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Dissertation

Models with Oscillator Terms in Noncommutative Quantum Field Theory

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Abstract

The main focus of this Ph.D. thesis is on noncommutative models involving oscillator terms in the action. The first one historically is the successful Grosse-Wulkenhaar (G.W.) model which has already been proven to be renormalizable to all orders of perturbation theory. Remarkably it is furthermore capable of solving the Landau ghost problem.
In a first step, we have generalized the G.W. model to gauge theories in a very straightforward way, where the action is BRS invariant and exhibits the good damping properties of the scalar theory by using the same propagator, the so-called Mehler kernel. To be able to handle some more involved one-loop graphs we have programmed a powerful Mathematica® package, which is capable of analytically computing Feynman graphs with many terms. The result of those investigations is that new terms originally not present in the action arise, which led us to the conclusion that we should better start from a theory where those terms are already built in.
Fortunately there is an action containing this complete set of terms. It can be obtained by coupling a gauge field to the scalar field of the G.W. model, integrating out the latter, and thus "inducing" a gauge theory. Hence the model is called Induced Gauge Theory. Despite the advantage that it is by construction completely gauge invariant, it contains also some unphysical terms linear in the gauge field. Advantageously we could get rid of these terms using a special gauge dedicated to this purpose. Within this gauge we could again establish the Mehler kernel as gauge field propagator. Furthermore we where able to calculate the ghost propagator, which turned out to be very involved.
Thus we were able to start with the first few loop computations showing the expected behavior. The next step is to show renormalizability of the model, where some hints towards this direction will also be given.

Kurzfassung

Der Hauptfokus dieser Dissertationsarbeit liegt auf nichtkommutativen Modellen mit Oszillatortermen in der Wirkung. Das historisch gesehen erste dieser Modelle ist das erfolgreiche Grosse-Wulkenhaar (G.W.) Modell, von welchem bereits gezeigt wurde, dass es zu allen Ordnungen der Störungstheorie renormierbar ist. Bemerkenswerterweise löst es außerdem das Landau Geist Problem.
In einem ersten Schritt haben wir das G.W. Modell direkt auf Eichtheorien verallgemeinert, wobei die Wirkung BRS invariant ist und die guten dämpfenden Eigenschaften der Skalartheorie beibehält, indem es denselben Propagator nutzt, den sogenannten Mehler Kern. Um manche aufwändigere Einschleifenrechnungen bewältigen zu können, haben wir ein Mathematica® Paket programmiert, welches in der Lage ist, Feynman Graphen mit vielen Termen analytisch zu berechnen. Das Ergebnis dieser Betrachtungen war, dass neue Terme, die ursprünglich nicht in der Wirkung vorhanden waren, entstehen, was uns zu dem Schluss führte, dass wir besser von einer Theorie wegstarten sollten bei der diese Terme bereits eingebaut sind.
Glücklicherweise gibt es eine Wirkung die diese vollständige Menge von Termen enthält. Sie kann erhalten werden, indem man ein Eichfeld an das Skalarfeld der G.W. Wirkung koppelt und dann Letzteres ausintegriert. Auf diese Art und Weise "induziert" man eine Eichfeldtheorie, welche deswegen Induzierte Eichfeldtheorie genannt wird. Trotz des Vorteils, dass sie per Konstruktion eichinvariant ist, enthält sie auch einige unphysikalische Terme, welche linear im Eichfeld sind. Vorteilhafterweise konnten wir diese Terme durch eine Eichung, die für diesen Zweck konstruiert wurde, loswerden. In dieser Eichung konnten wir wieder den Mehler Kern als Propagator für das Eichfeld etablieren. Weiters schafften wir es, den Geistpropagator zu berechnen, was sich als sehr aufwändig herausstellte.
Schließlich war uns deswegen die Möglichkeit gegeben, mit den ersten Schleifenrechnungen anzufangen, welche auch das erwartete Verhalten zeigten. Der nächste Schritt ist die Renormierbarkeit des Modells zu zeigen, wobei einige Hinweise in diese Richtung auch gegeben werden.

Acknowledgements

First of all I want to thank Prof. Schweda who made this work possible. His activeness and enthusiasm for physics despite of the fact that he is already retired, is inspiring. His first words to me when I started my research were: "Always be honest to yourself", and indeed it has always been the right guideline to question conclusions which don't seem perfectly clear.
Next I want to thank Prof. Grosse for the enlightening discussions. His knowledge on physics and mathematics is gigantic. He gave me some of the most useful impulses when I didn't know how to continue with solving a problem.
I specially thank my friends and colleagues Daniel Blaschke and Michael Wohlgenannt, who patiently explained me details I didn't know. They supported me in any thinkable kind of way and took the time to also discuss my most adventurous ideas. Without them, this Ph.D. thesis would have never been possible.
I would like to express my gratitude to Rene Sedmik for teaching me how to use Mathematica® and for exemplifying solid and clean mathematical derivations, as well as Arnold Rofner for the enthusiastic discussions and for his stories from all over the world.
I want to thank Franz Heindl, who never believed me anything. I always had to rigorously defend the mathematical derivations I was presenting to him which in turn helped me to better understand the deep underlying structure of the mathematics which is behind the physics we are doing.
Of course I also want to thank all my other colleagues, Thomas Reis, Albert Much, Thomas Ludwig, Daniela Klammer, Peter Somkuti and Axel de Goursac for the interesting discussions with sometimes even philosophical touch.
Last but not least I am very grateful for the private support of my family and my girlfriend who encouraged me to continue trying when I was stuck at some point or the other.

Contents

Chapter 1

Introduction

1.1 Outline of this Work

First, we will motivate noncommutativity in general by looking at three popular examples (Section 1.3). Next, we will introduce the noncommutative space on a formal mathematical level, which will allow us to formulate field theories thereon (Section 1.4). The work of people who put forward the first early models will be summarized in a small historical section (Section 1.5.1,1.5.2). It is there where for the first time the UV/IR mixing problem showed up, and solutions to it needed to be taken into concern (Section 1.5.3). Especially the so-called $1/p^2$ model will in this context be emphasized (Section 1.5.3), since the author has worked on this field also for quite some time. However, the main focus of this Ph.D. thesis is to cover the field of Euclidean noncommutative field models involving oscillator terms (Chapter 2), especially in the light of renormalization. Essentially three such models are known:

1. **The Grosse-Wulkenhaar (G.W.) model** (Section 2.1) Historically the first noncommutative model which has been shown to be renormalizable to all orders. Compared to commutative models it furthermore has the advantage that it kills the Landau ghost, which is roughly speaking a singular behavior of the beta function in certain regions. A special subsection will be devoted to the propagator of this model, the 'Mehler kernel' (Section 2.1.1).

2. **A simple gauge theory version of the latter**[1] (Section 2.2) The straightforward generalization of the G.W. model to gauge theories. It has also the favoured "Mehler kernel" as a propagator, and therefore one expects good damping properties like in the scalar case. However it will turn out that within the renormalization procedure, counterterms arise which weren't originally present in the action. This leads us to the next model in this list:

3. **Induced Gauge Theory** (Section 2.3) By coupling a gauge field to the scalar field of the G.W. model and by integrating the latter out, one naturally "induces" new gauge degrees of freedom. The corresponding action, referred to as 'Induced Gauge Theory', is by construction gauge invariant and features certain nice symmetries. Despite the advantages, it contains also unphysical terms linear in the gauge field.

[1] Also referred to as the "Mehler kernel gauge model".

Fortunately, we will be able to find a gauge fixing which is capable of getting rid of these terms (Section 2.3.2). The Feynman rules will be presented and some first loop calculations will be performed.

To complete this outline, it shall be pointed out that in the appendix of this work, apart from some detailed calculations associated with the respective chapters, one may also find the description of a very powerful Mathematica® package programmed by our team, with which we were able to compute the most involved Feynman graphs (Appendix B).

1.2 Conventions

We use the Einstein summation convention on Euclidean space,

$$\sum_{\mu=1}^{d} A_\mu B_\mu = A_\mu B_\mu = A^\mu B^\mu \, , \tag{1.1}$$

where d denotes the dimension of the Euclidean space.
Furthermore, whenever a d-vector is multiplied with another one in an exponential, the indices will be left out, that means

$$e^{A_\mu B_\mu} = e^{AB}. \tag{1.2}$$

We will use natural units, that is $\hbar = c = 1$.
In the following we will define some conventions which are listed here for completeness reasons. The definitions will make sense when deriving the structure of noncommutative Euclidean space, Section (1.4).
A tilde on a position/momentum variable means contraction with the deformation $\Theta_{\mu\nu}$, where the latter is describing noncommutativity of the coordinates ($\Theta_{\mu\nu}$ will be defined a posteriori in Section 1.4):

$$\tilde{x}_\mu = \Theta^{-1}_{\mu\nu} x_\nu \, , \quad \tilde{p}_\mu = \Theta_{\mu\nu} p_\nu \, . \tag{1.3}$$

Frequently we will use the $\times$ symbol to also indicate a contraction of the Theta matrix with two momenta:

$$p \times q = p\tilde{q} = p_\mu \tilde{q}_\mu = p_\mu \Theta_{\mu\nu} q_\nu \, , \tag{1.4}$$

and in position space we will use the $\wedge$ symbol likewise

$$x \wedge y = x\tilde{y} = x_\mu \tilde{y}_\mu = x_\mu \Theta^{-1}_{\mu\nu} y_\nu \, . \tag{1.5}$$

Further conventions concerning the noncommutative calculations will be made in Section 1.4.

1.3 Motivations for NCQFT

Why do we investigate noncommutative spaces? The three following examples might motivate the reader why noncommutativity is such a great feature to look at.
We will first start with the Landau problem which is about a particle moving in a strong

magnetic field. We will see that noncommutativity will emerge automatically when describing this particle. Next we will investigate the Planck scale, where physics has to be modified in order to avoid singularities. Noncommutativity is a good candidate fitting this purpose. Finally we will look at the rotation group. It will be visually illustrated that this group is noncommutative.

1.3.1 The Landau problem

The problem was first recognized by Landau in 1930 [1]. To describe it, we first start at the classical action of a charged particle in an external magnetic field

$$S[\vec{x}, \dot{\vec{x}}, \vec{A}] = \int dt \left(\frac{1}{2} m \dot{\vec{x}}^2 + e \vec{A} \dot{\vec{x}} \right) , \tag{1.6}$$

where e is the electric charge and $\vec{A}$ is the 3-dimensional vector potential. In order to get a constant magnetic field in the third spatial direction we can assume for example

$$\vec{A} = \begin{pmatrix} -\frac{B}{2} y \\ \frac{B}{2} x \\ 0 \end{pmatrix} , \tag{1.7}$$

because then $\operatorname{rot} \vec{A} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$.

From (1.7) we can also deduce that our problem is essentially two-dimensional. Plugging in the special form of the vector potential into the Lagrange function we get

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - e \frac{B}{2} \epsilon^{ij} x^i \dot{x}^j , \tag{1.8}$$

with ϵ^{ij} being the epsilon-tensor in 2 dimensions. In order to derive the Hamiltonian we have to perform a Legendre transformation. The definition of the conjugate momentum is

$$\frac{\partial L}{\partial \dot{x}} = p^i = m \dot{x}^i - \frac{eB}{2} \epsilon^{il} x^l , \tag{1.9}$$

which allows us to rewrite the Lagrange function from the original variables $(x, \dot{x})$ to the new ones (x, p), because from (1.9) we see that we can express all velocities in terms of (x, p):

$$\dot{x}^i = \frac{1}{m} \left(p^i + \frac{eB}{2} \epsilon^{il} x^l \right) . \tag{1.10}$$

Hence we can perform a Legendre transformation

$$
\begin{aligned}
H(x^i,p^j) =& p^i\dot{x}^i - L(x^i,\dot{x}^j)\\
=&\frac{p^i}{m}\left(p^i+\frac{eB}{2}\epsilon^{il}x^l\right)-\frac{m}{2}\frac{1}{m^2}\left(p^i+\frac{eB}{2}\epsilon^{il}x^l\right)^2+\frac{eB}{2}\epsilon^{ij}\frac{1}{m}\left(p^i+\frac{eB}{2}\epsilon^{il}x^l\right)x^j\\
=&\frac{1}{2m}p^ip^i+\frac{eB}{2m}\epsilon^{ij}p^ix^j+\frac{1}{8m}e^2B^2\epsilon^{ij}x^j\epsilon^{il}x^l \qquad (1.11)\\
=&\frac{1}{2m}\left(p^i+\frac{eB}{2}\epsilon^{ij}x^j\right)^2\\
=&\frac{1}{2m}\left(p^i-eA^i\right)^2 \, , \qquad (1.12)
\end{aligned}
$$

which shows that the vector potential is minimally coupled to the momentum. Defining this as our new <u>physical</u> momentum $\Pi^i = p^i - e\vec{A} = p^i + \frac{eB}{2}\epsilon^{ij}x^j$, we have

$$H = \frac{1}{2m}\Pi^i\Pi^i \, . \qquad (1.13)$$

Quantization The quantization is defined in replacing $x^i$and p^i by hermitian operators

$$
\begin{aligned}
x^i &\to \hat{x}^i\\
p^j &\to \hat{p}^j \, , \qquad (1.14)
\end{aligned}
$$

with

$$[\hat{p}^i,\hat{x}^j] = -i\delta^{ij} \, . \qquad (1.15)$$

This allows us to calculate the commutator of the physical momentum which we just defined:

$$
\begin{aligned}
[\hat{\Pi}^i,\hat{\Pi}^j] =& \left[\hat{p}^i+\frac{eB}{2}\epsilon^{il}\hat{x}^l,\hat{p}^j+\frac{eB}{2}\epsilon^{jm}x^m\right]\\
=&\frac{eB}{2}\epsilon^{il}\underbrace{[\hat{x}^l,\hat{p}^j]}_{\mathrm{i}\delta^{lj}}+\frac{eB}{2}\epsilon^{jm}\underbrace{[\hat{p}^i,\hat{x}^m]}_{-\mathrm{i}\delta^{im}} \, . \qquad (1.16)
\end{aligned}
$$

Thus,

$$[\hat{\Pi}^i,\hat{\Pi}^j] = \mathrm{i}eB\epsilon^{ij} \, . \qquad (1.17)$$

Similarly, we can deduce the commutation relation of the coordinates. For that, we plug (1.9) into (1.15)[2]

$$
\begin{aligned}
[p^i,x^j] = [m\dot{x}^i - \frac{eB}{2}\epsilon^{il}x^l,x^j] &= -\mathrm{i}\delta^{ij} \, ,\\
[\dot{x}^i,x^j] - \frac{eB}{2m}\epsilon^{il}[x^l,x^j] &= -\frac{\mathrm{i}}{m}\delta^{ij} \, . \qquad (1.18)
\end{aligned}
$$

[2]From now on we assume to have the Hamiltonian already quantized and leave out the hats.

In the case of a strong magnetic field (corresponding to the limit of vanishing mass, as justified in the next paragraph) the first term vanishes, and we get

$$\boxed{[x^k, x^j] = -\frac{2\mathrm{i}}{eB}\epsilon^{kj}}\,, \tag{1.19}$$

which is nonzero! Thus we have also found a noncommutativity relation of the coordinates.

Eigenvalues We may rewrite (1.11). For simplicity we set $y = 0$ and absorb a factor $\frac{1}{2}$ into the B-field[3]. Then

$$\begin{aligned} H =& \frac{1}{2m}p^i p^i + \frac{eB}{2m}\epsilon^{ij}p^i x^j + \frac{1}{8m}e^2B^2\epsilon^{ij}x^j\epsilon^{il}x^l \\ =& \frac{1}{2m}(p_x^2 + p_y^2) + \frac{eB}{2m}(p_x y - p_y x) + \frac{1}{8m}e^2B^2(x^2 + y^2) \\ \overset{y=0,\frac{B}{2}\to B}{=}& \frac{1}{2m}(p_x^2 + p_y^2) + \frac{eB}{m}(-p_y x) + \frac{1}{2m}e^2B^2x^2 \\ =& \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2\left(x - \frac{p_y}{m\omega_c}\right)^2\,, \end{aligned} \tag{1.20}$$

with the cyclotron frequency, $\omega_c = \frac{eB}{m}$. This describes a harmonic oscillator shifted in x. The corresponding eigenvalues are

$$E_n = \omega_c(n + \frac{1}{2})\,, \tag{1.21}$$

and are called the "Landau levels". The separation of the eigenvalues is given by

$$\Delta E = \omega_c = \frac{eB}{m}\,. \tag{1.22}$$

For a strong magnetic field, the separation energy is therefore also high so that the particles are essentially confined to the lowest Landau level. This also corresponds to take $m \to 0$, and a posteriori justifies this limit which we have taken to obtain (1.19). There can be much more said about the Landau problem, especially about the fractional Hall effect [2], but for now we leave it to the statement that indeed, as we have seen, the Landau problem motivates a more detailed study of noncommutativity.

1.3.2 The Planck scale

The Planck scale can be obtained by making the following gedankenexperiment: If we want to resolve a certain spatial region we have to deposit enough energy into that region. Making this region smaller and smaller we come to a scale where the energy needed can in principle create a black hole. This scale is called the "Planck scale", and in SI units it is around 10^{-35} meters corresponding to the energy 1.22×10^{19} GeV. A little more

[3]This corresponds to originally having chosen the gauge potential to be $\vec{A} = \begin{pmatrix} 0 \\ \frac{B}{2}x \\ 0 \end{pmatrix}$.

formally we can obtain the Planck length in the following way:
We start at an arbitrary mass M. This mass corresponds to a Compton wavelength

$$\lambda = \frac{h}{Mc}\,, \tag{1.23}$$

and to a Schwarzschild-radius

$$R_{\mathrm{S}} = \frac{2GM}{c^2}\,. \tag{1.24}$$

If we now demand this Schwarzschild radius to be of the order of the Compton wavelength $R_{\mathrm{S}} \sim \lambda$ we get

$$\begin{aligned} \frac{h}{Mc} &= \frac{2GM}{c^2} \\ M &= \sqrt{\frac{hc}{2G}} \sim \sqrt{\frac{hc}{G}} =: M_{\mathrm{Pl}}\,, \end{aligned} \tag{1.25}$$

where we have neglected the factor 2 because we are only talking about orders of magnitude anyway. Fascinatingly, we could now construct a mass purely in terms of constants of nature. Plugging in this mass into (1.23) or (1.24) we get the Planck length

$$l_{\mathrm{Pl}} = \sqrt{\frac{Gh}{c^3}}\,. \tag{1.26}$$

The situation has been analyzed in detail by Doplicher, Fredenhagen, Roberts in 1995 [3], where they have generalized the one-dimensional consideration we have done here to 4 spacetime directions. In particular they obtained a commutation relation between the coordinates, however in contrast to the Weyl-Moyal case we consider here (1.29) their Theta matrix is x-dependent and constructed in a way that it is Poincare-invariant[4].

1.3.3 The rotation group

When rotating a cuboid twice the outcome may differ depending on the order of the respective rotation applied. This is depicted in Figure 1.1.
The reason for this is that the rotation group $\mathcal{SO}(3)$ is noncommutative. The generators of infinitesimal rotations form a Lie algebra:

$$[T^i, T^j] = \mathrm{i}\epsilon^{ijk}T^k\,, \tag{1.27}$$

where $i, j, k = \{1, 2, 3\}$. A finite rotation $g(x)$ is given by

$$g(x) = e^{\mathrm{i}\alpha^i(x)T^i}\,, \tag{1.28}$$

where α is the rotation angle. Since the T^i's don't commute, the finite rotations $g(x)$ do neither.
Hence $\mathcal{SO}(3)$ is a good example for a noncommutative group and motivates to study noncommutativity quite generally.

[4] This means that it is invariant under Lorentz transformations, Lorentz boosts, Lorentz translations and time reversal.

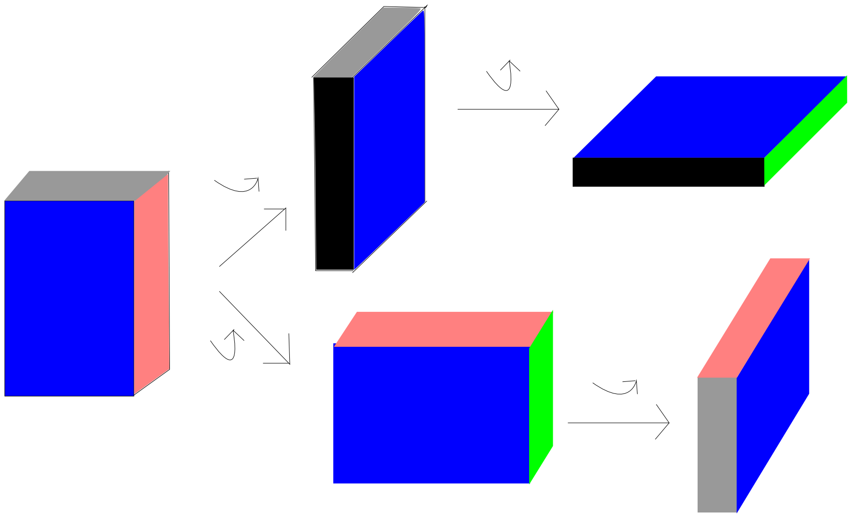


Figure 1.1: A rotated cuboid.

1.4 Mathematical Structure of NCQFT

In this section we will briefly introduce NCQFT on a formal level. We will take a closer look at the Weyl-Moyal star product, and finally we will compute the symmetry factors of Feynman graphs.

1.4.1 Structure of noncommutative Euclidean space

As motivated in the previous section, just like in quantum mechanics where one introduces a commutator between coordinates and momenta, one aims to introduce a nonvanishing commutator of the coordinates themselves[5]. This means that time does not commute with space any more, and space with space neither does. The simplest way to implement this is to put a constant, antisymmetric matrix on the r.h.s.[6] This is called the Moyal-Weyl case, formally

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i}\Theta_{\mu\nu}\,, \tag{1.29}$$

[5]Such an nonvanishing commutator was for the first time written down by Snyder [4].

[6]However, it should be mentioned here that one could start with different noncommutative algebras, for example one could start with a Lie-algebra ($[x_\mu, x_\nu] = iC^{\mu\nu}_\rho x^\rho$) or even with the quantum group space ($[x_\mu, x_\nu] = iR^{\mu\nu}_{\rho\sigma}x^\rho x^\sigma$). As another possible generalization the Theta matrix can in principle be x-dependent, as it is for example the case in emergent gravity models [5].

where the hat symbol shall hint to that we are dealing with noncommutative operator-valued objects. The deformation matrix $\Theta_{\mu\nu}$ is constant and antisymmetric, explicitly

$$\Theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \tag{1.30}$$

in 4 dimensions and

$$\Theta_{\mu\nu} = \theta\epsilon_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{1.31}$$

in 2 dimensions, with Theta having mass dimension -2. This simplest choice of $\Theta_{\mu\nu}$ is sufficient to describe complete noncommutativity in space and time because one can see from the block-diagonal form that it has full rank. It has to be antisymmetric because the the commutator on the l.h.s. of Equation (1.29) is too.
A product on noncommutative space can be defined by assuming the fields to be of Schwartz type, that is they decrease sufficiently fast at infinity. Then, a Fourier transform of them is mathematically well defined. Since the algebra (1.29) is central, the Baker-Campbell-Hausdorff formula terminates, and we find a closed expression which can be defined as a new product, the so-called Moyal-Weyl star product:

$$f(x) \star g(x) = e^{\mathrm{i}\partial^x_\mu \Theta_{\mu\nu} \partial^y_\nu} f(x) g(y) \Big|_{x=y} , \tag{1.32}$$

or with momentum variables

$$f(x) \star g(x) = \int dk \int dk' \, e^{\mathrm{i}(k+k')x - \frac{\mathrm{i}}{2} k_\mu k'_\nu \Theta^{\mu\nu}} f(k) g(k'). \tag{1.33}$$

This star product connects noncommutative fields with ordinary commutative ones. It has the following important properties

- **Star product of higher orders**

$$f_1(x) \star f_2(x) \star \ldots \star f_m(x)$$
$$= \int \frac{d^d k_1}{(2\pi)^{d/2}} \int \frac{d^d k_2}{(2\pi)^{d/2}} \cdots \int \frac{d^d k_m}{(2\pi)^{d/2}} e^{\mathrm{i} \sum_{i=1}^{m} k_{i\mu} x_\mu} e^{-\frac{\mathrm{i}}{2} \sum_{i<j}^{m} k_i \times k_j} \tilde{f}_1(k_1) \tilde{f}_2(k_2) \ldots \tilde{f}_m(k_m) \tag{1.34}$$

- **Cyclic permutation under the integral**

$$\int d^d x \, f_1(x) \star f_2(x) \star \ldots \star f_m(x) = \int d^d x \, f_2(x) \star \ldots \star f_m(x) \star f_1(x) \tag{1.35}$$

  This has the important consequence that when considering field theoretical models, in bilinear expressions the star drops out:

$$\int d^d x \, f_1(x) \star f_2(x) = \int d^d x \, f_1(x) f_2(x) \, . \tag{1.36}$$

  This implies that propagators are essentially the same as in the commutative world.

- **Associativity**

$$(f \star g) \star h = f \star (g \star h) \tag{1.37}$$

- **Variation**

$$\frac{\delta}{\delta f_1(y)} \int d^4x\, f_1(x) \star f_2(x) \star \ldots \star f_n(x) = f_2(y) \star \ldots \star f_n(y) \tag{1.38}$$

- **Star product of two exponentials**

$$e^{\mathrm{i}kx} \star e^{\mathrm{i}k'x} = e^{\mathrm{i}(k+k')x} e^{-\frac{\mathrm{i}}{2} k\Theta k'} \,.$$

All related proofs can be found e.g. in my diploma thesis [6].
To close the general part, let me just mention that we can now build a field theory by putting stars everywhere between the fields, which we will explicitly do in later chapters. In the following subsection, we will develop some other formulas commonly used for the star product, which are useful in one or the other context.

1.4.2 Other representations for the star product

One can write the kernel of the star product in terms of delta functions. To show this, we start with

$$(f \star g)(z) = f(z) \star g(z) = f(z) \star_z g(z) \,, \tag{1.39}$$

and just note that in this case, the star is an operator with respect to z. One can write each function, $g(z)$ and $f(z)$ individually in terms of delta functions

$$\begin{aligned} f(z) \star_z g(z) &= \int d^dx\, \delta^{(d)}(z-x) f(x) \star_z \int d^dy\, \delta^{(d)}(z-y) g(y) \\ &= \int d^dx \int d^dy\, \delta^{(d)}(z-x) \star_z \delta^{(d)}(z-y) f(x) g(y) \,. \end{aligned} \tag{1.40}$$

We can now interpret

$$\delta^{(d)}(z-x) \star_z \delta^{(d)}(z-y) =: K(x,y;z) \tag{1.41}$$

as the kernel of the star product operation

$$f(z) \star_z g(z) = \int d^dx \int d^dy\, K(x,y;z) f(x) g(y) \,. \tag{1.42}$$

In the following we will show why we can interpret the star product as a convolution. Our kernel can be written as

$$\begin{aligned} K(x,y;z) &= \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dp}{(2\pi)^d}\, e^{\mathrm{i}k(z-x)} \star_z e^{\mathrm{i}p(z-y)} \\ &= \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dp}{(2\pi)^d}\, e^{-\mathrm{i}kx} e^{-\mathrm{i}py} e^{\mathrm{i}kz} \star_z e^{\mathrm{i}pz} \,. \end{aligned} \tag{1.43}$$

The star product of two exponentials is given by (proof see my diploma thesis [6])

$$e^{\mathrm{i}kz} \star e^{\mathrm{i}pz} = e^{\mathrm{i}(k+p)z} e^{-\frac{\mathrm{i}}{2}k\Theta p}\,, \tag{1.44}$$

so we conclude

$$\begin{aligned} K(x,y;z) &= \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dp}{(2\pi)^d}\, e^{-\mathrm{i}kx} e^{-\mathrm{i}py} e^{\mathrm{i}(k+p)z} e^{-\frac{\mathrm{i}}{2}k\Theta p} \\ &= \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dp}{(2\pi)^d}\, e^{\mathrm{i}k(z-x)} e^{\mathrm{i}p(z-y+\frac{1}{2}\Theta k)} \\ &= \int \frac{d^dk}{(2\pi)^d}\, e^{\mathrm{i}k(z-x)} \delta^{(d)}(z-y+\frac{1}{2}\Theta k)\,. \end{aligned} \tag{1.45}$$

Hence, the star product becomes

$$f(z) \star_z g(z) = \int d^dx \int d^dy \int \frac{d^dk}{(2\pi)^d}\, e^{\mathrm{i}k(z-x)} \delta^{(d)}(z-y+\frac{1}{2}\Theta k) f(x) g(y)\,, \tag{1.46}$$

and by performing a shift $x \to x+z$, $y \to y+z$

$$\int d^dx \int d^dy \int \frac{d^dk}{(2\pi)^d}\, e^{-\mathrm{i}kx} \delta^{(d)}(-y+\frac{1}{2}\Theta k) f(x+z) g(y+z)\,. \tag{1.47}$$

The delta function over y solves the integral and we get

$$\boxed{f(z) \star_z g(z) = \int d^dx \int \frac{d^dk}{(2\pi)^d}\, e^{\mathrm{i}kx} f(z+x) g(z+\frac{1}{2}\Theta k)}\,, \tag{1.48}$$

which is exactly the shifted form of the star product given in Rivasseau's review [7] page 29, or in Douglas/Nekrasov [8]. In this form one can also recognize the form of a convolution, but slightly modified by the shift of $\frac{1}{2}\Theta k$. This is a so-called twisted convolution [7].

We now want to write the star product in x-space. We'll start again at formula (1.45) for our kernel

$$\int \frac{d^dk}{(2\pi)^d}\, e^{\mathrm{i}k(z-x)} \delta^{(d)}(z-y+\frac{1}{2}\Theta k)\,, \tag{1.49}$$

and substitute $\frac{1}{2}\Theta k \to k'$ with functional determinant $\frac{2^d}{|\det\Theta|}$. Then we arrive at

$$\frac{2^d}{|\det\Theta|} \int \frac{d^dk'}{(2\pi)^d}\, e^{2\mathrm{i}k'\Theta^{-1}(z-x)} \delta^{(d)}(z-y+k')\,. \tag{1.50}$$

The delta function solves the integral:

$$\begin{aligned} &\frac{1}{|\det\Theta|} \frac{1}{\pi^d}\, e^{2\mathrm{i}(y-z)\Theta^{-1}(z-x)} \\ =&\frac{1}{\theta^d} \frac{1}{\pi^d}\, e^{-2\mathrm{i}(y-z)\Theta^{-1}(x-z)}\,, \end{aligned} \tag{1.51}$$

where the special form of $\Theta_{\mu\nu}$ (1.30 and 1.31) has been used.
By reinserting this kernel into the star product formula (1.42) we end up at

$$f(z) \star_z g(z) = \frac{1}{\theta^d \pi^d} \int d^d x \int d^d y \, e^{-2\mathrm{i}(y-z)\Theta^{-1}(x-z)} f(x) g(y) \tag{1.52}$$

$$\boxed{= \frac{1}{\theta^d \pi^d} \int d^d x \int d^d y \, e^{-2\mathrm{i}y\Theta^{-1}x} f(x+z) g(y+z)} \, , \tag{1.53}$$

which coincides with the literature.
This star product in position space can be generalized to higher orders. A proof won't be given in this thesis, but for completeness reasons the formula shall be stated (taken from [7] and adapted to our conventions):

Lemma: For all $j \in [\![1, 2n+1]\!]$, let f_j be an element of the noncommutative algebra. Then

$$(f_1 \star \ldots \star f_{2n})(x) = \frac{1}{\pi^{2d} \det^2 \Theta} \int \prod_{j=1}^{2n} dx_j f_j(x_j) e^{-2\mathrm{i}x \times \sum_{i=1}^{2n} (-1)^{i+1} x_i} e^{-2\mathrm{i}\phi_{2n}} \, , \tag{1.54}$$

$$(f_1 \star \ldots \star f_{2n+1})(x) = \frac{1}{\pi^d \det \Theta} \int \prod_{j=1}^{2n+1} dx_j f_j(x_j) \delta\big(x - \sum_{i=1}^{2n+1} (-1)^{i+1} x_i\big) e^{-2\mathrm{i}\phi_{2n+1}} \, , \tag{1.55}$$

$$\forall p \in \mathbb{N}, \, \phi_p = \sum_{i<j}^{p} (-1)^{i+j+1} x_i \times x_j \, , \tag{1.56}$$

and if there is an integration present (like in the actions we consider), we are led to the following
Corollary: For all $j \in [\![1, 2n+1]\!]$, let f_j be an element of the noncommutative algebra. Then

$$\int d^d x \, (f_1 \star \ldots \star f_{2n})(x) = \frac{1}{\pi^d \det \Theta} \int \prod_{j=1}^{2n} dx_j f_j(x_j) \delta\big(\sum_{i=1}^{2n} (-1)^{i+1} x_i\big) e^{-2\mathrm{i}\phi_{2n}} \, , \tag{1.57}$$

$$\int d^d x \, (f_1 \star \ldots \star f_{2n+1})(x) = \frac{1}{\pi^d \det \Theta} \int \prod_{j=1}^{2n+1} dx_j f_j(x_j) e^{-2\mathrm{i}\phi_{2n+1}} \, , \tag{1.58}$$

$$\forall p \in \mathbb{N}, \, \phi_p = \sum_{i<j}^{p} (-1)^{i+j+1} x_i \times x_j \, . \tag{1.59}$$

1.4.3 Symmetry factors

We have found no literature telling how to compute symmetry factors of various graphs, especially of graphs with a higher number of external legs. Therefore we will give here an idea of how this is done.
Instead of giving a general rule for computing symmetry factors we will follow a more pedagogical point of view and explicitly calculate the symmetry factor step by step for one specific example, a 3-point graph, namely the one with a 4-photon and a 3-photon

vertex[7]. If one draws the disjoint pieces that build up the graph (Figure 1.2)

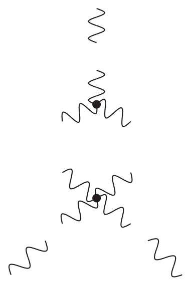


Figure 1.2: 3pt 4-photon 3-photon graph drawn disjoint.

one has 3×3 possibilities to connect one of the 3 external legs to the 3-photon vertex. Taking this factor into account we can already connect one external line (Figure 1.3)

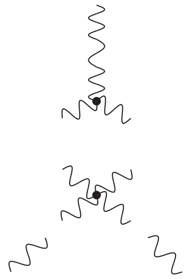


Figure 1.3: 3pt 4-photon 3-photon graph connected x1.

Now we have 4×3 possibilities to connect the remaining two external lines to the 4-photon vertex (Figure 1.4)

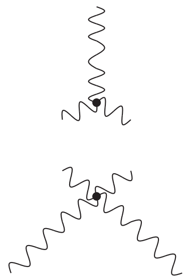


Figure 1.4: 3pt 4-photon 3-photon graph connected x3.

because when we have already connected one external line with one of the 4 ends of the vertex the remaining external line has only 3 possibilities left to connect to the vertex.

It remains to connect the 2 vertices to each other (Figure 1.5). We have two possibilities to do so (straightforward or crosswise).
All together we have so far the combinatorial factor $3 \times 3 \times 4 \times 3 \times 2$. We have to divide

[7]For a general rule how to derive the symmetry factors I recommend the Ph.D. thesis of Arnold Rofner [9], where this is explained in detail.

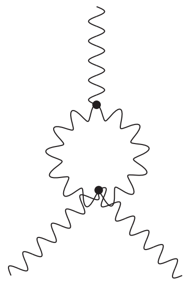


Figure 1.5: 3pt 4-photon 3-photon graph fully-connected.

this factor through 3! and 4!, for the corresponding vertex, respectively[8]. Those factors are already contained in the algebraic expression for the vertices at tree level. All in all our symmetry factor therefore is

$$\frac{3 \times 3 \times 4 \times 3 \times 2}{3! \times 4!} = \frac{3}{2} \,. \tag{1.60}$$

[8]Note that for graphs with n identical vertices one would have to divide through a factor $n!$ additionally.

1.5 Some Historical Introduction

In this section we will review what one could call the "naïve" attempts of introducing noncommutative actions, i.e. by considering those known from the commutative world and simply replacing pointwise by star products. We start with the noncommutative scalar ϕ^4 model and then continue to gauge theories. Linked to this we will point out the known solutions to the related UV/IR mixing problem. Furthermore we will briefly discuss an alternative approach known as Seiberg-Witten map. We will then round up this section by looking at renormalization schemes.

1.5.1 Scalar field theories

In replacing the ordinary pointwise product by the star product, a noncommutative extension to the scalar ϕ^4 model is given by

$$S = \int d^4x \left(\partial_\mu \phi \star \partial^\mu \phi + m^2 \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) . \tag{1.61}$$

The first one to consider this action was T. Filk [10] who derived the corresponding Feynman rules, noticing that — at least in Euclidean space — the propagator is exactly the same as in commutative space, i.e. $G^{\phi\phi}(k) = 1/(k^2 + m^2)$, while the vertex gains phase factors (in this case a combination of cosines) in the momenta. As a consequence, new types of Feynman graphs appear: In addition to the ones known from commutative space, where no phases depending on internal loop momenta appear and which exhibit the usual UV divergences, so-called nonplanar graphs come into the game which are regularized by phases depending on internal momenta. Other authors [11–15] performed explicit one-loop calculations and discovered the infamous UV/IR mixing problem: Due to the phases in the nonplanar graphs, their UV sector is regularized on the one hand, but on the other hand this regularization implies divergences for small external momenta instead.

For example the two point tadpole graph (in 4 dimensional Euclidean space) is approximately given by the integral

$$\Pi(\Lambda, p) \propto \lambda \int d^4k \frac{2 + \cos(k\tilde{p})}{k^2 + m^2} \equiv \Pi^{UV}(\Lambda) + \Pi^{IR}(p) . \tag{1.62}$$

The planar contribution is as usual quadratically divergent in the UV cutoff Λ, i.e. $\Pi^{UV} \sim \Lambda^2$, and the nonplanar part is regularized by the cosine to

$$\Pi^{\text{IR}} \sim \frac{1}{\tilde{p}^2} , \tag{1.63}$$

which shows that the original UV divergence is not present any more, but reappears when $\tilde{p} \to 0$ (where the phase is 1) representing a new kind of infrared divergence. Since both divergences are related to one another, one speaks of "UV/IR mixing". It is this mixing which renders the action (1.61) nonrenormalizable at higher loop orders.

1.5.2 Gauge field theories

In this section we will briefly overview the early attempts of understanding noncommutative gauge field theories. For a general overview over gauge theories we recommend

[16].
The pure star-deformed $\mathcal{U}(1)$ action of the Maxwell field is given by

$$S_{\mathrm{M}} = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} \right) , \tag{1.64}$$

where the field strength tensor is defined by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\, [A_\mu \overset{\star}{,} A_\nu] \\ &= -\mathrm{i}\, [\tilde{x}_\mu \overset{\star}{,} A_\nu] + \mathrm{i}\, [\tilde{x}_\nu \overset{\star}{,} A_\mu] - \mathrm{i}g\, [A_\mu \overset{\star}{,} A_\nu] \ . \end{aligned} \tag{1.65}$$

The corresponding Feynman rules for gauge field theories have been first worked out by C.P. Martin and D. Sánchez-Ruiz [17]. M. Hayakawa included fermions [18, 19], which leads to the action

$$S_{\mathrm{QED}} = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star \gamma^\mu \mathrm{i} D_\mu \psi - m\psi \star \bar{\psi} \right) , \tag{1.66}$$

with

$$D_\mu \psi = \partial_\mu \psi - \mathrm{i}g\, [A_\mu \overset{\star}{,} \psi] \ . \tag{1.67}$$

Hayakawa's loop calculations showed that UV/IR mixing is also present in gauge theories. Independently, A. Matusis *et al.* [13] derived the same result. Further early papers in this context are Refs. [20–23]. Explicitly, F. Ruiz Ruiz could even show that the quadratic and linear IR divergences in the $U(1)$ sector appear gauge independently[9] [24], and are therefore no gauge artefact. Furthermore, it was proven by using an interpolating gauge that quadratic IR divergences not only are independent of covariant gauges, but also of axial gauges [25]. As M. van Raamsdonk pointed out [26], the IR singularities have a natural interpretation in terms of a matrix model formulation of YM theories.

Regarding the group structure of the noncommutative YM theory, A. Armoni stressed the fact that $SU_\star(N)$ theory by itself is not consistent [27, 28], and one should rather consider $U_\star(N)$. In his computations, he showed that the planar sector leads to a β-function with negative sign, i.e. a running coupling g, and that the infamous UV/IR mixing arises only in those graphs which have at least one external leg in the $U_\star(1)$ subsector. Furthermore, in the limit $\theta \to 0$, $U_\star(N)$ does not converge to the usual $SU(N) \times U(1)$ commutative theory, which shows that the limit is nontrivial. One reason for this is that the β-function is independent from θ, meaning that the $U(1)$ coupling still runs in that limit.

Nevertheless, up to one loop order, $U_\star(N)$ YM theory is renormalizable in a BRST invariant way. Furthermore, the Slavnov-Taylor identity, the gauge fixing equation, and the ghost equation hold [29]. As in the naïve scalar model of the previous subsection, UV/IR mixing leads to nonrenormalizability at higher loop order.

Finally, the noncommutative two-torus has been studied by several authors [30–33].

A deformation of the Standard Model is discussed in [34]. The authors start with the gauge group $U_\star(3) \times U_\star(2) \times U_\star(1)$. In order to obtain the gauge group of the Standard Model one has to introduce a breaking and hence additional degrees of freedom. An alternative approach is to use Seiberg-Witten maps, described in the following subsection:

[9] However, one can improve the divergence behavior by introduction of supersymmetry.

Seiberg-Witten maps

The aim of this subsection is to touch the big subject of Seiberg Witten (S.W.) maps. The origin of the latter lies in String theory. When investigating $\mathcal{U}(1)$ gauge theories, Seiberg and Witten discovered [35] that different regularization schemes, cut-off reg. and point-split reg., lead to different gauge transformations[10],

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda \quad \text{commutative} \\ \hat{\delta}\hat{A}_\mu &= \partial_\mu \hat{\Lambda} + \mathrm{i}\left[\hat{\Lambda} \stackrel{\star}{,} \hat{A}_\mu\right] \quad \text{noncommutative}\,. \end{aligned} \tag{1.68}$$

Consequently they argued that since physics should not depend on the regularization scheme applied, there must be a map from one gauge transformation to the other one, and hence from noncommutative to commutative gauge theory. Thus this map, the S.W. map, must fulfill the gauge equivalence condition

$$\hat{A}[A + \delta_\lambda A] = \hat{A}[A] + \hat{\delta}_\lambda \hat{A}[A]\,, \tag{1.69}$$

where δ_α denotes a commutative gauge transformation, and $\hat{\delta}_\alpha$ a noncommutative one. By using this relation and by furthermore assuming that the S.W. map is an expansion in the formal parameter θ, one can derive the expansion for the gauge field itself

$$\hat{A}_\mu[A] = A_\mu - \frac{\theta}{4}\Theta^{\tau\nu}\{A_\tau, \partial_\nu A_\mu F_{\nu\mu}\} + \ldots \tag{1.70}$$

Using this relation, one can formulate noncommutative gauge field theories in terms of ordinary commuting fields. Explicitly, the noncommutative corrections then come in the form of new vertices (with increasing number of legs per order), but suppressed by the noncommutative parameter θ which decreases the contribution from order to order.
Finally it shall also be mentioned that this procedure can also be generalized to non-abelian gauge theory [36, 37], even a full standard model has been established [38–41]. Currently the work in this direction goes very far, one is even hoping (depending on the real value of θ) to find first signs of noncommutativity at the LHC collider at CERN.
This closes the short overview on S.W. maps, in the next section we will continue with the alternative approach on treating NCQFT's, i.e. we will look at

1.5.3 Solutions to the UV/IR mixing problem

In scalar theory, solutions to the UV/IR mixing problem are known. On the one hand there is the Grosse-Wulkenhaar model, which kills the IR divergences by the introduction of an oscillator-like term. On the other hand, the action of Gurau *et al.* [42] has also been proven to be renormalizable to all orders. A counterterm of the form $\phi\frac{1}{p^2}\phi$ is introduced, which is able to take care of the IR-divergences, and hence we shall refer to the model as the $\frac{1}{p^2}$-model.
This subsection is mainly devoted to the latter, because the Grosse-Wulkenhaar model will be broadly discussed in Section 2.1. For completeness reasons it should be mentioned that a third model, put forward by Grosse and Vignes-Tourneret [43], is capable of treating the UV/IR mixing problem too. In this model the authors started with an

[10] Quantities with hat fulfill noncommutative algebras.

antisymmetric Theta matrix with rank 2, but in 4 dimensions, hence it has not full rank. In this setting, the oscillator term alone is not sufficient to cure the UV/IR mixing mixing problem, but by additionally introducing a quadratic nonlocal term in the action they were able to show full renormalizability to all orders of perturbation theory.

The $\frac{1}{p^2}$ model

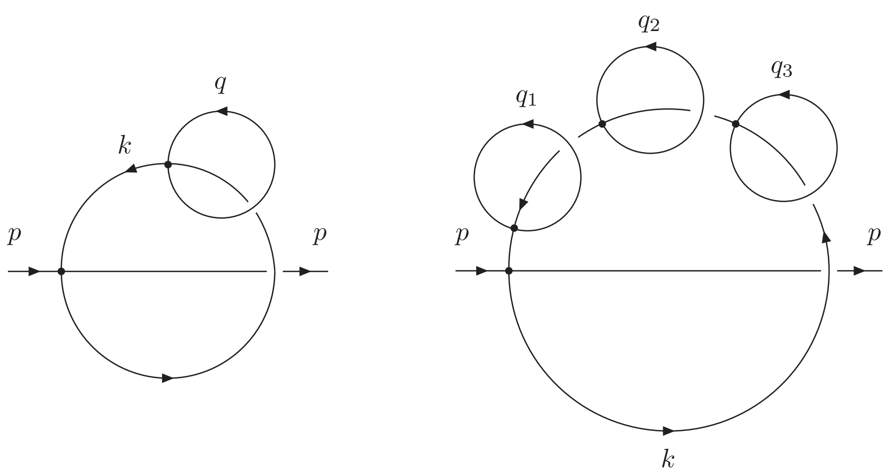
The action of the model is given by

$$S[\phi] = \int d^4p \left(\frac{1}{2} p_\mu \phi p^\mu \phi + \frac{1}{2}\mu^2 \phi\phi + \frac{1}{2} a \frac{1}{\theta^2 p^2} \phi\phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) , \tag{1.71}$$

where a is a positive dimensionless parameter. This action leads to the following propagator

$$\frac{1}{p^2 + m^2 + \frac{a}{\theta^2 p^2}} , \tag{1.72}$$

and one can already recognize from its form that taking the limit $p \to 0$ as well as $p \to \infty$ gives 0. This is the key feature of the model, which we will see will lead to a damping behavior at higher loops. The rough argument, which is explained in more detail in [44], will be explained in the following:
To one-loop order, the modified propagator doesn't change anything. The divergence is still of an $\frac{1}{\tilde{p}^2}$-type (see Expression 1.63). However, one-loop renormalization is never a problem (one can always add an appropriate counterterm to the action[11]), but higher loop renormalization indeed is. Explicitly, we will therefore take a look at a tadpole graph with n nonplanar insertions, depicted in Figure 1.6.



(a) with 1 insertion

(b) with 3 insertions

Figure 1.6: Non-planar 2-loop and 4-loop graphs.

[11]Which is in this case not even necessary, because $\int d^4k \frac{1}{k^2}$ is finite from a power-counting point of view anyway.

The mathematical expression for this Feynman graph is given by

$$\Pi^{n\ \text{npl-ins.}}(p) \equiv \lambda^2 \sum_{\eta=\pm 1} \int d^4k \, \frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{\left(\tilde{k}^2\right)^n \left[k^2 + m^2 + \frac{a^2}{k^2}\right]^{n+1}} \,, \tag{1.73}$$

where the $\frac{1}{\left(\tilde{k}^2\right)^n}$ factor comes from the n nonplanar insertions. For the naïve model (where $a = 0$), the integral (1.73) involves an IR divergence for $n \geq 2$, because the integrand behaves like $(k^2)^{-n}$ for $k^2 \to 0$. In contrast, for the model under consideration (where $a \neq 0$), the integrand behaves like

$$\frac{1}{\left(\tilde{k}^2\right)^n \left[\frac{a^2}{k^2}\right]^{n+1}} = \frac{\tilde{k}^2}{(a'^2)^{n+1}} \,. \tag{1.74}$$

Thus, the propagator (1.72) “damps” the IR-dangerous insertions and therefore cures potential IR problems in the integral (1.73). This is a nice demonstration of the mechanism leading to the renormalizability of the present model. In this respect we recall that its renormalizability has been proved in reference [42] using multiscale analysis.

Gauge versions of the $\frac{1}{p^2}$ model Several attempts of formulating gauge theories involving a $\frac{1}{p^2}$ term have been performed. I will present here step by step the essential ideas developed by our group.
The naïve generalization starting from scalar theory is given in reference [45]: A gauge invariant generalization of the $\phi\frac{1}{\square}\phi$ term is just

$$F^{\mu\nu} \star \frac{1}{D^2\widetilde{D}^2} \star F_{\mu\nu} \,. \tag{1.75}$$

However, the corresponding action suffers from an infinite number of gauge boson vertices coming from the expansion of the additional term (1.75). Fortunately one can cure this problem by the introduction of a multiplier field $B_{\mu\nu}$, leading to the following additional contribution to the action

$$S^{(\text{add})}_{\text{inv}} = \int d^4x \left[a' B_{\mu\nu} \star F_{\mu\nu} - B_{\mu\nu} \star \widetilde{D}^2 D^2 \star B_{\mu\nu}\right] \,, \tag{1.76}$$

see also [46]. Integrating the $B_{\mu\nu}$ field out leads back to (1.75). The main advantage is now that the number of vertices is only finite. In return, one has additionally the $B_{\mu\nu}$ field, which indeed has dynamical degrees of freedom. To avoid this problem, one should better guarantee that the additional field does not contribute to the Feynman rules. To achieve this in the present case, we have taken over a technique using BRST doublets from the Brazilian group Vilar *et al.* [47]. One introduces a BRST partner $\bar{\psi}_{\mu\nu}$ for the field $B_{\mu\nu}$ [48], and within the doublet structure it is guaranteed that all those fields are unphysical[12]. Thus one has achieved the $\frac{1}{p^2}$ damping behavior for the propagator like in the scalar case, but has no additional degrees of freedom.
Unfortunately the model turned out to have a from loop order to loop order increasing

[12]As an alternative one could for example as well introduce a BRST partner for the coupling constant of the nonlocal part of the action.

degree of divergence for certain graphs with external $B_{\mu\nu}$ field [49]. As a result of the discussion how to overcome these problems the so called BRSW model was born [50, 51]. It is much simpler than the former model because the damping is implemented only in the bilinear part. Therefore, no additional vertices than the standard ones are present, and hence no Feynman graphs involving the $B_{\mu\nu}$ field can be constructed. Thus the additional divergences encountered in the former model are simply not present. Furthermore, in the BRSW model the damping is implemented via a soft breaking term[13], which is in analogy to the Zwanziger solution [52] of the Gribov problem of QCD [53].
All in all the BRSW model seems to be a promising candidate for renormalization, and a respective proof using Multiscale Analysis is currently work in progress.

1.5.4 Renormalization schemes

We finish this introduction by touching the topic of renormalization, which is the holy grail of field theoreticians, because a renormalizable theory is a candidate of describing nature.
In the following, we will sketch some prominent renormalization schemes:

- **Algebraic Renormalization** (AR) is built around the idea that rather than calculating the necessary counterterms, all possible counterterms which the fields in the action permit are written down and ruled out by Ward identities (and other rules) exploiting the symmetry content of the theory. This relies on the so-called Quantum Action Principle. For a review of AR, we recommend [54].
  However, the situation is more complicated in noncommutative field theories, because the star product itself is nonlocal, and hence in principle an infinite number of counterterms is possible by introducing dimensionless expressions like $(\theta p^2)^n$. This situation is analyzed in detail in [55][56]. Fortunately power counting gets rid of the problem since it rules out contributions by constraining the number of powers of momenta.

- **Polchinski Approach** (PA) [57] is based on the work of Wilson [58]. In this approach the propagator is multiplied with a cutoff function depending on the scale[14]. The power counting degree of divergence of a graph then depends on the topological data of that graph and on the scale. In this way, suitable scaling dimensions provide a simple criterion to decide if a model has the chance of being renormalizable or not. In a next step scale transformations lead to flow equations connecting different loop graphs. These flow equations then need to be solved iteratively.

- **BPHZ** is not really a renormalization scheme itself. It does not provide a method of making an a priori infinite Feynman amplitude finite. However, when naively summing up the perturbation series (consisting of already renormalized finite contributions), subdivergences are not treated at all. In principle it can happen due to them that the series does not converge. But this is exactly what BPHZ is made

[13]The notion "soft" refers to the fact that the dimension of $\mathcal{L}_{\text{soft}}$ in $S_{\text{break}} = \int d^4x\gamma^2\mathcal{L}_{\text{soft}}$ is smaller than 4 (γ is a parameter of mass dimension 1). The notion "breaking" means that this term breaks BRST invariance, but as already discussed the latter can be restored by the introduction of BRST doublets.

[14]The scale idea due to Wilson is sketched in Appendix A.1.2.

to take care of. It rewrites the series using the so-called forest formula to be able to treat the subdivergences in a correct way. However, it still sums up too many finite contributions, leading to the so-called Renormalon problem. The latter is solved by Multiscale Analysis, described in the following.

- **Multiscale Analysis** (MA). In a first step within this procedure, propagators are "sliced", see Appendix A.1. By doing so, Feynman amplitudes can be up-bounded. Thus, the maximal degree of divergence for a graph of arbitrary loop order can be estimated. Hence all counterterms which could ever arise are known and when they are all qualitatively present in the initial action they can all in principle be absorbed into the constants.
  Furthermore, MA provides us with a scheme how to usefully sum up the perturbation series by classifying the graphs into dangerous and harmless ones. Instead of summing up the forests, like BPHZ does, the most economic forests are summed up. Further details can be found in [59][7].
  Concerning gauge theories, it is a fact that MA breaks gauge invariance (since it is dependent on a "scale"). However, by carefully introducing appropriate counterterms into the renormalized action, especially by respecting the magic formula relating the Z_i's, one is able to restore gauge invariance [60].

Chapter 2

Models with Oscillator Terms

In order to avoid the UV/IR mixing problem, several models which involve an oscillator like counterterm have been put forward. On the one hand such models explicitly break translation invariance (which does not necessarily need to be kept beyond the Standard model, that means when describing noncommutative photons[1]), but on the other hand they in general show a much better divergence behavior at higher loops or are even (in the case of the Grosse-Wulkenhaar model) proven to be renormalizable.
In the following we will present three such models.

2.1 The Grosse-Wulkenhaar Model

In 2004, the first renormalizable noncommutative scalar field model (in Euclidean $\mathbb{R}^4_\theta$) was introduced by H. Grosse and R. Wulkenhaar [61] (for a Minkowskian version see reference [62]). Their trick was to add a harmonic oscillator-like term to the scalar ϕ^4 action, i.e.:

$$\Gamma^0[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial_\mu\phi + \frac{\mu_0^2}{2}\phi \star \phi + \frac{\Omega^2}{2}(\tilde{x}\phi) \star (\tilde{x}\phi) + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right), \tag{2.1}$$

with $\tilde{x}_\mu = (\Theta_{\mu\nu})^{-1} x_\nu$ ($\Theta_{\mu\nu}$ constant and antisymmetric). This action cures the infamous UV/IR mixing problem. Indeed, for the bad IR-behavior found in the naïve model (triggered by the kinetic part of the action), the oscillator term acts as a sort of counter term. By exchanging $\tilde{x} \leftrightarrow p$ one can see that the action stays form invariant:

$$S[\phi; \mu_0, \lambda, \Omega] \mapsto \Omega^2 S[\phi; \frac{\mu_0}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]. \tag{2.2}$$

This symmetry is called Langman-Szabo duality [63] and at the self dual point, $\Omega = 1$, it is even exact.
The propagator of the model is the inverse of the operator $-\Delta + \Omega^2\tilde{x}^2 + \mu_0^2$ and is called the Mehler kernel. For the interested reader a detailed analysis of the Mehler kernel is performed in the next section. Here, we just state that the Mehler kernel is given by (in

[1]The author believes that there is probably some sort of phase transition from normal photons to noncommutative ones, when going down in scales towards the Planck length, but this is only speculation.

position space and 4 dimensions)

$$K_M(x,y) = \int_0^\infty d\alpha \, \frac{1}{8\pi^2\omega \sinh^2 \alpha} e^{-\frac{1}{4\omega}\left(u^2 \coth\frac{\alpha}{2} + v^2 \tanh\frac{\alpha}{2}\right) - \omega\mu_0^2\alpha}, \tag{2.3}$$

with $\omega = \frac{\theta}{\Omega}$, $u = x - y$ being a so-called *short variable* and $v = x + y$ being a *long variable.* This notation has been introduced by V. Rivasseau *et al.* [64]. They confirmed the renormalizability of the model by making use of a technique called Multiscale Analysis, additionally to the original renormalization proof of H. Grosse and R. Wulkenhaar which has been given in the matrix base employing the Polchinski approach.
Another beautiful aspect of the model can be illustrated by a quick glance at the beta function for λ, which we have taken from [65], see Fig. 2.1.

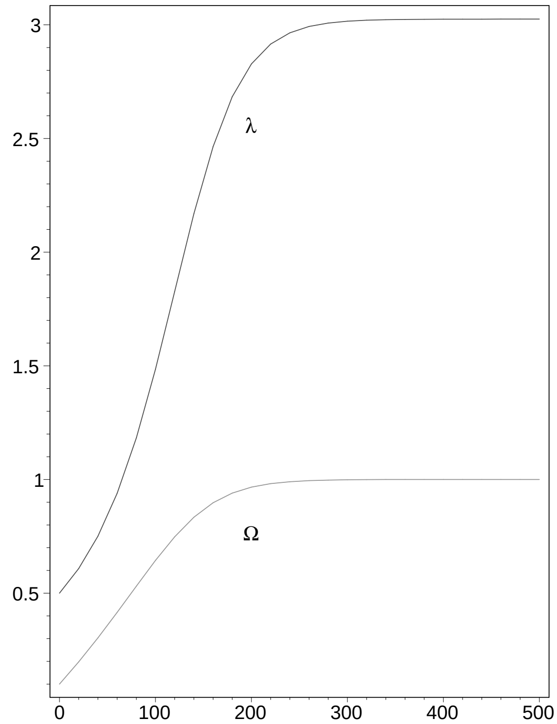


Figure 2.1: Beta function.

In contrast to the naïve scalar model (without oscillator term) the beta function becomes constant for high energies. Hence it does not diverge, and is therefore free of the Landau ghost problem [66–68].

2.1.1 The Mehler kernel

In this section we discuss some important properties of the Mehler kernel. For simplicity, the mass μ_0 is set to zero. Most generally, in position space and d dimensions the Mehler kernel is given by

$$K_M(x,y) = \int_0^\infty d\alpha \, \frac{\omega}{2(2\pi\omega \sinh \alpha)^{d/2}} e^{-\frac{1}{4\omega}\left((x-y)^2 \coth\frac{\alpha}{2} + (x+y)^2 \tanh\frac{\alpha}{2}\right)}. \tag{2.4}$$

In momentum space, it looks quite similar

$$K_M(p,q) = \int_0^\infty d\alpha \, \frac{\omega^{1+\frac{d}{2}}}{2(2\pi \sinh\alpha)^{d/2}} e^{-\frac{\omega}{4}\left((p-q)^2 \coth\frac{\alpha}{2} + (p+q)^2 \tanh\frac{\alpha}{2}\right)}, \tag{2.5}$$

which nicely illustrates the Langmann-Szabo duality.
We will in the following state some properties of the Mehler kernel. Several of them I have proven already in my diploma thesis [6]. For completeness reasons, I will however give a short summary:

- **Commutative limit**:
  In the limit $\Omega \to 0$ the Mehler kernel behaves like the ordinary propagator

  $$\lim_{\Omega\to 0} \int d^4p \, K_M(p,q) = \frac{1}{q^2}. \tag{2.6}$$

- **Kernel property**:
  One can see it best when renaming α to t (just for this paragraph), and defining

  $$K_M(x,y) = \int dt \, K_M(x,y,t). \tag{2.7}$$

  Exploiting the kernel property, which is

  $$\psi(x,t) = e^{iHt}\psi(x) = \int d^dy \, K_M(x,y,t)\psi(y), \tag{2.8}$$

  one can see that for $t \to 0$, the Mehler kernel must behave like a delta function, that is

  $$K_M(x,y,0) = \delta^{(d)}(x-y). \tag{2.9}$$

  One can see this also by directly taking the limit $\alpha \to 0$ in the explicit form of the Mehler kernel (2.4).

- **Bounds**:
  In the light of Multiscale Analysis, used by V. Rivasseau *et al.* to renormalize the G.W. model, the Mehler kernel can be sliced, $K_M(x-y, x+y) = \sum_{i=0}^{\infty} C^i$, with

  $$C^i(x-y,x+y) = \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha \, \frac{\omega}{2(2\pi\omega \sinh\alpha)^2} e^{-\frac{1}{\omega}\coth\left(\frac{\alpha}{2}\right)(x-y)^2 + \tanh\left(\frac{\alpha}{2}\right)(x+y)^2}. \tag{2.10}$$

  The zeroth slice usually gives only a finite contribution and the other slices can be bounded, namely for some constants K (large) and c (small) we get:

  $$C^i(x-y,x+y) \leq K M^{2i} e^{-c\left(M^i ||x-y|| + M^{-i}||x+y||\right)}. \tag{2.11}$$

  More mathematical details on Multiscale Analysis can be found in Appendix A.1.

In the following, I will list some other important properties of the Mehler kernel:

Damping behavior The Mehler kernel features a damping behavior for high momenta (UV) as well as for low momenta (IR). One can see this by comparison with the heat kernel, which is the inverse of $H_0 = -\Delta + \mu_0^2$ and has the form

$$H_0^{-1} = \int_0^\infty d\alpha \, \frac{1}{16\pi^2\alpha^2} e^{-\frac{(x-y)^2}{2\alpha} - \mu_0^2 \alpha} \, . \tag{2.12}$$

For $\mu_0 = 0$, one finds the well-known form of the undamped propagator after integrating over α and performing an expansion of the Bessel function

$$H_0^{-1} \simeq \frac{1}{4\pi^2 (x-y)^2} \, . \tag{2.13}$$

Equivalently, when setting $y = 0$ and $\mu_0 = 0$ in the Mehler kernel, one can perform the integration over the auxiliary Schwinger parameter and obtain

$$K_M(x) = \frac{e^{-\frac{x^2}{2\omega}}}{4\pi^2 x^2} \, , \tag{2.14}$$

which shows that the Mehler kernel has a much stronger convergence behavior for large values of x, corresponding to small values of p. However, the price to pay seems to be that translation invariance is broken, which can be seen directly in the action, because of the explicit x-dependence of the oscillator term $\tilde{x}^2\phi^2$. Recently it has been shown that this term can be interpreted as a coupling to the curvature of a noncommutative background space [69], giving it a nice geometrical interpretation.

Relation to the commutative propagator In a certain limit the Mehler kernel has a close relationship to the commutative propagator. To see this, we start at the Mehler kernel in momentum space (with for shortness reasons using the abbreviation of long and short variables, as already introduced in the previous section)

$$K_M(u,v) = \frac{\omega^3}{8\pi^2} \int_0^\infty d\alpha \frac{1}{\sinh^2 \alpha} \exp\left[-\frac{\omega}{4} u^2 \coth(\frac{\alpha}{2}) - \frac{\omega}{4} v^2 \tanh(\frac{\alpha}{2}) \right] . \tag{2.15}$$

We will now estimate the Mehler kernel for small parameters α. In zeroth order the Mehler kernel then becomes

$$K_M(u,v) = \frac{\omega^3}{8\pi^2} \int_0^\infty d\alpha \frac{1}{\alpha^2} \exp\left[-\frac{\omega}{4} u^2 \frac{2}{\alpha} - \frac{\omega}{4} v^2 \frac{\alpha}{2} \right] . \tag{2.16}$$

Using the Leibbrandt formula [70], page 368, formula 9

$$\int_0^\infty \frac{1}{\lambda^n} e^{-A\lambda - B\frac{1}{\lambda}} = 2 \left(\frac{A}{B} \right)^{\frac{n-1}{2}} K_{n-1}\left(2\sqrt{AB} \right) , \tag{2.17}$$

with K_{n-1} being a modified Bessel function of the second kind, one gets[2] for (2.16):

$$\frac{\omega^3}{8\pi^2}\sqrt{\frac{v^2}{u^2}}K_1\left(2\omega\sqrt{v^2u^2}\right)\,, \tag{2.18}$$

for $u^2 > 0$ and $v^2 \geq 0$. If the considered Mehler kernel is an external leg (like for example when calculating corrections to the propagator), we can expand the Bessel function into a series around small arguments[3]:

$$\frac{\omega^3}{8\pi^2}\sqrt{\frac{v^2}{u^2}}K_1\left(2\omega\sqrt{v^2u^2}\right) \approx \frac{\omega^2}{4\pi^2}\frac{1}{u^2} = \frac{\omega^2}{4\pi^2}\frac{1}{(k-k')^2}\,. \tag{2.19}$$

For $k' = 0$ this corresponds to the usual propagator in momentum space.

Parameter-independent form

We now want to find the form of the Mehler kernel when it is integrated out. In only 1 dimension, this is fully possible. In 2 or 4 dimensions, this can only be done when one of the two coordinates is set to zero.

1 Dimension The defining equation for the Mehler kernel reduces to

$$\begin{aligned}\left(-\frac{d^2}{d\bar{x}^2}+\frac{1}{\omega^2}\bar{x}^2\right)K_M(\bar{x},\bar{x}') &= \delta(\bar{x}-\bar{x}')\,,\\ \frac{1}{\sqrt{\omega}}\left(-\frac{d^2}{dx^2}+x^2\right)K_M(x,x') &= \delta(x-x')\,,\end{aligned} \tag{2.20}$$

where we have used dimensionless coordinates $\left(x = \frac{\bar{x}}{\sqrt{\omega}}\right)$. As a side remark we want to note here that this operator can be rewritten into Sturm-Liouville form:

$$L\,K_M(x,x') = \delta(x-x')\,, \tag{2.21}$$

$$L = \frac{1}{\sqrt{\omega}}\left(-\frac{d^2}{dx^2}+x^2\right) = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\,, \tag{2.22}$$

with $p(x) = \frac{1}{\sqrt{\omega}}$ and $q(x) = \frac{x^2}{\sqrt{\omega}}$.
In the following, we want to construct the solution of the inhomogeneous equation out of the solution of the homogeneous one. For this topic we follow the procedure described for example in the book of Arfken [71].
We will first solve the homogeneous equation

$$\left(-\frac{d^2}{dx^2}+x^2\right)K_M^{(0)}(x,x') = 0\,, \tag{2.23}$$

[2] In a theory without additional oscillator term the long variable v is zero and we are back at the heat kernel $K_M(u) = \frac{\omega^3}{8\pi^2}\int_0^\infty d\alpha \frac{1}{\alpha^2}\exp\left[-\frac{\omega}{4}u^2\frac{2}{\alpha}\right]$. However, integrating the latter out, we find the same result (2.19).

[3] If the Mehler kernel we treat here is describing internal momenta, we aren't allowed to do this because later on according to the Feynman rules one has to integrate over internal momenta, from $-\infty$ to $+\infty$, and we would make a significant error in this integration if we throw away the large momenta.

whose solution is

$$K_M^{(0)}(x,x') = C_1 D_{-\frac{1}{2}}\left(\sqrt{2}x\right) + C_2 D_{-\frac{1}{2}}\left(-\sqrt{2}x\right)\,, \tag{2.24}$$

where $D_{\pm\frac{1}{2}}$ are parabolic cylinder functions. The dependence on x' is hidden in the constants C_1 and C_2, which we will calculate in the following.
To include physical boundary conditions we require that $K_M(\infty, x') = K_M(-\infty, x') = 0$. By looking at the divergence behavior of the two linear independent parabolic cylinder functions we realize that to fulfill these boundary conditions we have to demand

$$K_M^{(0)}(x) = \begin{cases} G_1(x) = C_1 D_{-\frac{1}{2}}\left(-\sqrt{2}x\right) & \text{for} \quad x < t \\ G_2(x) = C_2 D_{-\frac{1}{2}}\left(\sqrt{2}x\right) & \text{for} \quad x > t\,, \end{cases} \tag{2.25}$$

with some arbitrary $t \in \mathbb{R}$. Additionally, in order to ensure the function to be continuous we have to require

$$\begin{aligned} G_2(x)\big|_t &= G_1(x)\big|_t\,, \\ C_2 D_{-\frac{1}{2}}\left(-\sqrt{2}x\right)\Big|_t &= C_1 D_{-\frac{1}{2}}\left(\sqrt{2}x\right)\Big|_t\,. \end{aligned} \tag{2.26}$$

Furthermore we demand that the derivations of our homogeneous solutions should be discontinuous at $x = t$, that is

$$\begin{aligned} \frac{d}{dx}G_2(x)\Big|_t - \frac{d}{dx}G_1(x)\Big|_t &= -\frac{1}{p(t)}\,, \\ \frac{d}{dx}C_2 D_{-\frac{1}{2}}\left(\sqrt{2}x\right) - \frac{d}{dx}C_1 D_{-\frac{1}{2}}\left(-\sqrt{2}x\right) &= -\sqrt{\omega}\,. \end{aligned} \tag{2.27}$$

Solving equations (2.26) and (2.27) allows us to determine the coefficients

$$C_1 \to \frac{\sqrt{\omega}e^{\frac{t^2}{2}} H_{-\frac{1}{2}}(t)}{\sqrt[4]{2}\left(H_{-\frac{1}{2}}(t)H_{\frac{1}{2}}(-t) + H_{-\frac{1}{2}}(-t)H_{\frac{1}{2}}(t)\right)}\,, \tag{2.28}$$

$$C_2 \to \frac{\sqrt{\omega}e^{\frac{t^2}{2}} H_{-\frac{1}{2}}(-t)}{\sqrt[4]{2}\left(H_{-\frac{1}{2}}(t)H_{\frac{1}{2}}(-t) + H_{-\frac{1}{2}}(-t)H_{\frac{1}{2}}(t)\right)}\,, \tag{2.29}$$

which become dependent on t by this way. However, by the above requirements, we can due to Arfken [71] conclude what the full solution is:

$$\begin{aligned} K_M(x,x') &= \int dt\, K_M^{(0)}(x,t) f(t,x') \\ &= \int dt\, K_M^{(0)}(x,t)\delta(x'-t) \\ &= K_M^{(0)}(x,x')\,. \end{aligned} \tag{2.30}$$

Putting everything together the complete solution is given by

$$K_M(x,x') = \begin{cases} \dfrac{\sqrt{\omega}e^{\frac{x'^2}{2}}H_{-\frac{1}{2}}(x')}{\sqrt[4]{2}\left(H_{-\frac{1}{2}}(x')H_{\frac{1}{2}}(-x') + H_{-\frac{1}{2}}(-x')H_{\frac{1}{2}}(x')\right)} D_{-\frac{1}{2}}\left(-\sqrt{2}x\right) & \text{for } x < x', \\ \dfrac{\sqrt{\omega}e^{\frac{x'^2}{2}}H_{-\frac{1}{2}}(-x')}{\sqrt[4]{2}\left(H_{-\frac{1}{2}}(x')H_{\frac{1}{2}}(-x') + H_{-\frac{1}{2}}(-x')H_{\frac{1}{2}}(x')\right)} D_{-\frac{1}{2}}\left(\sqrt{2}x\right) & \text{for } x > x'. \end{cases} \tag{2.31}$$

To illustrate that the result is correct we take a look at the special case $x' = 0$. Then the solution becomes

$$\begin{cases} \dfrac{\sqrt{\omega}\,\Gamma\left(\frac{1}{4}\right)}{2^{7/4}\sqrt{\pi}} D_{-\frac{1}{2}}\left(-\sqrt{2}x\right) & \text{for } x < 0, \\ \dfrac{\sqrt{\omega}\,\Gamma\left(\frac{1}{4}\right)}{2^{7/4}\sqrt{\pi}} D_{-\frac{1}{2}}\left(\sqrt{2}x\right). & \text{for } x > 0. \end{cases} \tag{2.32}$$

By using the absolute value of x we can write these two piecewise defined functions as one:

$$\frac{\sqrt{\omega}\,\Gamma\left(\frac{1}{4}\right)}{2^{7/4}\sqrt{\pi}} D_{-\frac{1}{2}}\left(\sqrt{2}|x|\right) = \frac{\sqrt{\omega}\,\Gamma\left(\frac{1}{4}\right)}{2^{7/4}\sqrt{\pi}} \sqrt{\frac{|x|}{\sqrt{2\pi}}} K_{1/4}\left(\frac{x^2}{2}\right) = \frac{\sqrt{\omega}\,\Gamma\left(\frac{1}{4}\right)}{4\pi} \sqrt{|x|} K_{1/4}\left(\frac{x^2}{2}\right), \tag{2.33}$$

and with the original coordinates $\bar{x} = x\sqrt{\omega}$ we end up at

$$\frac{\omega^{1/4}\,\Gamma\left(\frac{1}{4}\right)}{4\pi} \sqrt{|x|} K_{1/4}\left(\frac{x^2}{2\omega}\right). \tag{2.34}$$

This is exactly the Mehler kernel in 1 dimension when one integrates out α (for $x' = 0$), which will be proven in the next paragraph.

Higher Dimensions In higher dimensions (explicitly 2 and 4), we cannot use the method described above because e.g. in 2 dimensions one gets when solving the homogeneous problem 4 unknown constants, whereas one has only 2 matching conditions at hand.
Instead we will start from the already known form of the Mehler kernel

$$K_M(x,y) = \int_0^\infty d\alpha\, \frac{\omega}{2(2\pi\omega\sinh\alpha)^{d/2}} e^{-\frac{1}{4\omega}\left(u^2\coth\frac{\alpha}{2} + v^2\tanh\frac{\alpha}{2}\right)}, \tag{2.35}$$

where for simplicity we have set the mass equal to zero and $u = x - y$ as well as $v = x + y$, as usual. In general, this cannot be integrated, but when setting one variable equal to

zero, this is possible, which we will show in the following.
For $y = 0$, the Mehler kernel becomes

$$\begin{aligned} K_M(x,y) &= \int_0^\infty d\alpha \, \frac{\omega}{2(2\pi\omega \sinh\alpha)^{d/2}} e^{-\frac{1}{4\omega}\left(x^2 \coth\frac{\alpha}{2} + x^2 \tanh\frac{\alpha}{2}\right)} \\ &= \int_0^\infty d\alpha \, \frac{\omega}{2(2\pi\omega \sinh\alpha)^{d/2}} e^{-\frac{1}{2\omega}x^2 \coth\alpha} \, . \end{aligned} \tag{2.36}$$

By a substitution $\coth\alpha = \lambda$, with functional determinant $\frac{d\alpha}{d\lambda} = \frac{1}{1-\lambda^2}$, we get

$$\begin{aligned} &\frac{\omega}{2}\int_\infty^1 d\lambda \, \frac{1}{1-\lambda^2}\left(\frac{1}{2\pi\omega\sqrt{\lambda^2-1}}\right)^{d/2} e^{-\frac{1}{2\omega}x^2\lambda} \\ =&\frac{\omega}{2}\int_1^\infty d\lambda \, \frac{1}{(2\pi\omega)^{d/2}}\left(\lambda^2-1\right)^{\frac{d}{4}-1} e^{-\frac{1}{2\omega}x^2\lambda} \, , \end{aligned} \tag{2.37}$$

which can be integrated (e.g. by Mathematica®). In 1,2 and 4 dimensions, the result becomes rather simple, which one can summarize in a small table:

Dimension	Mehler kernel	Asymptotics for $x \ll 1$
1	$\frac{\omega^{1/4}\sqrt{\lvert x\rvert}}{4\pi} K_{\frac{1}{4}}\left(\frac{x^2}{2\omega}\right)\Gamma\left(\frac{1}{4}\right)$	$\sim$ const
2	$\frac{1}{4\pi} K_0\left(\frac{x^2}{2}\right)$	$\sim \ln x$
4	$\frac{e^{-\frac{x^2}{2\omega}}}{4\pi^2 x^2}$	$\sim \frac{1}{x^2}$

(2.38)

Hence, we have seen that it is indeed possible to integrate the Mehler kernel out, and that it takes a very nice form for $x' = 0$.

2.2 Extension to Gauge Theories

The aim is to obtain propagators for gauge models with a damping behavior similar to the Mehler kernel in the scalar case. Since an oscillator term $\Omega^2 \tilde{x}^2 A^2$ is not gauge invariant, there are more or less two possible ways to construct the model: either one adds further terms in order to make the action gauge invariant (which will be discussed in the following section) or one views the oscillator term as part of the gauge fixing. D. Blaschke, H. Grosse and M. Schweda put forward a model which follows the latter approach [72]. The action is given by[4]

$$\begin{aligned} \Gamma^{(0)} &= S_{\text{inv}} + S_{\text{m}} + S_{\text{gf}} \, , \\ S_{\text{inv}} &= \frac{1}{4}\int d^4x \, F_{\mu\nu} \star F_{\mu\nu} \, , \end{aligned}$$

[4]Throughout this section we will stick to 4-dimensional Euclidean space.

$$
\begin{aligned}
S_{\mathrm{m}} &= \frac{\Omega^2}{4}\int d^4x \left(\frac{1}{2}\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\star\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} + \{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\}\star\{\tilde{x}_\mu \stackrel{\star}{,} c\}\right) \\
&= \frac{\Omega^2}{8}\int d^4x \left(\tilde{x}_\mu \star \mathcal{C}_\mu\right) , \\
S_{\mathrm{gf}} &= \int d^4x \left[b\star\partial_\mu A_\mu - \frac{1}{2}b\star b - \bar{c}\star\partial_\mu s A_\mu - \frac{\Omega^2}{8}\,\widetilde{c}_\mu \star s\,\mathcal{C}_\mu\right] ,
\end{aligned}
\tag{2.39}
$$

with

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\left[A_\mu \stackrel{\star}{,} A_\nu\right] , \\
\mathcal{C}_\mu &= \Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} A_\nu\} + [\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \stackrel{\star}{,} c] + [\bar{c} \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} c\}]\Big) , \\
\tilde{x}_\mu &= \left(\Theta^{-1}\right)_{\mu\nu} x_\nu \,.
\end{aligned}
\tag{2.40}
$$

The gauge field A_μ transforms under the noncommutative generalization of a $U(1)$ gauge transformation which is infinite by construction of the noncommutative algebra. Once more, we denote the gauge group by $U_\star(1)$ in order to distinguish it from the commutative $U(1)$ gauge group. The multiplier field b implements a nonlinear gauge fixing[5]:

$$
\frac{\delta\Gamma^{(0)}}{\delta b} = \partial_\mu A_\mu - b + \frac{\Omega^2}{8}\Big([\{\tilde{x}_\mu \stackrel{\star}{,} c\} \stackrel{\star}{,} \widetilde{c}_\mu] - \{\tilde{x}_\mu \stackrel{\star}{,} [\widetilde{c}_\mu \stackrel{\star}{,} c]\}\Big) = 0\,. \tag{2.41}
$$

The field $\widetilde{c}_\mu$ is an additional multiplier field which guarantees the BRST-invariance of the action. The BRST-transformations are given by

$$
\begin{aligned}
sA_\mu &= D_\mu c = \partial_\mu c - \mathrm{i}g\left[A_\mu \stackrel{\star}{,} c\right] , & s\bar{c} &= b, \\
sc &= \mathrm{i}gc\star c, & sb &= 0, \\
s\widetilde{c}_\mu &= \tilde{x}_\mu, & s\tilde{x}_\mu &= 0 \\
s^2\varphi &= 0\ \forall\ \varphi \in \{A_\mu, b, c, \bar{c}, \widetilde{c}_\mu\} \ ,
\end{aligned}
\tag{2.42}
$$

Since $\widetilde{c}_\mu$ transforms into $\tilde{x}_\mu$, the part of the action including the Lagrange-multiplier field $\widetilde{c}_\mu$ exactly cancels with S_m under the application of the BRST-operator s onto the whole action. S_{inv} is BRST invariant anyway because it is even gauge invariant (hence the name S_{inv}), and the gauge fixing part that is left over is also easily seen to be BRST invariant, thus the full action fulfills this property, as it should. BRST invariance is generally a good feature to impose on gauge models because then the positive definiteness of the Hilbert space is guaranteed. Moreover, it kind of restores the old gauge invariance which has proven to be the correct way to describe real photons.
With these BRST transformations the action (2.39) can be written in the following beautiful form:

$$
\Gamma^{(0)} = \int d^4x \left(\frac{1}{4}F_{\mu\nu}\star F_{\mu\nu} + s\left(\frac{\Omega^2}{8}\tilde{c}_\mu\star\mathcal{C}_\mu + \bar{c}\star\partial_\mu A_\mu - \frac{1}{2}\bar{c}\star b\right)\right) . \tag{2.43}
$$

[5]Notice, that in the limit $\Omega \to 0$ this becomes the Feynman gauge.

2.2.1 Feynman rules

The action is full of stars and it is a priori not clear why they should all disappear in the bilinear part, since $\tilde{x}_\mu$ terms are involved. However, when we assume $\Theta_{\mu\nu}$ to be antisymmetric and constant, i.e.

$$(\Theta_{\mu\nu}) = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \tag{2.44}$$

as defined in Section 1.4.1, the following property holds:

$$\{A_\mu \stackrel{\star}{,} \tilde{x}_\mu\} = 2\tilde{x}_\mu A_\mu \,, \tag{2.45}$$

which can be directly verified by inserting the definition of the star product (1.32). It is therefore possible to reduce the bilinear parts of the action to one single star. The latter can be removed by exploiting the cyclic permutation property of the star product (1.35), and therefore the noninteracting part of the action is the same as in an undeformed model. Hence the propagators are more or less just the Mehler kernels, like in the scalar case (but without mass). In momentum space they are given by

$$\begin{aligned} G^{AA}_{\mu\nu}(p,q) &= (2\pi)^4 \widetilde{K}_M(p,q)\, \delta_{\mu\nu} \,, \\ G^{\bar{c}c}(p,q) &= (2\pi)^4 \widetilde{K}_M(p,q) \,, \end{aligned} \tag{2.46}$$

with the Mehler kernel in momentum representation

$$\widetilde{K}_M(p,q) = \frac{\omega^3}{8\pi^2} \int_0^\infty d\alpha \frac{1}{\sinh^2\alpha} e^{-\frac{\omega}{4}(p-q)^2 \coth\frac{\alpha}{2} - \frac{\omega}{4}(p+q)^2 \tanh\frac{\alpha}{2}} \,. \tag{2.47}$$

It has when compared to (2.3) a similar mathematical form, which is nothing else but an expression of the Langman-Szabo duality [63].

The $\widetilde{c}bc$-vertex involving the multiplier field $\widetilde{c}_\mu$ does not contribute to Feynman diagrams since a propagator connecting to that field does not exist. Similarly, a propagator does exist for b, but the only vertex connecting to it is the $\widetilde{c}bc$ vertex, which, as explained, does not contribute to loop diagrams. Hence, we will omit the related Feynman rules.

The vertices following from the action are just the usual noncommutative ones, as can be found for example in [19]. Equipped with the complete Feynman rules we can start deriving a power counting formula to estimate the worst degree of divergence of our graphs, which via UV/IR mixing is directly related to the degree of noncommutative IR divergence. A detailed derivation is given in Appendix A.2. Given the number of external legs for the various fields (denoted by $E_\varphi, \forall\ \varphi \in \{A_\mu, b, c, \bar{c}, \widetilde{c}_\mu\}$) the degree of UV divergence for an arbitrary graph in 4 dimensional space can be up-bounded by

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - E_{\widetilde{c}} \,. \tag{2.48}$$

This bound, however, represents merely a crude estimate. The true degree of divergence can (for certain graphs) be improved by BRST invariance. For example, for the one-loop boson self-energy graphs the power counting formula would predict at most a quadratic divergence, but gauge invariance usually reduces the sum of those graphs to be only logarithmically divergent. In our case we will show, however, that this does not happen due to a violation of translation invariance. The corresponding Ward identity will be worked out more explicitly in the next subsection.

2.2.2 Symmetries

In this subsection, we will take a closer look at the Ward identities (describing transversality) and the Slavnov-Taylor identities (describing BRST invariance). Every symmetry in general implies a conservation operator that gives zero when applied to the action. In the case of the BRST symmetry this is s. Regarding s as a total derivation of $\Gamma^{(0)}$ we can write

$$
\begin{aligned}
&s\Gamma^{(0)}[A_\mu, b, c, \bar{c}, \widetilde{c}_\mu] \\
&= \int d^4x \left(sA_\mu \star \frac{\delta\Gamma^{(0)}}{\delta A_\mu} + sb \star \frac{\delta\Gamma^{(0)}}{\delta b} + sc \star \frac{\delta\Gamma^{(0)}}{\delta c} + s\bar{c} \star \frac{\delta\Gamma^{(0)}}{\delta\bar{c}} + s\widetilde{c}_\mu \star \frac{\delta\Gamma^{(0)}}{\delta\widetilde{c}_\mu} \right) . \qquad (2.49)
\end{aligned}
$$

By introducing external sources ρ_μ and σ for sA_μ and sc, respectively

$$
\begin{aligned}
\Gamma &= \Gamma^{(0)} + \Gamma_{\text{ext}} , \\
\Gamma_{\text{ext}} &= \int d^4x \left(\rho_\mu \star sA_\mu + \sigma \star sc\right) , \qquad (2.50)
\end{aligned}
$$

and making use of (2.42) we can write the Slavnov-Taylor identity in a more convenient form:

$$
\mathrm{S}(\Gamma) = \int d^4x \left(\frac{\delta\Gamma}{\delta\rho_\mu} \star \frac{\delta\Gamma}{\delta A_\mu} + \frac{\delta\Gamma}{\delta\sigma} \star \frac{\delta\Gamma}{\delta c} + b \star \frac{\delta\Gamma}{\delta\bar{c}} + \tilde{x}_\mu \star \frac{\delta\Gamma}{\delta\tilde{c}_\mu} \right) = 0 . \qquad (2.51)
$$

To arrive now at the Ward identity describing transversality, one has to take as usual the functional derivative of the Slavnov-Taylor identity with respect to A_ρ and c and then one has to set all fields equal to zero. One immediately recognizes that the $\tilde{x}_\mu$-term which originates from the oscillator term in the action gives an additional contribution. The usual transversality is explicitly broken:

$$
\partial^z_\mu \frac{\delta^2\Gamma}{\delta A_\rho(y)\delta A_\mu(z)} = \int d^4x \left(\tilde{x}_\mu \frac{\delta^{(3)}\Gamma}{\delta c(z)\delta A_\rho(y)\delta\tilde{c}_\mu(x)} \right) \neq 0 . \qquad (2.52)
$$

The calculation leading to the explicit expression for the transversality breaking can be found in Appendix A.3.
Graphically, relation (2.52) can be depicted as shown in Fig. 2.2.

2.2.3 Loop calculations

In this section, the reader may find a more detailed version of the 1-loop calculations already presented in [73]:

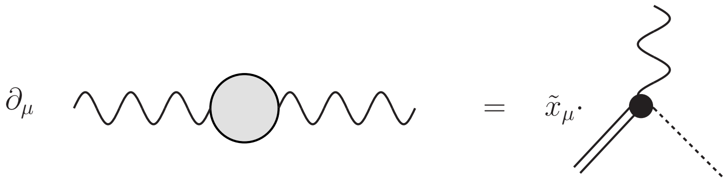


Figure 2.2: Ward identity replacing transversality.

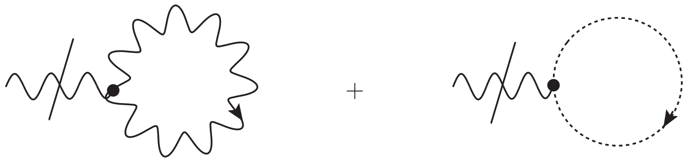


Figure 2.3: Tadpole graphs.

The simplest graphs one may construct are the (one-point) tadpoles, consisting of just one vertex and one internal propagator. They consist of two graphs which are depicted in Fig. 2.3. According to the Feynman rules, their sum is straightforwardly given by

$$\Pi_\mu(p) = 2\mathrm{i}g \int d^4k \int d^4k' \delta^4 \left(p + k' - k\right) \sin\left(\frac{k\tilde{p}}{2}\right) K_M(k,k') \left[2k_\mu + 3k'_\mu\right] . \tag{2.53}$$

We may now transform to "long and short" variables

$$u = k - k', \quad v = k + k' \quad \Rightarrow \quad k = \frac{v+u}{2}, \quad k' = \frac{v-u}{2}, \tag{2.54}$$

with functional determinant $\frac{1}{16}$. Moreover, since we want to rise everything up to the exponent in order to have simple Gaussian integrals, we make use of

$$\sin\left(\frac{k\tilde{p}}{2}\right) = \sum_{\eta=\pm1} \frac{\eta}{2\mathrm{i}} \exp\left(\frac{\mathrm{i}\eta}{2} k\tilde{p}\right) , \tag{2.55}$$

and plug in the explicit expression for the Mehler kernel (2.47). Altogether this leads to

$$\begin{aligned}\Pi^\varepsilon_\mu(p) &= \frac{g\omega^3}{2^8\pi^2} \sum_{\eta=\pm1} \int d^4v \left[5v_\mu - p_\mu\right] \int_\varepsilon^\infty d\alpha \frac{\eta e^{\frac{\mathrm{i}\eta}{4} v\tilde{p}}}{\sinh^2\alpha} \exp\left(-\tfrac{\omega}{4}\left[\coth\left(\tfrac{\alpha}{2}\right) p^2 + \tanh\left(\tfrac{\alpha}{2}\right) v^2\right]\right) \\ &= \frac{5\mathrm{i}g\tilde{p}_\mu}{64} \int_\varepsilon^\infty d\alpha \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh^5\left(\frac{\alpha}{2}\right)} \exp\left[-\frac{1}{4}\coth\left(\frac{\alpha}{2}\right)\left(\omega + \frac{\theta^2}{4\omega}\right)p^2\right] ,\end{aligned} \tag{2.56}$$

where in the last step the Gaussian integral has been solved and trigonometric identities have been used. Furthermore, we have introduced a cutoff $\varepsilon = 1/\Lambda^2$ which regularizes the integral.

Naïvely, one could simply integrate out α and discover a divergence structure of higher degree than expected, since it still contains a "smeared out" delta function. To make this clear, consider the usual commutative propagator, which depends on a second momentum only through a delta function, i.e. $G(k,k') \propto G(k)\delta^4(k-k')$. In the present case, due to the breaking of translational invariance, the delta function is replaced by something which might be described by a smeared out delta function, which is contained in the Mehler kernel, and hence one cannot simply split that part off. However, by integrating over one external momentum one can extract the divergence one is actually interested in. In some sense one can interpret this procedure as an expansion around the usual momentum conservation. This is the general procedure we will use to calculate the Feynman graphs. The 1-point tadpoles however are an exception: since they have only one external momentum, integrating the latter out would equally mean to set $p=0$. (One can see this by noticing that the integrand is antisymmetric in p, and the integration over the symmetric interval from $-\infty$ to ∞ would thus give zero.) With this procedure we would just hide the divergences. In conclusion, one can state that the integration over an external momentum is applicable for graphs with more than one external leg.

For the 1-point graphs, we use the trick of coupling an external field to the graph and expanding it around $p=0$:

$$\int \frac{d^4p}{(2\pi)^4}\,\Pi^\varepsilon_\mu(p)\left[A_\mu(0)+p_\nu\left(\partial^p_\nu A_\mu(p)\big|_{p=0}\right)+\frac{p_\nu p_\rho}{2}\left(\partial^p_\nu\partial^p_\rho A_\mu(p)\big|_{p=0}\right)+\right.$$
$$\left.+\frac{p_\nu p_\rho p_\sigma}{6}\left(\partial^p_\nu\partial^p_\rho\partial^p_\sigma A_\mu(p)\big|_{p=0}\right)+\dots\right]. \tag{2.57}$$

After smearing out the graph by coupling it to an external field, an integration over p is allowed. All terms of even order are zero for symmetry reasons. Of the other terms, we now show that only the first two, namely orders 1 and 3, diverge in the limit $\varepsilon\to 0$:

- *order 1:*

$$\int \frac{d^4p}{(2\pi)^4}\,p_\nu\Pi^\varepsilon_\mu(p)=\int\limits_\varepsilon^\infty d\alpha\frac{5ig\theta_{\mu\nu}}{32\pi^2\omega^3\left(1+\frac{\Omega^2}{4}\right)^3\sinh^2(\alpha)}$$
$$=\frac{5ig\theta_{\mu\nu}}{32\pi^2\omega^3\left(1+\frac{\Omega^2}{4}\right)^3}\left[\frac{1}{\varepsilon}-1+\mathcal{O}(\varepsilon)\right]. \tag{2.58}$$

With the external field, we obtain a counter term of the form

$$\left(\partial^p_\nu A_\mu(p)\big|_{p=0}\right)\int\frac{d^4p}{(2\pi)^4}\,p_\nu\Pi^\varepsilon_\mu(p)=$$
$$=\frac{5g\Omega^2}{32\pi^2\omega\left(1+\frac{\Omega^2}{4}\right)^3}\left[\frac{1}{\varepsilon}-1+\mathcal{O}(\varepsilon)\right]\int d^4x\,\tilde{x}_\mu A_\mu(x)\,. \tag{2.59}$$

- *order 3:*

$$\int\frac{d^4p}{(2\pi)^4}\,\frac{p_\alpha p_\beta p_\gamma}{6}\Pi^\varepsilon_\mu(p)=\frac{-5ig\left(\delta_{\alpha\beta}\theta_{\mu\gamma}+\delta_{\beta\gamma}\theta_{\mu\alpha}+\delta_{\alpha\gamma}\theta_{\mu\beta}\right)}{24\pi^2\omega^4\left(1+\frac{\Omega^2}{4}\right)^4}\left[\ln\varepsilon+\mathcal{O}(0)\right], \tag{2.60}$$

and with the external field we get the counter term

$$\left(\partial^p_\alpha\partial^p_\beta\partial^p_\gamma A_\mu(p)\big|_{p=0}\right)\int\frac{d^4p}{(2\pi)^4}\,\frac{p_\alpha p_\beta p_\gamma}{6}\Pi^\varepsilon_\mu(p) =$$

$$= \frac{5g}{8\pi^2}\frac{\Omega^4}{\left(1+\frac{\Omega^2}{4}\right)^4}\left[\ln\varepsilon+\mathcal{O}(0)\right]\int d^4x\,\tilde{x}_\mu\tilde{x}^2A_\mu(x)\,. \qquad (2.61)$$

- *order 5 and higher:*
These orders are *finite*. The contribution to order $5+2n$, $n\geq 0$ is proportional to

$$\int_0^\infty d\alpha\frac{\sinh^n\frac{\alpha}{2}}{\cosh^{n+4}\frac{\alpha}{2}} = \frac{4}{(n+1)(n+3)}\,. \qquad (2.62)$$

Notice, that all tadpole contributions vanish in the limit $\Omega\to 0$ as expected. However when $\Omega\neq 0$ the unphysical tadpole contributions are nonzero. Since this can certainly not describe nature, we must have started with a wrong vacuum. Furthermore, since we get additional counter terms of mathematical structure which were not initially present in the original action, we certainly need a new theory. Apparently this is the case here because Eqn. (2.59) and (2.61) reveal counter terms linear in A_μ. Ultimately this means that we will have to consider a whole new model, which will be the induced gauge theory, but more on that in Section 2.3.

2.2.4 Two-point functions at one loop level

Here we analyze the divergence structure of the gauge boson self-energy at one-loop level. The relevant graphs are depicted in Fig. 2.4.

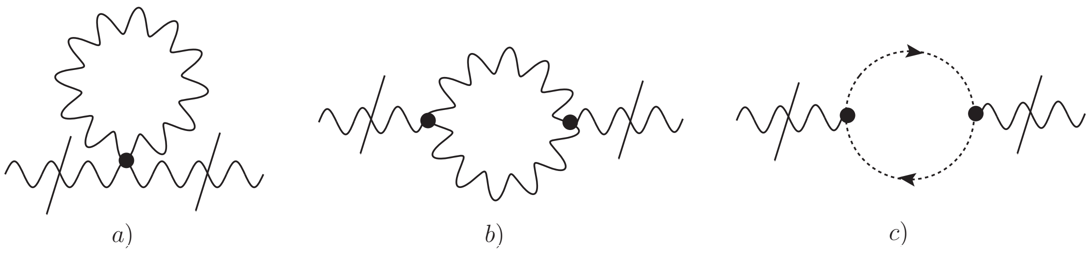


Figure 2.4: Gauge boson self-energy — amputated graphs.

As explained in the previous paragraph, we do not need to couple an external field and expand around it in this case. The notion of long and short variables has proven to be very useful and we will use it again here. In Appendix A.4.1 the graphs are calculated. In this form, we can easily sum up all three graphs a), b) and c). The sum yields the final result

$$\Pi^{\text{div}}_{\mu\nu}(p) = \frac{g^2\delta_{\mu\nu}\left(1-\frac{3}{4}\Omega^2\right)}{4\pi^2\omega\,\varepsilon\left(1+\frac{\Omega^2}{4}\right)^3}+\frac{3g^2\delta_{\mu\nu}\Omega^2}{8\pi^2\tilde{p}^2\left(1+\frac{\Omega^2}{4}\right)^2}+\frac{2g^2\tilde{p}_\mu\tilde{p}_\nu}{\pi^2(\tilde{p}^2)^2\left(1+\frac{\Omega^2}{4}\right)^2}$$
$$+\text{ logarithmic UV divergence}\,. \qquad (2.63)$$

In the limit $\Omega \to 0$ (i.e. $\omega \to \infty$), this expression reduces to the usual transversal result

$$\lim_{\Omega\to 0} \Pi^{\text{div}}_{\mu\nu}(p) = \frac{2g^2}{\pi^2} \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} + \text{logarithmic UV divergence}\,, \tag{2.64}$$

which is quadratically IR divergent[6] in the external momentum p and logarithmically UV divergent. The single graphs (A.50), (A.58), however, do not show this behavior, only the sum of all 3 graphs is transversal in the limit $\Omega \to 0$. When not taking this limit we can see from the general result (2.63) that not only transversality is broken due to the first two terms, but also that it has an ultraviolet divergence parameterized by ε, whose degree of divergence is higher compared to the (commutative) gauge model without oscillator term. Both properties are due to the term S_{m} in the action which breaks gauge invariance (cf. (2.52)).

2.2.5 Vertex corrections at one-loop level

Corrections to the 3-point vertex

Due to the vast amount of terms that arise when calculating the graphs depicted in Figure 2.5 it is practicable to use a computer. In fact, we taught Mathematica® to

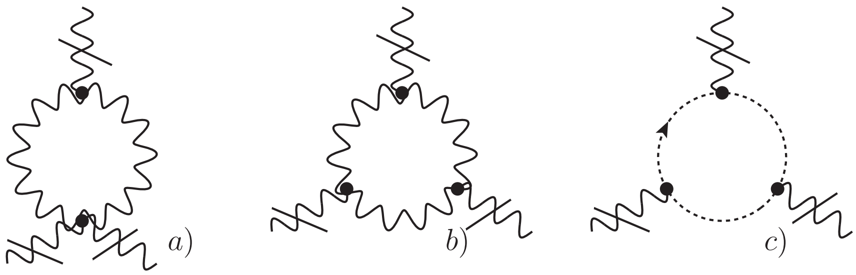


Figure 2.5: One loop corrections to the 3A-vertex.

perform the same steps as we would have done by hand. A short description of the functions used can be found in Appendix B.
Some important steps and calculation techniques, as well as a clear distinction into planar and nonplanar part can be found in Appendix A.4.2.
Performing the same steps as in the previous chapter the calculation gives as a sum of all 3 graphs

$$V^{3\text{A},\text{IR}}_{\mu\nu\rho}(p_1,p_2,p_3) = \frac{-8\text{i}g^3}{\pi^2\,(4+\Omega^2)^3} \sum_{i=1}^{3} \left[\frac{16\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{\tilde{p}_i^4} + \frac{3\Omega^2}{\tilde{p}_i^2} \left(\delta_{\mu\nu}\tilde{p}_{i,\rho} + \delta_{\mu\rho}\tilde{p}_{i,\nu} + \delta_{\nu\rho}\tilde{p}_{i,\mu}\right) \right], \tag{2.65}$$

which is linearly divergent.
Once more, this expression has additional terms due to the nonvanishing oscillator term

[6] In fact, this term is consistent with previous results [18, 24, 25] calculated in the naïve model, i.e. without any additional x-dependent terms in the action.

parameterized by Ω. However, in the limit $\Omega \to 0$ the usual result known from the literature is recovered[7] [13, 24, 27]:

$$\lim_{\Omega\to 0} V^{3\mathrm{A,IR}}_{\mu\nu\rho}(p_1,p_2,p_3) = \frac{-2\mathrm{i}g^3}{\pi^2}\sum_{i=1}^{3}\left[\frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{\tilde{p}_i^4}\right]. \tag{2.66}$$

In the ultraviolet, the graphs of Fig. 2.5 diverge only logarithmically.

Corrections to the 4-Photon Vertex

We essentially calculated the bubble graph (Fig. 2.6) as a representative of the 4 possible 4-pt. graphs.

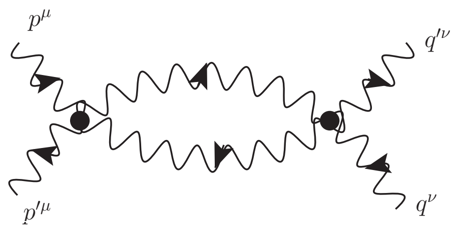


Figure 2.6: The bubble graph

It shows logarithmic behavior both in the ultraviolet as well as in the infrared sector, as expected from the power counting (2.48).

2.3 Induced Gauge Theory

Since in the previous section it has been shown that additional counter terms arise which were not present in the original action it is natural to start with an action that has those terms already built in, instead. Such an action is the "induced gauge theory" of Refs. [74, 75]. Its major advantage is that it is, by construction, completely gauge invariant.

2.3.1 Derivation from the G. W. model

Let us review how the induced action is derived.

One starts with the Grosse-Wulkenhaar model (2.1):

$$\Gamma^{(0)}[\phi] = \int d^4x \left(\frac{1}{2}\phi\star[\tilde{x}_\nu \stackrel{\star}{,} [\tilde{x}_\nu \stackrel{\star}{,} \phi]] + \frac{\Omega^2}{2}\phi\star\{\tilde{x}_\nu \stackrel{\star}{,} \{\tilde{x}_\nu \stackrel{\star}{,} \phi\}\} - \frac{\mu^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right)(x)\,, \tag{2.67}$$

where, in order to write the action in the previous form, the following important property has been used:

$$[\tilde{x}_\mu \stackrel{\star}{,} \phi] = \mathrm{i}\partial_\mu\phi\,. \tag{2.68}$$

[7] As in the case of the propagator corrections, the single graphs don't show this behavior, only the sum has the usual tensor structure in the limit $\Omega \to 0$, see Appendix A.4.2.

Now, one introduces external gauge fields by generalizing the ordinary coordinates x_μ to covariant ones[8] $\widetilde{X}_\mu$, with

$$\widetilde{X}_\mu = \tilde{x}_\mu + gA_\mu \,. \tag{2.69}$$

These coordinates have the nice property that they gauge transform covariantly, which is why they are named likewise. Therefore, the following action is gauge invariant by construction:

$$\int d^4x \left(\frac{1}{2}\phi \star \left[\widetilde{X}_\nu \stackrel{\star}{,} \left[\widetilde{X}_\nu \stackrel{\star}{,} \phi\right]\right] + \frac{\Omega^2}{2}\phi \star \left\{\widetilde{X}_\nu \stackrel{\star}{,} \left\{\widetilde{X}_\nu \stackrel{\star}{,} \phi\right\}\right\} \right.$$
$$\left. - \frac{\mu^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\right)(x) \,. \tag{2.70}$$

It can be shown either by performing a heat kernel expansion [75], or by explicit loop calculations [74] that to one loop order the action becomes

$$\Gamma^{(1l)}[A_\mu] = \int d^4x \left\{ \frac{3}{\theta}\left(1-\rho^2\right)\left(\tilde{\mu}^2 - \rho^2\right)\left(\widetilde{X}_\nu \star \widetilde{X}_\nu - \tilde{x}^2\right) \right.$$
$$\left. + \frac{3}{2}\left(1-\rho^2\right)^2 \left(\left(\widetilde{X}_\mu \star \widetilde{X}_\mu\right)^{\star 2} - \left(\tilde{x}^2\right)^2\right) + \frac{\rho^4}{4} F_{\mu\nu} \star F_{\mu\nu} \right\}, \tag{2.71}$$

where

$$\rho = \frac{1-\Omega^2}{1+\Omega^2} \,, \qquad \tilde{\mu}^2 = \frac{\mu^2\theta}{1+\Omega^2} \,. \tag{2.72}$$

Notice also, that the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\left[A_\mu \stackrel{\star}{,} A_\nu\right]$ can be written in terms of the covariant coordinates as

$$\mathrm{i}\left[\widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu\right] = \theta^{-1}_{\mu\nu} - gF_{\mu\nu} \,. \tag{2.73}$$

The field ϕ has been integrated out in order to arrive at the effective action (2.71), and A_μ has been considered as a background field. However, through its coupling to A_μ, the scalar field "induces" the effective one-loop action (2.71) above and A_μ becomes dynamical.

As already mentioned in Section 2.2, all (UV-divergent) terms that arise in the loop calculations of the previous model are present in the induced action. Hence, the chance that any unexpected new contributions arise during loop calculations is improbable, especially since the whole action is gauge invariant. This gives good hope concerning the renormalizability of the model. However, the problem that the tadpole graphs do not vanish, and that we therefore have a nontrivial vacuum, is still present. Furthermore, calculating the propagator of the induced gauge theory is a difficult task since the operator which has to be inverted is nonminimal (i.e. no Lorentz scalar). Additionally, calculating the propagator from the pure bilinear part seems not to be sufficient because, as already mentioned, linear (tadpole) terms in A_μ are also present in the action. All those severe problems are addressed in the following sections.

[8]Notice the slight difference to the Θ-expanded case where one usually introduces covariant coordinates without tilde [76].

2.3.2 Formulation in a way that the tadpole terms cancel

One may view (2.71) as a starting point for the new induced action. All prefactors and constants, as well as the gauge fixing, may however be chosen individually. In this section, we will present a gauge fixing which allows us to cancel the unphysical tadpole terms. Another nice effect of this gauge will be[9] that the photon propagator will become just the Mehler kernel, from which we already know that it has a nice damping behavior (see also Section 2.1.1). The induced action most generally consists of terms involving the commutator and the anticommutator of the covariant coordinates $\widetilde{X}_\mu = \tilde{x}_\mu + gA_\mu$, and is given by

$$S_{\text{inv}} = \int d^4x \left[-\frac{1}{4g^2} \left[\widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu \right] \star \left[\widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu \right] + \frac{\Omega^2}{4g^2} \left\{ \widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu \right\} \star \left\{ \widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu \right\} \right] . \quad (2.74)$$

At first glance one can already see that the action is subject to a symmetry exchanging the commutator and the anticommutator, and for $\Omega = 1$, this symmetry is exact. Indeed, this "extended Langmann-Szabo duality" has been investigated by J.C. Wallet, A. de Goursac and T. Masson [77].

Furthermore, all terms which turned out to be missing in the Mehler kernel gauge model (described in Section 2.2) are present here. However, (2.74) still contains so-called tadpole terms, i.e. terms linear in A_μ. We present in the following a gauge fixing designed to get rid of these terms. Additionally, we use (2.73) and leave out the constant term $\left(-\frac{\theta^4}{4}\right)$ because it can be absorbed into the normalization constant of the path integral. Then the complete action becomes

$$S = S_{\text{inv}} + S_{\text{FP}} + S_{\text{gf}} \,, \quad (2.75)$$

where the individual parts are given by

$$S_{\text{inv}} = \int d^dx \, \frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega^2}{4g^2} \left\{ \widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu \right\} \star \left\{ \widetilde{X}_\mu \stackrel{\star}{,} \widetilde{X}_\nu \right\} , \quad (2.76)$$

$$S_{\text{FP}} = \int d^dx \, \frac{\mathrm{i}}{2} \left\{ \bar{c} \stackrel{\star}{,} \tilde{x}_\mu \right\} \star D_\mu c \,, \quad (2.77)$$

$$S_{\text{gf}} = \int d^dx \left(-\frac{1}{8\alpha} \right) \left(\left\{ \tilde{x}_\mu \stackrel{\star}{,} A_\mu \right\} + \beta \left\{ \tilde{x}_\mu \stackrel{\star}{,} \tilde{x}_\mu \right\} \right)^{\star 2} . \quad (2.78)$$

The gauge fixing is some kind of modified Fock-Schwinger gauge [78]. The BRS-transformations are given by

$$s\bar{c} = -\frac{1}{2\alpha} \mathrm{i} \left(\left\{ \tilde{x}_\mu \stackrel{\star}{,} A_\mu \right\} + \beta \left\{ \tilde{x}_\mu \stackrel{\star}{,} \tilde{x}_\mu \right\} \right) , \quad sc = \mathrm{i} c \star c \,, \quad s\tilde{x}_\mu = 0 \,, \quad sA_\mu = D_\mu c \,, \quad (2.79)$$

where α and β are constant parameters, and applying s onto the whole action gives zero, as it should.

The coefficients α, Ω will in the following be chosen such that we get a simple form of our propagators. Additionally, β is chosen such that the tadpole terms, i.e. the ones linear in A_μ, vanish. That means we have to choose $\beta = 4\alpha \frac{\Omega^2}{g}$.

[9] Together with choosing $\Omega = 1$.

Varying the action leads to the e.o.m. (see appendix C.1)

$$\frac{\delta S^{\text{bil}}}{\delta A_\mu}: \quad \left(-\Box\delta_{\mu\nu} + (1-\Omega^2)\partial_\mu\partial_\nu + (-\frac{1}{\alpha} + 8\Omega^2)\tilde{x}_\mu\tilde{x}_\nu + 4\Omega^2\tilde{x}^2\delta_{\mu\nu}\right) A_\nu = -j_\mu \tag{2.80}$$

$$\frac{\delta S^{\text{bil}}}{\delta \bar{c}}: \quad \mathrm{i}\tilde{x}_\mu\partial_\mu c = -j_c\,. \tag{2.81}$$

In the following we will choose $\Omega = 1$ and $\alpha = \frac{1}{8\Omega^2}$, which yields $\beta = \frac{1}{2g}$. Then the photon propagator simply becomes the Mehler kernel[10]. Note that in the literature another convention, $\tilde{x}_\mu = 2\theta^{-1}_{\mu\nu}x_\nu$, is frequently used. In this case, other values for the constants α, β have to be chosen in order that the propagator becomes the Mehler kernel. But this is only a matter of convention and is not discussed any further in the following. A more involved subject is the ghost propagator which contains the inverse of the operator $\tilde{x}\partial$, which we will calculate in the next section.

2.3.3 The Ghost Propagator

We follow the method introduced by Kummer and Weiser [79]. The first step is to transform our Green function in just one variable:

$$\begin{aligned}
\mathrm{i}\tilde{x}\partial G(x,x') &= \delta(x-x')\,,\\
\mathrm{i}\tilde{x}\partial \int \frac{d^dk}{(2\pi)^d}\, e^{-\mathrm{i}kx} G(k,x') &= \int \frac{d^dk}{(2\pi)^d}\, e^{-\mathrm{i}k(x-x')}\,,\\
\mathrm{i}\int \frac{d^dk}{(2\pi)^d}\, G(k,x')(-\mathrm{i})k\tilde{x}e^{-\mathrm{i}kx} &= \int \frac{d^dk}{(2\pi)^d}\, e^{-\mathrm{i}k(x-x')}\,,\\
\mathrm{i}\int \frac{d^dk}{(2\pi)^d}\, G(k,x')k_\mu\Theta^{-1}_{\mu\nu}\partial^k_\nu e^{-\mathrm{i}kx} &= \int \frac{d^dk}{(2\pi)^d}\, e^{-\mathrm{i}k(x-x')}\,.
\end{aligned}$$

Now comes a partial integration (note that $\partial^k_\mu\tilde{k}_\mu = 0$)

$$-\mathrm{i}\int \frac{d^dk}{(2\pi)^d}\, e^{-\mathrm{i}kx}k_\mu\Theta^{-1}_{\mu\nu}\partial^k_\nu G(k,x') = \int \frac{d^dk}{(2\pi)^d}\, e^{-\mathrm{i}k(x-x')}\,. \tag{2.82}$$

Since the integration domain is the same on both sides we conclude

$$-\mathrm{i}k_\mu\Theta^{-1}_{\mu\nu}\partial^k_\nu G(k,x') = e^{\mathrm{i}kx'}\,. \tag{2.83}$$

The next step Kummer and Weiser take is to express everything only in terms of Lorentz scalars $G(k,x') = G(k^2, kx', x'^2, k\tilde{x}')$. However, in the case of our operator one runs into difficulties by continuing with this approach. Instead we will try to solve the differential equation directly. For this, we first limit ourselves to **2 dimensions**[11]. It is there where (2.83) becomes very simple. One can see this by switching to polar coordinates

$$k_0 = r\cos\phi\,,\quad k_1 = r\sin\phi\,,\quad r = \sqrt{k_0^2+k_1^2}\,,\quad \phi = \arctan\left(\frac{k_1}{k_0}\right)\,, \tag{2.84}$$

[10]with ω redefined as $\omega = \frac{\theta}{2\Omega}$

[11]Calculating the ghost propagator in 4 dimensions is work in progress.

which yields

$$\mathrm{i}\theta \frac{\partial}{\partial \phi} G(r, \phi, x_0', x_1') = e^{\mathrm{i}(x_0' r \cos\phi + x_1' r \sin\phi)}\,, \tag{2.85}$$

and then complete the Fourier transformation to end up at $G(k, k')$. The integral over x' can be carried out and one gets

$$G(r, \phi, k_0', k_1') = (-\mathrm{i}\theta) \int_c^{\phi} d\phi'\, \delta(r\cos\phi' - k_0')\delta(r\sin\phi' - k_1')\,. \tag{2.86}$$

We will now return to the original momentum variables

$$G(k_0, k_1, k_0', k_1') = (-\mathrm{i}\theta) \int_c^{\arctan\frac{k_1}{k_0}} d\phi'\, \delta\left(\sqrt{k_0^2 + k_1^2}\cos\phi' - k_0'\right) \delta\left(\sqrt{k_0^2 + k_1^2}\sin\phi' - k_1'\right)\,. \tag{2.87}$$

To solve the integral over ϕ' one needs to find the zeros of one of the delta functions. Unfortunately those are arbitrary many. Instead of trying to do so anyway, we will search for a clever substitution to make the problem simpler. For example if we substitute $\cos\phi' = \lambda$, we arrive at

$$G(k_0, k_1, k_0', k_1') = \frac{\mathrm{i}}{\theta} \int_{\cos c}^{\frac{k_0}{\sqrt{k_0^2+k_1^2}}} d\lambda\, \frac{1}{\sqrt{1-\lambda^2}} \delta\left(\sqrt{k_0^2 + k_1^2}\lambda - k_0'\right) \delta\left(\sqrt{k_0^2 + k_1^2}\sqrt{1-\lambda^2} - k_1'\right)\,. \tag{2.88}$$

The first delta function is now zero at $\lambda = \frac{k_0'}{\sqrt{k_0^2+k_1^2}}$. Applying the formula $\delta(f(x)) = \sum\limits_i \frac{\delta(x-x_i)}{|f'(x_i)|}$, where the x_i are the zeroes of $f(x)$, we get

$$G(k_0, k_1, k_0', k_1') = \frac{\mathrm{i}}{\theta} \int_{\cos c}^{\frac{k_0}{\sqrt{k_0^2+k_1^2}}} d\lambda\, \frac{1}{\sqrt{1-\lambda^2}} \frac{\delta\left(\lambda - \frac{k_0'}{\sqrt{k_0^2+k_1^2}}\right)}{\left|\sqrt{k_0^2 + k_1^2}\right|} \delta\left(\sqrt{k_0^2 + k_1^2}\sqrt{1-\lambda^2} - k_1'\right)\,. \tag{2.89}$$

By using the Heaviside Theta function Θ_{H} to extend the upper integral bound to infinity

$$G(k_0, k_1, k_0', k_1') = \frac{\mathrm{i}}{\theta} \int_{\cos c}^{\infty} d\lambda\, \left[1 - \Theta_{\mathrm{H}}\left(\lambda - \frac{k_0}{\sqrt{k_0^2+k_1^2}}\right)\right] \frac{1}{\sqrt{1-\lambda^2}} \frac{\delta\left(\lambda - \frac{k_0'}{\sqrt{k_0^2+k_1^2}}\right)}{\left|\sqrt{k_0^2 + k_1^2}\right|}$$
$$\delta\left(\sqrt{k_0^2 + k_1^2}\sqrt{1-\lambda^2} - k_1'\right)\,, \tag{2.90}$$

we are able to solve the integral with the Dirac delta function (The constant c can always be chosen such that the delta function lies within the integral bounds)

$$G(k_0,k_1,k_0',k_1') = \frac{\mathrm{i}}{\theta}\sqrt{\frac{1}{k_0^2+k_1^2-k_0'^2}}\,\delta\left(\sqrt{k_0^2+k_1^2-k_0'^2}-k_1'\right)\left[1-\Theta_{\mathrm{H}}\left(\frac{k_0'-k_0}{\sqrt{k_0^2+k_1^2}}\right)\right]. \tag{2.91}$$

This is equivalent to[12]

$$G(k_0,k_1,k_0',k_1') = \frac{\mathrm{i}}{\theta}\sqrt{\frac{1}{k_0^2+k_1^2-k_0'^2}}\,\delta\left(\sqrt{k_0^2+k_1^2-k_0'^2}-k_1'\right)\Theta_{\mathrm{H}}\left(k_0-k_0'\right). \tag{2.92}$$

We will now check whether this result is really right. For this aim we will apply the operator $i\theta\left(k_0\frac{\partial}{\partial k_1}-k_1\frac{\partial}{\partial k_0}\right)$ on (2.92). This gives

$$\frac{k_1\delta\left(k_0-k_0'\right)\delta\left(\sqrt{k_0^2-k_0'^2+k_1^2}-k_1'\right)}{\sqrt{k_0^2-k_0'^2+k_1^2}}. \tag{2.93}$$

Now we use the first delta function to set $k_0'=k_0$ everywhere else, which yields

$$\frac{k_1\delta\left(k_0-k_0'\right)\delta\left(\sqrt{k_1^2}-k_1'\right)}{\sqrt{k_1^2}}. \tag{2.94}$$

This is only the result we desire for $k_1\geq 0$. For $k_1<0$ a solution for the propagator is given by

$$G(k_0,k_1,k_0',k_1') = \frac{-\mathrm{i}}{\theta}\sqrt{\frac{1}{k_0^2+k_1^2-k_0'^2}}\,\delta\left(-\sqrt{k_0^2+k_1^2-k_0'^2}-k_1'\right)\Theta_{\mathrm{H}}\left(k_0-k_0'\right). \tag{2.95}$$

Both regions can be combined by pulling out k_1 of the square root:

$$G(k_0,k_1,k_0',k_1') = \frac{\mathrm{i}}{\theta}\frac{\Theta_{\mathrm{H}}(k_0-k_0')\delta\left(k_1\sqrt{\frac{k_0^2-k_0'^2}{k_1^2}+1}-k_1'\right)}{k_1\sqrt{\frac{k_0^2-k_0'^2}{k_1^2}+1}} \tag{2.96}$$

and indeed, when applying the operator $\tilde{x}\partial$ on it, this gives the 2-dimensional delta function.
By using the properties of the delta function one can also rewrite this as

$$\boxed{G(k,k') = \frac{\mathrm{i}\Theta_{\mathrm{H}}(k_0-k_0')\delta\left(k_1\sqrt{\frac{k_0^2-k_0'^2}{k_1^2}+1}-k_1'\right)}{\theta k_1\sqrt{\frac{k_1'^2}{k_1^2}}}}. \tag{2.97}$$

[12] $\Theta_{\mathrm{H}}\left(\frac{k_0'-k_0}{\sqrt{k_0^2+k_1^2}}\right)=\int dk_0'\,\delta\left(\frac{k_0'-k_0}{\sqrt{k_0^2+k_1^2}}\right)\frac{1}{\sqrt{k_0^2+k_1^2}}=\int dk_0'\,\delta(k_0'-k_0)=\Theta_{\mathrm{H}}(k_0'-k_0)$ and $1-\Theta_{\mathrm{H}}(x)=\Theta_{\mathrm{H}}(-x)$

One might argue that in this form, the propagator is not at all symmetric. However, by playing around with the delta function and symmetrization, one may find that

$$G(k,k') = \frac{\mathrm{i}}{\theta}\left(\mathrm{sign}(k_1)+\mathrm{sign}(k_1')\right)\delta\left(k_0^2+k_1^2-k_0'^2-k_1'^2\right)\Theta_{\mathrm{H}}\left(k_0-k_0'\right) \tag{2.98}$$

is also a valid form of the propagator[13]. Side remark: the two signum functions in the beginning take care of the other root of the delta function with the wrong overall sign, which evolved due to taking the square of the original argument, and thus implying additional solutions.

For later loop calculations, it may also prove wise to rewrite it once more, into a form where one sees that the at first glance awkward delta function in reality plays the role of overall conservation of the absolute value of the momenta:

$$G(k,k') = \frac{\mathrm{i}}{\theta\sqrt{k_0^2+k_1^2}}\left(\mathrm{sign}(k_1)+\mathrm{sign}(k_1')\right)\delta\left(\sqrt{k_0^2+k_1^2}-\sqrt{k_0'^2-k_1'^2}\right)\Theta_{\mathrm{H}}\left(k_0-k_0'\right)\,. \tag{2.99}$$

2.3.4 Vertices in momentum space

In this section we want to calculate the vertices of the induced action. To be as general as possible, and by taking into account that we will probably extend the ghost propagator to 4 dimensions one day, we calculate the vertices in arbitrary (Euclidean) dimensions.

There are different kinds of vertices appearing[14] (without any constants and prefactors)

- The 2 ghost 1 photon vertex $\tilde{x}_\mu \bar{c} \star [A_\mu \stackrel{\star}{,} c]$
- The standard 3 photon vertex $\partial_\mu A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu]$
- The local 3 photon vertex $(\tilde{x}_\mu A_\nu) \star \{A_\mu \stackrel{\star}{,} A_\nu\}$
- The standard 4 photon vertex $A_\mu \star A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu]$
- The symmetrized 4 photon vertex $A_\mu \star A_\nu \star \{A_\mu \stackrel{\star}{,} A_\nu\}$

In the following subsections we will calculate them all, where we sum up the respective 3 photon and 4 photon vertices.

The 2 ghost 1 photon vertex

Applying the star product formula yields

$$S_{\mathrm{int}}^{\bar{c}Ac} = \int d^dx\, \tilde{x}_\mu \bar{c} \star [A_\mu \stackrel{\star}{,} c] = \int d^dx \int \frac{d^dk_1 \ldots k_3}{(2\pi)^{3d}} e^{\mathrm{i}(k_1+k_2+k_3)x} \tilde{x}_\mu \tilde{\bar{c}}(k_1)\tilde{A}_\mu(k_2)\tilde{c}(k_3) \left[e^{-\frac{\mathrm{i}}{2}(k_1\times k_2+k_1\times k_3+k_2\times k_3)} - e^{-\frac{\mathrm{i}}{2}(k_1\times k_2+k_1\times k_3+k_3\times k_2)}\right]. \tag{2.100}$$

[13]Also the Heaviside Theta function may be symmetrized, but this complicates the expression.
[14]The expanded action may be found in Appendix C.1.

One can now use $k_3 \times k_2 = -k_2 \times k_3$ and in this way transform the exponential to a sine. Furthermore we leave out the tilde for the Fourier transformed quantities in order not to run into danger of misinterpreting it as a contraction with the Theta matrix, as it is the case for $\tilde{x}_\mu = \Theta^{-1}_{\mu\nu} x_\nu$,

$$S^{\bar{c}Ac}_{\text{int}} = \int d^d x \int \frac{d^d k_1 \dots k_3}{(2\pi)^{3d}} e^{\mathrm{i}(k_1+k_2+k_3)x} \tilde{x}_\mu \bar{c}(k_1) A_\mu(k_2) c(k_3)$$
$$e^{-\frac{\mathrm{i}}{2}(k_1 \times k_2 + k_1 \times k_3)} 2\mathrm{i} \sin\left(\frac{k_2 \times k_3}{2}\right) . \tag{2.101}$$

The position coordinate on the r.h.s. becomes a derivation with respect to k_1 acting on the first exponential. After this it is possible to exchange the integral over x with this derivation and thus integrate out the exponential into a delta function representing momentum conservation. However, the derivation still acts on this delta function:

$$S^{\bar{c}Ac}_{\text{int}} = \int \frac{d^d k_1 \dots k_3}{(2\pi)^{3d}} \bar{c}(k_1) A_\mu(k_2) c(k_3) e^{-\frac{\mathrm{i}}{2}(k_1 \times k_2 + k_1 \times k_3)}$$
$$\sin\left(\frac{k_2 \times k_3}{2}\right) 2\Theta^{-1}_{\mu\rho} \frac{\partial}{\partial k_1^\rho} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3) . \tag{2.102}$$

To be able to use the delta function to get rid of the exponential, we now need to perform a partial integration.

$$S^{\bar{c}Ac}_{\text{int}} = \int \frac{d^d k_1 \dots k_3}{(2\pi)^{3d}} e^{-\frac{\mathrm{i}}{2}(k_1 \times k_2 + k_1 \times k_3)} \left(-2\Theta^{-1}_{\mu\rho} \partial^{k_1}_\rho \bar{c}(k_1) - \mathrm{i}(k_2 + k_3)_\mu \bar{c}(k_1)\right)$$
$$A_\mu(k_2) c(k_3) \sin\left(\frac{k_2 \times k_3}{2}\right) (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3) . \tag{2.103}$$

Using the delta function eliminates the exponential and simplifies the expression in brackets

$$S^{\bar{c}Ac}_{\text{int}} = \int \frac{d^d k_1 \dots k_3}{(2\pi)^{3d}} \left(2\mathrm{i}\Theta^{-1}_{\mu\rho} \partial^{k_1}_\rho \bar{c}(k_1) + k_{1\mu} \bar{c}(k_1)\right) A_\mu(k_2) c(k_3)$$
$$\mathrm{i} \sin\left(\frac{k_2 \times k_3}{2}\right) (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3) . \tag{2.104}$$

Before varying this expression w.r.t. the different fields, one needs to partially integrate once more

$$S^{\bar{c}Ac}_{\text{int}} = \int \frac{d^d k_1 \dots k_3}{(2\pi)^{3d}} \bar{c}(k_1) A_\mu(k_2) c(k_3)$$
$$\mathrm{i} \sin\left(\frac{k_2 \times k_3}{2}\right) \left(-2\mathrm{i}\Theta^{-1}_{\mu\rho} \partial^{k_1}_\rho + k_{1\mu}\right) (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3) . \tag{2.105}$$

The variational process is very simple here because all 3 fields are different. The integrations just cancel and the vertex is enlarged by a factor $-(2\pi)^{3d}$. Additionally, with our convention to vary with respect to the momenta with a minus sign (which is the convention that all momenta of the vertex point inwards), we get delta functions with a plus sign and thus have to reverse the signs of the momenta in the end. In this case this leads to an additional overall minus sign:

$$\begin{aligned} V_\sigma^{\bar{c}Ac} &= -(2\pi)^{3d} \frac{\delta}{\delta c(-q_3)} \frac{\delta}{\delta A^\sigma(-q_2)} \frac{\delta}{\delta \bar{c}(-q_1)} S_{\text{int}}^{\bar{c}Ac} \\ &= \mathrm{i} \sin\left(\frac{q_2 \times q_3}{2}\right) \left(-2\mathrm{i}\Theta^{-1}_{\sigma\rho}\partial^{q_1}_\rho + q_{1\sigma}\right) (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3)\,. \end{aligned} \tag{2.106}$$

Together with its prefactor g the vertex is given by

$$V_\sigma^{\bar{c}Ac} = \mathrm{i}g \sin\left(\frac{q_2 \times q_3}{2}\right) \left(-2\mathrm{i}\Theta^{-1}_{\sigma\rho}\partial^{q_1}_\rho + q_{1\sigma}\right) (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3)\,. \tag{2.107}$$

In order to fit better to the conventions generally used, which is that the momenta are counted clockwise, we will rename the momenta $q_1 \leftrightarrow q_3$. Furthermore we will call $q_2 = k_2$ in order to emphasize that it is the momentum belonging to the photon leg.

$$V_\sigma^{\bar{c}Ac} = -\mathrm{i}g \sin\left(\frac{q_1 \times k_2}{2}\right) \left(-2\mathrm{i}\Theta^{-1}_{\sigma\rho}\partial^{q_3}_\rho + q_{3\sigma}\right) (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3)\,, \tag{2.108}$$

We conclude this subsection by remarking that in loop calculations, it will be possible to partially integrate and thus release the delta function.

The new 3 photon vertex

Together with its prefactor, the local 3 photon vertex is given by

$$S_{\text{int}}^{3A} = \int d^dx\, 2\Omega^2 g(\tilde{x}_\mu A_\nu) \star \{A_\mu \stackrel{\star}{,} A_\nu\}\ . \tag{2.109}$$

We apply the star product formula

$$\begin{aligned} S_{\text{int}}^{3A} = 2\Omega^2 g \int & d^dx d^2k_1 d^2k_2 d^2k_3\, e^{\mathrm{i}(k_1+k_2+k_3)x} \tilde{x}_\mu A_\nu(k_1) A_\mu(k_2) A_\nu(k_3) \\ & \left[e^{-\frac{\mathrm{i}}{2}(k_1\times k_2 + k_1 \times k_3 + k_2 \times k_3)} + e^{-\frac{\mathrm{i}}{2}(k_1\times k_2 + k_1 \times k_3 + k_3 \times k_2)}\right]\,. \end{aligned} \tag{2.110}$$

The next steps are identical to the last subsection, so we just get for the interaction term

$$\begin{aligned} S_{\text{int}}^{3A} =& 2\Omega^2 g \int \frac{d^dk_1 \dots k_3}{(2\pi)^{3d}} A_\nu(k_1) A_\mu(k_2) A_\nu(k_3) \\ & \cos\left(\frac{k_2 \times k_3}{2}\right) \left(-2\mathrm{i}\Theta^{-1}_{\mu\rho}\partial^{k_1}_\rho + k_{1\mu}\right) (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3)\,. \end{aligned} \tag{2.111}$$

The variational principle for this vertex is in our conventions

$$V_{\text{loc}}^{3A} = -(2\pi)^{3d} \frac{\delta}{\delta A^\tau(-k_3)} \frac{\delta}{\delta A^\sigma(-k_2)} \frac{\delta}{\delta A^\rho(-k_1)} S_{\text{int}}^{\bar{c}Ac}(q_1, q_2, q_3)\,, \tag{2.112}$$

which yields

$$\begin{aligned} V_{\text{loc}}^{3A} = -\int & d^dq_1 d^dq_2 d^dq_3\, 2\Omega^2 g \cos\left(\frac{q_2 \times q_3}{2}\right) \left(-2\mathrm{i}\Theta^{-1}_{\mu\epsilon}\partial^{q_1}_\epsilon + q_{1\mu}\right) (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3) \\ & \big[\delta_{\sigma\mu}\delta_{\rho\tau}\delta(k_1+q_1)\delta(k_2+q_2)\delta(k_3+q_3) + \delta_{\rho\sigma}\delta_{\tau\mu}\delta(k_1+q_1)\delta(k_2+q_3)\delta(k_3+q_2) \\ & + \delta_{\rho\mu}\delta_{\sigma\tau}\delta(k_1+q_2)\delta(k_2+q_1)\delta(k_3+q_3) + \delta_{\rho\mu}\delta_{\sigma\tau}\delta(k_1+q_2)\delta(k_2+q_3)\delta(k_3+q_1) \\ & + \delta_{\tau\mu}\delta_{\rho\sigma}\delta(k_1+q_3)\delta(k_2+q_1)\delta(k_3+q_2) + \delta_{\sigma\mu}\delta_{\rho\tau}\delta(k_1+q_3)\delta(k_2+q_2)\delta(k_3+q_1)\big]\,. \end{aligned} \tag{2.113}$$

We can solve the delta functions yielding

$$
\begin{aligned}
V^{3A}_{\text{loc}} = 2\Omega^2 g \cos\left(\frac{k_2 \times k_3}{2}\right)& \\
\Bigg[\delta_{\rho\tau}\Bigg(&\left(-2\mathrm{i}\Theta^{-1}_{\sigma\epsilon}\partial^{k_1}_{\epsilon} + k_{1\sigma}\right) + \left(-2\mathrm{i}\Theta^{-1}_{\sigma\epsilon}\partial^{k_3}_{\epsilon} + k_{3\sigma}\right)\Bigg) \\
&+ \delta_{\rho\sigma}\Bigg(\left(-2\mathrm{i}\Theta^{-1}_{\tau\epsilon}\partial^{k_1}_{\epsilon} + k_{1\tau}\right) + \left(-2\mathrm{i}\Theta^{-1}_{\tau\epsilon}\partial^{k_2}_{\epsilon} + k_{2\tau}\right)\Bigg) \\
&+ \delta_{\sigma\tau}\Bigg(\left(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}\partial^{k_2}_{\epsilon} + k_{2\rho}\right) + \left(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}\partial^{k_3}_{\epsilon} + k_{3\rho}\right)\Bigg)\Bigg] \\
&(2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3)\,,
\end{aligned}
\tag{2.114}
$$

and if we reorder the terms we arrive at a symmetric expression

$$
\begin{aligned}
V^{3A}_{\text{loc}} = 2\Omega^2 g (2\pi)^d \cos\left(\frac{k_2 \times k_3}{2}\right)& \\
\Bigg[\delta_{\rho\tau}\Big(&-2\mathrm{i}\Theta^{-1}_{\sigma\epsilon}(\partial^{k_1}_{\epsilon} + \partial^{k_3}_{\epsilon}) + k_{1\sigma} + k_{3\sigma}\Big) \\
&+ \delta_{\rho\sigma}\Big(-2\mathrm{i}\Theta^{-1}_{\tau\epsilon}(\partial^{k_1}_{\epsilon} + \partial^{k_2}_{\epsilon}) + k_{2\tau} + k_{1\tau}\Big) \\
&+ \delta_{\sigma\tau}\Big(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}(\partial^{k_2}_{\epsilon} + \partial^{k_3}_{\epsilon}) + k_{3\rho} + k_{2\rho}\Big)\Bigg] \\
&\delta^{(d)}(k_1 + k_2 + k_3)\,.
\end{aligned}
\tag{2.115}
$$

Combination with the standard 3 photon vertex

The standard vertex is well known in the literature and is given by

$$
\begin{aligned}
V^{3A}_{\text{std}} =& 2\mathrm{i}g(2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3) \sin\left(\frac{k_2 \times k_3}{2}\right) \\
&[(k_1 - k_3)_\sigma \delta_{\rho\tau} + (k_2 - k_1)_\tau \delta_{\rho\sigma} + (k_3 - k_2)_\rho \delta_{\sigma\tau}]\,.
\end{aligned}
\tag{2.116}
$$

For $\Omega = 1$, it is now possible to sum up both 3 photon vertices by expressing the trigonometric functions as exponentials,

$$
\begin{aligned}
V^{3A}_{\rho\sigma\tau} = 2\Omega^2 g (2\pi)^d& \\
\Bigg[\delta_{\rho\tau}\Big(&e^{\frac{\mathrm{i}}{2}k_2\times k_3}\Big(-2\mathrm{i}\Theta^{-1}_{\sigma\epsilon}(\partial^{k_1}_{\epsilon} + \partial^{k_3}_{\epsilon}) + k_{1\sigma}\Big) + e^{-\frac{\mathrm{i}}{2}k_2\times k_3} k_{3\sigma}\Big) \\
&+ \delta_{\rho\sigma}\Big(e^{\frac{\mathrm{i}}{2}k_2\times k_3}\Big(-2\mathrm{i}\Theta^{-1}_{\tau\epsilon}(\partial^{k_1}_{\epsilon} + \partial^{k_2}_{\epsilon}) + k_{2\tau}\Big) + e^{-\frac{\mathrm{i}}{2}k_2\times k_3} k_{1\tau}\Big) \\
&+ \delta_{\sigma\tau}\Big(e^{\frac{\mathrm{i}}{2}k_2\times k_3}\Big(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}(\partial^{k_2}_{\epsilon} + \partial^{k_3}_{\epsilon}) + k_{3\rho}\Big) + e^{-\frac{\mathrm{i}}{2}k_2\times k_3} k_{2\rho}\Big)\Bigg] \\
\delta^{(d)}(k_1 + k_2 + k_3)\,.&
\end{aligned}
\tag{2.117}
$$

The local 4 photon vertex

This vertex comes from the term

$$S_{\text{int}}^{4A} = \frac{\Omega^2 g^2}{2} \int d^d x \, A_\mu \star A_\nu \, \{A_\mu \stackrel{\star}{,} A_\nu\} \tag{2.118}$$

in the action. With our familiar star product formula this gives us

$$\begin{aligned} S_{\text{int}}^{4A} =& \frac{\Omega^2 g^2}{2} \int d^d x \int \frac{d^d k_1 ... k_4}{(2\pi)^{4d}} e^{\mathrm{i}(k_1+k_2+k_3+k_4)x} \tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2)\tilde{A}_\mu(k_3)\tilde{A}_\nu(k_4) \\ & \Big[e^{-\frac{\mathrm{i}}{2}(k_1\times k_2 + k_1\times k_3 + k_1\times k_4 + k_2\times k_3 + k_2\times k_4 + k_3\times k_4)} \\ & + e^{-\frac{\mathrm{i}}{2}(k_1\times k_2 + k_1\times k_3 + k_1\times k_4 + k_2\times k_3 + k_2\times k_4 + k_4\times k_3)} \Big] . \end{aligned} \tag{2.119}$$

We will leave out the tildes once more and use the momentum conservation to simplify everything. Hence we get

$$\begin{aligned} S_{\text{int}}^{4A} = \Omega^2 g^2 \int \frac{d^d k_1 ... k_4}{(2\pi)^{4d}} \delta(\sum_i k_i)(2\pi)^4 A_\mu(k_1)A_\nu(k_2)A_\mu(k_3)A_\nu(k_4) \\ e^{-\frac{\mathrm{i}}{2}(k_1\times k_2)} \cos\left(\frac{k_3\times k_4}{2}\right) . \end{aligned} \tag{2.120}$$

The variation

$$V_{\text{loc}}^{4A} = -(2\pi)^{4d} \frac{\delta}{\delta \tilde{A}_\lambda(-k_4)} \frac{\delta}{\delta \tilde{A}_\tau(-k_3)} \frac{\delta}{\delta \tilde{A}_\sigma(-k_2)} \frac{\delta}{\delta \tilde{A}_\rho(-k_1)} S_{\text{int}}^{4A} \tag{2.121}$$

symmetrizes the vertex and we get

$$\begin{aligned} V_{\text{loc}}^{4A} = & -4\Omega^2 g^2 \delta\left(\sum_i k_i\right) (2\pi)^d \\ & \Bigg[\cos\left(\frac{k_1\times k_2}{2}\right) \cos\left(\frac{k_3\times k_4}{2}\right) (\delta^{\rho\tau}\delta^{\sigma\lambda} + \delta^{\rho\lambda}\delta^{\sigma\tau}) \\ & + \cos\left(\frac{k_1\times k_3}{2}\right) \cos\left(\frac{k_2\times k_4}{2}\right) (\delta^{\rho\sigma}\delta^{\tau\lambda} + \delta^{\rho\lambda}\delta^{\sigma\tau}) \\ & + \cos\left(\frac{k_1\times k_4}{2}\right) \cos\left(\frac{k_2\times k_3}{2}\right) (\delta^{\rho\sigma}\delta^{\tau\lambda} + \delta^{\rho\tau}\delta^{\sigma\lambda}) \Bigg] . \end{aligned} \tag{2.122}$$

Combination with the standard 4 photon vertex

The term

$$\int d^d x \, \frac{-g^2}{2} A_\mu \star A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu] \tag{2.123}$$

in the action leads to the well known expression

$$V^{4A}_{\text{std}} = -4g^2 \delta\left(\sum_i k_i\right)(2\pi)^d$$
$$\left[\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)\left(\delta^{\rho\tau}\delta^{\sigma\lambda}-\delta^{\rho\lambda}\delta^{\sigma\tau}\right)\right.$$
$$+\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right)\left(\delta^{\rho\sigma}\delta^{\tau\lambda}-\delta^{\rho\lambda}\delta^{\sigma\tau}\right)$$
$$\left.+\sin\left(\frac{k_1\times k_4}{2}\right)\sin\left(\frac{k_2\times k_3}{2}\right)\left(\delta^{\rho\sigma}\delta^{\tau\lambda}-\delta^{\rho\tau}\delta^{\sigma\lambda}\right)\right]. \tag{2.124}$$

Summing this up with (2.122) leads to

$$V^{4A}_{\rho\sigma\tau\lambda} = -4g^2 \delta\left(\sum_i k_i\right)(2\pi)^d$$
$$\left[\left(\cos\left(\frac{k_1\times k_2}{2}\right)\cos\left(\frac{k_3\times k_4}{2}\right)+\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)\right)\delta^{\rho\tau}\delta^{\sigma\lambda}\right.$$
$$+\left(\cos\left(\frac{k_1\times k_2}{2}\right)\cos\left(\frac{k_3\times k_4}{2}\right)-\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)\right)\delta^{\rho\lambda}\delta^{\sigma\tau}$$
$$+\left(\cos\left(\frac{k_1\times k_3}{2}\right)\cos\left(\frac{k_2\times k_4}{2}\right)+\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right)\right)\delta^{\rho\sigma}\delta^{\tau\lambda}$$
$$+\left(\cos\left(\frac{k_1\times k_3}{2}\right)\cos\left(\frac{k_2\times k_4}{2}\right)-\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right)\right)\delta^{\rho\lambda}\delta^{\sigma\tau}$$
$$+\left(\cos\left(\frac{k_1\times k_4}{2}\right)\cos\left(\frac{k_2\times k_3}{2}\right)+\sin\left(\frac{k_1\times k_4}{2}\right)\sin\left(\frac{k_2\times k_3}{2}\right)\right)\delta^{\rho\sigma}\delta^{\tau\lambda}$$
$$\left.+\left(\cos\left(\frac{k_1\times k_4}{2}\right)\cos\left(\frac{k_2\times k_3}{2}\right)-\sin\left(\frac{k_1\times k_4}{2}\right)\sin\left(\frac{k_2\times k_3}{2}\right)\right)\delta^{\rho\tau}\delta^{\sigma\lambda}\right]. \tag{2.125}$$

We may now use trigonometric identities to finally get

$$V^{4A}_{\rho\sigma\tau\lambda} = -4g^2 \delta\left(\sum_i k_i\right)(2\pi)^d$$
$$\left[\left[\cos\left(\frac{k_1\times k_2 - k_3\times k_4}{2}\right)+\cos\left(\frac{k_1\times k_4 + k_2\times k_3}{2}\right)\right]\delta^{\rho\tau}\delta^{\sigma\lambda}\right.$$
$$+\left[\cos\left(\frac{k_1\times k_2 + k_3\times k_4}{2}\right)+\cos\left(\frac{k_1\times k_3 + k_2\times k_4}{2}\right)\right]\delta^{\rho\lambda}\delta^{\sigma\tau}$$
$$\left.+\left[\cos\left(\frac{k_1\times k_3 - k_2\times k_4}{2}\right)+\cos\left(\frac{k_1\times k_4 - k_2\times k_3}{2}\right)\right]\delta^{\rho\sigma}\delta^{\tau\lambda}\right]. \tag{2.126}$$

2.3.5 Vertices in position space

We will now in addition calculate all the vertices in position space, since there will in the end appear no derivation in the respective expressions (this will turn out to be a feature linked to the star product formula of 3 fields in position space), and thus loop calculations may become easier. Furthermore it is of general interest to compare the final loop calculation results in both spaces, on the one hand to check them, and on the other hand people who normally work in momentum space have often no idea how this would look like in position space and vice versa, so it is nice to once see both.

Vertices appearing in the action

In this action, we have some kinds of different vertices appearing (without integral):

- The 2 ghost 1 photon vertex $g(\tilde{x}_\mu \bar{c}) \star [A_\mu \stackrel{\star}{,} c]$
- The standard 3 photon vertex $-\mathrm{i}g\partial_\mu A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu]$
- The local 3 photon vertex $2\Omega^2 g\tilde{x}_\mu A_\nu \star \{A_\mu \stackrel{\star}{,} A_\nu\}$
- The standard 4 photon vertex $-\frac{g^2}{2} A_\mu \star A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu]$
- The symmetrized 4 photon vertex $\frac{\Omega^2 g^2}{2} A_\mu \star A_\nu \star \{A_\mu \stackrel{\star}{,} A_\nu\}$

In the following subsections we will calculate them all **in position space**, where we sum up the respective 3 photon and 4 photon vertices.

The 3 photon vertex

For a star product of 3 fields we have the following formula in position space

$$\int d^d x \, (f_1 \star f_2 \star f_3)(x) = \frac{1}{\pi^d \theta^d} \int d^d x d^d y d^d z \, f_1(x) f_2(y) f_3(z) e^{-2\mathrm{i}(x\wedge y + y \wedge z - x \wedge z)} \,. \quad (2.127)$$

The term in the action for the standard 3 photon vertex is given by $\int d^d x \, (-\mathrm{i}g)\partial_\mu A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu]$. Applying formula (2.127) yields

$$\frac{-\mathrm{i}g}{\pi^d \theta^d} \int d^d x d^d y d^d z \, \partial^x_\mu A_\nu(x) A_\mu(y) A_\nu(z) \underbrace{\left(e^{-2\mathrm{i}(x\wedge y+y\wedge z-x\wedge z)} - e^{-2\mathrm{i}(-x\wedge y-y\wedge z+x\wedge z)}\right)}_{-2\mathrm{i}\sin\left(2(x\wedge y+y\wedge z-x\wedge z)\right)} \,. \quad (2.128)$$

Partial integration leads to

$$\begin{aligned} &\frac{2g}{\pi^d \theta^d} \int d^d x d^d y d^d z \, A_\nu(x) A_\mu(y) A_\nu(z) \partial^x_\mu \sin\left(2(x\wedge y + y \wedge z - x \wedge z)\right) \\ =&\frac{4g}{\pi^d \theta^d} \int d^d x d^d y d^d z \, A_\nu(x) A_\mu(y) A_\nu(z) \cos\left(2(x\wedge y + y \wedge z - x \wedge z)\right)(\tilde{y} - \tilde{z})_\mu \,. \end{aligned} \quad (2.129)$$

The same procedure is now applied to the local 3 photon vertex $\int d^d x \, 2\Omega^2 g(\tilde{x}_\mu A_\nu) \star \{A_\mu \stackrel{\star}{,} A_\nu\}$. With $\Omega = 1$, it becomes

$$\begin{aligned} &\frac{2g}{\pi^d \theta^d} \int d^d x d^d y d^d z \, \tilde{x}_\mu A_\nu(x) A_\mu(y) A_\nu(z) \left(e^{-2\mathrm{i}(x\wedge y+y\wedge z-x\wedge z)} + e^{-2\mathrm{i}(-x\wedge y-y\wedge z+x\wedge z)}\right) . \\ =&\frac{4g}{\pi^d \theta^d} \int d^d x d^d y d^d z \, \tilde{x}_\mu A_\nu(x) A_\mu(y) A_\nu(z) \cos\left(2(x\wedge y + y \wedge z - x \wedge z)\right) . \end{aligned} \quad (2.130)$$

One can now combine the two interaction terms to one, namely

$$S_{\text{int}}^{3A} = \frac{4g}{\pi^d\theta^d}\int d^dxd^dyd^dz\, A_\nu(x)A_\mu(y)A_\nu(z)\cos\big(2(x\wedge y + y\wedge z - x\wedge z)\big)(\tilde{x}+\tilde{y}-\tilde{z})_\mu\,. \tag{2.131}$$

The corresponding vertex is now given by variation

$$V_{\rho\sigma\tau}^{3A} = -(2\pi)^{3d}\frac{\delta^{(3)}}{\delta A^\tau(-z)\delta A^\sigma(-y)\delta A^\rho(-x)}S_{\text{int}}^{3A}(x',y',z')\,, \tag{2.132}$$

where we have chosen to vary with respect to minus the position variables to coincide with the similar calculations in momentum space 2.3.4. The variational procedure gives

$$\begin{aligned} V_{\rho\sigma\tau}^{3A} = &-(2\pi)^{3d}\frac{4g}{\pi^d\theta^d}\int d^dx'd^dy'd^dz'\,\cos\big(2(x'\wedge y' + y'\wedge z' - x'\wedge z')\big)(\tilde{x}'+\tilde{y}'-\tilde{z}')_\mu \\ &\big[\delta_{\sigma\mu}\delta_{\rho\tau}\delta(x'+x)\delta(y'+y)\delta(z'+z) + \delta_{\rho\sigma}\delta_{\tau\mu}\delta(x'+x)\delta(z'+y)\delta(y'+z) \\ &+ \delta_{\rho\mu}\delta_{\sigma\tau}\delta(y'+x)\delta(x'+y)\delta(z'+z) + \delta_{\rho\mu}\delta_{\sigma\tau}\delta(y'+x)\delta(z'+y)\delta(x'+z) \\ &+ \delta_{\tau\mu}\delta_{\rho\sigma}\delta(z'+x)\delta(x'+y)\delta(y'+z) + \delta_{\sigma\mu}\delta_{\rho\tau}\delta(z'+x)\delta(y'+y)\delta(x'+z)\big]\,. \end{aligned} \tag{2.133}$$

Using the delta functions to solve the integrals leads to

$$\begin{aligned} V_{\rho\sigma\tau}^{3A} = (2\pi)^{3d}\frac{4g}{\pi^d\theta^d}\big[&\delta_{\rho\tau}\cos\big(2(x\wedge y + y\wedge z - x\wedge z)\big)(\tilde{x}+\tilde{y}-\tilde{z})_\sigma \\ &+ \delta_{\rho\sigma}\cos\big(2(x\wedge z + z\wedge y - x\wedge y)\big)(\tilde{x}+\tilde{z}-\tilde{y})_\tau \\ &+ \delta_{\sigma\tau}\cos\big(2(y\wedge x + x\wedge z - y\wedge z)\big)(\tilde{y}+\tilde{x}-\tilde{z})_\rho \\ &+ \delta_{\sigma\tau}\cos\big(2(z\wedge x + x\wedge y - z\wedge y)\big)(\tilde{z}+\tilde{x}-\tilde{y})_\rho \\ &+ \delta_{\rho\sigma}\cos\big(2(y\wedge z + z\wedge x - y\wedge x)\big)(\tilde{y}+\tilde{z}-\tilde{x})_\tau \\ &+ \delta_{\rho\tau}\cos\big(2(z\wedge y + y\wedge x - z\wedge x)\big)(\tilde{z}+\tilde{y}-\tilde{x})_\sigma\big]\,. \end{aligned} \tag{2.134}$$

One may now realize that the argument of the cosine is always the same $\big(\cos(-x) = \cos(x)\big)$ and we can therefore pull it out of the bracket. This gives

$$\begin{aligned} V_{\rho\sigma\tau}^{3A} =&(2\pi)^{3d}\frac{4g}{\pi^d\theta^d}\cos\big(2(x\wedge y + y\wedge z - x\wedge z)\big) \\ &\big[\delta_{\rho\tau}(\tilde{x}+\tilde{y}-\tilde{z})_\sigma + \delta_{\rho\sigma}(\tilde{x}+\tilde{z}-\tilde{y})_\tau \\ &+ \delta_{\sigma\tau}(\tilde{y}+\tilde{x}-\tilde{z})_\rho + \delta_{\sigma\tau}(\tilde{z}+\tilde{x}-\tilde{y})_\rho \\ &+ \delta_{\rho\sigma}(\tilde{y}+\tilde{z}-\tilde{x})_\tau + \delta_{\rho\tau}(\tilde{z}+\tilde{y}-\tilde{x})_\sigma\big]\,. \end{aligned}$$

One may sum up the terms to yield the final result:

$$\boxed{V_{\rho\sigma\tau}^{3A} = (2\pi)^{3d}\frac{8g}{\pi^d\theta^d}\cos\big(2(x\wedge y + y\wedge z - x\wedge z)\big)\big[\delta_{\rho\tau}\tilde{y}_\sigma + \delta_{\rho\sigma}\tilde{z}_\tau + \delta_{\sigma\tau}\tilde{x}_\rho\big]}\,. \tag{2.135}$$

The 4 photon vertex

For a star product of 4 fields we have the following formula in position space

$$\int d^dx\,(f_1\star f_2\star f_3\star f_4)(x) = \\ \frac{1}{\pi^d\theta^d}\int d^dxd^dyd^dz\,f_1(x)f_2(y)f_3(z)f_4(x-y+z)e^{-2\mathrm{i}(x\wedge y+y\wedge z-x\wedge z)}\,. \tag{2.136}$$

The contribution in the action is $\int d^dx\,\left(-\frac{g^2}{2}\right)A_\mu\star A_\nu\star[A_\mu\overset{\star}{,}A_\nu]+\frac{\Omega^2g^2}{2}A_\mu\star A_\nu\star\{A_\mu\overset{\star}{,}A_\nu\}$. For $\Omega=1$, this is just

$$g^2\int d^dx\,A_\mu\star A_\mu\star A_\nu\star A_\nu\,. \tag{2.137}$$

Applying formula (2.136) gives

$$S^{4A}_{\text{int}}=\frac{g^2}{\pi^d\theta^d}\int d^dxd^dyd^dz\,A_\mu(x)A_\mu(y)A_\nu(z)A_\nu(x-y+z)e^{-2\mathrm{i}(x\wedge y+y\wedge z-x\wedge z)}\,. \tag{2.138}$$

The rather lengthy variation is performed in Appendix C.2.1. The result is (C.24):

$$\boxed{\begin{aligned}V^{4A}&=-\frac{(2\pi)^{4d}g^2}{\pi^d\theta^d}4\cos\Big(2(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)\Big)\cdot\\ &\cdot\Big(\delta(x_1-x_2+x_3-x_4)+\delta(x_1-x_2-x_3+x_4)\Big)\Big(\delta^{\rho\tau}\delta^{\lambda\sigma}+\delta^{\sigma\tau}\delta^{\lambda\rho}+\delta^{\lambda\tau}\delta^{\rho\sigma}\Big)\end{aligned}} \tag{2.139}$$

The 2 ghost 1 photon vertex

The expression in the action is

$$S^{\bar{c}Ac}_{\text{int}}=\int d^dx\,g(\tilde{x}_\mu\bar{c})\star[A_\mu\overset{\star}{,}c]\,. \tag{2.140}$$

Using formula (2.127) gives

$$S^{\bar{c}Ac}_{\text{int}}=\frac{g}{\pi^d\theta^d}\int d^dxd^dyd^dz\,\tilde{x}_\mu\bar{c}(x)A_\mu(y)c(z)\underbrace{\left(e^{-2\mathrm{i}(x\wedge y+y\wedge z-x\wedge z)}-e^{2\mathrm{i}(x\wedge y+y\wedge z-x\wedge z)}\right)}_{-2\mathrm{i}\sin(2(x\wedge y+y\wedge z-x\wedge z))}\,. \tag{2.141}$$

The variational procedure gives the vertex

$$V^{\bar{c}Ac}_\sigma=-(2\pi)^{3d}\frac{\delta}{\delta c(-x_3)}\frac{\delta}{\delta A^\sigma(-x_2)}\frac{\delta}{\delta\bar{c}(-x_1)}S^{\bar{c}Ac}_{\text{int}}\,, \tag{2.142}$$

which is

$$V^{\bar{c}Ac}_\sigma=-(2\pi)^{3d}\frac{g}{\pi^d\theta^d}(-2i)\int d^dxd^dyd^dz\,\sin\left(2(x\wedge y+y\wedge z-x\wedge z)\right)\tilde{x}_\mu \\ \delta(x+x_1)\delta(y+x_2)\delta_{\sigma\mu}\delta(z+x_3)\,, \tag{2.143}$$

$$\boxed{V^{\bar{c}Ac}_\sigma=2\mathrm{i}g\frac{(2\pi)^{3d}}{\pi^d\theta^d}\sin\left(2(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)\right)\tilde{x}_{1\sigma}}\,. \tag{2.144}$$

2.3.6 The photon propagator for arbitrary Ω

Despite of the fact that we have now obtained all Feynman rules for the case $\Omega = 1$, it would be nice to have general expressions for $\Omega \neq 1$ too. This generalization only modifies the photon propagator. We will try to calculate the latter in this section and continue with the simpler case, $\Omega = 1$, in the next section where we will compute loop graphs.
We recall the e.o.m. for the gauge field (2.80):

$$\frac{\delta S^{\text{bil}}}{\delta A_\mu}: \quad \left(-\Box\delta_{\mu\nu} + (1-\Omega^2)\partial_\mu\partial_\nu + \left(-\frac{1}{\alpha} + 8\Omega^2\right)\tilde{x}_\mu\tilde{x}_\nu + 4\Omega^2\tilde{x}^2\delta_{\mu\nu}\right) A_\nu = -j_\mu \tag{2.145}$$

The expression in the brackets is a nonminimal operator in the Lorentz indices, and its inversion will yield the photon propagator.
This is a very complicated task and we will limit us here to at least formally calculating the propagator in terms of only Lorentz-minimal operators. For this we will use the method of the field equations. In the following we will review this procedure.
We start with (2.145) and contract it with several operators in order to render it a Lorentz scalar:

$$x_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad -x_\mu\Box A_\mu + (1-\Omega^2)(x\partial)(\partial A) + 4\Omega^2\tilde{x}^2(xA) = -(xj)\,, \tag{2.146a}$$

$$\tilde{x}_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad -\tilde{x}_\mu\Box A_\mu + (1-\Omega^2)(\tilde{x}\partial)(\partial A) + \left(-\frac{1}{\alpha} + 12\Omega^2\right)\tilde{x}^2(\tilde{x}A) = -(\tilde{x}j)\,, \tag{2.146b}$$

$$\partial_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad \Omega^2(-\Box + 4\tilde{x}^2)(\partial A) + \frac{8\Omega^2}{\theta^2}(xA) + \left(-\frac{1}{\alpha} + 8\Omega^2\right)(\tilde{x}\partial)(\tilde{x}A) = -(\partial j)\,, \tag{2.146c}$$

$$\widetilde{\partial}_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad (-\Box + 4\Omega^2\tilde{x}^2)(\widetilde{\partial}A) + \left(-\frac{1}{\alpha} + 8\Omega^2\right)\frac{1}{\theta^2}(d + x\partial)(\tilde{x}A) + \frac{8\Omega^2}{\theta^2}\tilde{x}A = -(\widetilde{\partial}j)\,. \tag{2.146d}$$

One can directly get the identities

$$\begin{aligned}
\partial_\mu(\tilde{x}_\rho A_\rho) &= -\widetilde{A}_\mu + \tilde{x}_\rho\partial_\mu A_\rho\,,\\
\Box(\tilde{x}_\rho A_\rho) &= \partial_\mu\left(-\widetilde{A}_\mu + \tilde{x}_\rho\partial_\mu A_\rho\right)\\
&= 2(\widetilde{\partial}A) + \tilde{x}_\rho\Box A_\rho\,,
\end{aligned} \tag{2.147}$$

i.e.

$$-\tilde{x}_\rho\Box A_\rho = 2(\widetilde{\partial}A) - \Box(\tilde{x}A)\,, \tag{2.148}$$

and similarly

$$-x_\rho\Box A_\rho = 2(\partial A) - \Box(xA)\,. \tag{2.149}$$

which we use to simplify (2.146):

$$x_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad \Big(2 + (1-\Omega^2)(x\partial)\Big)(\partial A) + (-\Box + 4\Omega^2 \tilde{x}^2)(xA) = -(xj)\,, \tag{2.150a}$$

$$\tilde{x}_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad 2(\widetilde{\partial} A) + (1-\Omega^2)(\tilde{x}\partial)(\partial A) + \bigg(-\Box + \Big(-\frac{1}{\alpha} + 12\Omega^2\Big)\tilde{x}^2\bigg)(\tilde{x}A) = -(\tilde{x}j)\,, \tag{2.150b}$$

$$\partial_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad \Omega^2(-\Box + 4\tilde{x}^2)(\partial A) + \frac{8\Omega^2}{\theta^2}(xA) + \Big(-\frac{1}{\alpha} + 8\Omega^2\Big)(\tilde{x}\partial)(\tilde{x}A) = -(\partial j)\,, \tag{2.150c}$$

$$\widetilde{\partial}_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad (-\Box + 4\Omega^2\tilde{x}^2)(\widetilde{\partial} A) + \Big(-\frac{1}{\alpha} + 8\Omega^2\Big)\frac{1}{\theta^2}(d + x\partial)(\tilde{x}A) + \frac{8\Omega^2}{\theta^2}\tilde{x}A = -(\widetilde{\partial} j)\,. \tag{2.150d}$$

In general one can now solve this system of 4 equations in the 4 variables $\tilde{x}A, xA, \partial A, \widetilde{\partial} A$. However, for simplicity reasons which are sufficient for the demonstration here, we set $\alpha = \frac{1}{8\Omega^2}$. This allows us to decouple the system, i.e. Equation (2.150a) and (2.150c) can be solved separately from the others. For this special choice of α the system of equations significantly reduces and we get

$$x_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad \Big(2 + (1-\Omega^2)(x\partial)\Big)(\partial A) + (-\Box + 4\Omega^2 \tilde{x}^2)(xA) = -(xj)\,, \tag{2.151a}$$

$$\tilde{x}_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad 2(\widetilde{\partial} A) + (1-\Omega^2)(\tilde{x}\partial)(\partial A) + (-\Box + 4\Omega^2\tilde{x}^2)(\tilde{x}A) = -(\tilde{x}j)\,, \tag{2.151b}$$

$$\partial_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad \Omega^2(-\Box + 4\tilde{x}^2)(\partial A) + \frac{8\Omega^2}{\theta^2}(xA) = -(\partial j)\,, \tag{2.151c}$$

$$\widetilde{\partial}_\mu \frac{\delta S^{bi}}{\delta A_\mu}: \quad (-\Box + 4\Omega^2\tilde{x}^2)(\widetilde{\partial} A) + \frac{8\Omega^2}{\theta^2}\tilde{x}A = -(\widetilde{\partial} j)\,. \tag{2.151d}$$

For the e.o.m. it is now sufficient to calculate ∂A. We use Equations (2.151a) and (2.151c) to express it. First Equation (2.151c) allows us to express xA

$$xA = \frac{-\theta^2}{8\Omega^2}\left(\partial j + \Omega^2(-\Box + 4\tilde{x}^2)(\partial A)\right)\,, \tag{2.152}$$

which we plug into (2.151a):

$$\bigg[2 + (1-\Omega^2)x\partial - \frac{\theta^2}{8}\big(-\Box + 4\Omega^2\tilde{x}^2\big)\big(-\Box + 4\tilde{x}^2\big)\bigg](\partial A) = -xj + \frac{\theta^2}{8\Omega^2}\partial j\,. \tag{2.153}$$

(∂A) is now in principle given by the inversion of the operator in the square brackets. We can insert it into the e.o.m.

$$\begin{aligned}
&\big(-\Box + 4\Omega^2\tilde{x}^2\big)A_\mu = -j_\mu - (1-\Omega^2)\partial_\mu(\partial A)\,,\\
&\big(-\Box + 4\Omega^2\tilde{x}^2\big)A_\mu = \\
&\quad -j_\mu - (1-\Omega^2)\partial_\mu\bigg[2 + (1-\Omega^2)x\partial - \frac{\theta^2}{8}\big(-\Box + 4\Omega^2\tilde{x}^2\big)\big(-\Box + 4\tilde{x}^2\big)\bigg]^{-1}\big(-xj + \frac{\theta^2}{8\Omega^2}\partial j\big)\,,
\end{aligned} \tag{2.154}$$

where now the difficulties of the calculation of the propagator have been reduced to the calculation of the inverse of the operator

$$2+(1-\Omega^2)x\partial-\frac{\theta^2}{8}\big(-\Box+4\Omega^2\tilde{x}^2\big)\big(-\Box+4\tilde{x}^2\big)\,. \tag{2.155}$$

In principle, we have shifted the problem of calculating the inverse of a nonminimal operator to a minimal one. Despite of this success, calculating the inverse of (2.155) is still an open question and work in progress. For simplicity we will therefore in the next section switch to the case $\Omega = 1$ where, as we have already seen, the gauge field propagator becomes very simple, namely the Mehler kernel.

2.3.7 Loop Calculations

Various loop calculations have already been performed for the Induced Gauge Theory, and all in all one can say that the graphs show the expected behavior. For simplicity the calculations have been performed in 2 dimensions.

The 1-pt. photonloop tadpole

Nothing surprising is happening here, except that the 3-photon vertex has an additional partial derivative, which has to be taken into account. Starting from

$$T_\mu=\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\int\frac{d^2k'}{(2\pi)^2}G^{AA}_{\nu\rho}(k,k')V^{3A}_{\mu\nu\rho}(k',p,-k)\,, \tag{2.156}$$

which is depicted by Figure (2.7)

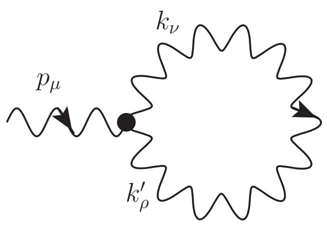


Figure 2.7: Photon tadpole

one can calculate that the graph is <u>logarithmically divergent</u>, as shown in Appendix C.3.1. The result is[15]

$$T^{(1)}_\mu=\frac{113}{25}\mathrm{i}\pi g\Theta^{-1}_{\rho\nu}\ln\left(\varepsilon\right)\,. \tag{2.157}$$

[15] The attentive reader may notice that there is a second Lorentz index ρ, which comes from the coupling to an external field and an expansion around it, as shown in Appendix C.3.1. The result written down here is the only divergent contribution, namely order one of this expansion.

Position space The same calculation can be performed in position space, where the number of terms is in principle bigger, but no derivations appear.
Algebraically, the graph is is given by

$$T_\mu = \frac{1}{2}\int \frac{d^2x}{(2\pi)^2}\int \frac{d^2y}{(2\pi)^2} G^{AA}_{\nu\rho}(x,y)V^{3A}_{\mu\nu\rho}(x,y,z)\,, \tag{2.158}$$

where $\frac{1}{2}$ is the symmetry factor. Plugging in the Feynman rules (2.135),(2.3) yields

$$\begin{aligned}T_\mu = \frac{1}{2}(2\pi)^2 \int \frac{d^2x}{(2\pi)^2}\int \frac{d^2y}{(2\pi)^2}\delta_{\nu\rho}\int\limits_0^\infty d\alpha\,\frac{1}{4\pi\sinh\alpha}e^{-\frac{1}{4\omega}\left((x-y)^2\coth\frac{\alpha}{2}+(x+y)^2\tanh\frac{\alpha}{2}\right)}\\ (2\pi)^6\frac{8g}{\pi^2\theta^2}\cos\left(2(x\times y+y\times z-x\times z)\right)\left[\delta_{\nu\rho}\tilde{x}_\mu+\delta_{\mu\rho}\tilde{y}_\nu+\delta_{\mu\nu}\tilde{z}_\rho\right],\end{aligned} \tag{2.159}$$

$$\begin{aligned}= \frac{4g}{\pi^2\theta^2}(2\pi)^4\int d^2x d^2y \int\limits_0^\infty d\alpha\,\frac{1}{4\pi\sinh\alpha}e^{-\frac{1}{4\omega}\left((x-y)^2\coth\frac{\alpha}{2}+(x+y)^2\tanh\frac{\alpha}{2}\right)}\\ \cos\left(2(x\times y+y\times z-x\times z)\right)\left[2\tilde{x}_\mu+\tilde{y}_\mu+\tilde{z}_\mu\right],\end{aligned} \tag{2.160}$$

The Gaussian integrals are computed with the help of our Mathematica® package. This gives

$$T_\mu = \int\limits_0^\infty d\alpha\,\frac{16\pi^3 g\left(\Omega^2+4\right)\operatorname{csch}(\alpha)\tilde{z}_\mu e^{-\frac{2z^2\Omega\tanh\left(\frac{\alpha}{2}\right)}{\Theta\Omega^2+\Theta}}}{\left(\Omega^2+1\right)^2}\,, \tag{2.161}$$

and when one expands it in α and integrates one gets the expected divergence:

$$T_\mu \simeq -\frac{16\pi^3 g\left(\Omega^2+4\right)\log(\varepsilon)\tilde{z}_\mu}{\left(\Omega^2+1\right)^2} \overset{\Omega\to 1}{=} -20\pi^3 g\ln(\varepsilon)\tilde{z}_\mu\,. \tag{2.162}$$

We can now conclude that calculations in position space do not show derivations in the initial expression, but the lack of the delta function (which in momentum space implements momentum conservation) makes them much larger in the amount of terms. It will be a task of future computations of higher loop graphs to check which basis will suit better to most efficiently end up at the result.
Furthermore we can conclude that Induced Gauge Theory indeed seems to be a good candidate for the first renormalizable gauge field model since it has all desired features and since the first loop calculations reflect the expected behavior.

Chapter 3

Conclusion

In this work we have first motivated noncommutative QFT theory (Section 1.3) with three interesting examples. We have then presented the mathematical structure of noncommutative Euclidean space in order to be able to formulate field theories on it. Explicitly, we have described three interesting models involving oscillator terms. Historically the Grosse Wulkenhaar model, Section 2.1, was the first one to solve the famous UV/IR mixing problem. Indeed the G.W. model has been proven to be renormalizable to all orders. This feature can also be nicely seen from the form of the respective propagator, the so-called Mehler kernel (Section 2.1.1).
The logical next step in this picture has been to continue with gauge theories. The straightforward generalization of the G.W. model to gauge theories is the Mehler kernel gauge model (Section 2.2). The symmetry content of the model as well as the Feynman rules have been examined. Again, the Mehler kernel is the propagator. Loop calculations have been performed (Section 2.2.3) using a self-programmed powerful Mathematica® package (Appendix B). One of the main results of this procedure was that new terms originally not present in the action arise. Those terms are already known from another model, induced gauge theory, which follows from the scalar G.W. model by "inducing" gauge fields (Section 2.3.1). It hence logically makes sense to already initially start from this model instead.
However, the great problem of the unphysical so-called tadpole terms (terms linear in A_μ) had to be solved. Indeed this has been achieved by using a gauge suited to this problem (Section 2.3.2). Also the nontrivial exercise of calculating the new ghost propagator has been managed (Section 2.3.3). Furthermore first loop calculations have been performed (Section 2.3.7).
Despite this success the loop calculations involving the ghost propagator tend to be rather involved and we have to admit that it will require still some effort to compute more complicated graphs. The final aim is now however to show full renormalizability of the induced gauge theory, where we are now one step closer to.

Appendix A

Mehler Kernel Related Calculations

A.1 Multiscale Analysis

A.1.1 Estimating the propagator

Let's start with an easy propagator, namely the one of the commutative ϕ^4 model. We work in an Euclidean world. In momentum space, the propagator is given by the well-known expression

$$\hat{C}(p,q) = \delta(q-p)\frac{1}{q^2+m^2}\,, \tag{A.1}$$

where the delta function represents momentum conservation. If we Fourier transform the propagator

$$\tilde{C}(x-y) = \int \frac{d^dp}{(2\pi)^d}\, e^{\mathrm{i}p(x-y)}\frac{1}{p^2+m^2}\,, \tag{A.2}$$

and use Schwinger parameterization

$$\tilde{C}(x-y) = \int \frac{d^dp}{(2\pi)^d}\int_0^\infty d\alpha\, e^{\mathrm{i}p(x-y)}e^{-\alpha(p^2+m^2)}\,, \tag{A.3}$$

we are able to solve the integral over p and come to the usual form of the propagator in position space:

$$C(x,y) = \int_0^\infty \frac{d\alpha}{(4\pi\alpha)^{d/2}}e^{-\frac{(x-y)^2}{4\alpha}-\alpha m^2}\,. \tag{A.4}$$

If we integrate over α and expand the resulting Bessel function, we will arrive at $C(x,y) = \frac{1}{(x-y)^2}$, but we want to stick at the above form and try to find an approximation for it. Multiscale Analysis does this job. It works as follows. We "slice" the propagator:

$$C^i(x,y) = \int_{M^{-2i}}^{M^{-2(i-1)}} \frac{d\alpha}{(4\pi\alpha)^{d/2}}e^{-\frac{(x-y)^2}{4\alpha}-\alpha m^2}\,, \tag{A.5}$$

with $i \in \mathbb{N}$ and $M > 1$. The zeroth slice is given by

$$C^0(x,y) = \int_1^\infty \frac{d\alpha}{(4\pi\alpha)^{d/2}} e^{-\frac{(x-y)^2}{4\alpha} - \alpha m^2} \,. \tag{A.6}$$

This splits the α-space in a very convenient way (Figure A.1).

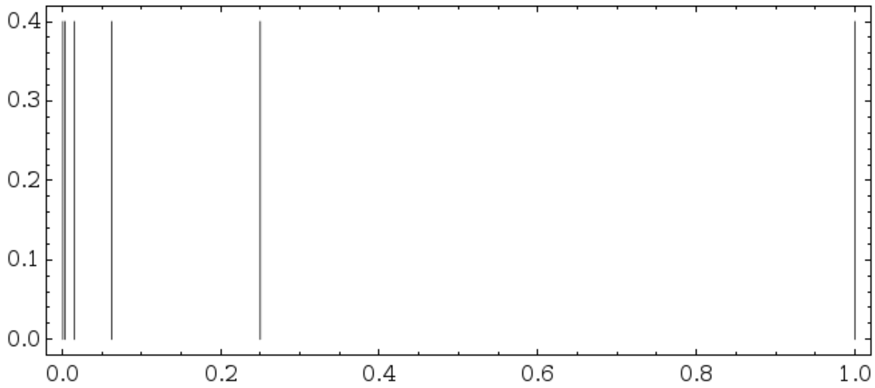


Figure A.1: Multiscale splitting of the space.

We can see that the dangerous region around $\alpha = 0$ is splitted finer and finer as we approach $\alpha \to 0$. Of course it is obvious that $C(x,y) = \sum_{i=0}^{\infty} C^i(x,y)$. One can even introduce an UV-cutoff M^{-2g} with $i \leq g$ at this stage.

The important point is now however that we can bound this sliced propagator, namely

$$C^i(x,y) \leq K_1 M^{(d-2)(i+1)} e^{-KM^{2(i+1)}(x-y)^2} \,, \tag{A.7}$$

where K and K_1 are constants.
Proof: We use the first mean value theorem for integration, which states that

$$\exists \xi \in [a,b], \text{ s.t. } \int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx \,, \tag{A.8}$$

where f is continuous and g integrable in $[a,b]$. For $g = 1$ this becomes

$$\int_a^b f(x)dx = f(\xi)|b-a| \,, \tag{A.9}$$

with $\xi \in (a,b)$ if f is continuous differentiable. If that is the case, it follows also that

$$\int_a^b f(x)dx \leq sup\big(f(x)\big)|b-a| \,, \tag{A.10}$$

where $f(\xi)$ in this case is the $\sup(f)$. If the function f is monotonically increasing and $b > a$, $sup\big(f(x)\big) = f(b)$, of course. That means

$$\int_a^b f(x)dx \le f(b)|b-a| \,. \tag{A.11}$$

In our case the function f is

$$\frac{1}{(4\pi\alpha)^{d/2}} e^{-\frac{(x-y)^2}{4\alpha} - \alpha m^2} \,, \tag{A.12}$$

which is monotonically increasing in α for small values of α (The mass part gives only a finite contribution and can be absorbed into the constants). Therefore we can state that

$$\begin{aligned} C^i(x,y) =& \int_{M^{-2i}}^{M^{-2(i-1)}} \frac{d\alpha}{(4\pi\alpha)^{d/2}} e^{-\frac{(x-y)^2}{4\alpha} - \alpha m^2} \\ \le& |M^{-2(i-1)} - M^{-2i}| \frac{1}{(4\pi M^{-2(i-1)})^{d/2}} e^{-\frac{1}{4M^{-2(i-1)}}(x-y)^2} \,, \end{aligned} \tag{A.13}$$

where we have plugged in the bigger integral bound ($M^{-2(i-1)} > M^{-2i}$) in the place of α. We can simplify this expression

$$\begin{aligned} C^i(x,y) \le& M^{-2i}(M^2-1)M^{di}(4\pi)^{-d/2} e^{-\frac{1}{4}M^{2(i-1)}(x-y)^2} \\ =& M^{(d-2)i}(M^2-1)(4\pi)^{-d/2} e^{-\frac{1}{4M^2}M^{2i}(x-y)^2} \,, \end{aligned} \tag{A.14}$$

and if we define $K = \frac{1}{4M^2}$ and $K_1 = (M^2-1)(4\pi)^{-d/2}$ we arrive as promised at

$$C^i(x,y) \le M^{(d-2)i} K_1 e^{-KM^{2i}(x-y)^2} \,, \tag{A.15}$$

which is the same as (A.7) if we shift i by 1.

We can do something similar with the zeroth slice:

$$C^0(x,y) = \int_1^\infty \frac{d\alpha}{(4\pi\alpha)^{d/2}} e^{-\frac{(x-y^2)}{4\alpha} - \alpha m^2} \,. \tag{A.16}$$

We split the mass part

$$C^0(x,y) = \int_1^\infty \frac{d\alpha}{(4\pi\alpha)^{d/2}} \underbrace{e^{-\frac{(x-y^2)}{4\alpha} - \frac{\alpha m^2}{2}}}_{f(\alpha)} \underbrace{e^{-\frac{\alpha m^2}{2}}}_{g(\alpha)} , \tag{A.17}$$

and use again the mean value theorem for integration (A.8),

$$C^0(x,y) \leq sup\left(e^{-\frac{(x-y)^2}{4\alpha}-\frac{\alpha m^2}{2}}\right)\int\limits_1^\infty e^{-\frac{\alpha m^2}{2}}d\alpha\,. \tag{A.18}$$

$\alpha_0 = \frac{|x-y|}{2m}$ does the job of being the supremum (check by differentiation and setting =0) and the remaining integral of $K_2 = \int\limits_1^\infty e^{-\frac{\alpha m^2}{2}}d\alpha = \frac{2}{m^2}e^{-\frac{m^2}{2}}$ is finite. Hence

$$C^0(x,y) \leq K_2 e^{-|x-y|m}\,. \tag{A.19}$$

Now we come to the point why this is called *Multiscale* Analysis. If we view the propagator (A.4) as a Gauss function (ignoring the massive part for the moment) we can state that it is concentrated in the region where $|x-y| < \alpha^{-1/2}$. If we consider our bound for the sliced propagator (A.7) and view it as a Gauss function we see that the main part of the function is concentrated in the region where $|x-y| \sim M^{-i}$. According to this, $|x-y|$ is directly related to i. Therefore we conclude that fixing the scale attribution (the size of x and y) means fixing the slice. Hence the name Multiscale Analysis.

Application of the slicing scheme to the Mehler kernel and the $1/p^2$ model

- The Mehler kernel:

$$K_M(u,v) = \frac{\omega^3}{8\pi^2}\int\limits_0^\infty d\alpha\,\frac{1}{\sinh^2(\alpha)}e^{-\frac{\omega}{4}\left(u^2\coth\left(\frac{\alpha}{2}\right)+v^2\tanh\left(\frac{\alpha}{2}\right)\right)-\omega\mu_0^2\alpha}\,. \tag{A.20}$$

  In the region $\alpha \in (0,1]$ the function is strictly monotonically increasing. Thus we need to insert the upper bound for α. Since the hyperbolic functions behave for small arguments like

$$\coth\left(\frac{\alpha}{2}\right) \approx \frac{2}{\alpha}$$
$$\tanh\left(\frac{\alpha}{2}\right) \approx \frac{\alpha}{2}\,, \tag{A.21}$$

  the estimation for the sliced Mehler kernel is

$$K_M^i(u,v) \leq KM^{2i}e^{-c(||u||M^i+||v||M^{-i})}\,, \tag{A.22}$$

  which can be cross-checked with Rivasseau's review [7].

- The $1/p^2$ propagator:

$$\Delta(p) = \frac{1}{p^2+m^2+\frac{a}{p^2}} = \int\limits_0^\infty d\alpha\,e^{-\alpha\left(p^2+m^2+\frac{a}{p^2}\right)}\,. \tag{A.23}$$

The integration kernel is here a monotonically decreasing function in α. Therefore the lower bound is our supremum.
The estimation for the sliced $\frac{1}{p^2}$-propagator is therefore

$$\Delta^i(p) \leq e^{-M^{-2i}\left(p^2+m^2+\frac{a}{p^2}\right)} . \tag{A.24}$$

A.1.2 Wilson's picture

If a certain scale is infinite one may come to a renormalized amplitude by subtracting

$$A_G^R(p^2,\mu^2) = A_G(p) - A_G(p^2=\mu^2) , \tag{A.25}$$

where μ^2 is the scale. How does this work in detail? We start at the partition function of the ϕ^4 model

$$Z = \int d\mu_c(\phi) e^{-\frac{\lambda}{4!}\int \phi^4} . \tag{A.26}$$

Our field ϕ is now a sum of fields at different scales

$$\phi = \phi_j + \phi^\rho , \tag{A.27}$$

where $\phi_j = \sum_{i=0}^{j} \phi_i$ and ϕ^ρ is our special scale. With this separation Z becomes

$$Z = \int d\mu_{c_j}(\phi_j) d\mu_{c^\rho}(\phi^\rho) e^{-\frac{\lambda}{4!}(\phi^\rho+\phi_j)^4} . \tag{A.28}$$

By writing $d\mu_{c^\rho}(\phi^\rho)$ in terms of $d\mu_{c^\rho}(\phi)$ we are able to fully separate the scale ρ.
Since $(-\phi_j+\phi)^2 = \phi_j^2 + \phi^2 - 2\phi_j\phi$ we have

$$d\mu_{c^\rho}(\phi^\rho) = e^{-\frac{1}{2}\int \phi^\rho C^{-1}\phi^\rho} = \underbrace{e^{\int \phi C^{-1}\phi}}_{d\mu_{c^\rho}(\phi)} \underbrace{e^{\int \phi (C_\rho)^{-1}\phi_j}}_{\text{source term}} e^{-\frac{1}{2}\int \phi_j (C^\rho)^{-1}\phi_j} . \tag{A.29}$$

We have now averaged over one scale (ρ) and are now at a higher scale with new Feynman rules. Graphically this looks like Figure A.2.

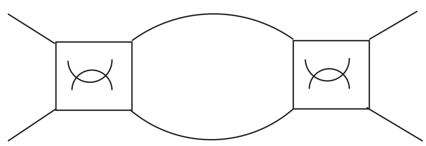


Figure A.2: Averaging a scale.

A.2 Power Counting

To derive the power counting formula (2.48), we first of all state that the degree of (ultraviolet) divergence is given by

$$d_\gamma = 4L - 6I_A - 6I_c + V_c + V_{3A} + V_{\tilde{c}} . \tag{A.30}$$

L is the number of integrations over 4-momenta. Since we are in 4 dimensions we therefore have to multiply L with 4 because we have per integration 4 momenta in the numerator. The propagator in a commutative theory behaves roughly like $\frac{1}{k^2}$. Here we have Mehler kernels which additionally to the $\frac{1}{k^2}$ factor are a smeared 4-dimensional delta function. Hence they behave like $\frac{1}{k^6}$. Thus we have to subtract from $4L$ the number of internal propagators times 6.
The ghost, the 3-photon and the $\tilde{c}$-vertices have each one momentum in their prefactor. Thus we have to add them once to our first degree of divergence estimation.

All these statements should be enough explanation to make formula (A.30) plausible.

$\mapsto$ Secondly we need to find an expression for L. For our specific model it is given by

$$L = 2I_A + 2I_c - (V_c + V_{3A} + V_{4A} + V_{\tilde{c}} - 1)\,. \tag{A.31}$$

The formula can be explained in the following way: The number of integrations over internal momenta is mainly given by the number of internal propagators times 2 (because in the Mehler kernel model we have no momentum conservation and therefore each propagator has one ingoing and one outgoing momentum).
The vertices however always kill one integration (via the delta function which is contained once in each vertex). Thus we have to subtract the number of vertices from our formula.

The little factor +1 in formula (A.31) can be at best understood by looking at a typical example, see figure (A.3): The number of integrations in this example would be 4, coming

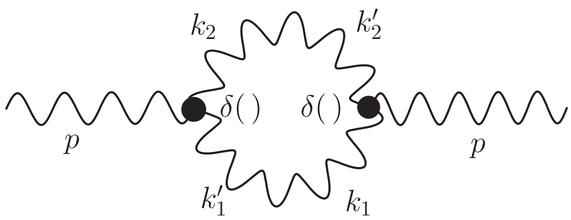


Figure A.3: A typical two point 1 loop graph.

purely from the propagators, but one could naively say that the 2 vertices would reduce the number of integrations down to 2 if one takes the 2 delta functions into account which come with the 2 vertices. However, one forgets in these considerations that one delta function has already been used to turn the outgoing momentum p' into p, or equivalently said, it has already been used to implement momentum conservation.

$\mapsto$ Thirdly we now have to find relations between the connection of propagators and vertices. They are given by:

$$E_{c/\bar{c}} + 2I_c = 2V_c + V_{\tilde{c}} \tag{A.32}$$

$$E_A + 2I_A = V_c + 3V_{3A} + 4V_{4A} + V_{\tilde{c}} \tag{A.33}$$

$$E_{\tilde{c}} = V_{\tilde{c}}\,. \tag{A.34}$$

This needs a little explanation:
Each external line is connected with one end to the graph, each internal line (=propagator) is connected with both ends. This number of connections must fit perfectly together with the number of vertices in the graph, because each vertex has to be connected with a (internal or external) propagator, of course. The formulas above describe per line the ghost, photon or $\tilde{c}$ - connections.
Note for example that despite the fact that the 2 ghost 1 photon vertex has 3 ends to be connected, in (A.33) it is only multiplied with a factor 1, because it has only one photon line for connection.

Using all the formulas we have listed so far, we can easily algebraically come to the following relation:

$$\begin{aligned} d_\gamma &= 2I_A + 2I_c - 3V_c - 3V_{3A} - 4V_{4A} - 3V_{\tilde{c}} + 4 \\ \Rightarrow &\quad \boxed{d_\gamma = 4 - E_A - E_{c/\bar{c}} - E_{\tilde{c}}} \;. \end{aligned} \tag{A.35}$$

A.3 Transversality Breaking

We can explicitly calculate how the right side of eq. (2.52) looks like:

$$\begin{aligned} &\int d^4x \left(\tilde{x}_\mu \frac{\delta^{(3)}\Gamma}{\delta c(z)\delta A_\rho(y)\delta\tilde{c}_\mu(x)} \right) \\ &= \int d^4x \left(-\frac{\Omega^2}{8}\tilde{x}_\mu \frac{\delta^{(2)}}{\delta c\delta A_\rho}\, s\Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} A_\nu\} \Big) \right) \\ &= \int d^4x \left(-\frac{\Omega^2}{8}\tilde{x}_\mu \frac{\delta^{(2)}}{\delta c\delta A_\rho}\, s\Big(\tilde{x}_\mu A_\nu A_\nu + 2A_\nu \xi_\mu A_\nu + A_\nu A_\nu \tilde{x}_\mu \Big) \right) \\ &= \int d^4x \left(-\frac{\Omega^2}{8}\tilde{x}_\mu \frac{\delta^{(2)}}{\delta c\delta A_\rho} \Big(\tilde{x}_\mu D_\mu c A_\nu + \tilde{x}_\mu A_\nu D_\nu c + 2D_\nu c \tilde{x}_\mu A_\nu \right. \\ &\qquad \left. + 2A_\nu \tilde{x}_\mu D_\nu c + D_\nu c A_\nu \tilde{x}_\mu + A_\nu D_\nu c \tilde{x}_\mu \Big) \right) \\ &= \int d^4x \left(-\frac{\Omega^2}{8}\tilde{x}_\mu \frac{\delta^{(2)}}{\delta c\delta A_\rho} \Big(\tilde{x}_\mu \partial_\nu c A_\nu + \tilde{x}_\mu A_\nu \partial_\nu c + 2\partial_\nu c \tilde{x}_\mu A_\nu \right. \\ &\qquad \left. + 2A_\nu \tilde{x}_\mu \partial_\nu c + \partial_\nu c A_\nu \tilde{x}_\mu + A_\nu \partial_\nu c \tilde{x}_\mu \Big) \right) . \end{aligned}$$

By partial integration and by using the relation $\partial_\nu \tilde{x}_\mu = \Theta^{-1}_{\mu\nu}$ we get

$$
\begin{aligned}
= \int d^4x \Bigg(& \frac{\Omega^2}{8} \tilde{x}_\mu \frac{\delta^{(2)}}{\delta c \delta A_\rho} \Big(\Theta^{-1}_{\mu\nu} c A_\nu + \tilde{x}_\mu c \partial_\nu A_\nu + \Theta^{-1}_{\mu\nu} A_\nu c + \tilde{x}_\mu \partial_\nu A_\nu c \\
& + 2c\Theta^{-1}_{\mu\nu} A_\nu + 2c\tilde{x}_\mu \partial_\nu A_\nu + 2A_\nu \Theta^{-1}_{\mu\nu} c + 2\partial_\nu A_\nu \tilde{x}_\mu c \\
& + cA_\nu \Theta^{-1}_{\mu\nu} + c\partial_\nu A_\nu \tilde{x}_\mu + A_\nu c \Theta^{-1}_{\mu\nu} + \partial_\nu A_\nu c \tilde{x}_\mu \Big) \\
& + \frac{\Omega^2}{8} \Theta^{-1}_{\mu\nu} \frac{\delta^{(2)}}{\delta c \delta A_\rho} \Big(\tilde{x}_\mu c A_\nu + \tilde{x}_\mu A_\nu c + 2c\tilde{x}_\mu A_\nu + 2A_\nu \tilde{x}_\mu c + cA_\nu \tilde{x}_\mu + A_\nu c \tilde{x}_\mu \Big) \Bigg) .
\end{aligned}
\tag{A.36}
$$

We can now execute the variation with respect to c and get

$$
\begin{aligned}
= \int d^4x \Bigg(& \frac{\Omega^2}{8} \tilde{x}_\mu \frac{\delta}{\delta A_\rho} \Big(\Theta^{-1}_{\mu\nu} A_\nu + \tilde{x}_\mu \partial_\nu A_\nu + \Theta^{-1}_{\mu\nu} A_\nu + \tilde{x}_\mu \partial_\nu A_\nu \\
& + 2\Theta^{-1}_{\mu\nu} A_\nu + 2\tilde{x}_\mu \partial_\nu A_\nu + 2A_\nu \Theta^{-1}_{\mu\nu} + 2\partial_\nu A_\nu \tilde{x}_\mu \\
& + A_\nu \Theta^{-1}_{\mu\nu} + \partial_\nu A_\nu \tilde{x}_\mu + A_\nu \Theta^{-1}_{\mu\nu} + \partial_\nu A_\nu \tilde{x}_\mu \Big) \delta^{(4)}(x-y) \\
& + \frac{\Omega^2}{8} \Theta^{-1}_{\mu\nu} \frac{\delta}{\delta A_\rho} \Big(\tilde{x}_\mu A_\nu + \tilde{x}_\mu A_\nu + 2\tilde{x}_\mu A_\nu + 2A_\nu \tilde{x}_\mu + A_\nu \tilde{x}_\mu + A_\nu \tilde{x}_\mu \Big) \delta^{(4)}(x-y) \Bigg) ,
\end{aligned}
\tag{A.37}
$$

and also with respect to A_ρ

$$
\begin{aligned}
= & \frac{\Omega^2}{8} \tilde{y}_\mu \Big(\Theta^{-1}_{\mu\nu} \delta_{\nu\rho} + \tilde{y}_\mu \partial_\nu \delta_{\nu\rho} + \Theta^{-1}_{\mu\nu} \delta_{\nu\rho} + \tilde{y}_\mu \partial_\nu \delta_{\nu\rho} \\
& + 2\Theta^{-1}_{\mu\nu} \delta_{\nu\rho} + 2\tilde{y}_\mu \partial_\nu \delta_{\nu\rho} + 2\delta_{\nu\rho} \Theta^{-1}_{\mu\nu} + 2\partial_\nu \delta_{\nu\rho} \tilde{y}_\mu \\
& + \delta_{\nu\rho} \Theta^{-1}_{\mu\nu} + \partial_\nu \delta_{\nu\rho} \tilde{y}_\mu + \delta_{\nu\rho} \Theta^{-1}_{\mu\nu} + \partial_\nu \delta_{\nu\rho} \tilde{y}_\mu \Big) \delta^{(4)}(y-z) \\
+ & \frac{\Omega^2}{8} \Theta^{-1}_{\mu\nu} \Big(\tilde{y}_\mu \delta_{\nu\rho} + \tilde{y}_\mu \delta_{\nu\rho} + 2\tilde{y}_\mu \delta_{\nu\rho} + 2\delta_{\nu\rho} \tilde{y}_\mu + \delta_{\nu\rho} \tilde{y}_\mu + \delta_{\nu\rho} \tilde{y}_\mu \Big) \delta^{(4)}(y-z)
\end{aligned}
\tag{A.38}
$$

$$
\begin{aligned}
= & \frac{\Omega^2}{8} \tilde{y}_\mu \Big(\Theta^{-1}_{\mu\rho} + \tilde{y}_\mu \partial_\rho + \Theta^{-1}_{\mu\rho} + \tilde{y}_\mu \partial_\rho \\
& + 2\Theta^{-1}_{\mu\rho} + 2\tilde{y}_\mu \partial_\rho + 2\Theta^{-1}_{\mu\rho} + 2\partial_\rho \tilde{y}_\mu \\
& + \Theta^{-1}_{\mu\rho} + \partial_\rho \tilde{y}_\mu + \Theta^{-1}_{\mu\rho} + \partial_\rho \tilde{y}_\mu \Big) \delta^{(4)}(y-z) \\
+ & \frac{\Omega^2}{8} \Theta^{-1}_{\mu\rho} \Big(\tilde{y}_\mu + \tilde{y}_\mu + 2\tilde{y}_\mu + 2\tilde{y}_\mu + \tilde{y}_\mu + \tilde{y}_\mu \Big) \delta^{(4)}(y-z)
\end{aligned}
\tag{A.39}
$$

$$
\begin{aligned}
& = \frac{\Omega^2}{8} \Big(16\tilde{y}_\mu \Theta^{-1}_{\mu\rho} + 8(\tilde{y}_\mu)^2 \partial_\rho \Big) \delta^{(4)}(y-z) \\
& \boxed{= \frac{\Omega^2}{\theta^2} \Big(-2y_\rho + y^2 \partial_\rho \Big) \delta^{(4)}(y-z)} .
\end{aligned}
\tag{A.40}
$$

We now evaluate the left hand side of equation (2.52):

$$
\begin{aligned}
&\partial^z_\mu \frac{\delta^{(2)}\Gamma}{\delta A_\rho(y)\delta A_\mu(z)} \\
=&\partial^z_\mu\bigg(\frac{\delta^{(2)}}{\delta A_\rho(y)\delta A_\mu(z)}\frac{\Omega^2}{8}\int d^4x\,\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\star\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\bigg) \\
=&\partial^z_\mu\bigg(\frac{\delta^{(2)}}{\delta A_\rho(y)\delta A_\mu(z)}\frac{\Omega^2}{8}\int d^4x\Big(\tilde{x}_\mu A_\nu \tilde{x}_\mu A_\nu + A_\nu \tilde{x}_\mu \tilde{x}_\mu A_\nu + \tilde{x}_\mu A_\nu A_\nu \tilde{x}_\mu + A_\nu \tilde{x}_\mu A_\nu \tilde{x}_\mu\Big)\bigg) \\
=&\partial^z_\mu\left(\frac{\Omega^2}{8} 8\tilde{z}^2\delta_{\rho\mu}\delta^{(4)}(z-y)\right) = \partial^z_\rho\Big(\Omega^2\tilde{z}^2\delta^{(4)}(z-y)\Big) \\
=&\Omega^2\Big(2\tilde{z}_\xi\Theta^{-1}_{\xi\rho} + \tilde{z}^2\partial_\rho\Big)\delta^{(4)}(z-y) = \boxed{\frac{\Omega^2}{\theta^2}\Big(-2z_\rho + z^2\partial_\rho\Big)\delta^{(4)}(z-y)}\,.
\end{aligned}
\tag{A.41}
$$

The two sides of the equation coincide, and so we have indirectly verified the consistency of the Slavnov Taylor identity with the action.

A.4 Calculation Details of Various Graphs

In this section we enlighten several details concerning the calculation of Feynman graphs in the Mehler kernel gauge model. We are able to distinguish between the planar and the nonplanar part already right from the start. Later on, we will however neglect the UV-divergent part which can be treated by standard renormalization schemes anyway, and focus on the IR region where the power of the oscillator damping term becomes important.

A.4.1 Propagator corrections

The 2-pt 4A graph

Here we calculate graph (a), depicted in Figure 2.4. At a very basic level the graph reads

$$
\Pi^a_{\mu\nu} = \int \frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\frac{1}{2}V^{4A}_{\mu\rho\sigma\nu}(p,-k_1,k_2,-p')(2\pi)^4 K_M(k_1-k_2,k_1+k_2)\delta_{\sigma\rho}\,, \tag{A.42}
$$

where the factor $\frac{1}{2}$ is the symmetry factor corresponding to the graph. If we now switch to long and short variables (functional determinant 1/16) this turns into

$$
\int\frac{d^4u}{(2\pi)^4}\int\frac{d^4v}{(2\pi)^4}\frac{1}{2^5}V^{4A}_{\mu\sigma\sigma\nu}(p,-\frac{v+u}{2},\frac{v-u}{2},-p')(2\pi)^4K_M(u,v)\,. \tag{A.43}
$$

Now we can simply insert the expression for the vertex and the Mehler kernel. One straightforwardly arrives at

$$
\begin{aligned}
\Pi^a_{\mu\nu} = -\frac{3g^2\delta_{\mu\nu}}{8}\int d^4v\,K_M(p-p',v)\Big[&\sin\Big(\tfrac{(v+p')\tilde{p}}{4}\Big)\sin\Big(\tfrac{(v+p)\tilde{p}'}{4}\Big) \\
&+\sin\Big(\tfrac{(v-p')\tilde{p}}{4}\Big)\sin\Big(\tfrac{(v-p)\tilde{p}'}{4}\Big)\Big]\,,
\end{aligned}
\tag{A.44}
$$

where we have eliminated the integral over u with the delta function coming from momentum conservation in the vertex. Some trigonometrics converts this to

$$-\frac{3g^2\delta_{\mu\nu}}{8}\int d^4v\, K_M(p-p',v)\left[-\cos\frac{v(\tilde{p}+\tilde{p}')}{4}+\cos\frac{v(\tilde{p}-\tilde{p}')}{4}\cos\frac{p'\tilde{p}}{4}\right]. \qquad \text{(A.45)}$$

where we can already state which is the planar and which the nonplanar part:
If one sets $p = p'$ one has momentum conservation. This corresponds to not having introduced the oscillator terms in the action, and hence to the naïve model, where it is particularly easy to distinguish between the planar and the nonplanar part because the latter is just the one where the phase remains. If we do this here we see that the right hand side becomes 1 and on the left hand side the phase remains. So, $-\cos\frac{v(\tilde{p}+\tilde{p}')}{4}$ limits towards the nonplanar part and $\cos\frac{v(\tilde{p}-\tilde{p}')}{4}\cos\frac{p'\tilde{p}}{4}$ limits towards the planar one. This notion of defining the nonplanar part as the one where the phase remains is not at all exact. In fact, a clear distinction can only be made at the end of the calculation (after the Gauss integrations have been done): when solving the integral over α the nonplanar part is the one where Bessel functions arise and the planar part is the one where Gamma function come out, after introducing a cutoff ε, of course. Thus the notion of planarness is coupled to the energy behavior: UV is planar and IR is nonplanar. Note, however, that this approach is not the same as Rivasseau *et al.* use in their papers, where they couple the distinction between planar and nonplanar to the genus of a graph. This goes in the direction of renormalizability questions. One-loop graphs e.g. are alway renormalizable.

The next steps in the calculation are the following:

- For the cosine we will use

$$\cos\left(\frac{k\tilde{p}}{2}\right)=\sum_{\eta=\pm1}\frac{1}{2}\exp\left(\frac{\mathrm{i}\eta}{2}k\tilde{p}\right). \qquad \text{(A.46)}$$

- We will approximate the hyperbolic functions in the Mehler kernel:

$$\coth\left(\frac{\alpha}{2}\right)\simeq\frac{2}{\alpha}, \quad \& \quad \tanh\left(\frac{\alpha}{2}\right)\simeq\frac{\alpha}{2}, \quad \& \quad \sinh(\alpha)\simeq\alpha\,, \qquad \text{(A.47)}$$

  exactly around the dangerous region $\alpha = 0$, where the kernel has a quadratic pole, in order to extract the divergent parts of our Feynman graphs. With these approximations the Mehler kernel simply becomes

$$\widetilde{K}_M(p,q)=\frac{\omega^3}{8\pi^2}\int_0^\infty d\alpha\frac{1}{\alpha^2}e^{-\frac{\omega}{4}(p-q)^2\frac{2}{\alpha}-\frac{\omega}{4}(p+q)^2\frac{\alpha}{2}}\,. \qquad \text{(A.48)}$$

- We will, additionally to the inner momenta, integrate over the external momentum p' in order to reveal the divergence structure of the general result without the "smeared out delta function".

With all these steps the momentum integrations are just Gaussian integrals. We end up at

$$\Pi^a_{\mu\nu}=\int_0^\infty d\alpha\,\frac{3g^2\omega^3\delta_{\mu\nu}}{4\alpha^2\pi^2\omega^4\left(1+\frac{\Omega^2}{4}\right)^2}\left(e^{-\frac{2\tilde{p}^2}{4\alpha\omega(1+\frac{\Omega^2}{4})}}-e^{-\frac{\tilde{p}^2\alpha}{2*4\omega(1+\frac{\Omega^2}{4})}}\right). \qquad \text{(A.49)}$$

The last integration over α gives a Bessel function in the case of the nonplanar part and a Gamma function (regularized by a cutoff ε for the lower bound of the integral) for the planar part. By expanding them for small values of p (this is the IR-region we are interested in) we get

$$\Pi^a_{\mu\nu} = -\frac{3g^2\delta_{\mu\nu}}{4\varepsilon\omega\left(1+\frac{\Omega^2}{4}\right)^2} + \frac{3g^2\delta_{\mu\nu}}{2\left(1+\frac{\Omega^2}{4}\right)\tilde{p}^2}\,, \tag{A.50}$$

where we have neglected logarithmic divergent and finite terms.

The 2-pt ghost- and photon loop graphs

Here we calculate the graphs (b)+(c), depicted in Figure 2.4. At the very basic level the Feynman rules for the sum of those two graphs lead to (in terms of long and short variables)

$$\begin{aligned}\Pi^{b,c}_{\mu\nu} = &\int \frac{d^4u_1}{(2\pi)^4}\int\frac{d^4u_2}{(2\pi^4)}\int\frac{d^4v_1}{(2\pi)^4}\int\frac{d^4v_2}{(2\pi)^4}\,(2\pi)^8 K_M(u_1,v_1)K_M(u_2,v_2)\\ &\left(\frac{1}{2}V^{3A}_{\tau\mu\sigma}\Big(\frac{v_2+u_2}{2},-p,-\frac{v_1+u_1}{2}\Big)V^{3A}_{\sigma\nu\tau}\Big(\frac{v_1-u_1}{2},p',-\frac{v_2-u_2}{2}\Big)\right.\\ &\left.-V^{A2c}_{\mu}\Big(\frac{v_2+u_2}{2},-p,-\frac{v_1+u_1}{2}\Big)V^{A2c}_{\nu}\Big(\frac{v_1-u_1}{2},p',-\frac{v_2-u_2}{2}\Big)\right),\end{aligned} \tag{A.51}$$

where the relative symmetry factors (-1 for the ghost loop and 1/2 for the photon loop) have already been included.
We now plug in the expressions for the vertices and use the delta functions to eliminate the integrals over u_2, v_2. We could equivalently have chosen the other two inner momenta. However we can't choose for example two long variables to be eliminated, which would be in favour of divergence behavior, but this is simple not possible due to the structure of the delta functions. A more general treatment of this topic goes under the name of *position routing* and is explained in detail e.g. in [80]. Additionally we will for simplicity call u_1 now u and v_1 v.
Anyway, performing this simple step leads us to

$$\begin{aligned}\Pi^{b,c}_{\mu\nu} = \frac{g^2}{16}&\int d^4u\, d^4v\, K_M(u,v)K_M(u+p-p',v+p+p')\sin\left(\tfrac{(v+u)\tilde{p}}{4}\right)\sin\left(\tfrac{(v-u)\tilde{p}'}{4}\right)\\ &\times\Big[\tfrac{7}{2}(v_\mu v_\nu - u_\mu u_\nu) + \tfrac{5}{2}(u_\mu v_\nu - v_\mu u_\nu) + \tfrac{1}{2}p'_\mu(v+u)_\nu + \tfrac{1}{2}(v-u)_\mu p_\nu\\ &\quad + 2p_\mu(v-u)_\mu + 4(v+u)_\mu p'_\nu + 2p_\mu p'_\nu - 4p'_\mu p_\nu\\ &\quad + \delta_{\mu\nu}\left(\tfrac{v^2-u^2}{2} + \tfrac{p'(v+u)}{2} + \tfrac{(v-u)p}{2} + 5pp'\right)\Big]\,.\end{aligned} \tag{A.52}$$

Before we continue, there is something interesting to be mentioned about the phase factors. The sines may be rewritten

$$
\begin{aligned}
&\sin\left(\tfrac{(v+u)\tilde{p}}{4}\right)\sin\left(\tfrac{(v-u)\tilde{p}'}{4}\right)\\
=&\frac{1}{4}\sum_{\xi=-1,1}\sum_{\eta=-1,1}\xi\eta e^{\frac{\mathrm{i}\eta}{4}(v-u)\tilde{p}'}e^{\frac{\mathrm{i}\xi}{4}(v+u)\tilde{p}'}\\
=&\frac{1}{4}\sum_{\xi=-1,1}\sum_{\eta=-1,1}\xi\eta e^{\frac{\mathrm{i}}{4}v\tilde{p}'\overbrace{(\eta+\xi)}^{\kappa}+\frac{\mathrm{i}}{4}u\tilde{p}'\overbrace{(\xi-\eta)}^{\delta}}\\
=&\frac{1}{4}\sum_{\kappa=-2,0,2}\sum_{\delta=-2,0,2}\frac{\delta^{(2)}-\kappa^2}{4}e^{\frac{\mathrm{i}\kappa}{4}v\tilde{p}'+\frac{\mathrm{i}\delta}{4}u\tilde{p}'}\\
=&\frac{1}{4}\sum_{\kappa=-1,0,1}\sum_{\delta=-1,0,1}(\delta^{(2)}-\kappa^2)e^{\frac{\mathrm{i}\kappa}{2}v\tilde{p}'+\frac{\mathrm{i}\delta}{2}u\tilde{p}'}\,.
\end{aligned}
\tag{A.53}
$$

The prefactor gives only nonzero when either κ or δ is zero. In these cases, the other variable is ±1 and yields a cosine:

$$
=\frac{1}{2}\Bigg[\underbrace{\cos\left(\frac{u\tilde{p}'}{2}\right)}_{\kappa=0}-\underbrace{\cos\left(\frac{v\tilde{p}'}{2}\right)}_{\delta=0}\Bigg]\,. \tag{A.54}
$$

In this form, we can make some statements about the planarity of those graphs. Since the short variable is approximately 0 (momentum conservation is only slightly violated in this model, where slightly refers to the energy scale), the first cosine is approximately 1 and should be the *planar part* which gives in the end the most crucial divergence, namely in zeroth order the $\frac{1}{\varepsilon}$ divergence. This case corresponds to $\kappa = 0$ or $\delta = \pm1$. Conversely, the other term will lead us to the nonplanar divergence. Anyway, let us keep on going with our main calculation.

The next steps are the same as in the case a) but technically much more difficult because on the one hand the expressions are much longer and on the other hand the Gaussian integrals often have a momentum attached to them in the beginning. Technically, especially when treating more complicated integrals like in the vertex renormalization case (where we used the computer) it has proven useful to us to use the following elegant trick:

$$
\int d^4p\,p_\mu e^{-f(p_\mu)}=\frac{d}{dz^\mu}\int d^4p\,e^{-f(p^2)+pz}\Big|_{z=0}\,. \tag{A.55}
$$

With this trick one can calculate the Gauss integral with a constant prefactor, where the difficulties have been handed over to the differentiation with respect to z afterwards.
Thereby we are able to integrate the momenta out. The integral over α leads again to a Bessel and a Gamma integral, like before. However, we have now two parameter integrals α_1 and α_2 (because we have two Mehler kernels). However, even for $n+2$ propagators (and hence $n+2$ parameter integrals)we can perform the following change of variables

$(\alpha_1, \dots, \alpha_{n+2}) \to (\xi_1, \dots, \xi_{n+1}, \lambda)$ with

$$\alpha_1 = \lambda \prod_{i=1}^{n+1} \xi_i \,, \quad \alpha_2 = \lambda(1-\xi_1) \prod_{i=2}^{n+1} \xi_i \,, \qquad \dots\,, \quad \alpha_k = \lambda(1-\xi_{k-1}) \prod_{i=k}^{n+1} \xi_i \,, \qquad \dots\,, \quad \alpha_{n+2} = \lambda(1-\xi_{n+1}) \,, \tag{A.56}$$

where $\xi_i \in [0,1]$ and $\lambda \in [0, \infty[$. The integration measure transforms as

$$\prod_{i=1}^{n+2} d\alpha_i = \lambda^{n+1} \prod_{l=1}^{n} (\xi_{l+1})^l \; d\lambda \prod_{j=1}^{n+1} d\xi_j \,, \tag{A.57}$$

with which the integral over λ gives again a Bessel or a Gamma function, and hence not only a calculation of the integral is easily possible, but also the distinction between planar and nonplanar is clear. The integration over ξ_1 is elementary. In the end we get

$$\Pi^{b,c}_{\mu\nu} = \frac{g^2 \delta_{\mu\nu}}{\pi^2 \varepsilon\omega \left(1+\frac{\Omega^2}{4}\right)^3} - \frac{3g^2\delta_{\mu\nu}}{2\pi^2\left(1+\frac{\Omega^2}{4}\right)^2 \tilde{p}^2} + \frac{2g^2 \tilde{p}_\mu \tilde{p}_\nu}{\pi^2 (\tilde{p}^2)^2 \left(1+\frac{\Omega^2}{4}\right)^2} \,. \tag{A.58}$$

A.4.2 Vertex corrections

The 3-pt 4A3A graph

This is graph (a) in Figure (2.5). The external legs are labelled by $p_1^\mu, p_2^\nu, p_3^\rho$. The graph is given by (in terms of long and short variables)

$$\begin{aligned} V^a_{\mu\nu\rho}(p_1,p_2,p_3) = \frac{3}{2}\frac{1}{2^8} \int \frac{d^4u_1}{(2\pi)^4} \int \frac{d^4v_1}{(2\pi)^4} \int \frac{d^4u_2}{(2\pi)^4} \int \frac{d^4v_2}{(2\pi)^4} \, (2\pi)^8 K_M(u_1,v_1) K_M(u_2,v_2) \\ V^{4A}_{\mu\sigma\tau\nu}\left(p_1, -\frac{v_1+u_1}{2}, \frac{v_2-u_2}{2}, p_2\right) V^{3A}_{\sigma\rho\tau}\left(\frac{v_1-u_1}{2}, p_3, -\frac{v_2+u_2}{2}\right) , \end{aligned} \tag{A.59}$$

where the factor $1/2^8$ is the functional determinant from the substitution into long and short variables and $\frac{3}{2}$ is the symmetry factor. Inserting the propagators and vertices results in

$$
\begin{aligned}
&\int d^4(u,v,u',v')\frac{3}{2^9}(2\mathrm{i}g)^3 K_M(u,v)K_M(u',v')\cancel{(-1)}\\
&\delta^{(4)}(p_1-\frac{v+u}{2}+\frac{v'+u'}{2}-p_2)\delta^{(4)}(\frac{v-u}{2}-p_3-\frac{v'-u'}{2})\\
&\Big[\Big(-\frac{v'-u'}{2}+p_3\Big)_\tau\delta_{\sigma\rho}+\Big(\frac{v-u}{2}+\frac{v'-u'}{2}\Big)_\sigma\delta_{\tau\rho}+\Big(-p_3-\frac{v-u}{2}\Big)_\rho\delta_{\tau\sigma}\Big]\\
&\sin\left(\frac{\frac{v-u}{2}\times\frac{v'-u'}{2}}{2}\right)\cancel{(-1)}\\
&\Bigg[(\delta_{\mu\sigma}\delta_{\tau\nu}-\delta_{\mu\nu}\delta_{\tau\sigma})\sin\left(\frac{p_1\times\frac{v+u}{2}}{2}\right)\sin\left(\frac{\frac{v'+u'}{2}\times p_2}{2}\right)\\
&+(\delta_{\mu\tau}\delta_{\sigma\nu}-\delta_{\mu\nu}\delta_{\tau\sigma})\sin\left(\frac{p_1\times\frac{v'+u'}{2}}{2}\right)\sin\left(\frac{\frac{v+u}{2}\times p_2}{2}\right)\\
&+(\delta_{\mu\tau}\delta_{\sigma\nu}-\delta_{\mu\sigma}\delta_{\tau\nu})\sin\left(\frac{p_1\times p_2}{2}\right)\sin\left(\frac{\frac{v+u}{2}\times\frac{v'+u'}{2}}{2}\right)\Bigg]\,. \qquad \text{(A.60)}
\end{aligned}
$$

As a reminder, the symbol $\times$ is defined in the section Conventions, 1.2.
As in the previous graphs, we now need to eliminate the delta functions. We start with eliminating v', but then realize that no other long variable can be eliminated. As a second variable to be eliminated we choose (for consistency also with a prime) the short variable u'. Hence the two delta functions may be rewritten in the following way

$$
\begin{aligned}
&\delta^{(4)}\Big(p_1-\frac{v+u}{2}+\frac{v'+u'}{2}+p_2\Big)\delta^{(4)}\Big(\frac{v-u}{2}-p_3-\frac{v'-u'}{2}\Big)\\
=&2^4\delta^{(4)}\Big(u'-(u-p_1-p_2+p_3)\Big)\delta^{(4)}\Big(v'-(v-p_1-p_2-p_3)\Big)\,. \qquad \text{(A.61)}
\end{aligned}
$$

Solving the delta functions yields

$$
\begin{aligned}
&\int d^4(u,v)\frac{3}{2^9}(2\mathrm{i}g)^3 K_M(u,v)K_M(u-p_1-p_2+p_3,v-p_1-p_2-p_3)\\
&\Big[\Big(-\frac{v-u-2p_3}{2}+p_3\Big)_\tau\delta_{\sigma\rho}+\Big(\frac{v-u}{2}+\frac{v-u-2p_3}{2}\Big)_\sigma\delta_{\tau\rho}+\Big(-p_3-\frac{v-u}{2}\Big)_\rho\delta_{\tau\sigma}\Big]\\
&\sin\left(\frac{\frac{v-u}{2}\times\frac{\cancel{v-u}-2p_3}{2}}{2}\right)\\
&\Bigg[(\delta_{\mu\sigma}\delta_{\tau\nu}-\delta_{\mu\nu}\delta_{\tau\sigma})\sin\left(\frac{p_1\times\frac{v+u}{2}}{2}\right)\sin\left(\frac{\frac{v+u-2(\cancel{p_2}+p_1)}{2}\times p_2}{2}\right)\\
&+(\delta_{\mu\tau}\delta_{\sigma\nu}-\delta_{\mu\nu}\delta_{\tau\sigma})\sin\left(\frac{p_1\times\frac{v+u-2(p_2+\cancel{p_1})}{2}}{2}\right)\sin\left(\frac{\frac{v+u}{2}\times p_2}{2}\right)\\
&+(\delta_{\mu\tau}\delta_{\sigma\nu}-\delta_{\mu\sigma}\delta_{\tau\nu})\sin\left(\frac{p_1\times p_2}{2}\right)\sin\left(\frac{\frac{v+u}{2}\times\frac{\cancel{v+u}-2(p_2+p_1)}{2}}{2}\right)\Bigg]\,. \qquad \text{(A.62)}
\end{aligned}
$$

We now do some trigonometric exercise using the formula $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$ and we rearrange the terms:

$$
\begin{aligned}
&\int d^4(u,v)\frac{3}{2^9}(2\mathrm{i}g)^3 K_M(u,v)K_M(u-p_1-p_2+p_3, v-p_1-p_2-p_3)\\
&\quad\Big[\Big(-\frac{v-u}{2}+2p_3\Big)_\tau \delta_{\sigma\rho} + \Big(v-u-p_3\Big)_\sigma \delta_{\tau\rho} + \Big(-\frac{v-u}{2}-p_3\Big)_\rho \delta_{\tau\sigma}\Big]\\
&\quad\sin\left(\frac{\frac{v-u}{2}\times p_3}{2}\right)\\
&\quad\Bigg[(\delta_{\mu\sigma}\delta_{\tau\nu}+\delta_{\mu\tau}\delta_{\sigma\nu}-2\delta_{\mu\nu}\delta_{\tau\sigma})\sin\left(\frac{p_1\times(v+u)}{4}\right)\sin\left(\frac{p_2\times(v+u)}{4}\right)\cos\left(\frac{p_1\times p_2}{2}\right)\\
&\quad+(2\delta_{\mu\sigma}\delta_{\tau\nu}-\delta_{\mu\tau}\delta_{\sigma\nu}-\delta_{\mu\nu}\delta_{\tau\sigma})\sin\left(\frac{p_1\times(v+u)}{4}\right)\sin\left(\frac{p_1\times p_2}{2}\right)\cos\left(\frac{p_2\times(v+u)}{4}\right)\\
&\quad+(\delta_{\mu\sigma}\delta_{\tau\nu}-2\delta_{\mu\tau}\delta_{\sigma\nu}+\delta_{\mu\nu}\delta_{\tau\sigma})\sin\left(\frac{p_1\times p_2}{2}\right)\sin\left(\frac{p_2\times(u+v)}{4}\right)\cos\left(\frac{p_1\times(v+u)}{4}\right)\Bigg].
\end{aligned}
\tag{A.63}
$$

Writing the phases in exponential form we may further compactify the expression:

$$
\begin{aligned}
&\int d^4(u,v)\frac{3}{2^9}(2\mathrm{i}g)^3 K_M(u,v)K_M(u-p_1-p_2+p_3, v-p_1-p_2-p_3)\\
&\quad\Big[\Big(-\frac{v-u}{2}+2p_3\Big)_\tau \delta_{\sigma\rho} + \Big(v-u-p_3\Big)_\sigma \delta_{\tau\rho} + \Big(-\frac{v-u}{2}-p_3\Big)_\rho \delta_{\tau\sigma}\Big]\\
&\quad\frac{i}{16}\sum_\kappa\sum_\xi\sum_\eta\sum_\zeta \kappa e^{\frac{i\kappa}{4}(v-u)\tilde{p}_3+\frac{i\xi}{4}(v+u)\tilde{p}_1+\frac{i\eta}{4}(v+u)\tilde{p}_2+\frac{i\zeta}{2}p_1\times p_2}\\
&\quad\Bigg[(\delta_{\mu\sigma}\delta_{\tau\nu}+\delta_{\mu\tau}\delta_{\sigma\nu}-2\delta_{\mu\nu}\delta_{\tau\sigma})\xi\eta\\
&\quad-(2\delta_{\mu\sigma}\delta_{\tau\nu}-\delta_{\mu\tau}\delta_{\sigma\nu}-\delta_{\mu\nu}\delta_{\tau\sigma})\xi\zeta\\
&\quad-(\delta_{\mu\sigma}\delta_{\tau\nu}-2\delta_{\mu\tau}\delta_{\sigma\nu}+\delta_{\mu\nu}\delta_{\tau\sigma})\eta\zeta\Bigg].
\end{aligned}
\tag{A.64}
$$

At this stage of our calculation we are pretty easily able to **distinguish between the planar and the nonplanar part**: when v vanishes in the phase factors we have a pure planar part, because setting $v = 0$ means taking the limit of the Mehler kernels to the Heat kernels, as explained in 2.1.1. This is exactly the case when either $\xi = \eta = 1\,\&\,\kappa = -1$ or when $\xi = \eta = -1\,\&\,\kappa = 1$.
The rest of the work to calculate this graph is performing the Gauss integrations, executing the sums and solving the parameter integrals[1] (of course after approximating the hyperbolic functions, like we did with the 2-pt graphs) which yield Bessel functions (for the nonplanar part) and Gamma functions (for the planar part). They may be expanded for small momenta or for a small UV-cutoff ε, respectively. The UV part can be handled

[1]This has been done by hand and with the help of our Mathematica® package as described in Appendix B.

by standard renormalization procedures. The final result for the IR part of graph (a) is

$$V^a_{\mu\nu\rho}(p_1,p_2,p_3) = -\frac{24\mathrm{i}g^3\left(\tilde{p}_{1,\rho}\delta_{\mu\nu}+\tilde{p}_{1,\nu}\delta_{\mu\rho}+\tilde{p}_{1,\mu}\delta_{\nu\rho}\right)\sin\left(\frac{(\Omega^2-4)(p_1\theta p_2)}{2(\Omega^2+4)}\right)}{\pi^2\Theta^2\left(\Omega^4-16\right)p_1^2(p_1\theta p_2)}$$
$$-\frac{24\mathrm{i}g^3\left(\tilde{p}_{2,\rho}\delta_{\mu\nu}+\tilde{p}_{2,\nu}\delta_{\mu\rho}+\tilde{p}_{2,\mu}\delta_{\nu\rho}\right)\sin\left(\frac{(\Omega^2-4)(p_1\theta p_2)}{2(\Omega^2+4)}\right)}{\pi^2\Theta^2\left(\Omega^4-16\right)p_2^2(p_1\theta p_2)}$$
$$-\frac{12\mathrm{i}g^3\cos\left(\frac{1}{2}(p_1\theta p_2)\right)\left(\tilde{p}_{3,\rho}\delta_{\mu\nu}+\tilde{p}_{3,\nu}\delta_{\mu\rho}+\tilde{p}_{3,\mu}\delta_{\nu\rho}\right)}{\pi^2\Theta^2\left(\Omega^2+4\right)^2p_3^2}, \tag{A.65}$$

and when one expands the trigonometric function for small arguments up to zeroth order, we get

$$V^a_{\mu\nu\rho} = \frac{-24\mathrm{i}g^3}{\pi^2\left(4+\Omega^2\right)^2}\left(\frac{\tilde{p}_{1,\tau}}{\tilde{p}_1^2}+\frac{\tilde{p}_{2,\tau}}{\tilde{p}_2^2}+\frac{\tilde{p}_{3,\tau}}{\tilde{p}_3^2}\right)\left(\delta_{\mu\nu}\delta_{\tau\rho}+\delta_{\mu\rho}\delta_{\tau\nu}+\delta_{\nu\rho}\delta_{\tau\mu}\right). \tag{A.66}$$

The 3-pt photonloop graph

This is graph (b) in Figure (2.5):

$$V^b_{\mu\nu\rho}(p_1,p_2,p_3) =$$
$$\frac{1}{2^{12}}\int\frac{d^4u_1}{(2\pi)^4}\int\frac{d^4v_1}{(2\pi)^4}\int\frac{d^4u_2}{(2\pi)^4}\int\frac{d^4v_2}{(2\pi)^4}\int\frac{d^4u_3}{(2\pi)^4}\int\frac{d^4v_3}{(2\pi)^4}K_M(u_1,v_1)K_M(u_2,v_2)K_M(u_3,v_3)$$
$$(2\pi)^{12}V^{3A}_{\epsilon\mu\sigma}\left(\frac{v_3-u_3}{2},p_1,-\frac{v_1+u_1}{2}\right)V^{3A}_{\sigma\rho\tau}\left(\frac{v_1-u_1}{2},p_3,-\frac{v_2+u_2}{2}\right)V^{3A}_{\tau\nu\epsilon}\left(\frac{v_2-u_2}{2},p_2,-\frac{v_3+u_3}{2}\right), \tag{A.67}$$

where the symmetry factor is 1. Due to the long expressions occurring when calculating this graph, we won't write it down here. The techniques used to derive the result of this graph are also nicely illustrated in the examples of graph (a) and (b). It shall be just mentioned that we have eliminated the following momenta with the delta function: u_1, u_2, u_3. The computer solves the task of integrating the Gaussians and the parameters[2], and we get in the end for this graph

$$V^b_{\mu\nu\rho} = \frac{-8\mathrm{i}g^3}{\pi^2\left(4+\Omega^2\right)^3}\Bigg[\frac{18\tilde{p}_{1,\mu}\tilde{p}_{1,\nu}\tilde{p}_{1,\rho}}{\tilde{p}_1^4}+\frac{18\tilde{p}_{2,\mu}\tilde{p}_{2,\nu}\tilde{p}_{2,\rho}}{\tilde{p}_2^4}+\frac{18\tilde{p}_{3,\mu}\tilde{p}_{3,\nu}\tilde{p}_{3,\rho}}{\tilde{p}_3^4}$$
$$-13\left(\frac{\tilde{p}_{1,\tau}}{\tilde{p}_1^2}+\frac{\tilde{p}_{2,\tau}}{\tilde{p}_2^2}+\frac{\tilde{p}_{3,\tau}}{\tilde{p}_3^2}\right)\left(\delta_{\mu\nu}\delta_{\tau\rho}+\delta_{\mu\rho}\delta_{\tau\nu}+\delta_{\nu\rho}\delta_{\tau\mu}\right)\Bigg]. \tag{A.68}$$

[2] with the package described in Appendix B,

The 3-pt ghostloop graph

This is graph (c) in Figure (2.5).

$$V^c_{\mu\nu\rho}(p_1,p_2,p_3) =$$
$$\frac{-2}{2^{12}}\int\frac{d^4u_1}{(2\pi)^4}\int\frac{d^4v_1}{(2\pi)^4}\int\frac{d^4u_2}{(2\pi)^4}\int\frac{d^4v_2}{(2\pi)^4}\int\frac{d^4u_3}{(2\pi)^4}\int\frac{d^4v_3}{(2\pi)^4}K_M(u_1,v_1)K_M(u_2,v_2)K_M(u_3,v_3)$$
$$(2\pi)^{12}V^c_\mu(\frac{v_3-u_3}{2},p_1,-\frac{v_1+u_1}{2})V^c_\nu(\frac{v_2-u_2}{2},p_3,-\frac{v_2+u_2}{2})V^c_\rho(\frac{v_1-u_1}{2},p_2,-\frac{v_2+u_2}{2})\,, \tag{A.69}$$

where the symmetry factor is (-2). Inserting the Feynman rules gives

$$-\frac{\mathrm{i}g^3\omega^9}{2^{21}\pi^6\alpha_1^2\alpha_2^2\alpha_3^2}\left(-u_{1,\mu}-v_{1,\mu}\right)\left(-u_{3,\nu}-v_{3,\nu}\right)\left(-u_{2,\rho}-v_{2,\rho}\right)$$
$$\sin\left(\frac{1}{2}\left(\frac{1}{2}(-u_3+v_3)\theta p\right)\right)\sin\left(\frac{1}{2}\left(\frac{1}{2}(-u_2+v_2)\theta p'\right)\right)\sin\left(\frac{1}{2}\left(\frac{1}{2}(-u_1+v_1)\theta q\right)\right)$$
$$\exp\left(-\frac{\omega u_1^2}{2\alpha_1}-\frac{\omega u_2^2}{2\alpha_2}-\frac{\omega u_3^2}{2\alpha_3}-\frac{1}{8}\alpha_1\omega v_1^2-\frac{1}{8}\alpha_2\omega v_2^2-\frac{1}{8}\alpha_3\omega v_3^2\right)\,, \tag{A.70}$$

where we have already inserted the approximation for the hyperbolic functions. We now eliminate with the delta function the following momenta: u_1,u_2,u_3. Then the graph becomes

$$-\frac{\mathrm{i}g^3\omega^9}{8192\pi^6\alpha_1^2\alpha_2^2\alpha_3^2}\left(-p_\mu+p'_\mu-q_\mu-v_{1,\mu}+v_{2,\mu}-v_{3,\mu}\right)\left(-p_\nu-p'_\nu+q_\nu+v_{1,\nu}-v_{2,\nu}-v_{3,\nu}\right)$$
$$\left(p_\rho-p'_\rho-q_\rho-v_{1,\rho}-v_{2,\rho}+v_{3,\rho}\right)$$
$$\sin\left(\frac{1}{2}\left(\frac{1}{2}(-p-p'+q+v_1-v_2+v_3)\theta p\right)\right)\sin\left(\frac{1}{2}\left(\frac{1}{2}(p-p'-q-v_1+v_2+v_3)\theta p'\right)\right)$$
$$\sin\left(\frac{1}{2}\left(\frac{1}{2}(-p+p'-q+v_1+v_2-v_3)\theta q\right)\right)$$
$$\exp\Bigg(-\frac{\omega(p+p'-q-v_1+v_2)^2}{2\alpha_3}-\frac{\omega(-p+p'+q+v_1-v_3)^2}{2\alpha_2}-\frac{\omega(p-p'+q-v_2+v_3)^2}{2\alpha_1}$$
$$-\frac{1}{8}v_1^2\alpha_1\omega-\frac{1}{8}v_2^2\alpha_2\omega-\frac{1}{8}v_3^2\alpha_3\omega\Bigg)\,. \tag{A.71}$$

The next steps are rewriting the trigonometric functions into exponential form and solving the Gaussian integrals[3]. In the end one performs the change of variables (A.56) and then gets as usual a Bessel function for the λ integral, which can be expanded, where the zeroth order contains the most severe divergence. The integrals over ξ_1, ξ_2 change only finite values and in the end we get

$$V^c_{\mu\nu\rho}=\frac{8\mathrm{i}g^3}{\pi^2\left(4+\Omega^2\right)^3}\Bigg[\frac{2\tilde{p}_{1,\mu}\tilde{p}_{1,\nu}\tilde{p}_{1,\rho}}{\tilde{p}_1^4}+\frac{2\tilde{p}_{2,\mu}\tilde{p}_{2,\nu}\tilde{p}_{2,\rho}}{\tilde{p}_2^4}+\frac{2\tilde{p}_{3,\mu}\tilde{p}_{3,\nu}\tilde{p}_{3,\rho}}{\tilde{p}_3^4}$$
$$-\left(\frac{\tilde{p}_{1,\tau}}{\tilde{p}_1^2}+\frac{\tilde{p}_{2,\tau}}{\tilde{p}_2^2}+\frac{\tilde{p}_{3,\tau}}{\tilde{p}_3^2}\right)\left(\delta_{\mu\nu}\delta_{\tau\rho}+\delta_{\mu\rho}\delta_{\tau\nu}+\delta_{\nu\rho}\delta_{\tau\mu}\right)\Bigg]\,., \tag{A.72}$$

[3]Which is a great job for the computer, see also Appendix B.

Sum of the Vertex Graphs

When we want to sum up all 3 vertex graphs (A.66, A.68 and A.72), we have to multiply graph (a) with a factor $(4+\Omega^2)$ in order to have the same denominator as the other 2 graphs. Thus, the numerical factor -24 gets multiplied by 4, at least when looking only at the term not proportional to Ω. One can now see nicely that those terms (i.e. the ones not proportional to Ω) in sum vanish,

$$-24 * 4 - 8 + 8 * 13 \stackrel{!}{=} 0\,, \tag{A.73}$$

as expected, because in the literature the usual term has the tensor structure $\sum\limits_{i=1}^{3} \frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{\tilde{p}_i^4}$, whereas we have here additional terms, which are indeed proportional to Ω, so they are obviously stemming from the oscillator potential.
Finally, the sum thus becomes

$$V^{3\mathrm{A,IR}}_{\mu\nu\rho}(p_1,p_2,p_3) = \frac{-8\mathrm{i}g^3}{\pi^2\left(4+\Omega^2\right)^3}\sum_{i=1}^{3}\left[\frac{16\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{\tilde{p}_i^4} + \frac{3\Omega^2}{\tilde{p}_i^2}\left(\delta_{\mu\nu}\tilde{p}_{i,\rho} + \delta_{\mu\rho}\tilde{p}_{i,\nu} + \delta_{\nu\rho}\tilde{p}_{i,\mu}\right)\right], \tag{A.74}$$

which is (2.65), as conjectured.

Appendix B

Description of the Mathematica® Package

In order to (analytically) calculate the more complicated graphs of the Mehler kernel gauge model, Rene I. P. Sedmik, D. N. Blaschke and myself have programmed the Mathematica® package "VectorAlgebra" (*http://sourceforge.net/projects/vectoralgebra/*) dedicated to this job. The basics of this package can be summarized as follows:

- In the beginning, Rene has "taught" Mathematica® to understand index notation for vectors and matrices. He has furthermore implemented in this context new versions of the Mathematica® internal functions *Limit, Simplify, Dot, Cross, D* (derivations) and *Series* by giving them the ability to treat objects with indices. This big part was the footing on which Daniel and I built up the rest of the package. More information on it can be found in Rene's Ph.D. thesis, [81].

- Daniel and I have programmed together the function "IntGauss" which performs d-dimensional Gauss integrals. This is the heart of the package. It can not only treat noncommutative expressions like $p_\mu \theta_{\mu\nu} q_\nu$ in the exponent but also prefactors which may carry indices or not, e.g.

$$\int d^4p \, p_\mu q_\nu p_\sigma e^{-a_1 p^2 + a_2 p \times q + p.q + \dots} \,. \tag{B.1}$$

  Especially nested expressions are treated in such a way that they don't need to be expanded. This was the key element for longer expressions which would have otherwise flooded the Mathematica® kernel.

- Various helpful functions have been programmed as well, like for example "SinToExp" which rewrites trigonometric functions into exponentials.

Below, the reader may find a list of all implemented functions and a short explanation of each one, as it can also be found in the usage documentation of the package:

SetDimension: SetDimension[] is used to set the protected global variable '$Dimension' representing the Euclidean space dimension (4 is default).
DefVec: DefVec[*symbol*] defines '*symbol*' to be a 4-vector. This must be done for each

vector appearing in subsequent calculations (Hint: DefVec also accepts a list of symbols.).
ClearVec: ClearVec[*symbol*] undefines '*symbol*' to be a 4-vector. Thereafter it can be used as a normal Mathematica® symbol without special meaning (Hint: ClearVec also accepts a list of symbols.).
In order to undefine all previously defined vectors, type ClearVec[] without any argument.
IsVec: IsVec[*symbol*] gives True if the given symbol has been defined to be a vector, False in any other case. IsVec is aware of nonvectorial factors and indices of the argument.
KDelta: KDelta[i,j] gives $Dimension (=4 unless changed with SetDimension[]) if i equals j, 0 otherwise. This modified version of the built-in KroneckerDelta[] is useful if Einstein's sum convention is presumed.
VCross: VCross[k,p] is a symbolic version of a matrix contracting two vectors. It acts solely on vectorial objects defined by DefVec and is antisymmetric. In index-style (see fct. 'IndexStyle[]') it is represented by the symbol Θ supplemented by two indices, i.e. IndexStyle[VCross[k,p]] is translated to $k_{\eta_1}\Theta_{\eta_1\eta_2}p_{\eta_2}$. Additional properties are controlled by 'SetupVCross[]'.
Vsimplify: Vsimplify[$expression$, $options : 0 \ldots$] does simplifications in the same way as Simplify does, but is aware of the vectorial calculus and sum convention. It takes any additional options Simplify takes with the exception of 'TransformationFunctions' and 'ComplexityFunction'.
VSimplify: VSimplify[$expression$, $options : 0 \ldots$] does simplifications in the same way as FullSimplify does, but is aware of the vectorial calculus and sum convention. It takes any additional options FullSimplify takes with the exception of 'TransformationFunctions' and 'ComplexityFunction'. A version using Simplify instead of FullSimplify is given by 'Vsimplify[]'.
See also VLSimplify[] which reduces computation time for very long expressions.
VLSimplify: VLSimplify[*expr*,*time*,*opt* : 1,$leaf$: 250,$verbose$: 0] is a version of VSimplify (→ see VSimplify) for very long expressions '*expr*' which crawls through the given formula piece by piece, thereby avoiding to give the whole expression to FullSimplify. This (in most cases) shortens computational times. Set the optional *verbose* argument to 1 to receive more progress information. The parameter $leaf$ controls the allowed complexity (i.e. the maximum LeafCount[]) of the subexpressions, *opt* $= 1, 2$ decides whether to use VSimplify or Vsimplify internally, and *time* is the TimeConstraint which is passed on to the internal Simplify-fcts.
(Hint: Calling VLSimplify[] a second time and/or increasing the value of the variable $leaf$ may in some cases lead to better results - but of course at the cost of increased computation time.)
VLimit: VLimit[$f(x)$, $x \to x_0$, opt:$OutputVect$] takes the limit $x \to x_0$ for the function f respecting all vectorial rules. Eventually the result contains the unit vector UV. If the limit is to be taken in a variable that is not known to be a vector, VLimit utilizes the Mathematica®-internal Limit function. The optional 3rd parameter is a Boolean indicating if the output is given in vec[] form (= 1) or in standard notation (= 0, default).
VSeries: VSeries[$f(x)$, x, x_0, ord] expands the function $f(x)$ into a series around x_0 up to order *ord*. The result is a regular Mathematica® expression (not a Series object as for the standard Series function). VSeries respects analytic vectorial computation rules

for defined vectors.
VD: VD is the vector analysis complement to the standard D derivation in Mathematica®. The syntax VD[$f(x)$, x, *index*] has an additional parameter '*index*' - therefore representing a partial derivative regarding x with index '*index*'. Furthermore, VD[$f(x)$, x,n, $i_1, \ldots, i_n$] can handle multiple derivations with respect to $x_{i_1} x_{i_2} \ldots x_{i_1}$. Additionally, the syntax VD[$f(x_1, \ldots, x_n), \{x1, \ldots, x_n\}$, $\{i_1, ..., i_n\}$] is accepted.
InternalVD: InternalVD is an internal function used by VD, see VD for more information.
θ: The symbol representing VCross[] in index style (cf. ?VCross and ?IndexStyle).
UV: Symbolizes a unit vector.
IndexStyle: IndexStyle[*expr*] transfers a given expression into a form with regular Times products, thereby writing all indices in an explicit form using the symbols η_1, η_2, etc. Note that this deactivates the automatic simplification of vectorial expressions. Use VectorStyle to retransform expressions into normal Dot and VCross syntax, and reactivate the auto-simplification.
VectorStyle: VectorStyle[*expr*] activates the automatic vector simplification rules, and transforms the given expression into a format writing Dot and VCross products wherever possible. (However, depending on the complexity of '*expr*', additional simplifications and subsequent replacements may be required to eliminate all previously introduced indices.)
SizeBrackets: SizeBrackets[*expr*] sizes brackets comparable to \left(and \right) in TeX. WARNING: The output of this function is for display purposes only and cannot be taken as an input to any further calculation!.
ColorBrackets: ColorBrackets[*expr*] colors each bracket level differently and sizes brackets comparable to \left(and \right) in TeX.
WARNING: The output of this function is for display purposes only and cannot be taken as an input to any further calculation!.
ApplyNCrules: ApplyNCrules[*expr*,$time$: 10] tries to simplify '*expr*' by applying several simplification rules. It leaves the expression unchanged when the time needed is larger than '*time*'.
ReleaseDeltas: ReleaseDeltas[expr] treats expressions like KDelta[m_1, s_1] $(3p_{s_1}+4q_{s_1}) \rightarrow (3p_{m_1} + 4q_{m_1})$.
DeltaInt: DeltaInt[*expr*,*arg*,*resolve*] integrates '*expr*' over '*resolve*' by using a Delta function with argument '*arg*'. If '*resolve*' is a vector, integration is 4 (or d) dimensional, else 1-dimensional from -∞ to $+\infty$. '*arg*' may be nonlinear in '*resolve*' provided '*resolve*' is a scalar (i.e. this is not implemented for vectors).
SinToExp: SinToExp[*expr*,opt : 0,ind:ζ,rec : 100,$start$: 0] combines several different functions depending on the option '*opt*':
By default, i.e. $opt = 0$, it rewrites all Sin[] and Cos[] functions in their exponential form (but not the hyperbolic functions).
If *opt* >0 the short notation with sum over ζ_i is used, but the 1/2*sums are not written out. (ζ can be replaced by any other symbol using '*ind*'.)
If $opt < 0$, the 1/2*sums over $\zeta_1 - \zeta_{100}$ are performed, i.e. this option is intended to be used after calling SinToExp with $opt > 0$. If Limit[] is needed for a certain ζ_i, that sum is skipped unless $opt < (-1)$. (The ζ_i sums can also be replaced by sums over '*ind*'$(1 +' start')$ to '*ind*'('*start*'+'*rec*').)
While working with expressions generated by SinToExp with $opt = 1$, it is often useful to do some simplifications such as $\zeta_i^2 \rightarrow 1$ etc. This can be done by calling

SinToExp[arg,$\{\zeta_1,\zeta_2,\ldots\}$].
InternalInt: InternalInt[$expr, p, time : 1, collect : 1, simplesum : 1, delete : 0, verbose : 0$] integrates '*expr*' over (Euclidean) d^4p, assuming it is a Gauss-integral with no occurence of p in front of Exp[].
'*time*' controls the time InternalInt gives VSimplify to simplify subcontributions.
'*collect*' toggles whether InternalInt collects parts of the prefactor after expanding it. '*simplesum*' toggles whether InternalInt internally calls 'Simplesum'.
'*delete*' toggles the call for 'DeleteZeroContribs'. '*verbose*' set to $\neq 0$ prints the Jacobian.
IntGauss: IntGauss[$expr, arg$] computes a 4 (or d) dimensional Gauss integral of expression '*expr*' with respect to the 4-momentum '*arg*'.
Advanced syntax: IntGauss[$expr, arg, time : 1, opt : 1, \{InternalInt{-}options\}, verbose : 0, tilde : 0$], where '*time*' can be a time-constraint on internal simplify-calls, '*opt*' decides which internal functions are used for integration (e.g. opt=2 calls faster but experimental algorithms which leave the expression unexpanded), the InternalIntoptions are passed on to InternalInt[], and '*tilde*' is an optional list of momenta InternalDifz[] uses to let ReplaceTilde[] simplify the differentiated exponent.
SimpleExpand: SimpleExpand[$expr, verbosity : 0, recursionlimit : 100$] represents a faster version of Expand[] for long expressions.
ScanIndices: ScanIndices[$expr, indexname : \eta, imax : 100, opt : 0$] scans '*expr*' for occurrences of '*indexnamei*', where '*i*' is a number between 1 and '*imax*' (defaults to 100), and prints the highest found value of i. If opt is not 0 the first found value of *indexnamei* will be returned.
SimpleSum: SimpleSum[$expr, indexname : \eta$] renames indices '*indexnamei*', where i are integers, to the lowest possible values.
Substitution: Substitution[$expression, vari : 2, varname : \alpha$] substitutes '*vari*' variables *varnamei* to λ and '$vari - 1$' $\{\xi_i\}$ in the expression, so that Sum[*varnamei*]=λ. Alternatively, '*vari*' may also be a list of variables (in which case '*varname*' is ignored).
BGIntDivergent: BGIntDivergent[$expr, arg$] integrates '*expr*' (which is assumed to have the form of a Bessel or Gamma integral) over '*arg*' from 0 to ∞ and returns the divergent part assuming small coefficients $B \to 0$ of $\exp(-B/arg)$, if that is part of '*expr*'.
BGIntComplete: BGIntComplete[$expr, arg, finite : 1$] integrates '*expr*' over '*arg*' from 0 to ∞, taking into account finite terms when $finite$ is set to 1 (=default value).
InternalRewritez: InternalRewritez[$expr, arg, dername : z, indexstart : 1$] rewrites the momentum in the prefactor by a derivation with respect to *dername*, namely as a symbol (for later Gauss integration and derivation).
InternalRewritezexp: Subfunction of InternalRewritez
Do not use separately.
InternalDifz: InternalDifz[$expr, time : 1, name : z1, verbose : 0$] explicitly performs the derivation with respect to '*name*' whenever there is an operator '*dername*',as well as simplifying the derivated expressions with time constraint '*time*'.
InternalDifzSingleExp: =Subfunction of InternalDifz
Do not use separately.
InternalDifzSlow: same as InternalDifz, only slower (and currently more stable) because it expands expressions. . .
CleverResetIndices: CleverResetIndices[$expr, index : \eta, start : 1, imax : (1000 + \$innerindex)$] resets all '*indexname*' to the lowest possible Integers,

starting with '*start*'. In sums, saturated indices are reused.
ResetIndices: ResetIndices[$expr, index : \eta, start : 1, imax : (1000 + \$innerindex)$] resets all '*indexname*' to the lowest possible Integers, starting with '*start*'. (See also CleverResetIndices[].)
ScanAllIndices: ScanAllIndices[$expr, indexname : \eta, imax : (1000 + \$innerindex)$] scans '*expr*' for occurences of '*indexnamei*', where '*i*' is a number between 1 and '*imax*', and prints a list of all found indices.
InternalLevelDepth: InternalLevelDepth[*expr*] counts how many Levels the function has in terms of Head=Times or Head=Plus
SchwingerPara: SchwingerPara[$expr, arg, x$] rewrites expr using Schwinger parametrization for occurrences of $1/arg^n$ using Schwinger parameter x.
DeleteZeroContribs: DeleteZeroContribs[$expr, arg$] is an experimental function to remove some finite contributions concerning Gauss integration of *expr* over arg.
ReplaceTilde: ReplaceTilde[$expr, arg : p$] introduces abbreviations Subscript[pt, η_1] for Subscript[θ,η_1,η_2] Subscript[p,η_2].
(Hint: 'arg' may also be a list of vectors instead of the default p.)
PickTerms: PickTerms[*expr*, *pattern*] picks out the terms of '*expr*' which contain '*pattern*'. (Another version of this function which does not expand the expression, hence only picking out the 'highest order' of a pattern, is given by 'CleverPickTerms'.)
See also the complementary function 'DropTerms'.
CleverPickTerms: CleverPickTerms[*expr*, *pattern*] picks out the terms of '*expr*' which contain '*pattern*' without expanding the expression. Hence, only the 'highest order' of a '*pattern*' is kept.
DropTerms: DropTerms[*expr*, *pattern*] drops the terms of '*expr*' which contain '*pattern*' and is hence equivalent to '*expr* - Pickterms[*expr*, *pattern*]'.
IntegrandSeries: IntegrandSeries[$f(p)$, p, 0, *ord*] is a function similar to VSeries, but it ignores Sin[] and Cos[] functions appearing in $f(p)$. It is meant to be used for extracting potentially divergent terms in loop-calculations for noncommutative models before integrating out the momenta.
VTeXForm: VTeXForm[*expr*] is an extended version of TeXForm[] which also handles symbols/abbreviations introduced in this package when converting to TeX code.
SetupVCross: SetupVCross[$blockdiagonal : 1, thetapara : \theta$] is used to control some properties of VCross[], i.e. SetupVCross[1] (=default) assumes block-diagonal Θ-matrix in even dimensions (with scalar parameter $thetaparam = \theta$) allowing for additional simplification rules. This may be turned off by invoking SetupVCross[0].
(Notice, that this function will likely be extended and hence its syntax change in future versions of VectorAlgebra.)
CheckSyntax: CheckSyntax[*expr*] scans '*expr*' for errors in the use of defined vectors, Dot[] and VCross[].

For readers interested in this package, it may be freely downloaded at
http://sourceforge.net/projects/vectoralgebra/
http://vectoralgebra.sourceforge.net

Appendix C

Induced Gauge Theory Related Calculations

C.1 Derivation of the E.o.M. of the Induced Action

C.1.1 Additional contributions to the dynamical part of the action

We will first compute the following expression

$$\int d^d x \, A_\mu \star A_\nu \star \tilde{x}_\mu \star \tilde{x}_\nu \,, \tag{C.1}$$

where for the star product we use the following formula

$$(f \star g)(x) = e^{\frac{\mathrm{i}}{2}\partial^x_\rho \theta_{\rho\sigma} \partial^y_\sigma} f(x) g(y) \Big|_{x=y} \,. \tag{C.2}$$

For this task let us first evaluate the star product between two coordinates:

$$\begin{aligned} \tilde{x}_\mu \star \tilde{x}_\nu &= e^{\frac{\mathrm{i}}{2}\partial^x_\rho \theta_{\rho\sigma} \partial^y_\sigma} \tilde{x}_\mu \tilde{y}_\nu \Big|_{x=y} = \left(1 + \frac{\mathrm{i}}{2}\partial^x_\rho \theta_{\rho\sigma} \partial^y_\sigma\right) \tilde{x}_\mu \tilde{y}_\nu \Big|_{x=y} \\ &= \tilde{x}_\mu \tilde{y}_\nu + \frac{\mathrm{i}}{2} \underbrace{\theta^{-1}_{\mu\rho} \theta_{\rho\sigma}}_{\delta_{\mu\sigma}} \theta^{-1}_{\nu\sigma} \Big|_{x=y} = \tilde{x}_\mu \tilde{x}_\nu - \frac{\mathrm{i}}{2} \theta^{-1}_{\mu\nu} \,. \end{aligned} \tag{C.3}$$

Taking additionally into account that the expression $\int d^4 x \, A_\mu \star A_\nu \star \theta^{-1}_{\mu\nu}$ is zero because a symmetric object is contracted with an antisymmetric one, expression (C.1) becomes

$$\int d^d x \, A_\mu \star A_\nu \star \tilde{x}_\mu \star \tilde{x}_\nu = \int d^d x \, A_\mu \star A_\nu \star \tilde{x}_\mu \tilde{x}_\nu \,. \tag{C.4}$$

We may now additionally leave out one star due to the cyclic property of the latter:

$$\int d^d x \, A_\mu \star A_\nu \star \tilde{x}_\mu \star \tilde{x}_\nu = \int d^d x \, A_\mu \big(A_\nu \star (\tilde{x}_\mu \tilde{x}_\nu)\big) \,, \tag{C.5}$$

and so the task left over is to calculate

$$
\begin{aligned}
&A_\nu \star \tilde{x}_\mu \tilde{x}_\nu \\
=&e^{\frac{\mathrm{i}}{2}\partial^x_\rho \theta_{\rho\sigma}\partial^y_\sigma} A_\nu(x)\tilde{y}_\mu \tilde{y}_\nu \Big|_{x=y} \\
=&\left(1+\frac{\mathrm{i}}{2}\partial^x_\rho \theta_{\rho\sigma}\partial^y_\sigma - \frac{1}{8}\partial^x_\rho \theta_{\rho\sigma}\partial^y_\sigma \partial^x_\lambda \theta_{\lambda\gamma}\partial^y_\gamma\right) A_\nu(x)\tilde{y}_\mu \tilde{y}_\nu \Big|_{x=y} \\
=&A_\nu(x)\tilde{y}_\mu\tilde{y}_\nu + \frac{\mathrm{i}}{2}\partial^x_\rho A_\nu(x)\theta_{\rho\sigma}\left(\theta^{-1}_{\mu\sigma}\tilde{y}_\nu + \theta^{-1}_{\nu\sigma}\tilde{y}_\mu\right) \\
&\quad - \frac{1}{8}\partial^x_\rho \partial^x_\lambda A_\nu(x)\theta_{\rho\sigma}\theta_{\lambda\gamma}\left(\theta^{-1}_{\mu\gamma}\theta^{-1}_{\nu\sigma} + \theta^{-1}_{\nu\gamma}\theta^{-1}_{\mu\sigma}\right)\Big|_{x=y} \\
=&A_\nu(x)\tilde{y}_\mu\tilde{y}_\nu - \frac{\mathrm{i}}{2}\partial^x_\rho A_\nu(x)\left(\delta_{\rho\mu}\tilde{y}_\nu + \delta_{\rho\nu}\tilde{y}_\mu\right) - \frac{1}{8}\partial^x_\rho\partial^x_\lambda A_\nu(x)\left(\delta_{\lambda\mu}\delta_{\rho\nu} + \delta_{\lambda\nu}\delta_{\rho\mu}\right)\Big|_{x=y} \\
=&\left(\tilde{x}_\mu\tilde{x}_\nu - \frac{\mathrm{i}}{2}\tilde{x}_\nu\partial_\mu - \frac{\mathrm{i}}{2}\tilde{x}_\mu\partial_\nu - \frac{1}{4}\partial_\mu\partial_\nu\right)A_\nu(x) \qquad \text{(C.6)}
\end{aligned}
$$

The imaginary contributions cancel one another, which can be seen by a partial integration

$$\int d^dx \left(-\frac{\mathrm{i}}{2}\right) A_\mu \tilde{x}_\mu \partial_\nu A_\nu = \int d^dx \left(-\frac{\mathrm{i}}{2}\right) A_\nu \tilde{x}_\nu \partial_\mu A_\mu \int d^dx \overset{\text{part. int.}}{=} \frac{\mathrm{i}}{2} A_\mu \tilde{x}_\nu \partial_\mu A_\nu\,, \quad \text{(C.7)}$$

So, all in all our desired formula is

$$\boxed{\int d^dx\, A_\mu \star A_\nu \star \tilde{x}_\mu \star \tilde{x}_\nu = \int d^dx\, A_\mu \tilde{x}_\mu \tilde{x}_\nu A_\nu - \frac{1}{4} A_\mu \partial_\mu \partial_\nu A_\nu}\,, \qquad \text{(C.8)}$$

which shows that apart from the obvious contribution (which one naively expects by leaving out the stars), there is an additional dynamical term, just like the terms coming from $F_{\mu\nu}F_{\mu\nu}$. As described in the main sections, indeed, taking Ω equal to its fixed point will thus lead to a very special form of the propagator.

C.1.2 Anticommutator of the covariant coordinates

The main modification of the action is of course the anticommutator of the covariant coordinates, which will lead to the desired oscillator term and ultimately to the Mehler kernel. In this subsection we will calculate how this term looks like when evaluating the stars. By plugging in the definition of the covariant coordinates we get

$$
\begin{aligned}
\int d^dx \left\{\tilde{X}_\mu \overset{\star}{,} \tilde{X}_\nu\right\}^{\star 2} &= \int d^dx \left\{\tilde{x}_\mu + gA_\mu \overset{\star}{,} \tilde{x}_\nu + gA_\nu\right\}^{\star 2} \\
&= \int d^dx \left(2\tilde{x}_\mu x_\nu + 2g\tilde{x}_\mu A_\nu + 2gA_\mu \tilde{x}_\nu + g^2\left\{A_\mu \overset{\star}{,} A_\nu\right\}\right)^{\star 2}\,, \quad \text{(C.9)}
\end{aligned}
$$

where we have used the formula $\{\tilde{x}_\mu \overset{\star}{,} f\} = 2\tilde{x}_\mu f$. When expanding this expression, we can leave out the star everywhere except for combinations with the last term $\{A_\mu \overset{\star}{,} A_\nu\}$, because it's the only star left (and due to the cyclic property of the star product we can neglect this last star). Thus we get

$$\int d^d x \left\{\widetilde{X}_\mu \overset{\star}{,} \widetilde{X}_\nu\right\}^{\star 2}$$
$$= \int d^d x \Big(4(\tilde{x}^2)^2 + 16g\tilde{x}^2(\tilde{x}A) + 8g^2(\tilde{x}A)^2 + 8g^2\tilde{x}^2A^2 + 4g^2\tilde{x}_\mu\tilde{x}_\nu \star \{A_\mu \overset{\star}{,} A_\nu\}$$
$$+ 4g^3\tilde{x}_\mu A_\nu \star \{A_\mu \overset{\star}{,} A_\nu\} + 4g^3 A_\mu\tilde{x}_\nu \star \{A_\mu \overset{\star}{,} A_\nu\} + g^4\{A_\mu \overset{\star}{,} A_\nu\}\{A_\mu \overset{\star}{,} A_\nu\}\Big). \quad \text{(C.10)}$$

For the fifth term we can use the formula developed in the previous subsection, (C.8). The next two terms are identical, which one can see by renaming the indices $\mu \leftrightarrow \nu$. Also the last term can be simplified using this trick. Hence we are left over with

$$\int d^d x \left\{\widetilde{X}_\mu \overset{\star}{,} \widetilde{X}_\nu\right\}^{\star 2} = \int d^d x \Big(4(\tilde{x}^2) + 16g\tilde{x}^2(\tilde{x}A) + 16g^2(\tilde{x}A)^2 + 8g^2\tilde{x}^2A^2 - 2g^2A_\mu\partial_\mu\partial_\nu A_\nu$$
$$+ 8g^3\tilde{x}_\mu A_\nu \star \{A_\mu \overset{\star}{,} A_\nu\} + 2g^4 A_\mu \star A_\nu \star \{A_\mu \overset{\star}{,} A_\nu\}\Big), \quad \text{(C.11)}$$

where we recognize one tadpole term (which we will bring away with the gauge fixing), some propagator contributions, and two terms producing a new 3-pt. and a new 4-pt vertex, respectively.

C.1.3 Alltogether

We are already used to the terms the F^2 part produces, so I will just state them:

$$\int d^d x\, F_{\mu\nu}F_{\mu\nu}$$
$$= \int d^d x \Big(2A_\mu\left(-\Box\delta_{\mu\nu} + \partial_\mu\partial_\nu\right)A_\nu - 4\mathrm{i}g\partial_\mu A_\nu \star [A_\mu \overset{\star}{,} A_\nu] - 2g^2 A_\mu \star A_\nu \star [A_\mu \overset{\star}{,} A_\nu]\Big). \quad \text{(C.12)}$$

Together with the additional anticommutator of the covariant coordinates and by taking into account all prefactors, the invariant part of the action (2.76) takes the form

$$S_{\text{inv}} = \int d^d x \frac{1}{4}\Big(2A_\mu\left(-\Box\delta_{\mu\nu} + \partial_\mu\partial_\nu\right)A_\nu - 4\mathrm{i}g\partial_\mu A_\nu \star [A_\mu \overset{\star}{,} A_\nu] - 2g^2 A_\mu \star A_\nu \star [A_\mu \overset{\star}{,} A_\nu]\Big)$$
$$+ \frac{\Omega^2}{4g^2}\Big(4(\tilde{x}^2) + 16g\tilde{x}^2(\tilde{x}A) + 16g^2(\tilde{x}A)^2 + 8g^2\tilde{x}^2A^2 - 2g^2A_\mu\partial_\mu\partial_\nu A_\nu$$
$$+ 8g^3\tilde{x}_\mu A_\nu \star \{A_\mu \overset{\star}{,} A_\nu\} + 2g^4 A_\mu \star A_\nu \star \{A_\mu \overset{\star}{,} A_\nu\}\Big). \quad \text{(C.13)}$$

From this part alone we can already conclude that for the fixed point, $\Omega = 1$, the not Lorentz minimal derivative part[1], $A_\mu\partial_\mu\partial_\nu A_\nu$, will vanish. For the other coefficients we need to compare this with the gauge fixing part (2.78):

$$S_{\text{gf}} = \int d^d x \left(-\frac{1}{8\alpha}\right)\left(\{\tilde{x}_\mu \overset{\star}{,} A_\mu\} + \beta\{\tilde{x}_\mu \overset{\star}{,} \tilde{x}_\mu\}\right)^{\star 2} = \int d^d x \left(-\frac{1}{2\alpha}\right)\left(\tilde{x}A + \beta\tilde{x}^2\right)^2$$
$$= \int d^d x \left(-\frac{1}{2\alpha}\right)\Big((\tilde{x}A)^2 + 2\beta\tilde{x}^2(\tilde{x}A) + \beta^2(\tilde{x}^2)^2\Big). \quad \text{(C.14)}$$

[1] nonminimal in the context of the e.o.m.

By looking at the coefficient of $(\tilde{x}A)^2$ we realize that we have to take $\alpha = \frac{1}{8\Omega^2}$ in order to have the simple form of the Mehler kernel for the photon propagator. Additionally, in order for the undesired tadpole term to vanish, we have to take $\beta = \frac{1}{2g}$, which one can see by looking at the coefficient of $(\tilde{x}^2\tilde{x}A)$.
It is also interesting to note that the term $(\tilde{x}^2)^2$ completely vanishes when choosing these values for the constants.
When taking these values for the constants in the gauge fixing, the sum of the invariant and the gauge fixing part becomes

$$
\begin{aligned}
S_{\text{inv}} + S_{\text{gf}} = \int d^dx\, \frac{1}{2}\Bigg(& A_\mu \left(-\Box\delta_{\mu\nu} + (1-\Omega^2)\partial_\mu\partial_\nu + 2\Omega^2\tilde{x}^2\delta_{\mu\nu}\right) A_\nu \\
& - 2\mathrm{i}g\partial_\mu A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu] + 4\Omega^2 g\tilde{x}_\mu A_\nu \star \{A_\mu \stackrel{\star}{,} A_\nu\} \\
& - g^2 A_\mu \star A_\nu \star [A_\mu \stackrel{\star}{,} A_\nu] + g^2\Omega^2 A_\mu \star A_\nu \star \{A_\mu \stackrel{\star}{,} A_\nu\}\Bigg) ,
\end{aligned} \tag{C.15}
$$

and when varying this with respect to the gauge field, one directly ends up at (2.80).

The last task we will perform in this section is to look at the ghost part (2.77):

$$
\begin{aligned}
S_{\text{FP}} &= \int d^dx\, \frac{\mathrm{i}}{2}\{\bar{c} \stackrel{\star}{,} \tilde{x}_\mu\} \star D_\mu c = \int d^dx\, \mathrm{i}\bar{c}\tilde{x}_\mu \star D_\mu c \\
&= \int d^dx\, \mathrm{i}\bar{c}\tilde{x}_\mu\partial_\mu c + g\bar{c}\tilde{x}_\mu \star [A_\mu \stackrel{\star}{,} c] ,
\end{aligned} \tag{C.16}
$$

where we recognize the ghost propagator which is largely discussed in this note (2.3.3), as well as a new vertex for the ghosts.

C.2 Derivation of the Vertices

C.2.1 The 4 photon vertex

We start at (2.138). The variational principle leads us to the corresponding vertex

$$
V^{4A} = -(2\pi)^{4d} \frac{\delta}{\delta A_\lambda(-x_4)} \frac{\delta}{\delta A_\tau(-x_3)} \frac{\delta}{\delta A_\sigma(-x_2)} \frac{\delta}{\delta A_\rho(-x_1)} S^{4A}_{\text{int}} . \tag{C.17}
$$

The first variation gives

$$
\begin{aligned}
V^{4A} = -\frac{(2\pi)^{4d}g^2}{\pi^d\theta^d} & \frac{\delta}{\delta A_\lambda(-x_4)} \frac{\delta}{\delta A_\tau(-x_3)} \frac{\delta}{\delta A_\sigma(-x_2)} \int d^dx d^dy d^dz\, e^{-2\mathrm{i}(x\wedge y + y\wedge z - x\wedge z)} \\
& \Big[\delta(x+x_1)A_\rho(y)A_\nu(z)A_\nu(x-y+z) + \delta(y+x_1)A_\rho(x)A_\nu(z)A_\nu(x-y+z) \\
& + \delta(z+x_1)A_\mu(x)A_\mu(y)A_\rho(x-y+z) + \delta(x-y+z+x_1)A_\mu(x)A_\mu(y)A_\rho(z)\Big] .
\end{aligned} \tag{C.18}
$$

The second variation gives

$$
\begin{aligned}
V^{4A} = &- \frac{(2\pi)^{4d} g^2}{\pi^d \theta^d} \frac{\delta}{\delta A_\lambda(-x_4)} \frac{\delta}{\delta A_\tau(-x_3)} \int d^d x d^d y d^d z \, e^{-2\mathrm{i}(x \wedge y + y \wedge z - x \wedge z)} \\
&\Bigg[\delta(x + x_1) \Bigg[\delta(y + x_2) \delta^{\rho\sigma} A_\nu(z) A_\sigma(x - y + z) \\
&\qquad\qquad + \delta(z + x_2) A_\rho(y) A_\sigma(x - y + z) \\
&\qquad\qquad + \delta(x - y + z + x_2) A_\rho(y) A_\sigma(z) \Bigg] \\
&+ \delta(y + x_1) \Bigg[\delta(x + x_2) \delta^{\rho\sigma} A_\nu(z) A_\nu(x - y + z) \\
&\qquad\qquad + \delta(z + x_2) A_\rho(x) A_\sigma(x - y + z) \\
&\qquad\qquad + \delta(x - y + z + x_2) A_\rho(x) A_\sigma(z) \Bigg] \\
&+ \delta(z + x_1) \Bigg[\delta(x + x_2) A_\sigma(y) A_\rho(x - y + z) \\
&\qquad\qquad + \delta(y + x_2) A_\sigma(x) A_\rho(x + y + z) \\
&\qquad\qquad + \delta(x - y + z + x_2) \delta^{\rho\sigma} A_\mu(x) A_\mu(y) \Bigg] \\
&+ \delta(x - y + z + x_1) \Bigg[\delta(x + x_2) A_\sigma(y) A_\rho(z) \\
&\qquad\qquad + \delta(y + x_2) A_\sigma(x) A_\rho(z) \\
&\qquad\qquad + \delta(z + x_2) \delta^{\rho\sigma} A_\mu(x) A_\mu(y) \Bigg] \Bigg] . \qquad \text{(C.19)}
\end{aligned}
$$

The third and the fourth variation give

$$
\begin{aligned}
V^{4A} =& -\frac{(2\pi)^{4d} g^2}{\pi^d \theta^d} \int d^d x d^d y d^d z \, e^{-2\mathrm{i}(x\wedge y + y\wedge z - x\wedge z)} \\
&\Bigg[\delta(x+x_1) \Bigg[\delta(y+x_2) \delta^{\rho\sigma} \delta^{\tau\lambda} \Big(\delta(z+x_3)\delta(x-y+z+x_4) + \delta(x-y+z+x_3)(z+x_4) \Big) \\
&\qquad + \delta(z+x_2) \Big(\delta^{\rho\tau}\delta^{\lambda\sigma} \delta(y+x_3)\delta(x-y+z+x_4) \\
&\qquad\qquad + \delta^{\sigma\tau}\delta^{\rho\lambda} \delta(x-y+z+x_3)\delta(y+x_4) \Big) \\
&\qquad + \delta(x-y+z+x_2) \Big(\delta^{\rho\tau}\delta^{\lambda\sigma} \delta(y+x_3)\delta(z+x_4) \\
&\qquad\qquad + \delta^{\sigma\tau}\delta^{\rho\lambda} \delta(z+x_3)\delta(y+x_4) \Big) \Bigg] \\
&+ \delta(y+x_1) \Bigg[\delta(x+x_2) \delta^{\rho\sigma} \delta^{\lambda\tau} \Big(\delta(z+x_3)\delta(x-y+z+x_4) + \delta(x-y+z+x_3)\delta(z+x_4) \Big) \\
&\qquad + \delta(z+x_2) \Big(\delta^{\rho\tau}\delta^{\lambda\sigma} \delta(x+x_3)\delta(x-y+z+x_4) \\
&\qquad\qquad + \delta^{\sigma\tau}\delta^{\rho\lambda} \delta(x-y+z+x_3)\delta(x+x_4) \Big) \\
&\qquad + \delta(x-y+z+x_2) \Big(\delta^{\rho\tau}\delta^{\lambda\sigma} \delta(x+x_3)\delta(z+x_4) + \delta^{\sigma\tau}\delta^{\rho\lambda} \delta(z+x_3)\delta(x+x_4) \Big) \Bigg] \\
&+ \delta(z+x_1) \Bigg[\delta(x+x_2) \Big(\delta^{\sigma\tau}\delta^{\rho\lambda} \delta(y+x_3)\delta(x-y+z+x_4) \\
&\qquad\qquad + \delta^{\rho\tau}\delta^{\lambda\sigma} \delta(x-y+z+x_3)\delta(y+x_4) \Big) \\
&\qquad + \delta(y+x_2) \Big(\delta^{\sigma\tau}\delta^{\rho\lambda} \delta(x+x_3)\delta(x-y+z+x_4) \\
&\qquad\qquad + \delta^{\rho\tau}\delta^{\lambda\sigma} \delta(x-y+z+x_3)\delta(x+x_4) \Big) \\
&\qquad + \delta(x-y+z+x_2) \delta^{\rho\sigma}\delta^{\lambda\tau} \Big(\delta(x+x_3)\delta(y+x_4) + \delta(y+x_3)\delta(x+x_4) \Big) \Bigg] \\
&+ \delta(x-y+z+x_1) \Bigg[\delta(x+x_2) \Big(\delta^{\sigma\tau}\delta^{\rho\lambda} \delta(y+x_3)\delta(z+x_4) + \delta^{\rho\tau}\delta^{\lambda\sigma} \delta(z+x_3)\delta(y+x_4) \Big) \\
&\qquad + \delta(y+x_2) \Big(\delta^{\sigma\tau}\delta^{\rho\lambda} \delta(x+x_3)\delta(z+x_4) + \delta^{\rho\tau}\delta^{\lambda\sigma} \delta(z+x_3)\delta(x+x_4) \Big) \\
&\qquad + \delta(z+x_2) \delta^{\rho\sigma}\delta^{\lambda\tau} \Big(\delta(x+x_3)\delta(y+x_4) + \delta(y+x_3)\delta(x+x_4) \Big) \Bigg] \Bigg] .
\end{aligned}
\tag{C.20}
$$

We rearrange now the terms with respect to the tensor structure

$$
\begin{aligned}
V^{4A} &= -\frac{(2\pi)^{4d}g^2}{\pi^d\theta^d}\int d^dxd^dyd^dz\, e^{-2\mathrm{i}(x\wedge y+y\wedge z-x\wedge z)} \\
&\Bigg[\delta^{\rho\tau}\delta^{\lambda\sigma}\Bigg[\delta(x+x_1)\Big(\delta(z+x_2)\delta(y+x_3)\delta(x-y+z+x_4) \\
&\qquad + \delta(x-y+z+x_2)\delta(y+x_3)\delta(z+x_4)\Big) \\
&\quad + \delta(y+x_1)\Big(\delta(z+x_2)\delta(x+x_3)\delta(x-y+z+x_4) \\
&\qquad + \delta(x-y+z+x_2)\delta(x+x_3)\delta(z+x_4)\Big) \\
&\quad + \delta(z+x_1)\Big(\delta(x+x_2)\delta(x-y+z+x_3)\delta(y+x_4) \\
&\qquad + \delta(y+x_2)\delta(x-y+z+x_3)\delta(x+x_4)\Big) \\
&\quad + \delta(x-y+z+x_1)\Big(\delta(x+x_2)\delta(z+x_3)\delta(y+x_4) \\
&\qquad + \delta(y+x_2)\delta(z+x_3)\delta(x+x_4)\Big)\Bigg] \\
&+\delta^{\sigma\tau}\delta^{\lambda\rho}\Bigg[\delta(x+x_1)\Big(\delta(z+x_2)\delta(x-y+z+x_3)\delta(y+x_4) \\
&\qquad + \delta(x-y+z+x_2)\delta(z+x_3)\delta(y+x_4)\Big) \\
&\quad + \delta(y+x_1)\Big(\delta(z+x_2)\delta(x-y+z+x_3)\delta(x+x_4) \\
&\qquad + \delta(x-y+z+x_2)\delta(z+x_3)\delta(x+x_4)\Big) \\
&\quad + \delta(z+x_1)\Big(\delta(x+x_2)\delta(y+x_3)\delta(x-y+z+x_4) \\
&\qquad + \delta(y+x_2)\delta(x+x_3)\delta(x-y+z+x_4)\Big) \\
&\quad + \delta(x-y+z+x_1)\Big(\delta(x+x_2)\delta(y+x_3)\delta(z+x_4) \\
&\qquad + \delta(y+x_2)\delta(x+x_3)\delta(z+x_4)\Big)\Bigg] \\
&+\delta^{\lambda\tau}\delta^{\rho\sigma}\Bigg[\delta(x+x_1)\delta(y+x_2)\Big(\delta(z+x_3)\delta(x-y+z+x_4)+\delta(x-y+z+x_3)\delta(z+x_4)\Big) \\
&\quad + \delta(y+x_1)\delta(x+x_2)\Big(\delta(z+x_3)\delta(x-y+z+x_4)+\delta(x-y+z+x_3)\delta(z+x_4)\Big) \\
&\quad + \delta(z+x_1)\delta(x-y+z+x_2)\Big(\delta(x+x_3)\delta(y+x_4)+\delta(y+x_3)\delta(x+x_4)\Big) \\
&\quad + \delta(x-y+z+x_1)\delta(z+x_2)\Big(\delta(x+x_3)\delta(y+x_4)+\delta(y+x_3)\delta(x+x_4)\Big)\Bigg]\Bigg].
\end{aligned}
\tag{C.21}
$$

Now we are able to solve the integrals with the help of the delta functions

$$
\begin{aligned}
V^{4A} = &-\frac{(2\pi)^{4d} g^2}{\pi^d \theta^d} \\
&\Bigg[\delta^{\rho\tau}\delta^{\lambda\sigma}\Big[e^{2\mathrm{i}(x_1\wedge x_3 + x_3\wedge x_2 - x_1\wedge x_2)}\delta(x_1 - x_3 + x_2 - x_4) \\
&\quad + e^{2\mathrm{i}(x_1\wedge x_3 + x_3\wedge x_4 - x_1\wedge x_4)}\delta(x_1 - x_3 + x_4 - x_2) \\
&\quad + e^{2\mathrm{i}(x_3\wedge x_1 + x_1\wedge x_2 - x_3\wedge x_2)}\delta(x_3 - x_1 + x_2 - x_4) \\
&\quad + e^{2\mathrm{i}(x_3\wedge x_1 + x_1\wedge x_4 - x_3\wedge x_4)}\delta(x_3 - x_1 + x_4 - x_2) \\
&\quad + e^{2\mathrm{i}(x_2\wedge x_4 + x_4\wedge x_1 - x_2\wedge x_1)}\delta(x_2 - x_4 + x_1 - x_3) \\
&\quad + e^{2\mathrm{i}(x_4\wedge x_2 + x_2\wedge x_1 - x_4\wedge x_1)}\delta(x_4 - x_2 + x_1 - x_3) \\
&\quad + e^{2\mathrm{i}(x_2\wedge x_4 + x_4\wedge x_3 - x_2\wedge x_3)}\delta(x_2 - x_4 + x_3 - x_1) \\
&\quad + e^{2\mathrm{i}(x_4\wedge x_2 + x_2\wedge x_3 - x_4\wedge x_3)}\delta(x_4 - x_2 + x_3 - x_1)\Big] \\
&+\delta^{\sigma\tau}\delta^{\lambda\rho}\Big[e^{2\mathrm{i}(x_1\wedge x_4 + x_4\wedge x_2 - x_1\wedge x_2)}\delta(x_1 - x_4 + x_2 - x_3) \\
&\quad + e^{2\mathrm{i}(x_1\wedge x_4 + x_4\wedge x_3 - x_1\wedge x_3)}\delta(x_1 - x_4 + x_3 - x_2) \\
&\quad + e^{2\mathrm{i}(x_4\wedge x_1 + x_1\wedge x_2 - x_4\wedge x_2)}\delta(x_4 - x_1 + x_2 - x_3) \\
&\quad + e^{2\mathrm{i}(x_4\wedge x_1 + x_1\wedge x_3 - x_4\wedge x_3)}\delta(x_4 - x_1 + x_3 - x_2) \\
&\quad + e^{2\mathrm{i}(x_2\wedge x_3 + x_3\wedge x_1 - x_2\wedge x_1)}\delta(x_2 - x_3 + x_1 - x_4) \\
&\quad + e^{2\mathrm{i}(x_3\wedge x_2 + x_2\wedge x_1 - x_3\wedge x_1)}\delta(x_3 - x_2 + x_1 - x_4) \\
&\quad + e^{2\mathrm{i}(x_2\wedge x_3 + x_3\wedge x_4 - x_2\wedge x_4)}\delta(x_2 - x_3 + x_4 - x_1) \\
&\quad + e^{2\mathrm{i}(x_3\wedge x_2 + x_2\wedge x_4 - x_3\wedge x_4)}\delta(x_3 - x_2 + x_4 - x_1)\Big] \\
&+\delta^{\lambda\tau}\delta^{\rho\sigma}\Big[e^{2\mathrm{i}(x_1\wedge x_2 + x_2\wedge x_3 - x_1\wedge x_3)}\delta(x_1 - x_2 + x_3 - x_4) \\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2 + x_2\wedge x_4 - x_1\wedge x_4)}\delta(x_1 - x_2 + x_4 - x_3) \\
&\quad + e^{2\mathrm{i}(x_2\wedge x_1 + x_1\wedge x_3 - x_2\wedge x_3)}\delta(x_2 - x_1 + x_3 - x_4) \\
&\quad + e^{2\mathrm{i}(x_2\wedge x_1 + x_1\wedge x_4 - x_2\wedge x_4)}\delta(x_2 - x_1 + x_4 - x_3) \\
&\quad + e^{2\mathrm{i}(x_3\wedge x_4 + x_4\wedge x_1 - x_3\wedge x_1)}\delta(x_3 - x_4 + x_1 - x_2) \\
&\quad + e^{2\mathrm{i}(x_4\wedge x_3 + x_3\wedge x_1 - x_4\wedge x_1)}\delta(x_4 - x_3 + x_1 - x_2) \\
&\quad + e^{2\mathrm{i}(x_3\wedge x_4 + x_4\wedge x_2 - x_3\wedge x_2)}\delta(x_3 - x_4 + x_2 - x_1) \\
&\quad + e^{2\mathrm{i}(x_4\wedge x_3 + x_3\wedge x_2 - x_4\wedge x_2)}\delta(x_4 - x_3 + x_2 - x_1)\Big]\Bigg].
\end{aligned}
\tag{C.22}
$$

We sort the expression and use the delta function to eliminate x_4:

$$
\begin{aligned}
V^{4A} = -\frac{(2\pi)^{4d}g^2}{\pi^d\theta^d}
&\Bigg[\delta^{\rho\tau}\delta^{\lambda\sigma}\Bigg[e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1+x_2-x_3-x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2-x_3+x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2+x_3-x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1-x_2+x_3+x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1+x_2-x_3-x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2-x_3+x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2+x_3-x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1-x_2+x_3+x_4)\Bigg]\\
&+\delta^{\sigma\tau}\delta^{\lambda\rho}\Bigg[e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1+x_2-x_3-x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2+x_3-x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2-x_3+x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1-x_2+x_3+x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1+x_2-x_3-x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2+x_3-x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2-x_3+x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1-x_2+x_3+x_4)\Bigg]\\
&+\delta^{\lambda\tau}\delta^{\rho\sigma}\Bigg[e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2+x_3-x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2-x_3+x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2+x_3-x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2-x_3+x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2+x_3-x_4)\\
&\quad + e^{2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(x_1-x_2-x_3+x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2+x_3-x_4)\\
&\quad + e^{-2\mathrm{i}(x_1\wedge x_2+x_2\wedge x_3-x_1\wedge x_3)}\delta(-x_1+x_2-x_3+x_4)\Bigg]\Bigg].
\end{aligned}
\tag{C.23}
$$

The exponential function can now be combined to a cosine which can be pulled out

$$\begin{aligned} V^{4A} =& - \frac{(2\pi)^{4d} g^2}{\pi^d \theta^d} 4 \cos\Big(2(x_1 \wedge x_2 + x_2 \wedge x_3 - x_1 \wedge x_3)\Big) \\ & \Big(\delta(x_1 - x_2 + x_3 - x_4) + \delta(x_1 - x_2 - x_3 + x_4)\Big)\Big(\delta^{\rho\tau}\delta^{\lambda\sigma} + \delta^{\sigma\tau}\delta^{\lambda\rho} + \delta^{\lambda\tau}\delta^{\rho\sigma}\Big). \end{aligned} \tag{C.24}$$

C.3 Calculation Details of Various Graphs

C.3.1 The 1-pt photonloop tadpole

We start at (2.156):

$$T_\mu = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2k'}{(2\pi)^2} G^{AA}_{\nu\rho}(k, k') V^{3A}_{\mu\nu\rho}(k', p, -k)\,, \tag{C.25}$$

where $\frac{1}{2}$ is the symmetry factor. Plugging in the Feynman rules (2.117,2.5) yields

$$\begin{aligned} \int d^2k d^2k' \, \frac{1}{2} \delta_{\mu\nu} \frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2 \coth\left(\frac{\alpha}{2}\right) - \frac{\omega}{4}(k+k')^2 \tanh\left(\frac{\alpha}{2}\right)} 2\Omega^2 g (2\pi)^2 \\ \Bigg[\delta_{\mu\rho}\Big(e^{-\frac{\mathrm{i}}{2}p\times k}\Big(-2\mathrm{i}\Theta^{-1}_{\nu\epsilon}(\partial^{k'}_\epsilon - \partial^k_\epsilon) + k'_\nu\Big) - e^{\frac{\mathrm{i}}{2}p\times k} k_\nu\Big) \\ + \delta_{\mu\nu}\Big(e^{-\frac{\mathrm{i}}{2}p\times k}\Big(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}(\partial^{k'}_\epsilon + \partial^p_\epsilon) + p_\rho\Big) + e^{\frac{\mathrm{i}}{2}p\times k} k'_\rho\Big) \\ + \delta_{\nu\rho}\Big(e^{-\frac{\mathrm{i}}{2}p\times k}\Big(-2\mathrm{i}\Theta^{-1}_{\mu\epsilon}(\partial^p_\epsilon - \partial^k_\epsilon) - k_\mu\Big) + e^{\frac{\mathrm{i}}{2}p\times k} p_\mu\Big)\Bigg] \\ \delta^{(d)}(k' + p - k)\,. \end{aligned} \tag{C.26}$$

Evaluating the Kronecker delta changes this to

$$\begin{aligned} \int d^2k d^2k' \, \frac{1}{2} \frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2 \coth\left(\frac{\alpha}{2}\right) - \frac{\omega}{4}(k+k')^2 \tanh\left(\frac{\alpha}{2}\right)} 2\Omega^2 g (2\pi)^2 \\ \Bigg[e^{-\frac{\mathrm{i}}{2}p\times k}\Big(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}(\partial^{k'}_\epsilon - \partial^k_\epsilon) + k'_\rho\Big) - e^{\frac{\mathrm{i}}{2}p\times k} k_\rho \\ + 2\Big(e^{-\frac{\mathrm{i}}{2}p\times k}\Big(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}(\partial^{k'}_\epsilon + \partial^p_\epsilon) + p_\rho\Big) + e^{\frac{\mathrm{i}}{2}p\times k} k'_\rho\Big) \\ + e^{-\frac{\mathrm{i}}{2}p\times k}\Big(-2\mathrm{i}\Theta^{-1}_{\rho\epsilon}(\partial^p_\epsilon - \partial^k_\epsilon) - k_\rho\Big) + e^{\frac{\mathrm{i}}{2}p\times k} p_\rho\Bigg] \\ \delta^{(d)}(k' + p - k)\,. \end{aligned} \tag{C.27}$$

One can now sum up some terms

$$\int d^2k d^2k' \, \frac{1}{2} \frac{\omega^2}{4\pi} \int\limits_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2 \coth\left(\frac{\alpha}{2}\right) - \frac{\omega}{4}(k+k')^2 \tanh\left(\frac{\alpha}{2}\right)} 2\Omega^2 g (2\pi)^2$$
$$\Bigg[2\mathrm{i} e^{-\frac{\mathrm{i}}{2} p\times k} \big(\underbrace{-3\Theta^{-1}_{\rho\epsilon} \partial^{k'}_\epsilon + 2\Theta^{-1}_{\rho\epsilon}\partial^k_\epsilon}_{①} \underbrace{-3\Theta^{-1}_{\rho\epsilon}\partial^p_\epsilon}_{②} \big)$$
$$+ \underbrace{\big(2e^{\frac{\mathrm{i}}{2} p\times k} + e^{-\frac{\mathrm{i}}{2} p\times k}\big) k'_\rho - 2\cos\left(\frac{p\times k}{2}\right) k_\rho + \big(e^{\frac{\mathrm{i}}{2} p\times k} + 2e^{-\frac{\mathrm{i}}{2} p\times k}\big) p_\rho}_{①} \Bigg]$$
$$\delta^{(d)}(k' + p - k) \, . \tag{C.28}$$

The next step is a partial integration to release the delta function. Unfortunately we have (not yet) an integration over d^2p on our disposal, so we will treat the ②-part separately in the end (C.3.1), when we will couple an external field and make an expansion around it. For now, we only calculate the ①-part, and perform the mentioned partial integration:

$$\int d^2k d^2k' \, \frac{1}{2} \delta^{(d)}(k' + p - k) 2\Omega^2 g (2\pi)^2$$
$$\Bigg[2\mathrm{i} \big(3\Theta^{-1}_{\rho\epsilon}\partial^{k'}_\epsilon - 2\Theta^{-1}_{\rho\epsilon}\partial^k_\epsilon \big) e^{-\frac{\mathrm{i}}{2} p\times k}$$
$$+ \big(2e^{\frac{\mathrm{i}}{2} p\times k} + e^{-\frac{\mathrm{i}}{2} p\times k}\big) k'_\rho - 2\cos\left(\frac{p\times k}{2}\right) k_\rho + \big(e^{\frac{\mathrm{i}}{2} p\times k} + 2e^{-\frac{\mathrm{i}}{2} p\times k}\big) p_\rho \Bigg]$$
$$\frac{\omega^2}{4\pi} \int\limits_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2 \coth\left(\frac{\alpha}{2}\right) - \frac{\omega}{4}(k+k')^2 \tanh\left(\frac{\alpha}{2}\right)} \, . \tag{C.29}$$

We let the derivations act on the right

$$\int d^2k d^2k' \, \frac{1}{2} \delta^{(d)}(k' + p - k) 2\Omega^2 g (2\pi)^2 \frac{\omega^2}{4\pi} \int\limits_0^\infty d\alpha$$
$$\Bigg[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon}(k-k')_\epsilon + 5\mathrm{i}\omega \tanh\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon}(k+k')_\epsilon + 2p_\rho \right) e^{-\frac{\mathrm{i}}{2} p\times k}$$
$$+ \big(2e^{\frac{\mathrm{i}}{2} p\times k} + e^{-\frac{\mathrm{i}}{2} p\times k}\big) k'_\rho - 2\cos\left(\frac{p\times k}{2}\right) k_\rho + \big(e^{\frac{\mathrm{i}}{2} p\times k} + 2e^{-\frac{\mathrm{i}}{2} p\times k}\big) p_\rho \Bigg]$$
$$\frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2 \coth\left(\frac{\alpha}{2}\right) - \frac{\omega}{4}(k+k')^2 \tanh\left(\frac{\alpha}{2}\right)} \, , \tag{C.30}$$

and are now able to use the delta function ($k' \to k - p$)

$$\int d^2k \, \frac{1}{2} 2\Omega^2 g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha$$
$$\Bigg[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} p_\epsilon + 5\mathrm{i}\omega \tanh\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} (2k-p)_\epsilon + 2p_\rho \right) e^{-\frac{\mathrm{i}}{2} p\times k}$$
$$+ \big(2e^{\frac{\mathrm{i}}{2}p\times k} + e^{-\frac{\mathrm{i}}{2}p\times k}\big)(k-p)_\rho - 2\cos\left(\frac{p\times k}{2}\right) k_\rho + \big(e^{\frac{\mathrm{i}}{2}p\times k} + 2e^{-\frac{\mathrm{i}}{2}p\times k}\big) p_\rho \Bigg]$$
$$\frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}p^2 \coth\left(\frac{\alpha}{2}\right) - \frac{\omega}{4}(2k-p)^2 \tanh\left(\frac{\alpha}{2}\right)}, \tag{C.31}$$

which enables us to further sum up terms

$$\int d^2k \, \frac{1}{2} 2\Omega^2 g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha$$
$$\Bigg[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} p_\epsilon + 5\mathrm{i}\omega \tanh\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} (2k-p)_\epsilon + 2p_\rho \right) e^{-\frac{\mathrm{i}}{2} p\times k}$$
$$+ e^{\frac{\mathrm{i}}{2}p\times k} k_\rho - 2\sin\left(\frac{p\times k}{2}\right) p_\rho \Bigg]$$
$$\frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}p^2 \coth\left(\frac{\alpha}{2}\right) - \omega(k-\frac{p}{2})^2 \tanh\left(\frac{\alpha}{2}\right)}. \tag{C.32}$$

In order to solve the Gauss integral we need to shift $k - \frac{p}{2} \to \bar{k}$

$$\int d^2k \, \frac{1}{2} 2\Omega^2 g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha$$
$$\Bigg[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} p_\epsilon + 10\mathrm{i}\omega \tanh\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} k_\epsilon \right) e^{-\frac{\mathrm{i}}{2} p\times k}$$
$$+ e^{\frac{\mathrm{i}}{2}p\times k} k_\rho + \big(-\frac{1}{2} e^{\frac{\mathrm{i}}{2}p\times k} + 3e^{-\frac{\mathrm{i}}{2}p\times k}\big) p_\rho \Bigg]$$
$$\frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}p^2 \coth\left(\frac{\alpha}{2}\right) - \omega k^2 \tanh\left(\frac{\alpha}{2}\right)}, \tag{C.33}$$

where we have again renamed $\bar{k} \to k$. We now collect all exponential factors and use the trick to rewrite the momenta in the prefactor as derivations with respect to z

$$\int d^2k \, \frac{1}{2} 2\Omega^2 g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha$$
$$\left[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} p_\epsilon + 10\mathrm{i}\omega \tanh\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} \frac{\partial}{\partial z^\epsilon} + 3p_\rho \right) e^{-\frac{\mathrm{i}}{2} p \times k} \right.$$
$$\left. + \left(\frac{\partial}{\partial z} - \frac{1}{2} p\right)_\rho e^{\frac{\mathrm{i}}{2} p \times k} \right]$$
$$\frac{1}{\sinh \alpha} e^{-\frac{\omega}{4} p^2 \coth\left(\frac{\alpha}{2}\right) - \omega k^2 \tanh\left(\frac{\alpha}{2}\right) + kz} \Bigg|_{z=0} . \tag{C.34}$$

We can solve the Gauss integral

$$\frac{1}{2} 2\Omega^2 g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{\pi}{\omega \tanh\left(\frac{\alpha}{2}\right)}$$
$$\left[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} p_\epsilon + 10\mathrm{i}\omega \tanh\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} \frac{\partial}{\partial z^\epsilon} + 3p_\rho \right) e^{\frac{(z - \frac{\mathrm{i}}{2}\tilde{p})^2}{4\omega} \coth\left(\frac{\alpha}{2}\right)} \right.$$
$$\left. + \left(\frac{\partial}{\partial z} - \frac{1}{2} p\right)_\rho e^{\frac{(z + \frac{\mathrm{i}}{2}\tilde{p})^2}{4\omega} \coth\left(\frac{\alpha}{2}\right)} \right]$$
$$\frac{1}{\sinh \alpha} e^{-\frac{\omega}{4} p^2 \coth\left(\frac{\alpha}{2}\right)} \Bigg|_{z=0} , \tag{C.35}$$

and execute the derivations

$$\frac{1}{2} 2\Omega^2 g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{\pi}{\omega \tanh\left(\frac{\alpha}{2}\right)}$$
$$\left[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} p_\epsilon + 10\mathrm{i}\omega \cancel{\tanh\left(\frac{\alpha}{2}\right)} \Theta^{-1}_{\rho\epsilon} \frac{1}{4\omega} \cancel{\coth\left(\frac{\alpha}{2}\right)} (2z - i\tilde{p})_\epsilon + 3p_\rho \right) e^{\frac{(z - \frac{\mathrm{i}}{2}\tilde{p})^2}{4\omega} \coth\left(\frac{\alpha}{2}\right)} \right.$$
$$\left. + \left(\frac{1}{4\omega} \coth\left(\frac{\alpha}{2}\right) (2z + \mathrm{i}\tilde{p}) - \frac{1}{2} p \right)_\rho e^{\frac{(z + \frac{\mathrm{i}}{2}\tilde{p})^2}{4\omega} \coth\left(\frac{\alpha}{2}\right)} \right]$$
$$\frac{1}{\sinh \alpha} e^{-\frac{\omega}{4} p^2 \coth\left(\frac{\alpha}{2}\right)} \Bigg|_{z=0} . \tag{C.36}$$

Setting $z = 0$ simplifies this to

$$\frac{1}{2}2\Omega^2 g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{\pi}{\omega \tanh\left(\frac{\alpha}{2}\right)}$$
$$\left[\left(5\mathrm{i}\omega \coth\left(\frac{\alpha}{2}\right) \Theta^{-1}_{\rho\epsilon} p_\epsilon + \frac{5}{2}\Theta^{-1}_{\rho\epsilon}\tilde{p}_\epsilon + 3p_\rho\right) e^{\frac{-\tilde{p}^2}{16\omega}\coth\left(\frac{\alpha}{2}\right)}\right.$$
$$\left. + \left(\frac{\mathrm{i}}{4\omega}\coth\left(\frac{\alpha}{2}\right)\tilde{p} - \frac{1}{2}p\right)_\rho e^{\frac{-\tilde{p}^2}{16\omega}\coth\left(\frac{\alpha}{2}\right)}\right]$$
$$\frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}p^2 \coth\left(\frac{\alpha}{2}\right)} \tag{C.37}$$
$$\stackrel{\Omega=1}{=} \frac{1}{2}2g(2\pi)^2 \frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{\pi}{\omega \tanh\left(\frac{\alpha}{2}\right)}$$
$$\left[\frac{19}{4}\mathrm{i}\coth\left(\frac{\alpha}{2}\right)\omega\Theta^{-1}_{\rho\epsilon}p_\epsilon + 5p_\rho\right]$$
$$\frac{1}{\sinh\alpha} e^{-\frac{5\theta}{16}p^2 \coth\left(\frac{\alpha}{2}\right)} . \tag{C.38}$$

By approximating the hyperbolic functions by their arguments this can be integrated, and yields

$$T_\mu = \frac{16\pi^2 g\omega p_\rho}{p^2\theta} + \frac{1216\mathrm{i}\pi^2 g\omega^2 p_\epsilon \Theta^{-1}_{\rho\epsilon}}{25p^4\theta^2} . \tag{C.39}$$

This high degree of divergence should not shock us since we haven't coupled an external field and integrated over the external momentum, like we did in the old model. This is used to compensate for the Mehler kernel being a smeared out delta function.
Thus, we couple an external field to (C.38) in the following way

$$\int \frac{d^dp}{(2\pi)^d}\,\Pi_\mu(p)\left[A_\mu(0) + p_\nu\left(\partial^p_\nu A_\mu(p)\big|_{p=0}\right) + \frac{p_\nu p_\rho}{2}\left(\partial^p_\nu\partial^p_\rho A_\mu(p)\big|_{p=0}\right) + \right.$$
$$\left. + \frac{p_\nu p_\rho p_\sigma}{6}\left(\partial^p_\nu\partial^p_\rho\partial^p_\sigma A_\mu(p)\big|_{p=0}\right) + \ldots\right] . \tag{C.40}$$

All even orders are zero for symmetry reasons. For the uneven orders, we get

- **order 1**:

$$\int_0^\infty d\alpha \left(-\frac{32\pi g\omega \tanh\left(\frac{\alpha}{2}\right)\delta_{\nu\rho}}{5\theta^2 \sinh\alpha} - \frac{152i\pi g\omega^2\Theta^{-1}_{\rho\nu}}{25\theta^2\sinh\alpha}\right) \tag{C.41}$$

  which gives when introducing a cutoff ε for the lower bound of α:

$$-\frac{32\pi g\omega\delta_{\nu\rho}}{5\theta^2}(1 - \tanh\varepsilon) + \frac{152\mathrm{i}\pi g\omega^2\Theta^{-1}_{\rho\nu}}{25\theta^2}\ln\left(\tanh\left(\frac{\varepsilon}{2}\right)\right) \tag{C.42}$$

$$\simeq \frac{152\mathrm{i}\pi g\omega^2\Theta^{-1}_{\rho\nu}}{25\theta^2}\ln(\varepsilon) + \text{finite contributions.} \tag{C.43}$$

  This is the expected logarithmic divergence.

- **order 3**: Will be computed together with the second part below.

The second part

The part ② of (C.28) we are discussing is given by

$$\int d^2k d^2k' \, \frac{1}{2}\frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2\coth\left(\frac{\alpha}{2}\right)-\frac{\omega}{4}(k+k')^2\tanh\left(\frac{\alpha}{2}\right)} 2\Omega^2 g(2\pi)^2 \left(- 6\mathrm{i}e^{-\frac{\mathrm{i}}{2}p\times k}\Theta^{-1}_{\rho\epsilon}\partial^p_\epsilon\right)\delta^{(2)}(k'+p-k)\,. \tag{C.44}$$

We now couple an external field to this (C.40).

- **Zeroth order**
  The zeroth order vanishes. This can be seen by two ways: one is directly: partial integration brings down $\tilde{k}$ which is replaced by $\tilde{k}'+\tilde{p}$ by the delta function. Integration over k' and p' respectively are zero for symmetry reasons. Another way to see this is that we could have equivalently taken $k\times k'$ as the argument of the phase when plugging in the vertex, since all legs are equal. In this case ∂^p is a total derivative and vanishes.
  Using a similar argument for the other orders we can alter the phase in the following way

  $$\int d^2k d^2k' \, \frac{1}{2}\frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2\coth\left(\frac{\alpha}{2}\right)-\frac{\omega}{4}(k+k')^2\tanh\left(\frac{\alpha}{2}\right)} 2\Omega^2 g(2\pi)^2 \left(- 6\mathrm{i}e^{-\frac{\mathrm{i}}{2}k\times k'}\Theta^{-1}_{\rho\epsilon}\partial^p_\epsilon\right)\delta^{(2)}(k'+p-k)\,, \tag{C.45}$$

  which makes the calculation for us easier.

- **First order**
  Partial integration gives

  $$\begin{aligned}
  &\int \frac{d^2p}{(2\pi)^2} d^2k d^2k' \, \frac{1}{2}\frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2\coth\left(\frac{\alpha}{2}\right)-\frac{\omega}{4}(k+k')^2\tanh\left(\frac{\alpha}{2}\right)} 2\Omega^2 g(2\pi)^2 \\
  &\qquad \delta^{(2)}(k'+p-k) 6\mathrm{i}e^{-\frac{\mathrm{i}}{2}k\times k'}\Theta^{-1}_{\rho\epsilon}\partial^p_\epsilon p_\nu \\
  &= \int d^2p d^2k d^2k' \, \frac{1}{2}\frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}(k-k')^2\coth\left(\frac{\alpha}{2}\right)-\frac{\omega}{4}(k+k')^2\tanh\left(\frac{\alpha}{2}\right)} 2\Omega^2 g \\
  &\qquad \delta^{(2)}(k'+p-k) 6\mathrm{i}e^{-\frac{\mathrm{i}}{2}k\times k'}\Theta^{-1}_{\rho\nu} \\
  &= \int d^2p d^2k \, \frac{1}{2}\frac{\omega^2}{4\pi} \int_0^\infty d\alpha \, \frac{1}{\sinh\alpha} e^{-\frac{\omega}{4}p^2\coth\left(\frac{\alpha}{2}\right)-\frac{\omega}{4}(2k-p)^2\tanh\left(\frac{\alpha}{2}\right)+\frac{i}{2}k\times p} 2\Omega^2 g 6\mathrm{i}\Theta^{-1}_{\rho\nu}\,.
  \end{aligned} \tag{C.46}$$

  Solving the integration over k leads us to

  $$-2\Omega^2 g 6\mathrm{i}\Theta^{-1}_{\rho\nu}\frac{1}{2}\frac{\omega^2}{4\pi}\int d^2p \int_0^\infty d\alpha \, \frac{\pi\coth\left(\frac{\alpha}{2}\right) e^{-\frac{p^2\coth\left(\frac{\alpha}{2}\right)\left(\Theta^2+4\omega^2\right)}{16\omega}}}{\omega\sinh\alpha}\,. \tag{C.47}$$

And the last Gauss integration over p evaluates to

$$\begin{aligned}
&\int_0^\infty d\alpha\,(-2)\Omega^2 g6\mathrm{i}\Theta^{-1}_{\rho\nu}\frac{1}{2}\frac{\omega^2}{4\pi}\frac{16\pi^2}{(\Theta^2+4\omega^2)\sinh\alpha} \\
&=\int_0^\infty d\alpha\; -6\pi\Omega^2 g\mathrm{i}\Theta^{-1}_{\rho\nu}\frac{1}{(1+\Omega^2)\sinh\alpha}\,, \qquad \text{(C.48)}
\end{aligned}$$

which yields when introducing a cutoff ε

$$6\pi\Omega^2 g\mathrm{i}\Theta^{-1}_{\rho\nu}\frac{1}{(1+\Omega^2)}\ln\left(\tanh\left(\frac{\varepsilon}{2}\right)\right)\simeq 6\pi\Omega^2 g\mathrm{i}\Theta^{-1}_{\rho\nu}\frac{1}{(1+\Omega^2)}\ln(\varepsilon)\overset{\Omega\to 1}{=}3\pi g\mathrm{i}\Theta^{-1}_{\rho\nu}\ln(\varepsilon)\,, \qquad \text{(C.49)}$$

which is again a logarithmic divergence.
The sum of both parts, (C.43) and (C.49), is

$$T^{(1)}_\mu=\frac{38\mathrm{i}\pi g\Theta^{-1}_{\rho\nu}}{25}\ln(\varepsilon)+3\pi g\mathrm{i}\Theta^{-1}_{\rho\nu}\ln(\varepsilon)=\frac{113}{25}\mathrm{i}\pi g\Theta^{-1}_{\rho\nu}\ln(\varepsilon)\,, \qquad \text{(C.50)}$$

as conjectured in (2.157).

- **Third order**:
We will compute it all in one, part ① and ② together. The third order in the expansion in the external field of (C.38) is

$$\begin{aligned}
&\int\frac{d^2p}{(2\pi)^2}\,\tilde{p}_\nu p_\rho p_\sigma\frac{1}{2}2g(2\pi)^2\frac{\omega^2}{4\pi}\int_0^\infty d\alpha\,\frac{\pi}{\omega\tanh\left(\frac{\alpha}{2}\right)} \\
&\left[\frac{19}{4}\mathrm{i}\coth\left(\frac{\alpha}{2}\right)\omega\Theta^{-1}_{\rho\epsilon}p_\epsilon+5p_\rho\right]\;\frac{1}{\sinh\alpha}e^{-\frac{5\theta}{16}p^2\coth\left(\frac{\alpha}{2}\right)}\,. \qquad \text{(C.51)}
\end{aligned}$$

Performing the Gaussian integral over p and then the parameter integral over α yields

$$\begin{aligned}
&\frac{8\pi g\omega}{125\theta^3}\Big(-4\mathrm{i}\left(-38\omega\left(\delta_{\rho\sigma}\Theta_{\mu\nu}+\delta_{\nu\sigma}\Theta_{\mu\rho}+2\delta_{\nu\rho}\Theta_{\mu\sigma}\right)+15(2\pi+\mathrm{i}\ln(2))\delta_{\mu\nu}\delta_{\rho\sigma}\right) \\
&\quad+45\delta_{\mu\nu}\delta_{\rho\sigma}+15(3-8\mathrm{i}\pi+\ln(16))\delta_{\mu\sigma}\delta_{\nu\rho}+15(3-8\mathrm{i}\pi+\ln(16))\delta_{\mu\rho}\delta_{\nu\sigma}\Big)\,, \qquad \text{(C.52)}
\end{aligned}$$

which is finite.

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Curriculum vitae

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Conferences

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May, 16-19, 2008	Bayrischzell Workshop
December 4-6, 2008	Supersymmetry and Noncommutative QFT Workshop in Memoriam Julius Wess
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May, 14-17, 2010	Bayrischzell Workshop
September 8-12, 2010	Satellite Workshop on Noncommutative Field Theoy and Gravity (Corfu)

Publications

1. D. N. Blaschke, F. Gieres, E. Kronberger, M. Schweda and M. Wohlgenannt, *Translation-invariant models for non-commutative gauge fields*, *J. Phys.* **A41** (2008) 252002, [arXiv:0804.1914].

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3. D. N. Blaschke, E. Kronberger, A. Rofner, M. Schweda, R. I. P. Sedmik and M. Wohlgenannt, *On the Problem of Renormalizability in Non-Commutative Gauge Field Models — A Critical Review*, *Fortschr. Phys.* **58** (2010) 364, [arXiv: 0908.0467].

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