Supervisor

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

THESIS

HAWKING RADIATION IN ETERNAL BLACK HOLE GEOMETRIES

at the Institute for Theoretical Physics

Vienna University of Technology

Supervised by

Privatdoz. Dipl.-Ing. Dr.techn. Herbert Balasin

Michael Torggler

Kolschitzkygasse 14-18/4/3, 1040 Wien

December 7, 2009

Graduand

Kurzfassung

Diese Diplomarbeit behandelt Quantenfeldtheorie in nicht-flachen Hintergrundgeometrien, wobei im Speziellen der Unruh- und der Hawking-Effekt untersucht werden. Der Unruh-Effekt besagt im Wesentlichen, dass ein gleichmäßig beschleunigter Beobachter im Minkokwski-Raum den Minkowski-Vakuum-Zustand als ein thermisches Bad wahrnimmt. Der Hawking-Effekt hingegen prognostiziert ein thermisches Spektrum der Strahlung, welche von schwarzen Löchern aufgrund quantenmechanischer Effekte emittiert wird. Die Behandlung des Letzteren (in der sphärisch reduzierten Schwarzschild Geometrie) stellt sich dabei aufgrund der konformen Invarianz der masselosen Klein-Gordon-Gleichung in 1+1 Dimensionen als besonders einfach dar. In dieser Arbeit wird Unruhs Resultat mittels analytischer Fortsetzung der Moden über den Horizont hergeleitet. Im Weiteren wird gezeigt, dass das Minkowski-Vakuum zwei charakteristische Eigenschaften von thermischen Zuständen besitzt: Zum einen stellt sich die entsprechende Dichtematrix (mittels einer partiellen Spurbildung über unbeobachtbare Freiheitsgrade) als thermische Dichtematrix heraus, d.h. ihre Einträge folgen einer Bose-Einstein Verteilung. Zum anderen zeigt sich, dass die Zweipunktfunktion des skalaren Feldes in Rindler-(Schwarzschild-) Koordinaten eine Periodizität in der imaginären Zeit erfüllt, wobei die Periode der inversen Temperatur entspricht.

Besonderes Augenmerk wird in der Folge auf die Zeitorientierung der entsprechenden Gleichzeitflächen gelegt. Die Verwendung des Killing-Vektors als Zeitrichtung ergibt bemerkenswerterweise ein Verschwinden der Unruh-Strahlung (wie auch der Hawking-Strahlung im Schwarzschild Fall). Dies wirft die Frage nach einer Änderung der Zeitorientierung der anderen asymptotischen Region der Schwarzschildgeometrie auf, da es sich bei diesen um (klassisch) vollständig kausal entkoppelte Bereiche handelt.

Abstract

This diploma thesis treats quantum field theory on non-flat background geometries. Especially the Unruh and Hawking effects are considered. The essence of the Unruh effect is that a constantly accelerated observer in Minkowski space will perceive the ordinary Minkowski vacuum state as being a thermal bath. The Hawking effect predicts a thermal spectrum of the radiation, which is emitted by a black hole due to quantum mechanical effects. The analysis of the latter turns out to be particularly easy because of the conformal invariance of the massless Klein-Gordon field equation in 1+1 dimensions. In this work Unruh's finding is rederived using his method of analytic continuation of the Rindler modes across the horizons. In the following we show that the Minkowski vacuum state satisfies two characteristic features of a thermal state: firstly, we realize that the corresponding density matrix (via a partial trace over unobservable degrees of freedom) is, in fact, a thermal density matrix, i.e. its entries follow a Bose-Einstein

distribution; and secondly, we observe that the two point function of the scalar field, written in terms of Rindler- (Schwarzschild-) coordinates, satisfies a certain periodicity in imaginary time, where the period corresponds to the inverse of the temperature.

In addition, special attention is paid to the time orientation of the corresponding equal time slicings. Usage of the Killing vector as direction of time yields a remarkable result, namely the cessation of the Unruh radiation (and also of the Hawking radiation in the Schwarzschild geometry). This brings up the question of a different time orientation in the other asymptotic region of the Kruskal spacetime, since these are - in a classical sense - causally completely decoupled.

Acknowledgements

I would like to express my gratitude to my supervisor Privatdoz. Dr. techn. Dipl. Ing. Herbert Balasin for all his support, guidance and patience in hours and hours of conversation and discussion (not only) over the matters of this thesis, as well as for his lectures in general relativity and loop quantum gravity, deepening my interest and understandings in these complex matters.

Furthermore, I would like to thank my parents and my sister for their neverending encouragement, personal and financial support before and during my times of study; without them this would never have been possible.

Contents

1 Introduction

Ever since Maxwell unified the electric and magnetic forces to the fundamental electromagnetic interaction, physicists have tried to do the same with the remaining forces of nature, and that with remarkable success, culminating in the standard model of particle physics. All these models are based on the fundament of quantum field theory. The gravitational interaction, however, could not be incorporated in this unification procedure, since, as it turns out, gravity is perturbatively non-renormalizable. For example, in quantum electrodynamics, one subtracts infinite constants, i.e. renormalizes the particle masses, charges and wavefunctions, thereby getting rid of the divergences and producing finite, experimentally checkable predictions. For gravity, on the contrary, this procedure does not seem to work. Renormalization of G, a possible cosmological constant Λ and two coupling constants of geometrical tensors does only suffice to render the resulting theory finite at the one-loop order (see, e.g. [1]). Renormalization of higher-loop terms would require the renormalization of further physical quantities, which are simply not present in the theory.

If one is to incorporate quantum and general relativistic effects in a theory, one has to deal with three constants of nature: c, the speed of light; G, the gravitational constant and $\hbar$, Planck's constant. Simple dimensional analysis then shows that there is one combination of these three constants with the dimension of length, namely the *Planck length,*

$$l_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-33} cm.$$

This Planck length determines the length scale at which quantum corrections to the classical theory of general relativity are believed to become important. Thus, one can take the position that classical general relativity is a very good approximation to nature as long as one considers system whose characteristic length scale is much greater than l_p, very much like classical electrodynamics is a good approximation to quantum electrodynamics as long as one considers high density photon fluxes.

Nevertheless, also without a full theory of quantum gravity, one expects the possibility of investigating semi-classical effects. As a first step in the investigation of how gravity affects field theory, one considers quantum fields on non-trivial backgrounds, e.g. the Schwarzschild or Kruskal spacetimes. Several interesting results have been found in this way, and one hopes that, if physicists are ever to find a satisfying theory of quantum gravity, these results would be reproduced by some sort of semiclassical limit. Thus, the procedure shortly described is very much like computing the Stark (or Zeeman) effect in atomic quantum theory, where the electrical (magnetic) field is treated classically, while the energy levels and transition rates of the electrons are computed using (ordinary) quantum mechanics.

Investigations in this setting, where quantum fields propagate in a fixed classical space-time (M, g_{ab}), have already been started in the late 1960s by Parker, Zel'dovich and co-workers, who looked into particle production by gravitational fields, especially in the context of cosmology. But probably the most important result was Stephen Hawking's [5] contribution (published 1975) that black holes are not, in fact, black, but rather emit a thermal spectrum of particles.

The road which led to the discovery of Hawking radiation could briefly be described as follows: In the 1970s, Roger Penrose found a process of energy extraction from rotating black holes (the Penrose process), sometimes also called superradiant scattering. This process is interpreted as (classical limit of) stimulated emission of particles from a Kerr black hole. From this point of view, it was only natural to look for a process corresponding to spontaneous emission, which was also found in the following. But, even more impact had Stephen Hawking's result (1975) that spontaneous emission does also occur in the vicinity of a non-rotating Schwarzschild black hole [5], since until then it was believed that particle creation by static black holes could occur only during the (dynamical) period of collapse. This process can heuristically be thought of as arising from virtual pair creation processes, where the particle can escape to infinity, while the anti-partner crosses the black hole horizon and is lost. However even more astonishing was the fact that Hawking predicted a perfect thermal spectrum of the emitted radiation, with its maximum corresponding to a temperature of ($M_\odot$ being the solar mass, M that of the black hole)

$$T = \frac{\hbar c^3}{8\pi k_B G M} \simeq 6 \times 10^{-8} \left(\frac{M_\odot}{M}\right) K.$$

This prediction led to a flood of other papers, trying to get more insight into this strange effect. Surely one of the most important of these was Robert Wald's paper (1975) [8], who showed that the density matrix of the outgoing state at infinity is exactly a thermal density matrix corresponding to the Hawking temperature, implying that the black hole behaves like a perfect blackbody. Another important paper is due to William Unruh (1976) [7], which discovered a closely related process, nowadays called the Unruh effect. While Hawking derived his result for the more realistic model of a collapsing star, Unruh considered the same process for an eternal black hole, that is in the Kruskal (the maximally extended Schwarzschild) solution. His observation was then that the underlying causal and topological structure was closely analogous to the one associated with a uniformly accelerated (Rindler) observer in ordinary Minkowski spacetime and that the notion of particle as arising from quantum field theory is highly ambiguous (and, as seen in the derivation of the Unruh effect, observer dependent).

The Unruh effect states that when considering ordinary quantum field theory in Minkowski space, the vacuum state of the quantum field (i.e. no particles are present for an inertial observer) is perceived by a constantly accelerated (non-inertial) observer as a thermal state corresponding to a temperature

$$T = \frac{\hbar a}{2\pi c k_B} \simeq 4 \times 10^{-21} \left(\frac{a}{m/s^2}\right) K,$$

where a is the (constant) acceleration of this observer. However, this temperature is surely too small to be measurable directly ($T \approx 4 \times 10^{-20} K$ for an acceleration corresponding to $a = 10\, m/s^2$).

The above mentioned results furnished the basis for the proposal of Bekenstein, already made in 1973, to further pursue another remarkable connection, namely the analogies between the laws of classical thermodynamics and the three laws of black hole mechanics. The following table (see e.g. [7]) lists the four laws of thermodynamics and their analogues in black hole mechanics (in units in which $G = \hbar = c = 1$).

Law	**Thermodynamics**	**Black Holes**
zeroth	T (the temperature) is constant throughout a system in thermal equilibrium	κ (the surface gravity) is constant over the horizon of a stationary black hole
first	$\delta E = T\delta S - p\delta V + ...$	$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + ...$
second	$\delta S \geq 0$ in any process	$\delta A \geq 0$ in any process
third	$T = 0$ cannot be achieved by a physical process	$\kappa = 0$ cannot be achieved by a physical process

Already from this table, it seems reasonable that one could associate a temperature to a black hole $T \propto \kappa$, its surface gravity, as well as an entropy $S \propto A$, its horizon area. A simple path integral argument in the framework of euclidean quantum gravity shows then [6], in accordance with the above stated results, that

$$T = \frac{\kappa}{2\pi},$$

and

$$S_{BH} = \frac{A}{4},$$

where the subscript BH stands for black hole and/or Bekenstein-Hawking, its discoverers.

Notation and conventions

Our notation for quantum field theory follows closely that of Bjorken & Drell (but with $G = \hbar = c = k_B = 1$), or Birrell & Davies [1], but it differs in the normalization conventions used (we use a Lorentz invariant normalization of the plane wave modes).

The notations used in tensor analysis will be those of Wald [9], our sign convention for the metric being $(-, +, +, +)$ (or, since we will mainly work in two dimensions, $(-, +)$).

2 Unruh Effect in 2D Minkowski Spacetime

In this section we consider quantum field theory in two-dimensional Minkowski spacetime (the obvious advantage of working in two dimensions is the conformal invariance of the massless Klein Gordon equation). The solution of the classical field can be carried out in two different coordinate systems, resulting in an inequivalent quantization and, by the principle of equivalence, can be seen as preparation for the analysis of the Schwarzschild case (but all the "hard" work is done in this chapter). This is the essence of the Unruh effect: the ordinary Minkowski vacuum state is perceived by a (constantly) accelerated observer as being a thermal state. We derive this result by analytically continuing the Rindler modes across the horizons (the "Unruh trick"). Along the lines we present some general results of quantum field theory in curved spacetime in their special form pertaining to the case considered. Two different conditions (at first clearly the thermal spectrum of the produced radiation, secondly the periodicity in imaginary time of Greens functions) implying the thermal nature of the system are shown to be satisfied. In the very last part of this section, we choose a different time slicing in the left Rindler wedge and it is shown that in that case, Unruh radiation comes to a stop.

2.1 Minkowski and Rindler coordinates

The metric of two dimensional Minkowski spacetime is given by

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 \\ &= -dudv, \end{aligned} \tag{2.1}$$

where we used Minkowski light cone coordinates:

$$\begin{aligned} u &= t - x \\ v &= t + x. \end{aligned}$$

Consider now an observer with constant acceleration of magnitude α in the positive

$x-$direction. Its trajectory will be given by

$$x^a(\tau) = \begin{pmatrix} t(\tau) \\ x(\tau) \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}\sinh(\alpha\tau) \\ \frac{1}{\alpha}\cosh(\alpha\tau) \end{pmatrix}, \tag{2.2}$$

as is seen by calculating the acceleration two-vector via

$$\begin{aligned} a^a &= \left(\dot{x}\cdot\partial\right)^2 x^a \\ &= \frac{\partial^2}{\partial\tau^2}x^a \\ &= \alpha\begin{pmatrix} \sinh(\alpha\tau) \\ \cosh(\alpha\tau) \end{pmatrix} \end{aligned}$$

and by determining its magnitude we find that α is the observers proper acceleration

$$\sqrt{a^b a_b} = \sqrt{\alpha^2\left[-\sinh^2(\alpha\tau) + \cosh^2(\alpha\tau)\right]} = \alpha.$$

The worldline of this observer therefore obeys the equation

$$x^2(\tau) = t^2(\tau) + \frac{1}{\alpha^2} \Leftrightarrow uv = -\frac{1}{\alpha^2},$$

which is a hyperbola in the $x-t$ plane asymptotic to the lines $x = \pm t$ (c.p. fig 2.1).

Now we introduce another set of coordinates more adapted to the accelerated observer, the so-called *Rindler coordinates* given by

$$\begin{cases} t &= \frac{1}{a}e^{a\xi}\sinh(a\eta) \\ x &= \frac{1}{a}e^{a\xi}\cosh(a\eta), \end{cases} \tag{2.3}$$

where the new coordinates ξ and η range over the whole $\mathbb{R}^2$, i.e. $\xi,\eta \in (-\infty,\infty)$.

Note, however, that these coordinates do not cover Minkowski space, but only the portion R

$$R = \left\{(t,x) \in \mathbb{R}^2 \,|\, x > |t|\right\} = \left\{(u,v) \in \mathbb{R}^2 \,|\, u < 0, v > 0\right\},$$

which will henceforth be denoted as right Rindler wedge (see also Fig.2.1).

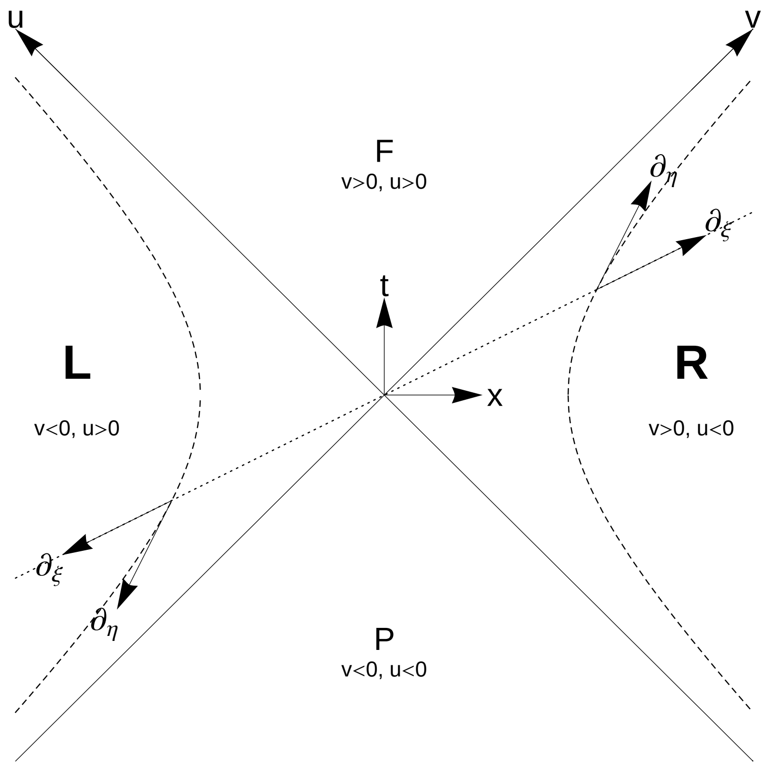

Figure 2.1: Minkowski spacetime and the Rindler wedges:
The diagonals are the lines $u = 0 = v$ (the Killing horizons); the dashed line in R is the worldline of a constantly accelerated observer ($\xi = const.$); the dotted line is one of $\eta = const.$

The path of the constantly accelerated observer (2.2) becomes in these coordinates

$$\begin{cases} \eta\left(\tau\right) & = \frac{\alpha}{a}\tau \\ \xi\left(\tau\right) & = \frac{1}{a}\ln\left(\frac{a}{\alpha}\right), \end{cases}$$

such that proper time is proportional to η (which will soon be seen to be “boost-time”), while ξ is constant along such a path and parametrizes the strength of the acceleration, i.e. α, the proper acceleration.

What we are ultimately interested in is doing quantum field theory, and in order to guarantee well-posedness of the Cauchy problem (see e.g. [9]), we need to cover also the left Rindler wedge L. Then we can take any line (of constant η) passing through the origin as Cauchy hypersurface on which the initial conditions have to be given.

Therefore, we define Rindler coordinates in L

$$\begin{cases} t &= -\frac{1}{a}e^{a\xi}\sinh(a\eta) \\ x &= -\frac{1}{a}e^{a\xi}\cosh(a\eta), \end{cases}$$

where again $-\infty < \xi, \eta < \infty$, and thus, as desired, they cover the left Rindler wedge

$$L = \left\{ x \in \mathbb{R}^2 \;|x < |t| \right\} = \left\{ (u, v) \in \mathbb{R}^2 \;|u > 0, v < 0 \right\}.$$

For completeness, we note that we could define Rindler coordinates to cover the future and past regions F and P, too, where

$$\begin{aligned} F &= \left\{ x \in \mathbb{R}^2 \;|t > |x| \right\} = \left\{ (u, v) \in \mathbb{R}^2 \;|u > 0, v > 0 \right\}, \\ P &= \left\{ x \in \mathbb{R}^2 \;|t < |x| \right\} = \left\{ (u, v) \in \mathbb{R}^2 \;|u < 0, v < 0 \right\}, \end{aligned}$$

but they will not be needed in the following so we can do without them.

The metric (2.1) in Rindler coordinates is then given by (valid in both L and R)

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2),$$

from which we can readily identify ∂_η^a as Killing vector

$$\begin{aligned} \partial_\eta^a &= \frac{\partial t}{\partial \eta}\partial_t^a + \frac{\partial x}{\partial \eta}\partial_x^a \\ &= a\left(x\partial_t^a + t\partial_x^a\right), \end{aligned}$$

and is seen to be the Killing vector of a boost in the x-direction, being timelike in the regions R and L and spacelike in P and F.

By taking the square root of its norm (in R and L), we obtain the (exponential) redshift factor

$$V = \sqrt{-\partial_\eta^2} = \sqrt{a^2\left(x^2 - t^2\right)} = \sqrt{a^2\frac{e^{2a\xi}}{a^2}} = e^{a\xi}.$$

Taking the derivative of this redshift factor and calculating its norm, one obtains the surface gravity

$$\nabla_a V = \frac{a^2}{V}\left(x dx_a - t dt_a\right),$$

$$\kappa := \sqrt{(\nabla V)^2} = \sqrt{\left(\frac{a^2}{V}\right)^2 (x^2 - t^2)} = a,$$

and we see that the magnitude of the acceleration takes the role of the surface gravity[1], thereby emphasizing our analogy with the Schwarzschild case.

2.2 Massless Klein Gordon field in Minkowski coordinates

The classical massless Klein Gordon equation in Minkowski light cone coordinates reads:

$$\Box\Phi = 0 = \partial_u \partial_v \Phi, \tag{2.4}$$

for which we can choose an orthonormal (with respect to a inner product which will be defined later on, cf. (2.6)) set of positive frequency modes

$$f_k = N_k \exp\left[ik \cdot x\right] = N_k \exp\left[i\left(-\omega t + kx\right)\right], \tag{2.5}$$

where $\omega = |k|$.

The definition of positive frequency in this case is dictated by the (global) time coordinate t, i.e. when taking the Lie derivative of the modes with respect to the global timelike future directed Killing vector field ∂_t^a one finds

$$\mathcal{L}_{\partial_t} f_k = -i\omega f_k.$$

The denotation of “positive frequency” arises since by the correspondence principle, upon quantization, the differential operator $i\mathcal{L}_{\partial_t}$ plays the role of “energy” which is then given by the positive $\omega = |k|$.

As inner product, we integrate the conserved current

$$j^a\left[\Phi, \Psi\right] = -i\left(\Phi \partial^a \Psi^* - \Psi^* \partial^a \Phi\right)$$

[1] This is in agreement with the definition of surface gravity given by Wald [9], as can easily be seen by taking the norm of the following expression

$$\begin{aligned}
\nabla_a V &= \nabla_a \sqrt{-\xi^2} = -\frac{1}{2\sqrt{-\xi^2}} \nabla_a \xi^2 \\
&= -\frac{1}{\sqrt{-\xi^2}} \xi^b \nabla_a \xi_b = \frac{1}{\sqrt{-\xi^2}} \xi^b \nabla_b \xi_a \\
&= \frac{1}{\sqrt{-\xi^2}} \kappa \xi_a.
\end{aligned}$$

over a Cauchy hypersurface (which in the two-dimensional case clearly reduces to a line), and by using Gauss' theorem one can show that this procedure is independent of the surface chosen (ω_h is the induced volume element on the hypersurface Σ' defined by its normal n^a)

$$\begin{aligned}(\Phi, \Psi) &= -i \int_{\Sigma'} \left(\Phi \partial_a \Psi^* - \Psi^* \partial_a \Phi\right) n^a \omega_h \\ &= -i \int_{\Sigma} \left(\Phi \partial_a \Psi^* - \Psi^* \partial_a \Phi\right) n^a \omega_h - i \int_{\Sigma' - \Sigma} \partial^a \left(\Phi \partial_a \Psi^* - \Psi^* \partial_a \Phi\right) \omega_h \\ &= -i \int_{\Sigma} \left(\Phi \partial_a \Psi^* - \Psi^* \partial_a \Phi\right) n^a \omega_h,\end{aligned} \tag{2.6}$$

where the second integral vanishes because of the field equations

$$\int_{\Sigma' - \Sigma} \partial^a \left(\Phi \partial_a \Psi^* - \Psi^* \partial_a \Phi\right) \omega_h = \int_{\Sigma' - \Sigma} \left(\underbrace{\partial^a \Phi \partial_a \Psi^* - \partial^a \Psi^* \partial_a \Phi}_{=0} + \Phi \underbrace{\Box \Psi^*}_{=0} - \Psi^* \underbrace{\Box \Phi}_{=0} \right) \omega_h .$$

Normalizing these modes (the *Minkowski modes*) gives, where for convenience we choose $t = 0$ as hypersurface,

$$\begin{aligned}(f_k,\, f_{k'}) &= -i \int dx \left\{ f_k \partial_t f^*_{k'} - f^*_{k'} \partial_t f_k \right\}_{t=0} \\ &= -N_k N^*_{k'} \left(\omega' + \omega\right) \int dx \exp\left[i \left(k - k'\right) x\right] \\ &= N_k N^*_{k'} \left(\omega' + \omega\right) (2\pi)\, \delta\left(k - k'\right) \\ &= |N_k|^2 \,(2\omega)\,(2\pi)\, \delta\left(k - k'\right) \\ &= 2\omega\,(2\pi)\, \delta\left(k - k'\right),\end{aligned}$$

and thus[2]

$$N_k = 1.$$

[2] This normalization convention assures the Lorentz invariance of the modes (which are then Lorentz scalars) as can be seen by considering a Lorentz boost on a momentum 4-vector p^a, whose components then transform as

$$E \to E' = \gamma \left(E + v_j p_j\right),$$

$$\begin{aligned}p_i \to p'_i &= \left(p_\perp\right)_i + \gamma \left[\left(p_\parallel\right)_i + v_i E\right] \\ &= p_i - \frac{p_j v_j}{v^2} v_i + \gamma \left[\frac{p_j v_j}{v^2} v_i + v_i E\right],\end{aligned}$$

and one finds

$$\frac{dp'_i}{dp_j} = \delta_{ij} + (\gamma - 1)\, v_j v_i v^{-2} + \gamma E^{-1} v_i p_j,$$

Returning to the solution of (2.4), we note that the negative energy modes have negative norm

$$\left(f_k^*,\, f_{k'}^*\right) \;\; = \;\; -\left(2\omega\right)\left(2\pi\right)\delta\left(k-k'\right),$$

while the mixed inner products vanish

$$\begin{aligned}\left(f_k,\, f_{k'}^*\right) &= -i\int dx\left\{f_k\partial_t f_{k'} - f_{k'}\partial_t f_k\right\}_{t=0}\\ &= N_{k'}N_k\left(\omega-\omega'\right)\int dx\exp\left[i\left(k+k'\right)x\right]=0.\end{aligned}$$

Now we perform a split in right and left movers

$$\begin{aligned}f_k &= \exp\left[-i\omega\left(t-x\right)\right]\Theta\left(k\right)+\exp\left[-i\omega\left(t+x\right)\right]\Theta\left(-k\right)\\ &= \exp\left[-i\omega u\right]\Theta\left(k\right)+\exp\left[-i\omega v\right]\Theta\left(-k\right),\end{aligned}$$

and since in section 2.5we will analytically continue the Rindler modes, we allow u and v to become complex. Therefore, one realizes that the right-movers

$$\exp\left[-i\omega u\right]=\exp\left[\omega\left(-i\Re u+\Im u\right)\right]$$

are *analytic* and *bounded* $\forall\Im u\leq 0$, while the left-movers

$$\exp\left[-i\omega v\right]=\exp\left[\omega\left(-i\Re v+\Im v\right)\right]$$

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whose determinant (the Jacobian of this transformation) is

$$|\frac{dp_i'}{dp_j}|=\frac{\gamma}{E}\left(E+v_kp_k\right)=\frac{E'}{E}.$$

The Lorentz invariant Dirac delta measure is then defined by the property

$$\begin{aligned}\int\frac{d^4k}{\left(2\pi\right)^4}2\pi\delta\left(k^ak_a\right)\theta\left(k^0\right) &= \int\frac{d^3k}{2k^0\left(2\pi\right)^3}\left[\delta\left(k^0-\omega\right)+\delta\left(k^0+\omega\right)\right]\theta\left(k^0\right)\\ &= \int\frac{d^3k}{2\omega\left(2\pi\right)^3}.\end{aligned}$$

The invariant Delta function then reads and transforms as

$$2\omega\left(2\pi\right)^3\delta^{(3)}\left(\overrightarrow{k}\right)\to 2\omega'\left(2\pi\right)^3\delta^{(3)}\left(\overrightarrow{k'}\right),$$

ensuring

$$\int\frac{d^3k}{2\omega\left(2\pi\right)^3}2\omega\left(2\pi\right)^3\delta^{(3)}\left(\overrightarrow{k}\right)=1\to\int\frac{d^3k'}{2\omega'\left(2\pi\right)^3}2\omega'\left(2\pi\right)^3\delta^{(3)}\left(\overrightarrow{k'}\right)=1.$$

are *analytic* and *bounded* $\forall \Im v \leq 0$.

Note that a superposition of only positive frequency modes (for simplicity, we consider only the right-movers) can be written as

$$f(u) = \int_0^\infty d\omega \widetilde{f}(\omega) \exp[-i\omega u],$$

and is easily seen to retain the above mentioned analyticity and boundedness conditions.

What we want to show now is the converse, i.e. any function which is bounded and analytic in the lower half complex u-plane, does contain only positive frequency components. An analytic function in the lower half plane satisfies the Cauchy-Riemann equations in this region, i.e. (with $u = u_1 + iu_2$)

$$\begin{aligned} \partial_{u_1}[\Re f(u)] &= \partial_{u_2}[\Im f(u)] \\ \partial_{u_2}[\Re f(u)] &= -\partial_{u_1}[\Im f(u)]; \quad \forall u_2 < 0, \end{aligned}$$

and being bounded implies

$$|f(u)| \leq C \; ; \; \forall\, u_2 < 0,$$

for some real constant $C > 0$.

An arbitrary, integrable function can be Fourier(-Laplace) transformed in time (a complex u in this case)

$$f(u) = \int_{-\infty}^\infty d\omega \widetilde{f}(\omega) \exp[-i\omega u_1 + \omega u_2],$$

and the analyticity properties

$$\begin{aligned} \partial_{u_1}[\Re f(u)] &= \partial_{u_1}\left[\int_{-\infty}^\infty d\omega \left(\Re\widetilde{f}(\omega)\cos\omega u_1 + \Im\widetilde{f}(\omega)\sin\omega u_1\right)\exp[\omega u_2]\right] \\ &= \left[\int_{-\infty}^\infty d\omega\omega \left(-\Re\widetilde{f}(\omega)\sin\omega u_1 + \Im\widetilde{f}(\omega)\cos\omega u_1\right)\exp[\omega u_2]\right] \\ \partial_{u_2}[\Im f(u)] &= \left[\int_{-\infty}^\infty d\omega\omega \left(-\Re\widetilde{f}(\omega)\sin\omega u_1 + \Im\widetilde{f}(\omega)\cos\omega u_1\right)\exp[\omega u_2]\right]; \\ \partial_{u_2}[\Re f(u)] &= \left[\int_{-\infty}^\infty d\omega\omega \left(\Re\widetilde{f}(\omega)\cos\omega u_1 + \Im\widetilde{f}(\omega)\sin\omega u_1\right)\exp[\omega u_2]\right] \\ -\partial_{u_1}[\Im f(u)] &= -\left[\int_{-\infty}^\infty d\omega \left(-\Re\widetilde{f}(\omega)\sin\omega u_1 - \Im\widetilde{f}(\omega)\cos\omega u_1\right)\exp[\omega u_2]\right], \end{aligned}$$

are seen to be fulfilled for all imaginary parts of u.

To show that the boundedness criterion restricts the support of $\widetilde{f}(\omega)$ to $\omega > 0$ we consider

$$|f(u)| = \left| \int_{-\infty}^{\infty} d\omega \widetilde{f}(\omega) \exp\left[-i\omega u_1 + \omega u_2\right] \right|,$$

which is seen to have an upper bound by using the triangle-inequality

$$\begin{aligned} |f(u)| &\leq \int_{-\infty}^{\infty} d\omega \left|\widetilde{f}(\omega)\right| \exp\left[\omega u_2\right] \\ &= \int_0^{\infty} d\omega \left|\widetilde{f}(\omega)\right| \exp\left[\omega u_2\right] + \int_0^{\infty} d\omega \left|\widetilde{f}(-\omega)\right| \exp\left[-\omega u_2\right], \end{aligned}$$

whose first integral kernel is easily seen to be exponentially damped for all $u_2 \leq 0$ and is therefore bounded, while the second one defines a monotonically increasing function of $-u_2$ and thus cannot fulfill the desired condition.

Hence, we have to impose

$$\widetilde{f}(\omega) = \widetilde{f}_0(\omega)\,\Theta(\omega),$$

and therefore established the above claimed result that a function, which is analytic and bounded in the lower half of the complex time plane, cannot contain negative frequency components. We will later employ this definition of positive frequency in the derivation of the Unruh modes.

Now we proceed as in ordinary field theory by expanding the field and its canonical conjugate in these modes

$$\begin{aligned} \Phi &= \int \frac{dk}{(2\omega)(2\pi)} \left[a_k f_k + a_k^\dagger f_k^*\right], \qquad (2.7) \\ \Pi &= \dot{\Phi} \\ &= \int \frac{dk}{(2\omega)(2\pi)} \left[a_k \dot{f}_k + a_k^\dagger \dot{f}_k^*\right] \\ &= -i \int \frac{dk}{(2\omega)(2\pi)} \omega \left[a_k f_k - a_k^\dagger f_k^*\right], \end{aligned}$$

where

$$\begin{aligned} a_{k'} &= (\Phi,\, f_{k'}) = -i \int dx \left\{\Phi \partial_t f_{k'}^* - f_{k'}^* \Pi\right\}_{t=0} \qquad (2.8) \\ a_{k'}^\dagger &= -(\Phi,\, f_{k'}^*) = i \int dx \left\{\Phi \partial_t f_{k'} - f_{k'} \Pi\right\}_{t=0} \end{aligned}$$

are the expansion coefficients. In order to assure that this is possible, we shall show that this set of Minkowski modes (2.5) is complete, i.e.

$$\begin{aligned}
\int d^2k\, f_k\left(t,x\right)\, f_k^*\left(t',x'\right) &= \int d\omega dk \exp\left[i\left(-\omega t+kx\right)\right]\exp\left[-i\left(-\omega t'+kx'\right)\right] \\
&= \int d\omega dk \exp\left[-i\omega\left(t-t'\right)\right]\exp\left[ik\left(x-x'\right)\right] \\
&= \delta\left(x-x'\right)\delta\left(t-t'\right).
\end{aligned}$$

Quantization is then achieved by imposing the canonical equal time commutation relations

$$\left[\Phi\left(x,t\right),\, \Pi\left(x',t\right)\right] = i\delta\left(x-x'\right),$$

and the expansion coefficients a_k $\left(a_k^\dagger\right)$ (2.8) are then interpreted as annihilation (creation) operators for quanta in mode k and obey the commutation relation

$$\left[a_k, a_{k'}^\dagger\right] = \left(2\omega\right)\left(2\pi\right)\delta\left(k-k'\right), \tag{2.9}$$

which is easily verified by inserting the expressions (2.8)

$$\begin{aligned}
\left[a_k, a_{k'}^\dagger\right] &= -\left[\left(\Phi,\, f_k\right),\left(\Phi,\, f_{k'}^*\right)\right] \\
&= \left[i\int dx\left\{\Phi\left(x\right)\partial_t f_k^*\left(x\right)-f_k^*\left(x\right)\Pi\left(x\right)\right\}_{t=0},\, i\int dx'\left\{\Phi\left(x'\right)\partial_{t'}f_{k'}\left(x'\right)-f_{k'}\left(x'\right)\Pi\left(x'\right)\right\}_{t=0}\right] \\
&= -\int dxdx'\left[\left\{\Phi\left(x\right)\partial_t f_k^*\left(x\right)-f_k^*\left(x\right)\Pi\left(x\right)\right\},\,\left\{\Phi\left(x'\right)\partial_{t'}f_{k'}\left(x'\right)-f_{k'}\left(x'\right)\Pi\left(x'\right)\right\}\right]_{t=0} \\
&= -\int dxdx'\left(-f_{k'}\left(x'\right)\partial_t f_k^*\left(x\right)\left[\Phi\left(x\right),\,\Pi\left(x'\right)\right]-f_k^*\left(x\right)\partial_{t'}f_{k'}\left(x'\right)\left[\Pi\left(x\right),\,\Phi\left(x'\right)\right]\right)_{t=0} \\
&= -\int dxdx' i\delta\left(x-x'\right)\left(-f_{k'}\left(x'\right)\partial_t f_k^*\left(x\right)+f_k^*\left(x\right)\partial_{t'}f_{k'}\left(x'\right)\right)_{t=0} \\
&= -i\int dx\left(f_{k'}\left(x\right)\partial_t f_k^*\left(x\right)-f_k^*\left(x\right)\partial_t f_{k'}\left(x\right)\right)_{t=0} \\
&= \left(f_{k'},\, f_k\right) \\
&= \left(2\omega\right)\left(2\pi\right)\delta\left(k-k'\right),
\end{aligned} \tag{2.10}$$

while all others turn out to be zero

$$\left[a_k, a_{k'}\right] = 0 = \left[a_k^\dagger, a_{k'}^\dagger\right].$$

2.3 Massless Klein Gordon field in Rindler coordinates

Because of the analogy between the Rindler wedges and the extended Schwarzschild spacetime mentioned in the introduction, we are now going to find a complete set of modes in the Rindler coordinate system. By the principle of equivalence we expect this procedure (the quantization in Rindler coordinates of a constantly accelerating observer) to model the situation of a stationary observer in Schwarzschild spacetime (which has to accelerate constantly to remain stationary, i.e. not to fall into the black hole).

The wave equation in Rindler coordinates reads

$$\begin{aligned} \Box\Phi &= e^{-2a\xi}\left(-\partial_\eta^2 + \partial_\xi^2\right)\Phi = 0 \\ \Rightarrow & \quad \left(-\partial_\eta^2 + \partial_\xi^2\right)\Phi = 0, \end{aligned}$$

where we used conformal invariance of the 2D Klein Gordon equation[3].

Since this has the same functional form as equation (2.4), we also have the same form for the modes, but with the replacements $x \to \xi\,;\, t \to \eta$. At this point, one has to be careful because the Rindler coordinates are not defined everywhere. Therefore we have to write down positive frequency modes for the L and R sectors, separately.

The latter are found to be

$$g_k = \exp\left[-i\omega\eta + ik\xi\right],$$

where now positive frequency is defined with respect to the future directed timelike Killing vector ∂_η^a, and the above modes are easily seen to satisfy

$$\mathcal{L}_{\partial_\eta} g_k = -i\omega g_k.$$

In the left Rindler wedge the modes are chosen as

$$g_k = \exp\left[i\omega\eta + ik\xi\right],$$

[3] A field equation in n-dimensional spacetime is said to be *conformally invariant* if there is a number $s \in \mathbb{R}$ (the conformal weight of the field Ψ), such that if Ψ is a solution to the equation with g_{ab}, then $\widetilde{\Psi} = \Omega^s\Psi$ is a solution to the equation with $\widetilde{g}_{ab} = \Omega^2 g_{ab}$.

One can then show that (see, e.g. [9]) for $n = 2$ and $s = 0$

$$g^{ab}\nabla_a\nabla_b\Psi = 0$$

implies

$$\widetilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\widetilde{\Psi} \quad = \quad 0.$$

and therefore $\Psi = \widetilde{\Psi}$. In our case, $\Omega = \exp[a\xi]$; $\nabla_a = \partial_a = (\partial_t\,,\,\partial_x)$ and $\tilde{\nabla}_a = (\partial_\eta\,,\,\partial_\xi)$.

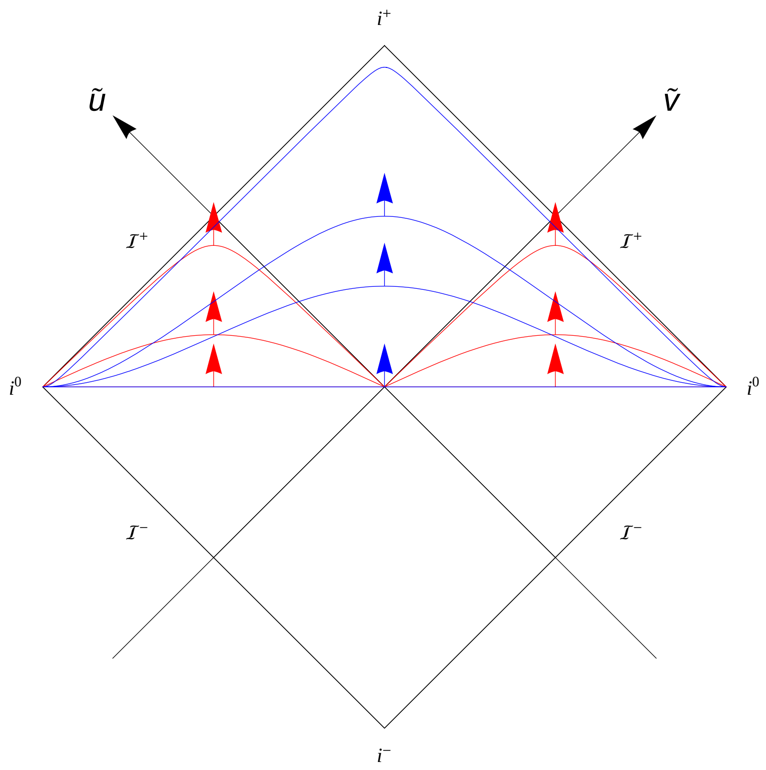


Figure 2.2: Compactified Minkowski spacetime and time slicings:
The blue lines are lines of constant Minkowski time t, the red ones correspond to constant Rindler η. Note that in L, the time slicings have been chosen in a way (i.e., reversed) to agree with the slicings defined by the inertial observer. For the compactification procedure carried out to obtain this graph, see section 3.1.

where now the Lie derivative is taken along the reversed Killing vector, i.e. $\left(-\partial_\eta^a\right)$, because, assuming a global time orientation, the vector field ∂_η^a is past directed in L (see figs. 2.1 and 2.2)

$$\mathcal{L}_{-\partial_\eta} g_k = -i\omega g_k.$$

Since, strictly speaking, the coordinates η and ξ are not the same in the two Rindler wedges, we restrict the support of the modes to their domain of definition by defining

$$g_k^{(1)} = \begin{cases} \exp\left[-i\omega\eta + ik\xi\right] & \text{in R} \\ 0 & \text{in L} \end{cases},$$

and

$$g_k^{(2)} = \begin{cases} \exp\left[i\omega\eta + ik\xi\right] & \text{in L} \\ 0 & \text{in R} \end{cases}.$$

By this construction, the modes $g_k^{(1)}$ and $g_k^{(2)}$ (from now on called the *Rindler modes*) form a complete orthonormal set in R and L, respectively

$$\left(g_k^{(i)}, g_{k'}^{(j)}\right) = \begin{cases} (2\pi)(2\omega)\,\delta(k-k') & ;\, i=j \\ 0 & ;\, i \neq j \end{cases} \quad i,j = 1,2.$$

Now one could, as before, expand the field operator in these modes, yielding

$$\Phi = \sum_{i=1,2} \int \frac{dk}{(2\pi)(2\omega)} \left[b_k^{(i)} g_k^{(i)} + b_k^{(i)\dagger} g_k^{(i)*}\right],$$

and completeness of the modes is easily seen to be fulfilled

$$\begin{aligned}
\int d^2k\Theta(-u)\,\Theta(v)\,g_k^{(1)}(\eta,\xi)\,g_k^{(1)*}(\eta',\xi') \quad &+ \quad \int d^2k\Theta(u)\,\Theta(-v)\,g_k^{(2)}(\eta,\xi)\,g_k^{(2)*}(\eta',\xi') \\
&= \quad \int d^2k\Theta(-u)\,\Theta(v)\exp\left[-i\omega\eta + ik\xi\right]\exp\left[i\omega\eta' - ik\xi'\right] \\
&\quad\; + \int d^2k\Theta(-v)\,\Theta(u)\exp\left[i\omega\eta + ik\xi\right]\exp\left[-i\omega\eta' - ik\xi'\right] \\
&= \quad \int d^2k\Theta(-u)\,\Theta(v)\exp\left[-i\omega\eta + i\omega\eta'\right]\exp\left[-ik\xi' + ik\xi\right] \\
&\quad\; + \int d^2k\Theta(-v)\,\Theta(u)\exp\left[-i\omega\eta' + i\omega\eta\right]\exp\left[-ik\xi' + ik\xi\right] \\
&= \quad \Theta(-u)\,\Theta(v)\,\delta(\xi-\xi')\,\delta(\eta-\eta') \\
&\quad\; + \Theta(-v)\,\Theta(u)\,\delta(\xi-\xi')\,\delta(\eta-\eta') \\
&= \quad \delta(\xi-\xi')\,\delta(\eta-\eta'),
\end{aligned}$$

as long as we consider only (ξ,η) and $(\xi',\eta') \in L \cup R$.

2.4 The Bogolubov transformation and two different vacua

From the sections 2.2 and 2.3 we know that we can expand the field operator Φ in two different sets of modes

$$\begin{aligned} \Phi &= \int \frac{dk}{(2\omega)(2\pi)} \left[a_k f_k + a_k^\dagger f_k^*\right] \\ &= \sum_{i=1,2} \int \frac{dk}{(2\pi)(2\omega)} \left[b_k^{(i)} g_k^{(i)} + b_k^{(i)\dagger} g_k^{(i)*}\right]. \end{aligned}$$

Since both sets of modes are complete, they can be expanded in terms of each other

$$f_k = \sum_{i=1,2} \int \frac{dk'}{(2\pi)(2\omega')} \left[\alpha_{k,k'}^{(i)} g_{k'}^{(i)} + \beta_{k,k'}^{(i)} g_{k'}^{(i)*}\right], \tag{2.11}$$

with the expansion coefficients

$$\begin{aligned} \alpha_{k,k'}^{(i)} &= \left(f_k\,,\, g_{k'}^{(i)}\right), \\ \beta_{k,k'}^{(i)} &= -\left(f_k\,,\, g_{k'}^{(i)*}\right), \end{aligned} \tag{2.12}$$

and viceversa

$$g_k^{(i)} = \int \frac{dk'}{(2\pi)(2\omega')} \left[\alpha_{k',k}^{(i)*} f_{k'} - \beta_{k',k}^{(i)} f_{k'}^*\right]. \tag{2.13}$$

The expansion coefficients $\alpha_{k,k'}^{(i)}$ and $\beta_{k,k'}^{(i)}$ are called the *Bogolubov coefficients* and satisfy their own normalization conditions, as can be seen by inserting (2.13) in (2.11):

$$\begin{aligned} f_k &= \sum_{i=1,2} \int \frac{dk'}{(2\pi)(2\omega')} \left[\alpha_{k,k'}^{(i)} g_{k'}^{(i)} + \beta_{k,k'}^{(i)} g_{k'}^{(i)*}\right] \\ &= \sum_{i=1,2} \int \frac{dk'}{(2\pi)(2\omega')} \int \frac{dk''}{(2\pi)(2\omega'')} \left[\begin{array}{c} \alpha_{k,k'}^{(i)} \left(\alpha_{k'',k'}^{(i)*} f_{k''} - \beta_{k'',k'}^{(i)} f_{k''}^*\right) \\ +\beta_{k,k'}^{(i)} \left(\alpha_{k'',k'}^{(i)} f_{k''}^* - \beta_{k'',k'}^{(i)*} f_{k''}\right) \end{array}\right] \\ &= \int \frac{dk''}{(2\pi)(2\omega'')} \sum_{i=1,2} \int \frac{dk'}{(2\pi)(2\omega')} \left[\begin{array}{c} \left(\alpha_{k,k'}^{(i)} \alpha_{k'',k'}^{(i)*} - \beta_{k,k'}^{(i)} \beta_{k'',k'}^{(i)*}\right) f_{k''} \\ + \left(\alpha_{k'',k'}^{(i)} \beta_{k,k'}^{(i)} - \alpha_{k,k'}^{(i)} \beta_{k'',k'}^{(i)}\right) f_{k''}^* \end{array}\right], \end{aligned}$$

and therefore

$$\begin{aligned}
\sum_{i=1,2}\int\frac{dk'}{(2\pi)(2\omega')}\left[\alpha^{(i)}_{k'',k'}\beta^{(i)}_{k,k'}-\alpha^{(i)}_{k,k'}\beta^{(i)}_{k'',k'}\right] &= 0,\\
\sum_{i=1,2}\int\frac{dk'}{(2\pi)(2\omega')}\left[\alpha^{(i)}_{k,k'}\alpha^{(i)*}_{k'',k'}-\beta^{(i)}_{k,k'}\beta^{(i)*}_{k'',k'}\right] &= 2\pi 2\omega\delta\left(k-k''\right).
\end{aligned}$$

They also describe the relations between the annihilation and creation operators associated with the two sets of modes

$$\begin{aligned}
a_{k'} &= (\Phi,\, f_{k'})\\
&= \sum_{i=1,2}\int\frac{dk}{(2\pi)(2\omega)}\left[b^{(i)}_{k}\alpha^{(i)*}_{k',k}-b^{(i)\dagger}_{k}\beta^{(i)}_{k',k}\right],
\end{aligned}\tag{2.14}$$

and its inverse

$$\begin{aligned}
b^{(j)}_{k'} &= \left(\Phi,\, g^{(j)}_{k'}\right)\\
&= \int\frac{dk}{(2\omega)(2\pi)}\left[a_k\alpha^{(j)}_{k,k'}+a^{\dagger}_{k}\beta^{(j)*}_{k,k'}\right].
\end{aligned}\tag{2.15}$$

Trough the expansion of the field operator in these two sets of modes, we have implicitly defined two types of vacuum, namely the *Minkowski vacuum*, which is annihilated by all the a_k's

$$a_k\left|0_M\right\rangle = 0\ \forall\ k\in\mathbb{R},$$

and the *Rindler vacuum*, which is annihilated by all the b_k

$$b^{(i)}_{k}\left|0_R\right\rangle = 0\ \forall\ k\in\mathbb{R},\ i=1,2.$$

Since there are two types of annihilation operators, along with them we have two (strictly speaking there are three) types of number operators, counting quanta in the mode k, i.e. the ordinary Minkowski number operator

$$N_{kM} = a^{\dagger}_{k}a_k,$$

and the Rindler number operators

$$N^{(i)}_{kR} = b^{(i)\dagger}_{k}b^{(i)}_{k},$$

which count the number of quanta in the $i=R,L$ sections.

Clearly we have

$$\langle 0_M \,|\, N_{kM} \,|\, 0_M \rangle = 0 = \left\langle 0_R \,\middle|\, N_{kR}^{(i)} \,\middle|\, 0_R \right\rangle,$$

but one can also consider the expectation value of the Rindler number operators in the Minkowski vacuum[4]

$$\begin{aligned}
\left\langle b_k^{(i)\dagger} b_k^{(i)} \right\rangle_M &= \int \frac{dk'}{(2\omega')(2\pi)} \int \frac{dk''}{(2\omega'')(2\pi)} \left\langle \left[a_{k''}^{\dagger} \alpha_{k'',k}^{(i)*} + a_{k''} \beta_{k'',k}^{(i)} \right] \left[a_{k'} \alpha_{k',k}^{(i)} + a_{k'}^{\dagger} \beta_{k',k}^{(i)*} \right] \right\rangle_M \\
&= \int \frac{dk'}{(2\omega')(2\pi)} \int \frac{dk''}{(2\omega'')(2\pi)} \beta_{k'',k}^{(i)} \beta_{k',k}^{(i)*} \left\langle \left[a_{k''} \,,\, a_{k'}^{\dagger} \right] \right\rangle_M \\
&= \int \frac{dk'}{(2\omega')(2\pi)} \int \frac{dk''}{(2\omega'')(2\pi)} \beta_{k'',k}^{(i)} \beta_{k',k}^{(i)*} \left\langle (2\omega')(2\pi)\, \delta(k' - k'') \right\rangle_M \qquad (2.16) \\
&= \int \frac{dk'}{(2\omega')(2\pi)} |\beta_{k',k}^{(i)}|^2,
\end{aligned}$$

and we deduce that, as long as $\beta_{k',k}^{(i)} \neq 0$, Minkowski vacuum does not appear to be empty to an accelerated observer.

One could now use the inner product (2.12) of the Minkowski and Rindler modes to calculate this last expression by brute force, but we are going to do this in the next section by a more elegant method due to Unruh [7].

[4] A careful analysis shows that this integral does not converge in this form because of the distributional nature of the ladder operators. Therefore one would have to regularize this expression by some point splitting procedure as $\lim_{q \to k} \left\langle b_k^{(i)\dagger} b_q^{(i)} \right\rangle_M$ or by introducing smeared creation and annihilation operators as follows

$$\begin{aligned}
b_f &= \int \frac{dk}{\sqrt{2\pi 2\omega}} f(k)\, b_k, \\
b_f^{\dagger} &= \int \frac{dk}{\sqrt{2\pi 2\omega}} f^*(k)\, b_k^{\dagger},
\end{aligned}$$

which renders the commutation relations

$$\begin{aligned}
\left[b_f \,,\, b_f^{\dagger} \right] &= \int \frac{dk}{\sqrt{2\pi 2\omega}} \frac{dk'}{\sqrt{2\pi 2\omega'}} f(k)\, f^*\left(k'\right) \left[b_k \,,\, b_{k'}^{\dagger} \right] \\
&= \int \frac{dk}{\sqrt{2\pi 2\omega}} \frac{dk'}{\sqrt{2\pi 2\omega'}} f(k)\, f^*\left(k'\right) 2\pi 2\omega \delta\left(k - k'\right) \\
&= \int dk \left| f(k)^2 \right| = 1
\end{aligned}$$

for normalized shape functions $f(k)$.

2.5 Unruh's method of analytic continuation of the Rindler modes

Unruh's insight consists essentially in analytically continuing the Rindler modes from region R to L, thereby finding a globally (at least in the Rindler wedges) valid expression for the Rindler modes in Minkowski coordinates. This procedure is carried out in a way that guarantees analyticity and boundedness in the lower half complex $u-$plane, resulting in the Unruh modes, which are then (by construction), as shown in paragraph 2.3, superpositions of purely positive frequency Minkowski modes. But this implies that the Unruh and Minkowski modes define the same vacuum state (though excited states may differ) and the Bogolubov coefficients may be read off the resulting expression.

We start with writing out the Rindler modes in Minkowski light cone coordinates

$$\begin{aligned} g_k^{(1)} &= \exp\left[-i\omega\eta + ik\xi\right] \\ &= \left[e^{-a(\eta-\xi)}\right]^{\frac{i\omega}{a}} \Theta(k) + \left[e^{a(\eta+\xi)}\right]^{-\frac{i\omega}{a}} \Theta(-k) \\ &= \left[-au\right]^{\frac{i\omega}{a}} \Theta(k) + \left[av\right]^{-\frac{i\omega}{a}} \Theta(-k) \\ &= \exp\left[\frac{i\omega}{a}\ln(-au)\right]\Theta(k) + \exp\left[-\frac{i\omega}{a}\ln(av)\right]\Theta(-k), \end{aligned} \tag{2.17}$$

where, since in R we have $u < 0$, $v > 0$, the logarithm is well defined as a real function of a positive, nonzero (the origin $u = 0 = v$ is not covered by the hyperbolic Rindler coordinates) argument. However, if we let $u, v \in \mathbb{C}$, we can consider the logarithm as complex function of a complex variable, thereby extending its domain to the entire complex plane with the origin removed. Now we recall from complex analysis that the logarithm function (as well as the power function) on complex domains is a multivalued function and therefore we have to restrict our calculations to one Riemann sheet and stay on this sheet troughout the rest of the computations.

Since we want our resulting mode to be analytic in the lower half complex $u-$plane, we choose the branch cut along the positive imaginary axis (connecting the two branch points at zero and infinity), thereby restricting the range of the the function $\arg z$ to (c.p. fig. 2.3)

$$\begin{aligned} \ln z &= \ln|z| + i\arg z, \\ \arg z &\in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right). \end{aligned}$$

The procedure is now as follows: We start with $\ln(-au)$ in R, where $u < 0$ and try

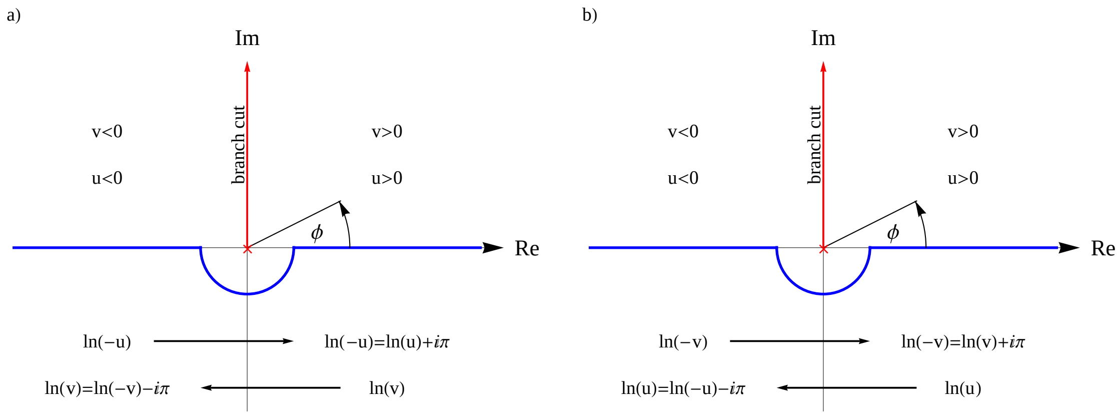


Figure 2.3: The complex $u-$ (and $v-$) plane:
Shown is the branch cut connecting the pole $(0,0)$ with complex infinity $(0, i\infty)$ and the path along which the modes are analytically continued, avoiding the origin and thereby picking up an imaginary part. Note that because of the branch cut, $\phi \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$.
a) shows the continuation of $\ln(-u)$ to positive (real) values of u as well as $\ln v$ to negative (real) values of v, used in the extension of $g_k^{(1)}$ from R to L
b) shows the continuation of $\ln(u)$ to negative (real) values of u as well as $\ln(-v)$ to positive (real) values of v, used in the extension of $g_k^{(2)}$from L to R

to relate it to the function $\ln au$, which is well defined for $u > 0$. This is achieved by following the path shown in fig.2.3, avoiding the branch cut connecting the origin with positive complex infinity $+i\infty$, thereby picking up a phase of $-i\pi$. This results in

$$\ln(-u)\,|_{u>0} = \ln u + i\pi,$$

and therefore

$$\begin{aligned} [-au]^{\frac{i\omega}{a}}\,|_{u>0} &= \exp\left[\frac{i\omega}{a}\ln(-au)\right]|_{u>0} \\ &= \exp\left[\frac{i\omega}{a}\ln(au)\right]\exp\left[-\frac{\pi\omega}{a}\right] \\ &= (au)^{\frac{i\omega}{a}}\exp\left[-\frac{\pi\omega}{a}\right]. \end{aligned}$$

The same procedure is carried out for the left- moving part of the Rindler mode

$$\begin{aligned} \ln(v)\,|_{v<0} &= \ln(-v) - i\pi; \\ [av]^{-\frac{i\omega}{a}}\,|_{v<0} &= \exp\left[-\frac{i\omega}{a}\ln(av)\right]|_{v<0} \\ &= \exp\left[-\frac{i\omega}{a}\ln(-av)\right]\exp\left[-\frac{\pi\omega}{a}\right] \\ &= (-av)^{-\frac{i\omega}{a}}\exp\left[-\frac{\pi\omega}{a}\right], \end{aligned}$$

and thus we have found the analytic continuation of $g_k^{(1)}$ to L (note, however, that the support of the modes $g_k^{(1)}$ and $g_k^{(2)}$ remains restricted to R, respectively L)

$$g_k^{(1)}|_{u>0\,;\,v<0} \quad\rightarrow\quad \exp\left[-\frac{\pi\omega}{a}\right]\left\{[au]^{\frac{i\omega}{a}}\,\Theta(k) + [-av]^{-\frac{i\omega}{a}}\,\Theta(-k)\right\}. \tag{2.18}$$

Now we write the second Rindler mode $g_k^{(2)}$ in Minkowski coordinates

$$\begin{aligned} g_k^{(2)} &= \exp[i\omega\eta + ik\xi] \\ &= \left[e^{a(\eta+\xi)}\right]^{\frac{i\omega}{a}}\Theta(k) + \left[e^{-a(\eta-\xi)}\right]^{-\frac{i\omega}{a}}\Theta(-k) \\ &= [-av]^{\frac{i\omega}{a}}\,\Theta(k) + [au]^{-\frac{i\omega}{a}}\,\Theta(-k)\,, \end{aligned}$$

which does not match the above expression for $g_k^{(1)}$. However, as Unruh found, its complex conjugate with the wave vector reversed does, i.e.

$$\begin{aligned} g_{-k}^{(2)*} &= \exp[-i\omega\eta + ik\xi] \\ &= \left[e^{-a(\eta-\xi)}\right]^{\frac{i\omega}{a}}\Theta(k) + \left[e^{a(\eta+\xi)}\right]^{-\frac{i\omega}{a}}\Theta(-k) \\ &= [au]^{\frac{i\omega}{a}}\,\Theta(k) + [-av]^{-\frac{i\omega}{a}}\,\Theta(-k)\,, \end{aligned}$$

does indeed, apart from the $\exp\left[-\frac{\pi\omega}{a}\right]$ factor, coincide with (2.18). Therefore, we can write

$$g_k^{(1)}|_{u>0\,;\,v<0} \rightarrow \exp\left[-\frac{\pi\omega}{a}\right] g_{-k}^{(2)*}$$

and consider $\exp\left[-\frac{\pi\omega}{a}\right] g_{-k}^{(2)*}$ as analytic extension of $g_k^{(1)}$ to the left Rindler wedge.

As a consistency check, we we try to apply the same procedure in the opposite direction, that is, whether $e^{-\frac{\pi\omega}{a}} g_{-k}^{(2)*}$, analytically continued to negative values of u, is proportional to $g_k^{(1)}$ and we conclude that the answer is in the affirmative

$$\begin{aligned}
e^{-\frac{\pi\omega}{a}} g_{-k}^{(2)*} &= e^{-\frac{\pi\omega}{a}} \left[[au]^{\frac{i\omega}{a}} \Theta(k) + [-av]^{-\frac{i\omega}{a}} \Theta(-k) \right] \\
&= e^{-\frac{\pi\omega}{a}} \left[\exp\left[\frac{i\omega}{a} \ln(au) \right] \Theta(k) + \exp\left[-\frac{i\omega}{a} \ln(-av) \right] \Theta(-k) \right] \\
&= e^{-\frac{\pi\omega}{a}} \left[\exp\left[\frac{i\omega}{a} (\ln(-au) - i\pi) \right] \Theta(k) + \exp\left[-\frac{i\omega}{a} (\ln(av) + i\pi) \right] \Theta(-k) \right] \\
&= \exp\left[\frac{i\omega}{a} \ln(-au) \right] \Theta(k) + \exp\left[-\frac{i\omega}{a} \ln(av) \right] \Theta(-k) \\
&= (-au)^{\frac{i\omega}{a}} \Theta(k) + (av)^{-\frac{i\omega}{a}} \Theta(-k) .
\end{aligned}$$

We further note that the same construction can be carried out for the Rindler mode $g_k^{(2)}$, which reads in Minkowski coordinates

$$\begin{aligned}
g_k^{(2)} &= \exp[i\omega\eta + ik\xi] \\
&= \left[e^{a(\eta+\xi)} \right]^{\frac{i\omega}{a}} \Theta(k) + \left[e^{-a(\eta-\xi)} \right]^{-\frac{i\omega}{a}} \Theta(-k) \\
&= [-av]^{\frac{i\omega}{a}} \Theta(k) + [au]^{-\frac{i\omega}{a}} \Theta(-k) \\
&= \exp\left[\frac{i\omega}{a} \ln(-av) \right] \Theta(k) + \exp\left[-\frac{i\omega}{a} \ln(au) \right] \Theta(-k) ,
\end{aligned}$$

where the logarithms are again well defined for arguments in $L = \{u, v \in \mathbb{C}^2 | u > 0;\ v < 0\}$. Following the same reasoning as above, we find that the analytic continuation of the logarithmic functions (and along with them, the continuation of the Rindler modes) to R :

$$\begin{aligned}
\ln(-v)\,|_{v>0} &= \ln v + i\pi, \\
[-av]^{\frac{i\omega}{a}}\,|_{v>0} &= \exp\left[-\frac{\pi\omega}{a} \right] \exp\left[\frac{i\omega}{a} \ln(av) \right], \\
[au]^{-\frac{i\omega}{a}}\,|_{u<0} &= \exp\left[-\frac{\pi\omega}{a} \right] \exp\left[-\frac{i\omega}{a} \ln(-au) \right],
\end{aligned}$$

$$g_k^{(2)} \to \exp\left[-\frac{\pi\omega}{a} \right] \left\{ [av]^{\frac{i\omega}{a}} \Theta(k) + [-au]^{-\frac{i\omega}{a}} \Theta(-k) \right\} = \exp\left[-\frac{\pi\omega}{a} \right] g_{-k}^{(1)*},$$

does agree with the complex conjugate of the first Rindler mode with the wave vector reversed.

Thus, we have found that the combinations

$$
\begin{aligned}
h_k^{(1)} &= N_k^{(1)} \left[g_k^{(1)} + e^{-\frac{\pi\omega}{a}} g_{-k}^{(2)*} \right], \\
h_k^{(2)} &= \tilde{N}_k^{(2)} \left[e^{-\frac{\pi\omega}{a}} g_{-k}^{(1)*} + g_k^{(2)} \right] \\
&= N_k^{(2)} \left[g_{-k}^{(1)*} + e^{\frac{\pi\omega}{a}} g_k^{(2)} \right],
\end{aligned}
$$

which from now on will be called the *Unruh modes*, are valid for all values of the coordinates in L *and* R. The factors $N_k^{(i)}$ are to be determined trough normalization and one finds

$$
\begin{aligned}
\left(h_k^{(1)}, h_{k'}^{(1)} \right) &= N_k^{(1)} N_{k'}^{(1)} \left(\left[g_k^{(1)} + e^{-\frac{\pi\omega}{a}} g_{-k}^{(2)*} \right], \left[g_{k'}^{(1)} + e^{-\frac{\pi\omega'}{a}} g_{-k'}^{(2)*} \right] \right) \\
&= N_k^{(1)} N_{k'}^{(1)} (2\omega)(2\pi)\, \delta(k-k') \left[1 - e^{-\pi\frac{\omega+\omega'}{a}} \right] \\
&= \left(N_k^{(1)} \right)^2 (2\omega)(2\pi)\, \delta(k-k') \left[1 - e^{-2\pi\frac{\omega}{a}} \right] \\
N_k^{(1)} &= \frac{1}{\sqrt{1 - e^{-2\pi\frac{\omega}{a}}}},
\end{aligned}
$$

$$
\begin{aligned}
\left(h_k^{(2)}, h_{k'}^{(2)} \right) &= N_k^{(2)} N_{k'}^{(2)} \left(\left[g_{-k}^{(1)*} + e^{\frac{\pi\omega}{a}} g_k^{(2)} \right], \left[g_{-k'}^{(1)*} + e^{\frac{\pi\omega'}{a}} g_{k'}^{(2)} \right] \right) \\
&= N_k^{(2)} N_{k'}^{(2)} (2\omega)(2\pi)\, \delta(k-k') \left[-1 + e^{\pi\frac{\omega+\omega'}{a}} \right] \\
&= \left(N_k^{(2)} \right)^2 (2\omega)(2\pi)\, \delta(k-k') \left[-1 + e^{2\pi\frac{\omega}{a}} \right] \\
N_k^{(2)} &= \frac{1}{\sqrt{-1 + e^{2\pi\frac{\omega}{a}}}}.
\end{aligned}
$$

Eventually, we can now write down the expressions for the *Normalized Unruh Modes* (cp. fig. 2.5, to be compared with fig. 2.4)

$$
\begin{aligned}
h_k^{(1)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}} \left[e^{\frac{\pi\omega}{2a}} g_k^{(1)} + e^{-\frac{\pi\omega}{2a}} g_{-k}^{(2)*} \right] \\
h_k^{(2)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}} \left[e^{-\frac{\pi\omega}{2a}} g_{-k}^{(1)*} + e^{\frac{\pi\omega}{2a}} g_k^{(2)} \right],
\end{aligned}
\tag{2.19}
$$

which are - by construction - analytic in the lower half complex u and v planes.

It remains to show that these modes are also bounded in these half plane, and therefore are superpositions of only positive frequency Minkowski modes as we have shown in section 2.2. For brevity, we check this boundedness criterion only for the right-moving (the $u-$dependent) part of $h_k^{(1)}$, which reads

$$\begin{aligned}
h_k^{(1)}\Theta(k) &= \frac{2\Theta(k)}{\sqrt{1-e^{-2\pi\frac{\omega}{a}}}}\left[-au\right]^{\frac{i\omega}{a}} \\
&= \frac{2\Theta(k)}{\sqrt{1-e^{-2\pi\frac{\omega}{a}}}}\exp\left[\frac{i\omega}{a}\ln(-au)\right] \\
&= \frac{2\Theta(k)}{\sqrt{1-e^{-2\pi\frac{\omega}{a}}}}\exp\left[\frac{i\omega}{a}\left\{\ln(-a)+\ln(u)\right\}\right] \\
&= \frac{2\Theta(k)}{\sqrt{1-e^{-2\pi\frac{\omega}{a}}}}\exp\left[\frac{\pi\omega}{a}+\frac{i\omega}{a}\ln a\right]\exp\left[-\frac{\omega}{a}\arg u+\frac{i\omega}{a}\ln(|u|)\right],
\end{aligned}$$

its absolute value being

$$\begin{aligned}
\left|h_k^{(1)}\Theta(k)\right| &= \frac{2\Theta(k)}{\sqrt{1-e^{-2\pi\frac{\omega}{a}}}}\exp\left[\frac{\pi\omega}{a}\right]\exp\left[-\frac{\omega}{a}\underbrace{\arg u}_{\in(-\pi,0)}\right] \\
&= 2\Theta(k)\sqrt{\frac{\exp\left[\frac{2\pi\omega}{a}\right]}{1-e^{-2\pi\frac{\omega}{a}}}}\exp\left(-\frac{\omega}{a}\phi\right),
\end{aligned}$$

which is clearly bounded from above for all u in the lower half complex plane (where $-\pi<\phi<0$).

For the second Unruh mode we find an exactly analogous result

$$\begin{aligned}
h_k^{(2)}\Theta(k) &= \frac{2}{\sqrt{-1+e^{2\pi\frac{\omega}{a}}}}\left[av\right]^{\frac{i\omega}{a}}\Theta(k) \\
&= \frac{2}{\sqrt{-1+e^{2\pi\frac{\omega}{a}}}}\exp\left[\frac{i\omega}{a}\left\{\ln a+\ln|v|\right\}-\frac{\omega}{a}\arg v\right]\Theta(k) \\
\left|h_k^{(2)}\Theta(k)\right| &= \frac{2}{\sqrt{-1+e^{2\pi\frac{\omega}{a}}}}\exp\left[-\frac{\omega}{a}\underbrace{\arg v}_{\in(-\pi,0)}\right]\Theta(k),
\end{aligned}$$

and hence we can conclude that the above defined Unruh modes do indeed define the same vacuum as the Minkowski modes.

As with any complete set of modes, we can expand the field operator, yielding

$$\Phi=\sum_{i=1,2}\int\frac{dk}{(2\omega)(2\pi)}\left[c_k^{(i)}h_k^{(i)}+c_k^{(i)\dagger}h_k^{(i)*}\right],$$

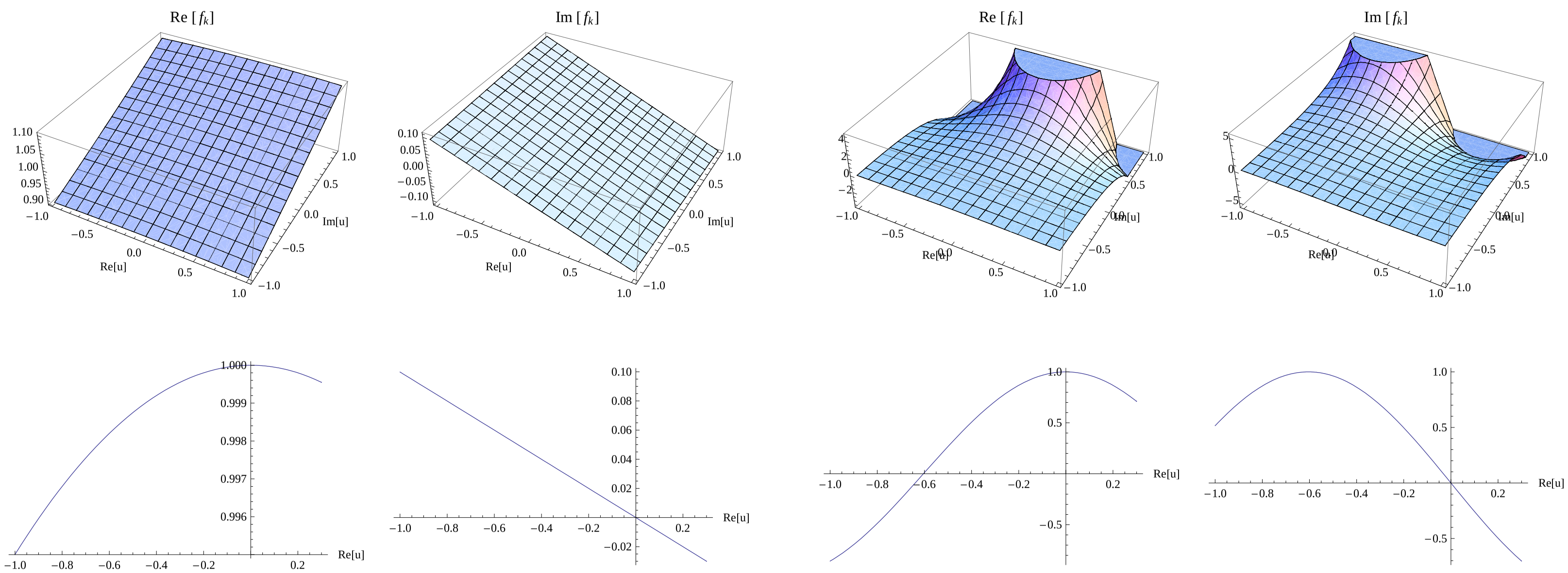


Figure 2.4: Real and imaginary parts (in arbitrary units) of the right-moving, normalized Minkowski mode f_k for two different energies $\omega = 0, 1$ and $\omega = 2, 6$.
In the second row we depicted the restrictions of the corresponding modes to real u.

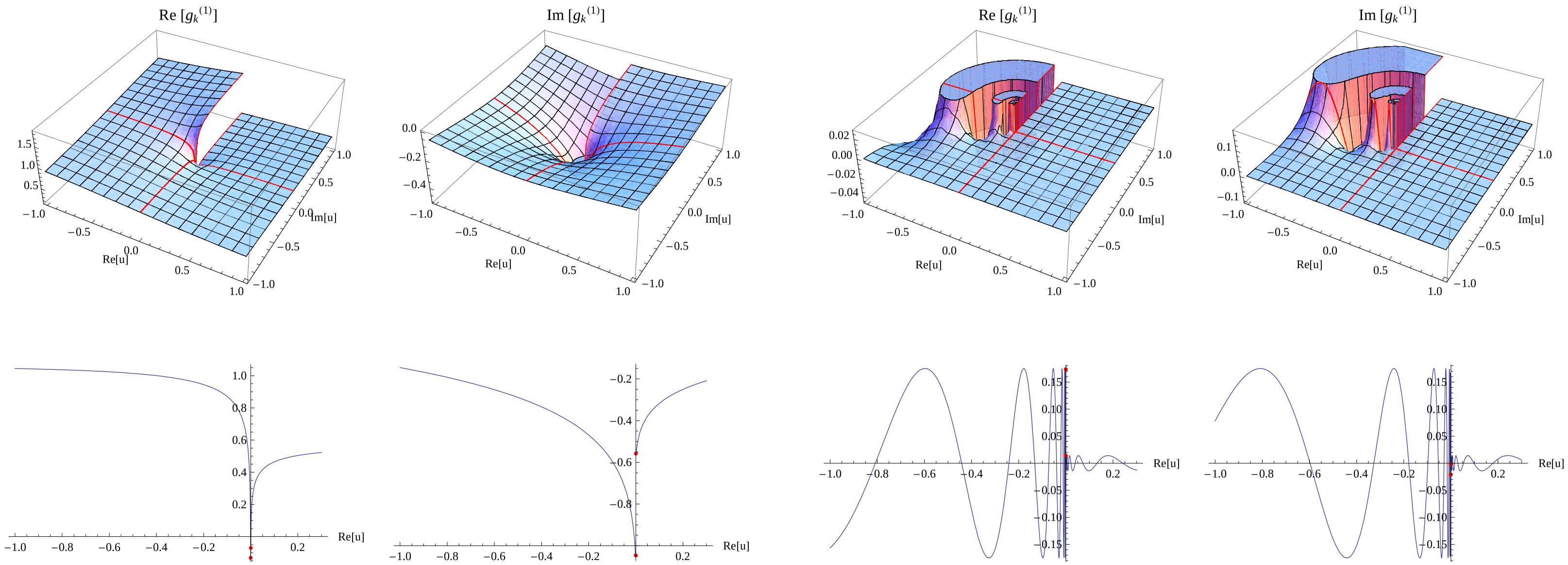


Figure 2.5: Real and imaginary parts (in arbitrary units) of the right-moving, normalized Unruh mode $h_k^{(1)}$ for two different energies $\omega = 0,1$ and $\omega = 2,6$.
In the second row we depicted the restrictions of the corresponding modes to real u. Clearly visible in the upper row is the branch cut, where the modes are not smooth.
Note that in the plot on the right hand side, corresponding to the higher energy mode, one can see the crowding of the oscillations as one approaches the horizon (reflecting the exponential redshift factor), and the functions in the region $\Re u > 0$ have been scaled with a factor 10^6 in order to make them visible.

where now the $c_k^{(i)}$, because of the analyticity and boundedness properties of the Unruh modes, annihilate the Minkowski vacuum

$$c_k^{(i)} \left|0_M\right\rangle = 0 \ \forall \ k \in \mathbb{R}, \ i = 1, 2.$$

Since the Unruh modes define the same vacuum as the Minkowski modes, the modes (2.19) replace the f_k's in (2.13), which reads in this case

$$h_k^{(i)} = \sum_{j=1,2} \int \frac{dk'}{(2\pi)(2\omega')} \left[\alpha_{k',k}^{(i)(j)*} g_{k'}^{(j)} - \beta_{k',k}^{(i)(j)} g_{k'}^{(j)*} \right],$$

and by comparing this with (2.19), we find for the Bogolubov coefficients

$$\begin{aligned}
\alpha_{k',k}^{(1)(1)} &= (2\pi)(2\omega)\, \delta\left(k - k'\right) \frac{e^{\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\frac{\pi\omega}{a}}} = \alpha_{k',k}^{(2)(2)} \\
\alpha_{k',k}^{(1)(2)} &= 0 = \alpha_{k',k}^{(2)(1)},
\end{aligned}$$

as well as

$$\begin{aligned}
\beta_{k',k}^{(1)(1)} &= 0 = \beta_{k',k}^{(2)(2)} \\
\beta_{k',k}^{(1)(2)} &= -\frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\frac{\pi\omega}{a}}} \delta\left(k + k'\right) = \beta_{k',k}^{(2)(1)}.
\end{aligned}$$

As mentioned in a footnote in the previous section, since the corresponding modes are not normalizable, i.e. they are generalized eigenfunctions, we cannot simply insert these expressions for the Bogolubov coefficients in (2.16) and evaluate that integral. Therefore we will calculate directly the matrix element $\left\langle b_k^{(i)\dagger} b_{k'}^{(i)} \right\rangle_M$.

To do so, we need to know the relation between the two sets of annihilation and creation operators. Luckily, we already have done all the work in section 2.4 (in the relations (2.14) and (2.15)) and by comparing this with equation (2.19) , we may readily write down the relations connecting the two sets of annihilation and creation operators

$$\begin{aligned}
b_k^{(1)} &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left[c_k^{(1)} e^{\frac{\pi\omega}{2a}} + c_{-k}^{(2)\dagger} e^{-\frac{\pi\omega}{2a}} \right] \\
b_k^{(2)} &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left[c_k^{(2)} e^{\frac{\pi\omega}{2a}} + c_{-k}^{(1)\dagger} e^{-\frac{\pi\omega}{2a}} \right].
\end{aligned} \tag{2.20}$$

Now we can complete this section by calculating the expectation value of Rindler number operator when the field state is Minkowski vacuum, giving

$$
\begin{aligned}
\left\langle b_k^{(1)\dagger} b_{k'}^{(1)} \right\rangle_M &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}\sqrt{2\sinh\left(\frac{\pi\omega'}{a}\right)}} \left\langle \left[c_k^{(1)\dagger} e^{\frac{\pi\omega}{2a}} + c_{-k}^{(2)} e^{-\frac{\pi\omega}{2a}}\right] \left[c_{k'}^{(1)} e^{\frac{\pi\omega}{2a}} + c_{-k'}^{(2)\dagger} e^{-\frac{\pi\omega}{2a}}\right] \right\rangle_M \\
&= \frac{e^{-\frac{\pi\omega}{a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}\sqrt{2\sinh\left(\frac{\pi\omega'}{a}\right)}} \left\langle 0_M \left| c_{-k}^{(2)} c_{-k'}^{(2)\dagger} \right| 0_M \right\rangle \\
&= \frac{\delta\left(k-k'\right)}{e^{\frac{2\pi\omega}{a}}-1} \left\langle 0_M \mid 0_M \right\rangle \\
&= \frac{\delta\left(k-k'\right)}{e^{\frac{2\pi\omega}{a}}-1},
\end{aligned}
$$

corresponding to a *Bose Einstein distribution at temperature*

$$T = \frac{a}{2\pi}, \tag{2.21}$$

which is Unruh‘s famous result stating that the Minkowski vacuum behaves like a thermal state for accelerating observers.

This temperature (2.21) is the temperature that would be measured by an observer with $\xi = 0$. To determine the temperature perceived by an arbitrary $\xi = const.$ observer, one can make use of the so-called Tolman relation (taking into account the conformal factor, see e.g. [1]), which in this case reads

$$
\begin{aligned}
T(\xi) &= \left(-g_{00}\right)^{-1/2} T \\
&= e^{-a\xi}\frac{a}{2\pi},
\end{aligned} \tag{2.22}
$$

and goes to zero as ξ tends to infinity. This result is also what one would expect, since at $\xi = \infty$ the observer becomes less accelerated with respect to the inertial (Minkowski) observer, and therefore defines the same vacuum as a Minkowski observer. Note that this result could also have been obtained by making use of the redshift factor determined in section 2.1, i.e. radiation emitted at ξ_1 with frequency ω_1 and observed at ξ_2 will be perceived as having a frequency

$$\omega_2 = \frac{V_1}{V_2}\omega_1 = e^{a(\xi_1-\xi_2)}\omega_1,$$

and since an observer at $\xi_1 = 0$ measures the temperature (2.21) we recover the above result.

Note that this temperature (2.22) diverges as $\xi \to -\infty$, which is quite reasonable since an observer, coming closer and closer to the horizons, will have to accelerate more and more to keep his motion stationary.

2.6 Minkowski vacuum as a thermal state

2.6.1 Minkowski vacuum as multi-particle Rindler state

Motivated by the above results, we show in this section that the Minkowski vacuum can be written as a multi-particle Rindler state. This can be seen as follows, starting with an Ansatz

$$|0_M\rangle = \prod_k \sum_{n_k, m_k=0}^{\infty} C_{n_k, m_k} |n_k^{(2)}\rangle |m_k^{(1)}\rangle , \tag{2.23}$$

where

$$|n_k^{(2)}\rangle |m_k^{(1)}\rangle \quad = \quad \frac{\left(b_{-k}^{(2)\dagger}\right)^{n_k} \left(b_k^{(1)\dagger}\right)^{m_k}}{\sqrt{n_k! m_k!}} |0_R\rangle ,$$

is a Rindler state containing $n_k^{(2)}$ $\left(m_k^{(1)}\right)$ quanta in mode k in the left (right) Rindler wedge.

Next, we show that quanta in the left and right region are always produced in pairs, i.e.

$$\left[b_k^{(1)\dagger} b_k^{(1)} - b_{-k}^{(2)\dagger} b_{-k}^{(2)}\right] |0_M\rangle = 0.$$

The proof is straightforward by inserting the expressions (2.20)

$$\begin{aligned}
b_k^{(1)\dagger} b_k^{(1)} |0_M\rangle &= \frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} b_k^{(1)\dagger} c_{-k}^{(2)\dagger} |0_M\rangle \\
&= \frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(\left[b_k^{(1)\dagger}, c_{-k}^{(2)\dagger}\right] + c_{-k}^{(2)\dagger} b_k^{(1)\dagger}\right) |0_M\rangle \\
&= \frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(\frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} + \frac{e^{\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} c_{-k}^{(2)\dagger} c_k^{(1)\dagger}\right) |0_M\rangle ,
\end{aligned}$$

$$\begin{aligned} b^{(2)\dagger}_{-k} b^{(2)}_{-k} \left|0_M\right\rangle &= \frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(\left[b^{(2)\dagger}_{-k},\, c^{(1)\dagger}_{k} \right] + c^{(1)\dagger}_{k} b^{(2)\dagger}_{-k} \right) \left|0_M\right\rangle \\ &= \frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(\frac{e^{-\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} + \frac{e^{\frac{\pi\omega}{2a}}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} c^{(2)\dagger}_{-k} c^{(1)\dagger}_{k} \right) \left|0_M\right\rangle , \end{aligned}$$

and therefore we find

$$\begin{aligned} \left[b^{(1)\dagger}_{k} b^{(1)}_{k} - b^{(2)\dagger}_{-k} b^{(2)}_{-k} \right] \left|0_M\right\rangle &= \prod_k \sum_{n_k, m_k = 0}^{\infty} C_{n_k, m_k} \left[m_k - n_k \right] \left|n^{(2)}_k\right\rangle \left|m^{(1)}_k\right\rangle = 0, \\ m_k &= n_k \;\; \forall k. \end{aligned}$$

This implies that the expansion matrix C is diagonal,

$$C_{n_k, m_k} = \delta_{n_k, m_k} K_{n_k},$$

and thus,

$$\left|0_M\right\rangle = \prod_k \sum_{n_k=0}^{\infty} K_{n_k} \left|n^{(2)}_k\right\rangle \left|n^{(1)}_k\right\rangle .$$

By plugging this into the equation defining the Minkowski vacuum one gets

$$\begin{aligned} 0 &= c^{(1)}_{k} \left|0_M\right\rangle \\ &\propto \left[b^{(1)}_{k} - e^{-\frac{\pi\omega}{a}} b^{(2)\dagger}_{-k} \right] \left|0_M\right\rangle , \end{aligned}$$

$$\begin{aligned} 0 &= \prod_q \sum_{n_q=0}^{\infty} K_{n_q} \left[b^{(1)}_{k} - e^{-\frac{\pi\omega}{a}} b^{(2)\dagger}_{-k} \right] \left|n^{(2)}_q\right\rangle \left|n^{(1)}_q\right\rangle \\ &= \prod_q \sum_{n_q=0}^{\infty} K_{n_q} \left[\sqrt{n_q}\, \delta_{n_q, n_k} \left|n^{(2)}_q\right\rangle \left|n^{(1)}_q - 1\right\rangle - e^{-\frac{\pi\omega}{a}} b^{(2)\dagger}_{-k} \sqrt{n_q + 1}\, \delta_{n_q, n_k} \left|n^{(2)}_q + 1\right\rangle \left|n^{(1)}_q\right\rangle \right] \\ &= \prod_k \sum_{n_k=0}^{\infty} K_{n_k} \left[\sqrt{n_k} \left|n^{(2)}_k\right\rangle \left|n^{(1)}_k - 1\right\rangle - e^{-\frac{\pi\omega}{a}} \sqrt{n_k + 1} \left|n^{(2)}_k + 1\right\rangle \left|n^{(1)}_k\right\rangle \right] . \end{aligned}$$

Projecting this equation with $\langle 0^{(1)}_k|$ yields

$$\begin{aligned} 0 &= \prod_k \sum_{n_k=0}^{\infty} \left[K_{n_k+1} \sqrt{n_k + 1} \left|n^{(2)}_k + 1\right\rangle \left|n^{(1)}_k\right\rangle - e^{-\frac{\pi\omega}{a}} K_{n_k} \sqrt{n_k + 1} \left|n^{(2)}_k + 1\right\rangle \left|n^{(1)}_k\right\rangle \right] \\ &= \left[K_1 - e^{-\frac{\pi\omega}{a}} K_0 \right] \left|1^{(2)}_k\right\rangle , \end{aligned}$$

and with $\langle m_k^{(1)}|$ we find, (where $m = 1, 2, 3, \ldots$)

$$\begin{aligned} 0 &= \prod_k \sum_{n_k=0}^{\infty} \left[K_{n_k} \sqrt{n_k}\, |n_k^{(2)}\rangle\, \delta_{m,n_k-1} - e^{-\frac{\pi\omega}{a}} K_{n_k} \sqrt{n_k+1}\, |n_k^{(2)}+1\rangle\, \delta_{m,n_k} \right] \\ &= \left[K_{m+1} - e^{-\frac{\pi\omega}{a}} K_m \right] \sqrt{m+1}\, |m^{(2)}\rangle\,. \end{aligned}$$

Hence,

$$\begin{aligned} K_{m+1} &= e^{-\frac{\pi\omega}{a}} K_m \\ K_1 &= e^{-\frac{\pi\omega}{a}} K_0 \\ K_m &= e^{-\frac{m\pi\omega}{a}} K_0, \end{aligned}$$

and expression (2.23) becomes

$$|0_M\rangle = \prod_k \sum_{n_k=0}^{\infty} e^{-n_k \frac{\pi\omega}{a}} K_{0_k}\, |n_k^{(2)}\rangle\, |n_k^{(1)}\rangle\,.$$

The factor K_0 is again trough the normalization condition

$$\begin{aligned} 1 &= \langle 0_M \,|\, 0_M \rangle \\ &= \prod_{k,q} \sum_{n_k,n_q=0}^{\infty} e^{-n_q \frac{\pi|q|}{a}} e^{-n_k \frac{\pi\omega}{a}} K_{0_k} K_{0_q}^* \left\langle n_q^{(1)} \,\middle|\, n_k^{(1)} \right\rangle \left\langle n_q^{(2)} \,\middle|\, n_k^{(2)} \right\rangle \\ &= \prod_k |K_{0_k}|^2 \sum_{n_k=0}^{\infty} e^{-2n_k \frac{\pi\omega}{a}} \\ &= \prod_k |K_{0_k}|^2 \frac{1}{1 - e^{-2\frac{\pi\omega}{a}}}, \\ K_{0_k} &= \sqrt{1 - e^{-2\frac{\pi\omega}{a}}}. \end{aligned}$$

Eventually, we have found the desired expression for the Minkowski vacuum state in terms of Rindler states:

$$|0_M\rangle = \prod_k \sqrt{1 - e^{-2\frac{\pi\omega}{a}}} \sum_{n_k=0}^{\infty} e^{-n_k \frac{\pi\omega}{a}}\, |n_k^{(2)}\rangle\, |n_k^{(1)}\rangle\,,$$

which will furnish the interpretation of the Minkowski vacuum in terms of a partial trace, as will be discussed in the following section.

2.6.2 The Minkowski vacuum density matrix

Since the two Rindler wedges are causally totally disconnected, that is an observer in R can never gather any information about processes and measurements in L, we are led to construct a density matrix, reflecting the fact that no complete information can be obtained by an observer in R.

The Minkowski vacuum is a pure state, therefore its density matrix is given by the dyadic product

$$\begin{aligned}\varrho &= |0_M\rangle\langle 0_M| \\ &= \prod_{k,q}\sqrt{1-e^{-2\frac{\pi\omega}{a}}}\sqrt{1-e^{-2\frac{\pi|q|}{a}}}\sum_{n_k,n_q=0}^{\infty} e^{-n_k\frac{\pi\omega}{a}}e^{-n_q\frac{\pi|q|}{a}}\,|n_k^{(2)}\rangle\,|n_k^{(1)}\rangle\,\langle n_q^{(2)}|\,\langle n_q^{(1)}|\,.\end{aligned}$$

If an observer is restricted to region R, no information about $|n_k^{(2)}\rangle$ can be gained, so we trace out these degrees of freedom

$$\begin{aligned}\rho_R &= \sum_{n_l=0}^{\infty}\langle n_l^{(2)}|0_M\rangle\,\langle 0_M|n_l^{(2)}\rangle \\ &= \prod_{k,q}\sqrt{1-e^{-2\frac{\pi\omega}{a}}}\sqrt{1-e^{-2\frac{\pi|q|}{a}}}\sum_{n_k,n_q,n_l=0}^{\infty} e^{-n_k\frac{\pi\omega}{a}}e^{-n_q\frac{\pi|q|}{a}}\,\langle n_l^{(2)}|n_k^{(2)}\rangle\,|n_k^{(1)}\rangle\,\langle n_q^{(1)}|\,\langle n_q^{(2)}|n_l^{(2)}\rangle \\ &= \prod_{k,q}\sqrt{1-e^{-2\frac{\pi\omega}{a}}}\sqrt{1-e^{-2\frac{\pi|q|}{a}}}\sum_{n_k,n_q,n_l=0}^{\infty} e^{-n_k\frac{\pi\omega}{a}}e^{-n_q\frac{\pi|q|}{a}}\,\delta_{n_l,n_k}\delta_{n_l,n_q}\,|n_k^{(1)}\rangle\,\langle n_q^{(1)}| \\ &= \prod_{k}\left(1-e^{-2\frac{\pi\omega}{a}}\right)\sum_{n_k=0}^{\infty} e^{-2n_k\frac{\pi\omega}{a}}\,|n_k^{(1)}\rangle\,\langle n_k^{(1)}|\,.\end{aligned}$$

By introducing some terminology

$$\begin{aligned}E_{n_k} &= n_k\omega \\ \beta &= \frac{2\pi}{a},\end{aligned}$$

we can make clearer the *thermal nature of this density matrix*

$$
\begin{aligned}
\rho_R &= \prod_k \left(1 - e^{-2\frac{\pi\omega}{a}}\right) \sum_{n_k=0}^{\infty} e^{-2n_k \frac{\pi\omega}{a}} \left|n_k^{(1)}\right\rangle \left\langle n_k^{(1)}\right| \\
&= \prod_k \left(\sum_{m=0}^{\infty} e^{-E_m \beta}\right)^{-1} \sum_{n_k=0}^{\infty} e^{-E_{n_k}\beta} \left|n_k^{(1)}\right\rangle \left\langle n_k^{(1)}\right| \\
&= \prod_k \sum_{n_k=0}^{\infty} \frac{e^{-E_{n_k}\beta}}{\sum_{m=0}^{\infty} e^{-E_m \beta}} \left|n_k^{(1)}\right\rangle \left\langle n_k^{(1)}\right| .
\end{aligned}
$$

2.7 Green functions and the KMS condition

2.7.1 The KMS condition

The probability for an arbitrary system in thermal equilibrium at temperature $T = \beta^{-1}$, governed by a Hamiltonian H, to be in a state $|\psi_i\rangle$ with eigenvalue E_i is (we neglect the chemical potential) exactly the Boltzmann distribution

$$
\begin{aligned}
\rho_i &= Z^{-1} \exp\left[-\beta E_i\right], \\
1 &= \sum_i \rho_i,
\end{aligned}
$$

and expectation values of operators A are determined via

$$
\langle A \rangle_\beta = \sum_i \rho_i \left\langle \psi_i \,|\, A \,|\, \psi_i \right\rangle .
$$

By introducing the density matrix operator

$$
\rho = Z^{-1} \exp\left[-\beta H\right],
$$

where $Z = tr \exp\left(-\beta H\right)$ and one sees that

$$
\rho_i = \left\langle \psi_i \,|\, \rho \,|\, \psi_i \right\rangle ,
$$

$$
tr\left(\rho\right) \equiv \sum_i \left\langle \psi_i \,|\, \rho \,|\, \psi_i \right\rangle = 1,
$$

ensures normalization and the expectation values are calculated via

$$
\langle A \rangle_\beta = tr\left(\rho A\right).
$$

A thermal density operator has two defining properties: first, it is clearly stationary since

$$[\rho, H] = 0,$$

and secondly expectation values in the state ρ possess certain symmetries under time translations in the complex $t-$plane, called the KMS condition (named after Kubo, Martin, Schwinger).

To pursue the second property, we recall the Heisenberg equation of motion

$$A(t) = e^{-itH} A(0) e^{itH},$$

and consider the following expectation value of two Heisenberg operators $A(t)$ and $B(t)$

$$\begin{aligned} \langle A(t) B(t) \rangle_\beta &= \frac{1}{Z} tr \left[e^{-\beta H} A(t) B(t) \right] \\ &= \frac{1}{Z} tr \left[e^{-\beta H} A(t) e^{\beta H} e^{-\beta H} B(t) \right] \\ &= \frac{1}{Z} tr \left[A(t - i\beta) e^{-\beta H} B(t) \right] \\ &= \frac{1}{Z} tr \left[B(t) A(t - i\beta) e^{-\beta H} \right] \end{aligned}$$

and therefore

$$\langle A(t) B(t) \rangle_\beta = \langle B(t) A(t - i\beta) \rangle_\beta \tag{2.24}$$

where we repeatedly used the invariance of the trace under cyclic permutations of its arguments. This periodicity with a twist (i.e. the reversed order in which the operators appear) is exactly the KMS condition. Note that the above derivation may not make sense if one of the operators A or B are unbounded. Nevertheless, the condition (2.24) can also be seen as *definition of thermal equilibrium at temperature* $T = \beta^{-1}$ and does also apply in cases where Z diverges (which is the case for our considerations in spatially infinite systems [3]).

2.7.2 The (thermal) Green function for the massless scalar field

To compare this general result with our results, we will calculate Green functions (to be accurate, we will consider Hadamard's elementary function) for the massless scalar field, which can be written as expectation values of products of field operators (see e.g. in [1]). Note that, strictly speaking, the Green function for the massless field in two dimensional Minkowski spacetime does not exist due to a non-integrable singularity at the origin in (2.26). However, one may choose a "liberal" point of view toward this problem, adopted also in [3], where the authors write:

> *Liberal:* A quantum field theory exists [for the massless Klein Gordon field in two-dimensional Minkowski spacetime], in a genuine Hilbert space, but the field operators are not defined on all test functions.

and, regarding the existence of the (Wightman) two-point function:

> (1) No two-point function exists for such a theory. (2) As two-point function one may choose any "regularization", remembering that its value on illicit test functions is irrelevant to physics.

We adopt the point of view (2), since we are not interested in extracting any numerical results from the Wightman function, but rather want to show that it satisfies the periodicity conditions in imaginary time characteristic of thermal states.

Hadamard's elementary function is given by the vacuum expectation value of the anti-commutator of two field configurations at different spacetime points:

$$\begin{aligned} D_M^{(1)}\left(x,x'\right) &= \left\langle 0_M \,\middle|\, \left\{\Phi\left(x\right)\, ,\, \Phi\left(x'\right)\right\} \,\middle|\, 0_M \right\rangle \\ &= \left\langle 0_M \,\middle|\, \Phi\left(x\right)\Phi\left(x'\right) \,\middle|\, 0_M \right\rangle + \left\langle 0_M \,\middle|\, \Phi\left(x'\right)\Phi\left(x\right) \,\middle|\, 0_M \right\rangle , \end{aligned} \tag{2.25}$$

the first term of which is called the Wightman function and is given by

$$\begin{aligned} \left\langle 0_M \,\middle|\, \Phi\left(x\right)\Phi\left(x'\right) \,\middle|\, 0_M \right\rangle &= \int \frac{dk}{\left(2\omega\right)\left(2\pi\right)}\frac{dk'}{\left(2\omega'\right)\left(2\pi\right)} f_k\left(x\right) f_{k'}^{*}\left(x'\right) \left\langle 0_M \,\middle|\, a_k a_{k'}^{\dagger} \,\middle|\, 0_M \right\rangle \\ &= \int \frac{dk}{\left(2\omega\right)\left(2\pi\right)}\frac{dk'}{\left(2\omega'\right)\left(2\pi\right)} f_k\left(x\right) f_{k'}^{*}\left(x'\right)\left(2\omega\right)\left(2\pi\right)\delta\left(k-k'\right) \\ &= \int \frac{dk}{\left(2\omega\right)\left(2\pi\right)} f_k\left(x\right) f_k^{*}\left(x'\right) \\ &= \frac{1}{4\pi}\int \frac{dk}{\omega}\exp\left[i\left(-\omega\left(t-t'\right)+k\left(x-x'\right)\right)\right] . \end{aligned} \tag{2.26}$$

When one naively interprets this expression as an ordinary function, this integral shows twofold divergence as can be seen by neglecting the exponential in the integrand and considering

$$\int \frac{dk}{4\pi\omega} = 2\int_0^\infty \frac{dk}{4\pi\omega} = \frac{1}{2\pi}\ln\omega\big|_0^\infty ,$$

which shows an ultraviolet divergence for large values of ω, as well as an infrared divergence for small ones.

This fact is remedied by considering (2.26) as a tempered distribution, i.e. $D_M^{(1)} \in \mathcal{S}'\left(\mathbb{R}^2\right)$ and let it act on a rapidly decreasing testfunction $\varphi \in \mathcal{S}\left(\mathbb{R}^2\right)$, which has the advantage that the Fourier-transformed test function is itself a test function, that is $\widetilde{\varphi} \in \mathcal{S}\left(\mathbb{R}^2\right)$.

Setting $x' = 0$ in equation (2.26) yields

$$\begin{aligned}\left(D_M^{(1)}\left(x^a,0\right)\,,\,\varphi\left(x^a\right)\right) &= \frac{1}{4\pi}\int\frac{d^2x dk}{|k|}\exp\left[i\left(-|k|\,x^0+kx^1\right)\right]\varphi\left(x^0,x^1\right)\\ &= \int\frac{dk}{2\,|k|}\widetilde{\varphi}\left(|k|\,,k\right),\end{aligned}$$

from which it is clear that the UV-divergence is irrelevant because of the rapid decrease of the test function, while the IR-divergence is avoided if one uses only testfunctions whose support does not include the origin in $k-$space (therefore, to remain in the terminology of the above lines, the illicit testfunctions are those whose support includes $k = 0$).

To get an explicit expression for equation (2.26), we regularize it by introducing an infrared cutoff λ as well as the usual $-i\epsilon$ prescription to assure damping of the integrand for large values of ω . Thus the integral becomes (where $\Delta u = u - u' = t - x - t' + x'$ and analogously for Δv)

$$\begin{aligned}\frac{1}{4\pi}\int\frac{dk}{\omega}\exp\left[i\left(-\omega\left(t-t'\right)+k\left(x-x'\right)\right)\right] &= \frac{1}{4\pi}\int_\lambda^\infty\frac{d\omega}{\omega}\exp\left[-i\omega\left(t-x-t'+x'-i\epsilon\right)\right]\\ &\quad+\frac{1}{4\pi}\int_\lambda^\infty\frac{d\omega}{\omega}\exp\left[-i\omega\left(t+x-t'-x'-i\epsilon\right)\right]\\ &= \frac{1}{4\pi}\int_\lambda^\infty\frac{d\omega}{\omega}\exp\left[-i\omega\left(\Delta u-i\epsilon\right)\right]\\ &\quad+\frac{1}{4\pi}\int_\lambda^\infty\frac{d\omega}{\omega}\exp\left[-i\omega\left(\Delta v-i\epsilon\right)\right].\end{aligned}$$

To evaluate this further, we rewrite those integrals as

$$\begin{aligned}\int_\lambda^\infty\frac{d\omega}{\omega}\exp\left[-i\omega\left(\Delta u-i\epsilon\right)\right] &= \int_\lambda^\infty\frac{d\omega}{\omega}\exp\left[-\omega\mu_u\right],\\ \text{where}\qquad &\mu_u = i\Delta u+\epsilon,\end{aligned}$$

and perform an integration by parts to arrive at

$$\begin{aligned}\int_\lambda^\infty\frac{d\omega}{\omega}\exp\left[-\omega\mu\right] &= \ln\omega\exp\left[-\omega\mu\right]_\lambda^\infty+\mu\int_\lambda^\infty d\omega\ln\omega\exp\left[-\omega\mu\right]\\ &= -\ln\lambda+\mu\int_\lambda^\infty d\omega\ln\omega\exp\left[-\omega\mu\right],\end{aligned}$$

where the upper limit gives no contribution because of $\Re\mu = \epsilon > 0$. For the same reason, the integral appearing on the right hand side above does also converge for $\lambda = 0$ and is tabulated (see e.g. 4.331.1 in [4])

$$\int_0^\infty d\omega \ln\omega \exp\left[-\omega\mu\right] = -\frac{1}{\mu}\left(\gamma + \ln\mu\right) \quad , \Im\mu > 0,$$

where $\gamma \approx 0,577216$ is Euler's constant. With these results, we find for the Wightman function

$$\begin{aligned}
\left\langle 0_M \,|\, \Phi(x)\,\Phi(x')\,|\, 0_M \right\rangle &= \frac{1}{4\pi}\int_\lambda^\infty \frac{d\omega}{\omega}\left\{\exp\left[-\omega\mu_u\right] + \exp\left[-\omega\mu_v\right]\right\} \\
&= \frac{1}{4\pi}\left\{-2\ln\lambda - 2\gamma - \ln\mu_u - \ln\mu_v\right\} \\
&= \frac{1}{4\pi}\left\{-2\ln\lambda - 2\gamma - \ln\left[\left(i\Delta u + \epsilon\right)\left(i\Delta v + \epsilon\right)\right]\right\} \\
&= \frac{1}{4\pi}\left\{-2\ln\lambda - 2\gamma - \ln\left[-\Delta u\Delta v\right]\right\} \\
&= \frac{1}{4\pi}\left\{-2\ln\lambda - 2\gamma - \ln\left[\Delta u\Delta v\right] + i\pi\right\}.
\end{aligned}$$

From the second Wightman function appearing in (2.25) we get exactly the same contribution

$$\begin{aligned}
D_M^-(x,x') &= \left\langle 0_M \,|\, \Phi(x')\,\Phi(x)\,|\, 0_M \right\rangle \\
&= \frac{1}{4\pi}\left\{-2\ln\lambda - 2\gamma - \ln\left(\Delta u\Delta v\right) + i\pi\right\},
\end{aligned}$$

so that Hadamard's elementary function becomes

$$\begin{aligned}
D_M^{(1)}(x,x') &= D_M^+(x,x') + D_M^-(x,x') \\
&= -\frac{1}{2\pi}\ln\left(\Delta u\Delta v\right) + C_{divergent},
\end{aligned} \tag{2.27}$$

where we absorbed all of the constants and the divergent $\lim_{\lambda\to 0}\ln\lambda$ terms in the constant $C_{divergent}$, to be dropped hereafter.

Now we write the arguments of the logarithms appearing in (2.27) in terms of Rindler coordinates (2.3) and find

$$\begin{aligned}
\Delta u &= u - u' \\
&= \frac{1}{a}\left[e^{a(\xi'-\eta')} - e^{a(\xi-\eta)}\right] \\
\Delta v &= \frac{1}{a}\left[e^{a(\xi+\eta)} - e^{a(\xi'+\eta')}\right], \\
\Delta u\Delta v &= \frac{2}{a^2}e^{a(\xi+\xi')}\left[\cosh\left(a\left[\eta-\eta'\right]\right) - \cosh\left(a\left[\xi-\xi'\right]\right)\right],
\end{aligned}$$

which renders (2.27)

$$\begin{aligned} D_M^{(1)}\left(x,x'\right) &= -\frac{1}{2\pi}\ln\left(\Delta u \Delta v\right) \\ &= -\frac{1}{2\pi}\ln\left[\cosh\left(a\left[\eta-\eta'\right]\right)-\cosh\left(a\left[\xi-\xi'\right]\right)\right] \\ &\quad -\frac{1}{2\pi}\ln\left[\frac{2}{a^2}e^{a\left(\xi+\xi'\right)}\right]. \end{aligned}$$

Now this expression is clearly invariant under the replacement

$$\eta \rightarrow \eta + \frac{2\pi n i}{a};\ n \in \mathbb{N},$$

such that Hadamard's elementary function is seen to be periodic in imaginary "boost time" η with period $2\pi/a$, i.e.

$$D_M^{(1)}\left(\xi-\xi'\,;\,\eta-\eta'\right) = D_M^{(1)}\left(\xi-\xi'\,;\,\eta-\eta'+\frac{2\pi n i}{a}\right), \tag{2.28}$$

which reflects precisely the periodicity in imaginary time found in (2.24), but without interchanging the operators in the expectation value[5].

[5]This is because of the symmetry of Hadamard's function, the Wightman functions still show this twist, as is easily seen by considering

$$\begin{aligned} D_\beta^+\left(x,x'\right) &= \left\langle \Phi\left(x\right)\Phi\left(x'\right)\right\rangle_\beta \\ D_\beta^-\left(x,x'\right) &= \left\langle \Phi\left(x'\right)\Phi\left(x\right)\right\rangle_\beta, \end{aligned}$$

and by using the Heisenberg equations of motion one finds

$$\begin{aligned} D_\beta^+\left(t,x\,;\,t',x'\right) &= Z^{-1}\,tr\left[\Phi\left(t,x\right)\Phi\left(t',x'\right)e^{-\beta H}\right] \\ &= Z^{-1}\,tr\left[e^{-\beta H}\Phi\left(t,x\right)e^{\beta H}e^{-\beta H}\Phi\left(t',x'\right)\right] \\ &= Z^{-1}\,tr\left[\Phi\left(t+i\beta,x\right)e^{-\beta H}\Phi\left(t',x'\right)\right] \\ &= Z^{-1}\,tr\left[\Phi\left(t',x'\right)\Phi\left(t+i\beta,x\right)e^{-\beta H}\right] \\ &= D_\beta^-\left(t+i\beta,x\,;\,t',x'\right), \end{aligned}$$

and analogously $D_\beta^-\left(t,x\,;\,t',x'\right) = D_\beta^+\left(t+i\beta,x\,;\,t',x'\right)$, such that

$$\begin{aligned} D_\beta^{(1)}\left(t,x\,;\,t',x'\right) &= D_\beta^+\left(t,x\,;\,t',x'\right)+D_\beta^-\left(t,x\,;\,t',x'\right) \\ &= D_\beta^-\left(t+i\beta,x\,;\,t',x'\right)+D_\beta^+\left(t+i\beta,x\,;\,t',x'\right) \\ &= D_\beta^{(1)}\left(t+i\beta,x\,;\,t',x'\right). \end{aligned}$$

So yet again we have found that Minkowski vacuum behaves to an accelerated observer like a thermal state at temperature

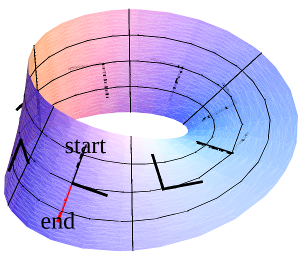
$$T = \beta^{-1} = 2\pi/a.$$

2.8 Killing-conformal choice of positive frequency

Since Minkowski spacetime is clearly globally time orientable[6] (as is Kruskal spacetime), the flipping of the sign (c.f. paragraph 3) when defining positive frequency in the left Rindler wedge, is justified by the requirement that the timelike Killing vector field be future pointing. One could call this procedure a Killing-anti-conformal choice of time slicing in the left Rindler wedge, i.e. the time direction in L is chosen as to agree with the time slicing defined by the Minkowski observer.

Now one can ask the question what happens when a choice of time direction coherent with the Killing vector in L was made? That is, we define positive frequency strictly with respect to the (boost-) Killing vector, be it future (in R) or past (in L) pointing. Being in Minkowski spacetime, this procedure seems rather artificial, however, as we perform the calculations of this section as preparation for the Schwarzschild spacetime, where the two wedges are causally totally decoupled (the two Rindler wedges are also causally decoupled, but the bifurcation point of the Killing horizons is completely arbitrary, that is, it could be relocated to any point (x_0, t_0) by a shift in the definition the Rindler coordinates (2.3) via $x \to x - x_0$, $t \to t - t_0$) and therefore this argumentation is more adept., since the choice of time-orientation in causally decoupled regions is not fixed a priori. Therefore we will consider Killing conformal slices in this section.

[6] A classical example of a non (time-)orientable manifold is given by the Möbius strip, where locally one can always define a forward lightcone. However, as one parallely transports this cone once around the Möbius strip, the forward lightcone is reverted, future and past having changed places.



Since (boost-) time η "runs backward" in L (compared to Minkowski time), one is led to define the positive frequency left Rindler modes with respect to ∂_η

$$g_k^{(2)} = \begin{cases} \exp\left[-i\omega\eta + ik\xi\right] & \text{in L} \\ 0 & \text{in R} \end{cases}.$$

Now something unexpected happens: Following exactly the same argumentation as in paragraph 5, we consider

$$\begin{aligned} g_k^{(1)} &= \exp\left[-i\omega\eta + ik\xi\right] \\ &= \left[e^{-a(\eta-\xi)}\right]^{\frac{i\omega}{a}} \Theta(k) + \left[e^{a(\eta+\xi)}\right]^{-\frac{i\omega}{a}} \Theta(-k) \\ &= \left[-au\right]^{\frac{i\omega}{a}} \Theta(k) + \left[av\right]^{-\frac{i\omega}{a}} \Theta(-k), \end{aligned}$$

and we find that, instead of $g_{-k}^{(2)*}$, the mode

$$\begin{aligned} g_k^{(2)} &= \exp\left[-i\omega\eta + ik\xi\right] \\ &= \left[e^{a(-\eta+\xi)}\right]^{\frac{i\omega}{a}} \Theta(k) + \left[e^{a(\eta+\xi)}\right]^{-\frac{i\omega}{a}} \Theta(-k) \\ &= \left[au\right]^{\frac{i\omega}{a}} \Theta(k) + \left[-av\right]^{-\frac{i\omega}{a}} \Theta(-k) \\ &= e^{\frac{\pi\omega}{a}} \left[-au\right]^{\frac{i\omega}{a}} \Theta(k) + e^{-\frac{\pi\omega}{a}} \left[av\right]^{-\frac{i\omega}{a}} \Theta(-k) \end{aligned}$$

can be used as analytic extension of $g_k^{(1)}$ into L.

Therefore, the Unruh modes become

$$\begin{aligned} \tilde{h}_k^{(1)} &= N_k^{(1)} \left[g_k^{(1)} + e^{-\frac{\pi\omega}{a}} g_k^{(2)}\right] \\ \tilde{h}_k^{(2)} &= N_k^{(2)} \left[g_{-k}^{(1)} + e^{\frac{\pi\omega}{a}} g_{-k}^{(2)}\right], \end{aligned}$$

where normalization yields

$$\begin{aligned} N_k^{(1)} &= \frac{1}{\sqrt{1 + e^{-2\frac{\pi\omega}{a}}}} \\ N_k^{(2)} &= \frac{1}{\sqrt{1 + e^{2\frac{\pi\omega}{a}}}}, \end{aligned}$$

and thus

$$\begin{aligned}
\tilde{h}_k^{(1)} &= \frac{1}{\sqrt{2\cosh\left(\frac{\pi\omega}{a}\right)}}\left[e^{\frac{\pi\omega}{2a}} g_k^{(1)} + e^{-\frac{\pi\omega}{2a}} g_k^{(2)}\right] \\
\tilde{h}_k^{(2)} &= \frac{1}{\sqrt{2\cosh\left(\frac{\pi\omega}{a}\right)}}\left[e^{-\frac{\pi\omega}{2a}} g_{-k}^{(1)} + e^{\frac{\pi\omega}{2a}} g_{-k}^{(2)}\right].
\end{aligned}$$

Yet another time we can expand the field operator in these modes

$$\Phi = \sum_{i=1,2} \int dk \left[c_k^{(i)} h_k^{(i)} + c_k^{(i)\dagger} h_k^{(i)*}\right],$$

and the relations between the annihilation operators are

$$\begin{aligned}
b_k^{(1)} &= \frac{1}{\sqrt{2\cosh\left(\frac{\pi\omega}{a}\right)}}\left[c_k^{(1)} e^{\frac{\pi\omega}{2a}} + c_{-k}^{(2)} e^{-\frac{\pi\omega}{2a}}\right] \\
b_k^{(2)} &= \frac{1}{\sqrt{2\cosh\left(\frac{\pi\omega}{a}\right)}}\left[c_{-k}^{(2)} e^{\frac{\pi\omega}{2a}} + c_k^{(1)} e^{-\frac{\pi\omega}{2a}}\right].
\end{aligned}$$

From the latter expressions we see that the $b_k^{(i)}$ contain only Rindler annihilation operators. But this implies that they define the same vacuum, i.e. the $b_k^{(i)}$ satisfy

$$b_k^{(i)} |0_M\rangle = 0.$$

Hence, we can conclude that *when adopting a Killing-conformal orientation of the time slicings in L, there is no more Unruh radiation* and in the next section, we will try to extend this result to the spherically reduced Schwarzschild geometry to check if the Hawking radiation comes also to a stop when defining positive frequency with respect to the past-pointing Killing vector.

3 Hawking Effect in spherically reduced (2D) Schwarzschild spacetime

In the first part of this section we are going to perform analogous coordinate (as in section 2.1 for the Minkowski/Rindler coordinates) transformations for the Schwarzschild metric, thereby extending the Schwarzschild manifold to the Kruskal spacetime and showing the very similar causal and topological properties. Subsequent to that, we will determine the analogues of Minkowski and Rindler (Unruh) modes in the static black hole spacetime, eventually deriving the Hawking temperature.

3.1 Schwarzschild and Kruskal coordinates

The two dimensional Schwarzschild line element reads (we simply suppress the angular part, or, equivalently, consider $\theta = const. = \phi$)

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2, \tag{3.1}$$

while the inverse metric is given by

$$g^{ba} = -\frac{1}{f}\partial_t^b \partial_t^a + f \partial_r^b \partial_r^a,$$

where $f = 1 - 2M/r$ will be used troughout this section.

Introduction of the so-called tortoise coordinate via $dr_* = f^{-1} dr$ (or in its integrated form $r_* = r + 2M \ln\left[\frac{r}{2M} - 1\right]$; cf. fig. 3.1) puts the above line element in the form

$$ds^2 = f\left(-dt^2 + dr_*^2\right), \tag{3.2}$$

and by transforming to light cone coordinates $u = t - r_*$ and $v = t + r_*$we find

$$ds^2 = -f du dv. \tag{3.3}$$

Note that now the coordinate function r appearing in $f = f(r)$ is implicitly determined by

$$r + 2M \ln\left[\frac{r}{2M} - 1\right] \quad = \quad \frac{v-u}{2},$$

and we find for

$$f = \left(1 - \frac{2M}{r}\right) = \frac{2M}{r} \exp\left[-\frac{r}{2M}\right] \exp\left[\frac{v-u}{4M}\right].$$

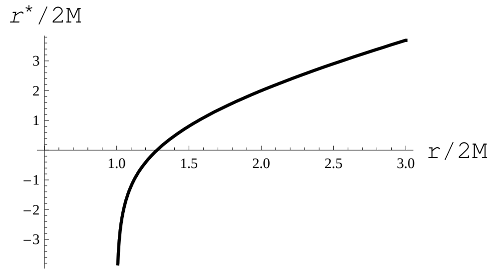


Figure 3.1: The Regge-Wheeler tortoise coordinate $r_* = r + 2M \ln\left[\frac{r}{2M} - 1\right]$.

By comparing this with the Rindler case, intuition tells us to define

$$\begin{aligned} U &= -e^{-\frac{u}{4M}} = -e^{-\kappa u} \\ V &= e^{\kappa v}, \end{aligned}$$

and for later use we write down the inverse transformation

$$\begin{aligned} \kappa u &= -\ln(-U) \\ \kappa v &= \ln V, \end{aligned}$$

so that (3.1) becomes

$$ds^2 = -\frac{32M^3 e^{-r/2M}}{r} dU dV. \tag{3.4}$$

To further pursue the analogy with the Rindler case, one is led to introduce another set of coordinates

$$\begin{cases} T &= \frac{U+V}{2} = e^{\kappa r_*} \sinh(\kappa t) \\ X &= \frac{V-U}{2} = e^{\kappa r_*} \cosh(\kappa t). \end{cases} \tag{3.5}$$

Note that this set of coordinates covers only part of the extended Schwarzschild space-time, namely $X > |T|$, as the coordinates t and r_* (for $r > 2M$) range over the whole $\mathbb{R}^2$.

In order to have coordinates describing also the left wedge of Schwarzschild spacetime, we proceed as in Rindler spacetime, by simply replacing $T \to -T$, $X \to -X$.

Now we digress a little by performing the calculations needed to compactify the Schwarzschild and Minkowski spacetimes. Since the Kruskal light cone coordinates U, V range over the whole of $\mathbb{R}^2$, one can never draw a picture of the whole spacetime. But this problem can be circumvented by mapping them to a compact set, i.e. the corresponding Penrose diagram, via

$$\begin{aligned} \tilde{U} &= \arctan U \\ \tilde{V} &= \arctan V, \end{aligned}$$

whose range is now given by

$$-\frac{\pi}{2} \leq \tilde{U}, \tilde{V} \leq \frac{\pi}{2}.$$

The same compactification procedure can also be carried out in the Minkowski spacetime by defining

$$\begin{aligned} \tilde{u} &= \arctan u \\ \tilde{v} &= \arctan v, \end{aligned}$$

and now one is able to draw the so-called Penrose diagrams, representing the whole of spacetime. In the following figure, we have drawn such sketches, and one immediately recognizes the substantial similarities between Kruskal and Minkowski spacetime (cf. fig. 3.2).

For completeness, we now determine the redshift factor and the surface gravity of Schwarzschild spacetime. The timelike Killing vector in the static black hole geometry is given by

$$\xi^a = (\partial_t)^a \,,$$

its norm being

$$\xi^2 = -f.$$

The square root of minus this norm is the redshift factor

$$V = \sqrt{-\partial_t^2} = \sqrt{f},$$

and by taking its derivative we find

$$\nabla_a V = \frac{2M}{2\sqrt{f}} \frac{1}{r^2} dr_a.$$

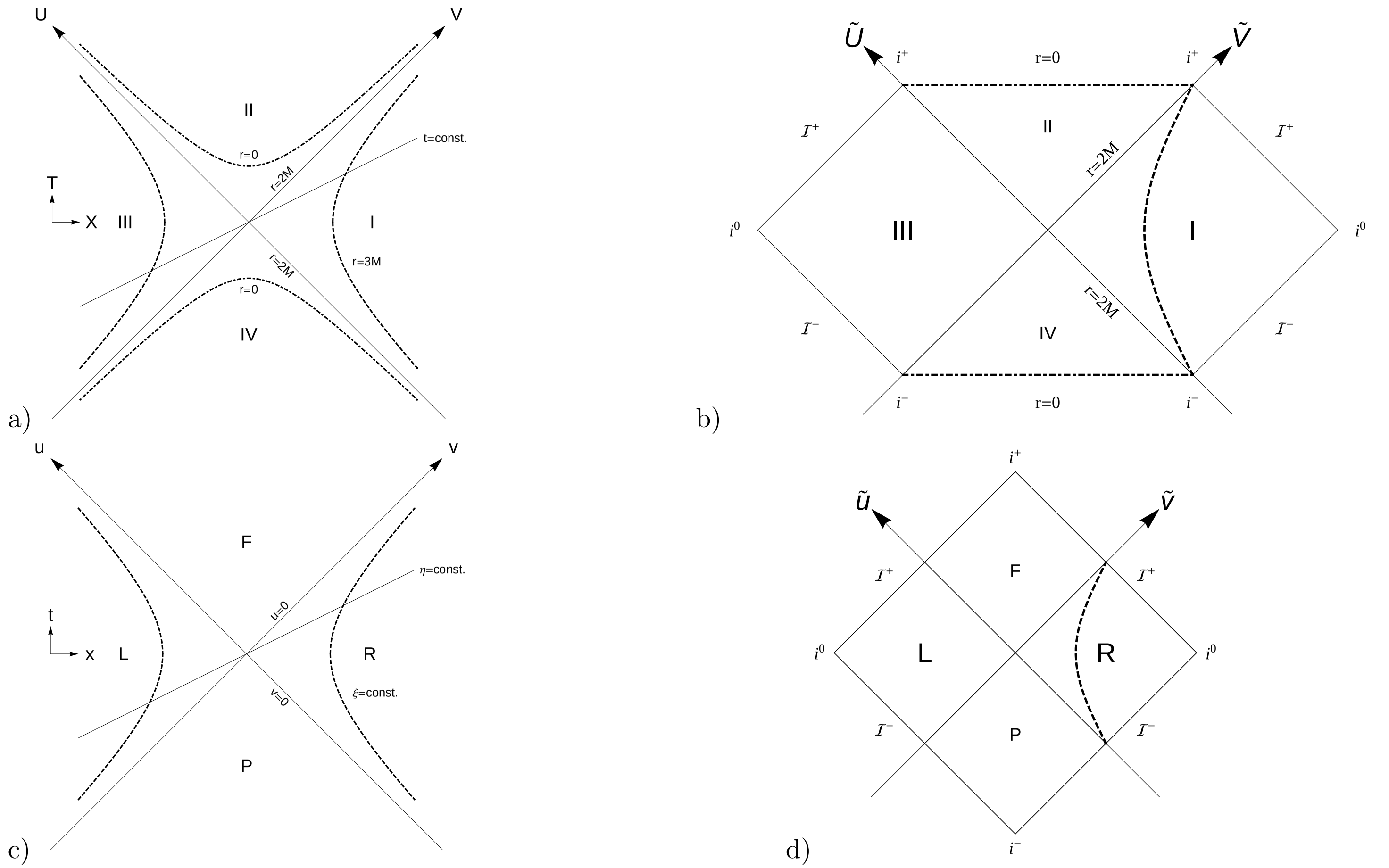


Figure 3.2: Kruskal ((a) and (b)) and Minkowski ((c) and (d)) spacetimes diagrams in their normal and compactified versions
Dotted lines represent worldlines of constantly accelerated observers ($r = const.$ resp. $\xi = const.$), while the dash-dotted lines in the Schwarzschild and Kruskal diagrams are the physical singularities ($r = 0$).

Its norm is then given by

$$(\nabla V)^2 = \frac{M^2}{r^4},$$

and by evaluating this expression at the horizon $r = 2M$, we have determined the surface gravity of the Schwarzschild black hole

$$\kappa = \sqrt{(\nabla V)^2 \,|_{r=2M}} = \frac{1}{4M}.$$

3.2 The massless Klein Gordon equation in the two coordinate systems

The massless Klein Gordon equation in Schwarzschild spacetimes in the $t-r$ coordinates is given by

$$\begin{aligned} g^{ab}\nabla_a \nabla_b \Phi &= \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} g^{ab} \partial_b\right) \Phi \\ &= \partial_a \left(g^{ab}\partial_b\right) \Phi \\ 0 &= -f^{-1}\partial_t^2 \Phi + \partial_r \left(f \partial_r\right) \Phi. \end{aligned} \tag{3.6}$$

Although this looks rather different from the flat-space wave equation, one can bypass this difficulty by using the form (3.2) or (3.3) of the Schwarzschild metric, which yields

$$\begin{aligned} \Box\Phi &= \frac{1}{f}\left(-\partial_t^2 + \partial_{r*}^2\right)\Phi, \\ &= -\frac{1}{f}\partial_u \partial_v \Phi = 0. \end{aligned}$$

Alternatively, we can also write down the wave equation in the coordinate systems (3.4) or (3.5), yielding

$$\begin{aligned} \Box\Phi &= \frac{r}{16M^3} e^{r/2M} \partial_U \partial_V \Phi, \\ &= \frac{r}{16M^3} e^{r/2M} (-\partial_T^2 + \partial_X^2)\Phi = 0. \end{aligned}$$

As can be seen from the above two expressions, these are conformally flat (which comes as to no surprise, since in two dimensions every metric is locally conformally flat [9]) and we can take over the results of part 2.5 with the appropriate replacements.

Therefore, we find (as analogue of the Minkowski modes) the *Kruskal modes*

$$\begin{aligned} f_k &= \exp\left[i\left(-\omega T + kX\right)\right], \\ &= \exp\left[-i\omega U\right]\Theta(k) + \exp\left[-i\omega V\right]\Theta(-k), \end{aligned} \tag{3.7}$$

and we have already performed the split in left and right movers. Note that these modes share the analyticity and boundedness properties of the Minkowski modes of section (2.3).

The analogues to the Rindler modes are then called the *Schwarzschild modes* and are given by

$$\begin{aligned} g_k^{(1)} &= \exp\left[-i\omega t + ikr_*\right] \\ &= \left[e^{-\kappa u}\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[e^{\kappa v}\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \\ &= \left[-U\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[V\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \end{aligned} \tag{3.8}$$

and (with U, V in III given by $U_L = e^{-\kappa u}$ and $V_L = -e^{\kappa v}$)

$$\begin{aligned} g_k^{(2)} &= \exp\left[i\omega t + ikr_*\right] \\ &= \left[e^{\kappa v}\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[e^{-\kappa u}\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \\ &= \left[-V\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[U\right]^{-\frac{i\omega}{\kappa}}\Theta(-k). \end{aligned} \tag{3.9}$$

Following the same lines of reasoning as in the Rindler case, we consider

$$\begin{aligned} g_{-k}^{(1)*} &= \exp\left[i\omega t + ikr_*\right] \\ &= \left[e^{\kappa v}\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[e^{-\kappa u}\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \\ &= \left[V\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[-U\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \\ &= (-1)^{\frac{i\omega}{\kappa}}\left[-V\right]^{\frac{i\omega}{\kappa}}\Theta(k) + (-1)^{-\frac{i\omega}{\kappa}}\left[U\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \end{aligned}$$

and

$$\begin{aligned} g_{-k}^{(2)*} &= \exp\left[-i\omega t + ikr_*\right] \\ &= \left[e^{-\kappa u}\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[e^{\kappa v}\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \\ &= \left[U\right]^{\frac{i\omega}{\kappa}}\Theta(k) + \left[-V\right]^{-\frac{i\omega}{\kappa}}\Theta(-k) \\ &= (-1)^{\frac{i\omega}{\kappa}}\left[-U\right]^{\frac{i\omega}{\kappa}}\Theta(k) + (-1)^{-\frac{i\omega}{\kappa}}\left[V\right]^{-\frac{i\omega}{\kappa}}\Theta(-k). \end{aligned}$$

Therefore we find for the *Unruh modes* (with $-1 = \exp(-i\pi)$ and thus $(-1)^{\pm\frac{i\omega}{\kappa}} =$

$\exp\left[\pm\pi\frac{\omega}{\kappa}\right])$

$$\begin{aligned}
h_k^{(1)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left[e^{\frac{\pi\omega}{2\kappa}}g_k^{(1)} + e^{-\frac{\pi\omega}{2\kappa}}g_{-k}^{(2)*}\right] \\
h_k^{(2)} &= \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left[e^{-\frac{\pi\omega}{2\kappa}}g_{-k}^{(1)*} + e^{\frac{\pi\omega}{2\kappa}}g_k^{(2)}\right].
\end{aligned} \tag{3.10}$$

Hence, we can readily take over the results of sections (2.5 and following) by simply making the replacements

$$\begin{aligned}
a &\rightarrow \kappa = 1/4M \\
(\eta,\, \xi) &\rightarrow (t,\, r_*) \\
(t,\, x) &\rightarrow (T,\, X)\,,
\end{aligned}$$

and we find that the Kruskal vacuum is perceived by a stationary observer as being a thermal state of temperature

$$T = \frac{1}{8\pi M}.$$

Note that, due to the different aymptotics of the redshift factors in the Rindler, respectively the Schwarzschild case, this temperature does *not* tend to zero as $r_* \rightarrow \infty$, which is easily seen by considering the Tolman relation

$$\begin{aligned}
T\left(r_*\right) &= \left(-g_{00}\right)^{-1/2} T \\
&= \frac{1}{\sqrt{1-\frac{2M}{r(r_*)}}}T,
\end{aligned}$$

and thus one finds

$$\lim_{r_* \rightarrow -\infty} T\left(r_*\right) \simeq T = \frac{1}{8\pi M}.$$

It does, however, show diverging behaviour when the observer approaches the horizon ($\lim_{r_* \rightarrow -\infty} r\left(r_*\right) = 2M$), with the same interpretation as in the Rindler case: the observer will then need an infinite amount of acceleration to keep up his stationary orbit.

3.3 Summary: Rindler vs. Schwarzschild

	Rindler	Schwarzschild
original metric	$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$	$ds^2 = \left(1 - \frac{2M}{r}\right)\left[-dt^2 + dr_*^2\right]$
extended metric	$ds^2 = -dt^2 + dx^2$	$ds^2 = \frac{32M^3 e^{-r/2M}}{r}\left(-dT^2 + dX^2\right)$
corresponding to	Minkowski spacetime	Kruskal extension
redshift factor	$V = \sqrt{-\partial_\eta^2} = e^{a\xi}$	$V = \sqrt{-\partial_t^2} = \sqrt{\left(1 - \frac{2M}{r}\right)}$
surface gravity	$\kappa = a$	$\kappa = \frac{1}{4M}$
accelerated observer	$\xi = const.$	$r = const. \geq 2M$
associated temperature	$T = a\,(2\pi)^{-1}$	$T = (8\pi M)^{-1}$

Using the above correspondence table, it is easy to convince oneself that choosing the Killing time orientation in the second asymptotic region results in a trivial Bogolubov transformation, thus telling us that also in the case of the (spherically reduced) Schwarzschild space-time the Hawking radiation ceases to exist, as might have been expected from paragraph 2.8.

4 Conclusion and Outlook

By investigating the massless Klein Gordon field in Minkowski spacetime from the perspective of two observers, namely an inertial and a constantly accelerated one, we found that the vacuum, as defined by the inertial observer, appears to be a thermal state to the accelerated observer. This is the essence of the Unruh effect. This was done by solving the wave equation in the Minkowski- and Rindler coordinate system, whereby we obtained two different sets of modes. Upon quantization, these two schemes turned out to be inequivalent, though connected via a Bogolubov transformation. We chose not to calculate the corresponding Bogolubov coefficients directly, instead we followed Unruh's method of analytically continuing the Rindler modes across the horizons and in doing so we found two new sets of modes, analytic and bounded in the lower half plane. This, in turn, implied that these Unruh modes define the same vacuum as the Minkowski observer and therefore the Bogolubov coefficients were very easily determined: we simply had to read them off the resulting expression. These coefficients could then be used to determine

$$\left\langle 0_M \left| b_k^{(1)\dagger} b_{k'}^{(1)} \right| 0_M \right\rangle = \frac{\delta\left(k-k'\right)}{\exp\left(2\pi\omega/a\right)-1},$$

the expectation value of the Rindler number operator (in R), when the state of the quantum field is the vacuum, and we found that this matches exactly a Bose-Einstein distribution for a system in thermal equilibrium at temperature (we restore the natural constants in the last expression)

$$T = \frac{a}{2\pi} = \frac{\hbar a}{2\pi c k_B}.$$

Furthermore, by considering the density matrix corresponding to the Minkowski vacuum in terms of Rindler states and carrying out a trace over the degrees of freedom "living" in the causally disconnected spacetime region L, we found that the resulting density matrix was indeed of thermal nature. Yet more evidence for the thermality of the vacuum was found when we investigated the two-point function of the scalar field. By regularizing its integral representation and discarding some infinite constants, we found that

$$\begin{aligned} D_M^{(1)}\left(\xi-\xi'\,;\,\eta-\eta'\right) &= \left\langle 0_M \,\middle|\, \left\{\Phi\left(x\right)\Phi\left(x'\right)\right\} \,\middle|\, 0_M \right\rangle \\ &= D_M^{(1)}\left(\xi-\xi'\,;\,\eta-\eta'+\frac{2\pi n i}{a}\right), \end{aligned}$$

i.e. Hadamard‘s elementary function is indeed periodic in imaginary time with period $2\pi/a$. This, however, is characteristic for a KMS state at temperature $T = a/2\pi$.

In the last chapter we explored the connections between the Rindler/Minkowski and Schwarzschild/Kruskal spacetimes, showing the profound causal similarities. This allowed us (due to the conformal invariance of the Klein Gordon field in two dimensions) to extend all of the results obtained in chapter 2 to this case, confirming that a black hole does indeed emit a thermal spectrum of scalar particles, as measured by a stationary observer which orbits at $r = const. \geq 2M$, at temperature

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} = \frac{\hbar c^3}{8\pi M k_B G}.$$

The investigations at the end of chapter 2 were done to explore the influence of the choice of time slicings in L, and we found that when we defined positive energy with respect to ∂_η (past-pointing in L), and therefore a Killing-conformal choice of time slicings, the Unruh radiation ceases to exist. Although from the point of view of Minkowski space this choice is rather artificial, the corresponding argument does not hold in the Schwarzschild context where changing the time orientation of the second asymptotic region is realizable due to the causal disconnection. This would yield a different ("super-selected") quantum field theory with vanishing Hawking temperature. A closer look at the calculation reveals however, that only the relative orientation of the time slicings in L is changed. Even though this leads to a vanishing temperature, i.e. a trivial Bogolubov transformation, this fact cannot be accounted for by the change of time orientation, since the Kruskal slicings retain their orientation in both of the wedges.

From a geometric point of view it may be argued that the "Killing conformal" orientation is more natural since the corresponding surfaces do not contain a jump in the surface normal at the bifurcation sphere of Schwarzschild as can be seen from fig. 4.1 or by considering the split in a spacelike hypersurface σ where $\eta = const.$ and a timelike normal vector n^a

$$g_{ab} = h_{ab} - n_a n_b,$$

where

$$d\sigma^2 = \exp[2a\xi]\, d\xi^2 = h_{ab} dx^a dx^b,$$

and we find for the unit normal ($n^2 = -1$) in R

$$\begin{aligned} n_a^{(1)} &= -\exp[a\xi]\, d\eta_a, \\ n^a_{(1)} &= \exp[-a\xi]\, \partial_\eta^a \end{aligned}$$

and in L, depending on the choice made (the subscript f is used to emphasize that we used the Killing anti-conformal time slicings to obtain a future pointing vector as well

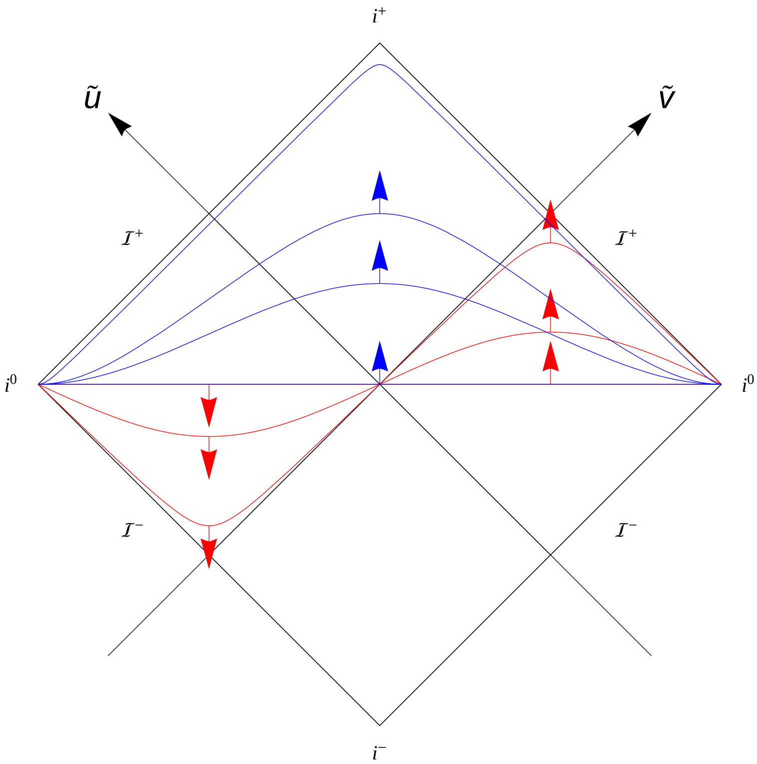


Figure 4.1: Killing-conformal choice of time slicings in the left Rindler wedge L:
In this case, the time slicings in L have not been altered by the requirement as to agree with the slicings defined by the Minkowski observer.

as p for the Killing conformal past directed slicings)

$$
\begin{aligned}
n^a_{(2),f} &= -\exp\left[-a\xi\right]\partial^a_\eta, \\
n^a_{(2),p} &= \exp\left[-a\xi\right]\partial^a_\eta.
\end{aligned}
$$

When evaluating these expressions (for simplicity) on the $t = 0$ surface, we find (with $\partial^a_\eta|_{t=0} = ax\partial^a_t$ and $\exp\left[-a\xi\right]|_{t=0} = (ax)^{-1}$)

$$
\begin{aligned}
n^a_{(1)}|_{t=0} &= \partial^a_t, \\
n^a_{(2),f}|_{t=0} &= -\partial^a_t, \\
n^a_{(2),p}|_{t=0} &= \partial^a_t,
\end{aligned}
$$

and we realize that with the Killing conformal choice ($n^a_{(2),p}$) we have a continuous normal vector to the $t = const.$ surfaces, whereas in the usual procedure ($n^a_{(2),f}$) we have a discontinuity in the normal vector field, with a jump when passing from R to L. In order to substantiate this point the corresponding Green-function calculation should be extended to cover this orientation, i.e. finding the corresponding boundary conditions.

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