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On the Development of Non-Commutative Translation-Invariant Quantum Gauge Field Models

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Abstract

Aiming to understand the most fundamental principles of nature one has to approach the highest possible energy scales corresponding to the smallest possible distances – the Planck scale. Historically, three different theoretical fields have been developed to treat the problems appearing in this endeavor: string theory, quantum gravity, and non-commutative (NC) quantum field theory (QFT). The latter was originally motivated by the conjecture that the introduction of uncertainty relations between space-time coordinates introduces a natural energy cutoff, which should render the resulting computations well defined and finite. Despite failing to fulfill this expectation, NC physics is a challenging field of research, which has proved to be a fruitful source for new ideas and methods. Mathematically, non-commutativity is implemented by the so called Weyl quantization, giving rise to a modified product — the Groenewold-Moyal product. It realizes an operator ordering, and allows to work within the well established framework of QFT on non-commutative spaces.
The main obstacle of NCQFT is the appearance of singularities being shifted from high to low energies. This effect, being referred to as 'UV/IR mixing', is a direct consequence of the deformation of the product, and inhibits or complicates the direct application of well approved renormalization schemes. In order to remedy this problem, several approaches have been worked out during the past decade which, unfortunately, all have shortcomings such as the breaking of translation invariance or an inappropriate alternation of degrees of freedom. Thence, the resulting theories are either being rendered 'unphysical', or considered *a priori* to be toy models. Nonetheless, these efforts have helped to analyze the mechanisms leading to UV/IR mixing and finally led to the insight that renormalizability can only be achieved by respecting the inherent connection of long and short distances (scales) of NCQFT in the construction of models
Attaching at these considerations, the present work aims to investigate and enhance a rather new ansatz, originally proposed by Gurau *et al.*. This model combines all positive features of recent approaches, as it is translation invariant and renormalizable. Starting at a simple scalar implementation the core achievement, being a damping mechanism which implements the demanded symmetry of scales, and thereby restricts the occurrence of UV/IR mixing, is analyzed. In a further step the theory is generalized to gauge models of the Yang-Mills type, where new problems appear, from which the need for additional modifications arises. A detailed investigation of the obstacles hindering a fully viable proof of renormalization is presented, and possible ways to overcome the current problems are identified. In a final step the insights, which have been gained, are utilized to construct a promising new gauge model — the BRSW model. Renormalizability is demonstrated by explicit computations at the one loop level. A general proof, however, will require a substantial effort in order to establish the required mathematical methods in the non-commutative regime prior to their application – a topic which unfortunately cannot be addressed within the framework of this thesis.

Kurzfassung

In dem Vorhaben, die grundlegenden Zusammenhänge der Natur zu verstehen, sind wir gezwungen zu den größtmöglichen Energien, und entsprechend zu den kleinstmöglichsten Distanzen – der Planck-Skala – vorzudringen. Historisch gesehen, entstanden in diesem Zusammenhang drei unabhängige theoretische Gebiete, welche sich der dabei auftretenden Probleme widmen: Stringtheorie, Quatengravitation und nichtkommutative (NC) Quantenfeldtheorie (QFT). Letztere wurde ursprünglich unter der Annahme eingeführt, dass das Auftreten von Unschärferelationen zwischen Raumzeit-Koordinaten eine natürliche Schranke für die Energie bedinge, deren Existenz wiederum bekanntlich zu wohldefinierten Integralen führt. Obwohl diese Hoffnung nicht erfüllt wurde, konnte sich NCQFT als eigenständiges und interessantes Forschungsgebiet etablieren, welches die Entwicklung neuer mathematischer Methoden zur Folge hatte, die in weiten Bereichen Anwendung finden. Mathematisch wird die Nichtkommutativität durch die sogenannte Weyl-Quantisierung verwirklicht, welche zu einem veränderten Produkt — dem Groenewold-Moyal Produkt fürhrt. Dieses ermöglicht es, im Rahmen der bekannten QFT auf deformierten Räumen zu arbeiten.

Das grundsätzliche Problem in NC Theorien stellt die Verschiebung bestimmter Divergenzen von hohen zu niedrigen Energiebereichen dar. Dieser Effekt, welcher allgemein als "UV/IR-Mischung" bezeichnet wird, ist eine direkte Konsequenz der Deformation des Produkts und verursacht Unsicherheiten bei der Anwendung, teilweise sogar die Unanwendbarkeit, vieler anerkannter Renormalisierungsschemata. Das vergangene Jahrzehnt brachte einige vielversprechende Ansätze zur Behebung dieses Problems, welche jedoch allesamt nicht vollkommen sind. Die resultierenden Modelle brechen die Translationsinvarianz oder führen zu unzulässigen Beschränkungen von Freiheitsgraden, weshalb die physikalische Aussagekraft stark eingeschränkt ist. Schlussendlich trugen diese Arbeiten jedoch zu einem tieferen Verständnis der Mechanismen, welche der UV/IR-Mischung zugrundeliegen, bei. Schließlich stellte sich heraus, dass der Weg zu renormierbaren, physikalischen Theorien nur über die Berücksichtigung der immanenten Verbindung von kurzen und langen Distanzen als Symmetrie in der Konstruktion entsprechender Modelle führt.

Ausgehend von diesen Entwicklungen analysiert und erweitert diese Arbeit den unlängst vorgeschlagenen Ansatz von Gurau *et al.*, welcher die positiven Eigenschaften der Translationsinvarianz und Renormierbarkeit in einem Modell vereint. Zunächst wird das Dämpfungsverhalten des Propagators, welches eine Schlüsselrolle in der Unterdrückung der UV/IR-Mischung, und der Verwirklichung der Symmetrie zwischen kurzen und langen Distanzen einnimmt, anhand einer einfachen skalaren Implementierung untersucht. Anschließend wird die Verallgemeinerung zu Yang-Mills Eichtheorien behandelt. Die dabei auftretenden Probleme erfordern weitreichende Modifikationen, welche anhand zweier Eichmodelle diskutiert werden. In einer Analyse zeigt sich, dass allgemeine Renormierungsbeweise unter der Berücksichtigung von Symmetrie und Nichtkommutativität nicht wohldefiniert sind und weiterer mathematischer Grundlagenarbeit bedürfen. In einem weiteren Schritt werden die gewonnenen Erkenntnisse angewendet, um ein verbessertes Eichmodell zu konstruieren, welches die bekannten grundsätzlichen Probleme umgeht. Renormierbarkeit wird explizit auf Einschleifenniveau gezeigt. Ein allgemeingültiger Beweis kann jedoch auf Grund des Umfanges und der fehlenden Verallgemeinerung mathematischer Beweise auf das Nichtkommutative, nicht im Rahmen dieser Arbeit gebracht werden.

On the Development of Non-Commutative Translation-Invariant Quantum Gauge Field Models

PhD Thesis

in Theoretical Physics

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evaluated by

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In mind — to mankind,

heartfelt — to Christine and Dominic.

All figures in this work have been created in Inkscape, Jaxodraw, OpenOffice Draw, or Wolfram Mathematica® on Debian Linux, which is all (with the exception of Mathematica®) free software under the GNU public license. The voluntary work of thousands of idealistically minded people behind these software projects is invaluable and greatly acknowledged. This document has been typeset in LaTeX.

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Contents

Chapter 1

Introduction

Our current understanding of the fundamentals governing the processes in the universe as we experience them today is based on two elementary ideas applying to distinct asymptotic regimes. At large distance scales, well exceeding those of our everyday world, physics is governed by the gravitational force manifesting itself in curvature of spacetime. In this regime we confidently apply the geometrically motivated equations of General Relativity [1]. It is also this limit which is accessible to a far extent to the experiment. From measurements of the cosmic background radiation [2] we are able to estimate the age of the universe to about 13.7 billion years, defining an upper bound for the (currently) largest distance of $4.4 \times 10^{26}\,\mathrm{m}$. The other end of the scale is set by the *Planck length* $\lambda_P = \sqrt{\hbar G/c^3} \approx 1.6 \times 10^{-35}\,\mathrm{m}$ (with the usual nomenclature) being a result of the second fundamental theory, quantum theory [3]. Due to the uncertainty principle [4], $\Delta x\, \Delta p \geq \hbar/2$, the localization of a wavefunction to a space interval Δx can only be achieved if the respective wavelength does not exceed this length, and hence, the assigned momentum obeys the relation $\Delta p \geq \hbar/2\Delta x$. As a direct consequence, a natural limit to the localization arises if the corresponding energy rises above the critical limit to generate a black hole with an event horizon λ_P. Curiously, this limit cannot be deduced from quantum mechanics alone, as the space curvature generating the hole is again described by General Relativity. In any case, we may only speculate about physics at such small distances, as our experimental capabilities allow us to investigate the structure of matter down to $10^{-19}\,\mathrm{m}$, leaving some kind of *terra incognita* [5] of 16 orders of magnitude down to the Planck length. However, there is no hope that this situation will improve in the near future but we may benefit from the measurements of cosmic gamma ray bursts, being rare occurrences but exhibiting the highest known accessible energies in our current universe.

Leaving aside any concerns about accessibility and experimental verification we may try to set up fundamental theories for the physics beyond the current limits. In principle, the presumption of a smallest possible length implies that space itself is somehow discretized or ‘grainy’. These ideas range back to the early days of quantum mechanics [6, 7] when Schrödinger and Heisenberg questioned the suitability of a continuous description of space at the smallest distances. However, the discussion remained rather philosophical. A consistent mathematical description, including commutator relations between coordinates was developed almost a decade later by Snyder [8, 9] but motivation was only given from the idea to implement a cutoff for renormalization of quantum field theory. The advent of non-commutative physics came half a century later with the work by Doplicher, Fredenhagen and Roberts [10], who finally realized non-commutativity of spacetime as a natural consequence of the uncertainty principle and General Relativity at the Planck scale. From the requirement that a measurement process shall not trigger black holes uncertainty relations are followed, and the respective coordinates are promoted to Hermitean operators. Independent of these achievements the mathematical notion of 'deformation' [11–13] led to the development of *fuzzy physics* [14, 15] and matrix algebras [16] which have become widely applied tools in non-commutative physics [17–19] (see also [15] for a good review). Also, it has been shown that Euclidean non-commutative QFT (NCQFT) can be obtained as an effective theory from closed D-brane strings [20–22], and even may be deduced from quantum gravity [23], which gave rise to additional motivation. However, the probably biggest contribution on the mathematical side has been given by A. Connes [24, 25] who elaborated on the differential calculus, represented by spectral triples on deformed spaces. His work is considered as the mathematical foundation of non-commutative physics.

The practical implementation of non-commutativity in field theoretical models is significantly simplified by quantization with Weyl operators [26]. This leads in a natural way to a deformed non-local product — the Groenewold Moyal star product [27, 28], allowing to work with ordinary fields and functions instead of operator-valued quantities. Unfortunately, the initial hope to cure the problem of ultraviolet divergences in QFT was not fulfilled. Instead it was discovered that the deformed product gives rise to a splitting of contributions to quantum corrections into a part showing exactly the same behavior as the respective commuting counterpart, and a part equipped with a regularizing phase factor being parametrized by external momenta. At low energies, the oscillating phases vanish, and the original divergences reappear but are mapped to the infrared which inspired the problem to be referred to as UV/IR mixing [29–31]. This effect stated a severe obstacle for the renormalization program since it was not known how to treat the new type of singularities. Finally, Grosse and Wulkenhaar (GW) found the way out of the maze when they realized that one needs to respect the symmetry between UV and IR regions which is imposed by the mixing phenomenon. For scalar $\phi_4^{\star 4}$ theory they added an oscillator term to the Lagrangian [32], thereby implementing a symmetry called Langmann Szabo duality [33]. Proofs for renormalizability have been achieved in the matrix base [32, 34] (which can be considered as an alternative to the Moyal-Weyl approach), by using the rather challenging Polchinsky renormalization group approach [35], and by multiscale analysis [36]. Another unique feature is the vanishing of the β function to all orders [37–39]. However, renormalizability was achieved for the price of abandoning translation invariance. Also, the classical limit with respect to deformation ($\theta \to 0$ limit) is singular.

Another proposal for a renormalizable QFT was given by Gurau *et al.* [40, 41], replacing the oscillator term of GW by an inverse squared derivative, which is why we shall refer to it as the non-commutative $1/p^2$ model. Such a term brings the advantage of maintaining translation invariance, but is non-local, and may eventually give rise to a violation of unitarity but this has not been proven yet. However, the model has been shown to be renormalizable up to all orders, allows for the limit $\theta \to 0$ [42], and there are indications that the β-function is proportional to its commutative counterpart [41]. For a review see [43].

The construction of gauge models on deformed spaces is much more involved than the scalar case due to two reasons. First, due to the non-commutativity, any symmetry becomes non-Abelian. Secondly, all of the established methods of ordinary quantum gauge field theory to prove renormalizability are based on locality which is inherently broken. Nevertheless many attempts in the direction of non-commutative gauge field theory (NCGFT) have been made. Early contributions in this respect neglected the UV/IR mixing, and focused on the planar contributions [44–46]. Different approaches have been proposed, such as the inclusion of a Slavnov term [47–50], leading to an over-reduction of the number of physical degrees of freedom, generalizations of QED [51–54], or the 'induced gauge theory' [55–58] being generated by minimally coupling of the non-commutative scalar theory to an exterior gauge potential. Direct generalizations of the successful GW model to gauge theories have also been attempted [59], but the complex form of the Mehler kernel represents a real obstacle in explicit calculations. Renormalizability has been achieved (in four dimensions) so far only in the Seiberg Witten map [60] (and for the induced gauge case). The rather simple form of the non-commutative $1/p^2$ model raises the hope that this approach is more amenable to a successful generalization to gauge theory. Starting at an initial approach [61] a series of papers marks the evolutionary steps in this endeavour[62–66], pushed by the group around Blaschke, Rofner, Schweda, Sedmik and Wohlgenannt. It is the aim of this thesis to summarize and extend these efforts in the search for a translation invariant renormalizable deformed U(1) gauge theory.

This work is organized as follows: After some definitions and a more detailed introduction into the subject in Sections 1.1–1.4 a thorough analysis of the scalar Gurau model including an explicit renormalization step at at the one loop order follows in Chapter 2. At the end of this chapter the functionality of the damping mechanism leading to renormalizability is revealed by an explicit analysis of the behaviour at higher loop levels. The knowledge gained to that point is applied to discuss the principal requirements for a generalization to gauge models in Section 2.2. Two successive models obtained by 'localization' are presented in Chapter 3, which is terminated by an extensive discussion of the lessons learned, and the unexpected subtleties arising in these first implementations of gauge symmetry. After an excursion to the theory of renormalization in Chapter 4 the apex of this work is set in Chapter 5 with the construction and analysis of the BRSW model which gives strong indications of being renormalizable. The proof of this conjecture, however, is another story which will require a substancial effort to be made in the future.

1.1 General Notation

Definitions are indicated by the notation $a := b$ (or $b =: a$) if the variable a is defined to symbolize the expression b. In contrast, the relation $b \equiv b$ is used to emphasize equivalence of the symbols or expressions a and b. Regarding the nomenclature, ambiguities are principally resolved by explanation in the text. More generally, capital size symbols are operators in the widest sense while lower case symbols indicate variables. Momenta are denoted by k, l, p, q and may be indexed by numbers or Greek letters.
Similar properties $P(E)$ and $P'(E)$ for several arguments E_i (which are variables or physical quantities) are given in the compact form $P(\{E_1, E_2, \ldots\}) = P'(\{E_1, E_2, \ldots\})$ where the lists $\{E_i\}$ are considered to be in an ordered form, i.e. to resolve to the one to one correspondence $P(E_1) = P'(E_1)$, $P(E_2) = P'(E_2), \ldots$.
Throughout this work the *Euclidean metric*

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.1}$$

in dimension $D = 4$ is premised. Since with this definition $v_\mu \equiv v^\mu$ for any vector $v \in \mathbb{R}^4$, η will not be written explicitly. For the sake of compactness the products

$$a\theta b := a_\mu \Theta^\mu{}_\nu b^\nu\,, \quad \widetilde{a}^\mu := \Theta^\mu{}_\nu a^\nu\,, \tag{1.2}$$

of vectors with the antisymmetric tensor $\Theta_{\mu\nu}$ are abbreviated[1]. Meaning will be given to these definitions subsequently in Section 1.3.3. Fourier transformed functions and operators are generally (with the exception of Section 1.3.4 where the tilde symbol is used) not decorated, but understood from the context of momentum space. Principally, we work in *natural units* $\hbar = c = G = 1$

[1]For the sake of completeness, we shall define $\Theta_{01} = -\Theta_{10} = \Theta_{23} = -\Theta_{32} = 1$, and all other elements vanish.

but there are exceptions in which remarks are given. The mass dimension of any quantity x is given by $d_m(x)$.

1.2 Motivation for Non-Commutativity

As already mentioned in the introduction above, the original motivation for non-commutativity has been to implement, in a natural way, an UV cutoff in momentum integrals [8]. The hope had been to avoid the appearence of singularities in the theory, and to find a solution for the renormalization problem of QFT. Although this initial motivation has been dimmished by the discovery of the UV/IR mixing problem (see Section 1.3.7), non-commutativity appears in many places in nature. From this arises the motivation to study the consequences arising from deformation in physics.

The simplest example in this respect is the non-commutative *Landau problem* [67] on Euclidean space where the introduction of a static external magnetic field $B_i = (0, 0, B_z)$ in the $z(= x_3)$ direction leads via the Lorentz gauge $(\partial_\mu A^\mu) = 0$ to a cyclotron movement of charged particles in the orthogonal (x, y) plane. Let us briefly review this in a little more detail. The classical Lagrangian for a particle with mass m and an external vector potential $A_i(x) = -\varepsilon_{ij} x_j \frac{B}{2}$ is given (in SI units) as [68]

$$\mathcal{L}_{\text{class}} = \frac{1}{2} m (\partial_0 x_i)^2 - \frac{e}{c} (\partial_0 x_i) A_i \,.$$

Performing the Legendre transformation $\mathcal{H}_{\text{class}} = p_i(\partial_0 x_i) - \mathcal{L}_{\text{class}}$ with the conjugate momentum $p_i = \partial\mathcal{L}/\partial(\partial_0 x_i)$ yields,

$$\mathcal{H}_{\text{class}} = \frac{1}{2m} \left(p_i - \frac{eB_z}{2c} \varepsilon_{ij} x_j \right)^2 ,$$

which motivates the definition $p'_i = p_i - \frac{eB_z}{2} \varepsilon_{ij} x_j$. Now, promoting x_i and p_j to Hermitean operators $\hat{x}_i$ and $\hat{p}_j$ respectively, and noting that $[\hat{x}_i(t), \hat{p}_j(t)] = \mathrm{i}\delta_{ij}$, we immediately obtain

$$\left[\hat{p}'_i(B), \hat{p}'_j(B) \right] = -\mathrm{i} e B_z \varepsilon_{ij} \,,$$

which implies a non-commutative structure. In the case $m \to 0$ the first term in the above Lagrangian vanishes and the equal time commutation relations yield[2]

$$[\hat{x}_i(t), \hat{p}_j(t)] \Big|_{m\to 0} = \frac{eB_z}{2c} \varepsilon_{ij} \left[\hat{x}_j, \hat{x}_k \right] = -\mathrm{i}\delta_{ik} \quad \Rightarrow \left[\hat{x}_i, \hat{x}_j \right] = 2\mathrm{i} \frac{c}{eB_z} \varepsilon_{ij} \,.$$

Hence, from the introduction of an external magnetic field follows the non-commutativity of coordinates in the resulting system.
Another similar example is the *Quantum Hall Effect* [69, 70] (for a comprehensive review see [71]). The intimate relation between theories on deformed space time, and the non-commutative effects described above could serve to find upper bounds for the value of θ, the constant of deformation [72–74].

However, the strongest motivation still comes from the simple argument given in the introduction, that the structure of space-time should become grainy at the Planck scale since localization below λ_P is definitely not possible. In Ref. [10] it has been shown that uncertainty relations between coordinates directly follow from the physical demand that spacetime should have an operational meaning, wich cannot be maintained in case of the gravitational collapse caused by

[2] It should be remarked that the following can be achieved by moving the coordinate origin out of the rotation axis of the B_z field, as the resulting shifted coordinates $\hat{x}'_i$ do not commute.

preparing a very sharply localized state. However, it is also shown that the same relations can be followed from the elements of the center (i.e. commutatior relations) of a $\mathcal{C}^*$ algebra. Hence, mathematically, the demand for a minimal length may be expressed by imposing non-vanishing commutator relations on the coordinates.

1.3 Basics of NC QFT

This section is intended to give the necessary definitions required for computations in non-commutative space in a brief but concise way. Furthermore the principal problems appearing in respective theories are discussed.

1.3.1 Definitions from QFT

In order to render this work (almost) self-contained some definitions from standard QFT shall be given at this point. Since sign conventions are rather confusing throughout the literature, an attempt is made at this point to give a consistent unified description. For this purpose a double notation $\left\{ {ek \atop mk} \right\}$ will be introduced, combining factors ek for Euclidean, and mk for Minkowski signatures[3].

The *generating functional* of all Green functions in a theory, being described by the Lagrange function $\mathcal{L}$ of quantized fields $\phi_1 \dots \phi_n$, being coupled to external *classical sources* (i.e. not quantized) $J_{a_1} \dots J_{a_n}$ with generic quantum configurations (indices) $a_1 \dots a_n$, and the time ordering symbol T is given by

$$\begin{aligned} Z[\phi_1, ..\phi_n] &= \langle 0| \left\{ {1 \atop \mathsf{T}} \right\} \mathrm{e}^{\left\{ {- \atop \mathrm{i}} \right\} \int \mathrm{d}^D x \, \mathcal{L}[\phi] + \left\{ {- \atop \mathrm{i}} \right\} \int \mathrm{d}^D x \sum\limits_{i=1}^{n} J_i \phi_i} |0\rangle \\ &= \sum_{i=0}^{\infty} \frac{\left\{ {(-1)^n \atop \mathrm{i}^n} \right\}}{n!} \int \left(\prod_{j=1}^{i} \mathrm{d}^D x_j \, J_{a_j}(x_j) \right) G_{a_1 \dots a_i}(x_1, ..x_i) \, . \end{aligned} \tag{1.3}$$

A general n-point Green function $G_{a_1 \dots a_n}(x_1, ..x_n)$ is obtained from Z by functional derivation with respect to the sources J,

$$G_{a_1 \dots a_n}(x_1, ..x_n) = \left\{ {(-1)^n \atop (-\mathrm{i})^n} \right\} \left. \frac{\delta^n Z[J]}{\delta J_{a_1}(x_1) \delta J_{a_2}(x_2) \dots J_{a_n}(x_n)} \right|_{J_i = 0 \forall i} . \tag{1.4}$$

The generating functional Z^c of connected Green functions G^c, is related to Z via

$$Z^c = \left\{ {- \atop -\mathrm{i}} \right\} \ln Z \, , \tag{1.5}$$

and likewise, the connected n-point Green functions are

$$G^c_{a_1 \dots a_n}(x_1, ..x_n) = \left\{ {(-1)^{n-1} \atop (-\mathrm{i})^{n-1}} \right\} \left. \frac{\delta^n Z^c}{\delta J_{a_1}(x_1) \delta J_{a_2}(x_2) \dots J_{a_n}(x_n)} \right|_{J_i = 0 \, \forall i} . \tag{1.6}$$

[3]The respective factors depend to a high degree on conventions but, of course, have to be consistent in order to generate the right results in loop calculations. For the inquest of the respective factors in this section the notations of several references have been analyzed. Minkowski: [75, 76], Euclidean: [76–79]. Most intriguing is the (Euclidean) sign of Eqn. (1.7) which has been derived by consistency checks (transversality of one-loop vacuum polarization, as discussed in Section 2.1.1).

Furthermore, one has the n-point vertex functions depending upon the classical fields $\phi^{\text{cl.}}(x)$ (with statistic $s=0$ for bosons and $s=1$ for fermions),

$$\Gamma_{a_1\ldots a_n}(x_1,..x_n) = \left\{\begin{smallmatrix}-1\\1\end{smallmatrix}\right\} \frac{\delta^n\Gamma[\phi^{\text{cl.}}]}{\delta\phi^{\text{cl.}}_{a_1}(x_1)\delta\phi^{\text{cl.}}_{a_2}(x_2)\ldots\phi^{\text{cl.}}_{a_n}(x_n)}\Bigg|_{J_i=J_i[\phi^{\text{cl.}}]\forall i} = \langle 0|\left\{\begin{smallmatrix}1\\\mathrm{i}\end{smallmatrix}\right\}\phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)|0\rangle^{\text{1PI}}, \tag{1.7}$$

with

$$\phi^{\text{cl.}}_a[J](x) = \frac{\delta Z^c[J]}{\delta J_a(x)}, \quad \text{and } J_a[\phi] = -(-)^s\frac{\delta\Gamma[\phi^{\text{cl.}}[J]]}{\delta\phi^{\text{cl.}}}, \tag{1.8}$$

derived from the *vertex functional* $\Gamma[\phi^{\text{cl.}}]$, itself being related to the generating functional $Z^c[J]$ via a Legendre transformation[4]

$$\Gamma[\phi^{\text{cl.}}] = Z^c[J] - \int \mathrm{d}^D x J_a(x)\phi^{\text{cl.}}_a(x)\big|_{J_a=J_a[\phi]}. \tag{1.9}$$

Note also that the vertex Green function $\Gamma_{a_1..a_n}$ relates to the *one particle irreducible* (1PI) graphs, being characterized by the fact that they stay connected upon the removal of any single internal line, as indicated in Eqn. (1.7). In this respect it has to be noted that the two-point vertex functionals represent the inverse of the two-point connected Green functions in the sense that (for bosonic statistic)

$$\begin{aligned}\int \mathrm{d}^D y\, \Gamma^{ab}(x,y)G^c_{bc}(y,z) &= \left\{\begin{smallmatrix}1\\\mathrm{i}\end{smallmatrix}\right\}\int \mathrm{d}^D y\, \frac{\delta^2\Gamma[\phi^{\text{cl.}}]}{\delta\phi_a(x)\delta\phi_b(y)}\frac{\delta^2 Z^c[J]}{\delta J_b(y)\delta J_c(z)} \\ &= \left\{\begin{smallmatrix}-1\\-\mathrm{i}\end{smallmatrix}\right\}\int \mathrm{d}^D y\, \frac{\delta J_a(x)}{\delta\phi_b(y)}\frac{\delta\phi_b(y)}{\delta J_c(z)} = \left\{\begin{smallmatrix}-1\\-\mathrm{i}\end{smallmatrix}\right\}\delta^a_c\delta^D(x-z).\end{aligned} \tag{1.10}$$

Physically, the local part of the vertex functional, $\Gamma_{\text{loc}}[\phi]$, equals the so called *effective action* which is constructed from the tree level action $S^0[\phi^{\text{cl}}] = \int \mathrm{d}^D\, \mathcal{L}[\phi^{\text{cl}}]$ and the sum of local quantum corrections $\Gamma^{(i)}$ to all orders in $\hbar$,

$$\Gamma_{\text{loc}}[\phi] \equiv \Gamma^{\text{eff}}[\phi^{\text{cl.}}] = S^{(0)} + \sum_{i=1}^{\infty}\hbar^i\Gamma^{(i)}_{\text{loc}}. \tag{1.11}$$

The full vertex functional may also contain non-local quantum corrections, i.e. $\Gamma[\phi] = \Gamma_{\text{loc}}[\phi] + \Gamma_{\text{nloc}}[\phi]$, a fact which will be picked up again at a later point. However, the tree level action $S^{(0)}$ equals the zero order vertex functional $\Gamma^{(0)}$.

Of special interest is the explicit computation of the two-point functions (propagators) and the vertex functions. With

$$G^c_{ab}[\phi_1(x),\phi_2(y)] = \left\{\begin{smallmatrix}-1\\-\mathrm{i}\end{smallmatrix}\right\}\frac{\delta^2 Z^c}{\delta J_a(x)\delta J_b(y)}\Bigg|_{J_a=J_b=0} \overset{Eqn.\ (1.8)}{=} \left\{\begin{smallmatrix}-1\\-\mathrm{i}\end{smallmatrix}\right\}\frac{\delta\phi^{\text{cl.}}_b[J_b](y)}{\delta J_a(x)}, \tag{1.12}$$

$$\Delta_{F,ab}(x-y) := \left\{\begin{smallmatrix}1\\\mathrm{i}\end{smallmatrix}\right\} G^c_{ab}[\phi_1(x),\phi_2(y)], \tag{1.13}$$

[4]Note that,

$$\frac{\delta\Gamma[\phi]}{\delta\phi(y)} = \int \mathrm{d}^D x\, \frac{\delta Z^c[J]}{\delta J(x)}\frac{\delta J(x)}{\delta\phi(y)} + \int \mathrm{d}^D x\, \delta^D(x-y)J(x) + \int \mathrm{d}^D x\, \phi(x)\frac{\delta J(x)}{\delta\phi(y)} = -J(y) + \int \mathrm{d}^D x\left(\frac{\delta Z^c[J]}{\delta J(x)} - \phi(x)\right)\frac{\delta J(x)}{\delta\phi(y)},$$

and the expression in brackets vanishes due to the first definition in Eqn. (1.8).

the procedure is clear. First the equations of motion Eqn. (1.8) are computed from the tree level action $\Gamma^{(0)} = S^{(0)}$ and solved for the fields ϕ_b. The propagators are then obtained by variation with respect to the sources J_a. Tree level n-point vertex functions can readily be derived from Eqn. (1.7) with $\Gamma[\phi]$ set to $S^{(0)}$[5].

It is generally reasonable to estimate the divergence behavior of a theory for a general (1PI) graph.

Definition 1. The UV *superficial degree of divergence* $d_\gamma \equiv d(J)$ of a 1PI graph γ with L internal loops, giving rise to the momentum space integral $J(p) = \left(\int \prod_{l=1}^{L} \mathrm{d}^D k_l \right) I(p, k_1..k_L)$, is defined by the limit

$$\lim_{p \to \infty} J(p) \sim p^{d_\gamma} .$$

More explicitly, for a general 1PI graph with V vertices v_i and E_ϕ external lines of type ϕ it takes the form[78, 79],

$$d_\gamma = D + \sum_{i=1}^{V} (d_m(v_i) - D) - \sum_{\phi} E_\phi d_m ,$$

where the v_i symbolize vertex Green functions.

For more detailed introductions to QFT please refer to the literature [75–77, 80–83].

1.3.2 Symmetry Factors

The topic of the explicit determination of symmetry factors for Feynman graphs is treated quite novercally in most text books and monographs. For this reason it seems worth it to spend some space in order to describe a generally applicable procedure to eliminate any ambiguity.

The origin of symmetry factors is twofold. For the first, a given interaction term of the action corresponds not only to a single diagram but to multiple graphs, depending on the number of possible Wick contractions.[6]. As an example, one may consider the artificial term $ABBA$ which allows for arbitrary fields A and B the contractions

$$\overline{AB}\,\overline{BA}\,, \quad \text{and } A\overline{BB}A\,,$$

giving rise to two different graphs. The result is a so called *multiplicity factor* M. Secondly, the actual symmetry factor can be ascribed to the structure of the graph:

Definition 2. The *symmetry factor* S of a given graph is equal to the number of operations which exchange specific lines or vertices, leaving the amplitude of the graph invariant.

More specifically, this definition leads to the following rules of thumb[85] for contributions to S.

▷ Factor 2 for tadpole lines without orientation that start and end at the same vertex.

▷ Factor $n!$ for n lines of the same field type (and direction, if a charge is involved) having a common starting and ending vertex.

▷ Factor $v!$ for v internal vertices (i.e. vertices without external legs) being exchangeable.

Finally, the actual factor which is multiplicatively added to the graph's amplitude is M/S.

[5]Note that we could define $\Delta_{F,ab}(x-y)$ with the opposite sign if the vertex functions in Eqn. (1.7) change their sign too.

[6]A nice explanation of this fact can be found in Ref. [84]

The above definitions may lead to ambiguities in some cases, and shall therefore be cast into a different form to give a cooking recipe applicable to any theory. The following scheme is based on the rules given by Veltman [86].

1. Draw the external lines without connections or internal vertices, and align them such as they appear in the desired graph
2. Draw the vertices at their correct places relative to each other but shifted below or above the external lines. The result is now a so called 'pregraph'.
3. Start with one of the vertices considered to be connected to a(n) external line(s) and count the possibilities how to do this. The resulting number is the starting point for M. Repeat this step for all vertices considered to be connected to external lines and count the possibilities. Note that the count will always be reduced by the external or internal lines already connected in previous steps.
4. Now connect the internal lines to form the complete graph. Account for all possibilities as done for the external connections in the previous step.
5. All multiplicities collected in steps 3 and 4 have now to be multiplied to give the final M
6. The first ingredient for the symmetry factor is the product of all internal symmetry factors of the vertices (i.e. a factor $n!$ for n legs of the same field type).
7. If there are v *identical* vertices (same number and type of fields) then S gains a factor $v!$
8. The final result is M/S.

For the purpose of demonstration this rather lengthy set of rules shall now be applied step by step to the graph in Fig. 1.1, which appears in the gauge models described in this work.

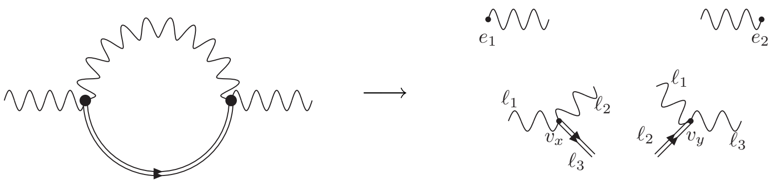


Figure 1.1: Splitting of a graph into a pregraph

1&2) See the pregraph in Fig. 1.1

3a) Options to connect internal vertex v_x to e_1 or e_2: 4 (by $\ell_1(v_x)$ and $\ell_2(v_x)$)

3b) Options to connect v_y to the remaining e_2 (or e_1, respectively): 2 (by $\ell_1(v_x)$ and $\ell_3(v_x)$)

4) Options to connect the remaining internal vertices in order to get the correct graph: 1

5) Collect the factors up to now to give $M = 4 \cdot 2 \cdot 1$.

6) Internal symmetry factor for each BAA vertex: 2!, two vertices v_x and v_y.

7) v_x and v_y are both of the same type (BAA) and, thus, can be interchanged $\Rightarrow$ factor 2!

8) Total factor $= \frac{M}{S} = \frac{4 \cdot 2 \cdot 1}{2! \cdot 2! \cdot 2!} = 1$

In practice, this procedure is amenable to the theories discussed in this thesis.

In order to automize the computation, a more general concept has been worked out and implemented as a package named `SymmetryFactor` in Wolfram Mathematica® 6. The general principle is to fix the vertices of a given diagram, and count the number of possible connections of vertex legs which are invariant with respect to symmetry operations as described in Ref. [85]. The result is then divided by the internal symmetry factors of the vertices. A detailed description of the algorithm as well as the source code is given in Appendix G.1.

1.3.3 Θ-Deformation

The so called Θ-*deformation* defines an associative but non-commutative C^*-algebra with generators $\hat{x}_\mu^\dagger = \hat{x}_\mu$ according to the relation

$$[\hat{x}_\mu, \hat{x}_\nu] \equiv \mathrm{i}\Theta_{\mu\nu} \quad \text{with } \mu, \nu \in \{1..D\}\,, \tag{1.14}$$

and the corresponding D-dimensional deformed space[7] shall be denoted by $\mathbb{R}^D_\theta$ (which is already a specialization to Euclidean space, since Eqn. (1.14) allows for an arbitrary metric). The choice of the totally antisymmetric tensor $\Theta_{\mu\nu} \in \mathbb{M}^n$ is not unique but can be classified with respect to its mass dimension d_m and contraction structure. Three options are generally found throughout the literature:

- ▷ $\Theta_{\mu\nu} = \theta_{\mu\nu} = \text{const.}$, $d_m(\theta) = -2$, constant case,
- ▷ $\Theta_{\mu\nu} = C^\rho_{\mu\nu}\hat{x}^\rho$, $d_m(C) = -1$, Lie group case,
- ▷ $\Theta_{\mu\nu} = C^{\rho\sigma}_{\mu\nu}\hat{x}^\rho\hat{x}^\sigma$, $d_m(C) = -0$, Quantum group case.

As most existing theories are constructed on flat Euclidean space, it seems reasonable to consider the simplest case of a constant, global Θ. In the framework of this thesis the special form

$$(\theta^{\mu\nu}) = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \text{with } \theta \in \mathbb{R}\,. \tag{1.15}$$

will be presumed, which obeys the following practical relations

$$\theta_{\mu\rho}\theta_{\rho\nu} = -\theta^2\delta_{\mu\nu}\,, \quad \tilde{k}^2 = \theta^2 k^2\,. \tag{1.16}$$

This definition will be kept for Chapters 1–4. In Chapter 5 a slightly modified version with separation of $\theta_{\mu\nu}$ into a dimensionful parameter ε and the pure tensor structure $\mathit{\Theta}_{\mu\nu}$ is introduced.

1.3.4 The Groenewold-Moyal-Weyl Star Product

In the point of view of physics it is convenient to require all (wave) functions $f(\hat{x})$ on $\mathbb{R}^D_\theta$ to be of the *Schwartz class* $\mathcal{S}$, i.e. that

$$f(\hat{x}) \in \mathcal{S} \Leftrightarrow \sup_{\hat{x}} \left(1 + |\hat{x}|^2\right)^{i+k_1+\dots k_D} \left|\partial_1^{k_1} \cdots \partial_D^{k_D} f(\hat{x})\right|^2 < \infty \quad \forall\, \{i, k_j\} \in \mathbb{Z}_0^+\,, \tag{1.17}$$

[7]Note that, in some sense, the properties and rules of computation in $\mathbb{R}^D_\theta$ are well known from $\mathrm{Mat}_n\,(C(M))$, i.e. the space of (eventually complex) $n \times n$ matrices.

i.e. that all functions and their derivatives vanish (faster than every polynomial) at infinity. For the latter definition to be complete a derivative ∂_μ has to be defined according to

$$[\partial_\mu, \hat{x}_\nu] = \delta_{\mu\nu}\,, \quad [\partial_\mu, \partial_\nu] = 0\,, \tag{1.18}$$

and the *Leibnitz rule*

$$\partial_i\left(f(\hat{x})g(\hat{x})\right) = \left(\partial_i f(\hat{x})\right) g(\hat{x}) + f(\hat{x})\partial_i g(\hat{x})\,. \tag{1.19}$$

Now it is rectified that every function f can be written in terms of its Fourier transformation[8]. In this respect, the *Weyl operator* [26] $\mathcal{W}[f]$ shall be introduced.

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{(2\pi)^D}\int \mathrm{d}^D x\, \mathrm{e}^{-\mathrm{i}kx} f(x) \\ &= \mathcal{W}^{-1}[f](k) := \frac{1}{(2\pi)^D}\int \mathrm{d}^D \hat{x}\, \mathrm{e}^{-\mathrm{i}k\hat{x}} f(\hat{x}) \\ f(\hat{x}) &= \mathcal{W}[f](\hat{x}) := \frac{1}{(2\pi)^D}\int \mathrm{d}^D k\, \mathrm{e}^{\mathrm{i}k\hat{x}} \tilde{f}(k)\,. \end{aligned} \tag{1.20}$$

Hence, $\mathcal{W}[\hat{f}]$ defines a map $f(x) \in \mathbb{R}^D \overset{\mathcal{W}[f]}{\longmapsto} \mathbb{R}^D_\theta \ni f(\hat{x})$ from fields (functions) to operators. Now, the product in $\mathbb{R}^D_\theta$ is (when considering plane waves, for example)

$$\mathrm{e}^{\mathrm{i}k\hat{x}}\mathrm{e}^{\mathrm{i}k'\hat{x}} = \mathrm{e}^{\mathrm{i}(k+k')\hat{x}+\frac{\mathrm{i}}{2}\theta_{\mu\nu}k_\mu k'_\nu} \tag{1.21}$$

according to the Baker-Campbell-Hausdorff formula. Note that the exponent is indeed complete since all further terms vanish due to the property $[\hat{x}_\nu, \theta_{\rho\sigma}] = 0$. Utilizing the definition (1.20) the notion of a *star product* can be introduced [26–28]

$$\mathcal{W}[f]\mathcal{W}[g] =: \mathcal{W}[f \star g]\,. \tag{1.22}$$

There are several explicit representations which are equivalent, if the definitions (1.14), (1.18), Eqn. (1.20), and (1.21) are applied. For the sake of completeness, the three most common forms which are direct consequences of the Weyl ordering [22] shall be stated at this place.

$$\begin{aligned} (f \star g)(x) &= \mathrm{e}^{\frac{\mathrm{i}}{2}\partial^x\theta\partial^y} f(x)g(y)|_{x=y} \\ &= \iint \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{d}^D k'}{(2\pi)^D} \tilde{f}(k)\tilde{g}(k)\mathrm{e}^{\mathrm{i}(k_\mu+k'_\mu)x_\mu}\mathrm{e}^{\frac{\mathrm{i}}{2}k\theta k'} \\ &= \int \frac{\mathrm{d}^D k}{(2\pi)^D}\int \mathrm{d}^D y\, f\big(x+\tfrac{\theta}{2}k\big)\, g(x+y)\mathrm{e}^{\mathrm{i}ky}\,, \end{aligned} \tag{1.23}$$

with the extension

$$f_1(x_1)\star\cdots\star f_n(x_n) = \prod_{i<j} \mathrm{e}^{\frac{\mathrm{i}}{2}\theta_{\mu\nu}\partial_\mu^{x_i}\partial_\nu^{x_j}} f_1(x_1)\ldots f_n(x_n)\,. \tag{1.24}$$

The latter equation shows (without proof) a remarkable property of the non-commutative product (1.21), namely that a multiplication by a plane wave leads to a translation $\hat{x}_\mu \to \hat{x}_\mu - \theta_{\mu\nu}k_\nu$ [22],

$$\mathrm{e}^{\mathrm{i}k\hat{x}} f(\hat{x})\mathrm{e}^{-\mathrm{i}k\hat{x}} \overset{(1.21)}{=} \mathrm{e}^{-\theta_{ij}k_j\partial_i} f(\hat{x}) = f(\hat{x}_\mu - \theta_{\mu\nu}k_\nu)\,. \tag{1.25}$$

[8] Fourier transformed quantities are indicated explicitly by a tilde (as in $\tilde{\phi}$) during this discussion. In subsequent sections this decoration will be omitted, and all functions on momentum space are assumed to be Fourier transformed quantities.

This can be interpreted as a manifestation of the inherent non-locality introduced by assuming a non-commutative product. By virtue of the star product it is possible to conduct computations on $\mathbb{R}^D_\theta$ while working with ordinary (wave-)functions being defined on commuting $\mathbb{R}^D$. As will be explained later on in more detail all effects of non-commutativity are covered entirely by the exponential phase factors in (1.24). Note also the following practical properties following from invariance under cyclic permutations under an integral,

$$\int d^D x\, f_1(x) \star \cdots \star f_n(x) = \int d^D x\, f_n(x) \star f_1(x) \star \cdots \star f_{n-1}(x)\,, \tag{1.26a}$$

$$\int d^D x\, f_1(x) \star f_2(x) = \int d^D x\, f_1(x) f_2(x)\,, \tag{1.26b}$$

and validity of the intuitive rule for functional variation

$$\frac{\delta}{\delta f_1(z)} \int d^D x\, f_1(x) \star f_2(x) \star \cdots \star f_n(x) = f_2(z) \star \cdots \star f_n(z)\,. \tag{1.27}$$

It should also be mentioned that Eqn. (1.14) can now be expressed entirely by commuting coordinates, as

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu}\,.$$

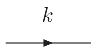
The star product has been investigated at a broad basis in the literature [87] and is, besides the matrix basis, the main mathematical tool for efficient work on non-commutative spaces.

1.3.5 A Naïve Model

The implications of non-commutativity can readily be analyzed when considering the simplest possible model, non-commutative scalar ϕ^4 theory in $\mathbb{R}^4_\theta$.

$$S[\phi] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi \star \partial_\mu\phi + \frac{m^2}{2}\phi^{\star 2} + \frac{\lambda}{4!}\phi^{\star 4}\right]\,. \tag{1.28}$$

This action represents a natural generalization of the well known commutative case with the usual product replaced by the star product. Due to the property (1.26b) the bilinear part (and, hence, the two point function) remains unaffected by deformation, (for details on the computation of two-point Green functions see Section 1.3.1 or the more explicit calculations in Appendix A.1,) and the propagator is



$$= G(k) = \frac{1}{k^2 + m^2}\,. \tag{1.29}$$

The situation is different for the interaction part $S_{\text{int}}[\phi] = \int d^4x\, \frac{\lambda}{4!}\phi^{\star 4}$ where the star products give rise to phase factors. The derivation is intuitive and shall be given in more detail at this point. First, $S_{int}[\phi]$ can be rewritten as,

$$\begin{aligned} S_{\text{int}}[\phi] &= \frac{\lambda}{4!}\int d^4x \int \left[\prod_{i=1}^{4} \frac{d^4k_i}{(2\pi)^4} e^{\mathrm{i}k_i x}\tilde{\phi}(k_i)\right] e^{\frac{\mathrm{i}}{2}(k_1\theta k_2 + k_1\theta k_3 + k_1\theta k_4 + k_2\theta k_3 + k_2\theta k_4 + k_3\theta k_4)} \\ &= \frac{\lambda}{4!(2\pi)^{12}} \int \left[\prod_{i=1}^{4} dk_i \tilde{\phi}(k_i)\right] \delta^4\left(k_1 + k_2 + k_3 + k_4\right) \underbrace{e^{\frac{\mathrm{i}}{2}(k_1\theta k_2 + k_3\theta k_4)}}_{\text{phase factor}}\,. \end{aligned} \tag{1.30}$$

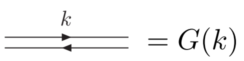
In the latter equality the delta function has been exploited in order to rewrite the exponential in a short form. Obviously, the only difference to the corresponding expression in ordinary commutative QFT is the *phase factor* which does not depend upon the fields. Aiming to derive the vertex expression one varies Eqn. (1.30) with respect to the fields[9]

$$\begin{aligned} V^{4\phi}(p_1,p_2,p_3,p_4) &= -\frac{\delta}{\delta\phi(-p_4)}\frac{\delta}{\delta\phi(-p_3)}\frac{\delta}{\delta\phi(-p_2)}\frac{\delta}{\delta\phi(-p_1)}S_{int}[\phi] \\ &= -\frac{\lambda}{4!(2\pi)^{12}}\int\left[\prod_{i=1}^{4}\mathrm{d}k_i\tilde{\phi}(k_i)\right]\delta^4\left(k_1+k_2+k_3+k_4\right) \\ &\times(2\pi)^{16}\sum_{i_{\{1..4\}}=\pi(1,2,3,4)}\delta^4(k_{i_1}+p_1)\delta^4(k_{i_2}+p_2)\delta^4(k_{i_3}+p_3)\delta^4(k_{i_4}+p_4) \\ &= -(2\pi)^4\frac{\lambda}{3}\Big[\cos\left(\tfrac{p_1\theta p_2}{2}\right)\cos\left(\tfrac{p_3\theta p_4}{2}\right)+\cos\left(\tfrac{p_1\theta p_3}{2}\right)\cos\left(\tfrac{p_2\theta p_4}{2}\right) \\ &\qquad+\cos\left(\tfrac{p_1\theta p_4}{2}\right)\cos\left(\tfrac{p_2\theta p_3}{2}\right)\Big], \end{aligned} \tag{1.31}$$

where the sum in the third line is taken over all possible permutations $\pi(\ldots)$ which, in this case, results in 4! terms. In the limit $|\theta|\to 0$ one reobtains the vertex known from the commutative scalar ϕ^4 model. However, the additional factor gives rise to several new effects unique to non-commutative theories. These will be discussed further in Section 1.3.7 after an excursion to graph topology.

1.3.6 Graph Topology

The cyclic property (1.26a) of the star product is well known from matrix models and it is convenient for the following discussion to introduce the same ribbon band graphical notation (for a more detailed introduction see for example the review [5]). Propagators are drawn as double lines with antipodally oriented momenta as shown in Fig. 1.2a. Ordering at vertices is automatically implemented by the alignment of lines. For instance, the factor associated with Fig. 1.2b is (as in Eqn. (1.30)) $\exp\left[\frac{i}{2}\left(k_1\theta k_2+k_3\theta k_4\right)\right]$. The main advantage of the ribbon



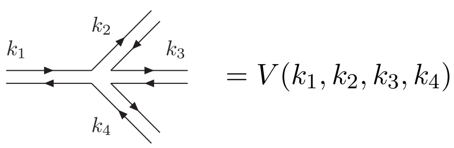


Figure 1.2: Generic parts of graphs in ribbon notation.

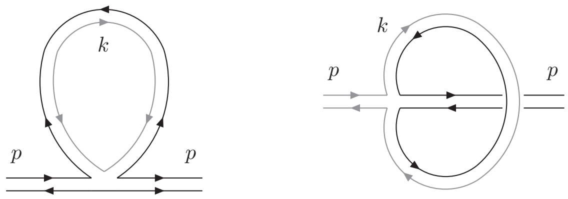
notation is that it eases the identification of the topological properties of the graph. The basics for classification are yielded by the *Euler Characteristic*

$$\chi = V - E + F \tag{1.32}$$

with V being the number of vertices, E is the number of *edges* (i.e. propagators), and F counts the faces. A *face* is a closed edge where the ends of external ribbons are considered to be connected. For instance, the (amputated) graph in Fig. 1.3b has $V=1$, $E=1$, and $F=2$ (as the grey and black lines each form a face), yielding $\chi=2$. The same is true for Fig. 1.3a. However, these two graphs are distinguished by another property which is the number of *faces*

[9]A more detailed version of this is given in Appendix A.1.

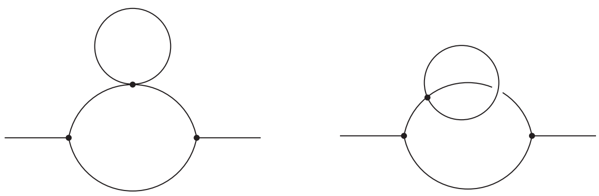
broken by external lines B. One can visualize this when considering Fig. 1.3a where the grey edge is closed by itself while the black one requires additional virtual connections between the two lines of each external ribbon (edge). Hence, only one face (the black one) is broken by external lines, i.e. $B = 1$ in Fig. 1.3a while for Fig. 1.3b $B = 2$ since both, the black and the grey lines, require external closures. Graphs with a single broken face are called *regular*, while such with $B > 1$ are *irregular*. Another important topological quantity is the *genus* g defined



(a) Planar tadpole (b) Non-planar tadpole

Figure 1.3: Two examples to visualize planarity.

by $\chi = 2 - 2g$. If $g \geq 1$ a graph is called *non-planar*. Although both graphs in Fig. 1.3 feature $g = 0$ the left one is called the *planar tadpole* and the right one is generally referred to as the *non-planar tadpole*. The *a priori* misleading naming convention becomes clear if higher loop insertions are considered as in Fig. 1.4 where for the addition of the planar tadpole in Fig. 1.4a results in $g = 0$ while for the non-planar tadpole in Fig. 1.4b the total genus becomes $g = 1$. The notion of planarity is heavily used in non-commutative field theory where it provides a way



(a) Planar graph ($V = 3$, $E = 4$, $F = 3$, $g = 0$) (b) Non-planar graph ($V = 3$, $E = 4$, $F = 1$, $g = 1$)

Figure 1.4: Different insertions of the tadpoles in Fig. 1.3 into a sunrise graph

to distinguish between IR and UV divergences. In addition to the above considerations, from the Euler characteristic follows a very useful relation

$$L = I - V + 1\,, \tag{1.33}$$

for the number of loops L in a graph.

1.3.7 UV/IR mixing

Attempting to compute a scattering amplitude in the naïve model (1.28) for the tadpole graph in Fig. 1.3 one obtains the expression

$$\Pi(p) = \Pi^{\mathrm{p}} + \Pi^{\mathrm{np}} = \frac{\lambda}{6} \int\limits_{\mathbb{R}^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{2 + \cos k\theta p}{k^2 + m^2}\,, \tag{1.34}$$

where the relation (F.18) has been used and the splitting in planar (pl) and non-planar (npl) parts corresponds to the contributions of the respective graphs in Figs. (1.3a) and (1.3b). Note that a symmetry factor $\frac{1}{2}$ has already been included. The integral Π^{pl} yields a quadratic UV divergence[10] in the cutoff Λ which can be expected from naïve power counting, similar to the commutative theory,

$$\begin{aligned}
\Pi^{\mathrm{p}} &= -\frac{\lambda}{6}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\\
&= -\frac{\lambda}{6}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\int\limits_0^\infty \mathrm{d}\alpha\,\mathrm{e}^{-\alpha\left(k^2+m^2\right)}\\
&= -\lim_{\Lambda\to\infty}\frac{\lambda}{6(2\pi)^4}\int\limits_0^\infty \mathrm{d}\alpha\frac{\pi^2}{\alpha^2}\mathrm{e}^{-\alpha m^2-\frac{1}{\Lambda^2\alpha}}\\
&= -\lim_{\Lambda\to\infty}\frac{\lambda}{54\pi^2}\sqrt{\Lambda^2m^2}\,\mathrm{K}_1\left(2\sqrt{\frac{m^2}{\Lambda^2}}\right)\\
&= -\lim_{\Lambda\to\infty}\frac{\lambda}{54\pi^2}\left[\frac{\Lambda^2}{2}+m^2\left(\gamma_E-\frac{1}{2}+\ln\sqrt{\frac{m^2}{\Lambda^2}}\right)+\mathcal{O}\left(\frac{1}{\Lambda^2}\right)\right]. \qquad (1.35)
\end{aligned}$$

Note that the second line has been obtained by Schwinger parametrization and in the last step the approximation (see also Eqn. (F.24))

$$\frac{1}{z}\,\mathrm{K}_1(z)=\frac{1}{z^2}+\frac{1}{2}\ln z+\frac{1}{2}\left(\gamma_E-\ln 2-\frac{1}{2}\right)+\frac{z^2}{16}\left(\ln z+\gamma_E-\ln 2-\frac{5}{4}\right)+\mathcal{O}(z^4)\,, \qquad (1.36)$$

for the Bessel function K_1 with the *Euler-Mascheroni* constant $\gamma_E\approx 0.577216$ has been applied. In contrast, Π^{np} is expected to be regularized by the oscillating cosine function.

$$\begin{aligned}
\Pi^{\mathrm{np}} &= -\frac{\lambda}{12}\sum_{\eta=\pm1}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}\eta k\theta p}}{k^2+m^2}\\
&= -\frac{\lambda}{12}\sum_{\eta=\pm1}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\int\limits_0^\infty \mathrm{d}\alpha\,\mathrm{e}^{-\alpha\left(k_\mu-\mathrm{i}\eta\frac{\theta_{\mu\nu}p_\nu}{2\alpha}\right)-\frac{(\eta\theta p)^2}{4\alpha}-\alpha m^2}\\
&= -\frac{\lambda}{6(2\pi)^4}\int\limits_0^\infty \mathrm{d}\alpha\frac{\pi^2}{\alpha^2}\mathrm{e}^{-\alpha m^2-\frac{(\eta\theta p)^2}{4\alpha}}\\
&= -\frac{\lambda}{54\pi^2}\sqrt{\frac{4m^2}{(\theta p)^2}}\,\mathrm{K}_1\sqrt{(\theta p)^2m^2}\\
&= -\frac{\lambda}{54\pi^2}\left[\frac{2}{(\theta p)^2}+m^2\left(\gamma_E-\frac{1}{2}+\ln\frac{1}{2}\sqrt{(\theta p)^2m^2}\right)+\mathcal{O}\left(p^2\right)\right]. \qquad (1.37)
\end{aligned}$$

The important fact to note at this place is that the phase factor introduced by non-commutativity indeed regularizes the UV region but gives rise to an IR singularity for vanishing external momentum p. Hence, nothing is won. Since the splitting into planar and non-planar parts is a general principle one always obtains both types of divergence for a single graph.

[10]Note also the logarithmic divergence in the mass m which may be problematic in gauge theories.

Mathematically the phase factors are regulating *a priori* ill-defined integrals by oscillation for high momenta. The involved exponential functions are parametrized with θ (which is intuitively clear, as in the commutative limit the phase has to vanish), and the external momentum, p. Hence it is not surprising that for vanishing regulator, i.e. $p \to 0$ or $\tilde{p} \to 0$, the divergences reappear. Therefore, in a more physical sense, they are nothing else but UV divergences being mapped to the infrared. This effect is commonly referred to as *UV/IR mixing* [29]. The picture of divergences being cast to the opposite energy limit is supported by the notion of an *effective cutoff* [29, 30] Λ_{eff},

$$\Lambda_{\text{eff}}^2 = \frac{1}{\tilde{p}^2 + \frac{1}{\Lambda^2}}\,, \tag{1.38}$$

which, in the limit $\tilde{p}^2 \to 0$, shows UV divergent behavior.

Another point which shall be mentioned is the inherent non-locality bound to the star product (1.22), and more general, to the assumption of a deformation (1.14). The latter relates different space-time points in a way which is not restricted to timelike or lightlike distances. Therefore, interaction outside the light cone is *a priori* allowed by definition. In the light of the star product non-locality manifests in the infrared divergences, as the ubiquitous $\tilde{p}^{-2}$ divergence in momentum space translates to $1/x^2$ in x-space. From this it is clear that the *entire space* contributes to results of loop calculations.
In commutative physics, locality is a basic requirement for physical relevance and renormalizability of theories. Therefore, it appears odd that this topic has not received much attention up to now [64]. It has to be remarked that, since the value of θ is assumed to be of the order of the Planck length λ_P, the non-locality imposed by Eqn. (1.14) is indeed restricted to very small distances. Unfortunately, despite almost a decade of research, up to now there is no satisfactory interpretation of non-locality introduced by deformation. The most promising ansatz is to replace the notion of (and general demand for) *locality* by the rather pragmatic approach to consider *almost localyity* [5] ,also called *Moyality*, i.e. to consider locality in dependence of the energy scale. See also Section 4.1.2, page 65.

1.4 Renormalizable non-commutative Models – An Overview

So far only a few models on deformed space have shown to be non-trivially renormalizable (i.e. models which are not super renormalizable in commuting space, or are studied only at a specific parameter configuration). In this respect one should mention the Grosse Wulkenhaar model [32] featuring the action

$$S[\phi] = \int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\right)^{\star 2} + \frac{\Omega^2}{2}\left(\tilde{x}_\mu\phi\right)^{\star 2} + \frac{m^2}{2}\phi^{\star 2} + \frac{\lambda}{4!}\phi^{\star 4}\right] \tag{1.39}$$

in Euclidean space, with $\tilde{x} = 2|\theta|^{-1}x$, and $\Omega \in [0,1]$ being a parameter. This model features a modified propagator, generally referred to as the *Mehler kernel* [88] which takes the form (in direct space)

$$G_{\text{M}}(x,y) = \frac{\Omega^2}{\theta^2\pi^2}\int\limits_0^\infty \frac{\mathrm{d}t}{\sinh^2 2\tilde{\Omega}t}\,\mathrm{e}^{-\frac{\tilde{\Omega}}{2}\left[(x-y)^2\coth(2\tilde{\Omega}t)-(x+y)^2\tanh(2\tilde{\Omega}t)\right]-m^2t} \tag{1.40}$$

This propagator softly violates momentum conservation, which is the reason why the Fourier transformed version depends on two different momenta, $G_{\text{M}}(k,k')$. It has been shown that the renormalizability of the model (1.39) is intimately connected to a symmetry in the asymptotic scales (Langmann Szabo duality [33]), which can be written as,

$$S\big[\phi; m, \lambda, \Omega\big] \;\mapsto\; \Omega^2 S\big[\phi; \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}\big]\,. \tag{1.41}$$

The point $\Omega = 1$ is called self-duality at which the symmetry (1.41) becomes trivial. Generally, in order to obtain renormalizability it seems to be necessary to pay tribute to the connection between short and long distances being introduced by the star product, and in consequence, the UV/IR mixing.
However, there is another model featuring a similar symmetry, the non-commutative $1/p^2$ model by Gurau *et al.* [40] being discussed extensively in the sequel of this thesis.

For the sake of completeness we should mention that there are several other approaches leading to renormalizable theories on non-commutative spaces. For example the $\phi^{\star 3}$ model in six [89] and four [90] dimensions which is trivially finite (superrenormalizable) but is likely to be unstable, or the LSZ model [91] featuring an oscillator-like term similar to the GW model. In addition, there are a number of non-commutative supersymmetric models, and the superrenormalizable $\phi_2^{\star 4}$ model [92].

Chapter 2

The Non-local $1/p^2$ Model

This chapter is devoted to the $\frac{1}{p^2}$ model which, besides the Grosse-Wulkenhaar approach, represents the second of the two main contributions for the construction of renormalizable non-commutative field models. The main idea is the insertion of a non-local counter term featuring the inverse of a second order derivative into the action. This leads to a damping behavior in the low momentum limit in all relevant propagators. Proofs for renormalizability, so far, have only been found for the simple $\phi^4_\star$ version by Gurau *et al.* [40]. Starting with their original scalar action in Section 2.1 the main features of the model are discussed. The stony road to a suitable gauge theory is entered in Section 2.2 where the problems of early approaches are discussed. More advanced implementations including BRST symmetry are addressed in Chapter 3. Exemplary computations and supplementary content to this chapter are collected in Appendix A.

2.1 Nonlocal Damping - A Scalar Model

The success of the Grosse-Wulkenhaar model with its oscillator term drew a lot of attention from the community but problems, such as the explicit breaking of translation invariance, where not solved. An alternative approach to tackle the problem of UV/IR mixing was proposed by Gurau *et al.* [40]. The main idea is to add a non-local term

$$S_{\text{nloc}}[\phi] = -\int \mathrm{d}^4x\, \phi(x) \star \frac{a^2}{\theta^2_x \Box} \star \phi(x)\,, \tag{2.1}$$

to the action (1.28), where a is a dimensionless constant. *A priori* the physical interpretation of the operator $\frac{1}{\Box}$ is difficult – especially in x-space one faces the inverse of a derivative which appears to be odd. In momentum space the situation becomes more intuitive since the inverse of the scalar function k^2 is well known[1]. More concisely,

$$\begin{aligned}
&\int \mathrm{d}^4x\, \phi(x) \frac{1}{\Box_x} \phi(x) \\
&\overset{\text{p.i.}}{=} \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int \frac{\mathrm{d}^4k'}{(2\pi)^4}\, \phi(k) e^{ik'x} \phi(k') \underbrace{\frac{1}{\Box_x} e^{ik'x}}_{-k'^2 e^{ik'x}} \\
&= -\int \frac{\mathrm{d}^4k}{(2\pi)^4} \phi(-k) \frac{1}{k^2} \phi(k)\,,
\end{aligned} \tag{2.2}$$

[1]Note that the four-dimensional Laplacian $\Box \equiv \partial_\mu \partial_\mu$ transforms to $-k^2$ in Euclidean space

and

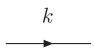
$$\Box_x \frac{1}{\Box'_x} \propto \delta^4(x - x')\,. \tag{2.3}$$

Hence, the new operator $\Box^{-1}$ is to be interpreted as the 'Green operator' of $\Box \equiv \partial_\mu \partial_\mu$.

The action including the non-local insertion reads

$$S[\phi] = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{1}{2}\left(k_\mu \phi(-k)^\mu \phi(k) + m^2\phi^2 + \frac{a^2}{2}\phi(-k)\frac{1}{\tilde{k}^2}\phi(k)\right) + \frac{\lambda}{4!}\phi^{\star 4}\right]. \tag{2.4}$$

Note that the $\star$ in the interaction part of Eqn. (2.4) does not make sense in momentum space, and shall represent the phase factor (see Appendix A.1) at this point. Variation of the bilinear part $S_{\mathrm{bil}}[\phi] = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{2}\left(k^2 + m^2 + \frac{a^2}{\tilde{k}^2}\right)\phi^2$ of the action with respect to ϕ immediately leads to the propagator

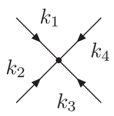


$$= G(k) = \frac{1}{k^2 + m^2 + \frac{a^2}{\tilde{k}^2}}\,. \tag{2.5}$$

This Green function is the core achievement of the approach by Gurau *et al.* since it features a damping behavior in the IR while not affecting the UV region, i.e.

$$\lim_{k\to 0} G(k) = \lim_{k\to\infty} G(k) = 0, \quad \forall m,\, \forall a \neq 0. \tag{2.6}$$

In contrast, the vertex functional



$$\begin{aligned} &= V(k_1, k_2, k_3, k_4) \\ &= -\frac{\lambda}{3}(2\pi)^4\delta^4\left(k_1 + k_2 + k_3 + k_4\right)\left[\cos\left(\tfrac{k_1\theta k_2}{2}\right)\cos\left(\tfrac{k_3\theta k_4}{2}\right)\right. \\ &\quad \left. + \cos\left(\tfrac{k_1\theta k_3}{2}\right)\cos\left(\tfrac{k_2\theta k_4}{2}\right) + \cos\left(\tfrac{k_1\theta k_4}{2}\right)\cos\left(\tfrac{k_2\theta k_3}{2}\right)\right], \end{aligned} \tag{2.7}$$

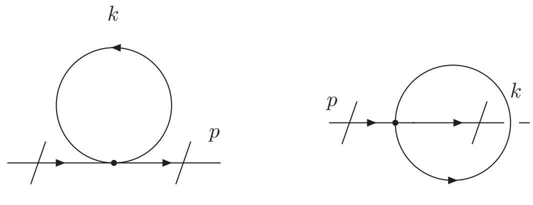
is not altered in comparison to the naïve model of Section 1.3.5. In the following it will be convenient to introduce a modified constant $a' := a/\theta$ of mass dimension 2.

2.1.1 One Loop Computations

Corrections to the Two-Point Function

The modified propagator (2.5) is expected to damp in the IR region. In order to study this, we consider the simple graph in Fig. 2.1. (The following discussion focuses on the main results; detailed computations are given in Appendix A.2.) It corresponds to the expression

$$\begin{aligned} \Pi(p) &= \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\, G(k)V(p, -k, k, -p) \\ &= -\frac{\lambda}{6}\int\limits_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{2 + \cos(k\tilde{p})}{k^2 + m^2 + \frac{a'^2}{k^2}} \equiv \Pi^{\mathrm{p}}(p) + \Pi^{\mathrm{np}}(p)\,. \end{aligned} \tag{2.8}$$



(a) planar (b) non-planar

Figure 2.1: One loop corrections to the photon propagator.

Here, Π^{p} and Π^{np} denote the *planar* and *non-planar parts*, (in correspondence with the different views in Figs. 2.1a, 2.1b, see Section 1.3.6 for a discussion of planarity) respectively. Facing the task to compute the integral in Eqn. (2.8) we have to cast the expression into a form amenable for Gaussian integration. We note that

$$\frac{1}{k^2+m^2+\frac{a'^2}{k^2}} = \frac{k^2}{\left(k^2+\frac{m^2}{2}\right)^2 - M^4} = \frac{1}{2}\sum_{\zeta=\pm1}\frac{1+\zeta\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta M^2}\,, \tag{2.9}$$

where $M^2 \equiv \sqrt{\frac{m^4}{4}-a'^2}$ (which may be real or purely imaginary depending on the value of a'). Taking into account Eqns. (2.9) and (F.21), the non-planar part can be evaluated straightforwardly by using Schwinger parametrization (see Appendix F):

$$\begin{aligned}\Pi^{\mathrm{np}}(p) &= -\frac{\lambda}{24}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{\eta,\zeta=\pm1}\frac{1+\zeta\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta M^2}\,\mathrm{e}^{\mathrm{i}\eta k\tilde{p}} \\ &= -\frac{\lambda}{48\pi^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m^2}{2M^2}\right)\sqrt{\frac{\frac{m^2}{2}+\zeta M^2}{\tilde{p}^2}}\,K_1\left(\sqrt{\tilde{p}^2\left(\frac{m^2}{2}+\zeta M^2\right)}\right)\,,\end{aligned} \tag{2.10}$$

where K_1 is the modified Bessel function. The result is finite for $\tilde{p}^2 \neq 0$, i.e. if $\theta \neq 0$ and $p \neq 0$.

In the following, we will focus on the IR behavior of the model, i.e. the limit $\tilde{p}^2 \to 0$. For small z, the function $\frac{1}{z}K_1(z)$ admits the expansion (1.36), as in the previous chapter (see also Eqn. (F.24) in Appendix F). Thus, for $\tilde{p}^2 \ll 1$, the expression (2.10) behaves like

$$\begin{aligned}\Pi^{\mathrm{np}}(p) = \frac{-\lambda}{6(4\pi)^2}\Bigg[&\frac{4}{\tilde{p}^2} + m^2\ln\left(\tilde{p}^2\sqrt{\frac{m^4}{4}-M^4}\right) \\ &+\left(M^2+\frac{m^4}{4M^2}\right)\ln\sqrt{\frac{\frac{m^2}{2}+M^2}{\frac{m^2}{2}-M^2}}\Bigg] + \mathcal{O}(1)\,.\end{aligned} \tag{2.11}$$

The latter expression involves a quadratic IR divergence (and a subleading logarithmic IR divergence). Moreover, for $a' \to 0$ (i.e. $M^2 \to \frac{m^2}{2}$) this result reduces to the one which is given in the literature for $a' = 0$ [29, 93], i.e. for the naïve model. From this result, we may conclude that the damping mechanism surprisingly does not act on the one loop level. This topic will be continued at a later point in Section 2.1.2 where higher order computations are discussed.

The integral defining the planar part does not contain a phase factor $\mathrm{e}^{\mathrm{i}k\theta p}$, and is therefore UV divergent. It can be regularized by introducing a cutoff Λ, and subsequently taking the limit $\tilde{p}^2 \to 0$, as explained in Appendix A.2.1. The final result can be expanded for large values of Λ, yielding

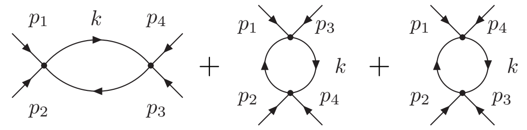
$$(\Pi^{\rm p})_{\rm regul.}(\Lambda) = \frac{-\lambda}{3(4\pi)^2}\left[4\Lambda^2 + m^2 \ln\left(\tfrac{1}{\Lambda^2}\sqrt{\tfrac{m^4}{4} - M^4}\right) + \left(M^2 + \tfrac{m^4}{4M^2}\right)\ln\sqrt{\frac{\frac{m^2}{2}+M^2}{\frac{m^2}{2}-M^2}}\right] + \mathcal{O}(1). \tag{2.12}$$

As expected, a quadratic and a subleading logarithmic divergence in the cutoff Λ are obtained.

Corrections to the Four Point Function

The basic one loop correction to the vertex is given by the three connected graphs that can be constructed with four external legs [81, 93, 94]:

$$V_{\text{1-loop}}(p_1,p_2,p_3,p_4) = \tfrac{1}{6}\Big[\quad \Big]. \tag{2.13}$$



This expression can be evaluated by proceeding along the lines of Ref. [93]. By applying the Feynman rules (2.5) and (2.7), and by taking advantage of the identity (2.9) we find that[2] (2.13) reads

$$\begin{aligned}
&\frac{\lambda^2}{27}\sum_{\zeta,\chi=\pm1}\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)}{k^2+\frac{m^2}{2}+\zeta M^2}\left[\left(1+\tfrac{1}{4}\sum_{i=2}^{4}e^{ik(\tilde p_1+\tilde p_i)}+\tfrac{1}{2}\sum_{i=1}^{4}e^{ik\tilde p_i}\right)\right.\\
&\quad\times\left(\frac{1}{(p_1+p_2-k)^2+\frac{m^2}{2}+\chi M^2}+\frac{1}{(p_1+p_3-k)^2+\frac{m^2}{2}+\chi M^2}+\frac{1}{(p_1+p_4-k)^2+\frac{m^2}{2}+\chi M^2}\right)\\
&\quad\left.+\tfrac{3}{4}\left(\frac{e^{ik(\tilde p_1+\tilde p_2)}}{(p_1+p_2-k)^2+\frac{m^2}{2}+\chi M^2}+\frac{e^{ik(\tilde p_1+\tilde p_3)}}{(p_1+p_3-k)^2+\frac{m^2}{2}+\chi M^2}+\frac{e^{ik(\tilde p_1+\tilde p_4)}}{(p_1+p_4-k)^2+\frac{m^2}{2}+\chi M^2}\right)\right].
\end{aligned} \tag{2.14}$$

Thus, we again have an expression involving planar and non-planar parts (the latter involving a phase factor of the form $e^{ik\theta q}$). The generic integral for the non-planar part is given by

$$\begin{aligned}
I(p,q) &\equiv \sum_{\zeta,\chi=\pm1}\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)e^{ik\theta(p+q)}}{\left(k^2+\frac{m^2}{2}+\zeta M^2\right)\left[(p-k)^2+\frac{m^2}{2}+\chi M^2\right]}\\
&= \sum_{\zeta,\chi}\left(1+\zeta\tfrac{m^2}{2M^2}\right)\left(1+\chi\tfrac{m^2}{2M^2}\right)\int_0^1 d\xi\,\frac{e^{i(1-\xi)p\theta q}}{8\pi^2}\\
&\qquad\times K_0\left(\sqrt{(\tilde p+\tilde q)^2\left[\xi(1-\xi)p^2+\frac{m^2}{2}+(\chi+\xi(\zeta-\chi))\,M^2\right]}\right).
\end{aligned} \tag{2.15}$$

Here, p denotes the total incoming momentum, and q represents one of the variables p_i (see Appendix A.2.2 for calculational details). For small arguments the modified Bessel function K_0 can be expanded according to (see Eqn. (F.24) in Appendix F)

$$K_0(z) = -\ln z + \ln 2 - \gamma_E + \mathcal{O}\left(z^2\right), \tag{2.16}$$

[2]For details on the computations see Appendix A.2.2.

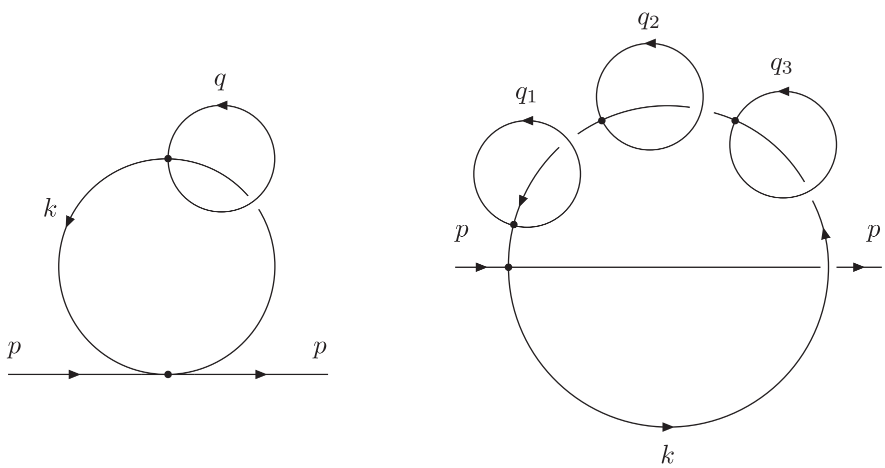
from which we can derive the following estimation for small external momenta p and q:

$$I(p,q) \approx \frac{1}{(2\pi)^2}\Bigg\{ \ln\left(\frac{(\tilde{p}+\tilde{q})^2}{4}\sqrt{\tfrac{m^4}{4}-M^4}\right) + 2\gamma_E - \frac{1}{2}\left(1-\tfrac{m^4}{4M^4}\right) + \left(3-\tfrac{m^4}{4M^4}\right)\tfrac{m^2}{4M^2}\ln\sqrt{\frac{m^2+2M^2}{m^2-2M^2}}\Bigg\}. \quad (2.17)$$

The planar part of expression (2.14) can again be evaluated by introducing a cut-off Λ (as is discussed for the propagator in Appendix A.2.1): the final result directly follows from (2.15) and (2.17) by replacing $(\tilde{p}+\tilde{q})^2$ with $1/\Lambda^2$. Note that the latter expression develops no divergence for $a \to 0$ (corresponding to the limit of the naïve model) as stated in [95], and can be checked by explicit computation.

2.1.2 Two and Higher Loop Orders

It turned out in Section 2.1.1 that the IR damping behavior of the modified propagator (2.5) does not alter the (divergent) result of one loop computations with respect to the naïve model. However, the regularizing effect becomes obvious when considering higher loop insertions, such as the non-planar tadpole graph with non-planar insertions (see Fig. 2.2a for one insertion, and Fig. 2.2b for several insertions, where external legs are considered to be amputated). Since we



(a) with 1 insertion (b) with 3 (symbolizing n) insertions

Figure 2.2: Non-planar 2 loop and 4 (resp. n) loop graphs.

are only concerned about the IR divergences, we limit ourselves to the first (i.e. most singular) term in the expansion (1.36) of $\Pi^{\text{n-pl}}$, hence [29] we consider the approximation $\Pi^{\text{n-pl}}(k) \propto 1/\tilde{k}^2$. Within this approximation, a graph with n non-planar insertions is described by the expression

$$\Pi^{n\ \text{np-ins.}}(p) \equiv \lambda^2 \sum_{\eta=\pm1} \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4}\, \frac{e^{i\eta k\theta p}}{\left(\tilde{k}^2\right)^n \left[k^2+m^2+\frac{a'^2}{k^2}\right]^{n+1}}\,. \quad (2.18)$$

For the naïve model (where $a = 0$), the integral of Eqn. (2.18) involves an IR divergence for $n \geq 2$, because the integrand behaves like $(k^2)^{-n}$ for $k^2 \to 0$. In contrast, for the model under consideration (where $a \neq 0$), the integrand behaves like

$$\frac{1}{\left(\tilde{k}^2\right)^n \left[\frac{a'^2}{\tilde{k}^2}\right]^{n+1}} = \frac{\tilde{k}^2}{(a'^2)^{n+1}} \,. \tag{2.19}$$

Thus, the propagator (2.5) 'damps' the IR-dangerous insertions, and therefore cures potential IR problems in the integral (2.18). This is a nice demonstration of the mechanism leading to the renormalizability of the $1/p^2$ model by Gurau *et al.*. In this respect it should be noted that its renormalizability has been proven up to all orders in reference [96] using multiscale analysis (see also Section 2.1.4).

Note that, in principle, the computation of an n loop graph requires to conduct the full renormalization programme up to order $n-1$. The above estimation (2.19) is based upon the tree level propagator (2.5) without any quantum corrections. Since, as will be shown in Section 2.1.3 explicitly for one loop, the corrections leave the form of the propagator invariant, this approximation is valid qualitatively.

In the following, a more detailed mathematical analysis of the IR behavior of the graph with n non-planar insertions will be given. The aim is to find out if the $1/k^2$ divergence of the one loop result (2.11) states a problem when being integrated over, upon insertion into higher loop graphs. The simplest test case is the *snowman graph* depicted in Fig. 2.2a (see Appendix A.3.1), which corresponds to the case $n = 1$ in Eqn. (2.18). One finds for the non-planar part

$$\begin{aligned}\Pi^{1\text{ np-ins.}}(p) \approx \frac{-\lambda^2}{3(4\pi)^4\theta^2M^6}\Bigg\{ & m^2\left[\sqrt{\frac{m_+^2}{\tilde{p}^2}}\,K_1\left(\sqrt{m_+^2\tilde{p}^2}\right) - \sqrt{\frac{m_-^2}{\tilde{p}^2}}\,K_1\left(\sqrt{m_-^2\tilde{p}^2}\right)\right] \\ & + M^2\left[m_+^2\,K_0\left(\sqrt{m_+^2\tilde{p}^2}\right) + m_-^2\,K_0\left(\sqrt{m_-^2\tilde{p}^2}\right)\right]\Bigg\} \,,\end{aligned} \tag{2.20}$$

where $m_\pm^2 := \frac{m^2}{2} \pm M^2$. Application of the expansion (F.24) for the Bessel functions reveals that all IR divergences in p cancel and the result is indeed finite for small momenta. This unambiguously shows that the insertion of the one loop $1/k^2$ divergence can be integrated out on two loop level and no additional divergences[3] appear.
Now let us consider the corresponding expression in the naïve model where we omit any numerical prefactors for simplicity

$$\begin{aligned}\Pi^{1\text{ np-ins. naïve}} &\propto \int\limits_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\,\frac{e^{ik\theta p}}{(k^2+m^2)^2} \\ &= \frac{1}{m^4(4\pi)^2}\left[\frac{4}{\tilde{p}^2} - 2m^2\,K_2\sqrt{\mu^2\tilde{p}^2}\right] \,,\end{aligned} \tag{2.21}$$

and again, all IR divergences cancel. This indicates, that the two loop level is not representative in the light of the investigation of possible divergences due to non-planar insertions into higher loop graphs. Indeed, from the estimation (2.18) and the discussion thereafter, it is clear that ill-defined integrals will not appear before the 2 loop level. Let us fortify this general argumentation

[3] It has to be noted that the $\frac{1}{\tilde{p}^2}$ divergence reappears order by order, but the occurrence of higher order poles is avoided.

by some explicit calculations. A detailed computation of the graph in Fig. 2.2b (with the help of Mathematica), shows the following

$$\Pi^{n\text{ np-ins. naïve}} \propto \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(\tilde{k}^2)^n} \frac{\mathrm{e}^{\mathrm{i}k\theta p}}{(k^2+m^2)^{n+1}}$$
$$= \frac{2}{(4\pi)^2} \int_0^\infty \mathrm{d}\lambda \int_0^1 \mathrm{d}\xi\, \lambda^{2n-2} \frac{\xi^n(1-\xi)^{n-1}}{\Gamma(n)\Gamma(n+1)} \mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\xi m^2-\mu^2\lambda}\,, \quad \text{for } n>1\,, \tag{2.22}$$

where a mass $\mu \to 0$ had to be introduced in order to regularize the integrals. This yields the following types of divergences[4] in $\tilde{p} \to 0$ and $\mu \to 0$, for the given number n of insertions

$$\begin{aligned} &n=2 && \ln\sqrt{\tilde{p}^2\mu^2},\ \ln\sqrt{\tilde{p}^2} \\ &n=3 && \frac{1}{\mu^2},\ \ln\sqrt{\tilde{p}^2\mu^2},\ \ln\sqrt{\tilde{p}^2} \\ &n=4 && \frac{1}{\tilde{p}^2},\ \frac{\tilde{p}^2}{\mu^2}\ln\sqrt{\tilde{p}^2\mu^2},\ \ln\sqrt{\tilde{p}^2} \end{aligned} \tag{2.23}$$

From the latter results it is clear that in the naïve model without damping we obtain ill-defined integrals from the third loop level on (i.e. more than one insertion of the type of Fig. 2.2b).

In the model by Gurau *et al.*, incorporating the additional damping term, the situation is different as divergences are suppressed. Due to the splitting (see Eqn. (2.9)) of the propagator the respective integrals become more involved. For generic n, the graph in Fig. 2.2b evaluates to $\Pi^{n\text{ np-ins.}}(p) = \lambda^2 J_n(p)$ with $J_n(p)$ being given by the integral (A.35) in Appendix A.3.2 (where some details of the computation can be found). By expanding the results for $n = 1..4$ insertions for small external momentum $\tilde{p}^2$, and introducing the abbreviation $m_\pm := \left(\frac{m^2}{2} \pm M^2\right)$, we obtain

$$\begin{aligned}
\Pi^{1\text{ np-ins.}}(p) &= -\frac{\lambda^2}{16\pi^2\theta^2M^6}\left[\left(m^4-4M^4\right)\ln\sqrt{\frac{m_+^2}{m_-^2}} - 4M^2\tfrac{m^2}{2}\right] + \mathcal{O}(\tilde{p}^2)\,, \\
\Pi^{2\text{ np-ins.}}(p) &= -\frac{\lambda^2}{2^8\pi^2\theta^4M^{10}}\left[\left(3m^4-4M^4\right)\ln\sqrt{\frac{m_+^2}{m_-^2}} - 12m^2M^2\right] + \mathcal{O}(\tilde{p}^2)\,, \\
\Pi^{3\text{ np-ins.}}(p) &= -\frac{\lambda^2}{2^{11}3\pi^2\theta^6M^{14}}\left[3\left(m^2-2M^2\right)\left(m^4-4M^4\right)\ln\sqrt{\frac{m_+^2}{m_-^2}}\right. \\
&\qquad\left. -4m^2M^2\left(15m^4-52M^4\right)\right] + \mathcal{O}(\tilde{p}^2)\,, \\
\Pi^{4\text{ np-ins.}}(p) &= \frac{\lambda^2}{2^{17}9\pi^2\theta^8M^{18}}\left[15\left(32M^4-56m^4\right)\ln\sqrt{\frac{m_+^2}{m_-^2}}\right. \\
&\qquad\left. -32M^2m^2\frac{105m^8-750m^4M^4+1296M^8}{\left(m^4-4M^4\right)^2}\right] + \mathcal{O}(\tilde{p}^2)\,.
\end{aligned} \tag{2.24}$$

In the limit $a \to 0$ (which eliminates the damping) all but the result for $n = 1$ diverge. This fact again illustrates that the propagator (2.5) regularizes graphs which diverge in the naïve model.

[4] Note that any numeric (even dimensionful) parameters are omitted.

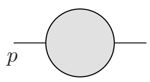
In the limit $m \to 0$ (i.e. for a massless field), the expressions (2.24) reduce to finite quantities

$$\begin{aligned}
\Pi^{1\ \text{np-ins.}}(p)\Big|_{m=0} &= \frac{\lambda^2}{2^6\pi|\theta a|} + \mathcal{O}(\tilde{p}^2)\,, \\
\Pi^{2\ \text{np-ins.}}(p)\Big|_{m=0} &= \frac{\lambda^2}{2^8\pi\theta^4|a^3|} + \mathcal{O}(\tilde{p}^2)\,, \\
\Pi^{3\ \text{np-ins.}}(p)\Big|_{m=0} &= \frac{\lambda^2}{2^9\pi\theta^6|a^5|} + \mathcal{O}(\tilde{p}^2)\,, \\
\Pi^{4\ \text{np-ins.}}(p)\Big|_{m=0} &= \frac{\lambda^2}{2^{10}9\pi\theta^8|a^7|} + \mathcal{O}(\tilde{p}^2)\,. \qquad (2.25)
\end{aligned}$$

Henceforth, in contrast to the naïve model (e.g. see reference [93]), the higher loop graphs in Fig. 2.2b do *not* diverge for $m \to 0$. In other words, the IR divergent insertion $1/\tilde{k}^2$ does not cause any harm in these higher loop graphs, even for a massless field. This is an important feature when bearing in mind a later generalization of the damping concept to gauge field theories.

2.1.3 An Attempt for IR Renormalization

Having at hand the explicit results of Eqns. (2.11), (2.12), and (2.17) we are in the position to start a renormalization procedure in order to achieve a one loop effective action $S^{(1)}_{\text{eff}}$. Applying a simple subtraction scheme (as it is discussed in the standard literature on QFT, see for example [75, 84]), the *dressed propagator at one loop level* is given by[5]



$$:= \Delta'(p) = \frac{1}{A} + \frac{1}{A}\,\Sigma(\Lambda, p)\,\frac{1}{A}\,, \qquad (2.26)$$

where

$$\begin{aligned}
A &:= p^2 + m^2 + \frac{a^2}{\tilde{p}^2}\,, \\
\Sigma(\Lambda, p) &:= (\Pi^{\text{p}})_{\text{regul.}}(\Lambda) + \Pi^{\text{n-pl}}(p)\,.
\end{aligned}$$

For $A \neq 0$, $\exists\ A^{-1}$, $A + B \neq 0$, and $\exists\ (A+B)^{-1}$ we can apply the formula

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}\,B\,\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}\,B\,\frac{1}{A} + \mathcal{O}(B^2)\,, \qquad (2.27)$$

which allows to rewrite expression (2.26) to order Σ (i.e. to order λ) as

$$\Delta'(p) = \frac{1}{p^2 + m^2 + \frac{a^2}{\tilde{p}^2} - \Sigma(\Lambda, p)}\,. \qquad (2.28)$$

Note that the contribution $\Pi^{\text{n-pl}}(p)$ to $\Sigma(\Lambda, p)$ is finite except for vanishing external momentum p. The expansion for small values of $\tilde{p}^2$, as given in Eqn. (2.11), reveals a quadratic and a logarithmic IR divergence[6] at $\tilde{p}^2 = 0$. The quadratically divergent term obviously has the same structure as the term $\frac{a^2}{\tilde{p}^2}$ appearing in the bare propagator (2.5). In fact, this has been the actual

[5] Remember the notation $a'^2 := a^2/\theta^2$, resulting in $a'^2/p^2 \equiv a^2/\tilde{p}^2$.

[6] In this respect, we should emphasize that these IR divergences are fundamentally different from the ones encountered in quantum field theories on commutative space since they are tied to the UV divergences and only appear in non-planar diagrams which are not present in usual QFT [22]. Thus, these divergences cannot be regularized by introducing an infrared regulator (like an additional mass parameter).

motivation for the introduction of the non-local term in the $1/p^2$ model by Gurau *et al.*, as it allows to absorb the typical quadratic IR divergence of non-commutative scalar theories by a finite renormalization of the parameter a'^2.

From the expansions (2.11) and (2.12) it follows that, to order λ, we have

$$\Delta'(p) = \frac{Z}{p^2 + m_r^2 + \frac{a_r^2}{p^2} + f(p^2)}\,, \tag{2.29}$$

where

$$\begin{aligned}
Z &:= 1 + \lambda\alpha\theta^2\,, \qquad\qquad (\alpha \in \mathbb{R})\,,\\
m_r^2 &:= m^2 + \frac{\lambda}{3(4\pi)^2}\left[4\Lambda^2 + m^2 \ln\left(\tfrac{1}{\Lambda^2}\sqrt{\tfrac{m^4}{4} - M^4}\right)\right] + \text{regular for } \Lambda \to \infty\,,\\
a_r^2 &:= a^2 + \lambda\left[\frac{2}{3(4\pi\theta)^2} + \alpha a^2\theta^2\right]\,,\\
f(p^2) &:= \frac{\lambda}{6(4\pi)^2}\left[m^2 \ln\left(\theta^2 p^2\right) + \mathcal{O}((\theta p)^4)\right]\,.
\end{aligned} \tag{2.30}$$

The quantities m_r and a_r represent the renormalized mass and a-parameter to one loop order, and the function $f(p^2)$ is analytic for $\theta \neq 0$ and $p^2 > 0$. The expression Z amounts to a finite wave function renormalization[7]. It has to be noted at this point that it is *a priori* not clear how to handle the logarithmic singularity of $f(p^2)$. In the literature, [96] this problem is treated by stating that for vanishing external momentum p the logarithm represents a 'mild divergence' which is unproblematic when computing physical amplitudes. This point can intuitively be understood since, in the denominator of Eqn. (2.29),

$$f(p) \overset{!}{<} \left[p^2 + m_r^2 + \frac{a_r^2}{p^2}\right]_{p\to\{0,\infty\}}\,,$$

but there is no rigorous interpretation by any means in the literature. The constant α appearing in Z and in a_r is determined by the numerical factor that occurs in the expansion of $\Sigma(\Lambda, p)$ at order $\tilde{p}^2$ (see Eqns. (F.24) and (2.11)). We have

$$\alpha\theta^2 = \frac{2}{3(16\pi)^2}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\left(m^4 - a^2\right)\,, \tag{2.31}$$

which is positive for $m^4 > a^2$. However, even in the case where $\alpha < 0$, the one loop renormalized parameter a_r^2 is positive, provided

$$\begin{aligned}
a^2 &< \frac{1}{2\theta^2}\left(\frac{B}{\lambda A} + \theta^2 m^4\right) + \frac{1}{\theta^2}\sqrt{\frac{1}{4}\left(\frac{B}{\lambda A} + \theta^2 m^4\right)^2 + B}\,,\\
\text{where} \qquad A &:= \frac{2}{3(4\pi)^2}\,, \quad B := \frac{16}{\ln 2 + \frac{5}{4} - \gamma_E} > 0\,.
\end{aligned} \tag{2.32}$$

Since θ is necessarily quite small on physical grounds, the dominating factor in the previous inequality is $1/\lambda$. Hence, even for $m = 0$, the parameter a_r^2 is positive for small values of the coupling constant λ (more precisely for $a^2 \lesssim 10^3/\left(\lambda\theta^2\right)$).

[7]For the ϕ^4-theory on commutative space, there is no wave function renormalization at one loop order, but this is a peculiarity of this theory [81].

The renormalized coupling constant λ_r at one loop order is obtained by considering the planar part[8] of Eqn. (2.14). One finds an expression which is similar to the one in the commutative theory. The non-planar part of Eqn. (2.14) again involves a logarithmic singularity (see Eqn. (2.17)). Further treatment of the one loop vertex correction, and a computation of the β function of the scalar $1/p^2$ model can be found in Ref. [41].

For any renormalizable model the stability of the action can be expressed in the form of *(re)normalization conditions*. For the current model we can express the 1PI vertex function for the free bilinear part of the action (2.4) by

$$\Gamma_2^{(0)}(k) = k^2 + m^2 + \frac{a'^2}{k^2}\,. \tag{2.33}$$

This expression leads to the following renormalization conditions (at the point[9] $k^2 = 0$), where $m_{\text{phys.}}$ denotes the applicable renormalized mass (see Section 4.1.2):

$$\begin{aligned} k^2\Gamma_2^{(0)}(k)\Big|_{k^2=0} &= a'^2\,, \\ \left(\Gamma_2^{(0)}(k) - \frac{a'^2}{k^2}\right)\Big|_{k^2=0} &= m^2_{\text{phys.}}\,, \\ \frac{\mathrm{d}}{\mathrm{d}k^2}\left(\Gamma_2^{(0)}(k) - \frac{a'^2}{k^2}\right)\Big|_{k^2=0} &= 1\,. \end{aligned} \tag{2.34}$$

In fact, the tree level action (2.4) as well as the respective one loop renormalized expression including the corrections of Eqn. (2.30) are compatible with these. Although no proof will be given in this work, we may conjecture that the conditions (2.34) hold up to all orders, and therefore guarantee stability of the action under quantum corrections.

2.1.4 General Proof of Renormalizability

The general proof of renormalizability has been given in a complete and concise way by Gurau *et al.* [40] using the technique of *Multiscale Analysis* (MA). It is the intent of this section to shed light on the idea of the procedure, not to present any rigorous derivation.

The main idea of Multiscale Analysis relies on the concept of scales. It is well known that an arbitrarily complex Feynman graph cannot simply be renormalized by just integrating over all internal momenta, and subtracting the resulting divergences. Instead, as was realized first by Bogoliubov [97] and Zimmermann [98], one has to iteratively subtract divergences of non-trivial subparts ('forests') of the graph (see Appendix D) for a short review of these techniques). These represent the contribution of a specific energy scale, i.e. a physical subprocess. The general topic how to select and handle scales has many facettes and approaches which can roughly be collected under the name *Renormalization Group* (RG). MA is a more general scheme to derive bounds for contributions of graphs, relying on the same ideas as RG. The basic concepts are the so called 'slicing' of propagators, and the approximation and bounding of graph amplitudes by application of a mean value theorem. The most complete description of the procedure and proofs can be found in the book by Rivasseau [99]. Now let us briefly sketch how MA works.

[8] In this case only the planar part is taken, since the logarithmic result is finite for any non-vanishing combination of external momenta $\tilde{p} + \tilde{q}$.

[9] Note that, in principle, the point at which these conditions are expressed, as well as the renormalization point for Eqn. (2.29) are arbitrary. A different choice is being discussed for instance in Ref. [41].

Any propagator $G(k)$ in k-space[10] can be 'sliced' with respect to the energy scales i as

$$G_\rho(k) = \lim_{\rho\to\infty} \sum_{i=0}^{\rho} G^i(k) \,, \tag{2.35}$$

where ρ is a UV cutoff. In the $1/p^2$ model, the slices explicitely read (see Appendix A.4)

$$G^i(k) = \int\limits_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha\, \mathrm{e}^{-\alpha\left(k^2+m^2+\frac{a'^2}{k^2}\right)} \,, \quad i \geq 1 \,, \tag{2.36}$$

$$G^0(k) = \int\limits_1^\infty \mathrm{d}\alpha\, \mathrm{e}^{-\alpha\left(k^2+m^2+\frac{a'^2}{k^2}\right)} . \tag{2.37}$$

Hence, the large convergent IR part[11] on $\alpha \in [1, \infty)$ is separated off, and the UV region $[M^{-2\rho}, 1]$ is covered by slices which all are finite by themselves. Note that the divergence at $\alpha \to 0$ (corresponding to the UV divergence in the momentum) is only reached in the limit $\rho \to \infty$; i.e. if the finite sum is extended to an infinite one. A key argument is the following

Theorem 1. *Every integral in* (2.36) *is bounded by*

$$G^i(k) \leq K \mathrm{e}^{-cM^{-2i}\left(k^2+m^2+\frac{a'^2}{k^2}\right)}, \qquad \textit{with } K > 1 \in \mathbb{R}^+, \textit{ and } c \in \mathbb{R}^+ \,, \tag{2.38}$$

and similarily

$$G^0(k) \leq K' \mathrm{e}^{-c'p^2}, \qquad \textit{with } K' > 1 \in \mathbb{R}^+, \textit{ and } c' \in \mathbb{R}^+ \,. \tag{2.39}$$

A proof is given in Appendix A.4. The amplitude $\mathcal{A}(\gamma)$ of an arbitrary N point Feynman graph γ can be expressed as

$$\begin{aligned}\mathcal{A}(\gamma, \{p_1..p_N\}) = \delta\Big(\sum_{i=1}^N p_i\Big) \mathrm{e}^{\frac{\mathrm{i}}{2}\sum\limits_{m<n}^{N} p_m\theta p_n} \int \Bigg[\prod_{j=1}^{L} \mathrm{d}^4k_j\, G(k_j) \\ \times \prod_{v\neq v_r} \delta\left(q_{v,1}+q_{v,2}+q_{v,3}+q_{v,4}\right) \mathrm{e}^{\frac{\mathrm{i}}{2}\sum\limits_{a,b=1}^{4} I_{ab} q_{v,a}\theta q_{v,b}} \Bigg],\end{aligned} \tag{2.40}$$

where the prefactor comes from the star products in the respective ($\phi^{\star 4}$) term in the action, the second factor is the contribution of the propagators in L loop integrations, and the last factor comes from momentum conservation (referenced to the external vertex v_r) and phases of the internal (and external) vertices v. The insertion of the bound (2.38) into Eqn. (2.40) gives (after a rather long computation) a global bound for the amplitudes. Since the general expression (2.40) is valid for *any* graph, a bound for $\mathcal{A}(\gamma, \{p_1..p_N\})$ indeed shows finiteness up to all orders. In the case of the scalar $1/p^2$ model the bound is (for graphs γ with genus 0, viewed at scale i, having connected components (subgraphs) γ_i^ρ depending on the scale attribution ρ (see Section 4.1.2 and Refs. [5, 99]).

$$\mathcal{A}^\rho(\gamma, \{p_1, ..p_N\}) \leq \prod_{i\in\rho,k} M^{-[N(\gamma_k^i)-4]} \,, \quad \text{with } M \in \mathbb{R}^+ \,, \tag{2.41}$$

[10] All of the concepts described here can also be translated to direct space.

[11] Note that the (Schwinger) parameter α corresponds to an inverse momentum k^{-2}, hence the limit $\alpha \to \infty$ relates to $k \to 0$

from which follows that only two and four point graphs γ, (with $N(\gamma) = 2$ and $N(\gamma) = 4$, resp.) give divergent contributions.

The final ingredient is the idea of RG steps, yielding an iterative procedure starting, in contrast to the 'classical' renormalization schemes (see the discussion in Chapter 4) from the highest energies (corresponding to $\rho \to \infty$), and ends at the completely renormalized effective action S^0. Assume, we have found a bound for a specific ρ, then we can define the field ϕ_ρ at this scale to be composed from a background $\phi_{\rho-1}$ and a fluctuating field φ_ρ representing the actual effects on the current scale, i.e. $\phi_\rho = \phi_{\rho-1} + \varphi_\rho$. Applying this notation we define a generating functional

$$Z_{\rho-1}[\phi_{\rho-1}] = \int \mathcal{D}\phi_\rho \, \mathrm{e}^{-S_\rho[\phi_{\rho-1}+\varphi_\rho]} \,,$$

from which, in turn, we gain the renormalized action at the next lower scale

$$S_{\rho-1}[\phi_{\rho-1}] = -\ln\left(Z_{\rho-1}[\phi_{\rho-1}]\right) \,.$$

The iterative procedure terminates at S_0 being the effective renormalized action. In practice, however, one will of course not explicitely compute the generating functional at each scale, but instead derive a rigorous power counting with the help of bounds for the amplitudes (see [40]). For the case of the non-commutative $1/p^2$ model this counting finally reads[12],

$$d(\gamma) = \begin{cases} 4 - E_\phi & \text{for } g(\gamma) = 0 \,, (\text{ i.e. planar graphs}) \\ -4 - E_\phi & \text{for } g(\gamma) > 0 \,, (\text{ i.e. non-planar graphs}) \end{cases} \,, \tag{2.42}$$

where $g(\gamma)$ is the genus of the respective graph γ (see Section 1.3.6) and $E_\phi \equiv N$ is the number of external legs. Explicitely, the counting (2.42) only tells us what types of graphs give divergent expressions (in this case the two and four point functions). The explicit renormalization contributions are then derived from approximations of Eqn. (2.40). For more details please refer to the monographs [5, 99], or [40] and references therein. The topic of renormalization and multiscale analysis will be picked up once again in Chapter 4.

2.2 The Way to a $U_\star(1)$ Gauge Model – Early Approaches

Being motivated by the successful proofs of renormalizability for the scalar models, the aim is now to generalize the concept of damping to gauge theories. As always, one starts with the simplest possible model, a free photon field, described by a $U(1)$ symmetry. As for the scalar case, there exists a naïve approach which is defined by the action

$$S_{\text{YM}}^{\text{naïve}} = \int \mathrm{d}^4x \, F_{\mu\nu} \star F_{\mu\nu} \,, \tag{2.43}$$

with the definitions

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} A_\nu\right] \,, \\ D_\mu\phi &= \partial_\mu\phi - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} \phi\right] \,, \forall \, \phi \,, \end{aligned} \tag{2.44}$$

for a field strength and covariant derivative respectively. Note that, due to the implicit x-dependence of the fields $A_\mu(x)$ the commutators do not vanish, as it is the case in commutative $U(1)$, i.e. QED or YM. Therefore, despite assuming an Abelian symmetry, the theory is rendered

[12] Note that in Ref. [40] $\omega(\gamma) = [d(\gamma)]^{-1}$.

non-Abelian by the Groenewold-Moyal star product Eqn. (1.22). In fact, the symmetry group is altered[13], and cannot be considered a $U(1)$ any more. Instead, the symmetry shall be named $U_\star(1)$ to indicate the deformation. More explicitly, one has

$$\begin{aligned}[A_\mu(x) \stackrel{\star}{,} A_\nu(y)] &= \left[A^a_\mu(x)T_a \stackrel{\star}{,} A^b_\nu(x)T_b\right] \\ &= \frac{1}{2}\Big(\left[A^a_\mu(x) \stackrel{\star}{,} A^b_\nu(y)\right]\{T_a,T_b\} + \left\{A^a_\mu(x) \stackrel{\star}{,} A^b_\nu(y)\right\}[T_a,T_b]\Big) \\ &= \frac{1}{2}\left[A^a_\mu(x) \stackrel{\star}{,} A^b_\nu(y)\right]\{T_a,T_b\} \\ &= \left[A^a_\mu \stackrel{\star}{,} A^a_\nu\right] \neq 0\,, \end{aligned} \tag{2.45}$$

where $[T_a, T_b] = 0$, and $\{T_a, T_b\} = 2\delta_{ab}$ has been applied for generators $T_a \in \mathbb{R}^1$ of $U(1)$.

Aiming to construct a physical theory, the following BRST transformations[14] are imposed

$$\begin{aligned} sA_\mu &= D_\mu c\,, & sc &= \mathrm{i}c \star c\,, \\ s\bar{c} &= b\,, & sb &= 0\,, \\ s^2\phi &= 0\,, \forall\phi \in \{A,b,c,\bar{c}\}\,. \end{aligned} \tag{2.46}$$

From these the properties

$$sF = \mathrm{i}g\left[c \stackrel{\star}{,} F\right]\,, \quad sD^2F = \mathrm{i}g\left[c \stackrel{\star}{,} D^2F\right]\,, \quad s\frac{1}{D^2}F = \mathrm{i}g\left[c \stackrel{\star}{,} \frac{1}{D^2}F\right]\,, \tag{2.47}$$

follow, which are proven in Appendix B.3, Theorems 4 and 5.

The model of Eqn. (2.43) has been discussed in great detail in Refs. [105, 106]. For the bosonic vacuum polarization at one loop order the result

$$\Pi_{\mu\nu} \propto \frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2}\,, \tag{2.48}$$

is obtained, which again represents a quadratic infrared divergence. It can easily be seen from the antisymmetry of the star product that $\Pi_{\mu\nu}$ is transversal with respect to external momenta p_μ, as $p_\mu\theta_{\mu\nu}p_\nu = -p_\nu\theta_{\nu\mu}p_\mu = 0$, as is physically required for a photon. Another notable point is that the quadratic divergence appears independently of the choice for the gauge parameter or Fadeev-Popov ghost terms [107–109]. Finally, the tensor structure of Eqn. (2.48) is of a type not appearing in commutative theory. This has to be accounted for in the renormalization programme, as is discussed further in Section 4.

From these considerations it is clear that the generalization from scalar to gauge models will not be straight forward in the non-commutative domain. In the literature, early contributions in this respect did not consider the modified IR behavior but solely took into account the UV divergent part for renormalization [44–46]. Research was also extended to more general topics, such as anomalies [110], restrictions on the representation of the algebra [104], and alternative

[13]It has been shown [100–104] that only enveloping algebras, such as $U(N)$ or $O(N)$ and $USp(2N)$, survive the introduction of a deformed product (in the sense that commutators of algebra elements are again algebra elements), while e.g. $SU(N)$ does not.

[14]In non-commutative theory the well known principle applies that a gauge boson propagator only exists if the gauge is explicitely broken by a fixing term. As can be found in many text books on the subject [75, 78, 79] the fixing requires the additional introduction of Grassmann-valued (Faddeev-Popov) ghost fields in order to leave invariant the functional integral. As has been recognized by Becchi, Rouet, Stora and Tyutin the resulting action remains invariant with respect to a nilpotent supersymmetric nonlinear transformation, represented by the BRST operator s with $s^2 = 0$.

deformations [111] but renormalizability was achieved so far only for $U(N)$ in the Seiberg Witten map [60, 112] (although in the same approximation other symmetries, such as the $U(1)\times SU(2)$ of QED do not share this feature [54]). More references and reviews can be found in Ref. [113]. For the sake of completeness it has to be mentioned that the non-commutative counterpart of QED including fermions has been treated in Refs. [51–53]. However, the effect of UV/IR mixing is not discussed there.

In the sequel the focus will lie on the construction of an extension to the non-commutative YM model (2.44) in order to implement the damping behavior of the $1/p^2$ model of Section 2.1.

2.2.1 The Search for Covariant Insertions

The aim of this section is to find an insertion to the non-commutative YM action similar to the non-local term in Eqn. (2.1). This new term, in addition to dimensional requirements, has to be invariant under gauge and BRST transformations.

From the form of the divergence appearing in the vacuum polarization Eqn. (2.48) one is led intuitively to the insertion

$$S_{\text{nloc}}^{\text{1st try}}[A] = \int \mathrm{d}^4x\, A_\mu(x) \star \frac{\widetilde{\partial}_\mu \widetilde{\partial}_\nu}{\widetilde{\Box}^2} \star A_\nu(x)\,.$$

In fact, $S_{\text{nloc}}^{\text{1st try}}$ is invariant under infinitesimal Abelian gauge variations $A_\mu \to A'_\mu = A_\mu + \delta A_\mu$, with $\delta A_\mu = \partial_\mu \Lambda$, and Λ being a local scalar parameter [114, 115], as can be seen by explicit computation. However, it is not invariant under the BRST transformations (2.46).
Noting that

$$\int \mathrm{d}^4x\, A_\mu(x) \star \frac{\widetilde{\partial}_\mu \widetilde{\partial}_\nu}{\widetilde{\Box}^2} \star A_\nu(x) = -\int \mathrm{d}^4x\, \widetilde{\partial}_\mu A_\mu(x) \star \frac{\widetilde{\partial}_\nu}{\widetilde{\Box}^2} \star A_\nu(x)\,,$$
$$\text{and } \widetilde{\partial}_\mu A_\mu = \theta_{\mu\rho}\partial_\rho A_\mu = \frac{1}{2}\theta_{\mu\rho}\left(\partial_\mu A_\rho - \partial_\rho A_\mu\right) \overset{\text{bilin.}}{\approx} \frac{1}{2}\widetilde{F}\,,$$

the next proposal is the insertion

$$S_{\text{nloc}}^{\text{2nd try}}[A] = \int \mathrm{d}^4x\, \widetilde{F}(x) \frac{1}{\widetilde{\Box}^2} \widetilde{F}(x)\,.$$

Again, gauge invariance is fulfilled but the $\frac{1}{\widetilde{\Box}^2}$ operator is not compatible with the BRST transformations (2.46). The only way to remedy this problem seems to be the replacement $\widetilde{\Box} \to \widetilde{D}^2 = \widetilde{D}_\mu \widetilde{D}_\mu = \theta^2 D^2$. Hence,

$$S_{\text{nloc}}^{\text{3rd try}}[A] = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \widetilde{F}(k) \frac{1}{\left(\widetilde{D}^2\right)^2} \widetilde{F}(-k)\,. \tag{2.49}$$

This insertion is completely invariant under all demanded symmetries, and features the right dimension. However, as being discussed in Refs. [61, 114], the resulting gauge propagator shows a quadratically IR divergent overall factor, i.e. $G^{AA} \propto \frac{1}{\tilde{k}^2} P_{\mu\nu}$, where $P_{\mu\nu}$ denotes the tensor structure which is not specified here. Hence, the term (2.49) cannot be utilized to implement the desired damping behavior (which would require an overall factor $\left(k^2 + \frac{\text{const.}}{\tilde{k}^2}\right)^{-1}$).

Finally, the solution is

$$S_{\text{nloc}}^{\text{final}}[A] = \int \mathrm{d}^4x\, F_{\mu\nu}(x) \frac{1}{D^2 \widetilde{D}^2} F_{\mu\nu}(x)\,. \tag{2.50}$$

The full tree-level action in position space then takes the form,

$$\begin{aligned}
S^{(0)} &= S_{\text{inv}} + S_{\text{gf}}\,, \\
S_{\text{inv}} &= \int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}\star F_{\mu\nu} + \frac{1}{4}F_{\mu\nu}\frac{1}{D^2\widetilde{D}^2}F_{\mu\nu}\right]\,, \\
S_{\text{gf}} &= s\int \mathrm{d}^4x\,\bar{c}\star\left[\left(1+\frac{1}{\square\widetilde{\square}}\right)\partial_\mu A_\mu - \frac{\alpha}{2}b\right] \\
&= \int \mathrm{d}^4x \left[b\star\left(1+\frac{1}{\square\widetilde{\square}}\right)\partial_\mu A_\mu - \frac{\alpha}{2}b\star b - \bar{c}\star\left(1+\frac{1}{\square\widetilde{\square}}\right)\partial_\mu D_\mu c\right]\,,
\end{aligned} \tag{2.51}$$

where the parameter α and the unphysical Lagrange multiplier field b have been introduced in order to fix the gauge. The insertion of the operators $\left(1+\frac{1}{\square\widetilde{\square}}\right)^{-1}$ (which are of the same type as in S^{inv}) in the gauge sector is motivated by the expectation of a damping for the ghost propagator $G^{\bar{c}c}$.

2.2.2 The Non-local Invariant Operator $1/D^2$

Having (with Eqn. (2.50)) found a gauge-compatible term to implement the damping behavior of the $1/p^2$ model the question arises how to interpret the new operator $\frac{1}{D^2}$. In contrast to the scalar version $\frac{1}{\square} = \frac{1}{\partial_\mu\partial_\mu}$, the covariant derivative includes the gauge field according to the definition (2.44). Since the inverse of a field cannot be defined in a reasonable way, an alternative representation for the new operator has to be found. Such is given [61] by the redefinition

$$\begin{aligned}
\widetilde{F} &= D^2\frac{1}{D^2}\widetilde{F} =: D^2\mathcal{Y} \\
&= \partial_\mu\partial_\mu\mathcal{Y} - \mathrm{i}g\partial_\mu\left[A_\mu \stackrel{\star}{,} \mathcal{Y}\right] - \mathrm{i}g\left[A_\mu \stackrel{\star}{,} \partial_\mu\mathcal{Y}\right] - g^2\left[A_\mu \stackrel{\star}{,} \left[A_\mu \stackrel{\star}{,} \mathcal{Y}\right]\right]\,.
\end{aligned}$$

Applying $\square^{-1}$ from the left and partially resolving for $\mathcal{Y}$ yields the relation,

$$\mathcal{Y} = \frac{1}{\square}\widetilde{F} + \mathrm{i}g\frac{\partial_\mu}{\square}\left[A_\mu \stackrel{\star}{,} \mathcal{Y}\right] + \mathrm{i}g\frac{1}{\square}\left[A_\mu \stackrel{\star}{,} \partial_\mu\mathcal{Y}\right] + g^2\frac{1}{\square}\left\{A_\mu \stackrel{\star}{,} \left\{A_\mu \stackrel{\star}{,} \mathcal{Y}\right\}\right\}\,, \tag{2.52}$$

which can be expanded according to the number of recursive insertions,

$$\begin{aligned}
\mathcal{Y}^{(0)} &= \frac{1}{\square}\widetilde{F} + \mathcal{O}(\mathcal{Y})\,, \\
\mathcal{Y}^{(1)} &= \frac{1}{\square}\widetilde{F} + \mathrm{i}g\frac{\partial_\mu}{\square}\left[A_\mu \stackrel{\star}{,} \frac{1}{\square}\right] + \mathrm{i}g\frac{1}{\square}\left[A_\mu \stackrel{\star}{,} \partial_\mu\frac{1}{\square}\right] + g^2\frac{1}{\square}\left[A_\mu \stackrel{\star}{,} \left[A_\mu \stackrel{\star}{,} \frac{1}{\square}\right]\right] + \mathcal{O}(\mathcal{Y})\,, \\
&\dots
\end{aligned} \tag{2.53}$$

Since $\frac{1}{\square}$ can be defined as in Eqn. (2.2) (see also Ref. [61]), the latter expression is defined as well. However, in terms of physics, the recursive procedure gives rise to an infinite number of gauge boson vertices. These, in turn are associated with an infinite number of parameters, therefore corresponding per definition to a power-counting non-renormalizable theory. It should also be remarked that any expression, involving $\mathcal{Y}^{(i)}$, for $i < \infty$, will not be gauge invariant since only the complete expression features this property.
For convenience, the following generally accepted definitions shall be given.

Definition 3. A theory is called *power counting renormalizable* if the number of divergent graphs being generated from it is bounded and only a finite number of counterterms are generated. This premises a finite number of vertices at tree level, each being bounded in its dimension on the upper side by the space-time dimension of the theory, and on the lower side by 1 (since dimensionless operators may be inserted to arbitrary powers).

Definition 4. A theory is called *power counting non-renormalizable* if the number of divergent graphs being generated from it increases infinitely with the order. Generally this is caused by vertices v with $d_m(v) > d$. A power counting non-renormalizable theory does not have any physical significance (but can only be interpreted as an *effective theory* which is valid up to a certain loop level).

From the discussion above it is clear that the model (2.51) will not be capable of giving any physical predictions since it is definitely of the power counting non-renormalizable category. Hence, no further computations will be presented for this approach. Instead, additional measures will be applied in order to obtain a *physical*, renormalizable model; a topic which is addressed in Chapter 3 below.

2.3 Summary

The translation invariant non-commutative $1/p^2$ model by Gurau *et al.* represents a modification of the naïve implementation of $\phi_4^{\star 4}$. It features an additional non-local quadratic term in the action which alters the propagator such that it vanishes in the UV as well as in the IR limit. In this way, a symmetry between high and low energies is implemented which, similar as in the successful oscillator Grosse Wulkenhaar model, seems to remedy the problem of UV/IR mixing.

We have analyzed the model via explicit one loop calculations in Section 2.1.1 giving corrections to the two and four point functions. The most important fact is, that the UV/IR mixing is not killed completely, as the infamous quadratic IR divergence still appears. An extended analysis in Section 2.1.2 revealed that the damping mechanism of the propagator becomes effective at the three loop level. Actually, an enhanced behavior is already achieved at two loops but the respective result for the naïve model in this case converges too. Hence, the effect is covered. Finally, in Section 2.1.3 we have utilized explicit one loop results to conduct a renormalization step. It appears that the quadratic IR divergences can be absorbed in the parameter of the new non-local term in the action, and the subleading logarithmic (UV) divergence enters a renormalized mass. Finally, one is left with a so called 'mild logarithmic divergence'. An interpretation of the latter cannot be found in the literature but it has been stressed that the respective term in the renormalized propagator is only a finite contribution for non-vanishing momenta and can be neglected in comparison with the squared terms for asymptotic values.

The great simplicity and structure of the scalar $1/p^2$ model have motivated the search for a suitable gauge generalization. Section 2.2 contains the discussion leading to an invariant term which is suspected to be capable of implementing the desired damping behavior of the propagator in a gauge invariant manner. However, the solution turned out to be problematic since it contains the inverse of covariant derivatives, and hence the inverse of fields. The latter is not well defined, and a reasonable interpretation can only (at first sight) be given in terms of an infinite series which gives rise to an (as well) infinite number of gauge boson vertices. The resolution of these problems will be subjected in the next chapter.

Chapter 3

Localized Gauge Models

Despite the inherent non-locality of the Groenewold-Moyal product it has been shown that renormalizability of non-commutative theory can be achieved in the scalar case by adding appropriate (counter)terms to the tree level action. These, in different ways, implement a damping mechanism which suppresses the UV/IR mixing and renders the theory finite. In the gauge case several constraints regarding symmetries and the tensorial structure have to be fullfilled by counterterms and insertions. As was discussed in Section 2.2, in the $1/p^2$ model one is forced to introduce the inverse of covariant derivatives which can only be interpreted in form of an infinite series, thereby inevitably leading to a power counting non-renormalizable theory. However, it turns out that there are alternative representations which 'localize' the problematic terms by coupling them to unphysical auxiliary fields. There are several ways to implement this, resulting in models with different properties, and even a modified physical content. In this respect we are led to the insight that only minimal couplings and the consequent construction of BRST doublet structures for all auxiliary fields result in a stable theory (even at tree level). Moreover, the consistent implementation of the damping behavior of the $1/p^2$ model requires the insertion of a so called 'soft breaking' term into the action; a method which is well known from the Gribov-Zwanziger approach to QCD (see Refs. [116–118] and Chapter 4).
This chapter describes basically two implementations being developed in order to localize the operator $(D^2\widetilde{D}^2)^{-1}$ discussed in Section 2.2 above. The aim is to construct a physical power-counting renormalizable model, based on non-commutative YM theory with gauge group $U_\star(1)$, and implementing the damping of the $1/p^2$ model. In Section 3.1 the first approach of localization with one real-valued auxiliary field is presented, which turns out to change physics in an unintended way. A more advanced version avoiding these problems is discussed in Section 3.2, where all auxiliary fields are rendered unphysical by the utilization of BRST doublet structures. Section 3.3 then contains a critical discussion of the lessons learned from the models analyzed up to that point. Based on these findings, after an excursion to the problem of renormalization in the presence of deformation in Section 4, the currently most advanced (BRSW) model will be constructed in Chapter 5.

3.1 Localization with a Real Auxiliary Field

The first ansatz in the construction of a renormalizable $U_\star(1)$ gauge version of the $1/p^2$ scalar model was the localization with a real-valued auxiliary field [61]. The following discussion will start from the non-local action (2.51) in Section 2.2.1. The methodology is clear. First, a localized action is constructed and analyzed in Section 3.1.1. A discussion of the UV power counting follows in Section 3.1.2, and the results of one loop calculations regarding the vacuum polarization, and the appearance of unphysical degrees of freedom are presented in Sections 3.1.3 and 3.1.4 respectively.

3.1.1 Construction of the Action

The idea is to localize the operator $\left(D^2\widetilde{D}^2\right)^{-1}$ (in the action (2.51), being denoted here by $S_{\rm inv}^{\rm nloc}$) by the introduction of an auxiliary real-valued antisymmetric field $B_{\mu\nu}$ of mass dimension two. This is achieved by replacing

$$S_{\rm inv}^{\rm nloc} \to S_{\rm inv}^{\rm loc}$$

$$\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}\star F_{\mu\nu} + \frac{1}{4}F_{\mu\nu}\star\frac{a^2}{D^2\widetilde{D}^2}\star F_{\mu\nu}\right] \to \int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}\star F_{\mu\nu} + aB_{\mu\nu}\star F_{\mu\nu} - B_{\mu\nu}\star D^2\widetilde{D}^2\star B_{\mu\nu}\right], \tag{3.1}$$

in the action, where a is a dimensionless operator motivated by the fact that a similar parameter was renormalized in the scalar $1/p^2$ model in Section 2.1.3. Regarding the notation, we will consequently omit the stars for the remainder of Section 3.1, and all products of fields can be considered to be of the form (1.24). Equivalence of the terms of the left and right hand side in Eqn. (3.1) can immediately be seen in the functional formalism [75, 77] by integrating out the new $B_{\mu\nu}$ field

$$\begin{aligned}
&\int \mathcal{D}A\mathcal{D}B\, \exp\left\{-\int \mathrm{d}^4x\left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + aB_{\mu\nu}F_{\mu\nu} - B_{\mu\nu}D^2\widetilde{D}^2B_{\mu\nu}\right]\right\}\\
&= \int \mathcal{D}A\mathcal{D}B\, \exp\left\{-\int \mathrm{d}^4x\left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu}\right.\right.\\
&\qquad \left.\left. -\left(B_{\mu\nu} - \frac{a}{2}\frac{1}{D^2\widetilde{D}^2}F_{\mu\nu}\right)D^2\widetilde{D}^2\left(B_{\mu\nu} - \frac{a}{2}\frac{1}{D^2\widetilde{D}^2}F_{\mu\nu}\right) + \frac{a^2}{4}F_{\mu\nu}\frac{1}{D^2\widetilde{D}^2}F_{\mu\nu}\right]\right\}\\
&= \int \mathcal{D}A\, \left(\det D^2\widetilde{D}^2\right)^{-2} \exp\left\{-\int \mathrm{d}^4x\,\frac{1}{4}F_{\mu\nu}\left(1+\frac{a^2}{D^2\widetilde{D}^2}\right)F_{\mu\nu}\right\}.
\end{aligned} \tag{3.2}$$

The inverse determinant depends on A and is therefore nontrivial. Its meaning, however, will be discussed later in Section 3.2.2. As an alternative approach for showing the equivalence of local and non-local action, the equation of motion for the $B_{\mu\nu}$ field yields

$$\begin{aligned}
\frac{\delta S_{\rm inv}^{\rm loc}}{\delta B_{\rho\sigma}} &= aF_{\rho\sigma} - 2\widetilde{D}^2D^2\star B_{\rho\sigma} = 0\\
\Rightarrow B_{\rho\sigma} &= \frac{a}{2}\frac{1}{D^2\widetilde{D}^2}F_{\rho\sigma}\,,
\end{aligned} \tag{3.3}$$

which may be reinserted into $S_{\rm inv}^{\rm loc}$, leading back to the original version $S_{\rm inv}^{\rm nloc}$.

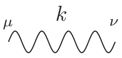
The new $B_{\mu\nu}$ field transforms BRST covariantly. For the sake of completeness the entire set of transformation rules is given at this point:

$$\begin{aligned}
sA_\mu &= D_\mu c\,, & sc &= \mathrm{i}cc\,,\\
s\bar{c} &= b\,, & sb &= 0\,,\\
sF_{\mu\nu} &= \mathrm{i}g\left[c, F_{\mu\nu}\right]\,, & sB_{\mu\nu} &= \mathrm{i}g\left[c, B_{\mu\nu}\right]\,,\\
s^2\phi &= 0\,, \forall\phi\in\{A,b,B,c,\bar{c}\}\,.
\end{aligned} \tag{3.4}$$

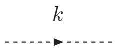
Finally, the action of this model takes the form

$$
\begin{aligned}
\Gamma^{(0)} &= S_{\text{inv}}^{\text{loc}} + S_{\text{gf}}\,, \\
S_{\text{inv}}^{\text{loc}} &= \int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + aB_{\mu\nu}F_{\mu\nu} - B_{\mu\nu}D^2\widetilde{D}^2B_{\mu\nu}\right]\,, \\
S_{\text{gf}} &= s\int \mathrm{d}^4x\; \bar{c}\left[\left(1+\frac{a^2}{\Box\widetilde{\Box}}\right)\partial^\mu A_\mu - \frac{\alpha}{2}b\right] \\
&= \int \mathrm{d}^4x \left[b\left(1+\frac{a^2}{\Box\widetilde{\Box}}\right)\partial^\mu A_\mu - \frac{\alpha}{2}bb - \bar{c}\left(1+\frac{a^2}{\Box\widetilde{\Box}}\right)\partial^\mu D_\mu c\right]\,.
\end{aligned}
\tag{3.5}
$$

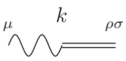
The equations of motion and a sketch of the derivation of the propagators are given in Appendix B.1. Results for the latter are collected in Eqns. (3.6a)–(3.6b) below.



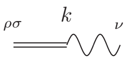
$$= G^{AA}_{\mu,\nu} = -\frac{\delta A_\mu}{\delta j^\nu_A} = \frac{1}{k^2+\frac{a^2}{\tilde{k}^2}}\left(-\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} - \alpha\frac{k_\mu k\nu}{k^2+\frac{a}{\tilde{k}^2}}\right)\,, \tag{3.6a}$$



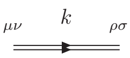
$$= G^{cc} = -\frac{\delta c}{\delta j_c} = \frac{-1}{k^2+\frac{a^2}{\tilde{k}^2}}\,, \tag{3.6b}$$



$$= G^{AB}_{\mu,\rho\sigma} = -\frac{\delta A^\mu}{\delta j^{\rho\sigma}_B} = \frac{-\mathrm{i}a}{2k^2\tilde{k}^2\left(k^2+\frac{a^2}{\tilde{k}^2}\right)}\left(k_\sigma\delta_{\rho\mu} - k_\rho\delta_{\sigma\mu}\right)\,, \tag{3.6c}$$



$$= G^{BA}_{\rho\sigma,\nu} = -\frac{\delta B^{\rho\sigma}}{\delta j^\nu_A} = \frac{\mathrm{i}a}{2k^2\tilde{k}^2\left(k^2+\frac{a^2}{\tilde{k}^2}\right)}\left(k_\sigma\delta_{\rho\nu} - k_\rho\delta_{\sigma\nu}\right)\,, \tag{3.6d}$$



$$
\begin{aligned}
= G^{BB}_{\mu\nu,\rho\sigma} = -\frac{\delta B_{\mu\nu}}{\delta j^{\rho\sigma}_B} = \frac{a}{4k^2\tilde{k}^2}\Bigg[&\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \\
&+ a\frac{k_\mu k_\sigma\delta_{\nu\rho} + k_\nu k_\rho\delta_{\mu\sigma} - k_\mu k_\rho\delta_{\nu\sigma} - k_\nu k_\sigma\delta_{\mu\rho}}{k^2\tilde{k}^2\left(k^2+\frac{a^2}{\tilde{k}^2}\right)}\Bigg]\,.
\end{aligned}
\tag{3.6e}
$$

The last three are antisymmetric in the index pairs corresponding to the $B_{\mu\nu}$ fields, i.e.

$$
\begin{aligned}
G^{AB}_{\rho,\sigma\tau}(k) &= -G^{AB}_{\rho,\tau\sigma}(k) = -G^{BA}_{\sigma\tau,\rho}(k)\,, \\
G^{BB}_{\rho\sigma,\tau\epsilon}(k) &= -G^{BB}_{\sigma\rho,\tau\epsilon}(k) = -G^{BB}_{\rho\sigma,\epsilon\tau}(k)\,.
\end{aligned}
\tag{3.7}
$$

Notice furthermore the relations

$$
\begin{aligned}
2k^2\tilde{k}^2G^{AB}_{\rho,\mu\nu}(k) &= \mathrm{i}ak_\mu G^{AA}_{\rho\nu}(k) - \mathrm{i}ak_\nu G^{AA}_{\rho\mu}(k)\,, \\
2k^2\tilde{k}^2G^{BB}_{\mu\nu,\rho\sigma}(k) &= \frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) + \mathrm{i}ak_\mu G^{BA}_{\rho\sigma,\nu}(k) - \mathrm{i}ak_\nu G^{BA}_{\rho\sigma,\mu}(k)\,,
\end{aligned}
\tag{3.8}
$$

which indicate that it is not possible to completely reduce the two point functions containing $B_{\mu\nu}$ to expressions containing solely G^{AA} (due to the antisymmetric unit $\frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right)$ in the second line of (3.8)). Obviously, the gauge propagator (3.6a), and the ghost propagator (3.6b) both feature the desired damping behavior which led to renormalizability in the scalar

$1/p^2$ model. The situation is different for all propagators involving B fields. In the IR limit the mixed functionals G^{AB} and G^{BA} diverge linearly, and G^{BB} even exhibits a quadratic singularity.

$$\lim_{k\to 0} G^{AB}(k) = -\lim_{k\to 0} G^{BA}(k) \propto \frac{1}{|k|}\,, \quad \lim_{k\to 0} G^{BB}(k) \propto \frac{1}{k^2\tilde{k}^2}\,.$$

These divergences will be discussed in more detail for another model in Section 3.2.5.

From the interactions in Eqn. (3.5) follow the tree level vertex functions by direct variation with respect to the fields. Due to the rather lengthy form of these expressions they are only listed in Appendix B.1.3. We may note at this place that there exist, besides the vertices V^{3A}, $V^{\bar{c}Ac}$, and V^{4A} known from naïve implementations of YM theory in non-commutative space [105, 106], the couplings V^{BAA}, V^{BBA}, V^{2B2A}, V^{2B3A}, and V^{2B4A}. These give rise to a large number of graphs even at the lowest orders in the perturbative expansion. Hence, being interested only in divergent contributions, it is important to be able to preselect graphs according to their expected behavior, i.e. to assess the degree of divergence without the need for the evaluation of loop integrals. Such a tool is given by a power counting formula which will be derived next.

3.1.2 UV Power Counting

In order to estimate the divergence behavior prior to explicit loop calculations we shall derive an expression for the superficial degree of divergence, as being defined in Section 1.3.1, for an arbitrary process in the model (3.5). For a general n-point 1PI graph γ with N vertices, L internal loops, I_ϕ internal lines of type ϕ (i.e. one of the two-point functions (3.6a)–Eqn. (3.6e)), and E_ϕ external lines of type $\phi \in \{A, B, c, \bar{c}\}$, we take into account the powers of internal momenta k each Feynman rule $\mathcal{R}$ contributes, i.e. $d(\mathcal{R}(k))$. In addition, each loop integral d^4k increases d_γ by four. Hence,

$$\begin{aligned} d_\gamma &= 4L - \sum_{\mathcal{R}} d(\mathcal{R}(k)) \\ &= 4L - 2I_{AA} - 2I_{\bar{c}c} - 5I_{AB} - 4I_{BB} + V_{\bar{c}Ac} + V_{3A} + 3V_{BBA} + 2V_{2B2A} + V_{2B3A}\,. \end{aligned} \tag{3.9}$$

From (1.33) follows

$$\begin{aligned} L = {} & I_{AA} + I_{\bar{c}c} + I_{AB} + I_{BB} - \\ & - (V_{\bar{c}Ac} + V_{3A} + V_{4A} + V_{BAA} + V_{BBA} + V_{2B2A} + V_{2B3A} + V_{2B4A} - 1)\,. \end{aligned}$$

Next, the connectedness implies that the number of free 'legs' for each field type ϕ has to sum up to zero for any Feynman graph. In this respect external lines always contribute only a single leg with the outer end being fixed[1]. All remaining internal elements (n-point functions) are accounted according to the number of legs they provide for each field type. Regarding the signs, external and internal lines are attributed positive, while vertex legs are considered to be negative. For the last two lines in (3.10) below the sign corresponds to the power of the respective parameter a Feynman rule contributes. This procedure yields,

$$\begin{aligned} E_{\bar{c}} + E_c + 2I_{\bar{c}c} &= 2V_{\bar{c}Ac}\,, \\ E_A + 2I_{AA} + I_{AB} &= V_{\bar{c}Ac} + 3V_{3A} + 4V_{4A} + 2V_{BAA} + V_{BBA} + 2V_{2B2A} + 3V_{2B3A} + 4V_{2B4A}\,, \\ E_B + 2I_{BB} + I_{AB} &= V_{BAA} + 2V_{BBA} + 2V_{2B2A} + 2V_{2B3A} + 2V_{2B4A}\,, \\ E_\theta &= 2I_{AB} + 2I_{BB} - 2V_{BBA} - 2V_{2B2A} - 2V_{2B3A} - 2V_{2B4A}\,, \\ E_a &= I_{AB} + V_{BAA}\,. \end{aligned} \tag{3.10}$$

[1] The scheme of counting and connecting legs is the same as used for the pregraphs discussed for the symmetry factors in Section 1.3.2, i.e. every internal or external leg has to be connected to one another.

Note that the E_c, $E_{\bar{c}}$, E_A, and E_B denote the number of external lines of the respective fields whereas E_θ and E_a count the total negative powers of θ and positive powers of a in a graph, respectively. The above set of equations can be resolved for d_γ. The last three lines of Eqn. (3.10) yield the relation $E_B + E_\theta = E_a$ which allows to write two alternative expressions for the power counting, reading

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - 2E_B - 2E_\theta \,, \tag{3.11a}$$

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - 2E_a \,. \tag{3.11b}$$

As expected from existing commuting theories, d_γ is reduced by the number of external legs weighted by the dimension of the respective fields (and parameters). In the first version (3.11a) the overall power of θ plays a role, indicating the effect of non-commutativity. However, since E_θ is not strictly positive, the form (3.11b) may be more intuitive, in practice.

3.1.3 One Loop Computations

As the form of the propagators in the gauge model (3.5) is in general more complicated than in the scalar model of Section 2.1.1 a few technical remarks regarding the details of computations are in order. First of all, the splitting of the propagator (c.f. Eqn. (2.9)) has to be modified due to the missing mass. A solution is given by,

$$\frac{1}{k^2 + \frac{1}{\tilde{k}^2}} = \frac{k^2}{\left(k^2 + \frac{\mathrm{i}}{\theta}\right)\left(k^2 - \frac{\mathrm{i}}{\theta}\right)} = \frac{1}{2}\left[\frac{1}{\left(k^2 + \frac{\mathrm{i}}{\theta}\right)} + \frac{1}{\left(k^2 - \frac{\mathrm{i}}{\theta}\right)}\right] . \tag{3.12}$$

The appearance of the imaginary terms gives rise to slightly more involved parameter integrals. For example consider the simplest possible expression

$$\begin{aligned}
\sum_{\eta=\pm1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{\eta \mathrm{i}k\tilde{p}}}{k^2+\frac{1}{\tilde{k}^2}} &= \frac{1}{2}\sum_{\eta,\xi=\pm1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^\infty \mathrm{d}\alpha \exp\left[-\alpha\left(k^2+\frac{\mathrm{i}\xi}{\theta}\right)+\mathrm{i}\eta k\tilde{p}\right] \\
&= \sum_{\xi=\pm1}\int_0^\infty d\alpha\frac{\pi^2}{\alpha^2}\exp\left[-\alpha\left(\frac{\mathrm{i}\xi}{\theta}\right)-\frac{\tilde{p}^2}{4\alpha}\right] \\
&= 2\pi^2\int_0^\infty d\alpha\frac{\cos\left(\frac{\alpha}{\theta}\right)}{\alpha^2}e^{-\frac{\tilde{p}^2}{4\alpha}} \\
&\overset{\text{Eqn. (F.13b)}}{=} 4\pi^2\frac{1}{\sqrt{\theta\tilde{p}^2}}\left[e^{\mathrm{i}\pi/4}K_{-1}\left(e^{\mathrm{i}\pi/4}\sqrt{\frac{\tilde{p}^2}{\theta}}\right)+e^{-\mathrm{i}\pi/4}K_{-1}\left(e^{-\mathrm{i}\pi/4}\sqrt{\frac{\tilde{p}^2}{\theta}}\right)\right] \\
&\overset{\text{Eqn. (F.24)}}{=} \frac{8\pi^2}{\tilde{p}^2}-\frac{\pi^3}{\theta}+\mathcal{O}(\tilde{p}^2)\,,
\end{aligned} \tag{3.13}$$

where in the last step the Bessel functions have been expanded for small argument according to Eqn. (F.24) (which is implied for the limit of small external momenta $\tilde{p} \ll 1$). However, computations involving the decomposition (3.12) will result in rather lengthy expressions, even at the one loop level. In fact, there exists a further option for simplification. Recall the discussion of UV/IR mixing in Section 1.3.7, which led to the insight that the IR divergences of loop calculations originate from the UV limit of the integrands. Since the high energy behavior is not affected by the damping term of the $1/p^2$ model, the following approximation for integrands

$\mathcal{I}$ is suggesting itself

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathcal{I}(p,k) \propto \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \varphi(p,k) \left(k^2 + \frac{a^2}{\tilde{k}^2}\right)^{-n} \approx \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \varphi(p,k) \frac{1}{\left(k^2\right)^n}\,, \tag{3.14}$$

where $\varphi(p,k)$ is a generic placeholder for constants, tensor structures, and phase factors; n is a positive integer-valued exponent. The validity of Eqn. (3.14) becomes obvious from an explicit recalculation of the example presented above,

$$\begin{aligned}\sum_{\eta=\pm 1} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{\eta i k\tilde{p}}}{k^2 + \frac{1}{\tilde{k}^2}} &\approx \sum_{\eta=\pm 1} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{\eta i k\tilde{p}}}{k^2} \\ &= 2\int_0^{\infty} \mathrm{d}\alpha \frac{\pi^2}{\alpha^2} \mathrm{e}^{-\frac{\tilde{p}^2}{4\alpha}} \\ &\overset{\text{Eqn. (F.12)}}{=} \frac{8\pi^2}{\tilde{p}^2}\,. \end{aligned} \tag{3.15}$$

In fact, the omission of the IR damping in the approximation only leads to an error in the finite contributions. The leading divergences, however, are exactly the same as in the (more) accurate result (3.13). It has to be mentioned that this simplification may be applied for any one loop graph but care has to be taken at two and higher loop orders since there the damping behavior of the propagators is essential in order to regularize the IR divergent insertions of one loop results[2].
Further technical issues are discussed for the vacuum polarization in Section 3.1.3. Detailed results and sketches of the relevant computations can be found in Appendix B.2.

Vanishing of Tadpole Graphs

The Feynman rules (3.6a)–(3.6e) and (B.6)–(B.16) give rise to four possible one loop tadpole graphs with external gauge boson lines being depicted in Figure 3.1. Each one of these graphs incorporates a phase factor $\sin\left(\frac{k\tilde{p}}{2}\right)$ with p and k being the external and internal momenta, respectively. Momentum conservation at the vertices implies $k - k + p = 0$, resulting in a phase $\sin 0 = 0$. Hence, all four graphs vanish identically.

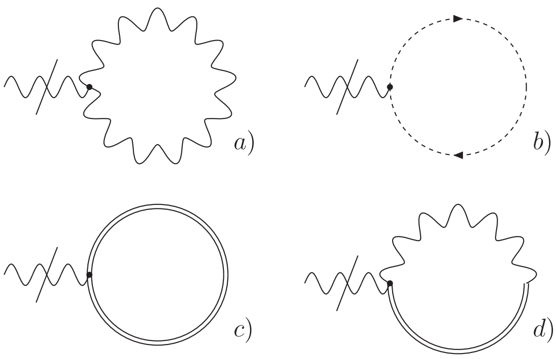


Figure 3.1: One-loop tadpole graphs

[2]Compare the discussion of the damping effect at higher loop orders for the scalar $1/p^2$ model in Section 2.1.2.

One Loop Vacuum Polarization

From the Feynman rules we can construct twelve 1PI one loop graphs with two external photon lines depicted in Fig. 3.2. The first three processes are known from commutative gauge theories with non-Abelian gauge groups, as for example pure YM or QCD [84], but also from non-commutative QED [51]. All remaining graphs are new and unique to this model. In contrast to

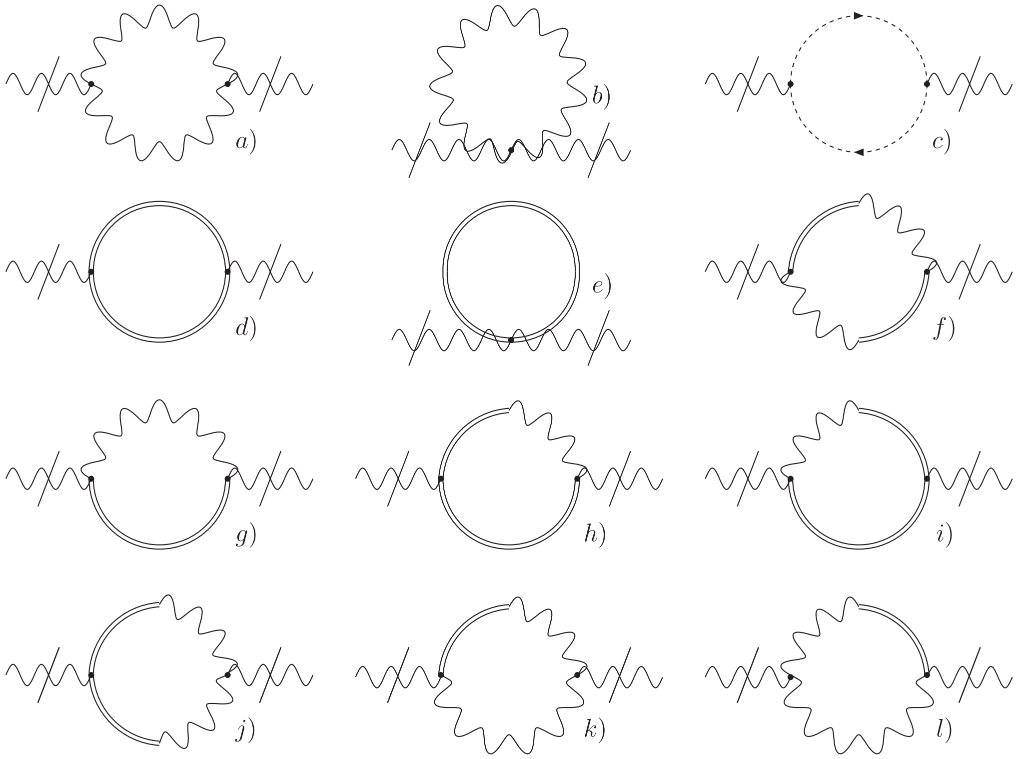


Figure 3.2: Processes contributing to the one loop boson vacuum polarization

the respective one loop amplitude (2.8) of scalar model in Section 2.1 the integrals corresponding to the processes in Fig. 3.2 are complicated functions of internal and external momenta k and p, and cannot be evaluated in a straight forward way. The general form of these expressions is

$$\Pi_{\mu\nu} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \mathcal{I}_{\mu\nu}(k,p) \sin^2\left(\frac{k\tilde{p}}{2}\right) . \tag{3.16}$$

Since we are interested mainly in the IR behavior of the theory the single-graph results are expanded for small external momenta p according to

$$\begin{aligned}\Pi_{\mu\nu} &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \mathcal{I}_{\mu\nu}(p,k) \sin^2\!\left(\tfrac{k\tilde{p}}{2}\right) \\ &\approx \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \sin^2\!\left(\tfrac{k\tilde{p}}{2}\right) \Big[\mathcal{I}_{\mu\nu}(0,k) + p_\rho \big[\partial_{p_\rho} \mathcal{I}_{\mu\nu}(p,k)\big]_{p\to 0} + \tfrac{p_\rho p_\sigma}{2} \big[\partial_{p_\rho}\partial_{p_\sigma} \mathcal{I}_{\mu\nu}(p,k)\big]_{p\to 0} + \mathcal{O}(p^3)\Big] .\end{aligned} \tag{3.17}$$

The phase factors are not expanded at this point in order not to lose the regularizing effects in the non-planar parts due to rapid oscillations for large k. In the integrands (see also Appendix B.2

Table 3.1: Symmetry factors for the graphs in Fig. 3.2

$s_a=\frac{1}{2}$	$s_b=\frac{1}{2}$	$s_c=1$
$s_d=\frac{1}{2}$	$s_e=\frac{1}{2}$	$s_f=1$
$s_g=1$	$s_h=1$	$s_i=1$
$s_j=\frac{1}{2}$	$s_k=1$	$s_l=1$

for the full expressions) the external momenta mostly appear in summations, together with k. Therefore, the above expansion is actually valid for the assumption $|p| \ll |k|$. Since k is being integrated over, there exists a small domain $|k| < \varepsilon \ll 1$ where this premise is not fulfilled. However, as has been discussed in Section 1.3.7, the IR divergences, which we are interested in here, originate from the UV sector of k where the above assumption applies, and everything is well defined. Hence, in the results being discussed subsequently, the error which is introduced by the approximation (3.17) may only be a finite contribution which is neglected with respect to divergent terms in p.

From the expansion (3.17) it can be expected that the highest divergences appear in the results obtained from the lowest order terms[3]. Explicit calculation reveals that only five $(a-e)$ of the graphs depicted in Fig. 3.2, show singular behavior for $\tilde{p} \to 0$. These read (after approximation for large k)

$$\Pi^{(\mathrm{a}),0}_{\mu\nu} \approx s_a \frac{8g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)^2} \left\{ 6k_\mu k_\nu + \alpha\, k^2 \frac{\left(k^2\delta_{\mu\nu}-k_\mu k_\nu\right)}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \right\}, \tag{3.18a}$$

$$\Pi^{(\mathrm{b}),0}_{\mu\nu} \approx -s_b \frac{8g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^2+\frac{a^2}{\tilde{k}^2}} \left[2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \alpha \frac{(k^2\delta_{\mu\nu}-k_\mu k_\nu)}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \right], \tag{3.18b}$$

$$\Pi^{(\mathrm{c}),0}_{\mu\nu} \approx -s_c \frac{4g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{k_\mu k_\nu}{k^4}, \tag{3.18c}$$

$$\Pi^{(\mathrm{d}),0}_{\mu\nu} \approx s_d \frac{192g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{k_\mu k_\nu}{k^4} \left(2 + \frac{a^2\left(\frac{a^2}{\tilde{k}^2}-2\right)}{\tilde{k}^2\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \right), \tag{3.18d}$$

$$\Pi^{(\mathrm{e}),0}_{\mu\nu} \approx -s_e \frac{24g^2}{(2\pi)^4} \int \mathrm{d}^4k\, \frac{\sin^2(\frac{k\tilde{p}}{2})}{k^4} [4k_\mu k_\nu + 2k^2\delta_{\mu\nu}] \left(2 - \frac{a^2}{\tilde{k}^2\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \right), \tag{3.18e}$$

where the symmetry factors s_i, $i \in \{\mathrm{a,b,c,d,e}\}$ are listed in Table 3.1. The remaining graphs (f)–(l) of Fig. 3.2 are found to be finite. This observation is consistent with the power counting formula (3.11b), as $E_A = 2$ for all graphs (a)–(l). In addition, the processes (g)–(l) come with two overall powers of a, i.e. $E_a = 2$, and graph (f) even has 4 powers of a, i.e. $E_a = 4$.

The final expression for the (leading order in the expansion (3.17) of the) vacuum polarization

[3]We can indeed expect to catch the most significant divergences in the lowest orders of the expansion since the point of approximation, actually, is the IR pole of the integrand. Mathematically, in order to stay within the borders of legality, we have to keep $\tilde{p} \neq 0$ for the time being, and analyze the limit $\tilde{p} \to 0$ not before the very end of our calculation. Otherwise the radius of convergence would vanish, and the expansion was not defined.

is obtained by summing up all (divergent) contributions of Eqns. (3.18a)–(3.18e), yielding

$$\begin{aligned}
\Pi^{\text{total},0}_{\mu\nu} &= \sum_{j=\{a,b,c,d,e\}} \Pi^{(\text{j}),0}_{\mu\nu} \\
&= \frac{4g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \left[\frac{-1}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \left(2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \alpha \frac{(k^2\delta_{\mu\nu} - k_\mu k_\nu)}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \right) - \frac{k_\mu k_\nu}{(k^2)^2} \right. \\
&\qquad + \frac{1}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)^2} \left(6k_\mu k_\nu + \alpha k^2 \frac{(k^2\delta_{\mu\nu} - k_\mu k_\nu)}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \right) \\
&\qquad \left. + \frac{12}{k^2} \left(2\frac{k_\mu k_\nu}{k^2} - \delta_{\mu\nu} \right) \right] \\
&= 14 \frac{g^2}{\pi^2} \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} + \text{finite terms.}
\end{aligned} \tag{3.19}$$

This result shows a quadratic IR divergence in leading order, as does the one loop result of the scalar models (c.f. Section 2.1.1, Eqn. (2.11)). The pole is independent of the gauge parameter α and transversal with respect to p_μ (i.e. $p_\mu \Pi^{\text{total},0}_{\mu\nu} = 0$), as would be required for a physical photon[4]. It has to be noted that the tensor structure $\tilde{p}_\mu \tilde{p}_\nu$ is of a completely different nature than the operator $(p^2 g_{\mu\nu} - p_\mu p_\nu)$ appearing in commuting gauge theories. However, the new structure does not replace the 'classical' one but represents an additional option to implement transversality. In fact, the hidden finite terms in Eqn. (3.19) contain either type as can (partly) be seen from the more detailed results in Appendix B.2.

It should also be mentioned that in the calculation of the integrals of Eqns. (3.18a)–(3.18e) the limit $a \to 0$ has been taken prior to the integration. This is motivated mainly by the fact that in this way any dependence on the gauge parameter α is eliminated, as is clear from the first two lines of Eqn. (3.19) where the α-dependent terms indeed cancel for $a \to 0$. In addition, the divergences originate from the UV limit in k, but the a-dependent (sub-)terms only affect the IR region via the damping mechanism[5]. Therefore,

$$\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{\left(k^2+\frac{a^2}{\tilde{k}^2}\right)} \approx \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2},$$

and it is not surprising that the result for the quadratic IR divergence in Eqn. (3.19) neither depends on a nor on α. Interestingly, the sum of the contributions from graphs (a), (b), and (c) in Fig. 3.2, which correspond to the processes known from commuting theories is transversal by itself (as is the contribution from the other two graphs).

Finally, it should be stressed that the zero order result Eqn. (3.19) does not contain any logarithmic IR divergences. These are obtained only in the second order result[6] (where α has been set to 0, corresponding to Landau gauge)

$$\Pi^{\text{total},(2)}_{\mu\nu}(p) = \frac{p_\rho p_\sigma}{2} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{\partial^2}{\partial p_\rho \partial p_\sigma} \Big(\sum_{i\in\{\text{a,c,d,e}\}} \mathcal{I}^{(\text{i}),(2)}_{\mu\nu}(k,p) \Big) \right]_{p=0}$$

[4]In case, the theory indeed describes the dynamics of electromagnetic waves in vacuum.

[5]The procedure of taking the limit $a \to 0$ equals the approximation for large k being discussed above on page 38 at the beginning of Section 3.1.3.

[6]Some details of the computation are given in Appendix B.2. Note that the first order identically vanishes due to a symmetric integration over an odd power of k in the integrand.

$$= \frac{g^2}{24\pi^2}\left(p_\mu p_\nu - p^2\delta_{\mu\nu}\right)\left(\ln\left(\Lambda^2\tilde{p}^2\right) + \text{finite terms}\right), \tag{3.20}$$

where again only divergent contributions have been considered, and the notion ‘finite’ refers to the limit $\tilde{p} \to 0$ as well as the respective limits of any cutoff or regulator appearing in hidden terms. The only logarithmic divergence is in the UV cutoff Λ coming from the planar parts. Hence, the result (3.20) is well-behaved for $p \to 0$, i.e. there is no logarithmic infrared divergence in the external momentum. However, the conditions of transversality with respect to p_μ, and independence of the gauge parameter α are fulfilled for the divergent part.

In conclusion, the one loop vacuum polarization in the model (3.5) reveals the quadratic divergence in the external momentum for $\tilde{p} \to 0$, as expected from the discussion of the UV/IR mixing in Section 1.3.7, and the results for the naïve non-commutative $U_\star(1)$ theory in Section 2.2. The transversal tensor structure $\tilde{p}_\mu\tilde{p}_\nu$ bound to this IR singularity is of a type unknown in commuting theories (since it contains contractions with $\theta_{\mu\nu}$). Since this structure does not appear in the tree level gauge propagator (3.6a) of this model problems in the renormalization can be expected. The planar parts of the integrals involved in the computation of the vacuum polarization contribute a logarithmic (UV) divergence in the cutoff $\Lambda \to \infty$, similar to the respective result in commuting YM theory. Hence, the one loop corrections for the photon propagator basically showed expected results.

At this point we could proceed, and start the renormalization process to obtain the one loop effective action. However, the above discussion has left one topic untouched which shall be discussed prior to any further calculation: the interpretation of the multiplier field $B_{\mu\nu}$ which has been introduced for the purpose of localization in Section 3.1.1. This point is discussed subsequently in Section 3.1.4.

3.1.4 Additional Degrees of Freedom

Regarding the interpretation of the new field $B_{\mu\nu}$ several points have to be considered. First, the original non-local term (2.50) in the action is parametrized by a. Hence, imposing the limit $a \to 0$ should lead back to the naïve non-commutative model in all relevant expressions. Contrary to this expectation, after localization according to Eqn. (3.1) the equation of motion for $B_{\mu\nu}$ (c.f. Eqn. (3.3)) is non-vanishing for $a \to 0$, and allows for non-trivial solutions. Another point is that the one loop corrections to the (physical) photon propagator, which are based on the results of the vacuum polarization (3.19) and (3.20), do contain contributions of the graphs (d) and (e) in Fig. 3.2 featuring loops of (intentionally unphysical) $B_{\mu\nu}$ fields. Finally, as has been indicated in the discussion of the Ward identities relating the propagators G^{AA}, G^{AB}, and G^{BB} (see page 35), the two point functions involving $B_{\mu\nu}$ cannot entirely be rewritten in terms of the photon propagator.

From these facts it is clear that the new field is more than a mere multiplier. It changes the physics expressed by the model, and is a dynamical quantum field. We may speculate about an interpretation as a new particle, but this was definitely not the intention at the time of its introduction. Bearing in mind the original target which was to find a non-commutative representation for YM $U_\star(1)$ theory, we will have to abandon this approach, and go back to the start.

3.1.5 Summary

We have seen in Section 3.1 that the problematic ill-defined non-local operator $(D^2\widetilde{D}^2)^{-1}$ (see Section 2.2) can be localized by the introduction of a real valued auxiliary tensorial field $B_{\mu\nu}$. Indeed, this allows to construct the action (3.5) which, apart from the deformed product, respects locality, and allows for explicit loop calculations as exercised in Section 3.1.3. However, the implementation in Eqn. (3.1) led to a new problem, namely the introduction of additional degrees of freedom. The origin of these is the missing coupling to a respective ghost field, leading to an additional factor in the path integral upon integrating out $B_{\mu\nu}$. There are numerous other indications for a non-equivalence between localized and non-local versions of the action which have been discussed in Section 3.1.4. In essence, the conclusion has to be that the implementation of the coupling with the auxiliary field is not physically equivalent to the orignial model, and therefore has to be abandoned.

3.2 Localization with BRST-Doublets

An alternative localization procedure has been suggested by Vilar *et al.* [119]. The main idea is to replace the real valued auxiliary field introduced in the model of Section 3.1 by a complex conjugated pair of fields, and respective ghosts, in such a way that BRST doublet structures are formed. Thereby, the localization of the problematic $1/D^2$ term in the action (c.f. Section 2.2) can be performed without introducing new degrees of freedom. In addition, the doublet structures give rise to a high degree of symmetry, being compatible[7] with the *Quantum Action Principle* (QAP) known from commutative theory (see Section 4.1.2, page 63, for an introduction to this subject). The hope is to apply the *Algebraic Renormalization* (AR) procedure in order to prove the renormalizability of the model. However, it will turn out that there are serious obstacles in non-commutative theory hindering a successful application of these techniques. Apart from this, the model by Vilar *et al.* is quite complicated due to a high number of Feynman graphs, even at the lowest order in perturmation theory. Hence, an enhanced version with a slightly modified localization, and a reduced number of auxiliary fields has been worked out [63]. It is the aim of this section to discuss the latter approach.

3.2.1 The Vilar Model

In Ref. [119] Vilar *et al.* proposed to rewrite the critical term S_{nloc} (see Eqn. (2.50)) by introducing two pairs of auxiliary complex conjugated antisymmetric tensorial fields $(B_{\mu\nu}, \bar{B}_{\mu\nu})$, and $(\chi_{\mu\nu}, \bar{\chi}_{\mu\nu})$ of mass dimension one,

$$\begin{aligned} S_{\text{nloc}} \to S_{\text{loc}} &= S_{\text{loc},0} + S_{\text{break}} \\ &= \int \mathrm{d}^4x \, \left(\bar{\chi}_{\mu\nu} \star D^2 B_{\mu\nu} + \bar{B}_{\mu\nu} \star D^2 \chi_{\mu\nu} + \gamma^2 \bar{\chi}_{\mu\nu} \star \chi_{\mu\nu}\right) \\ &\quad + \mathrm{i}\frac{\gamma}{2} \int \mathrm{d}^4x \, \left(\bar{B}_{\mu\nu} - B_{\mu\nu}\right) \star F_{\mu\nu} \,, \end{aligned} \tag{3.21}$$

with γ being a parameter of mass dimension one. The term S_{nloc} is now split into a BRST invariant part $S_{\text{loc},0}$, and a breaking term S_{break} as can be seen by explicit calculation with the definitions in Ref. [119]. The additional degrees of freedom are eliminated by following the ideas of Zwanziger [117] (see [120] for a more comprehensive review of the topic) to add a ghost for

[7] This statement remains questionable since it has been postulated but not proven by the authors of Ref. [119].

each auxiliary field in such a way that BRST doublet structures are formed. This results in a trivial BRST cohomology for $S_{\text{loc},0}$ from which follows [121] that

$$sS_{\text{loc},0} = 0 \quad \Rightarrow \quad S_{\text{loc},0} = s\hat{S}_{\text{loc},0}\,, \tag{3.22}$$

i.e. the part of the action depending on the auxiliary fields and their associated ghosts can be written as an exact expression with respect to the nilpotent BRST operator s.
Contrary to that, the breaking term S_{break} does not join this nice property due to a non-trivial cohomology (i.e. $sS_{\text{break}} \neq 0$). However, it is constructed such that the mass dimension of its field dependent part is smaller than $D = 4$, the dimension of the underlying Euclidean space $\mathbb{R}^4_\theta$. Such a breaking is referred to as 'soft' (c.f. Ref. [121]), and does not spoil renormalizability [118]. This latter fact becomes intuitively clear if we consider that a theory with vertices v having a canonical dimension $d_v < D$ is known to be superrenormalizable. Since the breaking term also features this dimensional property, it seems reasonable that it does not influence higher order quantum corrections corresponding to the high energy limit. Additionally, S_{break} is the actual origin of the suppression of UV/IR mixing featured by this theory, as it alters the IR sector while not affecting the UV part. The mechanism of soft breaking in combination with UV renormalization will be discussed in the subsequent sections below. Another important aspect of the model by Vilar *et al.* is the splitting of the operator $D^2\widetilde{D}^2$ from Section 2.2 into two separate parts, and an overall constant factor carrying the mass dimension of the parameter θ, i.e. $D^2\widetilde{D}^2 \to \theta^2(D^2)^2$. Such a splitting, however, is only possible in Euclidean space[8] if $\theta_{\mu\nu}$ has full rank, as has the special form of $\theta_{\mu\nu}$ being defined in Section 1.3.3, and allows for $\widetilde{D}^2 \equiv \theta^2 D^2$. Therefore, the proposed solution (3.21) will only exist in special cases, and cannot be considered as a general solution to the localization problem discussed in Section 2.2.

3.2.2 Construction of the Action

The starting point is the gauge invariant part S_{loc} of the action (3.5) which has originally been introduced to localize the term containing $1/D^2$ (c.f. Eqn. (2.50)), which in turn was motivated to implement the damping behavior of the $1/p^2$ model. As has been discussed in Section 3.1.4 (see also Ref. [62]) the real auxiliary field[9] $\mathcal{B}_{\mu\nu}$ appears to have its own dynamic properties, hence representing additional degrees of freedom. Following the ideas of Vilar *et al.* we shall now turn $\mathcal{B}_{\mu\nu}$ into a complex conjugated pair $(B_{\mu\nu}, \bar{B}_{\mu\nu})$ of fields and associate an additional pair of ghost and antighost fields $\psi_{\mu\nu}$ and $\bar{\psi}_{\mu\nu}$ to them. The localization corresponds to the following replacement

$$\begin{aligned} &\qquad\qquad S_{\text{nloc}} \longrightarrow S_{\text{loc}} \\ &\int \mathrm{d}^4x\, F_{\mu\nu}\frac{1}{D^2\widetilde{D}^2}F_{\mu\nu} \longrightarrow \int \mathrm{d}^4x \left[\frac{\lambda}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)F^{\mu\nu} - \mu^2\bar{B}_{\mu\nu}D^2\widetilde{D}^2B^{\mu\nu} + \mu^2\bar{\psi}_{\mu\nu}D^2\widetilde{D}^2\psi^{\mu\nu}\right], \end{aligned} \tag{3.23}$$

where (as in the remainder of this section) all field products are considered to be star products. The parameters λ and μ both have mass dimension $d_m = 1$ and replace the former dimensionless parameter a of Section 3.1.

[8]In Minkowski space non-commutativity with time leads to difficulties in the interpretation of time ordering and unitarity, and hence to rather new types of Feynman rules (see Refs. [122, 123] and references therein). Generally, the trend is therefore to restrict non-vanishing components of θ to the spacial part of the metric.

[9]Regarding the notation, from this point on, in order to avoid confusion, the real valued auxiliary field $B_{\mu\nu}$ of Section 3.1 shall be denoted by $\mathcal{B}_{\mu\nu}$.

We can immediately show the equivalence of this localized action and the original non-local version by employing the path integral formalism:

$$
\begin{aligned}
Z &= \int \mathcal{D}[\bar\psi\psi\bar B B A]\, \exp\left\{-\left(\int \mathrm{d}^4x\, \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + S_{\text{loc}}\right)\right\} \\
&= \int \mathcal{D}[\bar B B A] \det{}^4\left(\mu^2 D^2 \widetilde{D}^2\right) \exp\Bigg\{ -\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda}{2}\left(B_{\mu\nu}+\bar B_{\mu\nu}\right)F^{\mu\nu} \right.\\
&\qquad\qquad \left. - \mu^2 \bar B_{\mu\nu} D^2 \widetilde{D}^2 B^{\mu\nu}\right]\Bigg\} \\
&= \int \mathcal{D}[\bar B B A] \det{}^4\left(\mu^2 D^2 \widetilde{D}^2\right) \exp\Bigg\{ -\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{4\mu^2}F_{\mu\nu}\frac{1}{\widetilde{D}^2 D^2}F^{\mu\nu} - \right.\\
&\qquad \left. - \left(\bar B_{\mu\nu} - \frac{\lambda}{2\mu^2}\frac{1}{\widetilde{D}^2 D^2}F_{\mu\nu}\right)\mu^2 D^2 \widetilde{D}^2 \left(B^{\mu\nu} - \frac{\lambda}{2\mu^2}\frac{1}{\widetilde{D}^2 D^2}F^{\mu\nu}\right)\right]\Bigg\} \\
&= \int \mathcal{D}A \det{}^4\left(D^2\widetilde{D}^2\right) \det{}^{-4}\left(D^2\widetilde{D}^2\right) \exp\left\{ -\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}\left(1 + \frac{\lambda^2}{4\mu^2}\frac{1}{\widetilde{D}^2 D^2}F^{\mu\nu}\right)\right]\right\}. \qquad (3.24)
\end{aligned}
$$

Note that the prefactors generated by the determinant of the operator $D^2\widetilde{D}^2$ cancel each other due to the fact that in D dimensions the functional integration yields

$$
\int \mathcal{D}[\bar\psi\psi]\, \mathrm{e}^{\bar\psi\mathcal{O}\psi} = \left[\det(\mathcal{O})\right]^D , \quad \text{and} \quad \int \mathcal{D}[\bar B B]\, \mathrm{e}^{\bar B\mathcal{O}B} = \left[\det(\mathcal{O})\right]^{-D} ,
$$

for $(\bar\psi,\psi)$ and $(\bar B, B)$ being sets of fermionic and bosonic fields, respectively, and $\mathcal{O}$ being a bosonic operator. In this respect, strictly speaking, the equivalence between localized and non-local actions is not fullfilled in the model with a real auxiliary field since the compensating determinant factor is missing in Eqn. (3.2). This is again an indication that the localization applied in Section 3.1 is not physically correct, i.e. alters the physical content of the model.

Coming back to the current action with localization (3.23) the implementation of a Landau gauge fixing, i.e.

$$
S_{\phi\pi} = \int \mathrm{d}^4x\, \left(b\partial^\mu A_\mu - \bar c\partial^\mu D_\mu c\right) , \qquad (3.25)
$$

leads to the BRST transformation laws for the fields:

$$
\begin{aligned}
sA_\mu &= D_\mu c\,, & sc &= \mathrm{i}gcc\,,\\
s\bar c &= b\,, & sb &= 0\,,\\
sF_{\mu\nu} &= \mathrm{i}g\left[c, F_{\mu\nu}\right] , & & \qquad (3.26)
\end{aligned}
$$

and furthermore

$$
\begin{aligned}
s\bar\psi_{\mu\nu} &= \bar B_{\mu\nu} + \mathrm{i}g\left\{c, \bar\psi_{\mu\nu}\right\} , & s\bar B_{\mu\nu} &= \mathrm{i}g\left[c, \bar B_{\mu\nu}\right] ,\\
sB_{\mu\nu} &= \psi_{\mu\nu} + \mathrm{i}g\left[c, B_{\mu\nu}\right] , & s\psi_{\mu\nu} &= \mathrm{i}g\left\{c, \psi_{\mu\nu}\right\} . \qquad (3.27)
\end{aligned}
$$

Exploiting the BRST doublet structure of Eqn. (3.27) we can write

$$
S_{\text{loc}} = \int \mathrm{d}^4x \left[s\left(\frac{\lambda}{2}\bar\psi_{\mu\nu}F^{\mu\nu} - \mu^2\bar\psi_{\mu\nu}D^2\widetilde{D}^2 B^{\mu\nu}\right) + \frac{\lambda}{2}B_{\mu\nu}F^{\mu\nu}\right], \qquad (3.28)
$$

where the last term gives rise to a breaking of BRST invariance, as

$$
sS_{\text{break}} = \int \mathrm{d}^4x\, \frac{\lambda}{2}\psi_{\mu\nu}F^{\mu\nu}, \quad \text{with} \quad S_{\text{break}} = \int \mathrm{d}^4x\, \frac{\lambda}{2}B_{\mu\nu}F^{\mu\nu} . \qquad (3.29)
$$

As in the model by Vilar *et al.* sketched in Section 3.2.1 the mass dimension d_m of the field dependent part of S_{break} fulfills the condition $d_m\left(\psi_{\mu\nu}F^{\mu\nu}\right) = 3 < D = 4$, and thus can be considered as an implementation of soft breaking. However, in order to restore BRST invariance in the UV region (as is a prerequisite for an eventual future application of AR) an additional set of sources

$$\begin{aligned} s\bar{Q}_{\mu\nu\alpha\beta} &= \bar{J}_{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, \bar{Q}_{\mu\nu\alpha\beta}\right\}, & s\bar{J}_{\mu\nu\alpha\beta} &= \mathrm{i}g\left[c, \bar{J}_{\mu\nu\alpha\beta}\right], \\ sQ_{\mu\nu\alpha\beta} &= J_{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, Q_{\mu\nu\alpha\beta}\right\}, & sJ_{\mu\nu\alpha\beta} &= \mathrm{i}g\left[c, J_{\mu\nu\alpha\beta}\right], \end{aligned} \tag{3.30}$$

has to be coupled to the breaking term which then takes the form

$$\begin{aligned} S_{\text{break}} &= \int \mathrm{d}^4x\; s\left(\bar{Q}_{\mu\nu\alpha\beta}B^{\mu\nu}F^{\alpha\beta}\right) \\ &= \int \mathrm{d}^4x\; \left(\bar{J}_{\mu\nu\alpha\beta}B^{\mu\nu}F^{\alpha\beta} - \bar{Q}_{\mu\nu\alpha\beta}\psi^{\mu\nu}F^{\alpha\beta}\right). \end{aligned} \tag{3.31}$$

The original term Eqn. (3.29) is reobtained if the sources $\bar{Q}$ and $\bar{J}$ are assigned to their "physical values"

$$\begin{aligned} \bar{Q}_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0, & \bar{J}_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= \frac{\lambda}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right), \\ Q_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0, & J_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= \frac{\lambda}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right). \end{aligned} \tag{3.32}$$

Note that the Hermitian conjugate of the counterterm S_{break} in Eqn. (3.23), (i.e. the term $\int \mathrm{d}^4x\, \bar{B}_{\mu\nu}F^{\mu\nu}$) may also be coupled to external sources which is not required for BRST invariance but restores Hermiticity of the action.

$$\frac{\lambda}{2}\int \mathrm{d}^4x\, \bar{B}_{\mu\nu}F^{\mu\nu} \;\longrightarrow\; \int \mathrm{d}^4x\; s\left(J_{\mu\nu\alpha\beta}\bar{\psi}^{\mu\nu}F^{\alpha\beta}\right) = \int \mathrm{d}^4x\; J_{\mu\nu\alpha\beta}\bar{B}^{\mu\nu}F^{\alpha\beta}. \tag{3.33}$$

Including external sources Ω^{ϕ}, $\phi \in \{A, c, B, \bar{B}, \psi, \bar{\psi}, J, \bar{J}, Q, \bar{Q}\}$ for the non-linear BRST transformations the complete action with Landau gauge $\partial^\mu A_\mu = 0$ and general $Q/\bar{Q}$ and $J/\bar{J}$ reads:

$$S = S_{\text{inv}} + S_{\phi\pi} + S_{\text{new}} + S_{\text{break}} + S_{\text{ext}}, \tag{3.34}$$

with

$$\begin{aligned} S_{\text{inv}} &= \int \mathrm{d}^4x\, \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \\ S_{\phi\pi} &= \int \mathrm{d}^4x\; s\left(\bar{c}\,\partial^\mu A_\mu\right) = \int \mathrm{d}^4x\; \left(b\,\partial^\mu A_\mu - \bar{c}\,\partial^\mu D_\mu c\right), \\ S_{\text{new}} &= \int \mathrm{d}^4x\; s\left(J_{\mu\nu\alpha\beta}\bar{\psi}^{\mu\nu}F^{\alpha\beta} - \mu^2\bar{\psi}_{\mu\nu}D^2\widetilde{D}^2B^{\mu\nu}\right) \\ &= \int \mathrm{d}^4x\; \left(J_{\mu\nu\alpha\beta}\bar{B}^{\mu\nu}F^{\alpha\beta} - \mu^2\bar{B}_{\mu\nu}D^2\widetilde{D}^2B^{\mu\nu} + \mu^2\bar{\psi}_{\mu\nu}D^2\widetilde{D}^2\psi^{\mu\nu}\right), \\ S_{\text{break}} &= \int \mathrm{d}^4x\; s\left(\bar{Q}_{\mu\nu\alpha\beta}B^{\mu\nu}F^{\alpha\beta}\right) = \int \mathrm{d}^4x\; \left(\bar{J}_{\mu\nu\alpha\beta}B^{\mu\nu}F^{\alpha\beta} - \bar{Q}_{\mu\nu\alpha\beta}\psi^{\mu\nu}F^{\alpha\beta}\right), \\ S_{\text{ext}} &= \int \mathrm{d}^4x\; \Big(\Omega^A_\mu D^\mu c + \mathrm{i}g\,\Omega^c cc + \Omega^B_{\mu\nu}\left(\psi^{\mu\nu} + \mathrm{i}g\left[c, B^{\mu\nu}\right]\right) + \mathrm{i}g\,\Omega^{\bar{B}}_{\mu\nu}\left[c, \bar{B}^{\mu\nu}\right] \\ &\quad + \mathrm{i}g\,\Omega^{\psi}_{\mu\nu}\left\{c, \psi^{\mu\nu}\right\} + \Omega^{\bar{\psi}}_{\mu\nu}\left(\bar{B}^{\mu\nu} + \mathrm{i}g\left\{c, \bar{\psi}^{\mu\nu}\right\}\right) + \Omega^Q_{\mu\nu\alpha\beta}\left(J^{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, Q^{\mu\nu\alpha\beta}\right\}\right) \\ &\quad + \mathrm{i}g\,\Omega^J_{\mu\nu\alpha\beta}\left[c, J^{\mu\nu\alpha\beta}\right] + \Omega^{\bar{Q}}_{\mu\nu\alpha\beta}\left(\bar{J}^{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, \bar{Q}^{\mu\nu\alpha\beta}\right\}\right) + \mathrm{i}g\,\Omega^{\bar{J}}_{\mu\nu\alpha\beta}\left[c, \bar{J}^{\mu\nu\alpha\beta}\right]\Big). \end{aligned} \tag{3.35}$$

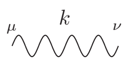
Table 3.2: Properties of fields and sources. ($g_\sharp$... ghost charge, f... fermionic, b... bosonic)

Field	A_μ	c	$\bar{c}$	$B_{\mu\nu}$	$\bar{B}_{\mu\nu}$	$\psi_{\mu\nu}$	$\bar{\psi}_{\mu\nu}$	$J_{\alpha\beta\mu\nu}$	$\bar{J}_{\alpha\beta\mu\nu}$	$Q_{\alpha\beta\mu\nu}$	$\bar{Q}_{\alpha\beta\mu\nu}$
$g_\sharp$	0	1	-1	0	0	1	-1	0	0	-1	-1
d_m.	1	0	2	1	1	1	1	1	1	1	1
Statistics	b	f	f	b	b	f	f	b	b	f	f
Source	Ω^A_μ	Ω^c	b	$\Omega^B_{\mu\nu}$	$\Omega^{\bar{B}}_{\mu\nu}$	$\Omega^\psi_{\mu\nu}$	$\Omega^{\bar{\psi}}_{\mu\nu}$	$\Omega^J_{\alpha\beta\mu\nu}$	$\Omega^{\bar{J}}_{\alpha\beta\mu\nu}$	$\Omega^Q_{\alpha\beta\mu\nu}$	$\Omega^{\bar{Q}}_{\alpha\beta\mu\nu}$
$g_\sharp$	-1	-2	0	-1	-1	-2	0	-1	-1	0	0
Mass dim.	3	4	2	3	3	3	3	3	3	3	3
Statistics	f	b	b	f	f	b	b	f	f	b	b

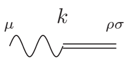
Tab. 3.2 summarizes properties of the fields and sources contained in the model (3.35).

Note that the mass μ is a physical parameter despite the fact that the variation of the action $\frac{\partial S}{\partial \mu^2} = s\left(\bar{\psi}_{\mu\nu} D^2 \widetilde{D}^2 B^{\mu\nu}\right)$ yields a BRST-exact form. Following the argumentation in Ref. [121] this is a consequence of the introduction of a soft breaking term. For vanishing Gribov-like parameter $\lambda = 0$ the contributions to the path integral of the μ dependent sectors of S_{new} in (3.35) cancel each other . (This latter fact will become obvious when considering explicit loop calculations in Section 3.2.5.) In the case $\lambda \neq 0$ we have to consider the additional breaking term which couples the gauge field A_μ to the auxiliary field $B_{\mu\nu}$ and the associated ghost $\psi_{\mu\nu}$. This mixing of the fields is the origin for the appearance of the damping factor $\left(k^2 + \frac{a^2}{\tilde{k}^2}\right)$ featured by the propagators (3.36a)–(3.36d). In fact, the IR-regularization vanishes entirely in the limit $a := \lambda/\mu \to 0$.

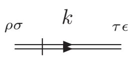
From the action (3.35) with $J/\bar{J}$ and $Q/\bar{Q}$ set to their physical values given by (3.32) we can derive the propagators



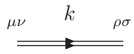
$$= G^{AA}_{\mu\nu}(k) = \frac{1}{\left(k^2 + \frac{a^2}{\tilde{k}^2}\right)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \tag{3.36a}$$



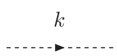
$$= G^{AB}_{\mu,\rho\sigma}(k) = \frac{\mathrm{i}a}{2\mu}\frac{(k_\rho\delta_{\mu\sigma} - k_\sigma\delta_{\mu\rho})}{k^2\tilde{k}^2\left(k^2 + \frac{a^2}{\tilde{k}^2}\right)} = G^{A\bar{B}}_{\mu,\rho\sigma}(k) = -G^{\bar{B}A}_{\rho\sigma,\mu}(k)\,, \tag{3.36b}$$



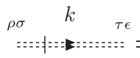
$$= G^{\bar{B}B}_{\mu\nu,\rho\sigma}(k) = \frac{-1}{2\mu^2k^2\tilde{k}^2}\left[\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - a^2\frac{k_\mu k_\rho\delta_{\nu\sigma} + k_\nu k_\sigma\delta_{\mu\rho} - k_\mu k_\sigma\delta_{\nu\rho} - k_\nu k_\rho\delta_{\mu\sigma}}{2k^2\tilde{k}^2\left(k^2 + \frac{a^2}{\tilde{k}^2}\right)}\right], \tag{3.36c}$$



$$= G^{BB}_{\mu\nu,\rho\sigma}(k) = \frac{a^2}{4k^2\tilde{k}^2}\left[\frac{k_\mu k_\rho\delta_{\nu\sigma} + k_\nu k_\sigma\delta_{\mu\rho} - k_\mu k_\sigma\delta_{\nu\rho} - k_\nu k_\rho\delta_{\mu\sigma}}{\mu^2k^2\tilde{k}^2\left(k^2 + \frac{a^2}{\tilde{k}^2}\right)}\right] = G^{\bar{B}\bar{B}}_{\mu\nu,\rho\sigma}(k)\,, \tag{3.36d}$$



$$= G^{\bar{c}c}(k) = -\frac{1}{k^2}\,, \tag{3.36e}$$



$$= G^{\bar{\psi}\psi}_{\mu\nu,\rho\sigma}(k) = \frac{(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})}{2\mu^2k^2\tilde{k}^2}\,, \tag{3.36f}$$

where the abbreviation $a = \lambda/\mu$ was used. From the form of these propagators we notice that both $G^{B\bar{B}}$ and $G^{\bar{\psi}\psi}$ scale with $1/k^2\tilde{k}^2$ in the ultraviolet. Furthermore, all vertices with one B, one $\bar{B}$, and an arbitrary number of A legs have exactly the same form as the ones with one ψ, one $\bar{\psi}$, and the same number of A legs. Therefore, considering the results of explicit one loop calculations in Section 3.1.3 (see also Ref. [62]), we expect all divergent contributions to the vacuum polarization coming from the ψ sector to exactly cancel those coming from the B sector. Of course this conjecture has to be proven by explicit calculations which are postponed to Section 3.2.5.

Note that the propagators obey the following symmetries and relations:

$$G^{AB}_{\mu,\rho\sigma}(k) = G^{A\bar{B}}_{\mu,\rho\sigma}(k) = -G^{BA}_{\rho\sigma,\mu}(k) = -G^{\bar{B}A}_{\rho\sigma,\mu}(k), \tag{3.37a}$$

$$G^{\phi}_{\mu\nu,\rho\sigma}(k) = -G^{\phi}_{\nu\mu,\rho\sigma} = -G^{\phi}_{\mu\nu,\sigma\rho}(k) = G^{\phi}_{\nu\mu,\sigma\rho}(k), \quad \text{for } \phi \in \{\bar{\psi}\psi, \bar{B}B, BB, \bar{B}\bar{B}\}, \tag{3.37b}$$

$$2k^2\tilde{k}^2 G^{AB}_{\rho,\mu\nu}(k) = \mathrm{i}\frac{a}{\mu}\left(k_\mu G^{AA}_{\rho\nu}(k) - k_\nu G^{AA}_{\rho\mu}(k)\right), \tag{3.37c}$$

$$\frac{1}{\mu^2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) = \mathrm{i}\frac{a}{\mu}\left(k_\mu G^{BA}_{\rho\sigma,\nu}(k) - k_\nu G^{BA}_{\rho\sigma,\mu}(k)\right) - 2k^2\tilde{k}^2 G^{B\bar{B}}_{\mu\nu,\rho\sigma}(k), \tag{3.37d}$$

$$0 = \mathrm{i}\frac{a}{\mu}\left(k_\mu G^{BA}_{\rho\sigma,\nu}(k) - k_\nu G^{BA}_{\rho\sigma,\mu}(k)\right) - 2k^2\tilde{k}^2 G^{BB}_{\mu\nu,\rho\sigma}(k), \tag{3.37e}$$

$$G^{B\bar{B}}_{\mu\nu,\rho\sigma}(k) = G^{\bar{\psi}\psi}_{\mu\nu,\rho\sigma}(k) + G^{BB}_{\mu\nu,\rho\sigma}(k). \tag{3.37f}$$

In fact, relations (3.37c)-(3.37f) follow directly from the equations of motion for $B_{\mu\nu}$ and $\bar{B}_{\mu\nu}$ (see Appendix E.1). Vertex expressions are, due to their rather lengthy form, collected in Appendix C.1. However, in the light of the above discussion it is worth mentioning that the symmetry betwen the $(\bar{B}, B)$ and $(\bar{\psi}, \psi)$ sectors is reflected accordingly in the vertices. Each vertex featuring a pair of $(\bar{B}, B)$ fields has a respective counterpart with $(\bar{\psi}, \psi)$. Particularly the relation

$$V^{\bar{B}B(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}) = -V^{\bar{\psi}\psi(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}),\ n \in \{1,2,3,4\}, \tag{3.38}$$

holds. The only exception to the symmetry is given by the vertices $V^{\bar{B}AA}_{\mu\nu,\rho\sigma}(q_1, k_2, k_3)$ and $V^{BAA}_{\mu\nu,\rho\sigma}(q_1, k_2, k_3)$ originating from the soft breaking term. Of course, these vanish for $\lambda \to 0$.

In addition we may note the following symmetries reflecting the antisymmetry of the fields

$$\begin{aligned} V^{\bar{\psi}\psi(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}) &= -V^{\psi\bar{\psi}(n\times A)}_{\rho\sigma,\mu\nu,\xi_1\ldots\xi_n}(q_2, q_1, k_{\xi_1}, \ldots k_{\xi_n}) \\ &= -V^{\bar{\psi}\psi(n\times A)}_{\nu\mu,\rho\sigma,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}), \\ &= -V^{\bar{\psi}\psi(n\times A)}_{\mu\nu,\sigma\rho,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}), \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} V^{\bar{B}B(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}) &= +V^{B\bar{B}(n\times A)}_{\rho\sigma,\mu\nu,\xi_1\ldots\xi_n}(q_2, q_1, k_{\xi_1}, \ldots k_{\xi_n}) \\ &= -V^{\bar{B}B(n\times A)}_{\nu\mu,\rho\sigma,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}) \\ &= -V^{\bar{B}B(n\times A)}_{\mu\nu,\sigma\rho,\xi_1\ldots\xi_n}(q_1, q_2, k_{\xi_1}, \ldots k_{\xi_n}), \end{aligned} \tag{3.40}$$

$$n \in 1,2,3,4. \tag{3.41}$$

This completes the discussion of the tree level properties of the model (3.35). However, before starting to compute loop corrections the discussion of the UV power counting, and the off-shell algebra of symmetries which is essential for an eventual application of AR have to be derived.

3.2.3 UV Power Counting

The superficial degree of UV divergence is determined by the number of external legs of the various fields ϕ denoted by E_ϕ. For an explicit example of the derivation see Section 3.1.2. Its explicit form is given by:

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - 2E_B - 2E_{\bar{B}} - 2E_{\psi\bar{\psi}} - 2E_\theta \tag{3.42a}$$

$$= 4 - E_A - E_{c/\bar{c}} - 2E_\lambda \,, \tag{3.42b}$$

where E_θ counts negative powers of θ. Since E_θ is not strictly positive it may be more intuitive to use the second version (counting E_λ, i.e. the overall powers of λ in a graph), as $E_\lambda \geq 0$ always holds.

From Eqn. (3.42b) it is clear that, not only external fields A_μ and ghosts $c/\bar{c}$ contribute to finiteness of graphs, (as in commutative YM theory,) but also appearances of λ. As mentioned above, this parameter is intimately tied to the soft breaking term which implements the damping of the $1/p^2$ model.

3.2.4 Symmetries

Aiming to apply the method of AR we explore the symmetry content of the current model. The Slavnov-Taylor identity is given by

$$\begin{aligned}\mathcal{B}(S) = \int \mathrm{d}^4x \Big[&\frac{\delta S}{\delta\Omega^A_\mu}\frac{\delta S}{\delta A^\mu} + \frac{\delta S}{\delta\Omega^c}\frac{\delta S}{\delta c} + b\frac{\delta S}{\delta\bar{c}} + \frac{\delta S}{\delta\Omega^B_{\mu\nu}}\frac{\delta S}{\delta B^{\mu\nu}} + \frac{\delta S}{\delta\Omega^{\bar{B}}_{\mu\nu}}\frac{\delta S}{\delta\bar{B}^{\mu\nu}} \\ &+ \frac{\delta S}{\delta\Omega^\psi_{\mu\nu}}\frac{\delta S}{\delta\psi^{\mu\nu}} + \frac{\delta S}{\delta\Omega^{\bar{\psi}}_{\mu\nu}}\frac{\delta S}{\delta\bar{\psi}^{\mu\nu}} + \frac{\delta S}{\delta\Omega^Q_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta Q^{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\Omega^J_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta J^{\mu\nu\alpha\beta}} \\ &+ \frac{\delta S}{\delta\Omega^{\bar{Q}}_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta\bar{Q}^{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\Omega^{\bar{J}}_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta\bar{J}^{\mu\nu\alpha\beta}}\Big] = 0\,. \end{aligned}\tag{3.43}$$

Furthermore we have the gauge fixing condition

$$\frac{\delta S}{\delta b} = \partial^\mu A_\mu = 0\,, \tag{3.44}$$

the ghost equation

$$\mathcal{G}(S) = \partial_\mu \frac{\delta S}{\delta\Omega^A_\mu} + \frac{\delta S}{\delta\bar{c}} = 0\,, \tag{3.45}$$

and the antighost equation

$$\bar{\mathcal{G}}(S) = \int \mathrm{d}^4x\, \frac{\delta S}{\delta c} = 0\,. \tag{3.46}$$

Following the notation of Ref. [119] the identity associated to the BRST doublet structure is given by

$$\begin{aligned}U^{(1)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \bigg(&\bar{B}_{\alpha\beta}\frac{\delta S}{\delta\bar{\psi}^{\mu\nu}} + \Omega^{\bar{\psi}}_{\mu\nu}\frac{\delta S}{\delta\Omega^{\bar{B}}_{\alpha\beta}} + \psi_{\alpha\beta}\frac{\delta S}{\delta B^{\mu\nu}} - \Omega^B_{\mu\nu}\frac{\delta S}{\delta\Omega^\psi_{\alpha\beta}} \\ &+ J_{\mu\nu\rho\sigma}\frac{\delta S}{\delta Q^{\alpha\beta}{}_{\rho\sigma}} + \Omega^Q_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta\Omega^J_{\mu\nu\rho\sigma}} + \bar{J}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\bar{Q}^{\alpha\beta}{}_{\rho\sigma}} + \Omega^{\bar{Q}}_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta\Omega^{\bar{J}}_{\mu\nu\rho\sigma}}\bigg) = 0\,. \end{aligned}\tag{3.47}$$

Interestingly the first two terms of the second line,

$$\int \mathrm{d}^4x \left(J_{\mu\nu\rho\sigma} \frac{\delta S}{\delta Q^{\alpha\beta}{}_{\rho\sigma}} + \Omega^{Q}_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta \Omega^{J}_{\mu\nu\rho\sigma}} \right) = 0 \,,$$

constitute a symmetry by themselves. These terms stem from the insertion of conjugated field partners J and Q for $\bar{J}$ and $\bar{Q}$, respectively, which are not necessarily required, but have been introduced in order to maintain Hermiticity of the action (see Section 3.2.2).

Furthermore, the action features the linearly broken symmetries $U^{(0)}$ and $\tilde{U}^{(0)}$:

$$U^{(0)}_{\alpha\beta\mu\nu}(S) = -\Theta^{(0)}_{\alpha\beta\mu\nu} = -\tilde{U}^{(0)}_{\alpha\beta\mu\nu}(S) \,, \tag{3.48}$$

with

$$\begin{aligned} U^{(0)}_{\alpha\beta\mu\nu}(S) = &\int \mathrm{d}^4x \left[B_{\alpha\beta} \frac{\delta S}{\delta B_{\mu\nu}} - \bar{B}_{\mu\nu} \frac{\delta S}{\delta \bar{B}_{\alpha\beta}} - \Omega^{B}_{\mu\nu} \frac{\delta S}{\delta \Omega^{B}_{\alpha\beta}} + \Omega^{\bar{B}}_{\alpha\beta} \frac{\delta S}{\delta \Omega^{\bar{B}}_{\mu\nu}} \right. \\ &\left. + J_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta J_{\mu\nu\rho\sigma}} - \bar{J}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \bar{J}_{\alpha\beta\rho\sigma}} - \Omega^{J}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \Omega^{J}_{\alpha\beta\rho\sigma}} + \Omega^{\bar{J}}_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta \Omega^{\bar{J}}_{\mu\nu\rho\sigma}} \right] , \end{aligned} \tag{3.49}$$

$$\begin{aligned} \tilde{U}^{(0)}_{\alpha\beta\mu\nu}(S) = &\int \mathrm{d}^4x \left[\psi_{\alpha\beta} \frac{\delta S}{\delta \psi_{\mu\nu}} - \bar{\psi}_{\mu\nu} \frac{\delta S}{\delta \bar{\psi}_{\alpha\beta}} - \Omega^{\psi}_{\mu\nu} \frac{\delta S}{\delta \Omega^{\psi}_{\alpha\beta}} + \Omega^{\bar{\psi}}_{\alpha\beta} \frac{\delta S}{\delta \Omega^{\bar{\psi}}_{\mu\nu}} \right. \\ &\left. + Q_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta Q_{\mu\nu\rho\sigma}} - \bar{Q}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \bar{Q}_{\alpha\beta\rho\sigma}} - \Omega^{Q}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \Omega^{Q}_{\alpha\beta\rho\sigma}} + \Omega^{\bar{Q}}_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta \Omega^{\bar{Q}}_{\mu\nu\rho\sigma}} \right] , \end{aligned} \tag{3.50}$$

$$\Theta^{(0)}_{\alpha\beta\mu\nu} = \int \mathrm{d}^4x \left[\bar{B}_{\mu\nu} \Omega^{\bar{\psi}}_{\alpha\beta} - \psi_{\alpha\beta} \Omega^{B}_{\mu\nu} + \bar{J}_{\mu\nu\rho\sigma} \Omega^{\bar{Q}}_{\alpha\beta\rho\sigma} - J_{\alpha\beta\rho\sigma} \Omega^{Q}_{\mu\nu\rho\sigma} \right] . \tag{3.51}$$

For the sake of completeness we should also mention that in the literature there appears a symmetry denoted by $U^{(2)}$ [119],

$$U^{(2)}_{\mu\nu\alpha\beta} = \int \mathrm{d}^4x \left(\psi_{\mu\nu} \frac{\delta S}{\delta \bar{\psi}_{\alpha\beta}} + \psi_{\alpha\beta} \frac{\delta S}{\delta \bar{\psi}_{\mu\nu}} - \Omega^{\bar{\psi}}_{\mu\nu} \frac{\delta S}{\delta \Omega^{\psi}_{\alpha\beta}} - \Omega^{\bar{\psi}}_{\alpha\beta} \frac{\delta S}{\delta \Omega^{\psi}_{\mu\nu}} \right) = 0. \tag{3.52}$$

The latter, however is not considered to contain any physical information, as it is generated solely by the exchange of indices. From the Slavnov-Taylor identity (3.43) one derives the linearized Slavnov operator.

$$\begin{aligned} \mathcal{B}_S = &\int \mathrm{d}^4x \left[\frac{\delta S}{\delta \Omega^{A}_{\mu}} \frac{\delta}{\delta A_\mu} + \frac{\delta S}{\delta A_\mu} \frac{\delta}{\delta \Omega^{A}_{\mu}} + \frac{\delta S}{\delta c} \frac{\delta}{\delta \Omega^{c}} + \frac{\delta S}{\delta \Omega^{c}} \frac{\delta}{\delta c} + b \frac{\delta S}{\delta \bar{c}} + \frac{\delta S}{\delta \Omega^{B}_{\mu\nu}} \frac{\delta}{\delta B_{\mu\nu}} + \frac{\delta S}{\delta B_{\mu\nu}} \frac{\delta}{\delta \Omega^{B}_{\mu\nu}} \right. \\ &+ \frac{\delta S}{\delta \Omega^{\bar{B}}_{\mu\nu}} \frac{\delta}{\delta \bar{B}_{\mu\nu}} + \frac{\delta S}{\delta \bar{B}_{\mu\nu}} \frac{\delta}{\delta \Omega^{\bar{B}}_{\mu\nu}} + \frac{\delta S}{\delta \Omega^{\psi}_{\mu\nu}} \frac{\delta}{\delta \psi_{\mu\nu}} + \frac{\delta S}{\delta \psi_{\mu\nu}} \frac{\delta}{\delta \Omega^{\psi}_{\mu\nu}} + \frac{\delta S}{\delta \Omega^{\bar{\psi}}_{\mu\nu}} \frac{\delta}{\delta \bar{\psi}_{\mu\nu}} + \frac{\delta S}{\delta \bar{\psi}_{\mu\nu}} \frac{\delta}{\delta \Omega^{\bar{\psi}}_{\mu\nu}} \\ &+ \frac{\delta S}{\delta \Omega^{Q}_{\mu\nu\alpha\beta}} \frac{\delta}{\delta Q_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta Q_{\mu\nu\alpha\beta}} \frac{\delta}{\delta \Omega^{Q}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta \Omega^{J}_{\mu\nu\alpha\beta}} \frac{\delta}{\delta J_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta J_{\mu\nu\alpha\beta}} \frac{\delta}{\delta \Omega^{J}_{\mu\nu\alpha\beta}} \\ &\left. + \frac{\delta S}{\delta \Omega^{\bar{Q}}_{\mu\nu\alpha\beta}} \frac{\delta}{\delta \bar{Q}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta \bar{Q}_{\mu\nu\alpha\beta}} \frac{\delta}{\delta \Omega^{\bar{Q}}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta \Omega^{\bar{J}}_{\mu\nu\alpha\beta}} \frac{\delta}{\delta \bar{J}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta \bar{J}_{\mu\nu\alpha\beta}} \frac{\delta}{\delta \Omega^{\bar{J}}_{\mu\nu\alpha\beta}} \right] . \end{aligned} \tag{3.53}$$

Furthermore, the $\mathcal{U}^{(0)}$ and $\tilde{\mathcal{U}}^{(0)}$ symmetries are combined to define the *reality charge*[10] operator $\mathcal{Q}$ as

$$\mathcal{Q} \equiv \delta_{\alpha\mu}\delta_{\beta\nu}\left(\mathcal{U}^{(0)}_{\alpha\beta\mu\nu} + \tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu}\right) . \tag{3.54}$$

Notice that the action is invariant under $\mathcal{Q}$, i.e. $\mathcal{Q}(S) = 0$, because of $\mathcal{U}^{(0)}_{\alpha\beta\mu\nu}(S) = -\tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu}(S)$.

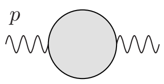
The next logical step in the AR procedure is to set up the off-shell algebra $\mathcal{E}$, yielding constraints to possible counter terms. In order to assure the completeness of the set of symmetries $\mathcal{E}$ (which is generated by these) has to close. Having defined the operators $\mathcal{B}_S$, $\bar{\mathcal{G}}$, $\mathcal{Q}$ and $\mathcal{U}^{(1)}$ we may derive the following set of graded commutators:

$$\begin{aligned}
&\left\{\bar{\mathcal{G}},\bar{\mathcal{G}}\right\} = 0\,, && \left\{\mathcal{B}_S,\mathcal{B}_S\right\} = 0\,, && \left\{\bar{\mathcal{G}},\mathcal{B}_S\right\} = 0\,,\\
&\left[\bar{\mathcal{G}},\mathcal{Q}\right] = 0\,, && \left[\mathcal{Q},\mathcal{Q}\right] = 0\,, && \left\{\bar{\mathcal{G}},\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0\,,\\
&\left\{\mathcal{B}_S,\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0\,, && \left\{\mathcal{U}^{(1)}_{\mu\nu\alpha\beta},\mathcal{U}^{(1)}_{\mu'\nu'\alpha'\beta'}\right\} = 0\,, && \left[\mathcal{U}^{(1)}_{\mu\nu\alpha\beta},\mathcal{Q}\right] = 0\,,\\
&\left[\mathcal{B}_S,\mathcal{Q}\right] = 0\,,
\end{aligned} \tag{3.55}$$

which shows, since all relations evaluate to 0, that the algebra indeed closes. Now, theoretically, the preparations for the AR procedure are completed. However, before a general proof procedure could be started we shall conduct some explicit checks for renormalizability. In this respect, it seems reasonable to investigate the behavior of the theory at the one loop level.

3.2.5 An Attempt for One Loop Renormalization

The aim of this section is to discuss the one loop corrections to the gauge boson propagator. Detailed computations of the relevant graphs are given in Appendix C.2. In the standard renormalization procedure, the dressed propagator at one loop level is given by



$$\equiv \Delta'(p) = \frac{1}{\mathsf{A}} + \frac{1}{\mathsf{A}}\,\Sigma(\Lambda,p)\,\frac{1}{\mathsf{A}}\,, \tag{3.56}$$

where (with abuse of the notation due to missing indices),

$$\begin{aligned}
&\frac{1}{\mathsf{A}} := G^{\mathrm{AA}}_{\mu\nu}(p)\,,\\
&\Sigma(\Lambda,p) := \left(\Pi^{\mathrm{p}}\right)_{\mathrm{reg.}}(\Lambda,p) + \Pi^{\mathrm{np}}(p) =: \mathsf{B}\,,
\end{aligned}$$

and the interpretation of the notation has to be $1/\mathsf{A} =: \mathsf{A}^{-1}$ in the sense that $\mathsf{A}^{-1}\mathsf{A} = \mathsf{A}\mathsf{A}^{-1} \equiv \mathbb{1}$, and similarily for $(\mathsf{A}+\mathsf{B})^{-1}$. For $\mathsf{A} \neq 0$, $\mathsf{A}+\mathsf{B} \neq 0$, $\exists \mathsf{A}^{-1}$, and $\exists(\mathsf{A}+\mathsf{B})^{-1}$, we can apply the formula

$$\frac{1}{\mathsf{A}+\mathsf{B}} = \frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}}\,\mathsf{B}\,\frac{1}{\mathsf{A}+\mathsf{B}} = \frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}}\,\mathsf{B}\,\frac{1}{\mathsf{A}} + \mathcal{O}(\mathsf{B}^2)\,, \tag{3.57}$$

which allows to rewrite expression (3.56) to order Σ as

$$\Delta'(p) = \frac{1}{\mathsf{A} - \Sigma(\Lambda,p)}\,, \tag{3.58}$$

[10]Note that $\mathcal{Q}$ is not a charge in the sense that it is the integral over some current but indeed if $\mathcal{Q} = 0$ reality of the action is guaranteed so we might interpret it as the ‘charge’ associated with reality, assigning the value 0 to reality.

and, if the model is indeed renormalizable, to absorb any divergences in the appropriate parameters of the theory (which are hidden in A). Remember that this procedure has successfully been executed for the scalar $1/p^2$ model in Section 2.1 (see also Ref. [95]).

In the model (3.35), the expansion (3.57) cannot be applied directly. Due to the existence of the mixed propagators G^{AB}, $G^{\mathrm{A\bar{B}}}$ (and their mirrored counterparts,) the propagator $\langle A_\mu A_\nu \rangle$ receives contributions from all graphs having two external fields out of the set $\{A, B, \bar{B}\}$. Denoting the sum of corrections with external fields ϕ_1 and ϕ_2 by $\Sigma^{\phi_1\phi_2}$, and the vacuum polarization by $\Pi_{\mu\nu} \equiv \Sigma^{\mathrm{AA}}_{\mu\nu}$, the complete correction can be written as

$$\begin{aligned} G^{\mathrm{AA,1l-ren}}_{\mu\nu}(p) = G^{\mathrm{AA}}_{\mu\nu}(p) &+ G^{\mathrm{AA}}_{\mu\rho}(p)\Pi_{\rho\sigma}(p)G^{\mathrm{AA}}_{\sigma\nu}(p) \\ &+ G^{\mathrm{AA}}_{\mu\rho}(p)2\Sigma^{\mathrm{AB}}_{\rho,\sigma1\sigma2}(p)G^{\mathrm{BA}}_{\sigma1\sigma2,\nu}(p) \\ &+ G^{\mathrm{AA}}_{\mu\rho}(p)2\Sigma^{\mathrm{A\bar{B}}}_{\rho,\sigma1\sigma2}(p)G^{\mathrm{\bar{B}A}}_{\sigma1\sigma2,\nu}(p) \\ &+ G^{\mathrm{AB}}_{\mu,\rho1\rho2}(p)\Sigma^{\mathrm{BB}}_{\rho1\rho2,\sigma1\sigma2}(p)G^{\mathrm{BA}}_{\sigma1\sigma2,\nu}(p) \\ &+ G^{\mathrm{AB}}_{\mu,\rho1\rho2}(p)2\Sigma^{\mathrm{B\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p)G^{\mathrm{\bar{B}A}}_{\sigma1\sigma2,\nu}(p) \\ &+ G^{\mathrm{A\bar{B}}}_{\mu,\rho1\rho2}(p)\Sigma^{\mathrm{\bar{B}\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p)G^{\mathrm{\bar{B}A}}_{\sigma1\sigma2,\nu}(p) + \mathcal{O}\left(g^4\right) . \end{aligned} \tag{3.59}$$

Note, that the factors 2 stem from the (not explicitly written) mirrored contributions $AB \leftrightarrow BA$, $A\bar{B} \leftrightarrow \bar{B}A$, and $B\bar{B} \leftrightarrow \bar{B}B$. Certainly the factor A must be the same for all summands. Since the tensor structure of the propagators (3.36a) and (3.36b) is incompatible, we have to use the Ward Identities (3.37a) and (3.37c), i.e.

$$\begin{aligned} G^{AB}_{\mu,\rho\sigma}(k) &= G^{A\bar{B}}_{\mu,\rho\sigma}(k) = -G^{BA}_{\rho\sigma,\mu}(k) = -G^{\bar{B}A}_{\rho\sigma,\mu}(k) , \\ 2k^2\tilde{k}^2 G^{AB}_{\rho,\mu\nu}(k) &= \mathrm{i}\frac{a'}{\mu}\left(k_\mu G^{AA}_{\rho\nu}(k) - k_\nu G^{AA}_{\rho\mu}(k)\right) , \end{aligned} \tag{3.60}$$

which allow us to express the (tree level) AB and $A\bar{B}$ propagators uniquely in terms of AA-propagators. This leads (in analogy to (3.57)) to the following representation for the dressed one loop gauge boson propagator:

$$G^{\mathrm{AA,1l-ren}}_{\mu\nu}(p) = \frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}}\Big(\sum \mathsf{B}_i\Big)\frac{1}{\mathsf{A}} , \tag{3.61}$$

where $1/\mathsf{A}$ once more stands for the tree level gauge boson propagator. The B_i's are given by the one loop corrections (with amputated external legs) of the two-point functions relevant for the dressed gauge boson propagator, multiplied by any prefactors coming from (3.60) and the occasional factor 2 (c.f. (3.59)). Thus, the full propagator is given by

$$\begin{aligned} G^{\mathrm{AA,1l-ren}}_{\mu\nu}(p) =& G^{\mathrm{AA}}_{\mu\nu}(p) + G^{\mathrm{AA}}_{\mu\rho}(p)\Pi_{\rho\sigma}(p)G^{\mathrm{AA}}_{\sigma\nu}(p) \\ &+ \left(\frac{\mathrm{i}a'}{\mu p^2\tilde{p}^2}\right)\left[2G^{\mathrm{AA}}_{\mu\rho}(p)\left(\Sigma^{\mathrm{AB}}_{\rho,\sigma1\sigma2}(p) + \Sigma^{\mathrm{A\bar{B}}}_{\rho,\sigma1\sigma2}(p)\right)p_{\sigma2}G^{\mathrm{AA}}_{\nu\sigma1}(p)\right] \\ &+ \left(\frac{\mathrm{i}a'}{\mu p^2\tilde{p}^2}\right)^2\left[p_{\rho1}G^{\mathrm{AA}}_{\mu\rho2}(p)\Big(\Sigma^{\mathrm{BB}}_{\rho1\rho2,\sigma1\sigma2}(p)\right. \\ &\left.\qquad + 2\Sigma^{\mathrm{B\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p) + \Sigma^{\mathrm{\bar{B}\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p)\Big)p_{\sigma2}G^{\mathrm{AA}}_{\nu\sigma1}(p)\right] . \end{aligned} \tag{3.62}$$

With the results (C.4), (C.6), (C.8), and (C.11) (computed in Appendix C.2) the correction $\mathsf{B} = \sum\limits_i \mathsf{B}_i$ is explicitly given by

$$\mathsf{B} = \frac{g^2}{8\pi^2\mu^4}\left\{\tilde{p}_\mu\tilde{p}_\nu\left(\frac{16\mu^4}{(\tilde{p}^2)^2} + \frac{\theta^4\lambda^4}{2(\tilde{p}^2)^4}\right) - 7\lambda^2\mu^2\frac{\theta^4}{(\tilde{p}^2)^4}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left(4 - \tilde{p}^2\Lambda^2\right)\right.$$

$$+ \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) \left[\ln 2 - \ln\tilde{p}^2 - \ln\Lambda\right] \left(\frac{5}{3}\mu^4 + \frac{3\lambda^2\mu^2\theta^2}{(\tilde{p}^2)^2} + \frac{\lambda^4\theta^4}{(\tilde{p}^2)^4}\right) \Bigg\}$$
$$+ \text{ finite terms}. \tag{3.63}$$

In contrast to commutative gauge models, and even though the vacuum polarization tensor $\Pi_{\mu\nu}$ only had a logarithmic UV divergence, the complete correction B diverges quadratically in the UV cutoff Λ. More intriguing is that, despite the fact that $\Pi_{\mu\nu}$ (see Eqn. (C.4)) exhibits the usual quadratic IR divergence, B behaves like $\frac{1}{(\tilde{p}^2)^3}$ in the IR limit. Both properties arise due to the existence (and the form) of the mixed AB and $A\bar{B}$ propagators, as these show an IR behavior $G^{AB,\bar{B}}(p) \underset{|p|\to 0}{\longrightarrow} |p|^{-1}\big|_{|p|\to 0} = \infty$. This is reflected in the factors $(p^2\tilde{p}^2)^{-1}p_\mu$ and $(p^2\tilde{p}^2)^{-2}p_\mu p_\nu$ being added to $\Sigma^{\{AB,\bar{B}\}}(p)$ and $\Sigma^{\{BB,B\bar{B},\bar{B}\bar{B}\}}(p)$, respectively, in Eqn. (3.62) and Eqn. (3.60). These additional poles originate from the difference in $\lim_{|p|\to 0} G^{\{AB,\bar{B}\}}(p)$ and $\lim_{|p|\to 0} G^{AA}(p)$ which has to be counterbalanced by the application of the Ward identities (3.60). Physically, the interpretation is clear. The actual one loop integrals corresponding to amputated graphs all feature the same expected $1/\tilde{p}^2$ behavior but upon dressing them with the IR divergent AB and $A\bar{B}$ propagators they pick up additional poles from the mixed propagators.
Regarding the renormalization this represents a serious obstacle. For the first, the form of the propagator is modified, thereby implying new counter terms in the effective action, i.e. the tree level action is not stable. Secondly, higher loop insertions of this expression may supposedly lead to IR divergent integrals, as will be discussed in the next section. However, we should also mention that all of the problems appearing at this point do not state a proof for non-renormalizability.

3.2.6 Higher loop calculations

In the light of renormalization it is important to investigate the IR behavior of expected integrands appearing at higher loop levels with insertions of the one loop corrections being discussed in Section 3.2.5. The aim is to identify possible poles at $\tilde{p}^2 = 0$. Hence, we consider chains of n non-planar insertions denoted by $\Xi^{\phi_1\phi_2}(p,n)$, each representing the sum of all divergent one loop contributions with external fields ϕ_1 and ϕ_2 (cf. Sections C.2.1–C.2.4). Due to the numerous possibilities of constructing such graphs, only a few exemplary configurations – especially those for which one expects the worst IR behavior, will be considered at this place.

To start with, let us state that amongst all types of one loop two point amplitudes ((C.4), (C.6), (C.8), and (C.11)), the vacuum polarization shows the highest, namely a quadratic IR divergence. Amongst the tree level propagators those with two external double-indexed legs, e.g. B or $\bar{B}$ feature the highest (quartic) divergence in the limit of vanishing external momenta. A chain of n vacuum polarizations $\Pi^{\rm np}_{\mu\nu}(p)$ (see Eqns. (C.3a) and (C.2b)) with $(n+1)$ AA-propagators ($(n-1)$ between the individual vacuum polarization graphs, and one at each end) leads to the following expression (for a graphical representation, see Fig. 3.3):

$$\begin{aligned}\Xi^{AA}_{\mu\nu}(p,n) &= \left(G^{AA}(p)\Pi^{\rm np}(p)\right)^n_{\mu\rho} G^{AA}_{\rho\nu}(p) \\ &= \left(\frac{2g^2}{\pi^2}\right)^n \frac{1}{\left(p^2 + \frac{a'^2}{\tilde{p}^2}\right)^{n+1}} \frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^{n+1}}\,.\end{aligned} \tag{3.64}$$

Note that, due to the transversality of $\Pi^{\rm np}_{\mu\nu}(p)$, from the propagator (3.36a) only the term with the Kronecker delta enters the calculation. For vanishing momenta, i.e. in the limit $\tilde{p}^2 \to 0$ the expression reduces to

$$\lim_{\tilde{p}^2\to 0} \Xi^{AA}(p,n) = \left(\frac{2g^2}{\pi^2}\right)^n \frac{\tilde{p}_\mu \tilde{p}_\nu}{a'^{2(n+1)}}\,, \tag{3.65}$$

exhibiting IR finiteness which is independent from the number of inserted loops.

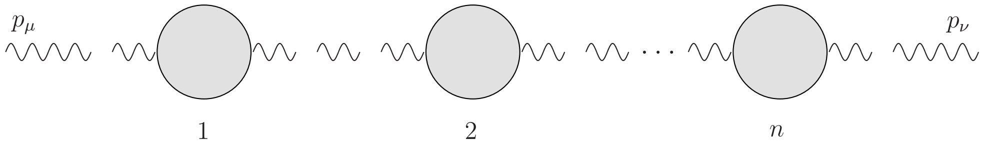


Figure 3.3: A chain of n non-planar insertions, concatenated by gauge field propagators.

Another representative is the chain

$$\Xi^{A\phi}(p,n) \equiv \left(G^{A\phi}(p)\Sigma^{\mathrm{np},\phi\mathrm{A}}(p)\right)^n_{\mu,\rho} G^{A\phi}_{\rho,\nu 1\nu 2}(p), \qquad \text{where} \quad \phi \in \{B, \bar{B}\},$$

which could replace any single G^{AB} (or $G^{A\bar{B}}$) line. Obviously, we have

$$\Xi^{A\phi}_{\mu,\nu 1\nu 2}(p,n) = \frac{\mathrm{i}a'}{2\mu}\left(-\frac{3g^2}{32\pi^2}a'^2\right)^n \frac{(p_{\nu 1}\delta_{\mu\,\nu 2} - p_{\nu 2}\delta_{\mu\,\nu 1})}{p^2\left[\tilde{p}^2\left(p^2 + \frac{a'^2}{\tilde{p}^2}\right)\right]^{n+1}} n \ln \tilde{p}^2\,, \tag{3.66}$$

which for $\tilde{p}^2 \ll 1$ (and neglecting dimensionless prefactors) behaves like

$$\Xi^{A\phi}(p,n) \approx n \frac{(p_{\nu 1}\delta_{\mu\,\nu 2} - p_{\nu 2}\delta_{\mu\,\nu 1})}{\mu p^2} \ln \tilde{p}^2\,. \tag{3.67}$$

The latter insertion can be regularized since the pole at $\tilde{p} = 0$ is independent of n. In contrast, higher divergences are expected for chain graphs being concatenated by propagators with four indices, i.e. $G^{\bar{B}B}_{\mu\nu,\rho\sigma}$, $G^{BB}_{\mu\nu,\rho\sigma}$, $G^{\bar{\psi}\psi}_{\mu\nu,\rho\sigma}$, due to the inherent quartic IR singularities. Let us discuss the combination $\Xi^{\bar{B}B}(p,n) \equiv \left(G^{\bar{B}B}(p)\Sigma^{\mathrm{p},\mathrm{B\bar{B}}}(p)\right)^n G^{\bar{B}B}(p)$. As before, we can approximate for $\tilde{p}^2 \ll 1$ and, omitting dimensionless prefactors and indices, find

$$\Xi^{\bar{B}B}(p,n) \underset{\tilde{p}^2 \ll 1}{\propto} \frac{n}{\mu^2}\frac{\ln \tilde{p}^2}{(p^2\tilde{p}^2)^n}\,, \tag{3.68}$$

which represents a singularity $\forall\, n > 1$ (since in any graph, at $n = 0$, the divergence is regularized by the phase factor being a sine function which behaves like p for small momenta). Regarding the index structures, no cancellations can be expected since the product of an arbitrary number of contracted, completely antisymmetric tensors (as is $G^{\{BB,B\bar{B},\bar{B}\bar{B}\}}$) is again an antisymmetric tensor with the outermost indices of the chain being free.

Exactly the same result is obtained for $\Xi^{BB}(p) \equiv \left(G^{BB}(p)\Sigma^{\mathrm{p,BB}}(p)\right)^n G^{BB}(p)$. From this it is clear that the damping mechanism seen in $\Xi^{AA}(p,n)$ fails for higher insertions of $B/\bar{B}$ (,and supposedly also $\psi/\bar{\psi}$) fields. This is now a more serious threat to the renormalization programme. In the following Section 3.3 we will collect the lessons learned from the two gauge implementations of the $1/p^2$ model.

3.2.7 Summary

In response of the proposition [62], of the model described in Section 3.1 giving rise to additional degrees of freedom, Vilar *et al.* [119] have shown that the task may be accomplished without changing the physical content of the theory by the introduction of two pairs of complex conjugated auxiliary fields and ghosts, such that BRST doublet structures are formed (see Section 3.2.1). It was then assumed that the theory could be renormalized within the framework of Algebraic Renormalization (described in Section 4.1.2) but no proof has been given that this method is indeed applicable on non-commutative spaces. Another important aspect is the assumed separation of $D^2\widetilde{D}^2 \to \theta^2(D^2)^2$ which is only possible for a tensor $\theta_{\mu\nu}$ with full rank in Euclidean space.
Avoiding the problematic presumptions of the model by Vilar *et al.*, the approach described in Section 3.2.2 introduces only one pair of auxiliary fields, and according ghosts. Besides this, the action (3.35) features as well a high degree of symmetry, and is far less complex than the one of Section 3.2. However, both models contain couplings of the gauge boson with auxiliary fields, leading to IR divergent propagators. These represent an obstacle to renormalization, as being discussed in Section 3.2.5. Indeed, the naïve approximations of Section 3.2.6 reveal that there exist types of graphs which diverge at higher loop level. In any case, this cannot be considered to be a proof for non-renormalizability since, still, cancellations may appear due to the high degree of symmetry, but it is an indication that there might appear problems. Basing solely on the facts which have been discussed up to now a decisive answer cannot be given. A more thorough analysis of the IR problems in this model follows in Section 3.3 below.

3.3 Lessons Learned

Now we have discussed two different implementations of the $1/p^2$ damping mechanism in gauge theories (see Section 3.1 and Section 3.2). Since in both of these different problems appeared we shall now analyze the obstacles to renormalization and the options we have to improve the situation.

3.3.1 Divergence Structures in Non-Commutative Gauge Field Models

In the literature several examples for implementations of non-commutative gauge theories can be found. The general concept is to take a commutative theory (YM or QED), and deform it by the introduction of a star product. Let us consider the situation for $U_\star(1)$ (see Section 2.2) gauge fields on $\mathbb{R}^4_\theta$. The gauge invariant Yang-Mills action (2.43) endowed with stars is repeated at this point for convenience,

$$S_{\mathrm{YM}\star} = \int \mathrm{d}^4x \, \frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} \,, \qquad \text{with}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \, [A_\mu \stackrel{\star}{,} A_\nu] \ .$$

This action has already been discussed in the early implementations of non-commutative QED by Hayakawa [51, 52] where the focus was set to the UV divergences of the planar part. IR phenomena, and corrections for two and three point functions can be found in Refs. [105, 106]. Respective one loop results obtained in these references principally coincide with those obtained in Sections 3.1.3 and 3.2.5, and take the form

$$\Pi_{\mu\nu}(p)|_{\tilde{p}\to 0} \propto \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} \,, \tag{3.69}$$

$$\Gamma_{\mu\nu\rho}(p_1,p_2,p_3)|_{\tilde{p}_i\to 0} \propto -\frac{2ig^3}{\pi^2}\cos(p_1\tilde{p}_2)\sum_{i=1,2,3}\frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{(\tilde{p}_i^2)^2}\,, \tag{3.70}$$

which is required by the demand for transversality, i.e. $p_{\{\mu,\nu\}}\Pi_{\mu\nu}(p) = 0$ for a two point function, and $p_{\{1,2,3\},\{\mu,\nu,\rho\}}\Gamma_{\mu\nu\rho}(p_1,p_2,p_3) = 0$ for a three point function. It is immediately clear that no term exists in the tree level actions presented to this point which is capable to absorb such divergences. Therefore, stability of the theory is violated, and there is no guarantee that the required counterterms do not lead to even new types of divergences.

More generally, restrictions to the possible form of divergences can only be derived from transversality, and power counting. In order to clarify this point let us consider for example the power counting of the model with BRST doublets in Section 3.2.3. Since every physical non-vanishing graph in this model has at least two external photon lines, $E_A \geq 2$, and insertions of $B/\bar{B}$ fields raise E_λ (either by the insertion of mixed propagators $G^{\{AB,A\bar{B}\}}$ or the vertex $V^{\{BAA,\bar{B}AA\}}$; other insertions, such as closed $\{B,\bar{B}\}$-lines are cancelled by respective insertions of $\{\psi,\bar{\psi}\}$) we can expect at most quadratic divergences. These may be only be represented by the following types of terms (where overall factors, even with non-vanishing mass dimension, have been omitted)

$$\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2}\,,\ \frac{\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho}{(\tilde{p}^2)^2}\,,\quad \frac{\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\sigma}{(\tilde{p}^2)^3}\,,$$
$$\frac{1}{\tilde{p}^2}(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2})\,,\quad \frac{\tilde{p}_\mu}{\tilde{p}^2}(\delta_{\rho\sigma}-\frac{p_\rho p_\sigma}{p^2})\,,\quad \frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^3}(\delta_{\rho\sigma}-\frac{p_\rho p_\sigma}{p^2})\,. \tag{3.71}$$

Note that, each of these divergences may be also be combined with logarithmically divergent terms. Transversality implies, that the multiplication with any momentum p, carrying a free index of the divergence, vanishes, i.e. p_μ(Eqn. (3.71)) $= 0$. Hence, the only allowed index structures are $\tilde{p}_{\mu 1}\ldots\tilde{p}_{\mu 4}$, $(p^2\delta_{\mu 1\,\mu 2} - p_{\mu 1}p_{\mu 2})$, and any combinations and permutations thereof which respect the degree of divergence. Finally, the divergence itself, i.e. overall negative powers of the momentum, can only appear in a form contracted with $\theta_{\mu\nu}$ since its origin is a phase factor which always contains a product $k\theta p$. From these considerations it is clear that the set of Eqn. (3.71) is indeed complete.

In contrast to the scalar theory, where renormalizability can be restored by adding a simple non-local term (see Section 2.1), gauge theories come up with an additional requirement for counter terms regarding the tensor structure. Hence, one of the main tasks in the construction of a renormalizable gauge field theory will be to provide terms on tree level which are suitable to absorb all expected types (3.71) of divergences.

3.3.2 Localization and Auxiliary Fields

We have seen in Section 2.2 that the $1/(D^2\widetilde{D}^2)$ term, implementing the damping behavior of the $1/p^2$ model in a gauge covariant way requires to be localized in order to be interpreted in any reasonable way. The first approach to achieve this by the introduction of a real-valued auxiliary field in Section 3.1 led to a model with additional physical degrees of freedom. In the light of Refs. [117, 118] this is an effect of a non-trivial cohomology. In fact, from the BRST transformations (3.4) we see that $\mathcal{B}_{\mu\nu}$ transforms covariant, i.e. $s\mathcal{B}_{\mu\nu} = ig\,[c \stackrel{\star}{,} \mathcal{B}_{\mu\nu}]$, rather than in a BRST doublet structure. Such would require the field to be assigned to respective ghost fields, as it has been done in Section 3.2.2. There, the (complex) field $B_{\mu\nu}$ (see Eqn. (3.27)) transforms as $sB_{\mu\nu} = \psi_{\mu\nu} + ig\,[c \stackrel{\star}{,} B_{\mu\nu}]$, and $s\psi_{\mu\nu} = ig\,\{c \stackrel{\star}{,} \psi_{\mu\nu}\}$. Note that the covariant (anti-)commutators in these relations do not contribute since they vanish under integrals due to the cyclic invariance of the star product (1.26a). Physical interactions can be excluded due to Eqn. (3.28), i.e. the doublet structure of the auxiliary fields allows to write the entire localization term (up to the soft breaking) as a BRST exact expression. As will be discussed in Chapter 4, the introduction of auxiliary fields in gauge theories has to be performed in a way allowing for a trivial cohomology of all affected terms $S_{\text{unphys.}}$ in the action. An unintended modification of the physical content of the theory can be eliminated *a priori* if new fields are introduced such that they form BRST doublet structure relations, and $S_{\text{unphys.}}$ obeys[78, 79]

$$S_{\text{unphys.}} = s\hat{S}_{\text{unphys.}} \Rightarrow sS_{\text{unphys.}} = 0\,. \tag{3.72}$$

Another point regarding the auxiliary fields is worth considering there. The failure of the one loop renormalization in Section 3.2.5 was due to the fact that the photon field A_μ was coupled to the auxiliary field $B_{\mu\nu}$ via mixed propagators $G^{\{AB,A\bar{B}\}}$, and respective vertices. This led to numerous corrections being dressed by external $G^{\{AB,A\bar{B}\}}$ lines. In this way, additional IR divergences up to sixth order in the external momentum where introduced which could not be absorbed by terms being present in the tree level propagator.

For the construction of new models two possible ways to avoid these problems can be identified. First, if couplings of gauge and auxiliary fields are present, one has to assure that all (pure and mixed) propagators have a finite IR behavior. Secondly, if the unphysical fields are uncoupled from the gauge sector there will exist no interactions and, hence, no corrections to the gauge field. The latter solution seems to be preferable.

Chapter 4

Renormalization

This chapter represents an excursion to the topic of renormalization with special focus on the amenabliliy of the discussed methods to gauge theories, and non-commutative physics. First, a very brief overview of the most successful schemes is given in Section 4.1. A short discussion of the background and the significance of the renormalization group follows in Section 4.2. A subject which has, so far, been neglected in the field of non-commutative physics is the appearance of (composite field-) operators having zero or even negative dimension. It is the aim of Section 4.3 to discuss the implications of these, and other problems being introduced by deformation. Finally, Section 4.4 attempts to find a way out of the misery of lacking an amenable method to prove renormalizability of non-commutative gauge models.

4.1 Excursion: Renormalization in a Nutshell

The early roots of renormalization can be traced back to the advent of QED when it was realized that in general, the integrals appearing in explicit calculations of scattering matrix elements do not converge [124]. The reason for this was immediately identified to be the mathematically ill-defined multiplication of propagators (being distributions) at the same point. A first step to success was the introduction of regularization procedures which render the integrations finite, and allow to extract the divergent contributions in an explicit way. However, the result still was not satisfactory, since the physical values (represented by the respective limits in regulators or cutoffs for which the divergences reappear) were infinite, and therefore unobservable in experiments. The final solution turned out to be the subtraction of the pole terms from the respective results, or equivalently (and preferred) the redefinition of the parameters of the theory by absorbing the singularities. Amazingly, and subjected to criticism by the mathematical community, one obtains physically sound and finite results by computing the (strictly speaking not well defined) difference of two infinite quantities. Over time physicists and mathematicians have become accustomed to renormalization, and its oddities. In fact, its history is guided (in company with the development of the standard model) by a series of remarkable successes, such as the almost perfect prediction[1] of the anomalous magnetic moment of the electron, the Lamb shift, fine splittings, and a large series of masses and cross sections measured in the large colliders from the 1970's until today.

[1] In fact, the value $g-2 = 1\,159\,652\,188.4 \pm 4.3 \times 10^{12}$ of the anomalous moment of the electron[125] shows agreement with theory up to 9 digits, which is one of the best experimental verifications known.

4.1.1 Regularization

After this quick 'tour de renormalisation' we will now take a closer look at the concepts, and their suitability for non-commutative physics. First of all, *regularization* of the integrands has to be achieved. This means to alter the integrands in a way that renders them well defined. After the subsequent renormalization has been executed any remains of the regularization may be removed since, if applied correctly, the physical values do not depend upon the form of regulator. Let us discuss some of the available schemes.
The simplest method is to introduce exponential cutoffs, as being described in Appendix A.2.1. This typically yields expressions contatining the cutoff(s) in a way, which allows to directly identify the degree of divergence. As an example, consider the planar one loop result of the tadpole in Eqn. (2.12) (page 20), being $(\Pi^{\rm p})_{\rm regul.}(\Lambda) \propto \frac{-\lambda}{3(2\pi)^2}\Lambda^2$. Obviously, this expression shows a quadratic divergence for the physical limit $\Lambda \to \infty$, and the method is perfectly suitable for non-commutative theory since the introduction of an exponential cutoff in an integral is possible without any precondition.
Another quite popular method is *dimensional regularization* which is applied mostly for (but not limited to) theories on Minkowski space. The key idea is to turn the dimensionality of the loop momentum integrals into a complex variable ε, i.e. $\int d^4k\, \mathcal{I}(p,k) \to \int d^\varepsilon k\, \mathcal{I}(p,k)$. For a typical Feynman integral with $n \in \mathbb{N}$ propagators, $\mathcal{P}(k,m)$ symbolizing a generic pole, depending on momenta, masses, etc., and $\epsilon \ll 1$ being chosen in accordence with some pole prescription we can write (without proof)

$$\int d^\varepsilon k \frac{1}{(k^2 - \mathcal{P}(k,m) + i\epsilon)^n} = i(-1)^n \pi^{\frac{\varepsilon}{2}} \frac{\Gamma(n-\frac{\varepsilon}{2})}{\Gamma(n)} \mathcal{P}^{\frac{\varepsilon}{2}-n} . \tag{4.1}$$

From the result (4.1) it is obvious that the divergences are now hidden in the Euler Gamma functions Γ, depending on the value of ε and reappear for $\varepsilon \to 4$ (depending on the value of n). Again, this method is applicable in non-commutative theories. More thorough descriptions, and replacement rules for the integrals can be found for example in the textbooks [75, 76, 84] or, for an application to deformed spaces, Ref. [126].

Different to the both schemes above is the method of *BPHZ*[2]. Its basic idea is to subtract the divergences prior to the actual integration. The assumption is that the relevant terms can be identified by a Taylor expansion $\mathcal{T}_n$ to the order n with respect to the (not integrated) external momenta[3] p_i of the UV divergent integrand $\mathcal{I}(k, p_i)$. Obviously, the number of necessary derivations, n, is determined by the superficial degree of divergence, as $n = d(\mathcal{I}(p_i, k)) + 1$. In commutative theories, the terms of the expansion can be identified order by order with poles of the original integrand. The regularized integral is then written as

$$J_{\rm reg.}(p) = \int d^4k \left[\mathcal{I}(p,k) - \mathcal{T}_n\big|_{p_i\to 0} \circ \mathcal{I}(p,k)\right] + \sum_{j=0}^{n} c_j (p_i)^j . \tag{4.2}$$

In a further step, the coefficients c_j, which already represent renormalizations of the respective parameters in the action, are determined from the solution of renormalization conitions (see Section 4.1.2 below). Hence, in some sense, BPHZ may be considered as a renormalization procedure. At this point we have to note that there are several prerequisites which are not fulfilled in non-commutative theories. First, the Taylor expansion approach only yields all divergences and regularizes the integrand, if the latter has its poles in the UV limit. This is

[2]Named after Bogoliubov, Parasiuk, Hepp, and Zimmermann.
[3]Note that the index i stands symbolically for any additional notation. The same is true for the summation index j in Eqn. (4.2).

clear since an IR divergence, which can intuitively be represented by a negative power, will not vanish upon differentiation but become even worse[4]. In fact, the expansion $\mathcal{T}$ then has a vanishing radius of convergence and is, hence, not well defined. However, this latter problem may be resolved by choosing a different point of expansion. The second severe obstacle in the presence of deformation is the inherent non-locality of the star product which, generally, invalidates most proofs which support the scheme. Therefore, the BPHZ method, in its original form is possibly not suitable for non-commutative theories.

4.1.2 Renormalization

In the following a brief review of some 'classical' renormalization schemes is given. The aim is to emphasize the (dis-)advantages with respect to an application on deformed spaces. First, however, the general process shall briefly be discussed. In QFT, the objects of desire are (scattering) $\mathcal{S}$*-matrix* elements, which give the probabilities and cross sections of processes which can (supposedly) be detected in colliders. The perturbation expansion originates from[5],

$$\begin{aligned}\mathcal{S} &= \langle 0|\mathsf{T}e^{-\mathrm{i}S_{\text{int.}}[\phi_{a-}]}|0\rangle \\ &= 1+\sum_{n=1}^{\infty}\frac{\lambda}{n!}\int\prod_{j=1}^{i}\mathrm{d}^4x_j\langle 0|\mathsf{T}\phi_{a-}(x_1)\dots\phi_{a-}(x_i)|0\rangle\,,\end{aligned} \tag{4.3}$$

where the notation of Section 1.3.1 has been used, λ is a generic coupling constant, and the $\phi_{a\pm}$ are the *asymptotically free states* before (-) and after (+) the collision. In order to express the matrix elements (for the transition from an asymptotic state with n incoming to m outgoing particles) we have to apply the *LSZ* formalism[6]

$$\begin{aligned}\mathcal{S}_{\alpha\beta}(\phi_{a-,n},\phi_{a+,m}) &= \langle\phi_{a+,m}|\mathcal{S}|\phi_{a-,n}\rangle \\ &= \mathrm{i}Z^{-\frac{1}{2}}\int\prod_{\{\alpha,\beta\}=1}^{\{n,m\}}\mathrm{d}^4x_\alpha\,\mathrm{d}^4y_\beta\,\mathrm{e}^{\mathrm{i}\left[\sum\limits_{\{k,l\}=1}^{\{n,m\}}p_{a-,k}x_k-p_{a+,l}y_l\right]} \\ &\quad\left(\Box_{y_\beta}+m^2\right)\left(\Box_{x_\alpha}+m^2\right)\langle 0|\mathsf{T}\phi(y_1)\dots\phi(y_m)\phi(x_1)\dots\phi(x_n)|0\rangle\,.\end{aligned} \tag{4.4}$$

Vacuum expectation values of time ordered field products $G_n(x_1..x_n)=\langle 0|\mathsf{T}\phi(x_1)\dots\phi(x_n)|0\rangle$ represent the Green functions (or, if analytically continued to Euclidean space) the Schwinger functions of the model, being described by the *Gell-Mann-Low formula*, or equivalently in the path integral formalism, by the *Feynman-Kac formula*

$$G_n(z_1..z,n)=Z^{-1}\int\mathcal{D}\phi\left[\prod_{i=1}^{n}\phi(z_i)\right]\mathrm{e}^{\mathrm{i}S[\phi]}\,,\quad\text{with } Z=\int\mathcal{D}\phi\,\mathrm{e}^{\mathrm{i}S[\phi]}\,, \tag{4.5}$$

where the time ordering is now hidden in the measure. This already represents some kind of renormalization since the division by Z effectively removes any 'vacuum bubbles', i.e. graphs without connection to external points z_i, which contribute to the infinite vacuum energy but not to any physical processes[7].

[4]Mathematically this becomes obvious for the non-planar integrals which contain a phase factor. The latter will, upon derivation, contribute one power of the loop momentum to be integrated out for each order in the Taylor series. Hence, the degree of divergence is not improved but worsened by the procedure.

[5]Note that the discussion in the general part of this section applies to Minkowski space. The respective Euclidean expressions can be obtained in the same way, as in Section 1.3.1 (by Wick rotation).

[6]Named after Lehmann, Symanzik and Zimmermann.

[7]In this respect it should be mentioned that this is not always true, since the (normally unobservable) vacuum energy may locally be reduced by imposing boundary conditions, thereby creating a negative potential well giving

However, in order to compute the n-fold time ordered product of fields, we have to apply the famous *Wick theorem*, yielding

$$\langle 0|\mathsf{T}\phi(x_1)\dots\phi(x_n)|0\rangle = \sum_{\text{pairings }\pi}\prod_{\gamma\in\pi}\Delta_F(x_{\gamma,1},x_{\gamma,2})\,. \tag{4.6}$$

In the latter expression the sum runs over all $n!!$ possible pairings ('contractions') of two fields each, and the product runs over the resulting two point functions (propagators) Δ_F. Now we have arrived at the very heart of the source for divergences, the illdefined multiple product of propagators at the same points. These products each correspond to a single Feynman graph. It has been recognized quite early that a cure to the problem of divergences can be found by redefining parameters, masses and wavefunctions by adding to them appropriate terms, corresponding to the singularities obtained by loop calculations. However, this requires the action to be stable, i.e. that for all types of divergences a respective term exists in the tree level action[8]. Otherwise, a 'counterterm' with an eventual *new* parameter has to be introduced for each new type of divergence[9]. In any case, the target is to include all of the terms appearing at tree level, and to arbitrary order in the perturbative expansion (4.3) in the form of divergences into the action. A physical interpretation is that the original masses, parameters, etc. on tree level are so called *bare* masses, parameters, etc. which are *a priori* infinite. The renormalized (physical) quantities, in contrast, are physically observable. Hence, denoting all available parameters by the symbol m, we have $m_{\text{phys.}} = m_{\text{bare}} - \sum m_{\text{corr.}}$, where the sum runs over loop orders. Stability of the action can be expressed mathematically in form of *(re)normalization conditions* which are constructed such that they are fulfilled solely for actions having an identical form as the one on tree level. Such relations have been derived for the scalar $1/p^2$ model in Section 2.1.3, page 26, and for the BRSW model in Section 5.4, page 84.

One more topic has to be mentioned. Graphs at higher loop orders do not simply contribute one overall divergence but, as has been recognized early by Bogoliubov, Zimmermann, Hepp, and others [98, 128], they form a non-trivial structure of singularities which have to be treated in a very specific way. This can be understood when considering that the integrands may diverge each separately, or in any possible combination. In order to untangle these nested or overlapping poles Zimmermann has derived his famous *forest formula* (see Appendix D). The heart of this procedure is to identify the substructures (referred to as 'forests') of Feynman graphs contributing to specific poles. Although solving the problem of *overlapping divergences* completely, the method is not perfect, as it turns out that too many terms are subtracted, leading to the so called *renormalon problem* (see page 66). Generally, this name describes the dramatic increase of (finite) amplitudes of Feynman graphs after renormalization. Hence, the rather depressing diagnosis is that neither the bare perturbation series nor the renormalized one is indeed correct at higher loop orders. Fortunately, as is described below there is a light at the end of the tunnel: Multiscale Analysis. For now we shall step back and consider explicit ways of renormalization.

Explicit Loop Calculations

The oldest and most straight-forward way of computing quantum corrections to an action is definitely to conduct explicit loop calculations. All of the regularization methods mentioned

rise to 'vacuum forces'. In fact, this effect can be measured (for a good review see for example [127]), and is named after H.B.G. Casimir.

[8] In this case, a generic parameter λ of a term t receiving a divergent contribution t' by renormalization is redefined according to $\lambda_{\text{ren}} = \lambda(t + t')$

[9] Hence, no renormalized constant can be defined, as the parameter of the divergence has to be introduced as a new constant, i.e. $\{\lambda, t\}\nexists$, new term $\lambda t'$.

in Section 4.1.1 may be applied to extract the divergent parts. In a further step, as has been exercised in Section 2.1.3, and Section 3.2.5, one attempts to absorb the singularities by suitable redefinition of the parameters of the theory. Since this scheme is treated in most (undergraduate) courses and textbooks, no further explanation will be given

Algebraic Renormalization

In case the subjected theory obeys a specific symmetry, the renormalizability may be improved due to the systematic cancellation of divergences. Hence, it is of vital importance that any applied renormalization procedure leaves invariant the symmetry properties in all its steps, although it has to be mentioned that there is an option to temporally break invariances during renormalization, and restore it afterwards on the quantum level in a rigorous way as for example is done by applying the Polchinski approach in Ref. [129]. In commutative theory, the so called *Quantum Action Principle* (QAP) [130–132] defines a set of conditions which, if fulfilled, guarantee this invariance and enable the application of the *Algebraic Renormalization* procedure (see also the textbooks [78, 79]). Let us briefly review this topic.
The general procedure is as follows:

- First, any symmetry of the tree level (starting) action $S^0[\phi_\alpha, Q]$ depending on generic (classical) fields[10] ϕ_α and composite local field operators Q_β (see below) is expressed by a Ward identity (WI) $\mathcal{W}_\rho[\phi]$. It is crucial for the AR procedure not to miss any WI. Hence, one may check for completeness[11] of the set of (anti-)commutators defining the off-shell algebra $\mathcal{A}_{\text{ext.}}$ of the model, $\left\{\left[\mathcal{W}_\rho[\phi_\alpha], \mathcal{W}_{\rho'}[\phi_\alpha]\right], .. \left\{\mathcal{W}_\rho[\phi_\alpha], \mathcal{W}_{\rho'}[\phi_\alpha]\right\}\right\}$, $\forall \rho \neq \rho'$, where the indices ρ, and ρ' run over all symmetries.
- In some way similar to the renormalization conditions mentioned above, the QAP gives a general set of conditions which impose restrictions to candidates for possible (counter-) terms which may freely be added to the action without altering the symmetry content. Before writing these in a generally applicable form we have to introduce some additional notation. Composite field operators Q_β, which may be arbitrary local field polynomials, are coupled to classical external sources ρ_β, such that $\delta S^0[\phi, Q_\beta]/\delta\rho_\beta = Q_\beta$. Counterterms are generally denoted by Δ and may carry generic indices to indicate their type. j_α are classical sources being coupled to classical fields ϕ_α, and λ denotes a generic parameter of the theory. Finally, the QAP takes the form

$$\frac{\delta S[\phi_\alpha, Q_\beta]}{\delta\rho_\beta} = \Delta_\alpha S[\phi_\alpha, Q_\beta]\,, \tag{4.7}$$

$$\frac{\delta S[\phi_\alpha, Q_\beta]}{\delta\lambda} = \Delta_\lambda S[\phi_\alpha, Q_\beta]\,, \tag{4.8}$$

$$\frac{\delta S[\phi_\alpha, Q_\beta]}{\delta\phi_\gamma} = \Delta_\gamma S[\phi_\alpha, Q_\beta]\,, \tag{4.9a}$$

$$\phi_\gamma\frac{\delta S[\phi_\alpha, Q_\beta]}{\delta\phi_\delta} = \Delta_{\gamma\delta} S[\phi_\alpha, Q_\beta]\,, \tag{4.9b}$$

$$\frac{\delta S[\phi_\alpha, Q_\beta]}{\delta\rho_\delta}\frac{\delta S[\phi_\alpha, Q_\beta]}{\delta\phi_\epsilon} = \Delta_{\delta\epsilon} S[\phi_\delta, Q_\epsilon]\,. \tag{4.9c}$$

[10]Note that any additional decoration, is suppressed here, and indices are introduced solely for the purpose of distinguishing fields.
[11]In fact, completeness of the symmetries is only given if all possible commutators and anticommutators identically vanish or yield an element of the algebra, i.e. the set closes. If this is not the case, we have most likely missed a symmetry.

The first two equations state that new composite field operators and parameters may only be inserted into the action in the linear forms $Q_\alpha \rho_\alpha$ and $\lambda\Delta_\lambda$, respectively. Linearity, in this respect, is an important property since non-linear terms may give rise to anomalies in symmetries as will be described below. The three equations (4.9a)–(4.9c) can be considered as a generalized form of the equations of motion which shall be retained all the way down to the quantum level.

▷ A rather tedious task is then to find all possible local field polynomials Δ being compatible with $\mathcal{A}_{\text{ext.}}$, Eqns. (4.7)–(4.9c), the Wess-Zumino consistency condition $s\Delta = 0$ (where s is the nilpotent BRST operator) of dimension $D = 4$, with odd C and even P symmetry[12]. After this major step has been achieved, one has obtained the complete set of counterterms which may freely be added to the action, and the actual proof of renormalizability can be started.

▷ Now linearity enters the game, since such proof will only succeed if all breakings and insertions feature this property. Indeed there is a way to check this in advance. Following Becchi *et al.* [133] we collect all symmetries contained in $\mathcal{A}_{\text{ext.}}$ to a single nilpotent[13] operator δ with raises the ghost charge $g_\sharp$ by one, and enables us to decide if the theory contains anomalies or not. Justification is given by computing the *cohomology* $\mathcal{H}(\delta) := \operatorname{Ker}\delta/\operatorname{Im}\delta$ in the sector $S_\mathcal{G}$ with $g_\sharp = 1$, i.e. the set of all non-invariant counterterms $\Delta = \delta\bar{\Delta}$. If $\mathcal{H}(\delta(S_\mathcal{G})) = 0$ the absence of anomalies is proven, and the resulting theory may be considered physical.
In addition, renormalizability can be guaranteed if the cohomology of the sector with $g_\sharp = 0$, i.e. the δ-invariant counterterms, is trivial too (which requires $\Delta = \mathcal{F} + \delta\bar{\Delta}$, with $\mathcal{F}$ being some integrated local functional with $\delta\mathcal{F} = 0$). Since it is not possible to give an appropriate coverage of the subject within the framework of this thesis please refer to the literature on the topic. See [133–135] for historic contributions on the application of the QAP, and classification of anomalies in YM and, more general, Lie Groups, [136, 137] for the application in commutative theory, and [138] for a more technical approach applying spectral sequences.
The most important fact to capture here is that trivial cohomology is intimately related to nilpotency of δ and exactness of the counterterms. For local integrated $x-$dependent field polynomials $\mathcal{F}$ (differential $n-$forms with strictly positive degree $n > 0$) and a nilpotent operator δ the relation (which, in essence is the *Algebraic Poincaré Lemma*)

$$\delta\mathcal{F} = 0 \quad \Longleftrightarrow \quad \mathcal{F} = \delta\bar{\mathcal{F}}\,, \tag{4.10}$$

holds [137]. The left hand side of this equation is called *cocycle condition* (which represents a constraint for possible counterterms Δ which have to fulfill $\delta\Delta \equiv 0$, compare the Wess Zumino consistency condition above), and the right hand side is referred to as the *coboundary condition.* For x-independent functionals (with $g_\sharp(\mathcal{F}) = 0$ the right hand side allows for not-trivial solutions $\mathcal{F} = \delta\bar{\mathcal{F}} + \Delta$ where Δ represents a general constant polynomial of positive form degree, corresponding to the case of δ-invariant counterterms above. More rigorous definitions and details on these relations can be found in Ref. [137].

▷ An explicit proof of renormalizability is achieved by induction, assuming that all symmetries are indeed fulfilled up to some order n in the perturbative expansion (1.11) in $\hbar$. Denoting a generical symmetry operator (which stands for any, not only the Slavnov

[12] The abbreviations refer to charge and parity transformations, respectively.
[13] The nilpotency property $\delta^2 = 0$ is of vital importance to achieve trivial cohomology.

Taylor operator) by $\mathcal{S}$, and a classical linear breaking by Δ, we write therefore

$$\begin{aligned}\mathcal{S}S^{(n)} &= \hbar^n\Delta + \mathcal{O}\left(\hbar^{n+1}\right) ,\\ &= \hbar^n\Delta + \hbar^{n+1}\Delta'_{n+1}\mathcal{O}\left(\hbar^{n+2}\right) ,\end{aligned} \tag{4.11}$$

and it remains to show that any term, which may be inserted for Δ_{n+1} and is compatible with the relations in $\mathcal{A}_{\text{ext.}}$, obeys

$$\mathcal{S}\Delta'_{n+1} = \hbar^{n+1}\Delta + \mathcal{O}\left(\hbar^{n+2}\right) . \tag{4.12}$$

Finally, the action at the next order is readily given by $S^{(n+1)} = S^{(n)} - \int \mathrm{d}^4x\, \Delta'_{n+1}(x)$. In the case of Eqn. (4.12) the symmetry $\mathcal{S}$ is linearly broken. In theories with BRST symmetry with an operator δ, as defined above, one principally has to solve the cohomologic problem $\delta\Delta' \equiv 0$ for Δ'. Regarding the solution we should consider the following

Definition 5. If the cocycle condition $\delta\Delta' = 0$ with a candidate counterterm (local integrated field polynomial) Δ' of ghost number one, and a nilpotent operator $\delta^2 = 0$, allows for solutions other than the trivial one, $\Delta' = \delta\hat{\Delta}'$, we speak of an *anomaly*.

See also Ref. [80] for an extensive review of this topic.

The method of AR has successfully been applied to many theories with gauge symmetry in the commutative case. However, the restrictions to counterterms from the QAP do not specify any coefficients. In fact, these have to be obtained by explicit loop calculations which is the reason why, in the light of the renormalon problem, this does not represent a satisfactory solution.

Being interested mainly in gauge extensions to renormalizable non-commutative QFT models we also have to analyze the AR scheme for its adequacy in this respect. Immediately we are confronted with the presumption of locality which seems to be critical. However, we may postpone any answer to this problem to Section 4.3 below.

Multiscale Analysis

The basic principles of this renormalization method have already been discussed in Section 2.1.4. However, we may catch up the topic and analyze the scheme with respect to renormalons, and a possible application to gauge theories. In fact, MSA seems to avoid the problems of 'classical' recursive renormalization schemes. The reason for this is that the classification of divergences is not made with respect to entire subgraphs, ordered in the form of trees (compare the Zimmermann formalism in Appendix D.1), but with respect to scales (as being discussed in Section 2.1.4). The principal idea is quite intuitive, to assign a specific energy scale to sub processes in Feynman graphs in an ordered way, i.e. the higher the (loop) order at which a specific subgraph appears, the higher its scales. Practically, each propagator effectively acquires a scale index which then appears in the summation over slices (c.f. Eqn. (2.35) on page 27). These scale attributions are in a further step used to define so called *rooted trees* which give a similar partition of the divergences of a given graph, as the trees in the Zimmermann approach. In the following we shall try to capture the interpretational content of the scheme. For mathematical rigor please refer to the textbook [99].
As we have seen in Theorem 1 the propagators are exponentially damped by the scale i. From this we are led to the insight that at a scale $j < i$ any contribution of i may be neglected. This motivates a redefinition of locality. We state the following

Definition 6. Denoting for a subgraph $\gamma \subseteq G$ of a proper Feynman (sub)graph G, the scale indices of its inner propagators $\ell \subset \gamma$ by $i(\ell)$, and those of its external propagators (i.e. those which connect γ to the rest of G) by $e(\ell)$, we define the *locality condition*,

$$i_\ell(\gamma) = \inf_{\lambda \subset \gamma} i(\ell) \quad e_\ell(\gamma) = \sup_{\lambda \subset G,\, \lambda \not\subseteq \gamma} e(\ell)\,, \tag{4.13}$$

$$i_\ell(\gamma) > e_\ell(\gamma)\,. \tag{4.14}$$

We call a subgraph γ obeying Eqn. (4.14) *almost local.*

In a more picturesque language we can say, a process (represented by propagators forming a subgraph) at scale i, $i+1$, .. appears at scale $j < i$ as a thick dot as indicated in Fig. 4.1. Similar to the view through the ocular of a microscope, at different resolutions (scales) we see different processes. In the literature the notion 'almost local' is often used synonymously with 'quasi local'.

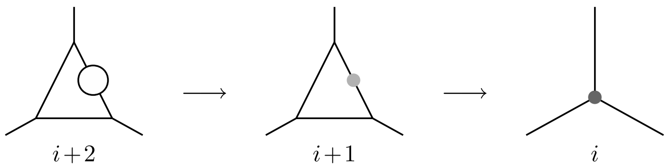


Figure 4.1: Almost locality — on lower scales (energies) the contributions of higher scales appear as a thick dot. Subgraphs are reduced to quasi local vertices.

As mentioned above, the scale attribution of the propagators in G is the basis for the definition of all possible quasi local subgraphs γ. Each subgraph then forms a so called 'rooted tree', where each vertex represents a divergent quasi local subgraph. The concept is quite simple, and we will pick up the example with overlapping divergences from Appendix D.1. In Fig. 4.2

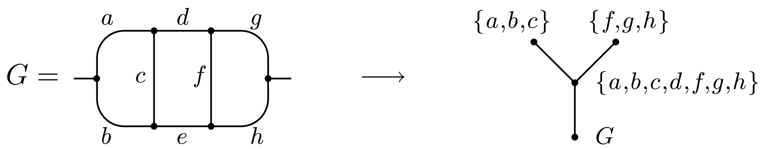


Figure 4.2: A divergent graph with labeled propagators, and the according rooted tree containing subdivergences for the scale assignation $\{a, b, c, f, g, h\} \to 3$, $d \to 2$, $e \to 1$, and external lines have per definition the scale -1.

a (rather arbitrary) assignment of scales to the propagators of G leads to the simple rooted tree on depicted the right, which contains solely quasi local subgraphs. In comparison to the Zimmermann result on page 124 we have 4 instead of 16 subtractions. In fact, only quasi local subgraphs give rise to divergences [99] but there are also finite ones with $i_\ell(\gamma) \leq e_\ell(\gamma)$. In the Zimmermann approach, no such distinction is made, and this is the actual origin of the *renormalon problem.* In the literature, non-divergent subtractions are referred to as 'useless' in the sense that, when traversing from the bare perturbation series to the renormalized one, not only divergences but also finite contributions are subtracted. The latter ones give rise to an $n!$ behaviour at perturbative loop order n in the renormalization group flow (see Section 4.2 below), i.e. all renormalized masses, wave functions and couplings are finite but scale factorially. This is the infamous *renormalon problem.* Indeed, MSA provides a viable solution, as the subtraction of finite counterterms is avoided by virtue of the locality condition (4.14).

In the point of view of non-commutative field theories the MSA is perfectly suitable, as is obvious from the paradigmatic examples in Refs. [36, 36, 40]. Moreover, the concept of quasi locality is enhanced on Moyal deformed space by the notion of *Moyality* [5]. It can be understood that in the same way as higher scale subgraphs appear local on lower scales, the manifest non-locality of the star product may be considered as a smearing of the vertex (represented by the 'fat dot' notation, as in Fig. 4.1 above). From lower energy scales the non-commutative vertices may therefore appear local, and the question arises if one can, approximatively, entirely neglect non-locality when considering the limit $p \to 0$. The latter assumption would, in principle, restore the validity of the BPHZ subtraction scheme, and supposedly some proofs of AR. However, it will not remedy the problem of UV/IR mixing. Again, an ultimate answer is still missing but, coming back to the MSA, the most important point to capture here is that the scheme is applicable to non-commutative theories, but in view of the discussions above and in Section 2.1.3, it is obvious that a generalization to gauge theories will require a substantial amount of work to retain all the proofs and bounds under consideration of the more complex set of fields and ghosts.

4.2 On the Renormalization Group

Besides the renormalization schemes discussed in Section 4.1.2 the *renormalization group* (RG) approach follows a rather different strategy. It does not attempt to renormalize a given action by explicit subtractions but provides a tool for the analysis of the dependence of renormalized parameters under a change of scale. Knowledge of this so called 'flow' allows to investigate the limit for large scales (respectively energy), which in turn yields information about physical couplings at high momenta.
Let us specify this in a little more detail. We consider scale transformations $p \to tp$ of renormalized n-point vertex functions $\Gamma_{\rm r}^{(n)}[p, \lambda_i]$ with a parameter t. In principle all renormalized parameters, denoted here generically by $\lambda_{{\rm r},i}$ depend upon t. Hence, for an unrenormalized n-point vertex function, $\Gamma^{(n)}$, we can state

$$\begin{aligned} 0 =& t\frac{\partial}{\partial t}\Gamma^{(n)}[p, \lambda_i] \\ =& t\frac{\mathrm{d}}{\mathrm{d}t}\left(Z^{-\frac{n}{2}}\Gamma_{\rm r}^{(n)}[tp, \lambda_{{\rm r},i}]\right) , \end{aligned} \tag{4.15}$$

where Z is a generic wave function renormalization ($\phi_r = Z^{-1/2}\phi$). We have to note that t may be replaced by any cutoff or mass which influences the energy scale of the theory. Specifically, this applies to masses in general, momentum cutoffs Λ for minimal subtraction, and the mass scaling parameter μ usually introduced in dimesional regularization to compensate the non-integer dimension of the integrals. Therefore, we may rewrite the second line of Eqn. (4.15) with an explicit mass m, and a coupling λ (where after exchanging the differentiation $\partial/\partial t \to \partial/\partial\Lambda$ we multiply from the right by $Z^{n/2}$),

$$\begin{aligned} 0 &= \left[n\Lambda\frac{\partial}{\partial\Lambda}\ln\sqrt{Z} + \Lambda\frac{\partial\lambda_{\rm r}}{\partial\Lambda}\frac{\partial}{\partial\lambda_{\rm r}} + \Lambda\frac{\partial m_{\rm r}}{\partial\Lambda}\frac{\partial}{\partial m_{\rm r}}\right]\Gamma_{\rm r}^{(n)}[p, \lambda_{\rm r}, m_{\rm r}] \,, \\ &= \left[-n\gamma + \beta\frac{\partial}{\partial\lambda_{\rm r}} + \gamma_m\frac{\partial}{\partial m_{\rm r}}\right]\Gamma_{\rm r}^{(n)}[p, \lambda_{\rm r}, m_{\rm r}] \,, \end{aligned} \tag{4.16}$$

and the new functions β, γ, and γ_m are defined by identification between the first and the second line. Eqn. (4.16) is generally referred to as the *renormalization group (RG) equation.* For the sake of completeness it has to be mentioned that the there exists an extension to the homogeneous equation (4.16) which is named after *Callan* and *Symanzik*. It is, in the current notation and for massless theories[14], given by

$$\left[-n\gamma + \beta\frac{\partial}{\partial\lambda_{\mathrm{r}}} + \gamma_m\frac{\partial}{\partial m_{\mathrm{r}}} + l\gamma_2\right]\Gamma_{\mathrm{r}}^{(n,l)}[p,\lambda_{\mathrm{r}},m_{\mathrm{r}}] = m_r^2\,(2-2\gamma)\,\Gamma_{\mathrm{r}}^{(n,l+1)}[p,\lambda_{\mathrm{r}},m_{\mathrm{r}}]\,. \tag{4.17}$$

The interesting fact is, that Eqn. (4.17) allows for a recursion over loop orders l, hence allowing for inductive proofs of renormalizability (for an extensive review see [77]). This reveals a principal feature of the renormalization group, which is that the effects of renormalization can be analyzed in a non-perturbative framework. In this respect it proved to be useful to analyze the *β function* of Eqn. (4.16) which yields information about the behaviour of the coupling λ_r. Let us define more generally,

$$\beta(\lambda(t)) := t\frac{\partial\lambda_{\mathrm{r}}(t)}{\partial t}\,. \tag{4.18}$$

In fact, if $\beta|_{t=0} = 0$, and is monotonically rising for $t \gg 1$ this indicates an asymptotically free coupling. If $\beta \to 0$ for $t \gg 1$ the respective coupling is unphysical, and vanishes. For more details refer to Ref. [75].

As mentioned above, the RG approach opens the door to a non-perturbative analysis and proof of renormalizability. An explicit method has been found by J. Polchinski [139], which shall be mentioned at this place only by its key points, as the particular form of the proof is rather lengthy. First, one recognizes, that in the flow of parameters, generated by Eqn. (4.17) the action can be split into relevant and negligible parts. The proof is constructed such, that the latter are shown to vanish in the high momentum limit, while the former converge to a finite limit. Mathematically, this is achieved by imposing appropriate cutoff functions which render all integrals finite. In a further step, one replaces the integrals by simple bounds (similar to the scheme in MSA), which allows finally to show that all vertex functionals have finite limits upon removal of the cutoff.
It should be mentioned that the original proof is given for a scalar ϕ_4^4 theory with simple momentum cutoffs. Remarks regarding gauge theories are also given and indicate, that the explicit cutoff scheme is incompatible with symmetries but that they should reappear upon going to the unconstrained limits at the end of the proof.

With respect to non-commutative theories, the Polchinski approach is definitively applicable, as locality is not presumed in any step. Moreover, it has been applied for the proof of renormalizability in the scalar GW approach [32]. The case of (commutative) spontaneously broken $SU(2)$ Yang-Mills theory has been discussed by C. Kopper and V. F. Müller [129]. Their starting point was the classical BRST invariant action including all (i.e. a finite set of) counter terms satisfying certain symmetry constraints. Since the regularization (which is required in the Polchinski approach) breaks the local gauge symmetry explicitly, the counter terms are only required to be invariant under a global $SO(3)$ isosymmetry. Nonetheless, the authors have shown that this ansatz solves the flow equations to all orders by induction. In the case of non-commutative gauge theories the set of all possible counter terms is infinite, but one could choose a restricted, finite set of counter terms instead. Renormalizability would be established, if it could be shown that this finite set solves the flow equations (which automatically induces that the set is also closed).

[14] Note that m_r may exist even though $m_0 = 0$, due to counterterms of the type $m_r\phi^2$.

4.3 Non-Commutativity and Locality

As has already indicated before, the introduction of the θ-deformation (see Section 1.3.3) inevitably leads to non-locality since the associated $\star$-product itself is non-local[15]. This property is not completely compatible with some of the classical renormalization schemes since the premise of locality is required to avoid the insertion of artificial (non-local) operators of zero or negative mass dimension, which in turn spoil renormalizability. Of course, this is exactly the same problem we are facing now in non-commutative QFT, and the question arises if renormalizability exists in the presence of non-locality. An affirmative answer can definitely be given for the renormalizable scalar models of Section 1.4, and the $1/p^2$ model of Section 2.1, but there are subtleties which require some additional argumentation.

There are two fundamental issues which will be discussed subsequently:

- ▷ The non-locality of the star product (1.22) invalidates several proofs underlying classical renormalization schemes.
- ▷ Due to the negative mass dimension of the parameter θ (see Section 1.3.3) arbitrary powers of dimensionless operators and composite fields may freely be inserted into the action.

Regarding the first point, we may consider that in commutative theory one generally follows the rule that only local terms are allowed in the action. This, at first sight rather arbitrary, restriction is motivated by the fact, that non-local insertions *do* allow for constructs, such as $\theta^n\phi^{2n}$, with $n \in \mathbb{N}$ and a generic field ϕ, which give rise to a possibly infinite number of additional terms, and thereby to a non-renormalizable theory[16].

For this reason, almost all proofs buried in classical renormalization schemes presume locality. As an example, let us review the situation for the ARprocedure. The general form of the QAP, Eqns. (4.7)–(4.9c), is only defined [78] for local actions. The reason is that highly non-linear insertions (as for instance the example $\theta^n\phi^{2n}$ mentioned above), despite obeying all demanded symmetries, may result in nontrivial solutions for the cocycle condition $\delta\Delta = 0$ (see Eqn. (4.10) above), and therefore induce anomalies. Regarding the classification with respect to cohomology one of the key theorems is the Algebraic Poincare Lemma@Algebraic Poincaré Lemma!and localityAlgebraic Poincaré Lemma (4.10), which again vitally depends on locality [137].

Despite these obstacles, some efforts for a generalization to non-commutative spaces have been made. For example, the notion of BRST cohomology and the Chern character have been introduced in [142] using Connes' notation of spectral triples [24, 143]. Another contribution has been the generalization of the descent equations, which describe Yang-Mills anomalies, to deformed spaces [144]. It has also been shown [45] that the symmetry content compatible with the QAP can be established for $U_\star(N)$ theories, and is invariant under an explicit one-loop UV renormalization.

Now, what in addition had to be done to restore rigor of the AR procedure on non-commutative spaces [64]? We may attempt to give an answer to this 'deep' question by stating that, first of all, the computation of the cohomology class has to be worked out rigorously for the ghost number 0 functionals $\mathcal{F}$, representing the most general quantum level action, to fulfill $s\mathcal{F} = 0$. In addition,

[15] We should mention, though, that attempts have been made to localize the star product by introducing a bifermionic non-commutativity parameter [140, 141].

[16] Strictly spoken such a theory is not power counting non-renormalizable since no vertex of mass dimension $d_m(v) > D$ appears, but an infinite number of terms cannot be considered to be physical. In any way, a practical treatment of such a theory seems impossible.

a proof for the triviality of the cohomology being sufficient to guarantee renormalizability in the presence of non-locality had to be achieved. However, this discussion shall only be considered as a provocation for thought, since it is constructed on rather thin ice.

The second point is more involved and of a quite general nature, as it applies to *all* non-commutative theories: It concerns the appearance of dimensionless operator insertions in the action. A parameter of non-commutativity θ with mass dimension -2 allows to freely add composite field operators[17] of zero mass dimension, such as $D^2\widetilde{D}^2$ or $\widetilde{F}^2$, to the action, where $\widetilde{D}_\mu = D_\nu\theta_{\mu\nu}$ is a contracted covariant derivative and $\widetilde{F} = F_{\mu\nu}\theta_{\mu\nu}$ is a field strength. Being invariant under all symmetries appearing in the QAP (and gauge transformations in general), there is no constraint or theorem preventing insertions of arbitrary powers of these operators both at tree level or as quantum corrections[18]. This is the reason why the sufficiency of a trivial cohomology class for renormalizability has been questioned above. It should furthermore be pointed out that, due to this problem, standard 'top down' renormalization schemes, as for example AR, supposedly do not work, as they start from the set of all possible counter terms, and restrict these by applying constraints. Since this set is *a priori infinite* in the presence of invariant dimensionless insertions, the attempt to achieve a finite number of counter terms will fail, independent of the cohomology.

4.4 Light on the Horizon

After all, the question is how renormalizability can be proven rigorously in the presence of deformation. Principally, three scenarios may be suggested.

1. A feasible path seems to re-establish the foundations for *AR* in the non-local case. As has already be mentioned in Section 4.3, the classification of anomalies by computation of the cohomology class $H^{(1)}$ of the BRST operator for general functionals (i.e. counterterms) with ghost number 1 has already been achieved [142] but the proof for ghost number 0, (i.e. the action) is missing. In addition it has to be assured in a rigorous way that trivial cohomology alone is sufficient to prove the absence of anomalies, i.e. renormalizability. And if this turns out not to be true, one has to find out which additional requirements are necessary.
   After all of these proofs have been achieved, one still has has to find constraints to limit the appearance of insertions of massless operators into the action. In this context also the issue of field redefinitions might be important and maybe some classes of insertions can be rewritten as such redefinitions (cf. [53] in the context of non-commutative $U_\star(1)$ gauge theory with Seiberg-Witten maps).

2. The second scenario is to apply the Polchinski approach for gauge theories. Being not limited to locality, it should be sufficient to pragmatically state that the set of terms implemented in the tree level action is complete, if stability is guaranteed. The principal eventuality of insertions of dimensionless operators to arbitrary powers is simply neglected, thus.
   However, explicitely breaking the gauge symmetry, one will have to proof that it is possible on the quantum level to restore any demanded symmetry. This latter task seems to be potentially nontrivial and extensive.

[17] Note that this also occurs in scalar field theories. For example, the non-local term $\Box^{-1}\phi^2$ could be inserted into the tree level action to arbitrary power.

[18] work in progress

3. Finally, the prosperous approach of MSA suffers from similar problems as the Polchinski method. Although gauge breaking may be restored at the end of the proof, one is still confronted with the so called 'Gribov problem' [116, 145, 146] stating a non-trivial ambiguity or 'remanent gauge freeness' . This can principally be treated by implementing a soft breaking mechanism, as has been done in Section 3.2. However, there is not much information available on this topic so a thorough study is essential before any decisive conclusions can be drawn.

In fact, the question for the renormalizability of non-commutative gauge theories, leads to a white spot on the map of explored theoretical aspects. Due to the rather constrained time for work on this thesis no definite answers can be given. Hopefully, one of the above ideas to start from will lead to success in future.

For now, we shall try to apply the knowledge gained in this chapter, and the analytic discussion of Section 3.3, to construct a *potentially* renormalizable gauge theory based on the non-commutative $1/p^2$ model; a topic which is addressed below in Chapter 5.

Chapter 5

The BRSW Model

Bearing in mind the insights of Section 3.3 and Chapter 4 we may now attempt to construct a model which avoids all of the described problems. The target is to achieve the following points

- ▷ The tree level action shall provide a counter term for the quadratic divergence $\frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}$.
- ▷ All propagators of fields shall be infrared finite and feature damping factors similar to the scalar $1/p^2$ model.
- ▷ If possible, any auxiliary fields and ghosts should be uncoupled from the gauge sector.
- ▷ The model shall be as simple as possible.

In the following sections a surprisingly simple solution to this ambitious list of requirements, the BRSW model[1], is presented.

5.1 Construction of the Action

The intention is to start from the localized action (3.23) of the model in Section 3.2, and modify it in order to achieve renormalizability and avoid the problems discussed in Section 3.3. In a first step, the interplay between terms of the action, and the form and type of propagators is analyzed. For details on this step see Appendix E.1. There are three main ideas leading to success.

First, in order to avoid (or at least restrict) the appearance of dimensionless derivative operators (as is discussed in Section 4.3) it is desirable to remove any explicit appearance of parameters with negative mass dimension from the action. However, this is impossible, since the effect of UV/IR mixing inevitably leads to divergences being contracted with $\theta_{\mu\nu}$ (as discussed in Section 3.3.1), which enter the action in the form of counter terms. A viable solution to this problem is to split the parameter of non-commutativity into a dimensionless tensor structure $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$, and a dimensionful scalar parameter ε, i.e.

$$\theta_{\mu\nu} \to \varepsilon\Theta_{\mu\nu}\,, \text{ with } d_m(\Theta_{\mu\nu}) = 0\,, \text{ and } d_m(\varepsilon) = -2\,. \tag{5.1}$$

In consequence, the appearance of ε in the tree level action is reduced by modifying our definition of contractions, $\widetilde{\square} := \Theta_{\mu\rho}\Theta_{\nu\sigma}\partial_\rho\partial_\sigma$, $\tilde{p}_\mu := p_\nu\Theta_{\mu\nu}$, for any vector p_μ, and $\widetilde{O}_{\mu_1\, m_2 \ldots \mu_n} :=$

[1]The name is an abbreviation for the list of contributors to the initial publication [66], Blaschke, Rofner, Schweda, Sedmik, Wohlgenannt.

$O_{\nu\,\mu_2\ldots\mu_n}\Theta_{\mu_1\nu}$ for a tensor with n indices. Hence, the only occurrence of the dimensionful ε is in the phase associated with the star product, which does not influence the bi-linear part according to the property (1.26b) (i.e. that the star may be omitted in bi-linear expressions under an integral). In this respect we note that operators such as $\widetilde{\Box}$ or $\widetilde{D}$ now come with their usually expected mass dimensions $d_m(\widetilde{\Box}) = 2$ and $d_m(\widetilde{D}) = 1$, respectively. Starting from the localized part of the action from Section 3.2.2, Eqn. (3.23), the remaining two steps can be written as

$$\int \mathrm{d}^4x \, \frac{a}{2} \left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right) F_{\mu\nu} - \bar{B}_{\mu\nu}\varepsilon^2 \widetilde{D}^2 D^2 B_{\mu\nu} \tag{5.2a}$$

$$\downarrow \text{step 1}$$

$$\int \mathrm{d}^4x \, \frac{\gamma^3}{2} \left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right) \frac{1}{\widetilde{\Box}} F_{\mu\nu} + \bar{B}_{\mu\nu}(\sigma - D^2) B_{\mu\nu} \tag{5.2b}$$

$$\downarrow \text{step 2}$$

$$\int \mathrm{d}^4x \, \frac{\gamma^2}{2} \left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right) \frac{1}{\widetilde{\Box}} \left(f_{\mu\nu} + \sigma \frac{\Theta_{\mu\nu}}{2} \tilde{f} \right) - \bar{B}_{\mu\nu} B_{\mu\nu} \tag{5.2c}$$

with several new definitions being explained subsequently. To understand the first step we note that the divergences in the $G^{\{AB,A\bar{B}\}}$, $G^{\{\bar{B}B,BB\}}$, and $G^{\bar{\psi}\psi}$ propagators are mainly caused by the appearance of the operator $D^2\widetilde{D}^2$ sandwiched between $\bar{B}_{\mu\nu}$ and $B_{\mu\nu}$. On the other hand this term is crucial to the construction of the correct damping factor for the gauge boson propagator G^{AA}. The analysis in Appendix E.1 leads to the insight that it is possible to move the problematic operator into the soft breaking term, thereby maintaining the desired damping while eliminating the divergences. Note also that, due to the redefinition of $\theta_{\mu\nu}$ in Eqn. (5.1) the dimensionful ε does not appear explicitly after the first step in Eqn. (5.2b). In the resulting action, the correct mass dimensions are restored by the new parameters γ and σ featuring $d_m(\gamma) = 1$ and $d_m(\sigma) = 2$, respectively.
In step 2, we note that the regularizing effects are solely implemented in the bi-linear part of the action, therefore opening the option to reduce the field strength tensor $F_{\mu\nu}$ in the soft breaking term to its bi-linear part $f_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. Noting furthermore, that the D^2 operator in the $\bar{B}/B$ sector is not required any more for the implementation of the damping mechanism we may entirely omit this derivative. Due to this reduction, any interaction (represented by n-point functions with $n \geq 3$) of A_μ with auxiliary fields and ghosts is eliminated. However, in order to restore the correct mass dimension for the altered terms we have to change d_m of the fields $B_{\mu\nu}$ and $\bar{B}_{\mu\nu}$ from 1 to 2. Finally, in order to implement a suitable term to absorb the θ-contracted one loop divergence (being discussed in Section 2.2) we further modify the breaking by the insertion of the term $\frac{\gamma^2}{4}\sigma\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)\frac{1}{\widetilde{\Box}}\Theta_{\mu\nu}\tilde{f}$, resulting in (5.2c).

Finally, the complete action takes the form,

$$\begin{aligned}
S &= S_{\text{inv}} + S_{\text{gf}} + S_{\text{aux}} + S_{\text{break}} + S_{\text{ext}} \,, \\
S_{\text{inv}} &= \int \mathrm{d}^4x \, \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \,, \\
S_{\text{gf}} &= \int \mathrm{d}^4x \; s\left(\bar{c}\,\partial_\mu A_\mu\right) = \int \mathrm{d}^4x \, \left(b\,\partial_\mu A_\mu - \bar{c}\,\partial_\mu D_\mu c\right) \,, \\
S_{\text{aux}} &= -\int \mathrm{d}^4x \; s\left(\bar{\psi}_{\mu\nu} B_{\mu\nu}\right) = \int \mathrm{d}^4x \, \left(-\bar{B}_{\mu\nu} B_{\mu\nu} + \bar{\psi}_{\mu\nu}\psi_{\mu\nu}\right) \,,
\end{aligned}$$

$$S_{\text{break}} = \int \mathrm{d}^4x \; s\left[\left(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\tilde{\Box}}}\left(f_{\alpha\beta} + \sigma\frac{\Theta_{\alpha\beta}}{2}\tilde{f}\right)\right] =$$

$$= \int \mathrm{d}^4x \left[\left(\bar{J}_{\mu\nu\alpha\beta}B_{\mu\nu} + J_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\tilde{\Box}}}\left(f_{\alpha\beta} + \sigma\frac{\Theta_{\alpha\beta}}{2}\tilde{f}\right) - \bar{Q}_{\mu\nu\alpha\beta}\psi_{\mu\nu}\frac{1}{\tilde{\tilde{\Box}}}\left(f_{\alpha\beta} + \sigma\frac{\Theta_{\alpha\beta}}{2}\tilde{f}\right)\right.$$

$$\left. - \left(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\tilde{\Box}}}\, s\left(f_{\alpha\beta} + \sigma\frac{\Theta_{\alpha\beta}}{2}\tilde{f}\right)\right],$$

$$S_{\text{ext}} = \int \mathrm{d}^4x \left(\Omega^A_\mu sA_\mu + \Omega^c sc\right), \tag{5.3}$$

where all products are implicitly assumed to be deformed Groenewold-Moyal products, and we have introduced the external sources Ω^A_μ and Ω^c. Due to the uncoupling of the gauge sector the form of the BRST transformations is simpler than the respective counterparts in Eqn. (3.26) for the model in Section 3.2[2].

$$\begin{aligned}
sA_\mu &= D_\mu c\,, & s\,c &= \mathrm{i}gcc\,,\\
s\,\bar{c} &= b\,, & s\,b &= 0\,,\\
s\,\bar{\psi}_{\mu\nu} &= \bar{B}_{\mu\nu}\,, & s\,\bar{B}_{\mu\nu} &= 0\,,\\
s\,B_{\mu\nu} &= \psi_{\mu\nu}\,, & s\,\psi_{\mu\nu} &= 0\,,\\
s\,\bar{Q}_{\mu\nu\alpha\beta} &= \bar{J}_{\mu\nu\alpha\beta}, & s\,\bar{J}_{\mu\nu\alpha\beta} &= 0\,,\\
s\,Q_{\mu\nu\alpha\beta} &= J_{\mu\nu\alpha\beta}, & s\,J_{\mu\nu\alpha\beta} &= 0\,,
\end{aligned} \tag{5.4}$$

As before, the additional pairs of sources $\{\bar{Q}_{\mu\nu\alpha\beta}, Q_{\mu\nu\alpha\beta}\}$ and $\{\bar{J}_{\mu\nu\alpha\beta}, J_{\mu\nu\alpha\beta}\}$, obeying the relations $\{\bar{Q}, Q, \bar{J}, J\}_{\mu\nu\alpha\beta} = -\{\bar{Q}, Q, \bar{J}, J\}_{\nu\mu\alpha\beta} = -\{\bar{Q}, Q, \bar{J}, J\}_{\mu\nu\beta\alpha}$, have been introduced in order to restore BRST invariance of the entire action (5.3) in the UV limit, i.e. $sS = 0$. In the IR limit the physical values

$$\begin{aligned}
\bar{Q}_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0\,, & \bar{J}_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= \frac{\gamma^2}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right),\\
Q_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0\,, & J_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= \frac{\gamma^2}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right),
\end{aligned} \tag{5.5}$$

lead back to a breaking term of the form of the first part of Eqn. (5.2c). Dimensions and ghost numbers of all fields and sources of the model are collected in Tab. 5.1. In accordance with the discussion in Appendix E.1.2 we obtain the following relevant propagators for the model

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\left(1 + \frac{\gamma^4}{(\tilde{k}^2)^2}\right)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\left(\sigma + \frac{\Theta^2}{4}\sigma^2\right)\gamma^4}{\left[\left(\sigma + \frac{\Theta^2}{4}\sigma^2\right)\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)\right]}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right), \tag{5.6}$$

$$G^{\bar{c}c}(k) = \frac{-1}{k^2}, \tag{5.7}$$

where the Landau gauge $\alpha \to 0$ has led to the omission of the term $-\alpha\frac{k_\mu k_\nu}{k^4}$.

[2]Since the (anti-)commutator relations can be omitted, thus.

Table 5.1: Properties of fields and sources in the BRSW model.

Field	A_μ	c	$\bar{c}$	$B_{\mu\nu}$	$\bar{B}_{\mu\nu}$	$\psi_{\mu\nu}$	$\bar{\psi}_{\mu\nu}$	$J_{\alpha\beta\mu\nu}$	$\bar{J}_{\alpha\beta\mu\nu}$	$Q_{\alpha\beta\mu\nu}$	$\bar{Q}_{\alpha\beta\mu\nu}$	Ω^A_μ	Ω^c	b
$g_\sharp$	0	1	-1	0	0	1	-1	0	0	-1	-1	-1	-2	0
Mass dim.	1	0	2	2	2	2	2	2	2	2	2	3	4	2
Statistics	b	f	f	b	b	f	f	b	b	f	f	f	b	b

Although there also exist two point functions $G^{\{AB,A\bar{B}\}}$, $G^{\{BB,\bar{B}B\}}$ and $G^{\bar{\psi}\psi}$ (listed in Appendix E.2.1) they will not contribute to any quantum correction since none of the vertex expressions ($V^{3A}_{\rho\sigma\tau}, V^{4A}_{\rho\sigma\tau\epsilon}$, and $V^{\bar{c}Ac}_{\mu}$ being listed in Appendix E.2.2) connects either of these to the gauge field. At this point we note a remarkable similarity of the Feynman rules of the BRSW model, and the respective expressions of the naïve implementation of NCQED in Ref. [52]. The quadratic divergence for $k \to 0$ in the ghost propagator (5.7) is typical for the Landau gauge $\alpha \to 0$. Alternatively, as has been done in Section 3.1 and Ref. [62] we could add a damping factor to the gauge fixing term $b(\partial A)$ and the ghost sector $\bar{c}\partial_\mu D^\mu c$. However, these damping insertions would inevitably appear in vertex expressions with an inverse power relative to the respective propagators and, thus, cancel each other. Moreover, these factors contribute to UV divergences at higher loop orders, and are omitted, hence.

The gauge boson two point function (5.6) fulfills all requirements which have been stated at the beginning of this chapter. It is finite in both, the IR limit $k^2 \to 0$, and the UV limit $k^2 \to \infty$. A simple analysis reveals that

$$G^{AA}_{\mu\nu}(k) \approx \begin{cases} \frac{\tilde{k}^2}{\gamma^4}\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\bar{\sigma}^4}{(\bar{\sigma}^4+\gamma^4)}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right], & \text{for } \tilde{k}^2 \to 0 \\ \frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), & \text{for } k^2 \to \infty \end{cases}, \tag{5.8}$$

where the abbreviation

$$\bar{\sigma}^4 \equiv 2\left(\sigma + \frac{\Theta^2}{4}\sigma^2\right)\gamma^4\,, \tag{5.9}$$

has been introduced for convenience[3]. Now, the form of G^{AA} should be stable under quantum corrections since it provides a suitable term $\propto \frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}$ to absorb expected divergences. This assumption will receive its confirmation later in Section 5.4 by explicit computation.

5.1.1 UV Power Counting

From the Feynman rules, it is straight forward to derive an expression for the UV power counting of the BRSW model. With the notation of Section 3.1.2 we obtain

$$\begin{aligned} L &= I_A + I_{c\bar{c}} - (V_{\bar{c}Ac} + V_{3A} + V_{4A} - 1)\,, \\ E_{c\bar{c}} + 2I_{c\bar{c}} &= 2V_{\bar{c}Ac}\,, \\ E_A + 2I_A &= 3V_{3A} + 4V_{4A} + V_{\bar{c}Ac}\,, \end{aligned} \tag{5.10}$$

and counting the UV powers of respective Feynman rules we have

$$d_\gamma = 4L - 2I_A - 2I_{c\bar{c}} + V_{3A} + V_{4A} + V_{\bar{c}Ac}\,. \tag{5.11}$$

This system of equations can be resolved by eliminating the I_ϕ and V_ϕ to yield

$$d_\gamma = 4 - E_A - E_{c\bar{c}}\,, \tag{5.12}$$

which, again, shows remarkable agreement with the respective relations for the naïve implementation of non-commutative $U_\star(1)$. Indeed, none of the auxiliary fields or respective parameters influences the power counting[4].

[3] Note that this requires the property $\tilde{k}^2 = \Theta^2 k^2$ which follows from the special block-diagonal form of Θ, as has been introduced in Section 1.15. Moreover, since $\Theta^2 = \Theta_{\mu\rho}\Theta_{\rho\nu} = \delta_{\mu\nu}$, we have indeed $\tilde{k}^2 \equiv k^2$.

[4] In comparison, the results of respective relations (3.11a) and (3.42b) for previous models are effectively reduced by the number of external legs of auxiliary fields and/or the parameter of the breaking (respectively damping) term.

5.1.2 Symmetries

Although, after the discussion of Section 4.3 it is questionable if the AR procedure is applicable on deformed spaces at all, we may state the set of symmetries, obeyed by the BRSW model. The Slavnov-Taylor identity describing the BRST symmetry content of the model is given by

$$\mathcal{B}(S) = \int \mathrm{d}^4x \left(\frac{\delta S}{\delta \Omega^A_\mu} \frac{\delta S}{\delta A_\mu} + \frac{\delta S}{\delta \Omega^c} \frac{\delta S}{\delta c} + b \frac{\delta S}{\delta \bar{c}} \right) = 0\,, \tag{5.13}$$

from which one derives the linearized Slavnov-Taylor operator

$$\mathcal{B}_S = \int \mathrm{d}^4x \left(\frac{\delta S}{\delta \Omega^A_\mu} \frac{\delta}{\delta A_\mu} + \frac{\delta S}{\delta A_\mu} \frac{\delta}{\delta \Omega^A_\mu} + \frac{\delta S}{\delta \Omega^c} \frac{\delta}{\delta c} + \frac{\delta S}{\delta c} \frac{\delta}{\delta \Omega^c} + b \frac{\delta}{\delta \bar{c}} \right)\,. \tag{5.14}$$

Furthermore we have the gauge fixing condition

$$\frac{\delta S}{\delta b} = \partial^\mu A_\mu = 0\,, \tag{5.15}$$

the ghost equation

$$\mathcal{G}(S) = \partial_\mu \frac{\delta S}{\delta \Omega^A_\mu} + \frac{\delta S}{\delta \bar{c}} = 0 \tag{5.16}$$

and the antighost equation

$$\bar{\mathcal{G}}(S) = \int \mathrm{d}^4x\, \frac{\delta S}{\delta c} = 0\,. \tag{5.17}$$

The identity associated to the BRST doublet structure of the auxiliary fields is given by

$$\mathcal{U}^{(1)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \left(\bar{B}_{\alpha\beta} \frac{\delta S}{\delta \bar{\psi}_{\mu\nu}} + \psi_{\mu\nu} \frac{\delta S}{\delta B_{\alpha\beta}} + J_{\mu\nu\rho\sigma} \frac{\delta S}{\delta Q_{\alpha\beta\rho\sigma}} + \bar{J}_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta \bar{Q}_{\mu\nu\rho\sigma}} \right) = 0\,, \tag{5.18}$$

and we finally also have the symmetries $\mathcal{U}^{(0)}$ and $\tilde{\mathcal{U}}^{(0)}$:

$$\mathcal{U}^{(0)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \left[B_{\alpha\beta} \frac{\delta S}{\delta B_{\mu\nu}} - \bar{B}_{\mu\nu} \frac{\delta S}{\delta \bar{B}_{\alpha\beta}} + J_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta J_{\mu\nu\rho\sigma}} - \bar{J}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \bar{J}_{\alpha\beta\rho\sigma}} \right] = 0\,, \tag{5.19a}$$

and

$$\tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \left[\psi_{\alpha\beta} \frac{\delta S}{\delta \psi_{\mu\nu}} - \bar{\psi}_{\mu\nu} \frac{\delta S}{\delta \bar{\psi}_{\alpha\beta}} + Q_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta Q_{\mu\nu\rho\sigma}} - \bar{Q}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \bar{Q}_{\alpha\beta\rho\sigma}} \right] = 0\,. \tag{5.19b}$$

The symmetry operators $\mathcal{U}^{(0)}$ and $\tilde{\mathcal{U}}^{(0)}$ may be combined to the operator $\mathcal{Q}$ [119] (describing the reality of the action) as

$$\mathcal{Q} \equiv \delta_{\alpha\mu}\delta_{\beta\nu} \left(\mathcal{U}^{(0)}_{\alpha\beta\mu\nu} + \tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu} \right)\,, \tag{5.20}$$

which obviously also generates a symmetry of the action (5.3). Note, that in contrast to the model with BRST doublets of Section 3.2 we do not find a $\mathcal{U}^{(2)}$ symmetry here, which is a consequence of omitting the non-bi-linear part of the field strength tensor in the soft breaking term of the action.

Having defined the operators $\mathcal{B}_S$, $\bar{\mathcal{G}}$, $\mathcal{Q}$ and $\mathcal{U}^{(1)}$ we may derive the following graded commutators:

$$\begin{aligned}
&\{\bar{\mathcal{G}},\bar{\mathcal{G}}\} = 0\,, && \{\mathcal{B}_S,\mathcal{B}_S\} = 0\,, && \{\bar{\mathcal{G}},\mathcal{B}_S\} = 0\,,\\
&[\bar{\mathcal{G}},\mathcal{Q}] = 0\,, && [\mathcal{Q},\mathcal{Q}] = 0\,, && \left\{\bar{\mathcal{G}},\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0\,,\\
&\left\{\mathcal{B}_S,\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0\,, && \left\{\mathcal{U}^{(1)}_{\mu\nu\alpha\beta},\mathcal{U}^{(1)}_{\mu'\nu'\alpha'\beta'}\right\} = 0\,, && \left[\mathcal{U}^{(1)}_{\mu\nu\alpha\beta},\mathcal{Q}\right] = 0\,,\\
&[\mathcal{B}_S,\mathcal{Q}] = 0\,,
\end{aligned} \tag{5.21}$$

which means these symmetry operators form a closed algebra.

5.2 Vacuum Polarization

The Feynman rules (5.6),(5.7), and (E.25a)–(E.25c) give rise to the three 'classical' graphs in Fig. 5.1 contributing to the vacuum polarization $\Pi_{\mu\nu}(p)$.

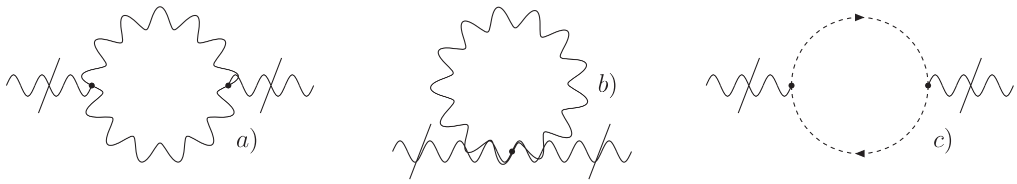


Figure 5.1: One loop corrections to the gauge boson propagator.

As already described in Section 3.1.3, we are interested here in the behavior for small external momenta p. This rectifies to expand the integrands according to Eqn. (3.17). In addition, since both, UV and IR divergences, originate from the high momentum limit of the integrated inner momentum k, the calculations can be simplified by using the propagator (5.8), limited for high momenta, i.e. $k \to \infty$.

Computation of the expressions corresponding to the Feynman graphs in Fig. 5.1 with the help of the Mathematica® package `VectorAlgebra` (see Appendix G.3; intermediate results are given in Appendix E.3.1) finally leads to

$$\Pi_{\mu\nu}(p) = \frac{2g^2}{\pi^2\varepsilon^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2} + \frac{13g^2}{3(4\pi)^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\left(\Lambda^2\right) + \text{finite terms}\,, \tag{5.22}$$

where Λ denotes an ultraviolet cutoff, and 'finite terms' collects contributions being finite in the (simultaneous) limits $\Lambda \to \infty$ and $\tilde{p}^2 \to 0$, respectively. The result meets our expectations in so far as it is quadratically IR divergent (with a subleading logarithmic UV divergence) and contains the Θ-contracted transversal tensor structure $\tilde{p}_\mu\tilde{p}_\nu$ in combination with the dimensionful parameter ε.

5.3 Vertex Corrections

Using the techniques described for the vacuum polarization in Appendix E.3.1, and simplifying the computations according to the discussion in Appendix E.3.2, we are able to calculate the one loop corrections to $V^{3A}_{\mu\nu\rho}(p_1,p_2,p_3)$ corresponding to the graphs depicted in Fig. 5.2.

Collecting terms after integration, and taking into account the symmetry factors $s_a = 1$, $s_b = \frac{1}{2}$, and $s_c = -2$, we obtain the following logarithmic UV divergence

$$\Gamma^{3A,\mathrm{UV}}_{\mu\nu\rho}(p_1,p_2,p_3) = \frac{ig^3}{\pi^2}\,\mathrm{K}_0\sqrt{\frac{M^2}{\Lambda^2}}\Bigg[\sin\frac{\varepsilon p_1\theta p_2}{2} \\ \Big((p_{2,\rho} - p_{1,\rho})\,\delta_{\mu\nu} + (p_{1,\nu} - p_{3,\nu})\,\delta_{\mu\rho} + (p_{3,\mu} - p_{2,\mu})\,\delta_{\nu\rho}\Big)\Bigg]\,, \tag{5.23}$$

where due to momentum conservation $p_3 = -p_1 - p_2$. Note that the planar result $\Gamma^{3A,\mathrm{UV}}_{\mu\nu\rho}(p_1,p_2,p_3)$ is contributed solely by graph a) of Fig. 5.2, as being discussed in Appendix E.3.2.
In the same way, the non-planar part can be computed. However, the respective results are much too large to be printed in their explicit form. Instead, we shall only discuss the types of

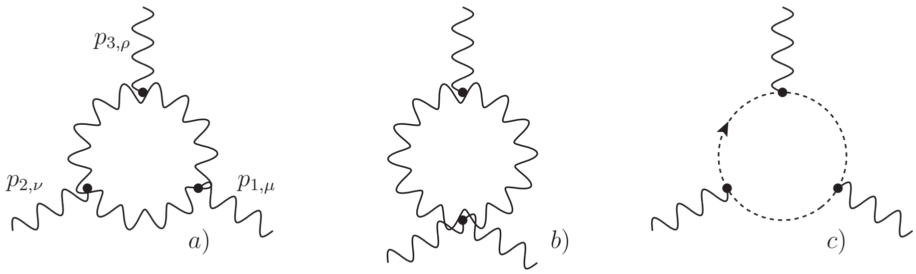


Figure 5.2: One loop corrections to the gauge boson vertex $V^{3A}_{\mu\nu\rho}(p_1, p_2, p_3)$. Conventions for external momenta and indices are defined in a).

divergences, and respective counter terms arising from them. According to the power counting formula (5.12) we expect a linear IR divergence, leaving several options for possible contractions. In fact, we obtain similar terms to those given the literature [105–107],

$$\Gamma^{3A,\mathrm{IR}}_{\mu\nu\rho}(p_1, p_2, p_3) = \left\{ \frac{\mathrm{i}g^3}{\varepsilon} \frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{(\tilde{p}_i^2)^2}, \frac{\tilde{p}_\alpha \delta_{\beta\gamma}}{\varepsilon \tilde{p}^2} \right\}, \text{ with } i \in \{1, 2, 3\}, \text{ and } \alpha \neq \beta \neq \gamma \in \{\mu, \nu, \rho\} \tag{5.24}$$

which give rise to the counter terms

$$\left\{ g^3 A_\mu A_\nu \frac{\widetilde{\partial}_\mu \widetilde{\partial}_\nu \widetilde{\partial}_\rho}{\varepsilon \widetilde{\square}^2} A_\rho,\, g^3 A^2 \frac{\widetilde{\partial} \cdot A}{\varepsilon \widetilde{\square}} \right\}. \tag{5.25}$$

It is important to remark that these terms do not give rise to any new interactions but solely scale the tree level function V^{3A}, (E.25a).

In addition, the Feynman rules (5.6),(5.7), and (E.25a)–(E.25c) give rise to four graphs contributing to a correction of the vertex $V^{4A}_{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4)$ depicted in Fig. 5.3. The numerous phase combinations appearing in these functions require tedious preprocessing in order to disentangle internal and external momenta. Together with the high number of permutations the total number of terms in the computations is extraordinary large. For this reason, we will consider the results (in particular for the non-planar IR divergences) in an abbreviated form, restricting ourselves to categories or types of terms without specifying all factors. The planar contribution of the four point graphs (which is given explicitely in Eqn. (E.34), Appendix E.3.4), may basically be deduced from the respective result obtained for Γ^{3A}. This becomes clear from the F^2

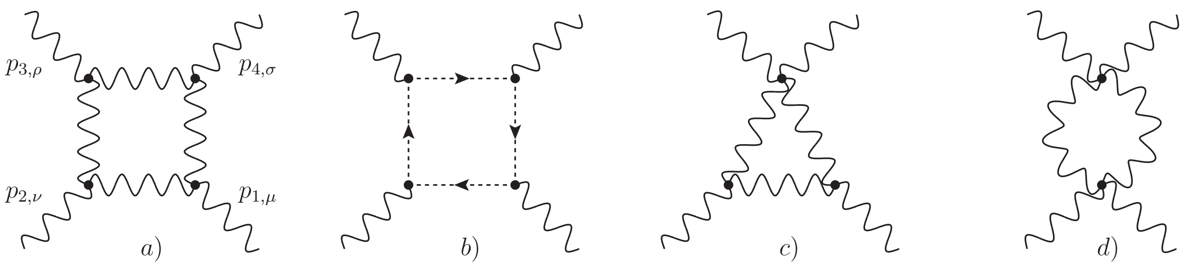


Figure 5.3: One loop corrections to the gauge boson vertex $V^{4A}_{\mu\nu\rho\rho}(p_1, p_2, p_3, p_4)$. Conventions for external momenta and indices are defined in a).

term of the action, which contains the field polynomials

$$\mathrm{i}g\,[A_\mu, A_\nu]\,\partial_\mu A_\nu\,, \text{ and } -g^2\,[A_\mu, A_\nu]^2\,.$$

The first term is the source for the vertex V^{3A}, while the latter corresponds to V^{4A}, and gauge invariance is only guaranteed for the particular combination of factors appearing in the tree level action $S^{(0)}$, i.e. g and g^2 as prefactors of the three and four point vertices, respectively. In the renormalized action $S^{(1)}_{\text{ren}}$ we have, respectively,

$$\mathrm{i}g_r \left[A^r_\mu, A^r_\nu\right] \partial_\mu A^r_\nu \,, \text{ and } -g_r^2 \left[A^r_\mu, A^r_\nu\right]^2 \,,$$

where we have anticipated that a wave function renormalization exists, which gives rise to a renormalized gauge field A^r_μ,

$$A^r_\mu = Z_A^{-1} A_\mu \,. \tag{5.26}$$

The vertex correction Γ^{3A} has been computed with unrenormalized fields A, and therefore gives rise to a counter term of the form

$$g(Z_g - 1) \left[A_\mu, A_\nu\right] \partial_\mu A_\nu \,.$$

But in the renormalized action we would write only renormalized quantities, i.e. a term of the form

$$g_r \left[A^r_\mu, A^r_\nu\right] \partial_\mu A^r_\nu \,.$$

Hence, in order to retain equivalence between these two forms, we have to define the coupling constant with one loop corrections as

$$g Z_g [A_\mu, A_\nu] \partial_\mu A_\nu = g Z_g Z_A^3 [A^r_\mu, A^r_\nu] \partial_\mu A^r_\nu =: g_r [A^r_\mu, A^r_\nu] \partial_\mu A^r_\nu \,, \tag{5.27}$$

where Z_g denotes the correction to the three-photon vertex. Furthermore, writing the renormalized vertices $V^{nA}_{\text{ren.}}$, $n \in \{3, 4\}$ at one loop level as

$$\begin{aligned}
V^{3A}_{\text{ren.}} &= V^{3A,(0)} - \Gamma^{3A,\text{UV}} \cong g \left(1 - \frac{g^2}{\pi^2} \ln \Lambda\right) \left(\frac{V^{3A,(0)}}{g}\right) &&=: g Z_g \left(\frac{V^{3A,(0)}}{g}\right) \\
& &&= g Z_g Z_A^3 \left(\frac{V^{3A}_{\text{ren.}}}{g_r}\right) = g_r \left(\frac{V^{3A}_{\text{ren.}}}{g_r}\right) \,, \\
V^{4A}_{\text{ren.}} &= V^{4A,(0)} - \Gamma^{4A,\text{UV}} \cong g^2 \left(1 - g^2 f_{4A} \ln \Lambda\right) \left(\frac{V^{4A,(0)}}{g^2}\right) &&=: g^2 Z_{4A} \left(\frac{V^{4A,(0)}}{g^2}\right) \\
& &&= g^2 Z_{4A} Z_A^4 \left(\frac{V^{4A}_{\text{ren.}}}{g_r^2}\right) = g_r^2 \left(\frac{V^{4A}_{\text{ren.}}}{g_r^2}\right) \,,
\end{aligned}$$

where we have written the prefactor of the correction to the tree level quantity $V^{4A,(0)}$ in general form f_{4A}, and the respective constant is Z_{4A}. Note, that the sign '$\cong$' is due to applying the approximation (F.24) for the Bessel function in Eqn. (5.23). In the light of the above discussion we have to demand,

$$\left(g Z_g Z_A^3\right)^2 \equiv g^2 Z_{4A} Z_A^4 \quad \Rightarrow \quad Z_{4A} = \left(Z_g Z_A\right)^2 \,.$$

And the correction to $V^{4A,(0)}$, i.e. has to be given by[5]

$$\begin{aligned}
\Gamma^{4A,\text{UV}}_{\mu\nu\rho}(p_1, p_2, p_3) &\cong f_{4A} V^{4A,(0)} \\
&= \frac{1}{13} \left(\frac{48}{\pi^2} + \frac{1225}{13 g^2 \ln \Lambda - 48\pi^2}\right) V^{4A,(0)} \,.
\end{aligned} \tag{5.28}$$

[5]For explicit definitions of Z_g and Z_A see Eqn. (5.45) below.

Note that this result, in fact, is not exact since the correction to V^{3A} is known only for one loop order and, additionally, has been approximated as series in g. However, in leading order, the correction to to V^{3A}, as computed for the graphs in Fig. 5.3 should obey (5.28).

Regarding the IR divergent non-planar part, we follow the same strategy as above for the three boson vertex. Due to the enormous number of terms ($\approx 41 \times 10^3$) in the respective result for $\Gamma^{4A,\mathrm{IR}}$ we restrict ourselves to analyzing types of divergences. From the power counting formula we expect at most logarithmic singularities. In fact, these are

$$\Gamma^{4A,\mathrm{IR}}_{\mu\nu\rho}(p_1,p_2,p_3) = g^4\,\mathrm{K}_0\sqrt{\mu^2\varepsilon^2\tilde{p}_i^2}\delta_{\alpha\beta}\delta_{\gamma\epsilon} \times \left[\cos\left(\varepsilon p_i\theta p_j\right)\cos\left(\varepsilon p_j\theta p_k\right)\cos\left(\varepsilon p_i\theta p_k\right)\right] \,, \qquad (5.29)$$
$$\text{with } \{\alpha\beta,\gamma,\epsilon\} \in \{\mu,\nu,\rho,\sigma\}, \text{ and } \{i,j,k\} \in \{1,..4\}\,.$$

Comparing to similar results in the literature [45] it is likely[6] that these divergences are eliminated by cancellations of the prefactors. However, this is (at the time of publication of this thesis) not known, and it has to be remarked that due to the high number of terms a reliable statement will require a considerable amount of computing time. Therefore, the reader should be referred to a forthcoming publication [66].
Similarly, it has to be mentioned that in the computation of the vertex corrections there is, at the moment, an uncertainty. The reason lies in the approximation (E.31), which has been applied to the denominators of integrands and led to the results stated in this thesis. One may, alternatively, consider series expansions

$$\mathcal{I}(k,p_1,..p_n) \approx \mathcal{I}(k,p_1,..p_n)\big|_{p_i\to 0\,\forall i} + \sum_{i=1}^{n}\left[\partial_{p_i}\mathcal{I}(k,p_1,..p_n)\right]_{p_j\to 0\,\forall j} p_i$$

which, in principle, equals the zeroth and first order of Eqn. (E.30). It has to be checked carefully that the limits $p_i \to 0\,\forall i$ commute (see Appendix E.3.2 for a more thorough discussion). Otherwise this series would not be well defined, and wrong. Hence, all of the prefactors given in this section, and in Section 5.5 represent the current status but may eventually change. Again, for a decisive answer the reader should refer to Ref. [66].

For the sake of completeness, we should also mention that there exist two graphs potentially contributing linearly divergent terms to the vertex $V^{\bar{c}Ac}$, as given in Appendix E.3.3. However, these are completely finite and not considered, thus.

5.4 One-loop Renormalization

For the renormalization of the two point function we choose the same ansatz as in Section 3.2.5 where the series of n-loop corrections has been approximated by the recursive Eqn. (2.27) which shall be repeated at this point for clarity[7]

$$\frac{1}{\mathsf{A}+\mathsf{B}} = \frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}}\,\mathsf{B}\,\frac{1}{\mathsf{A}+\mathsf{B}} = \frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}}\,\mathsf{B}\,\frac{1}{\mathsf{A}} + \mathcal{O}(\mathsf{B}^2)\,,$$

with,

$$\frac{1}{\mathsf{A}} := G^{AA}\,, \quad \text{and } \mathsf{B} := \Pi\,.$$

[6] A counterexample is Ref. [107], where Γ^{4A} contains $1/\epsilon$ corresponding in dimensional regularization in a logarithmic divergence, as well as a term $\ln\tilde{p}^2$. It is not entirely clear how to treat these divergences in the renormalization.

[7] Note that, since the entities A and B are tensors of rank two, the notation $1/\mathsf{A}$ has to be understood to represent the expression A^{-1} fulfilling $\mathsf{A}_{\mu\rho}(\mathsf{A}_{\rho\nu})^{-1} = (\mathsf{A}^{-1})_{\mu\rho}\mathsf{A}_{\rho\nu} = \delta_{\mu\nu}$, and similarly for B^{-1} and $(\mathsf{A}+\mathsf{B})^{-1}$.

With respect to the model of Section 3.2, where the corrections of the two point functions for $\bar{B}/B$ and $\bar{\psi}/\psi$ fields have contributed to B, the BRSW model, represents a considerable simplification since the corrections to G^{AA} are solely determined by the results of the vacuum polarization in Eqn. (5.22).

In the following, the task is to find the inverse of $\mathsf{A} = \left(G^{AA}\right)^{-1}$, add up with B, and invert the result again to obtain the expression $(\mathsf{A}+\mathsf{B})^{-1}$. Recall the tree-level gauge field propagator (5.6) which is now rewritten in the form

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\mathcal{D}}\left(\delta_{\mu\nu} - (1-\alpha\mathcal{D})\frac{k_\mu k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right), \tag{5.30}$$

where we have introduced the abbreviations

$$\begin{aligned}\mathcal{D}(k) &\equiv \left(1 + \frac{\gamma^4}{(\tilde{k}^2)^2}\right), \\ \mathcal{F}(k) &\equiv \frac{1}{\tilde{k}^2}\frac{\bar{\sigma}^4}{\left(k^2 + (\bar{\sigma}^4+\gamma^4)\frac{1}{\tilde{k}^2}\right)},\end{aligned} \tag{5.31}$$

and have generalized to $\alpha \neq 0$. The latter is necessitated by the fact that the inverse of the right side of Eqn. (5.30), being required in the subsequent calculation, does not exist. In this respect it has to be remarked that the quadratic IR divergence, and the result of the renormalization have to be independent of the gauge fixing [52, 107, 109]. Therefore, in the end, we will consider the limit $\alpha \to 0$ again. For now, we demand

$$\begin{aligned}\delta_{\mu\nu} &\equiv \mathsf{A}_{\mu\rho}\mathsf{A}^{-1}_{\rho\nu} \\ &= \frac{1}{k^2\mathcal{D}}\left(\delta_{\mu\rho} - (1-\alpha\mathcal{D})\frac{k_\mu k_\rho}{k^2} - \mathcal{F}\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\right)k^2\mathcal{D}\left(\delta_{\mu\nu} + a\frac{k_\rho k_\nu}{k^2} + b\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right),\end{aligned} \tag{5.32}$$

from which we obtain by comparison of the coefficients,

$$a = \frac{1-\alpha\mathcal{D}}{\alpha\mathcal{D}}, \tag{5.33}$$

$$b = \frac{\mathcal{F}}{1-\mathcal{F}}, \tag{5.34}$$

and finally arrive at the tree level two point vertex function

$$\Gamma^{AA,\text{tree}}_{\mu\nu}(k) = \left(G^{-1}_{AA}\right)_{\mu\nu}(k) = k^2\mathcal{D}\left(\delta_{\mu\nu} + \left(\frac{1}{\alpha\mathcal{D}} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}^4}{k^2\tilde{k}^2\mathcal{D}}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right). \tag{5.35}$$

The corrections (representing part B) are given by the results from Section 5.2, and can be rewritten in the form

$$\Gamma^{AA,\text{corr.}}_{\mu\nu}(k) = \mathit{\Pi}_1\frac{\tilde{k}_\mu\tilde{k}_\nu}{(\tilde{k}^2)^2} + \mathit{\Pi}_2\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right),$$

with

$$\mathit{\Pi}_1 = \frac{2g^2}{\pi^2\varepsilon^2}, \qquad \text{and } \mathit{\Pi}_2 = \frac{13g^2}{3(4\pi)^2}\ln\Lambda^2, \tag{5.36}$$

and Λ is an ultraviolet cutoff (see Appendix A.2.1, discussion on page 91). Hence, we can immediately write down the two point vertex function with one loop corrections[8].

$$\begin{aligned}\Gamma^{AA,\mathrm{ren}}_{\mu\nu}(k) &= \Gamma^{AA,\mathrm{tree}}_{\mu\nu}(k) - \Gamma^{AA,\mathrm{corr.}}_{\mu\nu}(k)\\ &= k^2(\mathcal{D}-\Pi_2)\left(\delta_{\mu\nu} + \left(\frac{1}{\alpha(\mathcal{D}-\Pi_2)} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar\sigma^4 - \Pi_1}{k^2\tilde k^2(\mathcal{D}-\Pi_2)}\frac{\tilde k_\mu \tilde k_\nu}{\tilde k^2}\right), \qquad (5.37)\end{aligned}$$

The next task in our agenda is to incorporate the additional terms in modified parameters of the theory, which shall enable us to rewrite the renormalized inverse propagator in exactly the same form as the tree level expression (5.35). If this can be achieved, we have shown stability of the theory with respect to one loop vacuum polarization corrections.
In Eqn. (5.37) the first intuitive step is motivated from the form of the third term, where stability can only be retained if

$$\bar\sigma^4_{\mathrm{r}} \propto \frac{\bar\sigma^4 - \Pi_1}{\mathcal{D}_{\mathrm{r}}}.$$

The denominator of this expression appears also as an overall prefactor in (5.37), whereas in the original form (5.35) we find $k^2\mathcal{D}$. Hence, we introduce a wave function renormalization $A_\mu \to A^{\mathrm{r}}_\mu = Z_A^{-1}A_\mu$. Now, the prefactor is rewritten as

$$D + \Pi_2 =: \mathcal{D}_{\mathrm{r}} Z_A^{-2},$$

and after inserting the explicit expressions we demand that the original form is reobtained by inserting a renormalized γ_{r},

$$\begin{aligned}1 + \frac{\gamma_{\mathrm{r}}^4}{k^2\tilde k^2} + \Pi_2 &= (1+\Pi_2)\left(1 + \frac{\gamma^4}{k^2\tilde k^2}\right)\\ &= 1 + \frac{\gamma^4}{k^2\tilde k^2}(1+\Pi_2) + \Pi_2\end{aligned}$$

which yields, by comparing coefficients,

$$\gamma_{\mathrm{r}}^4 = \frac{\gamma^4}{1+\Pi_2} = \frac{\gamma^4}{Z_A^{-2}}.$$

We have thus obtained three renormalized quantities, which are collected in Eqn. (5.38). Note, that the fact that all of these parameters are independent of the momentum, reflects (and proves) stability of the action under all corrections considered up to now,

$$\begin{aligned}Z_A &= \frac{1}{\sqrt{1-\Pi_2}},\\ \gamma_r^4 &= \gamma^4 Z_A^2,\\ \bar\sigma_r^4 &= \left(\bar\sigma^4 - \Pi_1\right) Z_A^2, \qquad (5.38)\end{aligned}$$

and the one-loop two-point vertex function is cast into the same form as its tree-level counter part, i.e.

$$\Gamma^{AA,\mathrm{ren}}_{\mu\nu}(k) = \frac{k^2\mathcal{D}_r}{Z_A^2}\left(\delta_{\mu\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar\sigma_r^4}{k^2\tilde k^2\mathcal{D}_r}\frac{\tilde k_\mu\tilde k_\nu}{\tilde k^2}\right),$$

[8] Regarding the notation, renormalized quantities are indicated by a superscript 'ren.' or a subscript 'r'.

$$\mathcal{D}_r(k) \equiv \left(1 + \frac{\gamma_r^4}{(\tilde{k}^2)^2}\right) . \tag{5.39}$$

Finally, we may also write $\bar{\sigma}_r$ in terms of the renormalized σ_r:

$$\begin{aligned} \bar{\sigma}_r^4 &= 2\left(\sigma_r + \frac{\Theta^2}{4}\sigma_r^2\right)\gamma^4 Z_A^2 , \\ \sigma_r &= -\frac{2}{\Theta^2} \pm 2\sqrt{\left(1 + \frac{\Theta^2}{2}\sigma\right)^2 - \frac{g^2\Theta^2}{\pi^2\gamma^4\varepsilon^2}} . \end{aligned} \tag{5.40}$$

For the sake of completeness, we also write the renormalized propagator in Landau gauge $\alpha \to 0$,

$$\begin{aligned} G_{\mu\nu}^{AA,\mathrm{ren}}(k) &= \frac{Z_A^2}{k^2\mathcal{D}_r}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \mathcal{F}_r\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right) , \\ \mathcal{F}_r &\equiv \frac{1}{\tilde{k}^2}\frac{\bar{\sigma}_r^4}{\left(k^2 + (\bar{\sigma}_r^4 + \gamma_r^4)\frac{1}{\tilde{k}^2}\right)} . \end{aligned} \tag{5.41}$$

In a renormalizable theory, the form of the action should be invariant under quantum corrections[9], and the parameters are fixed by normalization conditions. In the following, we will provide such conditions for the two-point vertex function Γ^{AA} of the gauge boson.

$$\Gamma_{\mu\rho}^{AA} = \Gamma^{AA,T}(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}) + (\Gamma^{AA,NC})\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2} + (\Gamma^{AA,L})\frac{k_\mu k_\rho}{k^2} .$$

This function allows for a splitting into transversal, longitudinal, and an as well transversal non-commutative part. Accordingly, we identify

$$\Gamma^{AA,T} = k^2\mathcal{D} , \qquad (\Gamma^{AA,L}) = \frac{k^2}{\alpha} , \qquad \Gamma^{AA,NC} = \frac{\bar{\sigma}^4}{\tilde{k}^2} , \tag{5.42}$$

and finally find the following *renormalization conditions*:

$$\left.\frac{(\tilde{k}^2)^2}{k^2}\Gamma^{AA,T}\right|_{k^2=0} = \gamma^4 , \tag{5.43a}$$

$$\left.\frac{1}{2k^2}\frac{\partial(k^2\Gamma^{AA,T})}{\partial k^2}\right|_{k^2=0} = 1 , \tag{5.43b}$$

$$\left.\tilde{k}^2\Gamma^{AA,NC}\right|_{k^2=0} = \bar{\sigma}^4 , \tag{5.43c}$$

$$\left.\Gamma^{AA,L}\right|_{k^2=0} = 0 , \tag{5.43d}$$

$$\left.\frac{\partial\Gamma^{AA,L}}{\partial k^2}\right|_{k^2=0} = \frac{1}{\alpha} . \tag{5.43e}$$

In fact, these are fulfilled by the tree level action (5.3) as well as the one loop renormalized action including the constants (5.38).

[9]Stability indeed forces renormalizability, but a specific 'unstable' tree level action may receive a finite number of counter terms types via perturbative corrections, and be, after all, renormalizable nonetheless.

5.5 β Function

The result for the UV correction to the three gauge boson vertex (5.23) exhibits the same structure[10] as the tree level expression (E.25a), and is amenable for a renormalization of the coupling constant g, thus. As described above, in Eqn. (5.38) we have employed the following convention for the renormalized gauge field A^r_μ:

$$A^r_\mu = Z_A^{-1} A_\mu \,,$$

giving rise to the renormalized coupling

$$g_r = g Z_g Z_A^3 \,, \tag{5.44}$$

where Z_g denotes the correction factor to the three photon vertex V^{3A}. In the present case, we have from Eqns. (5.22), (5.38), Eqn. (5.23), and after approximation of the Bessel function K_0 for small arguments according to Eqn. (F.24),

$$Z_A = \left(1 - \frac{13g^2}{3(4\pi)^2} \ln \Lambda^2\right)^{-1/2} \,, \tag{5.45a}$$

$$Z_g = 1 - \frac{g^2}{2\pi^2} \ln \Lambda \,. \tag{5.45b}$$

Consequently, we obtain

$$g_r = g\left(1 - \frac{5}{16\pi^2} g^2 \ln \Lambda\right) + \mathcal{O}(g^5) \,. \tag{5.46}$$

In the light of the discussion on page 80 on the current uncertainty in the correct prefactor of the correction in Z_g (due to the approximation in the calculation of $\Gamma^{3A,UV}$) we shall replace the factor $5/16\pi^2$ by a general factor f_β.

According to Eqn. (4.18) the β-function is given by the logarithmic derivative of the bare coupling g with respect to the cutoff, for fixed g_r (i.e. for $\partial g_r/\partial\Lambda := 0$):

$$\beta(g,\Lambda) = \Lambda \frac{\partial g}{\partial \Lambda}\Big|_{g_r \text{ fixed}} \,, \tag{5.47}$$

$$\text{and} \quad \beta(g) = \lim_{\Lambda\to\infty} \beta(g,\Lambda) \,. \tag{5.48}$$

In order to compute $\beta(g,\Lambda)$ we differentiate Eqn. (5.46) according to Eqn. (5.47)

$$\frac{\partial g_r}{\partial \Lambda} = 0 = \Lambda \frac{\partial g}{\partial \Lambda} + 3 f_\beta g^2 \Lambda \left(\frac{\partial g}{\partial \Lambda}\right) \ln \Lambda + f_\beta g^3 \tag{5.49}$$

$$= \beta(g,\Lambda) + 3 f_\beta g^2 \beta(g,\Lambda) \ln \Lambda + f_\beta g^3 \,. \tag{5.50}$$

Since the immediate solution $\beta(g,\Lambda) = -f_\beta g^3/(1 + 3 f_\beta g^2 \ln \Lambda)$ diverges for $\Lambda \to \infty$, we try to use an approximation for small g. The ansatz

$$\beta(g,\Lambda) = -f_\beta g^3 + \mathcal{O}(g^5) \,,$$

solves Eqn. (5.50) up to $\mathcal{O}\left(g^5\right)$. Hence,

$$\beta(g) = \lim_{\Lambda\to\infty} \beta(g,\Lambda) = -f_b g^3 = -\frac{5g^3}{16\pi^2} \,, \tag{5.51}$$

[10] Note that this fact is true, independent of the final coefficients of $V^{3A}_{\text{ren.}}$ and $V^{4A}_{\text{ren.}}$, as being discussed at the end of Section 5.3, on page 81 above.

where we have reinserted the definition of f_b. The value of β is negative (at the one loop level) which indicates asymptotic freedom, as is expected for YM theory (with $U_\star(1)$ symmetry) [107]. Note, that the corresponding result for commutative QED (also $U(1)$) is positive. Hence, this is a manifestation of the deformation, altering the gauge group to be non-Abelian.

Finally, it has to be remarked that the result (5.51) vitally depends upon the prefactor of the correction $\Gamma^{3A,UV}$ of the three boson graphs. As indicated above, there exists an uncertainty which unfortunately could not be resolved at the time of publication of this thesis. A final answer will be given in Ref. [66].

5.6 Summary

The non-commutative BRSW model is an advanced gauge implementation of the damping mechanism of the transversal $1/p^2$ model by Gurau *et al.*. In comparison to preceding models [63, 65, 95], it basically features two main enhancements:

▷ Any auxiliary fields are uncoupled from the gauge sector and, hence, do not give rise to any quantum corrections.

▷ The appearance of dimensionless non-local operators is, at least at tree level, avoided by separating off a dimensionful parameter ε from the tensorial structure $\Theta_{\mu\nu}$ describing the non-commutativity of the underlying space.

Due to the first point, the form of the action (5.3) is relatively simple, and the interactions (and hence the number and type of Feynman graphs to consider in loop calculations) are exactly the same as in naïve implementations of YM theory. Therefore, it is possible to compare results of respective calculations at one loop order to those given in the literature. The second point manifests itself in the dimensionality $d_m(\widetilde{\Box}) = 2$, which precludes operators such as $(\Box\widetilde{\Box})^n$. In addition, the action features a soft breaking term, which allows to include terms being not gauge invariant without any reduction of the symmetry content of the theory. This permits the addition of a suitable counter term for the well known quadratic IR divergence showing the transversal tensor structure $\tilde{p}_\mu\tilde{p}_\nu$.

Explicit one loop calculations for the vacuum polarization in Section 5.2 have given the expected results. Similar to the scalar $1/p^2$ model, the damping mechanism does not avoid the appearance of the typical quadratic IR divergence. However, it can be absorbed in the soft breaking term, which has been constructed exactly for this purpose. Corrections to the three and four point vertex functions at one loop level (in Section 5.3) have resulted in divergences being linear in the IR region, and logarithmic in the UV cutoff. While the latter can be absorbed into a renormalization of the coupling constant g, the first ones give rise to counter terms of the type $(\widetilde{\partial}\cdot A)^3/\widetilde{\Box}^2$, and $A^2(\widetilde{\partial}\cdot A)/\widetilde{\Box}$. These do not represent additional interactions, but result in multiplicative alternations of the tree level vertex functions. Therefore, they may be included into the tree level action of the model. Finally, it may be stated that the renormalization in Section 5.4 leaves invariant the form of the action which proves stability at the one loop level. The β function of the model from Section 5.5 is negative, and hence indicates asymptotic freedom of the coupling.

The damping behaviour of the gauge boson propagator and the absence of dimensionless operators in the tree level action, as well as the explicit one loop results, give a strong indication that the non-commutative BRSW model is indeed renormalizable. According to the discussions of Chapter 4 it shall be suggested to conduct a rigorous proof by Multiscale Analysis. Due to the appearance of a soft breaking term in combination with a high degree of symmetry, being compatible with the QAP, the Gribov problem should not appear in this respect.

Appendix A

Supplementary Content to the $1/p^2$ Scalar Model

This appendix contains supplementary calculations to the scalar model being discussed in Sections 2.1–2.2. Derivations of the Feynman rules are given in Appendix A.1. Subsequently, the basic methodology of computing loop integrations is discussed in Appendix A.2. Extensions to higher loop orders can be found in Appendix A.3, and the chapter is closed by the proof of Theorem 1 in Appendix A.4.

A.1 Feynman Rules

For the purpose of demonstration of the concepts presented in Section 1.3.1 the derivation of the Feynman rules is discussed in detail at this point. The bilinear part of the action (2.4) is

$$S_{\text{bil}} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{2} \phi(-k) \left(k^2 + m^2 + \frac{a^2}{\tilde{k}^2} \right) \phi(k) \,. \tag{A.1}$$

Hence, the propagator is computed according to,

$$\begin{aligned} -j &= \frac{\delta S_{\text{bil}}(p)}{\delta \phi} \\ &= \left(k^2 + m^2 - \frac{a^2}{\tilde{k}^2} \right) \phi(k) \,, \end{aligned} \tag{A.2}$$

and

$$G(k) = -\frac{\delta \phi(k)}{\delta j} = \frac{1}{k^2 + m^2 + \frac{a^2}{\tilde{k}^2}} \,. \tag{A.3}$$

The interaction term $S^{\text{int}}[\phi] = \frac{\lambda}{4!}\phi_\star^4$ of the tree level action gives rise to one vertex. Explicitly writing out the star product, the variation reads

$$\begin{aligned} &V(k_1, k_2, k_3, k_4) \\ &\quad = -\frac{\lambda}{4!} \frac{\delta}{\delta\phi(-k_1)\delta\phi(-k_2)\delta\phi(-k_3)\delta\phi(-k_4)} \int \mathrm{d}^4 p_{1..4} \delta^4(p_1 + p_2 + p_3 + p_4) \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \\ &\qquad \times \mathrm{e}^{\frac{\mathrm{i}}{2}(p_1\theta p_2 + p_1\theta p_3 + p_1\theta p_4 + p_2\theta p_3 + p_2\theta p_4 + p_3\theta p_4)} \end{aligned}$$

$$
\begin{aligned}
= -\frac{\lambda}{4!}\int \mathrm{d}^4 p_{1..4} \Big[& \delta(p_1+k_1)\big[\delta(p_2+k_2)\big(\delta(p_3+k_3)\delta(p_4+k_4)+\delta(p_4+k_3)\delta(p_3+k_4)\big) \\
& + \delta(p_3+k_2)\,(\ldots) + \delta(p_4+k_2)\,(\ldots)\,\big] \\
& + \delta(p_2+k_1)\big[\ldots\big] + \delta(p_3+k_1)\big[\ldots\big] + \delta(p_4+k_1)\big[\ldots\big]\Big] \\
& \times \mathrm{e}^{\frac{\mathrm{i}}{2}(p_1\theta p_2+p_1\theta p_3+p_1\theta p_4+p_2\theta p_3+p_2\theta p_4+p_3\theta p_4)}\,.
\end{aligned}
$$

Explicit evaluation of the δ functions with the exponential factor yields 24 terms which can be collected by applying the rule Eqn. (F.21) systematically. The remaining phase evaluates to 1 by utilization of the total momentum conservation $\delta^4(k_1+k_2+k_3+k_4)$. Finally, the result can be written as,

$$
\begin{aligned}
V(k_1,k_2,k_3,k_4) = & -\frac{\lambda}{3}\delta^4(k_1+k_2+k_3+k_4) \\
& \times\left[\cos\frac{k_1\theta k_2}{2}\cos\frac{k_3\theta k_4}{2}+\cos\frac{k_1\theta k_3}{2}\cos\frac{k_2\theta k_4}{2}+\cos\frac{k_1\theta k_4}{2}\cos\frac{k_2\theta k_3}{2}\right],
\end{aligned}
\tag{A.4}
$$

where the prefactor consists of the $(4!)^{-1}$ from $S^{\mathrm{int}}[\phi]$, a factor 4 from two cosinii, and a factor 2 since each term in Eqn. (A.4) appears twice in the derivation.

A.2 One Loop Calculations

A.2.1 Tadpole Graph

The integral in Eqn. (2.8) can explicitly be written as

$$
\Pi(p) = \frac{1}{2}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4 k}{(2\pi)^4}\,G(k)V(p,-k,k,-p) \tag{A.5}
$$

$$
\begin{aligned}
&= -\frac{\lambda}{6}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{2+\cos\frac{k\theta p}{2}}{k^2+m^2+\frac{a^2}{k^2}} \\
&= \Pi^{\mathrm{p}}(p)+\Pi^{\mathrm{np}}(p)\,,
\end{aligned}
\tag{A.6}
$$

where a symmetry factor $\frac{1}{2}$ has been included and the $\cos^2\frac{k\theta p}{2}$ coming from the vertex is transformed by Eqn. (F.18). The expression is then split into a planar and a non-planar part, denoted by Π^{p} and Π^{np} respectively. We start here with the *non-planar* one.
The first step is, to recast the denominator of Π^{np} into a form featuring only positive powers of the integrand k, i.e.

$$
\left(k^2+m^2+\frac{a'^2}{k^2}\right)^{-1} = k^2\left[\left(k^2+\frac{m^2}{2}\right)^2-\underbrace{\left(\frac{m^4}{4}-a'^2\right)}_{:=M^4}\right]^{-1}, \tag{A.7}
$$

which can be simplified further (equivalence can easily be seen when explicitly writing out the sum) by splitting the denominator,

$$
=\frac{1}{2}\sum_{\zeta=\pm1}\frac{1+\zeta\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta M^2}\,. \tag{A.8}
$$

The full integral then reads,

$$\begin{aligned}
\Pi^{\mathrm{np}}(p) &= -\frac{\lambda}{24}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{\eta,\zeta=\pm1}\frac{1+\zeta\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta M^2}\,\mathrm{e}^{\mathrm{i}\eta k\theta p}\\
&= -\frac{\lambda}{24}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\int\limits_0^{\infty}\mathrm{d}\alpha\sum_{\eta,\zeta=\pm1}\left(1+\zeta\frac{m^2}{2M^2}\right)\mathrm{e}^{-\alpha\left(k-\frac{\mathrm{i}\eta\tilde{p}}{2\alpha}\right)^2-\frac{\eta^2\tilde{p}^2}{4\alpha}-\alpha\left(\frac{m^2}{2}+\xi M^2\right)}\\
&= -\frac{\lambda}{24}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{\eta,\zeta=\pm1}\int\limits_0^{\infty}\mathrm{d}\alpha\frac{\pi^2}{\alpha^2}\left(1+\zeta\frac{m^2}{2M^2}\right)\mathrm{e}^{-\frac{\eta^2\tilde{p}^2}{4\alpha}-\alpha\left(\frac{m^2}{2}+\xi M^2\right)}\\
&\overset{(F.11)}{=} -\frac{\lambda}{6}\frac{1}{16\pi^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m^2}{2M^2}\right)\sqrt{\frac{2m^2+4\xi M^2}{\tilde{p}^2}}\,\mathrm{K}_1\sqrt{\tilde{p}^2\left(\frac{m^2}{2}+\xi M^2\right)}\,. \qquad \text{(A.9)}
\end{aligned}$$

Applying Eqn. (F.24) to expand the Bessel function for small p yields,

$$\begin{aligned}
\Pi^{\mathrm{np}}(p) \underset{|p|\to0}{=} -\frac{\lambda}{6(4\pi)^2}\Bigg[&\frac{4}{\tilde{p}^2}+m^2\ln\tilde{p}^2+m^2\left(\gamma_E-\frac{1}{2}-2\ln2+\ln a\right)\\
&+\left(M^2+\frac{m^4}{4M^2}\right)\ln\sqrt{\frac{m^2+2M^2}{m^2-2M^2}}\Bigg]+\mathcal{O}\left(\tilde{p}^2\right)\,. \qquad \text{(A.10)}
\end{aligned}$$

The *planar part* is described by the integral

$$\begin{aligned}
\frac{\lambda}{6}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{2}{k^2+m^2+\frac{a^2}{k^2}} &= \frac{\lambda}{12}\sum_{\xi=\pm1}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1+\xi\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\xi M^2}\\
&= \frac{\lambda}{6}\sum_{\xi=\pm1}\int\limits_{\mathbb{R}^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\int\limits_0^{\infty}\mathrm{d}\alpha\left(1+\xi\frac{m^2}{2M^2}\right)\mathrm{e}^{-\alpha\left(k^2+\frac{m^2}{2}+\xi M^2\right)}\\
&= \frac{\lambda}{6(4\pi)^2}\sum_{\xi=\pm1}\int\limits_0^{\infty}\mathrm{d}\alpha\left(1+\xi\frac{m^2}{2M^2}\right)\frac{1}{\alpha^2}\mathrm{e}^{-\frac{1}{4\Lambda^2\alpha}-\alpha\left(\frac{m^2}{2}+\xi M^2\right)},
\end{aligned}$$

where a UV cutoff $\Lambda\to\infty$ has been introduced to regularize the integral,

$$\begin{aligned}
&\overset{(F.11)}{=} \frac{\lambda}{6(4\pi)^2}\sum_{\xi=\pm1}\left(1+\xi\tfrac{m^2}{2M^2}\right)\sqrt{4\Lambda^2\left(\tfrac{m^2}{2}+\xi M^2\right)}\,\mathrm{K}_1\sqrt{\frac{1}{\Lambda^2}\left(\tfrac{m^2}{2}+\xi M^2\right)},\\
&\overset{(F.24)}{\approx} \frac{\lambda}{3(4\pi)^2}\Bigg[4\Lambda^2-m^2\ln\Lambda^2+m^2\left(\gamma_E-\frac{1}{2}-2\ln2+\ln a\right)\\
&\qquad+\left(M^2+\frac{m^4}{4M^2}\right)\ln\sqrt{\frac{m^2+2M^2}{m^2-2M^2}}\Bigg]+\mathcal{O}\left(\mu^2\right)\,.
\end{aligned}$$

The latter result corresponds to Eqn. (2.12) in the main text (with an overall $(-)$, which has been omitted here).

Obviously, the results for planar and non-planar parts can be converted into one another by the operation $\tilde{p}^2\leftrightarrow\frac{1}{\Lambda^2}$. This general principle has been used in the literature [29, 30, 147] to define a so called 'effective cutoff'

$$\Lambda^2_{\text{eff}} := \frac{1}{\tilde{p}^2+\frac{1}{\Lambda^2}} \to \begin{cases}\underset{\Lambda^2\to\infty}{\longrightarrow}\frac{1}{\tilde{p}^2}\,, & \text{non-planar}\\ \underset{\tilde{p}^2\to0}{\longrightarrow}\Lambda^2\,, & \text{planar}\end{cases}\,, \qquad \text{(A.11)}$$

which allows to generalize the results, independent of planarity. Indeed, by replacing $\tilde{p}^2 \to 1/\Lambda_{\text{eff}}^2$ in Eqn. (A.9) the respective planar or non-planar results can be obtained by taking the limits given in Eqn. (A.11). Generally, throughout this work the notation for cutoffs is as follows: A divergent integrand $\mathcal{I}_{\text{div}}(\alpha)$ is multiplicatively regularized to a convergent integrand $\mathcal{I}_{\text{reg.}}(\alpha)$ according to,

$$\int \mathrm{d}\alpha\, \mathcal{I}_{\text{div}}(\alpha) \overset{\text{regularization}}{\longrightarrow} \int \mathrm{d}\alpha\, \mathcal{I}_{\text{reg}}(\alpha) := \lim_{\substack{\mu^2\to 0\\ \Lambda^2\to\infty}} \int \mathrm{d}\alpha\, \mathcal{I}_{\text{div}}(\alpha) \mathrm{e}^{-\frac{1}{4\Lambda^2\alpha}-\mu^2\alpha}\,. \tag{A.12}$$

Since the generic parameter α mostly corresponds to the inverse of a momentum, $\mu \to 0$ regularizes the IR region, and $\Lambda \to \infty$ represents a UV *cutoff*. These factors are applied where necessary. It has to be remarked that the arbitrary factor 4 accompanying Λ^{-2} appears due to the fact that the $1/\tilde{p}^2$ divergence in non-planar results always appears with a similar factor, stemming from the completion of squares in exponential functions, prior to Gaussian integration. Generally, with the exception of Section 2.1, regularizion of integrals will not be indicated explicitely by any additional decoration. Instead, it is understood, that all quantities depending on masses μ or cutoffs Λ *are* indeed regularized versions of the original expressions with the same name. Correspondence is always given by taking the limits as indicated above.

A.2.2 Vertex Graphs

The vertex correction consists of three graphs, as indicated in Eqn. (2.13). However, since these are just permutations of one another with respect to the external momenta, it is sufficient to calculate the graph depicted in Fig. A.1. Setting $q \equiv p_1 + p_2 = p_3 + p_4$, it corresponds (after

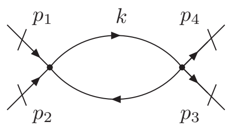


Figure A.1: One loop corrections to the vertex.

eliminating integrations over all internal momenta but k by conservation at the vertices,) to the expression,

$$\begin{aligned}
\Gamma(p_1,p_2,p_3,p_4) &= \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} V(p_1,-k,k-q,p_2)V(k,-p_3,-p_4,q-k)G(k)G(p_1+p_2-k)\\
&= \frac{\lambda^2}{9}\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\,\delta^4(p_1+p_2+p_3+p_4)\frac{1}{k^2+m^2+\frac{a^2}{k^2}}\frac{1}{(q-k)^2+m^2+\frac{a^2}{(q-k)^2}}\\
&\quad\left[\cos\tfrac{p_1\theta p_2}{2}\cos\tfrac{k\theta(p_1+p_2)}{2}+\cos\tfrac{p_2\theta k}{2}\cos\tfrac{p_1\theta(k-p_2)}{2}+\cos\tfrac{p_1\theta k}{2}\cos\tfrac{p_2\theta(k-p_1)}{2}\right]\\
&\quad\left[\cos\tfrac{p_3\theta p_4}{2}\cos\tfrac{k\theta(p_3+p_4)}{2}+\cos\tfrac{p_4\theta k}{2}\cos\tfrac{p_3\theta(k-p_4)}{2}+\cos\tfrac{p_3\theta k}{2}\cos\tfrac{p_4\theta(k-p_3)}{2}\right],
\end{aligned} \tag{A.13}$$

where use has been made of the symmetry of the cosine function with respect to the sign of its argument. In the following we will simplify the lengthy expression for the phase factor.

Expanding the multiplication yields

$$
\begin{aligned}
&\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\left(\cos\frac{(p_1+p_2)\theta k}{2}\cos\frac{(p_3+p_3)\theta k}{2}\right)\\
&+\cos\frac{p_1\theta p_2}{2}\left[\cos\frac{(p_1+p_2)\theta k}{2}\left(\cos\frac{p_3\theta k}{2}\cos\frac{(p_3-k)\theta p_4}{2}+\cos\frac{p_4\theta k}{2}\cos\frac{(p_4-k)\theta p_3}{2}\right)\right]\\
&+\cos\frac{p_3\theta p_4}{2}\left[\cos\frac{(p_3+p_4)\theta k}{2}\left(\cos\frac{p_1\theta k}{2}\cos\frac{(p_1-k)\theta p_2}{2}+\cos\frac{p_2\theta k}{2}\cos\frac{(p_2-k)\theta p_1}{2}\right)\right]\\
&+\left(\cos\frac{p_2\theta k}{2}\cos\frac{p_1\theta(k-p_2)}{2}+\cos\frac{p_1\theta k}{2}\cos\frac{p_2\theta(k-p_1)}{2}\right)\\
&\left(\cos\frac{p_4\theta k}{2}\cos\frac{p_3\theta(k-p_4)}{2}+\cos\frac{p_3\theta k}{2}\cos\frac{p_4\theta(k-p_3)}{2}\right)\,. \qquad \text{(A.14)}
\end{aligned}
$$

Using repeatedly the relations (F.18), (F.20), and $q_1+q_2=q_3+q_4$ this yields,

$$
\begin{aligned}
&\frac{1}{2}\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\frac{1}{2}\left(1+\cos(p_1+p_2)\theta k\right)\\
&+\frac{1}{4}\cos\frac{p_1\theta p_2}{2}\Big[\cos\frac{2(p_3+p_4)\theta k+p_3\theta p_4}{2}+\cos\frac{2p_3\theta k-p_3\theta p_4}{2}+\cos\frac{2p_4\theta k+p_3\theta p_4}{2}+\cos\frac{p_3\theta p_4}{2}\\
&\qquad+\cos\frac{2(p_3+p_4)\theta k-p_3\theta p_4}{2}+\cos\frac{2p_4\theta k+p_3\theta p_4}{2}+\cos\frac{2p_3\theta k-p_3\theta p_4}{2}+\cos\frac{p_3\theta p_4}{2}\Big]\\
&+\frac{1}{4}\cos\frac{p_3\theta p_4}{2}\Big[\{p_3,p_4\}\to\{p_1,p_2\}\Big]\\
&+\frac{1}{2}\Big[\cos\frac{(p_1+p_2)\theta k-p_1\theta p_2}{2}+2\cos\frac{p_1\theta p_2+k\theta(p_1-p_2)}{2}+\cos\frac{(p_1+p_2)\theta k+p_1\theta p_2}{2}\Big]\\
&\ \frac{1}{2}\Big[\cos\frac{(p_3+p_4)\theta k-p_3\theta p_4}{2}+2\cos\frac{p_3\theta p_4+k\theta(p_3-p_4)}{2}+\cos\frac{(p_3+p_4)\theta k+p_3\theta p_4}{2}\Big]\\
=&\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\frac{1}{2}\left(1+\cos(p_1+p_2)\theta k\right)\\
&+\frac{1}{2}\cos\frac{p_1\theta p_2}{2}\Big[\cos\frac{p_3\theta p_4}{2}\Big(1+\cos(p_3+p_4)\theta k\Big)+\cos\Big(\frac{p_3\theta p_4}{2}-p_3\theta k\Big)+\cos\Big(\frac{p_3\theta p_4}{2}+p_4\theta k\Big)\Big]\\
&+\frac{1}{2}\cos\frac{p_3\theta p_4}{2}\Big[\{p_3,p_4\}\to\{p_1,p_2\}\Big]\\
&+\Big[\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\cos^2\frac{(p_1+p_2)\theta k}{2}+\cos\frac{p_1\theta p_2}{2}\cos\frac{(p_1+p_2)\theta k}{2}\cos\frac{p_3\theta p_4+k\theta(p_3-p_4)}{2}\\
&+\cos\frac{p_3\theta p_4}{2}\cos\frac{(p_3+p_4)\theta k}{2}\cos\frac{p_1\theta p_2+k\theta(p_1-p_2)}{2}\\
&+\frac{1}{2}\sum_{\eta=\pm}\cos\frac{p_1\theta p_2+k\theta(p_1-p_2)+\eta\left(p_3\theta p_4+k\theta(p_3-p_4)\right)}{2}\Big]\\
=&2\cos\frac{p_1\theta p_2}{2}\cos\frac{p_3\theta p_4}{2}\frac{1}{2}\left(1+\cos(p_1+p_2)\theta k\right)\\
&+\cos\frac{p_1\theta p_2}{2}\Big[\cos\Big(\frac{p_3\theta p_4}{2}+k\theta p_3\Big)+\cos\Big(\frac{p_3\theta p_4}{2}-k\theta p_4\Big)\Big]\\
&+\cos\frac{p_3\theta p_4}{2}\Big[\cos\Big(\frac{p_1\theta p_2}{2}+k\theta p_1\Big)+\cos\Big(\frac{p_1\theta p_2}{2}-k\theta p_2\Big)\Big]\\
&+\frac{1}{2}\Big[\cos\Big(\frac{p_1\theta p_2+p_3\theta p_4}{2}+k\theta(p_1-p_4)\Big)+\cos\Big(\frac{p_1\theta p_2-p_3\theta p_4}{2}+k\theta(p_1-p_3)\Big)\Big]\,. \qquad \text{(A.15)}
\end{aligned}
$$

The latter result coincides with the respective form, Eqn. (B.2) in [93] when considering the different convention for the orientation of the external momenta $p_1..p_4$.

Further simplifications can be achieved, when writing the phase in exponential form. In doing so we follow Micu and Sheikh-Jabbari and omit all phase factors not depending on the internal momentum k (by setting $p_i \equiv 0\,\forall i$). This seems reasonable since in loop integrations these exponentials can be pulled out and their contribution is just a factor (being a function of the external momenta $p_1 \dots p_4$). Therefore the phase (A.15) becomes

$$\begin{aligned}
&2\left(1+\mathrm{e}^{\mathrm{i}k\theta(p_1+p_2)}+\mathrm{e}^{-\mathrm{i}k\theta(p_1+p_2)}\right)+\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\theta p_3}+\mathrm{e}^{-\mathrm{i}k\theta p_3}\right)+\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\theta p_4}+\mathrm{e}^{-\mathrm{i}k\theta p_4}\right)\\
&+\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\theta p_1}+\mathrm{e}^{-\mathrm{i}k\theta p_1}\right)+\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\theta p_2}+\mathrm{e}^{-\mathrm{i}k\theta p_2}\right)+\\
&+\frac{1}{4}\left(\mathrm{e}^{\mathrm{i}k\theta(p_1-p_4)}+\mathrm{e}^{-\mathrm{i}k\theta(p_1-p_4)}\right)+\frac{1}{4}\left(\mathrm{e}^{\mathrm{i}k\theta(p_1-p_3)}+\mathrm{e}^{-\mathrm{i}k\theta(p_1-p_3)}\right)\\
&=2+\sum_{\eta=\pm}\left[\frac{1}{2}\sum_{i=2}^{4}\mathrm{e}^{\mathrm{i}\eta k\theta(p_1+\sigma_i p_i)}+\frac{3}{2}\mathrm{e}^{\mathrm{i}k\theta(p_1+p_2)}+\frac{1}{2}\sum_{i=1}^{4}\mathrm{e}^{\mathrm{i}\eta\sigma_i k\theta p_i}\right], \qquad \text{(A.16)}
\end{aligned}$$

where $\sigma = (1, 1, -1, -1)$ comes from a switch in the directions of the momenta p_3 and p_4 relative to those in Fig. A.1 in order to cast the expression into a form compatible to the one in [93]. Due to the symmetric integrations over k any antisymmetric contribution has to vanish. For this reason one may omit the signs in the exponent and the sum over η just gives a factor 2.

Finally, one has to add the contributions of the two permutations of the graph as shown in the insert of Eqn. (2.13). These are generated by the index exchanges $(2 \leftrightarrow 4)$ for the second and, in addition, $(3 \leftrightarrow 4)$ for the third diagram[1]. Eqn. (A.13) can thus be written as

$$\begin{aligned}
\Gamma(p_1,p_2,p_3,p_4) = \frac{\lambda^2}{6\times 36}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\,&\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)}{k^2+\frac{m^2}{2}+\zeta M^2}\\
&\left\{\left(2+\frac{1}{2}\sum_{i=2}^{4}\mathrm{e}^{\mathrm{i}k\theta(p_1+p_i)}+\sum_{i=1}^{4}\mathrm{e}^{\mathrm{i}k\theta p_i}\right)\sum_{i=2}^{4}\frac{1}{(p_1+p_i-k)^2+\frac{m^2}{2}+\chi M^2}\right.\\
&\left.+\frac{3}{2}\sum_{i=2}^{4}\frac{\mathrm{e}^{\mathrm{i}k\theta(p_1+p_i)}}{(p_1+p_i-k)^2+\frac{m^2}{2}+\chi M^2}\right\}, \qquad \text{(A.17)}
\end{aligned}$$

where again the technique (2.9) has been applied to split the propagators (explained above in Appendix A.2.1) into a form which is convenient for Gaussian integration. The latter expression contains 48 integrals whose results can all be derived from two generic expressions

$$\begin{aligned}
I_1(p) &= \sum_{\zeta,\chi=\pm1}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)}{\left(k^2+\frac{m^2}{2}+\zeta M^2\right)\left((p-k)^2+\frac{m^2}{2}+\chi M^2\right)}, \quad \text{and}\\
I_2(p,q) &= \sum_{\zeta,\chi=\pm1}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)\mathrm{e}^{\mathrm{i}k\theta(p+q)}}{\left(k^2+\frac{m^2}{2}+\zeta M^2\right)\left((p-k)^2+\frac{m^2}{2}+\chi M^2\right)}, \qquad \text{(A.18)}
\end{aligned}$$

where p and q are generic external momenta now. Note that these two integrals correspond to the planar and non-planar contributions respectively. Introducing Schwinger parametrization

[1]It has to be remarked that the prefactor 1/3 appearing in Eqn. (2.13) is unmotivated and wrong, but kept here for maintaining consistency with Ref. [95].

with α and β, and abbreviating $f(\zeta,\chi) := \left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)$ we can write

$$\begin{aligned} I_1(p) &= \sum_{\zeta,\chi=\pm1} f(\zeta,\chi) \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \int_0^{\infty} \mathrm{d}\alpha \mathrm{d}\beta\, \mathrm{e}^{-\beta\left(k^2+\frac{m^2}{2}+\zeta M^2\right)-\alpha\left((p-k)^2+\frac{m^2}{2}+\zeta M^2\right)} \\ &= \sum_{\zeta,\chi=\pm1} f(\zeta,\chi) \int_0^{\infty} \frac{\mathrm{d}\lambda}{(2\pi)^4} \int_0^1 \mathrm{d}\xi\, \frac{\pi^2}{\lambda} \mathrm{e}^{-\lambda\left[\xi(1-\xi)p^2+\frac{m^2}{2}+(\xi\zeta+(1-\xi)\chi)M^2\right]-\frac{1}{\Lambda^2\lambda}}\,, \end{aligned} \tag{A.19}$$

where an infrared cutoff has been introduced by $\Lambda \to \infty$, and the integration variables have been changed according to

$$\left.\begin{aligned} \alpha &= \xi\lambda \\ \beta &= (1-\xi)\lambda \end{aligned}\right\} \quad \to \quad d\alpha d\beta = \lambda d\xi d\lambda\,. \tag{A.20}$$

Applying Eqn. (F.11) the result is,

$$I_1(p) = \sum_{\zeta,\chi=\pm1} f(\zeta,\chi) \int_0^1 \mathrm{d}\xi\, \frac{2\pi^2}{(2\pi)^4} \mathrm{K}_0 \sqrt{\frac{4}{\Lambda^2}\left(\xi(1-\xi)p^2+\frac{m^2}{2}+(\xi\zeta+(1-\xi)\chi)M^2\right)}\,. \tag{A.21}$$

In the same way as discussed above in Appendix A.2.1, the respective expression for $I_2(p,q)$ can immediately be obtained by replacing $\Lambda^2 \to \frac{4}{(\tilde{p}+\tilde{q})^2}$. Therefore, subsequently, a generic variable A will be used, standing for $\frac{(\tilde{p}+\tilde{q})^2}{4}$ in the non-planar case, and for $\frac{1}{4\Lambda^2}$ in the planar case. Note also that, due to the completion of the square in the exponent, there is an additional phase factor $\exp[\mathrm{i}\xi p\theta q] \approx 1$ in the result for $I_2(p,q)$, which can be neglected since we approximate for small external momenta.
The integral Eqn. (A.21) is not immediately solvable in an analytic way. Being interested in the behavior in the infrared ($p, q \to 0$ or, respectively, $\Lambda^2 \to \infty$) we may first expand the Bessel function up to first order in its argument (see Eqn. (F.24)). In addition, we now write out the sum over χ and ζ with the explicit form for $f(\chi,\zeta)$,

$$\begin{aligned} I_1(p) \underset{p,q\ll1}{\approx} \frac{-\pi^2}{(2\pi)^4}\int_0^1 \mathrm{d}\xi \Bigg\{ &\left(1-\frac{m^2}{2M^2}\right)^2 \left[\ln\frac{1}{4\Lambda^2}\left(p^2\xi(1-\xi)+\frac{m^2}{2}-M^2\right)+2\gamma_E\right] \\ &+\left(1+\frac{m^2}{2M^2}\right)^2 \left[\ln\frac{1}{4\Lambda^2}\left(p^2\xi(1-\xi)+\frac{m^2}{2}+M^2\right)+2\gamma_E\right] \\ &+\left(1-\frac{m^4}{4M^4}\right) \Bigg[\ln\frac{1}{4\Lambda^2}\left(p^2\xi(1-\xi)+\frac{m^2}{2}+(2\xi-1)M^2\right)+4\gamma_E \\ &+\ln\frac{1}{4\Lambda^2}\left(p^2\xi(1-\xi)+\frac{m^2}{2}-(2\xi-1)M^2\right)\Bigg]\Bigg\} + \mathcal{O}\left(\frac{1}{\Lambda^2}, p^2\right), \end{aligned}$$

and with $p^2\xi(1-\xi) \ll \frac{m^2}{2}$,

$$\approx \frac{-\pi^2}{(2\pi)^4}\int_0^1 \mathrm{d}\xi \Bigg\{ \left(1+\frac{m^2}{2M^2}\right)^2 \ln A\left(\frac{m^2}{2}+M^2\right) + \left(1-\frac{m^2}{2M^2}\right)^2 \ln A\left(\frac{m^2}{2}-M^2\right)$$

$$+\left(1-\frac{m^4}{4M^4}\right)\left[\ln A\left(\frac{m^2}{2}+(1-2\xi)M^2\right)+\ln A\left(\frac{m^2}{2}-(1-2\xi)M^2\right)\right]\Bigg\}. \tag{A.22}$$

The integrations can now be be performed by virtue of Eqn. (F.17b), yielding (after collecting terms)

$$\begin{aligned}
&\frac{-2\pi^2}{(2\pi)^4}\Bigg\{2\ln A+\left(1+\frac{m^4}{4M^4}\right)\ln\left(\frac{m^4}{4}-M^4\right)+\frac{m^2}{2M^2}\ln\frac{m^2+2M^2}{m^2-2M^2}\\
&\qquad+\frac{1}{2}\left(1-\frac{m^4}{4M^4}\right)\left[-2+\left(1+\frac{m^2}{2M^2}\right)\ln\left(\frac{m^2}{2}+M^2\right)+\left(1-\frac{m^2}{2M^2}\right)\ln\left(\frac{m^2}{2}-M^2\right)\right]\\
&\qquad+4\gamma_E\Bigg\}+\mathcal{O}\left(\frac{1}{\Lambda^2},p^2\right)\\
=&-\frac{1}{(2\pi)^2}\Bigg\{\ln\left(A\sqrt{\frac{m^4}{4}-M^4}\right)+\left(3-\frac{m^4}{M^4}\right)\frac{m^2}{M^2}\ln\sqrt{\frac{m^2+2M^2}{m^2-2M^2}}\\
&\qquad-\frac{1}{4}\left(1-\frac{m^4}{4M^4}\right)+2\gamma_E\Bigg\}+\mathcal{O}\left(\frac{1}{\Lambda^2},p^2\right).
\end{aligned} \tag{A.23}$$

The latter result is equal to the one given in the main text (see Eqn. (2.17)).

A.3 Higher Loop Calculations

A.3.1 Non-planar Two Loop Snowman Graph

The snowman graph in Fig. 2.2a on page 21 can be computed by using the decomposition (A.8) and Schwinger parametrization to evaluate the integral (2.18) for $n=1$. Hence $\Pi^{1\ \text{np-ins.}}(p)=\lambda^2 J_1(p)$ with

$$\begin{aligned}
J_1(p)&\equiv\frac{1}{4}\sum_{\zeta,\chi,\eta}\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\frac{e^{i\eta k\theta p}}{\tilde{k}^2}\frac{1+\zeta\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta M^2}\frac{1+\chi\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\chi M^2}\\
&=\frac{1}{4\theta^2}\sum_{\zeta,\chi,\eta}\left(1+\zeta\tfrac{m^2}{2M^2}\right)\left(1+\chi\tfrac{m^2}{2M^2}\right)\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\gamma\\
&\qquad\times\exp\left[-(\alpha+\beta+\gamma)k^2-(\alpha+\beta)\tfrac{m^2}{2}-(\alpha\zeta+\beta\chi)M^2+i\eta k\theta p\right].
\end{aligned} \tag{A.24}$$

After carrying out the integration over k and, changing variables according to

$$\begin{aligned}
&(\alpha,\beta,\gamma)\to(\lambda,\xi,\sigma)\,,\\
\text{with}\quad&\alpha=\lambda\xi\sigma\,,\\
&\beta=\lambda(1-\xi)\sigma\,,\\
&\gamma=\lambda(1-\sigma)\,,\\
\text{and}\quad&\lambda\in[0,\infty[\,,\ \xi\in[0,1]\,,\ \sigma\in[0,1]\,,
\end{aligned} \tag{A.25}$$

(which is a generalization of (A.20)) one obtains

$$J_1(p)=\sum_{\zeta,\chi}\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)}{4\theta^2(4\pi)^2}\int_0^\infty d\lambda\int_0^1 d\xi\int_0^1 d\sigma\sigma$$

$$\times \exp\Big[-\frac{\tilde{p}^2}{4\lambda} - \lambda\sigma\left(\tfrac{m^2}{2} + \xi\zeta M^2 + (1-\xi)\chi M^2\right)\Big]$$

$$= \frac{1}{\theta^2(4\pi)^2}\int_0^\infty \mathrm{d}\lambda \int_0^1 \mathrm{d}\xi \int_0^1 \mathrm{d}\sigma\sigma \left[\cosh\left(\lambda\sigma\xi M^2\right) - \frac{m^2}{2M^2}\sinh\left(\lambda\sigma\xi M^2\right)\right]$$
$$\times\left[\cosh\left(\lambda\sigma(1-\xi)M^2\right) - \frac{m^2}{2M^2}\sinh\left(\lambda\sigma(1-\xi)M^2\right)\right]\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\sigma\frac{m^2}{2}}\,. \tag{A.26}$$

The integration over ξ can be performend by virtue of the Eqns. (F.14a)–(F.14c),

$$= \frac{1}{\theta^2(4\pi)^2}\int_0^\infty \mathrm{d}\lambda \int_0^1 \mathrm{d}\sigma \frac{\sigma}{2}\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\sigma\frac{m^2}{2}}\Bigg[\left(1+\frac{m^4}{4M^4}\right)\cosh\left(\lambda\sigma M^2\right) - \frac{m^2}{M^2}\sinh\left(\lambda\sigma M^2\right)$$
$$\frac{1}{\lambda\sigma M^2}\left(1-\frac{m^4}{4M^4}\right)\sinh\left(\lambda\sigma M^2\right)\Bigg], \tag{A.27}$$

and Eqns. (F.15a), (F.16a), and (F.16b) readily allow to evaluate the integral over σ, yielding

$$= \frac{1}{\theta^2(4\pi)^2}\int_0^\infty \mathrm{d}\lambda\, \frac{\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\frac{m^2}{2}}}{4M^4\lambda^2}\left[\left(\frac{m^2}{M^2}+2M^2\lambda\right)\sinh\left(\lambda M^2\right) - m^2\lambda\cosh\left(\lambda M^2\right)\right]. \tag{A.28}$$

For the computation of the final λ integrals with Eqn. (F.11) it is necessary to rewrite the hyperbolic functions in their exponential form. This generates an integrability condition $M^2 = \sqrt{\frac{m^4}{4} - a^2} \leq \frac{m^2}{2}$ from the exponentials. Introducing the abbreviations $m_\pm := \left(\frac{m^2}{2} \pm M^2\right)$, the result writes

$$J_1(p) = \frac{-1}{32\pi^2 M^6}\Bigg\{M^2 m_+^2\,\mathrm{K}_0\sqrt{\tilde{p}^2 m_+^2} + M^2 m_-^2\,\mathrm{K}_0\sqrt{\tilde{p}^2 m_-^2}$$
$$+ m^2\sqrt{\frac{m_+^2}{\tilde{p}^2}}\,\mathrm{K}_1\sqrt{\tilde{p}^2 m_+^2} - m^2\sqrt{\frac{m_-^2}{\tilde{p}^2}}\,\mathrm{K}_1\sqrt{\tilde{p}^2 m_-^2}\Bigg\}, \tag{A.29}$$

which equals Eqn. (2.20) if the additional factor $\frac{4}{3(4\pi)^2}$ of the non-planar insertion in Eqn. (2.11) is considered.

A.3.2 Non-planar n-loop Graph

Basically, the calculation proceeds along the lines of the 2-loop integral discussed above in Appendix A.3.1. The integral (2.19) is given by $\Pi^{n\ \text{npl-ins.}}(p) = \lambda^2 J_n(p)$ with

$$J_n(p) \equiv \int_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{ik\theta p}}{\left(\tilde{k}^2\right)^n\left[k^2+m^2+\frac{a^2}{k^2}\right]^{n+1}}$$
$$= \frac{1}{2^{n+1}\theta^{2n}}\sum_{\{\zeta_1,..\zeta_{n+1}\}=\pm 1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{ik\theta p}}{(k^2)^n}\prod_{i=1}^{n+1}\left(\frac{1+\zeta_i\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta_i M^2}\right). \tag{A.30}$$

With all the ζ_i-dependent factors being independent of each other, a total of $n+2$ Schwinger parameters α_i are required to parameterize the denominators of the integrand (see Eqn. (F.22) on page 144),

$$\frac{1}{k^2+\frac{m^2}{2}+\zeta_i M^2} = \int_0^\infty d\alpha_i\, \mathrm{e}^{-\alpha_i\left(k^2+\frac{m^2}{2}+\zeta_i M^2\right)}\,, \qquad \text{for } i\in\{1,\ldots,n+1\}\,,$$

$$\frac{1}{(k^2)^n} = \frac{1}{\Gamma(n)}\int_0^\infty d\alpha_{n+2}\,(\alpha_{n+2})^{n-1}\mathrm{e}^{-\alpha_{n+2}k^2}\,, \qquad \text{for } k^2>0\,. \tag{A.31}$$

Generalizing from Eqn. (A.25) and Eqn. (A.20) we perform a change of variables $\{\alpha_1,..\alpha_{n+2}\}\mapsto\{\xi_1,..\xi_{n+1},\lambda\}$ with

$$\alpha_1 = \lambda\prod_{i=1}^{n+1}\xi_i\,,\quad \alpha_2=\lambda(1-\xi_1)\prod_{i=2}^{n+1}\xi_i\,,\quad ..\quad \alpha_k = \lambda(1-\xi_{k-1})\prod_{i=k}^{n+1}\xi_i\,, \quad ..\quad \alpha_{n+2} = \lambda(1-\xi_{n+1})\,, \tag{A.32}$$

where $\xi_i\in[0,1]$ and $\lambda\in[0,\infty)$. The integration measure transforms as

$$\prod_{i=1}^{n+2} d\alpha_i = \lambda^{n+1}\prod_{l=1}^{n}(\xi_{l+1})^l\; d\lambda\prod_{j=1}^{n+1}d\xi_j\,, \tag{A.33}$$

and from the definition (A.32) it follows that $\sum_{i=1}^{n+2}\alpha_i=\lambda$. The integration over k can be carried out by completing the square in the exponent, so that we arrive at

$$J_n = \frac{1}{\theta^{2n}2^{n+1}(4\pi)^2\Gamma(n)}\sum_{\zeta_1,..\zeta_{n+1}}\prod_{i=1}^{n+1}\left(1+\zeta_i\tfrac{m^2}{2M^2}\right)\int_0^\infty d\lambda\,\lambda^{2n-2}\prod_{j=1}^{n+1}\int_0^1 d\xi_j\prod_{l=1}^{n}(\xi_{l+1})^l$$
$$\times(1-\xi_{n+1})^{n-1}\exp\left[-\frac{\tilde{p}^2}{4\lambda}-\lambda\xi_{n+1}\frac{m^2}{2}-M^2\sum_1^{n+1}\zeta_i\alpha_i\right]\,, \tag{A.34}$$

where the coefficients $\alpha_1,..\alpha_{n+1}$ in the exponent are functions of the variables $\lambda,\xi_1,..\xi_{n+1}$ according to Eqn. (A.32). The sum over the ζ_i can be expressed in terms of hyperbolic functions (as was demonstrated above in Appendix A.3.1),

$$J_n = \frac{1}{\theta^{2n}(4\pi)^2\Gamma(n)}\prod_{j=1}^{n+1}\int_0^1 d\xi_j\,(1-\xi_{n+1})^{n-1}\prod_{l=1}^{n}(\xi_{l+1})^l\int_0^\infty d\lambda\,\lambda^{2n-2}$$
$$\times\,\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\xi_{n+1}\frac{m^2}{2}}\prod_{i=1}^{n+1}\left[\cosh\left(\alpha_i M^2\right)-\frac{m^2}{2M^2}\sinh\left(\alpha_i M^2\right)\right]\,. \tag{A.35}$$

Integration over $\xi_1,..\xi_{n+1}$ yields a sum of integrals over λ, which are once more given by modified Bessel functions.

The techniques for solving the above integrations for higher n are the same as described in Appendix A.3.1. However, since the expressions grow quite fast it is to be recommended to conduct all calculations aided by a computer, i.e. in Mathematica®.

A.4 Miscellanea

In the following the proof of Theorem 1 on page 27 shall be given,

Proof. Since the integrand $C(\alpha,k)$ with $G^i(k) = \int_{M^{-2i}}^{M^{-2(i-1)}} C(\alpha,k)\,\mathrm{d}\alpha$ in Eqn. (2.36) is monotonically falling and $M > 1$, it holds that $\sup\{C(\alpha) : \alpha \in [M^{-2i}, M^{-2(i-1)}]\} = C\left(M^{-2i}, k\right)$. Hence, the integral has an upper bound, given by a rectangular patch, $C^i \leq (M^{-2(i-1)} - M^{-2i})\, C\left(M^{-2i}\right)$ and,

$$\begin{aligned} G^i(k) \leq & \underbrace{(M^{-2(i-1)} - M^{-2i})}_{M^{-2i}(M^2-1)} \mathrm{e}^{-M^{-2i}\left(k^2+m^2+\frac{a'^2}{k^2}\right)} \\ = & K M^{-2i} \mathrm{e}^{-M^{-2i}\left(k^2+m^2+\frac{a'^2}{k^2}\right)}, \end{aligned}$$

and furthermore, since $M > 1$, we have $M^{-2i} < 1$, which finally gives

$$< K \mathrm{e}^{-cM^{-2i}\left(k^2+m^2+\frac{a'^2}{k^2}\right)}.$$

In the latter expression we have defined the constants

$$K := M^2 - 1 > 1, \qquad \text{and} \qquad c := \sup_i \left(-\ln M^{-2i}\right) = -\ln M^{-2} > 0\,.$$

For the slice $G^0(k)$ we can compute the explicit integral

$$\int_1^\infty \mathrm{d}\alpha\, \mathrm{e}^{-\alpha\left(k^2+m^2+\frac{a'^2}{k^2}\right)} = -\left(k^2 + m^2 + \frac{a'^2}{k^2}\right)^{-1} \mathrm{e}^{-\left(k^2+m^2+\frac{a'^2}{k^2}\right)},$$

and by virtue of the damping behaviour, i.e.

$$\lim_{k\to\{0,\infty\}} \left(k^2 + m^2 + \frac{a'^2}{k^2}\right) = 0\,,$$

the k-dependent factors can be approximated by constants κ. Hence the integral can entirely be bound by $G^0(k) < -\kappa^{-1}\mathrm{e}^{-\kappa} = \text{const.}$. However, we may also choose a tighter bound by only approximating the forefactor, and omitting in the exponent only terms which vanish for $k \to \infty$, leading to

$$G^0(k) < -\kappa^{-1}\mathrm{e}^{-ck^2}\,,$$

which completes the proof. □

Appendix B

Supplementary Content to the $1/p^2$ Gauge Model with Real Auxiliary Field

B.1 Feynman Rules

This section briefly describes the derivation of the Feynman rules in the $1/p^2$ model with localization by one real-valued auxiliary field as being discussed in Section 3.1.

B.1.1 Equations of Motion

The equations of motion for the action (3.5) are obtained by simple variation of the bilinear part S^{bi} with respect to the fields.

$$\frac{\delta S^{\text{bi}}}{\delta A^\mu} = -j^A_\mu \quad = \left(\partial_\mu \partial^\nu - \Box \delta^\nu_\mu\right) A_\nu + 2\partial^\nu B_{\mu\nu} - \partial_\mu b \star \left(1 + \frac{a}{\Box\widetilde{\Box}}\right), \tag{B.1a}$$

$$\frac{\delta S^{\text{bi}}}{\delta b} = -j^b \quad = \left(1 + \frac{a}{\Box\widetilde{\Box}}\right) \partial^\mu A_\mu - \alpha b = 0\,, \tag{B.1b}$$

$$\frac{\delta S^{\text{bi}}}{\delta c} = \quad j_c \quad = -\left(1 + \frac{a}{\Box\widetilde{\Box}}\right) \Box \bar{c} = 0\,, \tag{B.1c}$$

$$\frac{\delta S^{\text{bi}}}{\delta \bar{c}} = \quad j_{\bar{c}} \quad = \left(1 + \frac{a}{\Box\widetilde{\Box}}\right) \Box c = 0\,, \tag{B.1d}$$

$$\frac{\delta S^{\text{bi}}}{\delta B^{\mu\nu}} = -j^B_{\mu\nu} \quad = \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - 2\frac{\Box\widetilde{\Box}}{a} B_{\mu\nu}\,. \tag{B.1e}$$

B.1.2 Propagators

Eqn. (B.1e) involves three fields. The derivation shall be sketched at this point. b and $\partial_\nu B^{\mu\nu}$ are extracted from Eqns. (B.1b) and (B.1e) respectively, and inserted into Eqn. (B.1a) yielding

$$-j^\mu_A = U \left[\partial_\mu \left(\partial A\right) - \Box A_\mu - \frac{\partial_\mu}{\alpha} \left(U \left(\partial A\right) - j^b\right)\right] + \frac{a}{\Box\widetilde{\Box}} \partial^\nu j^B_{\mu\nu},$$

where the abbreviation $U := \left(1 + \frac{a}{\square\tilde{\square}}\right)$ has been used. Derivation of this result with respect to ∂^μ gives an expression for ∂A which can in turn be inserted into Eqn. (B.1a) again. We finally end up with

$$A_\mu = \frac{1}{U\square}\left[j^A_\mu + \partial_\mu j^b + \left(\frac{\alpha}{U} - 1\right)\frac{\partial_\mu}{\square}(\partial j_A) + \left(\frac{\alpha}{U} - \frac{1}{\square}\right)\frac{a}{\square\tilde{\square}}\partial_\mu\partial_\alpha\partial_\beta j_B^{\alpha\beta} + \frac{A}{\square\tilde{\square}}\partial^\alpha j^B_{\mu\alpha}\right]. \quad \text{(B.2)}$$

Insertion of A_μ into Eqn. (B.1e) yields,

$$B_{\mu\nu} = \frac{a}{2\square\tilde{\square}}\left[\frac{1}{U\square}\left(\partial_\mu j^A_\nu - \partial_\nu j^A_\mu + \frac{a}{\square\tilde{\square}}\partial^\alpha\left(\partial_\mu j^B_{\nu\alpha} - \partial_\nu j^B_{\mu\alpha}\right)\right) + j^B_{\mu\nu}\right]. \quad \text{(B.3)}$$

The expressions (B.2) and (B.3) form the starting point for the calculation of the propagators G^{AA}, G^{AB}, G^{BA}, and G^{BB} by variation with respect to the currents j^A_ρ and $j^B_{\rho\sigma}$. The ghost propagator $G^{\bar{c}c}$ is obtained by resolving Eqn. (B.1c) for c, and subsequent variation with respect to $j_{\bar{c}}$. After transformation to momentum space (by applying the simple substitution $\partial_\mu \to \mathrm{i}k_\mu$, and respecting the sign from partially integrating terms of the form $\partial_\mu j\cdot = -j\partial_\mu\cdot$) we arrive at Eqns. (3.6a)–(3.6b) in the main text on page 35 .

B.1.3 Vertices

The following vertex expressions are derived from the full form of the action (3.5) on page 35. Remember the conventions from Section 1.3.1.

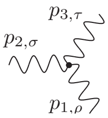
3A vertex: Three photon fields without other couplings are solely contained in the $\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu}$ term in Eqn. (3.5). In a more explicit form,

$$\int \mathrm{d}^4x\,\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu} = \int \mathrm{d}^4x\,\Big[\underbrace{-\frac{1}{2}A^\mu\left(\delta_\mu{}^\nu\square - \partial_\mu\partial^\nu\right)A_\nu}_{(*a)} + \underbrace{\mathrm{i}\frac{g}{4}\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right)\star\left[A_\mu \overset{\star}{,} A_\nu\right]}_{(*b)}$$
$$+ \underbrace{\frac{g^2}{4}\left[A^\mu \overset{\star}{,} A^\nu\right]\left[A_\mu \overset{\star}{,} A_\nu\right]}_{(*c)}\Big]. \quad \text{(B.4)}$$

For the sole trilinear term in $A_{\{\mu,\nu\}}$, $(*b)$ the variation reads

$$V^{3A}_{\rho,\sigma,\tau}(p_1,p_2,p_3) = -\frac{\mathrm{i}g}{4}(2\pi)^{12}\frac{\delta}{\delta A_\rho(-p_1)}\frac{\delta}{\delta A_\sigma(-p_2)}\frac{\delta}{\delta A_\tau(-p_3)}\int \mathrm{d}^4x \int \frac{\mathrm{d}k_{\{1-3\}}}{(2\pi)^{12}}\,\mathrm{e}^{\mathrm{i}x(k_1+k_2+k_3)}\times$$
$$\times\Big[\left(k_1^\nu A^\mu(k_1)\right)A_\mu(k_2)A_\nu(k_3)\mathrm{e}^{-\frac{\mathrm{i}}{2}(k_1\theta k_2 + k_1\theta k_3)}\left(\mathrm{e}^{-\frac{\mathrm{i}}{2}k_2\theta k_3} - \mathrm{e}^{-\frac{\mathrm{i}}{2}k_3\theta k_2}\right)$$
$$-\left(k_4^\mu A^\nu(k_4)\right)A_\mu(k_2)A_\nu(k_3)\mathrm{e}^{-\frac{\mathrm{i}}{2}(k_4\theta k_2 + k_4\theta k_3)}\left(\mathrm{e}^{-\frac{\mathrm{i}}{2}k_2\theta k_3} - \mathrm{e}^{-\frac{\mathrm{i}}{2}k_3\theta k_2}\right)\Big]. \quad \text{(B.5)}$$

The x-dependent exponential together with the d^4x integral results in $(2\pi)^4\delta^4(k_1 + k_2 + k_3)$ which can readily be applied to identify the remaining common exponential factors to be unity. Respecting Eqn. (F.21), the exponentials in the brackets can be written in terms of sine functions. After conducting the variations, momentum conservation unifies the arguments of the trigonometric functions. Finally, we are left with

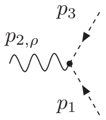


$$
\begin{aligned}
&= V^{3A}_{\rho,\sigma,\tau}(p_1,p_2,p_3) \\
&= 2\mathrm{i}g(2\pi)^4\delta^4(p_1+p_2+p_3)\left[(p_1-p_3)_\sigma\,\delta_{\rho\tau}+(p_2-p_1)_\tau\,\delta_{\rho\sigma}+(p_3-p_2)_\rho\,\delta_{\sigma\tau}\right]\sin\frac{p_1\theta p_2}{2}\,.
\end{aligned}
\tag{B.6}
$$

$\bar{c}Ac$ vertex From Eqn. (3.5) the only available term for a ghost interaction is the trilinear expression $-\bar{c}\star\left(1+\frac{a}{\square\widetilde{\square}}\right)\partial^\mu \mathrm{D}_\mu c$. By virtue of the partial integration property of the $\frac{1}{\square}$ operator (see Theorem 3 on page 112) the trilinear part of this expression can be brought into the form

$$
+\mathrm{i}g\left(1+\frac{1}{\square\widetilde{\square}}\right)\bar{c}(q_1)\star\partial^\mu\left[A_\mu(k_2)\stackrel{\star}{,}c(q_3)\right]. \tag{B.7}
$$

Temporarily denoting this expression by $(*r)$, appropriate variation gives,

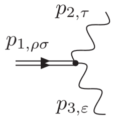


$$
\begin{aligned}
&= V^{\bar{c}Ac}_\rho(p_1,p_2,p_3) \\
&= -(2\pi)^{12}\frac{\delta}{\delta\bar{c}(-p_1)}\frac{\delta}{\delta A_\rho(-p_2)}\frac{\delta}{\delta c(-p_3)}\left[(*r)\right] \\
&= -2\mathrm{i}g(2\pi)^4\delta^4(p_1+p_2+p_3)p_{1,\rho}\left(1+\frac{a}{(p_3)^2(\tilde{p}_3)^2}\right)\sin\frac{p_1\theta p_3}{2}\,.
\end{aligned}
\tag{B.8}
$$

BAA vertex: The relevant term in the Lagrangian is $\int \mathrm{d}^4x\, a' B^{\mu\nu}\star F_{\mu\nu}$, where only the terms being bilinear in $A_{\{\mu,\nu\}}$ in $F_{\mu\nu}$ (i.e. the commutator terms) enter. The complete variation reads

$$
\begin{aligned}
V^{BAA}_{\rho\sigma,\tau,\varepsilon}(p_1,p_2,p_3) &= -ga'(2\pi)^{12}\frac{\delta}{\delta B_{\rho\sigma}(-p1)}\frac{\delta}{\delta A_\tau(-p2)}\frac{\delta}{\delta A_\varepsilon(-p3)}\int\mathrm{d}^4x\int\frac{\mathrm{d}q\,\mathrm{d}k_1\,\mathrm{d}k_2}{(2\pi)^{12}}\mathrm{e}^{\mathrm{i}x(q+k_1+k_2)}\times \\
&\times\left[B_{\mu\nu}(q)A_\mu(k_1)A_\nu(k_2)\mathrm{e}^{\frac{i}{2}k_1\theta k_2}-B_{\mu\nu}A_\nu(k_2)A_\mu(k_1)\mathrm{e}^{\frac{i}{2}k_2\theta k_1}\right]\mathrm{e}^{-\frac{\mathrm{i}}{2}(q\theta k_1+q\theta k_2)}.
\end{aligned}
\tag{B.9}
$$

Respecting the relation (F.21), and performing the variations finally results in



$$
= V^{BAA}_{\rho\sigma,\tau,\varepsilon}(p_1,p_2,p_3) = 2ga'\delta^4(p_1+p_2+p_3)\left(\delta_{\rho\tau}\delta_{\sigma\varepsilon}-\delta_{\rho\varepsilon}\delta_{\sigma\tau}\right)\sin\frac{p_2\theta p_3}{2}\,. \tag{B.10}
$$

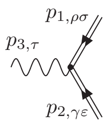
BBA vertex: The only available term of second order in $B_{\mu\nu}$ is $-B_{\mu\nu}\star\widetilde{D}^2D^2\star B_{\mu\nu}$. Performing two partial integrations the expression can be rewritten (see Theorem 2) as

$$
\begin{aligned}
-\left(\widetilde{D}^2B_{\mu\nu}\right)D^2B_{\mu\nu} = \Big\{&-\square B_{\mu\nu}\,\square B_{\mu\nu}+\underbrace{\mathrm{i}g\,\square B_{\mu\nu}\left[\partial_\eta A_\eta\stackrel{\star}{,}B_{\mu\nu}\right]}_{(a)}+\underbrace{2\mathrm{i}g\,\square B_{\mu\nu}\left[A_\eta\stackrel{\star}{,}\partial_\eta B_{\mu\nu}\right]}_{(b)}+ \\
&+\underbrace{g^2\,\square B_{\mu\nu}\left[A_\eta\stackrel{\star}{,}\left[A_\eta\stackrel{\star}{,}B_{\mu\nu}\right]\right]}_{(c)}+\underbrace{\mathrm{i}g\left[\partial_\eta A_\eta\stackrel{\star}{,}B_{\mu\nu}\right]\square B_{\mu\nu}}_{(d)}+ \\
&+\underbrace{g^2\left[\partial_\lambda A_\lambda\stackrel{\star}{,}B_{\mu\nu}\right]\left[\partial_\eta A_\eta\stackrel{\star}{,}B_{\mu\nu}\right]}_{(e)}+\underbrace{2g^2\left[\partial_\lambda A_\lambda\stackrel{\star}{,}B_{\mu\nu}\right]\left[A_\eta\stackrel{\star}{,}\partial_\eta B_{\mu\nu}\right]}_{(f)}-
\end{aligned}
$$

— continued —

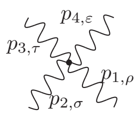
$$
\begin{aligned}
&- \underbrace{\mathrm{i}g^3 \left[\partial_\lambda A_\lambda \stackrel{\star}{,} B_{\mu\nu}\right] \left[A_\eta \stackrel{\star}{,} \left[A_\eta \stackrel{\star}{,} B_{\mu\nu}\right]\right]}_{(g)} + \underbrace{2\mathrm{i}g \left[A_\lambda \stackrel{\star}{,} \partial_\lambda B_{\mu\nu}\right] \Box B_{\mu\nu}}_{(h)} + \\
&+ \underbrace{2g^2 \left[A_\lambda \stackrel{\star}{,} \partial_\lambda B_{\mu\nu}\right] \left[\partial_\eta A_\eta \stackrel{\star}{,} B_{\mu\nu}\right]}_{(i)} + \underbrace{4g^2 \left[A_\lambda \stackrel{\star}{,} \partial_\lambda B_{\mu\nu}\right] \left[A_\eta \stackrel{\star}{,} \partial_\lambda B_{\mu\nu}\right]}_{(j)} - \\
&- \underbrace{2\mathrm{i}g^3 \left[A_\lambda \stackrel{\star}{,} \partial_\lambda B_{\mu\nu}\right] \left[A_\eta \stackrel{\star}{,} \left[A_\eta \stackrel{\star}{,} B_{\mu\nu}\right]\right]}_{(k)} + \underbrace{g^2 \left[A_\lambda \stackrel{\star}{,} \left[A_s \stackrel{\star}{,} B_{\mu\nu}\right]\right] \Box B_{\mu\nu}}_{(l)} - \\
&- \underbrace{\mathrm{i}g^3 \left[A_\lambda \stackrel{\star}{,} \left[A_\lambda \stackrel{\star}{,} B_{\mu\nu}\right]\right] \left[\partial_\eta A_\eta \stackrel{\star}{,} B_{\mu\nu}\right]}_{(m)} - \underbrace{2\mathrm{i}g^3 \left[A_\lambda \stackrel{\star}{,} \left[A_\lambda \stackrel{\star}{,} B_{\mu\nu}\right]\right] \left[A_\eta \stackrel{\star}{,} B_{\mu\nu}\right]}_{(n)} - \\
&- \underbrace{g^4 \left[A_\lambda \stackrel{\star}{,} \left[A_\lambda \stackrel{\star}{,} B_{\mu\nu}\right]\right] \left[A_\eta \stackrel{\star}{,} \left[A_\eta \stackrel{\star}{,} B_{\mu\nu}\right]\right]}_{(o)} \Bigg\} . \qquad \text{(B.11)}
\end{aligned}
$$

The relevant terms for the BBA-vertex are marked by (a), (b), (d), and (h) in Eqn. (B.11). Variation with respect to the fields, and subsequent summation leads to



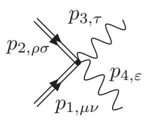
$$
\begin{aligned}
&= V^{BBA}_{\rho\sigma,\gamma\varepsilon,\tau}(p_1, p_2, p_3) \\
&= -(2\pi)^{12} \frac{\delta}{\delta B_{\rho\sigma}(-p_1)} \frac{\delta}{\delta B_{\gamma\varepsilon}(-p_2)} \frac{\delta}{\delta A_\tau(-p_3)} \left[(a) + (b) + (d) + (h)\right] \\
&= -2\mathrm{i}g\theta^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 + p_3\right) \left(\delta_{\rho\gamma}\delta_{\sigma\varepsilon} - \delta_{\rho\varepsilon}\delta_{\sigma\gamma}\right) \left({p_1}^2 + {p_2}^2\right) \left(p_1 - p_2\right)_\tau \sin \frac{p_1\theta p_2}{2} . \qquad \text{(B.12)}
\end{aligned}
$$

4A vertex: As for the $3A$ vertex the required interaction is given by the $\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu}$ term in the action (3.5). The $4A$ vertex is derived by variation of term $(*c)$ in Eqn. (B.4)



$$
\begin{aligned}
&= V^{4A}_{\rho,\sigma,\tau,\varepsilon}(p_1, p_2, p_3, p_4) \\
&= -(2\pi)^{16} \frac{\delta}{\delta A_\rho(-p_1)} \frac{\delta}{\delta A_\sigma(-p_2)} \frac{\delta}{\delta A_\tau(-p_3)} \frac{\delta}{\delta A_\varepsilon(-p_4)} \left[(*c)\right] \\
&= 4g^2 (2\pi)^4 \delta^4 (p_1 + p_2 + p_3 + p_4) \Bigg[\left(\delta_{\tau\rho}\delta_{\sigma\varepsilon} - \delta_{\tau\sigma}\delta_{\rho\varepsilon}\right) \sin \frac{p_1\theta p_2}{2} \sin \frac{p_3\theta p_4}{2} + \\
&\qquad + \left(\delta_{\varepsilon\tau}\delta_{\rho\sigma} - \delta_{\tau\sigma}\delta_{\rho\varepsilon}\right) \sin \frac{p_1\theta p_3}{2} \sin \frac{p_2\theta p_4}{2} + \\
&\qquad + \left(\delta_{\varepsilon\tau}\delta_{\rho\sigma} - \delta_{\tau\rho}\delta_{\sigma\varepsilon}\right) \sin \frac{p_1\theta p_4}{2} \sin \frac{p_2\theta p_3}{2} \Bigg] . \qquad \text{(B.13)}
\end{aligned}
$$

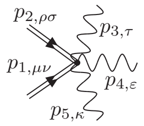
2B2A vertex: Six terms with two B fields and two photons can be found in Eqn. (B.11),



$$
\begin{aligned}
&= V^{2B2A}_{\rho\sigma,\tau\varepsilon,\gamma,\kappa}(p_{\{1..4\}}) \\
&= -(2\pi)^{16} \frac{\delta}{\delta B_{\rho\sigma}(-p_1)} \frac{\delta}{\delta B_{\tau\varepsilon}(-p_2)} \frac{\delta}{\delta A_\gamma(-p_3)} \frac{\delta}{\delta A_\kappa(-p_4)} \left[(c) + (e) + (f) + (i) + (j) + (l)\right]
\end{aligned}
$$

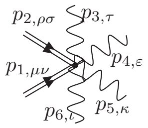
$$= 4g^2\theta^2(2\pi)^4\delta^4(p_1+p_2+k_3+k_4)\left(\delta_{\rho\tau}\delta_{\sigma\varepsilon}-\delta_{\rho\varepsilon}\delta_{\sigma\tau}\right)\times$$
$$\times\Bigg\{\left[p_{3,\gamma}p_{4,\kappa}+2\left(p_{1,\gamma}p_{4,\kappa}+p_{2,\kappa}p_{3,\gamma}\right)+4p_{1,\gamma}p_{2,\kappa}-\delta_{\kappa\gamma}\left({p_1}^2+{p_2}^2\right)\right]\sin\tfrac{p_1\theta p_3}{2}\sin\tfrac{p_2\theta p_4}{2}+$$
$$+\left[p_{3,\gamma}p_{4,\kappa}+2\left(p_{2,\gamma}p_{4,\kappa}+p_{1,\kappa}p_{3,\gamma}\right)+4p_{1,\kappa}p_{2,\gamma}-\delta_{\kappa\gamma}\left({p_1}^2+{p_2}^2\right)\right]\sin\tfrac{p_1\theta p_4}{2}\sin\tfrac{p_2\theta p_3}{2}\Bigg\}. \quad \text{(B.14)}$$

2B3A vertex: Two B fields and three A fields are contained in (g), (k), (m) and (n) of Eqn. (B.11).



$$= V^{2B3A}_{\mu\nu,\rho\sigma,\tau,\varepsilon,\kappa}(p_{\{1..5\}})$$
$$= -(2\pi)^{16}\frac{\delta}{\delta B_{\mu\nu}(-p_1)}\frac{\delta}{\delta B_{\rho\sigma}(-p_2)}\frac{\delta}{\delta A_\tau(-p_3)}\frac{\delta}{\delta A_\varepsilon(-p_4)}\frac{\delta}{\delta A_\kappa(-p_5)}\frac{1}{a}\left[(g)+(k)+(n)\right]$$
$$= -8\mathrm{i}g^3\theta^2(2\pi)^4\delta^4(p_1+p_2+p_3+p_4+p_5)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times$$
$$\times\Bigg\{\left[p_3+2p_1\right]_\tau\delta_{\epsilon\kappa}\sin\left(\tfrac{p_3\tilde{p}_1}{2}\right)\left[\sin\left(\tfrac{p_5\tilde{p}_2}{2}\right)\sin\left(\tfrac{p_4(\tilde{p}_5+\tilde{p}_2)}{2}\right)+(p_4\leftrightarrow p_5)\right]$$
$$+\left[p_4+2p_1\right]_\epsilon\delta_{\tau\kappa}\sin\left(\tfrac{p_4\tilde{p}_1}{2}\right)\left[\sin\left(\tfrac{p_5\tilde{p}_2}{2}\right)\sin\left(\tfrac{p_3(\tilde{p}_5+\tilde{p}_2)}{2}\right)+(p_5\leftrightarrow p_3)\right]$$
$$+\left[p_5+2p_1\right]_\kappa\delta_{\tau\epsilon}\sin\left(\tfrac{p_5\tilde{p}_1}{2}\right)\left[\sin\left(\tfrac{p_3\tilde{p}_2}{2}\right)\sin\left(\tfrac{p_4(\tilde{p}_3+\tilde{p}_2)}{2}\right)+(p_3\leftrightarrow p_4)\right]$$
$$+(p_1\leftrightarrow p_2)\Bigg\}. \quad \text{(B.15)}$$

2B4A vertex: Finally, the only term with two B fields and four A fields is (o) of Eqn. (B.11)



$$= V^{2B4A}_{\mu\nu,\rho\sigma,\tau,\varepsilon,\kappa,\iota}(p_{\{1..6\}})$$
$$= -(2\pi)^{16}\frac{\delta}{\delta B_{\mu\nu}(-p_1)}\frac{\delta}{\delta B_{\rho\sigma}(-p_2)}\frac{\delta}{\delta A_\tau(-p_3)}\frac{\delta}{\delta A_\varepsilon(-p_4)}\frac{\delta}{\delta A_\kappa(-p_5)}\frac{\delta}{\delta A_\iota(-p_6)}\left[(g)+(k)+(n)\right]$$
$$= 4g^4\theta^2(2\pi)^4\delta^4(p_1+p_2+p_3+p_4+p_5+p_6)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times$$
$$\times\Bigg\{2\delta_{\tau\epsilon}\delta_{\kappa\iota}\left[\sin\left(\tfrac{p_4\tilde{p}_1}{2}\right)\sin\left(\tfrac{p_3(\tilde{p}_4+\tilde{p}_1)}{2}\right)\sin\left(\tfrac{p_6\tilde{p}_2}{2}\right)\sin\left(\tfrac{p_5(\tilde{p}_6+\tilde{p}_2)}{2}\right)+(p_3\leftrightarrow p_4)+(p_5\leftrightarrow p_6)\right]$$
$$+\delta_{\tau\kappa}\delta_{\epsilon\iota}\left[\sin\left(\tfrac{p_5\tilde{p}_1}{2}\right)\sin\left(\tfrac{p_3(\tilde{p}_5+\tilde{p}_1)}{2}\right)\sin\left(\tfrac{p_6\tilde{p}_2}{2}\right)\sin\left(\tfrac{p_4(\tilde{p}_6+\tilde{p}_2)}{2}\right)+(p_3\leftrightarrow p_5)+(p_4\leftrightarrow p_6)\right]$$
$$+\delta_{\tau\iota}\delta_{\kappa\epsilon}\left[\sin\left(\tfrac{p_6\tilde{p}_1}{2}\right)\sin\left(\tfrac{p_3(\tilde{p}_6+\tilde{p}_1)}{2}\right)\sin\left(\tfrac{p_4\tilde{p}_2}{2}\right)\sin\left(\tfrac{p_5(\tilde{p}_4+\tilde{p}_2)}{2}\right)+(p_3\leftrightarrow p_6)+(p_5\leftrightarrow p_4)\right]$$
$$+(p_1\leftrightarrow p_2)\Bigg\}. \quad \text{(B.16)}$$

B.2 One Loop Calculations

This section contains exemplary computations and detailed results of the one loop vacuum polarization discussed in the main text in Section 3.1.3. The graphs in Fig. 3.2 on page 39 correspond to the expressions

$$\Pi^{(\mathrm{a})}_{\mu\nu} = s_a \frac{4g^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{\mathcal{G}(k)\mathcal{G}(k+p)} \left\{ \frac{k_\mu k_\nu}{k^2} \left(11k^2 - p^2 + 2kp + \frac{(p^2-k^2)^2}{(k+p)^2} \right) \right.$$
$$+ \delta_{\mu\nu} \left[\alpha \left(\frac{(k^2+2kp)^2}{\mathcal{G}(k)} + \frac{(k^2-p^2)^2}{\mathcal{G}(k+p)} \right) + k^2 + 5p^2 - 2kp - \frac{(k^2-p^2)^2}{(k+p)^2} - 4\frac{(kp)^2}{k^2} \right]$$
$$+ \alpha k_\mu k_\nu \left(\frac{p^2-k^2-2kp}{\mathcal{G}(k)} - \frac{\frac{p^4}{(k+p)^2}\left(\frac{a^2}{(\tilde{k}+\tilde{p})^2} - (k+p)^2(\alpha-1)\right)}{\mathcal{G}(k)\mathcal{G}(k+p)} - \frac{(k^2-p^2)^2}{k^2\mathcal{G}(k+p)} \right)$$
$$+ p_\mu p_\nu \frac{\left(\frac{a^2}{(\tilde{k}+\tilde{p})^2} - (k+p)^2(\alpha-1)\right)}{(\mathcal{G}(k+p))\,(k+p)^2} \left(\frac{k^2p^2+(kp)^2-2k^4}{k^2} - \alpha\frac{(kp)^2}{\mathcal{G}(k)} \right)$$
$$+ p_\mu p_\nu \left(-3 + \alpha\frac{k^2}{\mathcal{G}(k)} \right) + (k_\mu p_\nu + p_\mu k_\nu) \left(3\frac{kp+2k^2}{k^2} - \alpha\frac{3kp+k^2}{\mathcal{G}(k)} \right)$$
$$\left. + (k_\mu p_\nu + p_\mu k_\nu)(kp) \frac{\left(\frac{a^2}{(\tilde{k}+\tilde{p})^2} - (k+p)^2(\alpha-1)\right)}{\mathcal{G}(k+p)(k+p)^2} \left(\frac{k^2-p^2}{k^2} + \alpha\frac{p^2}{\mathcal{G}(k)} \right) \right\}, \quad \text{(B.17a)}$$

$$\Pi^{(\mathrm{b})}_{\mu\nu} = -s_b \frac{8g^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{\mathcal{G}(k)} \left[2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \alpha\frac{(k^2\delta_{\mu\nu} - k_\mu k_\nu)}{\mathcal{G}(k)} \right], \quad \text{(B.17b)}$$

$$\Pi^{(\mathrm{c})}_{\mu\nu} = -s_c \frac{4g^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\mu(k+p)_\nu}{k^2(k+p)^2} \sin^2\left(\frac{k\tilde{p}}{2}\right), \quad \text{(B.17c)}$$

$$\Pi^{(\mathrm{d})}_{\mu\nu} = s_d \frac{4g^2\theta^4}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{\left[(k+p)^2+k^2\right]^2 (2k+p)_\mu(2k+p)_\nu}{k^2\tilde{k}^2(k+p)^2(\tilde{k}+\tilde{p})^2} \times$$
$$\times \left[6 - \frac{3a^2}{\tilde{k}^2\mathcal{G}(k)} - \frac{3a^2}{(\tilde{k}+\tilde{p})^2\mathcal{G}(k+p)} + \frac{a^4\left(k^2(k+p)^2 + 2[k(k+p)]^2\right)}{k^2\tilde{k}^2(k+p)^2(\tilde{k}+\tilde{p})^2\mathcal{G}(k)\mathcal{G}(k+p)} \right], \quad \text{(B.17d)}$$

$$\Pi^{(\mathrm{e})}_{\mu\nu} = -s_e \frac{24g^2}{(2\pi)^4} \int \mathrm{d}^4k \, \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^4} [p_\mu p_\nu + 4k_\mu k_\nu + 2k^2\delta_{\mu\nu}] \left[2 - \frac{a^2}{\tilde{k}^2\mathcal{G}(k)} \right], \quad \text{(B.17e)}$$

$$\Pi^{(\mathrm{f})}_{\mu\nu} = s_f \frac{4\,a^4g^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{3k_\mu k_\nu + 2k_\mu p_\nu + k_\nu p_\mu}{k^2\tilde{k}^2(k+p)^2(\tilde{k}+\tilde{p})^2\mathcal{G}(k)\mathcal{G}(k+p)}, \quad \text{(B.17f)}$$

$$\Pi^{(\mathrm{g})}_{\mu\nu} = s_g \frac{4\,a^2g^2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^2\tilde{k}^2\mathcal{G}(k+p)} \left\{ 2\delta_{\mu\nu} + \frac{(k+p)_\mu(k+p)_\nu}{(k+p)^2} \right.$$
$$+ \frac{a^2}{k^2\tilde{k}^2\mathcal{G}(k)} \left[\delta_{\mu\nu}\left(\frac{[k(k+p)]^2}{(k+p)^2} - k^2 \right) - k_\mu k_\nu - \frac{k(k+p)}{(k+p)^2}\left(2k_\mu k_\nu + k_\mu p_\nu + p_\mu k_\nu\right) \right]$$
$$+ \frac{\alpha}{\mathcal{G}(k+p)\mathcal{G}(k)} \left(\delta_{\mu\nu} \left[k^2(k+p)^2 - a^2\frac{(kp)^2 - k^2p^2}{k^2\tilde{k}^2} \right] - k^2(k_\mu+p_\mu)(k_\nu+p_\nu) \right.$$
$$\left.\left. - \frac{a^2}{k^2\tilde{k}^2}\left(k^2p_\mu p_\nu + p^2k_\mu k_\nu - (kp)(k_\mu p_\nu + p_\mu k_\nu)\right) \right) \right\}, \quad \text{(B.17g)}$$

$$\Pi^{(\mathrm{h+i})}_{\mu\nu} = s_{\mathrm{h}} \frac{4\,a^2 g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{\tilde{k}^2 (k+p)^2 \mathcal{G}(k)} \left(\frac{1}{k^2} + \frac{1}{(k+p)^2}\right) (2k_\mu + p_\mu) \times$$
$$\times \left[3k_\nu - a^2 \frac{k_\nu[(k+p)^2 + 2k(k+p)] + 2p_\nu[k(k+p)]}{(k+p)^2(\tilde{k}+\tilde{p})^2\mathcal{G}(k+p)}\right] + \mu \leftrightarrow \nu, \tag{B.17h}$$

$$\Pi^{(\mathrm{j})}_{\mu\nu} = s_j \frac{4a^2 g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \left(\frac{1}{(k+p)^2} + \frac{1}{k^2}\right) \times$$
$$\times \frac{(2k+p)_\mu \left[(6k^2 + 6kp + 2p^2)k_\nu + (3k^2 + kp)p_\nu\right]}{\tilde{k}^2 (k+p)^2 \mathcal{G}(k)\mathcal{G}(k+p)}, \tag{B.17i}$$

$$\Pi^{(\mathrm{k+l})}_{\mu\nu} = s_{\mathrm{k}} \frac{4\,a^2 g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^2 \tilde{k}^2 \mathcal{G}(k)\mathcal{G}(k+p)} \Bigg\{ 3k_\mu k_\nu + 2p_\mu k_\nu + k_\mu p_\nu$$
$$+ \delta_{\mu\nu}\left[k(k-p) + k(k+p)\frac{(p^2-k^2)}{(k+p)^2} - \alpha \frac{k(k+p)(p^2-k^2)}{\mathcal{G}(k+p)}\right]$$
$$+ \frac{1}{(k+p)^2}\Big(k(k+p)(k_\mu k_\nu - p_\mu p_\nu) + (p^2 + 2k^2 + 3(kp))(k_\mu k_\nu + k_\mu p_\nu)\Big)$$
$$- \frac{\alpha}{\mathcal{G}(k+p)}\Big(k(k+p)(k_\mu k_\nu - p_\mu p_\nu) + (p^2 + 2k^2 + 3(kp))(k_\mu k_\nu + k_\mu p_\nu)$$
$$- (k+p)^2(2k_\mu k_\nu + k_\mu p_\nu)\Big) + \mu \leftrightarrow \nu \Bigg\}. \tag{B.17j}$$

where the Feynman rules (3.6a)–(3.6b), and (B.6)–(B.14) have been applied, and the abbreviation $\mathcal{G}(k) := \left(k^2 + \frac{a^2}{\tilde{k}^2}\right)$ has been introduced.

It is obvious that some of these integrals require three or more Schwinger parameters to be computed. In addition, the appearance of $\mathcal{G}(k+p)$ represents a serious obstacle for practical calculations. Therefore, the expansion (3.17) for small external momenta p is applied to the integrands without phase. In lowest order, the expressions (B.17a)–(B.17j) are (where the superscript 0 indicates the order)

$$\Pi^{(\mathrm{a}),0}_{\mu\nu} \approx s_a \frac{8g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{\mathcal{G}(k)^2} \left[6k_\mu k_\nu + \alpha\, k^2 \frac{(k^2\delta_{\mu\nu} - k_\mu k_\nu)}{\mathcal{G}(k)}\right], \tag{B.18a}$$

$$\Pi^{(\mathrm{b}),0}_{\mu\nu} \approx -s_b \frac{8g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{k^2 + \frac{a^2}{\tilde{k}^2}} \left[2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \alpha \frac{(k^2\delta_{\mu\nu} - k_\mu k_\nu)}{\mathcal{G}(k)}\right], \tag{B.18b}$$

$$\Pi^{(\mathrm{c}),0}_{\mu\nu} \approx -s_c \frac{4g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{k_\mu k_\nu}{k^4}, \tag{B.18c}$$

$$\Pi^{(\mathrm{d}),0}_{\mu\nu} \approx s_d \frac{192g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{k_\mu k_\nu}{k^4} \left(2 + \frac{a^2\left(\frac{a^2}{\tilde{k}^2} - 2\right)}{\tilde{k}^2\mathcal{G}(k)}\right), \tag{B.18d}$$

$$\Pi^{(\mathrm{e}),0}_{\mu\nu} \approx -s_e \frac{24g^2}{(2\pi)^4} \int \mathrm{d}^4k\, \frac{\sin^2(\frac{k\tilde{p}}{2})}{k^4} [4k_\mu k_\nu + 2k^2\delta_{\mu\nu}] \left(2 - \frac{a^2}{\tilde{k}^2\mathcal{G}(k)}\right), \tag{B.18e}$$

$$\Pi^{(\mathrm{f}),0}_{\mu\nu} \approx s_f \frac{12a^4g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{k_\mu k_\nu}{\left(k^2\tilde{k}^2\right)^2 \mathcal{G}(k)^2}\,, \tag{B.18f}$$

$$\Pi^{(\mathrm{g}),0}_{\mu\nu} \approx s_g \frac{4\,a^2g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^2\tilde{k}^2\mathcal{G}(k)} \left[2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} - \frac{3a^2k_\mu k_\nu}{k^2\tilde{k}^2\mathcal{G}(k)} + \alpha k^2 \frac{\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)}{\mathcal{G}(k)^2}\right]\,, \tag{B.18g}$$

$$\Pi^{(\mathrm{h+i}),0}_{\mu\nu} \approx s_h \frac{48\,a^2g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right) k_\mu k_\nu}{\tilde{k}^2k^4\mathcal{G}(k)} \left(1 - \frac{a^2}{\tilde{k}^2\mathcal{G}(k)}\right) + \mu \leftrightarrow \nu\,, \tag{B.18h}$$

$$\Pi^{(\mathrm{j}),0}_{\mu\nu} \approx s_j \frac{48\,a^2g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right) k_\mu k_\nu}{k^2\tilde{k}^2\mathcal{G}(k)^2}\,, \tag{B.18i}$$

$$\Pi^{(\mathrm{k+l}),0}_{\mu\nu} \approx s_k \frac{4\,a^2g^2}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^2\tilde{k}^2\mathcal{G}(k)^2} \left[6k_\mu k_\nu + \frac{\alpha k^2}{\mathcal{G}(k)}\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)\right] + \mu \leftrightarrow \nu\,. \tag{B.18j}$$

The notion $+\mu \leftrightarrow \nu$ refers to the addition of the expressions for the mirrored graphs (i) and (k) to $\Pi^{(\mathrm{h+i}),0}_{\mu\nu}$ and $\Pi^{(\mathrm{k+l}),0}_{\mu\nu}$, respectively. Explicitly, this summation is eased by the relations

$$\Pi^{(\mathrm{h}),0}_{\mu\nu} + \Pi^{(\mathrm{i}),0}_{\nu\mu} = 0\,, \quad \text{and } \Pi^{(\mathrm{k}),0}_{\mu\nu} + \Pi^{(\mathrm{l}),0}_{\nu\mu} = 0\,,$$

which motivates the collection into one term in Eqns. (B.18h) and (B.18j).
Power counting (with respect to k) reveals that only the integrals (B.18a)–(B.18e) (being also listed in Section 3.1.3) give divergent contributions. These integrals can be solved utilizing the rules in Appendix F.1.

Practically, the sum of graphs (a)–(e) is computed prior to integration in order to benefit from cancellations between different contributions. However, the high number of terms, and the stereotype way of repeated application of the integration rules motivates to conduct the calculations with the help of modern computing aids. For this purpose a package, called `VectorAlgebra`, has been written in Wolfram Mathematica® 6, capable of the simplification (i.e. collection, cancellation, transformation) of terms, partial differentiation, series expansion, and integration (for a specific range of integral types appearing in the current calculations) under consideration of the Einstein sum convention, and the specific properties of the Groenewold-Moyal product. The respective source code is listed in Appendix G.3.

After computation of the integrals and collection of terms with the help of `VectorAlgebra` we receive the results

$$\begin{aligned}\Pi^{\mathrm{p},0}_{\mu\nu} &= -\frac{g^2}{(4\pi)^2}\Lambda^2\left(2s_b(\alpha+5) - 2\left((\alpha+6)s_a + s_c + 48\left(s_d - s_e\right)\right)\right)\delta_{\mu\nu} + \text{finite}\\ &= 0 + \text{finite}\,,\end{aligned} \tag{B.19}$$

$$\begin{aligned}\Pi^{\mathrm{np},0}_{\mu\nu} &= \frac{g^2}{(2\pi)^2}\frac{1}{\tilde{p}^4}\Big[2\Big(2s_b(\alpha-1) - 2s_a\alpha - s_c + 12s_a - 48s_e + 96s_d\Big)\tilde{p}_\mu\tilde{p}_\nu\\ &\qquad + \Big(2s_b(\alpha+5) - 2\left(s_a(\alpha+6) + 48\left(s_d - s_e\right)\right) + s_c\Big)\tilde{p}^2\delta_{\mu\nu}\Big] + \text{finite}\\ &= \frac{14g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{\left(\tilde{p}^2\right)^2} + \text{finite}\,,\end{aligned} \tag{B.20}$$

where planar (p) and non-planar (np) results have been separated and the term 'finite' refers to contributions from integrals which are *a priori* finite, i.e. which are not considered here due to a dimension $d(\mathcal{I}) < -4$ obtained by power counting. Obviously, the quadratic divergence in $\Pi^{\text{p},0}_{\mu\nu}$ is eliminated by cancellations between the graphs upon insertion of the numerical values for the symmetry factors in Tab. 3.1 on page 40. In this way, $\Pi^0_{\mu\nu}$ boils down to the result Eqn. (3.19) in the main text. Note that, according to the discussion in Section 3.1.3 the finite contributions cannot be given accurately due to the approximation (3.14). Therefore, only divergent terms are given explicitly.

All integrals in the first order of expansion Eqn. (3.17) of the integrands of Eqns. (B.17a)–(B.17j) vanish identically due to the symmetric integration over an odd power (i.e. an antisymmetric function in k).

The explicit expressions for the second order terms with gauge fixing parameter $\alpha = 1$ (Feynman gauge) read

$$\Pi^{(\text{a}),2}_{\mu\nu} = s_a 4g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\frac{k\theta p}{2}}{(a^2+k^4)^5} k^{12} \Big[k^4\left(3p^2\delta_{\mu\nu} - 2p_\mu p_\nu\right) + 40(k.p)^2 k_\mu k_\nu$$
$$+ 2k^2\Big(2(kp)^2\delta_{\mu\nu} - 5\left[p^2 k_\mu k_\nu - kp\left(k_\nu p_\mu + k_\mu p_\nu\right)\right]\Big)\Big] + \text{finite}\,, \tag{B.21a}$$

$$\Pi^{(\text{b}),2}_{\mu\nu} = 0\,, \tag{B.21b}$$

$$\Pi^{(\text{c}),2}_{\mu\nu} = s_c 4g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \sin^2\frac{k\theta p}{2}\left[k_\mu k_\nu\left(\frac{p^2}{k^6} - 4\frac{(kp)^2}{k^8}\right) + 2p_\mu k_\nu \frac{kp}{k^6}\right], \tag{B.21c}$$

$$\Pi^{(\text{d}),2}_{\mu\nu} = s_d 96g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\frac{k\theta p}{2}}{(a^2+k^4)^4} k^8 \Big[k^2\left(k^2 p_\mu p_\nu - 4p^2 k_\mu k_\nu\right)$$
$$- 4k^2 kp\left(k_\nu p_\mu + k_\mu p_\nu\right) + 20(kp)^2 k_\mu k_\nu\Big] + \text{finite}, \tag{B.21d}$$

$$\Pi^{(\text{e}),2}_{\mu\nu} = -s_e 24g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{p_\mu p_\nu \sin^2\frac{k\theta p}{2}}{k^4}\left(2 - \frac{a'^2}{\tilde{k}^2\left(k^2+\frac{a^2}{\tilde{k}^2}\right)}\right). \tag{B.21e}$$

$$\Pi^{(\text{f}),2}_{\mu\nu} = -s_f 4a^4 g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\frac{k\theta p}{2}}{k^6\tilde{k}^6\left(k^2+\frac{a^2}{\tilde{k}^2}\right)^3} \Big[k^2\left(a^2+3k^4\right)\left[3p^2 k_\mu k_\nu + 2kp\left(k_\nu p_\mu + 2k_\mu p_\nu\right)\right]$$
$$- 12\frac{a^4 + 3a^2k^4 + 6k^8}{k^2\tilde{k}^2\left(k^2+\frac{a^2}{\tilde{k}^2}\right)}(kp)^2 k_\mu k_\nu\Big]\,, \tag{B.21f}$$

$$\Pi^{(\text{g}),2}_{\mu\nu} = -s_g 4a^2 g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\frac{k\theta p}{2}}{k^6\tilde{k}^8\left(k^2+\frac{a^2}{\tilde{k}^2}\right)^4} \Big[a^2k^2\left(a^2+k^4\right)^2 p_\mu p_\nu$$
$$- a^2\left(a^4 + a^2k^4 + 8k^8\right) kp\left(k_\nu p_\mu + k_\mu p_\nu\right)$$
$$+ p^2\Big[k^2(a^2-k^4)(a^4+2a^2k^4+3k^8)\delta_{\mu\nu} - a^2\left(a^4 - a^2k^4 + 2k^8\right)k_\mu k_\nu\Big]$$
$$+ \frac{(kp)^2}{a^2+k^4}\Big[\left(a^8 - 22a^6k^4 - 27a^4k^8 - 40a^2k^{12} + 12k^{16}\right)\delta_{\mu\nu}$$
$$+ 8a^2k^2\left(3a^4 + a^2k^4 + 4k^8\right)k_\mu k_\nu\Big]\Big]\,,$$

$$\Pi^{(\mathrm{h+i}),2}_{\mu\nu} = s_h 4a^2 g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\frac{k\theta p}{2}}{k^8\tilde{k}^6\left(a^2+k^4\right)^3} \Big[2a^2\left(a^2+k^4\right)k^2 p_\mu p_\nu - \left(4a^4+7a^2k^4-9k^8\right)p^2 k_\mu k_\nu$$
$$- kp\left[16a^2\left(a^2+2k^4\right)k_\mu p_\nu + \left(2a^4+5a^2k^4-9k^8\right)k_\nu p_\mu\right]$$
$$+ \frac{4(kp)^2}{k^2(a^2+k^4)}\left(5a^6+14a^4k^4+21a^2k^8-12k^{12}\right)k_\mu k_\nu\Big] + \mu\leftrightarrow\nu\,, \tag{B.21g}$$

$$\Pi^{(\mathrm{j}),2}_{\mu\nu} = s_j 8a^2 g^2 \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\frac{k\theta p}{2}}{k^2\left(a^2+k^4\right)^3} \Big[3k^2\left(a^2+k^4\right)p_\mu p_\nu - 2p^2\left(a^2+13k^4\right)k_\mu k_\nu$$
$$- 4kp\left[\left(a^2+7k^4\right)k_\mu p_\nu + 6k^4 k_\nu p_\mu\right]$$
$$+ \frac{12(kp)^2}{k^2(a^2+k^4)}\left(a^4-2a^2k^4+13k^8\right)k_\mu k_\nu\Big] \tag{B.21h}$$

$$\Pi^{(\mathrm{k+l}),2}_{\mu\nu} = 4a^2 g^2 s_k \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\frac{k\theta p}{2}}{\left(a^2+k^4\right)^5} \Big[\frac{a^6}{k^2}\left[4p^2k_\mu k_\nu + kp\left(4k_\nu p_\mu + 5k_\mu p_\nu\right)\right] - k^{12}p^2\delta_{\mu\nu}$$
$$+ a^4k^2\Big[k^2\left(7p^2\delta_{\mu\nu} - 2p_\mu p_\nu\right) + \left(2kp\left(2kp\,\delta_{\mu\nu} + 2k_\nu p_\mu - k_\mu p_\nu\right) - p^2 k_\mu k_\nu\right)\Big]$$
$$+ k^8\left(a^2\left(4p^2\delta_{\mu\nu} - p_\mu p_\nu\right) + 20(kp)^2 k_\mu k_\nu\right)$$
$$- a^2k^6\left(10p^2k_\mu k_\nu + kp\left(40kp\delta_{\mu\nu} + 4k_\nu p_\mu + 11k_\mu p_\nu\right)\right)$$
$$+ \left(2a^6p^2\delta_{\mu\nu} - a^6 p_\mu p_\nu - 68a^4(kp)^2 k_\mu k_\nu\right)$$
$$+ k^{10}\left(-5p^2k_\mu k_\nu + 4(kp)^2\delta_{\mu\nu} - 4kp\left(k_\nu p_\mu + k_\mu p_\nu\right)\right)\Big] + \mu\leftrightarrow\nu\,. \tag{B.21i}$$

As before, in order zero, the approximation (3.14) is applied in order to simplify the denominators of the expressions prior to integration. The resulting integrals can then be evaluated by utilization of the prescriptions in Appendix F.1. Furthermore, restricting ourselves to divergent contributions only, we do not have to consider the expressions of all graphs. Power counting with respect to k reveals a superficial degree of divergence $d\big(\mathcal{I}^{\{(a),(c),(d),(e)\},2}_{\mu\nu}\big) = 0$, corresponding to at most logarithmic singularities. All other integrals only give finite contributions, and need not be considered, i.e. $d\big(\mathcal{I}^{\{(b),(f),(g),(h+i),(j),(k+l)\},2}_{\mu\nu}\big) < 0$. The explicit results of the integration (again with the help of `VectorAlgebra`) read

$$\Pi^{\mathrm{p},2}_{\mu\nu}(p) + \Pi^{\mathrm{np},2}_{\mu\nu}(p) = \Pi^{(\mathrm{a}),2}_{\mu\nu} + \Pi^{(\mathrm{c}),2}_{\mu\nu} + \Pi^{(\mathrm{d}),2}_{\mu\nu} + \Pi^{(\mathrm{e}),2}_{\mu\nu} + \text{finite}\,, \tag{B.22}$$

$$\Pi^{\mathrm{p},2}_{\mu\nu} = \frac{g^2}{3(4\pi)^2}\mathrm{K}_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\Big[\left(s_a(50-12\alpha)+s_c-48s_d\right)p^2\delta_{\mu\nu}$$
$$+ 2p_\mu p_\nu\left(s_a(6\alpha-28)+s_c+96s_d-72s_e\right)\Big] + \text{finite}$$
$$= \frac{g^2}{3(4\pi)^2}(3\alpha-1)\Big[\ln\left(\frac{\mu^2}{4\Lambda^2}\right) + 2\gamma_E\Big]\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) + \text{finite}\,, \tag{B.23}$$

$$\Pi^{\mathrm{np},2}_{\mu\nu} = -\frac{g^2}{3(4\pi)^4}\Big[2\left((11-15\alpha)s_a - 2s_c - 168s_d\right)\frac{\tilde{p}_\mu\tilde{p}_\nu}{\theta^2}$$
$$+ \Big[p^2\delta_{\mu\nu}\left(s_a(50-12\alpha)+s_c-48s_d\right)$$
$$- 2p_\mu p_\nu\left(s_a(28-6\alpha)-s_c+72s_e-96s_d\right)\Big]K_0\left(\sqrt{\mu^2\tilde{p}^2}\right)\Big] + \text{finite} \tag{B.24}$$
$$= \frac{g^2}{3(4\pi)^2}\Big[2(3\alpha-17)\frac{\tilde{p}_\mu\tilde{p}_\nu}{\theta^2} + (1-3\alpha)\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left(\ln\left(\frac{\mu^2\tilde{p}^2}{4}\right) + 2\gamma_E\right)\Big] + \text{finite}\,. \tag{B.25}$$

Note that transversality can only be seen for the divergent terms at this place since all convergent contributions coming from graphs (b), and (f)–(l) have been omitted. In addition, the approximation (3.14) introduces indeterminate errors in the finite parts. Summing up the contributions (B.23) and (B.25) the μ-dependent divergences cancel each other, and we arrive at

$$\begin{aligned}\Pi^2_{\mu\nu} &= \frac{g^2}{3(4\pi)^2}\left[(6\alpha-34)\frac{\tilde{p}_\mu\tilde{p}_\nu}{\theta^2} - (p^2\delta_{\mu\nu} - p_\mu p_\nu)(3\alpha-1)\ln\left(\tilde{p}^2\Lambda^2\right)\right] + \text{finite} \\ &\approx -\frac{g^2}{3(4\pi)^2}\left[(p^2\delta_{\mu\nu} - p_\mu p_\nu)(3\alpha-1)\ln\left(\tilde{p}^2\Lambda^2\right)\right] + \text{finite}\,, \end{aligned} \tag{B.26}$$

where the approximation refers to the omission of finite terms (i.e. terms which remain finite in the limits $\tilde{p}\to 0$, $\mu\to 0$, and $\Lambda\to\infty$). The later result equals the result (3.20) stated in the main text for Feynman gauge $\alpha\to 1$.

B.3 Miscellanea

Theorem 2. *The covariant derivative D^n, $n\in\mathbb{N}^0$ fulfills the rule $\int \mathrm{d}^4x\, A\star(D^n\star B) = (-1)^i\int\mathrm{d}^4x\,(D^iA)\star D^{n-i}\star B$, with A, B being operators, $i\in\{0..n\}$, for partial integration.*

Proof. The proof is given by induction

$n=1$:

$$\begin{aligned}\int \mathrm{d}^Dx\, A^{\mu\nu}\star D_\rho\star B_{\mu\nu} &= \int\mathrm{d}^Dx\, A^{\mu\nu}\star\left(\partial_\rho\star B_{\mu\nu} - \mathrm{i}g\,[A_\rho \stackrel{\star}{,} B_{\mu\nu}]\right) \\ &= \int\mathrm{d}^Dx\, -(\partial_\rho A^{\mu\nu})\star B_{\mu\nu} - A^{\mu\nu}\star A_\rho\star B_{\mu\nu} + A^{\mu\nu}\star B_{\mu\nu}\star A_\rho \\ &= -\int\mathrm{d}^Dx\, -(D_\rho A^{\mu\nu})\,B_{\mu\nu}, \end{aligned} \tag{B.27}$$

where the cyclic permutation property of the star product has been applied in the last step.

$n=2$:

$$\begin{aligned}\int \mathrm{d}^Dx\, A^{\mu\nu}\star D^2\star B_{\mu\nu} &= \int\mathrm{d}^Dx\, A^{\mu\nu}\star\big(\partial^\rho\partial_\rho\star B_{\mu\nu} - \mathrm{i}g\partial^\rho\,[A_\rho \stackrel{\star}{,} B_{\mu\nu}] - \\ &\qquad - \mathrm{i}g\,[A^\rho \stackrel{\star}{,} (\partial_\rho B_{\mu\nu})] - g^2\,[A^\rho \stackrel{\star}{,} [A_\rho \stackrel{\star}{,} B_{\mu\nu}]]\big) \\ &= \int\mathrm{d}^Dx\, -(\partial^\rho B_{\mu\nu})\star\partial_\rho\star B_{\mu\nu} + \mathrm{i}g\,(\partial^\rho B_{\mu\nu})\star[A_\rho \stackrel{\star}{,} B_{\mu\nu}] + \\ &\qquad + \mathrm{i}g\,[(\partial_\rho\,(A_{\mu\nu}\star A_\rho))\star B_{\mu\nu} - (\partial_\rho\,(A_\rho\star A_{\mu\nu}))\star B_{\mu\nu}] + \\ &\qquad + g^2\,[A^\rho \stackrel{\star}{,} A^{\mu\nu}]\,[A_\rho \stackrel{\star}{,} A_{\mu\nu}] \\ &= -\int\mathrm{d}^Dx\,(D^\rho A^{\mu\nu})\,D_\rho B_{\mu\nu} \\ &= -\int\mathrm{d}^Dx\,\left(D^2A^{\mu\nu}\right)B_{\mu\nu}, \end{aligned} \tag{B.28}$$

where again the cyclic permutation property gave $[A^\rho \stackrel{\star}{,} [A_\rho \stackrel{\star}{,} B_{\mu\nu}]] = -[A^\rho \stackrel{\star}{,} A^{\mu\nu}][A_\rho \stackrel{\star}{,} A_{\mu\nu}]$, and the last step is rectified by redefining $A'^{\mu\nu} := (D^\rho A^{\mu\nu})$ and applying the rule (B.27).

$n > 2$: The right part with $n-1$ derivatives can be separated by virtue of the (presumed) associativity of the operator D and the star product. Redefining then $D^{n-1} \star B =: B'$, the case can be reduced to Eqn. (B.28),

$$\begin{aligned}\int \mathrm{d}^D x\, A^{\mu\nu} \star D^n \star B_{\mu\nu} &= \int \mathrm{d}^D x\, A^{\mu\nu} \star D \Big(\underbrace{D^{n-1} \star B_{\mu\nu}}_{:=B'_{\mu\nu}} \Big) \\ &= -\int \mathrm{d}^D x\, \left(D^\rho A^{\mu\nu}\right) \star B'_{\mu\nu} \\ &= -\int \mathrm{d}^D x\, \left(D^\rho A^{\mu\nu}\right) \star D^{n-1} \star B_{\mu\nu}\,. \qquad \text{(B.29)}\end{aligned}$$

$\square$

Theorem 3. *The operator* $\frac{1}{\square}$ *fulfills the rule for partial integration:*

$$\int \mathrm{d}x\, A \frac{1}{\square} B = \int \mathrm{d}^d x \left(\frac{1}{\square} A \right) B\,,$$

where $A, B \in \mathcal{S}\,(\mathrm{Mat})$ *in* d *dimensions.*

Proof. Generally, $\int \mathrm{d}x\, A \square B = +\int \mathrm{d}x\, (\square A) B$ for $A, B \in \mathcal{S}\,(\mathrm{Mat})$. Hence, with $\square_x \square_x^{-1} = \square_x^{-1} \square_x = \mathbb{1}_d$, i.e. $\left[\square_x, \square_x^{-1}\right] = 0$,

$$\int \mathrm{d}x\, A(x) B(x) = \int \mathrm{d}x\, A(x) \square_x^{-1} \square_x B = +\int \mathrm{d}x\, \left(\square_x A(x)\right) \square_x^{-1} B.$$

On the other hand,

$$\int \mathrm{d}x\, A(x) B(x) = \int \mathrm{d}x\, \left(\square_x^{-1} \square_x A(x)\right) B.$$

A redefinition finally leads to the desired result,

$$\begin{aligned}A'(x) &:= \left(\square_x A(x)\right) \\ \int \mathrm{d}x\, \left(\square_x^{-1} A'(x)\right) B(x) &= \int \mathrm{d}x\, A'(x) \square_x^{-1} B.\end{aligned}$$

Note that, due to Theorem 2, the inverse of the covariant derivative fulfills the partial integration rule too, i.e. $\int \mathrm{d}^d x\, A \frac{1}{D^2} B = \int \mathrm{d}^d x \left(\frac{1}{D^2} A\right) B$. $\square$

Theorem 4. *For any operator* B *transforming covariantly under the BRST transformation* s*, i.e.* $sB = \mathrm{i}g\,[c \stackrel{\star}{,} B]$ *the relation* $s\left(D_\mu^n B\right) = \mathrm{i}g\left[c \stackrel{\star}{,} D_\mu^n B\right]$, $\forall n \in \mathbb{N}_0$ *holds.*

Proof. The proof is given by induction. For $n = 1$ we have,

$$\begin{aligned}s\left(D_\mu B\right) &= s\partial_\mu - \mathrm{i}gs\,[A_\mu \stackrel{\star}{,} B] \\ &= \mathrm{i}g\partial_\mu\,[c \stackrel{\star}{,} B] - \mathrm{i}g\left[(D_\mu c)B + \mathrm{i}gA_\mu\,[c \stackrel{\star}{,} B] - \mathrm{i}g\,[c \stackrel{\star}{,} B]\,A_\mu - BD_\mu c\right] \\ &= \mathrm{i}g\left[c\partial_\mu B - (\partial_\mu B)c - \mathrm{i}g\left(cA_\mu B - \mathrm{i}gcBA_\mu + A_\mu Bc + BA_\mu c + 0\right)\right] \\ &= \mathrm{i}g\,[c \stackrel{\star}{,} D_\mu B]\,.\end{aligned}$$

For $n = 2$,

$$s\left(D^2 B\right) = s\big(D_\mu \underbrace{D_\mu B}_{:=B'}\big) \,,$$

and since, as proven for $n = 1$, $sB' = \mathrm{i}g\,[c \stackrel{\star}{,} B']$, this case can be reduced to the prior step. In the same way, for general n one defines

$$s\left(D_\mu^{n+1} B\right) =: s\left(D_\mu B'\right) = \mathrm{i}g\left[c \stackrel{\star}{,} D_\mu B'\right] = \mathrm{i}g\left[c \stackrel{\star}{,} D_\mu^{n+1} B\right] \,, \tag{B.30}$$

where $sB' = \mathrm{i}g\,[c \stackrel{\star}{,} B']$ per construction. □

Theorem 5. *The inverse of the covariant derivative fulfills Theorem 4, i.e.* $s\left(\frac{1}{D^{2n}} B\right) = \left[c \stackrel{\star}{,} \frac{1}{D^{2n}} B\right]$*, for all operators B transforming covariantly in the sense of Theorem 4, and $n \in \mathbb{N}_0$.*

Proof. From Theorem 4, we have

$$\begin{aligned}
& s\left(D^2 B\right) \\
&= \left(sD^2\right) B + \mathrm{i}g D^2 \left[c \stackrel{\star}{,} B\right] \\
&= \left(sD^2\right) B + \mathrm{i}g\left(\left[D^2 c \stackrel{\star}{,} B\right] + 2\left[D_\mu c \stackrel{\star}{,} D_\mu B\right] + \left[c \stackrel{\star}{,} D^2 B\right]\right) = \mathrm{i}g\left[c \stackrel{\star}{,} D^2 B\right] \,.
\end{aligned}$$

By comparing both sides we can deduce that $\left(sD^2\right) B = -\mathrm{i}g\left(\left[D^2 c \stackrel{\star}{,} B\right] - 2\left[D_\mu c \stackrel{\star}{,} D_\mu B\right]\right)$. Since, per definition $sB = \mathrm{i}g\,[c \stackrel{\star}{,} B]$, and $(D^2)^{-n}(D^2)^n = (D^2)^n(D^2)^{-n} = \mathbb{1}_d$, for $n = 1$ we can write

$$\begin{aligned}
sB = s\left(\mathbb{1}B\right) &= s\left(D^2 (D^2)^{-1} B\right) \\
&= \left(sD^2\right)(D^2)^{-1} B + D^2 s\left((D^2)^{-1} B\right) \\
&= -\mathrm{i}g\left[D^2 c \stackrel{\star}{,} (D^2)^{-1} B\right] - 2\mathrm{i}g\left[D_\mu c \stackrel{\star}{,} D_\mu (D^2)^{-1} B\right] + D^2 s\left((D^2)^{-1} B\right) \equiv \mathrm{i}g\left[c \stackrel{\star}{,} B\right] \,.
\end{aligned}$$

In order for the latter equivalence to hold, the identification $s\left((D^2)^{-1} B\right) \equiv \mathrm{i}g\left[c \stackrel{\star}{,} (D^2)^{-1} B\right]$ is required (as is seen by explicitly writing out the terms).
The rest of the proof proceeds along the lines of Theorem 4. For $n = 2$ one can redefine $(D^2)^{-2} D^4 B =: (D^2)^{-1} B'$, which again reduces the situation to the case $n = 1$. The same is immediately true for arbitrary $n > 0$. □

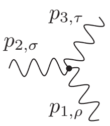
Appendix C

Supplementary Content to the $1/p^2$ Gauge Model with BRST Doublets

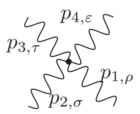
For the sake of completeness the Feynman rules of the model shall be listed in their full form in this section. The respective derivations are conducted in the same way as described in Appendix A.1 and Appendix B.1.

C.1 Feynman Rules

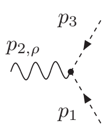
Conceptually, details of the computation of the equations of motion and propagators for the model of Section 3.2 is postponed to Appendix E.1 where a detailed analysis of the damping mechanisms is presented. Propagators are given in Eqns. (3.36a)–(3.36d), and vertices are listed below. The vertex expressions are obtained by direct variation of the action (3.35) with respect to the fields.



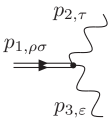
$$
\begin{aligned}
&= V^{3A}_{\rho\sigma\tau}(p_1,p_2,p_3) \\
&= 2\mathrm{i}g(2\pi)^4\delta^4(p_1+p_2+p_3)\sin\left(\tfrac{p_1\tilde{p}_2}{2}\right)\times \\
&\quad\times\left[(p_3-p_2)_\rho\delta_{\sigma\tau}+(p_1-p_3)_\sigma\delta_{\rho\tau}+(p_2-p_1)_\tau\delta_{\rho\sigma}\right],
\end{aligned}
\tag{C.1a}
$$



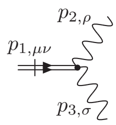
$$
\begin{aligned}
&= V^{4A}_{\rho\sigma\tau\epsilon}(p_1,p_2,p_3,p_4) \\
&= -4g^2(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\times \\
&\quad\times\Big[(\delta_{\rho\tau}\delta_{\sigma\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{p_1\tilde{p}_2}{2}\right)\sin\left(\tfrac{p_3\tilde{p}_4}{2}\right) \\
&\quad+(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{p_1\tilde{p}_3}{2}\right)\sin\left(\tfrac{p_2\tilde{p}_4}{2}\right) \\
&\quad+(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\tau}\delta_{\sigma\epsilon})\sin\left(\tfrac{p_2\tilde{p}_3}{2}\right)\sin\left(\tfrac{p_1\tilde{p}_4}{2}\right)\Big],
\end{aligned}
\tag{C.1b}
$$



$$
\begin{aligned}
&= V^{\bar{c}Ac}_{\mu}(p_1,p_2,p_3) \\
&= -2\mathrm{i}g(2\pi)^4\delta^4(p_1+p_2+p_3)p_{1\mu}\sin\left(\tfrac{p_1\tilde{p}_3}{2}\right),
\end{aligned}
\tag{C.1c}
$$

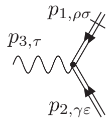


$$= V^{BAA}_{\mu\nu,\rho\sigma}(p_1, p_2, p_3) =$$

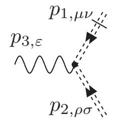


$$V^{\bar{B}AA}_{\mu\nu,\rho\sigma}(p_1, p_2, p_3)$$

$$= \lambda g (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) \sin\left(\frac{p_2\tilde{p}_3}{2}\right), \tag{C.1d}$$

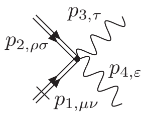


$$= V^{\bar{B}BA}_{\mu\nu,\rho\sigma\epsilon}(p_1, p_2, p_3) = -$$

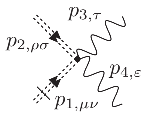


$$= -V^{\bar{\psi}\psi A}_{\mu\nu,\rho\sigma\epsilon}(p_1, p_2, p_3)$$

$$= -\mathrm{i}\mu^2 g (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) \times \\ \times \left((\tilde{p}_1)^2 + (\tilde{p}_2)^2\right) (p_1 - p_2)_\epsilon \sin\left(\frac{p_1\tilde{p}_2}{2}\right), \tag{C.1e}$$

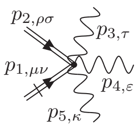


$$= V^{\bar{B}B2A}_{\mu\nu,\rho\sigma,\tau\epsilon}(p_1, p_2, p_3, p_4) =$$

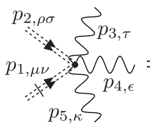


$$= V^{\bar{\psi}\psi 2A}_{\mu\nu,\rho\sigma,\tau\epsilon}(p_1, p_2, p_3, p_4)$$

$$= 2\mu^2 g^2 \theta^2 (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) \times \\ \times \Big\{ \left[p_{3,\tau}p_{4,\varepsilon} + 2\left(p_{1,\tau}p_{4,\varepsilon} + p_{2,\varepsilon}p_{3,\tau}\right) + 4p_{1,\tau}p_{2,\varepsilon} - \delta_{\varepsilon\tau}\left({p_1}^2 + {p_2}^2\right)\right] \sin\left(\tfrac{p_1\tilde{p}_3}{2}\right) \sin\left(\tfrac{p_2\tilde{k}_4}{2}\right) \\ + \left[p_{3,\tau}p_{4,\varepsilon} + 2\left(p_{2,\tau}p_{4,\varepsilon} + p_{1,\varepsilon}p_{3,\tau}\right) + 4p_{1,\varepsilon}p_{2,\tau} - \delta_{\varepsilon\tau}\left({p_1}^2 + {p_2}^2\right)\right] \sin\left(\tfrac{p_1\tilde{p}_4}{2}\right) \sin\left(\tfrac{p_2\tilde{p}_3}{2}\right) \Big\}, \tag{C.1f}$$

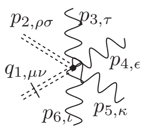


$$= V^{\bar{B}B3A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa}(p_1, p_2, p_3, p_4, p_5) = -$$

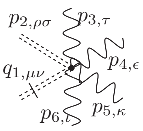


$$= -V^{\bar{\psi}\psi 3A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa}(q_1, q_2, p_3, p_4, p_5)$$

$$= -4\mathrm{i} g^3 \mu^2 \theta^2 (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4 + p_5) \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) \times \\ \times \Big\{ \left[p_3 + 2p_1\right]_\tau \delta_{\epsilon\kappa} \sin\left(\tfrac{p_3\tilde{p}_1}{2}\right) \left[\sin\left(\tfrac{p_5\tilde{p}_2}{2}\right) \sin\left(\tfrac{p_4(\tilde{p}_5+\tilde{p}_2)}{2}\right) + (p_4 \leftrightarrow p_5)\right] \\ + \left[p_4 + 2p_1\right]_\epsilon \delta_{\tau\kappa} \sin\left(\tfrac{p_4\tilde{p}_1}{2}\right) \left[\sin\left(\tfrac{p_5\tilde{p}_2}{2}\right) \sin\left(\tfrac{p_3(\tilde{p}_5+\tilde{p}_2)}{2}\right) + (p_5 \leftrightarrow p_3)\right] \\ + \left[p_5 + 2p_1\right]_\kappa \delta_{\tau\epsilon} \sin\left(\tfrac{p_5\tilde{p}_1}{2}\right) \left[\sin\left(\tfrac{p_3\tilde{p}_2}{2}\right) \sin\left(\tfrac{p_4(\tilde{p}_3+\tilde{p}_2)}{2}\right) + (p_3 \leftrightarrow p_4)\right] \\ + (p_1 \leftrightarrow p_2) \Big\}, \tag{C.1g}$$



$$= V^{\bar{B}B4A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa\iota}(p_1, p_2, p_3, p_4, p_5, p_6) = -$$



$$= -V^{\bar{\psi}\psi 4A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa\iota}(p_1, p_2, p_3, p_4, p_5, p_6)$$

$$= 2g^4 \mu^2 \theta^2 (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4 + p_5 + p_6) \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) \times \\ \times \Big\{ 2\delta_{\tau\epsilon}\delta_{\kappa\iota} \left[\sin\left(\tfrac{p_4\tilde{p}_1}{2}\right) \sin\left(\tfrac{p_3(\tilde{p}_4+\tilde{p}_1)}{2}\right) \sin\left(\tfrac{p_6\tilde{p}_2}{2}\right) \sin\left(\tfrac{p_5(\tilde{p}_6+\tilde{p}_2)}{2}\right) + (p_3 \leftrightarrow p_4) + (p_5 \leftrightarrow p_6)\right] \\ + \delta_{\tau\kappa}\delta_{\epsilon\iota} \left[\sin\left(\tfrac{p_5\tilde{p}_1}{2}\right) \sin\left(\tfrac{p_3(\tilde{p}_5+\tilde{p}_1)}{2}\right) \sin\left(\tfrac{p_6\tilde{p}_2}{2}\right) \sin\left(\tfrac{p_4(\tilde{p}_6+\tilde{p}_2)}{2}\right) + (p_3 \leftrightarrow p_5) + (p_4 \leftrightarrow p_6)\right] \\ + \delta_{\tau\iota}\delta_{\kappa\epsilon} \left[\sin\left(\tfrac{p_6\tilde{p}_1}{2}\right) \sin\left(\tfrac{p_3(\tilde{p}_6+\tilde{p}_1)}{2}\right) \sin\left(\tfrac{p_4\tilde{p}_2}{2}\right) \sin\left(\tfrac{p_5(\tilde{p}_4+\tilde{p}_2)}{2}\right) + (p_3 \leftrightarrow p_6) + (p_5 \leftrightarrow p_4)\right] \\ + (p_1 \leftrightarrow p_2) \Big\}. \tag{C.1h}$$

C.2 One Loop Calculations

C.2.1 Vacuum polarization

The model (3.35) gives rise to 23 graphs contributing to the two-point function $G^{AA}_{\mu\nu}(p)$. Omitting convergent expressions, there are 11 graphs left depicted in Fig. C.1 on page 118. Being interested in the IR divergent contributions we apply again the expansion (3.17) for $\tilde{p} \to 0$.

Summing up the contributions of the graphs in Fig. C.1, and denoting the result at order i for the planar (p) part by $\Pi^{(i),\mathrm{p}}_{\mu\nu}$, one is left with

$$\begin{aligned}
\Pi^{(0),\mathrm{p}}_{\mu\nu}(p) =& \frac{g^2}{16\pi^2}\Lambda^2\delta_{\mu\nu}\left(-10s_{\mathrm{c}} - 96s_{\mathrm{h}} - 96s_{\mathrm{j}} + 12s_{\mathrm{a}} + s_{\mathrm{b}} + 96s_{\mathrm{d}} + 96s_{\mathrm{f}}\right)\,, \\
=& 0\,, \qquad \text{(C.2a)} \\
\Pi^{(2),\mathrm{p}}_{\mu\nu}(p) =& -\frac{1}{3}\frac{g^2}{16\pi^2}\Big[\delta_{\mu\nu}p^2\left(22s_{\mathrm{a}} + s_{\mathrm{b}} + 48(s_{\mathrm{d}} + s_{\mathrm{f}})\right) \\
&+ 2p_\mu p_\nu\left(72(s_{\mathrm{h}} + s_{\mathrm{j}}) - 8s_{\mathrm{a}} + s_{\mathrm{b}} - 96(s_{\mathrm{d}} + s_{\mathrm{f}})\right)\Big]\mathrm{K}_0\sqrt{\frac{\mu^2}{\Lambda^2}}\,, \\
=& -\frac{5g^2}{12\pi^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)K_0\sqrt{\frac{\mu^2}{\Lambda^2}} \\
\approx& -\frac{5g^2}{12\pi^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\left(\frac{\Lambda^2}{\mu^2}\right) + \text{finite}\,, \qquad \text{(C.2b)}
\end{aligned}$$

where the symmetry factors in Tab. C.1 have been inserted and the approximation

$$\mathrm{K}_0(x) \underset{x\ll 1}{\approx} \ln\tfrac{2}{x} - \gamma_E + \mathcal{O}\left(x^2\right)\,,$$

for the modified Bessel function K_0 can be utilized for small arguments, i.e. vanishing regulator cutoffs[1] $\Lambda \to \infty$ and $\mu \to 0$. Finally, γ_E denotes the Euler-Mascheroni constant. Note, that the first order vanishes identically due to an odd power of k in the integrand which leads to a cancellation under the symmetric integration over the momenta.

Table C.1: Symmetry factors for the one loop vacuum polarization (where the factor (-1) for fermionic loops has been included).

s_{a}	$\frac{1}{2}$	s_{e}	1	s_{i}	1
s_{b}	-1	s_{f}	-1	s_{j}	-1
s_{c}	$\frac{1}{2}$	s_{g}	-1	s_{k}	-1
s_{d}	1	s_{h}	1		

[1]The cutoffs are introduced via a factor $\exp\left[-\mu^2\alpha - \frac{1}{\Lambda^2\alpha}\right]$ to regularize parameter integrals $\int_0^\infty d\alpha$. See Ref. [62] and Appendix A.2.1 for a more extensive description of the mathematical details underlying these computations.

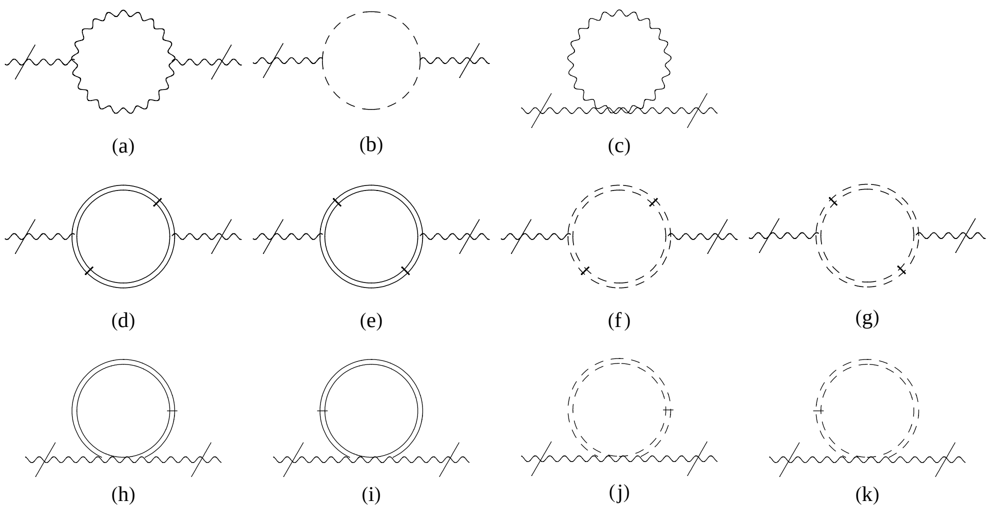


Figure C.1: One loop corrections for the gauge boson propagator

Of particular interest is the non-planar part (np) which for small p results to:

$$\begin{aligned}\Pi^{(0),\mathrm{np}}_{\mu\nu}(p) =& \frac{g^2}{4\pi^2\tilde{p}^2}\Big[\delta_{\mu\nu}\left(96(s_\mathrm{h}+s_\mathrm{j}-s_\mathrm{d}-s_\mathrm{f})-12s_\mathrm{a}-s_\mathrm{b}+10s_\mathrm{c}\right) \\ &\quad -2\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\left(48(s_\mathrm{h}+s_\mathrm{j})-96(s_\mathrm{d}+s_\mathrm{f})-12s_\mathrm{a}-s_\mathrm{b}+2s_\mathrm{c}\right)\Big] \\ =& \frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2}\,, \end{aligned} \tag{C.3a}$$

$$\begin{aligned}\Pi^{(2),\mathrm{np}}_{\mu\nu}(p) =& \frac{g^2}{48\pi^2\tilde{p}^2}\Bigg\{2\theta^2 p_\mu p_\nu p^2\left(72(s_\mathrm{h}+s_\mathrm{j})-8s_\mathrm{a}+s_\mathrm{b}-96(s_\mathrm{d}+s_\mathrm{f})\right)\mathrm{K}_0\left(\sqrt{\mu^2\tilde{p}^2}\right) \\ &\quad +\sqrt{\frac{\tilde{p}^2}{\mu^2}}p^2\Bigg[\sqrt{\frac{\tilde{p}^2}{\mu^2}}\left(22s_\mathrm{a}+s_\mathrm{b}+48(s_\mathrm{d}+s_\mathrm{f})\right)\mu^2\delta_{\mu\nu}\mathrm{K}_0\left(\sqrt{\mu^2\tilde{p}^2}\right) \\ &\qquad +2\mu^2\left(13s_\mathrm{a}+s_\mathrm{b}+120(s_\mathrm{d}+s_\mathrm{f})\right)\tilde{p}_\mu\tilde{p}_\nu\mathrm{K}_1\left(\sqrt{\mu^2\tilde{p}^2}\right) \\ &\qquad -3\sqrt{\frac{\mu^2}{\tilde{p}^2}}\left(16s_\mathrm{a}+s_\mathrm{b}+96(s_\mathrm{d}+s_\mathrm{f})\right)\tilde{p}_\mu\tilde{p}_\nu\Bigg]\Bigg\} \\ =& -\frac{g^2}{48\pi^2}\Bigg[\tilde{p}_\mu\tilde{p}_\nu\left(\frac{21}{\theta^2}-11p^2\sqrt{\frac{\mu^2}{\tilde{p}^2}}\mathrm{K}_1\left(\sqrt{\mu^2\tilde{p}^2}\right)\right) \\ &\quad -10\,\mathrm{K}_0\left(\sqrt{\mu^2\tilde{p}^2}\right)\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)\Bigg]\,. \end{aligned} \tag{C.3b}$$

Considering the limit $\tilde{p}^2 \to 0$ rectifies application of the approximation

$$\mathrm{K}_1(x) \underset{x\ll 1}{\approx} \tfrac{1}{x}+\tfrac{x}{2}\left(\gamma_E-\tfrac{1}{2}+\ln\tfrac{x}{2}\right)+\mathcal{O}\left(x^2\right)\,,$$

which reveals that the second order is IR finite (which is immediately clear from the fact that the terms of lowest order in p are $\mathcal{O}\left(p^2\right)$), apart from a $\ln(\mu^2)$-term which cancels in the sum of planar and non-planar contributions. Hence, collecting all divergent terms one is left with (in the limit $\mu \to 0$ and $\Lambda \to \infty$),

$$\Pi_{\mu\nu}(p) = \frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2} - \lim_{\Lambda\to\infty}\frac{5g^2}{12\pi^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\left(\Lambda^2\right) + \text{finite terms}\,, \tag{C.4}$$

which is independent of the IR-cutoff M. As expected, Eqn. (C.4) exhibits a quadratic IR divergence in $\tilde{p}^2$, and a logarithmic divergence in the cutoff Λ. Furthermore, the transversality condition $p_\mu\Pi_{\mu\nu}(p) = 0$ is fulfilled, which serves as a consistency check for the symmetry factors.

C.2.2 Corrections to the AB propagator

The action (3.35) gives rise to eight divergent graphs with one external A_μ and one $B_{\mu\nu}$ which are depicted in Fig. C.2. Applying an expansion of type (3.17) for small external momenta p, and summing up the divergent contributions of all graphs (all orders of an expansion similar to Eqn. (3.17)), one ends up with,

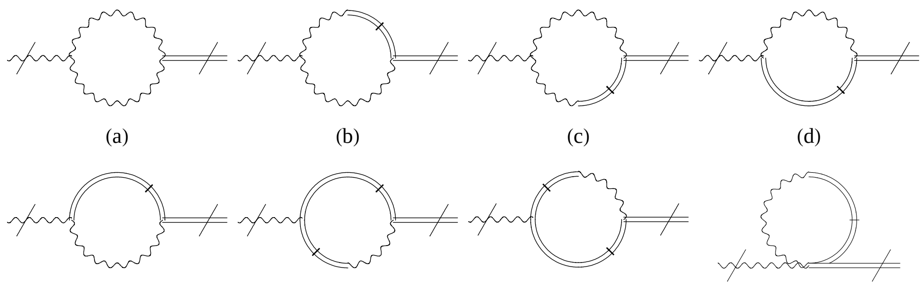


Figure C.2: One loop corrections for $\langle A_\mu B_{\nu_1\nu_2}\rangle$ (with amputated external legs).

Table C.2: Symmetry factors for the graphs depicted in Fig. C.2

(a)	1/2	(e)	1
(b)	1	(f)	1
(c)	1	(g)	1
(d)	1	(h)	1

$$\begin{aligned}\Sigma^{\text{p,AB}}_{\mu1,\nu1\nu2}(p) =& -\frac{3\text{i}g^2}{32\pi^2}\lambda\left(p_{\nu1}\delta_{\mu1\nu2} - p_{\nu2}\delta_{\mu1\nu1}\right)\text{K}_0\sqrt{\frac{\mu^2}{\Lambda^2}} + \text{finite}\\ \Sigma^{\text{np,AB}}_{\mu1,\nu1\nu2}(p) =& \frac{3\text{i}g^2}{32\pi^2}\lambda\,\text{K}_0\left(\sqrt{\mu^2\tilde{p}^2}\right)\left(p_{\nu1}\delta_{\mu1\nu2} - p_{\nu2}\delta_{\mu1\nu1}\right) + \text{finite}\,.\end{aligned} \tag{C.5}$$

Approximating the Bessel functions for small arguments (c.f. Eqn. (F.24)) and summing up planar and non-planar parts one finds the expression

$$\Sigma^{\text{AB}}_{\mu1,\nu1\nu2}(p) = \frac{3\text{i}g^2}{32\pi^2}\lambda\left(p_{\nu1}\delta_{\mu1\nu2} - p_{\nu2}\delta_{\mu1\nu1}\right)\left(\ln\Lambda + \ln|\tilde{p}|\right) + \text{finite}\,, \tag{C.6}$$

where the IR cutoff μ has cancelled, and which shows a logarithmic divergence for $\Lambda \to \infty$.

Due to the symmetry between B and $\bar{B}$ in the sense that both have identical interactions with the gauge field, it is obvious that $\Sigma^{\rm AB}_{\mu1,\nu1\nu2} \equiv \Sigma^{\rm A\bar{B}}_{\mu1,\nu1\nu2}$ and as implied by Eqn. (3.37a) it also holds that $\Sigma^{\rm BA}_{\mu1\mu2,\nu1} \equiv -\Sigma^{\rm AB}_{\nu1,\mu1\mu2}$.

C.2.3 Corrections to the BB propagator

The set of divergent graphs contributing to $\langle B_{\mu1\mu2} B_{\nu1\nu2}\rangle$ consists of those depicted in Fig. C.3. Making an expansion of type (3.17) for small external momenta p, and summing up the contributions of all nine graphs yields

$$\begin{aligned}\Sigma^{\rm p,BB}_{\mu1\mu2,\nu1\nu2}(p) =& \frac{g^2\lambda^2}{32\pi^2}\left(\delta_{\mu1\nu1}\delta_{\mu2\nu2} - \delta_{\mu2\nu1}\delta_{\mu1\nu2}\right) {\rm K}_0 \sqrt{\frac{\mu^2}{\Lambda^2}} + \text{finite}\,, \\ \Sigma^{\rm np,BB}_{\mu1\mu2,\nu1\nu2}(p) =& \frac{g^2\lambda^2}{64\pi^2}\Bigg(\frac{\delta_{\mu1\nu2}\tilde{p}_{\mu2}\tilde{p}_{\nu1} - \delta_{\mu1\nu1}\tilde{p}_{\mu2}\tilde{p}_{\nu2} - \delta_{\mu2\nu2}\tilde{p}_{\mu1}\tilde{p}_{\nu1} + \delta_{\mu2\nu1}\tilde{p}_{\mu1}\tilde{p}_{\nu2}}{\tilde{p}^2} \\ & + 2\,{\rm K}_0\left(\sqrt{\mu^2\tilde{p}^2}\right)\left(\delta_{\mu1\nu2}\delta_{\mu2\nu1} - \delta_{\mu1\nu1}\delta_{\mu2\nu2}\right)\Bigg) + \text{finite}\,,\end{aligned} \tag{C.7}$$

for the planar/non-planar part, respectively. Approximation of the Bessel functions for $p \ll 1$, i.e. small arguments, reveals cancellations of contributions depending on μ in the final sum. Hence, the divergent part boils down to

$$\Sigma^{\rm BB}_{\mu1\mu2,\nu1\nu2}(p) = \frac{g^2\lambda^2}{64\pi^2}\left(\delta_{\mu1\nu1}\delta_{\mu2\nu2} - \delta_{\mu2\nu1}\delta_{\mu1\nu2}\right)\left(\ln\Lambda^2 + \ln\tilde{p}^2\right) + \text{finite}\,, \tag{C.8}$$

leaving a logarithmic divergence for both the planar and the non-planar part.

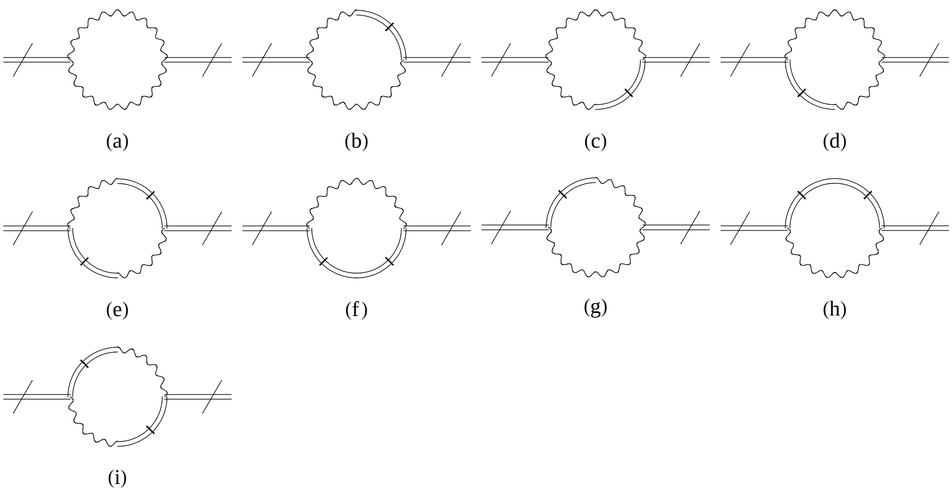


Figure C.3: One loop corrections for $\langle B_{\mu_1\mu_2} B_{\nu_1\nu_2}\rangle$ (with amputated external legs).

Due to symmetry reasons this result is also equal to the according correction to the $\bar{B}\bar{B}$ propagator, i.e.

$$\Sigma^{\rm \bar{B}\bar{B}}_{\mu1\mu2,\nu1\nu2}(p) = \Sigma^{\rm BB}_{\mu1\mu2,\nu1\nu2}(p)\,. \tag{C.9}$$

Table C.3: Symmetry factors for the graphs depicted in Fig. C.3

(a)	1/2	(d)	1	(g)	1
(b)	1	(e)	1	(h)	1
(c)	1	(f)	1	(i)	1

C.2.4 Corrections to the $B\bar{B}$ propagator

For the correction to $\langle B_{\mu 1\mu 2}\bar{B}_{\nu 1\nu 2}\rangle$ one finds the ten divergent graphs depicted in Fig. C.4.

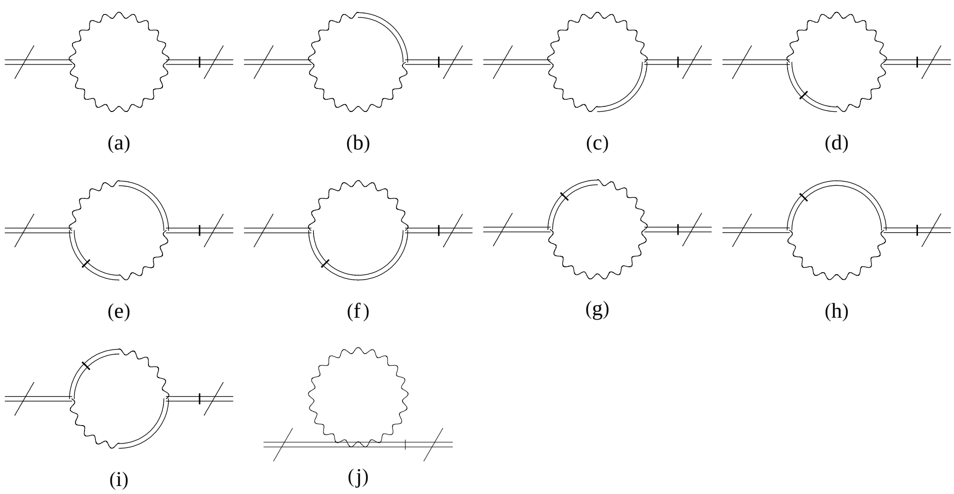


Figure C.4: One loop corrections for $\langle B_{\mu_1\mu_2}\bar{B}_{\nu_1\nu_2}\rangle$ (with amputated external legs).

Table C.4: Symmetry factors for the graphs depicted in Fig. C.4

(a)	1/2	(e)	1	(i)	1
(b)	1	(f)	1	(j)	1/2
(c)	1	(g)	1		
(d)	1	(h)	1		

Expansion for small external momenta p and summation of the integrated results yields

$$
\begin{aligned}
\Sigma^{\mathrm{p,B\bar{B}}}_{\mu 1\mu 2,\nu 1\nu 2}(p) =& \frac{g^2}{2\pi^2}\Lambda^2\mu^2\tilde{p}^2\left(\delta_{\mu 2\nu 1}\delta_{\mu 1\nu 2} - \delta_{\mu 1\nu 1}\delta_{\mu 2\nu 2}\right) \\
&+ \frac{g^2\lambda^2}{32\pi^2}\left(\delta_{\mu 1\nu 1}\delta_{\mu 2\nu 2} - \delta_{\mu 2\nu 1}\delta_{\mu 1\nu 2}\right)\mathrm{K}_0\sqrt{\frac{\mu^2}{\Lambda^2}} + \text{finite}\,, \\
\Sigma^{\mathrm{np,B\bar{B}}}_{\mu 1\mu 2,\nu 1\nu 2}(p) =& \frac{g^2\lambda^2}{64\pi^2}\Bigg(\frac{\delta_{\mu 1\nu 2}\tilde{p}_{\mu 2}\tilde{p}_{\nu 1} - \delta_{\mu 1\nu 1}\tilde{p}_{\mu 2}\tilde{p}_{\nu 2} - \delta_{\mu 2\nu 2}\tilde{p}_{\mu 1}\tilde{p}_{\nu 1} + \delta_{\mu 2\nu 1}\tilde{p}_{\mu 1}\tilde{p}_{\nu 2}}{\tilde{p}^2} \\
&+ 2\,\mathrm{K}_0\left(\sqrt{\mu^2\tilde{p}^2}\right)\left(\delta_{\mu 1\nu 2}\delta_{\mu 2\nu 1} - \delta_{\mu 1\nu 1}\delta_{\mu 2\nu 2}\right)\Bigg) + \text{finite}\,,
\end{aligned}
\tag{C.10}
$$

Hence, the divergent part is given by

$$\Sigma^{\mathrm{B}\bar{B}}_{\mu1\mu2,\nu1\nu2}(p) = \frac{g^2}{2\pi^2}\Lambda^2\mu^2\tilde{p}^2\left(\delta_{\mu2\nu1}\delta_{\mu1\nu2} - \delta_{\mu1\nu1}\delta_{\mu2\nu2}\right) + \frac{g^2\lambda^2}{64\pi^2}\left(\delta_{\mu1\nu1}\delta_{\mu2\nu2} - \delta_{\mu2\nu1}\delta_{\mu1\nu2}\right)\left(\ln\Lambda^2 + \ln\tilde{p}^2\right) + \text{finite}\,, \tag{C.11}$$

which is logarithmically divergent in $\tilde{p}^2$ and quadratically in Λ. Once more, μ has dropped out in the sum of planar and non-planar contributions. Furthermore, note that $\Sigma^{\mathrm{B}\bar{\mathrm{B}}}_{\mu1\mu2,\nu1\nu2} \equiv \Sigma^{\bar{\mathrm{B}}\mathrm{B}}_{\nu1\nu2,\mu1\mu2}$ as is obvious from the result (C.8).

Appendix D

A Short Story About Forests and Trees

This chapter aims to give a brief introduction to the treatment of overlapping divergences. Being split into two parts we will first focus to the Zimmermann approach, and then see the beauty of the mathematical structure of bi-algebras used by Epstein and Glaser, from which the same results emerge naturally.

D.1 Zimmermann Approach

For single loop integrands $\mathcal{I}(k)$ any UV divergence can directly be extracted and subtracted, as for instance has been discussed for the BPHZ scheme in Section 4.1.1. At higher loop order this does not work since the involved momenta may take their limits one at a time or simultaneously. Hence, the object of desire is a map $\mathcal{R} : G \mapsto \mathcal{R}(G)$ for a graph G, which extracts the finite content. Intuitively, this is $R(G) = G + C(G)$, where C is a function which returns the divergent contributions with opposite sign. With these definitions we can define Bogoliubov's $\bar{\mathcal{R}}$-operator

$$\bar{\mathcal{R}}(G) = G + \sum_{\gamma \subsetneq G} C_\gamma(G)\,. \tag{D.1}$$

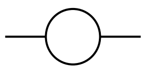
$$R(G) = \bar{\mathcal{R}}(G) + C(G)\,. \tag{D.2}$$

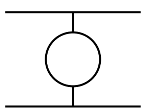
Next we define

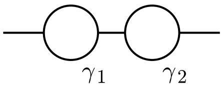
$$C_\gamma(G) = \begin{cases} -T \circ \bar{\mathcal{R}}(\gamma)\,, & \text{if } \gamma \text{ has an overall divergence} \\ 0\,, & \text{if } \gamma \text{ has no overall divergence} \end{cases}, \tag{D.3}$$

$$R(G) = \bar{\mathcal{R}}(G) - T \circ \bar{\mathcal{R}}(G)\,. \tag{D.4}$$

The operation T with $T^2 = T$ and $T \circ (\gamma_1\gamma_2) = (T \circ \gamma_1)(T \circ \gamma_2)$ gives the overall divergence of its argument with positive sign, i.e. $T \circ G = -C(G)$. This puzzling notation is kept only for historic reasons. Meaning is most easily given to these recursive definitions by some simple examples.

$G =$  no subdivergences, $R(G) = G - T \circ G = G + C(G)\,,$

$G =$  no overall, 1 subdiv. γ, $R(G) = G - T \circ \bar{\mathcal{R}}(\gamma) = G - T \circ \gamma = \bar{\mathcal{R}}(G),$

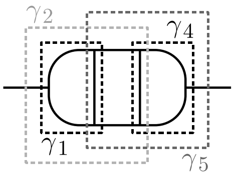
$G =$ 

no overall, 2 subdiv. γ_1, γ_2, $R(G) = G + C_{\gamma_1} + C_{\gamma_2} + C_{\gamma_1 \cup \gamma_2}$.

In the first line only the overall divergence needs to be subtracted. The graph of the second line has one subdivergence. Note that in this case the $\bar{\mathcal{R}}$ operation reduces to $\mathcal{R}$. For the third line we have the expression $C_{\gamma_1 \cup \gamma_2}$ for the simultaneous occurrence of the two independent subdivergences.

$$\begin{aligned} C_{\gamma_1 \cup \gamma_2} &= -T \circ \bar{\mathcal{R}}(\gamma_1 \cup \gamma_2) = -T \circ [\gamma_1 \gamma_2 + C_{\gamma_1} \gamma_2 + \gamma_1 C_{\gamma_2}] \\ &= (T \circ \gamma_1)(T \circ \gamma_2) - (T \circ T \circ \gamma_1)(T \circ \gamma_2) - (T \circ \gamma_1)(T \circ T \circ \gamma_2) = -T \circ (\gamma_1 \gamma_2) \,. \end{aligned}$$

The important point here is the independence of the divergences. More involved is the case of *overlapping divergences*, as explained for the next example

$G =$ 

$$\begin{aligned} R(G) &= \bar{\mathcal{R}}(G) - T \circ \bar{\mathcal{R}}(G) \\ &= G - \underbrace{\sum_\gamma T \circ \bar{\mathcal{R}}(\gamma)}_{\text{8 graphs}} - \underbrace{T \circ \bar{\mathcal{R}}}_{\text{8 graphs}} \,, \end{aligned}$$

$$\begin{aligned} C_{\gamma_1} \text{ and } C_{\gamma_4} &: \text{one loop divergences, simple subtraction,} \\ C_{\gamma_2} &= -T \circ [\gamma_2 - \gamma_2|_{\gamma_1 \to T \circ \gamma_1}] \,, \\ C_{\gamma_5} &= -T \circ [\gamma_5 - \gamma_5|_{\gamma_4 \to T \circ \gamma_4}] \,, \\ C_{\gamma_1 \cup \gamma_4} &= T \circ (\gamma_1 \gamma_4) \,. \end{aligned}$$

The graph G contains eight combinations which enter the $\bar{\mathcal{R}}$ operation (i.e. which are wrapped by a T function): single divergences for γ_1, γ_2, γ_4, γ_5, the combinations $\gamma_1 \& \gamma_2$, $\gamma_4 \& \gamma_5$ which overlap, independently γ_1 & γ_4, and all of them together. Each of these combinations (according to Eqn. (D.4)) appears once as single divergence, and together with the overall divergence of G, yielding a total of 16 counterterms. Now Zimmermann has combined these rather sketchy arguments into a rigorous subtraction scheme which represents a solution to Bogoliubov's recursion in a closed form. We have to introduce some additional definitions,

Definition 7. Two graphs $\gamma_1, \gamma_2 \subset G$ are called *overlapping* if they do not fulfill $\gamma_1 \subseteq \gamma_2 \vee \gamma_2 \subseteq \gamma_1 \vee \gamma_1 \cap \gamma_2 = 0$

Definition 8. A *forest* f is a set of non-overlapping subgraphs γ_i of G or the empty set e

Finally, we can write down the famous *Forest Formula*:

$$\mathcal{R}(G) = \sum_{f \in \mathcal{F}(G)} \prod_{\gamma \,\inf} [-T \circ \gamma] \,, \tag{D.5}$$

where $\mathcal{F} = \bigcup f \in G$ is the set[1] of all forests in G. Now, regarding the above example with overlapping divergences, we can immediately rewrite the subtractions in terms of forest, denoted by explicit sets: $\{\{\}, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_4\}, \{\gamma_5\}, \{\gamma_1, \gamma_2\}, \{\gamma_4, \gamma_5\}, \{\gamma_1, \gamma_4\}\}$, where the empty set corresponds to the overall divergence. For a detailed review see Ref. [148].

[1]Note that $\mathcal{F}$ gives rise to a tree structure [148].

D.2 Epstein Glaser Approach

Despite not being applied directly in this thesis, the beauty of the bi-algebra approach by Epstein and Glaser [149] shall briefly be discussed at this place. It shall be pointed out that the contents presented below can be found in a very didactive way in Ref. [150]. A more technical focus is given in to Ref. [151]. For a rudimentary understanding of the approach we have to introduce the notion of *rooted trees*, which are defined by a simply connected (i.e. there are no loops) set of oriented edges and vertices, where the latter, each, correspond to a divergent subdiagram[2] of the Graph G under consideration, i.e. i.e. $t \subseteq G$, and each edge connects exactly two vertices. The root is the only vertex without incoming edges and equals G. We define operations B_- and B_+ which remove and add the root nodes of a rooted tree t, respectively ($B_- : t \mapsto \{t_1..t_n\}$, $B_+ : \{t_1..t_n\} \mapsto t$, $B_- \circ B_+ = B_+ \circ B_- = \text{id}$, $B_-(\bullet) = \text{id}$, $B_+(\text{id}) = (\bullet)$). In addition, we have the notion of *cuts* on trees. We distinguish:
elementary cuts which simply split apart any graph  $\rightarrow$ { $\bullet$,  }
admissible cuts are sets of elementary cuts for which each vertex is separated from the root by at most one cut.
We further define $R(t) \equiv t$, and $P(t)$ to be to be t without its root t^0, i.e. t/t^0. Then, for the *full cut* we have $P(t) = e$, $R(t) = t$, and for the *empty cut*, $P(t) = t$, $R(t) = e$, where e is the empty cut.
On the mathematical side we have the

Definition 9. A *Hopf algebra* $\mathcal{H}$ on rooted trees is defined by the algebra properties

- ▷ associative product $m : \mathcal{H} \times \mathcal{H} \mapsto \mathcal{H}$
- ▷ unit $e_\mathcal{H} : m(e_\mathcal{H}, a) \equiv m(a, e_\mathcal{H}) \equiv a \,\forall a \in \mathcal{H}$
- ▷ inverse $a^{-1} : m(a^{-1}, a) \equiv m(a, a^{-1}) \equiv e_\mathcal{H}$

and the following coalgebra properties

- ▷ coassociative *coproduct* $\bar{m} : \mathcal{H} \mapsto \mathcal{H} \times \mathcal{H}$, with $(\bar{m}_\mathcal{H} \times \text{id})\bar{m}(a) \equiv (\text{id} \times \bar{m}_\mathcal{H})\bar{m}(a) \,\forall a \in \mathcal{H}$
- ▷ counit $\bar{e}_\mathcal{H} : (\bar{e}_\mathcal{H} \times \text{id}) \circ \bar{m}(a) \equiv (\text{id} \times \bar{e}_\mathcal{H}) \circ \bar{m}(a) \equiv a \,\forall a \in \mathcal{H}$
- ▷ coinverse (*antipode*) $\bar{s} : m \circ (\bar{s} \times \text{id}) \circ \bar{m}(a) = e_\mathcal{H} = \begin{cases} 1 & \text{if } a \equiv e_\mathcal{H} \\ 0 & a \text{ nontrivial} \end{cases}.$

Then, a direct application to rooted trees is given by letting m be a disjoint union of trees t, and the unit $e_\mathcal{H}$ be the empty set. Then the coproduct is given by the relations $\bar{m}(e_\mathcal{H}) = \bar{e}_\mathcal{H} \otimes \bar{e}_\mathcal{H}$, $\bar{m}(t_1, ..t_n) = \{\bar{m}(t_1), ..\bar{m}(t_n)\}$, and finally, $\bar{m}(t) = t \otimes \bar{e}_\mathcal{H} + (\text{id} \otimes B_+) \circ \bar{m}_\mathcal{H}(B_-(t))$. The latter equation can easily be understood when considering simple examples

$$\bar{m}(\bullet) = \bullet \otimes \bar{e}_\mathcal{H} + \bar{e}_\mathcal{H} \otimes \bullet \,,$$

$\bar{m}($$) =$ $\otimes \bar{e}_\mathcal{H} + \underbrace{\bullet \otimes \bullet}_{(\text{id} \otimes B_+) \circ \bar{m}(\bullet)} + \bar{e}_\mathcal{H} \otimes$,

$$\bar{m}(t) = \bar{e}_\mathcal{H} \otimes t + t \otimes \bar{e}_\mathcal{H} + \sum_{\text{admissible cuts } C \text{ of } t} P^C(t) \otimes R^C(t) \,,$$

[2] The vertices representing divergent subdiagrams are refered to as *Hepp sectors* in the literature.

from which follows a recursive splitting scheme of trees. In the same way, the antipode becomes $\bar{s}(\bar{e}_{\mathcal{H}}) = \bar{e}_{\mathcal{H}}$, $\bar{s}(t_1, ..t_n) = \bar{s}(t_1), ..\bar{s}(t_n)$ (disjoint union),

$$\bar{s}(t) = -t - \sum_{\text{admissible cuts } C \text{ of } t} P^C(t) R^C(t) = \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_c} P^C(t) R^C(t) \,,$$

where n_c is the number of elementary cuts of t. Again, the meaning becomes clear at hand of a simple example

$$\begin{aligned} \bar{s}(\bullet) &= -\bullet \,, \\ \bar{s}\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) &= -\genfrac{}{}{0pt}{}{\bullet}{\bullet} + (-1)^1 \, \bar{s}(\bullet) \bullet \,, \end{aligned}$$

from which we recognize the principle of multiplicative subtraction, similar to the Bogoliubov recursion (D.4). Defining finally a character $\phi : \mathcal{H} \mapsto V$ mapping from the Hopf algebra into a vector space V on which an action S can be defined, and a renormalization procedure $R : V \mapsto V$, the antipode on $\mathcal{H}$ induces the definitions

$$\begin{aligned} \bar{m}(t) &= t \otimes \bar{e}_{\mathcal{H}} + \bar{e}_{\mathcal{H}} \otimes t + \sum_{\text{admissible cuts } C \text{ of } t} P^C(t) \otimes R^C(t) \,, \\ \bar{s}_R(t) &:= -R \left[\phi(t) + \sum_C \bar{s}_R \left(P^C(t) \right) R^C(t) \right] \,, \end{aligned}$$

which translates, if we replace t by the n-point 1PI vertex functions Γ,

$$\begin{aligned} \bar{m}(\Gamma) &= \Gamma \otimes \bar{e}_{\mathcal{H}} + \bar{e}_{\mathcal{H}} \otimes \Gamma + \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma \,, \\ R(\Gamma) &= \bar{s}_R(\Gamma) \circ \phi(\Gamma) = \underbrace{\phi(\Gamma) + \sum \gamma \subseteq \Gamma \bar{s}_R(\gamma) \phi(\Gamma/\gamma)}_{:=\bar{\mathcal{R}}} + \bar{s}_R(\Gamma) \,. \end{aligned} \tag{D.6}$$

The latter equation resembles the forest formula (D.5) in a natural way, from the pure mathematical structure of the antipode contained in the Hopf algebra generated by 1PI proper graphs.

Appendix E

Supplementary Calculations to the BRSW Model

This chapter contains a step by step description of the construction of the BRSW model in Appendix E.1, listings of the Feynman rules in Appendix E.2, and detailed results for the vacuum polarization as well as a discussion of the extensive calculations leading to the corrections to the functions V^{3A} and V^{4A} in Appendix E.3. Regarding the first point, it is necessary to thoroughly review the interplay between the action and the propagators in the gauge model with BRST doublets. Therefore, actually, part of Appendix E.1 thematically belongs to Section 3.2 rather than Chapter 5. However, it seams reasonable to discuss the evolution of the BRSW model, which starts at the action of Section 3.2, in one piece.

E.1 In-Depth Analysis of Propagators

This section describes in detail the analysis of the interplay between terms in the action, and the resulting form of the IR divergent propagators of the gauge model with BRST doublets from Section 3.2. The aim is to find out which terms affect the poles in $G^{\{AB,A\bar{B}\}}$,G^{BB}, and $G^{B\bar{B}}$. These shall then be altered in order to achieve finiteness through the implementation of additional damping mechanisms.

First, we should state the bilinear part of the starting action (3.23)

$$S^{\text{bi}} = \int \mathrm{d}^4x \left[\frac{1}{2}A_\mu \left(\partial_\mu\partial_\nu - \Box\right) A_\nu + \frac{a}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)\left(\partial_\mu A_\nu + \partial_\nu A_\mu\right) - \bar{B}_{\mu\nu}\widetilde{\Box}\Box B_{\mu\nu} + \bar{\psi}_{\mu\nu}\widetilde{\Box}\Box\psi_{\mu\nu} \right.$$
$$\left. + b\left(\partial A\right) - \frac{\alpha}{2}b^2 - \bar{c}\Box c\right]. \tag{E.1}$$

The equations of motion are given by variation with respect to the fields.

$$\frac{\delta S^{\text{bi}}}{\delta A^\mu} = -j^A_\mu \quad = \left(\partial_\mu\partial^\nu - \Box\delta^\nu_\mu\right) A_\nu + a\partial_\nu\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right) - \partial_\mu b\,, \tag{E.2a}$$

$$\frac{\delta S^{\text{bi}}}{\delta b} = -j^b \quad = \partial^\mu A_\mu - \alpha b = 0, \tag{E.2b}$$

$$\frac{\delta S^{\text{bi}}}{\delta c} = j^c \quad = -\Box\bar{c} = 0\,, \tag{E.2c}$$

$$\frac{\delta S^{\text{bi}}}{\delta \bar{c}} = j^{\bar{c}} \quad = \Box c = 0, \tag{E.2d}$$

$$\frac{\delta S^{\mathrm{bi}}}{\delta B^{\mu\nu}} = -j^{B}_{\mu\nu} \quad = \frac{a}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - \Box\widetilde{\Box}\bar{B}_{\mu\nu}\,, \tag{E.2e}$$

$$\frac{\delta S^{\mathrm{bi}}}{\delta \bar{B}^{\mu\nu}} = -j^{\bar{B}}_{\mu\nu} \quad = \frac{a}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - \Box\widetilde{\Box}B_{\mu\nu}\,, \tag{E.2f}$$

$$\frac{\delta S^{\mathrm{bi}}}{\delta \psi^{\mu\nu}} = \quad j^{\psi}_{\mu\nu} \quad = -\Box\widetilde{\Box}\bar{\psi}_{\mu\nu}\,, \tag{E.2g}$$

$$\frac{\delta S^{\mathrm{bi}}}{\delta \bar{\psi}^{\mu\nu}} = \quad j^{\psi}_{\mu\nu} \quad = \Box\widetilde{\Box}\psi_{\mu\nu}\,, \tag{E.2h}$$

By resolving Eqn. (E.2e) for $\bar{B}_{\mu\nu}$, Eqn. (E.2f) for $B_{\mu\nu}$, and Eqn. (E.2b) for b, we immediately obtain the equalities

$$\{B_{\mu\nu}, \bar{B}_{\mu\nu}\} = \frac{a}{2\widetilde{\Box}\Box}\left[\partial_\mu A_\nu - \partial_\nu A_\mu + 2\frac{\{j^{\bar{B}}_{\mu\nu}, j^{B}_{\mu\nu}\}}{a}\right]\,, \tag{E.3}$$

$$-j^{A}_{\mu} = \left(1 + \frac{a^2}{\widetilde{\Box}\Box}\right)\left(\partial_\mu\partial_\nu - \Box\delta_{\mu\nu}\right)A_\nu + a\frac{\partial_\nu}{\widetilde{\Box}\Box}\left(j^{B}_{\mu\nu} + j^{\bar{B}}_{\mu\nu}\right) - \frac{1}{\alpha}\partial_\mu\left(\partial A\right) - \frac{\partial_\mu j^b}{\alpha}\,. \tag{E.4}$$

Applying ∂_μ from the left to $-j^{A}_{\mu}$, gives an expression for (∂A) which can be reinserted into Eqn. (E.4), yielding

$$A_\mu = \frac{1}{\Box}\left\{\left(1 + \frac{a^2}{\widetilde{\Box}\Box}\right)^{-1}\left[\underbrace{\frac{a}{\widetilde{\Box}\Box}}_{(*a)}\left(\delta_{\mu\rho}\partial_\sigma - \frac{\partial_\mu\partial_\rho\partial_\sigma}{\Box}\right)\left(j^{\bar{B}}_{\rho\sigma} + j^{B}_{\rho\sigma}\right) - \frac{1}{\Box}\left(\partial_\mu\partial_\rho - \Box\delta_{\mu\rho}\right)j^{A}_{\rho}\right] + t(\alpha)\right\}, \tag{E.5}$$

where $t(\alpha)$ symbolizes terms depending on the gauge parameter α. An explicit expression for $B_{\mu\nu}$ is then obtained by inserting A_μ into Eqn. (E.3).

Since the mixed propagator is given by $G^{\{AB,A\bar{B}\}}_{\mu,\rho\sigma} = -\delta A_\mu/\delta j^{\{B,\bar{B}\}}_{\rho\sigma}$, it is obvious that the term $(*a)$ in Eqn. (E.5) is responsible for the overall factor $(k^2\tilde{k}^2)^{-1}$. For the propagators $G^{\{B\bar{B},BB\}}$, the divergence results from the overall prefactor in Eqn. (E.3). It turns out that, with the only exception of terms involved in $G^{\psi\bar{\psi}}$, all occurrences of the operator $\widetilde{\Box}\Box$ originate from the term $-\bar{B}_{\mu\nu}\widetilde{\Box}\Box B_{\mu\nu}$ in the action (E.1). However, the appearance of $\widetilde{\Box}\Box$ does not only result in divergences but is actually required for the implementation of the desirable damping behavior of the $1/p^2$ model (which, initially, has been the cause for its introduction). This becomes clear when tracing the construction of the operator $\left(1 + \frac{a^2}{\widetilde{\Box}\Box}\right)^{-1}$ back to Eqn. (E.4). There it is constructed by a factor 1 stemming from the first term in the equation of motion for A_μ, Eqn. (E.2a), which in turn originates from the term $\frac{1}{4}F^2$ in the action. The second summand $a^2/\widetilde{\Box}\Box$ of the damping factor is introduced by the insertion of Eqn. (E.3) into the Eqn. (E.2a). Its actual origin is the soft breaking term $\frac{a}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)\left(\partial_\mu A_\nu + \partial_\nu A_\mu\right)$. Hence, we can unambiguously trace the damping factors, being required for the photon propagator, back to the breaking term of the action, and the operator $\widetilde{\Box}\Box$ sandwiched between the complex conjugated pair $\{B_{\mu\nu}, \bar{B}_{\mu\nu}\}$. The question is now, what can be done in order to avoid the overall divergences in mixed and pure propagators involving $\{B, \bar{B}\}$, while maintaining the desirable damping in the gauge boson propagator. The answer shall be constructed step by step in the subsequent Sections E.1.1 and E.1.2.

E.1.1 Evolutionary Step 1

A fundamental problem of non-commutative theories is (as is discussed in Section 4.3) the appearance of dimensionless derivative operators being contractions with $\theta_{\mu\nu}$. The reason for the existence of such objects is the negative dimension of θ, as defined in Section 1.3.3, Eqn. (1.15). We now try to avoid this problem by separating the dimension from the tensor structure,

$$\theta_{\mu\nu} \to \varepsilon\Theta_{\mu\nu}\,, \quad \text{with } d_m(\varepsilon) = -2\,, \text{ and } d_m(\Theta_{\mu\nu}) = 0\,,$$

as described in the main text. Contractions are then only performed with $\Theta_{\mu\nu}$, i.e. $\tilde{p}_\mu := p_\nu\Theta_{\mu\nu}$ and ε appears solely in terms stemming from integrations over phase factors representing star products.
The idea is now, to move the implementation of the damping part from the $\bar{B}B$ term to the breaking term. A possible action is given by

$$\begin{aligned} S_{\text{ev1}} = \int \mathrm{d}^4x \Big[&\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + b\partial\cdot A - \frac{\alpha}{2}b^2 - \bar{c}\partial_\mu D_\mu c \\ &+ \frac{\gamma^3}{2}\left(B_{\mu\nu}+\bar{B}_{\mu\nu}\right)\frac{1}{\widetilde{\Box}}F_{\mu\nu} + \bar{B}_{\mu\nu}\left(\sigma - D^2\right)B_{\mu\nu} - \bar{\psi}_{\mu\nu}\left(\sigma - D^2\right)P_{\mu\nu} + \text{source terms}\Big]\,, \end{aligned} \tag{E.6}$$

where we have introduced the new parameters γ and σ of mass dimension 1 and 2 respectively. Remember that the tilde symbol in $\widetilde{\Box}^{-1}$, in contrast to all models discussed up to this point, does not introduce any changes in the mass dimension. In the action (E.6) we have shifted the $\widetilde{D}^2$ from the $B/\bar{B}$ sector to the breaking term, where it appears as $\widetilde{\Box}^{-1}$. It is important to note that there is no need to use a covariant derivative here since the term states *a priori* a breaking to gauge and BRST symmetries. Due to this trick, all the problems linked to the interpretation and localization of the non-local $1/D^2$ operator described in Section 2.2 are eliminated at once.

We can now repeat the analysis given above in order to see the changes imposed by the new action (E.6). The relevant equations of motion are

$$\frac{\delta S^{\text{bi}}_{\text{ev1}}}{\delta A^\mu} = -j^A_\mu \quad = \left(\partial_\mu\partial^\nu - \Box\delta^\nu_\mu\right)A_\nu - \frac{\mu^2}{\widetilde{\Box}}\partial_\nu\left(B_{\mu\nu}+\bar{B}_{\mu\nu}\right) - \partial_\mu b\,, \tag{E.7}$$

$$\frac{\delta S^{\text{bi}}_{\text{ev1}}}{\delta\{B^{\mu\nu},\bar{B}^{\mu\nu}\}} = -j^{\{B,\bar{B}\}}_{\mu\nu} \quad = \frac{\mu^3}{2\widetilde{\Box}}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + Q\{\bar{B}_{\mu\nu},B_{\mu\nu}\}\,, \tag{E.8}$$

and, proceeding in the same way as described above, this yields (with the temporal notation $Q := (\sigma - \Box)$)

$$\{B_{\mu\nu},\bar{B}_{\mu\nu}\} = \frac{-1}{Q}\left[\frac{\gamma^2}{2\widetilde{\Box}}\partial_\mu A_\nu - \partial_\nu A_\mu + \{j^{\bar{B}}_{\mu\nu},j^B_{\mu\nu}\}\right]\,, \tag{E.9}$$

$$-j^A_\mu = \left(1+\frac{\gamma^6}{Q\widetilde{\Box}^2}\right)\left(\partial_\mu\partial_\nu - \Box\delta_{\mu\nu}\right)A_\nu - \frac{\gamma^3}{Q\widetilde{\Box}}\left(\partial_\sigma\delta_{\rho\mu} + \frac{\partial_\mu\partial_\rho\partial_\sigma}{\Box}\right)\left(j^B_{\rho\sigma}+j^{\bar{B}}_{\rho\sigma}\right) - \frac{\partial_\mu\partial_\rho}{\Box}j^A_\rho\,, \tag{E.10}$$

$$A_\mu = \frac{1}{\Box}\left(1+\frac{\gamma^6}{Q\widetilde{\Box}^2}\right)^{-1}\left[\left(\delta_{\mu\rho} - \frac{\partial_\mu\partial_\rho}{\Box}\right)j^A_\rho - \frac{\gamma^3}{\widetilde{\Box}Q}\left(\partial_\sigma\delta_{\rho\mu} + \frac{\partial_\mu\partial_\rho\partial_\sigma}{\Box}\right)\left(j^B_{\rho\sigma}+j^{\bar{B}}_{\rho\sigma}\right)\right] + t(\alpha)\,. \tag{E.11}$$

Now the overall factor $(\Box\widetilde{\Box})^{-1}$ of $\{\bar{B}_{\mu\nu},B_{\mu\nu}\}$ in Eqn. (E.3), which directly enters the propagators $G^{\{AB,A\bar{B}\}}$ and $G^{\{BB,\bar{B}B\}}$, is modified (in momentum space) according to $k^2\tilde{k}^2 \to (\sigma + k^2)$;

an expression having no poles in Euclidean space. The damping factor of the gauge boson propagator (E.12c) now takes the form $\left(1+\frac{\gamma^6}{(\sigma+k^2)(\tilde{k}^2)^2}\right)^{-1}$, which shows the same IR behavior as the respective counterpart in Eqn. (E.5). For completeness, the propagators of this model shall be given.

$$G^{\bar{c}c}(k) = -\frac{1}{k^2}\,, \tag{E.12a}$$

$$G^{\bar{\psi}\psi}_{\mu\nu,\rho\sigma}(k) = \frac{(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})}{2(\sigma+k^2)} = -G^{\psi\bar{\psi}}_{\mu\nu,\rho\sigma}(k)\,, \tag{E.12b}$$

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\left(1+\frac{\gamma^6}{\tilde{k}^4(\sigma+k^2)}\right)}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\alpha\frac{k_\mu k_\nu}{k^4}\,, \tag{E.12c}$$

$$\begin{aligned} G^{AB}_{\mu,\rho\sigma}(k) &= \frac{\mathrm{i}\gamma^3}{2}\frac{(k_\rho\delta_{\mu\sigma}-k_\sigma\delta_{\mu\rho})}{2k^2\left(\tilde{k}^4(\sigma+k^2)+\frac{\gamma^6}{\tilde{k}^4}\right)} \\ &= G^{A\bar{B}}_{\mu,\rho\sigma}(k) = -G^{\bar{B}A}_{\rho\sigma,\mu}(k)\,, \end{aligned} \tag{E.12d}$$

$$\begin{aligned} G^{BB}_{\mu\nu,\rho\sigma}(k) &= \frac{\gamma^6}{4k^2(\sigma+k^2)\left(\tilde{k}^4(\sigma+k^2)+\gamma^6\right)}\left[k_\mu k_\rho\delta_{\nu\sigma}+k_\nu k_\sigma\delta_{\mu\rho}-k_\mu k_\sigma\delta_{\nu\rho}-k_\nu k_\rho\delta_{\mu\sigma}\right] \\ &= G^{\bar{B}\bar{B}}_{\mu\nu,\rho\sigma}(k)\,, \end{aligned} \tag{E.12e}$$

$$G^{B\bar{B}}_{\mu\nu,\rho\sigma}(k) = G^{BB}_{\mu\nu,\rho\sigma}(k)+\frac{1}{2(\sigma+k^2)}\left[\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right]\,. \tag{E.12f}$$

(E.12g)

Note that all of these two-point functions are finite in the limit $\tilde{k}\to 0$. However, there are still some insufficiencies in this model. For the first, a shift of the divergent factors by $-\sigma$ is suitable for Euclidean space, where k^2 is strictly positive. But this trick will not work in a possible later implementation in Minkowski space. For the second, this model still lacks a dedicated term to absorb the expected divergences at the one loop level (see Section 3.3). Hence, we are motivated to enhance the solution (E.6).

E.1.2 Evolutionary step 2

We have seen above in Appendix E.1.1 that the negative potences of derivatives, which are required for the construction of damping terms in the $1/p^2$ model, can be implemented by direct insertion into a soft breakingterm (thereby avoiding any problems due to the interpretation of inverse powers of covariant derivatives). Based on the advanced model (E.6) several additional ideas can be implemented.

- ▷ Since the damping is implemented completely by the bilinear part of the action, and the necessary operator with negative powers of derivatives can be shifted to the breaking term, we may replace the operator sandwiched in the $\{\bar{B},B\}$ and $\{\bar{\psi},\psi\}$ sectors by a constant in Eqn. (E.6).
- ▷ In the same way, $F_{\mu\nu}$ in the breaking term may be reduced to its bilinear part, $f_{\mu\nu}$. This is valid since the breaking term, naturally, doesn't need to fulfill gauge invariance.
- ▷ In order to implement a counterterm for the expected $\tilde{p}_\mu\tilde{p}_\nu/(\tilde{p}^2)^2$ divergence, we may first analyze how the term $k_\mu k_\nu/k^2$ in the gauge propagator is constructed (since it carries a

similar index structure). It offsprings the expression $\partial_\mu\partial_\nu A_\nu$ appearing in the equations of motion (E.7) and (E.8). Hence, the conjecture is that a term $\tilde\partial_\mu\tilde\partial_\nu A_\nu$ will be required, in addition. In order to avoid unnecessary modification of the damping term which is influenced by both occurrences of $(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)A_\nu$ (from the $\frac{1}{4}F^2$ term, and the breaking term), we will have to take care when implementing any changes to these terms. The overall damping factor (which we denote now temporarily by $\mathcal{G}(k)$) is added automatically, if the final expression for j^A_μ contains an expression $\Box\mathcal{G}(k)A_\mu$ which, in turn, is guaranteed, if the combination $B_{\mu\nu}\tilde{\Box}^{-1}f_{\mu\nu}$ is unaltered.

From these considerations we are led to the action

$$S_{\text{ev2}} = \int \mathrm{d}^4x\,\Big[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + b\,(\partial A) - \frac{\alpha}{2}b^2 - \bar{c}\partial_\mu D_\mu c + \frac{\gamma^2}{2}\left(B_{\mu\nu}+\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\Box}}\left(f_{\mu\nu}+\sigma\frac{\Theta_{\mu\nu}}{2}\tilde{f}\right) - \mu^2\bar{B}_{\mu\nu}B_{\mu\nu} + \mu^2\bar{\psi}_{\mu\nu}\psi_{\mu\nu} + \text{source terms}\Big]\,, \tag{E.13}$$

with the definitions from the main text (page 74). Note ,that the mass dimension $d_m(\bar{B}) = d_m(B) = 2$ has changed. For the last time, we repeat the steps of the analysis for the propagators. The equations of motion now take the form

$$\frac{\delta S^{\text{bi}}_{\text{ev1}}}{\delta A^\mu} = -j^A_\mu = \left(\partial_\mu\partial^\nu - \Box\delta^\nu_\mu\right)A_\nu - \frac{\gamma^2}{2\tilde{\Box}}\left[2\partial_\sigma\delta_{\mu\rho} - \sigma\Theta_{\rho\sigma}\tilde{\partial}_\mu\right]\left(B_{\rho\sigma}+\bar{B}_{\rho\sigma}\right) - \partial_\mu b\,, \tag{E.14}$$

$$\frac{\delta S^{\text{bi}}_{\text{ev1}}}{\delta\{B^{\mu\nu},\bar{B}^{\mu\nu}\}} = -j^{\{B,\bar{B}\}}_{\mu\nu} = \frac{\gamma^2}{2\tilde{\Box}}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - \sigma\Theta_{\mu\nu}(\tilde{\partial}A)\right) - \{\bar{B}_{\mu\nu}, B_{\mu\nu}\}\,, \tag{E.15}$$

and finally,

$$\{B_{\mu\nu},\bar{B}_{\mu\nu}\} = \frac{\gamma^2}{2\tilde{\Box}}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - \sigma\Theta_{\mu\nu}(\tilde{\partial}A)\right) + \{j^B_{\mu\nu}, j^{\bar{B}}_{\mu\nu}\} \tag{E.16}$$

$$-j^A_\mu = \mathcal{G}(k)\left(\partial_\mu\partial_\nu - \Box\delta_{\mu\nu}\right)A_\nu - \frac{\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu\left(\tilde{\partial}A\right)\Sigma - \frac{\gamma^2}{\tilde{\Box}}\left(\partial_\sigma\delta_{\rho\mu} - \frac{\sigma}{4}\tilde{\partial}_\mu\varepsilon_{\rho\sigma}\right)\left(j^B_{\rho\sigma}+j^{\bar{B}}_{\rho\sigma}\right) + \frac{\partial_\mu}{\alpha}\left((\partial A)+j_b\right)\,, \tag{E.17}$$

$$\begin{aligned} A_\mu = \frac{1}{\Box\mathcal{G}(k)}\Bigg\{ &\frac{\partial_\mu}{\alpha}j^b - \frac{\gamma^2}{\tilde{\Box}}\left(\partial_\sigma\delta_{\rho\mu} - \frac{\sigma}{2}\tilde{\partial}_\mu\Theta_{\rho\sigma}\right)\left(j^B_{\rho\sigma}+j^{\bar{B}}_{\rho\sigma}\right) \\ &+ \frac{\gamma^4\Sigma}{\tilde{\Box}^2}\tilde{\partial}_\mu\left[\Sigma\frac{\gamma^4}{\tilde{\Box}} - \frac{\mathcal{G}(k)\Box}{2}\right]^{-1}\left[(\tilde{\partial}j^A) - \frac{\gamma^2}{\tilde{\Box}}\left(\tilde{\partial}_\rho\partial_\sigma - \frac{\tilde{\Box}}{4}\sigma\Theta_{\rho\sigma}\right)\left(j^B_{\rho\sigma}+j^{\bar{B}}_{\rho\sigma}\right)\right] \\ &- \frac{\partial_\mu}{\Box}\left[(\partial j^A) + \frac{\Box}{\alpha}j^b - \frac{\gamma^2}{\tilde{\Box}}\partial_\rho\partial_\sigma\left(j^B_{\rho\sigma}+j^{\bar{B}}_{\rho\sigma}\right)\right]\Bigg\} \\ &- \alpha\frac{\partial_\mu}{\Box}^2\left[(\partial j^A) + \frac{\Box}{\alpha} - \frac{\gamma^2}{\tilde{\Box}}\partial_\rho\partial_\sigma\left(j^B_{\rho\sigma}+j^{\bar{B}}_{\rho\sigma}\right)\right]\,, \end{aligned} \tag{E.18}$$

where the temporal abbreviation $\Sigma := \left(\sigma + \frac{\Theta^2}{4}\sigma^2\right)$ has been used. The rather lengthy form of these intermediate expressions leads to surprisingly simple forms for the propagators (E.19) – (E.24) in the next section. Despite, not being necessary any more due to the uncoupling, all of these (with the sole exception of the ghost propagator (E.20)) are IR finite. The gauge propagator now feature a suitable term $\propto \tilde{k}_\mu\tilde{k}_\nu/(\tilde{k}^2)^2$ to absorb the expected one loop divergence. Note again, that due to the complete uncoupling of the gauge sector from the $\{\bar{B}, B\}$ and

$\{\bar\psi,\psi\}$ sectors, naturally, any interactions of these fields with A_μ vanish. Indeed, although there exist two point functions $G^{\{AB,A\bar B\}}$, $G^{\{BB,\bar BB\}}$ and $G^{\bar\psi\psi}$, these may only result in unphysical disconnected vacuum bubbles. Further consequences and properties are discussed in the main text of Section 5.1.

E.2 Feynman Rules

E.2.1 Propagators

The propagators follow from the expressions given in Appendix E.1.2. Despite being partly stated in the main text (Section 5.1) they shall be repeated at this point for the sake of completeness.

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\left(1+\frac{\gamma^4}{(\tilde k^2)^2}\right)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\left(\sigma+\frac{\Theta^2}{4}\sigma^2\right)\gamma^4}{\left[\left(\sigma+\frac{\Theta^2}{4}\sigma^2\right)\gamma^4 + k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)\right]}\frac{\tilde k_\mu\tilde k_\nu}{\tilde k^2}\right), \tag{E.19}$$

$$G^{\bar cc}(k) = \frac{-1}{k^2}, \tag{E.20}$$

$$\begin{aligned} G^{BA}_{\mu\nu,\rho}(k) &= \frac{\mathrm{i}\gamma^2\left(k_\mu\delta_{\sigma\nu} - k_\nu\delta_{\sigma\mu} - \sigma\tilde k_\sigma\Theta_{\mu\nu}\right)}{2k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)}\left[\delta_{\rho\sigma} - \frac{\bar\sigma^4}{\left[\bar\sigma^4 + k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)\right]}\frac{\tilde k_\rho\tilde k_\sigma}{\tilde k^2}\right] \\ &= G^{\bar BA}_{\mu\nu,\rho}(k), \end{aligned} \tag{E.21}$$

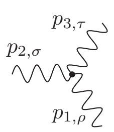
$$\begin{aligned} G^{BB}_{\mu\nu,\rho\sigma}(k) &= -\gamma^4\frac{(k_\mu k_\rho\delta_{\nu\sigma} + k_\nu k_\sigma\delta_{\mu\rho} - k_\mu k_\sigma\delta_{\nu\rho} - k_\nu k_\rho\delta_{\mu\sigma})}{2k^2\tilde k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)} \\ &\quad + \frac{\gamma^4}{4\tilde k^2\left[k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)+\bar\sigma^4\right]}\left[\sigma\Theta_{\mu\nu}\left(k_\rho\tilde k_\sigma - k_\sigma\tilde k_\rho\right) + \sigma\Theta_{\rho\sigma}\left(k_\mu\tilde k_\nu - k_\nu\tilde k_\mu\right)\right. \\ &\quad\left. - \sigma^2\tilde k^2\Theta_{\mu\nu}\Theta_{\rho\sigma} - \bar\sigma^4\frac{\left(k_\mu\tilde k_\nu\tilde k_\rho k_\sigma + k_\rho\tilde k_\sigma\tilde k_\mu k_\nu - k_\mu\tilde k_\nu\tilde k_\sigma k_\rho + k_\sigma\tilde k_\rho\tilde k_\mu k_\nu\right)}{k^2\tilde k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)}\right] \\ &= G^{\bar B\bar B}_{\mu\nu,\rho\sigma}(k), \end{aligned} \tag{E.22}$$

$$G^{B\bar B}_{\mu\nu,\rho\sigma}(k) = -\frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) + G^{BB}_{\mu\nu,\rho\sigma}(k), \tag{E.23}$$

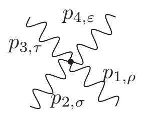
$$G^{\bar\psi\psi}_{\mu\nu\rho\sigma}(k) = -\frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right). \tag{E.24}$$

E.2.2 Vertices

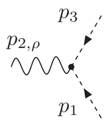
The vertex expressions result by direct variation of the action (5.3) with respect to the fields



$$\begin{aligned} &= V^{3A}_{\rho\sigma\tau}(k_1,k_2,k_3) = 2\mathrm{i}g(2\pi)^4\delta^4(k_1+k_2+k_3)\sin\left(\tfrac{\varepsilon}{2}k_1\tilde k_2\right)\times \\ &\quad\times\left[(k_3-k_2)_\rho\delta_{\sigma\tau} + (k_1-k_3)_\sigma\delta_{\rho\tau} + (k_2-k_1)_\tau\delta_{\rho\sigma}\right], \end{aligned} \tag{E.25a}$$



$$= V^{4A}_{\rho\sigma\tau\epsilon}(k_1,k_2,k_3,k_4) = -4g^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\times$$
$$\times\Big[(\delta_{\rho\tau}\delta_{\sigma\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_2\right)\sin\left(\tfrac{\varepsilon}{2}k_3\tilde{k}_4\right)$$
$$+(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_3\right)\sin\left(\tfrac{\varepsilon}{2}k_2\tilde{k}_4\right)$$
$$+(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\tau}\delta_{\sigma\epsilon})\sin\left(\tfrac{\varepsilon}{2}k_2\tilde{k}_3\right)\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_4\right)\Big], \qquad \text{(E.25b)}$$



$$= V^{\bar{c}Ac}_{\mu}(q_1,k_2,q_3) = -2\mathrm{i}g(2\pi)^4\delta^4(q_1+k_2+q_3)q_{3\mu}\sin\left(\tfrac{\varepsilon}{2}q_1\tilde{q}_3\right). \qquad \text{(E.25c)}$$

E.3 One Loop Calculations

E.3.1 Vacuum Polarization

The integrands of the expressions for the three graphs in Fig. 5.1 are expanded around $p = 0$ (see Section 3.1.3, Eqn. (3.17) on page 39), simplified by approximating for large k, and finally integrated according to the rules in Appendix F.1. The symmetry factors are determined to be $s_\mathrm{a} = 1/2$, $s_\mathrm{b} = 1/2$, $s_\mathrm{c} = -1$[1]. Denoting the result at order i for the planar (p) part by $\Pi^{(i),\mathrm{p}}_{\mu\nu}$ the detailed outcome of these calculations is

$$\Pi^{(0),\mathrm{p}}_{\mu\nu}(p) = \frac{g^2}{(4\pi)^2}\Lambda^2\delta_{\mu\nu}\left(12s_\mathrm{a}-10s_\mathrm{b}+s_\mathrm{c}\right)+\text{finite}$$
$$=0\,, \qquad \text{(E.26a)}$$
$$\Pi^{(2),\mathrm{p}}_{\mu\nu}(p) = \frac{g^2}{3(4\pi)^2}\,\mathrm{K}_0\sqrt{\frac{\mu^2}{\Lambda^2}}\left((50s_\mathrm{a}-s_\mathrm{c})p^2\delta_{\mu\nu}-2(28s_\mathrm{a}+s_\mathrm{c})p_\mu p_\nu\right)$$
$$=\frac{26g^2}{3(4\pi)^2}\,\mathrm{K}_0\sqrt{\frac{\mu^2}{\Lambda^2}}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)+\text{finite}\,, \qquad \text{(E.26b)}$$
$$\approx\frac{13g^2}{3(4\pi)^2}\ln\frac{\mu^2}{\Lambda^2}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)+\text{finite}\,. \qquad \text{(E.26c)}$$

Similar for the non-planar (np) part

$$\Pi^{(0),\mathrm{np}}_{\mu\nu}(p) = -\frac{g^2}{(2\pi)^2}\frac{1}{(\varepsilon^2\tilde{p}^2)^2}\left(\tilde{p}^2(12s_\mathrm{a}-10s_\mathrm{b}+s_\mathrm{c})\delta_{\mu\nu}-2\tilde{p}_\mu\tilde{p}_\nu(12s_\mathrm{a}-2s_\mathrm{b}+s_\mathrm{c})\right)+\text{finite}$$
$$=\frac{2g^2\tilde{p}_\mu\tilde{p}_\nu}{\pi^2(\varepsilon^2\tilde{p}^2)^2}\,, \qquad \text{(E.27a)}$$
$$\Pi^{(2),\mathrm{np}}_{\mu\nu}(p) = \frac{g^2}{3(4\pi)^2}\,\mathrm{K}_0\sqrt{\varepsilon^2\tilde{p}^2\mu^2}\left((-50s_\mathrm{a}+s_\mathrm{c})p^2\delta_{\mu\nu}+2(28s_\mathrm{a}+s_\mathrm{c})p_\mu p_\nu\right)+\text{finite}\,,$$
$$=-\frac{26g^2}{3(4\pi)^2}\,\mathrm{K}_0\sqrt{\varepsilon^2\tilde{p}^2\mu^2}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)+\text{finite}$$
$$\approx\frac{13g^2}{3(4\pi)^2}\ln\left(\tilde{p}^2\mu^2\right)\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)+\mathcal{O}\left(\frac{\mu^2}{\Lambda^2}\right)+\text{finite}\,. \qquad \text{(E.27b)}$$

[1] Verified by computation with the `SymmetryFactor` program; see Appendix G.1.

Note that the results in first order of the expansion in p vanish identically due to the symmetric integration over an antisymmetric function (represented by an odd power of the internal momentum k).

Combining all orders and parts we see that the regulating IR cutoff μ (see Section A.12) drops out in the sum if the Bessel functions K_0 are expanded according to Eqn. (F.24), and we are left with

$$\begin{aligned}\Pi_{\mu\nu}(p) &= \frac{g^2}{3\pi^2}\left[6\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\varepsilon\tilde{p}^2)^2} + \frac{13}{16}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left(\mathrm{K}_0\sqrt{\varepsilon^2\tilde{p}^2\mu^2} - \mathrm{K}_0\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\right] + \text{finite} \\ &\approx \frac{2g^2}{\pi^2\varepsilon^4}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2} + \frac{13g^2}{3(4\pi)^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\left(\Lambda^2\right) + \text{finite terms}\,, \end{aligned} \tag{E.28}$$

which equals Eqn. (5.22) in the main text.

E.3.2 3A Vertex

Principally, the vertex corrections corresponding to the graphs in Fig. 5.1 in the main text on page 78 are calculated in the same way as the vacuum polarization in Section E.3.1 above. However, there are a few subtleties which have to be considered. First, the form of the integrands is quite complicated due to the higher number of concatenated Feynman rules. For example, a typical expression to appear would be,

$$\int_{-\infty}^{\infty} \mathrm{d}^4k\, \frac{k_\mu k_\nu k_\rho k_\sigma \sin\left(\frac{p_1}{2}(\tilde{p}_2 - \tilde{k})\right)\sin\left(\frac{p_2\tilde{k}}{2}\right)\sin\left(\frac{(p_1+p_2)\tilde{k}}{2}\right)}{(-p_2+k)^4(-p_1-p_2+k)^4}\,. \tag{E.29}$$

An explicit computation requires three nested sums for the phase factors, two Schwinger parameters, and four derivatives. The resulting exponential can still be brought into a quadratic form, but the remaining terms are much to complicated to be integrated out. Application of the expansion (3.17) for small external momenta p_i is not applicable here since the limits $p_i \to 0$ and $p_j \to 0$ for $i \neq j$ do not commute. Let us examine this latter point in a little more detail. The multidimensional Taylor expansion $\mathcal{T}$ to order m of a function $f(\{p_1, ..p_n\})$ around the point $p_0 := (p_{0,1}, \dots p_{0,n}) = \{p_{0,1}, ..p_{0,n}\}$ is written in the form

$$\begin{aligned}\mathcal{T}(p_0)\, f(\{p_1,..p_n\}) &= f(\{p_1,..p_n\})|_{\{p_1,..p_n\}\to\{p_{0,1},..p_{0,n}\}} \\ &+ \sum_{i=1}^{n}\Big[\partial_i f(\{p_1,..p_n\})\Big]_{\{p_1,..p_n\}\to\{p_{0,1},..p_{0,n}\}}(p_i - p_{0,i}) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}\Big[\partial_i\partial_j f(\{p_1,..p_n\})\Big]_{\{p_1,..p_n\}\to\{p_{0,1},..p_{0,n}\}}(p_i - p_{0,i})(p_j - p_{0,j}) \\ &+ \sum_{i=1}^{n}\cdots\sum_{k=1}^{n}\Big[\partial_i\dots\partial_k f(\{p_1,..p_n\})\Big]_{\{p_1,..p_n\}\to\{p_{0,1},..p_{0,n}\}}\prod_{\alpha=i}^{k}(p_\alpha - p_{0,\alpha}) + \mathcal{R}_m\,, \end{aligned} \tag{E.30}$$

where $\mathcal{R}_m$ is the rest term at order m. Each term in this expansion requires the limit of all independent variables $p_1, ..p_n$ to be taken simultaneously. This, in turn, is well defined only for the case

$$\lim_{p_i\to p_{0,i}} f(\{p_1,..p_n\}) \lim_{p_j\to p_{0,j}} f(\{p_1,..p_n\}) \equiv \lim_{p_j\to p_{0,j}} f(\{p_1,..p_n\}) \lim_{p_i\to p_{0,i}} f(\{p_1,..p_n\})\,, \quad \forall i,j \in \{1,..n\}\,.$$

But this is not the case since all the p_i are vectorial quantities, and a limit of product expressions depends upon the relative angle. The problem can intuitively be understood when considering the simple function $g(x) = xy/(x^2 + y^2)$ which has no unique limit for $(x, y) \to (0, 0)$ since

$$\lim_{x\to 0}\left(\lim_{y\to 0} g(x,y)\right) = 0\,,$$

but

$$\lim_{y\to 0} g(x \to ty, y) = \frac{t}{1+t^2} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\,.$$

A similar example which appears in most computations presented in this work, is the term $(p_1 \cdot p_2)/(p_1^2 + p_2^2)$, with $p_{1,\mu}$ and $p_{2,\mu}$ being vectors.

In fact, the above discussion leads to the insight that, in general, the expansion (3.17) is only applicable if it is applied with respect to a single variable. This, however, is not the case for vertex graphs with an arbitrary number $E_A > 2$ of external gauge boson lines with equal preference. Hence, the expansion for small external momenta, as it has been applied in Appendices B.2, C.2, and E.3.1, cannot be utilized here.
An alternative approach for the simplification of the vertex integral expressions is motivated by the fact that, as has already been mentioned several times, all divergences of planar and non-planar parts originate from the UV limit of the integrands. Hence, expressions like Eqn. (E.29) may be reduced by[2]

$$\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{(k+q)^n} \underset{k\to\infty}{\to} \int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^n}\,. \tag{E.31}$$

Furthermore, we can rewrite the threefold phase factor by virtue of the relations (F.21), (F.19), and (F.20) as[3]

$$\begin{aligned}&\sin\frac{p_1(\tilde{p}_2-\tilde{k})}{2}\sin\frac{p_2\tilde{k}}{2}\sin\frac{p_3\tilde{k}}{2}\\ &=\frac{1}{4}\cos\frac{p_1\tilde{p}_2}{2}\left(\sin p_1\tilde{k}+\sin p_2\tilde{k}+\sin p_3\tilde{k}\right)-\frac{1}{4}\sin\frac{p_1\tilde{p}_2}{2}\left(1+\cos p_1\tilde{k}-\cos p_2\tilde{k}-\cos p_3\tilde{k}\right)\,,\end{aligned} \tag{E.32}$$

which enables to separate an overall phase factor depending solely on external momenta from a single phase of type $\sin k\tilde{p}_i$ or $\cos k\tilde{p}_i$ to be kept in the integrand. The remaining integrals can easily be solved by applying the integration rules of Appendix F.1. In addition, we have to consider permutations of external legs (similar as for the scalar model in Appendix A.2.2, page 94). These are collected in Tab. E.1 Using `VectorAlgebra` we arrive at the following expression for the planar part (decorated by 'UV' instead of 'p' here) of the sum of amplitudes in Fig. 5.1

$$\Gamma^{3A,\mathrm{UV}}_{\mu\nu\rho}(p_1,p_2,p_3) = \frac{ig^3}{\pi^2}\,\mathrm{K}_0\sqrt{\frac{M^2}{\Lambda^2}}\Bigg[\sin\frac{p_1\theta p_2}{2}\Big(\left(p_{2,\rho}-p_{1,\rho}\right)\delta_{\mu\nu}+\left(p_{1,\nu}-p_{3,\nu}\right)\delta_{\mu\rho}+\left(p_{3,\mu}-p_{2,\mu}\right)\delta_{\nu\rho}\Big)\Bigg]\,.$$

This result is contributed solely by one, namely a), of three graphs in Fig. 5.2. Note that none of these exists in commutative theories due to different reasons. First of all, the *Furry theorem*

[2]See also the discussion at the beginning of Section 3.1.3, and in Section 1.3.7.

[3]We temporally omit the dimensionful ε here.

Table E.1: Permutations functions and symmetry factors s of the graphs a)–c) in Fig. 5.1.

Gr.	s	Permutations
a)	1	$\{p_{1,\mu},p_{2,\nu},p_{3,\rho}\}\rightarrow\{p_{1,\mu},p_{2,\nu},p_{3,\rho}\},\ \{p_{1,\mu},p_{3,\rho},p_{2,\nu}\},\ \{p_{2,\nu},p_{1,\mu},p_{3,\rho}\},\ \{p_{2,\nu},p_{3,\rho},p_{1,\mu}\},$ $\{p_{3,\rho},p_{1,\mu},p_{2,\nu}\},\ \{p_{3,\rho},p_{2,\nu},p_{1,\mu}\}$
b)	1/2	$\{p_{1,\mu},p_{2,\nu},p_{3,\rho}\}\{p_{1,\mu},p_{2,\nu},p_{3,\rho}\},\ \{p_{1,\mu},p_{3,\rho},p_{2,\nu}\},\ \{p_{2,\nu},p_{3,\rho},p_{1,\mu}\}$
c)	-2	$\{p_{1,\mu},p_{2,\nu},p_{3,\rho}\}\rightarrow\{p_{1,\mu},p_{2,\nu},p_{3,\rho}\},\ \{p_{1,\mu},p_{3,\rho},p_{2,\nu}\},\ \{p_{2,\nu},p_{1,\mu},p_{3,\rho}\},\ \{p_{2,\nu},p_{3,\rho},p_{1,\mu}\},$ $\{p_{3,\rho},p_{1,\mu},p_{2,\nu}\},\ \{p_{3,\rho},p_{2,\nu},p_{1,\mu}\}$

states that, due to invariance of the action under charge (C) conjugation, any Green function with an odd number of external photon fields (E_A) vanishes. This argument affects all three graphs. However, the theorem does not exist on non-commutative spaces, and hence, the graphs do exist. Regarding the functions b) and c), in a theory obeying an undeformed $U(1)$ symmetry, no ghosts are present, and the commutator term of the field strength $F_{\mu\nu}$ vanishes. Since the planar contributions correspond, in some sense, to the undeformed part of the theory, it is not surprising that no planar corrections are generated by the graph with internal ghost loop, b). In the same way, the function c) of Fig. 5.1 contains a vertex V^{4A}, which originates solely from the double commutator term buried in the Yang Mills F^2 term. Accordingly, in the planar case it is intutitive, that this graph gives no divergent contribution.

The non-planar result $\Gamma^{3A,\text{IR}}_{\mu\nu\rho}(p_1,p_2,p_3)$ receives contributions from all three graphs. However, the explicit form is much to large to be printed here. Instead, we shall discuss the types of divergences and possible counterterms arising from them in the main text.

E.3.3 $cA\bar{c}$ Vertex

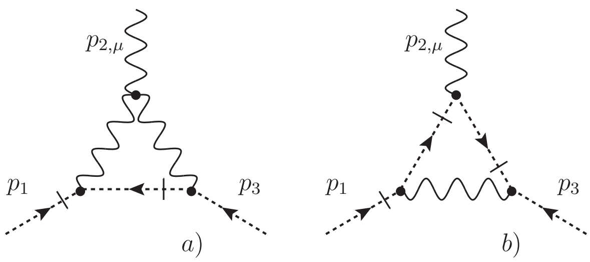


Figure E.1: One loop corrections to the ghost vertex vertex $V^{\bar{c}Ac}_{\mu}(p_1,p_2,p_3)$.

In fact, the graphs of Fig. E.1 do not give rise to any divergences. Hence, no correction to this vertex exists on the one loop level.

E.3.4 4A Vertex

For the gauge boson vertex with four external legs, we have the four types of graphs depicted in Fig. 5.3. The phase transformations are much more involved than for the $3A$ vertex but can still be derived by repeated application of the relations (F.19), (F.21), and (F.20). An examplary result is (where again ε has been omitted in the arguments of trigonometric functions)

$$\sin\left(\frac{p_1\tilde{k}}{2}\right)\sin\left(\frac{p_2\tilde{k}}{2}\right)\sin\left(\frac{p_3(\tilde{k}+\tilde{p}_2)}{2})\right)\sin\left(\frac{p_4(\tilde{k}+\tilde{p}_2}{2}\right)$$

$$
\begin{aligned}
=\frac{1}{8}\Big[\cos\left(\frac{p_3\tilde{p}_2}{2}\right)\cos\left(\frac{p_4\tilde{p}_2}{2}\right)&\Big(\cos((p_1+p_2)\tilde{k})+\cos((p_1+p_3)\tilde{k})+\cos((p_1+p_4)\tilde{k})\\
&-\cos(p_1\tilde{k})+\cos(p_2\tilde{k})+\cos(p_3\tilde{k})+\cos(p_4\tilde{k})+1\Big)\\
-\sin\left(\frac{p_3\tilde{p}_2}{2}\right)\cos\left(\frac{p_4\tilde{p}_2}{2}\right)&\Big(-\sin((p_1+p_2)\tilde{k})+\sin((p_1+p_3)\tilde{k})-\sin((p_1+p_4)\tilde{k})\\
&+\sin(p_1\tilde{k})+\sin(p_2\tilde{k})-\sin(p_3\tilde{k})+\sin(p_4\tilde{k})\Big)\\
-\cos\left(\frac{p_3\tilde{p}_2}{2}\right)\sin\left(\frac{p_4\tilde{p}_2}{2}\right)&\Big(-\sin((p_1+p_2)\tilde{k})-\sin((p_1+p_3)\tilde{k})+\sin((p_1+p_4)\tilde{k})\\
&+\sin(p_1\tilde{k})+\sin(p_2\tilde{k})+\sin(p_3\tilde{k})-\sin(p_4\tilde{k})\Big)\\
+\sin\left(\frac{p_3\tilde{p}_2}{2}\right)\sin\left(\frac{p_4\tilde{p}_2}{2}\right)&\Big(-\cos((p_1+p_2)\tilde{k})+\cos((p_1+p_3)\tilde{k})+\cos((p_1+p_4)\tilde{k})\\
&+\cos(p_1\tilde{k})+\cos(p_2\tilde{k})-\cos(p_3\tilde{k})-\cos(p_4\tilde{k})-1\Big)\Big]\,. \qquad \text{(E.33)}
\end{aligned}
$$

However, there exist combinations of arguments in the initial sine functions, which do not match this form. Hence, the respective transformations are computed in an automated way in Mathematica® by utilization of the functions `TrigReduce[]` and `TrigExpand[]`. The actual computation proceeds along the lines of Appendix E.3.2, where the respective symmetry factors and permutations are listed in Tab. E.2. After summing up all contributions and integration

Table E.2: Permutations functions and symmetry factors s of the graphs a)–c) in Fig. 5.1.

Gr.	s	Permutations
a)	1	$\{p_{1,\mu},p_{2,\nu},p_{3,\rho},p_{4,\sigma}\}\rightarrow\{p_{1,\mu},p_{2,\nu},p_{3,\rho},p_{4,\sigma}\}$, $\{p_{1,\mu},p_{2,\nu},p_{4,\sigma},p_{3,\rho}\}$, $\{p_{1,\mu},p_{3,\rho},p_{2,\nu},p_{4,\sigma}\}$,
		$\{p_{1,\mu},p_{3,\rho},p_{4,\sigma},p_{2,\nu}\}$, $\{p_{1,\mu},p_{4,\sigma},p_{2,\nu},p_{3,\rho}\}$, $\{p_{1,\mu},p_{4,\sigma},p_{3,\rho},p_{2,\nu}\}$,
		$\{p_{2,\nu},p_{1,\mu},p_{3,\rho},p_{4,\sigma}\}$, $\{p_{2,\nu},p_{1,\mu},p_{4,\sigma},p_{3,\rho}\}$, $\{p_{2,\nu},p_{3,\rho},p_{1,\mu},p_{4,\sigma}\}$,
		$\{p_{2,\nu},p_{3,\rho},p_{4,\sigma},p_{1,\mu}\}$, $\{p_{2,\nu},p_{4,\sigma},p_{1,\mu},p_{3,\rho}\}$, $\{p_{2,\nu},p_{4,\sigma},p_{3,\rho},p_{1,\mu}\}$,
		$\{p_{3,\rho},p_{1,\mu},p_{2,\nu},p_{4,\sigma}\}$, $\{p_{3,\rho},p_{1,\mu},p_{4,\sigma},p_{2,\nu}\}$, $\{p_{3,\rho},p_{2,\nu},p_{1,\mu},p_{4,\sigma}\}$,
		$\{p_{3,\rho},p_{2,\nu},p_{4,\sigma},p_{1,\mu}\}$, $\{p_{3,\rho},p_{4,\sigma},p_{1,\mu},p_{2,\nu}\}$, $\{p_{3,\rho},p_{4,\sigma},p_{2,\nu},p_{1,\mu}\}$,
		$\{p_{4,\sigma},p_{1,\mu},p_{2,\nu},p_{3,\rho}\}$, $\{p_{4,\sigma},p_{1,\mu},p_{3,\rho},p_{2,\nu}\}$, $\{p_{4,\sigma},p_{2,\nu},p_{1,\mu},p_{3,\rho}\}$,
		$\{p_{4,\sigma},p_{2,\nu},p_{3,\rho},p_{1,\mu}\}$, $\{p_{4,\sigma},p_{3,\rho},p_{1,\mu},p_{2,\nu}\}$, $\{p_{4,\sigma},p_{3,\rho},p_{2,\nu},p_{1,\mu}\}$
b)	-2	same as for a)
c)	1/2	$\{p_{1,\mu},p_{2,\nu},p_{3,\rho},p_{4,\sigma}\}\rightarrow\{p_{1,\mu},p_{2,\nu},p_{3,\rho},p_{4,\sigma}\}$, $\{p_{2,\nu},p_{1,\mu},p_{3,\rho},p_{4,\sigma}\}$, $\{p_{2,\nu},p_{4,\sigma},p_{3,\rho},p_{1,\mu}\}$
d)	1	$\{p_{1,\mu},p_{2,\nu},p_{3,\rho},p_{4,\sigma}\}\rightarrow\{p_{1,\mu},p_{2,\nu},p_{3,\rho},p_{4,\sigma}\}$, $\{p_{2,\nu},p_{1,\mu},p_{3,\rho},p_{4,\sigma}\}$, $\{p_{1,\mu},p_{3,\rho},p_{2,\nu},p_{4,\sigma}\}$,
		$\{p_{3,\rho},p_{1,\mu},p_{2,\nu},p_{4,\sigma}\}$, $\{p_{3,\rho},p_{2,\nu},p_{1,\mu},p_{4,\sigma}\}$, $\{p_{2,\nu},p_{3,\rho},p_{1,\mu},p_{4,\sigma}\}$,
		$\{p_{4,\sigma},p_{2,\nu},p_{3,\rho},p_{1,\mu}\}$, $\{p_{2,\nu},p_{4,\sigma},p_{3,\rho},p_{1,\mu}\}$, $\{p_{1,\mu},p_{4,\mu},p_{3,\rho},p_{2,\nu}\}$,
		$\{p_{4,\sigma},p_{1,\mu},p_{3,\rho},p_{2,\nu}\}$, $\{p_{4,\sigma},p_{3,\rho},p_{1,\mu},p_{2,\nu}\}$, $\{p_{3,\rho},p_{4,\sigma},p_{1,\mu},p_{2,\nu}\}$,

with `VectorAlgebra` we arrive at

$$
\begin{aligned}
\Gamma^{4A,UV}_{\mu\nu\sigma\tau}(k_1,k_2,k_3,k_4) = \text{const}\,\pi^2 g^4 (2\pi)^4\,\mathrm{K}_0\sqrt{\frac{\mu^2}{\Lambda^2}}\,&\delta^4(k_1+k_2+k_3+k_4)\\
&\Big[(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_2\right)\sin\left(\tfrac{\varepsilon}{2}k_3\tilde{k}_4\right)\\
&+(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_3\right)\sin\left(\tfrac{\varepsilon}{2}k_2\tilde{k}_4\right)\\
&+(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\rho}\delta_{\nu\sigma})\sin\left(\tfrac{\varepsilon}{2}k_2\tilde{k}_3\right)\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_4\right)\Big], \qquad \text{(E.34)}
\end{aligned}
$$

for the planar part (where the factor 'const' is unfortunately not definitely known at the time of publication, see [62]), and an expression of more than 4100 terms for the non-planar part. It does not seem reasonable to give the latter result in an explicit form here, but only to discuss the types of terms in the main text on page 81.

Appendix F

Useful Relations

F.1 Momentum Integrals

In the loop calculations discussed within the framework of this thesis several integrals of the generic type

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_{\eta_1} \dots k_{\eta_n}}{\left(k^2 + \frac{a^2}{k^2}\right)^m} \sin^2\left(\tfrac{1}{2} k\theta p\right) , \quad \text{with } m, n \in \mathbb{N}_0 \,,$$

appear. Since one is interested mainly in IR divergences in p which originate from the large momentum behavior in k (i.e. $k \to \pm\infty$) the integrands are approximated by considering $\left(k^2 + \frac{a^2}{k^2}\right) \approx k^2$. In addition, since $\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x)$, the integration can be split into planar and non-planar parts. The respective results are given separately (and all factors except the $(2\pi)^{-4}$ are included). In the results the cutoffs $\mu \to 0$ and $\Lambda \to \infty$ can be found. They stem from the insertion

$$\int_0^\infty \mathrm{d}\alpha\, \mathcal{I}_{\text{div}}(\alpha) \to \int_0^\infty \mathrm{d}\alpha\, \mathcal{I}_{\text{div}}(\alpha) \mathrm{e}^{-\mu^2\alpha - \frac{1}{4\Lambda^2\alpha}} \,,$$

which is applied where necessary in order to regularize a divergent integrand $\mathcal{I}_{\text{div}}(\alpha)$. Note that the prefactor 4 of Λ^2 is motivated by the factor 4 which always accompanies $\tilde{p}^2$ in non-planar integrals. In sum, this enables cancellations between planar and non-planar results.

▷ Integral $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \sin^2\left(\tfrac{1}{2} k\theta p\right)$:

planar part:

$$\frac{\pi^2}{2\mu^4} \,. \tag{F.1a}$$

non-planar part:

$$-\frac{\pi^2}{2\mu^4} \mathrm{e}^{-\frac{\tilde{p}^2}{4\mu^2}} \,. \tag{F.1b}$$

▷ Integral $\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{1}{2}k\theta p\right)}{k^2}$:

planar part:

$$2\pi^2\Lambda^2\,. \tag{F.2a}$$

non-planar part:

$$-\,2\pi^2\frac{1}{\tilde{p}^2}\,. \tag{F.2b}$$

▷ Integral $\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^2} \sin^2\left(\frac{1}{2}k\theta p\right)$:

planar part:

$$4\pi^2\Lambda^4\delta_{\mu\nu}\,. \tag{F.3a}$$

non-planar part:

$$-\,4\pi^2\left[\frac{\delta_{\mu\nu}}{\tilde{p}^4} - 4\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^6}\right]\,. \tag{F.3b}$$

▷ Integral $\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\sin^2\left(\frac{1}{2}k\theta p\right)}{k^4}$:

planar part:

$$\pi^2\,\mathrm{K}_0\left(\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\,. \tag{F.4a}$$

non-planar part:

$$-\,\pi^2\,\mathrm{K}_0\sqrt{\tilde{p}^2\mu^2}\,. \tag{F.4b}$$

▷ Integral $\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^4} \sin^2\left(\frac{1}{2}k\theta p\right)$:

planar part:

$$\pi^2\Lambda^2\delta_{\mu\nu}\,. \tag{F.5a}$$

non-planar part:

$$-\,\frac{\pi^2}{\tilde{p}^2}\left(\delta_{\mu\nu} - 2\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right)\,. \tag{F.5b}$$

▷ Integral $\int\limits_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^6} \sin^2\left(\frac{1}{2}k\theta p\right)$:

planar part:

$$\frac{\pi^2}{4}\delta_{\mu\nu}\,\mathrm{K}_0\left(\sqrt{\frac{\mu^2}{\Lambda^2}}\right) . \tag{F.6a}$$

non-planar part:

$$-\frac{\pi^2}{4}\left[\delta_{\mu\nu}\,\mathrm{K}_0\,\sqrt{\tilde{p}^2\mu^2} - \frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right] . \tag{F.6b}$$

▷ Integral $\int\limits_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma}{k^8} \sin^2\left(\frac{1}{2}k\theta p\right)$:

planar part:

$$\frac{\pi^2}{24}\left[\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho} + \delta_{\mu\sigma}\delta_{\nu\rho}\right]\mathrm{K}_0\left(\sqrt{\frac{\mu^2}{\Lambda^2}}\right) . \tag{F.7a}$$

non-planar part:

$$\begin{aligned}
-\frac{\pi^2}{24}\Big[&\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)\mathrm{K}_0\,\sqrt{\tilde{p}^2\mu^2} \\
&- \left(\tilde{p}_\mu\tilde{p}_\nu\delta_{\rho\sigma} + \tilde{p}_\mu\tilde{p}_\rho\delta_{\nu\sigma} + \tilde{p}_\mu\tilde{p}_\sigma\delta_{\nu\rho} + \tilde{p}_\nu\tilde{p}_\rho\delta_{\mu\sigma} + \tilde{p}_\nu\tilde{p}_\sigma\delta_{\mu\rho} + \tilde{p}_\rho\tilde{p}_\sigma\delta_{\mu\nu}\right)\frac{1}{\tilde{p}^2} \\
&+ 2\frac{\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\sigma}{\tilde{p}^4}\Big] .
\end{aligned} \tag{F.7b}$$

In addition, in the computation of $n > 2$ point functions on the one loop level, we need in addition the following integrals

▷ Integral $\int\limits_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma k_\tau \sin(k\theta p)}{k^8}$:

$$\begin{aligned}
\frac{\pi^2}{12}\Big[&\frac{1}{\tilde{p}^2}\Big(\tilde{p}_\mu\left(\delta_{\nu\rho}\delta_{\sigma\tau} + \delta_{\nu\sigma}\delta_{\rho\tau} + \delta_{\nu\tau}\delta_{\rho\sigma}\right) + \tilde{p}_\nu\left(\delta_{\mu\rho}\delta_{\sigma\tau} + \delta_{\mu\sigma}\delta_{\rho\tau} + \delta_{\mu\tau}\delta_{\rho\sigma}\right) \\
&\quad + \tilde{p}_\rho\left(\delta_{\mu\nu}\delta_{\sigma\tau} + \delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma}\right) + \tilde{p}_\sigma\left(\delta_{\mu\nu}\delta_{\rho\tau} + \delta_{\mu\rho}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\rho}\right) \\
&\quad + \tilde{p}_\tau\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)\Big) \\
&- \frac{2}{\tilde{p}^4}\Big(\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho\delta_{\sigma\tau} + \tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\sigma\delta_{\rho\tau} + \tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\tau\delta_{\rho\sigma} + \tilde{p}_\mu\tilde{p}_\rho\tilde{p}_\sigma\delta_{\nu\tau} + \tilde{p}_\mu\tilde{p}_\rho\tilde{p}_\tau\delta_{\nu\sigma} \\
&\quad + \tilde{p}_\mu\tilde{p}_\sigma\tilde{p}_\tau\delta_{\nu\rho} + \tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\sigma\delta_{\mu\tau} + \tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\tau\delta_{\mu\sigma} + \tilde{p}_\nu\tilde{p}_\sigma\tilde{p}_\tau\delta_{\mu\rho} + \tilde{p}_\rho\tilde{p}_\sigma\tilde{p}_\tau\delta_{\mu\nu}\Big) \\
&+ 8\frac{\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\sigma\tilde{p}_\tau}{\tilde{p}^6}\Big] .
\end{aligned} \tag{F.8}$$

▷ Integral $\int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho \sin(k\theta p)}{k^6}$:

$$2\pi^2 \left[2\frac{\tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho}{(\tilde{p}^2)^2} - \frac{1}{\tilde{p}^2} \left(\delta_{\mu\nu}\tilde{p}_\rho + \delta_{\mu\rho}\tilde{p}_\nu + \delta_{\nu\rho}\tilde{p}_\mu \right) \right] . \tag{F.9}$$

▷ Integral $\int_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\mu \sin(k\theta p)}{k^4}$:

$$-\pi^2 \tilde{p}_\mu \, \mathrm{K}_0 \sqrt{\tilde{p}^2 \mu^2} . \tag{F.10}$$

F.2 Integration Formulæ

▷ parameter integral formula

$$\int_0^\infty \mathrm{d}x \, x^{\nu-1} e^{-\frac{\beta}{x} - \gamma x} = 2 \left(\frac{\beta}{\gamma} \right)^{\frac{\nu}{2}} \mathrm{K}_\nu \left(2\sqrt{\beta\gamma} \right) , \quad \text{with } \mathrm{Re}(\beta) > 0 \cap \mathrm{Re}(\gamma) > 0 . \tag{F.11}$$

Source: Gradshteyn [152], Eqn. (3.471.9)

▷ negative potence, exponential rest of gauss integration

$$\int_0^u \mathrm{d}x \, \frac{\mathrm{e}^{-\frac{a}{x}}}{x^2} = \frac{\mathrm{e}^{-\frac{a}{u}}}{a} , \quad \int_0^\infty \mathrm{d}x \, \frac{\mathrm{e}^{-\frac{a}{x}}}{x^2} = \frac{1}{a} , \quad \int_0^\infty \mathrm{d}x \, \frac{\mathrm{e}^{-\frac{a}{x}}}{x^n} = a^{1-n} \Gamma(n-1), \quad \forall n \geq 2, a > 0 . \tag{F.12}$$

Source: Gradshteyn [152], Eqn. (3.471.1)

▷ negative potence, exponential rest of gauss integration and trigonometric functions

$$\int_0^\infty \mathrm{d}x x^{\mu-1} \exp\left[\frac{-\beta^2}{4x} \right] \sin(ax) = \\ = \frac{\mathrm{i}}{2^\mu} \beta^\mu a^{-\frac{\mu}{2}} \left[e^{-\frac{\mathrm{i}\pi}{4}\mu} K_\mu \left(\beta e^{\frac{\mathrm{i}\pi}{4}} \sqrt{a} \right) - e^{\frac{\mathrm{i}\pi}{4}\mu} K_\mu \left(\beta e^{-\frac{\mathrm{i}\pi}{4}} \sqrt{a} \right) \right] , \tag{F.13a}$$

$$\int_0^\infty \mathrm{d}x x^{\mu-1} \exp\left[\frac{-\beta^2}{4x} \right] \cos(ax) = \\ = \frac{1}{2^\mu} \beta^\mu a^{-\frac{\mu}{2}} \left[e^{-\frac{\mathrm{i}\pi}{4}\mu} K_\mu \left(\beta e^{\frac{\mathrm{i}\pi}{4}} \sqrt{a} \right) + e^{\frac{\mathrm{i}\pi}{4}\mu} K_\mu \left(\beta e^{-\frac{\mathrm{i}\pi}{4}} \sqrt{a} \right) \right] , \tag{F.13b}$$

with $a \in \mathbb{R} > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\mu) < 1$.
Source: Gradshteyn [152], Eqn. (3.957.1/.2)

▷ hyperbolic functions

$$\int \mathrm{d}x\, \sinh(ax+b)\sinh(ax+d) = -\frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)\,, \qquad \text{(F.14a)}$$

$$\int \mathrm{d}x\, \sinh(ax+b)\cosh(ax+d) = \frac{x}{2}\sinh(b-d) + \frac{1}{4a}\cosh(2ax+b+d)\,, \qquad \text{(F.14b)}$$

$$\int \mathrm{d}x\, \cosh(ax+b)\cosh(ax+d) = \frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)\,. \qquad \text{(F.14c)}$$

Source: Gradshteyn [152], Eqn. (2.425.4–6)

▷ hyperbolic functions with exponentials

$$\int \mathrm{d}x\, \mathrm{e}^{ax}\sinh(bx+c) = \frac{\mathrm{e}^{ax}}{a^2-b^2}\left[a\sinh(bx+c) - b\cosh(bx+c)\right]\,, \qquad \text{(F.15a)}$$

$$\int \mathrm{d}x\, \mathrm{e}^{ax}\cosh(bx+c) = \frac{\mathrm{e}^{ax}}{a^2-b^2}\left[a\cosh(bx+c) - b\sinh(bx+c)\right]\,, \qquad \text{(F.15b)}$$

with $a^2 \neq b^2$
Source: Gradshteyn [152], Eqn. (2.481.1/2)

▷ hyperbolic functions with exponentials and powers

$$\int \mathrm{d}x\, x\mathrm{e}^{ax}\sinh(bx+c) = \frac{\mathrm{e}^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\sinh(bx) - \left(bx - \frac{2ab}{a^2-b^2}\right)\cosh(bx)\right]\,, \qquad \text{(F.16a)}$$

$$\int \mathrm{d}x\, x\mathrm{e}^{ax}\sinh(bx+c) = \frac{\mathrm{e}^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\cosh(bx) - \left(bx - \frac{2ab}{a^2-b^2}\right)\sinh(bx)\right]\,, \qquad \text{(F.16b)}$$

with $a^2 \neq b^2$.
Source: Gradshteyn [152], Eqn. (2.483.1/2)

▷ logarithms

$$\int \mathrm{d}x\, x^m \ln(a+bx) = \frac{1}{m+1}\left(x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}}\right)\ln(a+bx) + \frac{1}{m+1}\sum_{i=1}^{m+1}\frac{(-1)^i x^{m-i+2}a^{i-1}}{(m-i+2)b^{i-1}}\,, \qquad \text{(F.17a)}$$

simplifies for $m = 0$, and with the limits $\big|_0^1$ to

$$\int_0^1 \mathrm{d}x\, \ln(a+bx) = \left(1+\frac{a}{b}\right)\ln(a+b) - \frac{a}{b}\ln a - 1\,. \qquad \text{(F.17b)}$$

Source: Gradshteyn [152], Eqn. (2.729.1)

F.3 Miscellanea

▷ Trigonometric identities

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)\,, \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)\,, \tag{F.18}$$

$$\begin{aligned}\sin(x_1 \pm x_2) &= \sin x_1 \cos x_2 \pm \sin x_2 \cos x_1\,,\\ \cos(x_1 \pm x_2) &= \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2\,,\end{aligned} \tag{F.19}$$

$$\begin{aligned}\sin x_1 \sin x_2 &= \frac{1}{2}\left(\cos(x_1-x_2)-\cos(x_1+x_2)\right)\,,\\ \sin x_1 \cos x_2 &= \frac{1}{2}\left(\sin(x_1-x_2)+\sin(x_1+x_2)\right)\,,\\ \cos x_1 \cos x_2 &= \frac{1}{2}\left(\cos(x_1-x_2)+\cos(x_1+x_2)\right)\,,\end{aligned} \tag{F.20}$$

$$\begin{aligned}\sin\frac{k\tilde{p}}{2} &= \frac{1}{2\mathrm{i}}\sum_{\eta=\pm1}\eta \mathrm{e}^{\frac{\mathrm{i}}{2}\eta k\tilde{p}},\\ \cos\frac{k\tilde{p}}{2} &= \frac{1}{2}\sum_{\eta=\pm1}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta k\tilde{p}}\,.\end{aligned} \tag{F.21}$$

Source: Bartsch [153], pp. 397–400

▷ Higher order Schwinger parametrization [154]

$$\frac{1}{q^{2N}} = \frac{1}{\Gamma(N)}\int_0^\infty \mathrm{d}\alpha\,\alpha^{N-1}e^{-\alpha q^2}, \quad \forall\, N \in \mathbb{N},\ \mathrm{Re}(q^2) > 0. \tag{F.22}$$

▷ Bessel K, general form

$$\begin{aligned}\mathrm{K}_n(z) =& \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{(n-k-1)!}{k!\left(\frac{z}{2}\right)^{n-2k}}\\ &+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!}\left[\ln\frac{z}{2}-\frac{1}{2}\left(\psi(k+1)+\psi(k+n+1)\right)\right].\end{aligned} \tag{F.23}$$

Source: Gradshteyn [152], Eqn. (8.446)

▷ Bessel K_n, up to 3rd order and $n \leq 4$

$$\mathrm{K}_0(x) \approx \ln\tfrac{2}{x} - \gamma_E - \tfrac{x^2}{4}\left(\gamma_E - 1 + \ln\tfrac{x}{2}\right) + \mathcal{O}\left(x^4\right)\,, \tag{F.24a}$$

$$\mathrm{K}_1(x) \approx \tfrac{1}{x} + \tfrac{x}{2}\left(\gamma_E - \tfrac{1}{2} + \ln\tfrac{x}{2}\right) + \tfrac{x^3}{16}\left(\gamma_E - \tfrac{5}{4} + \ln\tfrac{x}{2}\right)\mathcal{O}\left(x^5\right)\,, \tag{F.24b}$$

$$\mathrm{K}_2(x) \approx \tfrac{2}{x^2} - \tfrac{1}{2} - \tfrac{x^2}{8}\left(\gamma_E - \tfrac{3}{4} + \ln\tfrac{x}{2}\right) + \mathcal{O}\left(x^4\right)\,, \tag{F.24c}$$

$$\mathrm{K}_3(x) \approx \tfrac{8}{x^3} - \tfrac{1}{x} + \tfrac{x}{8} + \tfrac{x^3}{48}\left(\gamma_E - \tfrac{11}{12} + \ln\tfrac{x}{2}\right) + \mathcal{O}\left(x^5\right)\,, \tag{F.24d}$$

$$\mathrm{K}_4(x) \approx \tfrac{48}{x^4} - \tfrac{4}{x^2} + \tfrac{1}{4} - \tfrac{x^2}{48} + \mathcal{O}\left(x^4\right)\,. \tag{F.24e}$$

Appendix G

Code Listings

This appendix includes a very brief documentation of the Mathematica packages which have been written and utilized in the framework of this thesis.

G.1 SymmetryFactors.m

The `SymmetryFactor` package basically provides an option to compute the symmetry factors of any given graph. The scheme is, to fix the vertex positions as described for the pregraphs in Section 1.3.2, and then construct a list of all possible complete connections which are compatible with the available Feynman rules (propagators). Finally, the symmetry factor is determined by counting the instances of the desired target topology in the list, and dividing by internal symmetry factors of the vertices.

G.1.1 List of Commands

▷ **Command:** `SymmetryFactor`

`SymmetryFactor`[$\{E_1..E_n\}$,
$\{\{V_{11}, V_{12}, ..V_{1m_1}\}, ..\{V_{k1}, V_{k2}, ..V_{km_k}\}\}$,
$\{\{F_{11}, F_{12}, V_{1s}, V_{1e}\}, ..\{F_{j1}, F_{j2}, V_{js}, V_{je}\}\}$]

Description :

Computes the symmetry factor for a given graph topology. The first argument gives the external fields $E_1..E_n$ of the graph (these correspond to vertices $k+1..k+n$). The second argument gives a list of vertices $\{1..k\}$, each defined by a list of m_k fields $\{F_{k1}..F_{k\,m_k}\}$ (where m_k may be different for each vertex). The numbering continues, i.e. the first given vertex has number 1, the last one k, independent of the fact that some are internal, some external vertices. These numberings are required for the third argument which is a list of propagators between vertices. The structure is: {Field at start of prop., Field at end of prop., vertex ♯ at start, vertex ♯ at end}. All internal and external legs have to be connected. The function then counts all possible permutations which result in the given topology and divides by the internal symmetry of the vertices given by $(n_1!...n_i!)$ for i different bosonic fields of cardinality $n_1, ..n_i$ at each vertex.

Variables :

1:	$E_1..E_n$	comma-separated list of external fields
2:	$V_{\{1..k\}\{1..m_1\}}$	simply nested, comma separated list of fields entering the vertices. The outer list runs over the first index (vertices k) while the inner runs over fields m_k of each vertex
3:	$F_{\{1..j\}\{1,2,s,e\}}$	simply nested, comma separated list of propagator connections. The outer list runs over connections j, the inner gives in the order of appearance: the field type at the start (1), the field type at the end (2), vertex number at the start (s), vertex number at the end (e).

Return Value :

The function returns a fractional number representing the symmetry factor for the given graph, or 0 if an error in the definition has been found.

Example :

This simple example resembles the situation for the graph (c) of Fig. C.2, page 119.

```
In[1]:= SymmetryFactor[{A,B},{{B,A,A},{bB,B,A}},{{A,A,3,1},{B,B,2,4},{B,bB,1,2},{A,A,2,1}}]

Out[1]= 1
```

G.1.2 Loading of the Package

```
Get["SymmetryFactor.m", Path -> "⟨PATH⟩"];
```

G.1.3 Source Code

```
(* ::Package:: *)

(***********************************************************************
 *                                                                     *
 * Package SymmetryFactors                                             *
 *                                                                     *
 * Enables the computation of symmetry factors for the current        *
 * gauge model.                                                        *
 *                                                                     *
 * Author : Rene Sedmik                                                *
 *   Date : 2008-04-29                                                 *
 *                                                                     *
 * Version History:                                                    *
 * ______________________________________________________________     *
 * | Ver | Date       | Changes                                  |    *
 * +-----+------------+------------------------------------------+    *
 * | 0.1 | 2008-04-24 | Initial Version                          |    *
 * | 0.11| 2008-04-29 | Change in enumeration of vertices        |    *
 *                                                                     *
 ***********************************************************************)

 BeginPackage["SymFactors‘"];

 (**** usage Documentation ****)

  SymmetryFactor::usage = "SymmetryFactor[{E1...En},{{V11,V12,..V1m1},
```

```
   ..{Vk1,Vk2,...Vkmk}},{{F11,F12,V1s,V1e},...{Fj1,Fj2,Vjs,Vje}}]\n computes
   the symmetry factor for a given graph topology.\n The first argument gives
   the external fields E1..En of the graph (these correspond to vertices
   k+1..k+n).\n The second argument gives a list of vertices (1..k), each
   defined by a list of m_k Fields Fk1..Fkmk (, where m can be different
   for each vertex). The numbering continues, i.e. the first given vertex has
   number 1, the last one k. These numberings are required for the third
   argument which is a list of propagators between vertices. The structure
   is: {Field at start of prop., Field at end of prop., vertex # at start,
   vertex # at end}. All internal and external legs have to be connected.\n
   The function counts all possible permutations which result in the given
   topology and divides by the internal symmetry of the vertices given by
   prod_vertices(n1!...ni!) for i different bosonic fields of cardinality
   n1..ni at each vertex.";

 (**** Functions ****)

Off[General::spell1];


(* internal function needed by SymmetryFactor[], yields ttransformation rules*)
(* for propagators                                                           *)
Unprotect[PropOrdering];
PropOrdering={
HoldPattern[G[A,A,x_,y_]]:>G[A,A,Sort[{x,y}][[1]],Sort[{x,y}][[2]]],
HoldPattern[G[B,A,x_,y_]]:>G[A,B,y,x],
HoldPattern[G[bB,A,x_,y_]]:>G[A,bB,y,x],
HoldPattern[G[B,B,x_,y_]]:>G[B,B,Sort[{x,y}][[1]],Sort[{x,y}][[2]]],
HoldPattern[G[bB,bB,x_,y_]]:>G[bB,bB,Sort[{x,y}][[1]],Sort[{x,y}][[2]]],
HoldPattern[G[B,bB,x_,y_]]:>G[bB,B,y,x],
HoldPattern[G[c,bc,x_,y_]]:>G[bc,c,y,x],
HoldPattern[G[P,bP,x_,y_]]:>G[bP,P,y,x]};


(* Activate Mathemaitca-internal optimizations to speed up the rules         *)
Detach[PropOrdering];

Protect[PropOrdering];

(*****************************************************************)
(* Function SymmetryFactor, description see usage documentation     *)
Unprotect[SymmetryFactor];
SymmetryFactor[ExtFields_,Vertices_,Propagators_]:=Module[
{G,iVrtx,iLegs,iCnt,iCnt2,iPerm,iSym,iCorrFactor,LegList,GraphList,
PermList,sGraph,sTargetTopo,AllowedProp,AllowedVert},

  (* definition of error messages *)
  SymmetryFactor::odderr="The number of free legs in this graph ('1')
                               is odd which indicates an erroneous input.";

  (* definition of allowed propagators and vertices *)
  AllowedProp={{A,A},{A,B},{B,A},{A,bB},{bB,A},{B,B},{bB,B},{B,bB},{bB,bB},
               {bc,c},{c,bc},{bB,bB},{bP,P},{P,bP}};
  AllowedVert={{A,A,A},{A,A,A,A},{B,A,A},{bB,A,A},{bB,B,A},
               {bB,B,A,A},{bB,B,A,A,A},{bB,A,A,A,A}};

  (* ordering definitions for propagators                        *)
  (* only topological identities are required, no signs!          *)
  ClearAttributes[G,Orderless];

  (* allow only propagators in the list                          *)
```

```
G[A_,B_,x_,y_]:=If[!MemberQ[AllowedProp,{A,B} ],0];

(*** TARGET TOPOLOGY ***)
sTargetTopo="";
Clear[TargetTopo];
For[iCnt=1, iCnt<=Length[Propagators], iCnt++,
  sTargetTopo = sTargetTopo<>" G["<>ToString[Propagators[[iCnt,1]]]<>","
                <>ToString[Propagators[[iCnt,2]]]<>","
                <>ToString[Propagators[[iCnt,3]]]<>","
                <>ToString[Propagators[[iCnt,4]]]<>"]";
];

TargetTopo=(ToExpression[sTargetTopo])//.PropOrdering;

(*determines the inner correction factor of the vertices          *)
(* Hint: this works only for theories with a sole bosonic field A.*)
(*       For extensions, the process would have to be repeated for*)
(*       each field, and the factors multiplied                   *)
iCorrFactor=1;
For[iCnt=1, iCnt<=Length[Vertices], iCnt++,
  iCnt2 = Length[Position[Vertices[[iCnt]],A]];
  iCorrFactor *= If[iCnt2>0,iCnt2!,1,1];
];

(*** ALL POSSIBLE CONTRACTIONS ***)
(* assemble a list of all free legs                               *)
LegList={};
iLegs=0;

(*vertex legs *)
For [iCnt=1,iCnt<=Length[Vertices],iCnt++,
  For [iCnt2=1,iCnt2<=Length[Vertices[[iCnt]]],iCnt2++,
    LegList=Append[LegList,{"V",iCnt,Vertices[[iCnt,iCnt2]]}];
    iLegs++;
  ];
];

(* external legs *)
For [iCnt=1,iCnt<=Length[ExtFields],iCnt++,
  LegList=Append[LegList,{"E",iCnt+Length[Vertices],ExtFields[[iCnt]]}];
  iLegs++;
];

(* consistency check: there must be an even number of legs        *)
If[OddQ[iLegs],Message[SymmetryFactor::odderr,iLegs]];

GraphPermutations[verts_]:=Module[{ii,jj,PartList,RestList,Tmp,Ret},
  Ret={};
  For [ii=2,ii<=Length[verts],ii++,
    PartList={verts[[1]],verts[[ii]]};
    RestList=Delete[verts,{{1},{ii}}];
    If[Length[RestList]>2,
      Tmp=GraphPermutations[RestList];
      For[jj=1,jj<=Length[Tmp],jj++,
        Ret=Append[Ret,Union[{PartList},Tmp[[jj]]]];
      ],
      Ret=Append[Ret,Union[{PartList},{RestList}]];
    ](*endif*)
  ];(*endfor*)
  Ret
];(*end module*)
```

```

  (* generate all possible contractions as a list of permutations   *)
  PermList=GraphPermutations[Range[iLegs]];
  (**Print["Permutations: "<>ToString[Length[PermList]]];**)
  (* generate a list of expressions V1...VN,G1...GN for these       *)
  (* permutations                                                   *)
  GraphList={};
  For[iPerm=1,iPerm<=Length[PermList],iPerm++,
    sGraph="";
    For[iCnt=1,iCnt<=Length[PermList[[iPerm]]],iCnt++,
      sGraph=sGraph<>" G["<>ToString[LegList[[PermList[[iPerm,iCnt,1]],3]]
               <>","<>ToString[LegList[[PermList[[iPerm,iCnt,2]],3]]<>","
               <>ToString[LegList[[PermList[[iPerm,iCnt,1]],2]]]<>","
               <>ToString[LegList[[PermList[[iPerm,iCnt,2]],2]]]<>"]";
    ];
    GraphList=Append[GraphList,(ToExpression[sGraph])/.PropOrdering];
  ];
  (** DEBUGGING OUTPUT **)
  (**Print["TargetTopo: "];**)
  (**Print[TargetTopo];**)
  (**Print[GraphList];**)
  (**Print["Searching Symmetries..."];**)
  iSym=0;
  For[iCnt=1,iCnt<=Length[GraphList],iCnt++,
    If[TargetTopo==GraphList[[iCnt]],iSym++];
  ];
  (**Print[iSym];**)
  iSym/iCorrFactor
];
(* end SymmetryFactor *)

Protect[SymmetryFactor];

On[General::spell1];

EndPackage[ ];
(*** EOF ********************************************************************)
```

G.2 VariationalCalc.m

The `VariationalCalc` package has been designed to automate the computation of symmetries, as they appear in the gauge model with BRST doublets (see Section 3.2.4) or the BRSW model (see Section 5.1.2). One of the main achievements is that all functions in `VariationalCalc` respect the non-commutativity of the regular product, which is hard to achieve in Mathematica® since the respective option seems to be 'forgotten' from time to time. If applied to a new model, the definitions of `lAvFields` and `lFermiFields` in the region marked as `NON - GENERIC PART` in the code listing below (lines 130–156) have to be altered accordingly. These two lists are used to determine which variables are fields, and which have fermionic statistic. In addition, the explicit derivation rules for the respective fields in function `VarD[]` have to be adapted. In the following, descriptions for the most important functions are given. Besides, there exist the auxiliary functions `ACo[]`, `Co[]`, `Cyclic[]`, `IdxOf[]`, `GetPerms[]`, `IsDeriv[]`, `IsFermion[]`, `IsField[]`, `PwrOf[]`, `RmAll[]`, `RmIdx[]`, and `RmPwr[]`, which are not documented here, but may be useful in practice (they are heavily used internally by the main functions). As always, syntactic and behavioral descriptions are available by entering `?`⟨FUNCTION_NAME⟩`;` without arguments.
Note that it is essential to call `ClearAttributes[Times,Orderless];` immediately after loading the package since Mathematica® seems to reset the `Orderless` property of `Times` after returning from the package.

G.2.1 List of Commands

▷ **Command:** `IsEquiv`

`IsEquiv[` $e1$`,`$e2$`,[`$bPerm$`]]`

Description :

`IsEquiv[]` determines if two terms $e1$, and $e2$ equal each other under consideration of signs and in normalized form (see `NormalForm[]` below). The optional parameter $bPerm$ defines if cyclic permutations and partial derivative may also be taken into account, i.e. if an integral can be presumed. Note that this function is designed for checks on simple field polynomials only (as they appear in a local action, for example).

Variables :

1: $e1$ first term
2: $e2$ second term
3: $bPerm$ defines if an integral can be presumed
 0: integral is present, cyclic permutations are allowed
 1: integral is not present.

Output :

The output is binary, `False` if no equivalence is given, `True` if $e1 \equiv e2$.

Example :

```
In[1]:= IsEquiv[c bc,-bc c,0]

Out[1]= True

In[2]:= IsEquiv[c bc,-bc c,1]

Out[2]= False

In[3]:= IsEquiv[PD[c Subscript[A, \[Mu]],\[Rho],0] PD[Subscript[B, \[Rho],s],\[Sigma],0],
            -c Subscript[A, \[Mu]] PD[PD[Subscript[B, \[Rho],s],\[Sigma],0],\[Rho],0]]
Out[3]= True

In[4]:= IsEquiv[PD[c Subscript[A, \[Mu]],\[Rho],0] PD[Subscript[B, \[Rho],s],\[Sigma],0],
            -c Subscript[A, \[Mu]] PD[PD[Subscript[B, \[Rho],s],\[Sigma],0],\[Rho],0],1]
Out[4]= False
```

▷ **Command:** `NCCancel`

`NCCancel[` e`,`$bInt$`]`

Description :

`NCCancel[]` cancels terms in an expression e, under consideration of cyclic permutations and statistics of fields. The parameter $bInt$ defines if an integral can be presumed to enable cyclic permutations ($bInt = 1$) or not ($bInt = 0$). Internally, `NCCancel[]` calls `GetPerms[]` for each summand it finds in e. Subsequently, all possible combinations, resulting from alternative representations of the terms, are analyzed and the simplest one is selected. Accordingly, the timing behavior of this function is pretty bad, but improvements could be achieved by analyzing the terms in order to perform transformations selectively, instead of the brute force approach.

Variables :

1: *e* any expression

2: *bInt* defines if an integral can be presumed

0: no integral, no cyclic permutations

1: integral is present.

Output :

The function returns a simplified expression (if any cancellations are possible). Note that the computational effort scales factorial with the number of terms in *e*.

Example :

```
In[1]:= NCCancel[PD[c,\[Mu]] Subscript[A, \[Nu]]+c PD[Subscript[A, \[Nu]] ,\[Mu],0],1]
Out[1]= 0
In[2]:= NCCancel[PD[c,\[Mu]] Subscript[A, \[Nu]]+c PD[Subscript[A, \[Nu]] ,\[Mu],0],0]
Out[2]= c PD[Subscript[A, \[Nu]],\[Mu],1]+PD[c,\[Mu],1] Subscript[A, \[Nu]]
In[3]:= NCCancel[PD[c,\[Mu]] Subscript[A, \[Nu]] PD[c,\[Rho]]
          +c PD[Subscript[A, \[Nu]] PD[c,\[Rho]]+Subscript[A, \[Rho]] PD[c,\[Mu]],\[Mu],0],1]
Out[3]= c PD[Subscript[A, \[Rho]],\[CapitalSigma]1,1] PD[c,\[CapitalSigma]1,1]
        +c Subscript[A, \[Rho]] PD[PD[c,\[CapitalSigma]1,1],\[CapitalSigma]1,1]
```

▷ Command: NormalForm

`NormalForm[` *e*`[,`*bTop*`]]`

Description :

`NormalForm[]` transforms a given expression *e* into a predefined form and ordering, which allows to effectively compare and analyze terms. Any multiplication is expanded as far as possible. Scalars are shifted to the front and sorted. Fields are (due to non-commutativity) left in the order they appear but summation indices are replaced by 'Σn', where n is a running number. Care is taken with respect to contractions and powers (see examples below).

Variables :

1: *e* any expression

2: *bTop* is an optional parameter for internal use only

Output :

The function returns a normalized form of the input.

Example :

```
In[1]:= NormalForm[Subscript[A,\[Mu]] Subscript[A,\[Nu]] PD[Subscript[B,\[Rho]],\[Mu]]^2,\[Sigma],1] 4]
Out[1]= 4 Subscript[A,\[CapitalSigma]1] Subscript[A,\[Nu]]
          PD[Subscript[B,\[CapitalSigma]2,\[CapitalSigma]1]^2],\[Sigma],1]
In[2]:= NormalForm[Subscript[A,\[Mu]] Subscript[A,\[Nu]] const PD[4 Subscript[B,\[Rho],\[Tau]],\[Tau],1]]
Out[2]= 4 const Subscript[A,\[Mu]] Subscript[A,\[Nu]]
          PD[Subscript[B,\[Rho],\[CapitalSigma]1],\[CapitalSigma]1,1]
```

▷ Command: PD

`PD[` *e*`,`*v*`,`*ind*`[,`*bEval*`]]`

Description :

`PD[]` (PartialDerivative) implements a symbolic partial derivative, which respects the Leibnitz rule, transitivity, and non-commutativity but does not actually perform a derivative. Instead, it represents a functional dependence which is respected upon variation with `VarD[]`. The optional parameter *bEval* implements a switch which determines if `VD[]` shall be distributed under consideration of transitivity and Leibnitz rule over e.

Variables :

1:	e	any expression
2:	v	variable which is taken for differentiation
3:	ind	v's index
4:	$bEval$	[optional] defines if `VD[]` shall be distributed on e 0 (default): The only operation to be performed is pulling out of scalar factors. 1: Thread `VD[]` over sums, eliminate derivatives of scalars, and evaluate the Leibnitz rule. *bEval* is not propagated to nested calls of `PD[]`.

Output :

By default ($bEval = 0$) the function remains unevaluated. If $bEval = 1$, 0 is returned if e is a scalar and `VD[]` performs as described above.

Example :

```
In[1]:=  PD[PD[Subscript[A, \[Mu]] c,\[Rho],0],\[Sigma],1]

Out[1]=  PD[PD[Subscript[A, \[Mu]] c,\[Rho],0],\[Sigma],1]

In[2]:=  PD[const PD[Subscript[A, \[Mu]] c,\[Rho],1],\[Sigma],0]

Out[2]=  const PD[PD[Subscript[A, \[Mu]],\[Rho],1] c+Subscript[A, \[Mu]] PD[c,\[Rho],1],\[Sigma],0]
```

▷ **Command:** `VarD`

`VarD[` e`,`ϕ`,`ind`[,`$bNoInt$`]]`

Description :

`VarD[]` (VariationalDerivative) implements the functional variation as defined by Eqn. (1.27). However, this function is more general, as it is not limited to expressions under an integral but respects non-commutativity in general. The information if an integral is present, is yield by the optional parameter $bNoInt$. Partial integrations are performed automatically. Note however, that the actual variation rules for single fields are to be defined directly within the function (see source).

Variables :

1:	e	any expression
2:	ϕ	field which is taken for variation.
3:	ind	ϕ's index. For a scalar field give {} or omit the parameter.
4:	$bNoInt$	[optional] defines if an integral is present or not, to determine if partial integrations and cyclic permutations are permitted. 0 (default): an integral is present 1: no integral is present.

Output :

The output is the result of functional variation $\delta e/\delta\phi_{ind}$.

Example :

```
In[1]:= VarD[Subscript[A, \[Mu]] Subscript[A, \[Nu]],A,\[Alpha]]

Out[1]= KDelta[\[Nu],\[Alpha]] Subscript[A, \[Mu]]+KDelta[\[Mu],\[Alpha]] Subscript[A, \[Nu]]
```

In order to give an impression of the practical work with the package, a few more examples shall be given, which also make use of functions which are not documented here, but whose function is obvious. We have bosons A_μ, and fermions c, and $\bar{c}$ (bc).

```
In[1]:= Co[PD[Subscript[A,\[Nu]],\[Mu]],Subscript[A,\[Rho]]]

Out[1]= -Subscript[A,\[Rho]] PD[Subscript[A,\[Nu]],\[Mu]]+PD[Subscript[A,\[Nu]],\[Mu]] Subscript[A,\[Rho]]

In[2]:= NCCancel[%,1]

Out[2]= 0

In[3]:= FStrength[\[Mu],\[Nu]]

Out[3]= I PD[Subscript[A,\[Mu]],\[Nu],1]-I PD[Subscript[A,\[Nu]],\[Mu],1]
         -2 I g Subscript[A,\[Nu]] Subscript[A,\[Mu]]+2 I g Subscript[A,\[Mu]] Subscript[A,\[Nu]]
In[4]:= VarD[FStrength[\[Mu],\[Nu]], c]

Out[4]= 0
```

G.2.2 Loading of the Package

```
Get["VariationalCalc.m", Path -> "⟨PATH⟩"];
ClearAttributes[Times,Orderless];
```

G.2.3 Source Code

```
(* ::Package:: *)

(**********************************************************************
 *                                                                    *
 * Package VariationalCalc                                            *
 *                                                                    *
 * Enables simple variational calculations in nc spaces              *
 *                                                                    *
 * Author : Rene Sedmik                                               *
 *   Date : 2009-07-09                                                *
 *                                                                    *
 * Version History:                                                   *
 * _______________________________________________________________    *
 * | Ver | Date       | Changes                                  |    *
 * +-----+------------+------------------------------------------+    *
 * | 0.1 | 2009-05-29 | Initial Version                          |    *
 * | 0.2 | 2009-06-03 | Added NormalForm, Cyclic and NCCancel    |    *
 * | 1.0 | 2009-06-10 | Completely rewritten to act in X-Space   |    *
 * | 1.1 | 2009-06-16 | Debugged and tested version ready for use|    *
 * | 1.2 | 2009-06-18 | VarD and IsEquiv enhanced                |    *
 * | 1.21| 2009-07-09 | Bugfix for VarD                          |    *
 *                                                                    *
```

```
 *********************************************************************)

 BeginPackage["VariationalCalc`"];

(**** remove old definitions ****)
Unprotect[ACo];Clear[ACo];
Unprotect[Co];Clear[Co];
Unprotect[ContainsFields];Clear[ContainsFields];
Unprotect[Cyclic];Clear[Cyclic];
Unprotect[IdxOf];Clear[IdxOf];
Unprotect[GetPerms];Clear[GetPerms];
Unprotect[IsDeriv];Clear[IsDeriv];
Unprotect[IsEquiv];Clear[IsEquiv];
Unprotect[IsFermion];Clear[IsFermion];
Unprotect[IsField];Clear[IsField];
Unprotect[NCCancel];Clear[NCCancel];
Unprotect[NormalForm];Clear[NormalForm];
Unprotect[PartInt];Clear[PartInt];
Unprotect[PD];Clear[PD];
Unprotect[PwrOf];Clear[PwrOf];
Unprotect[RmAll];Clear[RmAll];
Unprotect[RmIdx];Clear[RmIdx];
Unprotect[RmPwr];Clear[RmPwr];
Unprotect[VarD];Clear[VarD];

Unprotect[lAvFields];Clear[lAvFields];
Unprotect[lFermiFields];Clear[lFermiFields];

(**** usage Documentation ****)

ACo::usage   = "ACo[expr1, expr2] gives the anticommutator of expr1 and expr2\n";
Cyclic::usage= "Cyclic[expr, num, [mode:0]] performs a rotation of expr by num
               places.\nnum > 0 corresponds to the direction in which the
               originally last element is\nrolled to the first position, num < 0
               respectively performs in the other direction.\nThe optional mode
               parameter determines if the counting considers arbitrary\nfactors
               (mode=1) or just defined fields (mode=0, default).\n The statistic
               of known fields are respected!\n";
Co::usage    = "Co[expr1, expr2] gives the commutator of expr1 and expr2\n";
ContainsFields::usage = "ContainsFields[expr] gives true if expr contains any
               defined field, and false otherwise.\nDeprecated function, use
               IsField instead."
GetPerms::usage = "GetPerms[expr] is applied to monomials of fields and derivatives.
               \nThe function gives a list of all equivalent cyclic permutations
               and partial integrations thereof.\n";
IdxOf::usage = "IdxOf[expr] gives a one dimensional list of all indices in an
               atomic expression.\nIndices can be single characters (any non-
               whitespace) with an arbitrary number of numbers\n attached to it.
               Any sequence of non-numeric non-whitespace characters is split.\n
               Purely numeric indices are not recognized.\n";
IsEquiv::usage = "IsEqual[expr1, expr2] determines if two expressions are equal
               under consideration of cyclic permutation and partial integration.\n";
IsDeriv::usage = "IsDeriv[expr] determines if a given expression constitutes a total
               derivative.\nIf yes, return = True, if not, return = False\n"
IsFermion::usage = "IsFermion[expr] determines if the expression contains defined
               fermi fields or not.\nIf fields are found, the effective number of
               fields is returned as an integer >0.\nIf no fermions are found the
               result is 0.\n";
IsField::usage = "IsField[expr] determines if the expression contains defined fields
               (return = True)\nor not (return = False).\n";
NCCancel::usage = "NCCancel[expr, bInt] cancels terms under consideration of cyclic
```

```
                permutations\nand statistics of fields. The parameter bInt defines if
                cyclic permutations are\nallowed (bInt=1) or not (bInt=0).\n";
NormalForm::usage = "NormalForm[expr] gives a sorted form of the input. Scalars and
                bosonic expressions\nare shifted to the left, each type sorted
                according to a canonical order. Fermions\nare shifted to the right,
                in their alignment the statistic is respected.\nFurthermore indices
                which are summed over are replaced by a normalized\nnotation
                \[CapitalSigma]i, where i is a counter for summation indices.\nUse
                this function in\norder to make expressions comparable\n";
PD::usage    = "PD[arg, ind [, eval]] symbolizes a partial derivative of arg with
                respect to x, index \[Mu].\nThe optional argument eval determines if
                the expression is to be expanded in terms of the Leibnitz rule
                (eval = 1) or not (eval=0,default).";
PwrOf::usage = "PwrOf[expr] gives the exponent of an atomic expression.\nThis
                function is intended to be applied only to simple atomic
                expressions!\n";
RmAll::usage = "RmAll[expr] removes any index or powers from expr.\nThis function
                is intended to be applied only to simple atomic expressions!\n";
RmIdx::usage = "RmIdx[expr] removes any indices from expr.\nThis function is
                intended to be applied only to simple atomic expressions!\n";
RmPwr::usage = "RmIdx[expr] removes any powers from expr.\nThis function is intended
                to be applied only to simple atomic expressions!\n";
VarD::usage  = "VarD[expr, var, indx, [NoInt]] performs a variation of expr with
                respect to the field var\nwith index indx.\n For scalar fields set
                indx={} or omit the argument. Statistics\nof known fields are
                respected!\n By default the existence of an integral in front of the
                expression is assumed\nwhich allows for partial integration and cyclic
                permutation of the terms.\nIf no integral is present set the optional
                argument NoInt to 1.";

Off[General::spell1];

(***** Start of functional code *****)
(* Better not to use the private package space since this rises numerous     *)
(*Begin["‘Private‘"];*)

(*Enable memory sharing for all expressions                                  *)
Share[];

(* disable canonical sorting of expressions with head Times *)
Unprotect[Times];
ClearAttributes[Times,Orderless];
Protect[Times];


(****************************************************************************)
(* NON - GENERIC PART                                                        *)
(* functions which are specific to the current model                         *)
Unprotect[lAvFields];Clear[lAvFields];
Unprotect[lFermiFields];Clear[lFermiFields];
Unprotect[CoD];Clear[CoD];
Unprotect[FStrength];Clear[FStrength];

CoD::usage = "CoD[expr, index] gives the covariant derivative of the expression
               expr with\nrespect to the given index as defined for our new nc
               gauge model\n";
FStrength::usage = "CoD[ind1, ind2] gives the field strength tensorcovariant
               derivative\nas defined for our new nc gauge model\n";

(*Listof defined fields for a specific model                                 *)
(* all fields *)
```

```
lAvFields = {A,bc,c,b,bB,B,bP,P,J,bJ,Q,bQ,OA,Oc,Obc,OB,ObB,OP,ObP,OJ,ObJ,OQ,ObQ};
lFermiFields ={bc,c,bP,P,Q,bQ,OA,OB,ObB,OJ,ObJ};              (* fermi fields *)

CoD[f_,ind_]:=I*PD[f,ind,0]-I*g*(Subscript[A, ind]*f-f*Subscript[A, ind]);
FStrength[ind1_,ind2_]:=(CoD[Subscript[A, ind1],ind2]-CoD[Subscript[A, ind2],ind1]);

Protect[lAvFields];
Protect[lFermiFields];
Protect[CoD];
Protect[FStrength];

(* END OF NON - GENERIC PART                                                   *)
(******************************************************************************)


(* some helper functions                                                       *)

(* Gives the power of non-numeric variables. Intended for single variables    *)
(* ONLY, does not work with composite expressions                              *)
PwrOf[expr_]:=If[Position[expr,Power]!={},expr//.{(c_:1)*Fun_[___,Power[a_,b_],
                                                 ___]:>b,(c_:1)*Power[a_,b_]:>b},1];
(* Gives a list of the indices of the current expression. Intended for        *)
(* single variables ONLY, does not work with composite expressions            *)
IdxOf[expr_]:=expr//.{(c_:1)*Fun_[___,Subscript[a_,i__],___]:>
                 ToExpression[Flatten[StringCases[ToString[#]&/@{i},
                              RegularExpression["\\S\\d*"]]]],
                      (c_:1)*Subscript[a_, i__]:>ToExpression[Flatten[
                 StringCases[ToString[#]&/@{i},RegularExpression["\\S\\d*"]]]],
                      others_:>{}/;Position[expr,Subscript]=={}};
(* a version for more complicated terms including PDs                          *)
IdxOf[expr_,1]:=Module[{aLocIdx,tmp,ii},
   aLocIdx={};
   tmp=If[Head[expr]===Times,Level[expr,1],{expr}];
   While[Position[tmp,PD]!={},
      (* add indices of free fields and derivatives *)
      (AppendTo[aLocIdx,#/.{HoldPattern[PD[arg_,ind_,_:0]]:>ind,c_:>IdxOf[c,1]/;
         Position[c,PD]=={}}])&/@tmp;
      (* simple expressions are now already processed -> delete *)
      For [ii=1,ii<=Length[tmp],ii++,
         If[Position[tmp[[ii]],PD]=={},tmp=Delete[tmp,ii];ii--;];
      ];
      (* dig deeper into the expression *)
      tmp=Flatten[(#/.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg)&/@tmp];
      (* split of multiplications in PD arguments *)
      tmp=Flatten[If[Head[#]===Times,List@@#,#]&/@tmp];
   ];(*end while*)
   (* pick up all remaining indices *)
   (AppendTo[aLocIdx,IdxOf[#]])&/@tmp;
   Flatten[aLocIdx]
];
(* Removes index and power decorations as well as derivatives                 *)
RmAll[HoldPattern[(c_:1)*PD[expr_,___]]] := RmAll[expr]/;(!ContainsFields[c]);
RmAll[HoldPattern[PD[expr_,___]]]        := RmAll[expr];
RmAll[expr_]:=expr//.{(c_:1)*Power[Subscript[a_,___],n_:1]/;(!ContainsFields[c])
:>a,
                      (c_:1)*Power[a_,n_:1]/;(!ContainsFields[c]):>a};
(* Removes inices *)
RmIdx[expr_]:=expr//.{(c_:1)*(Subscript[a_, __])^(n_:1):>c*a^n,
                      (c_:1)*(Subscript[a_, __])^(n_:1):>c*a^n};
(* Removes powers *)
RmPwr[expr_]:=expr//.{(c_:1)*Power[a_,n_:1]:>c*a};
```

```

Protect[PwrOf];
Protect[IdxOf];
Protect[RmAll];
Protect[RmIdx];
Protect[RmPwr];

(* commutator and anticommutator relations                                  *)
Co[a_,b_] :=a*b-b*a;
ACo[a_,b_]:=a*b+b*a;
Protect[Co];
Protect[Aco];

(* function to determine if an expression contains fields                   *)
ContainsFields[expr_]:=Module[{bRes},
   bRes=False;
   (If[Position[expr,#]!={},bRes=True])&/@lAvFields;
   bRes
];


(* Function to perform cyclic permutations. Statistics of known fermi/bose   *)
(* fields are respected.                                                     *)
Cyclic[expr_,num_,mode_:0]:=Module[{iCnt,iCnt2,iCnt3,iList,iSign,iNum,tHead},
   iNum=num;
   tHead=Head[expr];
   (* if the Head of the input is not recognized as List *)
   (* or Times, set num to 0 to bypass cycling operation.*)
   If[!(tHead===Times) && !(tHead===List),iNum=0;];
   iList=List@@expr;
   iCnt=0;iCnt2=0;iCnt3=0;iSign=1;
   (* the simple case*)
   If[iNum==0,expr,
   (*else*)
   While[(iCnt<Abs[iNum]) && (iCnt2<=Length[iList])&&(iCnt3<Abs[iNum]),
      If[iNum>0,
         (* roll the last element to the first place *)
         iList=RotateRight[iList];
         (* if the rolled element was a fermion, check the signs *)
         If[IsFermion[iList[[-1]]]>0,
            (iSign*=(-1)^(IsFermion[#]*IsFermion[iList[[1]]]))&/@iList[[2;;]];
         ];
         (* increment the field counter *)
         If[IsField[iList[[1]]],iCnt++];
      ,(*else, num<0*)
         (* roll the first element to the last place *)
         iList=RotateLeft[iList];
         (* if the rolled element was a fermion, check the signs *)
         If[IsFermion[iList[[-1]]]>0,
           (iSign*=(-1)^(IsFermion[#]*IsFermion[iList[[-1]]]))&/@iList[[;;-2]];
         ];
         (* increment the field counter *)
         If[IsField[iList[[-1]]],iCnt++];
      ];(*if num>0*)
      iCnt2++;
      If[mode>0,iCnt3++;];
   ];(*while*)
   (* just to be sure *)
   Unprotect[Times];
   ClearAttributes[Times,Orderless];
   Protect[Times];
```

```
   (*correct the sign*)
   iSign*tHead@@iList
   ](*endif simple case*)
]
Protect[Cyclic];

(* compute a list of all possible permutations and their partial derivatives *)
GetPerms[exprr_,bNoPI_:0]:=Module[{exprrInt,exprrOut,iCntr,iTmp,tmpr,tmpr2,tmpp,
jj,bFin,bFin2},
   Unprotect[Times];
   ClearAttributes[Times,Orderless];
   (* the initial expression in its MMTK-convenient form                    *)
   exprrInt = Evaluate[exprr];
   exprrOut = {};
   tmpr     = exprrInt;
   iCntr    = 0;
   (* loop over permutations                                                *)
   bFin=False;
   While[(!bFin),
      (* append the new permutation to the list                             *)
      AppendTo[exprrOut,tmpr];
      (* count permutations *)
      iCntr=Length[exprrOut];
      iTmp = 0;
      (* check for Partial Integration variants if the flag is given        *)
      If[bNoPI==0,
         Clear[tmpr2];
         If[Head[tmpr]===Times,tmpr2=List@@tmpr;,tmpr2={tmpr};];
         bFin2=False;
         (* add partially integrated terms as permutations                  *)
         While[((iTmp<=Length[Position[tmpr2,PD]])&&(bFin2==False)),
            (* check if the first field-containing term is a derivative     *)
            jj=1;
            (*Determine the position of the first field-like factor         *)
            If[Head[tmpr]===Times,
               (* get the position of the first field                       *)
               While[(!IsField[tmpr2[[jj]]])&&(jj<=Length[tmpr2]),jj++;];
            ];
            (* end the loop as soon as the first field has no more          *)
            (* derivatives                                                  *)
            If[IsDeriv[tmpr2[[jj]]],
               (* conduct P.I. and assign the modified term                 *)
               If[Head[exprrInt]===Times,
                  ClearAttributes[Times,Orderless];
                  tmpr2=Distribute[PartInt[Times@@tmpr2]];
                  (*check if the partial integration resulted in a sum      *)
                  If[Head[tmpr2]===Plus,
                     tmpp=List@@tmpr2;
                     (* replace every term of the sum by a list of its      *)
                     (* permutations                                        *)
                     tmpp=GetPerms[#,1]&/@tmpp;
                     (* compile all possible summations from the subsets*)
                     (*replace each variant of the first term by a list of  *)
                     (* possible summations with terms of the  jjth term*)
                     For[jj=2,jj<=Length[tmpp],jj++,
                        tmpp[[1]]=Flatten[ReplaceList[tmpp[[1]],{___,a_,___}:>
                                      {(a+ReleaseHold[#])&/@tmpp[[jj]]}]];
                     ];(*for jj*)
                     (* add the new variants to the list                    *)
                     exprrOut = Flatten[Append[exprrOut,tmpp[[1]]]];
                  ,(* else Head!= Plus*)
```

```
                     (* add the new variant of the term to the list          *)
                    exprrOut=Flatten[Append[exprrOut,GetPerms[Times@@tmpr2,1]]];
                  ];(* endif Head==Plus*)
               ,(*else Head!=Times*)
                  tmpr2={PartInt[tmpr2]};
                  (* add the new variant of the term to the list             *)
                  exprrOut=Flatten[Append[exprrOut,GetPerms[Times@@tmpr2,1]]];
               ];
               (* count P.I. variants *)
               iTmp++;
            ,(*else, no derivative*)
               bFin2=True;Break[];
            ]; (*end if IsDeriv *)
         ];(* end while bFin2 *)
      ];(* end if bNoPI *)
      Clear[tmpr];
      Unprotect[Times];
      ClearAttributes[Times,Orderless];
      Protect[Times];
      (* generate the next permutation                                      *)
      tmpr = Evaluate[Cyclic[exprrOut[[iCntr]],1]];
      If[IsEquiv[tmpr,exprrInt,2],bFin=True;];
   ];(* end while bFin *)
   exprrOut
];
(*** end of GetPerms ***)
Protect[GetPerms];


(* determine if a given expression is a total derivative                  *)
IsDeriv[expr3_]:=Module[{fNum,bTst,ii},
   bTst=False;
   (* rule out expressions that do not feature any PDs                    *)
   If[Position[expr3,PD]=={},bTst=True;
   ,(* else, PD is contained *)
      (* remove any factors *)
      fNum=expr3//.{(c_:1)*b_/;(Position[c,PD]=={}&&!ContainsFields[c]):>b};
      bTst=!(Head[fNum]===PD);
   ];
   !bTst
];

(* determine if fields are contained in a given expression                 *)
IsField[expr_]:=Or@@(Position[expr,#]!={}&/@lAvFields);

(* determine the fermionic power in a given expression                     *)
IsFermion[expr_]:=Module[{iCnt,iPos,iPos2,ii,bNot,iOut},
   iOut=0;
   For [iCnt=1,iCnt<=Length[lFermiFields],iCnt++,
      Clear[iPos];Clear[iPos2];
      (* first count occurrences with powers *)
      iPos=Position[expr,Power[Subscript[lFermiFields[[iCnt]],___],c_]/;c>1];
      (* necessary workaround for the subsequent /@ syntax *)
      If[iPos=={{}},iPos={{{}}}];
      (* count powers *)
      If[iPos!={},iOut+=Plus@@(PwrOf/@(Flatten[Extract[expr,#]&/@iPos]));];
      (* all occurrences *)
      iPos2=Position[expr,lFermiFields[[iCnt]]];
      If[(iPos2!={}&&iPos!={{{}}}),
         (* have to rule out those which have already been treated with pwrs *)
         (* loop the general occurrences *)
```

```
         For[ii=1,ii<=Length[iPos2],ii++,
            bNot=False;
            (* only check for equivalence if the compared term is longer than*)
            (* the first one                                                  *)
            (If[Length[iPos2[[ii]]]>=Length[#],
               If[(Take[iPos2[[ii]],Length[#]]==#),bNot=True;]])&/@iPos;
            (* if bNot has not been set, the term is unique                   *)
            (* =>one fermionic potence more                                   *)
            If[bNot==False,iOut+=1 ;];
         ](* end for ii*)
      ];(* end if iPos2!=0 *)
   ];(*end for iCnt*)
   iOut
];
(*** end of IsFermion ***)

Protect[IsField];
Protect[IsFermion];
Protect[IsDeriv];


(* checks for equivalence, if mode=0(or not given) permutations are taken    *)
(*into account. For mode>0 only equivalence under cyclic permutations of     *)
(* scalar (non-field) factors are considered.                                *)

IsEquiv[expr1_,expr2_,mode_:0]:=Module[{expr1Int,i1,ii,bRes,iCnt,aPerms,exprOut},
      Unprotect[Times];
      ClearAttributes[Times,Orderless];
      (* check for immediate equivalence                                      *)
      If[NormalForm[expr1]===NormalForm[expr2],
         bRes=True;
      ,(*else*)
         If[mode>0,
            (* security counter *)
            i1=0;
            (* internal backup to enable alternation                          *)
            expr1Int=expr1;
            If[!(Head[expr1Int]===Times),expr1Int={expr1Int}];
               (* bring the first expression into a defined state with        *)
               (* all factors rolled to the start                             *)
               While[(!IsField[Level[expr1Int,1][[-1]]])
                        &&(i1<=Length[Level[expr1Int,1]]),
                  expr1Int=Cyclic[expr1Int,1,1];
                  i1++;
               ];
               (* initial values *)
               bRes = False;i1=0;
               (* roll until a field-valued expr. appears at the 1st pos      *)
               While[(!IsField[Level[expr1Int,1][[1]]])
                       &&(i1<=Length[Level[expr1Int,1]]),
                  (* check for equivalence to the second term                 *)
                  If[Times@@expr1Int===expr2,bRes=True;];
                  expr1Int=Evaluate[Cyclic[expr1Int,-1,1]];
                  i1++;
               ];
         , (*else, bmode >0*)
            (* equivalence check with permutations: start with a difference  *)
            (* of the input terms-> must vanish for equivalence               *)
            expr1Int=(ReleaseHold[Distribute[(expr1-expr2)
             //.(HoldPattern[PD[arg_,ind_,0]]:>HoldForm[PD[arg,ind,1]])]]);
            Clear[ii];
```

```
            (* build a table of all summands with their possible cyclic      *)
            (* permutations                                                  *)
            expr1Int=If[Head[expr1Int]===Plus,(List@@(Distribute[expr1Int])),
                                              {expr1Int}];
            aPerms=Table[expr1Int[[ii]],{ii,Length[expr1Int]}];
            (* replace each single term by a list of its cyclic perms.       *)
            aPerms=GetPerms[#]&/@aPerms;
            (* build up a list of possible summations                        *)
            expr1Int={0};
            (* loop over level-0-summands                                    *)
            For[iCnt=1,iCnt<=Length[aPerms],iCnt++,
               ClearAttributes[Times,Orderless];
               expr1Int=(Flatten[ReplaceList[expr1Int,{___,ap_,___}:>
                        {(ap+ReleaseHold[#])&/@aPerms[[iCnt]]}]]);
            ];
            ClearAttributes[Times,Orderless];
            (* check for vanishing of any combination *)
            bRes=MemberQ[Evaluate/@(NormalForm/@expr1Int),0];
         ]; (* end if mode*)
      ];(*end if ===*)
      Protect[Times];
      bRes
   ];
(*** end of IsEquiv ***)
Protect[IsEquiv];

(*submodule for bringing a term into normal form:                            *)
(*scalars first in canonical order, then bosons, then fermions               *)
(*indices are replaced if summed up                                          *)
NormalForm[exprrr_,top_:1]:=Module[{exprrrInt,exprrrOut,iCntrr,aLocIdx,ii,lSF,
                                    iPos,iSumInd},
   Unprotect[Times];
   ClearAttributes[Times,Orderless];
   (* expand the term as far as possible                                     *)
   exprrrInt=(Evaluate[Distribute[exprrr//.HoldPattern[PD[arg_,ind_,0]]:>
              PD[arg,ind,1]]]);
   (* for sums thread over the summands                                      *)
   If[Head[exprrrInt]===Plus,exprrrOut={Plus@@(NormalForm[#,If[top==1,1,0]]&/@(
                                        List@@exprrrInt))};
   ,(*else*)
      (* generate a list of the terms                                        *)
      If [Head[exprrr]===Times, exprrrInt = List@@exprrr;, exprrrInt = {exprrr};];
      (* build lists for scalars and fields in the order they appear         *)
      iSumInd=1;
      lSF={};
      For[iCntrr=1,iCntrr<=Length[exprrrInt],iCntrr++,
         (* pick out scalar terms *)
         If[!IsField[exprrrInt[[iCntrr]]],
            AppendTo[lSF,exprrrInt[[iCntrr]]];
            exprrrInt=Delete[exprrrInt,iCntrr];
         ];(* end if*)
      ];(*end for iCntrr*)
      (*dig deeper into the expression*)
      exprrrInt=(exprrrInt/.HoldPattern[PD[arg_,idx_,eval_:1]]:>
                 ReleaseHold[PD[NormalForm[arg,0],idx,eval]]);
      (* compose the expression with ordered scalars                         *)
      exprrrOut = Flatten[Append[Sort[lSF],exprrrInt]];
      (* on top level (1st call of the function                              *)
      If[top==1,
         rep={};
         aLocIdx=IdxOf[Times@@exprrrOut,1];
```

```
         If[aLocIdx!={},
            (* create a list of indices which appear at least twice          *)
           aLocIdx=(ReplaceList[Sort[aLocIdx],{___,el_,el_,___}->el]
                                         //.{al___,el_,el_,bl___})->{al,el,bl};
           (* create a list of replacements for these                        *)
           (AppendTo[rep,#->Symbol["\[CapitalSigma]"<>ToString[iSumInd++]]])&/@aLocIdx;
           exprrrOut=exprrrOut//.rep;
         ];(* end if alocidx*)

         (* treat terms with powers *)
         Clear[aLocIdx];
         iPos=Position[exprrrOut,Power[Subscript[c_,ind__],n_]/;((n>1)&&IsField[c])];
         exprrrInt={};
         If[iPos!={},
            (*loop over occurrences*)
            For[ii=1,ii<=Length[iPos],ii++,
              Clear[aLocIdx];
              aLocIdx=Flatten[IdxOf[Extract[exprrrOut,{iPos[[ii]]}]]];
             (*loop over indices in an occurrence*)
             (If[(!StringMatchQ[ToString[#],RegularExpression["\[CapitalSigma]\\d+"]]),
             AppendTo[exprrrInt,#->Symbol["\[CapitalSigma]"<>ToString[iSumInd++]]]])&/
                                           @aLocIdx;
            ];(*end for ii*)
            If[exprrrInt!={},
               exprrrOut=exprrrOut//.exprrrInt;];
         ];(*end if rep*)
      ];(* end if top*)
   ];(*end if Head==Plus*)
   ClearAttributes[Times,Orderless];
   Protect[Times];
   ReleaseHold[Times@@exprrrOut]
];
(*** end of NormalForm ***)
Protect[NormalForm];

(* function to cancel (simplify) sum expressions under consideration of      *)
(* cyclic perm. and partial integration under an integral (bInt>0 given)     *)
NCCancel[expr_, bInt_] :=
Module[{exprInt,exprOut,aPerms,ii,iCnt,iCnt2,iCnts,iCnt2s,compList,ComplFun,tmp},
   (*just to be sure*)
   Unprotect[Times];
   ClearAttributes[Times,Orderless];
   Protect[Times];
   (*** submodule for evaluating the complexity of a term ***)
   ComplFun[exprr_]:=Module[{CFint,CFout,jj},
      If[Head[exprr]===Plus,CFint=Level[exprr,1],CFint={exprr}];
      CFout=0;
      For[jj=1,jj<=Length[CFint],jj++,
        If[Head[CFint[[jj]]]===Times,CFout+=20*Length[Level[CFint[[jj]],1]]+If[Re[(
            List@@CFint[[jj]])[[1]]]<0||Im[(List@@CFint[[jj]])[[1]]]<0,1,0,0],
            If[NumericQ[CFint[[jj]]],CFout+=0,CFout+=20]];
      ];(* end for jj*)
      CFout
   ];
   (*** end of submodules ***)
   (* bring into a maximally expanded form                                   *)
   exprInt=ReleaseHold[Distribute[(expr)//.(HoldPattern[PD[arg_,ind_,0]]:>
             HoldForm[PD[arg,ind,1]])]];
   Clear[ii];
   exprInt = If[Head[exprInt]===Plus,Level[Distribute[exprInt],1],{exprInt}];
   (* build a table of all summands with their possible cyclic permutations  *)
```

```
   aPerms = Table[exprInt[[ii]],{ii,Length[exprInt]}];

   (* if the expr. has an integral in front of it => cyclic perms possible   *)
   If[bInt==1,
      (* replace the single terms by a list of their cyclic permutations     *)
      For[iCnt = 1,iCnt<=Length[aPerms],iCnt++,
         aPerms[[iCnt]]=GetPerms[aPerms[[iCnt]]];
         If[Length[aPerms[[iCnt]]]<2,aPerms[[iCnt]]={aPerms[[iCnt]]}];
      ];
      exprInt={0};
      Unprotect[Times];
      ClearAttributes[Times,Orderless];
      Protect[Times];
      (* loop over level-0-summands, construct a list of all possible sum-   *)
      (* mations from the permutations of the single terms                   *)
      For[iCnt = 1,iCnt<=Length[aPerms],iCnt++,
         (* loop over existing sums in the sum list*)
         For[iCnts=1,iCnts<=Length[exprInt],iCnts++,
           exprInt[[iCnts]]=(exprInt[[iCnts]]+NormalForm[ReleaseHold[#]])&/
                                                                 @aPerms[[iCnt]];
         ];
         (* the above result is a nested list-> needs to be flattended out   *)
         exprInt=Flatten[exprInt];
      ];

      Print["Num of Sums: "<>ToString[Length[exprInt]]];
      (* compute the complexity for each term *)
      compList=ComplFun/@exprInt;
      (* get the the term with the smallest complexity *)
      exprOut=exprInt[[Ordering[compList,1][[1]]]];
   ,(*else, no int*)
      (* just try to eliminate and contract terms in normal form            *)
      aPerms=ReleaseHold/@(NormalForm/@aPerms);
      exprOut=Plus@@aPerms;
   ];(*end if int*)
   (* if any brackets are left, remove them and add up the result*)
   If[Head[exprOut]===List,
      exprOut=Plus@@exprOut;
   ];
   exprOut
];
(*** end of NCCancel ***)
Protect[NCCancel];

(* Perform a partial integration. Note that the argument has to have a         *)
(* partial derivative at the first place                                       *)
PartInt[(c_:1)*PD[expr_,ind_,eval_:0]*(rest_:1)]/;((!ContainsFields[c]):=
                                                  -c*expr*PD[rest,ind,eval]);
Protect[PartInt];


(* Partial derivative symbol                                                 *)
PD[arg_?NumericQ,ind_, _ : 0] := 0; (*    the deriv. of a scalar is 0        *)
                             (*      the derivative of a non-field is 0      *)
PD[arg_, ind_, _ : 0] :=  0 /; (!ContainsFields[arg]); (* transitivity       *)
PD[a_ + b_, ind_, 1] := PD[a, ind, 1] + PD[b,ind,1]; (*pulling out of factors*)
PD[Times[a_, b_], ind_, eval_: 0] :=  a*PD[b,ind,eval] /; (!ContainsFields[a]);
PD[Times[a_, b_], ind_, eval_: 0] :=  b*PD[a,ind,eval] /; (!ContainsFields[b]);
PD[Times[a_, b_], ind_, 1] := PD[a,ind,1]*b + a*PD[b,ind,1];   (*Leibniz rule*)

Protect[PD];
```

```
(* Function to perform functional variation in x-space                       *)
VarD[expr_,var_,indx_:{},bNoInt_:0]:=Module[{iCnt,iPwr,iTrm,iVars,iaPos,
                                exprIntern,lSubTerms,tTrm,tTrm2,fNum,fNumSub,res},
   (*just to be sure*)
   Unprotect[Times];
   ClearAttributes[Times,Orderless];
   Protect[Times];
   (*assure indx to be a list*)
   exprIntern=Expand[expr//.PD[arg_,ind_,0]->PD[arg,ind,1]];
   (*split the argument into summands and treat each summand separately      *)
   exprIntern = If[Head[exprIntern]===Plus,List@@exprIntern,{exprIntern}];
   (*****************************************************)
   (*** the actual variation function ***)
   Vari[trm_,fld_,idx_]:=Module[{ress,lIndices,tmp},
      (*Print["Vari:"];Print[trm];Print[fld];Print[idx];*)
      lIndices=IdxOf[trm];
      Unprotect[Times];
      ClearAttributes[Times,Orderless];
      Protect[Times];
      tmp:=1/4*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]])
              *(KDelta[lIndices[[3]],idx[[3]]]*KDelta[lIndices[[4]],idx[[4]]]
               -KDelta[lIndices[[3]],idx[[4]]]*KDelta[lIndices[[4]],idx[[3]]]);
      (*generate the permutations for each field*)
      ress=If[(!(RmAll[trm]===fld)),0, Switch[fld,
        A, KDelta[lIndices[[1]],idx],
        B, 1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
        bB,1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
        P, 1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
        bP,1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
        J, tmp,
        bJ,tmp,
        Q, tmp,
        bQ,tmp,
        c, 1,
        bc,1,
        b, 1,
        OA,KDelta[lIndices[[1]],idx],
        Oc,1,
        Obc,1,
        OB,1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
        ObB,1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
        OP,1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
        ObP,1/2*(KDelta[lIndices[[1]],idx[[1]]]*KDelta[lIndices[[2]],idx[[2]]]
               -KDelta[lIndices[[1]],idx[[2]]]*KDelta[lIndices[[2]],idx[[1]]]),
         OJ,tmp,
         ObJ,tmp,
         OQ,tmp,
         ObQ,tmp]
         ];
      ress
   ];
   (*** end Vari ***)
```

```
res=0;
(*loop over summands*)
For[iCnt=1,iCnt<=Length[exprIntern],iCnt++,
   (*remove numerical prefactors*)
   fNum=1;
   If[Head[exprIntern[[iCnt]]]===Times,
      Unprotect[Times];
      ClearAttributes[Times,Orderless];
      Protect[Times];
      fNum=Times@@Flatten[Cases[exprIntern[[iCnt]],a_/;!IsField[a]:>a,1]];
      exprIntern[[iCnt]]=Times@@Flatten[Cases[exprIntern[[iCnt]],a_/;
                                                  IsField[a]:>a,1]];
   ];(* end if *)
   (*determine the pos. of relevant terms in the current summand          *)
   iaPos=Position[exprIntern[[iCnt]],var];
   (*remove the index numbers of deeper levels, only take top one         *)
   If[iaPos!={},
      For[iTrm=1,iTrm<=Length[iaPos],iTrm++,
         If[Length[iaPos[[iTrm]]]==0,
            iaPos[[iTrm]]={1};
         ,(* else Length > 0 *)
            iaPos[[iTrm]]={iaPos[[iTrm,1]]};
         ];
      ];
   ,(* else iaPos == 0*)
    (* the term does not depend on the variable, skip to the next         *)
      Continue;
   ];
   (* apply an additional wrapping {} if we have only a single term       *)
   If[!(Head[exprIntern[[iCnt]]]===Times),
                              exprIntern[[iCnt]]={exprIntern[[iCnt]]};];
   (* check if an integral is present to enable rotations and PI.         *)
   If[bNoInt>0,
      Unprotect[Times];
      ClearAttributes[Times,Orderless];
      Protect[Times];
      (* split the multiplication                                         *)
      lSubTerms=Table[List@@exprIntern[[iCnt]],{Length[iaPos]}];
      (*loop over occurrences of the relevant variable in the             *)
      (* current summand                                                  *)
      For[iVars=1,iVars<=Length[iaPos],iVars++,
         (* extract any numeric prefactor                                 *)
         fNumSub=1;
         If[!IsField[lSubTerms[[iVars,1]]],
            fNumSub=lSubTerms[[iVars,1]];
            lSubTerms[[iVars]]=Delete[lSubTerms[[iVars]],1];
         ];
         (* pick up an occasional sign from pulling the variation         *)
         (* through a fermionic term -> store in fNumSub                  *)
         If[iaPos[[iVars,1]]>1, fNumSub=fNumSub*If[IsFermion[var]>0,
           (-1)^IsFermion[Times@@lSubTerms[[iVars,1;;(iaPos[[iVars,1]]-1)]]],1,1];
         ];
         (* check if the current term is a derivative -> use alternate    *)
         (* variation                                                     *)
         (* only the case of a derivative of a power is covered here      *)
         If[IsDeriv[lSubTerms[[iVars,iaPos[[iVars,1]]]]],
            If[IsDeriv[lSubTerms[[iVars,iaPos[[iVars,1]]]]/.HoldPattern[
                                          PD[arg_,ind_,eval_:0]]:>arg],
               (* double derivatives are boundary terms and vanish        *)
               lSubTerms[[iVars]]={0};
            ,(* else only single derivative*)
```

```
           iPwr=PwrOf[lSubTerms[[iVars,iaPos[[iVars,1]]]]];
           tTrm=RmPwr[lSubTerms[[iVars,iaPos[[iVars,1]]]]];
           If[iPwr>1,
              (* for a fermion: *)
              If[IsFermion[lSubTerms[[iVars,iaPos[[iVars,1]]]]]>0,
                 If[EvenQ[iPwr],
                    lSubTerms[[iVars]]={0}
                 ,(*else odd power*)
                    lSubTerms[[iVars,iaPos[[iVars,1]]]]=tTrm^(iPwr-1);
                    lSubTerms[[iVars]]={
 (-Vari[tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg,var,indx])
*PD[Times@@lSubTerms[[iVars]],tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>
 ind,tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>eval]};
                 ];(* end if EvenQ *)
              ,(*else boson*)
                 lSubTerms[[iVars,iaPos[[iVars]]]]=tTrm^(iPwr-1);
                 lSubTerms[[iVars]]={
 (-iPwr*Vari[tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg,var,indx])
*PD[Times@@lSubTerms[[iVars]],tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>
 ind,tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>eval]};
              ];
           ,(*else, no higher powers->take out the current term,      *)
              (* apply the derivative to the rest and attach          *)
              (* the result of the variation at the start             *)
             lSubTerms[[iVars]]=Delete[lSubTerms[[iVars]],iaPos[[iVars,1]]];
             lSubTerms[[iVars]]=-Vari[tTrm
                    /.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg,var,indx]
               *PD[Times@@lSubTerms[[iVars]],
                   tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>ind,
                   tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>eval];
           ];(*end if iPwr>1*)
        ];(* end if IsDeriv (2)*)
     ,(* else no derivative at all*)
        (* check if the current term is a power -> chain rule         *)
        iPwr=PwrOf[lSubTerms[[iVars,iaPos[[iVars,1]]]]];
        tTrm=RmPwr[lSubTerms[[iVars,iaPos[[iVars,1]]]]];
        If[iPwr>1,
           (* the case of a power of a derivative is not caugth       *)
           (* above; needs special treatment here                     *)
           If[Head[tTrm]===PD,
              (* derivation of double derivatives is 0                *)
             If[Head[tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg]===PD,
                 lSubTerms[[iVars]]={0};
              ,(*else power of a single derivative *)
                 (* for a fermion: *)
                 If[IsFermion[lSubTerms[[iVars,iaPos[[iVars,1]]]]]>0,
                    If[EvenQ[iPwr],
                       lSubTerms[[iVars]]={0}
                    ,(*else odd power*)
                       lSubTerms[[iVars,iaPos[[iVars,1]]]]=tTrm^(iPwr-1);
                       lSubTerms[[iVars]]={
         (-Vari[tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg,var,indx])
        *PD[Times@@lSubTerms[[iVars]],
          tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>ind,
          tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>eval]};
                    ];(* end if EvenQ *)
                 ,(*else boson*)
                 (* lower the power by 1 and apply the derivative     *)
                 (* to the entire term                                *)
                    lSubTerms[[iVars,iaPos[[iVars]]]]=tTrm^(iPwr-1);
                   lSubTerms[[iVars]]={
```

```
             (-iPwr*Vari[tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg,var,indx])
            *PD[Times@@lSubTerms[[iVars]],
              tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>ind,
              tTrm/.HoldPattern[PD[arg_,ind_,eval_:0]]:>eval]};
                     ]; (* end if fermion *)
                  ]; (* end if double derivative *)
               , (* else no PD *)
                  (* for a fermion: *)
                  If[IsFermion[lSubTerms[[iVars,iaPos[[iVars,1]]]]]>0,
                    lSubTerms[[iVars,iaPos[[iVars,1]]]]=
                      If[EvenQ[iPwr],0,(Vari[tTrm,var,indx])*tTrm^(iPwr-1)];
                  ,(*else boson*)
                    lSubTerms[[iVars,iaPos[[iVars,1]]]]=
                      iPwr*(Vari[tTrm,var,indx])*tTrm^(iPwr-1);
                  ];
               ];(*end if power of PD *)
            ,(*else, no higher powers*)
               lSubTerms[[iVars,iaPos[[iVars,1]]]]=Vari[tTrm, var, indx];
            ];(*end if iPwr>1*)
         ];(*end if IsDeriv (1)*)
         (* add the result of the current line                         *)
         res+=fNum*fNumSub*Times@@lSubTerms[[iVars]];
      ](*end for iVars*)
   ,(*else there is an Integral *)
      (* create a table, 1 line for each occurrence of the relevant       *)
      (* var, split the multiplication                                *)
      lSubTerms=Table[Level[Cyclic[exprIntern[[iCnt]],-iaPos[[ii,1]]+1,1],1],
                   {ii,Length[iaPos]}];
      (*loop over occurrences of the relevant variable in the          *)
      (*current summand                                                *)
      For[iVars=1,iVars<=Length[iaPos],iVars++,
         (* pick up an occasional sign from the rotation               *)
         (*-> store in fNumSub                                         *)
         fNumSub=1;
         If[!IsField[lSubTerms[[iVars,1]]],
            fNumSub=lSubTerms[[iVars,1]];
            lSubTerms[[iVars]]=Delete[lSubTerms[[iVars]],1];
         ];
         Unprotect[Times];
         ClearAttributes[Times,Orderless];
         Protect[Times];
         (* check if the current term is a derivative                  *)
         (* -> partial integration                                     *)
         While[Head[lSubTerms[[iVars,1]]]===PD,
            lSubTerms[[iVars]]=List@@(-PartInt[Times@@lSubTerms[[iVars]]]);
            (* put the relative sign into the separate forefactor       *)
            fNumSub*=-1;
         ];
         (* check if the current term is a power -> chain rule          *)
         iPwr=PwrOf[lSubTerms[[iVars,1]]];
         tTrm=RmPwr[lSubTerms[[iVars,1]]];
         If[iPwr>1,
            (* the case of a power of a derivative is not caugth         *)
            (* above; needs special treatment here                       *)
            If[Head[tTrm]===PD,
               Clear[tTrm2];
               tTrm2=tTrm;
               (* conduct partial integrations as often as necessary,     *)
               (* variate the rest and return the combined result         *)
               tTrm=Times@@Flatten[{tTrm^(iPwr-1),lSubTerms[[iVars,2;;]]}];
               While[Head[tTrm2]===PD,
```

```
                        (* Partial Integration *)
                        tTrm=-PD[tTrm,tTrm2/.HoldPattern[PD[arg_,ind_,eval_:0]]:>ind,
                                      tTrm2/.HoldPattern[PD[arg_,ind_,eval_:0]]:>eval];
                         (* the rest: *)
                         tTrm2=tTrm2/.HoldPattern[PD[arg_,ind_,eval_:0]]:>arg;
                      ];(* end while *)
                      (* compile result *)
                      (* for a fermion: *)
                      If[IsFermion[tTrm2]>0,
                        lSubTerms[[iVars]]=If[EvenQ[iPwr],0,(Vari[tTrm2,var,indx])*tTrm];
                      ,(*else boson*)
                         lSubTerms[[iVars]]=iPwr*(Vari[tTrm2,var,indx])*tTrm;
                      ]; (* end if fermi *)
                   , (* else no PD *)
                      If[IsFermion[lSubTerms[[iVars,1]]]>0,
                        lSubTerms[[iVars,1]]=If[EvenQ[iPwr],0,
                                              (Vari[tTrm,var,indx])*tTrm^(iPwr-1)];
                      ,(* else boson *)
                        lSubTerms[[iVars,1]]=iPwr*(Vari[tTrm,var,indx])*tTrm^(iPwr-1);
                      ];
                   ];
                ,(*else, no higher powers*)
                   lSubTerms[[iVars,1]]=Vari[tTrm, var, indx];
                ];(*end if iPwr>1*)
                (* add the result of the current line*)
                res+=fNum*fNumSub*Times@@lSubTerms[[iVars]];
             ](*end for iVars*)
          ];(*end if bNoInt*)
       ];(*end for iCnt*)
       res
    ]
    (* end VarD *)
    Protect[VarD];

    On[General::spell1];

    (* disable canonical sorting of expressions with head Times *)
    Unprotect[Times];
    ClearAttributes[Times,Orderless];
    Protect[Times];

    Print["IMPORTANT !!!\nCall 'ClearAttributes[Times,Orderless];'
                            now BEFORE working with the package!"];
    EndPackage[ ];

    (* necessary *)
    Unprotect[Times];
    ClearAttributes[Times,Orderless];
    Protect[Times];
    (*** EOF *********************************************************************)
```

G.3 VectorAlgebra.m

The `VectorAlgebra` package contains several functions and redefinitions which enable computations in Euclidean $\mathbb{R}^4_\theta$ vector space. The internal multiplication and addition are altered in order to be compatible with Einstein's sum convention. In addition, an inner product $a \cdot b := a_\mu b_\mu$, and an anti-symmetric tensorial product $a\theta b := a_\alpha b_\beta \theta_{\alpha\beta}$ are defined by the functions `Dot` and `VCross`, respectively. In the following, only the main functions of the package are listed. There

are several internal and auxiliary functions, which all feature a built-in help text, accessible by calling the function name with prepended `?` sign, and without arguments.
Vectors have to be defined prior to any calculation by calling the function `DefVec` with the new variable as argument. Any undefined vector will be treated as a regular scalar variable, and the functions of `VectorAlgebra` will not recognize it.

G.3.1 List of Commands

▷ **Command:** `DefVec`

`DefVec[`v`]`

Description :

This command defines a new vectorial variable. Double definitions are ignored but it may be unsafe to give names which are already defined in another context. This function has no output.

Variables :

1: v symbol of the new variable.

Example :

```
In[1]:= DefVec[k]
```

▷ **Command:** `UDefVec`

`UDefVec[`v`]`

Description :

This command removes the definition of a vectorial variable. If given an unknown variable, the input is ignored. This function has no output.

Variables :

1: v symbol of the variable to remove.

Example :

```
In[1]:= UDefVec[k]
```

▷ **Command:** `IsVec`

`IsVec[`v`]`

Description :

`IsVec` determines if the given variable is a vector, defined by a prior call to `DefVec`, and returns the result as Boolean values `True` or `False`.

Variables :

1: v symbol of the variable to check.

Output :

The function returns `True` if v is a defined vector, and `False` if not.

Example :

```
In[1]:= IsVec[k]
Out[1]= True
In[2]:= UDefVec[k];
In[3]:= IsVec[k]
Out[3]= False
```

▷ **Command:** `KDelta`

`KDelta[`i_1`,`i_2`]`

Description :

The function is an implementation for the four dimensional Kronecker delta δ_{i_1,i_2}, and enables index contractions with defined vectors. The Einstein sum convention is respected, no specific output is generated.

Variables :

1: i_1 first index
2: i_2 second index

Example :

```
In[1]:= KDelta[\[mu],\[nu]] Subscript[k,\[nu]]}
Out[1]= Subscript[k,\[mu]]
In[2]:= KDelta[\[mu],\[nu]]^2
Out[2]= 4
```

▷ **Command:** `VCross`

`VCross[`k_1`,`$k, 2$`]`

Description :

The function is an implementation of the anti-symmetric product appearing in phases, i.e. $k_\mu \theta_{\mu\nu} p_\nu$. Note that the ordering of parameters is respected, and interchange results in a relative sign. Transformation rules are *not* recognized upon regular input. Only `VSimplify[]` (see below) may cancel terms. No specific output is generated.

Variables :

1: k_1 a vector defined by `DefVec[]`
2: k_2 a vector defined by `DefVec[]`

Example :

```
In[1]:=  VCross[k,k]
Out[1]=  0
In[2]:=  VCross[p,k]+VCross[k,p]
Out[2]=  VCross[k,p]+VCross[p,k]
In[3]:=  VSimplify[VCross[p,k]+VCross[k,p]]
Out[3]=  0
```

▷ **Command:** `VSimplify`

`VSimplify[`*e*`,`*opt..*`]`

Description :

`VSimplify[]` (Vector Simplify) is a wrapper function for Mathematica®'s internally predefined `FullSimplify[]` function. For expressions e involving vectorial variables, it performs additional transformations, being specialized for this purpose (`simprule` and `comprule`). It also passes any additional option to `FullSimplify[]`. Therefore, please refer to the respective documentation in Mathematica® for details on the *opt* parameter.

Variables :

1: e any expression
2: $opts$ options which are passed through to `FullSimplify[]`

Output :

The function returns a simplified version of e.

Example :

```
In[1]:=  VSimplify[k.p-Subscript[k,\[Mu]] Subscript[p,\[Mu]]
         +(KDelta[\[Rho],\[Sigma]] Subscript[p,\[Rho]] Subscript[p,\[Sigma]])/(p.p),TimeConstraint->5]
Out[1]=  1
In[2]:=  VSimplify[Sin[VCross[2 k + p, k]] - Sin[VCross[p, k]]]
Out[1]=  0
```

▷ **Command:** `VLimit`

`VLimit[`e`,`v`->`v_0`[,`*vec*`[,`t`]]]`

Description :

This function performs limiting operations of an expression e with respect to the vectorial variable v, i.e. $\lim_{v\to v_0} e$. The strategy is to replace each vector by its absolute value times an unit vector carrying an index $v_\mu \to |v|e_\mu$. In a second step the Mathematica® internal `Limit[]` function is called with $|v| \to v_0$. This method, is not correct in the strict mathematical sense but gives the expected results in all cases which have been considered within the framework of this thesis. In general, one has to be aware of expressions like $(p \cdot k)/(p^2 + k^2)$, which do not have a well defined limit at $\{|k|, |p|\} \to 0$. Regarding the implementation, the function first checks, if the limiting can be performed by a simple replacement rule, i.e. without special rules such as l'Hospital. Only if this results in undefined or singular expressions, the actual `Limit[]` function is called. In this way, an enormous gain in speed is achieved for most expressions.

Variables :

1:	e	any expression
2:	v	the vectorial variable to be taken to a limit
3:	v_0	the (numerical) limit for v
4:	vec	[optional], binary parameter. 0(default): return a regular Mathematica® expression, 1: for internal use only.

Output :

The function returns the limit $\lim\limits_{v \to v_0} e$.

Example :

```
In[1]:= VLimit[(k.p) Subscript[k, \[Mu]] Subscript[p, \[Nu]]/(k.k),k->0]

Out[1]= (Subscript[p, \[Mu]] Subscript[p, \[Nu]])/4}
```

▷ **Command:** `VSeries`

`VSeries[`e`,{`v`,`v_0`,`o`}[,`$verb$`]`

overloaded

`VSeries[`e`,{{`v`},{`v_0`},`o`}[,`$verb$`]`

Description :

This function performs a Taylor series expansion of the expression e with respect to the vectorial variable v, at point v_0 to order o. Note that any trigonometric phase factors contained in e are *not* expanded. For a single indexed list for v and v_0 an expansion according to Eqn. (3.17) (see page 39) is performed. In the case of double indexed lists, the operation is defined as in Eqn. (E.30) on page 134. However, the implementation of the latter case is experimental, and it is not guaranteed (due to non-commuting limits for several variables) that the result is indeed correct (see discussion on page 134). This function is a completely independent implementation, and does not rely on the built in `Series[]` function.

Variables :

1:	e	any expression
2:	v, or $\{v_1..v_n\}$	the vectorial variable(s) defining the expansion.
3:	v_0, or $\{v_{0,1}..v_{0,n}\}$	the limiting point(s) of v
4:	o	the order up to which the expansion shall be performed, $o \in \mathbb{N}_0$
5:	$verb$	[optional] Boolean verbosity parameter: 0 (default): only print the result 1: print order by order intermediate results.

Output :

Both versions of the function returns a single indexed list containing $o+1$ elements $\{e_0, e_1, ..e_o\}$ corresponding to the results e_i obtained at each order i.

Example :

```
In[1]:= VSeries[KDelta[\[Mu],\[Nu]] (p.k)+Subscript[p,\[Mu]] Subscript[k,\[Nu]]/(k.k+m^2),{k,0,2}]

Out[1]= {0, k.p KDelta[\[Mu],\[Nu]]+(Subscript[k, \[Nu]] Subscript[p, \[Mu]])/m^2, 0}

In[1]:= VSeries[f[k,p],{{k,p},{0,0},2}]

Out[1]= {VLimit[VLimit[f[k,p],k->0,0],p->0,0],
         1/2 (VLimit[VLimit[VD[f[k,p],k,\[Tau]11],k->0,0],p->0,0] Subscript[k, \[Tau]11]
             +VLimit[VLimit[VD[f[k,p],p,\[Tau]12],k->0,0],p->0,0] Subscript[p, \[Tau]12]),
         1/6 (VLimit[VLimit[VD[VD[f[k,p],k,\[Tau]11],k,\[Tau]21],k->0,0],p->0,0]
              *Subscript[k, \[Tau]11] Subscript[k, \[Tau]21]
             +VLimit[VLimit[VD[VD[f[k,p],p,\[Tau]12],k,\[Tau]21],k->0,0],p->0,0]
              *Subscript[k, \[Tau]21] Subscript[p, \[Tau]12]
             +(VLimit[VLimit[VD[VD[f[k,p],k,\[Tau]11],p,\[Tau]22],k->0,0],p->0,0]*Subscript[k,\[Tau]11]
              +VLimit[VLimit[VD[VD[f[k,p],p,\[Tau]12],p,\[Tau]22],k->0,0],p->0,0]*Subscript[p,\[Tau]12])
              * Subscript[p, \[Tau]22])}
```

▷ **Command:** `VD`

`VD[`e`,`v`,`i`]`

Description :

`VD[]` performs vectorial differentiation of an expression e with respect to the vectorial variable v with index i. $\partial e(v)/\partial v_i$.

Variables :

1: e any expression
2: v the vectorial variable used as differential
3: i the index of v

Output :

The function returns the differential $\partial e(v)/\partial v_i$.

Example :

```
In[1]:= VD[Subscript[k,\[Mu]] Subscript[k,\[Nu]],k,\[Tau]

Out[1]= KDelta[\[Nu],\[Tau]] Subscript[k,\[Mu]]+KDelta[\[Mu],\[Tau]] Subscript[k,\[Nu]]
```

▷ **Command:** `IndexStyle`

`IndexStyle[`e`]`

Description :

This function is of high importance in practical calculations. It transforms the functions `Dot[]` and `VCross[]` to their explicit index representations. It generates inner indices ηi, where i is counted from 1 for each run. It may be a good starting point for enhancements to define a global index counter, as subsequent calls of `IndexStyle[]` may generate similar indices, which

eventually contract in an unintended way.
Note also that `IndexStyle[]` sets the global variable `iSimplifyIndices` to 0 to disable any automatic simplifications of index contractions.

Variables :

1: e any expression

Output :

The function returns a version of e in which all vectorial products (symmetric and anti-symmetric) are written in their explicit form.

Example :

```
In[1]:= IndexStyle[(k.p)+VCross[k,p]]

Out[1]= Subscript[k,\[Eta]1] Subscript[p,\[Eta]1]
        +Subscript[k,\[Eta]2] Subscript[p,\[Eta]3] Subscript[\[Theta],\[Eta]2,\[Eta]3]
```

▷ **Command:** `VectorStyle`

`VectorStyle[`e`]`

Description :

`VectorStyle[]` is the inverse function to `IndexStyle[]` as it attempts to rewrite expressions e containing indices in a form with inner products `Dot`, and anti-symmetric contractions with θ, `VCross[]`. However, in complex expressions this function may miss contractions which are separated. Hence, it is advisable to expand e prior to calling `VectorStyle[]`. Note also that `IndexStyle[]` sets the global variable `iSimplifyIndices` to 1 to enable any automatic simplifications of index contractions.

Variables :

1: e any expression

Output :

The function returns a version of e in which all vectorial products (symmetric and anti-symmetric) are written in their symbolized form.

Example :

```
In[1]:= VectorStyle[Subscript[k,\[Eta]1] Subscript[p,\[Eta]1]
        +Subscript[k,\[Eta]2] Subscript[p,\[Eta]3] Subscript[\[Theta],\[Eta]2,\[Eta]3]]
Out[1]= k.p+VCross[k,p]
```

▷ **Command:** `VSymmetricQ`

`VSymmetricQ[` e, v`]`

Description :

`VSymmetricQ[]` attempts to find out if a given function e is even or odd with respect to sign reversal of the variable v. Note, that this function is specialized to integrands appearing in this thesis which all have simple overall powers of momenta and trigonometric functions. The result is undefined for functions being neither even nor odd. Internally, this function utilizes `PowerCount[]`.

Variables :

1:	e	any expression
2:	v	a variable contained in e

Output :

The function returns a Boolean result
`False`: e is an odd function of v
`True`: e is an even function of v

Example :

```
In[1]:= VSymmetricQ[Subscript[k, \[Mu]] Sin[VCross[k,p]]/(k.k),k]
Out[1]= True
In[2]:= VSymmetricQ[Subscript[k, \[Mu]]/(k.k),k]
Out[2]= False
```

▷ **Command:** `PowerCount`

`PowerCount[` e, v,*lim*`]`

Description :

`PowerCount[]` performs a power counting according to the definitions in Section 1.3.1. It determines the superficial degree of divergence $d(e)$ of the expression e, depending on the vectorial moment v, in the limit *lim*. The latter is intended to give an option to analyze either the IR limit, $lim \to 0$ or the UV limit $lim \to \infty$. However, the current implementation only reliably works for $v \to \infty$, i.e. UV power counting. Note also that in the case that e does not depend on v, the result is $-\infty$, and that the function is defined solely for polynomial functions.

Variables :

1:	e	any expression
2:	v	a vectorial variable contained in e
3:	*lim*	the desired limit where the counting should be performed (i.e. $v \to lim$), $lim \in \{$`0,\[Infinity]`$\}$.

Output :

The function returns an integer $\in [-\infty, \infty)$ corresponding to $d(e)$.

Example :

```
In[1]:= PowerCount[Subscript[k,\[Mu]] Subscript[p,\[Nu]] Sin[VCross[k,p]]/(k.k)^2,k,\[Infinity]]

Out[1]= -3
```

▷ **Command:** `DivergentPart`

`DivergentPart[` e, v`]`

Description :

The function fully expands a given expression e and performs UV power counting on each summand. The result is a single indexed list of terms, corresponding to those summands s_i obeying $d(s_i) \geq -4$. Internally, this is achieved by utilization of the `PowerCount[]` function. Hence, all restrictions of this function apply.

Variables :

1: e any expression
2: v the momentum, which shall be utilized for power counting

Output :

The function returns a single indexed list of sub terms in e being potentially divergent upon integration.

Example :

```
In[1]:= DivergentPart[(Subscript[k, \[Mu]] Subscript[k, \[Nu]]+\[Theta] k^4)/k^8,k]

Out[1]= {\[Theta]/k^4}
```

▷ **Command:** `Int`

`Int[` e`[,` np`]]`

Description :

`Int[]` performs momentum integrations according to predefined schemes with k being the integration variable and p being an external momentum, per definition. It basically contains all (but the vertex integrals) of the replacement rules given in Appendix F.1. Note that this function is only applicable to the typical one loop integrals containing a $\sin^2(k\theta p)$ phase. The optional parameter np determines if the planar or non-planar result shall be computed. Note that this function is deprecated. Use `IntVert[]` instead.

Variables :

1: e any expression
2: np [optional] defines if planar or non-planar results shall be returned.
 0: planar results
 1 (default): non-planar results

Output :

The function returns the requested integration result or `UnknownInt[`e`,`np`]` if no rule matches for e.

Example :

```
In[1]:=  Int[Sin[1/2*VCross[k,p]]^2*Subscript[k, \[Mu]]*Subscript[k, \[Nu]]/(k.k)^2,0]

Out[1]=  \[Pi]^2 \[CapitalLambda]^2 KDelta[\[Mu],\[Nu]]

In[2]:=  Int[Sin[1/2*VCross[k,p]]^2*Subscript[k, \[Mu]]*Subscript[k, \[Nu]]/(k.k)^2,1]

Out[2]=  -((\[Pi]^2 (KDelta[\[Mu],\[Nu]]
           -(2 Subscript[p,\[Xi]1] Subscript[p,\[Xi]2] Subscript[\[Theta],\[Xi]1,\[Mu]]
             *Subscript[\[Theta],\[Xi]2,\[Nu]])
            /(p^2 \[Theta]^2)))/(p^2 \[Theta]^2))
```

▷ **Command: `IntVert`**

`IntVert[` e`]`

Description :

`IntVert[]` is the successor of Int. It does not presume a phase factor $\sin^2$ but contains all integrals in their actual form, as they could be found in an integral table. It contains all of the replacement rules given in Appendix F.1.

Variables :

1: e any expression

Output :

The function returns the requested integration result or `UnknownInt[`e`]` if no rule matches for e.

Example :

```
In[1]:=  IntVert[1/2 Subscript[k,\[Mu]]*Subscript[k,\[Nu]]/(k.k)^2]

Out[1]=  \[Pi]^2 \[CapitalLambda]^2 KDelta[\[Mu],\[Nu]]

In[2]:=  IntVert[-1/2 Cos[VCross[k,p]] Subscript[k,\[Mu]]*Subscript[k,\[Nu]]/(k.k)^2]

Out[2]=  -((\[Pi]^2 (KDelta[\[Mu],\[Nu]]
           -(2 Subscript[p,\[Xi]1] Subscript[p,\[Xi]2] Subscript[\[Theta],\[Xi]1,\[Mu]]
             *Subscript[\[Theta],\[Xi]2,\[Nu]])
            /(p^2 \[Theta]^2)))/(p^2 \[Theta]^2))
```

G.3.2 Loading of the Package

```
Get["VectorAlgebra.m", Path -> "⟨PATH⟩"];
```

G.3.3 Source Code

```
(* ::Package:: *)

```

```
(*********************************************************************
 *                                                                   *
 * Package VectorAlgebra                                             *
 *                                                                   *
 * Enables simple 4-vector calculations in Euclidean space           *
 *                                                                   *
 * Author : Rene Sedmik                                              *
 *   Date : 2009-10-27                                               *
 *                                                                   *
 * Version History:                                                  *
 * -------------------------------------------------------------    *
 * | Ver | Date       | Changes                                 |    *
 * +-----+------------+-----------------------------------------+    *
 * | 0.1 | 2008-10-08 | Initial Version                         |    *
 * | 0.11| 2008-10-09 | Bug fixes for cross product             |    *
 * | 0.12| 2008-10-14 | vec[ ] syntax and speed improvements    |    *
 * | 0.13| 2008-10-22 | VD, VSeries, VLimit, unit vector        |    *
 * | 0.14| 2008-10-23 | NumEquiv                                |    *
 * | 0.15| 2008-10-24 | Bug fixes                               |    *
 * | 0.16| 2008-10-28 | Humanize                                |    *
 * | 0.17| 2008-11-13 | Various bug fixes                       |    *
 * | 0.18| 2008-12-01 | More stability for VLimit and VSeries   |    *
 * | 0.19| 2008-12-03 | Bracket markup functions added          |    *
 * | 0.20| 2008-12-19 | Memory sharing and bug fixes            |    *
 * | 0.21| 2009-01-29 | DivergentPart,PowerCount, and Int added |    *
 * | 0.22| 2009-01-30 | Bug fix for Int                         |    *
 * | 0.23| 2009-02-12 | Bug fix for Int (signs) and IndexStyle  |    *
 * | 0.24| 2009-02-13 | Rule ptilde *p =0                       |    *
 * | 0.25| 2009-04-17 | Enhancements for Int                    |    *
 * | 0.26| 2009-04-21 | rule p\theta p=0                        |    *
 * | 0.27| 2009-04-28 | VLSimplify added                        |    *
 * | 0.28| 2009-05-06 | Several bug fixes and mem. optimizations|    *
 * | 0.29| 2009-06-09 | Bugfix for VSimplify                    |    *
 * | 0.30| 2009-06-24 | Bugfixes for VD                         |    *
 * | 0.31| 2009-09-17 | Enhancement for PowerCount and Int      |    *
 * | 0.32| 2009-06-24 | Bugfixes for VSeries                    |    *
 * | 0.33| 2009-10-13 | VSymmetricQ added                       |    *
 * | 0.34| 2009-10-27 | IntVert added                           |    *
 *                                                                   *
 *********************************************************************)

 BeginPackage["VectorAlgebra‘"];

 (**** clear all definitions ****)
Unprotect[DefVec,UDefVec,ClearDefVec,IsVec,KDelta,VCross,VSimplify,VLSimplify,
          VLimit,VSeries,VD,\[Theta],uv,IndexStyle,VectorStyle,PowerCount,
          DivergentPart,Int,NumEquiv,Humanize,SizeBrackets,ColorBrackets,
          VSymmetricQ,SubPwrCnt,IntVert];
Clear[DefVec,UDefVec,ClearDefVec,IsVec,KDelta,VCross,VSimplify,VLSimplify,
          VLimit,VSeries,VD,\[Theta],uv,IndexStyle,VectorStyle,PowerCount,
          DivergentPart,Int,NumEquiv,Humanize,SizeBrackets,ColorBrackets,
          VSymmetricQ,SubPwrCnt,IntVert];

 (**** usage Documentation ****)

  DefVec::usage = "DefVec[symbol] defines symbol to be a 4-vector. This must
             be done for each vector appearing in subsequent calculations.";
  UDefVec::usage = "UDefVec[symbol] undefines [symbol] to be a 4-vector.
             Thereafter it can be used as normal Mathematica symbol without
             special meaning.";
  ClearDefVec::usage = "ClearDefVec undefines all previously defined vectors.";
```

```
IsVec::usage = "IsVec[symbol] gives True if the given symbol has been
           defined to be a vector, False in any other case. IsVec is aware
           of non-vectorial factors and indices of the argument.";
KDelta::usage = "KDelta[i,j] gives 4 if i equals j, 0 otherwise. This
           modified version of the built-in KroneckerDelta[] is useful if
           Einstein's sum convention is presumed.";
VCross::usage = "VCross[k,p] is a symbolic version of Mathematica's Cross[]
           product. It acts solely on vectorial objects defined by DefVec
           and is antisymmetric.";
VSimplify::usage = "VSimplify[expression, options ..] does simplifications
           in the same way as FullSimplify does, but is aware of the
           vectorial calculus and sum convention. It takes any additional
           options FullSimplify takes with the exception of
           'TransformationFunctions' and 'ComplexityFunction'";
VLSimplify::usage = "VLSimplify[expr,[verbose, opts]] is a version of
           VSimplify (->see ?VSimplify) for very long expressions expr
           which crawls through the given formula piece by piece, thereby
           avoiding to give the whole expression to FullSimplify. This
           (in most cases) shortens computational times. Set the optional
           verbose argument to 1 to receive more progress information. The
           opts argument can be used to pass arguments to the inherited
           VSimplify function.";
VLimit::usage = "VLimit[f(x), x->x0, opt:OutputVect?] takes the limit x->x0
           for the function f respecting all vectorial rules. Eventually the
           result contains the unit vector uv. If the limit is to be taken
           in a variable that is not known to be a vector VLimit utilizes
           the Mathematica-internal Limit function. The optional 3rd
           parameter is a boolean indicating if the output is given in vec[]
           form (=1) or in standard notation (=0, default).";
VSeries::usage = "VSeries[f(x), {x, x0, ord}] expands the function f(x)
           into a series around x0 up to order ord. The result is a regular
           Mathematica expression (not a Series object as for the standard
           Series function). VSeries respects analytic vectorial computation
           rules for defined vectors.";
VSymmetricQ::usage = "VSymmetricQ[ f(x), x ] determines if the function f(x)
           is symmetric in x (return True) or not (False).\n If f(x) is a sum
           the result is a list of the outcomes for every summand.";
VD::usage = "VD is the vector analysis complement to the standard D
           derivation in Mathematica. The syntax VD[f(x), x, index] has an
           additonal parameter 'index' - therefore representing a partial
           derivative regarding x with index 'index'.";
\[Theta]::usage = "The non-commutative parameter theta.";
uv::usage = "Symbolizes a unit vector.";
IndexStyle::usage = "IndexStyle[expr] transfers a given expression into a
           form with regular Times products, thereby writing all indices in
           an explicit form. Note that this deactivates the automatic
           simplification of vectorial expressions. Use VectorStyle to
           retransform expressions into normal Dot and VCross syntax, and
           reactivate the auto-simplification.";
VectorStyle::usage = "VectorStyle[expr] activates the automatic vector
           simplification rules, and transforms the given expression into
           a format writing Dot and VCross products wherever possible.";
PowerCount::usage = "PowerCount[expr, var, lim] gives the effective power of
           var in expression expr for the limit lim. For the latter one 0 and
           \[Infinity] are supported. This function only works on 'simple'
           expressions, i.e. the output of Expand[].Note that PowerCount[0,..]
           =-\[Infinity]."
DivergentPart::usage = "DivergentPart[expr,var] gives a list of the summands
           of an expanded expression for expr whose power counting gives a
           degree of >=-4 in var."
Int::usage = "Int[expr[,np]] is a very specialized function applicable only
```

```
              in the p^-2 gauge model and gives a list of divergent summands
              integrated out according to known integral replacements.\n The
              optional parameter np defines if planar (np=0) or non-planar (np=1)
              results shall be computed.\nRegarding the cutoffs, M->0 is the
              prefactor of \[Alpha] in the exponent and is a mass,
              \[CapitalLambda]->\[Infinity] is the inverse prefactor of
              1/\[Alpha] and is a cutoff."
  IntVert::usage = "IntVert[expr] performs moemntum integrations by utilizing
              known integral replacements.\n Regarding the cutoffs, M->0 is
              the prefactor of \[Alpha] in the exponent and is a mass,
              \[CapitalLambda]->\[Infinity] is the inverse prefactor of
              1/\[Alpha] and is a cutoff."
  NumEquiv::usage = "NumEquiv[expr1, expr2] returns a 4x4 matrix corresponding
              to the numerical differences between expressions.
              I.e. a zero-matrix means equality."
  Humanize::usage = "Humanize[expr [,mode]] collects terms and tries to reshape
              a given equation. The optional parameter mode defines which terms
              are pulled out.\n 1: no variables are pulled out explicitely,\n 2:
              only alpha,\n 3: first alpha, then KDelta[mu,nu], \n, 4: first
              KDelta[mu,nu], then alpha,\n, 5: alpha, then KDelta[mu,nu], then
              k_mu k_nu\nThe default value is 1.";
  SizeBrackets::usage = "SizeBrackets[expr] sizes brackets comparable to
              \\left( and \\right) in TeX. \nWARNING: The output of this
              function is for display purposes only and cannot be taken as an
              input to any further calculation!.\n";
  ColorBrackets::usage = "ColorBrackets[expr] colors each bracket level
              differently and sizes brackets comparable to \\left( and \\right)
              in TeX. \nWARNING: The output of this function is for display
              purposes only and cannot be taken as an input to any further
              calculation!.\n";

(* turn off an annoying warning                                             *)
Off[General::spell1];

(***** Start of functional code *****)
(*Enable memory sharing for all expressions                                 *)
Share[];


(*Listof defined Vectors                                                    *)
DefinedVectors = {uv};                          (* unit vector uv           *)

(* enable global simplification and contraction of indices                  *)
iSimplifyIndices = 1;

(* List administration functions                                            *)
DefVec       := (DefinedVectors = Union[Append[DefinedVectors, #]];) &;
UDefVec      := (DefinedVectors=If[#!=uv,
                         DeleteCases[DefinedVectors,#],DefinedVectors];)&;
ClearDefVec  := (DefinedVectors = {uv});

(* internal conversion functions for simprule and VSimplify                 *)
                                   (* defined vector-> vec[def. vector]     *)
iVectorize[expr_]  := expr /. (k_ :> vec[k] /; MemberQ[DefinedVectors, k]);
iDeVectorize[expr_]:= expr //. (vec[a_] :> a); (* inverse fun. of iVectorize *)

(* Questioning function for vector definitions *)
IsVec[vec[k_]] :=IsVec[k];                      (* resolvation of vec       *)
IsVec[Subscript[k_, i_]]:=MemberQ[DefinedVectors,k]; (* index awareness     *)
IsVec[k_]      := MemberQ[DefinedVectors, k];   (* standard expressions     *)
IsVec[k_*p_]   := (IsVec[k] || IsVec[p]);       (* product arguments        *)
```

```
IsVec[Dot[_,_]]:= False;                          (* dot products-> scalars  *)
IsVec[k_ + p_] := (IsVec[k] && IsVec[p]);         (* sum arguments           *)

(* Definition of the customized Cross product*)
VCross[_, 0]           := 0;                              (* zero arguments  *)
VCross[0, _]           := 0;
VCross[n_.*a_, m_.*a_] := 0/;(IsVec[iDeVectorize[a]]);            (*linearity *)
                                    (* ordering of the arguments according   *)
                                    (* to their signature -> antisymmentry   *)
VCross[vec[k_],vec[p_]]:= (-1)*Unevaluated[
                               VCross[vec[p],vec[k]]]/;(Signature[{k, p}]!=1);
                                    (* if given vectors (lists) use the      *)
                                    (* standard built-in function            *)
VCross[a_, b_] := Cross[a, b] /; (MemberQ[a, List] && MemberQ[b, List]);
Protect[VCross];

(* Modification of the vector dot product                                    *)
(* Defining the attribute ' Orderless',  associativity and transitivity      *)
(* are defined for the first and second argument instantaneously            *)

Unprotect[Dot];
SetAttributes[Dot, {Orderless, Flat}];   (* commutativity and associativity  *)
Dot[0, _] := 0;                          (* zero argument                    *)
Protect[Dot];

(* Modification of the Cross Product                                         *)
Unprotect[Cross];
Cross[vec[k_], vec[p_]] := VCross[vec[k], vec[p]]; (*use VCross for defd. v. *)
Cross[k_, p_]           := VCross[k, p] /; (IsVec[k] && IsVec[p]);
Protect[Cross];

(* Simplification transformation rules:                                      *)
(* simprule expands all terms until only atomic arguments are found.         *)
(* comprule does some compression by applying inverse rules to simprule.     *)

(* dot product rules                                                         *)
simprule[Dot[vec[k_], vec[p_]]]      := Sort[Unevaluated[Dot[vec[k],vec[p]]]];
simprule[Dot[a_, b_ + d_]]           := simprule[Dot[a,b]]+simprule[Dot[a,d]];
simprule[HoldPattern[Dot[a_*vec[k_], b_.*vec[p_]]]] := a*b*Dot[vec[k], vec[p]];
simprule[Power[vec[k_],n_?EvenQ]]    := simprule[Power[Dot[vec[k],
                                         vec[k]],n/2]]/;IsVec[iDeVectorize[k]];
simprule[Subscript[Power[vec[k_],n_?EvenQ],i_]]     := simprule[Power[Dot[
                                   vec[k],vec[k]],n/2]]/;IsVec[iDeVectorize[k]];

(* cross product rules                                                       *)
simprule[HoldPattern[VCross[a_,c_+d_]]] := simprule[VCross[a,c]]+simprule[VCross[a,d]];
simprule[HoldPattern[VCross[a_+b_,c_]]] := simprule[VCross[a,c]]+simprule[VCross[b,c]];
simprule[HoldPattern[VCross[a_*k_,p_]]] := a*simprule[VCross[k, p]]/;(
                              IsVec[iDeVectorize[k]] && IsVec[iDeVectorize[p]]);
simprule[HoldPattern[VCross[k_,b_*p_]]] := b*simprule[VCross[k, p]]/;(
                              IsVec[iDeVectorize[k]] && IsVec[iDeVectorize[p]]);
simprule[HoldPattern[VCross[a_,b_]]]    := 0 /; (
                           !(IsVec[iDeVectorize[a]] && IsVec[iDeVectorize[b]]));

(* index rules                                                               *)
simprule[Subscript[(vec[k_]), i_]*Subscript[(vec[p_]), i_]]:=vec[k].vec[p];
simprule[HoldPattern[Subscript[(a_.*vec[k_]+b_),i_]]]:=a*Subscript[(vec[k]),i]
                             +simprule[Subscript[b, i]]/;IsVec[iDeVectorize[b]];
simprule[HoldPattern[Subscript[(a_*vec[p_]), i_]]]   :=a*Subscript[(vec[p]), i];
simprule[HoldPattern[Subscript[k_,i_]*Subscript[p_,i_]]]:=k.p/;(
                               IsVec[iDeVectorize[k]]&&IsVec[iDeVectorize[p]]);
```

```
simprule[HoldPattern[Subscript[vec[k_],i_]*Subscript[vec[p_],i_]]]:=vec[k].vec[p];
simprule[HoldPattern[KDelta[i_, j_]^n_?IntegerQ]]    := KDelta[i, i]/;n>1;

(* general term parsing                                                 *)
simprule[a_ + b_]      := simprule[a] + simprule[b];
simprule[a_ * b_]      := simprule[a] * simprule[b];
simprule[fun_[args_]] := fun[simprule[args]];
simprule[expr_]        := expr;

(* simprule is associative and projective                               *)
SetAttributes[simprule, {Flat, Listable, OneIdentity}];

(* compression of dot products                                          *)
comprule[HoldPattern[Dot[vec[k_], vec[p_]] + Dot[vec[m_], vec[p_]]]] :=
                                                  Dot[vec[k] + vec[m], vec[p]];

(* compression of indices for atomic vector sums                        *)
comprule[HoldPattern[a_?NumericQ*Subscript[(vec[k_]), i_]
              +b_?NumericQ*Subscript[(vec[p_]), i_]]]
                      :=Subscript[(a*vec[k]+b*vec[p]),i]/;IsVec[iDeVectorize[p]];

(* compression of cross products                                        *)
comprule[HoldPattern[Dot[vec[k_], vec[p_]] + Dot[vec[m_], vec[p_]]]]
                      := Dot[vec[k] + vec[m], vec[p]];
comprule[HoldPattern[a_?NumericQ*Subscript[(vec[k_]), i_]
              +b_?NumericQ*Subscript[(vec[p_]), i_]]]
                      :=Subscript[(a*vec[k]+b*vec[p]), i]/;IsVec[iDeVectorize[p]];
comprule[Plus[HoldPattern[(a_:1)*VCross[vec[k_],vec[p_]]],
              HoldPattern[(b_:1)*VCross[vec[m_],vec[p_]]]]]
                      :=VCross[a*vec[k]+b*vec[m],vec[p]]/;(NumericQ[a]&&NumericQ[b]);
comprule[Plus[HoldPattern[(a_:1)*VCross[vec[k_],vec[p_]]],
              HoldPattern[(b_:1)*VCross[vec[k_],vec[m_]]]]]
                      :=VCross[vec[k],a*vec[p]+b*vec[m]]/;(NumericQ[a]&&NumericQ[b]);
comprule[Plus[HoldPattern[(a_:1)*VCross[vec[k_],vec[p_]]],
             HoldPattern[(b_:1)*VCross[vec[p_],vec[m_]]]]]
                      :=VCross[vec[p],b*vec[m]-a*vec[k]]/;(NumericQ[a]&&NumericQ[b]);
comprule[Plus[HoldPattern[(a_:1)*VCross[vec[k_],vec[p_]]],
             HoldPattern[(b_:1)*VCross[vec[m_],vec[k_]]]]]
                      :=VCross[vec[k],a*vec[p]-b*vec[m]]/;(NumericQ[a]&&NumericQ[b]);

(* comprule is associative and projective ( comprule^n = comprule )     *)
SetAttributes[comprule, {Flat, Listable, OneIdentity}];

(* Activate Mathemaitca-internal optimizations to speed up the rules    *)
Dispatch[simprule]; Dispatch[comprule];

(* Index rules                                                          *)
(* These rules are valid at any time, thus, they are applied whenever   *)
(* Mathematica computes an expression. Some index transformations can be *)
(* deactivated by the global switch iSimplifyIndices set to 0           *)
Unprotect[Times];
Unprotect[Plus];
Unprotect[Power];
Subscript[(a_*p_), i_] := a*Subscript[p, i]/;((!IsVec[a])&&IsVec[p]);
Subscript[k_, i_]*Subscript[p_, i_]^:=k.p/;(IsVec[k]&&IsVec[p]
                                          &&(iSimplifyIndices==1));
Subscript[(k_+p_), i_]^:= Subscript[k, i]+Subscript[p, i]/;(IsVec[k]&&IsVec[p]);
Subscript[(vec[k_]+vec[p_]), i_]^:= Subscript[vec[k], i]
                                    +Subscript[vec[p], i]/;(IsVec[k]&&IsVec[p]);
Subscript[(a_*vec[p_]), i_]:=a*Subscript[vec[p], i]/;(!(isVec[iDeVectorize[a]]));
                                (Subscript[k_, i___])^(n_?EvenQ)^:=(k.k)^(n/2)/;
```

```
                                (isVec[iDeVectorize[k]]&&(iSimplifyIndices==1));

Subscript[0, i_]       := 0;
Subscript[uv, ___]^(n_?EvenQ)^:=1;
Subscript[uv, i_]*Subscript[uv, j_]:=KDelta[i,j]/4;
uv.uv                 ^:= 1;
Subscript[vec[0], _]   := 0;
vec[0]                 := 0;
Subscript[vec[k_], i_]*Subscript[vec[p_], i_]
                      ^:=vec[k].vec[p]/;(iSimplifyIndices==1);
Subscript[(vec[k_]+vec[p_]), i_]^:=Subscript[vec[k], i]+Subscript[vec[p], i]
(Subscript[vec[k_], i___])^(n_?EvenQ)
                      ^:=(vec[k].vec[k])^(n/2)/;(iSimplifyIndices==1);
(*** removed due to dramatic inc. in computing time for fun. such as Expand
a_*Subscript[vec[k_], i1_]*Subscript[vec[k_], i2_]*Subscript[\[Theta],i1_,i2_]
                      ^:=0/;(IsVec[iDeVectorize[k]]&&(iSimplifyIndices==1));
a_*Subscript[vec[k_], i1_]*Subscript[vec[k_], i2_]*Subscript[\[Theta],i2_,i1_]
                      ^:=0/;(IsVec[iDeVectorize[k]]&&(iSimplifyIndices==1));
a_*Subscript[k_, i1_]*Subscript[k_, i2_]*Subscript[\[Theta],i1_,i2_]
                      ^:=0/;(IsVec[iDeVectorize[k]]&&(iSimplifyIndices==1));
a_*Subscript[k_, i1_]*Subscript[k_, i2_]*Subscript[\[Theta],i2_,i1_]
                      ^:=0/;(IsVec[iDeVectorize[k]]&&(iSimplifyIndices==1));
***)
Protect[Power];
Protect[Plus];
Protect[Times];
Protect[uv];

(* Definition of an alternate Kronecker Delta                                *)
(* The sum convention is explicitely built in                                *)
    KDelta[i_, j_] /; i != j  := KroneckerDelta[i, j];
    KDelta[i_, j_] /; i === j := 4;
    SetAttributes[KDelta, Orderless];
    KDelta[i_, j_]*KDelta[j_, k_]       ^:= KDelta[i, k];
    KDelta[i_,j_]*Subscript[vec[p_], j_]^:= Subscript[vec[p], i];
    KDelta[i_,j_]*Subscript[p_, j_]     ^:= Subscript[p, i]/;IsVec[p];
    Power[KDelta[i_,j_],n_?IntegerQ]    ^:= KDelta[i,i]/;n>1;
    KDelta[i_,j_]*Subscript[o_,j_,k_]   ^:= Subscript[o,i,k];
    KDelta[i_,j_]*Subscript[o_,k_,j_]   ^:= Subscript[o,k,i];
Protect[KDelta];

(* Wrapper for FullSimplify                                                  *)
(* Expression are first 'vectorized', i.e. defined vectors written explicitly*)
(* in the vec[ ] syntax. Then the simprule expansion rules are applied and   *)
(* the results are fully simplified and contracted using comprule. Finally,  *)
(* the explicit vec[ ] syntax is transformed back to standard syntax.        *)
(* This function assures that the vectorial transformations are considered in*)
(* both directions                                                           *)
VSimplify := iDeVectorize[
               FullSimplify[
                 Together[FactorTerms[simprule[ iVectorize[#1]]]],
                 TransformationFunctions -> {Automatic,comprule}, ##2]] &;
Protect[VSimplify];

(* VectorStyle and IndexStyle                                                *)
(* These helper functions change the display style of expressions to be      *)
(* either with explicit indices or in dot/cross product form. They are       *)
(* inherently used by VSeries and VLimit functions.                          *)
(* Remark: Note that IndexStyle sets the global iSimplifyIndices to 0 which  *)
(*         disables automatic simplification of indexed vector structures.   *)
(*         To reactivate this feature use iSimplifyIndices=1 or VectorStyle. *)
```

```
Unprotect[VectorStyle];Clear[VectorStyle];
Unprotect[IndexStyle];Clear[IndexStyle];
VectorStyle[arg_:  1] := Module[{tmp},
   (* subfunction: expand all arguments *)
   RExpand[fun_]:=fun/.{f_[arrg__]:>f@@Expand[RExpand[#]&/@{arrg}]};
   tmp = RExpand[arg]//.{Times[Subscript[k1_,i1_], Subscript[k2_,i2_],
                                Subscript[\[Theta],i1_,i2_]]:>VCross[k1,k2]};
   iSimplifyIndices = 1;
   Evaluate[ReleaseHold[tmp]]
];
IndexStyle[arg_]:=Module[{i,pos,tmp},
     iSimplifyIndices=0;
     (* subfunction *)
     iindexrep[Dot[p_,q_]^(n_:1)]/;((IsVec[iDeVectorize[p]])&&(IsVec[iDeVectorize[q]]))
        :=Module[{iind,tmp2,j},
              tmp2 = 1;
              For[j=0, j<n, j++,
                tmp2*=HoldForm[Subscript[p, iind]*Subscript[q, iind]]
                        /.iind->Symbol["\[Eta]"<>ToString[i]];
              i=i+1;];
              tmp2
          ];
     iindexrep[VCross[p_,q_]]/;((IsVec[iDeVectorize[p]])&&(IsVec[iDeVectorize[q]]))
        :=Module[{iind,jind,tmp3},
               tmp3 = Unevaluated[Subscript[p, iind]*Subscript[\[Theta], iind,jind]
                        Subscript[q, jind]]/.{iind->Symbol["\[Eta]"<>ToString[i]],
                                              jind->Symbol["\[Eta]"<>ToString[i+1]]};
               i=i+2;
               tmp3
          ];
     i = 1;
     pos = Position[arg,(Dot[p_,q_]^(n_?IntegerQ)/;n>1)];
     tmp = MapAt[iindexrep,arg,pos];
     pos = Position[tmp,Dot[p_,q_]];
     tmp = MapAt[iindexrep,tmp,pos];
     pos = Position[tmp,VCross[k_,p_]];
     MapAt[iindexrep,tmp,pos]
];


(* Partial Derivative \partial_\mu as replacement for Mathematica D          *)
                                     (*       use standard D for non-vectors *)
VD[arg_, p_, ind_]      := D[arg, p] /; (! IsVec[iDeVectorize[p]]);
VD[arg_?NumericQ, _, _] := 0;        (*      the derivative of a scalar is 0 *)
VD[arg_, p_, _]         := 0 /; (! MemberQ[Level[arg, {-1}], p]);
                                     (* basic rule for derivative of a vector*)
VD[Subscript[(vec[p_]), i_],p_,ind_]    :=KDelta[i,ind];
VD[Subscript[p_, i_],p_,ind_]           :=KDelta[i,ind]/;IsVec[p];
                                     (* Leibnitz rule                        *)
VD[a_*b_, p_, ind_]                     := VD[a, p, ind]*b + a*VD[b, p, ind];
                                     (* Leibnitz rule for the dot product    *)
VD[HoldPattern[Dot[a_, b_]], p_, ind_]  := VectorStyle[VD[ReleaseHold[
                                  IndexStyle[Distribute[Dot[a, b]]]], p, ind]];
                                     (* transitivity regarding sum           *)
VD[a_ + b_, p_, ind_]                   := VD[a, p, ind] + VD[b, p, ind];
                                     (* Leibnitz rule for the cross product  *)
VD[VCross[a_, b_], p_, ind_]            := VectorStyle[VD[ReleaseHold[
                                  IndexStyle[Distribute[VCross[a, b]]]], p, ind]];
                                     (* Exp[] needs special treatment since  *)
                                     (* it is instantaneously converted to   *)
                                     (* Power[E, arg] which has different    *)
```

```
                                      (* derivation rules, however.           *)
VD[HoldPattern[Power[E, arg_]], p_, ind_] := VD[Unevaluated[Exp[arg]], p, ind];
                                      (* treatment for unevaluatable functions*)
VD[HoldForm[arg_], p_, ind_]             := HoldForm[VD[arg, p, ind]];
                                      (* derivative of a generic function     *)
VD[HoldPattern[fun_[arg__]], p_, ind_] /; (MemberQ[Level[{arg}, {-1}], p]) :=
    Module[{tmp=0, rep, icnt, targ=arg},
      If[(ToString[Definition[fun]]=="Null"),
        tmp=HoldForm[VD[rep,p,ind]]/.rep->Evaluate[If[Length[{targ}]>0,fun[targ],fun]],
        (*else*)
        For[icnt=1, icnt<=Length[{targ}], icnt++,
          tmp += ((D[fun[targ]/.{targ}[[icnt]]->rep, rep]/.rep->{targ}[[icnt]])*
                                                  VD[{targ}[[icnt]], p, ind]);
        ](*end for*)
      ];(*end if*)
      tmp
    ]
SetAttributes[VD,Listable];
Protect[VD];


(* Taylor Series expansion for vectorial objects                            *)
(* Remark: This function is not generally applicable since it leaves out any *)
(*         sin^2 expressions, multiplying them to the final result.         *)
VSeries[expr_,{p_,p0_,ord_},verb_:0]:=
(* Set the global Mathematica variable RecursionLimit.                      *)
(* Necessary for lengthy expressions taken to a limit (to incorp. simprule)  *)
    Module[{smex,zro,tmp,iind,sinfac,fac,o,APhases,i,aa,bb,nn},
      Off[General::spell1];
      $RecursionLimit=Infinity;
      CorrectPowerIndices[iexpr_, pwr_]:=(Dot[iexpr//.{Subscript[pp_,i_]/;
           IsVec[pp]->pp},iexpr//.{Subscript[pp_,i_]/;IsVec[pp]->pp}])^(pwr/2);
      zro[_] := 0;
      smex = Array[zro,ord+1];
      (* get all phase factors *)
      APhases = Cases[Level[(expr//.{(aa_:1)*(-1+Cos[bb_])->-2*aa*
                                          Sin[bb/2]^2}),1],Sin[_]^(n_:1)];
      (* treat only the rest *)
      o = ReleaseHold[iVectorize[iDeVectorize[expr//.{(aa_:1)*Sin[bb_]^(nn_:1)->aa,
                              (aa_:1)*(-1+Cos[bb_])^(nn_:1)->(-2)^(nn)*aa}]]];
      i = 0; fac = 1;
      (* take the limit in order 0 *)
      smex[[1]] = VLimit[o,p->p0,1];
      If[verb==1, Print["Order 0:"];Print[iDeVectorize[smex[[1]]]];];
      (* check if the limit was finited *)
      If[Position[smex[[1]],DirectedInfinity]!={},
         Print["ERROR: Limit returned an infinite result, Series expansion
                aborted, result up to now:"],
         For[i=1, i<=ord, i++,
           o = VD[o,p,Symbol["\[Tau]"<>ToString[i]]];
           iSimplifyIndices=1;
           If[o==0, Break[];];
           fac *= Subscript[(vec[p]-vec[p0]),iind]/.iind->Symbol["\[Tau]"<>ToString[i]];
           smex[[i+1]] = VLimit[o,p->p0,1]*fac/i!;
           ClearSystemCache[];
           If[verb==1, Print["Order "<>ToString[i]<>":"];
                     Print[iDeVectorize[smex[[i+1]]]];];
           If[Position[smex[[i+1]], DirectedInfinity]!={},
             Print["ERROR: Limit returned an infinite result, Series expansion
                   aborted, result up to now:"];Break;
           ];
```

```
          ];
       ];
       iSimplifyIndices=1;
       If[Position[smex,DirectedInfinity]!={},
         ((ReleaseHold/@(iDeVectorize/@(smex*Times@@APhases)))//.
           (Power[Subscript[(k_),i_],(n_?EvenQ)]->(k.k)^(n/2)))/.
           {Power[Subscript[k_, i_],(n_?EvenQ)]/;IsVec[k]:>(k.k)^(n/2),
                            Power[k_,(n_?EvenQ)]/;IsVec[k]:>CorrectPowerIndices[k,n]},
      (VSimplify[#,TimeConstraint->300,ExcludedForms->
            {((_).(_)+Divide[a^2,_])^(n_:1)}]&/@((ReleaseHold/@(iDeVectorize/@
            (smex*Times@@APhases)))//.(Power[Subscript[k_, i_],(n_?EvenQ)]->
            (k.k)^(n/2))))
           /.{Power[Subscript[k_, i_],(n_?EvenQ)]/;IsVec[k]:>(k.k)^(n/2),
          Power[k_,(n_?EvenQ)]/;IsVec[k]:>CorrectPowerIndices[k,n]}
       ]
     ];

(* experimental multi-variable version of VSeries                              *)
Unprotect[VSeries]; Clear[VSeries];
(* experimental multi-variable version of VSeries                              *)
VSeries[expr_,{{p__},{p0__},ord_},verbose_:0]:=
(* Set the global Mathematica variable RecursionLimit.                         *)
(* Necessary for lengthy expressions taken to a lim (to incorporate simprule)*)
    Module[{Plexpr,zro,uty,AOrds,APhases,Phas,ADerT,AFacT,ANumF,ALastDerT,uu,
                                    vv,ii,iOrd,iTrm,iDer,iTmp,iCnt,iind,fac,o},
       Off[General::spell1];
       $RecursionLimit=Infinity;
       VSeries::limerr="Limit returned an infinite result, Series expansion
                                       aborted, result up to now found below.";
       VSeries::paramerr="VSeries called with an unequal number of arguments
              for dependent variables {'1'} and expancion center points {'2'}.";
       (*input parameter check*)
       If[Length[{p}]!=Length[{p0}],Message[VSeries::paramerr,{p},{p0}],
          zro[_]:=0;uty[_]:=1;
          (*replace and remember phase factors*)
          APhases=Cases[Level[(expr//.{(aa_:1)*(-1+Cos[bb_])
                                         ->-2*aa*Sin[bb]^2}),1],Sin[_]^(n_:1)];
          If[APhases!={},
             If[verbose==1, Print["Phases: "];Print[APhases]];
             Phas  = Times@@APhases;
             Plexpr= ReleaseHold[iDeVectorize[DeleteCases[expr,Sin[_]^(n_:1)]]]
          ,
             Phas  = 1;
             Plexpr= ReleaseHold[iDeVectorize[expr]];
          ];(*end if APhases*)
          (*generate derivative expressions and prepare factors*)
          AOrds         = Array[0&,ord+1];
          ADerT         = Array[1&,ord+1];
          ALastDerT     = Array[1&,ord+1];
          ANumF         = Array[1&,ord+1];
          AFacT         = Array[1&,ord+1];
          ADerT[[1]]    = {Plexpr};
          ALastDerT[[1]]= {1};
          AFacT[[1]]    = {1};
          ANumF[[1]]    = {1};
          (*loop over orders 1..n*)
          For[iOrd=1, iOrd<=ord, iOrd++,
             (*allocate the lists for the current order*)
             Clear[iTmp];
             iTmp=ToExpression[NumTerms[iOrd,Length[{p}]]];
             (* the number of terms for non-commutative limits is the number  *)
```

```
            (* of terms in the previous order times the number of derivatives*)
            iTmp = Length[{p}]*Length[ADerT[[iOrd]]];
            ADerT[[iOrd+1]]    = Array[1&,iTmp];
            ALastDerT[[iOrd+1]]= Array[1&,iTmp];
            AFacT[[iOrd+1]]    = Array[1&,iTmp];
            ANumF[[iOrd+1]]    = Array[1&,iTmp];
            (*loop over terms in next lower order (for building the tree)    *)
            iCnt = 1;
            For[iTrm=1, iTrm<=Length[ADerT[[iOrd]]], iTrm++,
               For[iDer=1, iDer<=Length[{p}], iDer++;iCnt++,
                  ADerT[[iOrd+1,iCnt]] = VD[ADerT[[iOrd,iTrm]],{p}[[iDer]],
                            Symbol["\[Tau]"<>ToString[iOrd]<>ToString[iDer]]];
                  AFacT[[iOrd+1,iCnt]] = AFacT[[iOrd,iTrm]]*(Subscript[({p}[[iDer]]),
                            iind]-Subscript[({p0}[[iDer]]), iind])
                            /.iind->Symbol["\[Tau]"<>ToString[iOrd]<>ToString[iDer]];
                  AFacT[[iOrd+1,iCnt]] = AFacT[[iOrd+1,iCnt]]
                            //.Subscript[num_?NumericQ,_]->num;
               ];
            ];
            If[verbose==1,Print["Order "<>ToString[iOrd]<>" Derivatives: "];
                                                   Print[ADerT[[iOrd+1]]]];
         ];
         (* add the limits to each term *)
         i = 0; fac = 1;
         For[iOrd=1, iOrd<=ord+1, iOrd++,
            For[iTrm=1,iTrm<=Length[ADerT[[iOrd]]],iTrm++,
               AOrds[[iOrd]] += Fold[VLimit[#1,{p}[[#2]]->{p0}[[#2]]]&,
                                   ADerT[[iOrd,iTrm]],
                                   Table[jj,{jj,1,Length[{p}]}]]*AFacT[[iOrd,iTrm]];
               If[Position[AOrds[[iOrd]],DirectedInfinity]!={},
                                                 Message[VSeries::limerr]];
            ];
            AOrds[[iOrd]] = 1/(iOrd!)*Phas*Simplify[Plus@@AOrds[[iOrd]]];
         ];
         (* output *)
         AOrds
      ](*end if Length[{p}]!=Length[{p0}]*)
   ](*end Module VSeries*)


(* Vector-aware limit                                                      *)
VLimit[expr_,pxx_->plxx_,vector_:0,TimeConstr_:900]:=
    Module[{tmp,tmp2,out},
      Off[General::spell1];
      If[(ToString[Head[expr]]==ToString[HoldForm]),
        out = expr/.HoldPattern[HoldForm[tmp_]]->HoldForm[VLimit[tmp,pxx->plxx,vector]],
        (* else *)
        If[(ToString[Definition[Evaluate[Head[expr]]]]=="Null"),
          out=HoldForm[VLimit[expr,pxx->plxx,vector]],
          (*else*)
          iSimplifyIndices=0;
          (* try to perform the limit by a simple replacement and check for  *)
          (* any errors which would indicate that Limit has to be used.      *)
          Quiet[Check[
            tmp2=(ReleaseHold[IndexStyle[iDeVectorize[simprule[
            iVectorize[iDeVectorize[expr]/.{(k_)^n_?EvenQ/;IsVec[k]->(k.k)^(n/2)}]
            ]]]]/.{Subscript[pxx, i_]->pxx*Subscript[uv, i]})//.pxx->plxx,
            (* if check did not succeed use limit                            *)
            TimeConstrained[
              Print["try1"];tmp=ReleaseHold[IndexStyle[iDeVectorize[
              simprule[iVectorize[iDeVectorize[expr]/.{(k_)^n_?EvenQ
```

```
                /;IsVec[k]->(k.k)^(n/2)}]]]]]
               /.{Subscript[pxx, i_]->pxx*Subscript[uv, i]};
               tmp2 = Limit[tmp,pxx->plxx],TimeConstr,Print["try2"];
               tmp  = ReleaseHold[IndexStyle[iDeVectorize[simprule[
                  Expand[iVectorize[iDeVectorize[expr]
                  /.{(k_)^n_?EvenQ/;IsVec[k]->(k.k)^(n/2)}]]
                  ]]]]/.{Subscript[pxx, i_]->pxx*Subscript[uv, i]};
               tmp2 = Map[Limit[#,pxx->plxx]&,tmp,1]
             ] (* end TimeConstrained *)
           ]]; (* end check *)
           iSimplifyIndices = 1;
      If[IsVec[plxx],
             Print[tmp2];
             tmp2=tmp2//.{Times[v_,Subscript[uv,ind_]]/;IsVec[v]->Subscript[v, ind]}
           ];
           On[General::spell1];
           out= If[(Position[tmp2,DirectedInfinity]!={}),
                  If[vector>0,iVectorize[tmp2],tmp2,tmp2],
                  If[vector>0,iVectorize[Simplify[tmp2]],
                       Simplify[tmp2],VLSimplify[tmp2,0,TimeConstraint->300]]
                ];
           ];
         ];(*end if*)
     out]

Protect[VSeries];
Protect[VLimit];

(* Check for equivalence of two expressions                                *)
(* All trigonometric functions are replaced to their value at Pi/4, all    *)
(* vectors are replaced by discrete prime-numerical four-vectors. Indices \mu*)
(* and \n u are run from 0 to 3 and the results of both given expressions are*)
(* evaluated and compared. Equivalence is given if the result is a         *)
(* zero-matrix                                                             *)
NumEquiv[expr1_, expr2_] :=
    Module[{i, pos},
      Off[Part::"pspec"];
      tmp1 = StandardForm[VectorStyle[expr1]] /.{Sin[_] -> Sqrt[3]/2, Cos[_]
                              ->-1/2, Power[k_?IsVec, n_?EvenQ] -> (k.k)^(n/2)};
      tmp2 = StandardForm[VectorStyle[expr2]] /.{Sin[_] -> Sqrt[3]/2, Cos[_]
                              ->-1/2, Power[k_?IsVec, n_?EvenQ] -> (k.k)^(n/2)};
      numerize[Subscript[a_, i_]] := a[[i]];
      numerize[Subscript[a_, i_, j_]] :=
        a[[i, j]];(*numerize[Dot[a_, b_]] /; {! (Head[b] === Transpose)} :=
            Dot[Transpose[a], b];*)
      numerize[expr_] := expr;
      i = 1;
      indx := (i += 1);
      numrep[expr_] := (expr -> Array[Prime[10*indx + #] &, 4]);
      replacelst = numrep /@ DefinedVectors;
      res1 = ToExpression[ToBoxes[
               Table[Evaluate[numerize //@ (tmp1 /. replacelst)],
                     {\[Mu], 1, 4}, {\[Nu], 1, 4}]]]];
      res2 = ToExpression[ToBoxes[
               Table[Evaluate[numerize //@ (tmp2 /. replacelst)],
                     {\[Mu], 1, 4}, {\[Nu], 1, 4}]]]];
      On[Part::"pspec"];
      MatrixForm[N[Simplify[(res1 - res2)]]]
      ];
Protect[NumEquiv];
```

```
(* Collects terms like deltas, alpha, a, or potences of momenta              *)
Humanize[expr_,mode_:1]:=
    Module[{tmp},
       pull:=a_^(n_:1)*b_+a_^(m_:1)*c_->If[n<m,a^n*(b+c*a^(m-n)),a^m*(b*a^(n-m)+c)];
       tmp = FullSimplify[(Collect[#,{\[Alpha]^(n_:1),KDelta[\[Mu],\[Nu]],
                        Subscript[k, \[Mu]]*Subscript[k, \[Nu]],a^(n_:1)}]&
               //@(expr//.{(b_:1)*(a_^2+(Dot[k_,k_])^2)^(n_:1)->b*(k.k)^n*
                 (k.k+a^2/k.k)^n,a_^(n_:1)*b_+a_^(m_:1)*c_->If[n<m,a^n*
                 (b+c*a^(m-n)),a^m*(b*a^(n-m)+c)]})),
               TransformationFunctions->{pull,simprule,Collect[#,
                 {\[Alpha]^(n_:1)}]&,Collect[#,{KDelta[\[Mu],\[Nu]]}]&,Automatic},
               ExcludedForms->(Dot[k_,k_]+a^2/Dot[k_,k_])^(n_:1)];
       Switch[mode,
        1,tmp/.Dot[k_,k_]^(n_:1)->k^(2*n),
        2,(Collect[#,\[Alpha]^(n_:1)]&//@(tmp/.Dot[k_,k_]^(n_:1)->k^(2*n)))//.pull,
        3,(Collect[#,{\[Alpha]^(n_:1),KDelta[\[Mu],\[Nu]]}]&//@
                                    (tmp/.Dot[k_,k_]^(n_:1)->k^(2*n)))//.pull,
        4,(Collect[#,{KDelta[\[Mu],\[Nu]],\[Alpha]^(n_:1)}]&//@
                                     tmp/.Dot[k_,k_]^(n_:1)->k^(2*n))//.pull,
        5,(Collect[#,{\[Alpha]^(n_:1),KDelta[\[Mu],\[Nu]],Subscript[k, \[Mu]]
           *Subscript[k, \[Nu]]}]&//@tmp/.Dot[k_,k_]^(n_:1)->k^(2*n))//.pull]];
Protect[Humanize];

(* Conducts a power counting for a given variable and a given limit          *)
(*(0 or \infty supported) for simple expressions                             *)
PowerCount[expr1_, var_, lim_] :=
    Module[{tmp1,numr,denom},
      Off[Part::"pspec"];
      tmp1 = expr1//.{ Dot[a_,b_]->a*b,Subscript[var, i_]->var};
      numr =Exponent[Numerator[tmp1],var];
      denom=Exponent[Simplify[If[lim==0,Denominator[tmp1]//.{((c_:1)*var^(n_:1)+b_)->1},
              If[lim==\[Infinity],Denominator[tmp1]
                 //.{((c_:1)*var^(n_:1)+b_)->var^n}]]],var];
    numr-denom
];

(* Extracts the divergent part with power counting >=-4 of a given expression*)
DivergentPart[expr_,var_]:=
   Module[{tmp,mask},
     tmp=Expand[expr];
     If[ToString[Head[tmp]]=="Plus",
       mask=Select[Level[tmp,1],PowerCount[#,var,\[Infinity]]>=-4&],
       mask=Select[{tmp},PowerCount[#,var,\[Infinity]]>=-4&]]
     ];
Protect[PowerCount];
Protect[DivergentPart];

Clear[VSymmetricQ];
Clear[SubPwrCnt];
(* subfunction as wrapper for powercount to treat trigonometric functions    *)
SubPwrCnt[trm_,varr_]:=Module[{cntloc},
    cntloc=0;
    If[Position[trm,varr]!={},
      If[Head[trm]===Sin||Head[trm]===Tan,
        cntloc+=PowerCount[Level[trm,1],
        varr,
        \[Infinity]][[1]],
        (*else*)
        If[Head[trm]===Power,cntloc+=(trm/.Power[a_,b_]:>b*SubPwrCnt[a,varr]),
          (*else*)
          cntloc+=PowerCount[trm,varr,\[Infinity]]
```

```
          ]
        ];
        (* workaround, should trigger a warning                              *)
        If[!NumericQ[cntloc],cntloc=0]]; cntloc];

      (* Helper function for reliable determination if an integrand is         *)
      (* symmetric or not *)
      VSymmetricQ[arg_,var_]:=Module[{ret},
          (*check for trivial result *)
          If[Position[arg,var]=={},ret=True];
          (*Thread over sums*)
          If[Head[arg]===Sum,ret=VSymmetricQ[#,var]&/@arg;];
          (*Separate parts of the argument and compute power for each*)
          EvenQ[Plus@@(SubPwrCnt[ReleaseHold[#],var]&/@(Level[IndexStyle[arg],1]
                                                      //.Subscript[var,_]:>var))]
      ];

  (* Special Integration rules based on known integral replacements. The       *)
  (* function acts on expanded lists of known integrations. Any Cos-1          *)
  (* expressions have to be brought in the corresponding form with Sin^2.      *)
  (* The limit a->0(old model) in the denom. of the expressions is implied     *)
  (* automatically.                                                            *)
  (* Cutoffs: M->0 is the prefactor of \[Alpha] in the exponent, a mass        *)
  (*          \[CapitalLambda]->\[Infinity] is the inverse prefactor of        *)
  (*            1/[Alpha], a cutoff                                            *)
  (* Current model: \[Lambda]->0                                               *)
  (* Factors 1/2 for planar and -1/2 for non-planar are included!              *)

  Int[expr_,np_:1]:=
     Module[{tmp,intnpdefs,intpnpdefs,elimrule},
       Off[Part::"pspec",General::"spell"];
       elimrule=(a^2+(k.k)^2)->(k.k)^2;
       (* non-planar integration replacement rules                             *)
       intnpdefs={
          (c_:1)*Subscript[k,\[Mu]_]*Subscript[k,\[Nu]_]*Subscript[k,\[Rho]_]
           *Subscript[k, \[Sigma]_]*Cos[(__)]/(k.k)^4/;
           (Position[c,Subscript[k,_]]=={})
          ->1/12 c \[Pi]^2 BesselK[0,Sqrt[M \[Theta]^2 p.p]] (KDelta[\[Mu],
            \[Sigma]] KDelta[\[Nu],\[Rho]]+KDelta[\[Mu],\[Rho]] KDelta[\[Nu],
            \[Sigma]]+KDelta[\[Mu],\[Nu]] KDelta[\[Rho],\[Sigma]])+(\[Pi]^2
            Subscript[p, \[Chi]1] Subscript[p, \[Chi]2] Subscript[p, \[Chi]3]
            Subscript[p, \[Chi]4] Subscript[\[Theta], \[Mu],\[Chi]1]
            Subscript[\[Theta],\[Nu],\[Chi]2] Subscript[\[Theta],\[Rho],\[Chi]3]
            Subscript[\[Theta], \[Sigma],\[Chi]4])/(6 \[Theta]^4 (p.p)^2)-1/
            (12 \[Theta]^2 p.p) \[Pi]^2 (KDelta[\[Rho],\[Sigma]] Subscript[p,\[Chi]1]
            Subscript[p,\[Chi]2] Subscript[\[Theta],\[Mu],\[Chi]1]
            Subscript[\[Theta],\[Nu],\[Chi]2]+KDelta[\[Nu],\[Sigma]]
            Subscript[p, \[Chi]1] Subscript[p, \[Chi]3]
            Subscript[\[Theta], \[Mu],\[Chi]1] Subscript[\[Theta],\[Rho],\[Chi]3]
            +KDelta[\[Mu],\[Sigma]] Subscript[p, \[Chi]2] Subscript[p, \[Chi]3]
            Subscript[\[Theta], \[Nu],\[Chi]2] Subscript[\[Theta],\[Rho],\[Chi]3]
            +KDelta[\[Nu],\[Rho]] Subscript[p, \[Chi]1] Subscript[p, \[Chi]4]
            Subscript[\[Theta],\[Mu],\[Chi]1] Subscript[\[Theta],\[Sigma],\[Chi]4]
            +KDelta[\[Mu],\[Rho]] Subscript[p, \[Chi]2] Subscript[p, \[Chi]4]
            Subscript[\[Theta],\[Nu],\[Chi]2] Subscript[\[Theta],\[Sigma],\[Chi]4]
            +KDelta[\[Mu],\[Nu]] Subscript[p, \[Chi]3] Subscript[p, \[Chi]4]
            Subscript[\[Theta], \[Rho],\[Chi]3] Subscript[\[Theta], \[Sigma],\[Chi]4]),
          (c_:1)*Subscript[k,\[Alpha]_]*Subscript[k,\[Beta]_]*Subscript[k,\[Gamma]_]
            *Subscript[k,\[Delta]_]*Sin[1/2*(__)]^2/(k.k)^4/;
            (Position[c,Subscript[k,_]]=={})
          ->-c*\[Pi]^2/24*((KDelta[\[Alpha],\[Beta]]*KDelta[\[Gamma],\[Delta]]
```

```
        +KDelta[\[Alpha],\[Gamma]]*KDelta[\[Beta],\[Delta]]
        +KDelta[\[Alpha],\[Delta]]*KDelta[\[Gamma],\[Beta]])
        *BesselK[0,Sqrt[\[Theta]^2*p^2*M^2]]-(Subscript[p, \[Xi]1]
        *Subscript[\[Theta],\[Xi]1,\[Alpha]]*Subscript[p, \[Xi]2]
        *Subscript[\[Theta],\[Xi]2,\[Beta]]*KDelta[\[Gamma],\[Delta]]
        +Subscript[p,\[Xi]3]*Subscript[\[Theta],\[Xi]3,\[Alpha]]*Subscript[p,\[Xi]4]
        *Subscript[\[Theta],\[Xi]4,\[Gamma]]*KDelta[\[Beta],\[Delta]]
        +Subscript[p,\[Xi]5]*Subscript[\[Theta],\[Xi]5,\[Alpha]]*Subscript[p,\[Xi]6]
        *Subscript[\[Theta],\[Xi]6,\[Delta]]*KDelta[\[Beta],\[Gamma]]
         +Subscript[p,\[Xi]7]*Subscript[\[Theta],\[Xi]7,\[Beta]]*Subscript[p,\[Xi]8]
        *Subscript[\[Theta],\[Xi]8,\[Gamma]]*KDelta[\[Alpha],\[Delta]]
        +Subscript[p,\[Xi]9]*Subscript[\[Theta],\[Xi]9,\[Beta]]*Subscript[p,\[Xi]10]
        *Subscript[\[Theta],\[Xi]10,\[Delta]]*KDelta[\[Alpha],\[Gamma]]
        +Subscript[p,\[Xi]11]*Subscript[\[Theta],\[Xi]11,\[Gamma]]
        *Subscript[p,\[Xi]12]*Subscript[\[Theta],\[Xi]12,\[Delta]]
        *KDelta[\[Alpha],\[Beta]] )/(\[Theta]^2*p^2)+(Subscript[p,\[Xi]13]
        *Subscript[\[Theta],\[Xi]13,\[Alpha]]*Subscript[p,\[Xi]14]
        *Subscript[\[Theta],\[Xi]14,\[Beta]]*Subscript[p,\[Xi]15]
        *Subscript[\[Theta],\[Xi]15,\[Gamma]]*Subscript[p,\[Xi]16]
        *Subscript[\[Theta],\[Xi]16,\[Delta]])*2/(\[Theta]^4*p^4)),
      (c_:1)*Subscript[k,\[Alpha]_]*Subscript[k,\[Beta]_]*Sin[1/2*(__)]^2/(k.k)^3/;
        (Position[c,Subscript[k,_]]=={})
      ->-c*\[Pi]^2/4*(KDelta[\[Alpha],\[Beta]]*BesselK[0,Sqrt[\[Theta]^2*p^2*M^2]]
        -Subscript[p, \[Xi]1]*Subscript[\[Theta],\[Xi]1,\[Alpha]]*Subscript[p,\[Xi]2]
        *Subscript[\[Theta],\[Xi]2,\[Beta]]/(p^2*\[Theta]^2)),
      (c_:1)*Subscript[k,\[Alpha]_]*Subscript[k,\[Beta]_]*Sin[1/2*(__)]^2/(k.k)^2/;
        (Position[c,Subscript[k,_]]=={})
      ->-c*\[Pi]^2/(p^2*\[Theta]^2)*(KDelta[\[Alpha],\[Beta]]-2*Subscript[p,\[Xi]1]
        *Subscript[\[Theta],\[Xi]1,\[Alpha]]*Subscript[p, \[Xi]2]
        *Subscript[\[Theta],\[Xi]2,\[Beta]]/(p^2*\[Theta]^2)),
      (c_:1)*Sin[1/2*(__)]^2/(k.k)^2/;(Position[c,Subscript[k,_]]=={})
      ->-\[Pi]^2*c*BesselK[0,Sqrt[\[Theta]^2*p^2*M^2]],
      (c_:1)*Sin[1/2*(__)]^2/(k.k)/;(Position[c,Subscript[k,_]]=={})
      ->-2*\[Pi]^2*c/(p^2*\[Theta]^2),
      (c_:1)*Sin[1/2*(__)]^2/;(Position[c,k]=={})->-\[Pi]^2/(2*M^4)*c
        *Exp[-(p^2*\[Theta]^2)/(4*M^2)],
      someexpr_->UnknownInt[someexpr,1]
    };
  (* planar integration replacement rules                              *)
  intpnpdefs={
      (c_:1)*Subscript[k, \[Alpha]_]*Subscript[k, \[Beta]_]*
        Subscript[k, \[Gamma]_]*Subscript[k, \[Delta]_]*Sin[1/2*(__)]^2/(k.k)^4/;
        (Position[c,Subscript[k,_]]=={})
      ->c*\[Pi]^2/24*(KDelta[\[Alpha],\[Beta]]*KDelta[\[Gamma],\[Delta]]
         +KDelta[\[Alpha],\[Gamma]]*KDelta[\[Beta],\[Delta]]
         +KDelta[\[Alpha],\[Delta]]*KDelta[\[Gamma],\[Beta]])
         *BesselK[0,Sqrt[M^2/\[CapitalLambda]^2]],
      (c_:1)*Subscript[k, \[Alpha]_]*Subscript[k, \[Beta]_]*Sin[1/2*(__)]^2/(k.k)^3/;
         (Position[c,Subscript[k,_]]=={})
      ->c*\[Pi]^2/4*KDelta[\[Alpha],\[Beta]]*BesselK[0,Sqrt[M^2/\[CapitalLambda]^2]],
      (c_:1)*Subscript[k,\[Alpha]_]*Subscript[k,\[Beta]_]*Sin[1/2*(__)]^2/(k.k)^2/;
         (Position[c,Subscript[k,_]]=={})->c*\[Pi]^2*KDelta[\[Alpha],\[Beta]]
         *\[CapitalLambda]^2,
      (c_:1)*Sin[1/2*(__)]^2/(k.k)/;(Position[c,Subscript[k,_]]=={})
      ->c*\[Pi]^2*2*\[CapitalLambda]^2,
      (c_:1)*Sin[1/2*(__)]^2/(k.k)^2/;(Position[c,Subscript[k,_]]=={})
      ->c*\[Pi]^2*BesselK[0,Sqrt[M^2/\[CapitalLambda]^2]],
      (c_:1)/(k.k)^2/;(Position[c,Subscript[k,_]]=={})
      ->c*2*\[Pi]^2*BesselK[0,Sqrt[M^2/\[CapitalLambda]^2]],
      (c_:1)*Sin[1/2*(__)]^2/;(Position[c,k]=={})->c*\[Pi]^2/(2*M^4),
       someexpr_->UnknownInt[someexpr,0]
```

```
      };
    tmp=Simplify[(ReleaseHold[IndexStyle[expr/.elimrule]])/.
                                (Subscript[k_, a_])^(n_?EvenQ)->(k.k)^(n/2)];
    If[!VSymmetricQ[tmp,k],0
      ,(*else*)
      iSimplifyIndices=1;
      On[Part::"pspec",General::"spell"];
      If[np==1, tmp/.intnpdefs, tmp/.intpnpdefs]
    ]
];

(* Modified version of the Int function above to treat more general integrals*)
(* as they appear in vertex expressions at one loop level                    *)
IntVert[expr_]:=
    Module[{tmp,intnpdefs,intpnpdefs,elimrule},
      Off[Part::"pspec",General::"spell"];
      iSimplifyIndices=0;
      elimrule=(a^2+(k.k)^2):>(k.k)^2;
      intdefs = {
        Times[(c_:1),Subscript[k, \[Mu]_],Subscript[k, \[Nu]_],Subscript[k, \[Rho]_],
           Subscript[k, \[Sigma]_],Subscript[k, \[Tau]_],Sin[Times[Subscript[k,_],
           Subscript[p_,_],_]]]/(k.k)^4/;(Position[c,k]=={}):>c \[Pi]^2 (-(
           (2 Subscript[p, \[Chi]1] Subscript[p, \[Chi]2] Subscript[p, \[Chi]3]
           Subscript[p,\[Chi]4] Subscript[p,\[Chi]5] Subscript[\[Theta],\[Mu],\[Chi]1]
           Subscript[\[Theta], \[Nu],\[Chi]2] Subscript[\[Theta], \[Rho],\[Chi]3]
           Subscript[\[Theta], \[Sigma],\[Chi]4] Subscript[\[Theta], \[Tau],\[Chi]5])
           /(3 \[Theta]^6 (p.p)^3))+1/12 (-(1/(\[Theta]^2 p.p))((KDelta[\[Mu],\[Tau]]
           KDelta[\[Rho],\[Sigma]]+KDelta[\[Mu],\[Sigma]] KDelta[\[Rho],\[Tau]])
           Subscript[p, \[Chi]2] Subscript[\[Theta], \[Nu],\[Chi]2]+
           KDelta[\[Sigma],\[Tau]] (KDelta[\[Nu],\[Rho]] Subscript[p, \[Chi]1]
           Subscript[\[Theta],\[Mu],\[Chi]1]+KDelta[\[Mu],\[Rho]] Subscript[p,\[Chi]2]
           Subscript[\[Theta],\[Nu],\[Chi]2]+KDelta[\[Mu],\[Nu]] Subscript[p,\[Chi]3]
           Subscript[\[Theta],\[Rho],\[Chi]3])+KDelta[\[Nu],\[Sigma]] (
           KDelta[\[Rho],\[Tau]] Subscript[p,\[Chi]1] Subscript[\[Theta],\[Mu],\[Chi]1]
           +KDelta[\[Mu],\[Tau]] Subscript[p,\[Chi]3]
           Subscript[\[Theta],\[Rho],\[Chi]3])+(KDelta[\[Mu],\[Tau]]
           KDelta[\[Nu],\[Rho]]+KDelta[\[Mu],\[Nu]] KDelta[\[Rho],\[Tau]])
           Subscript[p, \[Chi]4] Subscript[\[Theta], \[Sigma],\[Chi]4]
           +KDelta[\[Nu],\[Tau]] (KDelta[\[Rho],\[Sigma]] Subscript[p, \[Chi]1]
           Subscript[\[Theta],\[Mu],\[Chi]1]+KDelta[\[Mu],\[Sigma]]
           Subscript[p, \[Chi]3] Subscript[\[Theta], \[Rho],\[Chi]3]+
           KDelta[\[Mu],\[Rho]] Subscript[p, \[Chi]4]
           Subscript[\[Theta],\[Sigma],\[Chi]4])+(KDelta[\[Mu],\[Sigma]]
           KDelta[\[Nu],\[Rho]]+KDelta[\[Mu],\[Rho]] KDelta[\[Nu],\[Sigma]]+
           KDelta[\[Mu],\[Nu]] KDelta[\[Rho],\[Sigma]]) Subscript[p, \[Chi]5]
           Subscript[\[Theta], \[Tau],\[Chi]5])-1/(\[Theta]^4 (p.p)^2)
           (-2 KDelta[\[Sigma],\[Tau]] Subscript[p, \[Chi]1] Subscript[p, \[Chi]2]
           Subscript[p,\[Chi]3] Subscript[\[Theta],\[Mu],\[Chi]1]
           Subscript[\[Theta],\[Nu],\[Chi]2] Subscript[\[Theta], \[Rho],\[Chi]3]
           -2 Subscript[p, \[Chi]4] (KDelta[\[Rho],\[Tau]] Subscript[p, \[Chi]1]
           Subscript[p, \[Chi]2] Subscript[\[Theta], \[Mu],\[Chi]1]
           Subscript[\[Theta], \[Nu],\[Chi]2]+Subscript[p, \[Chi]3] (
           KDelta[\[Nu],\[Tau]] Subscript[p, \[Chi]1]
           Subscript[\[Theta], \[Mu],\[Chi]1]+KDelta[\[Mu],\[Tau]]
           Subscript[p, \[Chi]2] Subscript[\[Theta], \[Nu],\[Chi]2])
           Subscript[\[Theta], \[Rho],\[Chi]3]) Subscript[\[Theta],\[Sigma],\[Chi]4]
           -2 Subscript[p, \[Chi]5] (KDelta[\[Rho],\[Sigma]] Subscript[p, \[Chi]1]
           Subscript[p,\[Chi]2] Subscript[\[Theta],\[Mu],\[Chi]1]
           Subscript[\[Theta], \[Nu],\[Chi]2]+Subscript[p, \[Chi]3]
           (KDelta[\[Nu],\[Sigma]] Subscript[p,\[Chi]1]
           Subscript[\[Theta],\[Mu],\[Chi]1]+KDelta[\[Mu],\[Sigma]]
```

```
  Subscript[p, \[Chi]2] Subscript[\[Theta], \[Nu],\[Chi]2])
  Subscript[\[Theta],\[Rho],\[Chi]3]+Subscript[p,\[Chi]4] (
  KDelta[\[Nu],\[Rho]] Subscript[p,\[Chi]1] Subscript[\[Theta],\[Mu],\[Chi]1]
  +KDelta[\[Mu],\[Rho]] Subscript[p,\[Chi]2] Subscript[\[Theta],\[Nu],\[Chi]2]
  +KDelta[\[Mu],\[Nu]] Subscript[p,\[Chi]3] Subscript[\[Theta],\[Rho],\[Chi]3]
  )Subscript[\[Theta],\[Sigma],\[Chi]4]) Subscript[\[Theta],\[Tau],\[Chi]5]))),
Times[(c_:1),Subscript[k,\[Alpha]_],Subscript[k,\[Beta]_],
  Subscript[k,\[Gamma]_],Subscript[k, \[Delta]_]]/(k.k)^4/;(Position[c,k]=={})
  :>c*\[Pi]^2/12*(KDelta[\[Alpha],\[Beta]]*KDelta[\[Gamma],\[Delta]]+
  KDelta[\[Alpha],\[Gamma]]*KDelta[\[Beta],\[Delta]]+KDelta[\[Alpha],\[Delta]]
  *KDelta[\[Gamma],\[Beta]])*BesselK[0,Sqrt[M^2/\[CapitalLambda]^2]],
Times[(c_:1),Subscript[k, \[Alpha]_],Subscript[k, \[Beta]_],
  Subscript[k, \[Gamma]_],Subscript[k, \[Delta]_],Cos[Times[Subscript[k,_],
  Subscript[p_,_],_]]]/(k.k)^4/;(Position[c,k]=={}):>c*\[Pi]^2/12*((
  KDelta[\[Alpha],\[Beta]]*KDelta[\[Gamma],\[Delta]]+KDelta[\[Alpha],\[Gamma]]
  *KDelta[\[Beta],\[Delta]]+KDelta[\[Alpha],\[Delta]]*KDelta[\[Gamma],\[Beta]]
  )*BesselK[0,Sqrt[\[Theta]^2*p^2*M^2]]-(Subscript[p, \[Xi]1]
  *Subscript[\[Theta],\[Xi]1,\[Alpha]]*Subscript[p, \[Xi]2]
  *Subscript[\[Theta],\[Xi]2,\[Beta]]*KDelta[\[Gamma],\[Delta]]
  + Subscript[p, \[Xi]3]*Subscript[\[Theta],\[Xi]3,\[Alpha]]
  *Subscript[p, \[Xi]4]*Subscript[\[Theta],\[Xi]4,\[Gamma]]
  *KDelta[\[Beta],\[Delta]] + Subscript[p, \[Xi]5]
  *Subscript[\[Theta],\[Xi]5,\[Alpha]]*Subscript[p, \[Xi]6]
  *Subscript[\[Theta],\[Xi]6,\[Delta]]*KDelta[\[Beta],\[Gamma]]
  + Subscript[p, \[Xi]7]*Subscript[\[Theta],\[Xi]7,\[Beta]]
  *Subscript[p, \[Xi]8]*Subscript[\[Theta],\[Xi]8,\[Gamma]]
  *KDelta[\[Alpha],\[Delta]] + Subscript[p, \[Xi]9]
  *Subscript[\[Theta],\[Xi]9,\[Beta]]*Subscript[p, \[Xi]10]
  *Subscript[\[Theta],\[Xi]10,\[Delta]]*KDelta[\[Alpha],\[Gamma]]
  + Subscript[p, \[Xi]11]*Subscript[\[Theta],\[Xi]11,\[Gamma]]
  *Subscript[p, \[Xi]12]*Subscript[\[Theta],\[Xi]12,\[Delta]]
  *KDelta[\[Alpha],\[Beta]] )/(\[Theta]^2*p^2)+(Subscript[p, \[Xi]13]
  *Subscript[\[Theta],\[Xi]13,\[Alpha]]*Subscript[p, \[Xi]14]
  *Subscript[\[Theta],\[Xi]14,\[Beta]]*Subscript[p, \[Xi]15]
  *Subscript[\[Theta],\[Xi]15,\[Gamma]]*Subscript[p, \[Xi]16]
  *Subscript[\[Theta],\[Xi]16,\[Delta]])*2/(\[Theta]^4*p^4)),
Times[(c_:1),Subscript[k, \[Mu]_],Subscript[k, \[Nu]_],Subscript[k,\[Rho]_],
  Sin[Times[Subscript[k,_],Subscript[p_,_],_]]]/(k.k)^3/;(Position[c,k]=={})
  :>-c*(\[Pi]^2 (-2 Subscript[p, \[Chi]1] Subscript[p, \[Chi]2]
  *Subscript[p, \[Chi]3] Subscript[\[Theta], \[Mu],\[Chi]1]
  *Subscript[\[Theta], \[Nu],\[Chi]2] Subscript[\[Theta], \[Rho],\[Chi]3]
  +\[Theta]^2 p.p (KDelta[\[Nu],\[Rho]] Subscript[p, \[Chi]1]
  *Subscript[\[Theta],\[Mu],\[Chi]1]+KDelta[\[Mu],\[Rho]] Subscript[p,\[Chi]2]
  *Subscript[\[Theta],\[Nu],\[Chi]2]+KDelta[\[Mu],\[Nu]] Subscript[p,\[Chi]3]
  *Subscript[\[Theta], \[Rho],\[Chi]3])))/(2 \[Theta]^4 (p.p)^2),
Times[(c_:1),Subscript[k, \[Alpha]_],Subscript[k, \[Beta]_],
  Cos[Times[Subscript[k,_],Subscript[p_,_],_]]]/(k.k)^3/;(Position[c,k]=={})
  :>c*\[Pi]^2/2*(KDelta[\[Alpha],\[Beta]]*BesselK[0,Sqrt[\[Theta]^2*p^2*M^2]]
  -Subscript[p,\[Xi]1]*Subscript[\[Theta],\[Xi]1,\[Alpha]]*Subscript[p,\[Xi]2]
  *Subscript[\[Theta],\[Xi]2,\[Beta]]/(p^2*\[Theta]^2)),
Times[(c_:1),Subscript[k, \[Alpha]_],Subscript[k, \[Beta]_]]/(k.k)^3/;
  (Position[c,k]=={}):>c*\[Pi]^2/2*KDelta[\[Alpha],\[Beta]]
  *BesselK[0,Sqrt[M^2/\[CapitalLambda]^2]],
Times[(c_:1),Subscript[k, \[Alpha]_],Subscript[k, \[Beta]_],
  Cos[Times[Subscript[k,_],Subscript[p_,_],_]]]/(k.k)^2/;(Position[c,k]=={})
  :>c*2*\[Pi]^2/(p^2*\[Theta]^2)*(KDelta[\[Alpha],\[Beta]]-2
  *Subscript[p,\[Xi]1]*Subscript[\[Theta],\[Xi]1,\[Alpha]]
  *Subscript[p,\[Xi]2]*Subscript[\[Theta],\[Xi]2,\[Beta]]/
  (p^2*\[Theta]^2)),
Times[(c_:1),Subscript[k, \[Alpha]_],Subscript[k, \[Beta]_]]/(k.k)^2/
  (Position[c,k]=={}):>c*2*\[Pi]^2*\[CapitalLambda]^2*KDelta[\[Alpha],\[Beta]],
```

```
          Times[(c_:1),Cos[Times[Subscript[k,_],Subscript[p_,_],_]]]/(k.k)^2/;
            (Position[c,k]=={}):>2*\[Pi]^2*c*BesselK[0,Sqrt[\[Theta]^2*p^2*M^2]],
          Times[(c_:1),Sin[Times[Subscript[k,_],Subscript[p_,_],_]],
            Subscript[k,\[Alpha]_]]/(k.k)^2/;(Position[c,k]=={}):>2 \[Pi]^2*c
            *Subscript[p, \[Sigma]1]*Subscript[\[Theta],\[Alpha],\[Sigma]1]
            /(\[Theta]^2 (p.p)),
          (c_:1)/(k.k)^2/;(Position[c,k]=={}):>2*\[Pi]^2*c
            *BesselK[0,Sqrt[M^2/\[CapitalLambda]^2]],
          Times[(c_:1),Subscript[k, \[Alpha]_],Subscript[k, \[Beta]_]]/(k.k)^2/;
            (Position[c,k]=={}):>c*2*\[Pi]^2*KDelta[\[Alpha],\[Beta]]*\[CapitalLambda]^2,
          Times[(c_:1),Cos[Times[Subscript[k,_],Subscript[p_,_],_]]]/(k.k)/;
            (Position[c,Subscript[k,_]]=={})->4*\[Pi]^2*c/(p^2*\[Theta]^2),
            (c_:1)/(k.k)/;(Position[c,k]=={}):>c*\[Pi]^2*4*\[CapitalLambda]^2,
          Times[(c_:1),Cos[Times[Subscript[k,_],Subscript[p_,_],_]]]/;(Position[c,k]=={})
            :>\[Pi]^2/(M^4)*c*Exp[-(p^2*\[Theta]^2)/(4*M^2)],
          (c_:1)/;(Position[c,k]=={}):>c*2*\[Pi]^2/(2*M^4),
          someexpr_:>UnknownInt[someexpr]
         };
     tmp=Simplify[(ReleaseHold[IndexStyle[expr/.elimrule]])
          /.(Subscript[k_, a_])^(n_?EvenQ):>(k.k)^(n/2)];
     If[!VSymmetricQ[tmp,k],iSimplifyIndices=1;0
       ,(*else*)
     On[Part::"pspec",General::"spell"];
       tmp=tmp/.intdefs;
       iSimplifyIndices=1;
       tmp
     ]
 ];
 Protect[Int];


 (* Functions for graphical markup of expressions.                           *)
 TransformBrackets[f_,expr_]:=DisplayForm[FindBrackets[ToBoxes[expr],f]];
 FindBrackets[startexpr_,f_]:=
     Fold[Function[{expr, location},
       ReplacePart[expr,f[expr[[Sequence @@ location]],Length[location]],location]],
         startexpr, Sort[Position[startexpr,
         RowBox[{a___,"{" | "[" | "(", b__, "]" | "}" | ")", c___}]],
         Length[#1] > Length[#2]&]];
 SizeBrackets[expr_]:=TransformBrackets[StyleBox[
         AdjustmentBox[#1,BoxMargins->{{0,0},{.5,.5}}], Background->White]&,expr]
 ColorBrackets[expr_]:=TransformBrackets[StyleBox[
         AdjustmentBox[#1,BoxMargins->{{0,0},{.5,.5}}], Background->
         Lighter[Blue,N[(1+0.3*#2)/10]]]&,expr];

 Protect[SizeBrackets];
 Protect[ColorBrackets];
 Protect[TransformBrackets];
 Protect[FindBrackets];

 On[General::spell1];

 EndPackage[ ];
 (***EOF*********************************************************************)
```

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Curriculum Vitae

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List of Publications

1. R. Sedmik, “Hall Thruster Plasma Interactions with Solar Cell Arrays on SMART-1”, Diploma Thesis, *University of Vienna*, Vienna, Austria, 2005
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9. Daniel N. Blaschke, Arnold Rofner, and Rene I. P. Sedmik. “In-Depth Analysis of the Localized Non-Commutative $1/p^2$ U(1) Gauge Model”, `[arXiv:0908.1743]`

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