ell|_T)^{2(\nu_{\mathrm{reg}}-1/2-\varepsilon)} C^2_{\mathrm{hot}}, \tag{4.71}$$

for all $\varepsilon > 0$. *Note that* $\nu_{\mathrm{reg}} - 1/2 - \varepsilon > 3/2$ *for* $\varepsilon > 0$ *sufficiently small. Considering the generic rate of convergence* $\mathcal{O}(h^{3/2})$ *of lowest-order BEM for Symm's integral equation and smooth solutions (cf. Theorem 3.3), we confirm that* hot_ℓ *is indeed a term of higher order.*

Proof of Theorem 4.4. We proved all the needed assumptions on the error estimator (E1), (E2), (E4) in Proposition 4.11, Proposition 4.12, and Proposition 4.9. The assumptions (R3) and (R4) are proven in Theorem 4.7 and Theorem 4.8, respectively. As mentioned in Chapter 2, the remaining assumptions are fulfilled trivially, or follow from the proven assumptions. Therefore, we may apply Theorem 2.7 to derive the equivalence

$$(\phi, f) \in \mathbb{A}_s^\eta \iff |||\phi - \Phi_\ell||| \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N}.$$

This proves the first statement of Theorem 4.4 With efficiency (E5) proved in Theorem 4.3, Equation (4.71) states

$$s_\star = \nu_{\mathrm{reg}} - 1/2 - \varepsilon \quad \text{for all } \varepsilon > 0.$$

Together with the characterization of $\mathbb{A}_s^\eta$ in Theorem 2.8, we prove the second statement of Theorem 4.4. □

CHAPTER 5

Approximation of Dirichlet Data

So far, we assumed that we are able to compute the right-hand side $(K+1/2)g$ of (4.1) exactly. However, in practical applications, this may turn out to be quite demanding. Therefore, we take a different approach and approximate the Dirichlet data g with a discrete function $G_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$. Due to the assumption $g \in H^1(\Gamma)$, Kondrachov's embedding theorem states that g is continuous on the 1D manifold Γ. Therefore, the simplest choice is $G_\ell := I_\ell g$, where I_ℓ is the nodal interpolation operator with respect to the nodes $\mathcal{N}_\ell$ of $\mathcal{T}_\ell$. According to [**12**, Theorem 1], G_ℓ satisfies the approximation property

$$\|g - G_\ell\|_{H^{1/2}(\Gamma)} \leq C_{12}\|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g - G_\ell)\|_{L^2(\Gamma)} \qquad \text{for all } \ell \in \mathbb{N} \tag{5.1}$$

with some constant $C_{12} > 0$ which depends only on $\kappa(\mathcal{T}_0)$. The well-known identity $\frac{\partial}{\partial s} I_\ell g = \Pi_\ell \frac{\partial}{\partial s} g$ in 1D gives the following elementwise best approximation result

$$\|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g - G_\ell)\|_{L^2(T)} = \inf_{V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g - V_\ell)\|_{L^2(T)} \quad \text{for all } \ell \in \mathbb{N},\, T \in \mathcal{T}_\ell. \tag{5.2}$$

For this section, we introduce the perturbed discrete formulation of (4.1) with approximated Dirichlet data, i.e.

$$\langle\!\langle \widetilde{\Phi}_\ell\, , \, \Psi_\ell \rangle\!\rangle = \langle (K + \tfrac{1}{2})G_\ell\, , \, \Psi_\ell \rangle_\Gamma \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell). \tag{5.3}$$

Next, we define the data oscillations

$$\mathrm{osc}_\ell := \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g - G_\ell)\|_{L^2(\Gamma)},$$

and the error indicators

$$\widetilde{\eta}_\ell := \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(V\widetilde{\Phi}_\ell - (K + \tfrac{1}{2})G_\ell)\|_{L^2(\Gamma)}.$$

The combination of error indicators and data oscillations gives the error estimator for the perturbed problem (5.3):

$$\widetilde{\rho}_\ell^{\,2} := \widetilde{\eta}_\ell^{\,2} + \mathrm{osc}_\ell^2.$$

As in the previous chapters, we denote the elementwise contributions by $\widetilde{\rho}_\ell(T)^2 = \widetilde{\eta}_\ell(T)^2 + \mathrm{osc}_\ell(T)^2$. To regain optimality of the adaptive algorithm, we use a slightly different approach and replace the Dörfler marking criterion with the stronger (cf. Lemma 5.4) modified Dörfler marking, which was proposed firstly in [**42**] and has been employed for adaptive FEM with inhomogeneous Dirichlet data in [**3**]. The adaptive algorithm now reads:

Algorithm 5.1. INPUT: *Initial mesh* $\mathcal{T}_0$*, adaptivity parameters* $0 < \theta_1, \theta_2, \vartheta < 1$*, counter* $\ell := 0$

(i) *Compute discrete solution* $\widetilde{\Phi}_\ell$ *of* (5.3) *corresponding to* $\mathcal{T}_\ell$ *and* $G_\ell = I_\ell g$*.*
(ii) *Compute refinement indicators* $\widetilde{\eta}_\ell(T)$ *and* $\mathrm{osc}_\ell(T)$ *for all* $T \in \mathcal{T}_\ell$*.*
(iii) *Provided that* $\mathrm{osc}_\ell^2 \le \vartheta \widetilde{\eta}_\ell^2$*, choose* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *of minimal cardinality such that*

$$\theta_1 \sum_{T\in\mathcal{T}_\ell} \widetilde{\eta}_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \widetilde{\eta}_\ell(T)^2. \tag{5.4}$$

(iv) *Provided that* $\mathrm{osc}_\ell^2 > \vartheta \widetilde{\eta}_\ell^2$*, choose* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *of minimal cardinality such that*

$$\theta_2 \sum_{T\in\mathcal{T}_\ell} \mathrm{osc}_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \mathrm{osc}_\ell(T)^2. \tag{5.5}$$

(v) *Refine (at least) marked elements* $T \in \mathcal{T}_\ell$ *to obtain new mesh* $\mathcal{T}_{\ell+1}$*.*
(vi) *Increase counter* $\ell \mapsto \ell + 1$ *and iterate.*

OUTPUT: *Discrete solutions* $\widetilde{\Phi}_\ell$ *and error estimators* $\widetilde{\rho}_\ell$ *for* $\ell \ge 0$*.*

Next, we state an inverse inequality similar to Lemma 4.10.

Lemma 5.2. *Let* $v \in H^1(\Gamma)$ *and* $\mathcal{T}_\ell \in \mathbb{T}$ *denote a mesh. Then, it holds*

$$C_{\mathrm{K}}^{-1} \| h_\ell^{1/2} \tfrac{\partial}{\partial s} K v \|_{L^2(\Gamma)} \le \|v\|_{H^{1/2}(\Gamma)} + \|h_\ell^{1/2} \tfrac{\partial}{\partial s} v\|_{L^2(\Gamma)}, \tag{5.6}$$

where $C_{\mathrm{K}} > 0$ *depends only on* $\kappa(\mathcal{T}_\ell)$ *and* Γ*.*

PROOF. Following the proof of [**5**, Theorem 1, Eq. (22)], we see that the statement (5.6) is proven as a preliminary result and no restriction on the data besides $v \in H^1(\Gamma)$ is needed. □

To recycle the analysis from the previous sections, we give some dependence between the solutions Φ_ℓ and $\widetilde{\Phi}_\ell$ of (4.2) and (5.3), respectively.

Proposition 5.3. *Let* $G_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ *satisfy* (5.1)*. Then, it holds*

$$C_{13}^{-1}\big(|||\phi - \Phi_\ell||| + \mathrm{osc}_\ell\big) \le |||\phi - \widetilde{\Phi}_\ell||| + \mathrm{osc}_\ell \le C_{13}\big(|||\phi - \Phi_\ell||| + \mathrm{osc}_\ell\big) \tag{5.7}$$

as well as

$$C_{13}^{-1}\Big(\sum_{T\in\mathcal{E}_\ell} \eta_\ell(T)^2 + \mathrm{osc}_\ell^2\Big) \le \sum_{T\in\mathcal{E}_\ell} \widetilde{\eta}_\ell(T)^2 + \mathrm{osc}_\ell^2 \le C_{13}\Big(\sum_{T\in\mathcal{E}_\ell} \eta_\ell(T)^2 + \mathrm{osc}_\ell^2\Big) \tag{5.8}$$

for all $\ell \in \mathbb{N}$ *and for each subset* $\mathcal{E}_\ell \subseteq \mathcal{T}_\ell$*. The constant* $C_{13} \ge 1$ *depends only on* $\kappa(\mathcal{T}_\ell)$ *and* Γ*.*

PROOF. We start with the first statement. The triangle inequality shows

$$|||\phi - \Phi_\ell||| \le |||\phi - \widetilde{\Phi}_\ell||| + |||\Phi_\ell - \widetilde{\Phi}_\ell|||.$$

The second term is treated by use of the stability of equation (5.3), i.e.

$$|||\Phi_\ell - \widetilde{\Phi}_\ell|||^2 = \langle (K + \tfrac{1}{2})(g - G_\ell)\,,\, \Phi_\ell - \widetilde{\Phi}_\ell \rangle_\Gamma \le \|g - G_\ell\|_{H^{1/2}(\Gamma)} \|\Phi_\ell - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)},$$

where we use the continuity $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$. With norm equivalence (3.12), we may cancel $|||\Phi_\ell - \widetilde{\Phi}_\ell|||$ on both sides and use the approximation property (5.1) to obtain

$$|||\phi - \Phi_\ell||| \lesssim |||\phi - \widetilde{\Phi}_\ell||| + \operatorname{osc}_\ell.$$

The converse estimate follows analogously.

To prove the second estimate, we apply the inverse inequalities from Lemma 4.10 and Lemma 5.2. This gives

$$\begin{aligned}\Big(\sum_{T\in\mathcal{M}} \eta_\ell(T)^2\Big)^{1/2} &\le \Big(\sum_{T\in\mathcal{M}} \widetilde{\eta}_\ell(T)^2\Big)^{1/2} + \|h_\ell^{1/2}\tfrac{\partial}{\partial s}V(\Phi_\ell - \widetilde{\Phi}_\ell)\|_{L^2(\bigcup\mathcal{M})}\\ &\qquad + \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(K+\tfrac12)(g-G_\ell)\|_{L^2(\bigcup\mathcal{M})}\\ &\lesssim \Big(\sum_{T\in\mathcal{M}} \widetilde{\eta}_\ell(T)^2\Big)^{1/2} + |||\Phi_\ell - \widetilde{\Phi}_\ell||| + \operatorname{osc}_\ell.\end{aligned}$$

Again, we apply (5.1) and the inverse inequality from [**32**, Theorem 3.6]. Stability of (5.3) proves the first inequality of (5.8). The converse inequality follows the same lines. □

The result above shows, that $\widetilde{\eta}_\ell$ is even locally equivalent to η_ℓ up to oscillation terms.

Lemma 5.4. *For arbitrary $\theta_1, \theta_2 \in (0,1)$ and sufficiently small $\vartheta \in (0,1)$, there is some $0 < \theta < 1$ such that the marking criterion (5.4)–(5.5) for $\widetilde{\rho}_\ell^2 = \widetilde{\eta}_\ell^2 + \operatorname{osc}_\ell^2$ implies the Dörfler marking*

$$\theta \sum_{T\in\mathcal{T}_\ell} \rho_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \rho_\ell(T)^2 \tag{5.9}$$

for $\rho_\ell^2 := \eta_\ell^2 + \operatorname{osc}_\ell^2$. The parameter $0 < \theta < 1$ depends on $0 < \theta_1, \theta_2, \vartheta < 1$ as well as on $C_{13} > 0$.

Proof. First, assume $\operatorname{osc}_\ell^2 \le \vartheta\widetilde{\eta}_\ell^2$. By use of Proposition 5.3, we see

$$\rho_\ell^2 \le (1 + C_{13}^2)(\widetilde{\eta}_\ell^2 + \operatorname{osc}_\ell^2) \le (1 + C_{13}^2)(1+\vartheta)\widetilde{\eta}_\ell^2. \tag{5.10}$$

Again, with Proposition 5.3 and (5.4), we conclude

$$\theta_1\widetilde{\eta}_\ell^2 \le \sum_{T\in\mathcal{M}_\ell} \widetilde{\eta}_\ell(T)^2 \le C_{13}^2\Big(\sum_{T\in\mathcal{M}_\ell}\eta_\ell(T)^2 + \operatorname{osc}_\ell^2\Big) \le C_{13}^2 \sum_{T\in\mathcal{M}_\ell}\rho_\ell(T)^2 + C_{13}^2\vartheta\widetilde{\eta}_\ell^2$$

and therefore

$$(\theta_1 - C_{13}^2\vartheta)\widetilde{\eta}_\ell^2 \le C_{13}^2 \sum_{T\in\mathcal{M}_\ell}\rho_\ell(T)^2.$$

The combination of (5.10) and the estimate above now yield

$$\theta_1 C_{13}^{-2}(1+C_{13}^2)^{-1}(1+\vartheta)^{-1}(\theta_1 - C_{13}^2\vartheta)\rho_\ell^2 \le \sum_{T\in\mathcal{M}_\ell}\rho_\ell(T)^2.$$

With $\vartheta > 0$ sufficiently small, we obtain $t_1 := \theta_1 C_{13}^{-2}(1+C_{13}^2)^{-1}(1+\vartheta)^{-1}(\theta_1 - C_{13}^2\vartheta) \in (0,1)$.

Second, assume $\mathrm{osc}_\ell^2 > \vartheta \widetilde{\eta}_\ell^2$. Now, we exploit (5.5) to see

$$\begin{aligned}\theta_2\rho_\ell^2 &\le \theta_2(1+C_{13}^2)(\widetilde{\eta}_\ell^2+\mathrm{osc}_\ell^2) \le \theta_2(1+C_{13}^2)(1+\vartheta^{-1})\mathrm{osc}_\ell^2\\ &\le (1+C_{13}^2)(1+\vartheta^{-1})\sum_{T\in\mathcal{M}_\ell}\mathrm{osc}_\ell(T)^2 \le (1+C_{13}^2)(1+\vartheta^{-1})\sum_{T\in\mathcal{M}_\ell}\rho_\ell(T)^2,\end{aligned}$$

where $t_2 := \theta_2(1+C_{13}^2)^{-1}(1+\vartheta^{-1})^{-1} \in (0,1)$ for $\vartheta > 0$ arbitrarily. Now, we define $\theta := \min\{t_1, t_2\}$ to prove (5.9). □

Lemma 5.5. *The extended error estimator $\rho_\ell^2 := \eta_\ell^2 + \mathrm{osc}_\ell^2$ satisfies the assumptions* (E1)–(E4).

Proof. First, note that (E1)–(E4) hold for η_ℓ (see Proposition 4.9, Proposition 4.11, Proposition 4.12). Therefore, the assumptions (E3) and (E4) follow obviously from $\eta_\ell(T)^2 \le \rho_\ell(T)^2$ for all $T \in \mathcal{T}_\ell$. To prove (E1) and (E2), it remains to show the assumptions for the oscillation term. Let therefore $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\ell) \in \mathbb{T}$. Then, it holds

$$\begin{aligned}&\Big|\Big(\sum_{T\in\mathcal{T}_\star\cap\mathcal{T}_\ell}\mathrm{osc}_\star^2(T)\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\star\cap\mathcal{T}_\ell}\mathrm{osc}_\ell^2(T)\Big)^{1/2}\Big|\\ &\qquad = \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g-G_\star)\|_{L^2(\bigcup\mathcal{T}_\star\cap\mathcal{T}_\ell)} - \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g-G_\ell)\|_{L^2(\bigcup\mathcal{T}_\star\cap\mathcal{T}_\ell)}\\ &\qquad \le \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(G_\star-G_\ell)\|_{L^2(\bigcup\mathcal{T}_\star\cap\mathcal{T}_\ell)} = 0,\end{aligned}$$

because $G_\star = I_\star g = I_\ell g = G_\ell$ on $\mathcal{T}_\star \cap \mathcal{T}_\ell$. This proves (E1). To see (E2), we exploit $h_\star|_T \le h_\ell|_T/2$ on refined elements $T \in \mathcal{T}_\ell \setminus \mathcal{T}_\star$. This gives

$$\sum_{T\in\mathcal{T}_\star\setminus\mathcal{T}_\ell}\mathrm{osc}_\star^2(T) \le \frac{1}{2}\|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g-G_\star)\|_{L^2(\mathcal{T}_\star\setminus\mathcal{T}_\ell)}^2 \le \frac{1}{2}\|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g-G_\ell)\|_{L^2(\mathcal{T}_\star\setminus\mathcal{T}_\ell)}^2,$$

where we used the $\mathcal{T}_\ell$-elementwise best approximation property (5.2) of the nodal interpoland $G_\star = I_\star g$. This concludes the proof. □

Let $\mathbb{A}_s^\rho$ denote the approximation class according to (5.3)

$$(\phi, f, g) \in \mathbb{A}_s^\rho \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \|(\phi,f,g)\|_{\mathbb{A}_s^\rho} := \sup_{N\in\mathbb{N}}\inf_{\mathcal{T}_\star\in\mathbb{T}_N} N^s\rho_\ell < \infty.$$

Theorem 5.6. *Let $\mathcal{T}_\ell$ denote the sequence of meshes generated by Algorithm 5.1. Then, for arbitrary $\theta_1, \theta_2, \vartheta \in (0,1)$, it holds convergence*

$$|||\phi - \widetilde{\Phi}_\ell||| \to 0, \qquad \text{as } \ell \to \infty. \tag{5.11}$$

Furthermore, for sufficiently small $0 < \theta_1, \vartheta < 1$ but arbitrary $0 < \theta_2 < 1$, Algorithm 5.1 guarantees the existence of a constant $C_{14} > 0$ such that

$$(\phi, f, g) \in \mathbb{A}_s^\rho \quad \Longleftrightarrow \quad \widetilde{\rho}_\ell \le C_{14}\,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \ge 0 \tag{5.12}$$

for all $s > 0$. The constant $C_{14} > 0$ depends only on $\|(\phi,f,g)\|_{\mathbb{A}_s^{\widetilde{\rho}}}$, the shape regularity $\kappa(\mathcal{T}_\ell)$, and Γ.

Proof. Lemma 5.5 shows that the extended error estimator ρ_ℓ satisfies (E1)–(E3). Therefore, Theorem 2.4 states for $\Delta_\ell := |||\phi - \Phi_\ell|||^2 + \gamma\rho_\ell^2$

$$\Delta_{\ell+1} \leq \kappa\Delta_\ell$$

for all $\ell \in \mathbb{N}$ and for some $\gamma, \kappa \in (0,1)$. Particularly, we obtain $\lim_{\ell\to\infty} \rho_\ell = 0$. Proposition 5.3 now yields

$$|||\phi - \widetilde{\Phi}_\ell|||^2 \simeq |||\phi - \Phi_\ell|||^2 + \mathrm{osc}_\ell^2 \lesssim \rho_\ell^2 \to 0, \quad \text{as } \ell \to \infty,$$

where we used reliability (E3) of ρ_ℓ (Lemma 5.5). This proves the first statement (5.11). To see (5.12), we apply Lemma 2.6 with $0 < \kappa_\star < 1$ arbitrary. For $(\phi, f, g) \in \mathbb{A}_s^\rho$, Lemma 2.6 guarantees the existence of a mesh $\mathcal{T}_\star \in \mathbb{T}$ such that

$$\#\mathcal{T}_\star - \mathcal{T}_\ell \lesssim \rho_\ell^{-1/s} \quad \text{and} \quad \rho_\star^2 \leq \kappa_\star\rho_\ell^2,$$

as well as Dörfler marking

$$\theta_\star\rho_\ell^2 \leq \sum_{T\in\mathcal{R}_\ell} \rho_\ell(T)^2,$$

where the set $\mathcal{R}_\ell \supset \mathcal{T}_\ell \setminus \mathcal{T}_\star$ satisfies $\#\mathcal{R}_\ell \lesssim \rho_\ell^{-1/s}$. Now, we want to show that this implies the modified Dörfler marking (5.4)–(5.5) for $\widetilde{\rho}$:

- In case of $\mathrm{osc}_\ell^2 \leq \vartheta\widetilde{\eta}_\ell^2$, we obtain

$$\theta_\star\widetilde{\eta}_\ell^2 \lesssim \theta_\star\rho_\ell^2 \leq \sum_{T\in\mathcal{R}_\ell} \rho_\ell(T)^2 \lesssim \sum_{T\in\mathcal{R}_\ell} \widetilde{\eta}_\ell(T)^2 + \vartheta\widetilde{\eta}_\ell^2.$$

  Put differently, we have

$$(\theta_\star C_{13}^{-2} - \vartheta)\widetilde{\eta}_\ell^2 \leq \sum_{T\in\mathcal{R}_\ell} \widetilde{\eta}_\ell(T)^2. \tag{5.13}$$

  For $\theta_1, \vartheta > 0$ sufficiently small, $\widetilde{\eta}_\ell$ satisfies the marking criterion (5.4).
- In case of $\mathrm{osc}_\ell^2 > \vartheta\widetilde{\eta}_\ell^2$, we use the local definition of osc_ℓ, i.e.

$$\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \mathrm{osc}_\ell(T)^2 = \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \mathrm{osc}_\star(T)^2 \leq \mathrm{osc}_\star^2 \leq \rho_\star^2 \lesssim \kappa_\star(1+\vartheta^{-1})\mathrm{osc}_\ell^2.$$

  This estimate yields

$$\big(1 - C_{13}\kappa_\star(1+\vartheta^{-1})\big)\mathrm{osc}_\ell^2 \leq \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_\star} \mathrm{osc}_\ell(T)^2 \leq \sum_{T\in\mathcal{R}_\ell} \mathrm{osc}_\ell(T)^2.$$

  For $\kappa_\star > 0$ small enough, osc_ℓ satisfies the marking criterion (5.5) with $\mathcal{R}_\ell \supset \mathcal{T}_\ell \setminus \mathcal{T}_\star$. Note carefully, that $\theta_\star$ and therefore ϑ (Equation (5.13)) depend on the the choice of $\kappa_\star$, but as one can check in Equation (2.23), $\theta_\star$ becomes larger for smaller $\kappa_\star$. Thus, we may fix $\vartheta \in (0,1)$ sufficiently small and choose $\kappa_\star \in (0,1)$ afterwards.

The conclusions above and the minimal choice of $\mathcal{M}_\ell$ in (5.4) and (5.5) show

$$\#\mathcal{M}_\ell \le \#\mathcal{R}_\ell \lesssim \rho_\ell^{-1/s} \simeq \Delta_\ell^{-1/(2s)} \quad \text{for all } \ell \in \mathbb{N}.$$

With Proposition 5.3, we obtain equivalence

$$\widetilde{\rho}_\ell \simeq \rho_\ell \simeq \Delta_\ell.$$

With this, we prove analogously to the proof of Theorem 2.7

$$\widetilde{\rho}_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-1/s} \quad \text{for all } \ell \in \mathbb{N}.$$

The converse implication in (5.12) is obvious. This concludes the proof. □

Corollary 5.7. *Let the given boundary data satisfy $g \in H^{s_{\rm reg}}(\Gamma)$ for some $s_{\rm reg} > 2$. Let ϕ denote the solution of (4.1). Then, the error estimator $\widetilde{\rho}_\ell$ is efficient in the following sense*

$$C_{15}^{-1}\widetilde{\rho}_\ell \le \|\phi - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{osc}_\ell + \mathrm{hot}_\ell. \tag{5.14}$$

Here, $C_{15} > 0$ depends only on Γ and $\kappa(\mathcal{T}_0)$. The higher-order term hot_ℓ is given in detail in Definition 4.19. For all $\varepsilon > 0$, it satisfies

$$\mathrm{hot}_\ell = \Big(\sum_{T\in\mathcal{T}_\ell} \mathrm{hot}_\ell(T)^2\Big)^{1/2} \quad \text{and} \quad \mathrm{hot}_\ell(T) \le C_{\rm hot}(h_\ell|_T)^{\min\{s_{\rm reg},5/2\}-1/2-\varepsilon},$$

where $C_{\rm hot} > 0$ depends only on Γ, $\kappa(\mathcal{T}_\ell)$, and $s_{\rm reg} > 2$.

Proof. We apply Proposition 5.3 as well as Theorem 4.3 and conclude

$$\widetilde{\rho}_\ell \lesssim \eta_\ell + \mathrm{osc}_\ell \lesssim \|\phi - \widetilde{\Phi}_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{osc}_\ell + \mathrm{hot}_\ell$$

for all $\ell \in \mathbb{N}$. □

Now, we are able to characterize the approximation class $\mathbb{A}_s^\rho$ and refine the optimality result in Theorem 5.6. Therefore, we define

$$(\phi, g) \in \widetilde{\mathbb{A}}_s \quad \overset{\text{def}}{\Longleftrightarrow} \quad \|(\phi,g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{osc}} < \infty \quad \text{and} \quad \|(\phi,g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{err}} < \infty,$$

with

$$\|(\phi,g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{err}} := \sup_{N\in\mathbb{N}} \inf_{\mathcal{T}_\star\in\mathbb{T}_N} \inf_{\Psi_\star\in\mathcal{P}^0(\mathcal{T}_\star)} N^s |\!|\!|\phi - \psi_\star|\!|\!|$$

$$\|(\phi,g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{osc}} := \sup_{N\in\mathbb{N}} \inf_{\mathcal{T}_\star\in\mathbb{T}_N} \inf_{W_\star\in\mathcal{S}^1(\mathcal{T}_\star)} N^s \|h_\star^{1/2}\tfrac{\partial}{\partial s}(g - W_\star)\|_{L^2(\Gamma)}.$$

Note that additionally to the Galerkin error, only the oscillation term

$$\mathrm{osc}_\ell := \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g - G_\ell)\|_{L^2(\Gamma)} = \inf_{W_\ell\in\mathcal{S}^1(\mathcal{T}_\ell)} \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g - W_\ell)\|_{L^2(\Gamma)}$$

appears in the definition of the approximation class and these two contributions are decoupled, i.e. it is sufficient that there exist optimal sequences of meshes for the Galerkin error and for the oscillation term separately.

Corollary 5.8. *Let $(\widetilde{\Phi}_\ell)_{\ell\in\mathbb{N}}$ denote the sequence of solutions generated by Algorithm 5.1 driven by the weighted residual-based error estimator $\widetilde{\rho}_\ell$. Assume that the corresponding sequence of meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ is created by local refinement as stated in Algorithm 4.1 or Algorithm 4.5. Let the given boundary data additionally satisfy $g \in H^{s_{\rm reg}}(\Gamma)$ for some $s_{\rm reg} > 2$. Then, for $0 < s < \min\{s_{\rm reg}, 5/2\} - 1/2$ and sufficiently small parameters $\theta_1, \vartheta \in (0,1)$ but arbitray $\theta_2 \in (0,1)$, Algorithm 5.1 is optimal in the following sense*

$$(\phi, g) \in \widetilde{\mathbb{A}}_s \quad \Longleftrightarrow \quad (|||\phi - \widetilde{\Phi}_\ell||| + \mathrm{osc}_\ell) \le C_{16}(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \text{for all } \ell \in \mathbb{N},$$

where the constant $C_{16} > 0$ depends only on $\|(\phi, g)\|_{\widetilde{\mathbb{A}}_s}$ and Γ.

Proof. For this proof, we define

$$\|(\phi, g)\|_{\widetilde{\mathbb{A}}_s} := \sup_{N\in\mathbb{N}} \inf_{\mathcal{T}_\star\in\mathbb{T}_N} \inf_{\Psi_\star\in\mathcal{P}^0(\mathcal{T}_\star)} N^s(|||\phi - \psi_\star||| + \mathrm{osc}_\star) < \infty.$$

First, we show the following equivalence

$$\|(\phi, g)\|_{\widetilde{\mathbb{A}}_s} < \infty \quad \Longleftrightarrow \quad \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{osc}} < \infty \quad \text{and} \quad \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{err}} < \infty, \tag{5.15}$$

for all $s > 0$. The first implication is trivial due to

$$\max\left\{\|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{osc}}, \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{err}}\right\} \le \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s} < \infty.$$

To see the converse implication, fix $N \in \mathbb{N}$. The definition of $\widetilde{\mathbb{A}}_s$ guarantees meshes $\mathcal{T}_{\star,\mathrm{err}}, \mathcal{T}_{\star,\mathrm{osc}} \in \mathbb{T}_{N/2}$ with

$$\inf_{\Psi_{\star,\mathrm{err}}\in\mathcal{P}^0(\mathcal{T}_{\star,\mathrm{err}})} (N/2)^s |||\phi - \psi_{\star,\mathrm{err}}||| \le \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{err}} < \infty,$$
$$\inf_{W_{\star,\mathrm{osc}}\in\mathcal{S}^1(\mathcal{T}_{\star,\mathrm{osc}})} (N/2)^s \|h_{\star,\mathrm{osc}}^{1/2} \tfrac{\partial}{\partial s}(g - W_{\star,\mathrm{osc}})\|_{L^2(\Gamma)} \le \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{osc}} < \infty.$$

Now, consider the overlay $\mathcal{T}_N := \mathcal{T}_{\star,\mathrm{err}} \oplus \mathcal{T}_{\star,\mathrm{osc}}$. With (R4), we obtain

$$\#\mathcal{T}_N - \#\mathcal{T}_0 \le \#\mathcal{T}_{\star,\mathrm{err}} - \#\mathcal{T}_0 + \#\mathcal{T}_{\star,\mathrm{osc}} - \#\mathcal{T}_0 \le N.$$

Therefore, we have $\mathcal{T}_N \in \mathbb{T}_N$ and conclude with the best approximation property of $G_\ell = I_\ell g$ (5.2) and Céa's lemma

$$\begin{aligned}\inf_{\Psi_N\in\mathcal{P}^0(\mathcal{T}_N)} N^s(|||\phi - \psi_N||| + \mathrm{osc}_N) &\le \inf_{\Psi_{\star,\mathrm{err}}\in\mathcal{P}^0(\mathcal{T}_{\star,\mathrm{err}})} (N/2)^s |||\phi - \psi_{\star,\mathrm{err}}||| \\ &\quad + \inf_{W_{\star,\mathrm{osc}}\in\mathcal{S}^1(\mathcal{T}_{\star,\mathrm{osc}})} (N/2)^s \|h_{\star,\mathrm{osc}}^{1/2} \tfrac{\partial}{\partial s}(g - W_{\star,\mathrm{osc}})\|_{L^2(\Gamma)} \\ &\le \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{err}} + \|(\phi, g)\|_{\widetilde{\mathbb{A}}_s,\mathrm{osc}}.\end{aligned}$$

This proves (5.15). With efficiency of $\widetilde{\rho}_\ell \simeq \rho_\ell$ (Corollary 5.7) and the characterization of $\widetilde{\mathbb{A}}_s$ in (5.15) , the remainder of the proof follows analogously to the proof of Theorem 4.4. □

CHAPTER 6

Numerical Examples

6.1. Preliminaries

The aim of this chapter is on the one hand to underline the results from the previous chapters and on the other hand to provide a fair comparison between the adaptive approach from Algorithm 5.1 and the more obvious uniform approach from Algorithm 6.1 to approximate solutions. To achieve this, we consider the model problem (4.1) with different data g, and approximate the solution with Algorithm 5.1. To visualize the results, we plot the following quantities with respect to the number of elements:

- Instead of the energy norm error $|||\phi - \widetilde{\Phi}_\ell|||$ which can hardly be computed analytically, we plot the following reliable error bound:

$$\mathrm{err}_\ell := \|h_\ell^{1/2}(\phi - \widetilde{\Phi}_\ell)\|_{L^2(\Gamma)}.$$

Due to Céa's Lemma (2.5) and Proposition 5.3, we may prove

$$\begin{aligned} |||\phi - \widetilde{\Phi}_\ell||| &\le |||\phi - \Phi_\ell||| + \mathrm{osc}_\ell \le |||\phi - \Pi_\ell\phi||| + \mathrm{osc}_\ell \\ &\lesssim \|h_\ell^{1/2}(\phi - \Pi_\ell\phi)\|_{L^2(\Gamma)} + \mathrm{osc}_\ell \le \mathrm{err}_\ell + \mathrm{osc}_\ell, \end{aligned}$$

where the hidden constant depends only on the shape regularity $\kappa(\mathcal{T}_\ell)$. The integral is computed via Gauss-Legendre quadrature. Note that under the regularity assumptions of Theorem 4.3 resp. Corollary 5.7, we obtain that err_ℓ is up to terms of higher order and oscillation terms even a lower bound for the energy norm error, i.e.

$$\mathrm{err}_\ell \lesssim |||\phi - \widetilde{\Phi}_\ell||| + \mathrm{osc}_\ell + \mathrm{hot}_\ell$$

for all $\ell \in \mathbb{N}$.

- We plot the error indicator $\widetilde{\eta}_\ell = \|h_\ell^{1/2}\frac{\partial}{\partial s}(V\widetilde{\Phi}_\ell - (K + \frac{1}{2})G_\ell))\|_{L^2(\Gamma)}$. The functions $(V\widetilde{\Phi}_\ell)(x)$ and $(KG_\ell)(x)$ are computed analytically, which is possible, since $\widetilde{\Phi}_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ and $G_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ are discrete functions.

- An important quantity in our analysis is the term of higher order hot_ℓ from Definition 4.19. Even if we prescribe the solution ϕ, we do not know ϕ_0 in general. Therefore, we aim to visualize the behavior of hot_ℓ as follows: First,

$$\mathrm{hot}_{1,\ell} := \|h_\ell^{1/2}(1 - \Pi_\ell^{(1)})\phi\|_{L^2(\Gamma)}$$

is intuitively an upper bound of hot_ℓ because ϕ is less regular than ϕ_0. Second,

$$\mathrm{hot}_{2,\ell} := \|h_\ell^{1/2}(1-\Pi_\ell^{(1)})\phi\|_{L^2(\Gamma_{\mathrm{reg}})},$$

where $\Gamma_{\mathrm{reg}} = \Gamma \setminus \bigcup_{j=1}^m B_\delta(c_j)$. Here, $\delta > 0$ is small compared to the size of the domain (for the depicted domain sizes in Figure 1, we chosse $\delta = 0.01$) and c_j, $j = 1, \dots, m$ denote the corners of the boundary, i.e. the generic singularities of ϕ. From the expansion (4.62), we know that $\phi|_{\Gamma_{\mathrm{reg}}}$ has the same regularity as $\phi_0|_{\Gamma_{\mathrm{reg}}}$. Therefore, $\mathrm{hot}_{2,\ell}$ should give a good representation of hot_ℓ.

- The data oscillation term $\mathrm{osc}_\ell := \|h_\ell^{1/2}\frac{\partial}{\partial s}(g - G_\ell)\|_{L^2(\Gamma)}$ must be considered because we only compute the right-hand side of (4.1) for discrete functions.

Next, we give a brief illustration of the uniform strategy.

Algorithm 6.1 (Uniform mesh-refinement)**.** INPUT: *Initial mesh $\mathcal{T}_0$, counter $\ell := 0$*

(i) *Compute discrete solution $\widetilde{\Phi}^{(\ell)}$ of* (5.3) *corresponding to $\mathcal{T}^{(\ell)}$ and $G^{(\ell)} = I^{(\ell)}g$.*
(ii) *Refine all elements $T \in \mathcal{T}_\ell$ to obtain new mesh $\mathcal{T}^{(\ell+1)} = \texttt{refine}(\mathcal{T}^{(\ell)}, \mathcal{T}^{(\ell)})$.*
(iii) *Increase counter $\ell \mapsto \ell + 1$ and iterate.*

OUTPUT: *Discrete solutions $\widetilde{\Phi}^{(\ell)}$ for all $\ell \in \mathbb{N}$.*

To compare the adaptive approach presented in Algorithm 5.1 and the uniform approach above, we want to consider the computational times:

- The time t_{unif} to compute the solution $\widetilde{\Phi}^{(\ell)}$ of the uniform approach in Algorithm 6.1 is the time needed to perform ℓ uniform refinements of the initial mesh $\mathcal{T}_0$, plus the time needed to build and solve the linear system corresponding to $\mathcal{T}^{(\ell)}$. Obviously, the second contribution is vastly dominant.
- The time t_{adap} to compute the solution $\widetilde{\Phi}_\ell$ of the adaptive approach in Algorithm 5.1 is the time to build an solve the system corresponding to the mesh $\mathcal{T}_\ell$ plus the time needed to compute all the previous solutions, to compute the error estimators, to discretize the data g, and to mark and refine the meshes.

Although this definition seems to favor the uniform approach, we think that it provides a fair comparison between those strategies. All the following experiments where conducted by use of the MATLAB-BEM library `HILBERT`. See the web page

`http://www.asc.tuwien.ac.at/abem/hilbert/`

for detailed information. Throughout, all the occurring linear systems were solved directly with the MATLAB backslash operator.

6.2. Experiments

Unless stated otherwise, the adaptivity parameters in Algorithm 5.1 are set as

$$\theta_1 = \theta_2 = \vartheta = 1/2.$$

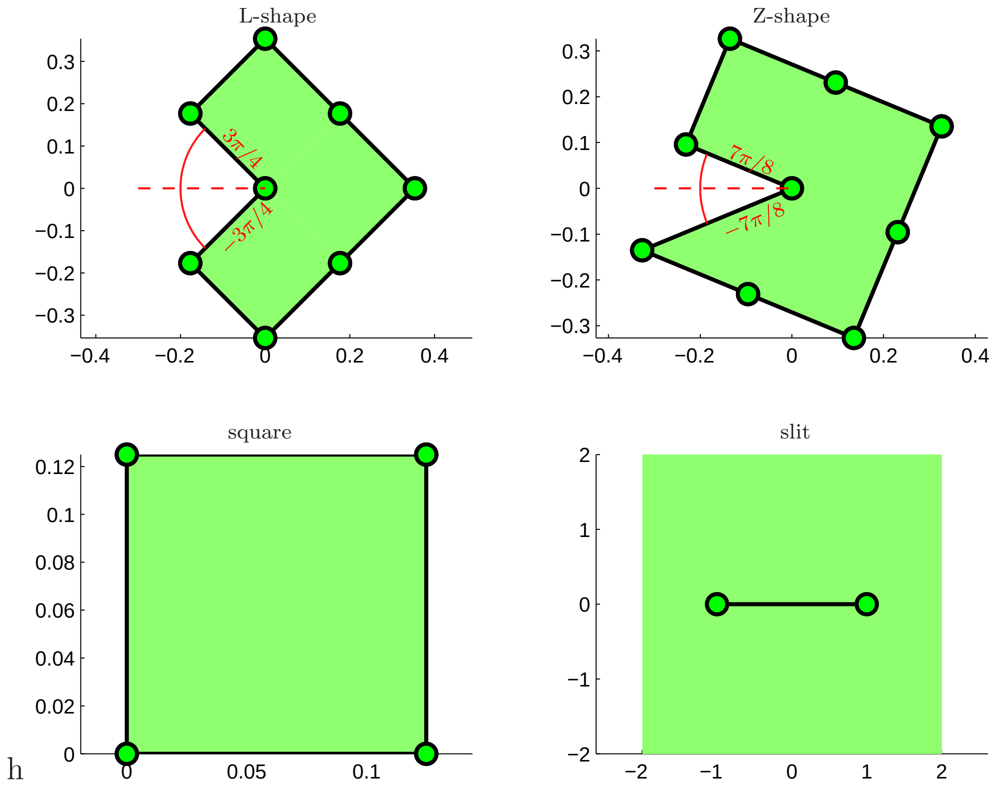

FIGURE 1. Different domains Ω with initial partitions of the boundary $\mathcal{T}_0$.

6.2.1. Experiment on L-shape with singular solution. Here, Γ is the boundary of the L-shaped domain Ω in Figure 1. We prescribe the solution u of

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{6.1}$$

as $u(x,y) := r^{2/3}\cos(2\alpha/3)$ with polar coordinates (r,α) with respect to $(0,0) \in \mathbb{R}^2$. It is easy to check that $u|_\Gamma = g$ is smooth and therefore meets the regularity assumptions of Theorem 4.3 resp. Corollary 5.7. We compute the data and solution thereof. Figure 2 shows that the error and the error estimator converge with optimal order $\mathcal{O}(N^{-3/2})$ on adaptively generated meshes. The terms $\mathrm{hot}_{1,\ell}$ and $\mathrm{hot}_{2,\ell}$ converge with even higher order, which underlines that the error estimator is efficient. Recall that $u \in H^{1+2/3-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ has a generic singularity in the reentrant corner. Therefore, uniform refinement leads to a suboptimal rate of convergence $\mathcal{O}(N^{-2/3})$. Figure 3 compares the two approaches in terms of computational time. We see that despite the computational overhead which comes with adaptive refinement, this strategy is superior to uniform refinement after only a few iterations.

In Figure 4, we vary the adaptivity parameters $\theta_1, \theta_2, \vartheta$ between zero and one. We observe that each choice yields the optimal rate of convergence $\mathcal{O}(N^{3/2})$. Although all

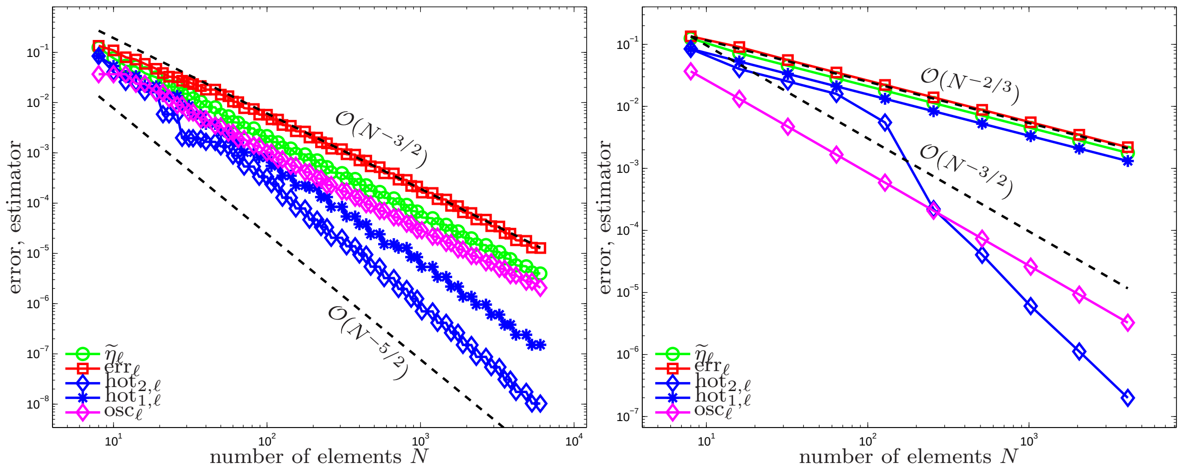

FIGURE 2. Experiment on L-shape with singular solution. The quantities err_ℓ, η_ℓ, $\mathrm{hot}_{1,\ell}$, $\mathrm{hot}_{2,\ell}$, and osc_ℓ are plotted versus the number of elements $N = \#\mathcal{T}_\ell$ for adaptive mesh-refinement (left) and uniform mesh-refinement (right).

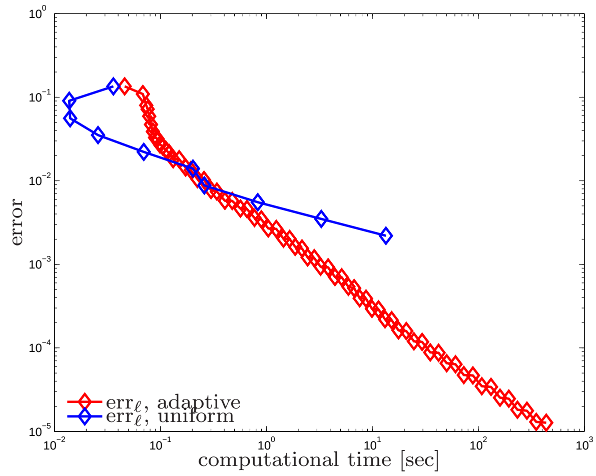

FIGURE 3. Experiment on L-shape with singular solution. Error of uniform and adaptive mesh-refinement is plotted over the computational time measured in seconds.

choices of the adaptivity parameters behave asymptotically similar, we see slight advantages for parameter choices $\theta_1 = \theta_2 = \vartheta \geq 1/2$.

6.2.2. Experiment on Z-shape with singular solution. Next, Γ is the boundary of the Z-shaped domain Ω in Figure 1. We prescribe the solution u of (6.1) as $u(r, \alpha) := r^{4/7} \cos(4\alpha/7)$. As in previous case, the boundary data g is smooth and fulfills the regularity assumptions of Theorem 4.3 resp. Corollary 5.7. Figure 5 and Figure 6 show

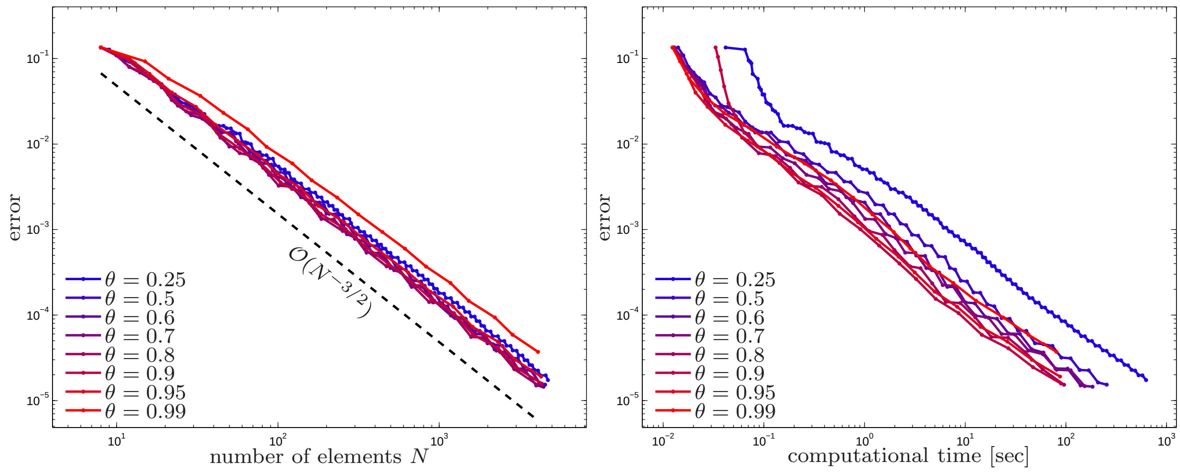

FIGURE 4. Experiment on L-shape with singular solution. Error of adaptive mesh-refinement is plotted over the number of elements $N = \#\mathcal{T}_\ell$ (left) and over the computational time measured in seconds (right). The adaptivity parameters are chosen equally as $\theta_1 = \theta_2 = \vartheta := \theta$, where θ varies in $(0, 1)$.

the results of the experiment. Now, the solution has an even stronger singularity than in the examples on the L-shape ($u \in H^{1+4/7}(\Omega)$). Therefore, the uniform mesh-refinement converges only with suboptimal rate $\mathcal{O}(N^{-4/7})$, whereas adaptive mesh-refinement converges with optimal rate $\mathcal{O}(N^{-3/2})$. Superiority of adaptive mesh-refinement can already be observed for meshes with approximately 200 elements.

6.2.3. Experiment on square with smooth solution. Here, Γ is the boundary of the square Ω in Figure 1. We prescribe the smooth solution u of (6.1) as $u(x, y) := \sinh(2\pi x)\cos(2\pi y)$. Figure 7 and Figure 8 show the results of the experiment. Note that for a smooth solution, uniform mesh-refinement is asymptotically the best strategy to approximate the solution. This can be easily confirmed with results from a priori analysis. Nevertheless, Figure 8 shows that adaptive mesh-refinement does not need significantly more computational time to reach the same accuracy.

6.2.4. Experiment on L-shape with singular solution and singular data. Again Γ is the boundary of the L-shaped domain Ω in Figure 1. We prescribe the solution u of (6.1) as $u(x, y) := v_{2/3}(x, y) + v_{7/8}(x - z_1, y - z_2)$, where $v_\delta(x, y) := r^\delta \cos(\delta\alpha)$ and $z = (z_1, z_2)$ is the uppermost corner of the L-shape in Figure 1. The solution ϕ has a generic singularity in the reentrant corner and in addition a singularity resulting from the singular data g. Note that $v_\delta \in H^{1+\delta-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Therefore, $g \in H^{1/2+7/8-\varepsilon}(\Gamma) \not\subseteq H^{2+\varepsilon}(\Gamma)$ for all $\varepsilon > 0$. Hence, g does not meet the regularity assumptions of Theorem 4.3 resp. Corollary 5.7. Nevertheless, Figure 9 and Figure 10 show that the error bound err_ℓ and the error estimator behave perfectly in case of adaptive refinement. Even $\mathrm{hot}_{1,\ell}$ and

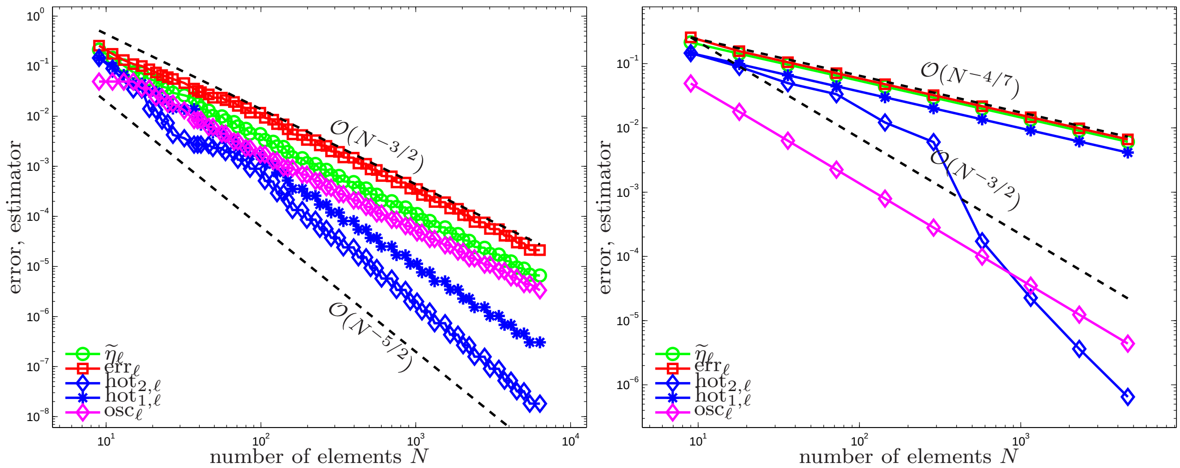

FIGURE 5. Experiment on Z-shape with singular solution. The quantities err_ℓ, η_ℓ, $\mathrm{hot}_{1,\ell}$, $\mathrm{hot}_{2,\ell}$, and osc_ℓ are plotted versus the number of elements $N = \#\mathcal{T}_\ell$ for adaptive mesh-refinement (left) and uniform mesh-refinement (right).

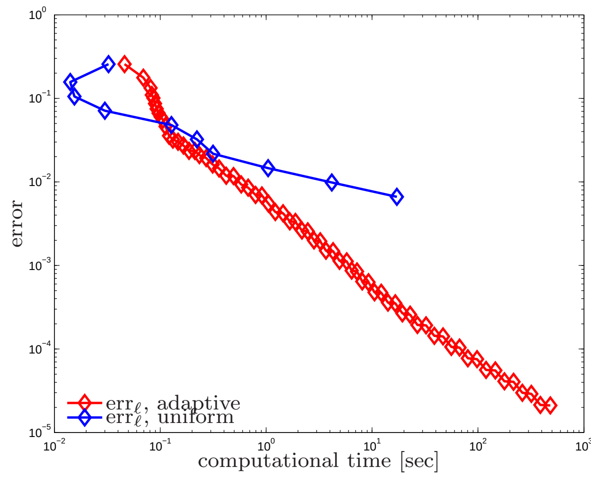

FIGURE 6. Experiment on Z-shape with singular solution. Error of uniform and adaptive mesh-refinement is plotted over the computational time measured in seconds.

$\mathrm{hot}_{2,\ell}$ converge with higher order, which gives us $\mathrm{err}_\ell \simeq |||\phi - \Phi_\ell|||$ for the computed steps. This indicates that the regularity assumptions in Theorem 4.3 are not fully necessary. The error for uniform mesh-refinement converges with suboptimal rate $\mathcal{O}(N^{-2/3})$ and the data oscillations show suboptimal rate $\mathcal{O}(N^{-7/8})$, too.

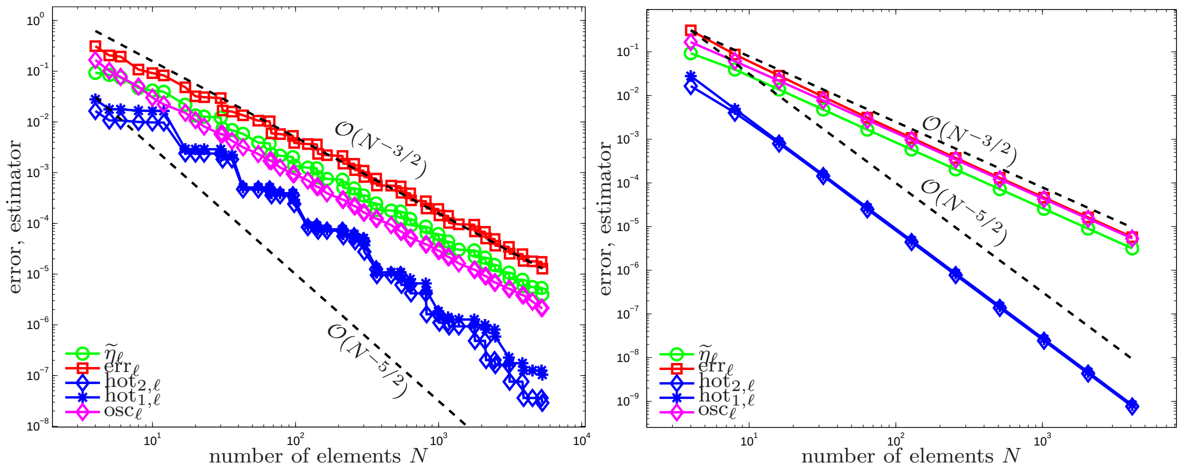

FIGURE 7. Experiment on square with smooth solution. The quantities err_ℓ, η_ℓ, $\mathrm{hot}_{1,\ell}$, $\mathrm{hot}_{2,\ell}$, and osc_ℓ are plotted versus the number of elements $N = \#\mathcal{T}_\ell$ for adaptive mesh-refinement (left) and uniform mesh-refinement (right).

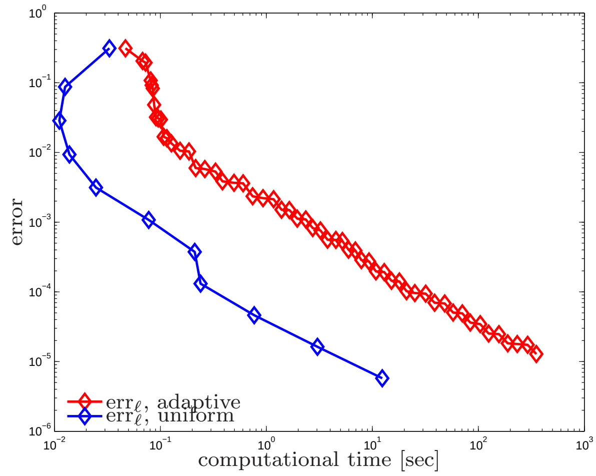

FIGURE 8. Experiment on square with smooth solution. Error of uniform and adaptive mesh-refinement is plotted over the computational time measured in seconds.

6.2.5. Experiment on slit. Finally, we define $\Gamma := [-1,1]$ as the boundary of $\mathbb{R}^2 \setminus [-1,1]$ in Figure 1. In contrast to our model problem, we consider indirect BEM, i.e.

$$V\phi = f,$$

with $f(x,y) := -x$ and $\phi(x,y) = -2x/\sqrt{1-x^2}$. Therefore, we do not need to approximate the right-hand side f and may use Algorithm 2.1 with $\theta = 1/2$ instead. Again, as

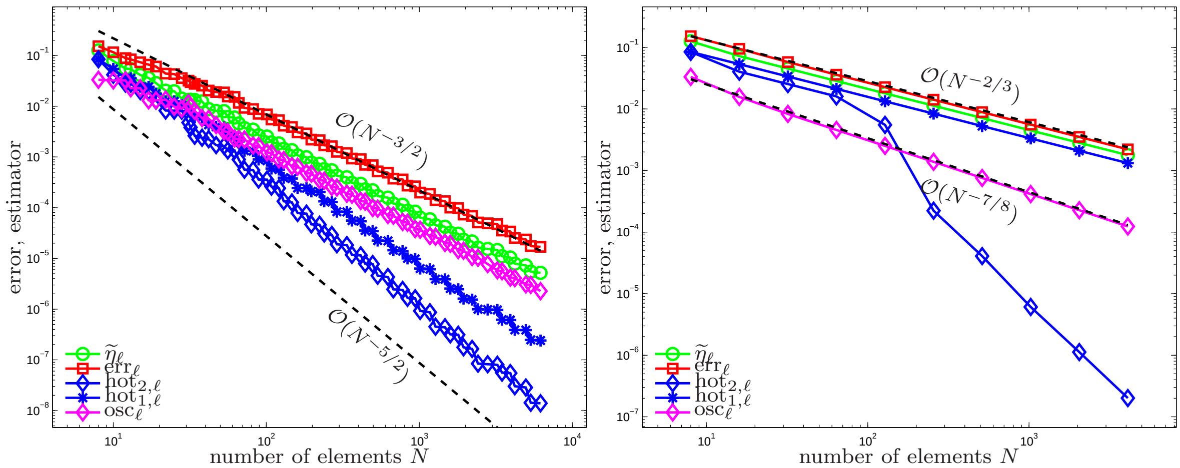

FIGURE 9. Experiment on L-shape with singular solution and singular data. The quantities err_ℓ, η_ℓ, $\mathrm{hot}_{1,\ell}$, $\mathrm{hot}_{2,\ell}$, and osc_ℓ are plotted versus the number of elements $N = \#\mathcal{T}_\ell$ for adaptive mesh-refinement (left) and uniform mesh-refinement (right).

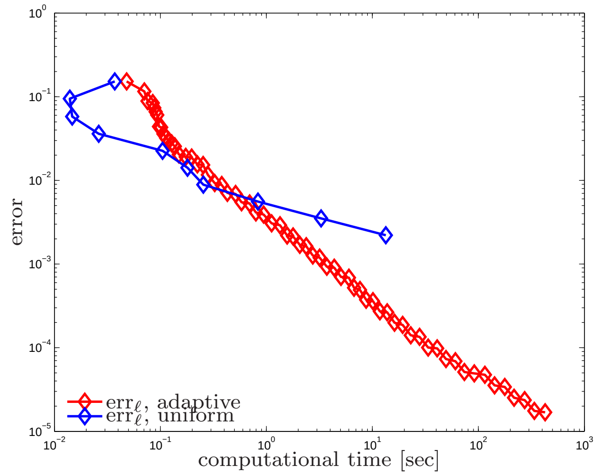

FIGURE 10. Experiment on L-shape with singular solution and singular data. Error of uniform and adaptive mesh-refinement is plotted over the computational time measured in seconds.

shown in Figure 11 and Figure 12, the adaptive strategy is much superior compared with uniform mesh-refinement.

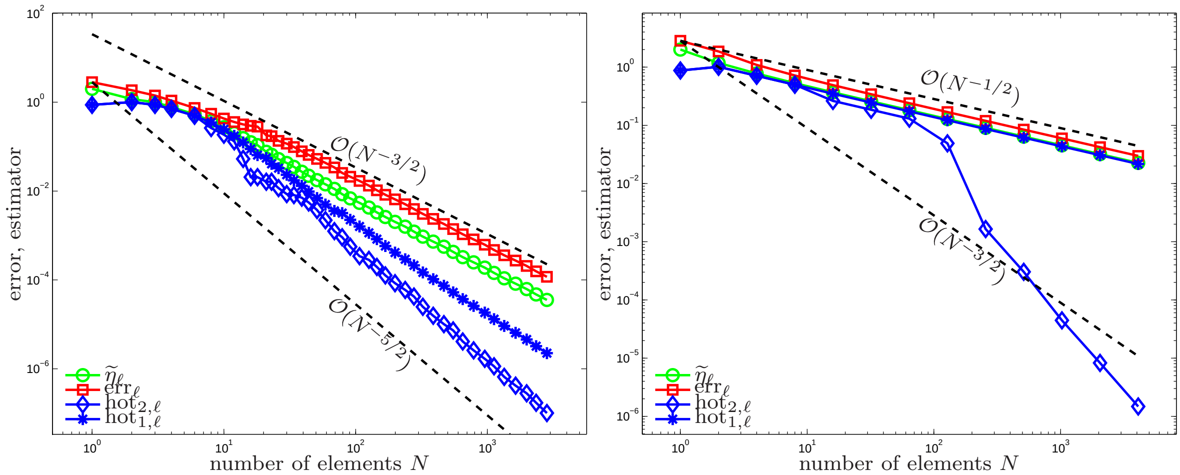

FIGURE 11. Experiment on slit. The quantities err_ℓ, η_ℓ, $\mathrm{hot}_{1,\ell}$, and $\mathrm{hot}_{2,\ell}$ are plotted versus the number of elements $N = \#\mathcal{T}_\ell$ for adaptive mesh-refinement (left) and uniform mesh-refinement (right).

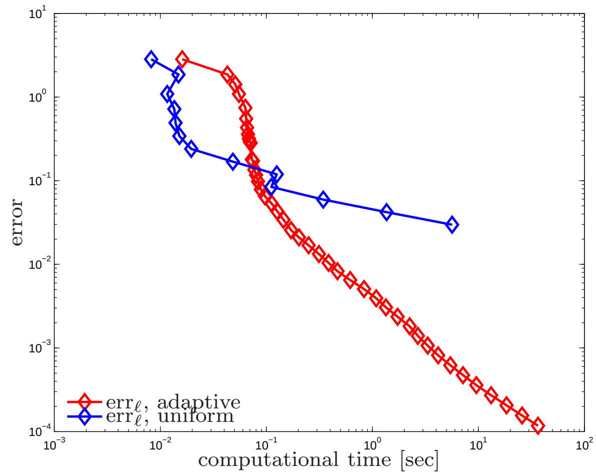

FIGURE 12. Experiment on slit. Error of uniform and adaptive mesh-refinement is plotted over the computational time measured in seconds.

APPENDIX A

Some Remarks on the Saturation Assumption

The saturation assumption for the Boundary Element Method states that there exists $q \in (0,1)$ such that

$$|||\phi - \Phi_{\ell,1}||| \le q|||\phi - \Phi_\ell||| \quad \text{for all } \ell \in \mathbb{N}, \tag{A.1}$$

where $\Phi_{\ell,1}$ is the Galerkin solution with respect to the uniformly refined mesh $\mathcal{T}_{\ell,1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{T}_\ell)$. In terms of $(h - h/2)$ based error estimators as proposed in [**4**], the saturation assumption (A.1) is equivalent to the reliability of these error estimators. Therefore, it is of certain interest to confirm this assumption. Up to data oscillation terms, (A.1) was proved for the finite element method and the Poisson problem, but still remains open for BEM. In this appendix, we attempt to prove a slightly weaker version of (A.1).

We assume the given boundary data to satisfy $g \in H^{s_{\rm reg}}(\mathcal{T}_0)$ for some $s_{\rm reg} > 2$ throughout the whole section.

Lemma A.1. *Let $\mathcal{T}_\ell \in \mathbb{T}$ denote a mesh and let ϕ denote the solution of* (4.1)*. Then, it holds the following discrete efficiency estimate*

$$C_{17}^{-1}\eta_\ell \le |||\Phi_{\ell,k} - \Phi_\ell||| + \mathrm{hot}_\ell,$$

where $k \in \mathbb{N}$ and $C_{17} \ge 1$ depend only on $\kappa(\mathcal{T}_\ell)$ and Γ. Here, $\Phi_{\ell,k}$ denotes the solution of (4.2) *with respect to the mesh $\mathcal{T}_{\ell,k} := \mathrm{unif}^{(k)}(\mathcal{T}_\ell)$.*

Proof. Recall the Céa Lemma and norm equivalence (3.12), to see

$$\|\phi - \Phi_{\ell,k}\|_{H^{-1/2}(\Gamma)} \lesssim |||\phi - \Phi_{\ell,k}||| \le |||(1 - \Pi_{\ell,k})\phi||| \lesssim \|(1 - \Pi_{\ell,k})\phi\|_{H^{-1/2}(\Gamma)}.$$

With the approximation properties of the L^2-projection (see [**14**, Theorem 4.1]), we conclude

$$\|\phi - \Phi_{\ell,k}\|_{H^{-1/2}(\Gamma)} \lesssim \|h_{\ell,k}^{1/2}(1 - \Pi_{\ell,k})\phi\|_{L^2(\Gamma)} \le 2^{-k/2}\|h_\ell^{1/2}(1 - \Pi_\ell)\phi\|_{L^2(\Gamma)}. \tag{A.2}$$

Now, we argue as in the proof of Theorem 4.3 and conclude together with (A.2)

$$\begin{aligned}
\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} &\lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell \\
&\le \|\phi - \Phi_{\ell,k}\|_{H^{-1/2}(\Gamma)} + \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell \\
&\lesssim 2^{-k/2}\|h_\ell^{1/2}(1 - \Pi_\ell)\phi)\|_{L^2(\Gamma)} + \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell \\
&\lesssim 2^{-k/2}\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} + \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell.
\end{aligned}$$

Hence, for $k \in \mathbb{N}$ sufficiently large, there holds

$$\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} \lesssim \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell. \tag{A.3}$$

With (4.36) and the approximation properties of the Galerkin solution, we prove

$$\begin{aligned}\eta_\ell &\lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}\\ &\lesssim \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}\\ &\lesssim \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell,\end{aligned}$$

where we inserted (A.3) to obtain the last estimate. Norm equivalence (3.12) proves the result. □

Now, we are able to prove the following result.

Proposition A.2 (weak saturation assumption)**.** *There exist constants $k \in \mathbb{N}$ and $0 < q < 1$ which depend only on $\kappa(\mathcal{T}_0)$ and Γ such that for all $\mathcal{T}_\ell \in \mathbb{T}$ with corresponding Galerkin solution Φ_ℓ, it holds*

$$|||\phi - \Phi_{\ell,k}|||^2 \le q|||\phi - \Phi_\ell|||^2 + \mathrm{hot}_\ell^2.$$

Proof. We combine reliability (2.15), Lemma A.1, and the Galerkin orthogonality to see

$$\begin{aligned}|||\phi - \Phi_{\ell,k}|||^2 &= |||\phi - \Phi_\ell|||^2 - |||\Phi_{\ell,k} - \Phi_\ell|||^2\\ &\le |||\phi - \Phi_\ell|||^2 - \tfrac{1}{2}C_{17}^{-2}\eta_\ell^2 + \mathrm{hot}_\ell^2\\ &\le |||\phi - \Phi_\ell|||^2 - \tfrac{1}{2}C_{17}^{-2}C_{\mathrm{rel}}^{-2}|||\phi - \Phi_\ell|||^2 + \mathrm{hot}_\ell^2\\ &\le q|||\phi - \Phi_\ell|||^2 + \mathrm{hot}_\ell^2.\end{aligned}$$

for $0 < q := 1 - \frac{1}{2}C_{17}^{-2}C_{\mathrm{rel}}^{-2} < 1$. Here, we used $C_{\mathrm{rel}}, C_{17} \ge 1$ to guarantee $q > 0$. □

In contrast to (A.1), the result above needs a certain number of uniform refinements to achieve a contraction. This raises the question if one could construct examples in which one uniform refinement is actually not sufficient. Obviously, one can construct examples, for which assumption (A.1) fails to hold for an arbitrarily large number of steps by choosing

$$\phi \in \mathcal{P}^0\big(\mathcal{T}^{(n+1)}\big)^\perp,$$

where $\mathcal{T}^{(n+1)} := \mathrm{unif}^{(n+1)}(\mathcal{T}_0)$. Then, there holds $\Phi_{\ell,1} = \Phi_\ell = 0$ for at least all meshes $\mathcal{T}_\ell$ with $\ell \le n$.

We conclude this appendix with a numerical example which underlines this result. Consider the example on the L-shape from Section 6.2.1. We plot the following quantities:

- The ratio between the two Galerkin error bounds err_ℓ and $\mathrm{err}_{\ell,1}$.
- The ratio between the higher order terms $\mathrm{hot}_{1,\ell}$ and $\mathrm{hot}_{2,\ell}$ and the Galerkin error bound err_ℓ.

For the definition of err_ℓ and $\mathrm{hot}_{1,\ell}$, $\mathrm{hot}_{2,\ell}$, see Chapter 6. Figure 1 shows the results of the experiment. We see that the ratio $\mathrm{err}_{\ell,1}/\mathrm{err}_\ell$ levels out at approximately $q = 0.35$ in case of the adaptive algorithm. Combined with the observation that the ratios $\mathrm{hot}_{i,\ell}/\mathrm{err}_\ell$, $i = 1, 2$ tend to zero, this indicates that the saturation assumption (A.1) holds for this example. For uniform mesh refinement, we see that the ratio is approximately $q = 0.63$. A heuristic computation as proposed in [**30**] may even predict the value of q. Assume therefore that the adaptive scheme has reached the asymptotic regime, i.e. $|||\phi - \Phi_\ell||| \approx C(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}$ for all $\ell \geq \ell_0$ and some $s > 0$. Then, it is plausible to calculate

$$q = \frac{|||\phi - \Phi_{\ell,1}|||}{|||\phi - \Phi_\ell|||} \approx \frac{\big(2(\#\mathcal{T}_\ell - \#\mathcal{T}_0)\big)^{-s}}{(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}} = 2^{-s}.$$

In case of the L-shape experiment from 6.2.1, we saw that the adaptive algorithm converges with $s = 3/2$. Therefore, this calculation predicts $q \approx 0.3536$. Analogously, we get $s = 2/3$ and therefore $q \approx 0.6299$ for uniform mesh refinement. Both cases coincide very well with the experimental results.

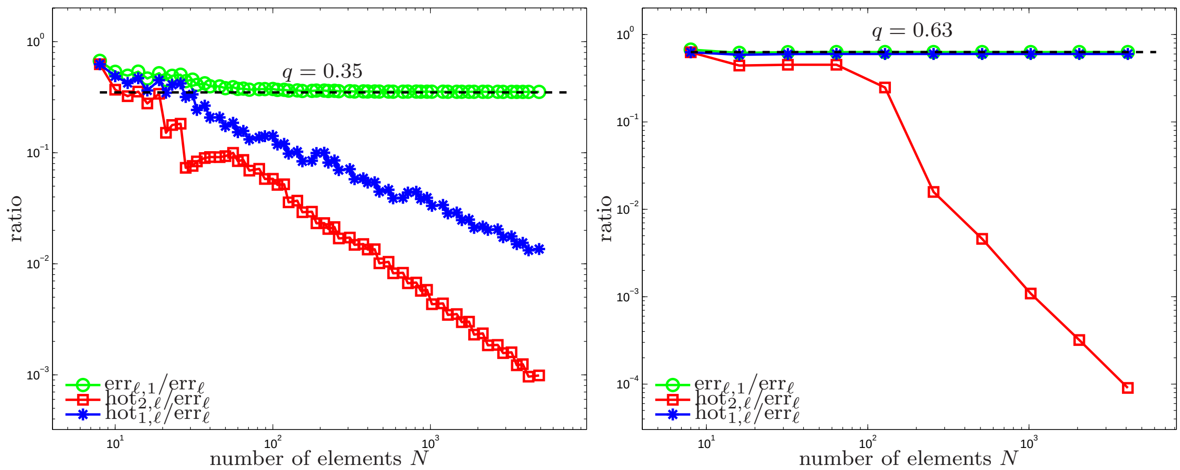

FIGURE 1. Test for saturation on the L-shape with adaptive mesh refinement (left) and uniform mesh refinement (right). The corresponding quantities are plotted over the number of elements N.

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