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Systematic Proof Theory for Non-Classical Logics: Advances and Implementation

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Abstract

Non-classical logics are logics different from classical, boolean logic. They encompass, amongst others, the family of intermediate, fuzzy, and substructural logics. During the past few years, these logics have gained importance, especially in many fields of computer science, artificial intelligence, and knowledge representation. By now, there are many useful and interesting non-classical logics and practitioners in various fields keep on introducing new logics to fulfill their needs.

Non-classical logics are usually defined by adding Hilbert axioms to known systems. The usefulness of these logics, however, heavily depends on the availability of analytic calculi for them, i.e., of calculi where the derivations proceed by step-wise decomposition of the formulas that should be proved. An analytic calculus for a logic is usually defined by first choosing a suitable formalism and then providing the right rules for formalizing the logic. Furthermore, soundness, completeness and cut-elimination for the defined calculus have to be proved. This leads to the introduction of a large number of such proofs and papers which increases with the definition of new logics. An automated procedure to introduce analytic calculi in a systematic way would therefore be very desirable.

In this thesis we extended the scope of the systematic procedure introduced in [15], which translates Hilbert axioms into equivalent analytic calculi, to capture the family of axioms known as (Bd_k) with $k \geq 1$, see [9]. These axioms are especially interesting as intuitionistic logic extended with (Bd_2) is the only interpolable intermediate logic to which the procedure in [15] does not apply to. To adapt the procedure to these axioms, we introduced a new formalism, called Non-Commutative Hypersequent Calculus, and identified suitable rules corresponding to (Bd_k). For these rules we provided uniform proofs of soundness, completeness and cut-elimination. In addition to this theoretical investigation, the systematic procedure in [15] has been translated into an algorithm which was implemented in PROLOG.

Zusammenfassung

Nichtklassische Logiken nahmen in den letzten Jahren in vielen Gebieten, vor allem in der Informatik, der Künstlichen Intelligenz oder der Wissensrepräsentation, an Bedeutung zu. Mittlerweile existiert bereits eine große Anzahl nützlicher und interessanter nichtklassischer Logiken, wie beispielsweise die Familie der intermediären Logiken, Fuzzy-Logiken oder Substrukturelle Logiken. Zudem führen Wissenschaftler verschiedener Fachbereiche weiterhin neue Logiken ein, um ihre Anforderungen zu erfüllen.

Nichtklassische Logiken definiert man üblicherweise, indem man Hilbert-Axiome zu bereits bekannten Systemen hinzufügt. Die Nützlichkeit dieser Logiken hängt jedoch stark von der Verfügbarkeit sogenannter analytischer Kalküle ab, in welchen Beweise durch schrittweises Zerlegen der Formeln geführt werden. Den analytischen Kalkül einer Logik definiert man durch die Wahl eines passenden Formalismus, sowie das Festlegen geeigneter Regeln zur Formalisierung der Logik. Zusätzlich müssen Korrektheit, Vollständigkeit und Schnittelimination des definierten Kalküls bewiesen werden. Mit der Definition neuer Logiken wächst daher auch die Anzahl solcher Beweise und eine automatisierte Prozedur zur Einführung analytischer Kalküle in systematischer Weise wäre somit erstrebenswert.

In dieser Diplomarbeit erweiterten wir die systematische Prozedur, die in [15] eingeführt wurde und Hilbert-Axiome in äquivalente analytische Kalküle umformt, auf die Familie der Axiome (Bd_k) mit $k \geq 1$, siehe [9]. Diese Axiome sind insbesondere interessant, da die Erweiterung intuitionistischer Logik mit dem Axiom (Bd_2) die einzige interpolierbare, intermediäre Logik ist, auf die man die Prozedur in [15] nicht anwenden kann. Um die Prozedur auf diese Axiome auszuweiten, definierten wir zunächst einen neuen Formalismus, den Nichtkommutativen Hypersequenzkalkül, und erstellten geeignete Regeln entsprechend den Axiomen (Bd_k). Für diese Regeln bewiesen wir Korrektheit, Vollständigkeit und Schnittelimination. Zusätzlich entwickelten wir einen Algorithmus auf Basis der systematischen Prozedur in [15] und implementierten diesen in PROLOG.

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Chapter 1

Introduction

During the past few years, non-classical logics — i.e., logics different from classical, boolean logic — have gained importance in many fields of computer science, artificial intelligence, or knowledge representation. By now, there are many useful and interesting non-classical logics, such as modal logics [9], linear logic [23], relevance logics [1, 2], or fuzzy logics [26]. These logics provide languages for formal modeling and reasoning, e.g. about dynamic data structures, time, or resources. Moreover, practitioners in various fields keep on introducing new logics to fulfill their needs.

Non-classical logics are usually defined by adding Hilbert axioms to known systems. The usefulness of these logics, however, heavily depends on the availability of calculi in which the cut-elimination theorem holds. These calculi are indeed a prerequisite for the development of automated reasoning methods, and also the key to establish essential properties of the formalized logics. Cut-elimination — i.e., any proof containing applications of the so-called cut-rule can be transformed into a proof that does not make use of the cut-rule — is a fundamental result to establish in proof theory for several reasons. One of the main factors is that cut-elimination gives analyticity in the sense that all intermediate statements in a proof are subformulas of the formulas to be proved. For a computer scientist, this means that the computational search for proofs in cut-free calculi is feasible, as opposed to non-analytic calculi where the allowance of the cut-rule adds non-determinism to the proof search.

An analytic calculus for a logic is usually defined by first choosing a suitable formalism, e.g. Sequent Calculus [22], Hypersequent Calculus [5], Display Logic [7], or the Calculus of Structure [25], and then providing the right rules for formalizing the logic. Furthermore, soundness, completeness and cut-elimination for the defined calculus have to be proved. The invention of new logics leads therefore to the introduction of a large number of such proofs. Hence, an automated procedure to introduce analytic calculi in a systematic and uniform way would be very desirable.

First steps in this direction have been made in Ciabattoni et al. [14, 15] where a new research direction called "algebraic proof theory" was introduced. Algebraic proof theory emerges from combining systematic proof theory for non-classical logics and universal algebra in an innovative way. Within this new field of research, a systematic and uniform procedure that translates Hilbert axioms into equivalent analytic calculi was developed in [14, 15]. This procedure works up to certain classes of a novel classification of Hilbert axioms called substructural hierarchy. In this thesis, the scope of this systematic procedure has been extended to capture the family of axioms known as (Bd_k) with $k \geq 1$ which are semantically characterized by Kripke models with depth $\leq k$, see [9]. These axioms are especially interesting as intuitionistic logic extended with (Bd_2) is the only one of the seven interpolable intermediate logics — i.e., logics between intuitionistic and classical logic — to which the procedure in [15] does not apply to. To capture these axioms and extend the procedure in [15, 17], we introduced a new formalism, more powerful than sequent and hypersequent calculi, and identified suitable rules corresponding to (Bd_k). For these rules we provided a uniform proof of soundness, completeness and cut-elimination. In addition to this theoretical investigation, the systematic procedure in [15] has been transformed into an algorithm which was implemented in PROLOG.

1.1 Overview

Chapter 2 provides basic definitions and an introduction to sequent and hypersequent calculi, as well as to cut-elimination. In Chapter 3, we outline the theoretical basis for this thesis. We explain the substructural hierarchy and the systematic procedure in [15] which transforms Hilbert axioms into equivalent analytic rules. Furthermore, we provide an overview of the results that have been established in this research direction so far. The translation of the systematic procedure into an algorithm and its implementation in PROLOG is described in Chapter 4. Chapter 5 constitutes a central part of this thesis as it contains the extension of the systematic procedure to the axioms (Bd_k). We introduce the Non-Commutative Hypersequent Calculus and adapt the procedure in [15] to deal with the axioms (Bd_k). Proofs of soundness, completeness and cut-elimination are provided for the defined non-commutative hypersequent calculi in a uniform way. We summarize the main results of this thesis and discuss further research directions in Chapter 6.

CHAPTER **2**

Proof Theory in Non-Classical Logics: Preliminaries

Proof theory studies the formalization and the structure of mathematical proofs. Hilbert is widely recognized as the initiator of proof theory since his attempt to provide a proof of the consistency of mathematics. Nowadays, there exist roughly two branches within proof theory: interpretational proof theory and structural proof theory. The latter has been first developed by Gentzen in [22] and is of particular interest for us as it is based on the combinatorial analysis of proof structure. Examples for applications of structural proof theory are automated theorem proving or logic programming. [35, 37]

In this chapter, we introduce some basic notions of structural proof theory, in particular the concepts of sequent and hypersequent calculi and cut-elimination. The main references for this chapter are the books and articles by Takeuti [35], Troelstra and Schwichtenberg [37], Metcalfe et al. [28], Gentzen [22] and Avron [4, 5].

2.1 Basic Definitions

Definition 2.1.1. [15, 35, 37] Atomic *formulas are propositional variables* $p, q, \ldots$. *The notion of* formulas *is defined inductively:*

(1) Every atomic formula is a formula.

(2) The logical constants 1 *(unit),* $\bot$ *(false),* $\top$ *(true) and* 0 *are formulas.*

(3) If A *and* B *are formulas, then* $(A \wedge B), (A \vee B)$ *and* $(A \rightarrow B)$ *are formulas.*
Metavariables $A, B, C, \ldots$ *denote formulas,* Π, Θ *stand for* stoups*, i.e., either a formula or an empty set, and* $\Gamma, \Delta, \Sigma, \ldots$ *for finite, possibly empty, lists of formulas.*

Definition 2.1.2. [35, 37] Sequents *are expressions of the form* $\Gamma \Rightarrow \Delta$ *with* Γ, Δ *being finite lists of formulas.* Γ *is called the* antecedent *and* Δ *is called the* succedent *of the sequent.*
Sequents, where the succedent contains at most one formula, are called single-conclusion *and they are called* multiple-conclusion*, otherwise.*

Intuitively, a sequent $A_1, \dots, A_m \Rightarrow B_1, \dots, B_n$ with $m, n \geq 1$ is interpreted as: if $A_1 \wedge \cdots \wedge A_m$, then $B_1 \vee \cdots \vee B_n$.

Definition 2.1.3. [37] *We use the symbol* $\vdash$ *to denote* derivability *in a formalism. For sequents and formulas derived in a formalism* **S**, *we write*

$\mathbf{S} \vdash \Gamma \Rightarrow \Delta$	*or*	$\vdash_{\mathbf{S}} \Gamma \Rightarrow \Delta$	*for sequents*
$\mathbf{S} \vdash A$	*or*	$\vdash_{\mathbf{S}} A$	*for formulas*

For formalisms based on sequents, $\mathbf{S} \vdash A$ *coincides with* $\mathbf{S} \vdash \Rightarrow A$*, i.e., the sequent* $\Gamma \Rightarrow A$ *with* Γ *being empty.*

2.2 Sequent and Hypersequent Calculus

One of the first systems introduced for writing formal deductions was the *(Frege-)Hilbert system.* Such a system is usually characterized by a set of axioms (or axiom schemes) and a small number of inference rules, e.g. modus ponens [8]. However, Hilbert systems are not analytic and therefore cannot be used for computational proof search. In 1935, Gentzen [22] introduced the sequent calculi **LK** and **LJ** as formalisms for classical and intuitionistic logic. Proofs in these systems are analytic which proved to be more suitable for automated deduction.

Below we will provide a short introduction to sequent and hypersequent calculus, substructural logics and some basic explanations regarding cut-elimination.

2.2.1 Sequent Calculus

We start presenting Gentzen's calculus for intuitionistic logic **LJ**. The system **LJ** consists of single-conclusion sequents (see Definition 2.1.2) of the form $\Gamma \Rightarrow \Pi$ with Γ being a finite, possibly empty, list and Π being either a formula or empty. Sequent calculi consist of initial axioms and inference rules, which are further divided into logical rules, structural rules and the cut-rule. For every logical and structural rule there exists a left and a right rule depending on which side of the sequent, i.e. left or right, is modified. The structural rules introduced by Gentzen (for the calculus **LJ**) are:

$$\frac{\Gamma, A \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\ (e,l) \qquad \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w,l) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \Pi}\ (w,r) \qquad \frac{\Gamma, A \Rightarrow \Pi}{\Gamma, A, A \Rightarrow \Pi}\ (c,l)$$

The cut-rule is different from the other rules as it contains a formula A, called *cut-formula*, in both premises which does not occur in the conclusion:

$$\frac{\Gamma \Rightarrow A \qquad \Sigma, A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi}\ (cut)$$

The rules of the **LJ** calculus are contained in Table 2.2.1.

Axioms

$A \Rightarrow A \qquad \bot \Rightarrow A$

Cut Rule

$$\frac{\Gamma \Rightarrow A \qquad A, \Sigma \Rightarrow \Pi}{\Sigma, \Gamma \Rightarrow \Pi}$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (w,\ l) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \Pi}\ (w,\ r)$$

$$\frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi}\ (c,\ l) \qquad \frac{\Gamma, A \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\ (e,\ l)$$

Logical Rules

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\to,\ r) \qquad \frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Pi}{\Gamma, A \to B \Rightarrow \Pi}\ (\to,\ l)$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\wedge,\ r) \qquad \frac{\Gamma, A_i \Rightarrow \Pi}{\Gamma, A_1 \wedge A_2 \Rightarrow \Pi}\ (\wedge_i,\ l)_{i=1,2}$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (\vee_i,\ r)_{i=1,2} \qquad \frac{\Gamma, A \Rightarrow \Pi \qquad \Gamma, B \Rightarrow \Pi}{\Gamma, A \vee B \Rightarrow \Pi}\ (\vee,\ l)$$

Table 2.1: Sequent Calculus **LJ** [22]

Substructural Logics

Substructural logics are non-classical logics where the structural rules are absent and which are therefore weaker than classical or intuitionistic logic. The base system **FL**, *Full Lambek Calculus*, is obtained from the sequent calculus **LJ** when all structural rules are dropped, e.g. see [30]. In this sense, instead of **LJ** we could also write **FLewc**: *Full Lambek Calculus* with the exchange **e**, weakening **w** and contraction **c** rule. Some important substructural logics are relevant logics, Full Lambek calculus, or linear logic, which are motivated, amongst others, by considerations from philosophy, linguistics, or computer science.

When some of the structural rules are missing, the commas in the antecedent of sequents do not behave like the additive conjunction $\wedge$ any more, but like the conjunction $\cdot$, called *fusion* or *multiplicative conjunction* [30]:

$$\frac{\Gamma, A, B, \Sigma \Rightarrow \Pi}{\Gamma, A \cdot B, \Sigma \Rightarrow \Pi}\ (\cdot, l) \qquad \frac{\Gamma \Rightarrow A \qquad \Sigma \Rightarrow B}{\Gamma, \Sigma \Rightarrow A \cdot B}\ (\cdot, r)$$

Sometimes, the propositional constants 1 and 0 are introduced in systems for substructural logics. Usually, we use $\bot$ and $\top$ to denote *false* and *true* which are introduced as follows (with Γ and Π possibly empty):

$$\frac{}{\bot, \Gamma \Rightarrow \Pi}\ (\bot, l) \qquad \frac{}{\Gamma \Rightarrow \top}\ (\top, r)$$

When a system lacks the weakening rules, these initial sequents cannot be replaced with $\Rightarrow \top$ and $\bot \Rightarrow$ any more. But if the system does not contain those weaker initial sequents, the constants defined by them behave differently. Therefore, the following rules for 0 and 1 are introduced:

$$\frac{}{0 \Rightarrow}\ (0, l) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0, r) \qquad \frac{\Gamma \Rightarrow \Pi}{1, \Gamma \Rightarrow \Pi}\ (1, l) \qquad \frac{}{\Rightarrow 1}\ (1, r)$$

Note that the negation $\neg A$ is used as an abbreviation for $A \to 0$.

2.2.2 Hypersequent Calculus

Gentzen's sequent calculus is a powerful framework which is capable of dealing with many interesting logics. However, many logics seem to escape a cut-free sequent formalization. For this reason, Avron introduced a simple generalization of the sequent calculus called hypersequent calculus (see e.g. [5]).

Definition 2.2.1. [5] *A* hypersequent *is a structure of the form*

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

where every $\Gamma_i \Rightarrow \Delta_i$ *is an ordinary sequent and is called a* component *of the hypersequent. If all components of a hypersequent are single-conclusion, the hypersequent is called* single-conclusion*, and* multiple-conclusion *otherwise.*

Note that the symbol "|" is usually interpreted as disjunction:

Definition 2.2.2. *A hypersequent of the form*

$$\mathcal{G} := \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

is interpreted as follows:

$$\mathcal{G}^I := (\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee (\bigwedge \Gamma_2 \to \bigvee \Delta_2) \vee \cdots \vee (\bigwedge \Gamma_n \to \bigvee \Delta_n)$$

where $\bigwedge \Gamma_i$ *is the conjunction of the formulas in* Γ_i *or* $\top$*, when* Γ_i *is empty, and* $\bigvee \Delta_i$ *is either the disjunction of the formulas in* Δ_i *or* $\bot$*.*

As in the case of sequent calculus, the hypersequent calculus consists of initial axioms, logical rules, the cut-rule and structural rules. Initial axioms, logical rules and the cut-rule are essentially the same as in the sequent calculus. The only difference is that a (possibly empty) side hypersequent G may occur in hypersequents. Consequently, to obtain the base calculus **HLJ**, the rules from the base calculus **LJ** in Table 2.2.1 are extended with the side hypersequent G, e.g. the hypersequent rules for $(\to)$ are as follows:

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \to B}\ (\to, r) \qquad\qquad \frac{G \mid \Gamma \Rightarrow A \qquad G \mid B, \Gamma \Rightarrow \Pi}{G \mid \Gamma, A \to B \Rightarrow \Pi}\ (\to, l)$$

The structural rules are divided into two groups: *external* structural rules and *internal* structural rules. The internal structural rules are the standard structural rules for exchange, weakening, and contraction and are applied to the formulas within a component. External rules manipulate the components of a hypersequent and are as follows:

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi}\ (ew) \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ (ec) \qquad \frac{G \mid \Gamma_2 \Rightarrow \Pi_2 \mid \Gamma_1 \Rightarrow \Pi_1}{G \mid \Gamma_1 \Rightarrow \Pi_1 \mid \Gamma_2 \Rightarrow \Pi_2}\ (ee)$$

In hypersequent calculi, there is the possibility to define rules which act on several components of one or more hypersequents in parallel. An example for such a rule is the communication-rule introduced by Avron in [4]:

$$\frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \qquad G \mid \Gamma_2, \Delta_2 \Rightarrow \Theta}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Theta}\ (com)$$

Consider as an example the hypersequent calculus **GLC** depicted in Table 2.2.2, which consists of the hypersequent version of the rules for **LJ**, the external structural rules and the communication rule [4]. With this calculus, we are able to handle Gödel logic [24] (also known as Dummett's **LC** [19], or Intuitionistic Fuzzy Logic [36]) which is the extension of **LJ** with the prelinearity axiom $(A \to B) \lor (B \to A)$.

Example 2.2.3. *The prelinearity axiom can indeed be proved in* ***GLC*** *as follows:*

$$\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{\Rightarrow A \to B \mid \Rightarrow B \to A}\ (\to, r)}{\Rightarrow (A \to B) \lor (B \to A) \mid \Rightarrow (A \to B) \lor (B \to A)}\ (\lor, r)}{\Rightarrow (A \to B) \lor (B \to A)}\ (ec)$$

2.2.3 Cut Elimination

Cut-elimination is a fundamental result to establish in proof theory. It corresponds to the removal of lemmas (the cuts) from proofs, resulting in a proof which is analytic in the sense of Leibniz, i.e., all statements in the proof are already contained in the conclusion.

There are various ways to prove for a calculus that the cut-elimination theorem holds. The better known method is the one introduced by Gentzen [22] for sequent calculus where he removes the *uppermost* cut in a derivation by a double induction on

Axioms

$$A \Rightarrow A \qquad\qquad \bot \Rightarrow A$$

Cut Rule

$$\frac{G \mid \Gamma \Rightarrow A \qquad H \mid A, \Sigma \Rightarrow \Pi}{G \mid H \mid \Sigma, \Gamma \Rightarrow \Pi}$$

Internal Structural Rules

$$\frac{G \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma, A \Rightarrow \Pi}\,(w,\ l) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \Pi}\,(w,\ r)$$

$$\frac{G \mid \Gamma, A, A \Rightarrow \Pi}{G \mid \Gamma, A \Rightarrow \Pi}\,(c,\ l) \qquad \frac{G \mid \Gamma, A \Rightarrow \Pi}{G \mid A, \Gamma \Rightarrow \Pi}\,(e,\ l)$$

External Structural Rules

$$\frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\,(ec) \qquad \frac{G}{G \mid \Gamma \Rightarrow \Pi}\,(ew)$$

$$\frac{G \mid \Gamma_2 \Rightarrow \Pi_2 \mid \Gamma_1 \Rightarrow \Pi_1}{G \mid \Gamma_1 \Rightarrow \Pi_1 \mid \Gamma_2 \Rightarrow \Pi_2}\,(ee)$$

Logical Rules

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \to B}\,(\to,\ r) \qquad \frac{G \mid \Gamma \Rightarrow A \qquad G \mid B, \Gamma \Rightarrow \Pi}{G \mid \Gamma, A \to B \Rightarrow \Pi}\,(\to,\ l)$$

$$\frac{G \mid \Gamma \Rightarrow A \qquad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B}\,(\wedge,\ r) \qquad \frac{G \mid \Gamma, A_i \Rightarrow \Pi}{G \mid \Gamma, A_1 \wedge A_2 \Rightarrow \Pi}\,(\wedge_i,\ l)_{i=1,2}$$

$$\frac{G \mid \Gamma \Rightarrow A_i}{G \mid \Gamma \Rightarrow A_1 \vee A_2}\,(\vee_i,\ r)_{i=1,2} \qquad \frac{G \mid \Gamma, A \Rightarrow \Pi \qquad G \mid \Gamma, B \Rightarrow \Pi}{G \mid \Gamma, A \vee B \Rightarrow \Pi}\,(\vee,\ l)$$

Communication Rule

$$\frac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \qquad G \mid \Gamma_2, \Delta_2 \Rightarrow \Theta}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Theta}\,(\mathrm{com})$$

Table 2.2: Hypersequent Calculus **GLC** [4, 5]

the complexity of the cut formula — i.e., the number of its connectives — and on the sum of its (left and right) ranks. The left (right, respectively) rank is defined as the number of consecutive sequents which contain the cut formula, counting upward from the left (right, respectively) upper sequent of the cut [35]. In the proof, the cut is either pushed up (see Example 2.2.4) or the cut formula is replaced by a smaller one (see Example 2.2.5).

Example 2.2.4. *Consider the following instance of a cut, where the cut is pushed up:*

$$\dfrac{\Gamma \Rightarrow X \qquad \dfrac{\Sigma, X \Rightarrow A}{\Sigma, X \Rightarrow A \vee B}\,(\vee, r)}{\Gamma, \Sigma \Rightarrow A \vee B}\,(cut) \qquad \longrightarrow \qquad \dfrac{\dfrac{\Gamma \Rightarrow X \qquad \Sigma, X \Rightarrow A}{\Gamma, \Sigma \Rightarrow A}\,(cut)}{\Gamma, \Sigma \Rightarrow A \vee B}\,(\vee, r)$$

Example 2.2.5. *Consider the following instance of a cut, where the cut is replaced with a smaller one:*

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,(\vee, r) \qquad \dfrac{A, \Sigma \Rightarrow C \qquad B, \Sigma \Rightarrow C}{A \vee B, \Sigma \Rightarrow C}\,(\vee, l)}{\Gamma, \Sigma \Rightarrow C}\,(cut) \qquad \longrightarrow \qquad \dfrac{\Gamma \Rightarrow A \qquad A, \Sigma \Rightarrow C}{\Gamma, \Sigma \Rightarrow C}\,(cut)$$

When repeating these two steps, the derivation will either end in an application of the cut rule with at least one of the upper sequents being an initial sequent, e.g.

$$\dfrac{\Gamma \Rightarrow A \qquad A \Rightarrow A}{\Gamma \Rightarrow A}\,(cut)$$

or in an application of the weakening rule where the cut formula is introduced. Either way, the proof can be made without using the cut rule. But Gentzen noticed that an application of the internal contraction rule to the cut formula might result in a problem as the contraction does not necessarily make the cut smaller or push it up in the derivation.

Example 2.2.6. *Consider the following instance of a cut, where the cut formula is contracted by an application of* (c, l)*:*

$$\dfrac{\Gamma \Rightarrow A \qquad \dfrac{A, A, \Sigma \Rightarrow C}{A, \Sigma \Rightarrow C}\,(c, l)}{\Gamma, \Sigma \Rightarrow C}\,(cut) \qquad \longrightarrow \qquad \dfrac{\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\Gamma \Rightarrow A \qquad A, A, \Sigma \Rightarrow C}{A, \Gamma, \Sigma \Rightarrow C}\,(cut_1)}{\Gamma, \Gamma, \Sigma \Rightarrow C}\,(cut_2)}{\Gamma, \Sigma \Rightarrow C}\,(c, l)$$

Note that the size of the cuts does not decrease in (cut_1) *or* (cut_2)*.*

To solve this problem, he introduced the multi-cut rule which generalizes the cut rule in the sense that more than one formula can be cut in one application of the rule:

$$\dfrac{\Gamma \Rightarrow A \qquad \Sigma \Rightarrow \Pi}{\Gamma, \Sigma_A \Rightarrow \Pi}\,(mcut)$$

Σ contains at least one occurrence of A and Σ_A is Σ with some occurrences of A deleted.

Gentzen's method of cut elimination was also generalized to the hypersequent calculus by Avron [4]. He ran into a similar problem when the cut formula was contracted using the external contraction rule (ec). Analogously to Gentzen's solution, Avron introduced an adequate version of the multi-cut rule for the hypersequent setting:

$$\dfrac{G \mid \Gamma \Rightarrow A \qquad H \mid \Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n}{G \mid H \mid \Gamma, \Sigma_1^A \Rightarrow \Pi_1 \mid \cdots \mid \Gamma, \Sigma_n^A \Rightarrow \Pi_n}\,(mcut)$$

Σ_i contains at least one occurrence of A and Σ_i^A is Σ_i with some occurrences of A deleted.

An alternative method to prove cut-elimination was introduced by Schütte [32] and Tait [34]. We will describe and adapt this method in Chapter 5 to prove cut-elimination for the non-commutative hypersequent calculus.

Chapter **3**

From Axioms to Analytic Rules: A Systematic Procedure

In this chapter, we describe the systematic procedure introduced by Ciabattoni, Galatos and Terui in [15] that transforms Hilbert axioms into equivalent analytic inference rules in sequent and hypersequent calculi. The uniform proofs of soundness, completeness and cut-elimination can be found in [14] and [15]. Cut-elimination shows that the (hyper)sequent calculus for **FLe** extended with rules obtained by the procedure is indeed analytic. We start with the definition of the substructural hierarchy which constitutes the foundation for the algorithm. In Section 3.2, we provide a description of the aforementioned procedure. In Section 3.3, we give an overview of the state of the art in this research field.

3.1 The Substructural Hierarchy

The foundation for the systematic procedure in [15] is the *substructural hierarchy* which is a novel classification of Hilbert axioms based on the connectives of **FL** (see e.g. [29]). The substructural hierarchy, which is similar to the arithmetical hierarchy Σ_n, Π_n, is based on the concept of polarity of logical connectives [3]. The logical connectives of **FL** can be divided into two groups of negative ($\rightarrow, \wedge, 0, \top$) and positive ($\cdot, \vee, 1, \bot$) connectives, depending on whether the right or left logical rule is invertible, i.e., the conclusion of a rule implies its premise(s). Axioms with a leading logical connective of positive (negative) polarity belong to a positive class $\mathcal{P}$ (negative class $\mathcal{N}$) of the hierarchy.

Definition 3.1.1. (Substructural Hierarchy) [15] *Let $\mathcal{A}$ be a set of atomic formulas. For $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined as follows:*

$$\mathcal{P}_0 ::= \mathcal{N}_0 ::= \mathcal{A}$$
$$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \bot$$
$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

A graphical representation of the substructural hierarchy is depicted in Figure 3.1. Note that the arrows $\rightarrow$ stand for inclusions $\subseteq$ of the classes.

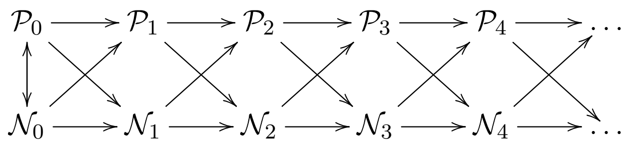

Figure 3.1: The substructural hierarchy by Ciabattoni et al. [15]

Each class of the hierarchy contains an infinite number of axioms.

Example 3.1.2. (Axioms in the substructural hierarchy)

- $\mathcal{N}_2$: *$A \to 1, 0 \to A$ (weakening, [22]), $A \to A \cdot A$ (contraction, [22])*
- $\mathcal{P}_2$: *$A \vee \neg A$ (excluded middle, [5]), $(A \to B) \vee (B \to A)$ (prelinearity, [4])*
- $\mathcal{N}_3$: *$((A \to B) \to B) \to ((B \to A) \to A)$ (Łukasiewicz axiom)*
- $\mathcal{P}_3$: *$\neg A \vee \neg\neg A$ (weak excluded middle, [13]), $\neg(A \cdot B) \vee (A \wedge B \to A \cdot B)$ (weak nilpotent minimum, [20])*
- $\mathcal{P}_4$: *$A \vee (A \to (B \vee \neg B))$ (Bd_2, [9])*

A normal form of the axioms of each class is given in the following proposition:

Proposition 3.1.3. [15] *Every axiom $A \in \mathcal{P}_{n+1}$ is equivalent to an axiom of the form $\bigvee_{1 \leq i \leq m} B_i$ where each B_i is a fusion of formulas in $\mathcal{N}_n$. Every axiom $A \in \mathcal{N}_{n+1}$ is equivalent to an axiom of the form $\bigwedge_{1 \leq i \leq m} C_i \to B_i$ where each B_i is either* 0 *or a formula in $\mathcal{P}_n$ and each C_i is a fusion of formulas in $\mathcal{N}_n$.*

Then the notion of $\mathcal{N}_2$-normal axioms is defined as follows:

Definition 3.1.4. [15] *An axiom A is called* $\mathcal{N}_2$-normal *if it is of the form $A_1 \cdots A_n \to B$ where B is either* 0 *or $\bigvee_{1 \leq i \leq m} B_i$ with every B_i a fusion of propositional variables and each A_i is of the form $\bigwedge_{1 \leq j \leq m_i} C_i^j \to B_i^j$ where B_i^j is* 0 *or a propositional variable and C_i^j is a fusion of propositional variables.*

For systems without weakening, e.g. **FLe**, it is hard to deal with axioms of the class $\mathcal{P}_3$. To solve this problem, a subclass of $\mathcal{P}_3$ called $\mathcal{P}'_3$ is considered in the substructural hierarchy [15]:

$$\mathcal{P}'_3 ::= \mathcal{N}_2 \wedge 1 \mid \mathcal{P}'_3 \cdot \mathcal{P}'_3 \mid \mathcal{P}'_3 \vee \mathcal{P}'_3 \mid 1 \mid \bot$$

We abbreviate a formula $A \wedge 1 \in \mathcal{P}'_3$ as $(A)_{\wedge 1}$. Then we can say that:

Lemma 3.1.5. [15]*: Every formula in $\mathcal{P}_3'$ ($\mathcal{P}_3$, respectively) is equivalent to a finite set of formulas $(A_1)_{\wedge 1} \vee \cdots \vee (A_n)_{\wedge 1}$ ($A_1 \vee \cdots \vee A_n$, respectively) where each A_i is $\mathcal{N}_2$-normal.*

3.2 The Transformation Procedure

We describe the systematic procedure in [15] which transforms axioms that belong to the classes $\mathcal{N}_2$ and $\mathcal{P}_3'$ ($\mathcal{P}_3$, respectively) into equivalent analytic rules with **FLe** and **HFLe** (**HFLew**, respectively) as base calculi. The axioms are first transformed into sets of equivalent structural sequent and hypersequent rules — structural and hyperstructural rules, for short — , i.e., rules not involving any connective. In a second step these (hyper)structural rules are transformed into analytic rules by applying a completion procedure. Examples for such transformation of axioms into their equivalent analytic rules — i.e., rules that preserve cut-elimination when added to a calculus — are depicted in Table 3.1.

Class	Name	Axiom	Rule
$\mathcal{N}_2$	weakening	$A \to 1$	$\dfrac{\Gamma \Rightarrow \Pi}{\Delta, \Gamma \Rightarrow \Pi}$ (w)
		$0 \to A$	$\dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \Pi}$ (w')
	contraction	$A \to A \cdot A$	$\dfrac{\Delta, \Delta, \Gamma \Rightarrow \Pi}{\Delta, \Gamma \Rightarrow \Pi}$ (c)
	weak contraction	$\neg(A \wedge \neg A)$	$\dfrac{\Gamma, \Gamma \Rightarrow}{\Gamma \Rightarrow}$ (wc)
$\mathcal{P}_2$	excluded middle	$A \vee \neg A$	$\dfrac{G \mid \Gamma, \Delta \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \mid \Delta \Rightarrow \Pi}$ (em)
	prelinearity	$(A \to B) \vee (B \to A)$	$\dfrac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Pi_2}{G \mid \Gamma_2, \Delta_1 \Rightarrow \Pi_1 \mid \Gamma_1, \Delta_2 \Rightarrow \Pi_2}$ (com)
$\mathcal{P}_3'$	linearity	$((A \to B) \wedge 1) \vee ((B \to A) \wedge 1)$	(com)
$\mathcal{P}_3$	weak excluded middle	$\neg A \vee \neg\neg A$	$\dfrac{G \mid \Gamma, \Delta \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Delta \Rightarrow}$ (lq)
	Kripke models,width $\leq k$	$\bigvee_{i=0}^{k}(p_i \to \bigvee_{j \neq i} p_j)$	$\dfrac{\{G \mid \Gamma_i, \Gamma_j \Rightarrow \Pi_i\}_{0 \leq i,j \leq k, i \neq j}}{G \mid \Gamma_0 \Rightarrow \Pi_0 \mid \cdots \mid \Gamma_k \Rightarrow \Pi_k}$ (Bwk)

Table 3.1: Some axioms and their equivalent analytic rules

The systematic procedure to obtain structural sequent (hypersequent, respectively) rules for axioms of the class $\mathcal{N}_2$ ($\mathcal{P}_3$ and $\mathcal{P}_3'$, respectively) is described in the following section(s).

3.2.1 Step 1: $\mathcal{N}_2$-axioms and sequent rules

Before describing the procedure to obtain structural sequent rules from $\mathcal{N}_2$-normal axioms, we observe the following equivalences:

Lemma 3.2.1. [15]*: The rule* $\dfrac{S_1 \quad \cdots \quad S_m}{A_1, \ldots, A_n \Rightarrow C}$ (r_0) *is equivalent to each of the rules*

$$\frac{\vec{S} \;\; Y_1 \Rightarrow A_1 \quad \cdots \quad Y_n \Rightarrow A_n}{Y_1, \ldots, Y_n \Rightarrow C}\,(r_1) \qquad\qquad \frac{\vec{S} \;\; C \Rightarrow X}{A_1, \ldots, A_n \Rightarrow X}\,(r_2)$$

where $\vec{S} = S_1 \cdots S_m$ *and* $Y_1, \ldots, Y_n, X$ *are fresh metavariables for formulas.*

The procedure to transform axioms within the class $\mathcal{N}_2$ into equivalent structural rules works as follows:

It suffices to consider $\mathcal{N}_2$-normal axioms A where $A = A_1 \cdots A_n \to C$ and every $A_i = \bigwedge_{1 \leq j \leq m_i} B_i^j \to C_i^j$ for $i = 1, \ldots, n$. We first use the invertible rules $(\to, r), (\cdot, l), (1, l)$ on the formula in the conclusion until no further rule applications are possible. Due to Lemma 3.2.1, we replace the antecedent of the conclusion $A_1, \ldots, A_n \to C$ with fresh metavariables $Y_1, \ldots, Y_n$:

$$\frac{Y_1 \Rightarrow A_1 \quad \cdots \quad Y_n \Rightarrow A_n}{Y_1, \ldots, Y_n \Rightarrow C}$$

Then we apply the invertible rules $(\wedge, r), (\to, r), (\cdot, l), (1, l)$ and $(0, r)$ to the premises. This way, we obtain a set $\mathcal{S}$ of sequents which consists only of metavariables without connectives.

C is either 0 or has the form $C_1 \vee \cdots \vee C_k$. In the first case, we remove C from the conclusion by applying $(0, r)$ to get a structural rule. In the second case, we replace C in the conclusion with a fresh metavariable X (by Lemma 3.2.1) and afterwards apply $(\vee, l)$ to the premises:

$$\frac{\mathcal{S} \qquad C_1 \Rightarrow X \quad \cdots \quad C_k \Rightarrow X}{Y_1, \ldots, Y_n \Rightarrow X}$$

Finally we use the invertibility of $(\cdot, l)$ and $(1, l)$ to obtain an equivalent structural rule.

Example 3.2.2. *We use the algorithm to obtain an equivalent structural rule from the axioms* $A \to A \cdot A$ *(contraction) and* $\neg(A \wedge \neg A)$ *(weak contraction):*

$A \to A \cdot A$*:* $\qquad \dfrac{}{A \Rightarrow A \cdot A} \quad \longrightarrow^{(r_2)} \quad \dfrac{A \cdot A \Rightarrow X}{A \Rightarrow X} \quad \longrightarrow^{(\cdot,l)} \quad \dfrac{A, A \Rightarrow X}{A \Rightarrow X}\,(c_0)$

$\neg(A \wedge \neg A)$*:* $\qquad \dfrac{}{\Rightarrow \neg(A \wedge \neg A)} \longrightarrow^{(\to,r)} \dfrac{}{A \wedge \neg A \Rightarrow} \longrightarrow^{(r_1)} \dfrac{Y \Rightarrow A \wedge \neg A}{Y \Rightarrow}$

$$\longrightarrow^{(\wedge,r)} \frac{Y \Rightarrow A \quad Y \Rightarrow \neg A}{Y \Rightarrow} \qquad \longrightarrow^{(\to,r)} \frac{Y \Rightarrow A \quad Y, A \Rightarrow}{Y \Rightarrow}\,(wc_0)$$

3.2.2 Step 1: $\mathcal{P}_3$-axioms and hypersequent rules

To handle axioms of the classes $\mathcal{P}_3'$ and $\mathcal{P}_3$, we have to shift to the hypersequent setting. A more detailed description of the hypersequent calculus can be found in Section 2.2. Recall that a hypersequent is a multiset $S_1 \mid \cdots \mid S_n$, where the symbol "|" denotes a disjunction on a meta-level. This correspondence is used in the following definition of the interpretation function that retrieves axioms from hypersequents:

Definition 3.2.3. [15] *Let $S = S_1 \mid \cdots \mid S_n$ be a hypersequent and $S_i = A_1, \ldots, A_n \Rightarrow C$ and $S_j = A_1, \ldots, A_n \Rightarrow$ components of S. They are interpreted as follows:*
$(A_1, \ldots, A_n \Rightarrow C)^I = A_1 \cdots A_n \to C$
$(A_1, \ldots, A_n \Rightarrow)^I = A_1 \cdots A_n \to 0$
$(S_1 \mid \cdots \mid S_n)^I = S_1^I \vee \cdots \vee S_n^I$

Example 3.2.4. *Let $S = G \mid A_1, A_2 \Rightarrow C \mid B_1, B_2 \Rightarrow$ be a hypersequent. S is then interpreted as follows:*

$$S^I = G^I \vee (A_1, A_2 \Rightarrow C)^I \vee (B_1, B_2 \Rightarrow)^I = G^I \vee (A_1 \cdot A_2 \to C) \vee (B_1 \cdot B_2 \to 0)$$

Analogously to Lemma 3.2.1 for $\mathcal{N}_2$, the following equivalences can be observed for structural hypersequent rules:

Lemma 3.2.5. [15]*: Let $\Phi, \Phi_1, \ldots, \Phi_m$ be (meta)hypersequents consisting of metavariables. The hypersequent rule $\dfrac{G \mid \Phi_1 \quad \cdots \quad G \mid \Phi_m}{G \mid \Phi \mid A_1, \cdots, A_n \Rightarrow C}\ (hr_0)$ is equivalent to each of the rules*

$$\frac{G \overrightarrow{\mid} \Phi \quad G \mid \Upsilon_1 \Rightarrow A_1 \quad \cdots \quad G \mid \Upsilon_n \Rightarrow A_n}{G \mid \Upsilon_1, \cdots, \Upsilon_n \Rightarrow C}\ (hr_1) \qquad \frac{G \overrightarrow{\mid} \Phi \quad G \mid C, \Upsilon \Rightarrow \Psi}{G \mid \Phi \mid A_1, \cdots, A_n, \Upsilon \Rightarrow \Psi}\ (hr_2)$$

where $G \overrightarrow{\mid} \Phi = (G \mid \Phi_1, \cdots G \mid \Phi_m)$, Υ_i is a fresh metavariable Y_i or Γ_i, and $\Upsilon \Rightarrow \Psi$ is either $\Rightarrow X$ or $\Sigma \Rightarrow \Pi$ with fresh X, Σ, Π.

The procedure to transform axioms within $\mathcal{P}_3'$ ($\mathcal{P}_3$, respectively) into equivalent hyperstructural rules works as follows:

We consider axioms in $\mathcal{P}_3'$ ($\mathcal{P}_3$, respectively) which are equivalent to a finite set $(A_1)_{\wedge 1} \vee \cdots \vee (A_n)_{\wedge 1}$ ($A_1 \vee \cdots \vee A_n$, respectively) where $A_1, \ldots, A_n$ are $\mathcal{N}_2$-normal axioms. By Definition 3.2.3 we directly obtain $\Phi = G \mid\Rightarrow A_1 \mid \cdots \mid\Rightarrow A_n$ from these axioms with G being an empty hypersequent. Analogously to the procedure for $\mathcal{N}_2$-normal axioms, we first apply the invertible rules $(\to, r), (\cdot, l), (1, l)$ to every component of the hypersequent Φ. Then we use the equivalences from Lemma 3.2.5 and introduce fresh metavariables to the antecedent of every component in the conclusion:

$$\frac{G \mid \Upsilon_1 \Rightarrow A_1 \quad \cdots \quad G \mid \Upsilon_n \Rightarrow A_n}{G \mid \Upsilon_1, \cdots, \Upsilon_n \Rightarrow C}$$

Afterwards we use the invertibility of $(\wedge, r), (\rightarrow, r), (\cdot, l), (1, l)$ and $(0, r)$ on the premises to obtain a set $\mathcal{S}_H$ of hypersequents without connectives.

The succedents of the components are either 0 or of the form $C_1 \vee \cdots \vee C_k$. In the first case, we remove C from a component with an application of $(0, r)$. Otherwise, we use Lemma 3.2.5 and the invertibility of $(\vee, l)$ to shift the succedent from a component to the premise:

$$\frac{\mathcal{S}_H \qquad G \mid C_1, \Upsilon \Rightarrow \Psi \quad \cdots \quad G \mid C_n, \Upsilon \Rightarrow \Psi}{G \mid \Upsilon_1, \cdots, \Upsilon_n, \Upsilon \Rightarrow \Psi}$$

Finally we use $(\cdot, l)$ and $(1, l)$ to obtain an equivalent hyperstructural rule.

Example 3.2.6. *We use the algorithm to obtain hyperstructural rules from the axioms $(A \rightarrow B) \vee (B \rightarrow A)$ (prelinearity) and $\neg A \vee \neg\neg A$ (weak excluded middle):*

$(A \rightarrow B) \vee (B \rightarrow A)$: $\quad \dfrac{}{G \mid A \Rightarrow B \mid B \Rightarrow A}\ (lin_0)$

$\neg A \vee \neg\neg A$: $\quad \dfrac{}{G \mid \Rightarrow \neg A \mid \Rightarrow \neg\neg A} \quad \longrightarrow^{(\rightarrow, r)} \quad \dfrac{}{G \mid A \Rightarrow \mid \neg A \Rightarrow}$

$\longrightarrow^{(hr_1)} \quad \dfrac{G \mid Y \Rightarrow \neg A}{G \mid A \Rightarrow \mid Y \Rightarrow} \quad \longrightarrow^{(\rightarrow, r)} \quad \dfrac{G \mid Y, A \Rightarrow}{G \mid A \Rightarrow \mid Y \Rightarrow}\ (wem_0)$

3.2.3 Step 2: The completion procedure

A three-step-completion procedure has been defined in [15] to obtain equivalent analytic rules from (hyper)structural rules. It applies to any (hyper)structural rule with a base calculus that admits weakening (e.g., **HFLew**) or to any acyclic (hyper)structural rule with a base calculus that does not admit weakening (e.g., **HFLe**).

The rules that we have obtained in Step 1 of the algorithm are equivalent to their corresponding axioms in $\mathcal{N}_2, \mathcal{P}_3'$ (with the base calculus **(H)FLe**) and $\mathcal{P}_3$ (with the base calculus **HFLew**) but they do not necessarily preserve cut admissibility. For those rules, we only need to apply steps (2) and (3) of the completion procedure to retrieve the equivalent analytic rules.

Definition 3.2.7. [15] *A hyperstructural rule (hr) is* acyclic *if the* dependency graph D(hr) *is acyclic: Given a hyperstructural rule* $\dfrac{G \mid \Upsilon_1' \Rightarrow \Psi_1' \quad \cdots \quad G \mid \Upsilon_n' \Rightarrow \Psi_n'}{G \mid \Upsilon_1 \Rightarrow \Psi_1 \mid \cdots \mid \Upsilon_m \Rightarrow \Psi_m}\ (hr)$ *, D(hr) is then built as follows:*

- *The vertices of D(hr) are the metavariables for formulas occurring in the premises* $G \mid \Upsilon_1' \Rightarrow \Psi_1', \cdots, G \mid \Upsilon_n' \Rightarrow \Psi_n'$.

- *There is a directed edge $A \longrightarrow B$ in D(hr) if and only if there is a premise $G \mid \Upsilon'_i \Rightarrow \Psi'_i$ such that A occurs in Υ'_i and $B = \Psi'$.*

Example 3.2.8. *The rules $(c_0), (wc_0)$ and (wem_0) from Example 3.2.2 and Example 3.2.6 are acyclic. The rule* $\dfrac{G \mid A \Rightarrow B \quad G \mid B \Rightarrow C \quad G \mid C \Rightarrow A}{G \mid A \Rightarrow C}$ *is cyclic.*

In general, the completion procedure is divided into the following three parts: a preliminary step, a restructuring step and a cutting step. Note that the preliminary step can be skipped for rules generated by the algorithm in Step 1.

(1) Preliminary Step. This step ensures that a given (hyper)structural rule contains neither Γ nor Π before applying steps (2) and (3). We skip this step when a (hyper)structural rule does not contain Γ or Π, which is the case for all rules that are obtained by the algorithm described in the previous sections. If a (hyper)structural rule contains Γ or Π, we replace it with a fresh metavariable B_Γ or B_Π. This step preserves the acyclicity of a rule.

Example 3.2.9. *[15, 5] The cyclic rule (S_I) from [5] is restructured as follows:*

$$\frac{G \mid \Gamma, \Delta \Rightarrow A}{G \mid \Gamma \Rightarrow A \mid \Delta \Rightarrow A}\,(S_I) \longrightarrow \frac{G \mid B_\Gamma, B_\Delta \Rightarrow A}{G \mid B_\Gamma \Rightarrow A \mid B_\Delta \Rightarrow A}\,(S_I)$$

(2) Restructuring. We proceed with (hyper)structural rules containing only metavariables for formulas. Let $\Gamma_1, \ldots, \Gamma_n, \Sigma_X, \Pi_X$ be mutually distinct, fresh metavariables. We replace:

- every component $(Y_1, \ldots, Y_n \Rightarrow X)$ in the conclusion with $(\Gamma_1, \ldots, \Gamma_n, \Sigma_X \Rightarrow \Pi_X)$ and add $n+1$ premises of the form $(G \mid \Gamma_1 \Rightarrow Y_1), \ldots, (G \mid \Gamma_n \Rightarrow Y_n), (G \mid X, \Sigma_X \Rightarrow \Pi_X)$.
- every component $(Y_1, \ldots, Y_n \Rightarrow\ )$ in the conclusion with $(\Gamma_1, \ldots, \Gamma_n \Rightarrow\ )$ and add n premises of the form $(G \mid \Gamma_1 \Rightarrow Y_1), \ldots, (G \mid \Gamma_n \Rightarrow Y_n)$.

Notice that the context $G \mid$ is only added to hyperstructural rules.
This step preserves the acyclicity of a rule. The resulting rules are equivalent to the axioms by Lemma 3.2.1 and Lemma 3.2.5 and satisfy the following properties [15]:

- linear-conclusion: Each metavariable occurs at most once in the conclusion.

- separation: No metavariable occurring in the antecedent (succedent, respectively) of a component of the conclusion occurs in the succedent (antecedent, respectively) of a premise.
- coupling: The metavariables of each pair (Σ_X, Π_X), which is associated to the same occurrence of X, occur in the same premise.

Example 3.2.10. *We apply the restructuring step to the rule* (c_0) *from Example 3.2.2 which we have obtained for the contraction axiom:*

$$\frac{A, A \Rightarrow X}{A \Rightarrow X}\,(c_0) \longrightarrow \frac{\Gamma \Rightarrow A \qquad A, A \Rightarrow X}{\Gamma \Rightarrow X}\,(c_1) \longrightarrow \frac{\begin{array}{c}\Gamma \Rightarrow A \\ X, \Sigma \Rightarrow \Pi\end{array} \qquad A, A \Rightarrow X}{\Gamma, \Sigma \Rightarrow \Pi}\,(c_2)$$

Example 3.2.11. *We apply the restructuring step to the rule* (lin_0) *from Example 3.2.6 which we have obtained for the prelinearity axiom:*

$$\frac{}{G \mid A \Rightarrow B \mid B \Rightarrow A}\,(lin_0) \qquad \longrightarrow \frac{G \mid \Gamma_1 \Rightarrow A \qquad G \mid \Gamma_2 \Rightarrow B}{G \mid \Gamma_1 \Rightarrow B \mid \Gamma_2 \Rightarrow A}\,(lin_1)$$

$$\longrightarrow \frac{\begin{array}{c}G \mid \Gamma_1 \Rightarrow A \\ G \mid B, \Sigma_1 \Rightarrow \Pi_1\end{array} \qquad \begin{array}{c}G \mid \Gamma_2 \Rightarrow B \\ G \mid A, \Sigma_2 \Rightarrow \Pi_2\end{array}}{G \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}\,(lin_2)$$

(3) Cutting. In the last step of the completion procedure, we remove the redundant metavariables, i.e., metavariables that occur in the premise but not in the conclusion [15].

Let A be a redundant metavariable. Let $\mathcal{G}_1 = \{G \mid \Upsilon'_i \Rightarrow A, 1 \leq i \leq k\}$ be the subset of the premises where A is in the succedent and $\mathcal{G}_2 = \{G \mid \Upsilon_j, A, \ldots, A \Rightarrow \Psi_j, 1 \leq j \leq m\}$ the subset of the premises with one or more A in the antecedent.

If $k = 0$, i.e., $\mathcal{G}_1$ is empty and, thus, A only occurs in the antecedent of some premises, we remove subset $\mathcal{G}_2$ from the premises of the rule. Similarly, if $m = 0$, i.e., A only occurs in the succedent, we remove subset $\mathcal{G}_1$ from the premises of the rule. If we deal with a base calculus that admits weakening, e.g. **HFLew** for the class $\mathcal{P}_3$, we also take a subset $\mathcal{G}_3 = \{G \mid \Upsilon, A, \ldots, A \Rightarrow A\}$ into consideration and remove it from the premises immediately.

Otherwise, if $k > 0$ and $m > 0$, we create a new subset of premises $\mathcal{G}^{cut} = \{G \mid \Upsilon_j, \Upsilon'_{i_1}, \ldots, \Upsilon'_{i_p} \Rightarrow \Psi_j, 1 \leq j \leq m, 1 \leq i_1, \ldots, i_p \leq k\}$. We replace $\mathcal{G}_1 \cup \mathcal{G}_2$ with $\mathcal{G}^{cut}$ and repeat this cutting step until we obtain a (hyper)structural rule without redundant metavariables.

A (hyper)structural rule is *completed* when all three steps of the completion procedure have been applied to it. Every completed (hyper)structural rule is equivalent to the original (hyper)structural rule and has the following properties [15]:

- linear-conclusion
- coupling
- strong subformula property: Every metavariable that occurs in the antecedent (succedent, respectively) of a premise, also occurs in the antecedent (succedent, respectively) of the conclusion.

Example 3.2.12. *We apply the cutting step to the rule* (c_2) *equivalent to the contraction axiom (see Example 3.2.10):*

$$\frac{\begin{array}{c}\Gamma \Rightarrow A \\ X, \Sigma \Rightarrow \Pi\end{array} \qquad A, A \Rightarrow X}{\Gamma, \Sigma \Rightarrow \Pi}\,(c_2) \longrightarrow^{A} \frac{X, \Sigma \Rightarrow \Pi \qquad \Gamma, \Gamma \Rightarrow X}{\Gamma, \Sigma \Rightarrow \Pi}\,(c_3) \longrightarrow^{X} \frac{\Gamma, \Gamma, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi}\,(c)$$

Example 3.2.13. *We apply the restructuring step to the rule* (lin_2) *equivalent to the prelinearity axiom (see Example 3.2.11):*

$$\frac{\begin{array}{c}G \mid \Gamma_1 \Rightarrow A \\ G \mid B, \Sigma_1 \Rightarrow \Pi_1\end{array} \qquad \begin{array}{c}G \mid \Gamma_2 \Rightarrow B \\ G \mid A, \Sigma_2 \Rightarrow \Pi_2\end{array}}{G \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}\,(lin_2) \qquad \longrightarrow^{A} \frac{G \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_2 \qquad \begin{array}{c}G \mid \Gamma_2 \Rightarrow B \\ G \mid B, \Sigma_1 \Rightarrow \Pi_1\end{array}}{G \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}\,(lin_2)$$

$$\longrightarrow^{B} \frac{G \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_2 \qquad G \mid \Gamma_2, \Sigma_1 \Rightarrow \Pi_1}{G \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}\,(com)$$

Indeed, adding the (com)*-rule to the hypersequent version of* ***LJ*** *yields the calculus* ***GLC*** *(see Table 2.2.2) introduced by Avron in [4] for Gödel-logic, i.e., intuitionistic logic extended with the prelinearity axiom.*

3.3 Related Work

The topic of this thesis is situated in a new research direction called "algebraic proof theory" which emerges from joining the two research fields systematic proof theory and universal algebra in an innovative way. Algebraic proof theory aims at integrating and investigating various methods of both research fields in a uniform way. In the first steps into this new direction, the substructural hierarchy was introduced by Ciabattoni et al. in [14, 15] as a novel classification of axioms. It constitutes the foundation for a first systematic procedure that allows the transformation of axioms into equivalent analytic rules. The general aim is to cover the whole hierarchy bottom up as far as possible.

The first results in [14] cover all axioms up to the class $\mathcal{N}_2$ and transform them into structural rules of the single-conclusion, sequent calculus. The authors provided an algebraic proof of cut-elimination for the base calculus **FL** extended with completed, acyclic rules obtained by the presented procedure. Moreover, they showed that the corresponding algebraic equations, which coincide with axioms in proof theory, are closed under the Dedekind-MacNeille completion [14].

In [15], Ciabattoni et al. extended the procedure to the axioms up to the classes $\mathcal{P}_3'$ ($\mathcal{P}_3$, respectively) of the hierarchy with **FLe** (**FLew**, respectively) as base calculus. The axioms are transformed into structural rules of the single-conclusion, hypersequent calculus. They provided a uniform, semantic cut-elimination proof for the base calculi **FLe** and **FLew** extended with completed rules obtained by the algorithm described before. An implementation of this systematic procedure for axioms up to the classes $\mathcal{N}_2$ and $\mathcal{P}_3$ can be found in Chapter 4 of this thesis.

In [17] the procedure was shifted to the classic, multiple-conclusion setting which caused a deconstruction of the hierarchy. As a consequence, some axioms that belong to higher classes in the intuitionistic, single-conclusion setting are brought down to lower classes in the classic, multiple-conclusion setting.

An extension of this procedure to certain axioms that reside in classes above $\mathcal{P}_3$ is provided in Chapter 5.

CHAPTER **4**

An implementation for the classes $\mathcal{N}_2$ and $\mathcal{P}_3$

We implemented in PROLOG the systematic procedure in [15]. As [15] does not contain an algorithmic description of this procedure but the theoretical foundation expressed in various theorems, we first had to transform the theoretical results into a real algorithm. In this chapter, we describe the implementation of this algorithm which works for axioms up to the classes $\mathcal{N}_2$ and $\mathcal{P}_3$ of the substructural hierarchy in the presence of weakening. To provide further detail, we illustrate the descriptions with code snippets. The program, called AxiomCalc, is available online at `http://www.logic.at/people/lara/axiomcalc.html` where you can either download the source code or use the program via a web interface.

4.1 General Information

PROLOG is a declarative logic programming language [33]: It only contains a single data type, a *term*, to construct a logic program which consists of Horn clauses, i.e., a clause with at most one positive literal. A term is either a *constant*, i.e., an integer or an atom, a *variable* or a *compound term*, i.e., an atom called *functor* and a number of one or more *arguments* which are again terms. A logic program consists of a finite set of clauses which are divided into the two groups of *rules* and *facts*. A rule is a statement of the form

$A \leftarrow B_1, \ldots, B_n$ with $n \geq 0$

where A is called the *head* and the conjunction of all B_i's are the *body* of the rule. A fact is a rule with an empty body, i.e., $n = 0$. To compute a logic program we use queries which are conjunctions of the form:

$A_1, \ldots, A_n?$ with $n > 0$

where every A_i is called a *goal*. For more information on PROLOG, consult one of the main references from Sterling and Shapiro [33] or Deransart et al. [18].

Among the many free PROLOG implementations, we chose SWI-PROLOG (see `http://www.swi-prolog.org`) by Jan Wielemaker for our implementation because it is one of the most famous implementations and available for many platforms.

According to the systematic procedure described in Chapter 3, we split the program into three parts:

(1) Identification of the class of the substructural hierarchy the input axiom belongs to.

(2) Transformation of the axiom into an equivalent structural sequent (hypersequent, respectively) rule if it is within the class $\mathcal{N}_2$ ($\mathcal{P}_3$, respectively).

(3) Completion procedure to transform the (hyper)structural rule into an analytic rule preserving cut-elimination when added to **(H)FLe(w)**.

A more detailed description of these parts is provided in the following sections.

4.1.1 Input: Syntax of AxiomCalc

In the first step, the user has to provide a Hilbert axiom which will be transformed into an analytic rule. The input axiom may consist of

- the letters `[a-z]` for atomic formulas
- `0, 1, bot` and `top` for logical constants
- for logical connectives:
  - `&` ... additive and
  - `v` ... or
  - `*` ... fusion/multiplicative and
  - `-` ... negation
  - `->` ... implication

Example 4.1.1. *Examples for axioms according to the input syntax would be:*

`a -> 1, 0 -> a, (a*a) -> a, -(a & -a), (a -> b) v (b -> a)`

4.1.2 Data structure

For the representation of hypersequents and rules we chose lists as data structure.

Hypersequents

Recall that we deal with single-conclusion hypersequents of the form $G \mid A_1, \ldots, A_n \Rightarrow C$ or $G \mid A_1, \ldots, A_n \Rightarrow$. To simplify matters, we omit the side hypersequent $G \mid$ in the list representation of a hypersequent and only print it in the output. One component of a

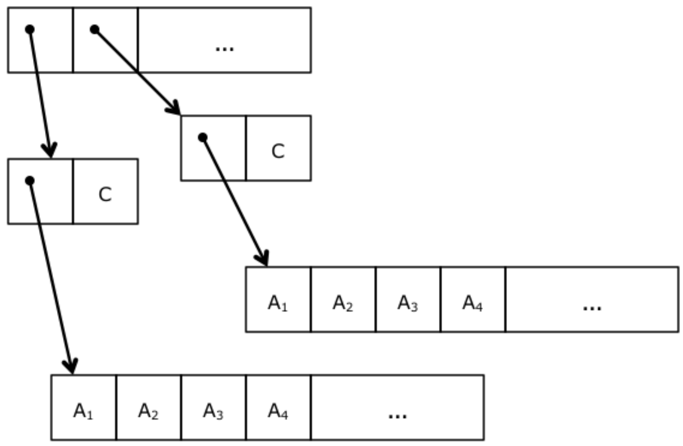

Figure 4.1: List representation of hypersequents

hypersequent is represented as a list, containing another list for the antecedent and one variable for the succedent:

```
[[ _, _, _, _, _, ... ], _ ]
```

A hypersequent is then considered a list of such lists. The list representation of a hypersequent is illustrated schematically in Figure 4.1.

Example 4.1.2. *Let $S = G \mid a, b \Rightarrow c \mid d \Rightarrow$ be a hypersequent. In our list representation, it is written as follows:*

```
[ [[ a, b], c ], [[ d ], '' ]]
```

Rules

The representation of the premises and the conclusion of a rule is the same as for hypersequents: the premises and the conclusion are handled as two lists of hypersequents.

Example 4.1.3. *Consider the rule $(\vee, l)$:*

$$\frac{G \mid X, A \Rightarrow C \qquad G \mid X, B \Rightarrow C}{G \mid X, A \vee B \Rightarrow C} \; (\vee, l)$$

In list representation, it is written as follows:

```
[[[X, A], C], [[X, B], C]]
--------------------------- (v,l)
[[[X, A v B], C]]
```

List Manipulation
For list manipulation we use the methods `append` for adding list items, `remove` for removing list items, `reverseList` for reversing the list and `replaceElements` for replacing list elements with new items.

4.2 Part 1: Identifying the axiom class

In the first part, we recall the definition of the substructural hierarchy and reformulate it with PROLOG clauses.

Definition 4.2.1. (Substructural Hierarchy) [15] *Let $\mathcal{A}$ be the set of atomic formulas. For $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined as follows:*

$$\mathcal{P}_0 ::= \mathcal{N}_0 ::= \mathcal{A}$$
$$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid 1 \mid \bot$$
$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid 0 \mid \top$$

Propositional variables are considered atomic whereas `0, 1, bot, top` reside on level ≥ 1 of the classes $\mathcal{P}$ or $\mathcal{N}$ (`A` is the axiom):

```
is_atom(A) :-
      atomic(A),                        % propositional variables are atomic
      \+ member(A,[0,1,bot,top]).       % except 0, 1, bot, top
```

Then we provide a recursive definition of the positive (`is_pos_axiom`) and the negative (`is_neg_axiom`) classes of the substructural hierarchy. The first parameter `A` stands for the axiom, the second argument `N` is the variable for calculating the level of the class in the hierarchy. E.g. for the definition of positive classes we write (analogously for the negative levels):

```
is_pos_axiom(A, 1) :-
      is_atom(A).
is_pos_axiom(1, N) :-        % 1 is situated on any level P_n, n > 0
      N > 0.
is_pos_axiom(bot, N) :-      % bot is situated on any level P_n, n > 0
      N > 0.
is_pos_axiom(A, N1) :-        % any axiom on a level P_n is on the level N_n-1
      N1 > 0,
      N is N1-1,
      is_neg_axiom(A, N).
```

```
is_pos_axiom(A v B, N) :-       % if A v B is situated on level P_n,
      N > 0,                    % then A and B are both also on level P_n
      is_pos_axiom(A, N),
      is_pos_axiom(B, N).
is_pos_axiom(A * B, N) :-
      N > 0,
      is_pos_axiom(A, N),
      is_pos_axiom(B, N).
```

This way we determine the levels $\mathcal{N}_{n_n}$ and $\mathcal{P}_{n_p}$ for the input axiom. If $n_p < n_n$, the axiom belongs to the class $\mathcal{P}_{n_p}$, otherwise it belongs to the class $\mathcal{N}_{n_n}$. If the axiom belongs to a class above $\mathcal{N}_2$ or $\mathcal{P}_3$, the program calculates the class level but does not proceed further. Instead, an information message is printed on the screen, e.g.

```
The axiom a v (a -> (b v -b)) is in the class: p(4)
The algorithm does not work for this class yet.
```

4.3 Part 2: From axioms to (hyper)structural rules

When the algorithm has determined the class of the input axiom and if it belongs to a class up to $\mathcal{N}_2$ or $\mathcal{P}_3$, the program generates a structural rule in sequent or hypersequent calculus equivalent to the original axiom. However, we do not distinguish between sequent and hypersequent rules in the program code because sequent rules are considered hypersequent rules with only one component and without the context $G \mid$ printed in the output. The program proceeds in the same way as it has been described in Section 3.2:

- Consider the interpretation function for hypersequents (Definition 3.2.3): We retrieve the components of the hypersequent by splitting the axiom at every $\vee$:

```
is_hypersequent(Ax1 v Ax2, HS1) :-
    is_hypersequent(Ax1, HS0),
    append([Ax2], HS0, HS1).

is_hypersequent(Ax, HS) :-
    atomic(Ax),
    append([Ax], [], HS).

is_hypersequent(Ax, HS) :-
    compound(Ax),
    append([Ax], [], HS).
```

Example 4.3.1. *Let* `A = -a v -(-a)` *be the input axiom.* `is_hypersequent` *then retrieves the following hypersequent:* `[[[], -a], [[], -(-a)]]`

- We use the invertible rules $(\to, r), (\cdot, l)$ and $(1, l)$ on every component in the conclusion of the hypersequent until no further rule-application is possible. E.g. the recursive application of the right-invertible rules $(\to, r)$ and $(\neg, r)$, the latter being basically a combined application of $(\to, r)$ and $(0, r)$, to the components of the conclusion is implemented as follows (note that the first two parameters, denoted by `_, _`, do not matter here; the third parameter is the succedent of the conclusion which we get as input, the fourth argument is the whole component of the conclusion which we get as a result):

```
apply_right(_, _, Axiom, [[], Axiom]).

% (-> r) rule: X =>  a -> b  ... a, X => b
apply_right(_, _, Ax1 -> Ax2, [P1, C0]) :-
    apply_right(_, _, Ax2, [P0, C0]),
    append(P0, [Ax1], P1).

% (-r) rule: X => -a  ...  a, X =>
apply_right(_, _, -C1, [[C1], '']).
```

- Then we use the equivalence lemmas, Lemma 3.2.1 and Lemma 3.2.5, to replace the antecedents A_i in the conclusion with fresh metavariables `Y+i` with `i` being an index and add `Y+i =>` A_i to the premises.
- We apply the invertible rules $(\wedge, r), (\to, r), (\cdot, l), (1, l)$ and $(0, r)$ to the premises until no further application is possible.
- We either apply $(0, r)$ to every component of the conclusion, or we use the equivalence lemmas to replace the succedents C of the conclusion with fresh metavariables `X+j` with `j` being an index and add `C => X+j` to the premises.
- We recursively apply the invertible rules $(\vee, l), (\cdot, l)$ and $(1, l)$ to the premises.

Thus we obtain a (hyper)structural rule equivalent to the input axiom.

4.4 Part 3: Applying the completion procedure

In the third part, the program transforms the (hyper)structural rule obtained in part (2) into an equivalent analytic rule. We implement step (2) and step (3) from the completion procedure described in Section 3.2, namely restructuring and cutting.

Given any (hyper)structural rule, we replace every metavariable `Y+i` in the antecedent of a component of the conclusion with `G+j`, which stands for Γ_j. Then we add `G+j => Y+i` to the premises. Similarly, we replace all metavariables `X+i` in the succedent of a component of the conclusion with `D+k => P+k` and add `X+i, D+k => P+k` to the premises:

```
%% restructuring(+Premises, -Premises, +Conclusion, -Conclusion)
restructuring(P0, P0, [], []).

restructuring(P0, P3, [[A0|['']]|T], C2) :-
    restructuring(P0, P1, T, C1),
    is_replaced_antecedent(A0, A1, P2),  %replace the antecedent
    append([[A1, '']], C1, C2),
    append(P1, P2, P3).

restructuring(P0, P4, [[A0|['X'+I]]|T], C2) :-
    restructuring(P0, P1, T, C1),
    is_replaced_antecedent(A0, A1, P2),
    append(['D'+I], A1, A2),
    append([[A2, 'P'+I]], C1, C2),
    append(P1, P2, P3),
    append([[['X'+I, 'D'+I], 'P'+I]], P3, P4).
```

Then we apply the cutting step until the premises are free of redundant metavariables (note that `is_metafree_premises(P0)` evaluates to true if there are no redundant metavariables in `P0`):

```
cutting(P0, P0) :-
    is_metafree_premises(P0).

cutting(P0, P2) :-
    \+ is_metafree_premises(P0),
    is_containing_metavariable(P0, MV),
    cut_metavariable(P0, P1, MV),
    cutting(P1, P2).
```

The rule that is obtained by this completion procedure does not contain any metavariables `X+i, Y+j` or any atoms, but only metavariables `G+i, D+j, P+j`.

Example 4.4.1. *The input axiom* (a -> b) v (b -> a) *results in the following analytic rule:*

```
Equivalent Analytic Rule:

G|G+1,D+2 => P+2  G|G+2,D+1 => P+1
-----------------------------
 G| D+2,G+2 => P+2| D+1,G+1 => P+1
```

CHAPTER **5**

From the axioms (Bd_k) to analytic rules

The systematic procedure of Ciabattoni et al. [15] to translate Hilbert axioms into equivalent analytic calculi works only up to the classes $\mathcal{N}_2$ and $\mathcal{P}_3$ of the substructural hierarchy. In this chapter, the scope of this procedure will be extended to capture the family of axioms known as (Bd_k) with $k \geq 1$ which are semantically characterized by Kripke models with depth $\leq k$, see [9]. The axiom scheme (Bd_k) is recursively defined as follows:

(Bd_1) $\quad A_1 \vee \neg A_1$

(Bd_{i+1}) $\quad A_{i+1} \vee (A_{i+1} \rightarrow (Bd_i))$

These axioms are especially interesting as intuitionistic logic extended with the axiom (Bd_2) is the only one of the seven interpolable intermediate logics — i.e., logics between intuitionistic and classical logic — to which the procedure in [15] does not apply to. Indeed all other six axioms are within the class $\mathcal{P}_3$ of the substructural hierarchy while (Bd_2) belongs to the class $\mathcal{P}_4$.

Example 5.0.2. *Consider the axiom* $(Bd_2) = A_2 \vee (A_2 \rightarrow (A_1 \vee \neg A_1))$*:* $(A_1 \vee \neg A_1)$ *is within* $\mathcal{P}_2$*,* $(A_2 \rightarrow (A_1 \vee \neg A_1))$ *is within* $\mathcal{N}_3$ *and due to the outermost connective* $\vee$*, the axiom* (Bd_2) *belongs to the class* $\mathcal{P}_4$*.*

The extension of the systematic procedure to cover the family of axioms known as (Bd_k) requires the introduction of a new, more powerful formalism and the identification of suitable rules corresponding to (Bd_k). Subsequently, the proofs of soundness, completeness and cut-elimination for the generated calculi in the new framework have to be provided.

5.1 The non-commutative hypersequent calculus

Here we introduce a new formalism, the non-commutative hypersequent calculus, and provide suitable rules corresponding to the axioms (Bd_k). Moreover, we give uniform proofs of soundness, completeness and cut-elimination for the introduced calculi in the framework.

Definition 5.1.1. *A non-commutative hypersequent (nc-hypersequent, for short) is an expression of the form*

$$\Gamma_1 \Rightarrow \Delta_1 \mid\lceil \ \ldots \ \mid\lceil \Gamma_n \Rightarrow \Delta_n$$

where for all $i = 1, \ldots, n, \Gamma_i \Rightarrow \Delta_i$ *is a sequent which is called* component *of the non-commutative hypersequent. A non-commutative hypersequent is* single-conclusion *if all of its components are single-conclusion and it is* multiple-conclusion *otherwise.*

As in the case of the hypersequent calculus [5], the nc-hypersequent calculus consists of initial axioms, logical rules, the cut-rule and (internal and external) structural rules. The main aspect where the non-commutative hypersequent calculus differs from the hypersequent calculus is the lack of the external-exchange structural rule (*ee*) (see Section 2.2.2), i.e., the order of the components of a non-commutative hypersequent matters. E.g., $\Gamma_1 \Rightarrow \Delta_1 \mid\lceil \Gamma_2 \Rightarrow \Delta_2$ is not the same as $\Gamma_2 \Rightarrow \Delta_2 \mid\lceil \Gamma_1 \Rightarrow \Delta_1$. Therefore, non-commutative hypersequents contain two (possibly empty) *side nc-hypersequents* G and G'.

Definition 5.1.2. *A non-commutative hypersequent of the form*

$$\mathcal{G} := \Gamma_1 \Rightarrow \Delta_1 \mid\lceil \Gamma_2 \Rightarrow \Delta_2 \mid\lceil \ \ldots \ \mid\lceil \Gamma_n \Rightarrow \Delta_n$$

is interpreted as follows:

$$\mathcal{G}^I := \bigwedge \Gamma_1 \to (\bigvee \Delta_1 \vee (\bigwedge \Gamma_2 \to (\bigvee \Delta_2 \vee (\cdots \vee (\bigwedge \Gamma_n \to \bigvee \Delta_n)))))$$

where $\bigwedge \Gamma_i$ *is the conjunction of the formulas in* Γ_i *or* $\top$*, when* Γ_i *is empty, and* $\bigvee \Delta_i$ *is either the disjunction of the formulas in* Δ_i *or* $\bot$*. For single-conclusion nc-hypersequents,* Δ_i *is either one formula or* $\bot$*.*

The difference between hypersequents and nc-hypersequents is illustrated best when comparing their interpretation functions. Recall the definition of the interpretation function for hypersequents (see Definition 3.2.3): A hypersequent $\mathcal{G}_c = \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2$ is interpreted as $\mathcal{G}_c^I = (\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee (\bigwedge \Gamma_2 \to \bigvee \Delta_2)$ which shows that the components are commutative and, consequently, the order does not matter. In contrast, the corresponding nc-hypersequent $\mathcal{G}_{nc} = \Gamma_1 \Rightarrow \Delta_1 \mid\lceil \Gamma_2 \Rightarrow \Delta_2$ is interpreted as

$\mathcal{G}^I_{nc} = \bigwedge \Gamma_1 \to (\bigvee \Delta_1 \vee (\bigwedge \Gamma_2 \to \bigvee \Delta_2))$ where the nesting of the components becomes obvious.

Notation. For the interpretations of side nc-hypersequents, we write $G^I[*]$ where $*$ is replaced with the interpreted nc-hypersequent components nested into G^I. Let $\mathcal{G}$ be a generic non-commutative hypersequent of the form

$$\mathcal{G} = G \mid\lceil \Gamma \Rightarrow \Delta \mid\lceil G'$$

$\mathcal{G}$ is interpreted as follows:

$$\mathcal{G}^I = (G \mid\lceil \Gamma \Rightarrow \Delta \mid\lceil G')^I = G^I[\bigwedge \Gamma \to (\bigvee \Delta \vee G'^I)]$$

Definition 5.1.3. [35, 37] *Given a logical rule, the* principal *formula is the formula in the conclusion of a rule in which the logical symbol is introduced. The formula(s) in the premise(s) from which the principal formula derives are called* active *formula(s). The formulas which remain unchanged are referred to as* side *formulas or* (internal) contexts*.*

Example 5.1.4. *Consider the rule* $(\wedge, r)$*:* $\dfrac{G \mid\lceil \Gamma \Rightarrow A \mid\lceil G' \qquad G \mid\lceil \Gamma \Rightarrow B}{G \mid\lceil \Gamma \Rightarrow A \wedge B \mid\lceil G'}\ (\wedge, r)$
Then $A \wedge B$ *is the* principal *formula,* A *and* B *are the* active *formulas.* Γ *is the* (internal) context *(or* side *formula).*

Calculus rules

We denote by $\mathbf{HLJ}^{nc}$ the non-commutative hypersequent calculus for **LJ**. $\mathbf{HLJ}^{nc}$ consists of the following rules:

Axioms

$$A \Rightarrow A \qquad\qquad \bot \Rightarrow A$$

Cut Rule

$$\frac{G \mid\!\lceil\; \Gamma \Rightarrow A \mid\!\lceil\; G' \qquad H \mid\!\lceil\; A, \Sigma \Rightarrow \Pi \mid\!\lceil\; H'}{G \mid\!\lceil\; H \mid\!\lceil\; \Sigma, \Gamma \Rightarrow \Pi \mid\!\lceil\; H' \mid\!\lceil\; G'}$$

External Structural Rules

$$\frac{G \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma \Rightarrow \Pi \mid\!\lceil\; G'}\ (ew) \qquad\qquad \frac{G \mid\!\lceil\; S \mid\!\lceil\; S \mid\!\lceil\; G'}{G \mid\!\lceil\; S \mid\!\lceil\; G'}\ (ec)$$

with $S = \Gamma_1 \Rightarrow \Pi_1 \mid\!\lceil\; \cdots \mid\!\lceil\; \Gamma_n \Rightarrow \Pi_n$

Internal Structural Rules

$$\frac{G \mid\!\lceil\; \Gamma \Rightarrow \Pi \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma, A \Rightarrow \Pi \mid\!\lceil\; G'}\ (w,\ l) \qquad\qquad \frac{G \mid\!\lceil\; \Gamma \Rightarrow\ \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma \Rightarrow \Pi \mid\!\lceil\; G'}\ (w,\ r)$$

$$\frac{G \mid\!\lceil\; \Gamma, A, A \Rightarrow \Pi \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma, A \Rightarrow \Pi \mid\!\lceil\; G'}\ (c,\ l) \qquad\qquad \frac{G \mid\!\lceil\; \Gamma, A \Rightarrow \Pi \mid\!\lceil\; G'}{G \mid\!\lceil\; A, \Gamma \Rightarrow \Pi \mid\!\lceil\; G'}\ (e,\ l)$$

Logical Rules

$$\frac{G \mid\!\lceil\; \Gamma, A \Rightarrow B}{G \mid\!\lceil\; \Gamma \Rightarrow A \rightarrow B \mid\!\lceil\; G'}\ (\rightarrow,\ r) \qquad\qquad \frac{G \mid\!\lceil\; \Gamma \Rightarrow A \mid\!\lceil\; G' \qquad G \mid\!\lceil\; B, \Gamma \Rightarrow \Pi \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma, A \rightarrow B \Rightarrow \Pi \mid\!\lceil\; G'}\ (\rightarrow,\ l)$$

$$\frac{G \mid\!\lceil\; \Gamma \Rightarrow A \mid\!\lceil\; G' \qquad G \mid\!\lceil\; \Gamma \Rightarrow B \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma \Rightarrow A \wedge B \mid\!\lceil\; G'}\ (\wedge,\ r) \qquad\qquad \frac{G \mid\!\lceil\; \Gamma, A_i \Rightarrow \Pi \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma, A_1 \wedge A_2 \Rightarrow \Pi \mid\!\lceil\; G'}\ (\wedge_i,\ l)_{i=1,2}$$

$$\frac{G \mid\!\lceil\; \Gamma \Rightarrow A_i \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma \Rightarrow A_1 \vee A_2 \mid\!\lceil\; G'}\ (\vee_i,\ r)_{i=1,2} \qquad\qquad \frac{G \mid\!\lceil\; \Gamma, A \Rightarrow \Pi \mid\!\lceil\; G' \qquad G \mid\!\lceil\; \Gamma, B \Rightarrow \Pi \mid\!\lceil\; G'}{G \mid\!\lceil\; \Gamma, A \vee B \Rightarrow \Pi \mid\!\lceil\; G'}\ (\vee,\ l)$$

Table 5.1: Non-commutative Hypersequent Calculus $\mathbf{HLJ}^{nc}$

We will show that $\vdash_{LJ} \Gamma \Rightarrow \Pi$ iff $\vdash_{HLJ^{nc}} \Gamma \Rightarrow \Pi$, i.e., if $\Gamma \Rightarrow \Pi$ is provable in **LJ**, it is also provable in $\mathbf{HLJ}^{nc}$ and vice versa (see Corollary 5.1.9).
Recall the hypersequent calculus **GLC** depicted in Table 2.2.2. Note that the calculus $\mathbf{HLJ}^{nc}$ lacks the external exchange rule (ee). Furthermore, in $\mathbf{HLJ}^{nc}$ we have to take care of the left *and* right contexts G and G', whereas we only have to consider the left

context G in **GLC**. There is also an important difference between the logical rule $(\to, r)$ and the other rules of $\mathbf{HLJ}^{nc}$ as the right context G' is omitted in the premise of $(\to, r)$. Thus, we have to be very careful when using this rule, especially when proving soundness, w.r.t. intuitionistic logic, and cut-elimination.

Lemma 5.1.5. (Completeness) $\quad S \vdash_{HLJ^{nc}} \Rightarrow S^I$

Proof. Let S be a generic non-commutative hypersequent. Assume that S is of the form $S = \Gamma_1 \Rightarrow \Pi_1 \mid\lceil \Gamma_2 \Rightarrow \Pi_2$. We show that $S \vdash_{HLJ^{nc}} \Rightarrow S^I$ — i.e., $S \vdash_{HLJ^{nc}} \Rightarrow \bigwedge \Gamma_1 \to (\Pi_1 \vee (\bigwedge \Gamma_2 \to \Pi_2))$ — as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma_1 \Rightarrow \Pi_1 \mid\lceil \Gamma_2 \Rightarrow \Pi_2}{\bigwedge\Gamma_1 \Rightarrow \Pi_1 \mid\lceil \bigwedge\Gamma_2 \Rightarrow \Pi_2}\;(c,l)(\wedge,l)}{\bigwedge\Gamma_1 \Rightarrow \Pi_1 \mid\lceil \bigwedge\Gamma_1, \bigwedge\Gamma_2 \Rightarrow \Pi_2}\;(w,l)}{\bigwedge\Gamma_1 \Rightarrow \Pi_1 \mid\lceil \bigwedge\Gamma_1 \Rightarrow \bigwedge\Gamma_2 \to \Pi_2}\;(\to,r)}{\bigwedge\Gamma_1 \Rightarrow \Pi_1 \vee (\bigwedge\Gamma_2 \to \Pi_2)) \mid\lceil \bigwedge\Gamma_1 \Rightarrow \Pi_1 \vee (\bigwedge\Gamma_2 \to \Pi_2)}\;(\vee,r)}{\Rightarrow \bigwedge\Gamma_1 \to (\Pi_1 \vee (\bigwedge\Gamma_2 \to \Pi_2)) \mid\lceil \Rightarrow \bigwedge\Gamma_1 \to (\Pi_1 \vee (\bigwedge\Gamma_2 \to \Pi_2))}\;(\to,r)}{\Rightarrow \bigwedge\Gamma_1 \to (\Pi_1 \vee (\bigwedge\Gamma_2 \to \Pi_2))}\;(ec)$$

The proof can easily be extended to a non-commutative hypersequent with n components.

Lemma 5.1.6. (Soundness) *If* $\vdash_{HLJ^{nc}} \Gamma_1 \Rightarrow \Pi_1 \mid\lceil \cdots \mid\lceil \Gamma_n \Rightarrow \Pi_n$ *then* $\vdash_{LJ} \Rightarrow (\Gamma_1 \Rightarrow \Pi_1 \mid\lceil \cdots \mid\lceil \Gamma_n \Rightarrow \Pi_n)^I$.

Proof. By induction on the length of the derivation. The base case is true since the claim holds for axioms ($\vdash_{HLJ^{nc}} A \Rightarrow A$ implies $\vdash_{LJ} \Rightarrow A \to A$ and therefore $\vdash_{LJ} A \Rightarrow A$, similarly for $\vdash_{HLJ^{nc}} \bot \Rightarrow A$). For the inductive case we proceed by showing that for each inference rule in $\mathbf{HLJ}^{nc}$ with premises

$$G \mid\lceil S_1 \mid\lceil G'', \ldots, G \mid\lceil S_n \mid\lceil G''$$

and conclusion $G \mid\lceil S_0 \mid\lceil G'$, the sequent

$$(G \mid\lceil S_1 \mid\lceil G'')^I, \ldots, (G \mid\lceil S_n \mid\lceil G'')^I \Rightarrow (G \mid\lceil S_0 \mid\lceil G')^I$$

is provable in **LJ** (with $n \in \{1, 2\}$ and G'' might be empty).

- **Logical rules.** We show the cases of the rules $(\to, r)$ and $(\to, l)$. The other logical rules can be proved analogously.

$(\to, r)$:

$$\frac{G \mid\lceil \Gamma, A \Rightarrow B}{G \mid\lceil \Gamma \Rightarrow A \to B \mid\lceil G'} (\to, r)$$

By induction hypothesis we have $\vdash_{LJ} \Rightarrow G^I[(\bigwedge \Gamma \wedge A) \to B]$. Since $\vdash_{LJ} G^I[(\bigwedge \Gamma \wedge A) \to B] \Rightarrow G^I[\bigwedge \Gamma \to ((A \to B) \vee G'^I]$, the claim follows by using (cut):

$$\frac{\Rightarrow G^I[(\bigwedge \Gamma \wedge A) \to B] \qquad G^I[(\bigwedge \Gamma \wedge A) \to B] \Rightarrow G^I[\bigwedge \Gamma \to ((A \to B) \vee G'^I)]}{\Rightarrow G^I[\bigwedge \Gamma \to ((A \to B) \vee G'^I)]} (cut)$$

Remark: The proof for the rule $(\to, r)$ does not work in the presence of a right context G' in the premise due to the nesting of the interpreted formula. Indeed we cannot prove $(\bigwedge \Gamma \wedge A) \to (B \vee G'^I) \Rightarrow \bigwedge \Gamma \to ((A \to B) \vee G'^I)$.

$(\to, l)$:

$$\frac{G \mid\lceil \Gamma \Rightarrow A \mid\lceil G' \qquad G \mid\lceil B, \Gamma \Rightarrow \Pi \mid\lceil G'}{G \mid\lceil \Gamma, A \to B \Rightarrow \Pi \mid\lceil G'} (\to, l)$$

By induction hypothesis we have $\vdash_{LJ} \Rightarrow G^I[\bigwedge \Gamma \to (A \vee G'^I)]$ and $\vdash_{LJ} \Rightarrow G^I[(B \wedge \bigwedge \Gamma) \to (\Pi \vee G'^I)]$. Since $\vdash_{LJ} G^I[\bigwedge \Gamma \to (A \vee G'^I)], G^I[(B \wedge \bigwedge \Gamma) \to (\Pi \vee G'^I)] \Rightarrow G^I[(\bigwedge \Gamma \wedge (A \to B)) \vee (\Pi \vee G'^I)]$, the claim follows by using (cut):

$$\frac{\Rightarrow G^I[\bigwedge \Gamma \to (A \vee G'^I)] \qquad \dfrac{\Rightarrow G^I[(B \wedge \bigwedge \Gamma) \to (\Pi \vee G'^I)] \qquad \begin{array}{c} G^I[\bigwedge \Gamma \to (A \vee G'^I)], G^I[(B \wedge \bigwedge \Gamma) \to (\Pi \vee G'^I)] \\ \Rightarrow G^I[(\bigwedge \Gamma \wedge (A \to B)) \to (\Pi \vee G'^I)] \end{array}}{G^I[\bigwedge \Gamma \to (A \vee G'^I)] \Rightarrow G^I[(\bigwedge \Gamma \wedge (A \to B)) \to (\Pi \vee G'^I)]} (cut)}{\Rightarrow G^I[(\bigwedge \Gamma \wedge (A \to B)) \to (\Pi \vee G'^I)]} (cut)$$

– **Structural rules.** We outline the proofs for the cases (ew) and (ec). The remaining cases are similar.

(ew):

$$\frac{G \mid\lceil G'}{G \mid\lceil \Gamma \Rightarrow \Pi \mid\lceil G'} (ew)$$

By induction hypothesis we have $\vdash_{LJ} \Rightarrow G^I[G'^I]$. Since $\vdash_{LJ} G^I[G'^I] \Rightarrow G^I[\bigwedge \Gamma \to (\Pi \vee G'^I)]$, the claim follows by using (cut):

$$\frac{\Rightarrow G^I[G'^I] \qquad G^I[G'^I] \Rightarrow G^I[\bigwedge \Gamma \to (\Pi \vee G'^I)]}{\Rightarrow G^I[\bigwedge \Gamma \to (\Pi \vee G'^I)]} (cut)$$

(ec):

$$\frac{G \mid\lceil S \mid\lceil S \mid\lceil G'}{G \mid\lceil S \mid\lceil G'} \ (ec)$$

Suppose $S = \Gamma \Rightarrow \Pi$. By induction hypothesis we have $\vdash_{LJ} \Rightarrow G^I[\bigwedge\Gamma \rightarrow (\Pi \vee (\bigwedge\Gamma \rightarrow (\Pi \vee G'^I)))]$. Since $\vdash_{LJ} G^I[\bigwedge\Gamma \rightarrow (\Pi \vee (\bigwedge\Gamma \rightarrow (\Pi \vee G'^I)))] \Rightarrow G^I[\bigwedge\Gamma \rightarrow (\Pi \vee G'^I)]$, the claim follows by using (cut):

$$\frac{\Rightarrow G^I[\bigwedge\Gamma \rightarrow (\Pi \vee (\bigwedge\Gamma \rightarrow (\Pi \vee G'^I)))] \qquad \begin{array}{c} G^I[\bigwedge\Gamma \rightarrow (\Pi \vee (\bigwedge\Gamma \rightarrow (\Pi \vee G'^I)))] \\ \Rightarrow G^I[\bigwedge\Gamma \rightarrow (\Pi \vee G'^I)] \end{array}}{\Rightarrow G^I[\bigwedge\Gamma \rightarrow (\Pi \vee G'^I)]} \ (cut)$$

– **(cut)-Rule.**

$$\frac{G \mid\lceil \Gamma \Rightarrow A \mid\lceil G' \qquad H \mid\lceil A, \Sigma \Rightarrow \Pi \mid\lceil H'}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'} \ (cut)$$

By induction hypothesis we have $\vdash_{LJ} \Rightarrow G^I[\bigwedge\Gamma \rightarrow (A \vee G'^I)]$ and $\vdash_{LJ} \Rightarrow H^I[(A \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I)]$. Since $\vdash_{LJ} G^I[\bigwedge\Gamma \rightarrow (A \vee G'^I)], H^I[(A \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I)] \Rightarrow G^I[H^I[(\bigwedge\Gamma \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I[G'^I])]]$, the claim follows by using (cut):

$$\frac{\Rightarrow H^I[(A \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I)] \qquad \dfrac{\Rightarrow G^I[\bigwedge\Gamma \rightarrow (A \vee G'^I)] \qquad \begin{array}{c} G^I[\bigwedge\Gamma \rightarrow (A \vee G'^I)], H^I[(A \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I)] \\ \Rightarrow G^I[H^I[(\bigwedge\Gamma \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I[G'^I])]] \end{array}}{H^I[(A \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I)] \Rightarrow G^I[H^I[(\bigwedge\Gamma \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I[G'^I])]]} \ (cut)}{\Rightarrow G^I[H^I[(\bigwedge\Gamma \wedge \bigwedge\Sigma) \rightarrow (\Pi \vee H'^I[G'^I])]]} \ (cut)$$

The reason why (ee) does not work in HLJ^{nc}:

$$\frac{G \mid\lceil G'}{G' \mid\lceil G} \ (ee)$$

Due to the definition of the interpretation function, we are not able to prove the external-exchange rule for **LJ**: Suppose $G = \Gamma \Rightarrow \Pi$ and $G' = \Gamma' \Rightarrow \Pi'$. Then we have to show $\vdash_{LJ} \bigwedge\Gamma \rightarrow (\Pi \vee (\bigwedge\Gamma' \rightarrow \Pi')) \Rightarrow \bigwedge\Gamma' \rightarrow (\Pi' \vee (\bigwedge\Gamma \rightarrow \Pi))$. After one rule application of $(\rightarrow, r)$ we are stuck. An application of $(\rightarrow, l)$ yields $\bigwedge\Gamma' \Rightarrow \bigwedge\Gamma$ which is not provable in general. When using $(\vee, r)$, we have to omit either Π' or $(\bigwedge\Gamma \rightarrow \Pi)$. Either way, we lose an important part of the formula which we will need later to finish the proof. Thus, the external exchange rule cannot be proved in **LJ** and an addition of this rule to the calculus would contradict Lemma 5.1.6.

Proposition 5.1.7. *For any non-commutative hypersequent S and any set of sequents $\mathcal{S} \cup \{S_0\}$, we have the following equivalences:*

$\{S\} \cup \mathcal{S} \vdash_{HLJ^{nc}} S_0$ *iff* $\{\Rightarrow S^I\} \cup \mathcal{S} \vdash_{HLJ^{nc}} S_0$ *iff* $\{\Rightarrow S^I\} \cup \mathcal{S} \vdash_{LJ} S_0$

Proof. $\{\Rightarrow S^I\} \cup \mathcal{S} \vdash_{LJ} S_0$ obviously implies $\{\Rightarrow S^I\} \cup \mathcal{S} \vdash_{HLJ^{nc}} S_0$ (if a sequent is provable in **LJ**, it is also provable in the non-commutative hypersequent framework **HLJ**^{nc}).

Due to Lemma 5.1.5, $\{\Rightarrow S^I\} \cup \mathcal{S} \vdash_{HLJ^{nc}} S_0$ implies $\{S\} \cup \mathcal{S} \vdash_{HLJ^{nc}} S_0$.

$\{S\} \cup \mathcal{S} \vdash_{HLJ^{nc}} S_0$ implies $\{\Rightarrow S^I\} \cup \mathcal{S} \vdash_{LJ} S_0^I$ by Lemma 5.1.6. The claim follows as S_0^I implies S_0 in **LJ**.

Lemma 5.1.8. (Bd$_k$) *Let S be the natural nc-hypersequent "corresponding" to (Bd_k), i.e., S is $\Rightarrow A_k \;||\!\lceil\; A_k \Rightarrow A_{k-1} \;||\!\lceil\; \cdots \;||\!\lceil\; A_2 \Rightarrow A_1 \;||\!\lceil\; \Rightarrow \neg A_1$. Then we can show for $k \geq 2$: $\vdash_{LJ} (Bd_k) \leftrightarrow S^I$.*

Proof. The proof tree is split in two parts, the first one showing the direction $\vdash_{LJ} (Bd_k) \to S^I$, the second one showing $\vdash_{LJ} S^I \to (Bd_k)$. This may be done since $\vdash_{LJ} (Bd_k) \leftrightarrow S^I$ is $\vdash_{LJ} ((Bd_k) \to S^I) \wedge (S^I \to (Bd_k))$ and due to the application of the $(\wedge, r)$ rule.

$\vdash_{LJ} (Bd_2) \to S^I$:

$$\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B \qquad \dfrac{\neg B \Rightarrow \neg B}{\neg B \Rightarrow \top \to \neg B}\;(\to,r)(w,l)}{B \vee \neg B \Rightarrow B \vee (\top \to \neg B)}\;(\vee,l)(\vee,r)}{A \to (B \vee \neg B) \Rightarrow A \to (B \vee (\top \to \neg B))}\;(\to,r)(\to,l)}{A \vee (A \to (B \vee \neg B)) \Rightarrow A \vee (A \to (B \vee (\top \to \neg B)))}\;(\vee,l)(\vee,r)}{\Rightarrow (A \vee (A \to (B \vee \neg B))) \to (A \vee (A \to (B \vee (\top \to \neg B))))}\;(\to,r)$$

$\vdash_{LJ} S^I \to (Bd_2)$:

$$\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B \qquad \dfrac{A \Rightarrow \top \qquad \neg B \Rightarrow \neg B}{\top \to \neg B, A \Rightarrow \neg B}\;(\to,l)}{B \vee (\top \to \neg B), A \Rightarrow B \vee \neg B}\;(\vee,l)(\vee,r)}{A \to (B \vee (\top \to \neg B)) \Rightarrow A \to (B \vee \neg B)}\;(\to,r)(\to,l)}{A \vee (A \to (B \vee (\top \to \neg B))) \Rightarrow A \vee (A \to (B \vee \neg B))}\;(\vee,l)(\vee,r)}{\Rightarrow (A \vee (A \to (B \vee (\top \to \neg B)))) \to (A \vee (A \to (B \vee \neg B)))}\;(\to,r)$$

The proof can easily be extended for $k > 2$ by induction, $k = 2$ being the base case.

Corollary 5.1.9. *For any nc-hypersequent P and any set $\mathcal{S} \cup \{S_0\}$ of sequents, we have $\mathcal{S} \vdash_{HLJ^{nc}+P} S_0$ iff $\mathcal{S} \vdash_{HLJ^{nc}+\{\Rightarrow P^I\}} S_0$ iff $\mathcal{S} \vdash_{LJ+\{\Rightarrow P^I\}} S_0$.*

In the following lemma, we show that the logical rule $(\rightarrow, r)$ is invertible when there is no right context G'. The invertibility of $(\rightarrow, r)$ holds only in this specific case because the right context is omitted in the premises of the rule (see Table 5.1).

Lemma 5.1.10. (Invertibility of $(\rightarrow, r)$) [6] *If $d \vdash_{HLJ^{nc}} G \mid\!\lceil\, \Gamma \Rightarrow A \rightarrow B$ then one can find $d_1 \vdash_{HLJ^{nc}} G \mid\!\lceil\, \Gamma, A \Rightarrow B$ such that $\rho(d_i) \leq \rho(d)$ and $|d_i| \leq |d|$ for $i = 1, 2$.*

Proof. We show the invertibility of $(\rightarrow, r)$ when there is no right context G' by induction on $|d|$. We consider the last inference R in d.

1. R is a logical rule.

   (a) $A \rightarrow B$ is the principal formula. Then we have:

   $$\dfrac{\vdots d' \atop G \mid\!\lceil\, \Gamma, A \Rightarrow B}{G \mid\!\lceil\, \Gamma \Rightarrow A \rightarrow B}$$

   The required derivation d_1 is d'.

   (b) $A \rightarrow B$ is not the principal formula. $A \rightarrow B$ is then propagated to one or two premises. Suppose that R is $(\vee, l)$:

   $$\dfrac{\vdots d' \atop G \mid\!\lceil\, \Gamma, X \Rightarrow A \rightarrow B \qquad \vdots d'' \atop G \mid\!\lceil\, \Gamma, Y \Rightarrow A \rightarrow B}{G \mid\!\lceil\, \Gamma, X \vee Y \Rightarrow A \rightarrow B}$$

   We apply the induction hypothesis to d' and d'' and get $d_1' \vdash_{HLJ^{nc}} G \mid\!\lceil\, \Gamma, X, A \Rightarrow B$ and $d_1'' \vdash_{HLJ^{nc}} G \mid\!\lceil\, \Gamma, Y, A \Rightarrow B$ with $|d_1'|, |d_1''| \leq |d|$ and $\rho(d_1'), \rho(d_1'') \leq \rho(d)$. The derivation d_1 can be retrieved by applying $(\vee, l)$ to d_1' and d_1''. The remaining cases are similar.

2. R is an internal or external structural rule.

   (a) $A \rightarrow B$ is neither weakened nor in the active component. The derivations can be retrieved by application of the induction hypothesis and subsequent application(s) of R.

   (b) R is (ec) and $A \rightarrow B$ is in the active component. Suppose that R is (ec) with $G = G_1 \mid\!\lceil\, G_2$ with $S = G_2 \mid\!\lceil\, \Gamma \Rightarrow A \rightarrow B$ being the contracted nc-hypersequent:

   $$\dfrac{\vdots d' \atop G \mid\!\lceil\, \Gamma \Rightarrow A \rightarrow B \mid\!\lceil\, G_2 \mid\!\lceil\, \Gamma \Rightarrow A \rightarrow B}{G \mid\!\lceil\, \Gamma \Rightarrow A \rightarrow B}$$

We apply the induction hypothesis to d' and get $d'_1 \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A \Rightarrow B$ or $d''_1 \vdash_{HLJ^{nc}} G \mid\lceil \Gamma \Rightarrow A \to B \mid\lceil G_2 \mid\lceil \Gamma, A \Rightarrow B$ with $|d'_1|, |d''_1| \leq |d|$ and $\rho(d'_1), \rho(d''_1) \leq \rho(d)$. The derivation d_1 is then d'_1 or can be retrieved from d''_1 by applying the induction hypothesis again to the second component.

3. R is (cut). Analogous to case 1.(b).

5.2 From the axioms (Bd_k) to nc-hypersequent rules

Here we describe how to adapt the systematic procedure in [15] to deal with the axioms (Bd_k). To obtain structural rules equivalent to the axioms (Bd_k), we suitably modify the transformation steps for hypersequent calculi described in [15] to deal with non-commutative hypersequents. In the first step of the algorithm, we transform an axiom into a non-commutative hypersequent structural rule, nc-hyperstructural rule for short. The second step consists of a completion procedure similar to the one provided in [15].

As described earlier, the axioms (Bd_k) have the form [9]:

(Bd_1) $A_1 \vee \neg A_1$
(Bd_{i+1}) $A_{i+1} \vee (A_{i+1} \to (Bd_i))$

By Lemma 5.1.8, we know that each axiom $(Bd_k), k \geq 2$ is equivalent to the non-commutative hypersequent:

$$\Rightarrow A_k \mid\lceil A_k \Rightarrow A_{k-1} \mid\lceil \cdots \mid\lceil \Rightarrow \neg A_1$$

Indeed, this is true by Proposition 5.1.7 which states that if some sequent is derivable in $\mathbf{HLJ}^{nc}$, the interpretation of this sequent is derivable in **LJ** and vice versa. After one application of $(\to, r)$ to the last component, this nc-hypersequent is in turn equivalent to the nc-hypersequent structural rule (with G being a possibly empty side nc-hypersequent) by the invertibility of $(\to, r)$ with no right context G' (see Lemma 5.1.10):

$$\frac{}{G \mid\lceil \Rightarrow A_k \mid\lceil A_k \Rightarrow A_{k-1} \mid\lceil \cdots \mid\lceil A_1 \Rightarrow}$$

Example 5.2.1. *The nc-hyperstructural rule* $(Bd_2 0)$ *for the axiom* $(Bd_2) = A_2 \vee (A_2 \to (A_1 \vee \neg A_1))$ *is:*

$$\frac{}{G \mid\lceil \Rightarrow A_2 \mid\lceil A_2 \Rightarrow A_1 \mid\lceil A_1 \Rightarrow}\ (Bd_2 0)$$

However, this nc-hyperstructural rule does not yet preserve cut admissibility when added to the calculus $\mathbf{HLJ}^{nc}$. To overcome this problem, we provide a completion procedure similar to the one in [15] to transform an nc-hyperstructural rule into an analytic rule. The transformation procedure consists of two steps, where the rule is first *restructured* and then *cut*.

(1) Restructuring

Given the nc-hypersequent structural rule derived from an axiom (Bd_k). We replace every component $(Y_1, \cdots, Y_n \Rightarrow X)$ in the conclusion with $(\Gamma_1, \cdots, \Gamma_n, \Sigma_X \Rightarrow \Pi_X)$ and add $n+1$ new premises of the form $(G \,||\!\lceil\, \Gamma_1 \Rightarrow Y_1), \cdots, (G \,||\!\lceil\, \Gamma_n \Rightarrow Y_n), (G \,||\!\lceil\, X, \Sigma_X \Rightarrow \Pi_X)$ with $\Gamma_1, \cdots, \Gamma_n, \Sigma_X, \Pi_X$ being fresh and mutually distinct metavariables. We replace every component $(Y_1, \cdots, Y_n \Rightarrow)$ with $(\Gamma_1, \cdots, \Gamma_n \Rightarrow)$ and add n new premises of the form $(G \,||\!\lceil\, \Gamma_1 \Rightarrow Y_1), \cdots, (G \,||\!\lceil\, \Gamma_n \Rightarrow Y_n)$. The resulting rules have the following properties [15]:

- linear conclusion: Each metavariable occurs at most once in the conclusion.
- separation: No metavariable occurring in the antecedent (succedent, respectively) of a component of the conclusion occurs in the succedent (antecedent, respectively) of a premise.
- coupling: The metavariables of each pair (Σ_X, Π_X), which is associated to the same occurrence of X, occur in the same premise.

Example 5.2.2. *Suppose we have the nc-hyperstructural rule for the axiom (Bd_2) from Example 5.2.1. By applying the restructuring step to all three components, we get the following nc-hyperstructural rule $(Bd_2 1)$:*

$$(1)\ \frac{G \,||\!\lceil\, A_2, \Sigma \Rightarrow \Pi}{G \,||\!\lceil\, \Sigma \Rightarrow \Pi \,||\!\lceil\, A_2 \Rightarrow A_1 \,||\!\lceil\, A_1 \Rightarrow}$$

$$(2)\ \frac{G \,||\!\lceil\, A_2, \Sigma \Rightarrow \Pi \qquad \begin{array}{c} G \,||\!\lceil\, \Gamma' \Rightarrow A_2 \\ G \,||\!\lceil\, A_1, \Sigma' \Rightarrow \Pi' \end{array}}{G \,||\!\lceil\, \Sigma \Rightarrow \Pi \,||\!\lceil\, \Gamma', \Sigma' \Rightarrow \Pi' \,||\!\lceil\, A_1 \Rightarrow}$$

$$(3)\ \frac{\begin{array}{c} G \,||\!\lceil\, \Gamma' \Rightarrow A_2 \\ G \,||\!\lceil\, A_2, \Sigma \Rightarrow \Pi \end{array} \qquad \begin{array}{c} G \,||\!\lceil\, \Gamma'' \Rightarrow A_1 \\ G \,||\!\lceil\, A_1, \Sigma' \Rightarrow \Pi' \end{array}}{G \,||\!\lceil\, \Sigma \Rightarrow \Pi \,||\!\lceil\, \Gamma', \Sigma' \Rightarrow \Pi' \,||\!\lceil\, \Gamma'' \Rightarrow}\ (Bd_2 1)$$

(2) Cutting

Given any nc-hypersequent structural rule derived from an axiom (Bd_k). We proceed with cutting the redundant metavariables, i.e., metavariables that occur in the premise but not in the conclusion [15].

Let A be such a redundant metavariable and $\mathcal{G}_1 = \{G \mid\lceil \Upsilon'_i \Rightarrow A, 1 \leq i \leq m\}$ be the subset of premises which have A in the succedent, $\mathcal{G}_2 = \{G \mid\lceil \Upsilon_j, A, \cdots, A \Rightarrow \Psi_j, 1 \leq j \leq n\}$ be the subset of premises which have at least one occurrence of A in the antecedent and A does not appear in Υ_j.

If $n = 0$, i.e., A only occurs in the succedent, we remove subset $\mathcal{G}_1$ from the premises. Analogously, if $m = 0$, i.e., A only occurs in the antecedent, we remove subset $\mathcal{G}_2$ from the premises. The resulting rule implies the original rule by instantiating A with $\bot$ (if $m = 0$) or $\top$ (if $n = 0$).

Otherwise, if $m > 0$ and $n > 0$, we create a new subset of premises $\mathcal{G}^{cut} = \{G \mid\lceil \Upsilon_j, \Upsilon'_{i_1}, \cdots, \Upsilon'_{i_s} \Rightarrow \Psi_j, 1 \leq j \leq n, 1 \leq i_1, \cdots, i_s \leq m\}$. Then we replace $\mathcal{G}_1 \cup \mathcal{G}_2$ with $\mathcal{G}^{cut}$.

We repeat this cutting step until we get an nc-hyperstructural rule without redundant metavariables.

Example 5.2.3. *We apply the cutting step to the nc-hyperstructural rule (*Bd_2*1) from Example 5.2.2 until we obtain the rule* (Bd_2)*. First, we apply the cutting procedure to* A_2*:*

$$\frac{\begin{array}{c} G \mid\lceil \Gamma'' \Rightarrow A_1 \\ G \mid\lceil \Gamma', \Sigma \Rightarrow \Pi \qquad G \mid\lceil A_1, \Sigma' \Rightarrow \Pi' \end{array}}{G \mid\lceil \Sigma \Rightarrow \Pi \mid\lceil \Gamma', \Sigma' \Rightarrow \Pi' \mid\lceil \Gamma'' \Rightarrow} \,(Bd_2 2)$$

Then we apply the cutting procedure to A_1*:*

$$\frac{G \mid\lceil \Gamma', \Sigma \Rightarrow \Pi \qquad G \mid\lceil \Gamma'', \Sigma' \Rightarrow \Pi'}{G \mid\lceil \Sigma \Rightarrow \Pi \mid\lceil \Gamma', \Sigma' \Rightarrow \Pi' \mid\lceil \Gamma'' \Rightarrow} \,(Bd_2)$$

Every nc-hyperstructural rule obtained by this procedure is called *completed*. Completed rules have the same properties as the rules after the restructuring step, except that the property of separation is strengthened to the "strong subformula property" [15]: Every metavariable that occurs in the antecedent (succedent, respectively) of a premise, also occurs in the antecedent (succedent, respectively) of the conclusion.

Lemma 5.2.4. *The rule* $\dfrac{G \mid\lceil \Phi_1 \mid\lceil G' \cdots G \mid\lceil \Phi_m \mid\lceil G'}{G \mid\lceil \Phi \mid\lceil A_1, \cdots, A_n \Rightarrow C \mid\lceil G'}\,(r_0)$ *is equivalent to the rules*

$$\dfrac{G \mid\lceil \vec{\Phi} \mid\lceil G' \qquad G \mid\lceil \Upsilon_1 \Rightarrow A_1 \mid\lceil G' \cdots G \mid\lceil \Upsilon_n \Rightarrow A_n \mid\lceil G'}{G \mid\lceil \Phi \mid\lceil \Upsilon_1, \cdots, \Upsilon_n \Rightarrow C \mid\lceil G'}\,(r_1)$$

and

$$\dfrac{G \mid\lceil \vec{\Phi} \mid\lceil G' \qquad G \mid\lceil C, \Upsilon \Rightarrow \Psi \mid\lceil G'}{G \mid\lceil \Phi \mid\lceil \psi_1, \cdots, \psi_n, \Upsilon \Rightarrow \Psi \mid\lceil G'}\,(r_2)$$

where $G \mid\lceil \vec{\Phi} \mid\lceil G' = (G \mid\lceil \Phi_1 \mid\lceil G', \cdots, G \mid\lceil \Phi_m \mid\lceil G')$ *with* $\Phi, \Phi_1, \ldots, \Phi_m$ *being nc-(meta)hypersequents consisting of metavariables.* Υ_i *is a fresh metavariable* Y_i *or* Γ_i *and* $\Upsilon \Rightarrow \Psi$ *is either* $\Rightarrow X$ *or* $\Sigma \Rightarrow \Pi$.

Proof.
$(r_0) \to (r_1)$: To show that (r_1) can be derived from (r_0), we use (cut):

$$\dfrac{\dfrac{G \mid\lceil \Upsilon_2 \Rightarrow A_2 \mid\lceil G' \qquad \dfrac{G \mid\lceil \Upsilon_1 \Rightarrow A_1 \mid\lceil G' \qquad \dfrac{G \mid\lceil \vec{\Phi} \mid\lceil G'}{G \mid\lceil \Phi \mid\lceil A_1, \cdots, A_n \Rightarrow C \mid\lceil G'}\,(r_0)}{G \mid\lceil \Phi \mid\lceil \Upsilon_1, A_2, \cdots, A_n \Rightarrow C \mid\lceil G'}\,(cut)}{G \mid\lceil \Phi \mid\lceil \Upsilon_1, \Upsilon_2, A_3 \cdots A_n \Rightarrow C \mid\lceil G'}\,(cut)}{G \mid\lceil \Phi \mid\lceil \Upsilon_1, \cdots, \Upsilon_n \Rightarrow C \mid\lceil G'}\,(cut)$$

$(r_1) \to (r_0)$: To show this direction, Υ_i has to be instantiated with A_i.
$(r_0) \to (r_2)$: To show that (r_2) can be derived from (r_0), we use (cut):

$$\dfrac{\dfrac{G \mid\lceil \vec{\Phi} \mid\lceil G'}{G \mid\lceil \Phi \mid\lceil A_1, \cdots, A_n \Rightarrow C \mid\lceil G'}\,(r_0) \qquad G \mid\lceil C, \Upsilon \Rightarrow \Psi \mid\lceil G'}{G \mid\lceil \Phi \mid\lceil A_1, \cdots, A_n, \Upsilon \Rightarrow \Psi \mid\lceil G'}\,(cut)$$

$(r_2) \to (r_0)$: To show this direction, $\Upsilon \Rightarrow \Psi$ has to be instantiated with $\Rightarrow C$.

Theorem 5.2.5. *Every nc-hypersequent structural rule can be transformed into a completed rule which is equivalent in* ***HLJ***^{nc}.

Proof. We have to show that every rule obtained by applications of the steps (1) "Restructuring" and (2) "Cutting" is equivalent to the initial nc-hypersequent structural rule of the axioms $(Bd_k), k \geq 2$.

- The equivalence of the initial nc-hypersequent structural rule and the nc-hypersequent structural rule after step (1) of the procedure follows by Lemma 5.2.4.
- We show that every rule r' obtained from r by replacing the premises $\mathcal{G}_1 \cup \mathcal{G}_2$ with $\mathcal{G}^{cut}$ in step (2) is equivalent to r as follows:
  $r \to r'$: To show that the premises of r' can be obtained from r, we use (cut) on the premises of r. E.g. suppose

$$\dfrac{G \,|\lceil\, \Gamma \Rightarrow A \qquad G \,|\lceil\, \Gamma' \Rightarrow A \qquad G \,|\lceil\, A, \Sigma \Rightarrow \Pi}{G \,|\lceil\, \Gamma, \Sigma \Rightarrow \Pi \,|\lceil\, \Gamma' \Rightarrow}\,(r)$$

$$\dfrac{G \,|\lceil\, \Gamma, \Sigma \Rightarrow \Pi \qquad G \,|\lceil\, \Gamma', \Sigma \Rightarrow \Pi}{G \,|\lceil\, \Gamma, \Sigma \Rightarrow \Pi \,|\lceil\, \Gamma' \Rightarrow}\,(r')$$

Then:

$$\dfrac{\dfrac{G \,|\lceil\, \Gamma \Rightarrow A \qquad G \,|\lceil\, A, \Sigma \Rightarrow \Pi}{G \,|\lceil\, \Gamma, \Sigma \Rightarrow \Pi} \qquad \dfrac{G \,|\lceil\, \Gamma' \Rightarrow A \qquad G \,|\lceil\, A, \Sigma \Rightarrow \Pi}{G \,|\lceil\, \Gamma', \Sigma \Rightarrow \Pi}}{G \,|\lceil\, \Gamma, \Sigma \Rightarrow \Pi \,|\lceil\, \Gamma' \Rightarrow}\,(r')$$

$r' \to r$: For this direction, we set $\bar{A} = \bigvee_{i=0}^{m} \Upsilon'_i$. Then we can prove the premises $G \,|\lceil\, \Upsilon'_i \Rightarrow \bar{A}$ from r. We can derive $G \,|\lceil\, \bar{A}, \Upsilon_j \Rightarrow \Psi_j$ by applying $(\vee, l)$ to the premises of r'. E.g. suppose the rules $(r), (r')$ as previously described. Then we have $\bar{A} = \Gamma \vee \Gamma'$ and:

$$\dfrac{G \,|\lceil\, \Gamma \Rightarrow \bar{A} \qquad G \,|\lceil\, \Gamma' \Rightarrow \bar{A} \qquad \dfrac{G \,|\lceil\, \Gamma, \Sigma \Rightarrow \Pi \qquad G \,|\lceil\, \Gamma', \Sigma \Rightarrow \Pi}{G \,|\lceil\, \bar{A}, \Sigma \Rightarrow \Pi}\,(\vee, l)}{G \,|\lceil\, \Gamma, \Sigma \Rightarrow \Pi \,|\lceil\, \Gamma' \Rightarrow}\,(r)$$

5.2.1 Completed rules resulting from the axioms (Bd_k)

We retrieve the following nc-hypersequent rules for the axioms (Bd_k) with the adapted transformation procedure:
For every axiom (Bd_k) of the form:

$$A_k \vee (A_k \to (A_{k-1} \vee (\cdots \vee (A_2 \to (A_1 \vee \neg A_1)))))$$

we get the following hyperstructural rule:

$$\dfrac{}{G \,|\lceil\, \Rightarrow A_k \,|\lceil\, A_k \Rightarrow A_{k-1} \,|\lceil\, \cdots \,|\lceil\, A_2 \Rightarrow A_1 \,|\lceil\, A_1 \Rightarrow}$$

After applying the completion procedure, we obtain the analytic rules:

$$\dfrac{\{G \,|\lceil\, \Gamma_i, \Sigma_{i+1} \Rightarrow \Pi_{i+1}\}_{1 \leq i \leq k}}{G \,|\lceil\, \Sigma_{k+1} \Rightarrow \Pi_{k+1} \,|\lceil\, \Gamma_k, \Sigma_k \Rightarrow \Pi_k \,|\lceil\, \cdots \,|\lceil\, \Gamma_2, \Sigma_2 \Rightarrow \Pi_2 \,|\lceil\, \Gamma_1 \Rightarrow}\,(Bd_k)$$

Example 5.2.6. *By applying to the axiom (Bd_2), i.e., $A_2 \vee (A_2 \to (A_1 \vee \neg A_1))$, the transformation procedure described in this chapter, we obtain the following analytic rule:*

$$\dfrac{G \,|\lceil\, \Gamma_2, \Sigma_3 \Rightarrow \Pi_3 \qquad G \,|\lceil\, \Gamma_1, \Sigma_2 \Rightarrow \Pi_2}{G \,|\lceil\, \Sigma_3 \Rightarrow \Pi_3 \,|\lceil\, \Gamma_2, \Sigma_2 \Rightarrow \Pi_2 \,|\lceil\, \Gamma_1 \Rightarrow}\,(Bd_2)$$

Consider the calculus $\mathbf{HLJ}^{nc}$ extended with the rule (Bd_2) from Example 5.2.6. We are then able to proof the axiom (Bd_2) as follows:

$$
\begin{array}{c}
A \Rightarrow A \qquad B \Rightarrow B \\
\hline
\Rightarrow A \mid\lceil A \Rightarrow B \mid\lceil B \Rightarrow \\
\hline
\Rightarrow A \mid\lceil A \Rightarrow B \mid\lceil A, B \Rightarrow \\
\hline
\Rightarrow A \mid\lceil A \Rightarrow B \mid\lceil A \Rightarrow \neg B \\
\hline\hline
\Rightarrow A \mid\lceil A \Rightarrow B \vee \neg B \mid\lceil A \Rightarrow B \vee \neg B \\
\hline
\Rightarrow A \mid\lceil A \Rightarrow B \vee \neg B \\
\hline
\Rightarrow A \mid\lceil \Rightarrow A \to (B \vee \neg B) \\
\hline\hline
\Rightarrow A \vee (A \to (B \vee \neg B)) \mid\lceil \Rightarrow A \vee (A \to (B \vee \neg B)) \\
\hline
\Rightarrow A \vee (A \to (B \vee \neg B))
\end{array}
\quad
\begin{array}{l}
(Bd_2) \\ (w, l) \\ (\to, r) \\ (\vee, r) \\ (ec) \\ (\to, r) \\ (\vee, r) \\ (ec)
\end{array}
$$

5.3 Proof of Cut-Elimination

Here we prove that the cut-elimination theorem holds for the non-commutative hypersequent calculus extended with completed rules which are obtained by the procedure described in the preceding section. One of the standard methods for cut-elimination was introduced by Gentzen in [22] and is shortly described in Chapter 2. However, we cannot use this method of cut-elimination for our calculus because of the asymmetry which arises by the logical rule $(\to, r)$ where the right context is omitted in the premises (see Example 5.3.1).

Example 5.3.1. *Consider the following instance of a cut:*

$$
\dfrac{\begin{array}{c}\vdots d_1 \\ G \mid\lceil \Gamma \Rightarrow X \mid\lceil G'\end{array} \qquad \dfrac{\begin{array}{c}\vdots d_2 \\ G \mid\lceil \Sigma, X, A \Rightarrow B\end{array}}{G \mid\lceil \Sigma, X \Rightarrow A \to B \mid\lceil G'}\,(\to, r)}{G \mid\lceil \Gamma, \Sigma \Rightarrow A \to B \mid\lceil G'}\,(cut)
$$

In this case, we cannot shift the cut rule over the premise d_2 due to the context G' in the premise $G \mid\lceil \Gamma \Rightarrow X \mid\lceil G'$.

Therefore, we use a method close to that introduced by Schütte-Tait [32, 34], in which the cuts are eliminated by shifting the cut formula upwards over only one premise. By using the invertibility of some of the logical rules, we replace the cuts with smaller ones in the exact same place.

Definition 5.3.2. *The* right *premise of the cut rule is the premise that contains the cut-formula in the succedent. Analogously, the* left *premise of the cut rule is the premise that contains the cut-formula in the antecedent.*

Example 5.3.3. *Consider the following cut:* $\dfrac{G \mid\lceil \Gamma \Rightarrow A \mid\lceil G' \qquad H \mid\lceil \Sigma, A \Rightarrow C \mid\lceil H'}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow C \mid\lceil H' \mid\lceil G'}\,(cut)$

Then $G \mid\lceil \Gamma \Rightarrow A \mid\lceil G'$ *is the right premise and* $H \mid\lceil \Sigma, A \Rightarrow C \mid\lceil H'$ *is the left premise of the cut.*

We generalize the Schütte-Tait-style cut-elimination in [6] for the non-commutative hypersequent calculus. Due to the fact that there are only two invertible rules, $(\wedge, l)$ and $(\vee, l)$, we have to be very careful when shifting the cut formula upwards. If the outermost connective of the cut formula is $\wedge$ or $\vee$, we proceed by shifting the cut formula over the right premise and replace the cuts with smaller ones by invertibility of the rules $(\wedge, l)$ and $(\vee, l)$. If the cut formula has $\rightarrow$ as the outermost connective, we have to first shift the cut formula over the right premise and then shift it upwards over the left premise. Note that in this case the right context of the right premise has already been deleted due to an application of $\rightarrow$ and the subsequent shift over the left premise may therefore be carried out.
To be able to trace the cut formula in the derivation, we introduce the notion of a decorated formula:

Definition 5.3.4. [6] *Let* $d \vdash_{HLJ^{nc}} H$ *and* A *be a formula in* H *that is not the cut formula of any cut in* d*. The* decoration *of* A *in* d *is inductively defined as follows: we denote by* A^* *any marked occurrence of* A*. Given a hypersequent H' in* d *with some (not necessarily all) marked A's. Let* R *be the rule introducing H'. We divide some cases according to* R*.*

1. R *is a logical rule:*

1.1 A *is principal in* R*, e.g.,*

$$\frac{G \mid\lceil \Gamma' \Rightarrow \Pi' \mid\lceil G'}{G \mid\lceil \Gamma \Rightarrow \Pi \mid\lceil G'}\,(R)$$

(a) Suppose that $A^* \in \Gamma$*.* $A^* \in \Gamma'$ *if and only if* A^* *is an occurrence of a formula in* Γ *which is not the principal formula. Moreover, the marked formulas of* G, G' *in the premise of* R *are as in the conclusion. That is, for each* $\{\Sigma \Rightarrow B\} \in \{G, G'\}$*,* $A^* \in \Sigma$ *if and only if* $A^* \in \Sigma$ *of the corresponding component belonging to the conclusion of* R*.*

(b) Suppose that Π *is* A^**. The marked formulas of* G, G' *in the premise of* R *are as in the conclusion.*

1.2 A *is not principal in* R*. The marked formulas of the premise of* R *are as in the conclusion.*

If R *is a rule with two premises, the definition is analogous.*

2. R is (ew)*,* (e,l) *or* (cut)*. The marked formulas of the premise(s) of R are as in the conclusion.*

3. R is (c,l)*.*

3.1 A^* *is the contracted formula, then* A^**,* A^* *belongs to the premise of R. The remaining formulas in the premise of R are marked as in the conclusion.*

3.2 A^* *is not the contracted formula. Analogous to case 2.*

4. R is (w,l) *or* (w,r)*. Analogous to case 1.*

5. R is (ec)*. Similar to case 3.*

Definition 5.3.5. [6] *The* complexity $|A|$ *of a formula A is inductively defined as:*
– $|A| = 0$ *if A is atomic*
– $|A \wedge B| = |A \vee B| = |A \rightarrow B| = max(|A|, |B|) + 1$
The length $|d|$ *of a derivation d is the maximal number of inference rules + 1 occurring on any branch of d. The* cut-rank $\rho(d)$ *of d is the maximal complexity of cut formulas in d + 1. (*$\rho(d) = 0$ *if d is cut-free).*

Definition 5.3.6. [10] *Any* $\mathbf{HLJ}^{nc}$*-rule R,* $n \geq 1$*:*

$$\frac{G \mid\lceil \Gamma_1 \Rightarrow \Pi_1 \mid\lceil G' \cdots G \mid\lceil \Gamma_n \Rightarrow \Pi_n \mid\lceil G'}{G \mid\lceil \Gamma \Rightarrow \Pi \mid\lceil G'} (R)$$

is said to be substitutive *whenever the following conditions hold:*
1. Let X be any formula occurring in Γ *that is not principal in R. Let P be the nc-hypersequent resulting by replacing some occurrences of X in* Γ *with any multiset of formulas* Δ*. P can then be derived from the premises of R, where every occurrence of X in each* $\Gamma_i, i = 1, \ldots, n$ *is substituted with* Δ*, by an application of R and, if needed, the structural rules of* $\mathbf{HLJ}^{nc}$*.*
2. If Π *is neither empty nor principal in R, the nc-hypersequent* $G \mid\lceil \Gamma, \Sigma \Rightarrow \Theta \mid\lceil G'$ *for any* Σ *and* Θ *is derivable only using R and the structural rules of* $\mathbf{HLJ}^{nc}$ *from the premises of R with* $\Gamma_i, \Sigma \Rightarrow \Theta$ *uniformly substituted for each* $\Gamma_i \Rightarrow \Pi_i$ *in which* $\Pi_i = \Pi$*.*

Lemma 5.3.7. *The calculus rules of* $\mathbf{HLJ}^{nc}$ *and the completed rules derived by the systematic procedure described in Section 5.2 are substitutive in the sense of Definition 5.3.6.*

Proof.

(1) Suppose that X is a formula in Γ that is not principal in R.

- R is a logical rule. The claim holds for all logical rules because the side formulas from the antecedent are propagated to the premise(s). Suppose we have $(\to, l)$ with $G \mid\lceil \Gamma, A \to B \Rightarrow \Pi \mid\lceil G'$ as the conclusion and $\Gamma = X, \Gamma'$. We are then able to derive P as follows:

$$\frac{G \mid\lceil \Gamma', \Delta \Rightarrow A \mid\lceil G' \qquad G \mid\lceil \Gamma', \Delta, B \Rightarrow \Pi \mid\lceil G'}{G \mid\lceil \Gamma', \Delta, A \to B \Rightarrow \Pi \mid\lceil G'}\ (\to, l)$$

Analogously for the other logical rules.

– R is an internal or external structural rule. The claim holds for all structural rules because the side formulas are propagated to the premise. Suppose we have (w, l) with $G \mid\lceil \Gamma, A \Rightarrow \Pi \mid\lceil G'$ as the conclusion and $\Gamma = X, \Gamma'$. Then we are able to derive P as follows:

$$\frac{G \mid\lceil \Gamma', \Delta \Rightarrow \Pi \mid\lceil G'}{G \mid\lceil \Gamma', \Delta, A \Rightarrow \Pi \mid\lceil G'}\ (w, l)$$

Analogously for the other structural rules.

– R is the (cut)-rule. The claim holds because the side formulas in the antecedent are either propagated to the left or to the right premise. Suppose we have $G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ with $\Sigma = X, \Sigma'$:

$$\frac{G \mid\lceil \Gamma \Rightarrow A \mid\lceil G' \qquad H \mid\lceil \Sigma', \Delta, A \Rightarrow \Pi \mid\lceil H'}{G \mid\lceil H \mid\lceil \Gamma, \Sigma', \Delta \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}\ (cut)$$

– R is a completed rule. Due to the "strong subformula property", every metavariable that occurs in the antecedent of a premise occurs also in the antecedent of the conclusion. Thus, the claim holds.

(2) Suppose that Π is neither empty nor principal in R.

– R is a logical left-rule. The claim holds for $(\wedge_i, l)_{i=1,2}$ and $(\vee, l)$ because the side formulas $\Gamma \Rightarrow \Pi$ (and, thus, $\Gamma, \Sigma \Rightarrow \Theta$) are propagated to the premises. Suppose we have $(\to, l)$ with $G \mid\lceil \Gamma, A \to B \Rightarrow \Pi \mid\lceil G'$. We are then able to derive P as follows:

$$\frac{\dfrac{G \mid\lceil \Gamma \Rightarrow A \mid\lceil G'}{G \mid\lceil \Gamma, \Sigma \Rightarrow A \mid\lceil G'}\ (w, l) \qquad G \mid\lceil \Gamma, \Sigma, B \Rightarrow \Theta \mid\lceil G'}{G \mid\lceil \Gamma, \Sigma, A \to B \Rightarrow \Theta \mid\lceil G'}\ (\to, l)$$

– R is an internal or external structural rule. The claim holds for all structural rules because the side formulas are propagated to the premises. Suppose we have (c, l) with $G \mid\lceil \Gamma, A \Rightarrow \Pi \mid\lceil G'$ as the conclusion. We derive P as follows:

$$\frac{G \mid\lceil \Gamma, A, A, \Sigma \Rightarrow D \mid\lceil G'}{G \mid\lceil \Gamma, A, \Sigma \Rightarrow D \mid\lceil G'}\ (c, l)$$

Analogously for the other structural rules.

– R is the (cut)-rule. The claim holds because the substituted metavariables are propagated to the premise with the cut formula in the antecedent. Suppose we have $G \mid\lceil H \mid\lceil \Gamma, \Gamma' \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$:

$$\frac{G \mid\lceil \Gamma \Rightarrow A \mid\lceil G' \qquad H \mid\lceil \Gamma', \Sigma, A \Rightarrow \Theta \mid\lceil H'}{G \mid\lceil H \mid\lceil \Gamma, \Gamma', \Sigma \Rightarrow \Theta \mid\lceil H' \mid\lceil G'}\ (cut)$$

– R is a completed rule. Due to the "strong subformula property", every metavariable that occurs in the succedent of a premise occurs also in the succedent of the conclusion. Additionally, the "coupling" property ensures that metavariables of a pair (Σ_X, Π_X) associated to the formula X, occur in the same premise. Thus, the claim also holds for completed rules.

Substitutivity ensures that cuts over side formulas can be shifted upwards over the premises.
In the following lemma, we show that the invertibility of the rules $(\wedge, l)$ and $(\vee, l)$ holds in general. This lemma differs considerably from the Lemma 5.1.10, which is restricted in the sense that $(\rightarrow, r)$ is only invertible when there is no right context G'.

Lemma 5.3.8. (Inversion) [6]

(i) If $d \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A \vee B \Rightarrow \Pi \mid\lceil G'$ *then one can find* $d_1 \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A \Rightarrow \Pi \mid\lceil G'$ *and* $d_2 \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, B \Rightarrow \Pi \mid\lceil G'$

(ii) If $d \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A \wedge B \Rightarrow \Pi \mid\lceil G'$ *then one can find* $d_1 \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A, B \Rightarrow \Pi \mid\lceil G'$.

such that $\rho(d_i) \leq \rho(d)$ *and* $|d_i| \leq |d|$ *for* $i = 1, 2$.

Proof. We show the invertibility of (i) and (ii) by induction on $|d|$. We consider the last inference R in d.
(i) invertibility of $(\vee, l)$:

1. R is a logical rule.

   (a) $A \vee B$ is the principal formula. Then we have:

$$\frac{\begin{array}{c}\vdots d' \\ G \mid\lceil \Gamma, A \Rightarrow \Pi \mid\lceil G'\end{array} \qquad \begin{array}{c}\vdots d'' \\ G \mid\lceil \Gamma, B \Rightarrow \Pi \mid\lceil G'\end{array}}{G \mid\lceil \Gamma, A \vee B \Rightarrow \Pi \mid\lceil G'}$$

   The required derivations d_1 and d_2 are then d' and d''.

(b) $A \vee B$ is not the principal formula. $A \vee B$ is then propagated to one or two premises. Suppose that R is $(\rightarrow, l)$:

$$\dfrac{\begin{array}{c}\vdots d' \\ G \mid\lceil \Gamma, A \vee B \Rightarrow X \mid\lceil G'\end{array} \qquad \begin{array}{c}\vdots d'' \\ G \mid\lceil \Gamma, A \vee B, Y \Rightarrow \Pi \mid\lceil G'\end{array}}{G \mid\lceil \Gamma, A \vee B, X \rightarrow Y \Rightarrow \Pi \mid\lceil G'}$$

We apply the induction hypothesis to d' and d'' and get $d_1' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A \Rightarrow X \mid\lceil G'$ and $d_2' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, B \Rightarrow X \mid\lceil G'$ and also $d_1'' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A, Y \Rightarrow \Pi \mid\lceil G'$ and $d_2'' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, B, Y \Rightarrow \Pi \mid\lceil G'$ with $|d_i'|, |d_i''| \leq |d|$ and $\rho(d_i'), \rho(d_i'') \leq \rho(d)$ for $i = 1, 2$. The derivations d_1 and d_2 can be retrieved by applying $(\rightarrow, l)$ to d_1' and d_1'' (d_2' and d_2'', respectively). The remaining cases are similar.

2. R is an internal or external structural rule.

   (a) $A \vee B$ is not in the principal component or the principal formula. The derivations can be retrieved by application of the induction hypothesis and subsequent application(s) of R.

   (b) R is (ec) or (c, l) and $A \vee B$ is in the principal component or the principal formula. Suppose that R is (ec) with $G = G_1 \mid\lceil G_2$ with $S = G_2 \mid\lceil \Gamma, A \vee B \Rightarrow \Pi$ being the contracted nc-hypersequent:

$$\dfrac{\begin{array}{c}\vdots d' \\ G \mid\lceil \Gamma, A \vee B \Rightarrow \Pi \mid\lceil G_2 \mid\lceil \Gamma, A \vee B \Rightarrow \Pi \mid\lceil G'\end{array}}{G \mid\lceil \Gamma, A \vee B \Rightarrow \Pi \mid\lceil G'}$$

   We apply the induction hypothesis to d' and get $d_1' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A \Rightarrow \Pi \mid\lceil G_2 \mid\lceil \Gamma, A \vee B \Rightarrow \Pi \mid\lceil G'$ and $d_2' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, B \Rightarrow \Pi \mid\lceil G_2 \mid\lceil \Gamma, A \vee B \Rightarrow \Pi \mid\lceil G'$ with $|d_i'| \leq |d|$ and $\rho(d_i') \leq \rho(d)$ for $i = 1, 2$. The derivations d_1 and d_2 can be retrieved by applying the induction hypothesis again to the second component and a subsequent application of (ec). Analogous for (c, l).

3. R is (cut). Analogous to case 1.(b).

4. R is a completed rule obtained by the algorithm described in Section 5.2. Analogous to case 1.(b).

(ii) invertibility of $(\wedge, l)$:

1. R is a logical rule.

(a) $A \wedge B$ is the principal formula. Then we have:

$$\dfrac{\vdots d'\\ G \mid\lceil \Gamma, A \Rightarrow \Pi \mid\lceil G'}{G \mid\lceil \Gamma, A \wedge B \Rightarrow \Pi \mid\lceil G'}$$

or

$$\dfrac{\vdots d''\\ G \mid\lceil \Gamma, B \Rightarrow \Pi \mid\lceil G'}{G \mid\lceil \Gamma, A \wedge B \Rightarrow \Pi \mid\lceil G'}$$

The required derivation d_1 contains both premises, d' and d''.

(b) $A \wedge B$ is not the principal formula. $A \wedge B$ is then propagated to one or two premises. Suppose that R is $(\rightarrow, l)$:

$$\dfrac{\vdots d' \quad\quad \vdots d''\\ G \mid\lceil \Gamma, A \wedge B \Rightarrow X \mid\lceil G' \qquad G \mid\lceil \Gamma, A \wedge B, Y \Rightarrow \Pi \mid\lceil G'}{G \mid\lceil \Gamma, A \wedge B, X \rightarrow Y \Rightarrow \Pi \mid\lceil G'}$$

We apply the induction hypothesis to d' and d'' and get $d_1' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A, B \Rightarrow X \mid\lceil G'$ and $d_1'' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A, B, Y \Rightarrow \Pi \mid\lceil G'$ with $|d_1'|, |d_1''| \leq |d|$ and $\rho(d_1'), \rho(d_1'') \leq \rho(d)$. The derivation d_1 can be retrieved by applying $(\rightarrow, l)$ to d_1' and d_1''. The remaining cases are similar.

2. R is an internal or external structural rule.

(a) $A \wedge B$ is not in the principal component or the principal formula. The derivations can be retrieved by application of the induction hypothesis and subsequent application(s) of R.

(b) R is (ec) or (c, l) and $A \wedge B$ is in the principal component or the principal formula. Suppose that R is (ec) with $G = G_1 \mid\lceil G_2$ with $S = G_2 \mid\lceil \Gamma, A \vee B \Rightarrow \Pi$ being the contracted nc-hypersequent:

$$\dfrac{\vdots d'\\ G \mid\lceil \Gamma, A \wedge B \Rightarrow \Pi \mid\lceil G_2 \mid\lceil \Gamma, A \wedge B \Rightarrow \Pi \mid\lceil G'}{G \mid\lceil \Gamma, A \wedge B \Rightarrow \Pi \mid\lceil G'}$$

We apply the induction hypothesis to d' and get $d_1' \vdash_{HLJ^{nc}} G \mid\lceil \Gamma, A, B \Rightarrow \Pi \mid\lceil G_2 \mid\lceil \Gamma, A \wedge B \Rightarrow \Pi \mid\lceil G'$ with $|d_1'| \leq |d|$ and $\rho(d_1') \leq \rho(d)$. The derivation d_1 can be retrieved by applying the induction hypothesis again to the second component and a subsequent application of (ec). Analogous for (c, l).

3. R is (cut). Analogous to case 1.(b)

4. R is a completed rule obtained by the algorithm described in Section 5.2. Analogous to case 1.(b).

In the following, $P[^B/_A]$ indicates the nc-hypersequent P in which we uniformly replace each occurrence of A by B.

Lemma 5.3.9. *Let $d \vdash_{HLJ^{nc}} G \mid\lceil \Gamma \Rightarrow A \mid\lceil G'$, where A is an atomic formula that is not the cut formula of any cut in d. One can find a derivation $d', H \mid\lceil A, \Sigma \Rightarrow \Pi \mid\lceil H' \vdash_{HLJ^{nc}} G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ with $\rho(d') = \rho(d)$.*

Notation. We write $\bar{\Gamma}$ when we refer to Γ which might have been changed by some rule applications during derivation. Analogously, we write $\bar{G}, \bar{G}'$ when we refer to the contexts G, G' after some rule applications to their components.

Proof. Let A be a decorated formula in d starting from $\Gamma \Rightarrow A^*$. We start with a stepwise derivation of $G \mid\lceil \Gamma \Rightarrow A^* \mid\lceil G'$. In every derivation step of the derivation tree d, we apply the following substitutions and insertions:

(a) Substitution of the decorated formula: We replace the component $\Psi \Rightarrow A^*$ with $\Psi, \Sigma \Rightarrow \Pi$. This may be done due to Lemma 5.3.7, part (2).

(b) Insertion of new contexts:

(b.1) We insert the component H between the context $\bar{G}$ and the component $\Psi, \Sigma \Rightarrow \Pi$ at the beginning of every nc-hypersequent. If the right context $\bar{G}'$ has not been completely omitted due to an application of $(\rightarrow, r)$, we insert H' between the component $\Psi, \Sigma \Rightarrow \Pi$ and $\bar{G}'$. If the component is contracted due to an application of (ec), the contexts have to be contracted as well, e.g. let $S = S_1 \mid\lceil \Psi, \Sigma \Rightarrow \Pi$ be the contracted component:

$$\dfrac{\vdots \atop \bar{G} \mid\lceil S_1 \mid\lceil H \mid\lceil \Psi, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil S_1 \mid\lceil H \mid\lceil \Psi, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}{\bar{G} \mid\lceil S_1 \mid\lceil H \mid\lceil \Psi, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\ (ec)$$

If the component is propagated to one or two premises by any other structural, logical or the cut-rule, the contexts have to be propagated accordingly, e.g. suppose an application of $(\rightarrow, l)$:

$$\dfrac{\vdots \atop \bar{G} \mid\lceil H \mid\lceil \Psi, \Sigma \Rightarrow X \mid\lceil H' \mid\lceil \bar{G}' \qquad \vdots \atop \bar{G} \mid\lceil H \mid\lceil \Psi, \Sigma, Y \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}{\bar{G} \mid\lceil H \mid\lceil \Psi, X \rightarrow Y, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\ (\rightarrow, l)$$

The contexts H, H' are also inserted when the component is weakened due to an application of (ew).

(b.2) For resulting nc-hypersequents of the form $H \mid\lceil B \Rightarrow B \mid\lceil H'$, we recover the original axioms $B \Rightarrow B$ of d by subsequent applications of (ew).

The insertion of the contexts does not harm the proof because the contexts are propagated in every derivation step of d. Step (b.2) ensures that the axioms of the original derivation d are also axioms of the new derivation tree. Nevertheless, depending on the origin of the decorated formula, we have to apply some "correction steps" which are described below.

To ensure a valid derivation tree, we have to apply the following "correction steps" according to the cases in which the decorated formula A^* originates:

(i) in an axiom: We get an nc-hypersequent of the form $\bar{G} \mid\lceil H \mid\lceil A, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'$. The axiom is then transformed into $H \mid\lceil A, \Sigma \Rightarrow \Pi \mid\lceil H'$ using several applications of (ew).

(ii) in an internal weakening after an application of (w, r): Then, the weakening of A* is replaced by several weakenings of the formulas $B \in \Sigma$ and Π. We add a subsequent application of (ew) to omit the contexts H, H' as they do not necessarily end in axioms. E.g.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \bar{G} \mid\lceil \bar{\Gamma} \Rightarrow\ \mid\lceil \bar{G}'\end{array}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow\ \mid\lceil H' \mid\lceil \bar{G}'}\text{(ew)}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow\ \mid\lceil H' \mid\lceil \bar{G}'}\text{(w,l)}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\text{(w,r)}$$

(iii) in an external weakening: Then, the weakening of the component $P = \bar{\Gamma} \Rightarrow A^*$ is replaced by a weakening of $P[^{\Sigma \Rightarrow \Pi}/_{A*}]$. We add a subsequent application of (ew) to omit the contexts H, H' as they do not necessarily end in axioms.

(iv) in the deleted context due to an application of $(\rightarrow, r)$ to a component in $\bar{G}$. Then the component $P[^{\Sigma \Rightarrow \Pi}/_{A*}], H$ and H' are also omitted when the rule $(\rightarrow, r)$ is applied to the same component in $\bar{G}$. E.g. suppose $\bar{G} = G_1 \mid\lceil \Delta \Rightarrow X \rightarrow Y$:

$$\dfrac{\begin{array}{c}\vdots\\ G_1 \mid\lceil \Delta, X \Rightarrow Y\end{array}}{G_1 \mid\lceil \Delta \Rightarrow X \rightarrow Y \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}$$

This procedure results in a derivation $d', H \mid\lceil A, \Sigma \Rightarrow \Pi \mid\lceil H' \vdash_{HLJ^{nc}} G \mid\lceil H \mid\lceil \Sigma, \Gamma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ with $\rho(d') = \rho(d)$.

Lemma 5.3.10. *In* $\mathbf{HLJ}^{nc}$ *non-atomic axioms can be derived from atomic axioms.*

Lemma 5.3.11. (Reduction) *Let* $d_r \vdash_{HLJ^{nc}} G \mid\lceil \Gamma \Rightarrow A \mid\lceil G'$ *and* $d_l \vdash_{HLJ^{nc}} H \mid\lceil A, \Sigma \Rightarrow \Pi \mid\lceil H'$ *both with cut-rank* $\rho(d_i) \leq |A|$*. Then we can find a derivation* $d \vdash_{HLJ^{nc}} G \mid\lceil H \mid\lceil \Sigma, \Gamma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ *with* $\rho(d) \leq |A|$*.*

Note: We could also derive $G \mid\lceil H \mid\lceil \Sigma, \Gamma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ by an application of (cut), but the resulting derivation would then have $\rho(d) = |A| + 1$ due to the definition of the cut-rank.

Notation. We write $\bar{\Gamma}, \bar{\Sigma}$ when we refer to Γ, Σ which might have been changed by some rule applications during derivation. Analogously, we write $\bar{G}, \bar{G}', \bar{H}, \bar{H}'$ when we refer to the contexts G, G', H, H' after some rule applications to their components.

Proof. We call d_r (d_l) the derivation of the *right* (*left*) premise of the cut formula. By Lemma 5.3.10, we assume that d_r and d_l have atomic axioms.
(1) A is atomic.
By Lemma 5.3.9 we can find a derivation $d', H \mid\lceil A, \Sigma \Rightarrow \Pi \mid\lceil H' \vdash_{HLJ^{nc}} G \mid\lceil H \mid\lceil \Sigma, \Gamma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ such that $\rho(d') = \rho(d_l)$. We get the derivation d by concatenation of the derivations d_r and d_l.
(2) A is not atomic.

- $A = B \wedge D$. We try to shift the cut over the right premise. Let us consider the decoration of A in d_r starting from $G \mid\lceil \Gamma \Rightarrow (B \wedge D)^* \mid\lceil G'$. We proceed with a stepwise derivation where we apply the substitutions and insertions described in steps (a) and (b) in Lemma 5.3.9. Although these replacements and insertions do not harm the proof, we might not have a valid derivation tree anymore. Therefore we have to apply some "correction steps" according to the cases in which the marked occurrence of $B \wedge D$ originates:

  (i) as principal formula of a logical inference. We get the following derivation tree ending in an application of the $(\wedge, r)$-rule:

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots \\ \bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow B \mid\lceil H' \mid\lceil \bar{G}' & & \bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow D \mid\lceil H' \mid\lceil \bar{G}'\end{matrix}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\,(\wedge, r)(*)}{\dfrac{\cdots}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}}$$

We proceed by replacing the premises of $(*)$ with the following, smaller cuts:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow B \mid\lceil H' \mid\lceil \bar{G}'\end{array}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow B \mid\lceil H' \mid\lceil \bar{G}'} \quad \dfrac{\dfrac{\begin{array}{c}\vdots \\ \bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow D \mid\lceil H' \mid\lceil \bar{G}'\end{array}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma, B \Rightarrow D \mid\lceil H' \mid\lceil \bar{G}'} \quad \dfrac{\begin{array}{c}\vdots d_l' \\ H \mid\lceil \Sigma, B, D \Rightarrow \Pi \mid\lceil H'\end{array}}{\bar{G} \mid\lceil H \mid\lceil D, \bar{\Gamma}, \Sigma, B \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}}{\bar{G} \mid\lceil H \mid\lceil B, \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\,(cut)}{\dfrac{\cdots}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}}
$$

The missing premise is obtained from d_l by the Inversion Lemma 5.3.8 for the rule $(\wedge, l)$.

(ii) in an internal weakening. The weakening of $(B \wedge D)^*$ is replaced with several weakenings of the formulas $X \in \Sigma$ and Π. We add a subsequent application of (ew) to omit the contexts H, H' as they do not necessarily end in axioms. E.g.

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \bar{G} \mid\lceil \bar{\Gamma} \Rightarrow \ \mid\lceil \bar{G}'\end{array}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow \ \mid\lceil H' \mid\lceil \bar{G}'}\,(\text{ew})}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \ \mid\lceil H' \mid\lceil \bar{G}'}\,(\text{w,l})}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\,(\text{w,r})
$$

(iii) in an external weakening. The weakening of the component $P = \bar{\Gamma} \Rightarrow (B \wedge D)^*$ is replaced with the external weakening of the component $P[^{\Sigma \Rightarrow \Pi}/_{(B \wedge D)^*}]$. We add a subsequent application of (ew) to omit the contexts H, H' as they do not necessarily end in axioms.

(iv) in the deleted context due to an application of $(\rightarrow, r)$ to a component in $\bar{G}$. Then the component $P[^{\Sigma \Rightarrow \Pi}/_{(B \wedge D)^*}]$, H and H' are also omitted when the rule $(\rightarrow, r)$ is applied to the same component in $\bar{G}$. E.g. suppose $\bar{G} = G_1 \mid\lceil \Delta \Rightarrow X \rightarrow Y$:

$$
\dfrac{\begin{array}{c}\vdots \\ G_1 \mid\lceil \Delta, X \Rightarrow Y\end{array}}{G_1 \mid\lceil \Delta \Rightarrow X \rightarrow Y \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}
$$

Then, we are able to check that this procedure results in a derivation $d' \vdash_{HLJ^{nc}} G \mid\lceil H \mid\lceil \Sigma, \Gamma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ with $\rho(d') \leq |A|$.

– $A = B \vee D$. We try to shift the cut over the right premise like in the case $A = B \wedge D$. Let us consider the decoration of A in d_r starting from $G \mid\lceil \Gamma \Rightarrow (B \vee D)^* \mid\lceil G'$. We proceed with a stepwise derivation where we apply the substitutions and insertions described in steps (a) and (b) in Lemma 5.3.9. Then we might not have a valid derivation tree anymore and have to apply some "correction steps" according to the cases in which the marked occurrence of $B \vee D$ originates. We only explain case (i) because the other cases are the same as in $A = B \wedge D$.

(i) as principal formula of a logical inference. We get the following derivation tree ending in an application of the $(\vee, r)$-rule:

$$\dfrac{\dfrac{\dfrac{\vdots}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow B \mid\lceil H' \mid\lceil \bar{G}'}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\,(\vee, r)(*)}{\dfrac{\cdots}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}}$$

We proceed by replacing the premise of $(*)$ with the following, smaller cut:

$$\dfrac{\dfrac{\dfrac{\vdots}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow B \mid\lceil H' \mid\lceil \bar{G}'}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow B \mid\lceil H' \mid\lceil \bar{G}'} \quad \dfrac{\dfrac{\vdots d_l'}{H \mid\lceil \Sigma, B \Rightarrow \Pi \mid\lceil H'}}{\bar{G} \mid\lceil H \mid\lceil B, \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}}{\dfrac{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}{\dfrac{\cdots}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}}}\,(cut)$$

The missing premise is obtained from d_l by the Inversion Lemma 5.3.8 for the rule $(\vee, l)$. This case works analogously when $\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma} \Rightarrow D \mid\lceil H' \mid\lceil \bar{G}'$ is derived by $(\vee, r)$ at $(*)$.

Then, we are able to check that this procedure results in a derivation $d' \vdash_{HLJ^{nc}} G \mid\lceil H \mid\lceil \Sigma, \Gamma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ with $\rho(d') \leq |A|$.

– $A = B \rightarrow D$. In this case, we cannot simply shift the cut over the right premise and use the Inversion Lemma because the rule $(\rightarrow, l)$ is not invertible. Instead, we first shift the cut over the right premise and, when A is introduced by an application of $(\rightarrow, r)$ as principal formula, we shift the cut over the left premise. We proceed as follows:

Let us consider the decoration of A in d_r starting from $G \mid\lceil \Gamma \Rightarrow (B \rightarrow D)^* \mid\lceil G'$. We proceed with a stepwise derivation where we apply the substitutions and insertions described in steps (a) and (b) in Lemma 5.3.9 until $(B \rightarrow D)^*$ is introduced. Then we consider the cases in which the marked occurrence of $(B \rightarrow D)^*$ is introduced – again, we only explain case (i) because the cases (ii), (iii) and (iv) are the same as in the previous cases:

(i) as principal formula of a logical inference. We get the following derivation tree where $(B \to D)^*$ is introduced by an application of $(\to, r)$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, B \Rightarrow D}}{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}\,(\to, r)(*)}{\dfrac{\cdots}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}}$$

We stop at $(*)$, add an application of (ew) to omit the right context $\bar{G}'$ and proceed by considering the decoration of A in d_l starting from $H \mid\lceil (B \to D)^*, \Sigma \Rightarrow \Pi \mid\lceil H'$. We proceed by a stepwise derivation until $(B \to D)^*$ is introduced, similar to steps (a) and (b) in Lemma 5.3.9:

(a') Substitution of the decorated formula: We replace the component $(B \to D)^*$ with $\bar{\Gamma}$. This may be done due to Lemma 5.3.7, part (1).

(b') Insertion of new contexts:

(b'.1) We add the context $\bar{G}$ at the beginning of the nc-hypersequent. If the component $\bar{\Gamma}, \Sigma \Rightarrow \Pi$ is contracted due to an application of (ec), we contract the context as well. Similarly, if the component is propagated to one or two premises by any other structural, logical or the cut-rule, the context has to be propagated accordingly at the beginning of the nc-hypersequent.

(b'.2) For resulting nc-hypersequents of the form $\bar{G} \mid\lceil B \Rightarrow B$, we recover the original axioms $B \Rightarrow B$ of d_l by a subsequent application of (ew).

The insertion of the context does not harm the proof as it is propagated to all components that are associated with the original component $\bar{\Gamma}, \Sigma \Rightarrow \Pi$ in the derivation tree d_l. Step (b'.2) ensures that the axioms of the original derivation are also axioms of the new derivation tree. Still, we have to apply some "correction steps".

We have to consider the cases in which the marked occurrence of $(B \to D)^*$ originates:

(i.i) as principal formula of a logical inference. We get the following derivation tree where $(B \to D)^*$ is introduced by an application of $(\to, l)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots d_l' \\ \bar{G} \mid\lceil \bar{H} \mid\lceil \bar{\Sigma} \Rightarrow B \mid\lceil \bar{H}'\end{array} \qquad \begin{array}{c}\vdots d_l'' \\ \bar{G} \mid\lceil \bar{H} \mid\lceil D, \bar{\Sigma} \Rightarrow \bar{\Pi} \mid\lceil \bar{H}'\end{array}}{\bar{G} \mid\lceil \bar{H} \mid\lceil \bar{\Gamma}, \bar{\Sigma} \Rightarrow \bar{\Pi} \mid\lceil \bar{H}'}\,(\rightarrow, l)(**)}{\cdots}}{\dfrac{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}{\cdots}}}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}
$$

We proceed by replacing the premises of $(**)$ with smaller cuts and get the following derivation tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots d_l' \\ \bar{G} \mid\lceil \bar{H} \mid\lceil \bar{\Sigma} \Rightarrow B \mid\lceil \bar{H}'\end{array}}{\bar{G} \mid\lceil \bar{H} \mid\lceil \bar{\Gamma}, \bar{\Sigma} \Rightarrow B \mid\lceil \bar{H}'} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots d_r' \\ \bar{G} \mid\lceil \bar{H} \mid\lceil \bar{\Gamma}, B \Rightarrow D\end{array}}{\bar{G} \mid\lceil \bar{H} \mid\lceil \bar{\Gamma}, \bar{\Sigma}, B \Rightarrow D \mid\lceil \bar{H}'} \qquad \dfrac{\begin{array}{c}\vdots d_l'' \\ \bar{G} \mid\lceil \bar{H} \mid\lceil D, \bar{\Sigma} \Rightarrow \bar{\Pi} \mid\lceil \bar{H}'\end{array}}{\bar{G} \mid\lceil \bar{H} \mid\lceil B, D, \bar{\Gamma}, \bar{\Sigma} \Rightarrow \bar{\Pi} \mid\lceil \bar{H}'}}{\bar{G} \mid\lceil \bar{H} \mid\lceil B, \bar{\Gamma}, \bar{\Sigma} \Rightarrow \bar{\Pi} \mid\lceil \bar{H}'}\,(cut)}{\bar{G} \mid\lceil \bar{H} \mid\lceil \bar{\Gamma}, \bar{\Sigma} \Rightarrow \bar{\Pi} \mid\lceil \bar{H}'}\,(cut)}{\cdots}}{\dfrac{\bar{G} \mid\lceil H \mid\lceil \bar{\Gamma}, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil \bar{G}'}{\cdots}}}{G \mid\lceil H \mid\lceil \Gamma, \Sigma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'}
$$

Note that any application of the rule $(\rightarrow, r)$ to Π or to a component of H' does not harm the proof because the context $\bar{G}'$ has already been deleted in d_r by the application of $(\rightarrow, r)$ to the cut formula.

(i.ii) in an internal weakening. The weakening on $(B \rightarrow D)^*$ is replaced with weakenings of the formulas $X \in \bar{\Gamma}$. We add a subsequent application of (ew) to omit the context $\bar{G}$.

(i.iii) in an external weakening. The weakening of the component P is replaced with the external weakening of the component $P[^{\bar{\Gamma}}/_{(B\rightarrow D)*}]$. The context $\bar{G}$ is also omitted using (ew).

(i.iv) in the deleted context due to an application of $(\rightarrow, r)$ to a component in $\bar{H}$. Then the component $P[^{\bar{\Gamma}}/_{(B\rightarrow D)*}]$ is also omitted when the rule $(\rightarrow, r)$ is applied to the same component in $\bar{H}$. The context $\bar{G}$ is omitted by a subsequent application of (ew). E.g. suppose $\bar{H} = H_1 \mid\lceil \Delta \Rightarrow X \rightarrow Y$:

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ H_1 \mid\lceil \Delta, X \Rightarrow Y\end{array}}{\bar{G} \mid\lceil H_1 \mid\lceil \Delta, X \Rightarrow Y}}{\bar{G} \mid\lceil H_1 \mid\lceil \Delta \Rightarrow X \rightarrow Y \mid\lceil \bar{\Gamma}, \bar{\Sigma} \Rightarrow \bar{\Pi} \mid\lceil \bar{H}'}
$$

Then, we are able to check that this procedure results in a derivation $d' \vdash_{HLJ^{nc}} G \mid\lceil H \mid\lceil \Sigma, \Gamma \Rightarrow \Pi \mid\lceil H' \mid\lceil G'$ with $\rho(d') \leq |A|$.

Theorem 5.3.12. (Cut-elimination) *If a hypersequent H is derivable in $\boldsymbol{HLJ}^{nc}$, then H is derivable in $\boldsymbol{HLJ}^{nc}$ without using the cut rule.*

Proof. Let $d \vdash_{\mathbf{HLJ}^{nc}} H$ and $\rho(d) > 0$. The proof proceeds by a double induction on $(\rho(d), n\rho(d))$, where $n\rho(d)$ is the number of cuts in d with cut-rank $\rho(d)$. Take an uppermost cut with cut-rank $\rho(d)$ in d. By application of the Reduction Lemma 5.3.11 to the premises, either $\rho(d)$ or $n\rho(d)$ decreases.

CHAPTER **6**

Summary and Future Work

In this thesis, we provide an introductory overview of a new systematic procedure which transforms Hilbert axioms into equivalent analytic rules and, hence, introduces analytic calculi for many non-classical logics in an automated way. We turned this procedure into an algorithm and implemented it in PROLOG. Moreover, we extended the scope of the systematic procedure to capture the family of axioms known as (Bd_k) with $k \geq 1$.

In [14, 15], the substructural hierarchy (see Figure 6.1) was introduced as the foundation for the aforementioned systematic procedure. It is a novel classification of Hilbert axioms based on the connectives of **FL**. The systematic procedure allows for an automated transformation of axioms up to the class $\mathcal{N}_2$ into equivalent analytic rules with **FL** as base calculus within the framework of Gentzen's sequent calculus. Axioms that belong to the classes $\mathcal{P}_3'$ ($\mathcal{P}_3$, respectively) can be captured with Avron's hypersequent calculus using **HFLe** (**HFLew**, respectively) as a base calculus. In [14, 15], uniform proofs of soundness, completeness and cut-elimination have been provided for the resulting calculi.

The main goal of this thesis was to extend the systematic procedure in [15] to the axioms (Bd_k) which are semantically characterized by Kripke models with depth $\leq k$. Due to the structure of the axioms (Bd_k), they belong to classes of the substructural hierarchy beyond $\mathcal{N}_2$ and $\mathcal{P}_3$. Indeed, even the axiom (Bd_2) resides at level $\mathcal{P}_4$. To capture the axioms (Bd_k), we introduced a new formalism called the non-commutative hypersequent calculus. In Chapter 5, we provided a uniform proof of soundness, completeness and cut-elimination for the rules in this new framework.

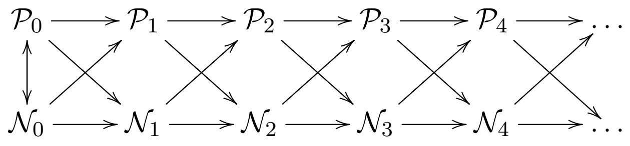

Figure 6.1: The substructural hierarchy by Ciabattoni et al. [15]

The second result established in this thesis is the PROLOG-implementation of the procedure in [15]. The implementation demanded a thorough understanding of the theorems in [15] and a translation of these theoretical results into an algorithm. The program, called *AxiomCalc*, is available online at `http://www.logic.at/people/lara/axiomcalc.html`. It works for axioms up to the classes $\mathcal{N}_2$ and $\mathcal{P}_3$ in presence of weakening and generates (hyper)structural, analytic rules automatically.

We have achieved the intended objectives of this thesis, and established new results for the framework. Still, some interesting questions remain open. Regarding the extension of our systematic procedure to capture the axioms (Bd_k), it would be very desirable to

(a) identify the general class of axioms for which we could use the non-commutative hypersequent calculus and the systematic procedure developed in this thesis.

(b) generalize the procedure to a base calculus with as few structural rules as possible, e.g. $\mathbf{HFLew}^{nc}$, $\mathbf{HFLe}^{nc}$, or even $\mathbf{HFL}^{nc}$.

Concerning the program *AxiomCalc*, it would be interesting to add features to the implementation for

(a) handling the axioms (Bd_k) (or the more general class of axioms) with the base calculus $\mathbf{HLJ}^{nc}$.

(b) generalizing the existing procedure to the base calculus **FL**.

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