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Anne-Sophie Bonnet-Ben Dhia

Thorsten Hohage

DISSERTATION

Analysis of radial complex scaling methods for scalar resonance problems in open systems

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Associate Prof. Dipl.-Math. Dr.rer.nat. Lothar Nannen
E101 - Institut für Analysis und Scientific Computing, TU Wien

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Dipl.-Ing. Martin Halla BSc
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Kurzfassung

Wir untersuchen die Approximationen von skalaren Resonanzproblemen durch radiale komplexe Skalierungsmethoden. Die Methoden basieren auf einer komplexen Skalierung der radialen Variable, so dass die Resonanzfunktionen exponentiell gedämpft werden und sich dadurch die Resonanzprobleme zu Eigenwertproblemen transformieren. Als Approximation wird das unbeschränkte Gebiet durch ein Endliches ersetzt und eine homogene Dirichlet Randbedingung am künstlichen Rand gefordert. Wegen des starken Abklingens der Eigenfunktionen wird erwartet, dass der begangene Fehler gering ist. Das entstandene Problem kann weiters mit gebräuchlichen numerischen Verfahren, wie zum Beispiel Finite Elemente Methoden, diskretisiert werden. Die Analysis des Letzteren kann auf ähnliche Weise wie für klassische auf endlichen Gebieten gestellte Eigenwertprobleme durchgeführt werden, während die Analyis der Gebietsstutzung typischer Weise spezielle Techniken erfordert.

Wir stellen ein neues Konzept basierend auf einigen Kernideen vor, um die Approximationen zu untersuchen. Zuallererst fassen wir die Gebietsstutzung als konforme Galerkinapproximation auf. Weiters greifen wir auf Literatur über die Analysis von Approximationen holomorpher Eigenwertprobleme zurück. Mittels Multiplikationsoperatoren konstruieren wir sogenannte T-Operatoren, um die untersuchten Probleme in die Form kompakter Störungen koerziver Operatoren zu bringen. Wir stellen ein Rahmenwerk vor, um die Approximationen von schwach T-koerziven Operaten zu untersuchen. Wir finden eine Bedingung an die Galerkinräume, sogenannte T-Kompatibilität, um die spektrale Konvergenz zu garantieren (inklusive Konvergenzraten von Eigenwerten und Eigenräumen, etc.). Wir wenden diese Theorie an, um Konvergenzresultate für Approximationen (basierend auf radialer komplexer Skalierung) von skalaren Resonanzproblemen zu erhalten.

Abstract

We consider the approximation of scalar resonance problems by means of radial complex scaling methods. The methods are based on a complex scaling of the radial variable so that resonance functions become exponentially damped and the resonance problems transform to linear eigenvalue problems. As an approximation the unbounded domain is truncated to a finite domain and a homogeneous Dirichlet boundary condition is imposed on the artificial boundary. Due to the rapid decay of eigenfunctions the error generated is expected to be small. Consequently the resulting eigenvalue problem can be discretized by standard numerical schemes such as finite element methods. The analysis of the latter can be performed similarly to that of classical eigenvalue problems posed on bounded domains, while the analysis of the domain truncation is typically more laborious.

We propose a new framework to analyze the domain truncation based on several ideas. At first we interpret the domain truncation as a conform Galerkin approximation. Secondly we apply theories of holomorphic operator function eigenvalue approximation. We further construct so called T-operators by means of multiplication operators to transform the investigated problems into the setup of compact perturbations of coercive operators. At last we establish a condition on the Galerkin spaces, which we call T-compatibility, sufficient to ensure spectral convergence (including convergence rates of eigenvalues and eigenfunctions, etc.). We apply this framework to obtain convergence results for approximations (based on radial complex scaling) of scalar resonance problems.

Contents

1 Introduction

We consider differential operator eigenvalue problems posed on unbounded domains. Such problems are usually referred to as resonance problems. They are relevant in various fields of physics and engineering [Zwo99]. We are interested in the numerical approximation of those. We focus on acoustic and electromagnetic problems. Standard numerical schemes such as finite element or finite difference methods require a bounded domain and hence cannot be applied directly to discretize resonance problems.

1.1 Transparent boundary conditions

Common numerical methods require a bounded domain. There are several ways to overcome the numerical obstacle of unbounded domains. The common approach is to consider a truncated domain and impose either explicitly or implicitly a suitable boundary condition at the artificial boundary. The artificial boundary condition has to be chosen so that solutions to the new problem equal solutions to the original problem in the truncated domain. Such boundary conditions are referred to as transparent, non-reflecting or radiation boundary conditions. As a second step one has to construct a numerically realizable approximation to the exact transparent boundary condition. There are different ideas to construct such transparent boundary conditions. In this section we will briefly review these methods.

Boundary element methods A classical approach is the reformulation of resonance problems as boundary integral equations (BIE) [CK98], [Kre99] which only involve the traces of functions. If the medium is homogeneous the problem can be formulated solely on the boundary, while a non-homogeneous medium requires the coupling with a volume equation [HM06]. If the boundary is finite standard Galerkin discretizations can be used to approximate the integral equations. We refer to [SS11] for so-called boundary element methods (BEM). One drawback of those is that the fundamental solution depends on the eigenvalue parameter and hence non-linear/holomorphic (matrix) eigenvalue problems have to be solved. Another drawback is that the arising system matrices are dense, which poses numerical difficulties to matrix eigenvalue solvers. To overcome the latter special matrix approximation techniques have been developed, see e.g. [Hac09]. Since [AST$^+$09], [Bey12] there exist very efficient holomorphic matrix eigenvalue solvers. We refer to [SU09], [WX13] for numerical experiments for scalar and electromagnetic resonance problems. We refer to [Ung09], [SU12] for a convergence analysis for scalar problems and to [Ung17] for a convergence analysis for electromagnetic problems.

Dirichlet-to-Neumann operators A more ad hoc technique is to truncate the domain and introduce a Dirichlet-to-Neumann (DtN) operator at the artificial boundary, which is based on an explicit formula of solutions. Dirichlet-to-Neumann operators are also referred to as Calderon operator [Ces96], boundary component map [CK13] or Poincaré- Steklov operator [Hip02]). Again the DtN operator depends holomorphicly on the eigenvalue parameter and therefore the linear structure is lost. Consequently discrete approximations to the DtN operator can be constructed as in [Mas87], [Giv92], [LVLH92]. While classical DtN operator approximations are based on a series representation, a rather new approach based on the Fourier transform has been used in [Ton15], [DFT18], [BBDFT18].

Infinite element methods Another ad hoc technique are infinite element (IE) methods [DG98], [Ihl98], [DI01], [DS06]. They are based on a representation of solutions in terms of Hankel functions. As the eigenvalue parameter enters the argument of the Hankel functions the arising system matrices are non-linear/holomorphic with respect to the eigenvalue parameter.

Absorbing layers A very simple technique are absorbing layers (AL) [CL08], [FLCB08]. Thereby the material coefficients are replaced by complex valued ones resulting in damped solutions. Consequently the domain is truncated and a homogeneous boundary condition is imposed at the artificial boundary. The method features the advantage that it preserves the linear eigenvalue problem structure. However, very large layers have to be used to achieve meaningful results. Moreover, there exists no mathematical justification for this approach.

Complex scaling/perfectly matched layer methods Since the 1970s a popular method has been used to compute resonances in molecular physics [Sim78], [Moi98]. This method is referred to by various names, e.g. complex scaling (CS), analytic dilation (AD) and spectral deformation (SD). As absorbing layer methods the complex scaling technique preserves the linear eigenvalue problem structure. In contrast to absorbing layer methods, this method offers a profound mathematical framework with the Aguilar-Balslev-Combes-Simon Theorem at its core. We refer to the book of Hislop and Sigal [HS96]. In the 1990s Bérenger introduced his perfectly matched layer (PML) method [Bér94], [Bér96b], [Bér96a] which became very popular for all kinds of wave propagation problems. In [CW94], [TC97], [CM98b] the PML method was recognized to be a complex scaling technique. We refer to [HHK04], [HHKS07] for resonance computations with PML/CS methods and to [Kim09], [KP09], [Kim14], [HN15b] for numerical analysis of PML/CS methods for resonance problems.

Hardy space infinite element methods A rather new method which also preserves the linear eigenvalue problem structure are Hardy space infinite element (HSIE) methods [HN09], [NHSS13], [RSS13], [HN15b], [Hal16], [HN18]. Despite the name, Hardy space infinite element methods are actually closer to PML/CS methods than to classical infinite element methods. They are of special interest for backward wave phenomena [HN15a], [HHNS16]. The connection between complex scaling methods and Hardy space infinite element methods is elaborated on in [NW19].

Absorbing boundary conditions At last we mention absorbing boundary conditions (ABC) [EM77], [GK95], [Giv04] which work fine for time-dependent and time-harmonic scattering problems. However, they are not meaningful to formulate or approximate resonance problems due to the exponential growth of resonance functions.

Inverted finite element methods For the same reason so-called inverted finite elements [Bou05], [BMBB15] fail. Therein the unbounded domain is deformed to a bounded domain and suitably weighted Sobolev spaces are employed.

1.2 Perfectly matched layer methods

Perfectly matched layer methods are based on a continuous complex coordinate transformation $x \mapsto \tilde{x}(x)$, the complex scaling, so that $\tilde{x}(x) = x$ in the domain of interest. For resonances in a suitable region of the complex plane the corresponding resonance functions become exponentially damped by the transformation $\tilde{u} = u \circ \tilde{x}$. Consequently a set of equations is derived for $\tilde{u}$ and due to the decay of $\tilde{u}$ the resonance problem transforms to an eigenvalue problem in a suitable standard Sobolev space. Furthermore the domain is truncated and a homogeneous boundary condition imposed at the artificial boundary. Due to the rapid decay of $\tilde{u}$ the committed error is expected to be small. The derived problem can consequently be discretized with standard numerical schemes such as finite element and finite difference methods. The literature on perfectly matched layer methods is extremely extensive. Nevertheless we try to give at least a rough survey and apologize to all authors who do not find their work listed herein.

History and applications Since the 1970s the method of complex scaling/analytic dilation/spectral deformation has been used in molecular physics [Sim73], [Sim78], [RS78], [Sim79], [CFKS87], [HS96], [Moi98], but did not attract much attention from scientists in other fields. In the 1990s Bérenger introduced his perfectly matched layer method for electromagnetic problems [Bér94], [Bér96b], [Bér96a], which is often described as a reflectionless sponge layer. In [CW94] the method was recognized to be a kind of complex scaling technique. The work of Bérenger had huge impact and PML methods were soon applied to all kinds of wave equations such as time-harmonic acoustics [QG98], [TY98], [HST00], advective acoustics [Hu96], [Hes98], [AGH99], [Hu01], linearized shallow water equations [NNH04], elastodynamics [CT01], [BC03], poroelastic media [ZHL01], heat and advection-diffusion equations [LN10], Schrödinger equations [NK11] and also non-linear wave equations [DH07], [AK07] to name but a few references from engineering literature. We mention the book [Bér07] of Bérenger. Recently PML methods have been applied to other kinds of problems as well. E.g. problems which are posed in bounded domains, but admit black hole phenomena [BBCCC16]. A further application of PMLs can be found in domain decomposition methods [EY11], [LY16].

Physically correct and stable PMLs We note that despite their popularity the construction of physically correct and stable PML methods has to be executed with care. In general PMLs have to be designed so that evanescent waves stay evanescent and propagative waves with positive group velocity become evanescent. While for some equations this poses no problem at all it can lead to serious difficulties for PML methods if the equation is anisotropic, advective or dispersive. Also heterogeneous domains like waveguides can generate such difficulties since they cause dispersive effects (although the equation may be dispersionless itself). While an unphysical PML for a time-dependent equation will result in temporally growing solutions and hence be conspicuous, an unphysical PML for a time-harmonic equation will result in unphysical solutions which may not look qualitatively differently to physical solutions. We refer to [Hu96], [Hu01], [Hes98], [TAC98], [AGH99], [Hag03], [DJ03], [BFJ03], [BBBL06], [Nat06] for advective acoustics/linearized Euler equations and to [BFJ03] for anisotropic elastodynamics. For the equations of those references a geometric stability condition is presented by Bécache, Fauqueux and Joly in [BFJ03]. We further refer to [Cum04] and the work of Kachanovska et. al. on dispersive materials [BJKV15], [BJV18], [BJK17], [BK17], [CJK17]. Analysis of PMLs for cylindrical waveguides is discussed in [BBBL04], [BBBL06], [SAC07], [BBCL14] and [HN15b].

Parameter optimization The application of a PML method involves the choice of many different parameters such as the profile of the absorption function, the scaling of the absorption function, the layer thickness and the kind of discretization. Attempts to find optimized parameters were made in [CM98a]. Druskin et. al. report [ADGK03], [DGK16] optimal finite difference grids for PML, while [MDKP07] proposes an automatic hp-adaptive discretization to improve the performance of the PML. Zschiedrich [Zsc09] proposes an algorithm which automatically determines the layer thickness and mesh width. On the other hand Bermudez et. al. use an absorption profile with blow up for their so-called exact PML [BHNPR04], [BHNPR06], [BHNPR07], [BHNPR08]. A different kind of absorption profile with blow up was also proposed in [HL05]. A blow up of the profile function leads to a formulation posed on a bounded domain and has the advantage that no domain truncation is necessary at the cost of fabricating singular coefficients. A performance comparison of different profiles and different discretization methods is reported in [MDG14]. The conclusion of this reference is that the proposed profiles by Bermudez et. al. admit an optimal scaling of the absorption function and are hence favorable. The performance of exact PMLs is also investigated in [CMH15]. An interpretation of Hardy space infinite element methods [Nan08] as composition of complex scaling and a "standard" infinite element method was proposed in [NW19]. These methods decrease the number of method parameters as well.

Comparison with other kinds of transparent boundary conditions We refer to [RGB10] for a performing comparison between PML and absorbing boundary conditions in the time-harmonic regime. The outcome of this reference is that these two methods perform quite similarly.

Multiple scattering objects Following [GK04] an efficient realization of PML for multiple very distant scattering objects is presented in [JZ12].

Coordinate systems In [CW94] it was recognized that the PML method proposed by Bérenger [Bér94] could be derived by a complex coordinate stretching. While the method of Bérenger is based on a Cartesian coordinate stretching, [TC97] and [CM98b] proposed coordinate stretchings based on cylindrical and spherical coordinate systems. The latter is also the approach taken in the molecular physics literature [HS96]. Generalizations of spherical scalings of the form

$$\tilde{\mathbf{x}}(r, \hat{\mathbf{x}}) = \tilde{r}(r)\mathbf{n}(\hat{\mathbf{x}}) + \boldsymbol{\gamma}(\hat{\mathbf{x}}).$$

are possible. Thereby r is the variable which is complex stretched to $\tilde{r}(r)$, $\hat{\mathbf{x}}$ runs through the unit sphere, $\mathbf{n}$ is the direction of the scaling and $\boldsymbol{\gamma}$ describes the inner boundary of the complex layer. Since only the variable r is stretched we refer to scalings of this form as radial scalings. Two reasons call for such radial scalings. The first reason is due to inhomogeneous materials, e.g. open waveguide structures [BBGH11]. For these the direction $\mathbf{n}$ of the scaling has to be altered to ensure that the PML implies a physically meaningful radiation condition. The second reason is that spherical scalings for elongated scattering objects lead to unnecessary large computational domains which poses a performance loss. For such geometries the inner boundary of the complex layer, i.e. the manifold described by $\boldsymbol{\gamma}$, can be adapted to a suitable hull containing the scatterer. We emphasize that these two variations of spherical scaling can be performed (almost) independently of each other, although they are commonly presented in a coupled form. For further reading on this topic we refer to [LS01], [KM97], [ZKSS06], [Tre10] and to the more recent papers [CLX13] and [CCZ13].

Surprisingly the mathematical properties of the derived formulations can be quite different depending on the choice of radial or Cartesian scaling.

We also mention [TC99], [LLS01] for derivation of PMLs via abstract differential forms.

1.2.1 Time-harmonic equations

For time-harmonic wave equations there exist two common problems of interest. For scattering problems a solution to the partial differential equation with given frequency and right hand side is sought. Differently for resonance problems a resonance frequency (also referred to as scattering pole) together with a non-trivial solution of the homogeneous partial differential equation is sought.

The formulation of any PML method to such problems starts with the derivation of a complex scaling formulation of the underlying problem. For a sufficient analysis the next step is to prove Fredholmness of the underlying operators. Further the domain is truncated to a bounded one and the convergence of solutions for increasing layer size and/or scaling parameter has to be established. At last the approximation of the truncated problem by a numerical scheme, e.g. finite element or finite difference methods, needs to be ensured. While the latter can be performed with standard techniques and does not need to be discussed in great detail, the approximation of the untruncated problem by the truncated problems is more intricate. The references given in this subsection are mostly concerned with the latter topic. We note that to our knowledge there exist no results on this issue in molecular physics research, i.e. on complex scaling/analytic dilation/spectral deformation (in contrast to perfectly matched layers).

Radial PML The first convergence analysis of spherical PML was conducted by Lassas and Somersalo in [LS98] for acoustic scattering problems. Therein the artificial boundaries are assumed to be spherical with increasing radii. The authors continued their work in [LS01] wherein they considered also more general radial scalings. Hohage et. al. analyzed radiation conditions for acoustic scattering problems with non-local radial symmetric potentials [HSZ03a] and extended the results of [LS98] to those equations in [HSZ03b]. Bao and Wu [BW05] established convergence for electromagnetic scattering problems. They considered spherical artificial boundaries and convergence with respect to increasing radii and/or profile scaling parameter. The analysis of [LS98], [HSZ03b], [BW05] indeed relies on a spherical truncation boundary and employs a series or integral representation of operators.

Bramble and Pasciak extended converge results for acoustic and electromagnetic scattering [BP07], [BP08] to non-spherical domain truncations. In [Kim09], [KP09] Kim and Pasciak established convergence for acoustic resonance problems with non-spherical layer truncations and increasing domains. Convergence of a PML formulation with a non-standard integration by parts is achieved in [BPT10] for constant coefficient elastic scattering problems. While the discretization analysis of the truncated PML equations is usually unconditionally covered by standard techniques, [BPT10] requires a smallness assumption on the profile scaling. The analysis of Pasciak et. al. is partially based on a reformulation of equations to satisfy a Gårding inequality. We note that all mentioned references require a smooth profile function for their analysis, although this condition seems unnecessary in practice. Apart from [LS01], [BW05] all former references require the scaling to be linear after a permissible transition zone. Hence scalings based on power functions are not covered by those.

Chen et. al. [CL05] proposed a scheme to match the errors committed by truncation and discretization for two dimensional Helmholtz scattering problems. They choose PML parameters based on an a priori estimate to obtain a neglectable error relative to the discretization error and apply an adaptive h-refinement lowest order finite element method. The authors consider scalings based on power functions, radial domain truncations, a fixed layer width, an increasing absorption and provide an analysis for their method. The scheme is expanded to electromagnetic scattering problems in [CC08]. The analysis however, requires a major assumption which is not proved. Generalizations to more general radial scalings by Chen et. al. can be found in [CLX13] and [CCZ13].

Cartesian PML For time-harmonic equations a major difference between radial and Cartesian scalings is that for the latter an explicit reformulation satisfying a Gårding inequality is difficult and not known. Hence different analysis techniques have to be employed. Cartesian PML methods for time-harmonic equations were mainly investigated by Chen et. al. and Pasciak, Bramble and Kim.

Chen and Wu proposed in [CW08] a scheme to match the errors committed by truncation and discretization for two dimensional Helmholtz scattering problems. They choose PML parameters based on an a priori estimate to obtain a neglectable error relative to the discretization error and apply an adaptive h-refinement lowest order finite element method. The authors consider scalings based on power functions and equal absorption strength in both directions. They further provide an analysis of their scheme whereby they consider a fixed layer width and an increasing absorption. However, to succeed they require a major assumption which is not proved. In [CGX09] the work is extended to high order discretizations and with their proposed refinement strategy the authors experience exponential convergence w.r.t. the number of degrees of freedom. Chen and Zheng consider in [CZ10] acoustic two dimensional scattering problems in two layered media. The authors consider affine scalings with equal absorption strength in both directions. They indeed prove a result which was merely assumed in the previous two works. For absorption larger than an explicitly known bound the authors achieve convergence with respect to the absorption scaling and/or the layer width.

Independently Kim and Pasciak undertook in [KP10b] a preliminary analysis for Cartesian PML for acoustic scattering problems in two dimensions. For such problems they achieved convergence with respect to the thickness of the layer in [Kim09], [KP10a]. The analysis therein requires the scaling to be smooth and linear after a permissible transition zone. The technique of Kim and Pasciak is different to that of Chen and Zheng and employs an iteration argument. Other kinds of scalings for acoustic scattering problems in two and three dimensions are considered by Bramble and Pasciak in [BP13] where the authors prove convergence with respect to the layer width and/or the profile scaling. For a certain profile form and increasing layer width convergence is proven in [BP12b] for electromagnetic scattering problems. By means of a special truncation boundary condition the reflection argument of Bramble and Pasciak is extended by Chen et. al. [CXZ16] to elastic scattering problems and convergence is obtained for a certain profile form and increasing layer width. Note that for elastic systems convergence of discretizations to the truncated problems cannot be obtained in a straightforward way. This is due to the “non-diagonal” nature of the differential operator involved. We further refer to [Kim14] for a convergence analysis of resonances for two-dimensional acoustic problems. Therein convergence is analyzed with respect to increasing layers under the assumption of a smooth scaling which is linear after a permissible transition zone.

Waveguides The following references deal with closed isotropic (asymptotic) cylindrical waveguides. Hence complex coordinate scaling has to be applied only in one direction. Differently to homogeneous domains solutions in waveguides admit a modal representation whereby each wave mode admits a different wavenumber. To treat the evanescent modes correctly the complex scaling has to be asymptotically linear.

Bécache, Bonnet-BenDhia and Legendre established convergence of PMLs with increasing layer width for scattering problems governed by convected Helmholtz [BBBL04] and linearized Galbrun equations with uniform flow [BBBL06]. To deal with the advective character of the equations the authors constructed problem adapted PMLs. Their analysis is based on the explicit computation of the Dirichlet-to-Neumann operator (in series form) generated by the truncated PML.

Hohage and Nannen considered Helmholtz scattering and resonance problems (without advection). They applied PMLs with Dirichlet layer termination as well as Hardy space infinite elements and achieved convergence with a unified framework for both methods [HN15b]. Their analysis is based on a reformulation of the equation to comply a Gårding inequality. For the analysis of the PML, they apply the interesting approach to consider the finite element space as subspaces of H^1 of the unbounded domain to avoid the introduction of a truncation analysis. This is the only work known to us which analyzes the domain truncation and the finite element discretization simultaneously.

In general, PMLs applied to elastic waveguides lead to unphysical solutions due to the existence of backward modes, which feature different signs of group and phase velocity. In order to overcome this obstacle, hybrid modal-PML methods were introduced by Skelton, Adams and Craster [SAC07] and Bonnet-BenDhia, Chambeyron and Legendre [BBCL14]. Generalized Hardy space infinite elements which do not rely on modal hybridization were developed by Halla, Nannen et. al. [HN15a], [HHNS16].

Kalvin considered Helmholtz scattering problems in waveguides which are only asymptotic cylindrical and established convergence of PMLs in [Kal12]. In [Kal13] he generalized his results to manifolds, i.e. the waveguide of interest consists only of an asymptotic cylindrical shell.

Gratings (Waveguides with periodic boundary conditions) PMLs can also serve as approximative transparent boundary condition for so called gratings, which are nothing but waveguides equipped with periodic boundary conditions. PMLs for such configurations have been studied by Wu, Chen, Bao et. al. [CW03], [BCW05], [CWW09] [BLW10] for Helmholtz as well as for Maxwell scattering problems with specific focus on a posteriori error estimates and adaptive refinement strategies. The properties of gratings and waveguides with natural/essential boundary conditions are very similar, compare the analysis of [BBBL04] and [BLW10]. The case of non-constant refractive index which stabilizes at infinity has been investigated by Kalvin [Kal11] who proves convergence for increasing layer sizes also in this more sophisticated case.

Resonance problems Due to the focus of this thesis we emphasize that the only literature on convergence analysis of PML methods for resonance problems known to us is [Kim09], [KP09], [Kim14], [HN15b].

1.2.2 Time-dependent equations

To gain basic insight of PML in the time-domain we refer to the instructive paper [Jol12] by Joly. The construction of PMLs for time-dependent equations starts in the time-harmonic regime. The derivation is performed similarly whereby the complex scaling is chosen frequency dependent to ensure a damping of waves independent of their frequency. Thereafter terms $-i\omega$ (ω denotes the frequency) are replaced by a time derivative. To avoid convolution terms and to obtain a partial differential equation system one usually introduces auxiliary fields. There is a lot of freedom in the choice of the auxiliary variables and hence lots of different formulations are possible. Note that in literature on *complex scaling* the scaling is classically chosen to be frequency independent and hence not equivalent to PML formulations as pointed out in [SSM14].

The analysis of PMLs for time-dependent equations includes different steps. The first one is to ensure the well-posedness [KL04] of the system. Well-posedness thereby means that the system admits a unique solution the norm in space of which can be bounded by a time-dependent constant times the norm of the data.

However the notation of well-posedness still allows for temporally exponentially growing solutions which is of course not desirable. The next step is therefore the derivation of stability. Strong stability means that the norm of solutions in space can be bounded by a temporally uniform constant times the norm of the data. The notion of weak stability allows for temporally algebraic growing constants and measures the data in a stronger norm.

Further similar results for the systems on the truncated domains need to be established and convergence of the truncated to the untruncated solutions proven. Finally one aims to derive convergence results also for the discretizations of the truncated problems.

Cartesian PML Bérenger constructed his PML [Bér94] with two steps. First he introduced auxiliary variables and reformulated Maxwells equation with them. Secondly he added a zero order absorption term to the equations. PML formulations related to Bérengers choice of auxiliary variables are usually referred to as split-field PML. Abarbanel and Gottlieb noted in [AG97] that Bérengers choice of auxiliary variables doesn't constitute a strongly well-posed system and hence the added absorption might lead to an ill-posed system, although numerical experiments did not indicate such. This gave rise to PMLs with different choices of auxiliary variables than Bérengers to allow the construction of well-posed systems. Such PMLs are usually referred to as unsplit PML, see the reviews by Turkel and Yefet [TY98] and Teixeira and Chew [TC01]. Bécache and Joly [BJ02] argued that although Bérengers auxiliary variables constitute only a weakly well-posed system, the added absorption term is not a perturbation exciting ill-posedness. Indeed they considered the problem solely posed in one layer with constant absorption and obtained stability of the physical fields therein.

A key remark to understand the vast literature on well-posedness and stability of PML systems is that for strongly hyperbolic systems with varying coefficients the question of well-posedness can be reduced to the corresponding frozen coefficient problems [KL04]. Hence for such systems it suffices to investigate well-posedness in each subdomain separately, i.e. the physical domain and the different parts of the layer. However, a similar conclusion is not valid concerning stability. Thus a derived stability result for the Cauchy problem solely posed in one layer can only be considered as a minimum requirement, but is by no means sufficient for a rigorous analysis. Well-posedness and stability analysis for separate parts of the PMlayer are reported e.g. by Bécache and Joly [BJ02], Bécache, Fauqueux, and Joly [BFJ03], Appelö and Hagstrom [AH06], [AH06], [AH09], Appelö, Hagstrom and Kreiss [AHK06] and Duru [Dur14].

The question of stable PML simulations is tricky and different kinds of instability can be introduced by various reasons. Abarbanel, Gottlieb and Hesthaven [AGH02] explain so-called long time instabilities with a wrong low frequency limit of equations in the layer. Bécache, Petropoulos and Gedney [BPG04] give further insight into this issue and propose as so-called complex frequency shift (CFS) PML as a cure. Also Bécache and Prieto [BP12a] report corner instabilities depending on the time discretization of auxiliary fields which can deteriorate the Courant-Friedrichs-Lewy stability condition. Duru, Kozdon and

Kreiss [DKK15] and Duru [Dur16] report instabilities due to the numerical treatment of boundary conditions.

We also mention the work of Kreiss et. al. [AK06], [DK12a], [KD13], [DK14a], [DKM14] on PML in elastic materials. Amongst others they suggest that coarse discretizations of PMLs which violate the geometric stability condition of Bécache, Fauqueux and Joly [BFJ03] may still produce acceptable results.

Chen and Wu [CW12] analyze PMLs for the scalar wave equation by a forth and back transformation of equations into the Laplace domain. Though the paper claims to report a convergence result its main Theorem 5.2 does not suffice. Indeed the right-hand-side of the theorem's inequality is not independent of the truncated PML solution $\hat{u}$.

At last we mention the few works which report actual stability and convergence results. By means of the Cagniard-de Hoop technique de Hoop, van den Berg and Remis [dHvdBR02] and Diaz and Joly [Dia05], [DJ06] achieve convergence in the case of a point-source for the scalar wave equation. In their analysis they explicitly construct the truncated and untruncated PML solution.

Radial PML Differently to time-harmonic equations radial PML have attracted only little attention of numerical analysts for time-dependent problems so far. For the latter PMLs in spherical and cylindrical coordinates were proposed by Collino and Monk [CM98b], Petropoulos [Pet00] and Teixeira and Chew [TC01]. Chen [Che09] performs an analysis of PML for the scalar wave equation, but although the paper claims to report a convergence result its main Theorem 5.3 does not suffice. Indeed the right-hand-side of the theorem's inequality is not independent of the truncated PML solution $\hat{u}$.

Waveguides Duru and Kreiss consider PMLs for isotropic acoustic waveguides [DK12b]. They conclude that the choice of auxiliary variables is essential to obtain a stable method. The authors continue their work in [DK14b] for elastic waveguides.

1.3 Main ideas of the thesis

In this thesis we propose a new framework to understand spherical perfectly matched layer approximations to time-harmonic wave propagation problems. The analysis is based on the following four main ideas.

1. We interpret the domain truncation as Galerkin approximation.
2. We apply literature on holomorphic Fredholm operator eigenvalue approximation theory to linear eigenvalue problems.
3. We ensure the regularity/stability of Galerkin approximations through the notion of weak T-coercivity and T-compatible approximations.
4. We construct T-operators through suitable multiplication and projection operators.

The first main idea goes back to Hohage and Nannen [HN15b]. Therein geometrically aligned finite element spaces are considered directly as subspaces of H^1 on the unbounded domain and the discretization is interpreted as Galerkin approximation. The imposed Dirichlet condition at the artificial boundary is thereby essential to ensure a conform approximation scheme. We consider two approaches for the approximation analysis: A direct one as in [HN15b] whereby the truncation and discretization are considered simultaneously. And a split one whereby we first analyze the domain truncation and secondly analyze the discretization. For the second approach, we adapt the idea of Hohage and Nannen and interpret the domain truncation as Galerkin approximation as well.

The standard approximation theory for linear eigenvalue problems is based on norm convergence of operators [BO91], [Bof10]. However in our case we approximate a non-compact operator with compact operators and hence norm convergence cannot hold true. Analysis techniques for such problems have been developed in [DNR78a], [DNR78b] aimed to magnetohydrodynamic problems [Rap77]. Non-compact operators are also encountered in electromagnetism where a vast progress has taken place [CFR00], [Buf05], [AFW06], [AFW10], [CW13]. Though different to electromagnetism the essential spectra of our investigated operators consist not only of a single point but of a continuum. Hence we cannot hope to reuse the techniques of the former references.

The second major idea to apply approximation theory on holomorphic Fredholm operator eigenvalue problems [Kar96a], [Kar96b] to the one in this thesis investigated linear eigenvalue problems is taken from our earlier work [Hal16] on Hardy space infinite elements.

Since we apply conform Galerkin approximations the assumptions of [Kar96a], [Kar96b] are reduced to the regularity of the approximations. We introduce an abstract framework to ensure the regularity. This framework is more restrictive than the regularity condition, but it serves useful for many applications. The analysis is based on the weak T-coercivity of the operator function associated to the eigenvalue problem. Thereby we say that an operator $A \in L(X)$ is weakly T-coercive, if $T \in L(X)$ is bijective and there exists a compact operator $K \in L(X)$ so that

$$|\langle Au, Tu\rangle + \langle Ku, u\rangle| \gtrsim \langle u, u\rangle$$

for all $u \in X$. The previous inequality can also be interpreted as generalized Gårding inequality and the operator T as an "inf-sup operator up to a compact perturbation". With T-compatibility of Galerkin approximations $A_n \in L(X_n)$ to A we mean the existence of linear bounded operators $T_n \in L(X_n)$ so that they converge to T in the discrete norm:

$$\|T - T_n\|_n := \sup_{u_n \in X_n \setminus \{0\}} \|(T - T_n)u_n\|_X / \|u_n\|_X \to 0 \quad \text{as} \quad n \to \infty.$$

A major result of this thesis is that T-compatible Galerkin approximations to weakly T-coercive operators are regular.

The notion of T-coercivity originates from [BBCZ10], [BBCC12], [BBCC14] and was introduced to analyze differential operators with sign-changing coefficients in the principle part. Such operators arise from wave propagation problems in composite materials consisting of classical and negative index materials. The discrete norm $\|\cdot\|_n$ previously appeared in [DNR78a], [DNR78b] and independently in [HN15b] and [Car15].

For our eigenvalue problem at hand (the radial complex scaled formulation of a Helmholtz resonance problem) the construction of a suitable T operator can be realized with a multiplication operator. For restricted kinds of scaling profiles this observation goes back to Bramble and Pasciak [BP07].

If we separate the approximation due to domain truncation and the approximation due to (finite element) approximations, the analysis becomes extremely simple: Since a multiplication operator does not increase the support of function, the Galerkin spaces (i.e. the Sobolev space on the truncated domains) are T-invariant. The approximation analysis of the truncated problems by finite element methods on the other hand is straight forward.

If we consider the approximation simultaneously, the Galerkin spaces are no longer T-invariant. We construct operators T_n by composition of T and a projection. By means of the discrete commutator property of Bertoluzza [Ber99], we obtain convergence of T_n to T in discrete norm $\|\cdot\|_n$.

1.4 Outline of the thesis

The remaining part of the thesis is structured as follows. In Chapter 2 we consider the Helmholtz equation and following [CK98] we derive in Section 2.1 well known solutions in terms of spherical Hankel functions and spherical harmonics. In Section 2.2 we specify the radiation condition and the resonance problem under consideration. In Section 2.3 we introduce the complex scaling. We attain that the complex scaled solution $\tilde{u}$ is well defined, derive a variational equation for it and show the correspondence to the original equation. In Section 2.4 we explain how domain truncation can be understood as a conform Galerkin approximation. We discuss that standard theory for approximations to linear eigenvalue problems [BO91] is not admissible.

In Chapter 3 we give a brief introduction into holomorphic Fredholm operator functions and holomorphic Fredholm operator eigenvalue problems. We introduce the notions of (weak) T($\cdot$)-coercivity of operator functions and T($\cdot$)-compatible Galerkin approximations. We derive that T($\cdot$)-compatible Galerkin approximations to weakly T($\cdot$)-coercive operator functions are always regular approximations in the sense of [Kar96a], [Kar96b]. Hence our introduced notions open the door to apply the powerful convergence results of [Kar96a], [Kar96b].

In Chapter 4 we continue the discussion of Chapter 2 and apply the framework of Chapter 3 to obtain convergence results for perfectly matched layer methods. In Section 4.1 we construct an appropriate T-operator. We prove weak T-coercivity on the admissible set of frequencies and characterize the essential spectrum. In Section 4.2 we discuss the approximation by domain truncation. In Section 4.3 we discuss the approximation by a subsequent discretization. In Section 4.4 we discuss the approximation by simulations domain truncation and discretization. In Section 4.5 we consider formulations which involve only a bounded domain and discuss the approximations. In Section 4.6 we give some remarks on how our theory extends to more involved situations.

2 Helmholtz resonance problems

In this chapter we discuss the framework of Helmholtz resonance problems in homogeneous (exterior) domains. We derive a formulation via the complex scaling method, establish its relation to the original problem and discuss the framework to analyze its PML approximations gained from domain truncations.

More specifically, in Section 2.1 we follow [CK98] and recall well known facts on spherical harmonics and Bessel functions. Since in [CK98] mostly only real arguments for Bessel functions are considered we pay attention to allow also complex arguments. In Lemma 2.2 we give a version of a less known estimate on spherical Hankel functions from [CL05]. In Section 2.2 we motivate and define the radiation condition in terms of spherical Hankel functions and specify the Helmholtz resonance problem under consideration. We recall standard results on the absence of resonances with non-negative imaginary part. In Section 2.3 we follow e.g. [CM98b], [LS98], [HSZ03b], [BP07] to define a complex scaling of the radial variable and obtain a variational formulation, whereby we adopt the notation of [BP07]. Differently to most works we allow complex scalings of rather general forms. In Lemma 2.12 we derive that solutions admit a representation in spherical harmonics and Hankel functions. In Lemma 2.13 we prove that the intuitive definition of the complex scaled solution is indeed proper and has common properties. In Lemma 2.14 and Lemma 2.15 we establish that complex scaled resonance functions are certainly solutions to the derived variational equation and vice-versa. In the proofs we proceed via the separated spherical Bessel equations (in contrast to argue with fundamental solutions). In Section 2.4 we present our new interpretation of domain truncation as Galerkin method and discuss why standard theory on eigenvalue approximation [BO91], [Bof10] cannot be applied. The former serves as motivation for Chapter 3.

For a Lipschitz domain $D \subset \mathbb{R}^l$, $l = 1, 2, 3$ denote ν the unit outward normal vector at ∂D. For $s > 0$ denote $L^2(D)$, $H^s(D)$ the standard Sobolev spaces of complex scalar valued functions with hermitian scalar products $\langle \cdot, \cdot \rangle_{L^2(D)}$, $\langle \cdot, \cdot \rangle_{H^s(D)}$ as defined e.g. in [Mon03]. Denote $H^{-s}(D)$ the dual space of $H^s(D)$. Further for $k \geq 0$ and $p \in [1, \infty]$ let $W^{k,p}(D)$ be the standard Sobolev space of scalar complex valued functions with norm $\| \cdot \|_{W^{k,p}(D)}$ the definition of which can also be found in [Mon03]. Let ∂D be the boundary of D. For Lipschitz domains D with finite boundary and $s \in [0, 1]$ let $L^2(\partial D)$ and $H^s(\partial D)$ be the respective standard boundary spaces with hermitian scalar products $\langle \cdot, \cdot \rangle_{L^2(\partial D)}$ and $\langle \cdot, \cdot \rangle_{H^s(\partial D)}$ as defined e.g. in [Mon03]. Further let $H^{-s}(\partial D)$ be the dual space of $H^s(\partial D)$. Let $H^1_0(D)$ be the subspace of $H^1(D)$ whose functions have vanishing Dirichlet traces at ∂D. Let $H^1_{\mathrm{loc}}(D)$ be the space of functions whose restrictions are in $H^1(\tilde{D})$ for every Lipschitz domain $\tilde{D}$ which is compact in D. Let $C^\infty(D)$ be the space of infinitely many times differentiable complex valued functions and $C^\infty_0(D)$ be the subspace of $C^\infty(D)$ whose functions have compact support in D. If $D \subset \mathbb{R}$ is an interval (r_1, r_2) we write

$L^2(r_1, r_2) := L^2((r_1, r_2))$, etc.. We denote the (partial) derivative of a function u with respect to a scalar variable x by $\partial_x u$. We denote the gradient, divergence, Laplacian and Jacobian as ∇, div, Δ and D.

2.1 Separation of variables

Let $S_r^2 := \{x \in \mathbb{R}^3 : |x| = r\}$ be the sphere with radius $r > 0$ and $S^2 := S_1^2$ be the unit sphere. Consider the standard parametrization

$$\hat{y}(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^\top \tag{2.1}$$

of S^2 and

$$Q(r, \hat{x}) := r\hat{x}, \quad r > 0, \hat{x} \in S^2. \tag{2.2}$$

For a frequency $\omega \in \mathbb{C} \setminus \{0\}$ we seek solutions $u\colon D \subset \mathbb{R}^3 \to \mathbb{C}$ to the Helmholtz equation

$$-\Delta u - \omega^2 u = 0. \tag{2.3}$$

In polar coordinates it holds

$$0 = -\Delta u \circ Q - \omega^2 u \circ Q = -r^{-2}\partial_r(r^2 \partial_r u \circ Q) - r^{-2}\Delta_{S^2} u \circ Q - \omega^2 u \circ Q. \tag{2.4}$$

Thus it is meaningful to discuss the spherical Laplacian Δ_{S^2}.

2.1.1 Spherical harmonics

With the associated Legendre functions $p_n^{|m|}$ [CK98, Equation (2.27)] it is known [CK98, Section 2.3] that the spherical harmonics

$$Y_n^m \circ \hat{y}(\theta, \phi) := \sqrt{\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} p_n^{|m|}(\cos\theta) e^{im\phi}, \qquad m = -n, \dots, n; \quad n =, 1, \dots \tag{2.5}$$

form a $L^2(S^2)$-orthonormal basis of eigenfunctions to Δ_{S^2} with eigenvalues $-n(n+1)$. It holds further [FGS98, § 5.1] after [Nan08, Page 34]

$$\|u\|^2_{H^s(S^2)} \quad \text{is equivalent to} \quad \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (n(n+1))^s |a_n^m|^2 \quad \text{for} \quad s \in \mathbb{R}, u = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m Y_n^m. \tag{2.6}$$

In addition the functions

$$U_n^m(\hat{x}) := \frac{1}{\sqrt{n(n+1)}} \nabla_{S^2} Y_n^m(\hat{x}), \qquad Z_n^m(\hat{x}) := \hat{x} \times U_n^m(\hat{x}), \quad m = -n, \dots, n; \quad n =, 1, \dots \tag{2.7}$$

form an orthonormal basis of $L_t^2(S^2)$ [CK98, Theorem 6.25].

Lemma 2.1. *Let f be a scalar differentiable function. Let*

$$v^0_{0,Y} \circ Q(r,\hat{x}) := f(r)Y^0_0(\hat{x})\hat{x}, \qquad v^m_{n,Y} \circ Q(r,\hat{x}) := f(r)Y^m_n(\hat{x})\hat{x}, \tag{2.8a}$$
$$v^m_{n,U} \circ Q(r,\hat{x}) := f(r)U^m_n(\hat{x}), \qquad v^m_{n,Z} \circ Q(r,\hat{x}) := f(r)Z^m_n(\hat{x}) \tag{2.8b}$$

for $m = -n,\dots,n$ and $n =,1,\dots$. Then

$$\operatorname{curl} v^0_{Y,0} \circ Q(r,\hat{x}) = 0, \tag{2.9a}$$
$$\operatorname{curl} v^m_{n,Y} \circ Q(r,\hat{x}) = -\frac{\sqrt{n(n+1)}f(r)}{r} Z^m_n(\hat{x}), \tag{2.9b}$$
$$\operatorname{curl} v^m_{n,U} \circ Q(r,\hat{x}) = \frac{\partial_r(rf(r))}{r} Z^m_n(\hat{x}), \tag{2.9c}$$
$$\operatorname{curl} v^m_{n,Z} \circ Q(r,\hat{x}) = -\frac{\sqrt{n(n+1)}f(r)}{r} Y^m_n(\hat{x})\hat{x} - \frac{\partial_r(rf(r))}{r} U^m_n(\hat{x}). \tag{2.9d}$$

for $m = -n,\dots,n$ and $n =,1,\dots$.

Proof. For the following computations we employ $r(x) := |x|$, $\hat{x}(x) := |x|^{-1}x$, $\hat{x} \times \hat{x} = 0$, $\operatorname{curl} x = 0$, $\operatorname{curl} \nabla = 0$ and the product rule

$$\operatorname{curl}(gG) = \nabla g \times G + g \operatorname{curl} G$$

for a scalar function g and a vectorial function G. We compute

$$\begin{aligned}
\operatorname{curl} v^m_{n,Y} &= \nabla(fr^{-1}Y^m_n) \times x + fr^{-1}Y^m_n \operatorname{curl} x \\
&= \partial_r(fr^{-1}Y^m_n)\hat{x} \times r\hat{x} + r^{-1}\nabla_{S^2}(fr^{-1}Y^m_n) \times r\hat{x} \\
&= -fr^{-1}\hat{x} \times \nabla_{S^2} Y^m_n.
\end{aligned}$$

Since Y^0_0 is constant the first two claims follow. We compute further

$$\begin{aligned}
\operatorname{curl} v^m_{n,U} &= \nabla f \times U^m_n + f \operatorname{curl} U^m_n \\
&= \partial_r f \hat{x} \times U^m_n + f \operatorname{curl}\Big(r\big(n(n+1)\big)^{-1/2}\nabla Y^m_n\Big) \\
&= \partial_r f Z^m_n + f\Big(\big(\nabla r\big)\big(n(n+1)\big)^{-1/2}\Big) \times \nabla Y^m_n\Big) + fr\big(n(n+1)\big)^{-1/2} \operatorname{curl} \nabla Y^m_n \\
&= \partial_r f Z^m_n + f\big(n(n+1)\big)^{-1/2}\hat{x} \times \nabla Y^m_n \\
&= \partial_r f Z^m_n + fr^{-1}\big(n(n+1)\big)^{-1/2}\hat{x} \times \nabla_{S^2} Y^m_n \\
&= \partial_r f Z^m_n + fr^{-1} Z^m_n.
\end{aligned}$$

We compute further

$$\begin{aligned}
\operatorname{curl} v^m_{n,Z} &= \nabla f \times Z^m_n + f \operatorname{curl} Z^m_n \\
&= \partial_r f \hat{x} \times Z^m_n + f \operatorname{curl}(\hat{x} \times U^m_n) \\
&= -\partial_r f U^m_n + f\big(n(n+1)\big)^{-1/2} \operatorname{curl}(\hat{x} \times \nabla_{S^2} Y^m_n).
\end{aligned}$$

We compute further

$$\begin{aligned}\operatorname{curl}(\hat{x} \times \nabla_{S^2} Y_n^m) &= -r^{-1}\nabla_{S^2} Y_n^m + r^{-1}(\Delta_{S^2} Y_n^m)\hat{x} \\ &= -r^{-1}\nabla_{S^2} Y_n^m - r^{-1} n(n+1) Y_n^m \hat{x}.\end{aligned}$$

Hence

$$\operatorname{curl} v_{n,Z}^m = -\partial_r f U_n^m - r^{-1} f U_n^m - r^{-1} f \big(n(n+1)\big)^{1/2} Y_n^m \hat{x}.$$

□

2.1.2 Spherical Hankel functions

Due to Subsection 2.1.1 it is meaningful to look for solutions to (2.4) of the form

$$u \circ Q(r, \hat{x}) = f(\omega r) Y_n^m(\hat{x}). \tag{2.10}$$

It follows that f has to satisfy the spherical Bessel equation

$$-r^{-2}\partial_r(r^2 \partial_r f) + n(n+1) r^{-2} f - f = 0, \quad r > 0. \tag{2.11}$$

From the quotient criterion it follows that the functions

$$j_n(z) := \sum_{l=0}^{\infty} z^{n+2l} \frac{(-1)^l 2^{n+l}(n+l)!}{2^l l!(2n+2l+1)!}, \tag{2.12a}$$

$$y_n(z) := -\frac{(2n)!}{2^n n!}\left(z^{-n-1} + \sum_{l=1}^{\infty} z^{2l-n-1} \frac{(-1)^{l+1}(2n-2l)!}{2^l l!(2n-1)!}\right) \tag{2.12b}$$

are well defined and holomorphic for $z \in \mathbb{C} \setminus \mathbb{R}_0^-$. By examining the behavior at zero it can be seen that for each $n = 0, 1, \ldots$ the functions j_n and y_n are linearly independent. By direct computation it can be seen that j_n and y_n solve the spherical Bessel equation. They are called spherical Bessel and spherical Neumann functions of the order n, respectively. The functions

$$h_n^1 := j_n + i y_n, \qquad h_n^2 := j_n - i y_n \tag{2.13}$$

are called spherical Hankel functions of the order n of first and second kind, respectively. Examining the definition of j_0 and y_0 shows that the first spherical Hankel functions are

$$h_0^1(z) = \frac{e^{iz}}{iz}, \quad h_0^2(z) = \frac{e^{-iz}}{-iz}. \tag{2.14}$$

From the definition of j_n, y_n and the Stirling formula $n! = \sqrt{2\pi n}(n/e)^n(1 + o(1))$ follow

$$j_n(z) = O\left(\left(\frac{2n}{ez}\right)^{-n}\right), \quad n \to \infty, \tag{2.15a}$$

$$y_n(z) = O\left(\left(\frac{2n}{ez}\right)^{n}\right), \quad n \to \infty, \tag{2.15b}$$

and thus

$$h_n^{1,2}(z) = O\left(\left(\frac{2n}{ez}\right)^n\right), \quad n \to \infty \tag{2.15c}$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}_0^-$. Direct computations lead for any $f_n = j_n, y_n, h_n^1, h_n^2$ to the formulas

$$f_n(z) = \frac{z}{2n+1}(f_{n+1}(z) + f_{n-1}(z)), \quad n = 1, 2, \dots, \tag{2.16a}$$

$$\partial_z f_n(z) = nz^{-1} f_n(z) - f_{n+1}(z), \quad n = 0, 1, \dots. \tag{2.16b}$$

From (2.16) and (2.14) follows

$$h_n^1(z) = (-i)^n \frac{e^{iz}}{iz}\left(1 + \sum_{m=1}^{n} a_{n,m} z^{-m}\right), \qquad h_n^2(z) = i^n \frac{e^{-iz}}{-iz}\left(1 + \sum_{m=1}^{n} \overline{a}_{n,m} z^{-m}\right) \tag{2.17}$$

for $n = 0, 1, \dots$ with coefficients $a_{n,m} \in \mathbb{C}$, $m = 1, \dots, n$.

Lemma 2.2. *Let $z \in \{z' \in \mathbb{C} \colon \Im(z') > 0\} \cup \mathbb{R}_0^+$ and $0 < z_0 \leq |z|$. Then the following estimates hold*

$$|h_n^1(z)| \leq e^{-\Im(z)\sqrt{1 - z_0^2/|z|^2}} |h_n^1(z_0)|, \quad n = 0, 1, \dots, \tag{2.18a}$$

$$|\partial_z h_n^1(z)| \leq \frac{n+2}{|z|} |h_n^1(z)|, \quad n = 0, 1, \dots. \tag{2.18b}$$

Proof. The first claim (2.18a) follows from Lemma 2.2 of [CL05] and the relation

$$h_n^1(z) = \sqrt{\frac{\pi}{2z}} H_{n+1/2}^1(z)$$

between cylindrical and spherical Hankel functions. Actually, the cited lemma is formulated only for $z \in \{z' \in \mathbb{C} \colon \Re(z'), \Im(z') > 0\}$. However its proof works for $z \in \{z' \in \mathbb{C} \colon \Im(z') > 0\} \cup \mathbb{R}_0^+$ as well. Equation (2.18) of [CL05] holds for any positive index greater equal one as the proof [CL05, Page 651] does so. Again, from the relation between cylindrical and spherical Hankel functions it follows $|h_{n-1}^1(r)| \leq |h_n^1(r)|$ for all $r > 0$, $n \geq 1$. The second claim (2.18b) follows now from (2.16). □

2.2 Resonance problem

Let $B_r \subset \mathbb{R}^3$ be the open ball with radius $r > 0$ centered at the origin, $B_r(x_0) \subset \mathbb{R}^3$ be the open ball with radius $r > 0$ centered at x_0 and $A_{r_1,r_2} \subset \mathbb{R}^3$ be the open annulus $B_{r_2} \setminus \overline{B_{r_1}}$ with radii $r_2 > r_1 > 0$:

$$B_r := \{x \in \mathbb{R}^3 \colon |x| < r\}, \tag{2.19a}$$

$$B_r(x_0) := \{x \in \mathbb{R}^3 \colon |x - x_0| < r\}, \tag{2.19b}$$

$$A_{r_1,r_2} := \{x \in \mathbb{R}^3 \colon r_1 < |x| < r_2\}. \tag{2.19c}$$

For a Lipschitz domain $D \subset \mathbb{R}^3$ let

$$\tilde{H}^1_{\mathrm{loc}}(D) := \{u \in H^1_{\mathrm{loc}}(D)\colon u|_{D\cap B_r} \in H^1(D \cap B_r) \text{ for all } r > 0 \text{ with } D \cap B_r \neq \emptyset\}. \quad (2.20)$$

For a Lipschitz domain $D \subset \mathbb{R}^3$ with finite boundary ∂D and $u \in H^1_{\mathrm{loc}}(D)$ the trace $u|_{\partial D} \in H^{1/2}(\partial D)$ is well defined. Hence let

$$\tilde{H}^1_{0,\mathrm{loc}}(D) := \{u \in \tilde{H}^1_{\mathrm{loc}}(D)\colon u|_{\partial D} = 0\}. \quad (2.21)$$

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain so that the complement Ω^c is compact and non-empty. We seek non-trivial solutions (ω, u) to

$$-\Delta u - \omega^2 u = 0 \quad \text{in } \Omega, \quad (2.22a)$$
$$u = 0 \quad \text{at } \partial\Omega, \quad (2.22b)$$

together with the abstract radiation condition (which will be specified in Definition 2.5)

$$u \text{ is outgoing} \quad (2.22c)$$

in the distributional sense. That is (ω, u) solves

$$\text{find } (\omega, u) \in \mathbb{C} \setminus \{0\} \times \tilde{H}^1_{0,\mathrm{loc}}(\Omega) \setminus \{0\} \text{ such that}$$
$$\langle \nabla u, \nabla u' \rangle_{L^2(\Omega)} - \omega^2 \langle u, u' \rangle_{L^2(\Omega)} = 0 \quad \text{for all } u' \in C^\infty_{00}(\Omega), \quad (2.23a)$$
$$u \text{ is outgoing.} \quad (2.23b)$$

We note that all our forthcoming theory extends to

$$-\operatorname{div} \varsigma \nabla u - \omega^2 \varrho u = 0 \quad \text{in } \Omega, \quad (2.24a)$$
$$u = 0 \quad \text{at } (\partial\Omega)_D, \quad (2.24b)$$
$$\partial_n u = 0 \quad \text{at } (\partial\Omega)_N, \quad (2.24c)$$

whereby ς is a real symmetric, measurable matrix function which is uniformly bounded, uniformly bounded from below and equal to the identity times a positive constant outside some bounded domain, ϱ is a real scalar measurable uniformly bounded function which is equal to a positive constant outside a bounded domain, ∂_n denotes the normal derivative and $(\partial\Omega)_D$, $(\partial\Omega)_N$ is a decomposition of $\partial\Omega$. We stick to (2.22) to ensure a convenient presentation.

We present the next two lemmata to motivate Radiation Condition 2.5. Lemma 2.3 is of a very general nature. It shows that the expansion in a suitable Fourier series is compatible with the derivative operators. The proof of Lemma 2.3 exploits the structure

$$H^1((r_1, r_2) \times S^2) = \big(H^1(r_1, r_2) \otimes L^2(S^2)\big) \cap \big(L^2(r_1, r_2) \otimes H^1(S^2)\big). \quad (2.25)$$

Lemma 2.4 on the other hand explicitly reports the Fourier coefficients for solutions to the Helmholtz equation.

Lemma 2.3. *Let* $0 < r_1 < r_2$ *and* $u \in H^1(A_{r_1,r_2})$. *Let*

$$f_n^m(r) := \langle u \circ Q(r,\cdot), Y_n^m \rangle_{L^2(S^2)}, \qquad r \in (r_1,r_2);\ n = 0,1,\dots;\ m = -n,\dots,n. \tag{2.26}$$

Then $f_n^m \in H^1(r_1,r_2)$ *for all* $n = 0,1,\dots$ *and* $m = -n,\dots,n$ *and*

$$u \circ Q(r,\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_n^m(r) Y_n^m(\hat{x}) \quad \text{in } L^2(r_1,r_2) \times L^2(S^2), \tag{2.27a}$$

$$\partial_r u \circ Q(r,\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \partial_r f_n^m(r) Y_n^m(\hat{x}) \quad \text{in } L^2(r_1,r_2) \times L^2(S^2), \tag{2.27b}$$

$$\nabla_{S^2} u \circ Q(r,\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_n^m(r) \nabla_{S^2} Y_n^m(\hat{x}) \quad \text{in } L^2(r_1,r_2) \times \left(L^2(S^2)\right)^3. \tag{2.27c}$$

Proof. 1. Step: Due to $u \in L^2(A_{r_1,r_2})$ and the transformation rule it holds $u \circ Q \in L^2(r^2; (r_1,r_2) \times S^2)$. We will repeatedly use the fact that the scalar products of $L^2(r^2; r_1, r_2)$ and $L^2(r_1,r_2)$ are equivalent. Due to Fubini's Theorem there hold the tensor product structure $L^2((r_1,r_2)\times S^2) = L^2(r_1,r_2) \times L^2(S^2)$ and $u \circ Q(r,\cdot) \in L^2(S^2)$ for almost all $r \in (r_1,r_2)$. Since Y_n^m with $n = 0,1,\dots$ and $m = -n,\dots,n$ form an orthonormal basis of $L^2(S^2)$ it follows with standard Hilbert space theory that $u \circ Q(r,\cdot) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_n^m(r) Y_n^m(\cdot)$ in $L^2(S^2)$ almost everywhere in (r_1,r_2). (2.27a) follows from Fubini's Theorem.

2. Step: Since $u \in H^1(A_{r_1,r_2})$ it holds by means of the chain and transformation rule $\partial_r u \circ Q = \hat{x} \cdot \nabla u \circ Q \in L^2(A_{r_1,r_2})$ and hence $\langle \partial_r u \circ Q(r,\cdot), Y_n^m \rangle_{L^2(S^2)} \in L^2(r_1,r_2)$. Again by Fubini's Theorem and standard Hilbert space theory we can expand

$$\partial_r u \circ Q(r,\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \langle \partial_r u \circ Q(r,\cdot), Y_n^m \rangle_{L^2(S^2)} Y_n^m(\hat{x}) \quad \text{in } L^2(r_1,r_2) \times L^2(S^2).$$

For $g \in C_{00}^{\infty}(r_1,r_2)$ it follows with Fubini's Theorem and integration by parts

$$\begin{aligned}
\langle f_n^m, \partial_r g \rangle_{L^2(r_1,r_2)} &= \langle \langle u \circ Q, Y_n^m \rangle_{L^2(S^2)}, \partial_r g \rangle_{L^2(r_1,r_2)} \\
&= \langle \langle u \circ Q, \partial_r g \rangle_{L^2(r_1,r_2)}, Y_n^m \rangle_{L^2(S^2)} \\
&= -\langle \langle \partial_r u \circ Q, g \rangle_{L^2(r_1,r_2)}, Y_n^m \rangle_{L^2(S^2)} \\
&= -\langle \langle \partial_r u \circ Q, Y_n^m \rangle_{L^2(S^2)}, g \rangle_{L^2(r_1,r_2)}.
\end{aligned}$$

Hence $f_n^m \in H^1(r_1,r_2)$ and $\partial_r f_n^m = \langle \partial_r u \circ Q, Y_n^m \rangle_{L^2(S^2)}$, i.e. (2.27b) holds.

3. Step: Since $u \in H^1(A_{r_1,r_2})$ it holds by means of the chain and transformation rule $\nabla_{S^2} u \circ Q = r^{-1}(\mathrm{I} - \mathrm{P}_{\hat{\mathrm{x}}}) \nabla u \circ Q \in L^2(r_1,r_2) \times \left(L^2(S^2)\right)^3$ and hence by Fubini's Theorem $\nabla_{S^2} u \circ Q(r,\cdot) \in \left(L^2(S^2)\right)^3$ almost every where in $r \in (r_1,r_2)$. Since $\hat{x} \cdot (\mathrm{I} - \mathrm{P}_{\hat{\mathrm{x}}}) = 0$ it holds even $\nabla_{S^2} u \circ Q(r,\cdot) \in L_t^2(S^2)$ and thus we can expand

$$\nabla_{S^2} u \circ Q(r,\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \langle \nabla_{S^2} u \circ Q(r,\cdot), U_n^m \rangle_{L^2(S^2)} U_n^m(\hat{x}) + \langle \nabla_{S^2} u \circ Q(r,\cdot), Z_n^m \rangle_{L^2(S^2)} Z_n^m(\hat{x})$$

in $L^2_t(S^2)$. Since $\langle \nabla_{S^2} u \circ Q(r,\cdot), Z_n^m\rangle_{L^2(S^2)} = -\langle u\circ Q(r,\cdot), \operatorname{div}_{S^2} Z_n^m\rangle_{L^2(S^2)}$, we compute further

$$\begin{aligned}\langle \nabla_{S^2} u \circ Q(r,\cdot), U_n^m\rangle_{L^2(S^2)} &= \langle \nabla_{S^2} u \circ Q(r,\cdot), (n(n+1))^{-1/2}\nabla_{S^2} Y_n^m\rangle_{L^2(S^2)}\\ &= \langle u\circ Q(r,\cdot), -\Delta_{S^2}(n(n+1))^{-1/2} Y_n^m\rangle_{L^2(S^2)}\\ &= \langle u\circ Q(r,\cdot), (n(n+1))^{1/2} Y_n^m\rangle_{L^2(S^2)}\\ &= (n(n+1))^{1/2} f_n^m(r),\end{aligned}$$

i.e. (2.27c) holds. □

Lemma 2.4. *Let $0<r_1<r_2$ and $(\omega,u)\in\mathbb{C}\setminus\{0\}\times H^1(A_{r_1,r_2})$ solve*

$$\langle \nabla u, \nabla u'\rangle_{L^2(A_{r_1,r_2})} - \omega^2\langle u,u'\rangle_{L^2(A_{r_1,r_2})} = 0 \quad \textit{for all} \quad u'\in C_0^\infty(A_{r_1,r_2}). \tag{2.28}$$

Let f_n^m be as in Lemma 2.3. Then

$$f_n^m(r) = a_n^m h_n^1(\omega r) + b_n^m h_n^2(\omega r) \tag{2.29}$$

with coefficients $a_n^m, b_n^m\in\mathbb{C}$ for all $n=0,1,\ldots$ and $m=-n,\ldots,n$.

Proof. 1. Step: We apply the coordinate transformation Q to the integrals in (2.28), apply Lemma 2.3, test with $u'\circ Q(r,\hat{x}) = g(r)Y_n^m(\hat{x})$ whereby $g\in C_0^\infty(r_1,r_2)$, to obtain that f_n^m solves

$$-\partial_r(r^2\partial_r f_n^m(r)) - n(n+1)f_n^m(r) + \omega^2 r^2 f_n^m(r) = 0, \quad r\in(r_1,r_2).$$

2. Step: It follows $f_n^m\in H^2(r_1,r_2)$. Due to the continuous embedding $H^1(r_1,r_2)\hookrightarrow C(r_1,r_2)$, $\partial_r\partial_r f_n^m$ is even continuous. It follows that f_n^m is twice continuously differentiable.
3. Step: f_n^m is a classical solution to a second order ordinary linear differential equation. The solution space of this equation is two dimensional and $h_n^1(\omega\cdot)$, $h_n^2(\omega\cdot)$ are linearly independent solutions. Hence the claim follows. □

Mathematically speaking we need to equip (2.23a) with a condition describing the behavior of u at infinity to obtain a well posed problem. We recall that for a solution u to $-\Delta u - \omega^2 u = 0$ a time-harmonic acoustic wave with frequency $\omega>0$ in a material with constant unit velocity is described by the pressure $p(x,t) = \Re(u(x)e^{-i\omega t})$ and the velocity $v(x,t) = \frac{1}{\omega}\nabla p$. Hence for a source at the origin, the average energy flux through the sphere S_r^2 with $r>0$ which is given by

$$J(r) := \frac{2\pi}{\omega}\int_0^{\frac{\omega}{2\pi}}\int_{S_r^2} \nu\cdot pv\,\mathrm{d}\hat{x}\mathrm{d}t \tag{2.30}$$

should be positive. A computation shows that $J(r)>0$ for $u\circ Q(r,\hat{x}) = h_n^1(\omega r)Y_n^m(\hat{x})$ and $J(r)<0$ for $u\circ Q(r,\hat{x}) = h_n^2(\omega r)Y_n^m(\hat{x})$ for any n,m. This motivates Definition 2.5 for real values ω. It is canonical to extend this condition in a continuous way to complex values of ω.

Definition 2.5 (Radiation condition). *Let $(\omega, u) \in \mathbb{C} \setminus \{0\} \times \tilde{H}^1_{0,\mathrm{loc}}(\Omega) \setminus \{0\}$ be a solution to* (2.23a)*. We call u to be outgoing if it admits a representation*

$$u \circ Q(r, \hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n^1(\omega r) Y_n^m(\hat{x}) \tag{2.31}$$

in $L^2(A_{r_1,r_2})$ for all $0 < r_1 < r_2$ with $\Omega^c \subset B_{r_1}$.

Remark 2.6. *It is known [Lig65] that the energy flux of waves is closely related to their group velocity and outgoing waves can be characterized by a positive group velocity. In our case group velocity and phase velocity coincide and hence outgoing waves have a positive phase velocity. However, for other kinds of partial differential equations and geometries waves with opposite signs of group and phase velocity can exist, see e.g. [SAC07], [BBCL14], [HN15a], [HHNS16] for guided elastic waves and [Cum04], [BJKV15], [BJV18] for dispersive materials. Hence for such problems a radiation condition which demands positive phase velocity of waves is physically not meaningful.*

Remark 2.7. *The limiting absorption principle [RT15] is another notion to derive radiation conditions rather than with energy flux arguments is. This principle introduces an artificial absorption, i.e. ω is replaced by $\omega + i\epsilon$ with $\epsilon > 0$, and attains well-posedness of scattering problems. Thereafter the existence and properties of the limit of those solutions $u_0 = \lim_{\epsilon \to 0} u_\epsilon$ is investigated in suitable function spaces. This limit function u_0 is then defined to be the outgoing solution of the original problem.*

We close this section with a standard result on the absence of resonances with non-negative imaginary part.

Theorem 2.8. *There exist no solutions (ω, u) to* (2.23) *with $\Im(\omega) = 0$.*

Proof. Let (ω, u) be a solution to (2.23) with $\Im(\omega) = 0$ and $r > 0$ so that $\Omega^c \subset B_r$. It follows

$$\langle \nabla u, \nabla u \rangle_{L^2(\Omega \cap B_r)} - \omega^2 \langle u, u \rangle_{L^2(\Omega \cap B_r)} - \langle \nu \cdot \nabla u, u \rangle_{H^{-1/2}(\partial B_r) \times H^{1/2}(\partial B_r)} = 0.$$

Since $u \circ Q(r, \hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n^1(\omega r) Y_n^m(\hat{x})$ in $B_{r_0}^c$ with $\Omega^c \subset B_{r_0}$ it follows

$$\langle \nu \cdot \nabla u, u \rangle_{H^{-1/2}(\partial B_r) \times H^{1/2}(\partial B_r)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a_n^m h_n^1(\omega r)|^2 z_n(\omega r)/r$$

with $z_n(r) := r \frac{\partial_r h_n^1(r)}{h_n^1(r)}$. The functions z_n have been analyzed in [Néd01, Theorem 2.6.1] where it is shown that $\Im(z_n) > 0$. (Actually [Néd01, Theorem 2.6.1] only states $\Im(z_n) \geq 0$. As noted in [MS11] the strict inequality follows from the positivity of the function q_l in [Néd01, Equation (2.6.34)].) Since

$$\begin{aligned} 0 &= \Im\big(\langle \nabla u, \nabla u \rangle_{L^2(\Omega \cap B_r)} - \omega^2 \langle u, u \rangle_{L^2(\Omega \cap B_r)} - \langle \nu \cdot \nabla u, u \rangle_{H^{-1/2}(\partial B_r) \times H^{1/2}(\partial B_r)}\big) \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a_n^m h_n^1(\omega r)|^2 \Im(z_n(\omega r))/r \end{aligned}$$

it follows $a_n^m = 0$ for all $n = 0, 1, \ldots$ and $m = -n, \ldots, n$. Hence $u = 0$ in B_r^c. By the unique continuation principle [LRL12] it follows $u = 0$ in Ω. □

Theorem 2.9. *There exist no solutions* (ω, u) *to* (2.23) *with* $\Im(\omega) > 0$.

Proof. Let (ω, u) be a solution to (2.23) with $\Im(\omega) > 0$. From Radiation Condition 2.5 and Lemma 2.2 it follows that $u \in H_0^1(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ it follows that u solves

$$\langle \nabla u, \nabla u' \rangle_{L^2(\Omega)} - \omega^2 \langle u, u' \rangle_{L^2(\Omega)} = 0$$

for all $u' \in H_0^1(\Omega)$. Since the sesquilinear form $\langle \nabla \cdot, \nabla \cdot \rangle_{L^2(\Omega)} - \omega^2 \langle \cdot, \cdot \rangle_{L^2(\Omega)}$ is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$ it follows $u = 0$. □

2.3 Complex scaling

We will define a complex change of the radial coordinate $\tilde{r}(r) = (1 + i\tilde{\alpha}(r))r$ in terms of a profile function $\tilde{\alpha}$. We make assumptions on this profile function as follows.

Assumption 2.10. *Let* $r_1^* > 0$ *be such that* Ω^c *is contained in the ball* $B_{r_1^*}$ *and* $\tilde{\alpha}\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be such that*

1. $\tilde{\alpha}(r) = 0$ *for* $r \le r_1^*$,
2. $\tilde{\alpha}$ *is continuous,*
3. $\tilde{\alpha}(r) > 0$ *for* $r > r_1^*$,
4. $\tilde{\alpha}$ *is non-decreasing,*
5. $\tilde{\alpha}$ *is twice continuously differentiable in* $(r_1^*, +\infty)$ *with continuous extensions of* $\tilde{\alpha}$, $\partial_r \tilde{\alpha}$, $\partial_r \partial_r \tilde{\alpha}$ *to* $[r_1^*, +\infty)$.

We will comment in detail on the reasons for the particular items of Assumption 2.10 in Remark 2.17.

Assumption 2.10 is very general. Later on in Chapter 4 we will require an additional Assumption 4.1 for the analysis therein. In particular Assumptions 2.10 and 4.1 are satisfied by profiles of the following kinds. The probably simplest complex scaling is

$$\tilde{r}(r) = r + i\alpha_0 (r - r_1^*), \quad r \ge r_1^* \tag{2.32a}$$

with a constant $\alpha_0 > 0$. It corresponds to

$$\tilde{\alpha}_{\text{affin}}(r) := \alpha_0 (1 - r_1^*/r), \quad r \ge r_1^*. \tag{2.32b}$$

A popular choice of complex scalings are power functions

$$\tilde{r}(r) = r + i\alpha_0 (r - r_1^*)^m, \quad r \ge r_1^* \tag{2.33a}$$

with a constant $\alpha_0 > 0$ and $m \in \mathbb{N}$. They correspond to

$$\tilde{\alpha}_{\text{power}}(r) := \alpha_0 (r - r_1^*)^m / r, \quad r \ge r_1^* \tag{2.33b}$$

with a constant $\alpha_0 > 0$. A profile which is more or less motivated by the aim to simplify analysis is

$$\begin{aligned}&\tilde{\alpha}_{\text{smooth}} \text{ non-decreasing and twice continuous differentiable in } \mathbb{R}^+,\\&\tilde{\alpha}_{\text{smooth}}(r) := 0 \quad \text{for} \quad r \leq r_1^* \quad \text{and} \quad \tilde{\alpha}_{\text{smooth}}(r) := \alpha_0 \quad \text{for} \quad r \geq r_2,\end{aligned} \tag{2.34}$$

with constants $\alpha_0 > 0$, $r_2 > r_1^*$. In particular, many authors (e.g. [LS98], [HSZ03b], [BP07], [KP09]) only consider profiles of the last kind for their analysis. An infinitely many times differentiable example of Kind (2.34) is

$$\tilde{\alpha}_\infty(r) := \alpha_0 \chi_2(r - r_1^*) \tag{2.35}$$

with constant $\alpha_0 > 0$, $r_2 = r_1^* + 1$ and

$$\chi_1(r) := \begin{cases} 0, & \text{for } r \leq 0,\\ \exp(-1/r), & \text{for } r > 0,\end{cases} \tag{2.36a}$$

$$\chi_2(r) := \begin{cases} 0, & \text{for } r \leq 0,\\ \frac{\chi_1(r)}{\chi_1(r)+\chi_1(1-r)}, & \text{for } 0 < r < 1,\\ 1, & \text{for } r \geq 1,\end{cases} \tag{2.36b}$$

$$\chi_3(r) := \begin{cases} 0, & \text{for } r \leq -1,\\ \chi_2\big(2(r+1)\big), & \text{for } -1 < r < -1/2,\\ 1, & \text{for } -1/2 \leq r \leq 1/2,\\ \chi_2\big(2(1-r)\big), & \text{for } 1/2 < r < 1,\\ 0, & \text{for } r \geq 1.\end{cases} \tag{2.36c}$$

It can easily be checked that χ_1, χ_2, χ_3 and $\tilde{\alpha}_\infty$ are infinitely many times differentiable and additionally $\partial_r \tilde{\alpha}_\infty \geq 0$.

In the following we introduce additional functions which will all depend on $\tilde{\alpha}$. These auxiliary functions will be necessary to formulate the forthcoming theory. We adopt the notation of Bramble and Pasciak [BP07]. Hence let

$$\tilde{d}(r) := 1 + i\tilde{\alpha}(r), \tag{2.37a}$$

$$\tilde{r}(r) := \tilde{d}(r)r, \tag{2.37b}$$

$$\alpha(r) := r\partial_r\tilde{\alpha}(r) + \tilde{\alpha}(r), \tag{2.37c}$$

$$d(r) := 1 + i\alpha(r), \tag{2.37d}$$

$$d_0 := \lim_{r\to+\infty}(\tilde{d}(r)/|\tilde{d}(r)|), \tag{2.37e}$$

$$\mathrm{P_x}(x) := |x|^{-2}xx^\top, \quad x \in \mathbb{R}^3, \tag{2.37f}$$

whereby $xx^\top$ denotes the dyadic product. The definitions of α and d have to be understood piece-wise. We note that the limes in (2.37e) exists in $\mathbb{C}$ due to Assumption 2.10. The function d is chosen such that $\partial_r\tilde{r}(r) = d(r)$. For $f = \tilde{\alpha}, \alpha, \tilde{d}, d, \tilde{r}$ we adopt the overloaded notation

$$f(x) := f(|x|), \quad x \in \mathbb{R}^3. \tag{2.37g}$$

Hence we write e.g. $f \circ Q(r, \hat{x}) = f(r)$.

Remark 2.11. *For time-dependent partial differential equations a frequency dependency of the complex scaling is essential to derive a real valued system of equations. For such the common ansatz and notation for the scaling is* $\tilde{r}(r) = r + \frac{1}{-i\omega}\int_0^r \sigma(\rho)\,\mathrm{d}\rho$*, i.e.* $\tilde{\alpha}(r) = \frac{1}{\omega r}\int_0^r \sigma(\rho)\,\mathrm{d}\rho$*, with a suitable function* σ*.*

For scattering problems the complex scaling is usually chosen frequency dependent too of the form $\tilde{r}(r) = r(1 + \frac{i}{\omega}\tilde{\alpha}(r))$ *to achieve an effective damping of solutions independent of the frequency.*

For resonance problems the complex scaling is commonly kept independent of the frequency to preserve the linear *structure of the resonance problem. This allows to apply a linear eigenvalue solver for the matrix eigenvalue problem which is obtained by the domain truncation and the discretization. Nevertheless, at our suggestion a frequency dependent complex scaling for resonance problems was studied in [NW18]. The extensive numerical experiments therein show an improved preasymptotic behavior of approximate eigenvalues for such scalings.*

2.3.1 The complex scaled eigenvalue problem

Consider a solution (ω, u) to (2.23). Formally we can define $\tilde{u} \circ Q(r, \hat{x}) := u \circ Q(\tilde{r}(r), \hat{x})$. Due to Assumption 2.10 and Lemma 2.2 we expect that $\tilde{u}$ is exponentially decreasing with respect to $|x|$. By means of the chain rule we can formally deduce that $(\omega, \tilde{u})$ solves $-\tilde{\Delta}\tilde{u} - \omega^2\tilde{u} = 0$ whereby

$$\tilde{\Delta} u \circ Q := (\tilde{dr})^{-2} d^{-1} \partial_r (\tilde{d}^2 r^2 d^{-1} \partial_r u \circ Q) + (\tilde{dr})^{-2} \Delta_{S^2} u \circ Q, \tag{2.38}$$

i.e.

$$\tilde{\Delta} u = (\tilde{d}^2 d)^{-1} \operatorname{div}\big((\tilde{d}^2 d^{-1}\,\mathrm{P_x} + d(\mathrm{I} - \mathrm{P_x}))\nabla u\big), \tag{2.39}$$

whereby I denotes the three by three identity matrix. Vice-versa we expect that for a solution $(\omega, \tilde{u})$ to $-\tilde{\Delta}\tilde{u} - \omega^2\tilde{u} = 0$ we can define u in reversal of $\tilde{u}$ and expect that (ω, u) solves $-\Delta u - \omega^2 u = 0$. However, since our coordinate transformation is complex valued we have to take utmost care to perform the above analysis in a mathematically proper way. In the remainder of this subsection we define the variational setting and formulate Eigenvalue Problem (2.43). In Subsection 2.3.2 we prove that for solutions (ω, u) to (2.23) $\tilde{u}(u)$ is well defined and solves (2.43). In Subsection 2.3.3 we prove the reverse, i.e. that for solutions $(\omega, \tilde{u})$ to (2.43) $u(\tilde{u})$ is well defined and solves (2.23).

For a Lipschitz domain $D \subset \Omega$ let

$$X(D) := \{u \in \tilde{H}^1_{0,\mathrm{loc}}(D) \colon \langle u, u\rangle_{X(D)} < \infty\}, \tag{2.40a}$$

$$\langle u, u'\rangle_{X(D)} := \langle (|\tilde{d}^2 d^{-1}|\,\mathrm{P_x} + |d|(\mathrm{I} - \mathrm{P_x}))\nabla u, \nabla u'\rangle_{L^2(D)} + \langle |\tilde{d}^2 d| u, u'\rangle_{L^2(D)}. \tag{2.40b}$$

and

$$a_D(\omega; u, u') := \langle (\tilde{d}^2 d^{-1}\,\mathrm{P_x} + d(\mathrm{I} - \mathrm{P_x}))\nabla u, \nabla u'\rangle_{L^2(D)} - \omega^2 \langle \tilde{d}^2 d u, u'\rangle_{L^2(D)} \tag{2.41}$$

for $\omega \in \mathbb{C}$ and $u, u' \in X(D)$. By definition of $X(D)$ the sesquilinearform $a_D(\omega;\cdot,\cdot)$ is bounded on $X(D) \times X(D)$. For $D = \Omega$ we set

$$X := X(\Omega), \qquad \langle\cdot,\cdot\rangle_X := \langle\cdot,\cdot\rangle_{X(\Omega)}, \qquad a(\cdot;\cdot,\cdot) := a_\Omega(\cdot;\cdot,\cdot). \tag{2.42}$$

Consider the eigenvalue problem to

$$\text{find } (\omega, \tilde{u}) \in \mathbb{C} \times X \setminus \{0\} \text{ such that } \quad a(\omega; \tilde{u}, u') = 0 \quad \text{for all } u' \in X. \tag{2.43}$$

We derive in the next lemmata the relation between (2.23) and (2.43). Note that the introduced space X is of importance only for profile functions with $\tilde{\alpha}, \alpha \notin L^\infty(\mathbb{R}^+)$ whereas X is reduced to the standard Sobolev space $H_0^1(\Omega)$ (equipped with an equivalent inner product) else wise.

2.3.2 The transformation $u \mapsto \tilde{u}(u)$

Lemma 2.12. *Let (ω, u) be a solution to (2.23). Then*

$$u \circ Q(r, \hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n^1(\omega r) Y_n^m(\hat{x}), \tag{2.44}$$

absolutely and uniformly for compact subsets of $(r_1, +\infty) \times S^2$ provided $\Omega^c \subset B_{r_1}$. The same holds for any derivative of u and the sum over the term by term derivatives in (2.44).

Proof. Due to Lemma 2.3, Radiation Condition 2.5 and the continuity of the trace map it follows (2.44) in $L^2(S^2_{r_1})$ for all $r_1 > 0$ so that $\Omega^c \subset B_{r_1}$. We proceed similar to the proof of [CK98, Theorem 2.14]. Let $r_1 > 0$ be so that $\Omega^c \subset B_{r_1}$ and (2.44) in $L^2(S^2_{r_1})$ and $r_1 < r_2 < r_3$. By Parseval's Equality

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a_n^m h_n^1(\omega r_1)|^2 < \infty.$$

First we show that (2.44) holds absolutely in $L^2(S^2_{\hat{r}})$ uniformly in $r_2 \le \hat{r} \le r_3$. From the asymptotic behavior (2.15c) follows the existence of n_0 so that $h_n^1(\omega r_1) \neq 0$ for all $n > n_0$. Let $n_1 > n_0$. Using the orthonormality of Y_n^m and the Schwarz inequality we obtain

$$\begin{aligned}\Big(\sum_{n=n_1}^{\infty} \sum_{m=-n}^{n} \|a_n^m h_n^1(\omega r) Y_n^m(\hat{x})\|^2_{L^2(S^2_{\hat{r}})}\Big)^2 &= \hat{r}^2 \Big(\sum_{n=n_1}^{\infty} \sum_{m=-n}^{n} |a_n^m h_n^1(\omega \hat{r})|^2\Big)^2 \\ &\le \hat{r}^2 \sum_{n=n_1}^{\infty} \sum_{m=-n}^{n} \left|\frac{h_n^1(\omega \hat{r})}{h_n^1(\omega r_1)}\right|^2 \sum_{n=n_1}^{\infty} \sum_{m=-n}^{n} |a_n^m h_n^1(\omega r_1)|^2.\end{aligned}$$

From the asymptotic behavior (2.15c) follows the existence of a constant $C(r_1, r_2, r_3) > 0$ so that

$$\sum_{n=n_1}^{\infty} \sum_{m=-n}^{n} \left|\frac{h_n^1(\omega \hat{r})}{h_n^1(\omega r_1)}\right|^2 \le C \sum_{n=n_1} (2n+1) \left(\frac{r_1}{\hat{r}}\right)^{2n}$$

which yields (2.44) absolutely in $L^2(S^2_{\hat{r}})$ uniformly in $r_2 \leq \hat{r} \leq r_3$. From the asymptotic behavior (2.15c) follows $h^1_{n+1}(z)/h^1_n(z) = O(n)$ uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}^-_0$. By using (2.16b) and reusing the previously used arguments the claim follows any derivative with respect to r. For derivatives with respect to $\hat{x}$ we apply (2.6)

$$\sum_{n=n_1}^{\infty}\sum_{m=-n}^{n} \|a^m_n h^1_n(\omega r) Y^m_n(\hat{x})\|^2_{H^s(S^2_{\hat{r}})} \leq C_s \hat{r}^2 \sum_{n=n_1}^{\infty}\sum_{m=-n}^{n} (n(n+1))^s |a^m_n h^1_n(\omega\hat{r})|^2$$

with a constant C_s only depending on $s > 0$. The previous used arguments can be reused to show that the former sum converges uniformly in $r_2 \leq \hat{r} \leq r_3$. Hence (2.44) holds in $H^s(A_{r_2,r_3}))$ for any $s > 0$. Finally, the claim follows by the Sobolev embedding. □

Lemma 2.13. *For a solution (ω, u) to (2.23) with $\Re(i\omega d_0) < 0$ and the expansion*

$$u \circ Q(r, \hat{x}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^m_n h^1_n(\omega r) Y^m_n(\hat{x}), \quad r \geq r^*_1, \hat{x} \in S^2 \tag{2.45}$$

as in Lemma 2.12 let

$$\tilde{u}(x) := \begin{cases} u(x), & \text{for } x \in \Omega \cap B_{r^*_1}, \\ \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^m_n h^1_n(\omega\tilde{r}(|x|)) Y^m_n(|x|^{-1}x), & \text{for } x \in B^c_{r^*_1}, \end{cases} \tag{2.46}$$

The sum in (2.46) converges absolutely and uniformly on bounded subsets of $B^c_{r^*_1}$. The same holds for derivatives of u with respect to r up to order two and for derivatives with respect to $\hat{x}$ of arbitrary order and the sum over the term by term derivatives. Moreover, for any $\epsilon > 0$ there exists a constant $C(\tilde{u}) > 0$ so that

$$\|\tilde{u}\|^2_{X(B^c_r)} \leq C(\tilde{u}) \int_r^{+\infty} e^{2(\Re(i\omega d_0)+\epsilon)\sqrt{1+\tilde{\alpha}(t)^2}t} \mathrm{d}t \quad \text{for all} \quad r \geq r^*_1. \tag{2.47}$$

In particular it holds $\tilde{u} \in X$. We write $\tilde{u} = \tilde{u}(u)$ to emphasize that $\tilde{u}$ is defined by means of u.

Proof. The uniform and absolute convergence of the sum and the sums over the term by term derivatives follows similarly to the proof of Lemma 2.12. Since $\tilde{u}|_{\Omega\cap B_{r^*_1}} \in H^1(\Omega\cap B_{r^*_1})$, $\tilde{u}|_{B^c_{r^*_1}} \in \tilde{H}^1_{\mathrm{loc}}(B^c_{r^*_1})$ and due to the continuity of $\tilde{d}$, it follows $\tilde{u} \in \tilde{H}^1_{0,\mathrm{loc}}(\Omega)$. From the definition of $\tilde{u}$, Lemma 2.2 and the techniques of the proof of Lemma 2.12 we obtain the estimates

$$\||\tilde{d}^2 d|^{1/2}\tilde{u}\circ Q\|^2_{L^2(S^2_r)} \leq \tilde{C}(\tilde{u}) r^2 |\tilde{d}(r)^2 d(r)| e^{2\Re(i\omega\tilde{d}(r)/|\tilde{d}(r)|)\sqrt{1-r^{*2}_1/|\tilde{r}(r)|^2}\sqrt{1+\tilde{\alpha}(r)^2}r},$$
$$\||d|^{1/2}\nabla_{S^2}\tilde{u}\circ Q\|^2_{L^2(S^2_r)} \leq \tilde{C}(\tilde{u}) r^2 |d(r)| e^{2\Re(i\omega\tilde{d}(r)/|\tilde{d}(r)|)\sqrt{1-r^{*2}_1/|\tilde{r}(r)|^2}\sqrt{1+\tilde{\alpha}(r)^2}r},$$
$$\||\tilde{d}^2/d|^{1/2}\partial_r\tilde{u}\circ Q\|^2_{L^2(S^2_r)} \leq \tilde{C}(\tilde{u}) r^2 |\tilde{d}(r)^2/d(r)| e^{2\Re(i\omega\tilde{d}(r)/|\tilde{d}(r)|)\sqrt{1-r^{*2}_1/|\tilde{r}(r)|^2}\sqrt{1+\tilde{\alpha}(r)^2}r}$$

with a constant $\tilde{C}(\tilde{u}) > 0$ for all $r \geq r^*_1$. We note that

$$\lim_{r\to+\infty} \Re(i\omega\tilde{d}(r)/|\tilde{d}(r)|)\sqrt{1 - r^{*2}_1/|\tilde{r}(r)|^2} = \Re(i\omega d_0).$$

The former estimates, the former limit, the domain transformation $\rho(r) := \tilde{\alpha}(r)r$ on $(r_1^*, +\infty)$ and the basic estimate $\sup_{x>0}(1+x^l)e^{-\delta x} < +\infty$ for any $l, \delta > 0$ and Assumption 2.10 allow to derive Estimate (2.47). In particular it follows $\|\tilde{u}\|_X < \infty$ and hence $\tilde{u} \in X$. ☐

Lemma 2.14. *For a solution (ω, u) to (2.23) with $\Re(i\omega d_0) < 0$ the pair $(\omega, \tilde{u}(u))$ with $\tilde{u}(u)$ as in Lemma 2.13 solves (2.43).*

Proof. Due to the chain rule it holds

$$-\frac{1}{\tilde{r}^2}\frac{1}{d}\partial_r\left(\tilde{r}^2\frac{1}{d}\partial_r f\right) - \frac{1}{\tilde{r}^2}\Delta_{S^2} f - \omega^2 f = 0$$

for any $f \circ Q(r, \hat{x}) = h_n^1(\omega\tilde{r}(r))Y_n^m(\hat{x})$. Due to Lemma 2.13 we can exchange sums and differentials and obtain

$$-\tilde{\Delta}\tilde{u} = -(\tilde{d}^2 d)^{-1} \operatorname{div}\left((\tilde{d}^2 d^{-1}\, \mathrm{P_x} + d(\mathrm{I} - \mathrm{P_x}))\nabla\tilde{u}\right) = 0$$

in $B_{r_1^*}^c$. Let $u' \in C_0^\infty(\Omega)$. We multiply $-\Delta\tilde{u} = 0$ with $\overline{u'}$, integrate over $\Omega \cap B_{r_1^*}$ and integrate by parts to obtain

$$a_{\Omega\cap B_{r_1^*}}(\tilde{u}, u') = -\langle \nu \cdot \nabla\tilde{u}, u'\rangle_{H^{-1/2}(\partial B_{r_1^*})\times H^{1/2}(\partial B_{r_1^*})}.$$

We multiply $-\tilde{\Delta}\tilde{u} = 0$ with $\tilde{d}^2 d\overline{u'}$, integrate over $B_{r_1^*}^c$ and integrate by parts to obtain

$$a_{B_{r_1^*}^c}(\tilde{u}, u') = -\langle \nu \cdot \tilde{d}^2 d^{-1}\nabla\tilde{u}, u'\rangle_{H^{-1/2}(\partial B_{r_1^*}^c)\times H^{1/2}(\partial B_{r_1^*}^c)}.$$

The boundary terms cancel each other out due to $d^{-1}\partial_r\tilde{u} \circ Q = \partial_r u \circ Q$ on $r = r_1^*$ and the continuity of $\tilde{d}$. Hence $a(\omega; u, u') = 0$. Since $C_0^\infty(\Omega)$ is dense in X, the claim follows. ☐

2.3.3 The transformation $\tilde{u} \mapsto u(\tilde{u})$

Lemma 2.15. *Let $(\omega, \tilde{u})$ be a solution to (2.43) with $\Re(i\omega d_0) < 0$. Then*

$$\tilde{u} \circ Q(r, \hat{x}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n h_n^1(\omega\tilde{r}(r))Y_n^m(\hat{x}), \quad r \geq r_1^*, \hat{x} \in S^2 \tag{2.48}$$

and the function

$$u(x) := \begin{cases} \tilde{u}(x), & \text{for } x \in \Omega \cap B_{r_1^*}, \\ \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n h_n^1(\omega|x|)Y_n^m(|x|^{-1}x), & \text{for } x \in B_{r_1^*}^c, \end{cases} \tag{2.49}$$

is well defined in $\tilde{H}^1_{0,\mathrm{loc}}(\Omega)$ and (ω, u) solves (2.23). We write $u = u(\tilde{u})$ to emphasize that u is defined by means of $\tilde{u}$.

Proof. Follows similarly to the proofs of Lemmata 2.3, 2.12, 2.13 and 2.14. We note that in the previous direction Radiation Condition 2.5 ensured that the coefficients b_n in (2.26) vanish. For the reverse direction, this condition is ensured by the integrability of $\tilde{u} \in X$ and f_n^m respectively. ☐

We collect the previous Lemmata in the next theorem.

Theorem 2.16. *Let (ω, u) be a solution to (2.23) with $\Re(i\omega d_0) < 0$. Then $(\omega, \tilde{u}(u))$ with $\tilde{u}(u)$ given by (2.46) solves (2.43) and Decay Estimate 2.47 holds. Vice-versa if $(\omega, \tilde{u})$ is a solution to (2.43) with $\Re(i\omega d_0) < 0$, then $(\omega, u(\tilde{u}))$ with $u(\tilde{u})$ given by (2.49) solves (2.23).*

Proof. Follows from Lemmata 2.12, 2.13, 2.14, 2.15. □

Remark 2.17. *We comment in the following on the reasons for the particular items of Assumption 2.10. We consider for solutions u to (2.23) their transformations $\tilde{u}(u)$ defined by Lemma 2.13. Item 1 of Assumption 2.10 ensures that $\tilde{u}$ coincides with u in $\Omega \cap B_{r_1^*}$. Item 2 and Item 5 of Assumption 2.10 ensure that $\tilde{u}$ is in $\tilde{H}^1_{\mathrm{loc}}(\Omega)$. In particular, Item 5 of Assumption 2.10 ensures that $\tilde{u}$ is twice continuously differentiable in $B^c_{r_1^*}$ and it is necessary for the proofs of Lemma 2.14 and Lemma 2.15. To be precise, we only require $\tilde{\alpha}$ to be twice continuously differentiable in $(r_1^*, +\infty)$ in this chapter. The continuous extensions of $\partial_r\tilde{\alpha}$ and $\partial_r\tilde{\alpha}$ will be necessary in Chapter 4. Item 4 of Assumption 2.10 ensures that the limit in (2.37e) exists and will be necessary for the proof of Lemma 4.2. Items 4 and 3 of Assumption 2.10 ensure an exponential decay of $\|\tilde{u}\|^2_{L^2(S_r^2)}$ and yield that $\tilde{u}$ is contained in the appropriate space X. In particular, they ensure a convenient decay and error bound, see (2.47).*

2.4 Domain truncation

For a solution $(\omega, \tilde{u})$ to (2.43) with $\Re(i\omega d_0) < 0$ it follows from Theorem 2.16 that $\tilde{u}$ decays exponentially to zero as $x \to \infty$. Thus it seems natural to approximate (2.43) by replacing the domain Ω with a bounded subdomain Ω_n and pose a homogeneous Dirichlet or Neumann boundary condition at the artificial boundary $\partial\Omega_n \setminus \partial\Omega$. As most authors we stick to Dirichlet boundary conditions. The resulting equation can then be discretized with a standard numerical scheme such as finite element methods. The question arises if and also how fast the solutions to this approximation converge to the solutions of the original Equation (2.43). It is a classical approach to separate the analysis into a truncation analysis and a discretization analysis whereby the latter can be performed similarly to classical problems posed on bounded domains. In the following we will introduce a new notion to perform the truncation analysis. To this end, we make our Assumptions on Ω_n more precise.

Assumption 2.18. *The sequence of subdomains $(\Omega_n)_{n\in\mathbb{N}}$ is such that for each $n \in \mathbb{N}$*

1. *Ω_n is a bounded Lipschitz domain,*
2. *$\Omega_n \subset \Omega$,*
3. *$\partial\Omega \subset \partial\Omega_n$,*
4. *$\partial\Omega_n \setminus \partial\Omega$ splits Ω into two connected parts,*

and for any $R > 0$ exists an index $n_0 \in \mathbb{N}$ such that $(\Omega \cap B_R) \subset \Omega_n$ for all $n > n_0$.

The PML approximation to (2.43) reads

$$\text{find } (\omega, u_n) \in \mathbb{C} \times X(\Omega_n) \setminus \{0\} \text{ so that } \quad a_{\Omega_n}(\omega; u_n, u_n') = 0 \quad \text{for all } u_n' \in X(\Omega_n). \tag{2.50}$$

We note that $\|\cdot\|_{X(\Omega_n)}$ is an equivalent norm to $\|\cdot\|_{H^1(\Omega_n)}$ and hence $X(\Omega_n) = H^1_0(\Omega_n)$. Let

$$X_n := \{u \in X : u = 0 \text{ in } \Omega \setminus \Omega_n\} \tag{2.51}$$

and consider the problem to

$$\text{find } (\omega, u_n) \in \mathbb{C} \times X_n \setminus \{0\} \text{ so that } \quad a(\omega; u_n, u_n') = 0 \quad \text{for all } u_n' \in X_n. \tag{2.52}$$

It is obvious that for every solution (ω, u_n) to (2.50) the extension $\hat{u}_n$ of u_n to $\Omega \setminus \Omega_n$ by zero is in X_n and $(\omega, \hat{u}_n)$ solves (2.52). Vice-versa for every solution (ω, u) to (2.52) $(\omega, u|_{\Omega_n})$ solves (2.50). However as X_n is a subspace of X we recognize (2.52) as conform Galerkin approximation to (2.43), which restores a common setup for numerical analysts. The former notion is an enhancement of [HN15b] where certain finite element spaces are considered directly as subspaces of X. Though as we will see the analysis simplifies if we perform the truncation analysis and the discretization analysis separately. We note that the choice of Dirichlet boundary condition at the artificial boundary is essential to ensure a conform approximation. Indeed an approximation with Neumann boundary condition could be analyzed as non-conform approximation to (2.43). We will not continue further in this direction as the analysis would be more intricate with barely additional gain.

Through our notion we can investigate the truncation error as Galerkin error. A classical way [BO91], [Bof10] to analyze Galerkin approximations to linear eigenvalue problems (such as ours) is to introduce solution operators

$$S : X \to X, \qquad S_n : X_n \to X_n \tag{2.53}$$

defined by

$$a(1; Su, u') = \langle \tilde{d}^2 du, u' \rangle_{L^2(\Omega)} \qquad \text{for all } u' \in X, \tag{2.54a}$$

$$a(1; S_n u_n, u_n') = \langle \tilde{d}^2 du_n, u_n' \rangle_{L^2(\Omega_n)} \qquad \text{for all } u_n' \in X_n. \tag{2.54b}$$

Of course it has to be ensured that S and S_n are well defined continuous operators (for sufficiently large n) through Equation (2.54). The spectra of (2.43) and (2.52) are connected to the spectra of S and S_n respectively by the transformation

$$\omega \mapsto \frac{1}{\omega^2 - 1}. \tag{2.55}$$

If S is a compact operator it can be deduced that S_n converges to S in operator norm which yields spectral convergence [BO91]. However, the essential spectrum of S equals $\{\frac{1}{z^2-1} : z \in \mathbb{C}, \Re(izd_0) = 0\}$ as we will see under an additional Assumption 4.1 on $\tilde{\alpha}$ in Proposition 4.7. Since the spectrum of a compact operator is discrete we deduce that S is not compact. Thus the standard theory [BO91] does not apply.

Differential operators with non-compact resolvent S occur e.g. in electromagnetism where sufficient conditions on the Galerkin spaces to ensure spectral convergence have been obtained e.g. in [CFR00], [Buf05]. The analysis therein is based on [DNR78a], [DNR78b], which state that

$$\|S - S_n\|_n := \sup_{u_n \in X_n \setminus \{0\}} \|(S - S_n)u_n\|_X / \|u_n\|_X \to 0 \quad \text{as} \quad n \to \infty \tag{2.56}$$

is sufficient to ensure spectral convergence. See also the very comprehensive works [AFW06], [AFW10], [CW13] for semi-definite operators. However, in the previous references the essential spectrum consists only of one isolated eigenvalue with infinite dimensional eigenspace whereas in our case it consists of a continuum. Thus the techniques of [DNR78a], [DNR78b], [AFW06], [AFW10], [CW13] cannot be applied for our analysis. Roughly speaking we cannot hope to approximate an operator with a non-discrete essential spectrum by operators with discrete spectrum in a uniform way. All we can hope for is that we obtain locally (with respect to the spectral parameter) converging approximations. Indeed local analysis techniques are the core of the holomorphic Fredholm operator approximation theory [Kar96a], [Kar96b] which is the topic of the forthcoming Chapter 3. We will apply this theory in Chapter 4 to obtain convergence results for PML truncations.

3 Holomorphic Fredholm theory

Holomorphic Fredholm operator theory serves as a suitable mathematical framework for many non-linear eigenvalue problems which dependent holomorphically on the eigenvalue parameter. In this chapter we briefly discuss how Chapter 2 relates to this framework and give a short introduction to holomorphic Fredholm operator theory in Section 3.1. The analysis of approximations for holomorphic Fredholm operator eigenvalue problems has a long history [GJ73], [VK74], [Vai76], [JW78], [Kar96a], [Kar96b] and is usually performed in the framework of discrete approximation schemes [Stu71] and regular approximations of operator functions [Gri73], [AT85]. In this framework a complete convergence analysis and asymptotic error estimates for eigenvalues are given by Karma in [Kar96a], [Kar96b]. If the discrete approximation scheme is chosen as a Galerkin scheme, then the assumptions of [Kar96a], [Kar96b] are reduced to a single non-trivial assumption: The Regular Approximation Property (see Definition 3.13 for the meaning of regularity). If the operator values are of the form "coercive+compact", then the regularity of Galerkin approximations is unconditionally satisfied. However, if the operator values are not of this kind the question of spectrally converging approximations is very delicate. This can already be observed for linear eigenvalue problems, see e.g. [BBG00], [AFW10]. Moreover the regularity condition is rather abstract and does not come with a toolbox to verify it. In Lemma 3.14 of Section 3.2 we establish a new condition on the Galerkin spaces to ensure the regularity of Galerkin approximations so that [Kar96a], [Kar96b] can be applied. This condition is stronger than the classical regularity condition, however it suffices for a wide variety of applications. Moreover in Lemma 3.16 we prove new asymptotic error estimates on eigenspaces for Galerkin approximations (which are not provided by [Kar96a], [Kar96b]). The latter is an improvement of [Ung09, Theorem 4.3.7]. At last we collect our new results in Theorem 3.17.

As preparation for the forthcoming concept of weakly T-coercive operators/operator functions we remind the reader how well-posedness of problems, i.e. Fredholmness of operators in our sense of the term, is usually established. In the case of coercive operators Fredholmness is trivial. The same holds for weakly coercive operators, i.e. A is a compact perturbation of a coercive operator. (If the compact perturbation is involves an embedding (usually from H^1 to L^2), the related estimate is commonly referred to as Gårding inequality.) Else we may construct an isomorphism T so that T^*A is weakly coercive (T^* denotes the adjoint operator of T), which yields the Fredholmness of A. The notion of T-coercivity originates from [BBCZ10], [BBCC12], [Cia12], [BBCC14] and was introduced to analyze differential operators with sign-changing coefficients in the principal part. For an operator A to be (weakly) T-coercive means that T^*A is already (weakly) coercive. However, the operator values will generally not be bijective in eigenvalue problems (precisely at the eigenvalues). Thus the nomenclature of T-coercivity is not meaningful for our purposes and we will rely on the term *weak* T-coercivity. In general, the Galerkin spaces will not be T-invariant and hence one cannot reproduce the above on the approximation level. An invariance condition is indeed not necessary, but can be relaxed. We will explain in which sense the Galerkin spaces have to interact with the operator T to ensure regularity. It will turn out that the existence of bounded linear operators T_n from the Galerkin spaces to themselves so that

$$\lim_{n\to\infty} \|T - T_n\|_n = 0, \tag{3.1}$$

whereby

$$\|T - T_n\|_n := \sup_{u_n \in X_n \setminus \{0\}} \|(T - T_n)u_n\|_X / \|u_n\|_X. \tag{3.2}$$

with Galerkin spaces X_n is sufficient. Note that the discrete norm (3.2) was already employed in [DNR78a], [DNR78b] and implicitly in [HN15b], [Car15], [BBCC18].

The original motivation for this chapter was to provide a framework for the convergence analysis of boundary element discretizations of boundary integral formulations of Maxwell eigenvalue problems [Ung17], see also [WX13] for further computational experiments. Although the Maxwell eigenvalue problem is of a linear nature, its formulation as boundary integral equation becomes non-linear due to the dependency of the fundamental solution on the frequency. We note that to understand the structure of Maxwell equations the clear presentation [Buf05] was most helpful to us. A further possible application of this work is the finite element error analysis for partial differential equations with sign-changing coefficients in the principal part of the differential operator. Therein the sign-change of the coefficient destroys the coercivity structure. Such problems occur if negative materials are coupled with classical ones [BBCZ10], [BBCC12], [BBCC14]. Note that the negative coefficients in these references stem from prefactors of the kind $(1-1/\omega^2)^{-1}$ with ω^2 being the eigenvalue parameter. Hence eigenvalue problems for such configurations are indeed non-linear. Also, the presented analysis suits as framework for [HN15b] wherein a T-operator (therein denoted $\mathcal{S}$) is constructed and analyzed for PML and Hardy space infinite element (HSIE) approximations to cylindrical waveguide problems. Casually, this chapter can serve as an intermediate reference for colleagues in the numerical analysis community who want to apply [Kar96a], [Kar96b] to conform Galerkin schemes, but want to avoid getting in touch with the extensive generality of the concept of discrete approximation schemes and its intricate notation.

In this chapter let X, Y be generic Hilbert spaces and let $L(X,Y)$ be the space of bounded linear operators from X to Y with norm $\|A\|_{L(X,Y)} := \sup_{u\in X\setminus\{0\}} \|Au\|_Y/\|u\|_X$ for $A \in L(X,Y)$. For $X = Y$ we write $L(X) := L(X,X)$. Similarly for a closed subspace X_n of X let $L(X_n)$ be the space of bounded linear operators from X_n to X_n with norm $\|A_n\|_{L(X_n)} := \sup_{u_n\in X_n\setminus\{0\}} \|A_n u_n\|_X/\|u_n\|_X$ for $A_n \in L(X_n)$. In the previous chapter we obtained a variational formulation of Helmholtz resonance problems in terms of a Hilbert space X and a ω-dependent bounded sesquilinearform $a(\omega;\cdot,\cdot)\colon X \times X \to \mathbb{C}$. For the forthcoming analysis it will be more suitable to work with operators instead of sesquilinearforms. Thus for a bounded ω-dependent sesquilinearform $a(\omega;\cdot,\cdot)\colon X \times X \to \mathbb{C}$ we associate with the Riesz representation theorem a ω-dependent operator $A(\omega) \in L(X)$ defined through

$$\langle A(\omega)u, u'\rangle_X = a(\omega; u, u') \qquad \text{for all} \quad u, u' \in X. \tag{3.3}$$

Eigenvalue Problem (2.43) can now be expressed as

$$\text{find} \quad (\omega, u) \in \mathbb{C} \times X \setminus \{0\} \quad \text{so that} \quad A(\omega)u = 0. \tag{3.4}$$

If X is approximated by a closed subspace X_n we can associate a ω-dependent operator $A_n(\omega) \in L(X_n)$ through

$$\langle A_n(\omega)u_n, u_n' \rangle_X = a(\omega; u_n, u_n') \qquad \text{for all} \quad u_n, u_n' \in X_n. \tag{3.5}$$

Similarly (2.52) can now be expressed as

$$\text{find} \quad (\omega, u_n) \in \mathbb{C} \times X_n \setminus \{0\} \quad \text{so that} \quad A_n(\omega)u_n = 0. \tag{3.6}$$

The operators $A(\omega)$ and $A_n(\omega)$ are related through $A_n(\omega) = P_n A(\omega)|_{X_n}$ whereby P_n denotes the orthogonal projection from X to X_n. In this chapter we assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of closed subspaces of X so that P_n converges point-wise to the identity, i.e. $\lim_{n\to\infty} \|u - P_n u\|_X = 0$ for all $u \in X$. In Lemma 3.14/Theorem 3.17 we establish a sufficient condition on the Galerkin spaces X_n to ensure convergence of eigenvalues and eigenspaces of (3.6) to (3.4).

3.1 Holomorphic Fredholm operator functions

We refer to the appendix of [KM99], [GGK90, Chapter XI] and [GL09, Chapter 1] for the theory on Fredholm operators and holomorphic operator functions. For $A \in L(X)$ we denote by $A^* \in L(X)$ the adjoint operator of A defined by

$$\langle u, A^*u' \rangle_X = \langle Au, u' \rangle_X \quad \text{for all } u, u' \in X. \tag{3.7}$$

Definition 3.1. *An operator $A \in L(X)$ is called Fredholm or Fredholm operator if*

1. $\ker A := \{u \in X \colon Au = 0\}$ *is finite dimensional,*

2. $\operatorname{ran} A := \{Au \colon u \in X\}$ *is closed and has a finite codimension (i.e. the dimension of the factor space $X/\operatorname{ran} A$).*

For a Fredholm operator A we define its index as $\operatorname{ind} A := \dim\ker A - \dim(X/\operatorname{ran} A)$.

Every invertible operator is Fredholm with index zero. Moreover if $A, B \in L(X)$ are Fredholm then [GGK90, Theorem 3.2, Theorem 4.2, Theorem 5.1]

1. AB is Fredholm and $\operatorname{ind}(AB) = \operatorname{ind} A + \operatorname{ind} B$,

2. $A + K$ is a Fredholm operator for every compact operator $K \in L(X)$ and $\operatorname{ind} A = \operatorname{ind}(A + K)$,

3. there exists an operator $T \in L(X)$ so that $\operatorname{Id} - TA$, $\operatorname{Id} - AT$ are compact operators.

Next we discuss a criteron for the Fredholmness of operators with the existence of so-called singular sequences. Although the knowledge of the following Theorem 3.3 seems widely spread, we are not aware of an appropriate literature reference. A discussion of Weyl and Zhislin sequences can be found in [HS96, Chapters 6 and 10].

Definition 3.2. *Let X be a Hilbert space and $A \in L(X)$. A sequence $(u_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ is called singular for A if*

1. $\|u_n\|_X = 1$ *for all* $n \in \mathbb{N}$,
2. $\lim_{n\in\mathbb{N}} Au_n = 0$,
3. *there exists no converging subsequence of* $(u_n)_{n\in\mathbb{N}}$.

Theorem 3.3. *Let X be a Hilbert space and $A \in L(X)$. If there exists a singular sequence for A, then A is not Fredholm.*

Proof. Let A be Fredholm and $(u_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a normalized sequence with $\lim_{n\in\mathbb{N}} Au_n = 0$. With [GGK90, Theorem 2.1] it follows $\lim_{n\in\mathbb{N}} \inf_{v\in\ker A} \|u_n - v\|_X = 0$. Since $\ker A$ is finite dimensional we can choose $v_n \in \ker A$ with $\|u_n - v_n\|_X = \inf_{v\in\ker A} \|u_n - v\|_X$. Since $\ker A$ is finite dimensional we can choose a subsequence $(v_{n'(n)})_{n\in\mathbb{N}}$ which converges to some $v \in \ker A$. It follows $\lim_{n\in\mathbb{N}} u_{n'(n)} = v$, i.e. $(u_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ is not singular for A. $\square$

Definition 3.4. *Let $A, T \in L(X)$ and T be bijective. The operator A is called*

1. *coercive, if* $\inf_{u\in X\setminus\{0\}} |\langle Au, u\rangle_X| / \langle u, u\rangle_X > 0$,
2. *weakly coercive, if there exists a compact operator $K \in L(X)$ so that $A+K$ is coercive,*
3. *T-coercive if T^*A is coercive,*
4. *weakly T-coercive if T^*A is weakly coercive.*

We say that a sesquilinear form $a(\cdot,\cdot)$ is (weakly) (T-)coercive, if the associated operator A defined through $\langle A\cdot,\cdot\rangle_X = a(\cdot,\cdot)$ admits the corresponding property. Due to the Lemma of Lax-Milgram every coercive operator is invertible. Also every coercive operator is weakly coercive and every weakly coercive operator is Id-weakly coercive. Every weakly T-coercive operator is Fredholm with index zero. However, for a (weakly) coercive operator A it is true that the Galerkin approximations $A_n = P_n A|_{X_n} \in L(X_n)$ inherit the (weak) coercivity, while for (weakly) T-coercive operators it is in general wrong. In the next section we will establish a criterion on the subspaces X_n to cope with this fact. But before we do that, we discuss an exceptional case in which the weak T-coercivity of $A \in L(X)$ can be inherited by its Galerkin approximation A_n.

We note that if T^*A is weakly coercive, then AT^{-1} is so too. Vice-versa if AT is weakly coercive, then so is $T^{-*}A$. Hence we could alternatively define A to be (weakly) *right* T-coercive, if AT is (weakly) coercive. However, we stick to the former variant because it is more convenient.

Lemma 3.5. *Let X be a Hilbert space and $A \in L(X)$ be weakly T-coercive. Let X_n be T-invariant and T^{-1}-invariant, i.e. $Tu_n, T^{-1}u_n \in X_n$ for all $u_n \in X_n$. Let $T_n := T|_{X_n} \in L(X_n)$. Then A_n is weakly T_n-coercive.*

Proof. $T_n \in L(X_n)$ is bijective by assumption. For $u_n, u_n' \in X_n$ we compute

$$\begin{aligned}\langle T_n^* A_n u_n, u_n'\rangle_{X_n} &= \langle A_n u_n, T_n u_n'\rangle_{X_n}\\ &= \langle A u_n, T u_n'\rangle_X\\ &= \langle T^* A u_n, u_n'\rangle_X.\end{aligned}$$

Hence $T_n^* A_n = P_n T^* A|_{X_n}$. Since $T^* A$ is weakly coercive, so is $P_n T^* A|_{X_n}$. Hence the claim is proven. □

For the remainder of this chapter let $\Lambda \subset \mathbb{C}$ be open and connected, and $A(\cdot)\colon \Lambda \to L(X)$ be an operator function.

Definition 3.6. *An operator function $A(\cdot)$ is called holomorphic, if it is complex differentiable, i.e.*

$$\lim_{\omega\to\omega_0} \frac{1}{\omega_0 - \omega}\big(A(\omega_0) - A(\omega)\big) \tag{3.8}$$

exists in $L(X)$ for all $\omega_0 \in \Lambda$. An operator function $A(\cdot)$ is called Fredholm, if $A(\omega)$ is Fredholm for all $\omega \in \Lambda$.

The spectrum and resolvent set of an operator function $A(\cdot)$ are defined as

$$\rho\big(A(\cdot)\big) := \{\omega \in \Lambda\colon A(\omega) \text{ is invertible}\}, \qquad \sigma\big(A(\cdot)\big) := \Lambda \setminus \rho\big(A(\cdot)\big). \tag{3.9}$$

For an operator function $A(\cdot)$ we denote by $A^*(\omega)$ the operator function defined by $A^*(\omega) := A(\omega)^*$ for all $\omega \in \Lambda$ and by $A^{-1}(\omega)\colon \rho\big(A(\cdot)\big) \to L(X)$ the operator function defined by $A^{-1}(\omega) := A(\omega)^{-1}$ for all $\omega \in \rho\big(A(\cdot)\big)$. Note that for a holomorphic operator function the operator function defined by $\omega \mapsto A^*(\overline{\omega})$ is holomorphic as well. Some basic properties of holomorphic operator functions are the following [GL09, Theorem 1.6.1, Theorem 1.7.1, 1.1.2, Corollary 1.5.3, Theorem 1.8.5].

1. $A(\cdot)$ is holomorphic if and only if the scalar valued function $\langle A(\cdot)u, u'\rangle_X$ is holomorphic for all $u, u' \in X$,
2. $A^{-1}(\cdot)$ is holomorphic on $\rho\big(A(\cdot)\big)$,
3. $A(\cdot)$ is infinitely many times complex differentiable,
4. for $\omega_0 \in \Lambda$ and $r > 0$ so that $\{z \in \mathbb{C}\colon |z - k_0| < r\} \subset \Lambda$ it holds

   $$A(\omega) = \sum_{n=0}^{\infty} (\omega - \omega_0)^n / n! A^{(n)}(\omega_0)$$

   for $\omega \in \{z \in \mathbb{C}\colon |z - \omega_0| < r\}$ whereby the sum converges absolutely and $A^{(n)}$ denotes the nth derivative.

For holomorphic Fredholm operator functions $A(\cdot)$ it further holds [GGK90, Theorem 8.2] that if $A(\omega)$ is bijective for at least one $\omega \in \Lambda$ then $\sigma\big(A(\cdot)\big)$ is discrete, has no accumulation points in Λ and every $\omega \in \sigma\big(A(\cdot)\big)$ is an eigenvalue, i.e. there exists $u \in X$ so that $A(\omega)u = 0$. In this case we call u an eigenelement. An ordered collection of elements $(u_0, u_1, \dots, u_{m-1})$ in X is called a Jordan chain of ω if u_0 is an eigenelement corresponding to ω and if

$$\sum_{j=0}^{l} \frac{1}{j!} A^{(j)}(\omega) u_{l-j} = 0 \quad \text{for } l = 0, 1, \dots, m-1. \tag{3.10}$$

The elements of a Jordan chain are called generalized eigenelements and the closed linear hull of all generalized eigenelements of $A(\cdot)$ at ω is called the generalized eigenspace $G(A(\cdot), \omega)$ for $A(\cdot)$ at ω. For an eigenelement $u \in \ker A(\omega) \setminus \{0\}$ we denote by $\varkappa(A(\cdot), \omega, u)$ the maximal length of a Jordan chain of ω beginning with u and

$$\varkappa(A(\cdot), \omega) := \max_{u \in \ker A(\omega) \setminus \{0\}} \varkappa(A(\cdot), \omega, u). \tag{3.11}$$

The length of a Jordan chain is always finite, see e.g. [KM99, Lemma A.8.3]. Next, we extend the definition of weakly T-coercive operators to operator functions.

Definition 3.7. *Let $T(\cdot)\colon \Lambda \to L(X)$ be an operator function so that $T(\omega)$ is invertible for all $\omega \in \Lambda$. An operator function $A(\cdot)$ is called weakly $T(\cdot)$-coercive, if there exists an operator function $K(\cdot)\colon \Lambda \to L(X)$ so that $T^*(\omega)A(\omega) + K(\omega)$ is coercive and $K(\omega)$ is compact for all $\omega \in \Lambda$.*

We say that a ω-dependent sesquilinear form $a(\omega; \cdot, \cdot)$ is weakly $T(\cdot)$-coercive, if the associated operator function $A(\cdot)$ defined through $\langle A(\omega)\cdot, \cdot\rangle_X = a(\omega; \cdot, \cdot)$ is so.

3.2 T-compatible approximations

We define

$$\|T\|_n := \sup_{x_n \in X_n \setminus \{0\}} \frac{\|T x_n\|_X}{\|x_n\|_X}. \tag{3.12}$$

for an operator $T \in L(X)$ or $T \in L(X_n), n \in \mathbb{N}$ or a sum of such.

Definition 3.8. *Let $T \in L(X)$ and $(T_n)_{n\in\mathbb{N}}$ be a sequence of operators with $T_n \in L(X_n)$. We say that T_n converges to T in discrete norm, if*

$$\lim_{n\to\infty} \|T - T_n\|_n = 0. \tag{3.13}$$

Let us collect some basic facts, which immediately follow from the definition of convergence in discrete norm.

1. Let X_n be $T \in L(X)$ invariant for all $n \in \mathbb{N}$, i.e. $T u_n \in X_n$ for all $u_n \in X_n$ and all $n \in \mathbb{N}$. Let $T_n := P_n T|_{X_n}$. Then $\|T - T_n\|_n = 0$ for all $n \in \mathbb{N}$.

2. Let $K \in L(X)$ be compact and set $K_n := P_n K|_{X_n}$. Then K_n converges to K in discrete norm.

3. For $i = 1, 2$ let $T^i \in L(X)$, $(T^i_n)_{n\in\mathbb{N}}$ be sequences with $T^i_n \in L(X_n)$ so that T^i_n converges to T^i in discrete norm. Then for any $c \in \mathbb{C}$, $T^1_n + cT^2_n$ converges to $T^1 + cT^2$ in discrete norm.

In preparation of our forthcoming analysis we formulate the next lemma.

Lemma 3.9. *Let $T \in L(X) \setminus \{0\}$ and $(T_n)_{n\in\mathbb{N}}$ be a sequence of operators with $T_n \in L(X_n)$ and $\lim_{n\to\infty} \|T - T_n\|_n = 0$. Then there exist a constant $c > 0$ and an index $n_0 \in \mathbb{N}$ so that*

$$\|T_n\|_{L(X_n)}, \|T_n\|^{-1}_{L(X_n)} \leq c \tag{3.14}$$

for all $n > n_0$. If T is bijective, and $T_n, n \in \mathbb{N}$ are Fredholm operators with index zero, then there exist a constant $c > 0$ and an index $n_0 \in \mathbb{N}$ so that T_n is also bijective for all $n > n_0$ and

$$\|(T_n)^{-1}\|_{L(X_n)} \leq c. \tag{3.15}$$

Proof. Let $u_n \in X_n$. With the triangle inequality we deduce

$$\|T_n u_n\|_X \leq \|Tu_n\|_X + \|(T - T_n)u_n\|_X$$

and hence

$$\|T_n\|_{L(X_n)} \leq \|T\|_{L(X)} + \|T - T_n\|_n.$$

Since $\lim_{n\in\mathbb{N}} \|T - T_n\|_n = 0$ the right hand side of the previous inequality is bounded. Similarly, with the inverse triangle inequality we deduce

$$\|T_n u_n\|_X \geq \|Tu_n\|_X - \|(T - T_n)u_n\|_X$$

and hence

$$\|T_n\|_{L(X_n)} \geq \|T\|_n - \|T - T_n\|_n.$$

It hold $\lim_{n\in\mathbb{N}} \|T\|_n \to \|T\|_{L(X)} > 0$ and $\lim_{n\in\mathbb{N}} \|T - T_n\|_n = 0$. Thus let $n_0 > 0$ be so that $|\|T\|_n - \|T\|_{L(X)}| < \|T\|_{L(X)}/3$ and $\|T - T_n\|_n < \|T\|_{L(X)}/3$ for all $n > n_0$. It follows

$$\|T_n\|_{L(X_n)} \geq \|T\|_{L(X)}/3 > 0.$$

For the last claim let $n_0 > 0$ be so that $\|T - T_n\|_n < 1/(2\|T^{-1}\|_{L(X)})$ for all $n > n_0$. Again with the inverse triangle inequality and $\inf_{x\in X, \|x\|_X=1} \|Tx\|_X = 1/\|T^{-1}\|_{L(X)} > 0$ it follows

$$\begin{aligned}\inf_{x_n\in X_n, \|x_n\|_X=1} \|T_n x_n\|_X &\geq \inf_{x\in X, \|x\|_X=1} \|Tx\|_X - \|T - T_n\|_n \\ &\geq 1/(2\|T^{-1}\|_{L(X)})\end{aligned}$$

for all $n > n_0$. We deduce that T_n is injective. Since T_n is Fredholm with index zero its bijectivity follows. The norm estimate holds due to $\inf_{x_n\in X_n, \|x_n\|_X=1} \|T_n x_n\|_X = 1/\|T_n^{-1}\|_{L(X_n)}$. □

We define in the following what we mean by $T(\cdot)$-compatible approximations of weakly $T(\cdot)$-coercive operator functions.

Definition 3.10. *Let $A(\cdot)$ be a weakly $T(\cdot)$-coercive operator function. Then we call the sequence of Galerkin approximations*

$$(A_n(\cdot) := P_n A(\cdot)|_{X_n} \colon \Lambda \to L(X_n))_{n\in\mathbb{N}} \tag{3.16}$$

a $T(\cdot)$-compatible approximation of $A(\cdot)$, if $(A_n(\cdot))_{n\in\mathbb{N}}$ is a sequence of index zero Fredholm operator functions and there exists a sequence of index zero Fredholm operator functions $(T_n(\cdot))_{n\in\mathbb{N}}$ with $T_n(\cdot)\colon \Lambda \to L(X_n)$ for each $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \|T(\omega) - T_n(\omega)\|_n = 0 \tag{3.17}$$

for all $\omega \in \Lambda$.

Lemma 3.11. *Let $A(\cdot)$ be a weakly $T(\cdot)$-coercive operator function with $K(\cdot)$ as in Definition 3.7. Let $(A_n(\cdot))_{n\in\mathbb{N}}$ be a $T(\cdot)$-compatible approximation of $A(\cdot)$ and let $\tilde{K}(\omega) := (T(\omega)^*)^{-1}K(\omega)$ for $\omega \in \Lambda$. Then for every $\omega \in \Lambda$ there exist $n_0 \in \mathbb{N}$ and $c > 0$, so that $A_n(\omega) + P_n\tilde{K}(\omega)|_{X_n}$ is invertible and*

$$\|\big(A_n(\omega) + P_n\tilde{K}(\omega)|_{X_n}\big)^{-1}\|_{L(X_n)} \le c \tag{3.18}$$

for all $n > n_0$.

Proof. Let n be large enough so that $T_n(\omega)$ is bijective (see Lemma 3.9). We compute

$$\begin{aligned}
&\inf_{u_n\in X_n\setminus\{0\}} \sup_{u_n'\in X_n\setminus\{0\}} \frac{|\langle (A(\omega)+\tilde{K}(\omega))u_n, u_n'\rangle_X|}{\|u_n\|_X\|u_n'\|_X} \\
&\qquad \ge \inf_{u_n\in X_n\setminus\{0\}} \sup_{u_n'\in X_n\setminus\{0\}} \frac{|\langle (A(\omega)+\tilde{K}(\omega))u_n, T_n(\omega)u_n'\rangle_X|}{\|T_n(\omega)\|_{L(X_n)}\|u_n\|_X\|u_n'\|_X} \\
&\qquad \ge \inf_{u_n\in X_n\setminus\{0\}} \sup_{u_n'\in X_n\setminus\{0\}} \frac{|\langle ((A(\omega)+\tilde{K}(\omega))u_n, T(\omega)u_n'\rangle_X|}{\|T_n(\omega)\|_{L(X_n)}\|u_n\|_X\|u_n'\|_X} \\
&\qquad\quad - \frac{\|A(\omega)+\tilde{K}(\omega)\|_{L(X)}}{\|T_n(\omega)\|_{L(X_n)}}\|T(\omega)-T_n(\omega)\|_n \\
&\qquad = \inf_{u_n\in X_n\setminus\{0\}} \sup_{u_n'\in X_n\setminus\{0\}} \frac{|\langle T(\omega)^*(A(\omega)+\tilde{K}(\omega))u_n, u_n'\rangle_X|}{\|T_n(\omega)\|_{L(X_n)}\|u_n\|_X\|u_n'\|_X} \\
&\qquad\quad - \frac{\|A(\omega)+\tilde{K}(\omega)\|_{L(X)}}{\|T_n(\omega)\|_{L(X_n)}}\|T(\omega)-T_n(\omega)\|_n \\
&\qquad = \inf_{u_n\in X_n\setminus\{0\}} \sup_{u_n'\in X_n\setminus\{0\}} \frac{|\langle (T(\omega)^*A(\omega)+K(\omega))u_n, u_n'\rangle_X|}{\|T_n(\omega)\|_{L(X_n)}\|u_n\|_X\|u_n'\|_X} \\
&\qquad\quad - \frac{\|A(\omega)+\tilde{K}(\omega)\|_{L(X)}}{\|T_n(\omega)\|_{L(X_n)}}\|T(\omega)-T_n(\omega)\|_n \\
&\qquad \ge c(\omega)\|T_n(\omega)\|_{L(X_n)}^{-1} - \frac{\|A(\omega)+\tilde{K}(\omega)\|_{L(X)}}{\|T_n(\omega)\|_{L(X_n)}}\|T(\omega)-T_n(\omega)\|_n
\end{aligned}$$

with coercivity constant

$$c(\omega) := \inf_{u \in X\setminus\{0\}} |\langle (T(\omega)^* A(\omega) + K(\omega))u, u\rangle_X| / \|u\|_X^2 > 0.$$

Since $\|T_n(\omega)\|_{L(X_n)}$ is uniformly bounded from above and below (see Lemma 3.9) and $T_n(\omega)$ converges to $T(\omega)$ in discrete norm by assumption, the existence of $n_0 \in \mathbb{N}$ and $\tilde{c} > 0$ follows so that

$$\inf_{u_n \in X_n\setminus\{0\}} \sup_{u_n' \in X_n\setminus\{0\}} \frac{|\langle (A(\omega) + \tilde{K}(\omega))u_n, u_n'\rangle_X|}{\|u_n\|_X \|u_n'\|_X} \geq \tilde{c}$$

for all $n > n_0$. Since $A_n(\omega)$ is Fredholm with index zero and $K(\omega)$ is compact, $\tilde{K}(\omega)$ is compact too and thus $A_n(\omega) + P_n\tilde{K}(\omega)|_{X_n}$ is Fredholm with index zero too. The claim follows now from the Banach-Nečas-Babuška Theorem [EG04, Theorem 2.6]. □

Definition 3.12. *A sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in X$ is said to be compact, if a converging subsubsequence exists for every subsequence exists in turn.*

Definition 3.13. *Let $A \in L(X)$ and $(A_n)_{n\in\mathbb{N}}$ be its Galerkin approximation. The sequence $(A_n)_{n\in\mathbb{N}}$ is called regular if for every bounded sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in X_n, n \in \mathbb{N}$ the compactness of $(A_n u_n)_{n\in\mathbb{N}}$ already implies the compactness of $(u_n)_{n\in\mathbb{N}}$.*

Lemma 3.14. *Let $A(\cdot)$ be a weakly $T(\cdot)$-coercive operator function and $(A_n(\cdot))_{n\in\mathbb{N}}$ be a $T(\cdot)$-compatible approximation. Then for every $\omega \in \Lambda$, $(A_n(\omega))_{n\in\mathbb{N}}$ is regular.*

Proof. Let $\omega \in \Lambda$ be given. W.l.o.g. let $\big(A_n(\omega)u_n\big)_{n\in\mathbb{N}}$ and $u' \in X$ be so that $A_n(\omega)u_n \to u'$ in X for $n \to \infty$. Let $K(\cdot)$ be as in Definition 3.7 and let $\tilde{K}(\omega) := (T(\omega)^*)^{-1}K(\omega)$. Let $\tilde{A}(\omega) := A(\omega) + \tilde{K}(\omega)$ and $\tilde{A}_n(\omega) := P_n\tilde{A}(\omega)|_{X_n}$. Since $\tilde{K}(\omega)$ is compact and $\|u_n\|_X \leq 1$ for all $n \in \mathbb{N}$, there exist a subsequence $(u_{n(m)})_{m\in\mathbb{N}}$ and $u'' \in X$ so that $\tilde{K}(\omega)u_{n(m)} \to u''$ in X as $m \to \infty$. It follows

$$\lim_{m\to\infty} \|\tilde{A}_{n(m)}(\omega)u_{n(m)} - (u' + u'')\|_X = 0.$$

Due to Lemma 3.11 there exist $c > 0$ and $m_0 \in \mathbb{N}$, so that for all $m > m_0$ the operator $\tilde{A}_{n(m)}(\omega)$ is invertible and $\|\tilde{A}_{n(m)}(\omega)^{-1}\|_{L(X_{n(m)})} \leq c$. For $m > m_0$ we compute

$$\begin{aligned}
\|u_{n(m)} - &\tilde{A}(\omega)^{-1}(u' + u'')\|_X \\
&\leq \|u_{n(m)} - P_{n(m)}\tilde{A}(\omega)^{-1}(u' + u'')\|_X + \|(\mathrm{Id} - P_{n(m)})\tilde{A}(\omega)^{-1}(u' + u'')\|_X \\
&\leq c\|\tilde{A}_{n(m)}(\omega)u_{n(m)} - \tilde{A}_{n(m)}(\omega)P_{n(m)}\tilde{A}(\omega)^{-1}(u' + u'')\|_X \\
&\quad + \|(\mathrm{Id} - P_{n(m)})\tilde{A}(\omega)^{-1}(u' + u'')\|_X.
\end{aligned}$$

Since $P_{n(m)}$ converge point-wise to the identity and due to the definition of $\tilde{A}_n(\omega)$, the claim follows. □

Lemma 3.15. *Let $A(\cdot)\colon \Lambda \to L(X)$ be a holomorphic operator function and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed subspaces of X with orthogonal projections P_n onto X_n, so that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity. Then the Galerkin scheme $\big(P_n A(\cdot)|_{X_n}\big)_{n\in\mathbb{N}}$ is a discrete approximation scheme in the sense of [Kar96a].*

Proof. For a Galerkin scheme it holds with the notation of [Kar96a]

$$U = V = X, \qquad X_n = Y_n = X_n, \qquad A_n(\cdot) = P_n A(\cdot) P_n|_{X_n}, \qquad p_n = q_n = P_n.$$

Assumptions a1)-a4) of [Kar96a] follow all from the point-wise convergence of P_n. □

Lemma 3.16. *Let $\Lambda \subset \mathbb{C}$ be open, X be a Hilbert space and $L(X)$ be the space of bounded linear operators from X to X. Let $A(\cdot)\colon \Lambda \to L(X)$ be a holomorphic operator function with non-empty resolvent set and $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed subspaces of X with orthogonal projections P_n onto X_n, so that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity, i.e. $\lim_{n\to\infty} \|u - P_n u\|_X = 0$ for all $u \in X$. Let $A_n(\cdot)\colon \Lambda \to L(X_n)$ be the Galerkin approximation of $A(\cdot)$ defined by $A_n(\omega) := P_n A(\omega)|_{X_n}$ for every $\omega \in \Lambda$. Let the assumptions of [Kar96a, Theorem 2, Theorem 3] and [Kar96b, Theorem 2, Theorem 3] be satisfied. Let $\tilde\Lambda \subset \Lambda$ be a compact set with boundary $\partial\tilde\Lambda \subset \rho\big(A(\cdot)\big)$ and $\tilde\Lambda \cap \sigma\big(A(\cdot)\big) = \{\omega_0\}$. Then there exist $n_0 \in \mathbb{N}$ and $c > 0$ so that for all $n > n_0$*

$$\inf_{u_0 \in \ker A(\omega_0)} \|u_n - u_0\|_X \le c\Big(|\omega_n - \omega_0| + \max_{\substack{u_0' \in \ker A(\omega_0)\\ \|u_0'\|_X \le 1}} \inf_{u_n' \in X_n} \|u_0' - u_n'\|_X\Big) \tag{3.19}$$

for all $\omega_n \in \sigma\big(A_n(\cdot)\big) \cap \tilde\Lambda$ and all $u_n \in \ker A_n(\omega_n)$ with $\|u_n\|_X = 1$.

Proof. We proceed as in [Ung09]: Theorem 4.3.7 of [Ung09] requires a special form of the operator function $A(\cdot)$. However its proof uses this assumption only to apply Lemma 4.2.1 of [Ung09]. Hence we prove [Ung09, Lemma 4.2.1] without the assumption on the special form of $A(\cdot)$. Again the proof of [Ung09, Lemma 4.2.1] does not need the special form of $A(\cdot)$ to establish [Ung09, Equation (4.25)]. We continue the proof of [Ung09, Lemma 4.2.1] at this point.

Let $\{y_n\}$ now be an orthonormal basis of $\big(\operatorname{ran} A(\omega_0)\big)^\perp$ and $\{z_n\}$ be an orthonormal basis of $\ker A(\omega_0)$. Let $K \in L(X)$ be defined as $Ku := \sum_n y_n \langle u, z_n\rangle_X$, $u \in X$ and $\tilde A\colon \Lambda \to L(X)$ be defined as $\tilde A(\omega)u := A(\omega)u + Ku$. Then K is compact and $\tilde A(\cdot)$ is bijective at ω_0. Since $A(\cdot)$, $A_n(\cdot) := P_n A(\cdot) P_n$ fulfill the assumptions of [Kar96a, Theorem 2] so do $\tilde A(\cdot)$, $\tilde A_n(\cdot) := P_n \tilde A(\cdot) P_n$, see e.g. [Kar96a, page 367] for the regularity of $\tilde A_n(\cdot)$. Since the resolvent set of a holomorphic Fredholm operator function is open, [Kar96a, Theorem 2 (3)] yields $\|\tilde A_n(\omega)^{-1}\| \le c$ for a $c > 0$ and all ω in a neighborhood of ω_0. Thus for sufficiently large $n \in \mathbb{N}$ it holds

$$\begin{aligned}
\|x^0 - x_{n_l}\|_X &\le \|(\operatorname{Id} - P_{n_l})x^0\|_X + \|P_{n_l}(x^0 - x_{n_l})\|_X \\
&\le \|(\operatorname{Id} - P_{n_l})x^0\|_X + c\|\tilde A_{n_l}(\omega_{n_l}) P_{n_l}(x^0 - x_{n_l})\|_X \\
&\le \|(\operatorname{Id} - P_{n_l})x^0\|_X + c\|P_{n_l} A(\omega_{n_l}) P_{n_l}(x^0 - x_{n_l})\|_X \\
&\qquad + c\|P_{n_l} K P_{n_l}(x^0 - x_{n_l})\|_X.
\end{aligned}$$

The first term $\|(\operatorname{Id} - P_{n_l})x^0\|_X$ converges to zero due to the point-wise convergence of P_{n_l}. For the second term $c\|P_{n_l} A(\omega_{n_l}) P_{n_l}(x^0 - x_{n_l})\|_X$ it holds

$$\|P_{n_l} A(\omega_{n_l}) P_{n_l}(x^0 - x_{n_l})\|_X \le \|P_{n_l} A(\omega_{n_l}) P_{n_l} x^0\|_X + \|P_{n_l} A(\omega_{n_l}) x_{n_l}\|_X.$$

Since $x^0 \in \ker A(\omega_0)$ and due to the point-wise convergence of P_{n_l} it follows

$$\|P_{n_l} A(\omega_{n_l}) P_{n_l} x^0\|_X \to \quad \text{as } l \to \infty.$$

Further $\|P_{n_l} A(\omega_{n_l}) x_{n_l}\|_X \to 0$ by assumption of [Ung09, Lemma 4.2.1]. The third term $c\|P_{n_l} K P_{n_l}(x^0 - x_{n_l})\|_X$ can be estimated as follows

$$\begin{aligned}\|P_{n_l} K P_{n_l}(x^0 - x_{n_l})\|_X &\le \|K P_{n_l}(x^0 - x_{n_l})\|_X \\ &\le \|K(\operatorname{Id} - P_{n_l}) x^0\|_X + \|K(x^0 - x_{n_l})\|_X.\end{aligned}$$

Due to the point-wise convergence of P_{n_l} it holds $\|K(\operatorname{Id} - P_{n_l}) x^0\|_X \to 0$. Since K is compact $\|K(x^0 - x_{n_l})\|_X \to 0$ follows from the weak convergence of $x_{n_l} \rightharpoonup x^0$. Hence the claim is proven. ☐

Theorem 3.17. *Let $\Lambda \subset \mathbb{C}$ be open and connected, X be a Hilbert space and $L(X)$ be the space of bounded linear operators from X to X. Let $A(\cdot)\colon \Lambda \to L(X)$ be a holomorphic weakly $T(\cdot)$-coercive operator function (see Definition 3.7) with non-empty resolvent set $\rho\big(A(\cdot)\big) \neq \emptyset$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed subspaces of X with orthogonal projections P_n onto X_n, so that $(P_n)_{n\in\mathbb{N}}$ converges point-wise to the identity, i.e. $\lim_{n\to\infty} \|u - P_n u\|_X = 0$ for all $u \in X$. Let $A_n(\cdot)\colon \Lambda \to L(X_n)$ be the Galerkin approximation of $A(\cdot)$ defined by $A_n(\omega) := P_n A(\omega)|_{X_n}$ for every $\omega \in \Lambda$. Assume that $(A_n(\cdot))_{n\in\mathbb{N}}$ is a $T(\cdot)$-compatible approximation of $A(\cdot)$ (see Definition 3.10). Then the following results hold.*

i) For every eigenvalue ω_0 of $A(\cdot)$ exists a sequence $(\omega_n)_{n\in\mathbb{N}}$ converging to ω_0 with ω_n being an eigenvalue of $A_n(\cdot)$ for almost all $n \in \mathbb{N}$.

ii) Let $(\omega_n, u_n)_{n\in\mathbb{N}}$ be a sequence of normalized eigenpairs of $A_n(\cdot)$, i.e. $A_n(\omega_n)u_n = 0$ and $\|u_n\|_X = 1$, so that $\omega_n \to \omega_0 \in \Lambda$, then

a) ω_0 is an eigenvalue of $A(\cdot)$,

b) $(u_n)_{n\in\mathbb{N}}$ is a compact sequence and its cluster points are normalized eigenelements of $A(\omega_0)$.

iii) For every compact $\tilde{\Lambda} \subset \rho(A)$ the sequence $(A_n(\cdot))_{n\in\mathbb{N}}$ is stable on $\tilde{\Lambda}$, i.e. there exist $n_0 \in \mathbb{N}$ and $c > 0$ so that $\|A_n(\omega)^{-1}\|_{L(X_n)} \le c$ for all $n > n_0$ and all $\omega \in \tilde{\Lambda}$.

iv) For every compact $\tilde{\Lambda} \subset \Lambda$ with boundary $\partial\tilde{\Lambda} \subset \rho\big(A(\cdot)\big)$ exists an index $n_0 \in \mathbb{N}$ so that

$$\dim G(A(\cdot), \omega_0) = \sum_{\omega_n \in \sigma(A_n(\cdot)) \cap \tilde{\Lambda}} \dim G(A_n(\cdot), \omega_n). \tag{3.20}$$

for all $n > n_0$, whereby $G(B(\cdot), \omega)$ denotes the generalized eigenspace of an operator function $B(\cdot)$ at $\omega \in \Lambda$.

Let $\tilde{\Lambda} \subset \Lambda$ be a compact set with boundary $\partial\tilde{\Lambda} \subset \rho\big(A(\cdot)\big)$, $\tilde{\Lambda} \cap \sigma\big(A(\cdot)\big) = \{\omega_0\}$ and

$$\begin{aligned}\delta_n &:= \max_{\substack{u_0 \in G(A(\cdot), \omega_0) \\ \|u_0\|_X \le 1}} \inf_{u_n \in X_n} \|u_0 - u_n\|_X, \\ \delta_n^* &:= \max_{\substack{u_0 \in G(A^*(\bar{\cdot}), \omega_0) \\ \|u_0\|_X \le 1}} \inf_{u_n \in X_n} \|u_0 - u_n\|_X,\end{aligned} \tag{3.21}$$

whereby $\overline{\omega_0}$ denotes the complex conjugate of ω_0 and $A^*(\cdot)$ the adjoint operator function of $A(\cdot)$ defined by $A^*(\omega) := A(\omega)^*$ for all $\omega \in \Lambda$. Then there exist $n \in \mathbb{N}$ and $c > 0$ so that for all $n > n_0$

v)

$$|\omega_0 - \omega_n| \leq c(\delta_n \delta_n^*)^{1/\varkappa(A(\cdot),\omega_0)} \tag{3.22}$$

for all $\omega_n \in \sigma\big(A_n(\cdot)\big) \cap \tilde{\Lambda}$, where $\varkappa(A(\cdot),\omega_0)$ denotes the maximal length of a Jordan chain of $A(\cdot)$ at the eigenvalue ω_0,

vi)

$$|\omega_0 - \omega_n^{\text{mean}}| \leq c\delta_n \delta_n^* \tag{3.23}$$

whereby ω_n^{mean} is the weighted mean of all the eigenvalues of $A_n(\cdot)$ in $\tilde{\Lambda}$

$$\omega_n^{\text{mean}} := \sum_{\omega \in \sigma(A_n(\cdot)) \cap \tilde{\Lambda}} \omega \frac{\dim G(A_n(\cdot),\omega)}{\dim G(A(\cdot),\omega_0)}, \tag{3.24}$$

vii)

$$\inf_{u_0 \in \ker A(\omega_0)} \|u_n - u_0\|_X \leq c\Big(|\omega_n - \omega_0| + \max_{\substack{u_0' \in \ker A(\omega_0) \\ \|u_0'\|_X \leq 1}} \inf_{u_n' \in X_n} \|u_0' - u_n'\|_X\Big) \tag{3.25}$$

for all $\omega_n \in \sigma\big(A_n(\cdot)\big) \cap \tilde{\Lambda}$ and all $u_n \in \ker A_n(\omega_n)$ with $\|u_n\|_X = 1$.

Proof. The first three claims follow with [Kar96a, Theorem 2], if we can prove that the required assumptions are satisfied. First of all a Galerkin scheme is a discrete approximation scheme due to Lemma 3.15. The operator function $A(\cdot)$ is holomorphic by assumption. It follows that $A_n(\cdot) := P_n A(\cdot) P_n|_{X_n}$ is also holomorphic. Since $A(\cdot)$ is weakly $T(\cdot)$-coercive, it is Fredholm valued. $A_n(\cdot)$ is Fredholm valued by assumption (see Def. 3.10). Assumption b1 $\rho\big(A(\cdot)\big) \neq \emptyset$ is also an assumption of this theorem. Assumption b2 follows from Lemma 3.11 (at least for sufficiently large n). Assumption b3 follows from $\|A_n(\omega)\|_{L(X_n)} \leq \|A(\omega)\|_{L(X)}$. Assumption b4 follows from the point-wise convergence of the projections P_n. Assumption b5 follows from Lemma 3.14.

The fourth claim follows with [Kar96a, Theorem 3], if we can prove the required assumption (R). We can choose r_n as injection, i.e. $r_n x_n := x_n$. Hence $\|r_n\| = 1$. Since $p_n = P_n$ ii) follows from the point-wise convergence of the projections P_n.

The fifth and sixth claim follow with [Kar96b, Theorem 2, Theorem 3], if we can prove their required assumptions. Assumption a1-a4 are canonically satisfied by Galerkin schemes. We already have proved that Assumptions b1-b5 are satisfied. We can chose $p_n' = p_n = q_n' = q_n = P_n$. For [Kar96b, Theorem 3] we can choose the same r_n as before.

For the proof of the seventh claim we actually do not need the notion of this chapter and only require the assumptions of [Kar96a, Theorem 2, Theorem 3], [Kar96b, Theorem 2, Theorem 3] plus the Galerkin setting. Thus we refer to Lemma 3.16. ∎

4 Continuation of Chapter 2

In this chapter we continue the discussion of Chapter 2 and show how to apply the framework of Chapter 3. In Section 4.1 we introduce an additional Assumption 4.1 on the profile function $\tilde{\alpha}$. In Lemma 4.4 we construct a suitable ω-dependent multiplication operator $T(\omega) \in L(X)$ and prove in Theorem 4.5 that $A(\omega)$ is weakly $T(\omega)$-coercive if $\Re(i\omega d_0) \neq 0$. The construction of $T(\cdot)$ is based on an observation from [BP07]. In Theorem 4.6 we prove that the essential spectrum of $A(\cdot)$ equals $\{z \in \mathbb{C} \colon \Re(izd_0) = 0)\}$.

In Section 4.2 we continue the discussion of Section 2.4. In particular we prove in Lemma 4.8 that the Galerkin spaces (2.51) are $T(\omega)$-invariant. Hence in Theorem 4.9 we conclude that Theorem 3.17 applies. In Lemma 4.10 we prove that the best approximation error $\inf_{u_n \in X_n} \|u - u_n\|_X$ can be estimated by a constant times $\|u\|_{X(B^c_{\tilde{r}_n})}$ which itself can be estimated as in (2.47). Our convergence results extend [KP09] in the sense that we allow profile functions $\tilde{\alpha}$ of very general form and obtain convergence of eigenspaces as well as convergence rates.

In Section 4.3 we discuss the subsequent approximation of truncated PML approximations, which requires no extravagant ingredients.

In Section 4.4 we perform a direct approximation analysis in preparation for Section 4.5. The main difficulty hereby is that we consider Galerkin spaces which are not necessarily $T(\omega)$-invariant. We overcome this obstacle by means of the discrete commutator property [Ber99]. In Section 4.5 we consider reformulations of the resonance problem which involve only a bounded domain but unbounded coefficients. We consider the "exact PML" method of Bermudez et. al. [BHNPR08] and a newly introduced similar method. We perform an approximation analysis for both.

In Section 4.6 we show that our techniques can easily be adapted to variations of (2.22) such as radial symmetric potentials whose PML approximations for scattering problems were studied in [HSZ03a], [HSZ03b].

Since we have the framework of Chapter 3 at hand, we continue the discussion of Chapter 2 for Helmholtz Resonance Problem (2.43) and its Truncated PML Approximation (2.52) at this point. Thus let Ω, $\tilde{\alpha}$, α, $\tilde{d}$, d, $\tilde{r}$, a, X, Ω_n, X_n be as in Chapter 2 and let $\tilde{\alpha}$ suffice Assumption 2.10. As in Chapter 2 we adopt the overloaded notation $f(x) := f(|x|), x \in \Omega$ for $f = \tilde{\alpha}, \alpha, \tilde{d}, d, \tilde{r}$.

4.1 Weak $T(\cdot)$-coercivity and the essential spectrum

The key ingredient for our analysis is that weak $T(\cdot)$-coercivity of $a(\cdot;\cdot,\cdot)$ can be obtained with $T(\omega)$ being simple multiplication operators. For specific profiles of the Kind (2.34) this was already exploited implicitly in [BP07] from wherein the ansatz is taken and extended.

For our forthcoming analysis we additionally require the following assumption.

Assumption 4.1. *Let $\tilde{\alpha}$ and r_1^* be as is Assumption 2.10 and $\tilde{d}$, d be as in* (2.37). *Let*

1. $\lim_{r\to+\infty} \tilde{d}(r)|d(r)|/\big(|\tilde{d}(r)|d(r)\big) = 1$,
2. $\lim_{r\to+\infty} \big(\partial_r(\tilde{d}/|\tilde{d}|)\big)(r) = \lim_{r\to\infty} \big(\partial_r(d/|d|)\big)(r) = 0.$

Assumption 4.1.1 is necessary for Lemma 4.2 which will yield the essential argument to prove the "coercivity part" in Theorem 4.5. Assumption 4.1.2 on the other hand will be necessary to prove the "compactness part" in Theorem 4.5.

It can easily be seen that any $\tilde{\alpha}$ of the Kind (2.32b), (2.33b) and (2.34) suffices Assumption 4.1. In general, any reasonable profile function that comes to our mind suffices Assumption 4.1.

Next we introduce two lemmata which will be essential for our analysis. Let

$$\arg z\colon \mathbb{C}\setminus\{0\} \to [-\pi,\pi), \quad z = |z|\exp(i\arg z). \tag{4.1}$$

Lemma 4.2. *Let Assumptions 2.10 and 4.1 hold. Then there exists $\tau \in (0,\pi/2)$ so that $\arg\big(d(r)/\tilde{d}(r)\big) \in [0,\tau]$ for all $r > r_1^*$.*

Proof. Let $r > r_1^*$. Due $\tilde{\alpha}(r) \geq 0$, the definition of $\tilde{d}, d$ and Assumption 2.10.4 it holds $\arg\tilde{d}(r) \leq \arg d(r)$. Since $\arg\big(d(r)/\tilde{d}(r)\big) = \arg d(r) - \arg\tilde{d}(r)$ it follows $\arg\big(d(r)/\tilde{d}(r)\big) \in [0,\pi/2)$. Due to Assumption 2.10.5 $\tilde{d}/d$ is continuous. Together with Assumption 4.1.1 it follows $\sup_{r>r_1^*}\arg\big(d(r)/\tilde{d}(r)\big) < \pi/2$. Hence the claim is proven. ☐

Lemma 4.3. *Let $\eta_1\colon\Omega\to\mathbb{C}$ be measurable so that $\eta_1|_{\Omega\cap B_n} \in L^\infty(\Omega\cap B_n)$ for all $n\in\mathbb{N}$. Let $Y\subset L^2(\Omega)$ be a Hilbert space so that $\|\eta_1 u\|_{L^2(\Omega)} \leq C\|u\|_Y$ for a constant $C>0$ and all $u\in Y$ and so that the embedding and restriction operator $K_n\colon Y\to L^2(\Omega\cap B_n)\colon u\mapsto u|_{\Omega\cap B_n}$ is compact for each $n\in\mathbb{N}$. Let $\eta_2\in L^\infty(\Omega)$ be so that $\lim_{r\to\infty}\|\eta_2\|_{L^\infty(B_r^c)} = 0$. Then the multiplication and embedding operator $K_{\eta_1\eta_2}\colon Y\to L^2(\Omega)\colon u\mapsto\eta_1\eta_2 u$ is compact.*

Proof. Consider a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n\in Y, \|u_n\|_Y \leq 1$. We construct a Cauchy subsequence as follows. We choose a subsequence $N_1\colon\mathbb{N}\to\mathbb{N}$ so that $(K_1 u_{N_1(n)})_{n\in\mathbb{N}}$ converges. Iteratively for $m\in\mathbb{N}$ we choose subsequences $N_m\colon\mathbb{N}\to\mathbb{N}$ so that $(K_m u_{N_m(n)})_{n\in\mathbb{N}}$ converges. Via diagonalization we construct a subsequence $N(n) := N_n(n)$. Let $\epsilon>0$ and $n_1>0$ be so that $\|\eta_2\|_{L^\infty(B_{n_1}^c)} < \epsilon/(4C)$. Let $n_2>0$ be so that $\|K_{n_1}(u_{N(n)} - u_{N(n')})\|_{L^2(\Omega\cap B_{n_1})} < \epsilon/(2\|\eta_1\eta_2\|_{L^\infty(\Omega\cap B_{n_1})})$ for all $n,n'>n_2$. It follows

$$\|\eta_1\eta_2(u_{N(n)} - u_{N(n')})\|_{L^2(\Omega)} \leq \|\eta_1\eta_2\|_{L^\infty(\Omega)}\|u_{N(n)} - u_{N(n')}\|_{L^2(\Omega\cap B_{n_1})} + 2\|\eta_2\|_{L^\infty(B_{n_1}^c)} < \epsilon.$$

Hence the claim is proven. ☐

Our analysis will further require the following functions.

$$\hat{\alpha}(r) := \begin{cases} \lim_{\rho\to r_1^*+}\alpha(\rho) & \text{for } 0\leq r\leq r_1^*, \\ \alpha(r) & \text{for } r>r_1^*, \end{cases} \tag{4.2a}$$

$$\hat{d}(r) := 1 + i\hat{\alpha}(r), \quad r\geq 0. \tag{4.2b}$$

Again, we adopt the overloaded notation $f(x) := f(|x|), x\in\Omega$ for $f = \hat{\alpha}, \hat{d}$.

Lemma 4.4. *Let Assumptions 2.10 and 4.1 hold. For all $\omega \in \mathbb{C} \setminus \{0\}$ and all $u \in X$ let*

$$T(\omega)u := \begin{cases} \overline{\frac{|\hat{d}|}{\hat{d}}} u & \text{for } \arg(-\omega^2 d_0^2) \in [-\pi, 0), \\ \overline{\frac{\hat{d}}{\tilde{d}^2} \frac{|\tilde{d}|^2}{|\hat{d}|}} u & \text{for } \arg(-\omega^2 d_0) \in [0, \pi). \end{cases} \tag{4.3}$$

Then $T(\omega) \in L(X)$ is bijective for all $\omega \in \mathbb{C} \setminus \{0\}$.

Proof. For any $\eta \in W^{1,\infty}(\Omega)$ and $u \in X$ it holds

$$\begin{aligned} \|\eta u\|_X^2 &= \langle (|\tilde{d}^2 d^{-1}| \,\mathrm{P_x} + |d|(\mathrm{I} - \mathrm{P_x}))(\eta \nabla u + u \nabla \eta), \eta \nabla u + u \nabla \eta \rangle_{L^2(\Omega)} + \langle |\tilde{d}^2 d| \eta u, \eta u \rangle_{L^2(\Omega)} \\ &\le 3 \|\eta\|_{W^{1,\infty}(\Omega)}^2 \|u\|_X^2. \end{aligned}$$

Thus multiplication with η is bounded from $X \to X$. If $|\eta| = 1$ it follows $1/\eta \in W^{1,\infty}(\Omega)$ as well. Hence the inverse of multiplication with η, which is multiplication with $1/\eta$, is bounded from $X \to X$ as well.

Let $\eta = \overline{\frac{|\hat{d}|}{\hat{d}}}$ or $\eta = \overline{\frac{\hat{d}}{\tilde{d}^2} \frac{|\tilde{d}|^2}{|\hat{d}|}}$. It follows $|\eta| = 1$. Due to the definition of $\hat{d}$ (4.2) and Assumption 2.10, η is weakly differentiable. Due to Assumption 2.10.5 and Assumption 4.1.2 it follows $\nabla \eta \in L^\infty(\Omega)$ and hence $\eta \in W^{1,\infty}(\Omega)$. Thus the claim is proven. ☐

Theorem 4.5. *Let Assumptions 2.10 and 4.1 hold. Let $a(\cdot; \cdot, \cdot)$ and X be as in* (2.42)*, $A(\cdot)$ be as in* (3.3)*, $T(\cdot)$ be as in* (4.3)*, d_0 be as in* (2.37e) *and*

$$\Lambda_{d_0} := \{z \in \mathbb{C} \colon \Re(i z d_0) \neq 0\}. \tag{4.4}$$

Then $A(\cdot) \colon \Lambda_{d_0} \to L(X)$ is weakly $T(\cdot)$-coercive.

Proof. We consider the two cases $\arg(-\omega^2 d_0^2) \in (-\pi, 0)$ and $\arg(-\omega^2 d_0^2) \in [0, \pi)$ separately. We split the sesquilinear form $a(\omega; \cdot, T(\omega)\cdot)$ into a coercive part $a_1(\cdot, \cdot)$ and a compact part $a_2(\cdot, \cdot)$.

First case $\omega \in \Lambda_{d_0}$ with $\arg(-\omega^2 d_0^2) \in (-\pi, 0)$: A direct computation yields

$$a(\omega; u, T(\omega)u') = a_1(u, u') + a_2(u, u')$$

with

$$\begin{aligned} a_1(u, u') &:= \left\langle \left(\frac{\tilde{d}^2 |\hat{d}|}{d \hat{d}} \mathrm{P_x} + \frac{d |\hat{d}|}{\hat{d}} (\mathrm{I} - \mathrm{P_x}) \right) \nabla u, \nabla u' \right\rangle_{L^2(\Omega)} - \omega^2 d_0^2 \langle |\tilde{d}^2 d| u, u' \rangle_{L^2(\Omega)}, \\ a_2(u, u') &:= \left\langle \frac{\tilde{d}^2}{d} \partial_r u, u' \partial_r \overline{\left(\frac{|\hat{d}|}{\hat{d}} \right)} \right\rangle_{L^2(\Omega)} - \omega^2 \left\langle \left(\frac{\tilde{d}^2 d |\hat{d}|}{|\tilde{d}^2 d| \hat{d}} - d_0^2 \right) |\tilde{d}^2 d| u, u' \right\rangle_{L^2(\Omega)}. \end{aligned}$$

Recall that $\hat{d}(r) = d(r)$ for $r > r_1^*$ and $\hat{d}(r) = 1 + i \lim_{r \to r_1^* +} \alpha(r)$ for $r \le r_1^*$. Due to Assumptions 2.10.4 and 2.10.5 it holds $\arg \hat{d}(r_1^*) \in [0, \pi/2)$. Let $\tau \in (0, \pi/2)$ be as in Lemma 4.2 and

$$\tau_1 := \min\{-2\tau, -\arg \hat{d}(r_1^*), \arg(-\omega^2 d_0^2)\}.$$

It follows that $\tau_1 \in (-\pi, 0)$ and

$$\arg\left(\frac{\tilde{d}^2|\hat{d}|}{d\hat{d}}\right)(r),\ \arg\left(\frac{d|\hat{d}|}{\hat{d}}\right)(r),\ \arg(-\omega^2 d_0^2) \in [\tau_1, 0]$$

for all $r \geq 0$. Thus $\Re(ie^{-i(\pi+\tau_1)/2} a_1(u,u)) \geq \cos(\tau_1/2)\min\{1, |\omega^2|\}\|u\|_X^2$ for all $u \in X$, i.e. $a_1(\cdot,\cdot)$ is coercive. Further $a_2(u,u') = \langle (K_1^* L_1 - \omega^2 K_2^* L_2)u, u'\rangle_X$ with bounded operators

$$\begin{aligned}
L_1 &: X \to L^2(\Omega) \colon u \mapsto \frac{\tilde{d}}{d^{1/2}} \partial_r u, \\
K_1 &: X \to L^2(\Omega) \colon u \mapsto \left(\overline{\partial_r \left(\frac{|\hat{d}|}{\hat{d}}\right)}\right) \overline{\frac{\tilde{d}}{d^{1/2}}} u, \\
L_2 &: X \to L^2(\Omega) \colon u \mapsto |\tilde{d} d^{1/2}| u, \\
K_2 &: X \to L^2(\Omega) \colon u \mapsto \left(\frac{\tilde{d}^2 d|\hat{d}|}{|\tilde{d}^2 d|\hat{d}} - d_0^2\right) |\tilde{d} d^{1/2}| u.
\end{aligned}$$

From the definitions of d_0 and $\hat{d}$ it follows $\left(\frac{\tilde{d}^2 d|\hat{d}|}{|\tilde{d}^2 d|\hat{d}} - d_0^2\right)(r) \to 0$ as $r \to +\infty$. From Assumption 4.1.2 follows $\left(\overline{\partial_r \left(\frac{|\hat{d}|}{\hat{d}}\right)}\right)(r) \to 0$ as $r \to +\infty$. Lemma 4.3, $1/|d| \leq 1$ and the compact Sobolev embedding $H^1(D) \to L^2(D)$ for bounded Lipschitz domains D yield that K_1 and K_2 are compact. Hence A_2 ($\langle A_2 u, u'\rangle_X = a_2(u,u')$) is compact too.
Second case $\omega \in \Lambda_{d_0}$ with $\arg(-\omega^2 d_0^2) \in [0, \pi)$: A direct computation yields $a(\omega; u, T(\omega)u') = a_1(u,u') + a_2(u,u')$ with

$$\begin{aligned}
a_1(u,u') &:= \left\langle \left(\frac{\hat{d}|\tilde{d}^2|}{d|\hat{d}|} \mathrm{P_x} + \frac{d\hat{d}|\tilde{d}^2|}{\tilde{d}^2|\hat{d}|}(\mathrm{I} - \mathrm{P_x})\right) \nabla u, \nabla u' \right\rangle_{L^2(\Omega)} - \omega^2 d_0^2 \langle |\tilde{d}^2 d| u, u'\rangle_{L^2(\Omega)}, \\
a_2(u,u') &:= \left\langle \frac{\tilde{d}^2}{d} \partial_r u, u' \partial_r \overline{\left(\frac{\hat{d}|\tilde{d}^2|}{\tilde{d}^2|\hat{d}|}\right)} \right\rangle_{L^2(\Omega)} - \omega^2 \left\langle \left(\frac{\hat{d}|\tilde{d}^2|\tilde{d}^2 d}{\tilde{d}^2|\hat{d}||\tilde{d}^2 d|} - d_0^2\right) |\tilde{d}^2 d| u, u' \right\rangle_{L^2(\Omega)}.
\end{aligned}$$

As in the previous case we find that

$$\arg\left(\frac{\hat{d}|\tilde{d}^2|}{d|\hat{d}|}\right)(r),\ \arg\left(\frac{d\hat{d}|\tilde{d}^2|}{\tilde{d}^2|\hat{d}|}\right)(r),\ \arg(-\omega^2 d_0^2) \in [0, \tau_1]$$

for all $r \geq 0$ with $\tau_1 := \max\{2\tau, \arg \hat{d}(r_1^*), \arg(-\omega^2 d_0^2)\} \in [0, \pi)$. It follows

$$\Re(-ie^{i(\pi-\tau_2)/2} a_1(u,u)) \geq \cos(\tau_1/2)\min\{1, |\omega^2|\}\|u\|_X^2$$

for all $u \in X$, i.e. $a_1(\cdot,\cdot)$ is coercive. Further $a_2(u,u') = \langle (K_1^* L_1 - \omega^2 K_2^* L_2)u, u'\rangle_X$ with bounded operators

$$\begin{aligned}
&L_1 \colon X \to L^2(\Omega) \colon u \mapsto \frac{\tilde{d}}{d^{1/2}} \partial_r u, \\
&K_1 \colon X \to L^2(\Omega) \colon u \mapsto \overline{\left(\partial_r \left(\frac{\hat{d}|\tilde{d}^2|}{\tilde{d}^2|\hat{d}|} \right) \right)} \, \overline{\frac{\tilde{d}}{d^{1/2}}} u, \\
&L_2 \colon X \to L^2(\Omega) \colon u \mapsto |\tilde{d} d^{1/2}| u, \\
&K_2 \colon X \to L^2(\Omega) \colon u \mapsto \left(\frac{\hat{d}|\tilde{d}^2|\tilde{d}^2 d}{\tilde{d}^2|\hat{d}||\tilde{d}^2 d|} - d_0^2 \right) |\tilde{d} d^{1/2}| u.
\end{aligned}$$

From the definitions of d_0, $\hat{d}$ and Assumption 4.1.1 follows $\left(\frac{\hat{d}|\tilde{d}^2|\tilde{d}^2 d}{\tilde{d}^2|\hat{d}||\tilde{d}^2 d|} - d_0^2 \right)(r) \to 0$ as $r \to +\infty$. From Assumption 4.1.2 it follows $\overline{\left(\partial_r \left(\frac{\hat{d}|\tilde{d}^2|}{\tilde{d}^2|\hat{d}|} \right) \right)}(r) \to 0$ as $r \to +\infty$. Again, Lemma 4.3, $1/|d| \leq 1$ and the compact Sobolev embedding $H^1(D) \to L^2(D)$ for bounded Lipschitz domains D yield that K_1 and K_2 are compact. Hence A_2 ($\langle A_2 u, u' \rangle_X = a_2(u, u')$) is compact too. $\square$

It is less intuitive why we need to employ the multiplication operator $T(\omega)$. The matrix of the principle part of $a(\omega; \cdot, \cdot)$ is $\tilde{d}^2 d^{-1} \mathrm{P_x} + d(\mathrm{I} - \mathrm{P_x})$. The coefficients are bounded away from zero and only take values in the closed salient sector spanned by $(1 + i\alpha(r))^{\pm 1}$. However, as the domain is unbounded the (asymptotic) complex sign of the mass term $-\omega^2 d_0^3$ also has to be taken into account. Although there is no way to estimate $1 + i\alpha(r)$ in terms of d_0 without further assumptions on $\tilde{\alpha}$. Nonetheless the asymptotic complex sign of the matrix coefficients is d_0. Thus it is meaningful to suitably rotate the complex sign of the principle part especially in the preasymptotic regime of the coefficients. The rotation for the mass term in the preasymptotic regime can be neglected as mass integrals on bounded sets lead to compact operators. In [BP07] it was noted that a rotation by d^{-1} yields the desired properties. Be aware that for different dimensions and different equations other rotations are necessary. A choice leading to coefficients 1 and $\tilde{d}^2/d^2$ or 1 and $d^2/\tilde{d}^2$ (depending on the complex sign of $-\omega^2 d_0^2$) in the principle part of the equation usually does the job.

Theorem 4.6. *Let Assumptions 2.10 and 4.1 hold. Assume additionally that there exists a constant $C_d > 0$ so that*

$$|d| \leq C_d |\tilde{d}| \tag{4.5}$$

for all $r > 0$. Let $a(\cdot; \cdot, \cdot)$ be as in (2.42), $A(\cdot)$ be as in (3.3), Λ_{d_0} be as in (4.4) and $\omega \in \mathbb{C} \setminus \Lambda_{d_0} = \{\omega \in \mathbb{C} \colon \Re(-i\omega d_0) = 0\}$. Then $A(\omega)$ is not Fredholm, i.e. $\mathbb{C} \setminus \Lambda_{d_0}$ is part of the essential spectrum of $A(\cdot)$.

Proof. We construct a singular sequence for $A(\omega)$ (see Definition 3.2) and employ Theorem 3.3.

1. step (definition of u): Consider $\omega \neq 0$ and let

$$u(r) := h_0^1(\omega d_0 |\tilde{d}(r)| r).$$

A computation yields $\partial_r(|\tilde{d}(r)| r) = \check{d}$ with

$$\check{d} := d_0 |\tilde{d}| \left(1 + \frac{\tilde{\alpha} r \partial_r \tilde{\alpha}}{|\tilde{d}^2|}\right).$$

Since h_0^1 solves Bessel equation (2.11) with index $n = 0$

$$\langle (d_0^2 |\tilde{d}^2| \check{d}^{-1} \operatorname{P_x} \nabla u, \nabla u' \rangle_{L^2(\Omega)} - \omega^2 \langle d_0^2 |\tilde{d}^2| \check{d} u, u' \rangle_{L^2(\Omega)} = 0$$

for all $u' \in C_0^\infty(\Omega)$ with $\operatorname{supp} u' \in B_{r_1^*}^c$.

2. step (computation of coefficients): We compute

$$\frac{\tilde{d}^2}{d} - \frac{d_0^2 |\tilde{d}^2|}{\check{d}} = \frac{\tilde{d}^2}{d}\left(1 - \frac{d}{\check{d}} \frac{d_0^2 |\tilde{d}^2|}{\tilde{d}^2}\right)$$

Recall $d_0 := \lim_{r\to+\infty} \tilde{d}/|\tilde{d}|$ and hence $\lim_{r\to+\infty} \frac{d_0^2 |\tilde{d}^2|}{\tilde{d}^2} = 1$. Next we compute $\lim_{r\to+\infty} \frac{d}{\check{d}}$. If $\tilde{\alpha}$ is bounded, Assumption 4.1.1 yields that $\lim_{r\to+\infty} r \partial_r \tilde{\alpha}(r) = 0$ and hence $\lim_{r\to+\infty} \frac{d}{\check{d}} = 1$. If $\tilde{\alpha}$ is unbounded, we compute

$$\frac{d}{\check{d}} = \frac{\tilde{\alpha}}{|\tilde{d}|} \frac{1 + \frac{r\partial_r \tilde{\alpha}}{\tilde{\alpha}} + \frac{1}{i\tilde{\alpha}}}{1 + \frac{\tilde{\alpha}^2}{|\tilde{d}^2|} \frac{r \partial_r \tilde{\alpha}}{\tilde{\alpha}}}.$$

and hence it holds $\lim_{r\to+\infty} \frac{d}{\check{d}} = 1$ too. Thus it holds $\lim_{r\to+\infty} \left(1 - \frac{d}{\check{d}} \frac{d_0^2 |\tilde{d}^2|}{\tilde{d}^2}\right) = 0$. Similarly we compute

$$\tilde{d}^2 d - d_0^2 |\tilde{d}^2| \check{d} = \tilde{d}^2 d \left(1 - \frac{d_0^2 |\tilde{d}^2|}{\tilde{d}^2} \frac{\check{d}}{d}\right)$$

and $\lim_{r\to+\infty} \left(1 - \frac{d_0^2 |\tilde{d}^2|}{\tilde{d}^2} \frac{\check{d}}{d}\right) = 0$.

3. step (definition of $\tilde{u}_n$): For the last part of the proof we proceed as in in the proof of [BBCP18, Theorem 4.1]. Let χ be a smooth cut-off function defined on $\mathbb{R}$ so that $\chi(r) = 0$ for $|r| > 1$ and $\|\chi\|_{L^2(\mathbb{R})} = 1$. Let

$$\chi_n(r) := n^{-1/2} \chi\left(\frac{r - n^2}{n}\right) |\tilde{d}(r)|^{-1/2} \qquad \text{and} \qquad \tilde{u}_n(r) := \chi_n(r) u(r).$$

Due to $h_0^1(z) = e^{iz}/(iz)$ and $\omega \in \mathbb{C} \setminus \Lambda_{d_0}$ we compute

$$\langle |\tilde{d}^2 d| \tilde{u}_n, \tilde{u}_n \rangle_{L^2(\Omega)} \in \left[|\omega|^{-2}, |\omega|^{-2} C_d \right].$$

We note that it would be more natural to replace $\tilde{d}$ by d in the definition of χ_n. However, this way $\partial_r \tilde{u}_n$ would contain second derivatives of $\tilde{\alpha}$ which we could bind only by means of additional assumptions. Instead we impose (4.5) and employ χ_n as previously defined.

4. step (computation of integrals): We compute

$$\partial_r \tilde{u}_n(r) = u\partial_r \chi_n + \chi_n \partial_r u(r) = u(r) f_1(r) + u(r) f_2(r) + \chi_n(r) f_3(r)$$

with

$$\begin{aligned}
f_1(r) &:= n^{-3/2} \partial_r \chi\left(\frac{r-n^2}{n}\right) |\tilde{d}(r)|^{-1/2}, \\
f_2(r) &:= r^{-1} n^{-1/2} \chi\left(\frac{r-n^2}{n}\right) \frac{-1}{2} |\tilde{d}(r)|^{-3/2} \tilde{\alpha}(r) r \partial_r \tilde{\alpha}(r), \\
f_3(r) &:= \omega d_0 \check{d}(r) \partial_r h_0^1(\omega d_0 |\tilde{d}(r)| r).
\end{aligned}$$

Lemma 2.2 and a direct computation show the existence of $C > 0$ independent of $n \in \mathbb{N}$ so that

$$\begin{aligned}
\left\langle |\tilde{d}^2/d| u f_1, u f_1 \right\rangle_{L^2(\Omega)} &\leq C/n^2, \\
\left\langle |\tilde{d}^2/d| u f_2, u f_2 \right\rangle_{L^2(\Omega)} &\leq C/n^2, \\
\left\langle |\tilde{d}^2/d| \chi_n f_3, \chi_n f_3 \right\rangle_{L^2(\Omega)} &\leq C, \\
\left\langle |\tilde{d}^2/d| f_1 f_3, f_1 f_3 \right\rangle_{L^2(\Omega)} &\leq C/n^2, \\
\langle |1/d| f_2 f_3, f_2 f_3 \rangle_{L^2(\Omega)} &\leq C/n^2.
\end{aligned}$$

5. step (final part): It follows that there exists $c > 0$ independent of $n \in \mathbb{N}$ so that $\|\tilde{u}_n\|_X \in [1/c, c]$ for all $n \in \mathbb{N}$. We now employ

$$\nabla(\chi_n u) \cdot \nabla \overline{u'} = \nabla u \cdot \nabla(\overline{\chi_n u'}) + u \nabla \chi_n \cdot \nabla \overline{u'} - (\nabla u \cdot \nabla \chi_n) \overline{u'}$$

to compute

$$
\begin{aligned}
\langle A(\omega)\tilde{u}_n, u'\rangle_X &= \left\langle \frac{\tilde{d}^2}{d} \mathrm{P_x}\, \nabla u, \nabla(\overline{\chi_n} u') \right\rangle_{L^2(\Omega)} - \omega^2 \left\langle |\tilde{d}^2 d| u, \overline{\chi_n} u' \right\rangle_{L^2(\Omega)} \\
&\quad + \left\langle \frac{\tilde{d}^2}{d} \mathrm{P_x}\, u \nabla\chi_n, \nabla u' \right\rangle_{L^2(\Omega)} - \left\langle \frac{\tilde{d}^2}{d} \nabla u \cdot \nabla\chi_n, u' \right\rangle_{L^2(\Omega)} \\
&= \left\langle \frac{\tilde{d}^2}{d} \left(1 - \frac{d}{\check{d}} \frac{d_0^2|\tilde{d}^2|}{\tilde{d}^2}\right) \mathrm{P_x}\, \nabla u, \nabla(\overline{\chi_n} u') \right\rangle_{L^2(\Omega)} \\
&\quad - \omega^2 \left\langle \tilde{d}^2 d \left(1 - \frac{d_0^2|\tilde{d}^2|}{\tilde{d}^2} \frac{\check{d}}{d}\right) u, \overline{\chi_n} u' \right\rangle_{L^2(\Omega)} \\
&\quad + \left\langle \frac{\tilde{d}^2}{d} \mathrm{P_x}\, u \nabla\chi_n, \nabla u' \right\rangle_{L^2(\Omega)} - \left\langle \frac{\tilde{d}^2}{d} \nabla u \cdot \nabla\chi_n, u' \right\rangle_{L^2(\Omega)} \\
&= \left\langle \frac{\tilde{d}^2}{d} \left(1 - \frac{d}{\check{d}} \frac{d_0^2|\tilde{d}^2|}{\tilde{d}^2}\right) \mathrm{P_x}\, \nabla(\chi_n u), \nabla u' \right\rangle_{L^2(\Omega)} \\
&\quad - \omega^2 \left\langle \tilde{d}^2 d \left(1 - \frac{d_0^2|\tilde{d}^2|}{\tilde{d}^2} \frac{\check{d}}{d}\right) \chi_n u, u' \right\rangle_{L^2(\Omega)} \\
&\quad + \left\langle \frac{d_0^2|\tilde{d}^2|}{\check{d}} \mathrm{P_x}\, u \nabla\chi_n, \nabla u' \right\rangle_{L^2(\Omega)} - \left\langle \frac{d_0^2|\tilde{d}^2|}{\check{d}} \nabla u \cdot \nabla\chi_n, u' \right\rangle_{L^2(\Omega)}
\end{aligned}
$$

for $u' \in X$. The existence of $(\epsilon_n)_{n\in\mathbb{N}}$ with $\epsilon_n > 0$ for $n \in \mathbb{N}$ and $\lim_{n\in\mathbb{N}} \epsilon_n = 0$ so that

$$|\langle A(\omega)\tilde{u}_n, u'\rangle_X| \leq \epsilon_n \|u'\|_X$$

follows. Let $u_n := \tilde{u}_n / \|\tilde{u}_n\|_X$. Since the supports of u_n and u_m are disjoint for $n \neq m$ and $\|u_n\|_X = 1$ there exists no converging subsequence of $(u_n)_{n\in\mathbb{N}}$. Altogether we have constructed a singular sequence $(u_n)_{n\in\mathbb{N}}$ for $A(\omega)$ and hence the claim is proven for $\omega \neq 0$. For $\omega = 0$ we can consider $u(r) = 1/(d_0|\tilde{d}(r)|r)$ and repeat the proof. ☐

Proposition 4.7. *Let Assumptions 2.10 and 4.1 and* (4.5) *hold. Let* $a(\cdot;\cdot,\cdot)$ *be as in* (2.42), $A(\cdot)$ *be as in* (3.3) *and* Λ_{d_0} *be as in* (4.4). *Then* $A(\omega)$ *with* $\omega \in \mathbb{C}$ *is Fredholm if and only if* $\omega \in \Lambda_{d_0}$. *For* $\omega \in \Lambda_{d_0}$, $A(\omega)$ *has index zero.* $A(\omega)$ *is bijective for* $\omega \in \mathbb{C} \setminus \{0\}$ *with* $\arg\omega \in [-\pi, -\arg d_0) \cup [0, \pi - \arg d_0)$.

Proof. Follows from Theorems 4.5, 4.6, 2.16, 2.9 and 2.8. ☐

Our approach to compute the essential spectrum of $A(\cdot)\colon \mathbb{C} \to L(X)$ is quite direct. Instead one could also employ the theory of spectral deformation [HS96]. We refrain from this approach for two reasons. On the one hand we consider more general complex scalings than in [HS96] and hence the theory therein does not cover all our considered scalings. On the other hand the theory of [HS96] is quite intricate and we prefer to give a selfcontained presentation.

4.2 Approximation by domain truncation

We consider a sequence of finite subdomains $(\Omega_n)_{n\in\mathbb{N}}$ which suffices Assumption 2.18, corresponding subspaces X_n defined by (2.51) and corresponding operator functions $A_n(\cdot)$ defined by (3.5). We investigate the approximation of $A(\cdot)$ by $A_n(\cdot)$.

Lemma 4.8. *Let Assumptions 2.10, 4.1 and 2.18 hold. Let X_n be as in* (2.51) *and $T(\cdot)$ be as in* (4.3)*. Then X_n is $T(\omega)$-invariant and $T^{-1}(\omega)$-invariant for all $n \in \mathbb{N}$, $\omega \in \mathbb{C} \setminus \{0\}$, i.e. $T(\omega)u_n, T^{-1}(\omega)u_n \in X_n$ for all $u_n \in X_n$, $n \in \mathbb{N}$, $\omega \in \mathbb{C} \setminus \{0\}$.*

Proof. A multiplication operator does not increase the support of a function. □

Theorem 4.9 (Spectral convergence)**.** *Let Assumptions 2.10 and 4.1 hold. Let X and $a(\cdot;\cdot,\cdot)$ be as in* (2.42)*, $A(\cdot)$ be as in* (3.3)*, $T(\cdot)$ be as in* (4.3) *and*

$$\Lambda^{\pm}_{d_0} := \{z \in \mathbb{C} \colon \pm\Re(izd_0) < 0\}. \tag{4.6}$$

Let Assumption 2.18 hold. Let X_n be as in (2.51) *and $A_n(\cdot)$ be as in* (3.5)*.*

Then $A(\cdot)\colon \Lambda^{\pm}_{d_0} \to L(X)$ is a weakly $T(\cdot)$-coercive holomorphic Fredholm operator function with non-empty resolvent set $\rho\big(A(\cdot)\big)$ and $A_n(\cdot)\colon \Lambda^{\pm}_{d_0} \to L(X_n)$ is a $T(\cdot)$-compatible approximation, i.e. Theorem 3.17 applies.

Proof. Since $A(\omega)$ is a polynomial in ω it is holomorphic. Due to Theorems 4.5, 2.16, 2.9 and 2.8 $A(\cdot)\colon \Lambda^{\pm}_{d_0} \to L(X)$ is a weakly $T(\cdot)$-coercive holomorphic Fredholm operator function with non-empty resolvent set $\rho\big(A(\cdot)\big)$. Due to Lemma 4.8 X_n is $T(\omega)$-invariant for all $n \in \mathbb{N}$, $\omega \in \Lambda^{\pm}_{d_0}$. Hence with $T_n(\omega) := T(\omega)|_{X_n}$ it hold $T_n(\omega), T_n^{-1}(\omega) \in L(X_n)$ and $\|T(\omega) - T_n(\omega)\|_n = 0$ for all $n \in \mathbb{N}$, $\omega \in \Lambda^{\pm}_{d_0}$ and $A_n(\omega)$ and $T_n(\omega)$ are Fredholm with index zero for all $\omega \in \Lambda^{\pm}_{d_0}$ due to Lemma 3.5. □

Theorem 4.9 yields via Theorem 3.17 convergence rates with respect to the best approximation errors (3.21). To estimate these we introduce the next Lemma 4.10.

Lemma 4.10. *Let Assumptions 2.10, 4.1 and 2.18 hold. Let X be as in* (2.42) *and X_n be as in* (2.51)*. Let $r_n > 0$ be so that $\Omega \cap B_{r_n+1} \subset \Omega_n$. Then there exists a constant $C > 0$ independent of n so that*

$$\inf_{u_n \in X_n} \|u - u_n\|_X \leq C\|u\|_{X(B^c_{r_n})} \tag{4.7}$$

for all $u \in X$.

Proof. We choose $u_n(x) := \chi_2(1 + r_n - |x|)u(x) \in X_n$ with χ_2 as in (2.36b) and compute

$$\|u - u_n\|_X = \|u - u_n\|_{X(\Omega\setminus B_{r_n})} \leq \|u\|_{X(\Omega\setminus B_{r_n})} + C\|u\|_{X(A_{r_n,r_n+1})}$$

with a constant $C > 0$ independent of u and n. □

Due to Theorem 2.16 $\|u\|_{X(B^c_{r_n})}$ can be estimated to decay exponentially for eigenfunctions u, i.e. $A(\omega)u = 0$. For generalized eigenfunctions (also called root functions) $\|u\|_{X(B^c_{r_n})}$ can also be estimated to decay exponentially due to Lemma 6.1 of [KP09].

For solutions (ω, u) to $A(\omega)u = 0$ the quantity of interest is actually only $(\omega, u|_{\Omega\cap B_{r_1^*}})$ where as $u|_{B^c_{r_1^*}}$ could be called an auxiliary variable. It is indeed possible to improve the error estimate obtained by Theorem 4.9 and Lemma 4.10 for the eigenspaces if the error is only measured in $\|\cdot\|_{X(\Omega\cap B_{r_1^*})}$. A hand waving explanation is that $A_n(\cdot)$ differs from $A(\cdot)$ only by a distortion at $B^c_{r_2^*}$ and as "the error propagates" towards $\Omega \cap B_{r_1^*}$ "the error decays". This argumentation can be made rigorous by a comparison of the Dirichlet-to-Neumann operators generated by the complex scaling in the (un)truncated domains. For details see e.g. [HN18, Section 4.3].

4.3 Approximation by subsequent discretization

In the previous section we considered a sequence of bounded subdomains $(\Omega_n)_{n\in\mathbb{N}}$ and approximation of (2.43) by (2.52). In this section we consider for a fixed index $n \in \mathbb{N}$ a subsequent approximation of (2.52) by the following. We consider a sequence of subspaces $\left(X_n^{h(m)}\right)_{m\in\mathbb{N}}$, $X_n^{h(m)} \subset X_n$, $m \in \mathbb{N}$, so that the orthogonal projections $P_n^{h(m)} \colon X_n \to X_n^{h(m)}$ converge point-wise to the identity in X_n and eigenvalue problem

$$\text{find } (\omega, u_{h(m)}) \in \mathbb{C} \times X_n^{h(m)} \setminus \{0\} \text{ so that } \quad a(\omega; u_{h(m)}, u'_{h(m)}) = 0 \quad \text{for all } u'_{h(m)} \in X_n^{h(m)}. \tag{4.8}$$

We note that restricted to X_n the norm $\|\cdot\|_X$ is equivalent to $\|\cdot\|_{H^1(\Omega)}$ and hence $X_n = \{u \in H^1_0(\Omega) \colon u = 0 \text{ in } \Omega \setminus \Omega_n\}$. It holds further that $A_n(\omega) \in L(X_n)$ is already weakly coercive as Lemma 4.11 shows. Altogether the approximation of (2.52) by (4.8) can already be performed with common techniques [BO91]. Of course our framework can be applied as well, which we formulate in the following.

Lemma 4.11. *Let Assumptions 2.10, 4.1 and 2.18 hold. Let X and $a(\cdot;\cdot,\cdot)$ be as in* (2.42), *$A(\cdot)$ be as in* (3.3), *$n \in \mathbb{N}$, X_n be as in* (2.51) *and $A_n(\cdot)$ be as in* (3.5). *Then $A_n(\omega) \in L(X_n)$ is weakly coercive for all $\omega \in \mathbb{C}$.*

Proof. It holds $A_n(\omega) = B - K$ with

$$\begin{aligned}
\langle Bu_n, u'_n\rangle_{X_n} &= \langle (\tilde{d}^2 d^{-1}\,\mathrm{P_x} + d(\mathrm{I} - \mathrm{P_x}))\nabla u_n, \nabla u'_n\rangle_{L^2(\Omega_n)} + \langle |\tilde{d}^2 d| u_n, u'_n\rangle_{L^2(\Omega_n)},\\
\langle Ku_n, u'_n\rangle_{X_n} &= \langle (\omega^2 \tilde{d}^2 d + |\tilde{d}^2 d|) u_n, u'_n\rangle_{L^2(\Omega_n)}
\end{aligned}$$

for all $u_n, u'_n \in X_n$. It follows that $\Re(\langle Bu_n, u_n\rangle_{X_n}) \geq c\|u_n\|^2_X$ for all $u_n \in X_n$ and $c := \cos(\tau), \tau := \max_{x\in\Omega_n} \arg d(x)$. Due to Assumptions 2.10, 4.1 and 2.18 it holds $\tau \in [0, \pi/2)$ and hence $c > 0$. Further $K = K_1^* L_1 K_1$ with bounded operators

$$\begin{aligned}
K_1 &\colon X_n \to L^2(\Omega_n) \colon u \mapsto u,\\
L_1 &\colon L^2(\Omega_n) \to L^2(\Omega_n) \colon u \mapsto (\omega^2 \tilde{d}^2 d + |\tilde{d}^2 d|)u.
\end{aligned}$$

K_1 is compact due to the compact Sobolev embedding $H^1(D) \to L^2(D)$ for bounded Lipschitz domains D. ∎

Lemma 4.12 (Spectral convergence)**.** *Let Assumptions 2.10, 4.1 and 2.18 hold. Let X and $a(\cdot;\cdot,\cdot)$ be as in (2.42), $A(\cdot)$ be as in (3.3), $n \in \mathbb{N}$, X_n be as in (2.51) and $A_n(\cdot)$ be as in (3.5). Let $\big(X_n^{h(m)}\big)_{m\in\mathbb{N}}$, $X_n^{h(m)} \subset X_n$, $m \in \mathbb{N}$, be a sequence of subspaces of X_n so that the orthogonal projections $P_n^{h(m)}\colon X_n \to X_n^{h(m)}$ converge point-wise to the identity and let $A_n^{h(m)}(\cdot)$ be defined by (3.5).*

Then $A_n(\cdot)\colon \mathbb{C} \to L(X_n)$ is a weakly id*-coercive holomorphic Fredholm operator function with non-empty resolvent set $\rho\big(A_n(\cdot)\big)$ and $A_n^{h(m)}(\cdot)\colon \mathbb{C} \to L(X_n^{h(m)})$ is a* id*-compatible approximation, i.e. Theorem 3.17 applies.*

Proof. $A_n(\cdot)$ is weakly coercive due to Lemma 4.11. Since $A_n(\cdot)$ is polynomial in k, it is holomorphic. $A_n(0)$ is injective and hence $0 \in \rho\big(A_n(\cdot)\big)$. Since $A_n(\cdot)$ is weakly coercive so is $A_n^{h(m)}(\cdot)$. □

The profile function $\tilde{\alpha}$ limits the regularity of solutions. However, to achieve optimal approximations rates of solutions by general finite element spaces smooth solutions are necessary. Yet, if $\tilde{\alpha}$ is piece-wise smooth optimal rates can be restored if the meshes of the finite element spaces are aligned to the jumps in the derivatives of $\tilde{\alpha}$. If this is not possible, e.g. because the finite element code is limited to polytopial meshes, it is desirable to chose a globally smooth profile function. Of course for finite element spaces with fixed maximal polynomial degree one can construct $\tilde{\alpha}$ with appropriate smoothness as piece-wise polynomial. However, in this case it seems more natural to us to construct $\tilde{\alpha} \in C^\infty(\mathbb{R}^+)$ in the first place, e.g. as in (2.35).

4.4 Approximation by simultaneous truncation and discretization

In the previous two sections we considered a sequence of bounded subdomains $(\Omega_n)_{n\in\mathbb{N}}$ as in Assumption 2.18, an approximation of (2.43) by (2.52) and subsequent a sequence of subspaces $\big(X_n^{h(m)}\big)_{m\in\mathbb{N}}$, $X_n^{h(m)} \subset X_n$ and an approximation of (2.52) by (4.8). The two key ingredients which allowed a pretty simple analysis were the $T(\cdot)$-invariance of X_n and the weak coercivity of $A_n(\cdot)$. This way we avoided to discuss the issue of the non-$T(\cdot)$-invariance of $X_n^{h(m)}$ and the construction of an appropriate $T_n^{h(m)}(\cdot)$ operator function.

In this section we consider a direct approximation of (2.43) through non-$T(\cdot)$-invariant subspaces of X, e.g. the diagonal sequence $\big(X_n^{h(n)}\big)_{n\in\mathbb{N}}$. The reason for this is twofold. On the one hand we want to foster that this approach which originates from [HN15b] is indeed a legit one. On the other hand we provide some new ideas and techniques which can serve for more intricate situations as e.g. in Section 4.5.

To conduct our analysis we introduce an operator function $T_\epsilon(\cdot)$ which is a slight modification of (4.3) in Lemmata 4.13 and 4.14. This new operator function has some favorable properties and is so that $A(\cdot)$ is still weakly $T_\epsilon(\cdot)$-coercive. We consider finite dimensional Galerkin spaces $X^{h(m)} \subset X$ which suffice two Assumptions 4.15 and 4.16. In Theorem 4.17 we prove that under such assumptions we can construct appropriate operator functions $T_\epsilon^{h(m)}(\cdot)\colon \mathbb{C} \to L(X^{h(m)})$ which converge to $T_\epsilon(\cdot)$ in discrete norm at each $\omega \in \mathbb{C}$, i.e. the Approximation $\big(A^{h(m)}(\cdot)\colon \Lambda_{d_0} \to L(X^{h(m)})\big)_{m\in\mathbb{N}}$ is $T_\epsilon(\cdot)$-compatible. A key ingredient for the analysis is a variant of the discrete commutator property of Bertoluzza [Ber99]. Finally in Theorem 4.18 we once again state that under the previous assumptions Theorem 3.17 applies.

Lemma 4.13. *Let $r_1 \in \mathbb{R}$ and $r_2 \in \mathbb{R} \cup \{+\infty\}$ with $r_1 < r_2$. Let $\eta\colon [r_1, r_2) \to \mathbb{C}$ be continuous so that $\lim_{r\to r_2-} \eta(r) =: \eta(r_2)$ exists in $\mathbb{C}$. Then for each $\epsilon > 0$ exist $\eta_\epsilon\colon [r_1, r_2) \to \mathbb{C}$ and $\hat{r}_1, \hat{r}_2 \in (r_1, r_2)$ so that*

1. $\|\eta - \eta_\epsilon\|_{L^\infty(r_1,r_2)} < \epsilon$,
2. $\eta_\epsilon(r) = \eta(r_1)$ *for* $r \le \hat{r}_1$,
3. $\eta_\epsilon(r) = \eta(r_2)$ *for* $r \ge \hat{r}_2$,
4. η_ϵ *is infinitely many times differentiable.*

Proof. Since η is continuous and $\lim_{r\to r_2-} \eta(r)$ exists we can choose $\check{r}_1, \check{r}_2 \in (r_1, r_2)$ so that $\|\eta - \eta(r_1)\|_{L^\infty(r_1,\check{r}_1)} < \epsilon/2$ and $\|\eta - \eta(r_2)\|_{L^\infty(\check{r}_2,r_2)} < \epsilon/2$. Since $C^\infty(r_1, r_2)$ is dense in $L^\infty(r_1, r_2)$ we can choose $\hat{\eta} \in C^\infty(r_1, r_2)$ with $\|\eta - \hat{\eta}\|_{L^\infty(r_1,r_2)} < \epsilon/2$. Let $\hat{r}_1 \in (r_1, \check{r}_1)$, $\hat{r}_2 \in (\check{r}_2, r_2)$ and

$$\eta_\epsilon := \begin{cases} \eta(r_1), & r \le \hat{r}_1, \\ \left(1 - \chi_2(\frac{r-\hat{r}_1}{\check{r}_1-\hat{r}_1})\right)\eta(r_1) + \chi_2(\frac{r-\hat{r}_1}{\check{r}_1-\hat{r}_1})\hat{\eta}(r), & \hat{r}_1 < r \le \check{r}_1, \\ \hat{\eta}(r), & \check{r}_1 < r < \check{r}_2, \\ \left(1 - \chi_2(\frac{r-\check{r}_1}{\hat{r}_2-\check{r}_2})\right)\hat{\eta}(r) + \chi_2(\frac{r-\check{r}_1}{\hat{r}_2-\check{r}_2})\eta(r_2), & \hat{r}_2 < r \le \check{r}_2, \\ \eta(r_2), & r \ge \hat{r}_2, \end{cases}$$

with χ_2 as in (2.36b). From the triangle inequality and $\chi_2(r) \in [0, 1]$ for all $r \in \mathbb{R}$ it follows $\|\eta - \eta_\epsilon\|_{L^\infty(r_1,r_2)} < \epsilon$. By construction η_ϵ suffices also the last three criteria. □

Lemma 4.14. *Let Assumptions 2.10 and 4.1 hold. Let X and $a(\cdot;\cdot,\cdot)$ be as in (2.42) and $A(\cdot)$ be as in (3.3). For $\epsilon > 0$ and $\omega \in \mathbb{C} \setminus \{0\}$ let*

$$T_\epsilon(\omega)u := \eta_\epsilon u \qquad \textit{with} \qquad \eta := \begin{cases} \overline{\frac{|\hat{d}|}{\hat{d}}} & \textit{for } \arg(-\omega^2 d_0^2) \in [-\pi, 0), \\ \overline{\frac{\hat{d}}{\tilde{d}^2}\frac{|\tilde{d}|^2}{|\hat{d}|}} & \textit{for } \arg(-\omega^2 d_0^2) \in [0, \pi) \end{cases} \tag{4.9}$$

and $\eta_\epsilon|_{(r_1^*,+\infty)}$ as in Lemma 4.13 with $r_1 = r_1^*, r_2 = +\infty$ and $\eta_\epsilon|_{[0,r_1^*]} := \eta_\epsilon(r_1^*)$.

For each $\omega \in \mathbb{C} \setminus \{0\}$ there exists $\epsilon_0(\omega) > 0$ so that for each $\epsilon \le \epsilon_0(\omega)$, $T_\epsilon(\omega) \in L(X)$ is bijective and $A(\omega)\colon \Lambda_{d_0} \to L(X)$ is weakly $T_\epsilon(\omega)$-coercive.

Proof. $T_\epsilon(\omega) \in L(X)$ and its bijectivity can be proven for a sufficiently small ϵ as in the proof of Lemma 4.4. Similarly the weak $T_\epsilon(\omega)$-coercivity of $A(\omega)$ can be proven for a sufficiently small ϵ as in the proof of Theorem 4.5. □

Assumption 4.15. *There exists a sequence $\big(h(n)\big)_{n\in\mathbb{N}} \in (\mathbb{R}^+)^\mathbb{N}$ with $\lim_{n\in\mathbb{N}} h(n) = 0$. There exist bounded linear projection operators $\Pi_{h(n)}\colon X \to X^{h(n)}, n \in \mathbb{N}$ that act locally in the following sense: there exist constants $C_1, R^* > 1$ so that for $n \in \mathbb{N}$, $s \in \{1, 2\}$, $x_0 \in \Omega$, if $B_{R^*h(n)}(x_0) \subset \Omega$, $u \in X$ and $u|_{B_{R^*h(n)}(x_0)} \in H^s(B_{R^*h(n)}(x_0))$, then*

$$\|u - \Pi_{h(n)}u\|_{H^1(B_{h(n)}(x_0))} \le C_1 h(n)^{s-1}\|u\|_{H^s(B_{R^*h(n)}(x_0))}. \tag{4.10}$$

Assumption 4.16. *For any $D \subset \Omega$ which is compact in Ω exists $n_0 > 0$ so that for each $n \in \mathbb{N}, n > n_0$ there exists $u_{D,n} \in X^{h(n)}$ with $u_{D,n}|_D = 1$.*

Theorem 4.17. *Let Assumptions 2.10 and 4.1 hold. Let X be as in (2.42), $\left(X^{h(n)}\right)_{n\in\mathbb{N}}$ be sequence of finite dimensional subspaces $X^{h(n)} \subset X$ so that the orthogonal projections from X onto $X^{h(n)}$ converge point-wise to the identity in X and so that Assumptions 4.15 and 4.16 hold. Let $\epsilon_0(\omega)$ be as in Lemma 4.14, $T_{\epsilon_0}(\omega) := T_{\epsilon_0(\omega)}(\omega)$ be as in (4.9) and $\|\cdot\|_n$ be as in (3.2). For $n \in \mathbb{N}$ let $\Pi_{h(n)}$ be as in Assumptions 4.15 and*

$$T^{h(n)}_{\epsilon_0}(\omega) := \Pi_{h(n)} T_{\epsilon_0}(\omega)|_{X^{h(n)}} \tag{4.11}$$

for $\omega \in \mathbb{C} \setminus \{0\}$. Then $T^{h(n)}_{\epsilon_0}(\omega) \in L(X^{h(n)})$ is Fredholm with index zero and

$$\lim_{n\in\mathbb{N}} \|T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega)\|_n = 0 \tag{4.12}$$

for all $\omega \in \mathbb{C} \setminus \{0\}$.

Proof. Let $\omega \in \mathbb{C} \setminus \{0\}$. It is straightforward to see $T^{h(n)}_{\epsilon_0}(\omega) \in L(X^{h(n)})$. Since $X^{h(n)}$ is finite dimensional, $T^{h(n)}_{\epsilon_0}(\omega)$ is Fredholm with index zero. Further, we note that if $n \in \mathbb{N}$, $x_0 \in \Omega$, $B_{R^*h(n)}(x_0) \subset \Omega$ and $u, \hat{u} \in X$ with $u|_{B_{R^*h(n)}(x_0)} = \hat{u}|_{B_{R^*h(n)}(x_0)}$, then also $(T^{h(n)}_{\epsilon_0}(\omega)u)|_{B_{h(n)}(x_0)} = (T^{h(n)}_{\epsilon_0}(\omega)\hat{u})|_{B_{h(n)}(x_0)}$. Indeed from Assumption 4.15 follows

$$\begin{aligned}\|\Pi_{h(n)}(\eta_{\epsilon_0}(u-\hat{u}))\|_{H^1(B_{h(n)}(x_0))} &= \|\eta_{\epsilon_0}(u-\hat{u}) - \Pi_h(\eta_{\epsilon_0}(u-\hat{u}))\|_{H^1(B_{h(n)}(x_0))}\\ &\le C_1\|\eta_{\epsilon_0}(u-\hat{u})\|_{H^1(B_{C_1h(n)}(x_0))} = 0.\end{aligned}$$

So let $\hat{r}_1, \hat{r}_2$ be as in Lemma 4.13. Let $r_2^* > \hat{r}_2$, $h_0 > 0$ with $h_0 < \min\{\hat{r}_1 - r_1^*, r_2^* - \hat{r}_2\}/C_1$ and $n_0 > 0$ be so that $h(n) < h_0$ for all $n > n_0$. Let $n > n_0$ and $u_n \in X^{h(n)}$. Since $\Pi_{h(n)}$ is linear and a projection it follows

$$\begin{aligned}(T^{h(n)}_{\epsilon_0}(\omega)u_n)|_{\Omega\cap B_{r_1^*}} = \big(\Pi_{h(n)}(\eta_{\epsilon_0}u_n)\big)|_{\Omega\cap B_{r_1^*}} &= \big(\Pi_{h(n)}(\eta_{\epsilon_0}(r_1^*)u_n)\big)|_{\Omega\cap B_{r_1^*}}\\ &= (\eta_{\epsilon_0}(r_1^*)\Pi_{h(n)}u_n)|_{\Omega\cap B_{r_1^*}}\\ &= (\eta_{\epsilon_0}(r_1^*)u_n)|_{\Omega\cap B_{r_1^*}}\\ &= (\eta_{\epsilon_0}u_n)|_{\Omega\cap B_{r_1^*}} = (T_{\epsilon_0}(\omega)u_n)|_{\Omega\cap B_{r_1^*}}.\end{aligned}$$

Likewise $(T^{h(n)}_{\epsilon_0}(\omega)u_n)|_{\Omega\cap B^c_{r_2^*}} = (T_{\epsilon_0}(\omega)u_n)|_{\Omega\cap B^c_{r_2^*}}$. Hence

$$\begin{aligned}&\|(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n\|^2_X\\ &\quad= \langle(|\tilde{d}^2 d^{-1}|\,\mathrm{P_x} + |d|(\mathrm{I}-\mathrm{P_x}))\nabla(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n, \nabla(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n\rangle_{L^2(\Omega)}\\ &\quad+ \langle|\tilde{d}^2 d|(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n, (T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n\rangle_{L^2(\Omega)}\\ &\quad= \langle(|\tilde{d}^2 d^{-1}|\,\mathrm{P_x} + |d|(\mathrm{I}-\mathrm{P_x}))\nabla(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n, \nabla(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n\rangle_{L^2(A_{r_1^*,r_2^*})}\\ &\quad+ \langle|\tilde{d}^2 d|(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n, (T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n\rangle_{L^2(A_{r_1^*,r_2^*})}\\ &\quad\le C^2\|(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n\|^2_{H^1(A_{r_1^*,r_2^*})}\end{aligned}$$

with $C^2 := \sup_{x\in A_{r_1^*,r_2^*}} \max\{|\tilde{d}^2 d^{-1}|, |d|, |\tilde{d}^2 d|\} < \infty$. Now we are in the position to apply the analysis of Bertoluzza [Ber99]. Although we cannot apply [Ber99, Theorem 2.1] directly since neither has η_{ϵ_0} compact support in Ω nor is the constant function included in $X^{h(n)}$ (due to the incorporated homogeneous Dirichlet boundary condition). Nevertheless, we can repeat the proof of [Ber99, Theorem 2.1] line by line as follows.

Let $h_0 > 0$ be so that $A_{r_1^*-R^*h_0,r_2^*+R^*h_0} \subset \Omega$ (with R^* as in Assumption 4.15) and let $n_0 > 0$ be so that $h(n) < h_0$ for all $n > n_0$. For each $n \in \mathbb{N}$, $n > n_0$ we consider a collection of balls $\{B_{h(n)}(x), x \in Z\}$ with $Z \subset A_{r_1^*,r_2^*}$ so that $A_{r_1^*,r_2^*} \subset \bigcup_{x\in Z} B_{h(n)}(x)$ and so that any point $y \in \Omega$ belongs to at most $m \in \mathbb{N}$ (with m independent of $n \in \mathbb{N}$, $n > n_0$) balls of the collection $\{B_{R^*h(n)}(x), x \in Z\}$. This implies the existence of a constant $\tilde{C}_1 > 0$ so that

$$\sum_{x\in Z} \|u\|^2_{H^s(B_{h(n)}(x))} \leq \tilde{C}_1 \|u\|^2_{H^s(\bigcup_{x\in Z} B_{h(n)}(x))}$$

for $s \in \{0, 1, 2\}$ and all $u \in H^s(\bigcup_{x\in Z} B_{h(n)}(x))$, $n \in \mathbb{N}$, $n > n_0$. Hence for $u_n \in X^{h(n)}$ we estimate

$$\begin{aligned}\|(T_{\epsilon_0}(\omega) - T^{h(n)}_{\epsilon_0}(\omega))u_n\|^2_{H^1(A_{r_1^*,r_2^*})} &= \|(1 - \Pi_{h(n)})\eta_{\epsilon_0} u_n\|^2_{H^1(A_{r_1^*,r_2^*})} \\ &\leq \sum_{x\in Z} \|(1 - \Pi_{h(n)})\eta_{\epsilon_0} u_n\|^2_{H^1(B_{h(n)}(x))}.\end{aligned}$$

For each $x \in Z$ Assumption 4.16 allows us to appropriately choose $u_{x,n} \in X^{h(n)}$ so that $u_{x,n}|_{B_{R^*h(n)}(x)}$ is constant,

$$\|u_{x,n}\|_{L^2(B_{R^*h(n)}(x))} \leq \|u_n\|_{L^2(B_{R^*h(n)}(x))}$$

and

$$\|u_n - u_{x,n}\|_{H^1(B_{R^*h(n)}(x))} \leq \tilde{C}_2 R^* h(n) \|u_n\|_{H^1(B_{R^*h(n)}(x))}$$

with a constant $\tilde{C}_2 > 0$ independent of $u_n \in X^{h(n)}$ and $n \in \mathbb{N}$, $n > n_0$. Thus we estimate further

$$\begin{aligned}\|(1 - \Pi_{h(n)})\eta_{\epsilon_0} u_n\|_{H^1(B_{h(n)}(x))} &\leq \|(1 - \Pi_{h(n)})\eta_{\epsilon_0}(u_n - u_{x,n})\|_{H^1(B_{h(n)}(x))} \\ &\quad + \|(1 - \Pi_{h(n)})\eta_{\epsilon_0} u_{x,n}\|_{H^1(B_{h(n)}(x))}.\end{aligned}$$

Since $u_{x,n}|_{B_{R^*h(n)}(x)}$ is constant it follows with Assumption 4.15

$$\begin{aligned}\|(1 - \Pi_{h(n)})\eta_{\epsilon_0} u_{x,n}\|_{H^1(B_{h(n)}(x))} &\leq C_1 h(n) \|\eta_{\epsilon_0} u_{x,n}\|_{H^2(B_{R^*h(n)}(x))} \\ &\leq C_1 h(n) \|\eta_{\epsilon_0}\|_{W^{2,\infty}(B_{R^*h(n)}(x))} \|u_{x,n}\|_{L^2(B_{R^*h(n)}(x))}.\end{aligned}$$

On the other hand, since $(u_n - u_{x,n}) \in X^{h(n)}$ and $\Pi_{h(n)}$ is a projection onto $X^{h(n)}$ it follows that $(1 - \Pi_{h(n)})\eta_{\epsilon_0}(x)(u_n - u_{x,n}) = 0$. Together with Assumption 4.15 we estimate

$$\begin{aligned}\|(1 - \Pi_{h(n)})\eta_{\epsilon_0}(u_n - u_{x,n})\|_{H^1(B_{h(n)}(x))} &= \|(1 - \Pi_{h(n)})(\eta_{\epsilon_0} - \eta_{\epsilon_0}(x))(u_n - u_{x,n})\|_{H^1(B_{h(n)}(x))} \\ &\leq C_1 \|(\eta_{\epsilon_0} - \eta_{\epsilon_0}(x))(u_n - u_{x,n})\|_{H^1(B_{R^*h(n)}(x))}.\end{aligned}$$

We compute

$$\begin{aligned}\|(\eta_{\epsilon_0}-\eta_{\epsilon_0}(x))(u_n-u_{x,n})\|^2_{H^1(B_{R^*h(n)}(x))} \leq{}& \|(\eta_{\epsilon_0}-\eta_{\epsilon_0}(x))(u_n-u_{x,n})\|^2_{L^2(B_{R^*h(n)}(x))}\\ &+2\|(\eta_{\epsilon_0}-\eta_{\epsilon_0}(x))\nabla u_n)\|^2_{L^2(B_{R^*h(n)}(x))}\\ &+2\|(\nabla\eta_{\epsilon_0})(u_n-u_{x,n})\|^2_{L^2(B_{R^*h(n)}(x))}\end{aligned}$$

and estimate

$$\begin{aligned}\|(\eta_{\epsilon_0}-\eta_{\epsilon_0}(x))(u_n-u_{x,n})\|_{L^2(B_{R^*h(n)}(x))} &\leq R^*h(n)\|\eta_{\epsilon_0}\|_{W^{1,\infty}(\Omega)}\|u_n\|_{L^2(B_{R^*h(n)}(x))},\\ \|(\eta_{\epsilon_0}-\eta_{\epsilon_0}(x))\nabla u_n\|_{L^2(B_{R^*h(n)}(x))} &\leq R^*h(n)\|\eta_{\epsilon_0}\|_{W^{1,\infty}(\Omega)}\|u_n\|_{H^1(B_{R^*h(n)}(x))},\\ \|(\nabla\eta_{\epsilon_0})(u_n-u_{x,n})\|_{L^2(B_{R^*h(n)}(x))} &\leq \tilde{C}_2R^*h(n)\|\eta_{\epsilon_0}\|_{W^{1,\infty}(\Omega)}\|u_n\|_{H^1(B_{R^*h(n)}(x))}.\end{aligned}$$

Altogether we obtain

$$\|(T_{\epsilon_0}(\omega)-T^{h(n)}_{\epsilon_0}(\omega))u_n\|_{H^1(A_{r_1^*,r_2^*})} \leq \tilde{C}_3h(n)\|u_n\|_{H^1(A_{r_1^*-R^*h_0,r_2^*+R^*h_0})}$$

with a constant $\tilde{C}_3>0$ independent of $n\in\mathbb{N}$, $n>n_0$, $u_n\in X^{h(n)}$. It remains to note

$$\|u_n\|_{H^1(A_{r_1^*-R^*h_0,r_2^*+R^*h_0})} \leq \tilde{C}_4\|u_n\|_X$$

for a constant $\tilde{C}_4>0$ independent of $n\in\mathbb{N}$, $n>n_0$, $u_n\in X^{h(n)}$. □

Theorem 4.18 (Spectral convergence)**.** *Let Assumptions 2.10 and 4.1 hold. Let X and $a(\cdot;\cdot,\cdot)$ be as in (2.42), $A(\cdot)$ be as in (3.3) and $\Lambda^{\pm}_{d_0}$ be as in (4.6). Let $\big(X^{h(n)}\big)_{n\in\mathbb{N}}$ be sequence of finite dimensional subspaces $X^{h(n)}\subset X$ so that the orthogonal projections from X onto $X^{h(n)}$ converge point-wise to the identity in X and so that Assumptions 4.15 and 4.16 hold. Let $A^{h(n)}(\cdot)$ be defined by (3.5) and $T_{\epsilon_0}(\cdot)$ be as in Theorem 4.17.*

Then $A(\cdot)\colon\Lambda^{\pm}_{d_0}\to L(X)$ is a weakly $T_{\epsilon_0}(\cdot)$-coercive holomorphic Fredholm operator function with non-empty resolvent set $\rho\big(A(\cdot)\big)$ and $A^{h(n)}(\cdot)\colon\Lambda^{\pm}_{d_0}\to L(X^{h(n)})$ is a $T_{\epsilon_0}(\cdot)$-compatible approximation, i.e. Theorem 3.17 applies.

Proof. Follows from Theorem 4.9 and Theorem 4.17. □

All three assumptions are fulfilled by common finite element spaces, see e.g. [BS08].

4.5 Approximation by truncationless discretizations

As previously discussed, the classical approach to approximate (2.43) is to first choose a bounded subdomain $\Omega_n\subset\Omega$ and secondly to choose a convenient Galerkin space $X^h\subset H^1_0(\Omega_n)$, e.g. a finite element space. However, if the approximation is not satisfactory enough and a better approximation is desired, it is in general not enough to increase the dimension of the finite element space, but also the size of the domain Ω_n needs to be increased. The latter involves a new domain and the generation of a new mesh. This may be undesirable for some people who would prefer to work with a fixed domain and solely increase discretization parameters. There are at least two concepts to achieve this goal.

One is the implementation of infinite elements into the code. I.e. the fixed domain is $\Omega \cap B_{r_1^*}$ and the exterior domain $\Omega \setminus B_{r_1^*}$ is not explicitly meshed. Instead tensor product (finite element) functions with respect to polar coordinates can be used. This can indeed be implemented without the explicit generation of a mesh for $\Omega_n \setminus B_{r_1^*}$. Of course it is possible to also use non-classical basis functions with respect to the radial variable, e.g. as $\exp(-r)p(r)$ with polynomials p. We mention the recent work [NW19] wherein the Hardy space infinite element method introduced in [HN09] is framed as a complex scaling infinite element method. We note that the analysis thereof is already covered by [Hal16].

A different approach is to derive a formulation of the eigenvalue problem which involves only a bounded domain (but singular coefficients) and subsequently to apply a classical finite element discretization. To our knowledge Bermúdez et. al. were the first to consider a variant of this idea in [BHNPR04] and subsequently in [BHNPR06], [BHNPR07], [BHNPR08]. Their idea is to use a profile function $\tilde{\alpha}$ which is unbounded on $(0, r_2^*)$ with $r_2^* > r_1^*$. This leads to a formulation of the eigenvalue problem on the bounded domain $\Omega \cap B_{r_2^*}$. Since in this case the formulation (and subsequently the discretization) is posed on a bounded domain without committing a truncation error, Bermúdez et. al. coined their method "exact PML". We will discuss their method in Subsection 4.5.2. Another variant is to consider the formulation derived in Chapter 2 and subsequently apply a *real* domain transformation $B^c_{r_1^*} \to A_{r_1^*, r_2^*}$ [HL05], [Nan16]. We will discuss this second method in Subsection 4.5.1.

There is a noteworthy alternative interpretation to both methods [Nan16]. Namely the formulation can be transformed (back) to the unbounded domain Ω. If this happens after the discretization one obtains a discretization of the problem posed in Ω. This way one implicitly applies basis functions with unbounded support. Thus these mentioned "exact" methods could also validly be called "infinite element" methods. However, we will stick to the formulations on bounded domains for convenience.

A difference between these two methods is that the method of Subsection 4.5.1 still allows the choice of d_0 and hence a control of the essential spectrum $\{d_0^{-1}x \colon x \in \mathbb{R}\}$, while for the method of Subsection 4.5.2 the essential spectrum is implicitly set to $\{-ix \colon x \in \mathbb{R}\}$. This is of importance if one seeks to apply these techniques to problems which involve evanescent waves which occur e.g. for waveguide geometries. The technique of Subsection 4.5.1 can be applied successfully to such problems, while the technique of Subsection 4.5.2 fails.

We perform an approximation analysis for both methods in the following subsections. The analysis involves no new concepts but only slightly adapts the techniques of the previous sections, in particular the technique of Section 4.4.

4.5.1 Reparametrization of $B^c_{r_1^*}$

We derive from Eigenvalue Problem (2.43) by means of a real domain transformation x_{e_1} (see (4.13a) and Assumption 4.19) the related Eigenvalue Problem (4.17) [HL05], [Nan16]. We discuss how the results of Sections 4.1 and 4.4 easily translate to the new setting. Finally we discuss how appropriate finite element spaces fit the derived theory.

We consider real domain transformations r_{e_1} of the following kind.

Assumption 4.19. *Let r_1^* be as in Assumption 2.10 and $r_2^* > r_1^*$. Let $r_{e_1}\colon (0, r_2^*) \to \mathbb{R}^+$ be bijective, continuous, $r_{e_1}|_{(r_1^*, r_2^*)}$ be continuously differentiable and so that $r_{e_1}(r) = r$ for $r \le r_1^*$.*

Let $\tilde\alpha$ suffice Assumption 2.10 and Assumption 4.1. Let r_{e_1} suffice Assumption 4.19, $\tilde d$, $\tilde r$, α, d be as in (2.37), $\hat d$ be as in (4.2b) and

$$\begin{aligned}
x_{e_1}(x) &:= r_{e_1}(|x|)/|x| x, && (4.13a)\\
\gamma_{e_1}(x) &:= (\partial_r r_{e_1})(|x|), && (4.13b)\\
\tilde\gamma_{e_1}(x) &:= r_{e_1}(|x|)/|x|, && (4.13c)\\
\tilde\alpha_{e_1} &:= \tilde\alpha \circ r_{e_1}, && (4.13d)\\
\tilde d_{e_1} &:= \tilde d \circ r_{e_1}, && (4.13e)\\
\tilde r_{e_1} &:= \tilde r \circ r_{e_1}, && (4.13f)\\
\alpha_{e_1} &:= \alpha \circ r_{e_1}, && (4.13g)\\
d_{e_1} &:= d \circ r_{e_1}, && (4.13h)\\
\hat d_{e_1} &:= \hat d \circ r_{e_1}. && (4.13i)
\end{aligned}$$

As hitherto we adopt the overloaded notation (2.37g) also for the new quantities r_{e_1}, $\tilde r_{e_1}$, γ_{e_1}, $\tilde\gamma_{e_1}$, $\tilde\alpha_{e_1}$, α_{e_1}, $\tilde d_{e_1}$, d_{e_1}, $\hat d_{e_1}$. We compute

$$\begin{aligned}
\mathrm{D}\, x_{e_1} &= \gamma_{e_1} \mathrm{P_x} + \tilde\gamma_{e_1}(\mathrm{I} - \mathrm{P_x}), && (4.14a)\\
(\mathrm{D}\, x_{e_1})^{-1} &= \gamma_{e_1}^{-1} \mathrm{P_x} + \tilde\gamma_{e_1}^{-1}(\mathrm{I} - \mathrm{P_x}), && (4.14b)\\
\det \mathrm{D}\, x_{e_1} &= \gamma_{e_1}\tilde\gamma_{e_1}^2. && (4.14c)
\end{aligned}$$

We consider the bounded domain

$$\Omega_{e_1} := \Omega \cap B_{r_2^*}, \tag{4.15}$$

subsequently set

$$\begin{aligned}
a_{e_1}(\omega; u, u') := {}& \langle (\tilde\gamma_{e_1}^2 \gamma_{e_1}^{-1} \tilde d_{e_1}^2 d_{e_1}^{-1} \mathrm{P_x} + \gamma_{e_1} d_{e_1}(\mathrm{I} - \mathrm{P_x}))\nabla u, \nabla u'\rangle_{L^2(\Omega_{e_1})}\\
&- \omega^2 \langle \tilde\gamma_{e_1}^2 \gamma_{e_1} \tilde d_{e_1}^2 d_{e_1} u, u'\rangle_{L^2(\Omega_{e_1})},
\end{aligned} \tag{4.16a}$$

$$\begin{aligned}
X_{e_1} &:= \{u \in H^1_{\mathrm{loc}}(\Omega_{e_1})\colon \langle u, u\rangle_{X_{e_1}} < \infty, u|_{\partial\Omega} = 0\}, && (4.16b)\\
\langle u, u'\rangle_{X_{e_1}} &:= \langle u, u'\rangle_{X_{e_1}(\Omega_{e_1})}, && (4.16c)
\end{aligned}$$

and

$$\begin{aligned}
\langle u, u'\rangle_{X_{e_1}(D)} := {}& \langle (|\tilde\gamma_{e_1}^2 \gamma_{e_1}^{-1}| \tilde d_{e_1}^2 d_{e_1}^{-1}| \mathrm{P_x} + \gamma_{e_1}|d_{e_1}|(\mathrm{I} - \mathrm{P_x}))\nabla u, \nabla u'\rangle_{L^2(D)}\\
&+ \langle \tilde\gamma_{e_1}^2 \gamma_{e_1} |\tilde d_{e_1}^2 d_{e_1}| u, u'\rangle_{L^2(D)},
\end{aligned} \tag{4.16d}$$

for $\omega \in \mathbb{C}$, $u, u' \in X_{e_2}$ and $D \subset \Omega_{e_1}$ and consider the eigenvalue problem to

$$\text{find } (\omega, u) \in \mathbb{C} \times X_{e_1} \setminus \{0\} \text{ so that } \quad a_{e_1}(\omega; u, u') = 0 \quad \text{for all } u' \in X_{e_1}. \tag{4.17}$$

Due to the transformation rule and the chain rule it is clear that

$$F_{e_1}u := u \circ x_{e_1} \tag{4.18}$$

is a linear bijective Hilbert space isomorphism, i.e. $F_{e_1} \in L(X, X_{e_1})$, F_{e_1} is bijective and

$$\langle u, u' \rangle_X = \langle F_{e_1}u, F_{e_1}u' \rangle_{X_{e_1}} \tag{4.19}$$

for all $u, u' \in X$ (with X as in (2.42)). Further it holds

$$a(\omega; u, u') = a_{e_1}(\omega; F_{e_1}u, F_{e_1}u') \tag{4.20}$$

for all $u, u' \in X$. Thus we can simply deduce the properties of $A_{e_1}(\cdot)$ (defined through (3.3)) from $A(\cdot)$. In particular it holds that $(\omega, u) \in \mathbb{C} \times X \setminus \{0\}$ is a solution to $A(\omega)u = 0$ if and only if $A_{e_1}(\omega)F_{e_1}u = 0$. $A_{e_1}(\omega)$ is Fredholm if and only if $\omega \in \Lambda_{d_0}$ (with Λ_{d_0} as in (4.4)). Further $A_{e_1}(\cdot)|_{\Lambda_{d_0}}$ is weakly $T_{e_1}(\cdot)$-coercive with

$$T_{e_1}(\omega)u = F_{e_1}T(\omega)F_{e_1}^{-1}u = (\eta \circ x_{e_1})u \tag{4.21}$$

for $u \in X_{e_1}$ and η being the symbol of $T(\omega)$ as in (4.3). Further $A_{e_1}(\omega)$ is bijective for all $\omega \in \mathbb{C} \setminus \{0\}$ with $\arg \omega \in [-\pi, -\arg d_0) \cup [0, \pi - \arg d_0)$.

It remains to discuss the approximation of (4.17). Hence we first adapt Lemma 4.3 to our current setting in Lemma 4.20. Then we proceed as in Section 4.4 and construct an operator function $T_{e_1,\epsilon}(\cdot)$ with appropriate properties.

Lemma 4.20. *Let $(r_n)_{n\in\mathbb{N}}$ with $r_n \in (r_1^*, r_2^*)$ for all $n \in \mathbb{N}$ be a monotonically increasing sequence with limes r_2^*. Let $\eta_1 \colon \Omega_{e_1} \to \mathbb{C}$ be mesuarable so that $\eta_1|_{\Omega_{e_1} \cap B_{r_n}} \in L^\infty(\Omega_{e_1} \cap B_{r_n})$ for all $n \in \mathbb{N}$. Let $Y \subset L^2(\Omega_{e_1})$ be a Hilbert space so that $\|\eta_1 u\|_{L^2(\Omega_{e_1})} \leq C\|u\|_Y$ with $C > 0$ for all $u \in Y$ and so that the embedding and restriction operator $K_n \colon Y \to L^2(\Omega_{e_1} \cap B_{r_n}) \colon u \mapsto u|_{\Omega_{e_1} \cap B_{r_n}}$ is compact for each $n \in \mathbb{N}$. Let $\eta_2 \in L^\infty(\Omega_{e_1})$ be so that $\lim_{r \to r_2^*-} \|\eta_2\|_{L^\infty(\Omega_{e_1} \setminus B_r)} = 0$. Then the multiplication and embedding operator $K_{\eta_1\eta_2} \colon Y \to L^2(\Omega_{e_1}) \colon u \mapsto \eta_1\eta_2 u$ is compact.*

Proof. Proceed as in the proof of Lemma 4.3. □

Lemma 4.21. *Let Assumptions 2.10, 4.1 and 4.19 hold. Let X_{e_1} be as in* (4.16b), *$a_{e_1}(\cdot; \cdot, \cdot)$ be as in* (4.16a) *and $A_{e_1}(\cdot)$ be as in* (3.3). *For $\epsilon > 0$ and $\omega \in \mathbb{C} \setminus \{0\}$ let*

$$T_{e_1,\epsilon}(\omega)u := \eta_{e_1,\epsilon}u \qquad \textit{with} \qquad \eta_{e_1} = \begin{cases} \overline{\frac{|\hat{d}_{e_1}|}{\hat{d}_{e_1}}} & \textit{for } \arg(-\omega^2 d_0^2) \in [-\pi, 0), \\ \frac{\hat{d}_{e_1}}{\check{d}_{e_1}^2} \frac{|\tilde{d}_{e_1}|^2}{|\hat{d}_{e_1}|} & \textit{for } \arg(-\omega^2 d_0^2) \in [0, \pi). \end{cases} \tag{4.22}$$

with $\eta_{e_1,\epsilon}|_{(r_1^*, r_2^*)}$ as in Lemma 4.13 with $r_1 = r_1^*, r_2 = r_2^*$ and $\eta_{e_1,\epsilon}|_{[0,r_1^*]} := \eta_{e_1,\epsilon}(r_1^*)$.

There exists $\epsilon_0(\omega) > 0$ so that for each $\epsilon \leq \epsilon_0(\omega)$, $T_{e_1,\epsilon}(\omega) \in L(X_{e_1})$ is bijective for all $\omega \in \mathbb{C} \setminus \{0\}$ and $A_{e_1}(\cdot) \colon \Lambda_{d_0} \to L(X_{e_1})$ is weakly $T_{e_1,\epsilon}(\cdot)$-coercive.

Proof. Proceed as in the proof of Lemma 4.14 with Lemma 4.3 replaced by Lemma 4.20. □

Next we consider a sequence of finite dimensional subspaces $(X_{e_1}^{h(n)})_{n\in\mathbb{N}}$, $X_{e_1}^{h(n)} \subset X_{e_1}$, $n \in \mathbb{N}$ so that the orthogonal projections onto $X_{e_1}^{h(n)}$ converge point-wise to the identity in X_{e_1}. Further let

$$\text{find } (\omega, u) \in \mathbb{C} \times X_{e_1}^{h(n)} \setminus \{0\} \text{ so that } \quad a_{e_1}(\omega; u, u') = 0 \quad \text{for all } u' \in X_{e_1}^{h(n)} \tag{4.23}$$

be the Galerkin approximation to (4.17). As in Section 4.4 we make two additional assumptions on the Galerkin spaces $X_{e_1}^{h(n)}$.

Assumption 4.22. *There exists a sequence $\big(h(n)\big)_{n\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$ with $\lim_{n\in\mathbb{N}} h(n) = 0$. There exist bounded linear projection operators $\Pi_{h(n)}^{e_1}\colon X_{e_1} \to X_{e_1}^{h(n)}, n \in \mathbb{N}$ that act locally in the following sense: there exist constants $C_1, R^* > 1$ so that for $n \in \mathbb{N}$, $s \in \{1, 2\}$, $x_0 \in \Omega_{e_1}$, if $B_{R^*h(n)}(x_0) \subset \Omega_{e_1}$, $u \in X_{e_1}$ and $u|_{B_{R^*h(n)}(x_0)} \in H^s(B_{R^*h(n)}(x_0))$, then*

$$\|u - \Pi_{h(n)}^{e_1} u\|_{H^1(B_{h(n)}(x_0))} \le C_1 h(n)^{s-1} \|u\|_{H^s(B_{R^*h(n)}(x_0))}. \tag{4.24}$$

Assumption 4.23. *For any $D \subset \Omega_{e_1}$ which is compact in Ω_{e_1} exists $n_0 > 0$ so that for each $n \in \mathbb{N}, n > n_0$ there exists $u_{D,n} \in X_{e_1}^{h(n)}$ with $u_{D,n}|_D = 1$.*

Theorem 4.24. *Let Assumptions 2.10, 4.1 and 4.19 hold. Let X_{e_1} be as in* (4.16b)*, $(X_{e_1}^{h(n)})_{n\in\mathbb{N}}$ be a sequence of finite dimensional subspaces $X_{e_1}^{h(n)} \subset X_{e_1}$ so that the orthogonal projections onto $X_{e_1}^{h(n)}$ converge point-wise to the identity and so that Assumptions 4.22 and 4.23 hold. Let $T_{e_1,\epsilon_0}(\omega) := T_{e_1,\epsilon_0(\omega)}(\omega)$ be as in Lemma 4.21 and $\|\cdot\|_n$ be as in* (3.2)*. For $n \in \mathbb{N}$ let $\Pi_{h(n)}^{e_1}$ be as in Assumptions 4.22 and*

$$T_{e_1,\epsilon_0}^{h(n)}(\omega) := \Pi_{h(n)}^{e_1} T_{e_1,\epsilon_0}(\omega)|_{X_{e_1}^{h(n)}} \tag{4.25}$$

for $\omega \in \mathbb{C} \setminus \{0\}$. Then $T_{e_1,\epsilon_0}^{h(n)}(\omega) \in L(X_{e_1}^{h(n)})$ is Fredholm with index zero and

$$\lim_{n\in\mathbb{N}} \|T_{e_1,\epsilon_0}(\omega) - T_{e_1,\epsilon_0}^{h(n)}(\omega)\|_n = 0 \tag{4.26}$$

for all $\omega \in \mathbb{C} \setminus \{0\}$.

Proof. Proceed as in the proof of Theorem 4.24. □

Theorem 4.25 (Spectral convergence)**.** *Let Assumptions 2.10 and 4.1 hold. Let r_{e_1} fulfill Assumption 4.19 and X_{e_1}, $a_{e_1}(\cdot;\cdot,\cdot)$ be as defined in* (4.16)*. Let $A_{e_1}(\cdot)\colon \Lambda \to L(X_{e_1})$ be defined through* (3.3)*, $T_{e_1,\epsilon_0}(\cdot)$ as in Theorem 4.24 and $\Lambda_{d_0}^{\pm}$ be as in* (4.6)*. Let $\big(X_{e_1}^{h(n)}\big)_{n\in\mathbb{N}}$ be a sequence of finite dimensional subspaces $X_{e_1}^{h(n)} \subset X_{e_1}$ so that the orthogonal projections from X_{e_1} onto $X_{e_1}^{h(n)}$ converge point-wise to the identity in X_{e_1} and so that Assumptions 4.22 and 4.23 hold. Let $A_{e_1,h(n)}(\cdot)$ be defined by* (3.5) *and $T_{e_1,\epsilon_0}(\cdot)$ be as in Theorem 4.40.*

Then $A_{e_1}(\cdot)\colon \Lambda_{d_0}^{\pm} \to L(X_{e_1})$ is a weakly $T_{e_1,\epsilon_0}(\cdot)$-coercive holomorphic Fredholm operator function with non-empty resolvent set $\rho\big(A_{e_1}(\cdot)\big)$ and $A_{e_1,h(n)}(\cdot)\colon \Lambda_{d_0}^{\pm} \to L(X_{e_1}^{h(n)})$ is a $T_{e_1,\epsilon_0}(\cdot)$-compatible approximation, i.e. Theorem 3.17 applies.

Proof. Proceed as in the proof of Theorem 4.18. $\square$

Finally we discuss how to choose appropriate parameters $\tilde{\alpha}$, r_{e_1} and an appropriate sequence of subspaces $(X_{e_1}^{h(n)})_{n\in\mathbb{N}}$, $X_{e_1}^{h(n)} \subset X_{e_1}$. To this end we introduce two lemmata.

Lemma 4.26. *Assume that*

$$\sup_{x\in\Omega_{e_1}} \frac{1}{(r_2^* - |x|)\gamma_{e_1}(x)|d_{e_1}(x)|} < +\infty. \tag{4.27}$$

Let $(X_{e_1}^{h(n)})_{n\in\mathbb{N}}$, $X_{e_1}^{h(n)} \subset X_{e_1}$ *be so that for any* $\delta > 0$ *and* $u \in X_{e_1}$ *with* $u|_{A_{r_2^*-\delta,r_2^*}} = 0$ *it holds*

$$\lim_{n\in\mathbb{N}} \inf_{u'\in X_{e_1}^{h(n)}} \|u - u'\|_{X_{e_1}} = 0. \tag{4.28}$$

Then (4.28) *holds for any* $u \in X_{e_1}$.

Proof. For $\delta > 0$ consider

$$g_\delta(x) := \chi_2\big(|x|/\delta - (r_2^* - 2\delta)/\delta\big)$$

with χ_2 as in (2.36b). Let $u \in X_{e_1}$ and $\epsilon > 0$ be given. By means of the product rule, the triangle inequality, the properties of g_δ and the chain rule we compute

$$\begin{aligned}
\|g_\delta u\|_{X_{e_1}} &\leq 2\langle \tilde{\gamma}_{e_1}^2 \gamma_{e_1}^{-1} |\tilde{d}_{e_1}^2 d_{e_1}^{-1}| \, \mathrm{P_x}\, u\nabla g_\delta, u\nabla g_\delta\rangle_{L^2(\Omega_{e_1})} \\
&\quad + 2\langle \tilde{\gamma}_{e_1}^2 \gamma_{e_1}^{-1} |\tilde{d}_{e_1}^2 d_{e_1}^{-1}| g_\delta^2 \, \mathrm{P_x}\, \nabla u, \nabla u\rangle_{L^2(\Omega_{e_1})} \\
&\quad + \langle \gamma_{e_1} |d_{e_1}| g_\delta^2 (\mathrm{I} - \mathrm{P_x})) \nabla u, \nabla u\rangle_{L^2(\Omega_{e_1})} \\
&\quad + \langle \tilde{\gamma}_{e_1}^2 \gamma_{e_1} |\tilde{d}_{e_1}^2 d_{e_1}| g_\delta^2 u, u\rangle_{L^2(\Omega_{e_1})} \\
&\leq 2\Big(1 + \big(\sup_{x\in A_{r_2^*-2\delta,r_2^*-\delta}} |\nabla g_\delta|^2 (\gamma_{e_1} d_{e_1})^{-2}\big)\Big) \|u\|^2_{X_{e_1}(A_{r_2^*-2\delta,r_2^*})} \\
&\leq 2\Big(1 + \|\partial_r \chi_2\|^2_{L^\infty(0,1)} \big(\sup_{x\in A_{r_2^*-2\delta,r_2^*-\delta}} (\delta\gamma_{e_1} d_{e_1})^{-2}\big)\Big) \|u\|^2_{X_{e_1}(A_{r_2^*-2\delta,r_2^*})} \\
&\leq 2\Big(1 + \|\partial_r \chi_2\|^2_{L^\infty(0,1)} \big(\sup_{x\in A_{r_2^*-2\delta,r_2^*-\delta}} ((r_2^* - |\cdot|)\gamma_{e_1} d_{e_1})^{-2}\big)\Big) \|u\|^2_{X_{e_1}(A_{r_2^*-2\delta,r_2^*})} \\
&\leq 2\Big(1 + \|\partial_r \chi_2\|^2_{L^\infty(0,1)} \big(\sup_{x\in \Omega_{e_1}} ((r_2^* - |\cdot|)\gamma_{e_1} d_{e_1})^{-2}\big)\Big) \|u\|^2_{X_{e_1}(A_{r_2^*-2\delta,r_2^*})} \\
&=: C\|u\|^2_{X_{e_1}(A_{r_2^*-2\delta,r_2^*})}.
\end{aligned}$$

Due to $\lim_{\delta\to0+} \|u\|^2_{X_{e_1}(A_{r_2^*-2\delta,r_2^*})} = 0$ we can choose $\delta > 0$ so that $C\|u\|^2_{X_{e_1}(A_{r_2^*-2\delta,r_2^*})} < \epsilon/2$. Since $1 - g_\delta(x) = 0$ for $x \geq r_2^* - \delta$ we can choose $n_0 \in \mathbb{N}$ be so that

$$\inf_{u'\in X_{e_1}^{h(n)}} \|(1 - g_\delta)u - u'\|_{X_{e_1}} < \epsilon/2$$

for all $n > n_0$. It follows for all $n > n_0$

$$\begin{aligned}\inf_{u' \in X_{e_1}^{h(n)}} \|u - u'\|_{X_{e_1}} &\leq \|g_\delta u\|_{X_{e_1}} + \inf_{u' \in X_{e_1}^{h(n)}} \|(1 - g_\delta)u - u'\|_{X_{e_1}} \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was chosen arbitrarily it follows that $\lim_{n\in\mathbb{N}} \inf_{u' \in X_{e_1}^{h(n)}} \|u - u'\|_{X_{e_1}} = 0$. □

Lemma 4.27. *Let $\tilde{\alpha}$ be of Kind* (2.32b) *or* (2.34)*. Let r_{e_1} be so that for $r \in (r_1^*, r_2^*)$ either*

$$r_{e_1}(r) = -(\ln(r_2^* - r) - \ln(r_2^* - r_1^*)) + r_1^* \tag{4.29a}$$

or

$$r_{e_1}(r) = (r_2^* - r)^\beta - (r_2^* - r_1^*)^\beta + r_1^* \tag{4.29b}$$

with $\beta \in (-2/3, 0)$. Then (4.27) *holds. If $u \in H_0^1(\Omega_{e_1})$ is so that $|u(x)| \leq C(r_2^* - |x|)$ for a constant $C > 0$ and all $x \in \Omega_{e_1}$, then $u \in X_{e_1}$.*

Proof. For r_{e_1} as in (4.29a) it holds $\gamma_{e_1}(x) = (r_2^* - |x|)^{-1}$. For r_{e_1} as in (4.29b) it holds $\gamma_{e_1}(x) = -\beta(r_2^* - 1)^{\beta-1}$. Since $|d_{e_1}| \geq 1$ it easiliy follows (4.27) in both cases. Due to the choice of $\tilde{\alpha}$ the coefficients $|\tilde{d}_{e_1}^2/d_{e_1}|$, $|d_{e_1}|$, $|\tilde{d}_{e_1}^2 d_{e_1}|$ are uniformly bounded. For r_{e_1} as in (4.29a) we compute

$$\tilde{\gamma}_{e_1}(x)^2/\gamma_{e_1}(x) = \big(-(\ln(r_2^* - |x|) - \ln(r_2^* - r_1^*)) + r_1^*\big)^2 |x|^{-2}(r_2^* - |x|), \tag{4.30a}$$

$$\gamma_{e_1}(x)(r_2^* - |x|)^2 = (r_2^* - |x|), \tag{4.30b}$$

$$\tilde{\gamma}_{e_1}(x)^2\gamma_{e_1}(x)(r_2^* - |x|)^2 = \big(-(\ln(r_2^* - |x|) - \ln(r_2^* - r_1^*)) + r_1^*\big)^2 |x|^{-2}(r_2^* - |x|). \tag{4.30c}$$

For r_{e_1} as in (4.29b) we compute

$$\tilde{\gamma}_{e_1}(x)^2/\gamma_{e_1}(x) = \big((r_2^* - |x|)^\beta - (r_2^* - r_1^*)^\beta + r_1^*\big)^2 |x|^{-2}(-\beta)^{-1}(r_2^* - |x|)^{-\beta+1}, \tag{4.31a}$$

$$\gamma_{e_1}(x)(r_2^* - |x|)^2 = -\beta(r_2^* - |x|)^{\beta+1}, \tag{4.31b}$$

$$\tilde{\gamma}_{e_1}(x)^2\gamma_{e_1}(x)(r_2^* - |x|)^2 = \big((r_2^* - |x|)^\beta - (r_2^* - r_1^*)^\beta + r_1^*\big)^2 |x|^{-2}(-\beta)(r_2^* - |x|)^{\beta+1}. \tag{4.31c}$$

It follows that each function in (4.30) and (4.31) is uniformly bounded in $x \in A_{r_1^*, r_2^*}$. It follows $\|u\|_{X_{e_1}} < +\infty$. □

Consider r_{e_1} and $\tilde{\alpha}$ as in Lemma 4.27. Due to Lemma 4.27 common finite element spaces are indeed subspaces of X_{e_1}. Due to Lemma 4.26 $(X_{e_1}^{h(n)})_{n\in\mathbb{N}}$ is asymptotically dense in X_{e_1} if it is so in $H_0^1(\Omega_{e_1})$. Hence with the stated choice of parameters $\tilde{\alpha}$, r_{e_1} a reliable discretization of (4.17) can be constructed straightforwardly.

4.5.2 Profile function with blow up

In [BHNPR04], [BHNPR08] a so-called exact PML method with profile function $\tilde{\alpha}$ so that $\tilde{\alpha}(r) \to \infty$ as $r \to r_2^* > r_1^*$ was proposed and similar Cartesian variants published in [BHNPR06], [BHNPR07]. In [BHNPR08] the authors proved the equivalence between the derived formulation and the original problem by means of integral representations of solutions. As the method of Subsection 4.5.1 this method has the advantage that no domain truncation is necessary with the cost of fabricating singular coefficients.

First we formulate the method and argue its relation to (2.43) as in Chapter 2. Next we derive weak $T(\cdot)$-coercivity results as in Section 4.1. Then we perform an approximation analysis as in Section 4.4 and Subsection 4.5.1. At last we comment on the choice of appropriate parameters and appropriate subspaces.

We consider a domain Ω as in Chapter 2 and make an assumption on $\tilde{\alpha}_{e_2}$ similar to Assumption 2.10.

Assumption 4.28. *Let $r_1^* > 0$ be so that Ω^c is contained in the ball $B_{r_1^*}$, $r_2^* > r_1^*$ and $\tilde{\alpha}_{e_2}\colon (0, r_2^*) \to \mathbb{R}_0^+$ be so that*

1. *$\tilde{\alpha}_{e_2}(r) = 0$ for $r \leq r_1^*$,*
2. *$\tilde{\alpha}_{e_2}$ is continuous,*
3. *$\tilde{\alpha}_{e_2}(r) > 0$ for $r > r_1^*$,*
4. *$\tilde{\alpha}_{e_2}$ is non-decreasing,*
5. *$\tilde{\alpha}_{e_2}$ is twice continuous differentiable in (r_1^*, r_2^*) with continuous extensions to $[r_1^*, r_2^*)$,*
6. *$\lim_{r\to r_2^*-} \tilde{\alpha}_{e_2}(r) = +\infty$.*

Subsequently we consider the bounded domain

$$\Omega_{e_2} := \Omega \cap B_{r_2^*} \tag{4.32}$$

and set

$$\tilde{d}_{e_2}(r) := 1 + i\tilde{\alpha}_{e_2}(r), \tag{4.33a}$$
$$\tilde{r}_{e_2}(r) := \tilde{d}_{e_2}(r)r, \tag{4.33b}$$
$$\alpha_{e_2}(r) := r\partial_r\tilde{\alpha}_{e_2}(r) + \tilde{\alpha}_{e_2}(r), \tag{4.33c}$$
$$d_{e_2}(r) := 1 + i\alpha_{e_2}(r), \tag{4.33d}$$

and

$$\hat{\alpha}_{e_2}(r) := \begin{cases} \lim_{r\to r_1^*+} \alpha_{e_2}(r) & \text{for } 0 \leq r \leq r_1^*, \\ \alpha_{e_2}(r) & \text{for } r > r_1^*, \end{cases} \tag{4.34a}$$
$$\hat{d}_{e_2}(r) := 1 + i\hat{\alpha}_{e_2}(r), \quad r \geq 0. \tag{4.34b}$$

Again, we adopt the overloaded notation $f(x) := f(|x|), x \in \Omega$ for $f = \tilde{\alpha}_{e_2}$, $\tilde{d}_{e_2}$, $\tilde{r}_{e_2}$, α_{e_2}, $\hat{\alpha}_{e_2}$, $\hat{d}_{e_2}$. Let

$$
\begin{aligned}
a_{e_2}(\omega; u, u') &:= \langle \tilde{d}_{e_2}^2 d_{e_2}^{-1} \mathrm{P_x} + d_{e_2}(\mathrm{I} - \mathrm{P_x}))\nabla u, \nabla u' \rangle_{L^2(\Omega_{e_2})} - \omega^2 \langle \tilde{d}_{e_2}^2 d_{e_2} u, u' \rangle_{L^2(\Omega_{e_2})}, && (4.35a)\\
X_{e_2} &:= \{u \in H^1_{\mathrm{loc}}(\Omega_{e_2})\colon \langle u, u\rangle_{X_{e_2}} < \infty, u|_{\partial\Omega} = 0\}, && (4.35b)\\
\langle u, u' \rangle_{X_{e_2}} &:= \langle u, u' \rangle_{X_{e_2}(\Omega_{e_2})}, && (4.35c)\\
\langle u, u' \rangle_{X_{e_2}(D)} &:= \langle (|\tilde{d}_{e_2}^2 d_{e_2}^{-1}| \mathrm{P_x} + |d_{e_2}|(\mathrm{I} - \mathrm{P_x}))\nabla u, \nabla u' \rangle_{L^2(D)} + \langle |\tilde{d}_{e_2}^2 d_{e_2}| u, u' \rangle_{L^2(D)}, && (4.35d)
\end{aligned}
$$

for $\omega \in \mathbb{C}$, $u, u' \in X_{e_2}$ and $D \subset \Omega_{e_2}$. Consider the eigenvalue problem to

$$
\text{find } (\omega, \tilde{u}) \in \mathbb{C} \times X_{e_2} \setminus \{0\} \text{ so that } \quad a_{e_2}(\omega; \tilde{u}, u') = 0 \quad \text{for all } u' \in X_{e_2}. \tag{4.36}
$$

The relation between (2.23) and (4.36) is as follows.

Lemma 4.29. *For a solution (ω, u) to (2.23) with $\Re(\omega) > 0$ and the expansion*

$$
u \circ Q(r, \hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n^1(\omega r) Y_n^m(\hat{x}), \quad r \geq r_1^*, \hat{x} \in S^2 \tag{4.37}
$$

as in Lemma 2.12 let

$$
\tilde{u}(x) := \begin{cases} u(x), & \textit{for } x \in \Omega_{e_2} \cap B_{r_1^*}, \\ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n^1(\omega \tilde{r}_{e_2}(|x|)) Y_n^m(|x|^{-1}x), & \textit{for } x \in A_{r_1^*, r_2^*}, \end{cases} \tag{4.38}
$$

The sum in (4.38) converges absolutely and uniformly on compact subsets of $A_{r_1^*, r_2^*}$. The same hold for derivatives of u with respect to r up to order two and for derivatives with respect to $\hat{x}$ of arbitrary order and the sum over the term by term derivatives. Moreover it holds $\tilde{u} \in X_{e_2}$. We write $\tilde{u} = \tilde{u}(u)$ to emphasize that $\tilde{u}$ is defined through u.

Proof. For the first part we proceed as in the proof of Lemma 2.13. It remains to show that $\|\tilde{u}\|_{X_{e_2}(A_{r_1^*, r_2^*})} < +\infty$. To this end we estimate as in the proof of Lemma 2.13

$$
\begin{aligned}
&\| |\tilde{d}_{e_2}^2 \tilde{d}_{e_2}|^{1/2} \tilde{u} \circ Q \|^2_{L^2(S_r^2)} \\
&\qquad \leq \tilde{C}(\tilde{u}) r^2 |\tilde{d}_{e_2}(r)^2 d_{e_2}(r)| e^{2\Re(i\omega \tilde{d}_{e_2}(r)/|\tilde{d}_{e_2}(r)|)\sqrt{1 - r_1^{*2}/|\tilde{r}_{e_2}(r)|^2}\sqrt{1+\tilde{\alpha}_{e_2}(r)^2} r}, \\
&\| |\tilde{d}_{e_2}|^{1/2} \nabla_{S^2} \tilde{u} \circ Q \|^2_{L^2(S_r^2)} \\
&\qquad \leq \tilde{C}(\tilde{u}) r^2 |d_{e_2}(r)| e^{2\Re(i\omega \tilde{d}_{e_2}(r)/|\tilde{d}_{e_2}(r)|)\sqrt{1 - r_1^{*2}/|\tilde{r}_{e_2}(r)|^2}\sqrt{1+\tilde{\alpha}_{e_2}(r)^2} r}, \\
&\| |\tilde{d}_{e_2}^2 / \tilde{d}_{e_2}|^{1/2} \partial_r \tilde{u} \circ Q \|^2_{L^2(S_r^2)} \\
&\qquad \leq \tilde{C}(\tilde{u}) r^2 |\tilde{d}_{e_2}(r)^2 d_{e_2}(r)| e^{2\Re(i\omega \tilde{d}_{e_2}(r)/|\tilde{d}_{e_2}(r)|)\sqrt{1 - r_1^{*2}/|\tilde{r}_{e_2}(r)|^2}\sqrt{1+\tilde{\alpha}_{e_2}(r)^2} r}
\end{aligned}
$$

with a constant $\tilde{C}(\tilde{u}) > 0$ for all $r \in (r_1^*, r_2^*)$. We note that

$$
\lim_{r \to +\infty} \Re(i\omega \tilde{d}_{e_2}(r)/|\tilde{d}_{e_2}(r)|)\sqrt{1 - r_1^{*2}/|\tilde{r}_{e_2}(r)|^2} = -\Re(\omega).
$$

The former estimates, the former limit, the domain transformation $\rho(r) := \tilde{\alpha}(r)r$ on (r_1^*, r_2^*), the basic estimate $\sup_{x>0}(1+x^l)e^{-\delta x} < +\infty$ for any $l, \delta > 0$ and Assumption 4.28 allow to bound

$$\||\tilde{d}_{e_2}^2 \tilde{d}_{e_2}|^{1/2}\tilde{u} \circ Q\|^2_{L^2(S_r^2)} + \||\tilde{d}_{e_2}|^{1/2}\nabla_{S^2}\tilde{u} \circ Q\|^2_{L^2(S_r^2)} + \||\tilde{d}_{e_2}^2/\tilde{d}_{e_2}|^{1/2}\partial_r \tilde{u} \circ Q\|^2_{L^2(S_r^2)}$$

by a constant $C < +\infty$ uniformly in $r \in (r_1^*, r_2^*)$. Hence $\|\tilde{u}\|_{X_{e_2}} < +\infty$. □

Lemma 4.30. *For a solution (ω, u) to (2.23) with $\Re(\omega) > 0$ the pair $(\omega, \tilde{u}(u))$ with $\tilde{u}(u)$ as in Lemma 4.29 solves (4.36).*

Proof. Proceed as in the proof of Lemma 2.14. □

Lemma 4.31. *Let $(\omega, \tilde{u})$ be a solution to (4.36) with $\Re(\omega) > 0$. Then*

$$\tilde{u} \circ Q(r, \hat{x}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n h_n^1(\omega \tilde{r}_{e_2}(r))Y_n^m(\hat{x}), \quad r \in (r_1^*, r_2^*), \hat{x} \in S^2 \tag{4.39}$$

and the function

$$u(x) := \begin{cases} \tilde{u}(x), & \text{for } x \in \Omega_{e_2} \cap B_{r_1^*}, \\ \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n h_n^1(\omega|x|)Y_n^m(|x|^{-1}x), & \text{for } x \in B_{r_1^*}^c, \end{cases} \tag{4.40}$$

is well defined in $\tilde{H}^1_{0,\mathrm{loc}}(\Omega)$ and (ω, u) solves (2.23). We write $u = u(\tilde{u})$ to emphasize that u is defined through $\tilde{u}$.

Proof. Proceed as in the proof of Lemma 2.15. □

Theorem 4.32. *Let (ω, u) be a solution to (2.23) with $\Re(\omega) > 0$. Then $(\omega, \tilde{u}(u))$ with $\tilde{u}(u)$ given by (4.38) solves (4.36). Vice-versa if $(\omega, \tilde{u})$ is a solution to (4.36) with $\Re(\omega) > 0$, then $(\omega, u(\tilde{u}))$ with $u(\tilde{u})$ given by (4.40) solves (2.23).*

Proof. Follows from Lemmata 4.29, 4.30, 4.31. □

Assumption 4.33. *Let $\tilde{\alpha}$, r_1^* and r_2^* be as is Assumption 4.28. Let*

1. $\lim_{r \to r_2^*-} \tilde{d}_{e_2}(r)|d_{e_2}(r)|/\big(|\tilde{d}_{e_2}(r)|d_{e_2}(r)\big) = 1,$

2. $\lim_{r \to r_2^*-} \big(\partial_r(\tilde{d}_{e_2}/|\tilde{d}_{e_2}|)\big)(r) = \lim_{r \to r_2^*-} \big(\partial_r(d_{e_2}/|d_{e_2}|)\big)(r) = 0.$

Lemma 4.34. *Let Assumptions 4.28 and 4.33 hold. Then there exists $\tau \in (0, \pi/2)$ so that $\arg\big(d_{e_2}(r)/\tilde{d}_{e_2}(r)\big) \in [0, \tau]$ for all $r \in (r_1^*, r_2^*)$.*

Proof. Proceed as in the proof of Lemma 4.2. □

Lemma 4.35. *Let Assumptions 4.28 and 4.33 hold. For $\omega \in \mathbb{C} \setminus \{0\}$ let*

$$T_{e_2}(\omega)u := \begin{cases} \overline{\frac{|\hat{d}_{e_2}|}{\hat{d}_{e_2}}} u & \text{for } \arg(\omega^2) \in [-\pi, 0), \\ \overline{\frac{\hat{d}_{e_2}}{\tilde{d}_{e_2}^2} \frac{|\tilde{d}_{e_2}|^2}{|\hat{d}_{e_2}|}} u & \text{for } \arg(\omega^2) \in [0, \pi). \end{cases} \tag{4.41}$$

Then $T_{e_2}(\omega) \in L(X)$ is bijective for all $\omega \in \mathbb{C} \setminus \{0\}$.

Proof. Proceed as in the proof of Lemma 4.4. □

Theorem 4.36. *Let Assumptions 4.28 and 4.33 hold. Let $a_{e_2}(\cdot;\cdot,\cdot)$ and X_{e_2} be as in (4.35b), $A_{e_2}(\cdot)$ be as in (3.3), $T_{e_2}(\cdot)$ be as in (4.41) and Λ_{d_0} be as in (4.4). Then $A_{e_2}(\cdot)\colon \Lambda_i \to L(X_{e_2})$ is weakly $T_{e_2}(\cdot)$-coercive.*

Proof. Proceed as in the proof of Theorem 4.5 with Lemma 4.20 instead of Lemma 4.3. □

Lemma 4.37. *Let Assumptions 4.28, 4.33 hold. Let $a_{e_2}(\cdot;\cdot,\cdot)$ and X_{e_2} be as in (4.35b) and $A_{e_2}(\cdot)$ be as in (3.3). For $\epsilon > 0$ and $\omega \in \mathbb{C} \setminus \{0\}$ let*

$$T_{e_2,\epsilon}(\omega)u := \eta_{e_2,\epsilon} u \qquad \text{with} \qquad \eta_{e_2} = \begin{cases} \overline{\frac{|\hat{d}_{e_2}|}{\hat{d}_{e_2}}} & \text{for } \arg(\omega^2) \in [-\pi, 0), \\ \overline{\frac{\hat{d}_{e_2}}{\tilde{d}_{e_2}^2} \frac{|\tilde{d}_{e_2}|^2}{|\hat{d}_{e_2}|}} & \text{for } \arg(\omega^2) \in [0, \pi). \end{cases} \tag{4.42}$$

with $\eta_{e_2,\epsilon}|_{(r_1^*,r_2^*)}$ as in Lemma 4.13 with $r_1 = r_1^*, r_2 = r_2^*$ and $\eta_{e_2,\epsilon}|_{[0,r_1^*]} := \eta_{e_1,\epsilon}(r_1^*)$.

There exists $\epsilon_0(\omega) > 0$ so that for each $\epsilon \le \epsilon_0(\omega)$, $T_{e_2,\epsilon}(\omega) \in L(X_{e_2})$ is bijective for all $\omega \in \mathbb{C} \setminus \{0\}$ and $A_{e_2}(\cdot)\colon \Lambda_i \to L(X_{e_2})$ is weakly $T_{e_2,\epsilon}(\cdot)$-coercive.

Proof. Proceed as in the proof of Lemma 4.21. □

Next we consider a sequence of finite dimensional subspaces $(X_{e_2}^{h(n)})_{n\in\mathbb{N}}$, $X_{e_2}^{h(n)} \subset X_{e_2}$, $n \in \mathbb{N}$ so that the orthogonal projections onto $X_{e_2}^{h(n)}$ converge point-wise to the identity in X_{e_2}. Let

$$\text{find } (\omega, u) \in \mathbb{C} \times X_{e_2}^{h(n)} \setminus \{0\} \text{ so that } \quad a_{e_2}(\omega; u, u') = 0 \quad \text{for all } u' \in X_{e_2}^{h(n)} \tag{4.43}$$

be the Galerkin approximation to (4.36). As in Section 4.4 and Subsection 4.5.1 we make two additional assumptions on the Galerkin spaces $X_{e_2}^{h(n)}$.

Assumption 4.38. *There exists a sequence $\big(h(n)\big)_{n\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$ with $\lim_{n\in\mathbb{N}} h(n) = 0$. There exist bounded linear projection operators $\Pi_{h(n)}^{e_2}\colon X_{e_2} \to X_{e_2}^{h(n)}, n \in \mathbb{N}$ that act locally in the following sense: there exist constants $C_1, R^* > 1$ so that for $n \in \mathbb{N}$, $s \in \{1, 2\}$, $x_0 \in \Omega_{e_2}$, if $B_{R^*h(n)}(x_0) \subset \Omega_{e_1}$, $u \in X_{e_2}$ and $u|_{B_{R^*h(n)}(x_0)} \in H^s(B_{R^*h(n)}(x_0))$, then*

$$\|u - \Pi_{h(n)}^{e_2} u\|_{H^1(B_{h(n)}(x_0))} \le C_1 h(n)^{s-1} \|u\|_{H^s(B_{R^*h(n)}(x_0))}. \tag{4.44}$$

Assumption 4.39. *For any $D \subset \Omega_{e_2}$ which is compact in Ω_{e_2} exists $n_0 > 0$ so that for each $n \in \mathbb{N}, n > n_0$ there exists $u_{D,n} \in X_{e_2}^{h(n)}$ with $u_{D,n}|_D = 1$.*

Theorem 4.40. *Let Assumptions 4.28 and 4.33 hold. Let X_{e_2} be as in (4.35b), $\big(X_{e_2}^{h(n)}\big)_{n\in\mathbb{N}}$ be a sequence of finite dimensional subspaces $X_{e_2}^{h(n)} \subset X_{e_2}$ so that the orthogonal projections onto $X_{e_2}^{h(n)}$ converge point-wise to the identity and so that Assumptions 4.38 and 4.39 hold. Let $T_{e_2,\epsilon_0}(\omega) := T_{e_2,\epsilon_0(\omega)}(\omega)$ be as in Lemma 4.37 and $\|\cdot\|_n$ be as in (3.2). For $n \in \mathbb{N}$ let $\Pi_{h(n)}^{e_2}$ be as in Assumptions 4.22 and*

$$T_{e_2,\epsilon_0}^{h(n)}(\omega) := \Pi_{h(n)}^{e_2} T_{e_2,\epsilon_0}(\omega)|_{X_{e_2}^{h(n)}} \tag{4.45}$$

for $\omega \in \mathbb{C} \setminus \{0\}$. Then $T_{e_2,\epsilon_0}^{h(n)}(\omega) \in L(X_{e_2}^{h(n)})$ is Fredholm with index zero and

$$\lim_{n\in\mathbb{N}} \|T_{e_2,\epsilon_0}(\omega) - T_{e_2,\epsilon_0}^{h(n)}(\omega)\|_n = 0 \tag{4.46}$$

for all $\omega \in \mathbb{C} \setminus \{0\}$.

Proof. Proceed as in the proof of Theorem 4.24. □

Theorem 4.41 (Spectral convergence)**.** *Let Assumptions 4.28 and 4.33 hold. Let X_{e_2}, $a_{e_2}(\cdot;\cdot,\cdot)$ be as defined in (4.35). Let $A_{e_2}(\cdot)\colon \Lambda \to L(X_{e_2})$ be defined through (3.3), T_{e_2,ϵ_0} be as in Lemma 4.37 and $\Lambda_{d_0}^{\pm}$ be as in (4.6). Let $\big(X_{e_2}^{h(n)}\big)_{n\in\mathbb{N}}$ be a sequence of finite dimensional subspaces $X_{e_2}^{h(n)} \subset X_{e_2}$ so that the orthogonal projections from X_{e_2} onto $X_{e_2}^{h(n)}$ converge point-wise to the identity in X_{e_2} and so that Assumptions 4.38 and 4.39 hold. Let $A_{e_2,h(n)}(\cdot)$ be defined by (3.5) and $T_{e_2,\epsilon_0}(\cdot)$ be as in Theorem 4.40.*

Then $A_{e_2}(\cdot)\colon \Lambda_{d_0}^{\pm} \to L(X_{e_2})$ is a weakly $T_{e_2,\epsilon_0}(\cdot)$-coercive holomorphic Fredholm operator function with non-empty resolvent set $\rho\big(A_{e_2}(\cdot)\big)$ and $A_{e_2,h(n)}(\cdot)\colon \Lambda_{d_0}^{\pm} \to L(X_{e_2}^{h(n)})$ is a $T_{e_2,\epsilon_0}(\cdot)$-compatible approximation, i.e. Theorem 3.17 applies.

Proof. Proceed as in the proof of Theorem 4.25. □

Finally we discuss how to choose an appropriate parameter $\tilde{\alpha}_{e_2}$ and an appropriate sequence of subspaces $(X_{e_2}^{h(n)})_{n\in\mathbb{N}}$, $X_{e_2}^{h(n)} \subset X_{e_2}$. As in the previous subsection we state two lemmata to this end.

Lemma 4.42. *Assume that*

$$\sup_{x\in\Omega_{e_2}} \frac{1}{(r_2^* - |x|)|d_{e_2}(x)|} < +\infty. \tag{4.47}$$

Let $(X_{e_2}^{h(n)})_{n\in\mathbb{N}}$, $X_{e_2}^{h(n)} \subset X_{e_2}$ be so that for any $\delta > 0$ and $u \in X_{e_2}$ with $u|_{A_{r_2^*-\delta,r_2^*}} = 0$ it holds

$$\lim_{n\in\mathbb{N}} \inf_{u'\in X_{e_2}^{h(n)}} \|u - u'\|_{X_{e_2}} = 0. \tag{4.48}$$

Then (4.48) holds for any $u \in X_{e_2}$.

Proof. Proceed as in Lemma 4.26. ☐

Lemma 4.43. *Let $\tilde{\alpha}_{e_2}$ be so that for $r \in (r_1^*, r_2^*)$ either*

$$\tilde{\alpha}_{e_2}(r) = -(\ln(r_2^* - r) - \ln(r_2^* - r_1^*)) + r_1^* \tag{4.49a}$$

or

$$\tilde{\alpha}_{e_2}(r) = (r_2^* - r)^\beta - (r_2^* - r_1^*)^\beta + r_1^* \tag{4.49b}$$

with $\beta \in (-2/3, 0)$. Then (4.47) holds. If $u \in H_0^1(\Omega_{e_2})$ is so that $|u(x)| \leq C(r_2^* - |x|)$ for a constant $C > 0$ and all $x \in \Omega_{e_2}$, then $u \in X_{e_2}$.

Proof. Proceed as in Lemma 4.27. ☐

Hence the situation is as in Subsection 4.5.1. Consider $\tilde{\alpha}_{e_2}$ as in Lemma 4.43. Due to Lemma 4.43 common finite element spaces are indeed subspaces of X_{e_2}. Due to Lemma 4.42 $(X_{e_2}^{h(n)})_{n\in\mathbb{N}}$ is asymptotically dense in X_{e_2} if it is so in $H_0^1(\Omega_{e_2})$. Hence with the stated choice of parameter $\tilde{\alpha}_{e_2}$ a reliable discretization of (4.36) can be constructed straightforwardly.

4.6 Asymptotically constant potentials

In this section we discuss which of our results still hold if we include a potential in Differential Equation (2.22a). I.e. we consider the problem to find $(\omega, u) \in \mathbb{C} \setminus \{0\} \times H^1_{\mathrm{loc}}(\Omega) \setminus \{0\}$ so that

$$-\Delta u - \omega^2(1 + p)u = 0 \quad \text{in } \Omega, \tag{4.50a}$$

$$u = 0 \quad \text{at } \partial\Omega, \tag{4.50b}$$

$$u \text{ is outgoing.} \tag{4.50c}$$

We assume $0 \in \Omega^c$ and that the potential p is of the form $p(x) = q(|x|)$ with q being a real analytic function of the form $q(r) = \sum_{n=2}^\infty q_n r^{-n}$ so that $q(r^{-1})$ has convergence radius $1/a_p$ with $a_p \in (0, \infty]$. In this case the formulation and study of meaningful radiation conditions are more intricate. For $\omega > 0$ radiation conditions for this equation were studied in [HSZ03a]. More specifically, in [HSZ03a] Hohage, Schmidt and Zschiedrich prove the equivalence of the Sommerfeld's Condition to a pole condition. In [HSZ03b] they continue their work and prove the equivalence to a complex scaling radiation condition as well as the convergence of a PML method for scattering problems. We also mention [Hei18] which studies a one-dimensional problem and allows q to be even meromorphic. Since the functional framework for the derivation of the radiation conditions is quite intricate, we do not intend to elaborate in this direction. It seems reasonable that a similar result as stated in Theorem 2.16 holds, though. We content ourselves with a discussion of the application of our theory as developed in this chapter.

Let $\tilde{p}(x) := q \circ \tilde{r}(|x|)$. With notation as hitherto the complex scaled formulation of (4.50) reads

$$\text{find } (\omega, \tilde{u}) \in \mathbb{C} \times X \setminus \{0\} \text{ so that} \quad a_p(k; \tilde{u}, u') = 0 \quad \text{for all } u' \in X, \tag{4.51}$$

with sesquilinear form

$$a_p(\omega; u, u') := \langle (\tilde{d}^2 d^{-1}\,\mathrm{P_x} + d(\mathrm{I} - \mathrm{P_x}))\nabla u, \nabla u' \rangle_{L^2(\Omega)} - \omega^2 \langle \tilde{d}^2 d(1+\tilde{p})u, u' \rangle_{L^2(\Omega)} \tag{4.52}$$

and associated operator function $A_p(\cdot)$ defined through (3.3). In our new setting Lemma 4.4 remains true since $T(\cdot)$ is independent of $A(\cdot)$. Since

$$a(\omega; u, u') - a_p(\omega; u, u') = \omega^2 \langle \tilde{d}^2 d\tilde{p}u, u' \rangle_{L^2(\Omega)} \tag{4.53}$$

and due to Lemma 4.3, $A_p(\omega)$ is a compact distortion of $A(\omega)$. As in [HSZ03b, Lemma 4.1] it follows that the resolvent set of $A_p(\cdot)$ is non-empty. Hence Theorem 4.5 and Theorem 4.6 still apply to $A_p(\cdot)$. Lemma 4.8 is independent of $A(\cdot)$ and $A_p(\cdot)$ and hence remains true. Theorem 4.9 holds for $A_p(\cdot)$ too. Lemma 4.10 remains true too. However to obtain an exponential decay one would need to extend the results from [HSZ03b] to hold also for complex valued ω. The approximation results from Sections 4.3 and 4.4 remain true. There are no obstacles to successfully apply the techniques of Section 4.5 either.

Nomenclature

Domains

Ω 18
Ω_{e_1} 59
Ω_{e_2} 64
Ω_n 28

Functions (general)

arg 44
χ_1 23
χ_2 23
χ_3 23
h_n^1 16
h_n^2 16
I 24
j_n 16
ν 13
$\mathrm{P_x}$ 23
$p_n^{|m|}$ 14
Q 14
U_n^m 14
y_n 16
Y_n^m 14
$\hat{y}$ 14
Z_n^m 14

Functions (PML related)

α 23
α_{e_1} 59
α_{e_2} 64
$\hat{\alpha}$ 44
$\tilde{\alpha}$ 22, 44
$\tilde{\alpha}_{e_1}$ 59
$\tilde{\alpha}_{e_2}$ 64
$\tilde{\alpha}_{\text{affin}}$ 22
$\tilde{\alpha}_{\infty}$ 23
$\tilde{\alpha}_{\text{power}}$ 22
$\tilde{\alpha}_{\text{smooth}}$ 23
d 23
d_0 23
d_{e_1} 59
d_{e_2} 64
$\hat{d}$ 44
$\hat{d}_{e_1}$ 59
$\hat{d}_{e_2}$ 64
$\tilde{d}$ 23
$\tilde{d}_{e_1}$ 59
$\tilde{d}_{e_2}$ 64
γ_{e_1} 59
$\tilde{\gamma}_{e_1}$ 59
r_{e_1} 59
$\tilde{r}$ 23
$\tilde{r}_{e_1}$ 59
$\tilde{r}_{e_2}$ 64
x_{e_1} 59

Norms

$\|\cdot\|_n$ 36

Numbers

$\varkappa(A(\cdot),\omega,u)$ 36
$\varkappa(A(\cdot),\omega)$ 36
$\rho\big(A(\cdot)\big)$ 35
$\sigma\big(A(\cdot)\big)$ 35

Operators

$A(\omega)$ 32
$A_{e_1}(\omega)$ 61
$A_{e_2}(\omega)$ 67
$A^{h(n)}(\omega)$ 57
$A_{e_1,h(n)}(\omega)$ 61
$A_{e_2,h(n)}(\omega)$ 68
$A^{-1}(\omega)$ 35
$A_n(\omega)$ 33
$A_p(\omega)$ 70
A^* 33
$A^*(\omega)$ 35
F_{e_1} 60

Bibliography

[ADGK03] Sergey Asvadurov, Vladimir Druskin, Murthy N. Guddati, and Leonid Knizhnerman. On optimal finite-difference approximation of PML. *SIAM J. Numer. Anal.*, 41(1):287–305 (electronic), 2003.

[AFW06] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006.

[AFW10] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc. (N.S.)*, 47(2):281–354, 2010.

[AG97] Saul Abarbanel and David Gottlieb. A mathematical analysis of the PML method. *J. Comput. Phys.*, 134(2):357–363, 1997.

[AGH99] S. Abarbanel, D. Gottlieb, and J.S. Hesthaven. Well-posed perfectly matched layers for advective acoustics. *Journal of Computational Physics*, 154(2):266 – 283, 1999.

[AGH02] S. Abarbanel, D. Gottlieb, and J. S. Hesthaven. Long time behavior of the perfectly matched layer equations in computational electromagnetics. In *Proceedings of the Fifth International Conference on Spectral and High Order Methods (ICOSAHOM-01) (Uppsala)*, volume 17, pages 405–422, 2002.

[AH06] Daniel Appelö and Thomas Hagstrom. Construction of stable PMLs for general 2×2 symmetric hyperbolic systems. In *Hyperbolic problems: theory, numerics and applications. I*, pages 263–270. Yokohama Publ., Yokohama, 2006.

[AH09] Daniel Appelö and Thomas Hagstrom. A general perfectly matched layer model for hyperbolic-parabolic systems. *SIAM J. Sci. Comput.*, 31(5):3301–3323, 2009.

[AHK06] Daniel Appelö, Thomas Hagstrom, and Gunilla Kreiss. Perfectly matched layers for hyperbolic systems: general formulation, well-posedness, and stability. *SIAM J. Appl. Math.*, 67(1):1–23, 2006.

[AK06] Daniel Appelö and Gunilla Kreiss. A new absorbing layer for elastic waves. *J. Comput. Phys.*, 215(2):642–660, 2006.

[AK07] Daniel Appelö and Gunilla Kreiss. Application of a perfectly matched layer to the nonlinear wave equation. *Wave Motion*, 44(7-8):531–548, 2007.

[AST$^+$09] Junko Asakura, Tetsuya Sakurai, Hiroto Tadano, Tsutomu Ikegami, and Kinji Kimura. A numerical method for nonlinear eigenvalue problems using contour integrals. *JSIAM Letters*, 1:52–55, 2009.

[AT85] P. M. Anselone and M. L. Treuden. Regular operator approximation theory. *Pacific J. Math.*, 120(2):257–268, 1985.

[BBBL04] Éliane Bécache, Anne-Sophie Bonnet-BenDhia, and Guillaume Legendre. Perfectly matched layers for the convected Helmholtz equation. *SIAM Journal on Numerical Analysis*, 42:409–433, 2004.

[BBBL06] Éliane Bécache, Anne-Sophie Bonnet-BenDhia, and Guillaume Legendre. Perfectly matched layers for time-harmonic acoustics in the presence of a uniform flow. *SIAM J. Numer. Anal.*, 44(3):1191–1217, 2006.

[BBCC12] Anne-Sophie Bonnet-BenDhia, Lucas Chesnel, and Patrick Ciarlet. T-coercivity for scalar interface problems between dielectrics and metamaterials. *Math. Mod. Num. Anal.*, 46:363–1387, 2012.

[BBCC14] Anne-Sophie Bonnet-BenDhia, Lucas Chesnel, and Patrick Ciarlet. T-coercivity for the maxwell problem with sign-changing coefficients. *Communications in Partial Differential Equations*, 39:1007–1031, 2014.

[BBCC18] Anne-Sophie Bonnet-BenDhia, Camille Carvalho, and Patrick Ciarlet. Mesh requirements for the finite element approximation of problems with sign-changing coefficients. *Numerische Mathematik*, 138(4):801–838, Apr 2018.

[BBCCC16] A.-S. Bonnet-BenDhia, C. Carvalho, L. Chesnel, and P. Ciarlet. On the use of Perfectly Matched Layers at corners for scattering problems with sign-changing coefficients. *J. Comput. Phys.*, 322:224–247, 2016.

[BBCL14] Anne-Sophie Bonnet-BenDhia, Colin Chambeyron, and Guillaume Legendre. On the use of perfectly matched layers in the presence of long or backward guided elastic waves. *Wave Motion*, 51(2):266–283, 2014.

[BBCP18] Anne-Sophie Bonnet-BenDhia, Lucas Chesnel, and Vincent Pagneux. Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 474(2213), 2018.

[BBCZ10] Anne-Sophie Bonnet-BenDhia, Patrick Ciarlet, and Carlo Maria Zwölf. Time harmonic wave diffraction problems in materials with sign-shifting coefficients. *Journal of Computational and Applied Mathematics*, 234(6):1912–1919, 2010.

[BBDFT18] Anne-Sophie Bonnet-Ben Dhia, Sonia Fliss, and Yohanes Tjandrawidjaja. Numerical analysis of the Half-Space Matching method with Robin traces on a convex polygonal scatterer. working paper or preprint, May 2018.

[BBG00] Daniele Boffi, Franco Brezzi, and Lucia Gastaldi. On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form. *Math. Comp.*, 69(229):121–140, 2000.

[BBGH11] Anne-Sophie Bonnet-BenDhia, Benjamin Goursaud, and Christophe Hazard. Mathematical analysis of the junction of two acoustic open waveguides. *SIAM J. Appl. Math.*, 71(6):2048–2071, 2011.

[BC03] Ushnish Basu and Anil K. Chopra. Perfectly matched layers for time-harmonic elastodynamics of unbounded domains: theory and finite-element implementation. *Computer Methods in Applied Mechanics and Engineering*, 192(11–12):1337–1375, 2003.

[BCW05] Gang Bao, Zhiming Chen, and Haijun Wu. Adaptive finite-element method for diffraction gratings. *J. Opt. Soc. Amer. A*, 22(6):1106–1114, 2005.

[Bér94] Jean-Pierre Bérenger. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 114(2):185–200, 1994.

[Bér96a] Jean-Pierre Bérenger. Perfectly matched layer for the fdtd solution of wave-structure interaction problems. *Antennas and Propagation, IEEE Transactions on*, 44(1):110–117, 1996.

[Bér96b] Jean-Pierre Bérenger. Three-dimensional perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 127(2):363–379, 1996.

[Ber99] Silvia Bertoluzza. The discrete commutator property of approximation spaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(12):1097–1102, 1999.

[Bér07] Jean-Pierre Bérenger. *Perfectly Matched Layer (PML) for Computational Electromagnetics*. Synthesis lectures on computational electromagnetics. Morgan & Claypool Publishers, 2007.

[Bey12] Wolf-Jürgen Beyn. An integral method for solving nonlinear eigenvalue problems. *Linear Algebra Appl.*, 436(10):3839–3863, 2012.

[BFJ03] Éliane Bécache, Sandrine Fauqueux, and Patrick Joly. Stability of perfectly matched layers, group velocities and anisotropic waves. *J. Comput. Phys.*, 188(2):399–433, 2003.

[BHNPR08] A. Bermúdez, L. Hervella-Nieto, A. Prieto, and R. Rodríguez. An exact bounded perfectly matched layer for time-harmonic scattering problems. *SIAM J. Sci. Comput.*, 30(1):312–338, 2007/08.

[BHNPR04] A. Bermúdez, L. Hervella-Nieto, A. Prieto, and R. Rodríguez. An exact bounded PML for the Helmholtz equation. *C. R. Math. Acad. Sci. Paris*, 339(11):803–808, 2004.

[BHNPR06] A. Bermúdez, L. Hervella-Nieto, A. Prieto, and R. Rodríguez. Numerical simulation of time-harmonic scattering problems with an optimal PML. *Sci. Ser. A Math. Sci. (N.S.)*, 13:58–71, 2006.

[BHNPR07] A. Bermúdez, L. Hervella-Nieto, A. Prieto, and R. Rodríguez. An optimal perfectly matched layer with unbounded absorbing function for time-harmonic acoustic scattering problems. *Journal of Computational Physics*, 223(2):469–488, 2007.

[BJ02] Élianeliane Bécache and Patrick Joly. On the analysis of Bérenger's perfectly matched layers for Maxwell's equations. *M2AN Math. Model. Numer. Anal.*, 36(1):87–119, 2002.

[BJK17] Éliane Bécache, Patrick Joly, and Maryna Kachanovska. Stable perfectly matched layers for a cold plasma in a strong background magnetic field. *Journal of Computational Physics*, 341:76–101, 7 2017.

[BJKV15] Éliane Bécache, Patrick Joly, Maryna Kachanovska, and Valentin Vinoles. Perfectly matched layers in negative index metamaterials and plasmas. In *CANUM 2014—42e Congrès National d'Analyse Numérique*, volume 50 of *ESAIM Proc. Surveys*, pages 113–132. EDP Sci., Les Ulis, 2015.

[BJV18] Éliane Bécache, Patrick Joly, and Valentin Vinoles. On the analysis of perfectly matched layers for a class of dispersive media and application to negative index metamaterials. *Mathematics of Computations*, 87:2775–2810, 11 2018.

[BK17] Bécache, Éliane and Kachanovska, Maryna. Stable perfectly matched layers for a class of anisotropic dispersive models. part i: necessary and sufficient conditions of stability. *ESAIM: M2AN*, 51(6):2399–2434, 2017.

[BLW10] Gang Bao, Peijun Li, and Haijun Wu. An adaptive edge element method with perfectly matched absorbing layers for wave scattering by biperiodic structures. *Math. Comp.*, 79(269):1–34, 2010.

[BMBB15] T. Z. Boulmezaoud, S. Mziou, B. Boudjedaa, and M. M. Babatin. Inverted finite elements for degenerate and radial elliptic problems in unbounded domains. *Jpn. J. Ind. Appl. Math.*, 32(1):237–261, 2015.

[BO91] I. Babuška and J. Osborn. Eigenvalue problems. In *Finite Element Methods (Part 1)*, volume 2 of *Handbook of Numerical Analysis*, pages 641 – 787. Elsevier, 1991.

[Bof10] Daniele Boffi. Finite element approximation of eigenvalue problems. *Acta Numer.*, 19:1–120, 2010.

[Bou05] Tahar Zamène Boulmezaoud. Inverted finite elements: a new method for solving elliptic problems in unbounded domains. *M2AN Math. Model. Numer. Anal.*, 39(1):109–145, 2005.

[BP07] James H. Bramble and Joseph E. Pasciak. Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems. *Math. Comp.*, 76(258):597–614 (electronic), 2007.

[BP08] James H. Bramble and Joseph E. Pasciak. Analysis of a finite element PML approximation for the three dimensional time-harmonic Maxwell problem. *Math. Comp.*, 77(261):1–10 (electronic), 2008.

[BP12a] Élianeliane Bécache and Andrés Prieto. Remarks on the stability of Cartesian PMLs in corners. *Appl. Numer. Math.*, 62(11):1639–1653, 2012.

[BP12b] James H. Bramble and Joseph E. Pasciak. Analysis of Cartesian PML approximation to the three dimensional electromagnetic wave scattering problem. *Int. J. Numer. Anal. Model.*, 9(3):543–561, 2012.

[BP13] James H. Bramble and Joseph E. Pasciak. Analysis of a Cartesian PML approximation to acoustic scattering problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. *J. Comput. Appl. Math.*, 247:209–230, 2013.

[BPG04] Élianeliane Bécache, Peter G. Petropoulos, and Stephen D. Gedney. On the long-time behavior of unsplit perfectly matched layers. *IEEE Trans. Antennas and Propagation*, 52(5):1335–1342, 2004.

[BPT10] James H. Bramble, Joseph E. Pasciak, and Dimitar Trenev. Analysis of a finite PML approximation to the three dimensional elastic wave scattering problem. *Math. Comp.*, 79(272):2079–2101, 2010.

[BS08] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[Buf05] Annalisa Buffa. Remarks on the discretization of some noncoercive operator with applications to heterogeneous maxwell equations. *SIAM Journal on Numerical Analysis*, 43(1):1–18, 2005.

[BW05] Gang Bao and Haijun Wu. Convergence analysis of the perfectly matched layer problems for time-harmonic Maxwell's equations. *SIAM J. Numer. Anal.*, 43(5):2121–2143, 2005.

[Car15] Camille Carvalho. *Etude mathématique et numérique de structures plasmonique avec des coins.* PhD thesis, 12 2015.

[CC08] Junqing Chen and Zhiming Chen. An adaptive perfectly matched layer technique for 3-D time-harmonic electromagnetic scattering problems. *Math. Comp.*, 77(262):673–698, 2008.

[CCZ13] Zhiming Chen, Tao Cui, and Linbo Zhang. An adaptive anisotropic perfectly matched layer method for 3-D time harmonic electromagnetic scattering problems. *Numer. Math.*, 125(4):639–677, 2013.

[Ces96] Michel Cessenat. *Mathematical methods in electromagnetism*, volume 41 of *Series on Advances in Mathematics for Applied Sciences*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. Linear theory and applications.

[CFKS87] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer-Verlag, Berlin, study edition, 1987.

[CFR00] Salvatore Caorsi, Paolo Fernandes, and Mirco Raffetto. On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems. *SIAM J. Numer. Anal.*, 38(2):580–607 (electronic), 2000.

[CGX09] Zhiming Chen, Benqi Guo, and Yuanming Xiao. An *hp* adaptive uniaxial perfectly matched layer method for Helmholtz scattering problems. *Commun. Comput. Phys.*, 5(2-4):546–564, 2009.

[Che09] Zhiming Chen. Convergence of the time-domain perfectly matched layer method for acoustic scattering problems. *Int. J. Numer. Anal. Model.*, 6(1):124–146, 2009.

[Cia12] Patrick Ciarlet, Jr. T-coercivity: application to the discretization of Helmholtz-like problems. *Comput. Math. Appl.*, 64(1):22–34, 2012.

[CJK17] Maxence Cassier, Patrick Joly, and Maryna Kachanovska. Mathematical models for dispersive electromagnetic waves: An overview. *Computers and Mathematics with Applications*, 74 (11):2792–2830, 2017.

[CK98] David Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 1998.

[CK13] David Colton and Rainer Kress. *Integral equation methods in scattering theory*, volume 72 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013. Reprint of the 1983 original [MR0700400].

[CL05] Zhiming Chen and Xuezhe Liu. An adaptive perfectly matched layer technique for time-harmonic scattering problems. *SIAM J. Numer. Anal.*, 43(2):645–671 (electronic), 2005.

[CL08] Michel Castaings and Michael Lowe. Finite element model for waves guided along solid systems of arbitrary section coupled to infinite solid media. *The Journal of the Acoustical Society of America*, 123(2):696–708, 2008.

[CLX13] Zhiming Chen, Chao Liang, and Xueshuang Xiang. An anisotropic perfectly matched layer method for Helmholtz scattering problems with discontinuous wave number. *Inverse Probl. Imaging*, 7(3):663–678, 2013.

[CM98a] F. Collino and P. B. Monk. Optimizing the perfectly matched layer. *Comput. Methods Appl. Mech. Engrg.*, 164(1-2):157–171, 1998. Exterior problems of wave propagation (Boulder, CO, 1997; San Francisco, CA, 1997).

[CM98b] Francis Collino and Peter Monk. The perfectly matched layer in curvilinear coordinates. *SIAM J. Sci. Comput.*, 19(6):2061–2090 (electronic), 1998.

[CMH15] Radu Cimpeanu, Anton Martinsson, and Matthias Heil. A parameter-free perfectly matched layer formulation for the finite-element-based solution of the Helmholtz equation. *J. Comput. Phys.*, 296:329–347, 2015.

[CT01] Francis Collino and Chrysoula Tsogka. Application of the perfectly matched absorbing layer model to the linear elastodynamic problem in anisotropic heterogeneous media. *Geophysics*, 66(1):294–307, 2001.

[Cum04] S. A. Cummer. Perfectly matched layer behavior in negative refractive index materials. *IEEE Antennas and Wireless Propagation Letters*, 3(1):172–175, Dec 2004.

[CW94] W. C. Chew and W. H. Weedon. A 3d perfectly matched medium from modified Maxwell's equations with stretched coordinates. *Microwave Optical Tech. Letters*, 7:590–604, 1994.

[CW03] Zhiming Chen and Haijun Wu. An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures. *SIAM J. Numer. Anal.*, 41(3):799–826, 2003.

[CW08] Zhiming Chen and Xinming Wu. An adaptive uniaxial perfectly matched layer method for time-harmonic scattering problems. *Numer. Math. Theory Methods Appl.*, 1(2):113–137, 2008.

[CW12] Zhiming Chen and Xinming Wu. Long-time stability and convergence of the uniaxial perfectly matched layer method for time-domain acoustic scattering problems. *SIAM J. Numer. Anal.*, 50(5):2632–2655, 2012.

[CW13] Snorre H. Christiansen and Ragnar Winther. On variational eigenvalue approximation of semidefinite operators. *IMA J. Numer. Anal.*, 33(1):164–189, 2013.

[CWW09] Jie Chen, Desheng Wang, and Haijun Wu. An adaptive finite element method with a modified perfectly matched layer formulation for diffraction gratings. *Commun. Comput. Phys.*, 6(2):290–318, 2009.

[CXZ16] Zhiming Chen, Xueshuang Xiang, and Xiaohui Zhang. Convergence of the PML method for elastic wave scattering problems. *Math. Comp.*, 85(302):2687–2714, 2016.

[CZ10] Zhiming Chen and Weiying Zheng. Convergence of the uniaxial perfectly matched layer method for time-harmonic scattering problems in two-layered media. *SIAM J. Numer. Anal.*, 48(6):2158–2185, 2010.

[DFT18] Anne-Sophie Bonnet-Ben Dhia, Sonia Fliss, and Antoine Tonnoir. The halfspace matching method: A new method to solve scattering problems in infinite media. *Journal of Computational and Applied Mathematics*, 338:44 – 68, 2018.

[DG98] L Demkowicz and K Gerdes. Convergence of the infinite element methods for the Helmholtz equation in separable domains. *Numer. Math.*, 79:11–42, 1998.

[DGK16] Vladimir Druskin, Stefan Güttel, and Leonid Knizhnerman. Near-Optimal Perfectly Matched Layers for Indefinite Helmholtz Problems. *SIAM Rev.*, 58(1):90–116, 2016.

[DH07] Tomáš Dohnal and Thomas Hagstrom. Perfectly matched layers in photonics computations: 1D and 2D nonlinear coupled mode equations. *J. Comput. Phys.*, 223(2):690–710, 2007.

[dHvdBR02] A. T. de Hoop, P. M. van den Berg, and R. F. Remis. Absorbing boundary conditions and perfectly matched layers - an analytic time-domain performance analysis. *IEEE Transactions on Magnetics*, 38(2):657–660, Mar 2002.

[DI01] Leszek Demkowicz and Frank Ihlenburg. Analysis of a coupled finite-infinite element method for exterior Helmholtz problems. *Numer. Math.*, 88(1):43–73, 2001.

[Dia05] Julien Diaz. *Approches analytiques et numériques de problèmes de transmission en propagation d'ondes en régime transitoire. Application au couplage fluide-structure et aux méthodes de couches parfaitement adaptées*. PhD thesis, Université Paris 6, 2005.

[DJ03] Julien Diaz and Patrick Joly. Stabilized perfectly matched layer for advective acoustics. In *Mathematical and numerical aspects of wave propagation—WAVES 2003*, pages 115–119. Springer, Berlin, 2003.

[DJ06] Julien Diaz and Patrick Joly. A time domain analysis of pml models in acoustics. *Computer Methods in Applied Mechanics and Engineering*, 195, 29-32:3820–3853, 2006.

[DK12a] Kenneth Duru and Gunilla Kreiss. On the accuracy and stability of the perfectly matched layer in transient waveguides. *J. Sci. Comput.*, 53(3):642–671, 2012.

[DK12b] Kenneth Duru and Gunilla Kreiss. A well-posed and discretely stable perfectly matched layer for elastic wave equations in second order formulation. *Commun. Comput. Phys.*, 11(5):1643–1672, 2012.

[DK14a] Kenneth Duru and Gunilla Kreiss. Boundary waves and stability of the perfectly matched layer for the two space dimensional elastic wave equation in second order form. *SIAM J. Numer. Anal.*, 52(6):2883–2904, 2014.

[DK14b] Kenneth Duru and Gunilla Kreiss. Numerical interaction of boundary waves with perfectly matched layers in two space dimensional elastic waveguides. *Wave Motion*, 51(3):445–465, 2014.

[DKK15] Kenneth Duru, Jeremy E. Kozdon, and Gunilla Kreiss. Boundary conditions and stability of a perfectly matched layer for the elastic wave equation in first order form. *J. Comput. Phys.*, 303:372–395, 2015.

[DKM14] Kenneth Duru, Gunilla Kreiss, and Ken Mattsson. Stable and high-order accurate boundary treatments for the elastic wave equation on second-order form. *SIAM J. Sci. Comput.*, 36(6):A2787–A2818, 2014.

[DNR78a] Jean Descloux, Nabil Nassif, and Jacques Rappaz. On spectral approximation. part 1. the problem of convergence. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 12(2):97–112, 1978.

[DNR78b] Jean Descloux, Nabil Nassif, and Jacques Rappaz. On spectral approximation. part 2. error estimates for the galerkin method. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 12(2):113–119, 1978.

[DS06] L. Demkowicz and Jie Shen. A few new (?) facts about infinite elements. *Comput. Methods Appl. Mech. Engrg.*, 195(29-32):3572–3590, 2006.

[Dur14] Kenneth Duru. A perfectly matched layer for the time-dependent wave equation in heterogeneous and layered media. *J. Comput. Phys.*, 257(part A):757–781, 2014.

[Dur16] Kenneth Duru. The role of numerical boundary procedures in the stability of perfectly matched layers. *SIAM J. Sci. Comput.*, 38(2):A1171–A1194, 2016.

[EG04] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[EM77] Björn Engquist and Andrew Majda. Absorbing boundary conditions for numerical simulation of waves. *Proc. Nat. Acad. Sci. U.S.A.*, 74(5):1765–1766, 1977.

[EY11] Björn Engquist and Lexing Ying. Sweeping preconditioner for the Helmholtz equation: moving perfectly matched layers. *Multiscale Model. Simul.*, 9(2):686–710, 2011.

[FGS98] W. Freeden, T. Gervens, and M. Schreiner. *Constructive approximation on the sphere*. Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, 1998. With applications to geomathematics.

[FLCB08] Z. Fan, M. J. S. Lowe, M. Castaings, and C. Bacon. Torsional waves propagation along a waveguide of arbitrary cross section immersed in a perfect fluid. *The Journal of the Acoustical Society of America*, 124(4):2002–2010, 2008.

[GGK90] Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek. *Classes of linear operators. Vol. I*, volume 49 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1990.

[Giv92] Dan Givoli. *Numerical methods for problems in infinite domains*, volume 33 of *Studies in Applied Mechanics*. Elsevier Scientific Publishing Co., Amsterdam, 1992.

[Giv04] D. Givoli. High-order local non-reflecting boundary conditions: a review. *Wave Motion*, 39:319–326, 2004.

[GJ73] RolfDieter Grigorieff and Hansgeorg Jeggle. Approximation von Eigenwertproblemen bei nichtlinearer Parameterabhängigkeit. *manuscripta mathematica*, 10(3):245–271, 1973.

[GK95] Marcus J. Grote and Joseph B. Keller. On nonreflecting boundary conditions. *J. Comput. Phys.*, 122(2):231–243, 1995.

[GK04] Marcus J. Grote and Christoph Kirsch. Dirichlet-to-Neumann boundary conditions for multiple scattering problems. *J. Comput. Phys.*, 201(2):630–650, 2004.

[GL09] Israel Gohberg and Jürgen Leiterer. *Holomorphic operator functions of one variable and applications*, volume 192 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2009. Methods from complex analysis in several variables.

[Gri73] Rolf Dieter Grigorieff. Zur Theorie linearer approximationsregulärer Operatoren. I, II. *Math. Nachr.*, 55:233–249; ibid. 55 (1973), 251–263, 1973.

[Hac09] W. Hackbusch. *Hierarchische Matrizen: Algorithmen und Analysis*. Springer Berlin Heidelberg, 2009.

[Hag03] Thomas Hagstrom. A new construction of perfectly matched layers for hyperbolic systems with applications to the linearized Euler equations. In *Mathematical and numerical aspects of wave propagation—WAVES 2003*, pages 125–129. Springer, Berlin, 2003.

[Hal16] Martin Halla. Convergence of Hardy space infinite elements for Helmholtz scattering and resonance problems. *SIAM J. Numer. Anal.*, 54(3):1385–1400, 2016.

[Hei18] Thomas Heitzinger. Complex scaling for one-dimensional resonance problems in inhomogeneous exterior domains. Master's thesis, TU Wien, 2018.

[Hes98] J. S. Hesthaven. On the analysis and construction of perfectly matched layers for the linearized Euler equations. *J. Comput. Phys.*, 142(1):129–147, 1998.

[HHK04] Stefan Hein, Thorsten Hohage, and Werner Koch. On resonances in open systems. *J. Fluid Mech.*, 506:255–284, 2004.

[HHKS07] Stefan Hein, Thorsten Hohage, Werner Koch, and Joachim Schöberl. Acoustic resonances in high lift configuration. *J. Fluid Mech.*, 582:179–202, 2007.

[HHNS16] Martin Halla, Thorsten Hohage, Lothar Nannen, and Joachim Schöberl. Hardy space infinite elements for time harmonic wave equations with phase and group velocities of different signs. *Numer. Math.*, 133(1):103–139, 2016.

[Hip02] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002.

[HL05] Jean Paul Hugonin and Philippe Lalanne. Perfectly matched layers as nonlinear coordinate transforms: a generalized formalization. *J. Opt. Soc. Amer. A*, 22(9):1844–1849, 2005.

[HM06] R. Hiptmair and P. Meury. Stabilized FEM-BEM coupling for Helmholtz transmission problems. *SIAM J. Numer. Anal.*, 44(5):2107–2130, 2006.

[HN09] Thorsten Hohage and Lothar Nannen. Hardy space infinite elements for scattering and resonance problems. *SIAM J. Numer. Anal.*, 47(2):972–996, 2009.

[HN15a] Martin Halla and Lothar Nannen. Hardy space infinite elements for time-harmonic two-dimensional elastic waveguide problems. *Wave Motion*, 59:94 – 110, 2015.

[HN15b] Thorsten Hohage and Lothar Nannen. Convergence of infinite element methods for scalar waveguide problems. *BIT Numerical Mathematics*, 55(1):215–254, 2015.

[HN18] Martin Halla and Lothar Nannen. Two scale hardy space infinite elements for scalar waveguide problems. *Advances in Computational Mathematics*, 44(3):611–643, Jun 2018.

[HS96] P. D. Hislop and I. M. Sigal. *Introduction to spectral theory*, volume 113 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. With applications to Schrödinger operators.

[HST00] Isaac Harari, Michael Slavutin, and Eli Turkel. Analytical and numerical studies of a finite element pml for the helmholtz equation. *Journal of Computational Acoustics*, 8(01):121–137, 2000.

[HSZ03a] Thorsten Hohage, Frank Schmidt, and Lin Zschiedrich. Solving time-harmonic scattering problems based on the pole condition. I. Theory. *SIAM J. Math. Anal.*, 35(1):183–210, 2003.

[HSZ03b] Thorsten Hohage, Frank Schmidt, and Lin Zschiedrich. Solving time-harmonic scattering problems based on the pole condition. II. Convergence of the PML method. *SIAM J. Math. Anal.*, 35(3):547–560, 2003.

[Hu96] Fang Q. Hu. On Absorbing Boundary Conditions for Linearized Euler Equations by a Perfectly Matched Layer. *Journal of Computational Physics*, 129(1):201–219, 1996.

[Hu01] Fang Q Hu. A Stable, Perfectly Matched Layer for Linearized Euler Equations in Unsplit Physical Variables. *Journal of Computational Physics*, 173(2):455–480, 2001.

[Ihl98] Frank Ihlenburg. *Finite element analysis of acoustic scattering*, volume 132 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[Jol12] Patrick Joly. An elementary introduction to the construction and the analysis of perfectly matched layers for time domain wave propagation. *S⃗eMA J.*, 57:5–48, 2012.

[JW78] Hansgeorg Jeggle and Wolfgang Wendland. On the discrete approximation of eigenvalue problems with holomorphic parameter dependence. *Proc. Roy. Soc. Edinburgh Sect. A*, 78(1-2):1–29, 1977/78.

[JZ12] Xue Jiang and Weiying Zheng. Adaptive perfectly matched layer method for multiple scattering problems. *Comput. Methods Appl. Mech. Engrg.*, 201/204:42–52, 2012.

[Kal11] Victor Kalvin. Perfectly matched layers for diffraction gratings in inhomogeneous media. Stability and error estimates. *SIAM J. Numer. Anal.*, 49(1):309–330, 2011.

[Kal12] Victor Kalvin. Limiting absorption principle and perfectly matched layer method for Dirichlet Laplacians in quasi-cylindrical domains. *SIAM J. Math. Anal.*, 44(1):355–382, 2012.

[Kal13] Victor Kalvin. Analysis of perfectly matched layer operators for acoustic scattering on manifolds with quasicylindrical ends. *J. Math. Pures Appl. (9)*, 100(2):204–219, 2013.

[Kar96a] Otto Karma. Approximation in eigenvalue problems for holomorphic Fredholm operator functions. I. *Numer. Funct. Anal. Optim.*, 17(3-4):365–387, 1996.

[Kar96b] Otto Karma. Approximation in eigenvalue problems for holomorphic Fredholm operator functions. II. (Convergence rate). *Numer. Funct. Anal. Optim.*, 17(3-4):389–408, 1996.

[KD13] Gunilla Kreiss and Kenneth Duru. Discrete stability of perfectly matched layers for anisotropic wave equations in first and second order formulation. *BIT*, 53(3):641–663, 2013.

[Kim09] Seungil Kim. *Analysis of a Pml Method Applied to Computation of Resonances in Open Systems and Acoustic Scattering Problems*. PhD thesis, College Station, TX, USA, 2009. AAI3384266.

[Kim14] Seungil Kim. Cartesian PML approximation to resonances in open systems in $\mathbb{R}^2$. *Appl. Numer. Math.*, 81:50–75, 2014.

[KL04] Heinz-Otto Kreiss and Jens Lorenz. *Initial-boundary value problems and the Navier-Stokes equations*, volume 47 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2004. Reprint of the 1989 edition.

[KM97] M. Kuzuoglu and R. Mittra. Investigation of nonplanar perfectly matched absorbers for finite-element mesh truncation. *IEEE Transactions on Antennas and Propagation*, 45(3):474–486, Mar 1997.

[KM99] Vladimir Kozlov and Vladimir Maz′ya. *Differential equations with operator coefficients with applications to boundary value problems for partial differential equations*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[KP09] Seungil Kim and Joseph E. Pasciak. The computation of resonances in open systems using a perfectly matched layer. *Math. Comp.*, 78(267):1375–1398, 2009.

[KP10a] Seungil Kim and Joseph E. Pasciak. Analysis of a Cartesian PML approximation to acoustic scattering problems in $\mathbb{R}^2$. *J. Math. Anal. Appl.*, 370(1):168–186, 2010.

[KP10b] Seungil Kim and Joseph E. Pasciak. Analysis of the spectrum of a Cartesian perfectly matched layer (PML) approximation to acoustic scattering problems. *J. Math. Anal. Appl.*, 361(2):420–430, 2010.

[Kre99] Rainer Kress. *Linear integral equations*, volume 82 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1999.

[Lig65] M. J. Lighthill. Group velocity. *J. Inst. Math. Appl.*, 1:1–28, 1965.

[LLS01] Matti Lassas, Jukka Liukkonen, and Erkki Somersalo. Complex Riemannian metric and absorbing boundary conditions. *J. Math. Pures Appl. (9)*, 80(7):739–768, 2001.

[LN10] Nicolas Lantos and Frédéric Nataf. Perfectly matched layers for the heat and advection-diffusion equations. *J. Comput. Phys.*, 229(24):9042–9052, 2010.

[LRL12] Jérôme Le Rousseau and Gilles Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 18(3):712–747, 2012.

[LS98] M. Lassas and E. Somersalo. On the existence and the convergence of the solution of the pml equations. *Computing*, 60:229–241, 1998.

[LS01] M. Lassas and E. Somersalo. Analysis of the pml equations in general convex geometry. *Proceedings of the Royal Society of Edinburgh. Sect. A. Mathematics 131*, 5:1183–1207, 2001.

[LVLH92] M Lenoir, M Vullierme-Ledard, and C Hazard. Variational formulations for the determination of resonant states in scattering problems. *SIAM J. Math. Anal.*, 23:579–608, 1992.

[LY16] Fei Liu and Lexing Ying. Additive sweeping preconditioner for the Helmholtz equation. *Multiscale Model. Simul.*, 14(2):799–822, 2016.

[Mas87] M. Masmoudi. Numerical solution for exterior problems. *Numer. Math.*, 51(1):87–101, 1987.

[MDG14] A. Modave, E. Delhez, and C. Geuzaine. Optimizing perfectly matched layers in discrete contexts. *Internat. J. Numer. Methods Engrg.*, 99(6):410–437, 2014.

[MDKP07] C. Michler, L. Demkowicz, J. Kurtz, and D. Pardo. Improving the performance of perfectly matched layers by means of *hp*-adaptivity. *Numer. Methods Partial Differential Equations*, 23(4):832–858, 2007.

[Moi98] N Moiseyev. Quantum theory of resonances: Calculating energies, width and cross-sections by complex scaling. *Physics reports*, 302:211–293, 1998.

[Mon03] Peter Monk. *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[MS11] J. Melenk and S. Sauter. Wavenumber explicit convergence analysis for galerkin discretizations of the helmholtz equation. *SIAM Journal on Numerical Analysis*, 49(3):1210–1243, 2011.

[Nan08] Lothar Nannen. *Hardy-Raum Methoden zur numerischen Lösung von Streu- und Resonanzproblemen auf unbeschränkten Gebieten*. PhD thesis, Georg-August-Universität Göttingen, Der Andere Verlag, Tönning, 2008.

[Nan16] Lothar Nannen. personal communication, 2016.

[Nat06] Frédéric Nataf. A new approach to perfectly matched layers for the linearized Euler system. *J. Comput. Phys.*, 214(2):757–772, 2006.

[Néd01] J C Nédélec. *Acoustic and Electromagnic Equations: Integral Respresentations for Harmonic Problems*. Springer, New York, 2001.

[NHSS13] Lothar Nannen, Thorsten Hohage, Achim Schädle, and Joachim Schöberl. Exact Sequences of High Order Hardy Space Infinite Elements for Exterior Maxwell Problems. *SIAM J. Sci. Comput.*, 35(2):A1024–A1048, 2013.

[NK11] Anna Nissen and Gunilla Kreiss. An optimized perfectly matched layer for the Schrödinger equation. *Commun. Comput. Phys.*, 9(1):147–179, 2011.

[NNH04] IM Navon, Beny Neta, and MY Hussaini. A perfectly matched layer approach to the linearized shallow water equations models. *Monthly Weather Review*, 132(6):1369–1378, 2004.

[NW18] Lothar Nannen and Markus Wess. Computing scattering resonances using perfectly matched layers with frequency dependent scaling functions. *BIT Numerical Mathematics*, 58(2):373–395, Jun 2018.

[NW19] Lothar Nannen and Markus Wess. Complex scaled infinite elements for exterior helmholtz problems. Technical report, 2019. https://arxiv.org/abs/1907.09746.

[Pet00] Peter G. Petropoulos. Reflectionless sponge layers as absorbing boundary conditions for the numerical solution of Maxwell equations in rectangular, cylindrical, and spherical coordinates. *SIAM J. Appl. Math.*, 60(3):1037–1058 (electronic), 2000.

[QG98] Quan Qi and Thomas L Geers. Evaluation of the perfectly matched layer for computational acoustics. *Journal of Computational Physics*, 139(1):166–183, 1998.

[Rap77] J. Rappaz. Approximation of the spectrum of a non-compact operator given by the magnetohydrodynamic stability of a plasma. *Numer. Math.*, 28(1):15–24, 1977.

[RGB10] Daniel Rabinovich, Dan Givoli, and Eliane Bécache. Comparison of high-order absorbing boundary conditions and perfectly matched layers in the frequency domain. *Int. J. Numer. Methods Biomed. Eng.*, 26(10):1351–1369, 2010.

[RS78] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[RSS13] Daniel Ruprecht, Achim Schädle, and Frank Schmidt. Transparent boundary conditions based on the pole condition for time-dependent, two-dimensional problems. *Numer. Methods Partial Differential Equations*, 29(4):1367–1390, 2013.

[RT15] Igor Rodnianski and Terence Tao. Effective limiting absorption principles, and applications. *Comm. Math. Phys.*, 333(1):1–95, 2015.

[SAC07] Elizabeth A. Skelton, Samuel D. M. Adams, and Richard V. Craster. Guided elastic waves and perfectly matched layers. *Wave Motion*, 44(7-8):573–592, 2007.

[Sim73] B Simon. The theory of resonances for dilation analytic potentials and the foundations of time dependent perturbation theory. *Ann. Math.*, 97:247–274, 1973.

[Sim78] Barry Simon. Resonances and complex scaling: A rigorous overview. *International Journal of Quantum Chemistry*, 14(4):529–542, 1978.

[Sim79] B. Simon. The definition of molecular resonance curves by the method of exterior complex scaling. *Phys. Lett. A*, 71A(2, 3), 1979.

[SS11] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original.

[SSM14] A. Scrinzi, H. P. Stimming, and N. J. Mauser. On the non-equivalence of perfectly matched layers and exterior complex scaling. *J. Comput. Phys.*, 269:98–107, 2014.

[Stu71] Friedrich Stummel. Diskrete Konvergenz linearer Operatoren. I. *Math. Ann.*, 190:45–92, 1970/71.

[SU09] O. Steinbach and G. Unger. A boundary element method for the dirichlet eigenvalue problem of the laplace operator. *Numerische Mathematik*, 113(2):281–298, 2009.

[SU12] O. Steinbach and G. Unger. Convergence analysis of a galerkin boundary element method for the dirichlet laplacian eigenvalue problem. *SIAM Journal on Numerical Analysis*, 50(2):710–728, 2012.

[TAC98] Tam Christopher K.W., Auriault Laurent, and Cambuli Francesco. Perfectly Matched Layer as an Absorbing Boundary Condition for the Linearized Euler Equations in Open and Ducted Domains. *Journal of Computational Physics*, 144(1):213–234, jul 1998.

[TC97] F.L. Teixeira and W.C. Chew. Systematic derivation of anisotropic PML absorbing media in cylindrical and spherical coordinates. *Microwave and Guided Wave Letters, IEEE*, 7(11):371–373, 1997.

[TC99] F.L. Teixeira and W.C. Chew. Unified analysis of perfectly matched layers using differential forms. *Microwave and Optical Technology Letters*, 20(2):124–126, 1999.

[TC01] F.L. Teixeira and W.C. Chew. Advances in the theory of perfectly matched layers. In *Fast and Efficient Algorithms in Computational Electromagnetics*, pages 283–346. Artech House, Boston, 2001.

[Ton15] Antoine Tonnoir. *Transparent conditions for the diffaction of elastic waves in anisotropic media*. Theses, ENSTA ParisTech, June 2015.

[Tre10] Dimitar Vasilev Trenev. *Spatial scaling for the numerical approximation of problems on unbounded domains*. PhD thesis, 2010.

[TY98] E. Turkel and A. Yefet. Absorbing PML boundary layers for wave-like equations. *Appl. Numer. Math.*, 27(4):533–557, 1998. Absorbing boundary conditions.

[Ung09] Gerhard Unger. *Analysis of Boundary Element Methods for Laplacian Eigenvalue Problems*. PhD thesis, TU Graz, Graz, Austria, 2009.

[Ung17] Gerhard Unger. Convergence analysis of a galerkin boundary element method for electromagnetic eigenvalue problems. Technical Report 2017/2, Institute of Computational Mathematics, Graz University of Technology, 2017. https://www.numerik.math.tugraz.at/berichte/Bericht0217.pdf.

[Vai76] Genadi Vainikko. *Funktionalanalysis der Diskretisierungsmethoden*. B. G. Teubner Verlag, Leipzig, 1976. Mit Englischen und Russischen Zusammenfassungen, Teubner-Texte zur Mathematik.

[VK74] G. M. Vaĭnikko and O. O. Karma. The rate of convergence of approximation methods for an eigenvalue problem in which the parameter occurs nonlinearly. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 14:1393–1408, 1628, 1974.

[WX13] Ch. Wieners and J. Xin. Boundary element approximation for maxwell's eigenvalue problem. *Math. Methods Appl. Sci.*, 36:2524–2539, 2013.

[ZHL01] YangQing Zeng, JianQi He, and QingHuo Liu. The application of the perfectly matched layer in numerical modeling of wave propagation in poroelastic media. *Geophysics*, 66(4):1258–1266, 2001.

[ZKSS06] Lin Zschiedrich, Roland Klose, Achim Schädle, and Frank Schmidt. A new finite element realization of the perfectly matched layer method for Helmholtz scattering problems on polygonal domains in two dimensions. *J. Comput. Appl. Math.*, 188(1):12–32, 2006.

[Zsc09] Lin Zschiedrich. *Transparent boundary conditions for Maxwell's equations: Numerical concepts beyond the PML method*. PhD thesis, FU Berlin, Berlin, Germany, 2009.

[Zwo99] Maciej Zworski. Resonances in physics and geometry. *Notices Amer. Math. Soc.*, 46(3):319–328, 1999.

Martin Halla

Geburtsdatum	29.01.1989
Geburtsort	Wien
Nationalität	Österreich
E-Mail	`martin.halla@protonmail.com`

Ausbildung

Seit 03/2013	**Doktoratsstudium**, Technische Mathematik, *TU Wien.*
04/2012–01/2013	**Masterstudium**, Technische Mathematik, *TU Wien.*
10/2007–03/2012	**Bachelorstudium**, Technische Mathematik, *TU Wien.*
09/1999–06/2007	**Gymnsasium**, *GRGXVI Maroltingergasse, Wien.*
09/1995–06/1999	**Volksschule**, *Josefinum, Wien.*

Publikationen

M. Halla. *Electromagnetic Stekloff eigenvalues: existence and behavior in the selfadjoint case.* arXiv:1909.01983, 2019.

M. Halla. *Electromagnetic Stekloff eigenvalues: approximation analysis.* arXiv:1909.00689, 2019.

M. Halla. *Galerkin approximation of holomorphic eigenvalue problems: weak T-coercivity and T-compatibility.* arXiv:1908.05029, 2019.

M. Halla und L. Nannen. *Two scale Hardy space infinite elements for scalar waveguide problems.* Advances in Computational Mathematics, Volume 44, Issue 3, Pages 611–643, 2018.

M. Halla. *Convergence of Hardy space infinite elements for Helmholtz scattering and resonance problems.* SIAM J. Numer. Anal., Volume 54(3), Pages 1385–1400, 2016.

M. Halla, T. Hohage, L. Nannen und J. Schöberl. *Hardy Space Infinite Elements for Time Harmonic Wave Equations with Phase and Group Velocities of Different Signs.* Numerische Mathematik, Volume 133(1), Pages 103-139, 2016.

M. Halla und L. Nannen. *Hardy space infinite elements for time-harmonic two-dimensional elastic waveguide problems.* Wave Motion, Volume 59, Pages 94–110, 2015.