



TECHNISCHE
UNIVERSITÄT
WIEN
Vienna University of Technology



MASTERARBEIT

New anomalies of Generalized Massive Gravity

Ausgeführt am Institut für
Theoretische Physik
der Technischen Universität Wien

unter der Anleitung von Daniel Grumiller

durch

Thomas Zojer

Gauermannstr. 40,
2542, Kottlingbrunn.

25.10.2010

Datum

Unterschrift

Ich bestätige, dass ich diese Arbeit alleine, ohne andere als die von mir angeführten Quellen und Hilfsmittel, verfasst habe.

Einleitung/Kurzfassung

Die Allgemeine Relativitätstheorie von Albert Einstein ist momentan die letzte fundamentale Wechselwirkung, zu der keine vollständig verstandene Quantentheorie existiert. Verschiedene Ansätze dazu kommen aus der Stringtheorie, die bisher einzige Theorie, die alle bekannten fundamentalen Wechselwirkungen vereint. Aufgrund konzeptueller Schwierigkeiten versucht man jedoch auf einem einfacheren Weg eine Quantentheorie der Gravitation zu erlangen. Dies beinhaltet die Auseinandersetzung mit niederdimensionalen Gravitationstheorien, z.B. dreidimensional.¹ Im Rahmen dieser Arbeit werden Theorien betrachtet, die dual zu sogenannten konformen Feldtheorien (CFT) sind. CFTs sind (Quanten-)feldtheorien und stellen eine Möglichkeit dar, eine quantisierte Theorie der Allgemeinen Relativitätstheorie zu erhalten. Wir betrachten insbesondere eine spezielle dreidimensionale Gravitationstheorie, Generalized Massive Gravity (GMG). In dieser Arbeit werden nun anhand der sogenannten AdS/CFT Dualität, die Lösungen der Allgemeinen Relativitätstheorie und konforme Feldtheorien verbindet, die zentrale Ladung der CFT und die neue Anomalie der logarithmischen CFT (LCFT) ausgerechnet, d.h. ausgesuchte Charakteristika der dualen Feldtheorie untersucht. Dies basiert auf der Annahme, dass GMG an speziellen, kritischen Punkten tatsächlich dual zu einer LCFT ist. Weiters verwenden wir eine neue Idee die Anomalien (zentrale Ladung und neue Anomalie) auf der CFT Seite untereinander in Verbindung zu setzen, andererseits berechnen wir die Anomalien auch auf der Gravitationsseite um Sicherheit bezüglich der Ergebnisse und des neuen Formalismus zu erhalten.

¹In der Sprache der Allgemeinen Relativitätstheorie ist die Welt in der wir leben, das *klassische* Limit der gesuchten Gravitationstheorie, vierdimensional.

Abstract

We conjecture that Generalized Massive Gravity (GMG) in three dimensions is dual to a (two dimensional) logarithmic conformal field theory (LCFT). We find the new anomalies for the putative LCFT duals that arise from different limits of the independent coupling constants of GMG. We also derive a shortcut that allows us to calculate the new anomaly on the (L)CFT side. To confirm our results we evaluate two point correlators also on the gravity side and find matching results.

Acknowledgements

I am grateful to my supervisor Daniel Grumiller for his invaluable support. I also want to thank Niklas Johansson who patiently answered all my questions.

In addition I want to thank Sabine Ertl, Michal Michalcik, Nils-Ole Walliser, Andrea Puhm, Andreas Braun and Rashid Ahmad for enlightening discussions.

I was supported by the project Y 435-N16 of the Austrian Science Foundation.

Contents

1	Introduction	7
1.1	AdS – Anti de Sitter spacetimes	10
1.2	CFT – conformal field theories	14
1.3	The AdS/CFT-correspondence	18
2	LCFT – Logarithmic conformal field theories	20
2.1	Logarithmic Points – The new anomaly	20
2.2	LCFTs with Jordan cell of rank two and three	22
3	From the Central Charge to Anomalies – a Derivation of Derivatives	24
3.1	Rank two LCFTs	26
3.2	Rank three LCFTs	29
4	GMG – an application	36
4.1	The model	36
4.2	Anomalies	40
4.2.1	Standard degeneration: $\mathcal{D}^L = \mathcal{D}^{m-}$	40
4.2.2	Doubly Logarithmic degeneration: $\mathcal{D}^L = \mathcal{D}^{m-} = \mathcal{D}^{m+}$	41
5	Two point correlators on the gravity side	42
5.1	Central charges and the new anomaly	43
5.2	The generalized new anomaly	44
5.3	Massive degeneration: $\mathcal{D}^{m-} = \mathcal{D}^{m+}$	47
6	Conclusion	54
A	The doubly logarithmic mode	56

1 Introduction

Quantum theories are understood as the generalizations of classical theories such as electrodynamics or gravity to fundamental scales. These quantum theories allow for structures and phenomena, such as the tunneling effect, that we do not see in the 'real' world (nevertheless we can see traces in experiments), or that we cannot describe with the classical laws of physics in mechanics, electromagnetism or thermodynamics. The interactions and forces we observe in everyday life are obtained via so-called classical limits of the quantum theories. Altogether the following four fundamental forces are known: electromagnetic, weak, strong and gravitational interactions. By describing them as gauge theories, quantum theories for the first three of them have been found. Gravity is also a gauge theory as it is invariant under coordinate changes, diffeomorphisms. Attempts for quantizing gravity are available in the realm of string theory. However, due to technical and conceptual difficulties there are many open problems. Therefore one wants to find a theory for quantum gravity via another, easier way that will hopefully fit predictions from string theory. A theory of pure gravity in three dimensions would provide insights into the complexities of higher dimensional theories of gravity.

Since 2007 there has been renewed interest in three-dimensional gravity. Witten [1] discussed possible quantum theories of gravity emerging from extremal conformal field theories (CFTs) as CFT duals to three-dimensional gravity theories. CFTs and CFT duals to gravity theories are understood in the sense of the AdS/CFT correspondence [2]. The introduction of these concepts will be subject of the first section. The gravity models considered in [1] are pure Einstein gravity with cosmological constant and, to some extent,² topologically massive gravity (TMG) [4], a higher (metric-)derivative theory that is not invariant under parity transformations. One of the assumptions under which the partition function was derived by Witten, was that the central charge of the left and the right-moving sector of the CFT had to be the same.³ Also holomorphic factorization of the partition function was assumed. In the same year in a paper with Maloney [5] they showed that the conjectured factorization of the partition function was in fact impossible for usual three-dimensional gravity with cosmological constant. In 2008 Li, Song and

²As mentioned in [3] the deformation of pure gravity considered by Witten is purely topological and does not contain local propagating degrees of freedom, therefore it is not equivalent to TMG.

³The discussion on gauge theory descriptions of gravity via a Chern-Simons Lagrangian was general. It was only later that the *topological*, see footnote above, 'coupling' to the left and the right sector were set equal, thus excluding TMG like Lagrangians.

Strominger [3] showed that with a special tuning of the parameters of TMG, a gravity theory where Einstein gravity is supplemented by a topological metric derivative term, it is possible to tune one of the central charges to zero, whereas the other remains finite. The hope was that one of the left or right-moving primary modes was pure gauge and the partition function would in fact be only right-moving. The theory was dubbed chiral gravity referring to the partition function that was conjectured to be chiral, i.e. it would be a purely holomorphic function without anti-holomorphic contributions. A chiral theory of gravity would be preferred because a quantization via path integral formalisms would be simple if the partition function is chiral.

By explicit calculation of partition functions of gravity theories conjectured to be chiral, the conjecture of Witten and Maloney, that the partition function actually does not factorize was proven [6]. Up to now no consistent chiral theory of gravity is found, though the definition of *chiral* gravity has been refined [7, 8]. New candidates for theories which factorize holomorphically were recently found in massless higher spin theories [9].

Nevertheless a lot of interest and a lot investigation was pointed towards the chiral tuning of TMG. Grumiller and Johansson [10] conjectured that TMG at its chiral point is dual to a *logarithmic* CFT (LCFT) [11]. Since then more gravity duals to logarithmic CFTs were conjectured, sometimes with quite strong evidence in their favor [12, 13]. For topological massive gravity and new massive gravity this conjecture is corroborated. For a review on chiral gravity and gravity duals to logarithmic CFTs, dubbed log-gravity see [8]. However, many open questions still remain. We know that log-gravity as a non-unitary theory of gravity is of marginal interest as a toy model for quantum gravity. But chiral gravity might exist as a unitary subsector of log-gravity. The point is that this subsector might not be dual to a local CFT. Therefore the quantum theory might not have the properties of a CFT desired for quantization.

The goal of this thesis is to corroborate the conjecture that generalized massive gravity (GMG), another three dimensional theory of gravity, is also dual to an LCFT at its critical points. The action of GMG is

$$S_{GMG} = \frac{1}{\kappa} \int d^3x \sqrt{-g} \left\{ \sigma R - 2\lambda m^2 + \frac{1}{m^2} K + \frac{1}{\mu} L_{CS} \right\}, \quad (1)$$

where

$$L_{CS} = \frac{1}{2} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^{\alpha} \left[\partial_{\mu} \Gamma_{\alpha\nu}^{\sigma} + \frac{2}{3} \Gamma_{\nu\tau}^{\sigma} \Gamma_{\nu\alpha}^{\tau} \right] \quad (2)$$

$$K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2. \quad (3)$$

The Chern-Simons term L_{CS} (2) introduced in [4] breaks parity invariance due to the explicit appearance of the Levi-Civita tensor $\varepsilon^{\lambda\mu\nu}$. This (topological) term is characteristic for Topologically Massive Gravity (TMG). The extension K (3), proposed in [14] is quadratic in the Ricci tensor, i.e. includes fourth derivatives of the metric. The action above (1) without the Chern-Simons term L_{CS} is called New Massive Gravity (NMG). The constant λ is proportional to the cosmological constant Λ in (4) that we will see in the introduction to Anti de Sitter spacetimes below. TMG (and NMG) can be obtained from GMG by the limits $m^2 \rightarrow \infty$ with λm^2 finite (and $\mu \rightarrow \infty$). The gravitational coupling constant $\kappa = 16\pi G$ will not play a major role in our analysis, though it will enter as the overall scale of the central charges and new anomalies.

The critical points of GMG can be obtained by tuning the parameters Λ , m^2 and μ . In the spirit of [12] we will compute two point correlators on the AdS and CFT side and compare the results via the AdS/CFT duality.

The interest in investigating LCFTs and dual theories is not only motivated by gravity duals, but has been triggered earlier by models in condensed matter physics. Due to their scale invariance CFTs are *the* model to describe physical systems at fixed points. However, CFTs fail to describe systems at fixed/critical points once materials with impurities, defects and disorders are considered. For some of them, e.g. quenched random magnets [15], polymers and percolation [16], logarithmic corrections to power law behavior [17] or logarithmic terms in correlations functions appear. LCFT correlation functions and the meaning of the new anomaly for $c = 0$ CFTs for critical systems with quenched disorder are nicely investigated in e.g. [18, 19].

The layout of this work is as follows. I will give a brief introduction to the two main theories this work is based on, AdS/gravity and conformal field theories, supplemented by a short comment on the AdS/CFT duality. In the second section we introduce *logarithmic* CFTs and show their significant differences to ordinary CFTs. We proceed with a technical discussion on how we can relate characteristics of CFTs to those of LCFTs in section three. The main goal of the thesis will then be to apply this formalism to GMG and compute the new anomaly and the generalized new anomaly for GMG in section four. To put our conjecture for (L)CFT duals to GMG on firm ground we calculate the central charges and new anomalies of GMG also on the gravity side in section five. In section six we conclude and discuss our findings.

1.1 AdS – Anti de Sitter spacetimes

Handling theories of gravity, e.g. finding all their solutions, is a difficult or even impossible task, due to their complexity. Therefore, to find such solutions, theories of gravity are hardly ever considered in full generality. One always reduces problems to the most simple form, the most simple model with the desired properties. The hope is to generalize the solutions or features found for these so-called toy models to more complex theories. Therefore one (very) small step towards a consistent quantum theory of gravity is to consider candidates in three spacetime dimensions, i.e. one time and two spatial dimensions.⁴ The simplest model of gravity is Einstein gravity. The next step to introduce additional structure is to add a cosmological constant to the action. The action is then given by

$$S = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} (R - 2\Lambda). \quad (4)$$

Here $\kappa^2 = 16\pi G_N$ where G_N is Newtons constant, describing the strength of gravitational forces in the classic limit. Λ is Einstein's famous cosmological constant. The Ricci scalar R is the fully contracted Riemann tensor $R_{\mu\nu\rho\sigma}$ and basically describes the curvature of spacetime. The only variable in this action is the metric which appears explicitly in the determinant and also in the Riemann tensor. Varying the action w.r.t. the metric yields the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - \Lambda) = 0. \quad (5)$$

To simplify the task of finding solutions to this equation one tries to find spacetimes/solutions which are maximally symmetric. Then R is constant. Depending on the value of Λ , R can be positive, negative or zero. Maximally symmetric spacetimes with positive curvature are called de Sitter (dS), spacetimes with negative curvature Anti de Sitter (AdS) and spacetimes without curvature are (called) flat. We will further focus on anti de Sitter spacetimes in three dimensions (AdS₃) because these are the only known spacetimes for which a dual conformal field theory (CFT) formulation exists. This is convenient because a consistent quantization would be possible in the context of CFTs, and our long term goal is to find a consistent quantum theory of gravity.

We consider spacetimes which are only asymptotically AdS₃, i.e. an AdS spacetime plus a small deviation that vanishes when we approach the so-called AdS boundary. Our ansatz is $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ denotes

⁴One also refers to such theories as 2+1 dimensional.

the AdS₃ metric and $h_{\mu\nu}$ is the small perturbation on this AdS background. Using cylindrical coordinates τ , ϕ and ρ the metric $\bar{g}_{\mu\nu}$ reads

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = \ell^2(-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2). \quad (6)$$

The constant ℓ denotes the AdS length, a quantity that roughly speaking parameterizes the curvature of the spacetime. It is related to the cosmological constant Λ given in the action by $\Lambda = -1/\ell^2$. In order to get an AdS space in the limit $\rho \rightarrow \infty$ the field $h_{\mu\nu}$ has to fall off at infinity as compared to the background. The AdS background actually diverges as $e^{2\rho}$ for $\rho \rightarrow \infty$ so any linear divergence $h \sim \mathcal{O}(\rho)$ is small compared to the background metric.

We plug the ansatz $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ into the equations of motion (5) and linearize in $h_{\mu\nu}$. This yields equations for $h_{\mu\nu}$, which we want to solve. Since they are differential equations we have to impose boundary conditions upon the field $h_{\mu\nu}$. The right choice of boundary conditions is crucial at this point. If they are too strict, we will not find solutions at all. On the other hand too weak boundary conditions would allow for unphysical solutions with infinite energy. A suitable choice was first given by Brown and Henneaux. For these particular conditions the asymptotic symmetry group, discussed below, of possible solutions is isomorphic to the conformal group.

Asymptotic symmetry group

For the following discussion it is convenient to perform some coordinate changes.⁵ First we introduce $x^\pm = (\phi \pm t)/2$ which change the metric to

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = \ell^2(- (dx^+)^2 - (dx^-)^2 + 2 \cosh(2\rho) dx^+ dx^- + d\rho^2). \quad (7)$$

A further substitution $y = e^{-\rho}$ shifts the AdS boundary from $\rho \rightarrow \infty$ to $y \rightarrow 0$. $d\rho^2$ goes to dy^2/y^2 and also $2 \cosh 2\rho \rightarrow e^{2\rho} = 1/y^2$. For vanishing y we can neglect the $(dx^\pm)^2$ differentials because all other terms diverge as $y \rightarrow 0$.

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu \approx \ell^2 \frac{dx^+ dx^- + dy^2}{y^2} \quad (8)$$

The so-called Brown-Henneaux boundary conditions in this coordinate patch are [20]

$$h_{\mu\nu} = \begin{pmatrix} h_{++} = \mathcal{O}(1) & h_{+-} = \mathcal{O}(1) & h_{+y} = \mathcal{O}(y) \\ h_{--} = \mathcal{O}(1) & h_{-y} = \mathcal{O}(y) & h_{yy} = \mathcal{O}(1) \end{pmatrix}. \quad (9)$$

⁵We will use similar coordinates in the appendix to denote explicitly the components of the solutions $h_{\mu\nu}$. In the appendix we will denote x^\pm by $v/2$ and $u/2$ respectively to ease comparison with other literature.

They allow for an infinite set of diffeomorphisms respecting these boundary conditions.

$$\begin{aligned}
\xi^+ &= \epsilon_+(x^+) - \frac{y^2}{2} \partial_-^2 \epsilon_- + \mathcal{O}(y^4) \\
\xi^- &= \epsilon_-(x^-) - \frac{y^2}{2} \partial_+^2 \epsilon_+ + \mathcal{O}(y^4) \\
\xi^\rho &= \frac{y}{2} (\partial_+ \epsilon_+ + \partial_- \epsilon_-) + \mathcal{O}(y^3)
\end{aligned} \tag{10}$$

Here ϵ_+ and ϵ_- are arbitrary functions of the coordinates $x^\pm = t \pm \varphi$ respectively. A vector field ξ^μ acts on the metric via the Lie derivative. A diffeomorphism changes the metric in the following way. $g_{\mu\nu} \rightarrow g'_{\mu\nu}$ where $g'_{\mu\nu} = \bar{g}_{\mu\nu} + h'_{\mu\nu}$ ⁶ and

$$h'_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu} \partial_\mu \xi^\sigma + g_{\mu\sigma} \partial_\nu \xi^\sigma. \tag{11}$$

The point is that if $h_{\mu\nu}$ obeyed the b.c. (9) any $h'_{\mu\nu}$ that is generated by ξ^μ will also obey the boundary conditions.

Of course there are other symmetry transformations that do not change the metric $h_{\mu\nu}$ to leading order. These are called trivial transformations. The asymptotic symmetry group is defined as the set of all diffeomorphisms that preserve the boundary conditions (9), modulo (i.e. without) all trivial symmetry transformations.

In our case the asymptotic symmetry group is generated by two functions $\epsilon_\pm(x^\pm)$ and it is the conformal group in two dimensions. To see this we Fourier-expand the functions ϵ_\pm

$$\epsilon_+ = \sum_n \epsilon_n^+ e^{-inx^+} \tag{12a}$$

$$\epsilon_- = \sum_n \epsilon_n^- e^{-inx^-}. \tag{12b}$$

We define the generators

$$\xi_n = \xi(\epsilon_n^+) \quad \text{and} \quad \bar{\xi}_n = \bar{\xi}(\epsilon_n^-) \tag{13}$$

where only the ϵ_n^\pm are non zero. Under Lie brackets these fulfill the algebra

$$\{\xi_n, \xi_m\}_{\text{L.B.}} = i(n-m)\xi_{m+n} \tag{14}$$

⁶Note the difference between the AdS background metric $\bar{g}_{\mu\nu}$ and the full metric $g_{\mu\nu}$ (background plus perturbation $h_{\mu\nu}$).

and equivalently for the $\bar{\xi}$. We will see in the next section that this algebra is the conformal algebra without central charge, also known as the Witt algebra. Furthermore the modes ξ_n and $\bar{\xi}_m$ commute, i.e. $\{\xi_n, \bar{\xi}_m\} = 0$. Therefore we have two independent, identical copies of the Witt algebra.

The global conformal group, which consists of the six generators $\{\xi_{\pm 1}, \xi_0\}$ and $\{\bar{\xi}_{\pm 1}, \bar{\xi}_0\}$ is the isometry⁷ group of the AdS background (6). We will learn soon that the algebra related to this group admits so-called central extensions, or central charges. Later on these will be the main point of our interest. We want to find the central charges of the isometry group/algebra of solutions of GMG that are asymptotically AdS.

Note:

We are looking for fields $h_{\mu\nu}$ on three-dimensional anti-de Sitter space. These solutions shall be asymptotically AdS (9). There are infinitely many solutions obeying the b.c. (9) namely each solution corresponds to an element of the conformal group.

The algebra generated by the diffeomorphisms ξ^μ , the conformal algebra, admits a central extension, central charges.

We are interested in the central charges, and generalizations thereof, of the solutions of GMG.

⁷I.e. diffeomorphisms that leave the background invariant, i.e. $L_\xi \bar{g}_{\mu\nu} = 0$, as opposed to diffeomorphisms that induce changes within the defined boundary conditions $L_\xi \bar{g}_{\mu\nu} = h_{\mu\nu}$, which belong to the asymptotic symmetry group.

1.2 CFT – conformal field theories

Conformal field theories describe a wide amount of special quantum field theories. They are special in the sense that they fulfill certain symmetry conditions. CFTs are a very important tool in condensed matter physics where they are used to describe physical systems at critical points. Renormalization group flows are studied with CFTs not only in physics, e.g. gravity duals, condensed matter physics. . . , but also in statistics and economy. CFTs brought a lot of insight into physics and, vice versa, motivated (mostly) by string theory, new interest was triggered in that branch of mathematics. The short introduction to CFTs that follows is mostly based on [21].

Conformal Transformations

Many theories in physics, e.g. general relativity, are simply based on symmetry arguments. Usually a theory or model that describes certain interactions ought to be invariant under transformations. These symmetries can basically be anything, like exchange of coordinates (parity), reversing time-flow or charge conjugation. It should not be a big surprise that the starting point of our discussion of CFTs is again a symmetry. Conformal theories are invariant under transformations that preserve angles. These include translations, rotations, boosts, dilatations ('blowing up spacetime', i.e. multiplying the coordinates describing our spacetime by a number) and special conformal transformations⁸. Mathematically these conditions are expressed in the form that a transformation $x \rightarrow x'$ should not change the metric of the spacetime up to a multiplicative factor:

$$g_{\mu\nu}(x) \rightarrow \Omega(x') g'_{\mu\nu}(x') \quad (15)$$

Conformal transformations are those that fulfill this equation. The conformal group in two dimensions is infinite, i.e. those CFTs have to be invariant under infinitely many transformations. Loosely speaking this imposes 'infinitely' many constraints on the CFT and therefore is very restrictive.

Consider a two dimensional theory with coordinates x_0 and x_1 and Euclidean metric. We introduce complex variables $z = x_0 + ix_1$ and $\bar{z} = x_0 - ix_1$. Infinitesimal conformal transformations are parameterized by

$$z \rightarrow f(z) = z + \epsilon(z) \quad \text{and} \quad \bar{z} \rightarrow g(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z}) \quad (16)$$

with (anti-)holomorphic functions $f(z)$ and $g(\bar{z})$. These are functions that only depend on z or \bar{z} , treating them as independent variables. Using a

⁸Roughly speaking they consist of an inversion of coordinates on a sphere/circle followed by a translation/boost and then reinverting the coordinates.

Laurent expansion we can rewrite

$$f(z) = z + \epsilon(z) = z - \sum_{n \in \mathbb{Z}} \epsilon_n(z^{n+1}) \quad (17)$$

and identify the generators corresponding to the transformation for a particular n

$$l_n = -z^{n+1} \partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}. \quad (18)$$

The commutation relations of these generators yield the algebra of infinitesimal conf. trans. in two dimensions

$$\begin{aligned} [l_n, l_m] &= -z^{n+1} \partial_z (-z^{m+1} \partial_z) + z^{m+1} \partial_z (-z^{n+1} \partial_z) \\ &= (n+1) z^{m+n+1} \partial_z - (m+1) z^{m+n+1} \partial_z \\ &= (n-m) z^{m+n+1} \partial_z \\ [l_n, l_m] &= (n-m) l_{m+n}, \quad (19a) \\ [\bar{l}_n, \bar{l}_m] &= (n-m) \bar{l}_{m+n}, \quad (19b) \\ [\bar{l}_n, l_m] &= 0. \quad (19c) \end{aligned}$$

Due to the vanishing of the mixed commutator we obtain two independent algebras for the generators l_n and \bar{l}_m . Thus we have two copies of the Witt algebra.

The Witt algebra (19) admits a so-called central extension. If we add the extension we usually denote the generators by capital L_n and call them Virasoro generators. The Virasoro algebra is obtained by adding a constant to the commutation relation (19)

$$[L_n, L_m] = (n-m) L_{n+m} + c(m, n). \quad (20)$$

The Virasoro algebra is a Lie algebra and as such (20) has to fulfill the Jacobi identity

$$[[L_n, L_m], L_k] + [[L_m, L_k], L_n] + [[L_k, L_n], L_m] = 0. \quad (21)$$

With $c(m, n) = \frac{c}{2} \delta_{m+n,0} m(m-1)(m+1)$, where c is some constant, we can fulfill (21). The constants c and \bar{c} , with the normalization introduced below, are called the central charges of the algebra. The respective algebras and charges are

$$[L_n, L_m] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}, \quad (22)$$

$$[\bar{L}_n, \bar{L}_m] = (m-n) \bar{L}_{m+n} + \frac{\bar{c}}{12} (m^3 - m) \delta_{m+n,0}, \quad (23)$$

where c and \bar{c} are arbitrary constants that commute with all generators. In fact these constants will become very important for us since they, and generalizations thereof in logarithmic CFTs, are precisely the quantities we are interested in. We will denote the independent L_n and \bar{L}_m algebras as left and right-moving sector of the CFT and distinguish left and right central charges.

(Quasi-) primary fields and conformal weight

A field $\mathcal{O}(z, \bar{z})$ that transforms under conf. trans. $z \rightarrow f(z), \bar{z} \rightarrow g(\bar{z})$ according to

$$\mathcal{O}(z, \bar{z}) \rightarrow \mathcal{O}'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{g}}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}(z, \bar{z}) \quad (24)$$

is called a primary field of conformal dimensions or conformal weights (h, \bar{h}) . If (24) does not hold for arbitrary f but only for some subgroup of the conformal group (e.g. $f \in SL(2, \mathbb{C})/\mathbb{Z}_2$) the field is called *quasi*-primary ($SL(2, \mathbb{C})$ -primary).

The energy-momentum tensor

The energy-momentum tensor of any field theory can be deduced from the variation of the action with respect to the metric. If we consider specific variations that correspond to conformal transformations, and demand the theory be invariant under them, we get constraints on the energy-momentum tensor. Using invariance of the theory under arbitrary coordinate transformations $x^\mu \rightarrow x^\mu + \epsilon^\mu$, i.e. $\delta g_{\mu\nu} = \partial_{(\mu} \epsilon_{\nu)}$, implies⁹

$$\delta S \sim \int d^d x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} = \int d^d x \sqrt{-g} T_{\mu\nu} \partial^{(\mu} \epsilon^{\nu)} \stackrel{!}{=} 0 \quad (25)$$

for any ϵ^μ . Thus we demand

$$\partial^\mu T_{\mu\nu} = 0, \quad (26)$$

or $\nabla^\mu T_{\mu\nu} = 0$ if we are on a curved background. Considering conformal transformations of the metric (15) the variation becomes

$$\delta S \sim \int d^d x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} = \int d^d x \sqrt{-g} T_{\mu\nu} (\Omega g^{\mu\nu}) \stackrel{!}{=} 0 \quad (27)$$

⁹We denote symmetrization of indices by brackets: $\partial_{(\mu} \epsilon_{\nu)} := \frac{1}{2}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$

and we infer that for CFTs the energy-momentum tensor must be traceless

$$T_{\mu}{}^{\mu} = 0. \quad (28)$$

Additionally, in two dimensions, the only non-vanishing components of the energy-momentum tensor are chiral and anti-chiral fields

$$T_{zz}(z) =: T(z) \quad \text{and} \quad T_{\bar{z}\bar{z}}(\bar{z}) =: \bar{T}(\bar{z}). \quad (29)$$

Later on we will denote these two components of the energy-momentum tensor as \mathcal{O}^L and \mathcal{O}^R and refer to them as left and right-moving primaries.

Two-point functions

In any field theory interactions between two points/two particles, or more generally *events*, are given by so-called two-point functions. These functions describe how information *propagates* from one event to another; if they are correlated and how (and if) they interact. That is why two-point functions go under various different names, e.g. correlators, propagators and some more. The names mostly depend on the feature one wants to emphasize.

Correlators of CFT-fields or CFT-operators have to be invariant under conformal transformations. This is so restrictive that one can immediately write down the form of a two point function of any (quasi-primary) field

$$\langle \mathcal{O}_i(z) \mathcal{O}_j(w) \rangle = \frac{d_{ij} \delta_{h_i, h_j}}{(z - w)^{2h_i}}. \quad (30)$$

We see that the correlator is always zero if the fields have different conformal weights. If we plug the energy-momentum tensor into this relation d_{ij} is given by the central charge. Since two point correlators in CFTs always take the same form and the energy-momentum tensor exists in every CFT the central charge is one way to distinguish different CFTs. Of course one has to take into account the field content of the CFT because scalar and fermionic fields contribute differently to the central charge. So equal c is an indicator for similar CFTs only if they contain the same fields. Another objection to this comes from logarithmic CFTs where the central charge of different LCFTs can be equal. However, in LCFTs another characteristic, the new anomaly, discriminates one LCFT against another.

Note:

Two point correlators of the energy-momentum tensor yield the central charge of the CFT. This quantity is one characteristic for a CFT.

We refer to the chiral $T(z)$ and anti-chiral $\bar{T}(\bar{z})$ mode of the energy-momentum tensor as left and right-moving fields/modes.

In LCFTs, more complicated variants of CFTs that we will encounter in the next section, we find an additional quantity, the new anomaly. For our purposes - calculating two point functions - the new anomaly in LCFTs will be the LCFT equivalent of the central charge in CFTs.

1.3 The AdS/CFT-correspondence

AdS/CFT-duality [2] has found its way to many different applications in physics, mainly in theoretical physics. The most popular examples being string theory and quantum chromodynamics (AdS/QCD). The duality predicts a relation between a d dimensional bulk theory with gravity, string theory or pure gravity, and a $d - 1$ dimensional gauge theory, a CFT, that lives at the boundary. For example Brown and Henneaux [20] indicated that any quantum theory of gravity on AdS_3 has a holographic dual CFT in two dimensions.

AdS/CFT also relates strongly coupled field theories to weakly coupled gauge theories and vice versa. This admits to use perturbative techniques for both limits, strongly and weakly coupled. Because of its variety of different applications this duality is of vital importance.

For our purposes the main point is that solutions/fields in the gravity theory are dual to operators in the CFT. For example $T_{\mu\nu}$ in the CFT is sourced by non-normalizable solutions $h_{\mu\nu}^{\text{non-norm.}}$ to the linearized equations of motion on the gravity side (non-normalizable gravitons). One can choose whether to calculate e.g. correlators in the gravity, or in the CFT context. This already establishes the only aspect of AdS/CFT that we will use; the duality of two-point functions. We will derive them on the CFT and on the gravity side. We consider the CFT side first, because this is easier. Then we calculate the same correlators on the gravity side to check if our conjectures about characteristics such as central charges or new anomalies are true, and to corroborate our conjecture about the existence of dual LCFTs.

Two point correlators on the (L)CFT side have only one characteristic quantity. This is either the central charge or the new anomaly. We will postulate a relation between these two quantities, i.e. a way to derive the new anomaly

from the central charge. Taking the value of c from literature we derive the new anomaly on the CFT side, and then check the result by explicit calculations on the gravity side.

The prescription that follows from AdS/CFT is that CFT propagators are dual to inserting the non-normalizable solutions to the eoms into the second variation of the action:

$$\langle \mathcal{O}^1(z) \mathcal{O}^1(0) \rangle = \delta S^{(2)}(h_{\mu\nu}^1, h_{\mu\nu}^1) \quad (31)$$

Here \mathcal{O}^1 and $h_{\mu\nu}^1$ are dual operators/fields. We will later see how the positions z are encoded in (the 'conformal weights' of) the fields $h_{\mu\nu}^1$. Since the duality holds at the AdS boundary $\lim \rho \rightarrow \infty$, the correlator will be given by boundary terms. Therefore we would have to use the full, holographically renormalized action to obtain all boundary terms. Holographic renormalization is what makes this a very lengthy calculation. We are not going to do this renormalization (which has not been done for GMG so far) but use a short-cut that allows us to relate results from Einstein gravity to GMG results. However, this short-cut will not work directly for all correlators.

2 LCFT – Logarithmic conformal field theories

The most defining characteristic of LCFTs is the appearance of logarithms in correlation functions, see (42c). An LCFT arises when two CFT operators degenerate and their respective 'charges' (the constants d_{ii} in the two point function (30) of the respective operators) approach zero. We describe how this happens in the next section, or see [13]. Therefore in an LCFT one operator of the CFT acquires a partner. We mostly consider cases where the energy-momentum tensor acquires a logarithmic partner operator.¹⁰ For convenience we always choose the operator \mathcal{O}^M to degenerate with \mathcal{O}^L . We define a new operator, called the logarithmic operator of the theory, in the following way. If we parameterize the degeneracy by the variable ε

$$\lim_{\varepsilon \rightarrow 0} \mathcal{O}^M(\varepsilon) \rightarrow \mathcal{O}^L \quad (32)$$

we define

$$\mathcal{O}^{\log} = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{O}^M(\varepsilon) - \mathcal{O}^L}{\varepsilon} \quad (33)$$

as the new logarithmic operator. We will show this in more detail in section 3.

The primaries \mathcal{O}^L and \mathcal{O}^R have conformal weights $(h, \bar{h}) = (2, 0)$ and $(h, \bar{h}) = (0, 2)$ respectively. For the massive operator we usually parameterize the weights by $(h, \bar{h}) = (2 + \varepsilon, \varepsilon)$. Hence we get the two point correlators

$$\langle \mathcal{O}^L(z) \mathcal{O}^L(0) \rangle = \frac{c_L}{2 z^4} \quad (34a)$$

$$\langle \mathcal{O}^R(z) \mathcal{O}^R(0) \rangle = \frac{c_R}{2 z^4} \quad (34b)$$

$$\langle \mathcal{O}^M(z, \bar{z}) \mathcal{O}^M(0, 0) \rangle = \frac{\hat{B}}{2 z^{4+2\varepsilon} \bar{z}^{2\varepsilon}}. \quad (34c)$$

Now we will consider the limit $\varepsilon \rightarrow 0$, i.e. $\mathcal{O}^M \rightarrow \mathcal{O}^L$.

2.1 Logarithmic Points – The new anomaly

In order to show how correlators of the form (34) come about we make a short calculation (different to section 3!) where the notation is adapted

¹⁰In our cases this partner is almost always the energy-momentum tensor. LCFTs may also arise from degeneration of two arbitrary fields, where none of them is the energy-momentum tensor.

to [12]. Note that we define the logarithmic operator here different to (33). We send ε and c_L to zero such that the following limits exist:

$$b_L = -\lim_{c_L \rightarrow 0} \frac{c_L}{\varepsilon} \neq 0 \quad B := \lim_{c_L \rightarrow 0} \left(\hat{B} + \frac{2}{c_L} \right) \neq 0 \quad (35)$$

The logarithmic operator is given by the combination

$$\mathcal{O}^{\log} = \frac{b_L}{c_L} \mathcal{O}^L + \frac{b_L}{2} \mathcal{O}^M. \quad (36)$$

ε parameterizes the degeneration of the massive operator \mathcal{O}^M with the left-moving primary \mathcal{O}^L . Due to the choice $(h, \bar{h}) = (2 + \varepsilon, \varepsilon)$ for the weights of the massive branch the small parameter denotes the difference of the conformal weights of the left-moving primary and the massive mode. It follows that

$$\bar{h} = \varepsilon + \mathcal{O}(\varepsilon^2) \quad (37)$$

defines our 'normalization' of ε .

Now we calculate the 2-point correlators of the operators \mathcal{O}^L and \mathcal{O}^{\log} . We write (35) in the form

$$c_L = -\lim_{\varepsilon \rightarrow 0} \varepsilon b_L \quad \hat{B} = \lim_{\varepsilon \rightarrow 0} B - \frac{2}{\varepsilon b_L}. \quad (38)$$

Furthermore we expand the two point function of the massive operator, using $z^{-\varepsilon} = 1 - \varepsilon \ln z + \mathcal{O}(\varepsilon^2)$:

$$\langle \mathcal{O}^M(z) \mathcal{O}^M(0) \rangle = \frac{\hat{B}}{2z^4} (1 - 2\varepsilon \ln(z^2) + \mathcal{O}(\varepsilon^2)) \quad (39)$$

Since \mathcal{O}^L and \mathcal{O}^M have different weights the 2-point function $\langle \mathcal{O}^L \mathcal{O}^M \rangle$ vanishes. With (38) and (34) we get

$$\langle \mathcal{O}^{\log}(z) \mathcal{O}^L(0) \rangle = \frac{b_L}{c_L} \langle \mathcal{O}^L \mathcal{O}^L \rangle = \frac{b_L}{c_L} \frac{c_L}{2z^4} = \frac{b_L}{2z^4} \quad (40)$$

$$\begin{aligned} \langle \mathcal{O}^{\log}(z) \mathcal{O}^{\log}(0) \rangle &= \left(\frac{b_L}{c_L} \right)^2 \langle \mathcal{O}^L \mathcal{O}^L \rangle + \left(\frac{b_L}{2} \right)^2 \langle \mathcal{O}^M \mathcal{O}^M \rangle = \\ &= \lim_{\varepsilon \rightarrow 0} -\frac{b_L}{2\varepsilon z^4} + \frac{b_L^2}{4z^4} \left(B + \frac{2}{\varepsilon b_L} \right) (1 - 2\varepsilon \ln(z^2) + \mathcal{O}(\varepsilon^2)) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{b_L^2 B - 4b_L \ln(z^2)}{4z^4} + \mathcal{O}(\varepsilon). \end{aligned} \quad (41)$$

If we set $b_L B = -4 \ln m_L^2$ we finally arrive at the following two-point correlators

$$\langle \mathcal{O}^L(z) \mathcal{O}^L(0) \rangle = 0 \quad (42a)$$

$$\langle \mathcal{O}^{\log}(z) \mathcal{O}^L(0) \rangle = \frac{b_L}{2 z^4} \quad (42b)$$

$$\langle \mathcal{O}^{\log}(z) \mathcal{O}^{\log}(0) \rangle = -\frac{b_L \ln(m_L^2 |z|^2)}{z^4}. \quad (42c)$$

The coefficient b_L appearing in the two point functions is called the new anomaly of the logarithmic conformal field theory. Its value is characteristic for the LCFT, similarly to the central charge for ordinary CFTs. We infer that we get a new anomaly when the central charge vanishes and two operators degenerate. We investigate the case where three operators degenerate below.

2.2 LCFTs with Jordan cell of rank two and three

For comparison with the formulas we will derive in section 3 we give here the results for two point correlators of an LCFT with a Jordan cell of rank two and three. The rank of the Jordan cell denotes how many operators degenerate. We already derived the correlators for a Jordan cell of rank two in the last section when we introduced the new anomaly. Now we are simply going to write down the correlators for a Jordan cell of rank three. We will use a slightly different notation. The primary of each cell is \mathcal{O}^0 and the logarithmic partner fields are \mathcal{O}^{\log} and \mathcal{O}^{\log^2} . The characteristic form of the correlators in the simplest LCFT is

$$\langle \mathcal{O}^0(z) \mathcal{O}^0(0) \rangle = 0 \quad (43a)$$

$$\langle \mathcal{O}^0(z) \mathcal{O}^{\log}(0) \rangle = \frac{b_0}{2 z^4} \quad (43b)$$

$$\langle \mathcal{O}^{\log}(z) \mathcal{O}^{\log}(0) \rangle = \frac{b_1 - b_0 \ln |z|^2}{z^4}. \quad (43c)$$

The constant b_1 is unphysical since it can be shifted to an arbitrary value by the well known freedom of redefining the logarithmic fields

$$\mathcal{O}^{\log} \rightarrow \mathcal{O}^{\log} + \gamma \mathcal{O}^0. \quad (44)$$

We will show this explicitly in the case of an LCFT with a Jordan cell of rank three. There the two point functions are generically of the form [22–24]

$$\langle \mathcal{O}^0(z) \mathcal{O}^0(0) \rangle = \langle \mathcal{O}^0(z) \mathcal{O}^{\log}(0) \rangle = 0 \quad (45a)$$

$$\langle \mathcal{O}^{\log}(z) \mathcal{O}^{\log}(0) \rangle = \langle \mathcal{O}^0(z) \mathcal{O}^{\log^2}(0) \rangle = \frac{a_0}{z^4} \quad (45b)$$

$$\langle \mathcal{O}^{\log}(z) \mathcal{O}^{\log^2}(0) \rangle = \frac{a_1 - 2a_0 \ln |z|^2}{z^4} \quad (45c)$$

$$\langle \mathcal{O}^{\log^2}(z) \mathcal{O}^{\log^2}(0) \rangle = \frac{a_2 - 2a_1 \ln |z|^2 + 2a_0 \ln^2 |z|^2}{z^4}. \quad (45d)$$

Once again we can add primaries \mathcal{O}^0 to \mathcal{O}^{\log} and both \mathcal{O}^0 and \mathcal{O}^{\log} to \mathcal{O}^{\log^2} to shift the values of the two constants a_1 and a_2 . Note, however, that if we perform the shifts

$$\mathcal{O}'^{\log} \rightarrow \mathcal{O}^{\log} + \alpha \mathcal{O}^0 \quad (46)$$

$$\mathcal{O}'^{\log^2} \rightarrow \mathcal{O}^{\log^2} + \beta \mathcal{O}^{\log} + \gamma \mathcal{O}^0 \quad (47)$$

we get a condition on α and β , while we can choose γ freely:

$$\langle \mathcal{O}'^{\log} \mathcal{O}'^{\log} \rangle = \langle \mathcal{O}^{\log} \mathcal{O}^{\log} \rangle \quad (48)$$

$$i) \quad \langle \mathcal{O}'^{\log^2} \mathcal{O}'^{\log} \rangle = \langle \mathcal{O}^{\log^2} \mathcal{O}^{\log} \rangle + \alpha \langle \mathcal{O}^{\log^2} \mathcal{O}^0 \rangle + \beta \langle \mathcal{O}^{\log} \mathcal{O}^{\log} \rangle \quad (49)$$

$$ii) \quad \langle \mathcal{O}'^{\log^2} \mathcal{O}'^{\log^2} \rangle = \langle \mathcal{O}^{\log^2} \mathcal{O}^{\log^2} \rangle + 2\gamma \langle \mathcal{O}^{\log^2} \mathcal{O}^0 \rangle + 2\beta \langle \mathcal{O}^{\log^2} \mathcal{O}^{\log} \rangle + \beta^2 \langle \mathcal{O}^{\log} \mathcal{O}^{\log} \rangle \quad (50)$$

Comparing the coefficients of the logarithmic terms shows that we shift the parameters a_i according to

$$i) \quad a'_1 = a_1 + (\alpha + \beta) a_0 \quad (51)$$

$$-2a'_0 = -2a_0 \quad (52)$$

$$ii) \quad a'_2 = a_2 + (2\gamma + \beta^2) a_0 + 2\beta a_1 \quad (53)$$

$$-2a'_1 = -2a_1 - 4\beta a_0 = -2(a_1 + 2\beta a_0) \quad (54)$$

$$2a'_0 = 2a_0. \quad (55)$$

We note that a_1 and a_2 are changed, whereas a_0 remains the same. We further demand $\alpha = \beta$. The only relevant parameter left is a_0 , which appears in all non-vanishing correlators. This parameter shall therefore be our defining characteristic for LCFTs with Jordan cell of rank three. We call a_0 the generalized¹¹ new anomaly of the LCFT. In section 4 we are going to compute it for GMG.

¹¹In order to name all possible anomalies for LCFTs with arbitrary rank Jordan cell one would have to invent an infinite tower of adjectives. But for every rank we have only one physical quantity. Jordan cells of rank two were first considered and the name *new anomaly* introduced. (Central charges are *anomalies* as well.) Higher rank Jordan cells generalize this new anomaly, hence we call the defining quantity *generalized* new anomaly.

3 From the Central Charge to Anomalies – a Derivation of Derivatives

In this section we define logarithmic partner fields as limits of degenerating but mutually different operators, see also [24]. We follow [13] but give more detailed calculations.

Consider the three operators \mathcal{O}^A , \mathcal{O}^B and \mathcal{O}^C . They shall all have different conformal weights and 'central charges'. We parameterize these quantities by one dimensionless parameter that we call m . We denote their respective two point correlators as

$$\langle \mathcal{O}^i(z, \bar{z}) \mathcal{O}^i(0) \rangle = \frac{c^i(m_i)}{2} z^{-2h(m_i)} \bar{z}^{-2\bar{h}(m_i)}. \quad (56)$$

Later, namely in the context of gravity duals to (L)CFTs, we will consider degenerations of modes that are induced by the degenerations of the masses of the modes. Therefore in our case the parameter m will mainly correspond to the mass. I will, however, point out explicitly where I make use of this choice. Thus the formulas remain valid in general, although we call our parameter a mass.

We look at the cases where two or three of these operators degenerate separately, with the degeneration parameterized by the parameters m_i . We will use the mass differences as our small parameters. Setting $m_B = 1$ we can draw the following picture where we consider the limits ϵ_1 and ϵ_2 to zero.

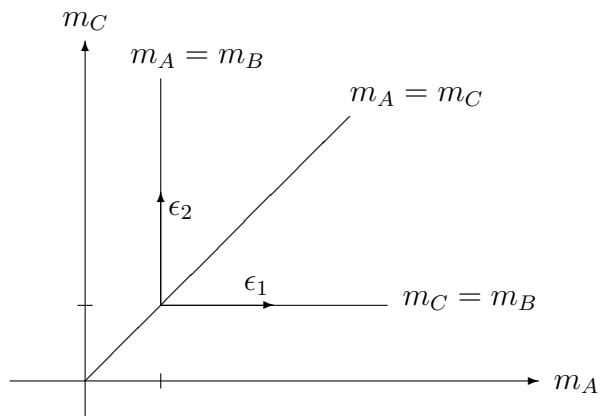


Figure 1: Degenerations of the masses.

But before we start let us comment on another freedom, that of re-

normalization of the modes. A redefinition of the operators by an arbitrary¹² function f

$$\mathcal{O} \rightarrow \sqrt{f} \mathcal{O} \quad (57)$$

changes the normalization of the 2-point correlator according to

$$c_i \rightarrow f c_i. \quad (58)$$

This will become important because we can use this freedom to tune the charges, or more importantly, derivatives thereof, to have the same value. Consequently charges c_i and derivatives thereof are only defined up to a multiplicative factor. When we consider gravity duals we will fix the normalization of one of the modes, usually of the left-moving primary, by its coupling to the energy-momentum tensor.¹³ The normalization of the logarithmic and doubly logarithmic modes a priori is undefined.¹⁴

Now we can expand f and c_i in terms of the epsilons. We show that we can rescale derivatives of the c_i by choosing matching functions f_i .

Later we will consider cases where

$$c_i(\epsilon_j = 0) = 0 \quad i = 1, 2, 3 \quad j = 1, 2 \quad \text{and} \quad (59)$$

$$\partial_{\epsilon_j} c_i(\epsilon_l = 0) = 0 \quad i = 1, 2, 3 \quad j, l = 1, 2. \quad (60)$$

Using these identities we denote the derivatives of the rescaled charges as

$$f c_i|_{\epsilon=0} \rightarrow 0 \quad (61a)$$

$$\partial_{\epsilon_j} (f_i c_i)|_{\epsilon=0} \rightarrow f_i \partial_{\epsilon_j} c_i \quad (61b)$$

$$\partial_{\epsilon_j} \partial_{\epsilon_k} (f_i c_i)|_{\epsilon=0} \rightarrow f_i \partial_{\epsilon_j} \partial_{\epsilon_k} c_i \quad (61c)$$

$$\partial_{\epsilon_j} \partial_{\epsilon_k} \partial_{\epsilon_l} (f_i c_i)|_{\epsilon=0} \rightarrow f_i \partial_j \partial_k \partial_l c_i + \partial_{(j} f_i \partial_{|k} \partial_{l)} c_i. \quad (61d)$$

In the limits (61a) and (61b) we used the condition (59) and similarly the condition (60) is used to arrive at (61c) and (61d). In the last line we introduced abbreviations $\partial_{\epsilon_j} \rightarrow \partial_j$ and denoted symmetrization of the indices/derivatives ∂_j , ∂_k and ∂_l by brackets, so there are three terms of the form $\partial f \partial \partial c$ with j, k and l exchanged.

From (61b) we see that we can tune the first derivative of the charge c_i to any value of our liking, simply by multiplying the respective operator \mathcal{O}_i with the right function $\sqrt{f_i}$. Or we can similarly tune the second derivative

¹²The only restriction of f is that it should not vanish if two masses degenerate, $f(\epsilon_i = 0) \neq 0$.

¹³This was done in [25].

¹⁴Later we will fix it by the action of the operators \mathcal{D}^L , e.g. $\mathcal{D}^L \psi^{\log} = -2\psi^L$.

of c_i due to (61c). Having chosen a particular value for f_i at $\epsilon_j = 0$ we can still tune the first derivative at this point. Thus we can also change the third derivative of c_i to our wishes, see (61d).

Of course we cannot use the relations (61b) and (61c) simultaneously to change $\partial_j c_i$ and $\partial_{jk}^2 c_i$ to different values. However, these cases will not interfere since looking at the degeneration of three operators $\partial_j c_i$ will vanish anyway, so we need only use (61c) and (61d).

3.1 Rank two LCFTs

Taking the limit $\mathcal{O}^B \rightarrow \mathcal{O}^A$ via $m_B \rightarrow m_A$ we will be left with what we know to be a logarithmic theory since two of the operators degenerate. Therefore we have to consider a new set of operators where we replace one of \mathcal{O}^A or \mathcal{O}^B by a logarithmic operator. We make the following ansatz¹⁵

$$\left\{ \mathcal{O}^A, \mathcal{O}^{\text{diff}} = a \frac{\mathcal{O}^A - \mathcal{O}^B}{m_A - m_B}, \mathcal{O}^C \right\} \xrightarrow{m_B \rightarrow m_A} \left\{ \mathcal{O}^A, \mathcal{O}^{\text{log}}, \mathcal{O}^C \right\} \quad (62)$$

and formally compute all two point functions. a stands for some arbitrary normalization that we will fix later.

From now on let us denote $m_B - m_A$ by Δ_{BA} or even just Δ since we do not consider any further mass differences in the rank two case.

In order to take the limit $\Delta \rightarrow 0$ we parameterize the masses as $m_B = m_A + \Delta$. We calculate the two point correlators under the assumptions

$$1. \quad c^A(\Delta = 0) = c^B(\Delta = 0) = 0 \quad \text{and} \quad (63a)$$

$$2. \quad \partial_\Delta c^A|_{\Delta=0} = -\partial_\Delta c^B|_{\Delta=0} \quad (63b)$$

which translate to series expansions

$$c^A = (\partial_\Delta c^A)|_{\Delta=0} \Delta + \mathcal{O}(\Delta^2) \quad (64)$$

$$c^B = -(\partial_\Delta c^A)|_{\Delta=0} \Delta + \mathcal{O}(\Delta^2). \quad (65)$$

The first assumption (63a) is, to the best of our knowledge, not restrictive since in all known cases the two point function $\langle \mathcal{O}^A \mathcal{O}^A \rangle$ becomes zero once the operator \mathcal{O}^A acquires a logarithmic partner. The second assumption (63b) is also with no loss of generality.

First we note that it does not matter if we set ϵ_1 or Δ to zero, as indicated

¹⁵Note that $\mathcal{O}^{\text{log}} = a \frac{d}{dm} \mathcal{O}^m|_{m_A}$ if we parameterize m_B by $m_A + \Delta$ and take the limit $\Delta \rightarrow 0$ as suggested in e.g. [23, 25].

in Fig. 1. Additionally we can always fulfill the second requirement (63b) by choosing $f_B|_{\epsilon=0} = -c_1^A/c_1^B$ with the expansion coefficients

$$c^i(\epsilon) = c^i(0) + \epsilon c_1^i(0) + \mathcal{O}(\epsilon^2). \quad (66)$$

For the correlators the first requirement (63a) simply leads to

$$\langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^A(0) \rangle = \langle \mathcal{O}^B(z, \bar{z}) \mathcal{O}^B(0) \rangle = 0. \quad (67)$$

For the second two point function we insert the definition (62) of $\mathcal{O}^{\text{diff}}$. Since we defined the operators to have different weights, we can use the identity $\langle \mathcal{O}^A \mathcal{O}^B \rangle = 0$ and get

$$\begin{aligned} \langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^{\text{log}}(0) \rangle &= a \lim_{\Delta \rightarrow 0} \frac{\langle \mathcal{O}^A \mathcal{O}^A \rangle}{\Delta} \\ &= \frac{a}{2} \lim_{\Delta \rightarrow 0} \left(\frac{c^A/\Delta}{z^{2h_A} \bar{z}^{2\bar{h}_A}} \right) \\ &= \frac{a}{2} \lim_{\Delta \rightarrow 0} \left(\frac{(\partial_\Delta c^A)|_{\Delta=0}}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \mathcal{O}(\Delta) \right) \\ &= \frac{a}{2} \frac{(\partial_\Delta c^A)|_{\Delta=0}}{z^{2h_A} \bar{z}^{2\bar{h}_A}}. \end{aligned} \quad (68)$$

$\langle \mathcal{O}^B \mathcal{O}^{\text{log}} \rangle$ yields the same result because the different sign in the expansion compensates for the sign from $\mathcal{O}^{\text{diff}}$.

The third correlator is more difficult, but we can simplify the calculation by using the relation between the mass and the weights of massive operators. It follows from locality that

$$\lim_{\Delta \rightarrow 0} \frac{dh}{dm} = \lim_{\Delta \rightarrow 0} \frac{d\bar{h}}{dm} =: \hat{B} \quad (69)$$

with some constant \hat{B} . Therefore

$$\begin{aligned}
\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log}(0) \rangle &= a^2 \lim_{\Delta \rightarrow 0} \frac{\langle \mathcal{O}^A \mathcal{O}^A \rangle + \langle \mathcal{O}^B \mathcal{O}^B \rangle}{\Delta^2} \\
&= \frac{a^2}{2} \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left(\frac{c^A/\Delta}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \frac{c^B/\Delta}{z^{2h_B} \bar{z}^{2\bar{h}_B}} \right) \\
&= \frac{a^2}{2} \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left(- \frac{(\partial_{\Delta} c^A)|_{\Delta=0}}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \frac{(\partial_{\Delta} c^A)|_{\Delta=0}}{z^{2h_B} \bar{z}^{2\bar{h}_B}} \right. \\
&\quad \left. + \frac{\Delta(\partial_{\Delta}^2 c^A)|_{\Delta=0}}{2 z^{2h_A} \bar{z}^{2\bar{h}_A}} + \frac{\Delta(\partial_{\Delta}^2 c^B)|_{\Delta=0}}{2 z^{2h_B} \bar{z}^{2\bar{h}_B}} + \mathcal{O}(\Delta^2) \right) \\
&= \frac{a^2}{2} \frac{(\partial_{\Delta}^2 c^A)|_{\Delta=0} + (\partial_{\Delta}^2 c^B)|_{\Delta=0}}{2 z^{2h_A} \bar{z}^{2\bar{h}_A}} \\
&\quad - \frac{a^2}{2} (\partial_{\Delta} c^A)|_{\Delta=0} \lim_{\Delta \rightarrow 0} \frac{z^{-2h_A} \bar{z}^{-2\bar{h}_A} - z^{-2h_B} \bar{z}^{-2\bar{h}_B}}{\Delta} \\
&= \frac{a^2}{2} \frac{cst}{2 z^{2h_A} \bar{z}^{2\bar{h}_A}} + \frac{a^2}{2} (\partial_{\Delta} c^A)|_{\Delta=0} \frac{d}{dm} \frac{1}{z^{2h} \bar{z}^{2\bar{h}}} \Big|_{m=m_A} \\
&= \frac{a^2}{2} \frac{cst - 2\hat{B}(\partial_{\Delta} c^A)|_{\Delta=0} \ln |z|^2}{2 z^{2h_A} \bar{z}^{2\bar{h}_A}}. \tag{70}
\end{aligned}$$

To summarize we get the two point correlators

$$\langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^A(0) \rangle = 0 \tag{71a}$$

$$\langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^{\log}(0) \rangle = \frac{a}{2} \frac{\partial c^A}{\partial \Delta} \frac{1}{z^{2h(m_A)} \bar{z}^{2\bar{h}(m_A)}} \tag{71b}$$

$$\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log}(0) \rangle = \frac{a^2}{2} \frac{cst - 2\hat{B} \frac{\partial c^A}{\partial \Delta} \ln |z|^2}{z^{2h(m_A)} \bar{z}^{2\bar{h}(m_A)}}, \tag{71c}$$

from which we can infer that

$$b_L = a \frac{\partial c^A}{\partial \Delta} = a^2 \hat{B} \frac{\partial c^A}{\partial \Delta} \tag{72}$$

by comparing with (42) and identifying $\mathcal{O}^A = \mathcal{O}^L$. We also see that $a = 1/\hat{B}$ so the normalization depends on the derivatives of the conformal weights w.r.t. the mass. As we will see later this gives a factor $\text{sign}(m)\ell/2$ and hence the new anomaly corresponds to

$$b_L = a \frac{\partial c^A}{\partial \Delta} \Big|_{\Delta=0} = \text{sign}(m) \frac{2}{\ell} \frac{\partial c^A}{\partial \Delta} \Big|_{\Delta=0}. \tag{73}$$

3.2 Rank three LCFTs

Similar to the limit we took in the previous section, we are now going to calculate the correlators for three degenerating masses $m_A = m_B = m_C = \bar{m}$. Another operator $\mathcal{O}^{\text{diff}2}$ has to be defined as some sort of second derivative of the primaries \mathcal{O}^i . We also choose a more symmetric combination for $\mathcal{O}^{\text{diff}}$ than before. Our ansatz takes the form

$$\left\{ \mathcal{O}^A, \mathcal{O}^{\text{diff}} = \frac{a}{3} \left(\frac{\mathcal{O}^A - \mathcal{O}^B}{\Delta_{AB}} + \frac{\mathcal{O}^B - \mathcal{O}^C}{\Delta_{BC}} + \frac{\mathcal{O}^C - \mathcal{O}^A}{\Delta_{CA}} \right), \right. \\ \left. \mathcal{O}^{\text{diff}2} = b \frac{\Delta_{CB}\mathcal{O}^A + \Delta_{AC}\mathcal{O}^B + \Delta_{BA}\mathcal{O}^C}{\Delta_{AB}\Delta_{AC}\Delta_{CB}} \right\} \quad (74)$$

$$m_A, m_B, m_C \rightarrow \bar{m} \left\{ \mathcal{O}^A, \mathcal{O}^{\log}, \mathcal{O}^{\log 2} \right\}.$$

For the degeneration of the masses we have to take different limits depending on which of the operators \mathcal{O}^A , \mathcal{O}^B or \mathcal{O}^C we consider. We also have additional assumptions:

$$1. \quad c^A = c^B = c^C = 0 \quad (75a)$$

$$2. \quad \frac{\partial c^A}{\partial \Delta_{Aj}} = \frac{\partial c^B}{\partial \Delta_{Bj}} = \frac{\partial c^C}{\partial \Delta_{Cj}} = 0 \quad (75b)$$

$$3. \quad \frac{\partial^2 c^i}{(\partial \Delta_{ij})^2} = 0 \quad \text{i.e. no sum over } i \text{ and } i \neq j \quad (75c)$$

$$4. \quad \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} = \frac{\partial^2 c^B}{\partial \Delta_{BA} \partial \Delta_{BC}} = \frac{\partial^2 c^C}{\partial \Delta_{CA} \partial \Delta_{CB}} \quad (75d)$$

Setting all Δ s to zero is understood in these equations. We can again summarize all these assumptions by the expansion of e.g. c^C in terms of our small Δ parameters

$$c^C = \frac{\partial^2 c^C}{\partial \Delta_{CA} \partial \Delta_{CB}} \Delta_{CA} \Delta_{CB} + \mathcal{O}(\Delta^3), \quad (76)$$

since all other terms of order one or two vanish. Again assumptions one (75a) and two (75b) are always fulfilled for a rank three LCFT and requirements three (75c) and four (75d) can be achieved by the rescalings (61).

We take a closer look on how these constraints, especially those for the second derivatives, emerge. The charges generically fulfill

$$\text{a) } c^A(0, \epsilon_2) = c^B(0, \epsilon_2) = 0,$$

$$\text{b) } c^B(\epsilon_1, 0) = c^C(\epsilon_1, 0) = 0,$$

c) $c^A = c^C$ 'somewhere',

d) $\partial_{\epsilon_j} c^i|_{\epsilon=0} = 0$.

From a) and b) it follows that all second and higher derivatives vanish as well:

$$\frac{\partial^n c^{A,B}}{(\partial \epsilon_2)^n} \Big|_{(0,\epsilon_2)} = 0 \quad \text{and} \quad \frac{\partial^n c^{B,C}}{(\partial \epsilon_1)^n} \Big|_{(\epsilon_1,0)} = 0 \quad (77)$$

By construction, Fig. 1, the expansions of Δ_{ij} are

$$\Delta_{AB} = \epsilon_1 \Delta_{AB}^1 + \mathcal{O}(\epsilon^2) \quad (78a)$$

$$\Delta_{BC} = \epsilon_2 \Delta_{BC}^1 + \mathcal{O}(\epsilon^2) \quad (78b)$$

$$\Delta_{AC} = \Delta_{AB} + \Delta_{BC} = \epsilon_1 \Delta_{AB}^1 + \epsilon_2 \Delta_{BC}^1 + \mathcal{O}(\epsilon^2). \quad (78c)$$

The quantity c^A in terms of epsilons, or, using (78), in terms of Δ is

$$\begin{aligned} c^A(\epsilon_1, \epsilon_2) &= c_{11}^A \epsilon_1^2 + c_{12}^A \epsilon_1 \epsilon_2 + \mathcal{O}(\epsilon^3) \\ &= c_{11}^A \frac{(\Delta_{AB})^2}{(\Delta_{AB}^1)^2} + c_{12}^A \frac{\Delta_{AB} \Delta_{BC}}{\Delta_{AB}^1 \Delta_{BC}^1} + \mathcal{O}(\Delta^3) \\ &= c_{11}^A \frac{(\Delta_{AB})^2}{(\Delta_{AB}^1)^2} + c_{12}^A \frac{\Delta_{AB} (\Delta_{AC} - \Delta_{AB})}{\Delta_{AB}^1 \Delta_{BC}^1} + \mathcal{O}(\Delta^3) \\ &= (\Delta_{AB})^2 \left(\frac{c_{11}^A}{(\Delta_{AB}^1)^2} - \frac{c_{12}^A}{\Delta_{AB}^1 \Delta_{BC}^1} \right) + c_{12}^A \frac{\Delta_{AB} \Delta_{AC}}{\Delta_{AB}^1 \Delta_{BC}^1} + \mathcal{O}(\Delta^3). \end{aligned} \quad (79)$$

It seems as if c^A would have a non-vanishing $(\Delta_{AB})^2$ expansion term which contradicts our assumption three (75c). But it turns out that this term is identically zero. We can show this as follows.

Remember the second degeneration $\mathcal{O}^A = \mathcal{O}^C$ along which the charge also vanishes, but which we did not parameterize with ϵ_1 or ϵ_2 , but which would rather be combination of them.¹⁶ This combination of ϵ_1 or ϵ_2 manifests itself in relating the coefficients c_{11}^A and c_{12}^A in such a way that the unwanted term becomes exactly zero.

Using (78) with $\Delta_{AC} = 0$ and $c^A(\Delta_{AC} = 0) = 0$ we get

$$\begin{aligned} c^A(\epsilon_1, \epsilon_2) &= c_{11}^A \epsilon_1^2 + c_{12}^A \epsilon_1 \epsilon_2 + \mathcal{O}(\epsilon^3) \\ 0 &= c_{11}^A \frac{(\Delta_{AB})^2}{(\Delta_{AB}^1)^2} - c_{12}^A \frac{\Delta_{AB} \Delta_{AB}}{\Delta_{AB}^1 \Delta_{BC}^1} + \mathcal{O}(\Delta^3). \end{aligned} \quad (80)$$

¹⁶We chose our parameterization such that the c^B expansion is of the form we want. With c^C a similar story to the one we are discussing now goes through.

This is exactly the $(\Delta_{AB})^2$ expansion term and apparently it vanishes identically. Its appearance is only an artifact of our parameterization. Now we have shown that our initial assumptions on the expansion of the charges with respect to the various Δ are correct and in no way restrictive.

Coming back to the derivation of correlators we note that our additional assumptions set more terms to zero, for example we already derived

$$\langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^{\log}(0) \rangle \approx \frac{\partial c^A}{\partial \Delta} = 0. \quad (81)$$

Now we compute the first two non-zero correlators

$$\begin{aligned} \langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle &= \frac{b}{2} \lim_{\substack{\Delta_{AB} \rightarrow 0 \\ \Delta_{AC}}} \frac{c^A(\Delta)}{\Delta_{AB} \Delta_{AC}} z^{-2h_A} \bar{z}^{-2\bar{h}_A} \\ &= \frac{b}{2} \lim_{\substack{\Delta_{AB} \rightarrow 0 \\ \Delta_{AC}}} \left(\frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} + \mathcal{O}(\Delta) \right) z^{-2h_A} \bar{z}^{-2\bar{h}_A} \\ &= \frac{b}{2} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} z^{-2h_A} \bar{z}^{-2\bar{h}_A} \end{aligned} \quad (82)$$

and

$$\begin{aligned} \langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log}(0) \rangle &= \frac{a^2}{18} \left\{ \lim_{\substack{\Delta_{AB} \rightarrow 0 \\ \Delta_{AC}}} \left(\frac{1}{\Delta_{AB}} + \frac{1}{\Delta_{AC}} \right)^2 \frac{c^A(\Delta)}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \right. \\ &\quad + \lim_{\substack{\Delta_{BC} \rightarrow 0 \\ \Delta_{BA}}} \left(\frac{1}{\Delta_{BC}} + \frac{1}{\Delta_{BA}} \right)^2 \frac{c^B(\Delta)}{z^{2h_B} \bar{z}^{2\bar{h}_B}} + \\ &\quad \left. + \lim_{\substack{\Delta_{CA} \rightarrow 0 \\ \Delta_{CB}}} \left(\frac{1}{\Delta_{CA}} + \frac{1}{\Delta_{CB}} \right)^2 \frac{c^C(\Delta)}{z^{2h_C} \bar{z}^{2\bar{h}_C}} \right\} \\ &= \frac{a^2}{18} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \lim_{\Delta \rightarrow 0} \left\{ \frac{\Delta_{AC}}{\Delta_{AB}} + 2 + \frac{\Delta_{AB}}{\Delta_{AC}} + \right. \\ &\quad \left. + \frac{\Delta_{BA}}{\Delta_{BC}} + 2 + \frac{\Delta_{BC}}{\Delta_{BA}} + \frac{\Delta_{CB}}{\Delta_{CA}} + 2 + \frac{\Delta_{CA}}{\Delta_{CB}} + \mathcal{O}(\Delta) \right\} \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} \\ &= \frac{a^2}{18} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \lim_{\Delta \rightarrow 0} \left\{ 9 + \mathcal{O}(\Delta) \right\} \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} \\ &= \frac{a^2}{2} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}}. \end{aligned} \quad (83)$$

These two already tell us something about the normalization b in (74). Comparing the correlators (82) and (83) to general form (45) we observe $a^2 = b$

and the generalized new anomaly will be proportional to the second derivative of the charges. We proceed with

$$\begin{aligned}
\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle &= \frac{ab}{6} \left\{ \lim_{\substack{\Delta_{AB} \rightarrow 0 \\ \Delta_{AC}}} \left(\frac{1}{\Delta_{AB}^2 \Delta_{AC}} + \frac{1}{\Delta_{AB} \Delta_{AC}^2} \right) \frac{c^A(\Delta)}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \right. \\
&+ \lim_{\substack{\Delta_{BC} \rightarrow 0 \\ \Delta_{BA}}} \left(\frac{1}{\Delta_{BA}^2 \Delta_{BC}} + \frac{1}{\Delta_{BA} \Delta_{BC}^2} \right) \frac{c^B(\Delta)}{z^{2h_B} \bar{z}^{2\bar{h}_B}} + \\
&+ \left. \lim_{\substack{\Delta_{CA} \rightarrow 0 \\ \Delta_{CB}}} \left(\frac{1}{\Delta_{CA}^2 \Delta_{CB}} + \frac{1}{\Delta_{CA} \Delta_{CB}^2} \right) \frac{c^C(\Delta)}{z^{2h_C} \bar{z}^{2\bar{h}_C}} \right\} \\
&= \frac{ab}{6} \lim_{\Delta \rightarrow 0} \left\{ \left(\frac{1}{6} \frac{\partial^3 c^A}{\partial \Delta_{AB}^3} \left(\frac{\Delta_{AB}}{\Delta_{AC}} + \frac{\Delta_{AB}^2}{\Delta_{AC}^2} \right) + \frac{1}{2} \frac{\partial^3 c^A}{\partial \Delta_{AB}^2 \partial \Delta_{AC}} \left(1 + \frac{\Delta_{AB}}{\Delta_{AC}} \right) + \right. \right. \\
&+ \frac{1}{2} \frac{\partial^3 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}^2} \left(\frac{\Delta_{AC}}{\Delta_{AB}} + 1 \right) + \left. \frac{1}{6} \frac{\partial^3 c^A}{\partial \Delta_{AC}^3} \left(\frac{\Delta_{AC}^2}{\Delta_{AB}^2} + \frac{\Delta_{AC}}{\Delta_{AB}} \right) \right) \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} \\
&+ \left. \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \left(\frac{1}{\Delta_{AB}} + \frac{1}{\Delta_{AC}} \right) \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \mathcal{O}(\Delta) \right\} + \text{B,C}. \quad (84)
\end{aligned}$$

As indicated we take the limits $\Delta_{AB}, \Delta_{AC}, \Delta_{BC} \rightarrow 0$ simultaneously. Therefore taking this limit of a fraction of any two Δ yields an arbitrary number, depending just on how we parameterize the limiting procedure. All terms that are third derivatives of the c^i functions add up to some constant. Since this constant depends only on our parameterization it cannot have any physical meaning. Indeed this shows once more that logarithmic operators \mathcal{O}^{\log} are only defined up to an arbitrary number of $\mathcal{O}^{A/B}$ operators and similarly for \mathcal{O}^{\log^2} .

When we write out all second derivative terms we get three terms of the form

$$\frac{1}{\Delta_{ij}} \left(\frac{1}{z^{2h_i} \bar{z}^{2\bar{h}_i}} - \frac{1}{z^{2h_j} \bar{z}^{2\bar{h}_j}} \right) \rightarrow \frac{d}{dm} \frac{1}{z^{2h_i} \bar{z}^{2\bar{h}_i}} \Big|_{m=\bar{m}} = -2\hat{B} \frac{\ln |z|^2}{z^{2h_i} \bar{z}^{2\bar{h}_i}}. \quad (85)$$

The final result for $\langle \mathcal{O}^{\log} \mathcal{O}^{\log^2} \rangle$ is

$$\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{ab}{6} \left(\frac{3 \text{cst}_1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} - 6\hat{B} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \frac{\ln |z|^2}{z^{2h_A} \bar{z}^{2\bar{h}_A}} \right). \quad (86)$$

When calculating the next correlator we try to obtain the same third derivative terms that add up to cst_1 . In addition the same argument – no divergences, everything adds up to a constant – holds for all terms that are fourth

order derivatives of the c^i .

$$\begin{aligned}
\langle \mathcal{O}^{\log^2}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle &= \frac{b^2}{2} \left\{ \lim_{\substack{\Delta_{AB} \rightarrow 0 \\ \Delta_{AC}}} \frac{c^A(\Delta)}{\Delta_{AB}^2 \Delta_{AC}^2} z^{-2h_A} \bar{z}^{-2\bar{h}_A} + \right. \\
&+ \lim_{\substack{\Delta_{BA} \rightarrow 0 \\ \Delta_{BC}}} \frac{c^B(\Delta)}{\Delta_{BA}^2 \Delta_{BC}^2} z^{-2h_B} \bar{z}^{-2\bar{h}_B} + \left. \lim_{\substack{\Delta_{CA} \rightarrow 0 \\ \Delta_{CB}}} \frac{c^C(\Delta)}{\Delta_{CA}^2 \Delta_{CB}^2} z^{-2h_C} \bar{z}^{-2\bar{h}_C} \right\} = \\
&= \frac{b^2}{2} \lim_{\Delta \rightarrow 0} \left\{ \left(\frac{1}{24} \frac{\partial^4 c^A}{\partial \Delta_{AB}^4} \frac{\Delta_{AB}^2}{\Delta_{AC}^2} + \frac{1}{6} \frac{\partial^4 c^A}{\partial \Delta_{AB}^3 \partial \Delta_{AC}} \frac{\Delta_{AB}}{\Delta_{AC}} + \frac{1}{4} \frac{\partial^4 c^A}{\partial \Delta_{AB}^2 \partial \Delta_{AC}^2} + \right. \right. \\
&+ \frac{1}{6} \frac{\partial^4 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}^3} \frac{\Delta_{AC}}{\Delta_{AB}} + \left. \frac{1}{24} \frac{\partial^4 c^A}{\partial \Delta_{AC}^4} \frac{\Delta_{AC}^2}{\Delta_{AB}^2} \right) \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \\
&+ \left(\frac{1}{6} \frac{\partial^3 c^A}{\partial \Delta_{AB}^3} \frac{\Delta_{AB}}{\Delta_{AC}^2} + \frac{1}{2} \frac{\partial^3 c^A}{\partial \Delta_{AB}^2 \partial \Delta_{AC}} \frac{1}{\Delta_{AC}} + \frac{1}{2} \frac{\partial^3 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}^2} \frac{1}{\Delta_{AB}} + \right. \\
&+ \left. \frac{1}{6} \frac{\partial^3 c^A}{\partial \Delta_{AC}^3} \frac{\Delta_{AC}}{\Delta_{AB}^2} \right) \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \\
&+ \left. \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \frac{1}{\Delta_{AB} \Delta_{AC}} \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \mathcal{O}(\Delta) \right\} + \text{B,C} \tag{87}
\end{aligned}$$

Unfortunately there is no simple way to rewrite the third order term. Simple counting of Δ suggests that it will yield a derivative dz/dm . The prefactor should ideally match cst_1 from the $\langle \mathcal{O}^{\log} \mathcal{O}^{\log^2} \rangle$ correlator. If it does not, we have to choose a different \mathcal{O}^{\log^2} by adding further \mathcal{O}^{\log} operators. Now we have to show how the derivative dz/dm comes about and that the result (87) is actually finite.

We start by 'symmetrizing' each c^A , c^B and c^C factor, i.e. we pull out a factor $1/\Delta + 1/\Delta$. For c_A this is $(1/\Delta_{AB} + 1/\Delta_{AC})$. Then we are left with three terms of the exact same form as those that added up to $3cst_1$. All of them are third derivatives of the 'charge' c_i and are multiplied a different factor $(1/\Delta_{ij} + 1/\Delta_{ik})/z^i$. At this point a useful observation is that we can change the third order term of the expansion of the c^i by choosing another normalization of the operators \mathcal{O}^i . The point is that we change the expansion of the quantity $f \cdot c^B$ in the third order according to (61d)

$$(\partial^3 c^B \Delta^3) + 3 \frac{\partial^2 c^B}{\partial \Delta_{BA} \partial \Delta_{BC}} \Delta_{BA} \Delta_{BC} \left(\frac{\partial f}{\partial \Delta_{BA}} \Delta_{BA} + \frac{\partial f}{\partial \Delta_{BC}} \Delta_{BC} \right). \tag{88}$$

Here $(\partial^3 c^B \Delta^3)$ denotes all usual expansion terms, e.g. the ones we wrote down explicitly in all glory detail for c^A . We use this freedom to normalize \mathcal{O}^B and \mathcal{O}^C such that $(\partial^3 c^A \Delta^3) = (\partial^3 (c^B f) \Delta^3) = (\partial^3 (c^C g) \Delta^3)$.

The aforementioned 'symmetrization' yield a factor $1/2$, thus the third order

terms are

$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{6} \frac{\partial^3 c^A}{\partial \Delta_{AB}^3} \left(\frac{\Delta_{AB}}{\Delta_{AC}} + \frac{\Delta_{AB}^2}{\Delta_{AC}^2} \right) + \frac{1}{2} \frac{\partial^3 c^A}{\partial \Delta_{AB}^2 \partial \Delta_{AC}} \left(1 + \frac{\Delta_{AB}}{\Delta_{AC}} \right) + \dots \right) \times \\ & \left[\left(\frac{1}{\Delta_{AB}} + \frac{1}{\Delta_{AC}} \right) \frac{1}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \left(\frac{1}{\Delta_{BA}} + \frac{1}{\Delta_{BC}} \right) \frac{1}{z^{2h_B} \bar{z}^{2\bar{h}_B}} + \right. \\ & \left. + \left(\frac{1}{\Delta_{CA}} + \frac{1}{\Delta_{CB}} \right) \frac{1}{z^{2h_C} \bar{z}^{2\bar{h}_C}} \right] \rightarrow \frac{cst_2}{2} \frac{3 dz}{dm}. \end{aligned} \quad (89)$$

We showed that the result is finite and yields a derivative. The constant cst_2 equals the previous cst_1 with c^B and c^C renormalized by f and g respectively. Now we go on with the full correlator

$$\begin{aligned} \langle \mathcal{O}^{\log^2} \mathcal{O}^{\log^2} \rangle &= \frac{b^2}{2} \left(\frac{cst_3}{z^{2h_A} \bar{z}^{2\bar{h}_A}} - 3\hat{B} cst_2 \frac{\ln |z|^2}{z^{2h_A} \bar{z}^{2\bar{h}_A}} + \right. \\ & \left. + \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \lim_{\Delta \rightarrow 0} \left(\frac{z^{-2h_A} \bar{z}^{-2\bar{h}_A}}{\Delta_{AB} \Delta_{AC}} + \frac{z^{-2h_B} \bar{z}^{-2\bar{h}_B}}{\Delta_{BA} \Delta_{BC}} + \frac{z^{-2h_C} \bar{z}^{-2\bar{h}_C}}{\Delta_{CA} \Delta_{CB}} \right) \right). \end{aligned} \quad (90)$$

We show now that the last terms is just the second derivative. We parameterize the Δ as $\Delta_{AB} = \Delta_{BC} = \epsilon$ so that $m_A = m_B + \epsilon$ and $m_C = m_B - \epsilon$. It follows

$$\begin{aligned} & \lim_{\Delta \rightarrow 0} \left(\frac{z^A}{\Delta_{AB} \Delta_{AC}} + \frac{z^B}{\Delta_{BA} \Delta_{BC}} + \frac{z^C}{\Delta_{CA} \Delta_{CB}} \right) = \\ & \lim_{\epsilon \rightarrow 0} \left(\frac{z(m_B + \epsilon)}{2\epsilon^2} - \frac{z(m_B)}{\epsilon^2} + \frac{z(m_B - \epsilon)}{2\epsilon^2} \right) = \frac{1}{2} \frac{d^2 z}{dm^2} \Big|_{m=m_B}, \end{aligned} \quad (91)$$

and the last term becomes $\frac{d^2 z}{dm^2} = 2\hat{B} \frac{\ln^2 |z|^2}{z^{2h} \bar{z}^{2\bar{h}}}$. Finally we arrive at

$$\langle \mathcal{O}^{\log^2}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{b^2 cst_3 - 3\hat{B} cst_2 \ln |z|^2 + 2\hat{B}^2 \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \ln^2 |z|^2}{z^{2h_A} \bar{z}^{2\bar{h}_A}}.$$

In summary we get the following non-zero two point functions

$$\langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{b}{2} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \frac{1}{z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}} \quad (92a)$$

$$\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log}(0) \rangle = \frac{a^2}{2} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \frac{1}{z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}} \quad (92b)$$

$$\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{ab cst_2 - 2\hat{B} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \ln |z|^2}{2 z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}} \quad (92c)$$

$$\langle \mathcal{O}^{\log^2}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{b^2 cst_3 - 3\hat{B} cst_2 \ln |z|^2 + 2\hat{B}^2 \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \ln^2 |z|^2}{2 z^{2h(m)} \bar{z}^{2\bar{h}(m)}}. \quad (92d)$$

We already noticed $a^2 = b$. Inserting $a = 1/\hat{B} = 2$ we parameterize the modes by the (dimensionless) mass. The generalized new anomaly a_0 is given by

$$a_0 = \frac{2}{\ell^2} \frac{\partial^2 c^A}{\partial \Delta_{AB} \partial \Delta_{AC}} \Big|_{\Delta_{AB}=\Delta_{AC}=0}. \quad (93)$$

The two point correlators become

$$\langle \mathcal{O}^A(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{a_0}{z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}} \quad (94a)$$

$$\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log}(0) \rangle = \frac{a_0}{z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}} \quad (94b)$$

$$\langle \mathcal{O}^{\log}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{4cst_2 - 2a_0 \ln |z|^2}{z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}} \quad (94c)$$

$$\langle \mathcal{O}^{\log^2}(z, \bar{z}) \mathcal{O}^{\log^2}(0) \rangle = \frac{cst_3 - 12cst_2 \ln |z|^2 + 2a_0 \ln^2 |z|^2}{z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}}. \quad (94d)$$

They do not match (45) to the point, but this means that our definitions of $\mathcal{O}^{\text{diff}}$ and $\mathcal{O}^{\text{diff}^2}$ were not the same as in (45). Addition of further \mathcal{O}^A and $\mathcal{O}^{\text{diff}}$ alters the non-physical constants as described in section 2. Then the 2-point functions (94) match precisely the 2-point functions (45).

4 GMG – an application

4.1 The model

GMG was introduced together with new massive gravity (NMG) in [14]. In this paper, however, only NMG was discussed.¹⁷ As mentioned in the introduction, GMG is a combination of topologically massive gravity (TMG) and new massive gravity (NMG).

TMG was introduced in [4] and later reconsidered by Li, Song and Strominger in [3]. Their goal was to present a chiral theory of gravity. At the chiral/logarithmic¹⁸ point the holographic dual boundary CFT was conjectured to be a chiral CFT. *Chiral gravity* would essentially be characterized by a chiral (right-moving) partition function, so it factorizes trivially holomorphically. Since one of the central charges vanishes the hope was to discard the left-moving excitations as pure gauge. The CFT obtained in this way would be a chiral CFT. The references [10] and [27] showed that it is impossible to discard *all* left-moving excitations. Moreover it was shown that the partition function does not factorize [6], neither for TMG, nor for NMG.

Nevertheless TMG earned a lot of interest in the last two years. Though relatively simple it exhibits black hole solutions and propagates massive gravity waves/gravitons. The special properties of TMG compared to Einstein gravity arise due to the Chern-Simons term L_{CS} . This term adds further derivatives of the metric and discriminates left-moving modes against right-moving modes and renders the model parity non-invariant. The coupling of the Chern-Simons term introduces a massive scale and parameterizes the propagating (massive) waves.

In 2009 NMG was introduced by Bergshoeff, Hohm and Townsend, see e.g. [14]. The idea was to find a three-dimensional gravity model that propagates massive spin two modes of helicity ± 2 .¹⁹ Gravitons, gauge bosons that mediate the gravitational force, have these properties. So NMG is a gravity model that exhibits solutions which we could interpret as gravitons. It was constructed as an extension to the Pauli-Fierz theory for massive particles with spin 2 in 3D.

In summary GMG is a gravity model that exhibits two independent massive graviton solutions. They can degenerate with the right and left-moving

¹⁷The formulas in this section are from [26], but I am using the conventions from [14].

¹⁸In the original paper [3] TMG was conjectured that the dual CFT be chiral. Therefore the point where the dual CFT becomes an LCFT is often called 'chiral' point. As it turned out the dual CFT is not chiral. But one always obtains a logarithmic CFT. Thus we prefer to speak of *logarithmic* points.

¹⁹Since NMG is parity preserving there have to be two modes, which is not the case for TMG. In this parity violating theory (TMG) only a single mode of helicity $+2$ propagates.

modes separately, as in T/NMG. Additionally we get a doubly logarithmic point when both massive solution degenerate with one of the primaries. New interesting physics arises when the two massive modes degenerate, while the central charges stay finite. LCFTs of this form only appeared in solid state physics. GMG is the first gravity model with this property.

Here we compute the new anomaly b_L and the generalized new anomaly a_0 for GMG. As GMG is a combination of TMG and NMG we can get their respective results for b_L by taking different limits of the GMG parameters to arrive at the logarithmic points of TMG and NMG. We show that the new anomaly, which we get with the formalism derived in the latter section, precisely matches the results found previously [25, 28]. Additionally we calculate a_0 for GMG at the doubly logarithmic point.

We start with the GMG action given in (1)

$$S_{GMG} = \frac{1}{\kappa} \int d^3x \sqrt{-g} \left\{ \sigma R - 2\lambda m^2 + \frac{1}{m^2} K + \frac{1}{\mu} L_{CS} \right\}, \quad (95)$$

with

$$L_{CS} = \frac{1}{2} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^\alpha \left[\partial_\mu \Gamma_{\alpha\nu}^\sigma + \frac{2}{3} \Gamma_{\nu\tau}^\sigma \Gamma_{\nu\alpha}^\tau \right] \quad (96)$$

$$K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2. \quad (97)$$

The eoms of GMG are

$$\sigma G_{\mu\nu} + \lambda m^2 g_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0. \quad (98)$$

For an asymptotic AdS solution we make an ansatz with an AdS background metric (6) plus a small variation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (99)$$

Using transverse traceless gauge, i.e. $h = h_\mu{}^\mu = \nabla^\mu h_{\mu\nu} = 0$, the linearized eoms simplify considerably. They are given by

$$\sigma G_{\mu\nu}^{(1)} + \lambda m^2 h_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu}^{(1)} + \frac{1}{\mu} C_{\mu\nu}^{(1)} = 0, \quad (100)$$

where the superscript denotes the first order expansion in $h_{\mu\nu}$. The various tensors read

$$G_{\mu\nu}^{(1)} = R_{\mu\nu}^{(1)} + \frac{3}{\ell^2} h_{\mu\nu} \quad (101)$$

$$C_{\mu\nu}^{(1)} = \varepsilon_\mu{}^{\alpha\beta} \nabla_\alpha (R_{\beta\nu}^{(1)} + \frac{2}{\ell^2} h_{\beta\nu}) \quad (102)$$

$$K_{\mu\nu}^{(1)} = 2\nabla^2 R_{\mu\nu}^{(1)} + \frac{4}{\ell^2} \nabla^2 h_{\mu\nu} + \frac{5}{\ell^2} R_{\mu\nu}^{(1)} - \frac{19}{2\ell^4} h_{\mu\nu}. \quad (103)$$

The linearized Ricci tensor is

$$R_{\mu\nu}^{(1)} = -\frac{1}{2}\nabla^2 h_{\mu\nu} - \frac{3}{\ell^2}h_{\mu\nu}. \quad (104)$$

Now we have to insert the formulas above into the linearized eoms (100). With the cosmological constant λ given by the AdS length ℓ and the mass m^2 ,

$$\lambda = \frac{1}{m^4\ell^4}\left(\frac{1}{4} - \sigma m^2\ell^2\right), \quad (105)$$

we can factorize the eoms into

$$\left(\nabla^2 + \frac{2}{\ell^2}\right)\left(\nabla^2 h_{\mu\nu} + \frac{m^2}{\mu}\varepsilon_\mu^{\alpha\beta}\nabla_\alpha h_{\beta\nu} + \left(\sigma m^2 + \frac{5}{2\ell^2}\right)h_{\mu\nu}\right) = 0. \quad (106)$$

We can rewrite these equations using the four commuting first order operators $\mathcal{D}^{L/R}$ and $\mathcal{D}^{m\pm}$. These operators were introduced in [3].

$$\left(\mathcal{D}^{L/R}\right)_\mu^\nu = \delta_\mu^\nu \pm \ell\varepsilon_\mu^{\alpha\nu}\nabla_\alpha \quad \text{and} \quad \left(\mathcal{D}^{m\pm}\right)_\mu^\nu = \delta_\mu^\nu + \frac{1}{m_\pm}\varepsilon_\mu^{\alpha\nu}\nabla_\alpha \quad (107)$$

A short calculation shows that $(\mathcal{D}^L\mathcal{D}^R h)_{\mu\nu}$ is proportional to the first bracket in (106):

$$\begin{aligned} (\mathcal{D}^L\mathcal{D}^R h)_{\mu\nu} &= (\delta_\mu^\alpha + \ell\varepsilon_\mu^{\beta\alpha}\bar{\nabla}_\beta)(\delta_\alpha^\gamma - \ell\varepsilon_\alpha^{\delta\gamma}\bar{\nabla}_\delta)h_{\gamma\nu} \\ &= h_{\mu\nu} - \ell^2\varepsilon_\mu^{\beta\alpha}\bar{\nabla}_\beta\varepsilon_\alpha^{\delta\gamma}\bar{\nabla}_\delta h_{\gamma\nu} \\ &= h_{\mu\nu} + \ell^2(\delta_\mu^\delta\bar{g}^{\beta\gamma} - \delta_\mu^\gamma\bar{g}^{\beta\delta})\bar{\nabla}_\beta\bar{\nabla}_\delta h_{\gamma\nu} \\ &= h_{\mu\nu} + \ell^2\bar{\nabla}_\beta\bar{\nabla}_\mu h_\nu^\beta - \ell^2\bar{\nabla}^2 h_{\mu\nu} \\ &= h_{\mu\nu} - 3h_{\mu\nu} - \ell^2\bar{\nabla}^2 h_{\mu\nu} \\ &= -\ell^2(\bar{\nabla}^2 h_{\mu\nu} + \frac{2}{\ell^2}h_{\mu\nu}) \end{aligned} \quad (108)$$

We used the identity

$$[\bar{\nabla}_\beta, \bar{\nabla}_\alpha]h^\beta{}_\nu = -\frac{3}{\ell^2}h_{\alpha\nu} \quad (109)$$

that follows from our gauge condition. A similar calculation shows

$$\begin{aligned} (\mathcal{D}^{m+}\mathcal{D}^{m-}h)_{\mu\nu} &= \frac{1}{m_+m_-}\left(\bar{\nabla}^2 h_{\mu\nu} + (m_+ + m_-)\varepsilon_\mu^{\alpha\beta}\nabla_\alpha h_{\beta\nu} + \right. \\ &\quad \left. + (m_+m_- + \frac{3}{\ell^2})h_{\mu\nu}\right). \end{aligned} \quad (110)$$

The eoms can be written in the form

$$(\mathcal{D}^L \mathcal{D}^R \mathcal{D}^{m_+} \mathcal{D}^{m_-} h)_{\mu\nu} = 0 \quad (111)$$

with m_{\pm} given by

$$m_+ + m_- = \frac{m^2}{\mu} \quad \text{and} \quad m_+ m_- = \sigma m^2 - \frac{1}{2\ell^2} \quad (112)$$

or

$$\ell m_{\pm} = \frac{m^2 \ell^2}{2\mu\ell} \pm \sqrt{\frac{m^4 \ell^4}{4\mu^2 \ell^2} - \sigma m^2 \ell^2 + \frac{1}{2}}. \quad (113)$$

Since all operators commute mutually, the solutions to the eoms are simply given by modes $h_{\mu\nu}$ that obey

$$(\mathcal{D}^i h^i)_{\mu\nu} = 0, \quad (114)$$

where we call $h_{\mu\nu}^i$ left-,right- or massive modes. At the logarithmic points, where one or two operators degenerate with the left mode, we get further solutions of the form

$$(\mathcal{D}^L \mathcal{D}^L h^{\text{log}})_{\mu\nu} = 0 \quad \text{but} \quad (\mathcal{D}^L h^{\text{log}})_{\mu\nu} \neq 0. \quad (115)$$

At the logarithmic points ℓm_{\pm} have to be either plus or minus one.²⁰ It is also possible that m_+ and m_- are the same. The central charges and conformal weights of the normalizable massive branches are

$$\begin{aligned} c_{L/R} &= \frac{3\ell}{2G} \sigma \left(1 + \frac{1}{2\sigma m^2 \ell^2} \mp \frac{1}{\sigma \mu \ell} \right) \\ &= \frac{3\ell}{2G} \sigma \left(1 + \frac{1 \mp 2\ell(m_+ + m_-)}{1 + 2\ell^2 m_+ m_-} \right) \end{aligned} \quad (116)$$

$$\begin{aligned} m_{\pm} > 0: \quad (h, \bar{h}) &= \left(\frac{3 + \ell m_{\pm}}{2}, \frac{-1 + \ell m_{\pm}}{2} \right) \\ m_{\pm} < 0: \quad (h, \bar{h}) &= \left(\frac{-1 - \ell m_{\pm}}{2}, \frac{3 - \ell m_{\pm}}{2} \right). \end{aligned} \quad (117)$$

NMG and TMG can be recovered from this model by taking the limits $\mu \rightarrow \infty$ or $m^2 \rightarrow \infty$, i.e. $m_+ \rightarrow \infty$ or $m_+ \rightarrow -m_-$, respectively. The following picture shows the possible operator degenerations of \mathcal{D}^L and $\mathcal{D}^{m_{\pm}}$ in GMG.

²⁰In contrast to [14] we define $m_+ > m_-$ not necessarily bigger than zero.

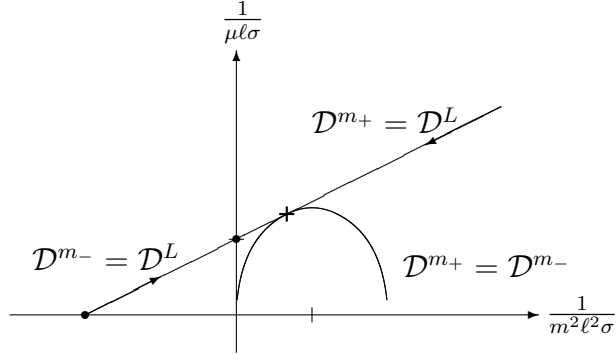


Figure 2: The logarithmic limits of General Massive Gravity.

4.2 Anomalies

Now we want to use the formulas derived in the sections two and three to compute the new and the generalized new anomaly of GMG. First we need to identify the c^A quantity from (56), which in our case at hand corresponds exactly to the left central charge $c^A = c_L$. The Δ are the differences between the masses of the respective operators, e.g. for the $\mathcal{D}^{m_-} = \mathcal{D}^L$ case $\Delta = 1/\ell - m_-$. We can thus simply replace in (73) the derivative ∂_Δ by $-\partial_{m_-}$. According to the weights (117) the $(d/dm)(z^{-2h(m)}\bar{z}^{-2\bar{h}(m)})$ terms yield a factor $-\ell \ln|z|^2$ for each derivative.

4.2.1 Standard degeneration: $\mathcal{D}^L = \mathcal{D}^{m_-}$

For the new anomaly we get from (73)

$$b_L = -a \frac{\partial c_L}{\partial m_-} \Big|_{m_- = 1/\ell} = a \frac{3\ell^2 \sigma}{G} \frac{1 - \ell m_+}{1 + 2\ell m_+}. \quad (118)$$

We observe that b_L vanishes at the doubly logarithmic point $m_+ = 1/\ell$. Note that the denominator cannot be zero. This would happen at $m_+ = -1/2\ell$ but we defined $m_+ \geq m_-$ and we already set $m_- = 1/\ell$.

The limits where we approach the logarithmic points of TMG and NMG are $m_+ \rightarrow \infty$ and $m_+ \rightarrow -1/\ell$. When we compare the values for these limits we note once more that we have to choose $a = 2/\ell$ to get the new anomalies

$$b_L^{\text{TMG}} = -\frac{3\ell}{G} \sigma \quad \text{and} \quad b_L^{\text{NMG}} = -\frac{12\ell}{G} \sigma. \quad (119)$$

The new anomaly of GMG finally reads

$$\begin{aligned}
b_L^{\text{GMG}} &= \frac{6\ell\sigma}{G} \frac{1 - \ell m_+}{1 + 2\ell m_+} \\
&= \frac{3\ell\sigma}{G} \left(\frac{3}{2m^2\ell^2\sigma} - 1 \right) = \frac{3\ell\sigma}{G} \left(\frac{3}{\mu\ell\sigma} - 4 \right). \tag{120}
\end{aligned}$$

4.2.2 Doubly Logarithmic degeneration: $\mathcal{D}^L = \mathcal{D}^{m_-} = \mathcal{D}^{m_+}$

From (93) we get the generalized new anomaly of GMG

$$a_0 = \frac{a^2}{2} \frac{\partial^2 c_L}{\partial m_+ \partial m_-} \Big|_{m_{\pm}=1/\ell} = \frac{2\ell}{G} \sigma. \tag{121}$$

5 Two point correlators on the gravity side

In order to have a non-trivial test for our results of the anomalies on the CFT side we calculate the two point correlators of the modes $\psi_{\mu\nu}^L$, $\psi_{\mu\nu}^{\log}$ and $\psi_{\mu\nu}^{\log^2}$ on the gravity side. The AdS/CFT recipe tells us that we have to insert the non-normalizable solutions of the eoms – sourcing the operators in the corresponding CFT – into the second variation of the action. Next we only have to consider the asymptotic behavior, i.e. the large ρ limit. Then we have to take the limit of large weights $h \rightarrow \infty$ and $\bar{h} \rightarrow -\infty$.

In spirit we follow the calculations in [25, 28] so here we will only denote the differences. These will mainly concern prefactors, however for the two-point function between two doubly logarithmic modes we will (be forced to) go into some more detail and refer to a paper by Skenderis, Taylor and van Rees [29]. As a warm up we use the techniques for the log-log² correlator too, which is the same calculation as in [29].

Let me give a very short review of the ideas that lead Grumiller and Sachs, and that we will pursue also.

To derive the correlators we take a short-cut and instead of inserting the modes into the full second variation of GMG and compute all holographic counterterms to this action, we refer to what Grumiller and Sachs called the Einstein result. The following formula is the simplest (and only) non-trivial two point correlator in Einstein gravity:

$$\begin{aligned} \delta^{(2)}S(\psi_1, \psi_2) &= -\frac{1}{16^2\pi G} \int d^3x \sqrt{-g} \psi_1^{\mu\nu*} (EOM_{\psi_1})_{\mu\nu} \\ &= -\frac{1}{16^2\pi G} \int d^3x \sqrt{-g} \psi_1^{\mu\nu*} \mathcal{G}(\psi_2)_{\mu\nu} \end{aligned} \quad (122)$$

We relate this integral (plus contributions from counterterms) to the Brown-Henneaux central charge $c_{BH} = 3\ell/2G$. In the following we calculate the difference of the two point correlators in GMG to this ‘Einstein’ result. Once we are left with a result of the same form as (122), we know what the contribution of the holographic counterterms is and do not have to construct them from scratch.

I denote here some identities that will simplify our task of comparing the results. The eoms (106) for GMG can be expressed in the form

$$\frac{m_+ m_-}{2 m^2 \ell^2} (\mathcal{D}^L \mathcal{D}^R \mathcal{D}^{m_+} \mathcal{D}^{m_-} h)_{\mu\nu} = 0. \quad (123)$$

We further use the on-shell identity

$$2 \ell^2 \mathcal{G}(\psi^1)_{\mu\nu} = (\mathcal{D}^L \mathcal{D}^R \psi^1)_{\mu\nu}. \quad (124)$$

Now the second variation of the GMG action can be written in the form

$$\begin{aligned}\delta^{(2)}S(\psi_1, \psi_2) &= -\frac{1}{16^2\pi G} \frac{m_+m_-}{m^2} \int d^3x \sqrt{-g} \psi_1^{\mu\nu*} (\mathcal{D}^{m_+} \mathcal{D}^{m_-} \mathcal{G}(\psi_1))_{\mu\nu} \\ &= -\frac{m_+m_-}{16^2\pi G m^2} \int d^3x \sqrt{-g} (\mathcal{D}^{m_+} \mathcal{D}^{m_-} \psi_1^{\mu\nu*}) \mathcal{G}(\psi_2)_{\mu\nu} \quad +\text{boundary terms.}\end{aligned}$$

We neglect²¹ the boundary terms and compare the bulk integrals.

Furthermore we fix the normalization of the logarithmic modes to $a = 2/\ell$

$$\psi^{\log} := \frac{2}{\ell} \lim_{m \rightarrow 1/\ell} \frac{\psi^L - \psi^m}{1/\ell - m} \quad (125)$$

$$\psi^{\log 2} := \frac{4}{\ell^2} \lim_{\substack{m_+ \rightarrow 1/\ell \\ m_- \rightarrow 1/\ell}} \frac{(m_+ - m_-) \psi^L + (1/\ell - m_+) \psi^{m_-} + (m_- - 1/\ell) \psi^{m_+}}{(1/\ell - m_-)(1/\ell - m_+)(m_+ - m_-)} \quad (126)$$

and we use the identities

$$\mathcal{D}^L \psi^{\log} = -2 \psi^L, \quad \mathcal{D}^L \psi^{\log 2} = -2 \psi^{\log} \quad \text{and} \quad \mathcal{D}^{m_+} \psi^L = \frac{\ell m_+ - 1}{\ell m_+} \psi^L. \quad (127)$$

5.1 Central charges and the new anomaly

As a warm up we compute the central charge of GMG on the gravity side. We take the two point correlator of two left or right-moving primaries

$$\begin{aligned}\delta^{(2)}S(\psi_R^L, \psi_R^L) &= -\frac{1}{16^2\pi G} \frac{m_+m_-}{m^2} \int d^3x \sqrt{-g} (\mathcal{D}^{m_+} \mathcal{D}^{m_-} \psi_R^L)^{\mu\nu*} \mathcal{G}(\psi_R^L)_{\mu\nu} \\ &= -\frac{1}{16^2\pi G} \frac{2\sigma\ell^2 m_+ m_-}{1 + 2\ell^2 m_+ m_-} \left(1 \mp \frac{1}{\ell m_+}\right) \left(1 \mp \frac{1}{\ell m_-}\right) \int d^3x \sqrt{-g} \psi_R^{L,*} \mathcal{G}(\psi_R^L).\end{aligned} \quad (128)$$

The upper sign corresponds to the left modes and the lower one to the right modes. The central charges are

$$c_R^L = \frac{3\ell}{2G} \sigma \frac{2\ell^2 m_+ m_-}{1 + 2\ell^2 m_+ m_-} \left(1 \mp \frac{1}{\ell m_+}\right) \left(1 \mp \frac{1}{\ell m_-}\right) \quad (129)$$

which coincides with the known result (116).

²¹More precisely we do not care about them because they will be of the same form as the boundary terms in the Einstein-correlators. Since we already know these boundary terms the difference of the calculations is just the multiplicative factor in front of the action.

The new anomaly is derived similarly via $m_- \rightarrow 1/\ell$ and the respective degeneration of differential operators $\mathcal{D}^{m_-} \rightarrow \mathcal{D}^L$.

$$\begin{aligned}
\delta^{(2)}S(\psi^{\log}, \psi^L) &= -\frac{1}{16^2\pi G} \frac{m_+}{m^2\ell} \int d^3x \sqrt{-g} (\mathcal{D}^{m_+} \mathcal{D}^L \psi^{\log})^{\mu\nu} * \mathcal{G}(\psi^L)_{\mu\nu} \\
&= -\frac{1}{16^2\pi G} \frac{-2m_+}{m^2\ell} \int d^3x \sqrt{-g} (\mathcal{D}^{m_+} \psi^L)^{\mu\nu} * \mathcal{G}(\psi^L)_{\mu\nu} \\
&= -\frac{1}{16^2\pi G} \frac{4\sigma(1-\ell m_+)}{1+2\ell m_+} \int d^3x \sqrt{-g} \psi^{L,\mu\nu} * \mathcal{G}(\psi^L)_{\mu\nu}
\end{aligned} \tag{130}$$

Thus,

$$b_L^{\text{GMG}} = \frac{4\sigma(1-\ell m_+)}{1+2\ell m_+} \frac{3\ell}{2G} = \frac{6\ell\sigma}{G} \frac{1-\ell m_+}{1+2\ell m_+}. \tag{131}$$

5.2 The generalized new anomaly

To compute the value of the generalized new anomaly we consider

$$\begin{aligned}
\delta^{(2)}S(\psi^{\log^2}, \psi^L) &= -\frac{1}{16^2\pi G} \frac{2\sigma}{3} \int d^3x \sqrt{-g} (\mathcal{D}^L \mathcal{D}^L \psi^{\log^2})^{\mu\nu} * \mathcal{G}(\psi^L)_{\mu\nu} \\
&= -\frac{1}{16^2\pi G} \frac{8\sigma}{3} \int d^3x \sqrt{-g} \psi^{L,\mu\nu} * \delta\mathcal{G}(\psi^L)_{\mu\nu}.
\end{aligned} \tag{132}$$

It differs from the Einstein value by a factor of $8\sigma/3$, so we arrive at²²

$$a_0 = \frac{4\sigma}{3} \frac{3\ell}{2G} = \frac{2\ell\sigma}{G}. \tag{133}$$

Similarly for $\langle \mathcal{O}^{\log} \mathcal{O}^{\log} \rangle$:

$$\begin{aligned}
\delta^{(2)}S(\psi^{\log}, \psi^{\log}) &= -\frac{1}{16^2\pi G} \frac{2\sigma}{3} \int d^3x \sqrt{-g} (\mathcal{D}^L \psi^{\log})^{\mu\nu} * \mathcal{D}^L \mathcal{G}(\psi^{\log})_{\mu\nu} \\
&= -\frac{1}{16^2\pi G} \frac{8\sigma}{3} \int d^3x \sqrt{-g} \psi^{L,\mu\nu} * \mathcal{G}(\psi^L)_{\mu\nu}.
\end{aligned} \tag{134}$$

Since $\mathcal{G} \approx \mathcal{D}^L \mathcal{D}^R$ it commutes with \mathcal{D}^L and we obtain the known Einstein result up to boundary terms due to partial integration of one \mathcal{D}^L operator.

²²Note that we defined the correlator $\langle \mathcal{O}^0 \mathcal{O}^0 \rangle = a_0/z^4$, whereas the central charge is defined by $\langle \mathcal{O}^L \mathcal{O}^L \rangle = c_L/2z^4$ so we have to divide by a factor of two to get the right result.

For the $\langle \mathcal{O}^{\log^2} \mathcal{O}^{\log} \rangle$ correlator we proceed in the same way it was done in [25] for $\langle \mathcal{O}^{\log} \mathcal{O}^{\log} \rangle$. We get

$$\begin{aligned} \delta^{(2)} S(\psi^{\log^2}, \psi^{\log}) &= -\frac{1}{16^2 \pi G} \frac{2\sigma}{3} \int d^3x \sqrt{-g} (\mathcal{D}^L \mathcal{D}^L \psi^{\log^2})^{\mu\nu} * \mathcal{G}(\psi^{\log})_{\mu\nu} \\ &= -\frac{1}{16^2 \pi G} \frac{8\sigma}{3} \int d^3x \sqrt{-g} \psi^{L,\mu\nu} * \delta \mathcal{G}(\psi^{\log})_{\mu\nu} \\ &\sim \frac{1}{16^2 \pi G} \frac{4\sigma}{3} \lim_{\rho \rightarrow \infty} \int_{t_0}^{t_1} dt \int_0^{2\pi} d\phi \psi_{ij}^{L*} \bar{g}^{ik} \bar{g}^{jl} \nabla_\rho \psi_{kl}^{\log}. \end{aligned} \quad (135)$$

If we keep only terms that do not vanish in the limit $\rho \rightarrow \infty$ we get

$$\begin{aligned} \delta^{(2)} S(\psi^{\log^2}, \psi^{\log}) &\sim \frac{\sigma}{6G} \lim_{\rho \rightarrow \infty} \int_{t_0}^{t_1} dt [\psi_{vv}^{L*} \partial_\rho (\psi_{uu}^{\log} e^{-2\rho}) + \psi_{uu}^{L*} \partial_\rho (\psi_{vv}^{\log} e^{-2\rho})] \\ &\sim \frac{\sigma}{3G} \lim_{\rho \rightarrow \infty} \int_{t_0}^{t_1} dt \frac{h(h^2 - 1)}{\bar{h}} [(\psi(h-1) + \psi(-\bar{h})) + \\ &\quad + \tilde{\alpha}(\rho + it) + \tilde{\beta}]. \end{aligned} \quad (136)$$

Here $\tilde{\alpha}$ and $\tilde{\beta}$ are weight independent functions and ψ is the digamma function. It goes like $\psi(h) \sim \ln(h)$ for large h . Taking the limit of large weights we obtain

$$\lim_{h, -\bar{h} \rightarrow \infty} \delta^{(2)} S(\psi^{\log^2}, \psi^{\log}) \sim \frac{2\sigma}{3G} \ln(\sqrt{-h\bar{h}}) \frac{h^3}{\bar{h}} \int_{t_0}^{t_1} dt. \quad (137)$$

Transforming this integral to coordinate space and fixing the time integral to $2\pi i$ we get

$$\langle \psi^{\log^2}(z, \bar{z}) \psi^{\log}(0) \rangle = \frac{2\sigma}{3G} 4\pi i \ln(m^2 \sqrt{-\partial\bar{\partial}}) \frac{\partial^3}{\partial} \delta^{(2)}(z, \bar{z}). \quad (138)$$

Comparing this result to [29] shows that $\delta^{(2)} S(\psi^{\log^2}, \psi^{\log})$ indeed yields a term proportional to $(\ln|z|^2)/z^4$:

In coordinate space we use the identities

$$-2i \delta^{(2)}(z, \bar{z}) = \delta(\tau) \delta(x) \quad 4\partial\bar{\partial} = \partial_\tau^2 + \partial_x^2. \quad (139)$$

Taking the limit $\alpha \rightarrow 2$ of the following integral

$$\frac{1}{4\pi^2} \int d\omega dk \frac{e^{i\omega\tau + ikx}}{(\omega^2 + k^2)^{\alpha/2}} = \frac{1}{\pi} 2^{-\alpha} \frac{\Gamma(1 - \alpha/2)}{\Gamma(\alpha/2)} (\tau^2 + x^2)^{-1 + (\alpha/2)} \quad (140)$$

we see

$$\frac{1}{\partial\bar{\partial}}\delta^{(2)}(z,\bar{z}) = \frac{2i}{\partial_\tau^2 + \partial_x^2}\delta^{(2)}(\tau,x) = \frac{i}{2\pi}\ln(m^2(\tau^2+x^2)) = \frac{i}{2\pi}\ln(m^2|z|^2). \quad (141)$$

If we differentiate (140) once w.r.t. α and set $\alpha = 2$ we get

$$\ln(\sqrt{-\partial\bar{\partial}})\frac{1}{\partial\bar{\partial}}\delta^{(2)}(z,\bar{z}) = -\frac{i}{8\pi}\ln^2(m^2|z|^2). \quad (142)$$

Since we are interested in the coefficient of the $\ln(m^2|z|^2)/z^4$ term we substitute

$$-\partial^4\frac{i}{8\pi}\ln^2(m^2|z|^2) \rightarrow \frac{3i}{2\pi}\frac{\ln(m^2|z|^2)}{z^4} \quad (143)$$

and finally obtain

$$\langle\psi^{\log^2}(z,\bar{z})\psi^{\log}(0)\rangle \sim \frac{2\sigma}{3G}4\pi i\frac{3i}{2\pi}\frac{\ln(m^2|z|^2)}{z^4} = -\frac{4\sigma}{G}\frac{\ln(m^2|z|^2)}{z^4}. \quad (144)$$

In [25] and [29] ℓ was set to one. If we reintroduce it we get $-2a_0 = -4\sigma\ell/G$ (see e.g. (45)).

To calculate the $\langle\mathcal{O}^{\log^2}\mathcal{O}^{\log^2}\rangle$ correlator on the gravity side we need to know the ψ^{\log^2} solution at least asymptotically. This derivation is rather technical and is of no great importance to us. See appendix A for details.

$$\begin{aligned} \delta^{(2)}S(\psi^{\log^2},\psi^{\log^2}) &= -\frac{1}{16^2\pi G}\frac{2\sigma}{3}\int d^3x\sqrt{-g}(\mathcal{D}^L\mathcal{D}^L\psi^{\log^2})^{\mu\nu*}\mathcal{G}(\psi^{\log^2})_{\mu\nu} \\ &= -\frac{1}{16^2\pi G}\frac{8\sigma}{3}\int d^3x\sqrt{-g}\psi^{L,\mu\nu*}\delta\mathcal{G}(\psi^{\log^2})_{\mu\nu} \\ &\sim \frac{1}{16^2\pi G}\frac{4\sigma}{3}\lim_{\rho\rightarrow\infty}\int_{t_0}^{t_1}dt\int_0^{2\pi}d\phi\psi_{ij}^{L*}\bar{g}^{ik}\bar{g}^{jl}\nabla_\rho\psi_{kl}^{\log^2}. \end{aligned} \quad (145)$$

Again we keep only terms that do not vanish when we take the limit $\rho \rightarrow \infty$

$$\begin{aligned} \delta^{(2)}S(\psi^{\log^2},\psi^{\log^2}) &\sim \frac{\sigma}{6G}\lim_{\rho\rightarrow\infty}\int_{t_0}^{t_1}dt(\psi_{vv}^{L*}\partial_\rho(\psi_{uu}^{\log^2}e^{-2\rho}) + \psi_{uu}^{L*}\partial_\rho(\psi_{vv}^{\log^2}e^{-2\rho})) \\ &= \frac{\sigma}{6G}\frac{h(h^2-1)}{\bar{h}}\lim_{\rho\rightarrow\infty}\int_{t_0}^{t_1}dt\left(-(\rho+it) + \beta(h,\bar{h})(\rho-it) + \frac{1}{2}(\rho-it)^2 + \right. \\ &\quad \left. + \alpha_1 + \alpha_2(\psi(h-1) + \psi(-\bar{h})) - 2(\psi'(h-1) - \psi'(-\bar{h})) + \right. \\ &\quad \left. + 2(\psi(h-1) + \psi(-\bar{h}))^2\right) \end{aligned} \quad (146)$$

The \sim means equality up to contact terms and the α_i are numerical constants. β is a function of the weights h and \bar{h} including terms $\psi(h-1)$. Now taking the limit of large weights $h \rightarrow \infty$ and $\bar{h} \rightarrow -\infty$ we obtain

$$\lim_{h, -\bar{h} \rightarrow \infty} \delta^{(2)} S(\psi^{\log^2}, \psi^{\log^2}) \sim \frac{4\sigma}{3G} \frac{h^3}{\bar{h}} \ln^2 \sqrt{-h\bar{h}} \int_{t_0}^{t_1} dt. \quad (147)$$

Following the same steps as before fixing the time integral and going to coordinate space yields

$$\langle \psi^{\log^2}(z, \bar{z}) \psi^{\log^2}(0) \rangle = \frac{4\sigma}{3G} 4\pi i \ln^2(m^2 \sqrt{-\partial\bar{\partial}}) \frac{\partial^3}{\partial} \delta^{(2)}(z, \bar{z}). \quad (148)$$

Differentiating the integral (140) twice w.r.t. α and setting $\alpha = 2$

$$\ln^2(\sqrt{-\partial\bar{\partial}}) \frac{1}{\partial\bar{\partial}} \delta^{(2)}(z, \bar{z}) = \frac{i}{24\pi} \ln^3(m^2 |z|^2). \quad (149)$$

For the $\ln^2(m^2 |z|^2)$ term we substitute

$$\partial^4 \frac{i}{24\pi} \ln^3(m^2 |z|^2) \rightarrow -\frac{3i}{4\pi} \frac{\ln^2(m^2 |z|^2)}{z^4} \quad (150)$$

and arrive at

$$\langle \psi^{\log^2}(z, \bar{z}) \psi^{\log^2}(0) \rangle \sim \frac{4\sigma}{G} \frac{\ln^2(m^2 |z|^2)}{z^4}. \quad (151)$$

Hence we recover the correct 2-point correlator (45d) and the generalized new anomaly if given by

$$a_0 = 2\sigma\ell/G. \quad (152)$$

5.3 Massive degeneration: $\mathcal{D}^{m_-} = \mathcal{D}^{m_+}$

Now we are interested in the parameter region where $m_+ \rightarrow m_-$. In order to have a (L)CFT we expect two point functions of the following form:

$$\langle \psi^{m_\pm}(z, \bar{z}) \psi^{m_\pm}(0) \rangle = \frac{c_{m_\pm}}{2 z^{2h(m_\pm)} \bar{z}^{2\bar{h}(m_\pm)}} \quad (153a)$$

$$\langle \psi^{m_+}(z, \bar{z}) \psi^{m_-}(0) \rangle = 0 \quad (153b)$$

Then the limit $m_+ \rightarrow m_- = \bar{m}$ should yield correlators of the form

$$\langle \psi^{\bar{m}}(z, \bar{z}) \psi^{\bar{m}}(0) \rangle = 0 \quad (154a)$$

$$\langle \psi^{\bar{m}}(z, \bar{z}) \psi^{\text{lm}}(0) \rangle = \frac{b_{\bar{m}}}{2 z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}} \quad (154b)$$

$$\langle \psi^{\text{lm}}(z, \bar{z}) \psi^{\text{lm}}(0) \rangle = -\frac{b_{\bar{m}} \ln |z|^2}{z^{2h(\bar{m})} \bar{z}^{2\bar{h}(\bar{m})}}. \quad (154c)$$

The formalism we derived in section 3 tells us how we can derive $b_{\bar{m}}$ from $c_{\bar{m}}$, but only if all two point functions are of the form above, i.e. a dual (L)CFT exists. Since we do not know if this is the case we would like to check the correlators (153) on the gravity side and if they match, calculate $b_{\bar{m}}$, i.e. (154), on the gravity and LCFT side. It turns out, however, that this calculation is very tedious and lengthy, i.e. we would have to use the full holographically renormalized action because the short-cut we have used until now will fail. Thus we are not going to derive c_{\pm} in this from.

This is not as big an obstacle as it might sound. The only thing of interest to us is if c_{\pm} really vanishes when we take the limits $\mathcal{D}^{m\pm} \rightarrow \mathcal{D}^{L/R}$ or $\mathcal{D}^{m+} \rightarrow \mathcal{D}^{m-}$ and whether or not it is nonzero at all. In other words we want to find the zeros of the function c_{\pm} .

A good estimate for what the normalization of the two point correlator c_{\pm} will look like is given by the energy of the massive modes. To compute the energy we have to change from our Lagrangian description of the action to a Hamiltonian one. In other words we have to derive the Hamiltonian density of the system.

Classically if we have a Lagrangian as a function of the variables q_i and \dot{q}_i the Hamiltonian is obtained by a Legendre transformation

$$L = L(q_i, \dot{q}_i), \quad p^i = \frac{\partial L}{\partial \dot{q}_i} \quad \rightarrow \quad H = \sum_i p^i \dot{q}_i - L. \quad (155)$$

We generalize the above statement since we are working in a field theory and thus have to consider Lagrangian- and Hamiltonian *densities*, as well as functional derivatives thereof instead of partial ones. One important point, however, is that this procedure to get the Hamiltonian of the system only works straightforwardly if our Lagrangian includes no terms that are time derivatives of second or higher order. Now our Lagrangian under consideration is the second variation of the action. This is a theory with *four* differential operators (schematically $S^{(2)} \sim h\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}h$), so we will have up to four time derivatives. Since the action is quadratic in $h_{\mu\nu}$ and we do not consider boundary terms²³, we can partially integrate to get an action of the form $S^{(2)} \sim \mathcal{D}\mathcal{D}h\mathcal{D}\mathcal{D}h$. There is no way to get rid of more derivatives, so we have to find the Hamiltonian for a system with two time derivatives.

The way to do this was first described by Ostrogradsky [30] and I will give a short overview of it. To keep matters simple I will not do this for a full field theory but in a very simple form. The generalization of the results to field theories is straightforward.

²³When we compute the energy of the modes we have to insert the normalizable modes. By definition they fall off sufficiently fast near the boundary.

The Ostrogradsky formalism

Consider a Lagrangian of the form $L = L(q, \dot{q}, \ddot{q})$. To get a system with only first order time derivatives we introduce a new variable $k = \dot{q}$ so that $\ddot{q} = \dot{k}$. The two variables q and k are not independent of course. We treat them as independent variables nevertheless, but add the constraint $k = \dot{q}$ via a Lagrange multiplier λ .

$$L = L(q, \dot{q}, \ddot{q}) \rightarrow L^* = L(q, k, \dot{k}) + \lambda(k - \dot{q}) \quad (156)$$

Variation w.r.t. λ will allow us to replace all k and we are left with a Hamiltonian that depends only on q and its time derivatives. The equations of motion and the canonical momenta are

$$\frac{\delta L^*}{\delta \lambda} = k - \dot{q} = 0, \quad \frac{\delta L^*}{\delta k} = \frac{\partial L}{\partial k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{k}} + \lambda = 0 \quad \text{and} \quad (157)$$

$$p_k = \frac{\partial L^*}{\partial \dot{k}} = \frac{\partial L}{\partial \dot{k}}, \quad p_q = \frac{\partial L^*}{\partial \dot{q}} = -\lambda. \quad (158)$$

On shell the Hamiltonian becomes

$$H = p_k \dot{k} + p_q \dot{q} - L^* = \frac{\partial L}{\partial \ddot{q}} \ddot{q} + \left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \right) \dot{q} - L. \quad (159)$$

Now we have all the tools we need to derive the energy of the massive mode. As a cross check we also compute the energy of the left mode and compare it to the left central charge.

We start with an action of the form

$$S = \frac{1}{16\pi G} \frac{1}{2m^2} \int d^3x \sqrt{-g} h^{\mu\nu} \mathcal{L}(h_{\mu\nu}) \quad (160)$$

with the Lagrangian density $\mathcal{L}(h)$ given by the linearized eoms (106)

$$\left(\nabla^2 + \frac{2}{\ell^2} \right) (\nabla^2 h_{\mu\nu} + (m_+ + m_-) \varepsilon_\mu^{\alpha\beta} \nabla_\alpha h_{\beta\nu} + (m_+ m_- + \frac{3}{\ell^2}) h_{\mu\nu}) = 0. \quad (161)$$

Partial integration yields a theory with two time derivatives at most. The action then is proportional to

$$\begin{aligned} \mathcal{L} \sim & \nabla^2 h^{\mu\nu} \nabla^2 h_{\mu\nu} + \nabla^2 h^{\mu\nu} (m_+ + m_-) \varepsilon_\mu^{\alpha\beta} \nabla_\alpha h_{\beta\nu} - \\ & - \nabla^\lambda h^{\mu\nu} (m_+ m_- + \frac{5}{\ell^2}) \nabla_\lambda h_{\mu\nu} + h^{\mu\nu} \frac{2}{\ell^2} (m_+ + m_-) \varepsilon_\mu^{\alpha\beta} \nabla_\alpha h_{\beta\nu}. \end{aligned} \quad (162)$$

We choose the canonical variables $h_{\mu\nu}$ and $k_{\mu\nu} = \nabla_0 h_{\mu\nu}$. The momenta are similar to the previous case, but generalized to field theory,

$$\Pi^{(2),\mu\nu} := \frac{\partial \mathcal{L}}{\partial(\partial_0 \nabla_0 h_{\mu\nu})} = 2g^{00} \nabla^2 h^{\mu\nu} + g^{00} (m_+ + m_-) \varepsilon^{\mu\alpha}{}_{\beta} \nabla_{\alpha} h^{\beta\nu} \quad (163)$$

and

$$\begin{aligned} \Pi^{(1),\mu\nu} &:= \frac{\partial \mathcal{L}}{\partial(\partial_0 h_{\mu\nu})} - \partial_0 \Pi^{(2),\mu\nu} = -2\nabla^0 \nabla^2 h^{\mu\nu} - 2(m_+ m_- + \frac{5}{\ell^2}) \nabla^0 h^{\mu\nu} \\ &\quad - (\nabla^2 + \frac{2}{\ell^2}) (m_+ + m_-) \varepsilon^{\mu 0}{}_{\beta} h^{\beta\nu} - (m_+ + m_-) \nabla^0 \varepsilon^{\mu\alpha}{}_{\beta} \nabla_{\alpha} h^{\beta\nu}. \end{aligned} \quad (164)$$

The Hamiltonian density is then determined by

$$\mathcal{H} = \Pi^{(1),\mu\nu} \partial_0 h_{\mu\nu} + \Pi^{(2),\mu\nu} \partial_0 \nabla_0 h_{\mu\nu} - \mathcal{L}. \quad (165)$$

The on-shell value of the energy of the modes is then simplified by using the eoms and integrating over a spatial volume. With the eoms the last term in \mathcal{H} vanishes identically, since these basically *are* the eoms. In the second term we would like to shift the ∂_0 from $\partial_0 \nabla_0 h_{\mu\nu}$ to act on the momentum $\Pi^{(2),\mu\nu}$ to get terms of the same form as in $\Pi^{(1),\mu\nu} \partial_0 h_{\mu\nu}$. We could then simplify the momenta further by reintroducing the differential operators \mathcal{D}^i and using the eoms.

$$\begin{aligned} (\nabla^2 + \frac{2}{\ell^2}) h^{\mu\nu} &= -\frac{1}{\ell^2} (\mathcal{D}^L \mathcal{D}^R h)^{\mu\nu} \quad \text{and} \\ \nabla^2 h^{\mu\nu} + (m_+ + m_-) \varepsilon^{\mu\alpha}{}_{\beta} \nabla_{\alpha} h^{\beta\nu} + (m_+ m_- + \frac{3}{\ell^2}) h^{\mu\nu} &= m_+ m_- (\mathcal{D}^{m_+} \mathcal{D}^{m_-} h)^{\mu\nu} \end{aligned}$$

The momentum $\Pi^{(1),\mu\nu}$ takes the form

$$\begin{aligned} \Pi^{(1),\mu\nu} &= \frac{m_+ + m_-}{\ell^2} \mathcal{D}^L \mathcal{D}^R \varepsilon^{\mu 0}{}_{\beta} h^{\beta\nu} + \frac{2}{\ell^2} \nabla^0 \mathcal{D}^L \mathcal{D}^R h^{\mu\nu} + \nabla^0 \nabla^2 h^{\mu\nu} - \\ &\quad - m_+ m_- \mathcal{D}^{m_+} \mathcal{D}^{m_-} h^{\mu\nu} - (m_+ m_- + \frac{3}{\ell^2}) \nabla^0 h^{\mu\nu}. \end{aligned} \quad (166)$$

Now we should point out that we calculate the energy by integrating \mathcal{H} over a *spatial* volume, so we should actually not be allowed to partially integrate time derivatives. However, we allow us a little trick here. Since our theory is time translation invariant, we know by Nöther's theorem that there has to be a conserved charge that is time independent. This charge, by definition, is the Hamiltonian. Therefore we must have

$$\frac{d}{dt} H = \frac{d}{dt} \int d^2x \mathcal{H} = 0. \quad (167)$$

Another way of looking at this is to define a Hamiltonian that fulfills the relation (167) and then assume that the Hamiltonian obtained via the Ostrogradsky procedure coincides with the one from (167).

With (167) the relation

$$H = \frac{1}{2T} \int_{-T}^T dt H \quad (168)$$

is true for any T . Thus we introduce a time integral by hand in order to partially integrate time derivatives ∂_0 . This will yield boundary terms evaluated at $\pm T$. Now the boundary terms that we obtain are of the form $h^{\mu\nu} \dot{h}_{\mu\nu}$ and as the $h^{\mu\nu}$ are bounded in t (the only t dependence is in the exponential $\exp(-ihu - \bar{h}v)$ where u and v are functions of t and φ) the integral yields a finite term. But since (168) holds for any T we can choose $T \rightarrow \infty$. Then the prefactor $1/(2T)$ will kill all boundary terms and we can happily ignore the time integral for all other terms.

The energy of the left mode becomes

$$\begin{aligned} E_L &= \frac{1}{32\pi G m^2} \int d^2x \sqrt{-g} \{ \Pi_L^{(1),\mu\nu} \partial_0 h_{\mu\nu}^L + \Pi_L^{(2),\mu\nu} \partial_0 \nabla_0 h_{\mu\nu}^L \} = \\ &= \frac{1}{32\pi G m^2} \int d^2x \sqrt{-g} \{ \partial_0 h_{\mu\nu}^L [-m_+ m_- (1 - \frac{1}{\ell m_+}) (1 - \frac{1}{\ell m_-}) \nabla^0 h_L^{\mu\nu} - \\ &\quad - \frac{2}{\ell^2} \nabla^0 h_L^{\mu\nu} - (m_+ m_- + \frac{3}{\ell^2}) \nabla^0 h_L^{\mu\nu}] + \\ &\quad + \partial_0 \nabla_0 h_{\mu\nu}^L [-g^{00} \frac{4}{\ell^2} h_L^{\mu\nu} - g^{00} \frac{m_+ + m_-}{\ell} h_L^{\mu\nu}] \} = \\ &= \frac{1}{32\pi G m^2} \int d^2x \sqrt{-g} \{ \partial_0 h_{\mu\nu}^L [-m_+ m_- (1 - \frac{1}{\ell m_+}) (1 - \frac{1}{\ell m_-}) \nabla^0 h_L^{\mu\nu} - \\ &\quad - (\frac{2}{\ell^2} + m_+ m_- + \frac{3}{\ell^2} - \frac{4}{\ell^2} - \frac{m_+ + m_-}{\ell}) \nabla^0 h_L^{\mu\nu}] \} = \\ &= \frac{1}{32\pi G m^2} \frac{-2(1 - \ell m_+)(1 - \ell m_-)}{\ell^2} \int d^2x \sqrt{-g} \dot{h}_{\mu\nu}^L \nabla^0 h_L^{\mu\nu} . \end{aligned} \quad (169)$$

Similarly for the m_- -mode we get

$$\begin{aligned}
E_- &= \frac{1}{32\pi Gm^2} \int d^2x \sqrt{-g} \left\{ \partial_0 h_{\mu\nu}^- \left[\frac{m_+ + m_-}{\ell^2} (1 - \ell^2 m_-^2) \varepsilon^{\mu 0}{}_{\beta} h_-^{\beta\nu} + \right. \right. \\
&\quad + \frac{2}{\ell^2} (1 - \ell^2 m_-^2) \nabla^0 h_-^{\mu\nu} - \left. \left(\frac{2}{\ell^2} + \frac{1 - \ell^2 m_-^2}{\ell^2} \right) \nabla^0 h_-^{\mu\nu} - \right. \\
&\quad - (m_+ m_- + \frac{3}{\ell^2}) \nabla^0 h_-^{\mu\nu} \left. \right\} + \partial_0 \nabla_0 h_{\mu\nu}^- \left[2g^{00} \left(-\frac{2}{\ell^2} - \frac{1 - \ell^2 m_-^2}{\ell^2} \right) h_-^{\mu\nu} + \right. \\
&\quad \left. + g^{00} (m_+ + m_-) (-m_-) h_-^{\mu\nu} \right] \left. \right\} = \\
&= \frac{1}{32\pi Gm^2} \int d^2x \sqrt{-g} \left\{ \partial_0 h_{\mu\nu}^- \left[\frac{m_+ + m_-}{\ell^2} (1 - \ell^2 m_-^2) \varepsilon^{\mu 0}{}_{\beta} h_-^{\beta\nu} + \right. \right. \\
&\quad + \left(\frac{1 - \ell^2 m_-^2}{\ell^2} - \frac{2}{\ell^2} - m_+ m_- - \frac{3}{\ell^2} + \frac{4}{\ell^2} + 2 \frac{1 - \ell^2 m_-^2}{\ell^2} + \right. \\
&\quad \left. \left. + (m_+ + m_-) m_- \right) \nabla^0 h_-^{\mu\nu} \right] \left. \right\} = \\
&= \frac{1}{32\pi Gm^2} (1 - \ell^2 m_-^2) \int d^2x \sqrt{-g} \left\{ \partial_0 h_{\mu\nu}^- \left[\frac{m_+ + m_-}{\ell^2} \varepsilon^{\mu 0}{}_{\beta} h_-^{\beta\nu} + \right. \right. \\
&\quad \left. \left. + \frac{2}{\ell^2} \nabla^0 h_-^{\mu\nu} \right] \right\}. \tag{170}
\end{aligned}$$

Here we can use the eom of the m_- -mode by contracting it with an $\varepsilon^{\delta 0}{}_{\mu}$ in order to express the $\nabla^0 h_-^{\mu\nu}$ term:

$$m_- \varepsilon^{\mu 0}{}_{\delta} (\mathcal{D}^- h_-)^{\delta\nu} = 0 \quad \rightarrow \quad \nabla^0 h_-^{\mu\nu} = \nabla^{\mu} h_-^{0\nu} - m_- \varepsilon^{\mu 0}{}_{\beta} h_-^{\beta\nu} \tag{171}$$

The resulting $\partial_0 h_{\mu\nu}^- \nabla^{\mu} h_-^{0\nu}$ vanishes after partial integration of ∇^{μ} due to our gauge choice $\nabla^{\mu} h_{\mu\nu} = 0$.²⁴ The energy of the massive mode hence becomes

$$\begin{aligned}
E_- &= \frac{1}{32\pi Gm^2} (1 - \ell^2 m_-^2) \int d^2x \sqrt{-g} \left\{ \partial_0 h_{\mu\nu}^- \left[\frac{m_+ - m_-}{\ell^2} \varepsilon^{\mu 0}{}_{\beta} h_-^{\beta\nu} + \right. \right. \\
&\quad \left. \left. + 2 \nabla^{\mu} h_-^{0\nu} \right] \right\} = \\
&= \frac{1}{32\pi Gm^2} (1 - \ell^2 m_-^2) \frac{m_+ - m_-}{\ell^2} \int d^2x \sqrt{-g} \dot{h}_{\mu\nu}^- \varepsilon^{\mu 0}{}_{\beta} h_-^{\beta\nu}. \tag{172}
\end{aligned}$$

The resulting integral depends on the normalization of the massive modes. Therefore we derived the energy to within a numerical constant due to overall normalization. We might also have to multiply with another function that depends on the masses m_+ and m_- of the modes. However, there cannot be any further zeros, i.e. the integral will always be finite for all possible values

²⁴Here the careful reader might object that we are not (necessarily) allowed to throw away boundary terms obtained via partial integration of ∇_{μ} . For reasons mentioned below (168) neither of the boundaries give any contribution.

of m_+ and m_- . This is enough for our purpose of deriving the new anomaly, since we are only interested in the behavior of c_{\pm} in the vicinity of its zeros. Now we ignore the integral and compare the coefficients in the energy E_L to the left central charge. Note that the m^2 in the denominator is a function of m_+ and m_- too. Inserting $2\sigma\ell^2m^2 = 1 + 2\ell^2m_+m_-$ we obtain a difference of $3\ell/2$. Thus our conjecture for the massive charge is

$$c_{m_-} = \frac{3\ell\sigma}{2G} \frac{(1 - \ell^2m_-^2)(\ell m_+ - \ell m_-)}{1 + 2\ell^2m_+m_-}. \quad (173)$$

When \mathcal{D}^{m_+} and \mathcal{D}^{m_-} degenerate we get the following new anomaly $b_{m_{\pm}}$

$$b_{m_{\pm}} = -\text{sign}(m_{\pm}) \frac{3\ell\sigma}{G} \frac{1 - \ell^2m_{\pm}^2}{1 + 2\ell^2m_{\pm}^2}. \quad (174)$$

6 Conclusion

The goal of our analysis was to find the new anomalies of GMG. GMG is a theory that depends on two essential coupling parameters m^2 and μ . Thus we have two possible degenerations of modes $\mathcal{D}^{m_+} \rightarrow \mathcal{D}^L$ and $\mathcal{D}^{m_-} \rightarrow \mathcal{D}^L$ where we end up with an LCFT similar to those found at the logarithmic points of TMG [10] and NMG. An interesting feature of the theory is possible degenerations of the massive modes $\mathcal{D}^{m_+} \rightarrow \mathcal{D}^{m_-}$. This leads to an LCFT with non-vanishing central charge. GMG is the first known gravity dual that has this property.²⁵

Another new aspect is that the new anomaly is not a constant but a function of the leftover massive parameter. For example if we consider $\mathcal{D}^{m_-} \rightarrow \mathcal{D}^L$, then $b_L^{m_-}$ is a function of m_+ and we can tune it to zero. Then we obtain the maximal degeneration of GMG, a point where three operators degenerate. Our conjecture was that GMG might again be dual to a (L)CFT at this critical *point*. However, the LCFT located at that point has a bigger Jordan cell than the equivalent LCFTs at its critical *lines*. The possibility of obtaining bigger Jordan cells due to degeneration of more operators generalizes the notion of the new anomaly. One interpretation of the central charge is to describe the number of massless degrees of freedom in the theory. If c_L vanishes b_L in turn counts the massless degrees of freedom [18]. For rank three Jordan cells we conjecture that a_0 will count these degrees of freedom, etc. Therefore we called the characteristic of the arising LCFT *generalized* new anomaly. It constitutes the characteristic that can be used to distinguish between different LCFTs with the same rank of the Jordan cell. In the case of TMG and NMG the corresponding quantity is the new anomaly, because there is only one critical point.

In order to be dual to an LCFT the modes of GMG – the solutions of the equations of motion – on the gravity side, had to be dual to CFT operators whose two point functions take exactly the form of an LCFT. In order to prove this we calculated all two point functions at various levels of degeneration on both, the CFT and the gravity side. We derived a formalism to deduce the new and the generalized new anomaly from the central charge and the weights of the degenerating modes. The result from the two point functions on one hand are a proof that our formalism works, and on the other hand they are a non-trivial check of our (educated) conjecture that GMG is dual to a logarithmic CFT at its logarithmic lines $\mathcal{D}^{m_\pm} \rightarrow \mathcal{D}^{L/R}$ and at its logarithmic point $\mathcal{D}^{m_+} \rightarrow \mathcal{D}^{m_-} \rightarrow \mathcal{D}^L$.

²⁵Actually such LCFTs with non-zero vanishing central charge arise also in NMG for the special tuning $m_+ = m_- = 0$, see also [13]. The novel feature in GMG is that the degenerating operators can have arbitrary mass.

We conclude that GMG at the logarithmic and at the doubly logarithmic point appears to be dual to a logarithmic CFT (with Jordan cells of different sizes).

The line along which only the two mass parameters degenerate requires further investigations. Again we think that there exists a dual LCFT. We provided a value for the 'massive charge' and for the new anomaly. We leave the direct calculation on the gravity side of the correlators of massive modes for the future. It would be interesting to apply our findings in LCFTs with bigger Jordan cells to some recently proposed models that generalize NMG or GMG, e.g. to look at extended NMG [31–33], Born-Infeld extended NMG [34] or Born-Infeld-Chern-Simons theories [35]. These are models that are higher derivatives in curvature. However, expanding the Born-Infeld-(Chern-Simons) action in powers of curvature one finds exactly the terms K (and I_{CS}) from NMG and TMG plus further terms that are fifth or sixth order derivatives of the metric (and arbitrary higher order derivatives of the metric). Thus one possibly finds further doubly, triply etc. logarithmic points. Although it seems these Born-Infeld-Chern-Simons gravity theories allow for infinitely many LCFT duals in perturbation theory, the full, not perturbed action is *not* dual to an LCFT. We also suggest to calculate the various 'anomalies' we derived on the gravity side in a 'more precise' way to prove or to falsify our results. Further directions for future research are outlined in [13].

A The doubly logarithmic mode

In [25] the logarithmic mode was derived by linearizing the eoms for the massive mode. The parameter which is small is the mass μ of the massive mode parameterized by ε . Vanishing of ε will yield a left-moving graviton mode. Hence the degeneration of the two modes yields the left-moving primary and a logarithmic partner field.

The eoms for the massive mode are [3]

$$\psi_{\mu\nu} + \frac{1}{\mu} \varepsilon_\mu^{\alpha\beta} \nabla_\alpha \psi_{\beta\nu} = 0. \quad (175)$$

The separation ansatz

$$\psi_{\mu\nu}(h, \bar{h}) = e^{-i(hu + \bar{h}v)} \begin{pmatrix} F_{uu}(\rho) & F_{uv}(\rho) & F_{u\rho}(\rho) \\ & F_{vv}(\rho) & F_{v\rho}(\rho) \\ & & F_{\rho\rho}(\rho) \end{pmatrix} \quad (176)$$

leads to four algebraic equations for the $F_{\mu\nu}$ s and two coupled differential equations:

$$\bar{h}F_{uu} - hF_{uv} = \frac{\mu - 1}{4i} \sinh(2\rho) F_{u\rho} \quad (177a)$$

$$\bar{h}F_{uv} - hF_{vv} = \frac{\mu + 1}{4i} \sinh(2\rho) F_{v\rho} \quad (177b)$$

$$\bar{h}F_{u\rho} - hF_{v\rho} = \frac{i}{\sinh(2\rho)} (F_{vv}(\mu + 1) + F_{uu}(\mu - 1) - 2\mu \cosh(2\rho) F_{uv}) \quad (177c)$$

$$F_{\rho\rho} = \frac{4}{\sinh^2(2\rho)} (2 \cosh(2\rho) F_{uv} - F_{uu} - F_{vv}) \quad (177d)$$

$$\frac{dF_{uv}}{d\rho} = \frac{\mu + 1}{\sinh(2\rho)} \left(F_{uv} \left(\frac{4h\bar{h}}{(\mu + 1)^2} - \cosh(2\rho) \right) + F_{vv} \left(1 - \frac{4h^2}{(\mu + 1)^2} \right) \right) \quad (177e)$$

$$\frac{dF_{vv}}{d\rho} = -\frac{\mu + 1}{\sinh(2\rho)} \left(F_{vv} \left(\frac{4\bar{h}h}{(\mu + 1)^2} - \cosh(2\rho) \right) + F_{uv} \left(1 - \frac{4\bar{h}^2}{(\mu + 1)^2} \right) \right) \quad (177f)$$

We introduce ε by reparameterizing the coupling constant μ by

$$\mu = 1 - 2\varepsilon. \quad (178)$$

In this section we set $\ell = 1$. The limit $\varepsilon \rightarrow 0$, i.e. $\mu \rightarrow 1$, implies the degeneration $\mathcal{D}^M \rightarrow \mathcal{D}^L$. Using the abbreviation

$$x := \cosh(2\rho) \quad (179)$$

we can decouple the differential equations (177e) and (177f) and solve them to get

$$F_{vv} = a_2 (x-1)^{(h-\bar{h})/2} (x+1)^{(h+\bar{h})/2} {}_2F_1(h-\varepsilon+2, h+\varepsilon-1; h-\bar{h}+1; \frac{1-x}{2})$$

$$\text{with } a_2 = 2^{1-h-\varepsilon} \frac{\Gamma(2+h-\varepsilon)\Gamma(2-\bar{h}-\varepsilon)}{(h-\bar{h})!\Gamma(3-2\varepsilon)} \quad (180)$$

as the solution for F_{vv} . The function ${}_2F_1$ is the Gauss hypergeometric function. The overall normalization a_2 is determined by demanding the coefficient of the highest order term in x be one. All other components can be derived from this one algebraically or by differentiation of F_{vv} w.r.t. ρ . For completeness we denote here also the large ρ/x expansion of the left mode, which we used to calculate the two point functions on the gravity side

$$\psi_{vv}^L = e^{-i(hu+\bar{h}v)} \left[h\bar{h} + x + \mathcal{O}\left(\frac{\ln x}{x}\right) \right] \quad (181a)$$

$$\psi_{uv}^L = e^{-i(hu+\bar{h}v)} \left[1 - h^2 + \mathcal{O}\left(\frac{\ln x}{x}\right) \right] \quad (181b)$$

$$\psi_{uu}^L = e^{-i(hu+\bar{h}v)} \left[\frac{h(1-h^2)}{\bar{h}} + \mathcal{O}\left(\frac{\ln x}{x}\right) \right] \quad (181c)$$

$$\psi_{u\rho}^L = \mathcal{O}\left(\frac{\ln x}{x}\right) \quad (181d)$$

$$\psi_{v\rho}^L = -e^{-i(hu+\bar{h}v)} \left[2ih + \mathcal{O}\left(\frac{\ln x}{x}\right) \right] \quad (181e)$$

$$\psi_{\rho\rho}^L = e^{-i(hu+\bar{h}v)} \left[\frac{4}{x}(1-2h^2) + \mathcal{O}\left(\frac{\ln x}{x^2}\right) \right]. \quad (181f)$$

The logarithmic mode was derived by shifting the weights $h \rightarrow \varepsilon$ and $\bar{h} \rightarrow \varepsilon$ to impose a mass-dependence on the weights and linearizing the equations (177) for small ε ; i.e. setting $F_{\mu\nu} = F_{\mu\nu}^L + \varepsilon F_{\mu\nu}^{\log}$ and considering the eoms (175) to the linear order in ε . The log-mode follows from

$$\psi_{\mu\nu}^{\log} = i(u+v)\psi_{\mu\nu}^L - F_{\mu\nu}^{\log} e^{-i(hu+\bar{h}v)}. \quad (182)$$

For large ρ/x it takes the form

$$\psi_{vv}^{\log} = e^{-i(hu+\bar{h}v)} \left[(h\bar{h} + x)(\ln x + i(u+v)) - h - \bar{h} - h\bar{h} \right] + \mathcal{O}\left(\frac{\ln x}{x}\right) \quad (183a)$$

$$\psi_{uv}^{\log} = e^{-i(hu+\bar{h}v)} \left[(1-h^2)(\ln x + i(u+v)) - (1-3h)(1+h) \right] + \mathcal{O}\left(\frac{\ln x}{x}\right) \quad (183b)$$

$$\begin{aligned} \psi_{uu}^{\log} = & e^{-i(hu+\bar{h}v)} \frac{h(h^2-1)}{\bar{h}} \left[\left(\ln \frac{x}{4} - i(u+v)\right) + \right. \\ & \left. + 2(\psi(h-1) + \psi(-\bar{h}) - \frac{3}{2} + 2\gamma) \right] + \mathcal{O}\left(\frac{\ln x}{x}\right) \end{aligned} \quad (183c)$$

$$\psi_{u\rho}^{\log} = \mathcal{O}\left(\frac{\ln x}{x}\right) \quad (183d)$$

$$\psi_{v\rho}^{\log} = -e^{-i(hu+\bar{h}v)} 2i \left[h(\ln x + i(u+v)) - 1 - h \right] + \mathcal{O}\left(\frac{\ln x}{x}\right) \quad (183e)$$

$$\psi_{\rho\rho}^{\log} = e^{-i(hu+\bar{h}v)} \frac{4}{x} \left[(1-2h^2)(\ln x + i(u+v)) - 2(1-3h)(1+h) \right] + \mathcal{O}\left(\frac{\ln x}{x^2}\right). \quad (183f)$$

Another way to obtain this mode would be to i) make the shifts $h \rightarrow \varepsilon$ and $\bar{h} \rightarrow \varepsilon$ in (180) and ii) take the derivative of the full massive mode w.r.t. $-\varepsilon$, i.e. calculate all F_{ij} from the new F'_{vv} and then take the derivative thereof. Finally set $\varepsilon = 0$.

In a similar fashion we derive the doubly logarithmic mode by doing i) and for ii) we generalize the first derivative to

$$\psi^{\log 2} = \frac{1}{2} \frac{d^2 \psi^M}{d\varepsilon^2} \Big|_{\varepsilon=0}. \quad (184)$$

The prefactor one half comes from our choice of normalization. We demand $\mathcal{D}^L \psi^{\log 2} = -2\psi^{\log}$.

We obtain the following asymptotic result:

$$\begin{aligned} \psi_{vv}^{\log 2} = & e^{-i(hu+\bar{h}v)} \left[\frac{h\bar{h}+x}{2} (\ln x + i(u+v))^2 - \right. \\ & - (h+\bar{h}+h\bar{h})(\ln x + i(u+v)) + \\ & \left. + (1+h+\bar{h}+h\bar{h}) \right] + \mathcal{O}\left(\frac{\ln^2 x}{x}\right) \end{aligned} \quad (185a)$$

$$\begin{aligned} \psi_{uv}^{\log 2} = & e^{-i(hu+\bar{h}v)} \left[\frac{1-h^2}{2} (\ln x + i(u+v))^2 - \right. \\ & - (1-3h)(1+h)(\ln x + i(u+v)) + \\ & \left. + (1-7h)(1+h) \right] + \mathcal{O}\left(\frac{\ln^2 x}{x}\right) \end{aligned} \quad (185b)$$

$$\begin{aligned} \psi_{uu}^{\log 2} = & e^{-i(hu+\bar{h}v)} \frac{h(1-h^2)}{\bar{h}} \left[\frac{1}{2} (\ln \frac{x}{4} - i(u+v))^2 - \right. \\ & - 2(\psi(h-1) + \psi(-\bar{h}) - \frac{3}{2} + 2\gamma)(\ln \frac{x}{4} - i(u+v)) + \\ & + (7+4\gamma(-3+2\gamma)) + 2[(-3+4\gamma)(\psi(h-1) + \psi(-\bar{h})) + \\ & \left. + (\psi(h-1) + \psi(-\bar{h}))^2 - (\psi'(h-1) - \psi'(-\bar{h}))] \right] + \mathcal{O}\left(\frac{\ln^2 x}{x}\right) \end{aligned} \quad (185c)$$

$$\psi_{u\rho}^{\log 2} = \mathcal{O}\left(\frac{\ln^2 x}{x}\right) \quad (185d)$$

$$\begin{aligned} \psi_{v\rho}^{\log 2} = & -e^{-i(hu+\bar{h}v)} 2i \left[\frac{h}{2} (\ln x + i(u+v))^2 - (1+h)(\ln x + i(u+v)) + \right. \\ & \left. + (1+h) \right] + \mathcal{O}\left(\frac{\ln^2 x}{x}\right) \end{aligned} \quad (185e)$$

$$\begin{aligned} \psi_{\rho\rho}^{\log 2} = & e^{-i(hu+\bar{h}v)} \frac{4}{x} \left[\frac{1-2h^2}{2} (\ln x + i(u+v))^2 - \right. \\ & \left. - 2(1-3h)(1+h)(\ln x + i(u+v)) + 2(1-7h)(1+h) \right] + \mathcal{O}\left(\frac{\ln^2 x}{x^2}\right) \end{aligned} \quad (185f)$$

One can check explicitly that this mode obeys

$$\mathcal{D}^L \mathcal{D}^L \mathcal{D}^L \psi^{\log 2} = \mathcal{D}^L \mathcal{D}^L (-2\psi^{\log}) = \mathcal{D}^L (4\psi^L) = 0. \quad (186)$$

References

- [1] E. Witten, “Three-Dimensional Gravity Revisited,” 0706.3359.
- [2] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231–252, hep-th/9711200.
O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” *Phys. Rept.* **323** (2000) 183–386, hep-th/9905111.
- [3] W. Li, W. Song, and A. Strominger, “Chiral Gravity in Three Dimensions,” *JHEP* **04** (2008) 082, 0801.4566.
- [4] S. Deser, R. Jackiw, and S. Templeton, “Three-dimensional massive gauge theories,” *Phys. Rev. Lett.* **48** (1982) 975–978.
S. Deser, R. Jackiw, and S. Templeton, “Topologically massive gauge theories,” *Ann. Phys.* **140** (1982) 372–411. *Erratum-ibid.* **185** (1988) 406.
- [5] A. Maloney and E. Witten, “Quantum Gravity Partition Functions in Three Dimensions,” *JHEP* **02** (2010) 029, 0712.0155.
- [6] D. Grumiller, D. Vassilevich, and M. R. Gaberdiel, “Graviton 1-loop partition function for 3-dimensional massive gravity,” 1007.5189.
- [7] A. Strominger, “A Simple Proof of the Chiral Gravity Conjecture,” 0808.0506.
- [8] A. Maloney, W. Song, and A. Strominger, “Chiral Gravity, Log Gravity and Extremal CFT,” *Phys. Rev.* **D81** (2010) 064007, 0903.4573.
- [9] M. R. Gaberdiel, R. Gopakumar, and A. Saha, “Quantum \mathcal{W} -symmetry in AdS_3 ,” 1009.6087.
- [10] D. Grumiller and N. Johansson, “Instability in cosmological topologically massive gravity at the chiral point,” *JHEP* **07** (2008) 134, 0805.2610.
S. Ertl, D. Grumiller, and N. Johansson, “Erratum to ‘Instability in cosmological topologically massive gravity at the chiral point’, arXiv:0805.2610,” 0910.1706.
- [11] V. Gurarie, “Logarithmic operators in conformal field theory,” *Nucl. Phys.* **B410** (1993) 535–549, hep-th/9303160.

- [12] D. Grumiller and N. Johansson, “Gravity duals for logarithmic conformal field theories,” *J. Phys. Conf. Ser.* **222** (2010) 012047, 1001.0002.
- [13] D. Grumiller, N. Johansson, and T. Zojer, “Short-cut to new anomalies in gravity duals to logarithmic conformal field theories,” 1010.4449.
- [14] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, “Massive Gravity in Three Dimensions,” *Phys. Rev. Lett.* **102** (2009) 201301, 0901.1766.
E. A. Bergshoeff, O. Hohm, and P. K. Townsend, “More on Massive 3D Gravity,” *Phys. Rev.* **D79** (2009) 124042, 0905.1259.
- [15] M. Reza Rahimi Tabar, “Quenched Averaged Correlation Functions of the Random Magnets,” *Nucl. Phys.* **B588** (2000) 630–637, cond-mat/0002309.
- [16] J. L. Cardy, “Logarithmic Correlations in Quenched Random Magnets and Polymers,” cond-mat/9911024.
- [17] J. S. Caux, I. I. Kogan, and A. M. Tsvelik, “Logarithmic operators and hidden continuous symmetry in critical disordered models,” *Nucl. Phys.* **B466** (1996) 444–462, hep-th/9511134.
- [18] V. Gurarie and A. W. W. Ludwig, “Conformal algebras of 2D disordered systems,” *J. Phys.* **A35** (2002) L377–L384, cond-mat/9911392.
- [19] V. Gurarie and A. W. W. Ludwig, “Conformal field theory at central charge $c = 0$ and two-dimensional critical systems with quenched disorder,” hep-th/0409105.
- [20] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” *Commun. Math. Phys.* **104** (1986) 207–226.
- [21] R. Blumenhagen and E. Plauschinn, “Introduction to conformal field theory,” *Lect. Notes Phys.* **779** (2009) 1–256.
- [22] M. Flohr, “Operator product expansion in logarithmic conformal field theory,” *Nucl. Phys.* **B634** (2002) 511–545, hep-th/0107242.
- [23] M. R. Rahimi Tabar, A. Aghamohammadi, and M. Khorrami, “The logarithmic conformal field theories,” *Nucl. Phys.* **B497** (1997) 555–566, hep-th/9610168.

- [24] J. Rasmussen, “Jordan cells in logarithmic limits of conformal field theory,” *Int. J. Mod. Phys.* **A22** (2007) 67–82, [hep-th/0406110](#).
- [25] D. Grumiller and I. Sachs, “AdS₃/LCFT₂ – Correlators in Cosmological Topologically Massive Gravity,” *JHEP* **03** (2010) 012, [0910.5241](#).
- [26] Y. Liu and Y.-W. Sun, “On the Generalized Massive Gravity in AdS₃,” *Phys. Rev.* **D79** (2009) 126001, [0904.0403](#).
- [27] G. Giribet, M. Kleban, and M. Porrati, “Topologically Massive Gravity at the Chiral Point is Not Unitary,” *JHEP* **10** (2008) 045, [0807.4703](#).
- [28] D. Grumiller and O. Hohm, “AdS₃/LCFT₂ – Correlators in New Massive Gravity,” *Phys. Lett.* **B686** (2010) 264–267, [0911.4274](#).
- [29] K. Skenderis, M. Taylor, and B. C. van Rees, “Topologically Massive Gravity and the AdS/CFT Correspondence,” *JHEP* **09** (2009) 045, [0906.4926](#).
- [30] M. Ostrogradsky *Mem. Acad. St. Petersburg* **VI** (1850) 385.
- [31] A. Sinha, “On the new massive gravity and AdS/CFT,” *JHEP* **06** (2010) 061, [1003.0683](#).
- [32] M. F. Paulos, “New massive gravity, extended,” [1005.1646](#).
- [33] A. Sinha, “On higher derivative gravity, c-theorems and cosmology,” [1008.4315](#).
- [34] I. Gullu, T. Cagri Sisman, and B. Tekin, “Born-Infeld extension of new massive gravity,” *Class. Quant. Grav.* **27** (2010) 162001, [1003.3935](#).
- [35] M. Alishahiha, A. Naseh, and H. Soltanpanahi, “On Born-Infeld Gravity in Three Dimensions,” *Phys. Rev.* **D82** (2010) 024042, [1006.1757](#).