TU WIEN

TECHNISCHE UNIVERSITÄT WIEN

VIENNA UNIVERSITY OF TECHNOLOGY

# DISSERTATION

# The Method of Forcing with a Category of Conditions and Allegory Axioms for Algebraic Set Theory

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines Doktors der technischen Wissenschaften unter der Leitung von

**a.o. Univ.-Prof. Dipl.-Ing. Dr.techn. Martin Goldstern**
Institut für Diskrete Mathematik und Geometrie (E104)
Technische Universtität Wien

eingereicht an der Technischen Universität Wien
bei der Fakultät für Mathematik und Geoinformation

von

**Wolfgang E. Eppenschwandtner**
weppens@dmg.tuwien.ac.at
Matrikelnummer: 9725925
Promenadegasse 57/D/5
1170 Wien

Wien, im Oktober 2006

# Kurzfassung


**Die Methode des Forcing mit einer Kategorie von Bedingungen und Allegorien-Axiome für die algebraische Mengentheorie**

Die Methode des Forcing für Unabhängigkeitsbeweise wird in abgewandelter Form untersucht: Statt einer partiellen Ordnung wird eine Kategorie von Bedingungen verwendet.

Zunächst wird eine Theorie der $\mathcal{C}$-Namen für eine Kategorie $\mathcal{C}$ entwickelt, inklusive der Definition einer Auswertung von $\mathcal{C}$-Namen und einer Forcing-Relation. Auf diese Weise können generische Forcing-Erweiterungen und Permutationsmodelle in einen gemeinsamen Rahmen gestellt werden. Weiters wird gezeigt, daß hinter $\mathcal{C}$-Namen und Garben bezüglich der dichten Überdeckung auf $\mathcal{C}$ im wesentlichen dasselbe Konzept steht.

Mit der Freydschen Darstellung kann die Methode des Forcing mit einer Kategorie von Bedingungen als Kombination von Forcing mit einer partiellen Ordnung $P$ und der Methode der Permutationsmodelle gesehen werden. Die Darstellung von Schranken für Grothendieck Topoi, die in diesem Text entwickelt wird, führt zu mehr Flexiblität in der Auswahl der partiellen Ordnung $P$.

Abschließend werden Axiome der Algebraischen Mengentheorie im Kontext abstrakter Relationen (Allegorien) angegeben. Dieser Zugang erlaubt eine Axiomatisierung von Familien von Mengen, die durch Klassen indiziert werden, ohne den üblichen Fokus auf disjunkte Mengen und führt zu einer verdichteten Formulierung der Algebraischen Mengentheorie.

# Abstract

**The Method of Forcing with a Category of Conditions and Allegory Axioms for Algebraic Set Theory**

The method of forcing for set theory independence proofs is examined in a modified version with a category of conditions rather then a partial order of conditions.

A theory of $\mathcal{C}$-names is developed for a category $\mathcal{C}$, including a definition of an evaluation of $\mathcal{C}$-names and a forcing relation. This way, generic forcing extensions and permutation models fit into one framework. It is shown that $\mathcal{C}$-names and $\neg\neg$-sheaves on $\mathcal{C}$ are essentially the same concept.

With the Freyd representation, forcing with a category of conditions can be seen as a combination of conventional forcing on a partial order $P$ and permutation model method. The representation of prebounds for Grothendieck topoi given in this text leads to more flexibility to choose the partial order $P$.

Finally, a set of axioms for Algebraic Set Theory is presented, based on the allegory setting of abstract relations. Axiomising class indexed families of sets, this approach allows to drop the usual focus on families of disjoint sets and leads to a more condensed formulation of Algebraic Set Theory.

# Preface and Acknowledgement

I am deeply grateful to my colleagues at the Institute of Discrete Mathematics and Geometry at Vienna University of Technology, especially to my supervisor MARTIN GOLDSTERN, who did their best to provide me with optimal and motivating working conditions. In these two years when I was employed as project assistant, I was never hampered in my work by artificial hierarchies within the institute. In contrary, I felt rather naturally as a member of the institute with equal rights. This is not a matter of course in Austrian university system yet.

Other institutions of Vienna University of Technology deserve to be mentioned here, too. The library procured nearly half a meter of books all together on my request, for when I started getting acquainted to this field at the end of my master study, category theory was quite underrepresented in their inventory.

A scholarship of Vienna University of Technology enabled me to gain new experiences in a research period at the *Université Catholique de Louvain* in Louvain-La-Neuve, Belgium in 2005. I am indebted to FRANCIS BORCEUX who invited me to his department, and to those category theorists which I met in Belgium and who gave me advice. In 2004, I attended the PSSL79 near Utrecht, I am grateful to the organisers of the who kindly offered me a waiver of the conference fee.

My special commendations and thanks go to the *Austrian Science Fund (FWF)*. In the last two years, I was employed in the research project *Application of Algebra to Logic* (P17627) funded by the FWF. This way, I could concentrate on my research. Additional activities I pursued in those two years were therefore based on voluntary engagement with the aim to develop complementary skills and competences and not based on a financial necessity.

Since the very beginning of my studies in Technical Mathematics/Mathematical Computer Science at Vienna University of Technology, I was involved in several organisations: FSTM, HTU, PRO SCIENTIA, EURODOC, and DOKTORAT.AT, the latter which was founded with my contribution in the time of my doctoral course. There is insufficient space to thank all my friends met there and elsewhere and my colleagues individually, but I value those times we spent together.

Finally, I thank my family for their love and support without which I would never have got here in the first place.

Wolfgang E. Eppenschwandtner, Wien, 24. October 2006

# Chapters and Contents

# Chapter 1

# Introduction and Motivation

Independent statements, i.e. statements that are neither provable nor refutable within a consistent system of axioms, have a singular attractive position in mathematics.

It is a consequence of KURT GÖDELs revolutionary *Incompleteness Theorem* that no consistent system of axioms that has enough power of expression to contain the theory of natural numbers can prove all true[1] sentences. It is a further different challenge to actually prove that a given statement is independent, though.

One major method for such independence proofs in set theory is *forcing*. Since its invention by PAUL COHEN (for the famous proof of the independence of the Continuum Hypothesis in 1963), it has been refined in many directions, e.g. iterate forcing constructions.
In rough terms, the aim of the technique of forcing is to build, starting with a ground model of ZFC-set theory, a new model which satisfies a given statement, thus implying that this statement cannot be disproved within ZFC. (see e.g. [Jec03] for the meta-mathematical justification).

As the main intermediate step, special sets called *$P$-names* are singled out from the sets of the ground model. Those names are recursively built using elements of a *"partial order of conditions"* $P$ which has to be chosen appropriately to the specific problem that needs to be solved.

In 1971 WILLIAM LAVWERE and MYLES TIERNEY added an alternative view to this construction. In [Tie72] they reformulated the proof of the consistence

[1]true is meant either in the platonic sense or relative to a ground model on which we will base our considerations. See [GJ95] for an introduction to the incompleteness phenomenon.

of Continuum Hypothesis into a category theory[2] setting. They showed that for every cardinality $\kappa$, there is a category $\mathrm{Sh}_{\neg\neg}(P)$ of sheaves (or continuous sets) over $P$, in which there are more reals (subsets of $\mathbb{N}=\omega$) than $\kappa$, and cardinals are being preserved by the embedding. See also [MLM92] and [MLM94, VI.2].
It was not the centrepiece of their work on sheaf and topos theory, they rather wanted to *"check the usefulness of the axioms"*([McL90]) (a variant of) which now constitute the basis of *topos theory.*

Later, MICHEAL FOURMAN ([Fou80]) and independently SUSUMU HAYASHI ([Hay81]) gave the proper justification that this *"reformulation of forcing"* is really capable to comply with $ZFC$-set theory (and not with a weaker theory, only). Growing understanding of the theory of toposes, especially of the internal language and contributions like [Bun74] or [Sce84] showed that transferring independence proofs into category theory language is always possible, but it also showed that there is little hope that mere reformulating of existing proofs that use a partial order of conditions will add new quality to these proofs.

However, in a category theory setting, it is rather natural to extend the technique by using a category instead of a partial order as the collection of conditions. In more technical words, general categories of sheaves $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$ (or continuous sets) over a category, also known as *Grothendieck toposes*, come in. So more interesting is the question, whether this additional latitude leads to new insights into the theory of forcing. It is mainly the aim of this thesis to shed some more light on *"forcing with a category of conditions"*.

## Forcing with Categories of Conditions and Partial Orders

Historically, the first obvious challenge was to find out whether in principle such an extended version of the technique of forcing could yield intrinsic new generality. To be more precise, the question was if there could be any statement whose consistency could only be proved by using a category of conditions, i.e. whether there could be a proof using forcing with a category of conditions that cannot be rewritten in terms of forcing over a partial order.

PETER FREYD answered this particular question negatively in the article [Fre87] with the conspicuous title *"All topoi are localic or why permutation models prevail"*. He showed that every Grothendieck topos is equivalent to an exponential variety in a topos of internal canonical $L$-sheaves in $\mathrm{Cont}(G)$, where $G$ is a topo-

[2]This might be a proper place to stress that categories in our sense are unrelated to the Baire category notion of meagre and non-meagre sets.

logical group, $\mathrm{Cont}(G)$ is the category of sets equipped with a $G$-continuous action and equivariant maps between them, and $L$ is a locale in $\mathrm{Cont}(G)$. (A locale is complete partial order with a distributivity condition.)

Bearing in mind that the cumulative hierarchy operates only in the smallest exponential variety (section 2.2), this means, reformulated in set theory terms, that for any category $\mathcal{C}$, *"forcing with a category of conditions"* is equivalent to a combination Fraenkel-Mostowski permutation model method and (conventional) forcing with a partial order. See [Bru96] for a survey on Fraenkel-Mostowski-type models. Section 5.2 contains a brief outline of the theory behind the proof of the Freyd representation in modern language using classifying toposes.

In fact, PETER FREYD showed more than that: the group of automorphisms of $\mathbb{N}$ equipped with the product topology suffices, i.e. only the Basic Fraenkel Model is needed[3]. See [Bru90] for the corresponding result obtained independently by NORBERT BRUNNER directly, within set theory setting.

However, even if in principle, in theory, every proof using forcing with a category of conditions can be reduced to more conventional methods, this does by far not imply that all has been said. It is well known that forcing over a partial order can be reduced to the special case when $P$ is a complete boolean algebra, yet partial orders are still in practice. The reason is that often they arise more direct from the applications and the corresponding boolean algebras tend to be more complicated in some case.

So far, ANDREAS BLASS and ANDRE SCEDROV were the first and only, who followed PETER FREYDs result and applied it to concrete examples. In [BS89], they examined the topos theory models of [Fre80] in which in the (internal) Axiom of Choice is false. They explicitly gave a dense subset $P$ of the locale $L$ for these examples. The proof of the Freyd representation relies on a special object $B \in \mathrm{Sh}_{\neg\neg}(\mathcal{C})$, called a *prebound* (or progenitor). ANDREAS BLASS and ANDRE SCEDROV took the most obvious (pre)bound, the sum over all representables for calculating the partial order $P$.

Their work was target-oriented for the models $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$ that arise in [Fre80]; they gave set theoretic generators for these models, but their article they did not aim at furthering the theory and describing the general case for arbitrary (pre)bounds.

In Chapter 5, a representation for collection of prebounds which we call the collection of small prebounds is established. Any other prebound contains a

[3] $\mathrm{Cont}(G)$ is then equivalent to $\mathrm{Sh}_{\neg\neg}(\mathcal{N}_m^{op})$ and to $\mathcal{S}[\mathbb{D}_\infty]$, the classifying topos for infinite, decidable sets.

small prebound as a subobject. Applying the Freyd representation, we then give a dense subset of the locale $L$ corresponding to a small prebound $B$.

## Names and Forcing with a category of conditions

In connection with *"forcing with a category of conditions"*, basically three main lines can be discerned

(1) Either, one could work directly within the topos theory setting, building a category $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$ of sheaves or continuous sets over $\mathcal{C}$, with the category $\mathcal{S}et$ of sets (of the ground model) as basis. If desired, a ZF-set theory model can be obtained from $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$ applying the method described in section 2.2 afterwards.

(2) Or, one could transliterate forcing with a category of conditions into forcing with partial order over a permutation model following the Freyd representation. A refinement of this line using *"smaller prebounds"* is presented in the Chapter 5.

(3) Finally, one could rework the notion of $P$-names to attain a definition $\mathcal{C}$-names while still remaining in a set theory setting. Other concepts like filter, evaluation, forcing relation, etc. have to be adopted as well, naturally.

The latter approach was not mentioned so far, indeed it was not followed by anyone in literature so far. In Chapter 4 this leak is closed. Moreover we show that $\mathcal{C}$-names and $\neg\neg$-sheaves on $\mathcal{C}$ constitute different viewpoints on same concept.

## Algebraic Set Theory

Whereas axioms like the powerset axiom, the separation axiom, or even the axiom of choice have their direct counterpart formulation in the categorical setting of a topos, the replacement axiom seems to be of different nature in this respect. It is true that for a *complete* boolean topos $\mathcal{E}$ over $\mathcal{S}et$ all ZF-axioms including the replacement scheme can be interpreted. This construction mainly works by mimicking the cumulative hierarchy in the topos; the main idea will be reviewed in section 2.2. Yet still there is no *natural* reformulation of the replacement axiom in this setting.

Replacement can either be formalised by an infinite collection of axioms like in usual presentations of a Zermelo Fraenkel set theory, or, alternatively, as one single axiom using classes like in Gödel-Bernays set theory. In the latter approach, the axioms declare which classes *"small enough"* to be sets. An analogue of the latter approach in the language of category theory, known as *Algebraic Set Theory*, will be followed in chapter 6.

Algebraic Set Theory was mostly contrived in the beginning 1990ties by ANDRÉ JOYAL and IEKE MOERDIJK, who compiled their work in the book [JM95]. Their main idea can be sketched as follows. We are given a category $\mathcal{C}$ with rather weak constraints, named to be the category of classes. Mostly it will be a regular category but depending on the needs of later applications also weaker requirements might suffice. A set of axioms then singles out a collection of those maps which are intended to be *small*, i.e. which are intended to have small fibres. In fact, those maps stand for families of disjoint nonempty sets indexed by classes.

With the machinery of indexed categories that will be presented in section 2.3, the collection of small maps can be conceived as a sort of *"subcategory"* of the category $\mathcal{C}$, while working *internally* within $\mathcal{C}$. In this viewpoint, axiomising small maps comes as a natural approach.

On the other hand it seems to be questionable why those families of sets indexed by classes should be disjoint by definition. This is rather unnatural from a set theory perspective. CARSTEN BUTZ chose a different approach. His article [But03] contains a set of axioms for *families of subsets of a given class X indexed by classes.*

In this thesis, a set of axioms for general *families of sets indexed by classes* is proposed. For this aim, small relations instead of small maps need to be considered. In fact, in chapter 6, we go one step further and work with allegories instead of categories to axiomise smallness.

Allegories are an abstract setting for relations. They have been explored first by ROSANNA SUCCI CRUCIANI and PETER FREYD the latter who is also responsible for the name ([Kel76], see also [FS90]). With few additional constraints, an allegory is equivalent to a category of relations of a regular category, thus enabling us to compare the set of axioms given in 6.2.1 with other versions of Algebraic Set Theory.

We mainly discuss relations to the variant of Algebraic Set Theory axioms of ALEX SIMPSON in [Sim99] (see also [ABSS03] for a quite similar set of axioms).

## 1.1 Notation and Prerequisites

As the essence of this thesis lies in the intersection of category theory and set theory, the reader will need some prior knowledge from both of these two fields.

On category theory side, we will require acquaintance with basic concepts like limits, adjoints, different types of morphisms, etc. which can be found in any deeper introduction into category theory such as [AHS90] or [Bor94]. Other concepts like indexed categories, (pre)sheaves or geometric morphisms will be defined in this text, yet not primarily for the sake of an introduction, but rather to add a different viewpoint to these concepts.

Introductions into the theory of indexed categories and fibrations, for example, are mostly structured along the commonplace that they are generalising the category of families of sets. The relation to internal categories is explored later, if at all. A converse approach is advocated in section 2.3 where an outline of the concept of indexed categories starts with internal categories as a source of motivation for indexed categories. For metamathematical contemplations this viewpoint of indexed categories as *"generalised internal categories"* is important, because it indicates that indexed categories (or fibrations) are categories which are not small enough to lie within $\mathcal{S}$, but can be examined from the perspective of an observer *"living in"* $\mathcal{S}$.

On set theory side, the reader should have a basic knowledge on the technique of forcing, she/he should be acquainted with the concept of $P$-names for a partial order $P$, the definition of dense sets, generic filters, etc. See [Jec03] for a profound treatment or [Eas05], [Cho01] for popular introductions.

The words *morphism*, *map* and *function* will be treated as synonyms with the major exception of section 3.1 and chapter 6, where the notion map will be reserved for morphisms of the category $\mathrm{Map}(\mathcal{A})$. The notation for the set of morphisms from $A$ to $B$ in $\mathcal{C}$ is $(A, B)_{\mathcal{C}}$. A partial order is just a category with one object only.[4]

For composition of maps we will use traditional functional notation, also called postfix or left-to-right notation throughout this text, i.e. $g \circ f$ shall be the composition of $f : A \to B$ followed by $g : B \to C$.

A *generalised element* $x$ of an object $A$ on the domain $T$ is a map $x : T \to A$. More briefly, we will occasionally also say $x$ is a $T$-element or simply an element of $A$. Any morphism $u : S \to T$ to the domain of $x$ acts on $x$ simply by composition

[4] We will not claim that $p \leq q \leq p \Rightarrow p = q$ for a partial order.

$x \cdot u$. Occasionally we will also write $u^*x$ for this expression. By the Yoneda-Lemma, any map $h : A \to B$ is uniquely given by a map on elements. To be more precise: It is given by (set)-maps $(h_T)_{T \in \mathcal{C}}$ between $T$-elements that are compatible, i.e. $u^*h_S(x) = h_T(u^*(x))$ for any morphism.

A *variable set* or a *presheaf* over a category $\mathcal{C}$ in $\mathcal{S}$ is a functor $F : \mathcal{C}^o \to \mathcal{S}$. Given a $\mathcal{C}$-morphism $f$ to $C \in \mathcal{C}$ and an (set-) element $x \in F(C)$, we write $x \cdot f$ for the action of $f$ on $x$, that is $(Ff)(x)$. This is reminiscent to the one object case, where a variable set is just a set equipped with a monoid action, and in fact we will conceive variable sets more as generalised monoid actions than as functors. Covariant functors $G : \mathcal{C} \to \mathcal{S}$ will also occur in this text, to avoid misinterpretations we will write $f \star x$ for the covariant action induced by $G$.

We could think of $F(C)$ as showing us how $F$ looks like in different worlds $C \in \mathcal{C}$, at different stages or under different conditions. Any set $I$ induces a presheaf $\mathsf{c}^*I$ with $\mathsf{c}^*I(C) = I$ and $\mathsf{c}^*I(f) = id_I$. These presheaves will be called *constant sets*; Especially in Chapter 3 and 5 we will write lower Greek letters $\alpha, \beta$ for constant sets. In general, if $h : D \to C$, we think of $D$ being a stronger condition than $C$, or containing more information that $C$ and $F(C)$ is a more definite version of $F$ than $F(D)$. In compliance with prevalent notation, we also say that $D$ is smaller than $C$. The morphism $f$ is a witness for this property. It determines in which way $F(C)$ is being incorporated $F(D)$, in which way the additional information of $C$ gives rise to a modification $F(D)$ of $F(C)$, which new elements come in and which are being identified.

$\mathcal{C}$ can fully be embedded in the category of variable sets on $\mathcal{C}$, denoted by $[\mathcal{C}^o, \mathcal{S}]$, with $C \mapsto \mathsf{y}C = (-, C)_{\mathcal{C}}$ and $\Omega(C) = \mathrm{Sub}\,\mathsf{y}C$. A subset $R$ of $\mathsf{y}C$ corresponds to a *sieve*, that is a downwards closed set of morphisms with common domain $C$. Here downwards closed means that if $f \in R$, then also $f \circ g$. Occasionally in Chapter 5 we will need to consider subobjects of $\mathsf{y}C_0 \times \cdots \times \mathsf{y}C_{n-1}$ for categories which do not necessarily contain products. Those subobjects correspond to sets of tuples $(f_0, \ldots, f_{n-1})$ of morphisms $f_i : D \to C_i$ which again are closed under composing with morphisms $h : E \to D$. They will be called sieves as well.

A *continuous set* or a *sheaf* over a category $\mathcal{C}$ is a variable set which *"does not vary too much"*. In different contexts, a different notion of variability is appropriate. In analogy to topology, continuity depends on a certain structure on $\mathcal{C}$. This structure used to be called Grothendieck topology, but we will follow [Joh02] to call it a *coverage* on $\mathcal{C}$ for the analogy to topology cannot be carried over too far. For each object $C$ a coverage declares whether a set of maps $M$ with common codomain $C$ is a cover of $C$. This notion of a cover has to be stable, i.e. if $M$ is a cover of $C$ and $h : D \to C$, then the set $h^*(M) = \{g : \exists f \in M \wedge \exists r : f \circ r = h \circ g\}$ has to be a cover of $D$. It suffices to consider only those sets of morphisms that are closed under compositions from the right. A continuous set has to fulfil the condition that any $x \in F(C)$ can be uniquely recovered by its compatible set of images $x \cdot f$ where $f$ ranges in a set of maps covering $C$. See 2.1.1 and [Joh02] for the precise definition.

The category of continuous sets $\mathrm{Sh}(\mathcal{C})$ forms a subcategory of $[\mathcal{C}^\circ, \mathcal{S}]$, in fact it is a reflective subcategory with reflector $\mathsf{a}\colon [\mathcal{C}^\circ, \mathcal{S}] \to \mathrm{Sh}(\mathcal{C})$. Instead of a notion of a coverage, a local operator $j\colon \Omega \to \Omega$ or a closure operator on the subobjects or $\mathsf{a}$ itself can be used as a basis of a definition of sheaves; see A.2.6 in the appendix for details.

The theory on prebounds in section 5.1 will be developed for a general notion of a coverage, but for the other parts the we will mostly use two coverages, which are defined as follows:

- $M$ covers $C$ in the *dense coverage* if for any morphism $g$ there is an $h \in M$ and two morphisms $r, s$ such that $g \circ r = h \circ s$. The local operator is the double negation operator $\neg\neg\colon \Omega \to \Omega$ in this case. We will write $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$ for the category of $\neg\neg$-sheaves, that are sheaves for the dense coverage. For a partial order $P$, the sets $M$ which cover a $p \in P$ in the dense coverage are the sets which are (pre)dense below $p$.

- $M$ covers $C$ in the *canonical coverage* if the colimit of $M{\downarrow}C \to C$ is $C$. Note that this particular definition of the canonical coverage is only appropriate if the colimits are stable under pullback in $\mathcal{C}$. All representables $\mathsf{y}C$ are sheaves for the canonical coverage. We will write $\mathrm{Sh}_{\mathrm{c}}(\mathcal{C})$ for the category of canonical sheaves, that are sheaves for the canonical coverage.

A category $\mathcal{C}$ is *separated* if two morphisms $r, s : D \to C$ in $\mathcal{C}$ are equal if and only if there is a set $M$ that covers $D$ such that $\forall f \in M \; r \circ f = s \circ f$. In that case, there is a canonical inclusion[5] $\eta_A : A \hookrightarrow \mathsf{a}(A)$.

Suppose $A$ is decidable, i.e. there is a relation $\neq \subseteq A \times A$ that is complementary to equality. Then we denote with $\langle A \rangle^n$ the object $n$-tuples of mutually distinct elements, i.e.

$$\langle A \rangle^n = \{(a_0, a_1, \ldots, a_{n-1}) \mid a_i \neq a_j \text{ for } i{\neq}j\}.$$

[5]To be more precise, the unit $\eta_A$ of the adjunction $\mathsf{a} \dashv \iota$ is a monomorphism iff $\mathcal{C}$ is separated.

# Chapter 2

# Spaces, Logic and Meta Theory

Category Theory can be conceived as an abstract setting for mathematical structures and structure preserving maps, that is the viewpoint which is most popular among mathematicians today. But categories can also be seen as generalised partial orders (*"partial orders with proofs"*), generalised monoids (*"monoids with multiple identities"*) or generalised graphs (*"graphs with an equivalence of paths"*). Motivation can be drawn from any of those germs.

Likewise, the theory on toposes has many faces. In the preface of his recent opus [Joh02] PETER JOHNSTONE counts thirteen of them. We pick two out of the list, slightly reformulated: *"Toposes as models of a theory of sets"* and *"toposes as generalised topological spaces"*.

A *topos* is a category which, roughly spoken, behaves like the category of sets and functions *Set*. In a topos, one can form products, unions, subobjects $\{\underline{x}|\varphi(\underline{x})\}$, where $\varphi(\underline{x})$ is a formula in some type theoretic language, power objects $\mathsf{P}A$, and there is an object of truth values $\Omega = \mathsf{P}1$ called the subobject classifier. The difference to prevalent set theory is that the inherent underlying logic is typically non-classical (though one can of course single out those toposes with carry boolean logic) and that there is only a local element relation $\in_X \subseteq X \times \mathsf{P}(X)$. Thus a topos, for its own, or an elementary topos to be precise, is an embodiment of intuitionistic higher order type theory. We will later explore the relations to conventional set theory.

But that is not the way topos theory did emerge as COLIN MCLARTHY keenly argues in [McL90]. The concept of a topos was contrived by ALEXANDER GROTHENDIECK who needed a generalisation of topological spaces for problems in algebraic geometry.

## 2.1 Generalised Spaces and Geometric Morphisms

General topology is sometimes known as *Point-Set Topology* which may indicate that in other branches of topology like algebraic topology, geometric topology, and differential topology points do not play such a prominent role. It is rather the interplay between continuous maps and spaces which makes up topology. In fact, there is an alternative, more algebraic recasting of the concept of topological spaces that is not based on sets of points. *Point-free Topology*, as it is called, has some advantage in a metamathematical point of view as it is constructive and the axiom of choice can be avoided in some situations (see [Joh91], [Epp04] for a discussion). One could think of applications where points are not tangible in a direct way, too.

The axioms for point-free topology can be obtained by direct transference of properties of the partial order of open sets in a topological space. In fact, the collection of all open sets $\mathcal{O}(\mathcal{X})$ of a topological space $\mathcal{X}$ is a complete lattice in which arbitrary unions distribute over finite intersections. Sometimes such a lattice is called a *frame*. A frame homomorphism $\mathcal{X} \to \mathcal{Y}$ is then a map preserving arbitrary unions and finite intersections. Observe that the inverse image $f^* = f^{-1}$ of a continuous map is such a frame homomorphism.

Leaving concrete topological spaces, we define a *locale* $\mathcal{X}$ to be such a frame, and map of locales $\mathcal{X} \to \mathcal{Y}$ to be a frame homomorphism $\mathcal{Y} \to \mathcal{X}$, or in other words, the category of locales is the opposite category of the category of frames. Most topological properties can be reformulated without mentioning points, and even points can be reintroduced as complete filters in this setting.

Indeed, there is an adjunction between the category of topological spaces and the category of locales. It restricts to an equivalence between spaces that are *sober*, that is a property in between $T_0$ and $T_1$, and locales that have enough points, called *spatial* locales. For a definition of the latter condition and an overview on locale theory see [Joh91] or [PPT04]. See also Table A.3.3 in the appendix for a condensed picture on representations of locale theory, operations on sublocales and notions like denseness.

The set-theoretic image $f(U)$ is not an open set unless $f$ is an open continuous map between topological spaces. Nevertheless, in any case, there exists a canonical map $f_*$ in the covariant direction $\mathcal{X} \to \mathcal{Y}$ that is perfectly defined also in a point-free setting. This order preserving map, usually called direct or dual image map, is determined as being the right adjoint to $f^*$. It can be explicated by the formula

$$f_*(U) = \bigvee \{V : f^{-1}(V) \subseteq U\}.$$

Every locale gives rise to a category $\mathrm{Sh_c}(\mathcal{X})$ of sheaves (or continuous sets).

**Definition 2.1.1.** *A* sheaf *or* continuous set *$F$ on a locale $\mathcal{X}$ is a variable set on $\mathcal{X}$, such that for each family $(U_i)_{i\in I}$ of elements of $\mathcal{X}$ which is covering, i.e. $\bigvee_i U_i = U$, the following condition, called the* sheaf condition *for $(U_i)_{i\in I}$, holds:*
*For each compatible family $(x_{U_i})_{i\in I}$ of elements $x_{U_i} \in F(U_i)$ there is exactly one element $x \in U$ such that $x|_{U_i} = x_{U_i}$ for all $i \in I$.*
*Here a family $x_U$ is called* compatible *if $x_U|_W = x_V|_W$ for every $W$ such that $W \leq U$ and $W \leq U$.*

The locale $\mathcal{X}$ can be recovered as $\mathrm{Sub}_{\mathrm{Sh_c}(\mathcal{X})}(1) = (1, \Omega)$, where $\Omega$ is the subobject classifier $\Omega(U) = \{V : V \subseteq U\}$. See [MLM94, II] for this fact and other representations of the category of sheaves on a locale.

The locale structure of $\mathcal{X}$ is touched only once in this definition, namely when the family $(U_i)_{i\in I}$ is required to cover $U$. It therefore can be extended to arbitrary categories equipped with a notion of a coverage.

**Definition 2.1.2.** *A* continuous set *or* sheaf *$F$ on a category $\mathcal{C}$ with respect to a notion of a coverage is a variable set on $\mathcal{C}$, such that for each family $(f_i)_{i\in I}$ of morphisms $f_i : C_i \to C$ in $\mathcal{C}$ which is covering, the following condition, called the* sheaf condition *for $(f_i)_{i\in I}$, holds:*
*For each compatible family $(x_i)_{i\in I}$ of elements $x_i \in F(C_i)$ there is exactly one element $x \in U$ such that $x \cdot f_i = x_i$ for all $i \in I$.*
*Here a family $x_i$ is called* compatible *if $x_i \cdot h = x_j \cdot k$ for every $h, k$ such that $f_i \circ h = f_j \circ k$.*
*The category of all continuous sets (or sheaves) will be denoted by* $\mathrm{Sh}(\mathcal{C})$

Those categories which arise as $\mathrm{Sh}(\mathcal{C})$ for some category $\mathcal{C}$ (and some coverage) are called *Grothendieck toposes.*

For every morphism $f : \mathcal{X} \to \mathcal{Y}$ in the category of locales, the inverse image map $f^*$ uniquely extends to a functor $f^* : \mathrm{Sh_c}(\mathcal{X}) \to \mathrm{Sh_c}(\mathcal{Y})$ that preserves colimits and finite limits. It is therefore natural to extend this definition to arbitrary Grothendieck toposes:

**Definition 2.1.3.** *Let $\mathcal{E}$ and $\mathcal{F}$ be Grothendieck toposes. A* geometric morphism *$f : \mathcal{E} \to \mathcal{F}$ is given by a functor $f^* : \mathcal{F} \to \mathcal{E}$ that preserves colimits and finite limits.*

This way, Grothendieck toposes can be conceived as generalised spaces and geometric morphisms as generalised continuous maps. The category of locales is a full subcategory in the category of toposes and geometric morphisms.

It is an easy corollary from the Special Adjoint Functor Theorem (see A.2.3 in the Appendix for a formulation or [Bor94, Th 3.3.4]), that a functor $f^*$ between Grothendieck toposes has a right adjoint (which we call $f_*$) iff it preserves colimits and finite limits.

So far, we had the generalisation from topological spaces to locales and from locales to Grothendieck toposes. There is one more level of generalisation to arbitrary toposes. An (elementary) *topos* is a category that has finite limits and power objects, see e.g. [Joh02]. Any Grothendieck topos fulfils these two conditions; in fact they have been singled out to be powerful enough to build a sufficiently strong theory that still reflects the major properties of Grothendieck toposes. To extend the notion of a geometric morphism to arbitrary toposes, it is necessary to include the functor $f_*$ in the definition.

While we will formulate the general theory for elementary toposes in this text when possible, the major results involve Grothendieck toposes only. So the reader could read Grothendieck topos whenever topos is written.

Several classes of geometric morphisms like inclusions, surjections, closed and open maps, etc. can be introduced for geometric morphisms in direct extension of the corresponding properties of continuous maps. In section 5.1 we will meet the notion of localic and bounded geometric morphisms. Those properties of geometric morphisms get trivial for geometric morphisms $\mathrm{Sh_c}(\mathcal{X}) \to \mathrm{Sh_c}(\mathcal{Y})$, that is for maps between locales $\mathcal{X}$ and $\mathcal{Y}$.

## 2.2 Internal Logic and Cumulative Hierarchy

The proper structure of truth values for classical propositional logic is a boolean algebra. For weaker systems, other partial orders may take over this role, like Heyting algebras for intuitionistic propositional logic; for a very weak propositional system with conjugations only a lower semilattice suffices. Categories, in contrast, provide an algebraic framework for systems of typed *predicate* logic. The general principle for categories (and partial orders) is: The more conditions we impose on a category, the more structure it carries, the more logical substance it can grasp.

In fact, properties of a category itself, of its morphisms can be expressed with

| | | |
|---|---|---|
| Partial Orders | are to | Categories like |
| Boolean Algebras | to | Boolean Toposes like |
| Heyting Algebras | to | Toposes like |
| Complete Heyting-algebras = Locales | to | Grothendieck toposes |
| | | |
| Classic. Propositional Logic | is to | Boolean Algebra like |
| Intuitionistic Prop. Logic | to | Heyting Algebras like |
| Higher Order Intuitionistic Logic | to | Toposes like |
| Regular Logic ($\exists, \wedge$ only) | to | Regular Categories, etc. like |
| Prop. Calculus with $\wedge$ only | to | Lower Semilattices like |
| ... | | |

Figure 2.1: Similarity of concepts: relations between partial orders versus categories and logical systems versus categories.

logical means. It is therefore said that each category comes with an attached *internal logic.*

There are a few different expositions and approaches to the internal logic of categories. Some authors put more stress on an algebraic view, they introduce only those operations that are really essential and express others in terms of them (e.g. [Awo00]). On the other hand, in the context of indexed categories which we will briefly overview in 2.3, categorical logic arises very natural in the subobject indexed category $\mathbb{S}ub_{\mathcal{C}}$ of a category $\mathcal{C}$. Intersection and union can be seen as finite products and coproducts in this setting whereas existence and universal quantifiers arise as indexed coproducts and products, i.e. adjoints to the substitution functor (see also [Jac99]).

In more elementary terms, for a description of the internal logic we first need a typed language. There are basic types or sorts $X, Y, \ldots$ and those types that are build with type constructors, e.g. $X \times Y$ or $\mathsf{P}X$. For the moment, this is just a language, so all these types are formal expressions only. The language may also contain some function symbols. They are typed, i.e. information on their domain and their codomain type is always included. Also relation symbols may occur in this setting. Function symbols build up terms, which also have to be typed, understood. As usual, formulas are built recursively. But note that a formula always comes with a *context*, that is the sequence $x_0.X_0, \ldots x_{n-1}.X_{n-1}$ of the types of its variables. In fact for the full language suitable for toposes we also need a further level of recursivity: types of the form $\{x.X|\varphi(x)\}$, are allowed, too, where $\varphi$ is a formula in a lower level of recursion.

An *interpretation* $\mathcal{M}$ in a category $\mathcal{C}$ is then an assignment of types $X$ to objects $X^{\mathcal{M}}$ of $\mathcal{C}$ translating type constructors into their intended meaning, function symbols $f : X \to Y$ to morphisms $f^{\mathcal{M}} : X^{\mathcal{M}} \to Y^{\mathcal{M}}$ of $\mathcal{C}$ and relation symbols $R \hookrightarrow X \times Y$ to relations $R^{\mathcal{M}} \hookrightarrow X^{\mathcal{M}} \times Y^{\mathcal{M}}$. The interpretation of a formula $\varphi$ is modelled by the subobject the formula singles out from the interpretation of its context:

$$\{x_0.X_0, \dots x_{n-1}.X_{n-1} \mid \varphi(x_0 \dots x_{n-1})\}^{\mathcal{M}} \hookrightarrow X_0^{\mathcal{M}} \times \dots \times X_{n-1}^{\mathcal{M}}$$

The meaning of $\{x_0.X_0, \dots x_{n-1}.X_{n-1} \mid \varphi(x_0 \dots x_{n-1})\}^{\mathcal{M}}$ is defined recursively, using generalised elements, for example; see section A.3.1 on page 86 in the appendix. The general motto from above applies here: The more structure the category carries, the more logical operations are allowed for formulas $\varphi$. In a topos, full higher order logic is allowed, for Grothendieck toposes also infinite disjunction $\bigvee_I$ for a set $I$.

A sequent $\varphi \vdash \psi$ is fulfilled in the interpretation $\mathcal{M}$ in $\mathcal{C}$ iff the inclusion $\{x.X|\varphi(x)\}^{\mathcal{M}} \subseteq \{x.X|\psi(x)\}^{\mathcal{M}}$ holds in $\mathcal{C}$. If we consider a *theory* $\mathbb{T}$ as a set of sequents in some language, this describes a *model* $\mathcal{M}$ of the theory $\mathbb{T}$ in $\mathcal{C}$. A theory $\mathbb{T}$ is a *geometric theory* if only $\wedge, \exists, \bigvee$ are involved in building the formulas for the sequents of $\mathbb{T}$ (and not $\forall, \Rightarrow$).

The *internal logic* of a category can now be described as follows: The objects of the category $\mathcal{C}$ themselves serve as types, the morphisms of $\mathcal{C}$ as function symbols, and every valid equation $f \circ g = h$ induces a sequent $\top \vdash f(g(x)) = h(x)$. In a topos, moreover, a typed element relation $\in_X \hookrightarrow \mathsf{P}(X) \times X$ neatly fits into this setting. As a conclusion, we may therefore state that a topos is an embodiment of a *"local set theory"* ([Bel88]).

## Cumulative Hierarchy and Global Logic

In a *complete* topos (over $\mathcal{S}et$), there is also an interpretation of formulas in a *global* set-theoretic language. In other words, there is a translation from topos theory setting to conventional set-theoretical setting. We construct an interpretation of the axioms of Zermelo Fraenkel set theory within a boolean Grothendieck topos over the base category $\mathcal{S}et$. As it is more appropriate to topos theory, we will base our considerations on IZF, i.e. the underlying logic will be intuitionistic. The boolean case will arise as a corollary.

The interpretation works roughly by mimicking the set-theoretical cumulative hierarchy within the topos, and by interpreting unbounded quantifiers by the infinum resp. supremum of the values in the stages of the hierarchy. This construction (for the general case of IZF with atoms) was first published by M. Fourman [Fou80] and independently by S. Hayashi [Hay81], thus it is know as the *Fourman-Hayashi interpretation.* Details and the proof that the axioms of IZF are indeed fulfilled can be found there, we only record the basic construction:

$$V_0 = 0, \; V_\beta = \mathsf{P}(\varinjlim_{\alpha<\beta} V_\alpha)$$

There are more ordinals than objects, thus the sequence of these values stabilises. So we can define

$$\exists x \varphi = \bigvee_\alpha \exists_{x_\alpha} \varphi$$
$$\forall x \varphi = \bigwedge_\alpha \forall_{x_\alpha} \varphi$$

Note that this construction touches only a part of the topos.
Let an *exponential variety* in a topos $\mathcal{E}$ be a subcategory of $\mathcal{E}$ which is closed under exponentiation, subobjects and finite limits (and, as a consequence also under power objects and finite colimits, see [Joh02, C5.4] and [Fre87]). The Fourman-Hayashi construction only operates within the smallest nontrivial exponential variety, that is the *well-founded part* of the topos. If the only nontrivial exponential variety in a topos $\mathcal{E}$ is the topos $\mathcal{E}$ itself, $\mathcal{E}$ is called a *well-founded topos.* In anticipation of chapter 5 we note that the topos of variable sets or of continuous sets over a partial order $P$ is always well-founded. It is shown there that any object in such topos is a quotient of a subobject of a coproduct of $1 = V_1$.

Starting with an object $W$ rather then with 0, the hierarchy can be extended to

$$V_0 = W, \; V_{\alpha+1} = W + \mathsf{P}(V_\alpha), \; V_\delta = W + \mathsf{P}(\varinjlim_{\alpha<\delta} V_\alpha).$$

This way, we obtain the smallest exponential variety containing $W$, also referred as the exponential variety generated by $W$. In Lemma 4.3.1 the hierarchy will play a key role in the translation of names to continuous sets.

## 2.3 Internal and Indexed Categories

The concept of a conventional $\mathcal{S}et$-based category can easily be extended to the notion of an *internal* category in any cartesian base category $\mathcal{S}$ simply by using generalised elements instead of conventional set-elements in the definition of a category. Given two objects of $\mathcal{S}$, $C_0$ and $C_1$, objects of an internal category $\mathbb{C}$ are now $T$-elements $o \in (T, C_0)$, morphisms are $T$-elements $r \in (T, C_1)$. Moreover, the definition has to comply with a change of domains of elements.

Rewritten in other words: an *internal category* $\mathbb{C}$ in $\mathcal{S}$ consists of

- a (conventional) category $\mathcal{C}^T$ for each $T$ with set of objects $(T, C_0)$ and set of morphisms $(T, C_1)$ and
- for each map $u : S \to T$ to the domain $T$ a functor $u^* : \mathcal{C}^T \to \mathcal{C}^S$ with $(u \circ v)^* = v^* \circ u^*$.

Of course, by the Yoneda-Lemma, this definition can be formulated completely within the category $\mathcal{S}$ with four maps $C_1 \overset{c}{\underset{d}{\overset{\longleftarrow e}{\rightrightarrows}}} C_0$ and $m : C_1 \times_{C_0} C_1 \to C_1$ satisfying some axioms, but that is not our aim, the introduction of internal categories rather should lead by analogy to the more general concept of indexed categories:

An indexed category $\mathbb{C}$ over $\mathcal{S}$ consists of

- a category $\mathcal{C}^T$, for each $T$ and
- for each map $u : S \to T$ to the domain $T$ a functor $u^* : \mathcal{C}^T \to \mathcal{C}^S$ with $(u \circ v)^* \simeq v^* \circ u^*$.

Every internal category in $\mathcal{S}$ hence is an indexed category over $\mathcal{S}$, but the notion of an indexed category is much more general. In short, *indexed categories are those which are to large to be an internal category.* In the category of sets, the category of sets itself is an indexed category over $\mathcal{S}et$ as well as other set models extending the ground model $\mathcal{S}et$.

In general, every category $\mathcal{S}$ is an indexed category over itself (also called canonical indexing of $\mathcal{S}$ and denoted by $\mathbb{S}$): For each domain $T$, the slice category $\mathcal{S}/T$ serves as the category $\mathcal{S}^T$, and the functor $u^* : \mathcal{S}/T \to \mathcal{S}/S$ is defined by pullback. Let us check that this definition indeed meets with the intuitive perception of an *"element of $Ob(\mathcal{S})$ with domain $T$"*. Suppose the base category is the category of variable sets on a partial order, for example, then an object of $[P^\circ, \mathcal{S}]^{yp} = [P^\circ, \mathcal{S}]/yp$ is a map $h : C \to yp$. This map forces $C$ to be nonempty if $q \leq p$ only, thus $C$ is indeed an object of $[\mathrm{P}^\circ, \mathcal{S}]$ *"with truth value $p$"*. Likewise, an 2-element $(A, B)$ of $Ob(\mathcal{S})$ is internally represented as the morphism $(A + B \to 2) \in \mathcal{S}/2$. In this way, the elements of $\mathcal{S}^T$ can be alternatively considered as families of (disjoint) sets indexed by $T$.

Every category also induces an *indexed category of elements.* For each domain $T$, the coslice category $T/\mathcal{S}$ serves as the category $\mathcal{S}^T$ this time, and the functor $u^* : T/\mathcal{S} \to S/\mathcal{S}$ is defined by composition of arrows. Put in other words: the introduction of *generalised* elements in the notation section 1.1 describes exactly the indexed category of elements over a category.

In a more succinct form, the definition above delineates an indexed category as a pseudo-functor $\mathcal{S}^{op} \to \mathcal{C}at$. For a pseudo-functor, we require that composition and identity are only preserved up to a suitable natural isomorphism and $\mathcal{C}at$ is a sufficiently large category of categories. From a foundational point of view, this might be an unsatisfactory definition, since it involves 2-categorical language (natural isomorphisms) and a category of categories. Fibrations provide an alternative setting without these deficiencies.

Given an indexed category $\mathbb{C}$ over $\mathcal{S}$, all categories $\mathcal{C}^T$ can be packed into one total category Tot$\mathbb{C}$ equipped with a functor $p$ : Tot$\mathbb{C} \to \mathcal{S}$ in a way such that we can recover the categories $\mathcal{C}^T$ as consisting of those objects that are being mapped to $T$ by the functor $p$ and of those morphisms that are being mapped to the identity on $T$. Such a functor $p$ is called a fibration (with cleavage).

Fibrations can be axiomised in an elementary way without using pseudo-functors and thus provide a sound base for foundations in a pure category theory setting. See [Pho92], [Jac99] or [Joh02] for the formal definition of fibrations and more information on the theory of indexed categories and fibrations.

In the following, however, we will certainly not stick on a severe formalism, we will use indexed categories and base categories in a rather flexible way.

Indexed categories also provide a framework for general (non-finite) sums and products which does not depend on the particular category $\mathcal{S}et$. For motivating the definition of indexed products, consider for $h : I \to 1$ the evident functor $h^*$: $\mathcal{S}et \to \mathcal{S}et^I = \mathcal{S}et/I$. The functor mapping a family $(A_i)_{i\in I}$ of sets to their sum (respectively to their product) is the left adjoint (respectively right adjoint) to the functor $h^*$. Thus, in extension to this observation, an indexed category is said to have *indexed sums* (respectively *indexed products*) iff for every $u : S \to T$ there is a left adjoint $\sum_u$ to $u^*$ (respectively a right adjoint $\prod_u$ to $u^*$).

Given a geometric morphism $\gamma : \mathcal{E} \to \mathcal{S}$, the topos $\mathcal{E}$ can be seen as an indexed

category over $\mathcal{S}$ by setting $\mathcal{E}^T = \mathcal{E}/\gamma^*(T)$. The functor $u^* : \mathcal{E}/\gamma^*(T) \to \mathcal{E}/\gamma^*(S)$ likewise is defined by pullback.
Moreover, in this framework, $\gamma$ itself can described as $\gamma^*(I) = \sum_I 1$ where the sum is the indexed sum over $\mathcal{S}$ described in the last paragraph. This is true since for $h : I \to 1$ the left adjoint to $h^* : \mathcal{E}/\gamma^*(1) = \mathcal{E}/1 = \mathcal{E} \to \mathcal{E}/\gamma^*(I)$ is simply the map $\mathcal{E}/\gamma^*(I) \to \mathcal{E}$ which maps $A \to \gamma^*(I)$ to $A$. The sum of $\gamma^*(I) \to \gamma^*(I)$, which represents the family $(1)_{j\in\gamma^*(I)}$ is thus $\gamma^*(I)$.

**Fact 2.3.1.** *Suppose $\gamma : \mathcal{E} \to \mathcal{S}$ is a geometric morphism between two toposes. We can work then within $\mathcal{S}$ as a base category and consider $\mathcal{E}$ as a $\mathcal{S}$-complete topos over $\mathcal{S}$. The products are then no longer $\mathbb{S}$et-indexed, but $\mathcal{S}$-indexed. Using the internal language in $\mathcal{S}$, we can argue in the same way as if $\mathcal{E}$ were a topos over $\mathcal{S}$ as long as we use constructive arguments, only.*

An indexed subcategory $\mathbb{B}$ of $\mathbb{C}$ is given by subcategories $\mathcal{B}^T \leq \mathcal{C}^T$ such that the indexing functors $u^*$ restrict to the subcategories. As mentioned above, the purpose of the canonical indexing over $\mathcal{S}$ is to grasp $\mathcal{S}$ itself from an inner viewpoint in $\mathcal{S}$. Suppose $a$ is a map $A \to C_0$, which may be best considered as a family $(A_i)_{i\in C_0}$ this time, we now describe how the full subcategory of $\mathcal{S}$ generated by this family can be conceived as an indexed subcategory of the canonical indexing $\mathbb{S}$ over $\mathcal{S}$.

**Definition 2.3.2.** *Let $\mathbb{S}$ be the canonical indexing over a category $\mathcal{S}$. The* full indexed subcategory *$\mathbb{F}_{\mathcal{S}}(a)$ of $\mathcal{S}$ generated by $a$, where $a$ is a map $A \to C_0$, i.e. an object of $\mathcal{S}^{C_0} = \mathcal{S}/C_0$, is defined as follows:*
*$\mathbb{F}_{\mathcal{S}}(a)^T$ is the full subcategory of $\mathcal{S}/T$ of generated by of all those maps $A' \to T$ that arise as $u^*(a)$ for some $u : T \to C_0$.*

*If $\mathcal{S}$ is a topos, it can be shown that $\mathbb{F}_{\mathcal{S}}(a)$ is an internal category (with object of objects $C_0$), therefore leading to the more common notion of a* full internal subcategory.

Other immediate examples of indexed categories over $\mathcal{S}$ include the subobject indexing $\mathbb{S}ub_{\mathcal{S}}$, which is also an indexed subcategory of the canonical indexing. It is given by the collection of partial orders $(\mathrm{Sub}(T))_{T\in\mathcal{S}}$, i.e. $(\mathbb{S}ub_{\mathcal{S}})^T = \mathrm{Sub}(T)$. In a topos $\mathcal{S}$ this subcategory is by definition of the subobject classifier $\Omega$ again a full internal subcategory, the full internal subcategory generated by $\top : 1 \to \Omega$. The relational indexing over $\mathcal{S}$ describes the category of relations in $\mathcal{S}$. We will introduce and work with this example in section 6.2.

# Chapter 3

# Relations, Filters and Classifying Toposes

## 3.1 Relations and Allegories

A relation in a category $\mathcal{C}$ is a subobject $R \subseteq A \times B$ or equivalently a span $(k, l)$ of jointly monic morphisms $k : R \to A$ and $l : R \to B$ (the latter definition has the advantage of not referring to products).

As stated in the beginning, we use traditional functional notation i.e the composition of $f : A \to B$ and $g : B \to C$ is denoted by $g \circ f$. Relations will compose in the same way, this implies that we also have to consider relations in some sense as acting from right to left. The composition of relations can be expressed in internal language as $SR = \{(z, x) : \exists y\ (z, y) \in R \wedge (y, x) \in S\}$ or, using a diagram,

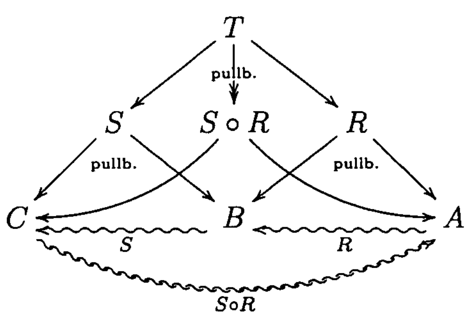


Figure 3.1: Composition for relations $R : A \rightsquigarrow B$ and $S : B \rightsquigarrow C$

as the image of the morphism $T \to C \times A$ where $T$ is constructed as a pullback. Thus we need images in $\mathcal{C}$. In fact, for composition it suffices that $\mathcal{C}$ possesses images of spans. The proper setting for relations thus is a regular category or at least a *nearly regular category*, where the latter notion should stand for a category that has pullbacks and (stable) images for spans, but not necessarily a terminal object 1.

Note that in other texts on relational categories as [FS90, Kaw95] arrows compose the other way.

A category of relations carries more structure than mere composition. In the following, this structure is axiomised to provide an abstract setting of relations called allegories. Allegories are to categories like relations to functions.

**Definition 3.1.1.** *An* allegory *is a category $\mathcal{A}$ together with a binary operation $\cap$ (intersection) on arrows with same target and source and a unary operation $^\circ$ (converse) reversing target and source, subject to the following equations:*

- *intersection induces a semilattice structure, i.e.* $(R \cap S) \cap T = R \cap (S \cap T), \quad R \cap S = S \cap R, \quad R \cap R = R$
- *converse and composition are monotone, i.e. if $R \subseteq S$ and $R' \subseteq S'$ then $RR' \subseteq SS'$ and $R^\circ \subseteq S^\circ$*
- *converse is a contravariant involution, i.e. $R^{\circ\circ} = R$ and $(SR)^\circ = R^\circ S^\circ$*
- *modular law $TR \cap S \subseteq T(R \cap T^\circ S)$*

*where the partial order on $\mathcal{A}$ is defined by $A \subseteq B$ iff $A \cap B = A$. We call the morphisms in an allegory relations and denote the identity on $A$ with $\Delta_A$ resp. $\Delta$ for short.*

Examples which do not come from a category of relations include graphs $R$ with edges labelled by real numbers and equipped with a distinguished ordered set $X$ of input vertices and a set $Y$ of output vertices. Such a graph $R : X \rightsquigarrow Y$ is intended to model a flow, thus composition $SR$ is defined by serially soldering the input vertices of $R$ with the output vertices of $S$, and intersection is given by parallel linking. Other non-trivial example are given by modular lattices, as they are exactly one-object allegories with $xy = x \vee y$, $x \cap y = x \wedge y$ and $x^\circ = x$.

$\mathcal{L}$-valued relations for a locale $\mathcal{L}$ provide an example for an allegory, too. Composition of two $\mathcal{L}$-valued relations can be visualised as a matrix-product where sum means supremum and multiplication infinum. The allegory of $\mathcal{L}$-valued relations roughly corresponds to the category of relations of continuous sets $\mathrm{Sh}_c(\mathcal{L})$.

A relation $R$ is

- *univalent* if $RR^\circ \subseteq \Delta$ (that is if $k$ is mono for $(l,k) : R \hookrightarrow X \times Y$ in a category $\mathcal{C}$)
- *total* if $R^\circ R \supseteq \Delta$ (that is if $k$ is regular epi for relations in a category $\mathcal{C}$)
- *one-to-one* if $R^\circ R \subseteq \Delta$ (that is if $l$ is mono for relations in a category $\mathcal{C}$)
- *full* if $RR^\circ \supseteq \Delta$ (that is if $l$ is regular epi for relations in a category $\mathcal{C}$)

Maps can be recovered in the abstract setting as total univalent relations. For any relation we can define its domain $\operatorname{dom} R = R^\circ R \cap \Delta$ and its image $\operatorname{im} R = RR^\circ \cap \Delta$. For maps we also set $\ker f := f^\circ f$.
Two maps $f$ and $g$ are *jointly monic* if $f^\circ f \cap g^\circ g = \ker f \cap \ker g = \Delta$, in the same manner we can also define a joint version of univalentness; $R$ and $S$ are *jointly univalent* if $RR^\circ \cap SS^\circ \subseteq \Delta$.

The next lemma collects consequences of the axioms of allegories, some of which follow immediately from the definition. Proofs for the others can be found in [FS90].

**Lemma 3.1.2.** *Let $R, S, T$ be morphisms in an allegory $\mathcal{A}$. Then*

- $(R \cap S)^\circ = R^\circ \cap S^\circ$ *and* $T(R \cap S) \subseteq TR \cap TS$ *as well as* $(R \cap S)T \subseteq RT \cap ST$.
- *The modular law implies $R \subseteq RR^\circ R$. If $F$ is univalent then*
  $F^\circ R \cap F^\circ S = F^\circ(R \cap S)$ *and dually* $RF \cap SF = (R \cap S)F$.
- *The rule $RS \cap T \subseteq (R \cap TS^\circ) \circ (S \cap R^\circ T)$ (D-rule) is equivalent to the modular law, $\operatorname{dom} R \cap S = R^\circ S \cap \Delta$ is an easy consequence. Therefore:*
  *$R \cap S$ is a function if $\Delta \subseteq R^\circ S$, and $R,S$ are (jointly) univalent or $RS^\circ \subseteq \Delta$.*
- *For any map $f$ we have*
  $Rf^\circ \subseteq S \Leftrightarrow R \subseteq Sf$ *and dually* $fR \subseteq S \Leftrightarrow R \subseteq f^\circ S$
  *The order relation on maps is trivial, i.e. for two maps we have $f \subseteq g$ iff $f = g$.*
  *As a consequence $fk = gl$ iff $lk^\circ \subseteq g^\circ f$*

**Proof.** Only few remarks on the second item to complement [FS90]: Set in the modular law $R \mapsto \Delta$ and $T, S \mapsto R$ then $R \subseteq RR^\circ R$ follows. For the second of the two dual equalities in the item under discussion note that one inclusion is generally valid, for the other we apply the modular law with $S \mapsto F^\circ S$. ∎

In which case an allegory is equivalent to a category of relations of some category? The answer to this obvious question is not much surprising:

**Lemma 3.1.3.** *An allegory $\mathcal{A}$ is induced by a category of relations iff for every morphism $R$ there is a jointly monic span $k, l$ of maps (called a* tabulation of $R$*) such that $R = l \circ k^\circ$.*

$$\begin{tikzcd} & \overline{R} \arrow[dl, "k"'] \arrow[dr, "l"] & \\ X \arrow[rr, squiggly, "R"'] & & Y \end{tikzcd}$$

Figure 3.2: Tabulation of $R$

*In fact, $\mathcal{A}$ is then induced by the category of* Map($\mathcal{A}$) *which is a nearly regular category in this case.*

*If furthermore* Map($\mathcal{A}$) *has a terminal object* 1 *(which we call* unit *of $\mathcal{A}$) then $Map(\mathcal{A})$ also has stable images and hence is regular.*

*The pullback of $f$ and $g$ corresponds to the tabulation of $g^\circ f$ and for $R = lk^\circ$ and $S = ts^\circ$*

$$R \subseteq S \Leftrightarrow \exists i \quad ti = l \wedge si = k \text{ (} i \text{ is monic then)}$$

**Proof.** The proof can be found in [FS90]. Let as for illustration proof give a direct proof that the tabulation $k$, $l$ of $g^\circ f$ extends $f$, $g$ to a pullback diagram.

From $lk^\circ = g^\circ f$ we follow by Lemma 3.1.2 that $fk = gl$, i.e. that the diagram commutes. For the pullback property, we suppose that $\bar{k}$ and $\bar{l}$ are two candidate maps with $f\bar{k} = g\bar{l}$. We simply define the universal morphism by $H := k^\circ \bar{k} \cap l^\circ \bar{l}$. Lemma 3.1.2 makes it easy to check that $H$ is indeed a function; it is moreover unique because the maps $k$ and $l$ are jointly monic. For the last statement, define $i := t^\circ l \cap s^\circ k$ and check with Lemma 3.1.2 that $i$ indeed is a map which commutes as required. ■

This proof shows very well the particular flavour of allegory theory: A relation or morphism that should be proved to exist can be simply defined, it remains to show that it indeed fulfils the desired conditions, though.

Relations which are smaller than the diagonal should intuitively correspond to subobjects. These relations are symmetric and idempotent, in fact their intersection $A \cap B$ is the same as composition $A \circ B$. Dually, those relations that contain the diagonal and are symmetric and transitive (and hence idempotent), i.e. equivalence relations, should correspond to quotient objects. In fact in any category, there is a procedure of formally introducing a class of idempotent morphisms as new objects; this way every allegory can be embedded in an allegory in which every sub-diagonal relation (every equivalence relations) uniquely determines a subobject (a quotient object) up to isomorphism, see [Joh02] or [FS90].

## 3.2 Classifying Toposes and Examples

Let $\mathbb{T}$ be a geometric theory in a typed language. As sketched in 2.2, the notion of a model of $\mathbb{T}$ can be expounded in any topos, in fact there is a category $\mathrm{Mod}(\mathcal{E}, \mathbb{T})$ of $\mathbb{T}$-models in a topos $\mathcal{E}$. For the applications in this thesis, only a vague idea of a category of models in a topos is needed; see [Joh02, D] for a detailed treatise of models and logic in toposes.

**Definition 3.2.1.** *A topos $\mathcal{S}[\mathbb{T}]$ is called the* classifying topos *for the theory $\mathbb{T}$ if*

$$\mathrm{Geo}(\mathcal{E}, \mathcal{S}[\mathbb{T}]) \cong \mathrm{Mod}(\mathcal{E}, \mathbb{T}),$$

*in other words if there is a model $U$ in $\mathcal{S}[\mathbb{T}]$ such that any model $M$ in any Grothendieck topos $\mathcal{E}$ arises as $f^*(U)$ for a unique geometric morphism $f$.*

Let $\mathbb{O}$ be the theory of one type $X$ only and no sequents or other data. A model for $\mathbb{O}$ in $\mathcal{E}$ is just an object $X^{\mathcal{M}}$, a morphism between models is just a normal morphism in $\mathcal{E}$. So the classifying topos $\mathcal{S}[\mathbb{O}]$ for the theory of objects, called the *object classifying topos*, is determined by the property that geometric morphisms from $\mathcal{E}$ to $\mathcal{S}[\mathbb{O}]$ uniquely correspond to objects of $\mathcal{E}$. That is the topos $\mathcal{S}[\mathbb{O}]$ contains a universal object $G$ such that any object $B$ in any topos $\mathcal{E}$ can be represented as $B = X^{\mathcal{M}} = f_B^*(G)$ for a unique $f_B \colon \mathcal{E} \to \mathcal{S}[\mathbb{O}]$.

Let $\mathbb{D}$ be the theory of one type, one relation and sequents expressing that the relation is complementary to equality. A model for $\mathbb{D}$ in $\mathcal{E}$ is thus a decidable object, a morphism between models is a monomorphism in $\mathcal{E}$.
The classifying topos $\mathcal{S}[\mathbb{D}]$ for the theory $\mathbb{D}$ is determined by the property that geometric morphisms from $\mathcal{E}$ to $\mathcal{S}[\mathbb{D}]$ uniquely correspond to decidable objects of $\mathcal{E}$.

The theory $\mathbb{D}_\infty$ of infinite decidable sets (cf. also 3.3.2) additionally contains for any natural number $n$ a sequent which states that there are more than $n$ different elements:

$$\top \;\vdash\; \exists x : x \neq x_0 \wedge \cdots \wedge x \neq x_{n-1}.$$

By adding the sequent $\top \vdash \exists x.x \in X$ to $\mathbb{O}$ we obtain the theory $\mathbb{O}_1$ of inhabited sets.

From [Joh02, D3] we take the explicit representations of $\mathcal{S}[\mathbb{O}]$, $\mathcal{S}[\mathbb{D}]$ and $\mathcal{S}[\mathbb{D}_\infty]$ as toposes of variable and continuous sets. Let $\mathcal{N}$ be the category of natural numbers and maps between them, i.e. a morphism from $n$ to $m$ is a map from {0,1,...,$n$-1} to {0,1,...,$m$-1} and let $\mathcal{N}_m$ the subcategory of injective maps between natural numbers. Then

$$\mathcal{S}[\mathbb{O}] = [\mathcal{N}, \mathcal{S}], \quad \mathcal{S}[\mathbb{D}] = [\mathcal{N}_m, \mathcal{S}] \text{ and } \mathcal{S}[\mathbb{D}_\infty] = \mathrm{Sh}_{\neg\neg}(\mathcal{N}_m{}^{\circ})$$

For the latter, the category of sheaves on the dense coverage is meant, but in this special case, the dense coverage is it is equivalent to the so called atomic coverage: Every nonempty family of morphisms with common domain covers. The difference between $\mathcal{S}[\mathbb{O}]$ and $\mathcal{S}[\mathbb{O}_1]$ is only that the latter is equivalent to $[\mathcal{N}^x, \mathcal{S}]$, where $\mathcal{N}^x$ is $\mathcal{N}$ without the object 0.

In $\mathcal{S}[\mathbb{O}]$, the universal model $G$ is defined by $n \mapsto$ {0,1,...,$n$-1}, which is $\mathsf{y}1$ in fact. And note that $\mathsf{y}n(k) = \{0,1,...,k\text{-}1\}^n = \mathsf{y}1^{\ n}$, thus $\mathsf{y}n = G^n$.

To the end of this chapter, we sketch an alternative description of $\mathcal{S}[\mathbb{O}]$, which may contribute to a more comprehensive picture of this topos. An *$\mathbb{O}$-expression* on a set $R$ is a formal expression of the form $r(x_0, \ldots, x_{n-1})$ where $r \in R$ and $x_i$ are formal variables (which could be coded as natural numbers, understood). Any substitution of variables $[x_i/x_j]$ can be carried over to $\mathbb{O}$-expressions with $r(x_0, \ldots, x_j, \ldots, x_{n-1})[x_i/x_j] = r(x_0, \ldots, x_i, \ldots, x_{n-1})$. For the following, *"substitution"* should stand for both replacing variables and adding dummy variables. Any map $h$ : {0,1,...,$m$-1} $\longrightarrow$ {0,1,...,$n$-1} induces a substitution transferring an expression $t(z_0, \ldots, z_{m-1})$ into the expression $t(x_{h(0)}, \ldots, x_{h(m-1)})$ with $n$ variables $x_0, \ldots, x_{n-1}$, some of which are possibly dummy.

These $\mathbb{O}$-expressions are different to terms as there is not any substitution other than variable substitution in general. They are no formulas, because they are not intended to have any interpretation attaching a truth value to it.

The category $\mathcal{F}$ is defined as follows: objects are sets $X$ of $\mathbb{O}$-expressions that are closed under substitution, equipped with a relation $R \subseteq X \times X$ on $X$ which respects substitution. A morphism between two such pairs $(X, R)$ and $(Y, S)$ is a function $X \longrightarrow Y$ mapping an $\mathbb{O}$-expression in $X$ to a $\mathbb{O}$-expression which is compatible with substitution and moreover moves $R$-equivalent expressions to $S$-equivalent expressions.

**Lemma 3.2.2.** *Suppose $V$ is an infinite set of variables, $y_0, y_1, \ldots,$ is a fixed infinite sequence of mutually distinct variables in $V$ and the category $\mathcal{F}$ is defined* *as above. Then the assignments*

$$\begin{array}{rcl} \mathcal{S}[\mathbb{O}] = [\mathcal{N}^\circ, \mathcal{S}] & \rightleftarrows & \mathcal{F} \\ A & \mapsto & (\{t(x_0, \ldots, x_{n-1}) : t \in A(n), x_i \in V\}, R) \\ \overline{A}/R & \leftarrow & (U, R) \end{array}$$

*with* $\overline{A}(n) = \{(t(y_0, \ldots, y_{n-1}) : t(y_0, \ldots, y_{n-1}) \in U)\}$, *extend to an equivalence between* $\mathcal{F}$ *and* $\mathcal{S}[\mathbb{O}]$

The category $\mathcal{S}[\mathbb{D}]$ has a similar representation as sets $X$ of $\mathbb{O}$-expressions and relations, with the only difference that those sets have to be closed under adding dummy variables only, not necessarily under substitution. As an application, the geometric morphism $l : \mathcal{S}[\mathbb{D}] \to \mathcal{S}[\mathbb{O}]$ can be described in a direct way. The inverse image part of $l^*$ is simply the inclusion. The functor $l_*$ maps a set $A$ to the set of those expressions whose substitutes are also in $A$. There is moreover also a (not finite limit preserving) left adjoint to $l^*$, which maps every set $A$ to closure of $A$ under substitution. The universal object $G$ is the set of $\mathbb{O}$-expressions which are generated by a single variable, only. It has the same description in $\mathcal{S}[\mathbb{O}]$ and in $\mathcal{S}[\mathbb{D}]$.

A *clone* is a set of finite-place operations on a set $A$ which is closed under (general, not only variable) substitution. Put differently, a clone is the set of term functions for a universal algebra on $A$. Every clone is in $\mathcal{S}[\mathbb{O}]$. (but the notion of subobject for sets of $\mathbb{O}$-expression is much weaker than for clones.)

## 3.3 Filters and Denseness

A filter in a partial order $P$ can be equivalently seen as an order preserving map $P \to 2$, which preserves intersections *"even if $P$ has none"*, i.e. filteredness is an extension the notion of preserving intersections. Likewise, a filtered functor $F : \mathcal{C} \to \mathcal{S}$ will be designed as an extension of a finite limit preserving functor.

**Definition 3.3.1.** *Let $F : \mathcal{C} \to \mathcal{S}'$ be a functor and $\mathcal{S}'$ be a 2-valued topos. Then:*

*$F$ is* filtered $\Leftrightarrow$

*(1) For $a_1 \in F(C_1)$ and $a_2 \in F(C_2)$ there need not necessarily exist a unique $(a_1, a_2)$ but at least an element $c \in F(D)$ and two morphisms $f_1$ and $f_2$ that transfer $c$ to $a_1$ resp. $a_2$ and*

*(2) if $F(h)(a) = h \star a = h' \star a = F(h')(a)$ there has to be a (not necessarily unique) map $u$ and an element $c \in F(D)$ that is transfered to $a$ by the covariant action of $F$ and moreover $h \circ u = h' \circ u$ .*

$$\begin{tikzcd} C_1 & \\ & D \arrow[ul, dotted, "f_1"'] \arrow[dl, dotted, "f_2"] \\ C_2 & \end{tikzcd} \qquad \begin{tikzcd} E & C \arrow[l, shift right, "h'"'] \arrow[l, shift left, "h"] & D \arrow[l, dotted, "u"'] \end{tikzcd}$$

*$F$ is* generic $\Leftrightarrow$ *it moves the dense coverage on $\mathcal{C}$ to the canonical coverage on $\mathcal{S}'$, i.e. if*

> *for any element $a \in F(C)$ and any set of morphisms $M$ which is dense below $C$ there exists an $f : D \to C$ in $M$ and an $c \in F(D)$ such that $f \star c = a$.*

*We call a filtered functor $F$ simply a* filter *on $\mathcal{C}$ if $F(C)$ is either $0 \approxeq \emptyset$ or $1 \approxeq \{\emptyset\}$.*

With a combination of (1) and (2), similar conditions on $F$ and can be derived for any finite diagram. For example, generalising (1), for a finite set of elements $a_0, \ldots, a_{n-1}$ there is an element $c$ and morphisms $f_i$ moving it to $a_i$.

Especially for Lemma 4.2.5 we will see that it is necessary to consider a weaker version of the notion of generic filtered functors, namely filtered functors which are *generic with respect to a submodel* $\mathsf{M}'$. For that property we do no longer require that an appropriate $c \in F(D)$ exists for *all* dense sets $M$ below $C$ but only for those dense sets which are in $\mathsf{M}'$.

**Lemma 3.3.2.** *Suppose $\mathsf{M}'$ is a countable submodel of the ground model and $C$ is an object in $\mathcal{C} \in \mathsf{M}'$. Then there is always an $\mathsf{M}'$-generic filtred functor with $F(C) \neq \emptyset$.*

**Proof.** Starting with $C = C_0$ we first choose an infinite sequence of objects $(C_n)_{n \in \mathbb{N}}$ and a sequence of morphisms $(f_n)_{n \in \mathbb{N}}$ with $f_i : C_{n+1} \to C_n$ and $C_{n+1} \neq C_n$; let $f_{m,n}$ be an abbreviation for $f_{m-1} \circ \cdots \circ f_n$. We choose this sequence in such a way that $f_n$ is always in $M_n$, which is a dense set below $C$ which did not yet occur, i.e. which is other than $f^*_{m,n}(M_m)$ for all $m$. There are only countably many dense sets below $C$, therefore for any dense $M$ below $C$, there is an $n$ such that $f^*_{0,n} M = M_n$. If such an infinite sequence $f_i$ does not exist, then $F = \mathrm{y}D$ where $D$ is the object at which the construction terminates. Otherwise, set $F(D)$ consists of all infinite paths through $\mathcal{C}$, starting as a subsequence of $f_i$ and ending at $D$, they are identified if finite subsequences correspond to the same maps. $F$ is clearly filtered, but it is also generic. Given any dense set and sequence $\underline{g} = \underline{h} \circ \underline{f'}$

where $\underline{g}$ does not belong to $\underline{f}$. Then the map $g$ associated to $\underline{g}$ transfers $\underline{f'}$ to $\underline{g}$ and moreover $g^*M$ is dense. So it suffices to consider only dense sets below an object of the form $C_i$, but for those the condition for being generic is fulfilled by construction of $(f_n)_{n\in\mathbb{N}}$. ∎

As every object in $\mathcal{N}$ can be written as a coproduct of 1, it is trivial that filtered functors on $\mathcal{N}^{op}$ are all of the form $n \mapsto A^n$ for some set $A$. A bit more involved is the analogous result for $\mathcal{N}_m^{op}$:

**Lemma 3.3.3.** *The only filtered functors on $\mathcal{N}_m^{op}$ are the functors $n \mapsto \langle A\rangle^n$ for a set $A$. They are generic iff $A$ is infinite (or $A = \emptyset$).*

**Proof.** Consider the injections $\jmath_i : 1 \hookrightarrow m$ for $i < m$. Put together, they induce a map $\tau : F(m) \to \langle A\rangle^m$. That is true because suppose $\jmath_l \star x = \jmath_k \star x$, then there exists a map $h : r \to m$ with $h \circ \jmath_l = h \circ \jmath_k$, but as all morphisms in $\mathcal{N}_m$ are mono, this implies that $l = k$.

For $a_0, \dots, a_{m-1}$ with $a_i \in A = F(1)$ there is a common extension $c \in F(n)$ for some $n$ with $f_l \star c = a_l$. If all elements $(a_l)_{l<m}$ are mutually distinct, then, all injections $f_l : 1 \hookrightarrow n$ have to be distinct, too. They hence factor through the injections $\jmath_l : 1 \hookrightarrow m$ by a morphism $u$. Then also $u \star c$ is such a common extension, i.e. an element with $\jmath_l \star (u \star c) = a_l$ or put differently, $u \star c$ is an element of $F(m)$ such that $\tau(u \star c) = (a_0, \dots, a_{m-1})$.

It thus remains to show that $\tau$ is injective. But for $c$ and $d$ with equal image under $\tau$, i.e. with $\jmath_l \star c = a_l = \jmath_l \star d$ for all $l < m$, there is an element $e \in F(n)$ for some $n$ with $f_1 \star e = c$ and $f_2 \star e = d$. With the same argument as before, it can assumed to be in $F(m)$ with $\jmath_l \star e = a_l$ and $\jmath_l \circ f_i = \jmath_l$. But then $f_i = id$ and $c = e = d$. The set of all inclusions from $m$ to a bigger $n$ is dense, so for $\overline{a} \in \langle A\rangle^m$ there is a $c \in \langle A\rangle^n$ for some $n > m$, thus $\langle A\rangle^n$ is nonempty which suffices to deduce that $A$ is infinite if it is not empty. ∎

Every geometric morphism $f : \mathcal{E} \to \mathrm{Sh}(\mathcal{C})$ is determined by the behaviour of $f^*$ at the representables $\mathsf{ay}C$. If we set $F(C) = f^*(\mathsf{ay}C)$, then $f^*$ can be recovered as

$$f^*(X) = f^*(\varinjlim \mathsf{ay}C) = \varinjlim f^*(\mathsf{ay}C) = \varinjlim F(C)$$

where the colimit is the usual representation of sheaf as colimit of representables. Both $f^*$ and $\mathsf{ay}$ preserve covers and are filtered, thus also $F$. Conversely, given such a functor $F$, it extends to a geometric morphism the with the same formula as above, see for example [Joh02] for details.

If $P$ is a partial order, every filtered functor $F$ to $\mathcal{S}'$ is a filter, as $F$ preserves 1 and monomorphisms. The following lemma describes for general categories $\mathcal{C}$ in which case a filtered functor is a functor in terms of the geometric morphism it induces.

**Lemma 3.3.4.** *A geometric morphism $f : \mathcal{S}' \to \mathrm{Sh}(\mathcal{C})$ is an inclusion iff it is induced by a filter on $\mathcal{C}$.*

**Proof.** We have to check whether the $f_*$ is full and faithful. $f_*$ is given by $A \mapsto (F-, J)_{\mathcal{S}'}$. If a $\mathcal{S}'$-function $\varphi : f_*(I) \to f_*(J)$ really is induced by a map on $\mathcal{C}$, we obtain this map as a map of 1-elements because $(F1, I)_{\mathcal{S}'} = (1, I)_{\mathcal{S}'} = I$. This fact also implies that $f_*$ is always faithful. Suppose the $F$ only maps to $\{0, 1\}$, then $(FC, A)_{\mathcal{S}'}$ is either 0 or $A$, so there is no doubt that $f_*$ is full in this case.

Conversely, assume that $f_*$ is an inclusion, $\mathcal{S}'$ then arises as a $r$-sheaf on $\mathcal{C}$ for some local operator $r$. The truth object $\Omega$ at $C$ on consists on the one hand of all $r$-closed subobjects of $yC$, i.e. of all subobjects of $\mathsf{a}_r\mathsf{y}C = F(C)$. On the other hand $\Omega$ is 2 as $\mathcal{S}'$ is supposed to be classically 2-valued, so $\Omega(C) \subseteq 2$. ■

**Lemma 3.3.5.** *Suppose $\mathcal{C}$ is a regular category, $f, g \in \mathcal{C}/C$, and the incompatibility relation $\perp$ is defined by $g \perp h$ if there are no maps $f_1$, $f_2$ with common domain other than 0 such that $g \circ f_1 = h \circ f_2$.*
*Then*

$$f \perp g \Leftrightarrow \mathrm{im}(f) \perp g \Leftrightarrow \mathrm{im}(f) \perp \mathrm{im}(g).$$

*As a consequence, a set $M$ of arrows with common codomain $C$ is dense below $C$ iff the set of all images of these maps is dense in the partial order* $\mathrm{Sub}(C)$

**Proof.** Suppose $R$ is the pullback of $\mathrm{im}(f)$ and $g$. Under this assumption $R'$ is the pullback of $f$ and $g$ iff the left square in the diagram

$$\begin{tikzcd}
R' \arrow[r, two heads, "e'"] \arrow[d] & R \arrow[r, hook] \arrow[d] & E \arrow[d, "g"] \\
D \arrow[r, two heads, "e"] \arrow[rr, bend right, "f"'] & I \arrow[r, hook, "\mathrm{im}(f)"] & C
\end{tikzcd}$$

is a pullback. Hence $f \perp g$, that is $R' = 0$, if and only if $R = 0$ (and $\mathrm{im}(f) \perp g$) because $e'$ is regular epi as a pullback of a regular epi. ■

# Chapter 4

# Names and Forcing with Categories

In the category theory view on forcing with a partial order $P$ of conditions, the extended model is built by considering the category $\mathrm{Sh}_{\neg\neg}(P)$ of continuous sets. Generalising from a partial order of conditions to a category of conditions is natural and straightforward in this formulation.

In this section, we examine whether it is also possible to work with a category $\mathcal{C}$ of conditions while *still persisting in* the prevalent set theory formulation of the method of forcing with *names.* So we seek to introduce the notion of a $\mathcal{C}$-name for a category $\mathcal{C}$ in such a way that $P$-names for a partial order $P$ arise as a special case. In this chapter, a germane definition of a forcing relation for $\mathcal{C}$-names and an notion of an evaluation at $(a, F)$ is given, where $F$ is a generic filtered functor and $a \in F(C)$. In succession, it is proved that definable and the external definition of forcing coincides, and that a statement is true if it can be forced by some object $C$. Finally, we show that $\mathcal{C}$-names and sheaves are in fact just different approaches to the same concept – this is made precise in Theorem 4.3.3.

But we start with $\mathcal{C}$-names where $\mathcal{C}$ is a particular example of a category, namely the category $\mathcal{N}_m^{op}$ of natural numbers and injections with point to the opposite direction. This category also appeared in the definition of the classifying topos for decidable sets. The model $\mathsf{M}[\langle A\rangle^n]$ which is built by evaluating of those names at the filtered functor $n \mapsto \langle A\rangle^n$ is the Basic Fraenkel permutation model. This way, forcing with categories of conditions also encompasses permutation models.

## 4.1 A Different View on Permutation Models

Suppose $A'$ is a countable infinite set. Starting from a ground model $\mathsf{M}$, the *Basic Fraenkel Model* $\mathsf{M}(A)$ is built in two steps. First, atoms are added, see [Bru96, 3.1] for the construction which adds in fact a copy $A$ of $A'$ as the set of atoms.

For a permutation $\pi$ on the set of atoms, the set $\pi X$ is defined by hereditarily permuting atoms. A set is *symmetric* if there is a finite set $H \subseteq A$ such that $\pi X = X$ for all $\pi$ with $\pi|_H = id_H$. The Basic Fraenkel Model $\mathsf{M}(A)$ then consists of those sets $X$ which are *hereditarily symmetric*, i.e. which are symmetric and have hereditarily symmetric elements. The smallest set of atoms $H$ such that $X$ is symmetric is called the *support* of $X$. See [Bru96, Bru90] and [BS89, A.1.] for details on the Basic Fraenkel Model and other permutation models.

Externally, within the ground model, we choose an enumeration of names $i \mapsto a_i$. The *support index* is then $m = \max\{i : a_i \in \mathrm{supp}(X)\} + 1$.

In the following, we introduce the notion of $\mathcal{N}_m^{op}$-names below a natural number $m$, and define the evaluation $\underset{\sim}{X}[a_0, \ldots, a_{n-1}]$ of a $\mathcal{N}_m^{op}$-name $\underset{\sim}{X}$ with values in $\mathsf{M}(A)$. Then a name $\mathfrak{f}(X)$ below $m$ can be assigned to every set $X$ in $\mathsf{M}(A)$ with support index $m$ in a way such that the evaluation of $\mathfrak{f}(X)$ recovers $X$.

**Definition 4.1.1.** *Let $\mathcal{N}_m$ be the category of natural numbers and injective maps. Recall that $\langle A\rangle^n = \{(a_0, a_1, \ldots, a_{n-1}) : a_i \neq a_j \text{ for } i \neq j\}$*
*An $\mathcal{N}_m^{op}$-*name of rank 0 below $m$ *is a natural number*[1] $k < m$ *or* $\emptyset$,
*an $\mathcal{N}_m^{op}$-*name of rank $\kappa > 0$ below $m$ *is a (nonempty) subset of*

$$\{(\underset{\sim}{x}, h) : \; h{:}m \hookrightarrow n \text{ and } \underset{\sim}{x} \text{ is an } \mathcal{N}_m^{op}\text{−name below } n \text{ with rank } \lambda < \kappa \}$$

*Suppose $A$ is a countable infinite set, $\underset{\sim}{X}$ is an $\mathcal{N}_m^{op}$−name below $m$. The* evaluation *$\underset{\sim}{X}[a_0, \ldots, a_{m-1}]$ of $\underset{\sim}{X}$ at $(a_0, \ldots, a_{m-1}) \in \langle A\rangle^m$ is defined as follows:*

$$\begin{aligned} \underset{\sim}{X}[a_0, \ldots, a_{m-1}] &= \{\underset{\sim}{x}[a'_0, \ldots, a'_{n-1}], \text{ where } (\underset{\sim}{x}, h) \in \underset{\sim}{X}, \; \underline{a}' {\in} \langle A\rangle^n \text{ and } a'_{h(i)} {=} a_i\} \\ k[a_0, \ldots, a_{m-1}] &= a_k \end{aligned}$$

Suppose now $X \in \mathsf{M}(A)$, $m \geq \max\{i : a_i \in \mathrm{supp}\, X\} + 1$. Then $X$ can be written as $X = X' \cup H \cup K$ where $H$ is a finite set of atoms with all indices smaller than $m$, the set $K$ either empty or $K = A \backslash \{a_0, \ldots, a_{m-1}\}$ and $X'$ has no atoms among

[1] Here we assume a model of natural numbers with $0 \in \mathbb{N}$ such that $0 \neq \emptyset$.

its elements. Consider the assignment

$$\begin{array}{rcl} \mathfrak{f}_m(): M_m \subseteq \mathsf{M}(A) & \to & \mathcal{N}_m^{\varphi}\text{-names below } n \\ a_k & \mapsto & k \\ H & \mapsto & \{(k, id_m) : a_k \in H\} \\ K & \mapsto & \{(m, \iota_m^{m+1})\} \text{ if } K \text{ is nonempty} \\ X' & \mapsto & \big\{(\mathfrak{f}_n(x), \iota_m^n) : x \in X',\ n = \max\{m, i{+}1 : a_i \in \operatorname{supp} x\}\big\} \\ \mathfrak{f}(): \mathsf{M}(A) & \to & \mathcal{N}_m^{\varphi}\text{-names} \\ X & \mapsto & \mathfrak{f}_m(X) \text{ where } m = \max\{i : a_i \in \operatorname{supp} X\} + 1 \end{array}$$

Then $\mathfrak{f}_m(X)[a_0, \ldots, a_{m-1}] = X$ for such a name $\mathfrak{f}_m(X)$. That statement is clear for single atoms and finite sets of atoms, but it is also true for the cofinite case for a nonempty $K$:

$$\begin{aligned} \{(m, \iota_m^{m+1})\}[a_0, \ldots, a_{m-1}] &= \{m[a_0, \ldots, a_{m-1}, a'_m] : a'_m \in A, a'_m \neq a_i \text{ for } i < m\} \\ &= \{a'_m : a'_m \in A, a'_m \neq a_i \text{ for } i < m\} = K. \end{aligned}$$

Note that $\mathfrak{f}(X)$ is not the only $\mathcal{N}_m^{\varphi}$-name which represents $X$, though. If $\sigma$ is a permutation on $\{0,1,\ldots,m\text{-}1\}$ moving $H$ to $\{0,1,\ldots,|H|\text{-}1\}$, then for example also $\{(0, \sigma), (1, \sigma), \ldots, (|H| - 1, \sigma)\}$ would be an admissible translation of $H$. Later in this chapter, a equivalence relation on names will be defined, and in Theorem 4.3.6 we will see that a properly defined category of names is equivalent to the category of sets and maps (with fixed support index) of $\mathsf{M}(A)$.

As an illustration of the interplay between $\mathcal{N}_m^{\varphi}$-names and hereditarily symmetric sets, consider the following set in $\mathsf{M}(A)$:

$$\big\{a_2, \{a_3, \emptyset\}\big\} \cup \big\{\{a_i, a_j\} : 4 < i < j\big\} \cup \big\{\{\emptyset, a_i : i \in I \backslash \{2, 3, 4, 6\}\}\big\}$$

Its support index is 7, the associated name $\mathfrak{f}(X)$ is hence

$$\begin{aligned} &\Big\{(2, id_7), (\{(3, id_7), \emptyset\}, id_7)\Big\} \cup \\ &\Big\{(\{(i, id_n), (j, id_n)\}, \iota_m^n) \text{ where } n = \max\{5, i, j\} + 1 \text{ and } 5 < i < j\Big\} \cup \\ &\Big\{(\{\emptyset, (1, id_7), (5, id_7), (7, \iota_7^8)\}, id_7)\Big\} \end{aligned}$$

## 4.2 Names and Categories

There are basically two conceptual difficulties we have to overcome when introducing $\mathcal{C}$-names for a category $\mathcal{C}$. Firstly, in contrast to names on a partial order, it is now more essential attach the context to a name, i.e. every name has to be defined *below* some object $C$. For a partial order $P$, a name below $p \in P$ is a name involving only elements of $P$ that are smaller than $p$. And every $P$-name $\underset{\sim}{x}$ can be converted into an equivalent name $\overline{\underset{\sim}{x}}$ such that for every element $(\underset{\sim}{y}, q)$ of $\underset{\sim}{x}$, the name $\underset{\sim}{y}$ is a defined below $q$.

Secondly, for a general recursive definition of $\mathcal{C}$-names, we first need to identify starting values, i.e. names of rank 0. The only grain on which $P$-names are usually developed is the empty set $\emptyset$. If the ground model is a set theory with atoms, then the construction of $P$-names is modified in such a way that, in the outset, one can choose from a set of atoms rather being restricted to $\emptyset$ as the only basic building element.

In the definition of $\mathcal{N}_m^{\varphi}$-names, starting values are no longer independent from the objects below which the name of rank 0 defined. Names of rank 0 below $n$ are natural numbers $k \in \mathbb{N}$ with $k < n$ in that case. In general, $\mathcal{C}$-names will be build on a variable set $W$ of starting values. For $P$-names, we had $W(p) = \emptyset$ or $W(p) = \Delta A(p) = A$, for $\mathcal{N}_m^{\varphi}$-names $W = G$, i.e. $W(n) = \{0,1,\ldots,n\text{-}1\}$.

**Lemma 4.2.1.** *For a generic filtered functor $F$ and a variable set $W$ of starting values, there is an equivalence relation*

$$\approx \text{ on } \sum_{C \in \mathcal{C}} (W(C) \times F(C)) \text{ which is generated by } (w \cdot h, a) \approx (w, h \star a).$$

*This relation can be made explicit as*

$(w, a) \approx (v, b)$ *iff there is an* $h, g, c$ *such that* $h \star c = a,\ g \star c = b$ *and* $w \circ h = v \circ g$.
$(w, a) \approx (v, a)$ *iff there is an* $h, c$ *such that* $h \star c = a$ *and* $w \circ h = v \circ h$.

*Moreover,* $(w, a) \approx (v, a)$ *for all generic filtered functors $F$ and elements* $a \in F(C)$ *if the set* $\{h : w \cdot h = v \cdot h\}$ *is dense.*

**Proof.** The equivalence relation is build by transfinite induction with two basic building rules. The first one appears above as the explicit definition of $\approx$, indeed

$$(w, a) = (w, h \star c) = (w \circ h, c) = (v \circ g, c) = (v, g \star c) = (v, b)$$

if $w, v, h, g, a, c, b$ as above. Alternatively there might exist a $\lambda, g', h'$ such that

$$(w, a) = (\lambda \circ h', a) = (\lambda, h' \star a) = (\lambda, g' \star b) = (\lambda \circ g', b) = (v, b).$$

But if $(\lambda, h' \star a) = (\lambda, g' \star b)$, then, since $F$ is filtered, we know of the existence of an element $c$ and morphisms $g, f$ which reduce this rule to the first case. And a composition of two building rules of the same kind yields nothing new.

If $h \star c = a$, $g \star c = a$ then there is a $c', l$ such that $gl = hl$ and $l \star c' = c$, which justifies the simplification of the definition of $\approx$ for $b = a$. Finally, if $\{h : w \cdot h = v \cdot h\}$ is dense, the existence of $c$ follows as $F$ preserves covers. ■

The expression

$$\sum_{C \in \mathcal{C}} (W(C) \times F(C)) \Big/ \approx$$

is also called the *tensor product* $W \otimes F$ of $W$ and $F$. See [MLM94, VII.2] for more on tensor products of $\mathcal{S}et$-valued functors, [ML98] for a definition in terms of limits and ends and [Epp02] for a general framework for tensor products.
For the guiding examples of $P$-names and $\mathcal{N}_m^{op}$-names, this relation can be reduced considerably: it is trivial for $P$-names, i.e. $[(w, a)]_\approx = [(\emptyset, \emptyset)]_\approx$ which can be identified with $\emptyset$ and $(k, a_0, ..., a_{n-1}) \approx (0, a_k)$, so the set of equivalence classes can be identified with the set of atoms in that case.

**Definition 4.2.2.** *Let $\mathcal{C}$ be a category, $W$ a variable set on $\mathcal{C}$.*

*A* $\mathcal{C}$-name of rank 0 below $C$ *is an element $w \in W(C)$ or $\emptyset$,*
*a* $\mathcal{C}$-name of rank $\kappa > 0$ below $C$ *is a (nonempty) subset of*

$$\{(\underset{\sim}{x}, h) : \ h \colon D \to C \textit{ and } \underset{\sim}{x} \textit{ is a } \mathcal{C}\textit{-name below } D \textit{ with rank } \lambda < \kappa\ \}$$

*Any map $f : D \to C$ transfers a name $\underset{\sim}{X}$ below $C$ to a name $\underset{\sim}{X} \cdot g$ below $D$ by*

$$\underset{\sim}{X} \cdot g = \{(\underset{\sim}{x} \cdot g', h') : (\underset{\sim}{x}, h) \in \underset{\sim}{X} \textit{ and } g \circ h' = h \circ g'\}.$$

*Suppose $F$ is a generic filtered functor, $\underset{\sim}{X}$ is an $\mathcal{C}$-name below $C$. The* evaluation *$\underset{\sim}{X}[F, a]$ of $\underset{\sim}{X}$ at $a \in F(C)$ and $F$ is defined as follows:*

$$\begin{aligned} \underset{\sim}{X}[F, a] &= \{(\underset{\sim}{x}[F, a']), \textit{ where } (\underset{\sim}{x}, h) \in \underset{\sim}{X},\ a' \in F(D) \textit{ and } h \star a' = a\} \\ w[F, a] &= [(w, a)]_\approx \end{aligned}$$

**Lemma 4.2.3.** *Suppose $X$ is a $\mathcal{C}$-name below $C$, $F$ is a generic filtered functor, $b \in F(D)$ and $h$ a map $h : D \to C$. Then*

$$(\underset{\sim}{X} \cdot h)[F, b] = \underset{\sim}{X} \cdot [F, h \star b].$$

**Proof.** Consider the diagram

$$\begin{tikzcd}
 & & C & & \\
 & & a & & \\
D \arrow[uurr, "g"] & a' \arrow[ur, mapsto] & & b \arrow[ul, mapsto] & C' \arrow[uull, "h"'] \\
 & & c \arrow[ul, mapsto] \arrow[ur, mapsto] & & \\
 & & D' \arrow[uull, "g'"] \arrow[uurr, "h'"'] & &
\end{tikzcd}$$

Unravelling the definition of $\underset{\sim}{X} \cdot h$ and of the evaluation, we see that

$$\begin{aligned}
(\underset{\sim}{X} \cdot h)[F, b] &= \{(\underset{\sim}{x} \cdot h', g') : (x, g) \in \underset{\sim}{X}, g \circ h' = h \circ g'\}[F, b] = \\
&= \{(\underset{\sim}{x} \cdot h')[F, c] : (x, g) \in \underset{\sim}{X},\ g' \star c = b,\ c \in F(D'), gh' = hg'\} = \\
&= \{\underset{\sim}{x}[F, h' \star c] : (x, g) \in \underset{\sim}{X},\ g' \star c = b,\ c \in F(D'), gh' = hg'\}
\end{aligned}$$

where the last line follows by induction hypothesis. On the other hand,

$$(\underset{\sim}{X})[F, h \star b] = \{\underset{\sim}{x}[F, a'] : (x, g) \in \underset{\sim}{X},\ g \star a' = h \star b,\ a' \in F(D)\}.$$

The functor $F$ is filtered, so for each pair $(a', b)$ such that $g \star a' = h \star b$ there is always a morphism $h'$ and a morphism $g'$ such that $g \circ h' = h \circ g'$ and moreover an element $c \in F(D')$ with $h' \star c = a'$ and $g' \star c = b$. For names of rank 0, the lemma follows from the definition of the equivalence relation $\approx$. ■

## The Forcing Relation

In the following, we will introduce a forcing relation $C \Vdash \varphi(\underset{\sim}{x})$ in a definable way. The next lemmas will then relate the forcing relation to evaluation of names and to truth in the forcing extension. For that, we need to introduce a countable submodel of the ground model for meta-mathematical justification, just like in the partial order case. Such a model always exists see e.g. [Kun83]. The main argument there is that any possible contradiction would involve only finitely many axioms of ZFC. With Lemma 3.3.2 then a M-generic filtered functor always exists, while a generic filtered functor need not exist in general.

**Definition 4.2.4.** *Let $\underset{\sim}{X}$,$\underset{\sim}{Y}$, $\underset{\sim}{z}$ be $C$-names. The* forcing relation $C \Vdash \varphi(\underset{\sim}{x})$ *for $C$-names is defined by recursion both on the rank of the involved names and the complexity of the formula $\varphi$ as follows:*

$$
\begin{array}{lcl}
C \Vdash \underset{\sim}{X} \subseteq \underset{\sim}{Y} & \Leftrightarrow & \text{For any } (\underset{\sim}{x}, g) \in \underset{\sim}{X} \\
 & & \Big\{r : \exists(\underset{\sim}{y}, h) \in \underset{\sim}{Y} \ \wedge \ D \Vdash \ \underset{\sim}{x} \cdot r = \underset{\sim}{y} \cdot s\Big\} \text{ covers.} \\
 & \Leftrightarrow & \text{For any } (\underset{\sim}{x}, g) \in \underset{\sim}{X} \\
 & & \Big\{f : \Big(f = r \circ g \Rightarrow \Big(\exists(\underset{\sim}{y}, h) \in \underset{\sim}{Y} \ \wedge \ D \Vdash \underset{\sim}{x} \cdot r = \underset{\sim}{y} \cdot s\Big)\Big)\Big\} \\
 & & \text{covers.} \\
C \Vdash w = v & \Leftrightarrow & \{f : w \cdot f = v \cdot f\} \text{ covers.} \\
C \Vdash \underset{\sim}{z} \in \underset{\sim}{X} & \Leftrightarrow & \{f : \exists(\underset{\sim}{x}, g) \in \underset{\sim}{X} \ \wedge \ D \Vdash \ \underset{\sim}{z} \cdot f = \underset{\sim}{x} \cdot r\} \text{ covers.}
\end{array}
$$

$$
\begin{array}{lcl}
C \Vdash \varphi(\underset{\sim}{x}) \wedge \psi(\underset{\sim}{x}) & \Leftrightarrow & C \Vdash \varphi(\underset{\sim}{x}) \text{ and } C \Vdash \psi(\underset{\sim}{x}) \\
C \Vdash \varphi(\underset{\sim}{x}) \Rightarrow \psi(\underset{\sim}{x}) & \Leftrightarrow & \{f : D \to C \ : \ D \Vdash \varphi(\underset{\sim}{x} \cdot f) \ \Rightarrow \ D \Vdash \psi(\underset{\sim}{x} \cdot f)\} \text{ covers} \\
 & \Leftrightarrow & \forall f : D \to C \ : \ D \Vdash \varphi(\underset{\sim}{x} \cdot f) \ \Rightarrow \ D \Vdash \psi(\underset{\sim}{x} \cdot f) \\
C \Vdash \neg\varphi(\underset{\sim}{x}) & \Leftrightarrow & \text{there is no } f : D \to C \text{ such that } D \Vdash \varphi(\underset{\sim}{x} \cdot f) \\
C \Vdash \forall \underset{\sim}{y} : \varphi(\underset{\sim}{y}, \underset{\sim}{x}) & \Leftrightarrow & \forall \underset{\sim}{y} : \ C \Vdash \varphi(\underset{\sim}{y}, \underset{\sim}{x}) \\
C \Vdash \exists \underset{\sim}{y} : \varphi(\underset{\sim}{y}, \underset{\sim}{x}) & \Leftrightarrow & \Big\{f : D \to C \ : \ \exists \underset{\sim}{y} : D \Vdash \varphi(\underset{\sim}{y}, \underset{\sim}{x} \cdot f)\Big\} \text{ covers} \\
 & \Leftrightarrow & \exists \underset{\sim}{y} : C \Vdash \varphi(\underset{\sim}{y}, \underset{\sim}{x})
\end{array}
$$

*where $f, g, h, r, s$ commute as in the diagram*

$$
\begin{tikzcd}
 & C & \\
S \arrow[ur, bend left, "g"] & & T \arrow[ul, dashed, "h"'] \\
 & D \arrow[uu, "f"] \arrow[ul, bend left, "r"] \arrow[ur, dashed, "s"'] &
\end{tikzcd}
$$

**Lemma 4.2.5.** *Suppose $\underset{\sim}{X}$ is a $C$-name below $C$ and $\mathsf{M}'$ is countable submodel of the ground model (which contains the sets defined above). Then the following statements are equivalent:*

*(1)* $C \Vdash \varphi(\underset{\sim}{X})$

*(2)* $\varphi(\underset{\sim}{X}[F,a])$ *holds for every pair* $(a, F)$ *where* $F$ *is a* $\mathsf{M}'$*-generic filtered functor and* $a \in F(C)$.

**Proof.** Suppose first that $F$ is a generic filtered functor, $a$ is an element of $F(C)$ and $C \Vdash \varphi(\underset{\sim}{X})$. We show that $\varphi(\underset{\sim}{X}[F,a])$ holds, i.e. that (1)$\Rightarrow$(2). As $F$ preserves covers, we know that for every cover $(f_i)_{i\in I}$, there is a $f_i : D \to C$ and an element $a_i \in F(D)$ such that $f_i \star a_i = a$.

So especially, if $C \Vdash \underset{\sim}{X} \subseteq \underset{\sim}{Y}$ and $\underset{\sim}{x}[F,b] \in \underset{\sim}{X}[F,a]$ (for some $(\underset{\sim}{x}, g) \in \underset{\sim}{X}$ and $g \star b = a$) there is an $r$ and an element $c \in F(D)$ such that $r \star c = b$ and $\exists(\underset{\sim}{y}, h) \in \underset{\sim}{Y}$ and $D \Vdash \underset{\sim}{x} \cdot r = \underset{\sim}{y} \cdot s$. Using the induction hypothesis with $F$ and $a'$, we obtain

$$\underset{\sim}{x}[F,b] = \underset{\sim}{x}[F, r \star c] = (\underset{\sim}{x} \cdot r)[F,c] = (\underset{\sim}{y} \cdot s)[F,c] = \underset{\sim}{y}[F, s \star c]$$

with $h$ and $s$ as in the diagram above as a consequence of $D \Vdash \underset{\sim}{x} \cdot r = \underset{\sim}{y} \cdot s$. And indeed, $\underset{\sim}{y}[F, s \star c]$ is in $\underset{\sim}{X}[F,a]$ as $h \star (s \star c) = f \star c = g \circ r \star c = a$. (For names of rank zero, the statement has been already proved in Lemma 4.2.1.)

Likewise, if $C \Vdash \underset{\sim}{z} \in \underset{\sim}{X}$ then there is an $f$ and $c \in F(D)$ such that $f \star c = a$ and $\exists(\underset{\sim}{x}, g) \in \underset{\sim}{X}$ and $D \Vdash \underset{\sim}{z} \cdot f = \underset{\sim}{x} \cdot r$. Hence

$$\underset{\sim}{z}[F,a] = \underset{\sim}{z}[F, f \star c] = (\underset{\sim}{x} \cdot f)[F,c] = (\underset{\sim}{x} \cdot r)[F,c] = \underset{\sim}{x}[F, r \star c]$$

where again $\underset{\sim}{x}[F, r \star c] \in \underset{\sim}{X}[F,a]$ because $g \star (r \star c) = a$.

For fixed $F$ and $a$ again we follow from the definition of $C \Vdash \exists \underset{\sim}{y} : \varphi(\underset{\sim}{y}, \underset{\sim}{x})$ that there is an $f\colon D \to C$ and a $c \in F(D)$ with $f \star c = a$ and moreover $\exists \underset{\sim}{y} : D \Vdash \varphi(\underset{\sim}{y}, \underset{\sim}{x} \cdot f)$. By induction this means that $\varphi(\underset{\sim}{y}[F,b], \underset{\sim}{x}[F,a])$ holds. In Lemma 4.2.7 it will be proved that density can be dropped. a Leading over to the converse direction, we show (1)$\Leftrightarrow$(2) for the remaining logical constructors. The rules for finite conjugation and the universal quantifier are trivial and negation is a special case of implication.

So for the implication rule, suppose then that $a \in F(C)$, the set

$$\{f\colon D \to C \;:\; D \Vdash \varphi(\underset{\sim}{x} \cdot f) \;\Rightarrow\; D \Vdash \psi(\underset{\sim}{x} \cdot f)\}$$

covers and $\varphi(\underset{\sim}{x}[F,a])$ holds. As $F$ is a generic filter, there is an element $b \in F(D)$ and a map $f\colon D \to C$ such that $f \star b = a$ and $D \Vdash \varphi(\underset{\sim}{x} \cdot f) \Rightarrow D \Vdash \psi(\underset{\sim}{x} \cdot f)$ By the induction hypothesis, we thus follow that $\psi(\underset{\sim}{x} \cdot f[F,b]) = \psi(\underset{\sim}{x}[F,a])$ holds as desired. Conversely, let $f\colon D \to C$ be an arbitrary morphism, and suppose $D \Vdash \varphi(\underset{\sim}{x} \cdot f)$. We know already that this means that $\varphi(\underset{\sim}{x} \cdot f[F,b])$ for every $b \in F(D)$, but also that $\psi(\underset{\sim}{x}[F, f \star b]) = \psi(\underset{\sim}{x} \cdot f[F,b])$ for $\varphi(\underset{\sim}{x} \cdot f[F,b]) = \varphi(\underset{\sim}{x}[F, f \star b])$ and $C \Vdash \varphi(\underset{\sim}{x}) \Rightarrow \psi(\underset{\sim}{x})$.

For the remaining part of the proof note that for $\neg(1) \Rightarrow \neg(2)$ it suffices to show that if $C$ does not force $\varphi(\underset{\sim}{x})$, then there is an $h\colon T \to C$ such that $T \Vdash \neg\varphi(\underset{\sim}{x} \cdot h)$. That is because once this is true, the first direction $(1) \Rightarrow (2)$ can be applied to the formula $\neg\varphi(\underset{\sim}{x} \cdot h)$. We know from Lemma 3.3.2 that a M-generic $F$ with an element $b \in F(C)$ exists thus $\varphi(\underset{\sim}{x}[F, h \star b])$ and the pair $(F, h \star c)$ is a counterexample to the premise that $\varphi(\underset{\sim}{x}[F,a])$ has to be true for all generic filtred functors $F$ and elements $a \in F(C)$.

Observe that $C \Vdash \varphi(\underset{\sim}{x})$ iff $\{f : D \to C : D \Vdash \varphi(\underset{\sim}{x} \cdot f)\}$ covers (Lemma A.2.8). So when $C$ does not force $\varphi(\underset{\sim}{x})$, the set $\{f : D \to C : D \Vdash \varphi(\underset{\sim}{x} \cdot f)\}$ does not cover or in other words, there is an $h$ with the property that there is no $s$ such that $D \Vdash \varphi(\underset{\sim}{x} \cdot h \circ s)$. But that is precisely the definition of $T \Vdash \neg\varphi(\underset{\sim}{x} \cdot h)$. ∎

**Lemma 4.2.6.** *Suppose $\underset{\sim}{X}$ is a $C$-name below $C$. Then the following statements are equivalent:*

*(1) There are maps $(f_i : D \to C_i)_{i=0\ldots n-1}$ and an element $b \in F(D)$ such that $f_i \star b = a_i$ and $D \Vdash \varphi(\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_{n-1})$*

*(2) $\varphi\left(\underset{\sim}{X}_0[F,a_0], \underset{\sim}{X}_1[F,a_1], \ldots, \underset{\sim}{X}_{n-1}[F,a_{n-1}]\right)$ holds.*

**Proof.** First note $(1) \Rightarrow (2)$ is immediate from Lemma 4.2.5 and that the statement of the Lemma can be reduced to the special case with all $a_i$ equal because $F$ is filtered.

Starting with equality, if $[(w,a)]_\approx = w[F,a] = v[F,a] = [(v,a)]_\approx$ in the case of names of rank 0 then there is an $h, c$ such that $h \star c = a$ and $w \circ h = v \circ h$ by Lemma 4.2.1.

Suppose now $\underset{\sim}{X}[F,a] \subseteq \underset{\sim}{Y}[F,a]$ for downward complete names then for each $(\underset{\sim}{x}', g') \in \underset{\sim}{X}$, $g' \star b' = a$ there is a $(\underset{\sim}{y}'', h'') \in \underset{\sim}{Y}$ and a $c''$ such that $h'' \star c'' = a$ and $\underset{\sim}{x}'[F,b] = \underset{\sim}{y}''[F,c]$., i.e. there exists an $l_1$ and an $l_2$ such that $D \Vdash \underset{\sim}{x}' \cdot l_1 = \underset{\sim}{y}'' \cdot l_2$.

Setting $\underset{\sim}{y}' = \underset{\sim}{y}'' \cdot l_2$, $r' = l_1$ and $f' = h'' \circ l_2 = g' \circ l_1$ this statement reformulates to

$$\forall (\underset{\sim}{x}', g') \in \underset{\sim}{X}, g' \star b' = a \quad \exists c', (\underset{\sim}{y}', f') \in \underset{\sim}{Y} : f' \star c' = a, f' = g' \circ r' \text{ and } D' \Vdash \underset{\sim}{x}' \cdot r' = \underset{\sim}{y}'$$

Define now the set $M$ by

$$M = \Big\{ f : \underset{\sim}{X} \cdot f \subseteq \underset{\sim}{Y} \cdot f \quad \text{or} \quad \exists (\underset{\sim}{x}, g) \in \underset{\sim}{X}, g \circ r = f : \forall (\underset{\sim}{y}, h) \in \underset{\sim}{Y},$$
$$\forall s_1, s_2 : h \circ s_2 = g \circ s_1 \; (E \Vdash \; \underset{\sim}{x} \cdot s_1 = \underset{\sim}{y} \cdot s_2 \Rightarrow f \perp h \circ s_2) \Big\}$$

If there is a $d$ such that $f \star d = a$ and $f$ in $M$, then $\underset{\sim}{X} \cdot f \subseteq \underset{\sim}{Y} \cdot f$, for else setting $h = f', s_2 = id, s_1 = r'$ yields a contradiction. So it remains to show that $M$ is dense. But for any $f$, if not $\underset{\sim}{X} \cdot f \subseteq \underset{\sim}{Y} \cdot f$, then, unravelling the definition, there is an $(\underset{\sim}{x}', g') \in \underset{\sim}{X} \cdot f'$ and an $f''$ such that for all $u$ such that $f'' \circ u = g' \circ r'$ and for all $(\underset{\sim}{y}', h') \in \underset{\sim}{Y}$ we have $\neg(D' \Vdash \underset{\sim}{x} \cdot r = \underset{\sim}{y} \cdot s)$. Then for $g = f'g'$ and $(\underset{\sim}{y}, h) \in \underset{\sim}{Y}$ we have $f \perp h \circ s_2$ for else there would be a contradiction.

Suppose $\underset{\sim}{z}[F, a] \in \underset{\sim}{X}[F, a]$, then there is a pair $(\underset{\sim}{x}, h) \in \underset{\sim}{X}$ such that $h \star a' = a$ and $\underset{\sim}{z}[F, a] = \underset{\sim}{x}[F, a']$ or, equivalently (by induction) such that $D \Vdash \underset{\sim}{x} \cdot l' = \underset{\sim}{z} \cdot l$ for some $g \circ l' = l$ and $c$ with $l' \star c = a'$. For all $f$ there is a pair $(\underset{\sim}{x}', g) \in \underset{\sim}{X} \; \wedge \; E \Vdash \underset{\sim}{z} \cdot f = \underset{\sim}{x}' \cdot r$. Indeed, this pair is given by $(x \cdot l' \circ f, f)$.

Furthering the induction, the case of existence quantifier and conjugation is straight-forward, for negation note that $\{f : D \Vdash \varphi(\underset{\sim}{x} \cdot f) \vee D \Vdash \neg\varphi(\underset{\sim}{x} \cdot f)\}$ is dense, so there exists an $a'$ and an $f$ which is an element of this set and $f \star a' = a$. In case $D \Vdash \neg\varphi(\underset{\sim}{x} \cdot f)$ we are ready, if $D \Vdash \varphi(\underset{\sim}{x} \cdot f)$, evaluating at $F$ and $a'$ leads to a contradiction. ■

**Lemma 4.2.7.** *Suppose $M$ is a set of morphisms with common codomain $C$. A family $(\underset{\sim}{x}_g)_{g \in M}$ of names is* compatible *if $E \Vdash \underset{\sim}{x}_h \cdot s = \underset{\sim}{x}_g \cdot r$ for every $g, h \in M$ and $r, s$ as in the diagram*

$$\begin{tikzcd} & C & \\ D \arrow[ur, "g"] & & D' \arrow[ul, "h"'] \\ & E \arrow[ul, "r"] \arrow[ur, "s"'] & \end{tikzcd}$$

*An antichain $A$ is a set of morphisms with common codomain $C$ which are mutually incompatible, so any family indexed by an antichain is compatible.*

*For any compatible family $(\underset{\sim}{x}_g)_{g \in M}$ of names with rank $> 0$ indexed by a dense set of morphisms $M$ or any family $(\underset{\sim}{x}_g)_{g \in A}$ of names with rank $> 0$ indexed by* *an antichain $A$, there is a name $\underset{\sim}{x}$ such that $D \Vdash \underset{\sim}{x} \cdot g = \underset{\sim}{x}_g$ holds. (pasting property)*

*As a consequence,*

$$C \Vdash \exists \underset{\sim}{y} : \varphi(\underset{\sim}{y}, \underset{\sim}{x}) \text{ and } \exists \underset{\sim}{z} \in \underset{\sim}{y} \quad \Leftrightarrow \quad \text{there is a name } \underset{\sim}{y} \text{ of rank } > 0 : C \Vdash \varphi(\underset{\sim}{y}, \underset{\sim}{x}).$$

**Proof.** $\underset{\sim}{x} = \{(\underset{\sim}{y}, h \circ u) : (y, u) \in \underset{\sim}{x}_h,\ h \in M\}$. Suppose $a \in F(D)$ is fixed, we then have to show that $\underset{\sim}{x}[g \star a] = \underset{\sim}{x}_g[a]$. So suppose $\underset{\sim}{y}[b'] \in \underset{\sim}{x}[g \star a]$, then $h \circ u \star b' = g \star a$ for some $(\underset{\sim}{y}, u) \in \underset{\sim}{x}_h$. Since $F$ is filtered, there is a $r, s$ such that $g \circ r = h \circ s$ and a $c$ with $r \star c = a$ and $s \star c = u \star b'$. As the family $(\underset{\sim}{x}_g)_{g \in M}$ is compatible

$$\underset{\sim}{x}_h[u \star b'] = \underset{\sim}{x}_h[s \star c] = (\underset{\sim}{x}_h \cdot s)[c] = (\underset{\sim}{x}_g \cdot r)[c] = \underset{\sim}{x}_g[r \star c] = \underset{\sim}{x}_g[a],$$

so it remains to show that $\underset{\sim}{y}[b'] \in \underset{\sim}{x}_h[u \star b']$ which follows by definition of the evaluation. Conversely, if $\underset{\sim}{y}[b'] \in \underset{\sim}{x}_g[a]$ then $t \star b' = a$ with $(\underset{\sim}{y}, t) \in \underset{\sim}{x}_g$ but also $g \circ t \star b' = g \star a$ which implies that $\underset{\sim}{y}[b'] \in \underset{\sim}{x}[g \star a]$.

Suppose that the set $\left\{f : D \to C \ : \ \exists \underset{\sim}{y}_f : D \Vdash \varphi(\underset{\sim}{y}_f, \underset{\sim}{x} \cdot f)\right\}$ covers, then these $\underset{\sim}{y}_f$ amalgamate to a single name $\underset{\sim}{y}$ such that $\{f : D \to C \ : \ D \Vdash \varphi(\underset{\sim}{y} \cdot f, \underset{\sim}{x} \cdot f)\}$ covers and hence $D \Vdash \varphi(\underset{\sim}{y}, \underset{\sim}{x})$. ■

## 4.3 Names and Sheaves do Coincide

For an object $D \in \mathcal{C}$, $D$-forced equality ($\underset{\sim}{x} \sim_D \underset{\sim}{y} \Leftrightarrow D \Vdash \underset{\sim}{x} = \underset{\sim}{y}$) is an equivalence relation on the class of $\mathcal{C}$-names below $C$. Let $[\underset{\sim}{x}]_D$ denote an equivalence class of $\sim_D$.

In a category, there might be many isomorphic objects, but they all have the same generalised elements up to isomorphism. Superficially[2] considered, names behave differently. Some of the elements $(\underset{\sim}{x}, f) \in \underset{\sim}{X}$ of a name might be redundant. In the proofs below, occasionally we will find it convenient have such redundant elements present, for that reason we introduce the following two operations:

The *saturation* $\underset{\sim}{X}_s$ of a name $\underset{\sim}{X}$ below $C$ is defined by

$$\underset{\sim}{X}_s = \left\{(\underset{\sim}{x}, f) \ : \ f : D \to C \text{ and } D \Vdash \underset{\sim}{x} \in \underset{\sim}{X} \cdot f\right\},$$

[2]In the *category of names* below $C$ as used in Theorem 4.3.6, isomorphic objects have the same generalised elements.

The *downward completion* $\underset{\sim}{X}_d$ of a name $\underset{\sim}{X}$ below $C$ is defined by

$$\underset{\sim}{X}_d = \{(\underset{\sim}{x} \cdot g, f \circ g) \ : \ (\underset{\sim}{x}, f) \in \underset{\sim}{X},\ f\colon D \to C \text{ and } g\colon E \to D\}.$$

For a downward complete name $\underset{\sim}{X}$, the definition of $\underset{\sim}{X} \cdot f$ can be simplified to $\underset{\sim}{X} \cdot f = \{(\underset{\sim}{x}, r) : (\underset{\sim}{x}, h \circ s) \in \underset{\sim}{X}\}$. With Lemma 4.2.5 it is immediate that $C$ forces $\underset{\sim}{X}_s$, $\underset{\sim}{X}_d$ and $\underset{\sim}{X}$ to be equal, i.e. that $\underset{\sim}{X}_s \sim_C \underset{\sim}{X}_d \sim_C \underset{\sim}{X}$.

**Lemma 4.3.1.** *Let $\underset{\sim}{X}, \underset{\sim}{Y}$ be a $C$-names of rank $> 0$ below $C$. Then*

$$\mathsf{s}(\underset{\sim}{X})(D) = \{([\underset{\sim}{x}]_D, f) \ : \ f\colon D \to C \text{ and } D \Vdash \underset{\sim}{x} \in \underset{\sim}{X} \cdot f\}$$

*defines a variable set $\mathsf{s}(\underset{\sim}{X}) \hookrightarrow X' \times \mathsf{y}C$ for some $X'$. It contains*

$$\mathsf{s}'(\underset{\sim}{X})(D) = \{([\underset{\sim}{x}]_D, f) : (\underset{\sim}{x}, f) \in \underset{\sim}{X}, f\colon D \to C\}$$

*as a subset and $\mathsf{s}(\underset{\sim}{X}) = \mathsf{s}'(\underset{\sim}{X_s})$. Moreover*

$$\mathsf{s}(\underset{\sim}{X}) \subseteq \mathsf{s}(\underset{\sim}{Y}) \Leftrightarrow C \Vdash \underset{\sim}{X} \subseteq \underset{\sim}{Y},$$

*hence the equivalence class $[\underset{\sim}{x}]_D$ can be identified with $\mathsf{s}(\underset{\sim}{x}_s)$ and as a consequence, there is an embedding $\mathsf{s}(\underset{\sim}{X}) \hookrightarrow V_\beta \times \mathsf{y}C$ where $\beta$ is the rank of $\underset{\sim}{X}$.*

*If $\underset{\sim}{z}$ is downwards closed, then $\mathsf{s}(\underset{\sim}{z} \cdot h) = \mathsf{s}(\underset{\sim}{z}) \cdot h$.*

**Proof.** Only the last but one statement need to be commented in detail. Names of rank 1 are by definition subobjects of $V_1^W \times \mathsf{y}C = W \times \mathsf{y}C$. If $\beta = \alpha + 1$, the embedding is defined by

$$\begin{array}{rlcl} \iota: & \mathsf{s}(\underset{\sim}{X}) & \hookrightarrow & V_{\alpha+1}^W \times \mathsf{y}C \\ & \mathsf{s}(\underset{\sim}{X})(D) & \hookrightarrow & (P(V_\alpha^W) \times \mathsf{y}C)\,(D) = \\ & & & \mathrm{Sub}(V_\alpha^W \times \mathsf{y}D) \times \mathsf{y}C(D) \\ & ([\underset{\sim}{x}]_D, f) & \mapsto & (\ \mathsf{s}(\underset{\sim}{x}) \hookrightarrow V_\alpha^W \times \mathsf{y}D\ ,\ f\ ) \end{array}$$

If $\beta$ rather is a limit ordinal, the embedding is defined in the same way, with the only exception that the elements $s(x)$ are no longer subobjects of a single $V_\alpha^W \times \mathsf{y}D$ but that there exists an $\alpha + 1 < \beta$ such that $\mathsf{s}(\underset{\sim}{x}) \hookrightarrow V_\alpha^W \times \mathsf{y}D$.

For the following, we identify $[\underset{\sim}{x}]_D$ and $\mathsf{s}(\underset{\sim}{x})$ without further notice. ■

Suppose $(x, f) \in X(D) \subseteq (V_\beta^W \times \mathsf{y}C)(D)$. Then there is an $\alpha$ such that $x$ is in $V_{\alpha+1}^W(D) = \mathsf{P}(V_\alpha^W)(D) = \mathrm{Sub}(\mathsf{y}D \times V_\alpha^W)$. When $x$ is conceived as a (sub)object in this way, it will be marked as $\overline{x}$.

For a sheaf $X \subseteq (V_\beta^W \times \mathrm{y}C)$, the associated name to $X$ is defined as

$$\mathsf{n}(X) = \{(\mathsf{n}(\overline{x}), f) \; : \; (x, f) \in X(D), \; f : D \to C, \; D \in \mathcal{C}\}.$$

**Lemma 4.3.2.** *Suppose $X$ is a continuous set and suppose $\mathsf{s}(\mathsf{n}(x)) = x$ for all continuous sets $x$ with lower rank than $X$.*

*If $D \Vdash \underset{\sim}{z} \in \mathsf{n}(X) \cdot f$ then there is an element $x$ and a map $f : D \to C$ such that $(x, f) \in X(D)$ and $D \Vdash \mathsf{n}(\overline{x}) = \underset{\sim}{z}$. In fact $x$ can be chosen to be $\mathsf{s}(\underset{\sim}{z})$.*

**Proof.** First note that $D \Vdash \underset{\sim}{z} \in \mathsf{n}(X) \cdot f$ can equivalently be replaced by the expression $D \Vdash \underset{\sim}{z} \in (\mathsf{n}(X) \cdot f)_d$ which in turn implies that

$$\left\{h : \exists(\underset{\sim}{y}, h) \in \mathsf{n}(X) \cdot f \text{ and } E \Vdash \underset{\sim}{z} \cdot h = \underset{\sim}{y}\right\} \text{ covers}$$

by definition of the forcing relation (set $\underset{\sim}{y} = \underset{\sim}{x} \cdot r$ there). Again, it does not harm if we replace $\mathsf{n}(X) \cdot f$ by $\mathsf{n}(X)_d \cdot f = \{(\mathsf{n}(x), h) : (x, f \circ h) \in X(E)\}$, the might only become bigger. Thus $\{h : (x, f \circ h) \in X(E) \text{ and } E \Vdash \underset{\sim}{z} \cdot h = \mathsf{n}(x)\}$ covers. Note that $E \Vdash \underset{\sim}{z} \cdot h = \mathsf{n}(x)$ iff $\mathsf{s}(\underset{\sim}{z} \cdot r) = \mathsf{s}(\mathsf{n}(x)) = x$ so we deduce that

$$\left\{h : \left(\mathsf{s}(\underset{\sim}{z} \cdot h), f \circ h\right) \in X(E_h)\right\} = \left\{h : \left(\mathsf{s}(\underset{\sim}{z}), f\right) \cdot h \in X(E_h)\right\} \text{ covers.}$$

But that is equivalent to $(\mathsf{s}(\underset{\sim}{z}), f) \in X(D)$ as $X$ is a sheaf. ∎

**Theorem 4.3.3.** *Suppose that $W \in \mathrm{Sh}_{\neg\neg}(\mathcal{C})$ is a continuous set such that $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$ is equal to the exponential variety generated by $W$. Names form a category up to $\sim_C$-equivalence, i.e. the objects of the* category of $\mathcal{C}$-names below $C$ *are $\sim_C$-equivalence classes of names, an map between $[\underset{\sim}{X}]_{\sim_C}$ and $[\underset{\sim}{Y}]_{\sim_C}$ is given by (the equivalence class of) a name $\underset{\sim}{f} \subseteq \underset{\sim}{Y} \times \underset{\sim}{X}$ such that $C \Vdash$ ($\underset{\sim}{f}$ is a map). Then:*

*The category of $\mathcal{C}$-names below $C$ is equivalent to the slice category $\mathrm{Sh}_{\neg\neg}(\mathcal{C})/\mathrm{y}C = \mathrm{Sh}_{\neg\neg}(\mathcal{C}/C)$. Especially, if $\mathcal{C}$ has a terminal object, the category of $\mathcal{C}$-names below 1 is equivalent to $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$, the category of continuous sets on $\mathcal{C}$.*

**Proof.** Rather then establishing the equivalence of the categories directly, we show that the corresponding allegories of relations are equivalent and apply Lemma 3.1.3. There is a one-to-one correspondence between subobjects $X \hookrightarrow X' \times I$ of $X' \times I$ and objects $r : X \to I$ of the slice category $\mathcal{C}/I$. Relations in $\mathcal{C}/I$ correspond then to relations $R$ in $\mathcal{C}$ which are fixed on $I$, i.e. which satisfy

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr] & X \times Y \arrow[r, hook] & (X' \times I) \times (Y' \times I) \\ & X \times Y \times I \arrow[ur, hook] & \end{tikzcd}$$

So it remains to show that: (1) every subobject of $\mathsf{y}C \times X'$ for some sheaf $X'$ corresponds to a $\mathcal{C}$-name below $C$ and vice versa in an invertible way (up to $\sim_C$) and (2) relations on names uniquely correspond to relations on subobjects of the form $\mathsf{y}C \times X'$ which are fixed on $\mathsf{y}C$, every relation arises that way and that the composition is preserved.

For $C \Vdash X = \mathsf{n}(\mathsf{s}(\underset{\sim}{X}))$, we may assume that $\underset{\sim}{X}$ is saturated. In that case, $\mathsf{s}'(\underset{\sim}{X}) = \mathsf{s}(\underset{\sim}{X})$ and hence

$$\begin{aligned} \mathsf{n}\Big(\mathsf{s}'(\underset{\sim}{X})\Big) &= \Big\{(\mathsf{n}(\overline{x}), f) : (x, f) \in \mathsf{s}'(\underset{\sim}{X})(D)\Big\} \\ &= \Big\{\Big(\mathsf{n}\Big(\mathsf{s}'(\underset{\sim}{x})\Big), f\Big) : (x, f) \in \underset{\sim}{X}\Big\} = \underset{\sim}{X}. \end{aligned}$$

Recall that $\mathsf{s}'(\mathsf{n}(X)) \subseteq \mathsf{s}(\mathsf{n}(X))$, therefore

$$\begin{aligned} \mathsf{s}'(\mathsf{n}(X))(D) &= \Big\{(\mathsf{s}(\underset{\sim}{y}), f) : \underset{\sim}{y} = \mathsf{n}(x), (x, f) \in X(D), f : D \to C\Big\} \\ &= \big\{(\mathsf{s}(\mathsf{n}(x)), f), (x, f) \in X(D), f : D \to C\big\} = X(D) \end{aligned}$$

implies that $X \subseteq \mathsf{s}(\mathsf{n}(X))$. The other inclusion follows with Lemma 4.3.2 because

$$\begin{aligned} \mathsf{s}(\mathsf{n}(X))(D) &= \Big\{(\mathsf{s}(\underset{\sim}{y}), f) : D \Vdash \underset{\sim}{y} \in \mathsf{n}(X) \cdot f\Big\} \\ &\subseteq \{(\mathsf{s}(\mathsf{n}(x)), f) : (x, f) \in X(D)\} = X(D). \end{aligned}$$

Assuming that $X$ is saturated, we know from Lemma 4.3.1 that subobjects of $\mathsf{s}(X)$ correspond to subobjects of $X$ (up to $\sim_C$). For downward complete relations $\underset{\sim}{R} \subseteq \underset{\sim}{Y} \times \underset{\sim}{X}$ and $\underset{\sim}{S} \subseteq \underset{\sim}{Z} \times \underset{\sim}{Y}$ we see from the definition of the composition

$$C \Vdash \Big((\underset{\sim}{z}, \underset{\sim}{x}) \in \underset{\sim}{S} \circ \underset{\sim}{R} \Leftrightarrow \exists \underset{\sim}{y}\ (\underset{\sim}{z}, \underset{\sim}{y}) \in \underset{\sim}{R} \text{ and } (\underset{\sim}{y}, \underset{\sim}{x}) \in \underset{\sim}{S}\Big)$$

that $(\underset{\sim}{S} \cdot f) \circ (\underset{\sim}{R} \cdot f) = (\underset{\sim}{S} \circ \underset{\sim}{R}) \cdot f$. Next, consider the following reformulations

$$\begin{aligned}
\mathsf{s}(\underset{\sim}{S} \circ \underset{\sim}{R})(D) &= \{([\underset{\sim}{z}, \underset{\sim}{x}]_D, f) \ : \ f\colon D \to C \text{ and } D \Vdash (\underset{\sim}{z}, \underset{\sim}{x}) \in (\underset{\sim}{S} \circ \underset{\sim}{R}) \cdot f\} = \\
&= \{([\underset{\sim}{z}, \underset{\sim}{x}]_D, f) : \ D \Vdash \exists \underset{\sim}{y}\, (\underset{\sim}{z}, \underset{\sim}{y}) \in \underset{\sim}{R} \cdot f \text{ and } (\underset{\sim}{y}, \underset{\sim}{x}) \in \underset{\sim}{S} \cdot f\} \\
&= \{([\underset{\sim}{z}, \underset{\sim}{x}]_D, f) : \ \exists \underset{\sim}{y}\ D \Vdash (\underset{\sim}{z}, \underset{\sim}{y}) \in \underset{\sim}{R} \cdot f \text{ and } D \Vdash (\underset{\sim}{y}, \underset{\sim}{x}) \in \underset{\sim}{S} \cdot f\} \\
&= \{([\underset{\sim}{z}, \underset{\sim}{x}]_D, f) : \ \exists \underset{\sim}{y}\ D \Vdash ([\underset{\sim}{z}, \underset{\sim}{y}]_D, f) \in \mathsf{s}(\underset{\sim}{R}) \text{ and } ([\underset{\sim}{y}, \underset{\sim}{z}]_D, f) \in \mathsf{s}(\underset{\sim}{S})\}
\end{aligned}$$

The latter expression is exactly the definition of the composition of the two relations $\mathsf{s}(\underset{\sim}{R})$ and $\mathsf{s}(\underset{\sim}{S})$ in the category of sheaves $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$. ∎

In the topos $\mathrm{Sh}_{\neg\neg}(P)$, every object $X$ is a quotient of a subobject of a sum of 1, i.e. $\mathrm{Sh}(P)$ is bounded by 1, a notion which will be further explored in Chapter 5. As a consequence, $\mathrm{Sh}(P)$ itself is also the only exponential variety which contains 1. Therefore:

**Corollary 4.3.4.** *Let $P$ be a partial order. Then:*
*The category of $P$-names is equivalent to the category of continuous sets on* $\mathrm{Sh}_{\neg\neg}(P)$.

This chapter started with an adumbration of the relation between $\mathcal{N}_m^{\varphi}$-names and the Basic Fraenkel Model. Having developed the general theory on $\mathcal{C}$-names a forcing relation, the next goal is to exhibit the equivalence between names and in a more definite formulation.

**Lemma 4.3.5.** *Suppose $\sigma$ is an invertible morphism. Then*

$$C \Vdash \{(\underset{\sim}{x} \cdot \sigma, h)\} = \{(\underset{\sim}{x}, h \circ \sigma^{-1})\} \tag{4.1}$$

*and every $\mathcal{N}_m^{\varphi}$-name is equivalent to a $\mathcal{N}_m^{\varphi}$-name which contains only elements of the form $(\underset{\sim}{x}, \iota_m^n)$. For such a name $\underset{\sim}{X}$*

$$\begin{aligned}
\underset{\sim}{X} \cdot \iota_m^l = \{\ & (\underset{\sim}{x} \cdot (\pi \circ \iota_n^k), \iota_l^k) : (\underset{\sim}{x}, \iota_m^n) \in \underset{\sim}{X}, \\
& \pi\colon k \to k \textit{ perm.},\ \pi|_m = id, k = \max\{l, n\}\ \}
\end{aligned}$$

*Moreover, for a permutation $\sigma$ on $\{0,1,\dots,m\text{-}1\}$ and a number $l \geq n$*

$$m \Vdash \mathfrak{f}_m(X) \cdot \sigma = \mathfrak{f}_m(\sigma^{-1} X) \ \textit{and}\ \ l \Vdash \mathfrak{f}_m(X) \cdot \iota_m^l = \mathfrak{f}_l(X).$$

**Proof.** The first statement follows straight-forward with Lemma 4.2.5.
Suppose $\underaccent{\sim}{X}$ is a name which contains only elements of the form $(\underaccent{\sim}{x}, \iota^n_m)$ and $r \cdot \iota^n_m = s \cdot \iota^l_m$. The morphisms $r$ and $s$ can be split as $r = \alpha \circ \iota^k_n$ and $s = \beta \circ \iota^k_n$, as in the diagram:

$$
\begin{tikzcd}
 & m \arrow[dl, hook', "\iota^n_m"'] \arrow[dr, hook, "\iota^l_m"] & \\
n \arrow[dr, hook', "\iota^k_n"'] & & l \arrow[dl, hook, "\iota^k_n"] \\
 & k \arrow[loop, out=240, in=280, distance=2em, "\alpha"'] \arrow[loop, out=260, in=300, distance=2em, "\beta"] &
\end{tikzcd}
$$

The equation $r \cdot \iota^n_m = s \cdot \iota^l_m$ then is equivalent to $\alpha|_m = \beta|_m$ or $\pi|_m = \mathit{id}_m$ for $\pi = \alpha^{-1} \circ \beta$, thus leading to the desired expression of $\underaccent{\sim}{X} \cdot \iota^l_m$ by applying (4.1).

Suppose $X$ has no atoms among its elements. Then consider

$$\mathfrak{f}_m(X) \cdot \sigma = \big\{(\mathfrak{f}_n(x), (\iota^n_m \circ \sigma^{-1})) : x \in X',\ n = \max\{m, i{+}1 : a_i \in \operatorname{supp} x\}\big\}$$

Introducing $\sigma_1^{-1} \colon n \to n$ with $\iota^n_m \circ \sigma^{-1} = \sigma_1^{-1} \circ \iota^n_m = \iota^n_m \cdot \sigma_1^{-1}$ this reformulates by (4.1) to

$$
\begin{aligned}
\mathfrak{f}_m(X) \cdot \sigma &\sim_m \{(\mathfrak{f}_n(x) \cdot \sigma_1, \iota^n_m) : x \in X',\ n = \max\{m, i{+}1 : a_i \in \operatorname{supp} x\}\} \\
&= \big\{(\mathfrak{f}_n(\sigma_1^{-1} x), \iota^n_m) : x \in X',\ n = \max\{m, i{+}1 : a_i \in \operatorname{supp} x\}\big\} \\
&= \big\{(\mathfrak{f}_n(y), \iota^n_m) : y \in \sigma^{-1} X',\ n = \max\{m, i{+}1 : a_i \in \operatorname{supp} y\}\big\} \\
&= \mathfrak{f}_m(\sigma^{-1} X)
\end{aligned}
$$

For the final equation, we note that if $X$ has no atoms among its elements, then $\mathfrak{f}_m(X)$ has only elements of the form $(\mathfrak{f}_n(x), \iota^n_m)$, therefore

$$
\begin{aligned}
\mathfrak{f}_m(X) \cdot \iota^l_m = \Big\{ &(\mathfrak{f}_n(x) \cdot (\pi \circ \iota^k_n), \iota^k_l) : x \in X, n = \max\{m, i+1 : a_i \in \operatorname{supp} x\}, \\
&\pi \text{ perm. on } \{0,1,\ldots,k\text{-}1\} \text{ with } \pi|_m = \mathit{id},\ k = \max\{l, n\} \Big\}
\end{aligned}
$$

We know that $\mathfrak{f}_n(x) \cdot (\pi \circ \iota^k_n) \sim_k \mathfrak{f}_k(x) \cdot \pi \sim_k \mathfrak{f}_k(\pi^{-1} x)$ and that $k = \max\{l, n\} = \max\{l, i+1 : a_i \in \operatorname{supp} x\} = \max\{l, i+1 : a_i \in \operatorname{supp} \pi^{-1} x\}$ thus the expression above reformulates to

$$
\begin{aligned}
\Big\{ &(\mathfrak{f}_k(\pi^{-1} x), \iota^k_l) : x \in X, \quad \pi \text{ perm. on } \{0,1,\ldots,k\text{-}1\} \text{ with } \pi|_m = \mathit{id}, \\
&k = \max\{l, i+1 : a_i \in \operatorname{supp} \pi^{-1} x\} \Big\}
\end{aligned}
$$

which is equal to $\mathfrak{f}(X)$ because $x \in X \Leftrightarrow y = \pi^{-1}x \in X$ for any symmetric set $X$. The remaining proofs for sets of atoms are straight-forward. ∎

**Theorem 4.3.6.** *The category of $\mathcal{N}_m^{\mathrm{op}}$-names below $m$ is equivalent to the category of sets and maps in the Basic Fraenkel Model* $\mathsf{M}(A)$ *with support index* $\leq m$.

**Proof.** Again, we show that the corresponding allegories of relations are equivalent. It is clear from Lemma 4.2.5 that evaluation at $(a_0, \ldots, a_{m-1})$ preserves both subobjects and composition of relations. It remains to show that (1) $m \Vdash \underset{\sim}{X} = \mathfrak{f}(\underset{\sim}{X}[a_0, \ldots, a_{m-1}])$ and (2) $X \subseteq Y \Rightarrow m \Vdash \mathfrak{f}_m(X) \subseteq \mathfrak{f}_m(Y)$

Suppose we have $\underline{b} = (b_0, \ldots, b_{m-1}) \in \langle A \rangle^m$ and $\underline{c} = (a_0, \ldots, a_{n-1})$ where $n$ is the maximum of $\{i+1 : b_j = a_i\}$. Choose a permutation $\sigma$ on $\{0,1,\ldots,n\text{-}1\}$ such that $\underline{b} = (\sigma \circ \iota_m^n) \star \underline{c} = \iota_m^n \star \sigma \star \underline{c}$. For any $X$ with support index $m$ we deduce that

$$(\mathfrak{f}(X))[\underline{b}] = (\mathfrak{f}_m(X)) \cdot \iota_m^n \cdot \sigma[\underline{c}] = (\mathfrak{f}_n(X)) \cdot \sigma[\underline{c}] = \mathfrak{f}_n(\sigma^{-1}(X))[\underline{c}] = \sigma^{-1}(X)$$

As a consequence,

$$(\mathfrak{f}(\underset{\sim}{X}[\underline{a}]))[\underline{b}] = \sigma^{-1}(\underset{\sim}{X}[\iota_m^n \star \underline{c}]) = (\underset{\sim}{X}[\iota_m^n \star \sigma \star \underline{c}]) = \underset{\sim}{X}[\underline{b}]$$

for any name $\underset{\sim}{X}$. As a second application, note $X \subseteq Y$ iff $\mathfrak{f}_m(X)[\underline{b}] \subseteq \mathfrak{f}_m(Y)[\underline{b}]$ for all $b = (b_0, \ldots, b_{m-1}) \in \langle A \rangle^m$ iff $\mathfrak{f}_m(X)[\underline{b}] \subseteq \mathfrak{f}_m(Y)[\underline{b}]$ for one $b \in \langle A \rangle^m$. But then also $m \Vdash \underset{\sim}{X} = \mathfrak{f}(\underset{\sim}{X}[\underline{a}])$ and $m \Vdash \mathfrak{f}_m(X) \subseteq \mathfrak{f}_m(Y)$ as desired. ∎

**Corollary 4.3.7.** *The category of sets and maps in the Basic Fraenkel Model* $\mathsf{M}(A)$ *with support index* $\leq m$ *is equivalent to the category* $\mathrm{Sh}_{\neg\neg}(\mathcal{N}_m^{\mathrm{op}}/m)$ *and equivalent to the category of sets with a continuous $G$-action* $\mathrm{Cont}(G)$, *where* $G = \mathrm{Aut}(\mathbb{N})$ *is group of permutations on* $\mathbb{N}$ *(which are fixed on* $\{0,1,\ldots,m\text{-}1\}$*) equipped with the product topology.*

**Proof.** The first equivalence follows from a combination of 4.3.3 and 4.3.6. A description of the relation to the category of sets with a continuous $\mathrm{Aut}(\mathbb{N})$-action can be found e.g. in [BS89] or [Bru96]. With that background, it is also clear that a result similar to Theorem 4.3.6 is also possible for permutation models other than the Basic Frankel model. For that aim, the category of transitive sets takes over the role of the category $\mathcal{N}_m^{\mathrm{op}}$. See [MLM94, III.9] or [BS89, 3C1] for a description of this category. ∎

# Chapter 5

# Prebounds and Partial Orders in $\mathcal{S}[\mathbb{D}_\infty]$

## 5.1 Describing and Comparing Prebounds

The absence of the axiom of choice in general topos setting requires a more discriminating approach when comparing the size of objects. When there is an inclusion $A \hookrightarrow B$, for two objects $A$ and $B$ of a category, there is no coercion that there is also an epi or extremal epi morphism $B \twoheadrightarrow A$ anymore. Therefore, in the order relation below, these two approaches to compare size or cardinality are combined to a relation we will call subquotient relation. Moreover, cardinality in this sense will not induce a linear order in general. The guiding analogon will be rather be the order on a set of functions induced by an order on their codomain.

### Subquotient relation and properties

Suppose $j$ is a local operator which corresponds to a notion of a coverage on $[\mathcal{C}^\circ, \mathcal{S}]$(c.f. A.2.6). A morphism $f$ is called *$j$-dense* if $\mathsf{a}f$ is extremal epi in $\mathrm{Sh}(\mathcal{C})$, that is if its image is a $j$-dense subobject. In other words $f$ is dense if whenever it factors through a subobject, this subobject is $j$-dense (but not necessarily an isomorphism like in the definition of extremal epiness).

**Definition 5.1.1.** *Let $A$ and $B$ be two objects in a category $\mathcal{E}$.*
*$A$ is a* subquotient *of $B$, abbreviated $A \preccurlyeq B$, if there is a total one-to-one relation*

*$R : A \rightsquigarrow B$, i.e. if there is a span*

$$\begin{tikzcd} & R \arrow[dl, two heads, "e"'] \arrow[dr, hook, "m"] & \\ A & & B. \end{tikzcd}$$

*$R$ is called a* witness *for $A \preccurlyeq B$.*

*Suppose $\mathcal{E} = [\mathcal{C}^\circ, \mathcal{S}]$ where $\mathcal{C}$ is a category equipped with a notion of a coverage. $A$ is* dense in a subquotient *of $B$, abbreviated $A \preccurlyeq B$, if there is a dense inclusion into a subquotient of $B$, or, put differently, if the morphism $e$ in the span above is only required to be a dense morphism rather than an extremal epi morphism.*

In our setting, epiness and denseness of morphisms will be always stable under pullback. With that provision, both relations on the set of objects of $\mathcal{E}$ are transitive by composition of spans (which is the same as composition of relations in this case, of course).

## Bounds, Generators, Prebounds

**Definition 5.1.2.** *Let $\mathcal{E}$ be a topos over $\mathcal{S}$ (via $\gamma$).*
*An object $B$ is called* bound *for $\mathcal{E}$ (over $\mathcal{S}$) if every object $A$ of $\mathcal{E}$ is a subquotient of $\gamma^*(I) \times B$.*
*In a more loose notation, we also write $A \preccurlyeq \alpha \cdot B$, i.e. every object has to be bounded by $B$ with respect to subquotient order up to a constant set.*
*An object $B$ is called* prebound *(or* progenitor*) for $\mathcal{E}$ (over $\mathcal{S}$) if $\sum_{n\in\mathbb{N}} B^n$ is a bound, that is if there exists a constant set $\alpha$ such that*

$$A \preccurlyeq \alpha \cdot \sum_{n\in\mathbb{N}} B^n.$$

The introduction of generalised elements comes as a rather natural generalisation when considering elements on small domains such as 2 (a generalised element is then a pair), or $R \hookrightarrow 1$ (one can think of an element relation with truth value $R$). In contrast it is more remote to pretend that the identity $id_X : X \to X$ fits into an intuitive conception of the notion of an element of $X$. Also, in general there are quite many domains, it would be much more desirable to work with a *set* of domains only. Those sets of domains on which generalised elements can still identify the properties of the category are called generating sets, and they are fairly related to prebounds.

**Definition 5.1.3.** *Let $\mathcal{C}$ be a category. A set of objects $(S_i)_{i\in I}$ is* generating *if properness of subobjects can be discerned by elements on a domain $S_i$ only. That is in that case, for any proper subobject $A$ there is an element $a \in_{S_i} X$ that is not in $A$. Positively formulated, for any $X$, the family of all morphisms from all objects $S_i$ to $X$ form an extremal epimorphic family.*

There are also weaker and stronger variants of this definition. For a separating set, the family of all morphisms to $X$ is required to be epic only. On the other hand, for a dense generating set $X$ has to be the limit of the objects $S_i$, more precisely the limit of $\mathcal{D}\downarrow X \to \mathcal{C}$ where $\mathcal{D}$ is the full subcategory of $\mathcal{C}$ induced by the set of objects $\{(S_i) : i \in I\}$. When $\mathcal{C} = \mathcal{E}$ is a topos, all these definitions coincide. In that case, $\mathcal{E} = \mathrm{Sh}_c(\mathcal{D})$. In fact, this definition is not restricted to a full subcategory, only. A subcategory $\mathcal{D}$ is called a *dense subcategory* of $\mathcal{C}$ if the limit of $\mathcal{D}\downarrow X \to \mathcal{C}$ is $X$ for all objects $X \in \mathcal{C}$. In a topos, again that is if $\mathcal{C} = \mathrm{Sh}_c(\mathcal{D})$, or equivalently if for any given map $h : X \to D_0$, properness of subobjects can be discerned by all elements $t \in_D X$ with $D$ and $h \circ t$ in $\mathcal{D}$. See also [Bor94, Vol.1, Ch.4] and [Joh02] for details and equivalent definitions.

**Lemma 5.1.4.** *Let $B$ an object in a topos $\mathcal{E}$ with has $\mathcal{S}$-indexed coproducts. Then:*
*$B$ is a prebound iff the set of all subobjects of all $B^n$ for $n\in\mathbb{N}$ is a generating set.*

*Even more, the subcategory $\mathcal{B}$ of $\mathcal{E}$ consisting of all subobjects $C \hookrightarrow B^n$ for some $n$ and those morphisms $\varphi$ which fit into a diagram of the form*

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\varphi"'] & B^n \arrow[d, "i^*"] & n \\ A \arrow[r, hook] & B^m & m \arrow[u, "i"] \end{tikzcd}$$

*is a dense subcategory. In consequence, $\mathcal{E} = \mathrm{Sh}_c(\mathcal{B})$.*

**Proof.** When the category $\mathcal{E}$ has $\mathcal{S}$-coproducts, one can sum up all those objects $(S_i)_{i\in I}$ that form the generating family to one object $R$. The family of morphisms into $A$ then gives rise to one single extremal epimorphism from $R$ to $A$. Note that it depends on $A$ whether one particular $S_i$ has to appear in the sum. But for sure, for any $A$, the object $R$ is a subobject of some $\alpha \times \sum_{i\in I} S_i$, thus $\sum_{i\in I} S_i$ is a bound. If all $S_i$ are contained in some $B^n$, then also $\sum_{i\in I} S_i \preccurlyeq \sum_n \alpha_n B^n$ for some $\alpha_n$, hence $B$ is a prebound.

Conversely, the set subobjects of finite powers of prebound always forms a generating set. To show that elements on the subobjects of $B^n$ can discern properness of a subobject of $A \preccurlyeq \sum B^n$ with witness $R$, it suffices to show this is true for elements of $R$ since $R \twoheadrightarrow A$ is epi. So for $U \subsetneq R \subseteq \sum B^n$, there is an inclusion $B^i \subseteq \sum B^n$ such that $U \cap B^i \subsetneq R \cap B^i \subseteq B^i$ where intersection is meant with respect to that inclusion. So $r : R \cap B^i \to R$ is an element which is in $R$ but not in $U$.

Finally we show that $\mathcal{B}$ is a generating subcategory of $C$. For any given map $h : X \to D_0 \hookrightarrow B^n$ in $\mathcal{E}$ and a proper subobject $R \hookrightarrow X$, then there is an element $t \in_D X$ which is not in $R$, i.e. the morphism $t \colon D \to X$ does not factor through $R$, and $D$ is a subobject of some $B^m$. We enlarge the domain $D$ to $D' = D \times \mathrm{im}(h \circ t) = \{(d, h \circ t(d)) | d \in D\} \hookrightarrow B^n {\times} B^m = B^{n+m}$. Then also $t' = t \circ \pi \notin^X_{D'} R$, and $h \circ t'$ is the inclusion of the projection of $D \times \mathrm{im}(h \circ t)$ to $\mathrm{im}(h \circ t)$ in $D$, thus a morphism in $\mathcal{B}$. ∎

This fact immediately implies that the topos of variable sets on a partial order is bounded by 1, as the representables $\mathrm{y}p$ are subobjects of 1 and they generate. And this is the only case in which 1 arises as a bound: take subobjects of 1 as locale $L$, then the category of sheaves on this locale is equivalent to a given topos that is bounded by 1.

In the following, we concentrate on general prebounds on the topos of variable sets and the topos of continuous sets with the final aim to obtain a more direct description.

**Lemma 5.1.5.** *Let $B$ be a variable set on a separated category $\mathcal{C}$.*

*Then:*

$$\begin{aligned}
\mathrm{a}B \text{ is a prebound in } \mathrm{Sh}(\mathcal{C}) \ &\Leftrightarrow\ \text{Every } A \in [\mathcal{C}^\circ, \mathcal{S}] \text{ is dense in a subquotient} \\
&\qquad \text{of } \sum_{C \in \mathcal{C}} \alpha_C \cdot B^{n_C} \text{for some } \alpha_C, \\
&\qquad \text{i.e. } A \preccurlyeq \sum_{C \in \mathcal{C}} \alpha_C \cdot B^{n_C} \text{ in } [\mathcal{C}^\circ, \mathcal{S}] \\
&\Leftrightarrow\ \text{Every } \mathrm{y}C \text{ is dense in a subquotient} \\
&\qquad \text{of some } B^{n_C}, \text{ i.e.} \\
&\qquad \forall C \in \mathcal{C}\ \exists n_C : \mathrm{y}C \preccurlyeq B^{n_C} \text{ in } [\mathcal{C}^\circ, \mathcal{S}]
\end{aligned}$$

**Proof.** First note that we only need to show the equivalence for the case that $A$ is a sheaf, because the unit $\eta_A : A' \to \mathsf{a}A'$ is monic for separated category $\mathcal{C}$ and thus $A' \preccurlyeq \mathsf{a}A' = A$. Now,

$$\sum_{C \in \mathcal{C}} \alpha_C \cdot \mathsf{a}B^{n_C} = \mathsf{a}\left(\sum_{C \in \mathcal{C}} \alpha_C \cdot B^{n_C}\right)$$

which we will abbreviate by $\mathsf{a}K$ for a moment. Given a span $(e, m) : R \hookrightarrow A \times \mathsf{a}K$, pull $m$ back along the unit $\eta_K$ to obtain another span $(e \circ \eta', m') : R' \hookrightarrow A \times K$. Finally, we check that $\mathsf{a}(e \circ \eta')$ is simply $e$, thus epi.

For the second statement, we use the fact that $A$ can be written as a colimit of representables $\mathsf{y}C$, thus there is an epi morphism from some sum $\sum yC$ to $A$, so $A \preccurlyeq \sum yC$ and also $A \preccurlyeq \sum yC$. The relation $\preccurlyeq$ (dense in a subquotient relation) is stable under summing up. ∎

**Theorem 5.1.6.** *Let $B$ be a variable set on a separated category $\mathcal{C}$.*

*Then:*

| | | |
|---|---|---|
| $\mathsf{a}B$ *is a prebound in* $\mathrm{Sh}(\mathcal{C})$ | $\Leftrightarrow$ | *For every $C \in \mathcal{C}$ there is a dense set $M$ of morphisms with common domain $C$ such that every map $f : D \to C \in M$ comes equipped with a number $n_f$ and an element $b_{(f)} \in_D B^{n_f}$, subject to the condition that* $b_{(f)} \cdot x = b_{(f)} \cdot y \Rightarrow f \circ x = f \circ y$ |

(Those elements $b_{(f)}$ need not be compatible, we do *not* claim that $b_{(f)} \circ g$ and $b_{(fg)}$ are equal).

**Proof.** Let $A_f$ be the image of the map $b_{(f)} : \mathsf{y}D \to B^{n_f}$, and $R$ be the sum of the family of all $(A_f)_{f \in M}$. It clearly is a subobject of some $\sum_{C \in \mathcal{C}} \alpha_C \cdot B^{n_C}$, it remains to show that there is a dense map to $\mathsf{y}C$. We define it component-wise as $A_f \to \mathsf{y}C$, $[id_D] \mapsto f$. This is indeed a map by the condition on the elements $b_{(f)}$, and put together, this map reaches every $f \in M$, hence it is dense.

Conversely, suppose $\mathsf{a}B$ is a prebound and $\varphi : R \to \mathsf{y}C$ is dense. For a generating set $M$ of the image of $\varphi$, choose, using the axiom of choice in $\mathcal{S}$, for every $f \in M$ an element $b_{(f)}$ that is mapped to $f$ by $\varphi$. By definition, these elements $b_{(f)}$ have to be in one $B^{n_f}$. By applying $\varphi$, we notice that desired condition on the elements

$b_{(f)}$ is fulfilled. ■

**Corollary 5.1.7.** *Let $B$ be a variable set on a category $\mathcal{C}$. Then:*

*$B$ is a prebound in $[\mathcal{C}^\circ, \mathcal{S}]$* $\Leftrightarrow$ *For every $C \in \mathcal{C}$ there is an element $b \in B^{n_C}$ for some $n_C$ which is faithful, that is*
*$b \cdot x = b \cdot y \Rightarrow x = y$.*
*In other words, there is an inclusion $\mathsf{y}C \hookrightarrow B^{n_C}$ induced by $b$.*

In the following, we reformulate the last theorem with the aim to finally obtain a representation of all small prebounds in terms of morphisms and relations on them only. Small is meant in the sense that we aim to confine ourselves to a set of bounds (which we call small bounds), such every other bound contains a small bound (by subobject inclusion).

Every element $b \in_D B^n$ corresponds to a morphism $\bar{b} : n \times \mathsf{y}D \to B$. The condition on an element $b_{(f)}$ occurring in the statement of 5.1.6 reformulates to

$$\bar{b}_{(f)}(n, x) = \bar{b}_{(f)}(n, y)\,\forall i < n_C \quad \Rightarrow \quad f \circ x = f \circ y$$

for $f : D \to C$ and $x, y \in_E \mathsf{y}D$. Next, while keeping track of this condition, we gather all these maps $\bar{b}_{(f)}$, i.e. we sum up their domains in order to obtain one single map. So for any variable set $B$ there has to be a map

$$b : \left( \sum_{\substack{f:D \to C \in M_C \\ C \in \mathcal{C}}} n_f \times \mathsf{y}D \right) \to B$$

We define those prebounds to be which arise as the image of such map to be *small prebounds.* As the map $b$ is then a (regular) epimorphism, we reap the following representation theorem:

**Theorem 5.1.8.** *Let $B$ be a variable set on a separated category $\mathcal{C}$. Then:*

$\mathsf{a}B$ *is a prebound in* $\mathrm{Sh}(\mathcal{C})$ $\Leftrightarrow$ *$B$ contains an object of the form*

$$\left( \sum_{\substack{f:D\to C\in M_C \\ C\in\mathcal{C}}} n_f \times \mathsf{y}D \right) \Big/ R$$

*where, for each $C$, $M_C$ is a given dense set of arrows with domain $C$ and $R$ is a relation that fulfils the condition that $(f,g,i)\ R\ (f,g',i)\ \forall i < n_f \Rightarrow f \circ g = f \circ g'$.*

*$B$ is a prebound in $[\mathcal{C}^\circ, \mathcal{S}]$* $\Leftrightarrow$ *$B$ contains an object of the form*

$$\left( \sum_{C\in\mathcal{C}} n_C \times \mathsf{y}C \right) \Big/ R$$

*where $R$ fulfils the condition that $(g,i)\ R\ (g',i)\ \forall i < n_C \Rightarrow g = g'$.*

In the following, for a number $n_C$, we write $D \xrightarrow{f,i} C$ for a map $f$ and a number $i < n_C$, the latter condition on $i$ thus will be considered as implicitly given whenever it is appropriate.

## Comparing Prebounds

We conclude this analysis on prebounds in the category of continuous and variable sets by giving a description of the subquotient relation $B_1 \preccurlyeq B_2$ for two prebounds $B_1$ and $B_2$.

**Lemma 5.1.9.** *Let $A$, $B$ be two objects in a topos $\mathcal{E}$, $R \subseteq A^2$, $S \subseteq B^2$ two relations on it.*

*Then:*

$A/R \preccurlyeq B/S$ *iff there is a relation $(\sigma,\tau) : N \hookrightarrow A \times B$ such that:*

- *$\sigma$ is epic and*
- *$\tau(m)\ S\ \tau(n) \Rightarrow \sigma(m)\ R\ \sigma(n)$.*

**Proof.** Given a subquotient relation $U$ between $A/R$ and $B/S$, we get $\sigma, \tau, N$, by pulling back along the canonical surjections $\kappa_R$ and $\kappa_S$, see figure. The fact

$$
\begin{tikzcd}
 & & N \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\text{pullb.}"] \arrow[ddll, bend right, two heads, "\sigma"'] \arrow[ddrr, bend left, "\tau"] & & \\
 & K \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\text{pullb.}"] & & L \arrow[dl, two heads] \arrow[dr, hook] \arrow[dd, phantom, "\text{pullb.}"] & \\
A \arrow[dr, two heads, "\kappa_R"'] & & U \arrow[dl, two heads] \arrow[dr, hook] & & B \arrow[dl, two heads, "\kappa_S"] \\
 & A/R & & B/S &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N \arrow[dl, bend right, two heads, "\sigma"'] \arrow[d, two heads, "f"] \arrow[dr, bend left, "\tau"] & \\
A \arrow[d, two heads, "\kappa_R"'] & I \arrow[dl, dotted, two heads] \arrow[dr, hook] & B \arrow[d, two heads, "\kappa_S"] \\
A/R & & B/S
\end{tikzcd}
$$

that $N$ indeed is a relation and that $\sigma(m)\ R\ \sigma(n)$ for any $n, m \in_T N$ with $\tau(m)\ S\ \tau(n)$ follows by a diagram chase. Conversely, let $f : N \to I$ in the right diagram be the image of $\kappa_S \circ \tau$. Then $f$ is the coequaliser of the kernel pair of $\kappa_S \circ \tau$, thus for the existence of the dotted map we only need to show that for $m, n \in N : \kappa_S \circ \tau(m) = \kappa_S \circ \tau(n) \Rightarrow \kappa_R \circ \sigma(m) = \kappa_R \circ \sigma(n)$. But that is exactly a reformulation of the second condition. It is trivial that the dotted arrow above is an epimorphism. ■

**Corollary 5.1.10.** *Let $\mathcal{C}$ be a separated category. Then*

$$\sum_{C \in \mathcal{C}} n_C \times \mathsf{y}C \Big/ R \quad \preccurlyeq \quad \sum_{E \in \mathcal{C}} k_E \times \mathsf{y}E \Big/ S$$

*iff*

*there is a relation* $\quad (\sigma, \tau) : \ N \hookrightarrow \left(\sum_{C \in \mathcal{C}} n_C \times \mathsf{y}C\right) \times \left(\sum_{E \in \mathcal{C}} k_E \times \mathsf{y}E\right)$

*such that:*

(1) *For each $(f, i)$ there is a $(h, j)$*

$$\begin{tikzcd}[row sep=small] & C \\ D \arrow[ur, "{f,i}"] \arrow[dr, dotted, "{h,j}"'] & \\ & E \end{tikzcd} \qquad \textit{with } (f,i)\ N\ (h,j)$$

(2) *For any $(f, i)\ N\ (h, j)$ and $(f', i')\ N\ (h', j')$ :*
$(h, j)\ S\ (h', j') \Rightarrow (f, i)\ R\ (f', i')$

## 5.2 Localic Morphisms and Partial Orders in the Topos $\mathcal{S}[\mathbb{D}_\infty]$

First in this section, some results are collected and established which relate the concept of a bound and a prebound to properties of geometric morphisms and classifying toposes.

A geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{F}$ is called *bounded by $B$* if $\mathcal{E}$ is bounded as a topos over $\mathcal{F}$, i.e. if for any $A \in \mathcal{E}$ there is a $I \in \mathcal{F}$ such that $A \preccurlyeq f^*(I) \times B$. And $f$ is *localic* if is bounded by 1. In the latter case, $\mathcal{E}$ is the topos of continuous sets which are constructed within $\mathcal{F}$.

A bound always has global support, that means that $B \longrightarrow 1$ is always epi. That is because every map from a subobject $R$ of $B$ to 1 trivially factors through the unique map $B \longrightarrow 1$, thus this map inherits (extremal) epiness from the family of all morphisms from all subobjects $R$ to 1. In internal language, global support means that the formula '$\exists x.x \in B$' is fulfilled, i.e. that $B$ is (internally) inhabited (which is classically the same as non-empty, naturally). In other words, every bound is a model of the theory of inhabited objects.

**Lemma 5.2.1.** *Suppose $B$ is a prebound on $\mathcal{F}$ over $\mathcal{S}$ and $f : \mathcal{E} \longrightarrow \mathcal{F}$ is a geometric morphism. Then:*

$$f \text{ is localic} \Leftrightarrow f^*(B) \text{ is a prebound.}$$

*Suppose $B$ is an object, $B' = \sum_{\mathbb{N}} B^n$ and $f_B : \mathcal{E} \longrightarrow \mathcal{S}[\mathbb{O}]$ and $f'_{B'} : \mathcal{E} \longrightarrow \mathcal{S}[\mathbb{O}_1]$ are the unique geometric morphisms such that ${f'}^*_{B'}(G) = B'$ and $f^*_B(G) = B$. Then:*

$$\begin{aligned} & B' \text{ is a bound of } \mathcal{F} \text{ over } \mathcal{S} \\ \Leftrightarrow\ & B \text{ is a prebound of } \mathcal{F} \text{ over } \mathcal{S} \\ \Leftrightarrow\ & f_B : \mathcal{E} \longrightarrow \mathcal{S}[\mathbb{O}] \text{ is localic} \\ \Rightarrow\ & f'_{B'} : \mathcal{E} \longrightarrow \mathcal{S}[\mathbb{O}_1] \text{ is localic} \end{aligned}$$

*Moreover, if $B$ is a decidable (and infinite) prebound then there is a localic geometric morphism $g_B : \mathcal{E} \longrightarrow \mathcal{S}[\mathbb{D}]$ ($g'_B : \mathcal{E} \longrightarrow \mathcal{S}[\mathbb{D}_\infty]$) such that $g^*_B(G) = B$ (${g'}^*_B(G) = B$).*

**Proof.** For $A \in \mathcal{F}$ we know that $A \preccurlyeq \gamma^* I \times \sum B^n$. Thus

$$f^*(A) \preccurlyeq f^*\Big(\gamma^* I \times \Big(\sum B^n\Big)\Big) = f^*(\gamma^* I) \times f^*\Big(\sum B^n\Big) = \gamma'^* I \times \sum (f^*\mathcal{B})^n$$

Fix $X \in \mathcal{E}$, if the geometric morphism $f$ is localic then $X \preccurlyeq f^*(A)$ for some $A$ and hence also $X \preccurlyeq \gamma'^* I \times \sum (f^*\mathcal{B})^n$ so $f^*\mathcal{B}$ is a prebound. Conversely, note that $X \preccurlyeq \gamma'^* I \times \sum (f^*\mathcal{B})^n$ implies $X \preccurlyeq f^*(A)$ for $A = \gamma^* I \times (\sum B^n)$. The remaining equivalences then follow immediately as $G$ is a prebound in $\mathcal{S}[\mathbb{O}]$. ■

**Lemma 5.2.2.** *Suppose $\mathcal{B}'$ and $\mathcal{B}$ are two categories equipped with a notion of a coverage and $F : \mathcal{B}' \to \mathcal{B}$ is a full functor which is the identity on objects and moreover preserves covers. Then the functor*

$$\begin{aligned} f^*\colon \mathrm{Sh}(\mathcal{B}) &\to \mathrm{Sh}(\mathcal{B}') \\ X &\mapsto X \circ F \end{aligned}$$

*is full and faithful. Moreover, it is the inverse image part of a geometric morphism.*

**Proof.** Suppose $t$ is a morphism $t\colon X \to Y$ for two sheaves $X$ and $Y$ in $\mathrm{Sh}(\mathcal{B})$. Then $f^*(t)$ is given by the family $(t_{FC})_{C \in \mathcal{C}}$. But as $C = F(C)$ also $t_C = t_{FC}$ and $f^*$ is trivially faithful. $(s_C)_{C \in \mathcal{B}'}$ is the data for a morphism $s : X \to Y$ in $\mathrm{Sh}(\mathcal{B}')$ iff $s_C \circ (XF(f)) = YF(f) \circ s_D$ for all morphisms $f : D \to C$ in $\mathcal{B}$. As $F$ is full, also $s_C \circ (Xh) = Yh \circ s_D$ for all morphisms $h : D \to C$ in $\mathcal{B}'$, so $s$ is a morphism in $\mathrm{Sh}(\mathcal{B})$, too. Finally, $f^*$ trivially preserves colimits and finite limits, thus it is the inverse image part of a geometric morphism. ■

Suppose $f : \mathcal{E} \to \mathcal{F}$ is a localic geometric morphism. In 2.1.1, we remarked that for a category $\mathrm{Sh}_c(L)$, the locale $L$ can be obtained as $\mathrm{Sub}_{\mathrm{Sh}(L)}(1) = (1, \Omega) = \gamma_*(\Omega)$. Thus when considering $\mathcal{E}$ as a topos over $\mathcal{F}$ as justified in 2.3.1, then $\mathcal{E}$ is equal to the category of internal sheaves in $\mathcal{F}$ on the locale $L = f_*\Omega$ in $\mathcal{F}$.
If moreover $\mathcal{F}$ is a topos of continuous sets $\mathrm{Sh}(\mathcal{D})$, then

$$(f_*\Omega)(D) = (\mathsf{ay}D, f_*\Omega) = (f^*\mathsf{ay}D, \Omega)_{\mathcal{E}} = \mathrm{Sub}_{\mathcal{E}}(f^*(\mathsf{ay}D)).$$

This leads directly to the next Lemma:

**Lemma 5.2.3.** *Suppose $B$ is a prebound on $\mathcal{E}$. Then the $\mathcal{E}$ is the category of internal sheaves in $\mathcal{S}[\mathbb{O}]$ on the locale $L$ given by $n \mapsto \mathrm{Sub}_{\mathcal{E}}(B^n)$. If $\mathcal{E}$ is decidable* *(e.g. Boolean), then* $\mathcal{E}$ *is the category of internal sheaves in* $\mathcal{S}[\mathbb{D}]$ *on the locale* $L'$ *given by* $n \mapsto \mathrm{Sub}_{\mathcal{E}}(\langle B\rangle^n)$*. If moreover* $B$ *is infinite, then the latter locale also exhibits* $\mathcal{E}$ *as the category of internal sheaves in* $\mathcal{S}[\mathbb{D}_\infty]$.

**Proof.** Let $f : \mathcal{E} \to \mathcal{S}[\mathbb{O}]$ and $g : \mathcal{E} \to \mathcal{S}[\mathbb{D}]$ be the localic geometric morphisms induced by $B$. Then $f^*(\mathsf{y}n) = f^*(G^n) = (f^*G)^n = B^n$ holds. In a decidable category, $\langle G\rangle^n$ is determined as the subobject of $G^n$ such that $G^n = \langle G\rangle^n + \bigvee_{\binom{n}{2}} \cdot G^n$. But this description involves only finite products and sums. Hence it is preserved by any geometric functor between decidable toposes. As a consequence, $L(n) = (f_*\Omega)(n) = \mathrm{Sub}_{\mathcal{E}}(B^n)$ and $L'(n) = (g_*\Omega)(n) = \mathrm{Sub}_{\mathcal{E}}(\langle B\rangle^n)$. ■

Clearly, in any Grothendieck topos $\mathcal{E}$, if $B$ is a (pre)bound then also the object $\mathbb{N}\times B$, which is infinite. Therefore in principle, the above result suffices to describe forcing with a category of conditions in terms of a combination of a Basic Fraenkel Model and a forcing extension with the partial order $L'$. (Recall that in Corollary 4.3.7 it was shown that $\mathcal{S}[\mathbb{D}_\infty]$ represents the basic Fraenkel model.)

But working with $\mathbb{N} \times B$ instead of $B$ has the drawback that the locale $L$ will be considerably larger and more complicated. We will choose a different infinite extension of $B$ which will not lie within $\mathcal{E}$ itself anymore but in an extension $\mathcal{E}'$. The category $\mathcal{E}$ will be contained in $\mathcal{E}'$ as an exponential variety. The cumulative hierarchy and hence the interpretation of a ZF set theory as described in 2.2 is the same in $\mathcal{E}'$ and $\mathcal{E}$. In the following, we construct such a topos $\mathcal{E}'$ which contains an infinite prebound $B'$ such that $\mathrm{Sub}_{\mathcal{E}'}(\langle B'\rangle^n) = \mathrm{Sub}_{\mathcal{E}}(B^n)$. This might be seen as the smallest infinite extension $B'$ of $B$ which is feasible in that generality. Choosing $B'$ infinite here means to ensure that $\mathrm{Sub}_{\mathcal{E}'}(\langle B'\rangle^n)$ is nonempty for any $n \in \mathbb{N}$.

Recall from Lemma 5.1.4 the dense generating subcategory $\mathcal{B}$ which represents a topos $\mathcal{E}$ as $\mathcal{E} = \mathrm{Sh}_{\mathrm{c}}(\mathcal{B})$. The objects of $\mathcal{B}$ are all subobjects of $C \hookrightarrow B^n$ for some $n$ and the morphisms are maps $\varphi$ between them which fit into a diagram of the form

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\varphi"'] & B^n \arrow[d, "i^*"] & n \\ A \arrow[r, hook] & B^m & m \arrow[u, "i"] \end{tikzcd}$$

Next, we introduce a modification $\mathcal{B}'$ of $\mathcal{B}$ which will then define the topos $\mathcal{E}'$ as $\mathcal{E} = \mathrm{Sh}_{\mathrm{c}}(\mathcal{B}')$. Note that the identity is the only endomorphism on the object $B \hookrightarrow B^2$ in $\mathcal{B}$. But there are two morphisms which testify that $id_B$ is in $\mathcal{B}$. In the category $\mathcal{B}'$, the pairs $(id_B, id_{B^2})$ and $(id_B, \tau_{(01)})$ will give rise to different endomorphisms, actually.

**Lemma 5.2.4.** *Suppose $B$ is a prebound in $\mathcal{E}$. Let $\mathcal{B}'$ be the category with inclusions $C \hookrightarrow B^n$ as objects and monomorphisms $i : m \hookrightarrow n \ \in \mathcal{N}_m$ which admit a morphism $\varphi$ fitting into*

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\varphi"'] & B^n \arrow[d, "i^*"] & n \\ A \arrow[r, hook] & B^m & m. \arrow[u, hook, "i"] \end{tikzcd}$$

*as morphisms. Then $\mathcal{B}'$ is a regular category, all morphisms in $\mathcal{B}'$ are epi and those with $\varphi$ (extremal)epi are the extremal epi morphisms. Moreover, $\mathcal{E}$ is an exponential variety in $\mathcal{E}' = \mathrm{Sh}_c(\mathcal{B}')$ and $B'$ is a decidable, infinite prebound in $\mathcal{E}'$ with* $\mathrm{Sub}_{\mathcal{E}'}(\langle B' \rangle^n) = \mathrm{Sub}_{\mathcal{E}}(B^n)$.

**Proof.** Define in $\mathrm{Sh}_c(\mathcal{B})$ the object $B' = \mathsf{ay}(B \hookrightarrow B)$. The subobjects of $\mathsf{a}(\mathsf{y}(B \hookrightarrow B))^n$ correspond to s to $(B \hookrightarrow B, \cdots, B \hookrightarrow B)$ in which are closed (for the canonical coverage, see A.2.7). In the following we write $B$ for $B \hookrightarrow B$. Suppose a morphism of the form

$$\begin{tikzcd} & C \arrow[r, hook] & B^r \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & & & r & \\ B & B & \cdots & B & 1 \arrow[ur, hook] & 1 \arrow[u, hook] \ \cdots & 1 \arrow[ul, hook] \end{tikzcd}$$

is in a sieve which represents a subobject of $\langle B \rangle^n$. If $r < n$, then at least two of these morphisms to $B$ are equal and the corresponding subobject is not in $\langle B' \rangle^n$. If on the other hand, $r > n$, then the morphisms $1 \hookrightarrow n$ factor through $r$ by a map $l : n \hookrightarrow r$ and

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\varphi"'] & B^r \arrow[d, "l^*"] & r \\ C' \arrow[r, hook] & B^n & n. \arrow[u, hook, "l"] \end{tikzcd}$$

where $C' \hookrightarrow B^n$ is the image of $C \hookrightarrow B^r \to B^n$. The morphism $l^*$ from $C \hookrightarrow B^r$ to $C' \hookrightarrow B^n$ is an extremal and hence regular epimorphism. If $e \circ f_i$ is an element of a closed sieve, for $e$ regular epi, then also $f_i$. Therefore, we can assume that whenever a sieve is not empty, it contains also morphisms of the form $C \hookrightarrow B^n$ to $(B, \ldots, B)$. Moreover, there are no morphisms which factor through a $C \hookrightarrow B^r$ for smaller $r$. In short, the sieves corresponding to subobjects of $\langle B \rangle^n$ are generated by morphisms from $C \hookrightarrow B^n$ to $(B, \ldots, B)$, thus proving the equivalence to subobjects of $B^n$ in $\mathcal{E}$.

Lemma 5.2.2 implies that $\mathcal{E}$ can be embedded in $\mathcal{E}'$ as full subcategory. The inclusion maps $X$ to $\mathsf{ay}_{\mathcal{E}}X$. Next, we show that $E$ is a logical subtopos, i.e. that $P(\mathsf{ay}_{\mathcal{E}}A)$ is in $\mathcal{E}$ for a $A \in \mathcal{E}$ and that it is a power object also in $\mathcal{E}$. Consider objects $A \hookrightarrow B^r$ ($A$ for short) and $E \hookrightarrow B^s$ ($B$ for short). Then we need to show that

$$\begin{aligned} P_{\mathcal{E}'}(\mathsf{ay}_{\mathcal{E}}A)(E) &= \mathrm{Sub}_{\mathcal{E}'}(\mathsf{ay}_{\mathcal{E}}A \times \mathsf{ay}_{\mathcal{E}'}E) \\ &= c\text{- closed-}\mathrm{Sub}_{[\mathcal{B}'^{\circ},\mathcal{S}]}(\mathsf{y}_{\mathcal{E}}A \times \mathsf{y}_{\mathcal{E}'}E) = \mathrm{Sub}_{\mathcal{E}}(A \times E) \\ &\qquad\qquad\qquad\qquad\qquad\quad = \mathrm{Sub}_{\mathcal{E}}(A \times \mathsf{ay}_{\mathcal{E}}E) = P_{\mathcal{E}}(A)(E) \end{aligned}$$

Let $(g,i)$ be a pair which represents a morphism from $C \hookrightarrow B^m$ to $A \hookrightarrow B^r$ (in $\mathcal{B}$) and $(h,j)$ be a morphism from $C \hookrightarrow B^m$ to $E \hookrightarrow B^s$ in $\mathcal{B}'$. We may assume that $m \leq r+s$. Else we can reduce to the latter case just like in the proof of $\mathrm{Sub}_{\mathcal{E}'}(\langle B' \rangle^n) = \mathrm{Sub}_{\mathcal{E}}(B^n)$ above. Suppose $f = (g,h)$, $i_p$ and $j_p$ are the canonical injections from $r$ and $s$ in the coproduct $r+s$ and $e$ is the unique map induced $i$ and $j$; moreover $l$ is a map such that $e \circ l = id$ and $j_p = l \circ j$ (but $i_p \neq l \circ i$ in general) as shown in the diagram

$$\begin{tikzcd} & r+s \arrow[d, shift right, "e"'] & \\ & m \arrow[u, dotted, shift right, "l"'] & \\ r \arrow[uur, bend left, "i_p"] \arrow[ur, "i"'] & & s \arrow[uul, bend right, "j_p"'] \arrow[ul, "j"] \end{tikzcd}$$

Then in the following sequence of diagrams, the first one commutes in $\mathcal{E}$ by definition of the product. In consequence, $f$ is a monomorphism and the second one commutes because $l^* \circ e^* = id_{B^m}$. The third one is just a different picture for the same equation of maps.

$$\begin{tikzcd} A \times E \arrow[r, hook] & B^{r+s} \\ C \arrow[u, "f"] \arrow[r, hook] & B^m \arrow[u, "e^*"'] \end{tikzcd} \qquad \begin{tikzcd} A \times E \arrow[r, hook] & B^{r+s} \arrow[d, "l^*"] \\ C \arrow[u, "f"] \arrow[r, hook] & B^m \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, hook, "f"] \arrow[d, "id_C"'] & A \times E \arrow[r, hook] & B^{r+s} \arrow[d, "l^*"] \\ C \arrow[rr, hook] & & B^m \end{tikzcd}$$

The latter diagram also represents a regular epic morphism in $\mathcal{B}'$. By construction, $(g,i) \circ (id_C, l)$ represents the same morphisms as $(g, i_p)$ in $\mathcal{B}$ and $(h,j) \circ (id_C, l)$ represents the same morphisms as $(h, j_p)$ in $\mathcal{B}'$ because $j_p = l \circ j$. As the sieve representing a subobject has to be closed for the canonical coverage, ((g,i),(h,j)) is in the sieve iff the span $((g, i_p), (g, j_p)$ with common domain $C \hookrightarrow A \times E \hookrightarrow B^{r+s}$ is. Thus subobjects of $\mathsf{ay}_{\mathcal{E}}A \times \mathsf{ay}_{\mathcal{E}'}E$ in $\mathcal{E}'$ correspond to subobjects of $A \times E$ in $\mathcal{E}$, which proves that $P_{\mathcal{E}'}(\mathsf{ay}_{\mathcal{E}}A)(E) = P_{\mathcal{E}}(A)(E)$.

The object $B' = \mathsf{ay}(B \hookrightarrow B)$ is decidable just because all morphisms are epi in $\mathcal{B}'$. It is infinite iff $\{0{=}1\} \neq \mathrm{Sub}_{\mathcal{E}'}((\langle B' \rangle^n) = \mathrm{Sub}_{\mathcal{E}}(B^n)$, which is true for any nontrivial topos $\mathcal{E}$. Finally, $B'$ is a prebound because its subobjects generate, in fact even a smaller subcategory, namely $\mathcal{B}'$ generates. ∎

**Corollary 5.2.5.** *Suppose $B$ is a prebound on $\mathcal{E}$. Then $\mathcal{E}$ is equivalent to an exponential variety of the category of internal sheaves in $\mathcal{S}[\mathbb{D}_\infty]$ on the locale $L$ given by $n \mapsto \mathrm{Sub}_{\mathcal{E}}(B^n)$. If $B$ is infinite and decidable, $\mathcal{E}$ is as well equivalent to the category of internal sheaves in $\mathcal{S}[\mathbb{D}_\infty]$ on the locale $L'$ given by $n \mapsto \mathrm{Sub}_{\mathcal{E}}(\langle B \rangle^n)$.*

## Towards a Dense Subset of $L$

Let $B$ be a prebound in a boolean Grothendieck topos $\mathcal{E} = \mathrm{Sh}_{\neg\neg}(\mathcal{C})$. Applying the theory on prebounds and locales established above, the next step is to identify dense subsets of the locales. The partial orders $P$ in $\mathcal{S}[\mathbb{D}_\infty]$ arising that way have a considerably neater description than the locales $L$ and $L'$ itself, and still $E$ is equivalent to the category of internal sheaves on $P$.

But first, observe that there is no need to examine $L$ and $L'$ separately. As well, by Lemma A.2.7, we consider $\neg\neg$-closed subobjects in $[\mathcal{C}^\circ, \mathcal{S}]$ for they are in one-to-one correspondence with subobjects of $\mathsf{a}X$ in $\mathrm{Sh}_{\neg\neg}(\mathcal{C})$. Therefore:

**Fact 5.2.6.**

$$\begin{aligned}
\mathrm{Sub}_{\mathrm{Sh}_{\neg\neg}(\mathcal{C})}\left(\langle \mathsf{a}B \rangle^n\right) &= \{R \in \mathrm{Sub}_{\mathrm{Sh}_{\neg\neg}(\mathcal{C})}\left(\mathsf{a}(B^n)\right) : \ \lambda \textit{ not mono } \Rightarrow R \cdot \lambda = 0\} \\
&= \big\{R \in \mathrm{Sub}_{[\mathcal{C}^\circ, \mathcal{S}]}\, B^n : \neg\neg R = R \textit{ and} \\
&\qquad\qquad \varphi \textit{ not mono } \Rightarrow R \cdot \varphi = 0\big\} = L'(n) \\
\mathrm{Sub}_{\mathcal{E}'}\left(\langle B' \rangle^n\right) &= \mathrm{Sub}_{\mathrm{Sh}_{\neg\neg}(\mathcal{C})}\left((\mathsf{a}B)^n\right) = \mathrm{Sub}_{\mathrm{Sh}_{\neg\neg}(\mathcal{C})}\left(\mathsf{a}(B^n)\right) \\
&= \big\{R \in \mathrm{Sub}_{[\mathcal{C}^\circ, \mathcal{S}]}\, B^n : \neg\neg R = R\big\} = L(n).
\end{aligned}$$

It thus suffices to calculate the locale $K \in \mathcal{S}[\mathbb{O}]$ which is defined by $K(n) = \mathrm{Sub}_{[\mathcal{C}^\circ, \mathcal{S}]}\, B^n$. The prebound $B$ is infinite if for any $n$ there is an element $R \in K(n)$ other than 0 such that $R \cdot \lambda = 0$ for any $\lambda$ which is not a monomorphism. Thus it is possible to work with the smaller partial order $L'$ iff it is not trivial.

For maintaining clarity, we first do not unravel the whole expression of Theorem

5.1.8 when calculating the locale $K$. We rather start with

$$B = \left(\sum_{k\in K} T_k\right) \Big/ R$$

where $(T_k)_{k\in I}$ is a family of objects in a complete topos $\mathcal{E}$. Then:

$$K(n) = \mathrm{Sub}(B^n) = \mathrm{Sub}\left(\sum_{k\in I} T_k \big/ R\right)^n = \mathrm{Sub}\left(\left(\sum_{\substack{|\underline{k}|=n\\ \underline{k}\in I^n}} T_{k_0} \times \cdots \times T_{k_{n-1}}\right) \Big/ R^n\right)$$

Note that for every any quotient map $A \twoheadrightarrow A/R$ there is a retraction in the category of partial orders as in the diagram

$$\mathrm{Sub}(A/R) \underset{e^*}{\overset{\mathrm{im}\, e}{\rightleftarrows}} \mathrm{Sub}(A)\,.$$

To express a partial order of the form $\mathrm{Sub}(A/R)$ in terms of subobjects of $A$ we could either take only those subobjects that arise as $e^*(M)$ or, more appropriate for our situation, define a congruence $U \sim V$ on the partial order with $U \lesssim V$ iff $e^*U \lesssim e^*(V)$ iff $U \lesssim \mathrm{im}(e) \circ e^*V$. Written in the language of (generalised) elements, this means that $U \lesssim V$ iff for any $u \in U$ there is a $v \in V$ such that $u\ R\ v$. It is evident that if a subset is dense in $\mathrm{Sub}(A)$, its image is also dense in $\mathrm{Sub}(A/R)$.
Continuing the description of $K$, this implies that

$$K(n) = \left(\mathrm{Sub} \sum_{\substack{|\underline{k}|=n\\ \underline{k}\in I^n}} (T_{k_0} \times \cdots \times T_{k_{n-1}})\right) \Big/ \sim_R$$

Suppose for a moment that $L$ is an internal locale in a category $[\mathcal{C}, \mathcal{S}]$. Then for $h : C \to D$, the covariant action map $L(h) : L(C) \to L(D)$ is a frame homomorphism, it thus has a left adjoint $L_o(h) : L(D) \to L(C)$. So any internal locale $L$ in $[\mathcal{C}, \mathcal{S}]$ is in one-to-one correspondence with a locale $L_o$ in $[\mathcal{C}^\circ, \mathcal{S}]$, or, put differently, there is also a contravariant action on $L$. If $P \subseteq L_o$, then $Q(D) = \{L(h)(l) : l \in P(C)\}$ is a subobject of $L$.

Coming back to the particular context $L = K$ and $\mathcal{C} = \mathcal{N}_m$, note that for an

inclusion like $1 \hookrightarrow 2$, the covariant action multiplies with $B$ as in $G \mapsto G \times B$ for $G \subseteq B$ whereas the contravariant action is the projection $B^2 \mapsto B$.
When translating to names or to sets in the Basic Fraenkel Model, the covariant action of the canonical inclusion $\iota^n_m \colon m \hookrightarrow n$ is only implicitly perceivable anymore: It is just considering a set with support index $\leq m$ as a set with support index $\leq n$. The order of $L$ can be described completely within $L_0$. For $i \colon m \hookrightarrow n$, $G \in K(m)$ and $H \in K(n)$, clearly $G \leq L(i)(H) \Leftrightarrow L_o(i)(G) \leq H$, that is just the adjointness property. But also

$$L(i)(G) \leq H \quad \text{iff} \quad \text{for all } H' \in K(n) : \; (L_0(H') \leq G \Rightarrow H' \leq H). \tag{5.1}$$

Next, we identify a dense subset of this locale $K_o$. Clearly, every inhabited subobject of $\sum(T_{k_0} \times \cdots \times T_{k_{n-1}})$ has to include at least one inhabited subobject of one factor $(T_{k_0} \times \cdots \times T_{k_{n-1}})$. Collecting these subobjects yields a dense subset of $K_o$:

$$P(n) = \left( \sum_{\substack{|\underline{k}|=n \\ \underline{k} \in I^n}} Sub(T_{k_0} \times \cdots \times T_{k_{n-1}}) \right) \Big/ \sim_R$$

For abbreviation, let $Q(n)$ be $\sum_{\substack{|\underline{k}|=n \\ \underline{k} \in I^n}} \mathrm{Sub}(T_{k_0} \times \cdots \times T_{k_{n-1}})$. It is time to specify the objects $(T_k)_{k \in I}$. Recall from Theorem 5.1.8 that we examine prebounds $B$ of the form

$$\left( \sum_{C \in \mathcal{C}} n_C \times \mathrm{y}C \right) \Big/ R$$

where $R$ fulfils the condition that $(g, i) \; R \; (g', i) \; \forall i < n_C \Rightarrow g = g'$.

Therefore, we set $I = \{(C, i) : i < n_C, \; C \in \mathcal{C}\}$ and $T_k = \mathrm{y}C$ for $k = (C, i)$. An element of $Q(n)$ is a subobject $S$ of some $(\mathrm{y}C_0 \times \cdots \times \mathrm{y}C_{n-1})$, i.e. sieves on $(C_0, \ldots, C_{n-1})$. If such a subobject $S$ is not empty, then it contains at least one $n$-tuple of maps $(g_j)_{j=0..n-1}$ where $f_j : D \to C_j$. The sieve generated by $(g_j)_{j=0..n-1}$ is then smaller than $S$ in $Q(n)$. Let $Q'(n)$ be the set of all sieves generated by single $n$-tuples of maps. Then $Q'(n)$ is a dense subset of $Q(n)$. In fact, $Q'$ is a subobject of $Q$, as for $\iota : m \to n$ the projection of a sieve generated by $(g_j)_{j=0..n-1}$ is just the sieve generated by $(g_{\iota(r)})_{r=0..m-1}$. We may assume that the elements of $Q'(n)$ are tuples of maps $(g_j)_{j=0..n-1}$ with a common domain while the order in $Q'(n)$ is given by $(g_j)_{j=0..n-1} \leq (h_j)_{j=0..n-1}$ iff there is a morphism $l$ such that $g_j = l \circ h_j$ for all $j$.

**Lemma 5.2.7.** *Suppose $R$ is given Theorem (5.1.8). The locale $L_o$ then contains a dense subset $P'$ whose $n$-elements are equivalence classes of $n$-tuples $(g_j, i_j)_{j=0..n-1}$ where $g_j : C \to D_j$, and $[(g_j, i_j)_{j=0..n-1}]_R \leq [(g'_j, i'_j)_{j=0..n-1})]_R$ iff there is a morphism $l$ such that for all $j$ the relation $(g_j, i_j)\ R\ (l \circ g'_j, i'_j)$ holds.*

**Theorem 5.2.8.** *Suppose $C$ is a separated category, $n_C$ is a natural number which is attached to each object of $C$. Moreover suppose $R$ is a relation on pairs $(g, i)$ with $g : D \to C$ and $i < n_C$, which satisfies the conditions*

- *If $(g, i)\ R\ (h, j)$ then $g$ and $h$ have a common domain $D$*
- *If $(g, i)\ R\ (h, j)$ then also $(g \circ f, i)\ R\ (h \circ f, j)$*
- *If $(g, i)\ R\ (g', i)\ \forall i < n_C$ then $g = g'$.*

*Let $P$ be the set consisting of all elements of the form*

$$(F, D, (C_a)_{a\in F}, (g_a)_{a\in F}, (i_a)_{a\in F})$$

*where $F$ is a finite set of atoms, $D$ and $C_a$ are objects of $C$, $g_a : D \to C_a$ morphisms of $C$ and $i_a < n_{D_a}$,*
*equipped with an order structure which is defined in a two step procedure:*

$$p = (F, D, (C_a)_{a\in F}, (g_a)_{a\in F}, (i_a)_{a\in F}) \leq_1 (F', D', (C'_a)_{a\in F'}, (g'_a)_{a\in F'}, (i'_a)_{a\in F'}) = p'$$

*if $F' \subseteq F$ and there exists an $l$: $D \to D'$ such that for all $a \in F'$ the relation $(g_a, i_a)\ R\ (l \circ g'_a, i'_a)$ holds.*

$$p = (F, D, (C_a)_{a\in F}, (g_a)_{a\in F}, (i_a)_{a\in F}) \leq (F', D', (C'_a)_{a\in F'}, (g'_a)_{a\in F'}, (i'_a)_{a\in F'}) = p'$$

*if either $F' \subseteq F$ and $p \leq_1 p'$, or $F' \supsetneq F$ and $p'' \leq_1 p$ follows $p'' \leq_1 p'$ for any other $p'' = ((F', D'', (C''_a)_{a\in F'}, (g''_a)_{a\in F'}, (i''_a)_{a\in F'})$. Then:*

*The interpretation of ZF set theory in $\mathrm{Sh}_{\neg\neg}(C)$ is equivalent to the model of pure sets of a generic extension of the Basic Fraenkel Model with the partial order $P$.*

*If $R$ has infinitely many equivalence classes, we might equivalently choose the partial order $P_d \subseteq P$ consisting of those $(F, D, (C_a)_{a\in F}, (g_a)_{a\in F}, (i_a)_{a\in F})$ in $P$ such that all $(g_a, i_a)_{a\in F}$ are in different equivalence classes.*

**Proof.** The proof is just a combination of the material established above. We apply the correspondence between $\mathrm{Sh}_{\neg\neg}(\mathcal{N}_m^{op})$ and the Basic Fraenkel Model, Corollary 5.2.5 and the definition of the order of $L$ in terms of the order of $L_o$ (equation 5.1). Finally, with Fact 5.2.6, we can choose $P_d$ if $P_d(n)$ is not trivial i.e. if there are infinitely many equivalence classes. ∎

# Chapter 6

# Smallness Axioms and Allegories

## 6.1 More on Allegories

This section introduces an additional operations on morphisms of an allegory $\mathcal{A}$ which will prove to be helpful to keep the text succinct and legible: Division of relations.

**Definition 6.1.1.** *Let $\mathcal{A}$ be an allegory and $R$ and $S$ two morphisms.*

*The* division[1] $\frac{R}{S}$ *between $R$ and $S$ is defined by $T \subseteq \frac{R}{S} \Leftrightarrow T \circ S \subseteq R$.*

$$\begin{tikzcd} & X \arrow[dl, squiggly, "\frac{R}{S}"'] \arrow[d, phantom, "\subseteq"] & \\ Y & & Z \arrow[ll, squiggly, "R"] \arrow[ul, squiggly, "S"'] \end{tikzcd}$$

Figure 6.1: Division $\frac{R}{S}$ - the diagram semicommutes.

*The* symmetric division *is defined by $\underset{S}{\overset{R}{=}} = \frac{R}{S} \cap \left(\frac{S}{R}\right)^\circ$, or alternatively by*

$$T \subseteq \underset{S}{\overset{R}{=}} \Leftrightarrow T \circ S \subseteq R \textit{ and } T^\circ \circ R \subseteq S$$

Evidently, such a division morphism $\frac{R}{S}$ is unique if it exists. The notion of a division can also be introduced for ordered categories as it only depends on

[1] The ordinary division is not symmetric – just as composition is not. Freyd and Scedrov [FS90] use $R/S$ and $R \setminus S$ for the right and left version. Our division corresponds their $R \setminus S$, but the equations look similar to $R/S$ simply because we have reversed arrow composition. Another notational variant is $R \div S$ as in [Kaw95].

the order on $\mathcal{A}$. In fact, every ordered category and hence every allegory is a 2-category (with no more than one 2-cell between each morphism or 1-cell). The division $\frac{R}{S}$ is then is the same as the *right Kan extension* of $R$ along $S$ in the 2-category $\mathcal{A}$, see also Definition A.2.11 in the appendix.

It might be illuminative to unravel the definition of the division of two relations $R$ and $S$ in the category $\mathcal{Set}$ of sets, in fact it is given by

$$\frac{R}{S} = \{(y, x) : \forall z : (x, z) \in S \Rightarrow (y, z) \in R\}$$

in that case. The symmetric division $\frac{\underline{R}}{S}$ of $R$ and $S$ is then explicitly given by

$$\begin{aligned}\frac{\underline{R}}{S} &= \{(y, x) : \forall z : (x, z) \in S \Leftrightarrow (y, z) \in R\} \\ &= \Big\{(y, x) : \{z : (x, z) \in S\} = \{z : (y, z) \in R\}\Big\}.\end{aligned}$$

For the following, we occasionally will make use of some rules for the division. These rules are summarised in the next two lemmas.

**Lemma 6.1.2.** *Let $R, S, E, T$ be relations in an allegory $\mathcal{A}$, $f$ be a map. Then the following rules can be derived:*

- $\frac{R}{S} \circ S \subseteq R$ *and* $T \subseteq \frac{TS}{S}$
- $\frac{R}{E} \circ \frac{E}{S} \subseteq \frac{R}{S}$ *and* $\frac{R}{TS} = \frac{\left(\frac{R}{S}\right)}{T}$
- $\frac{R}{f^\circ} = R \circ f$ *and* $\frac{f}{S} = \frac{\Delta}{S \circ f^\circ}$
- $\Delta \subseteq \frac{S}{E} \circ \frac{E}{S} \Rightarrow \frac{R}{E} \circ \frac{E}{S} = \frac{R}{S}$
- $\frac{R}{R} \circ \frac{R}{R} = \frac{R}{R}$
- *The division is monotone in the numerator and antimonotone in the denominator.*

**Proof.** Justifications are straight-forward for all items, like $\frac{R}{S} \subseteq \frac{R}{S} \circ \frac{S}{E} \circ \frac{E}{S} \subseteq \frac{R}{E} \circ \frac{E}{S} \subseteq \frac{R}{S}$ for the forth item. ∎

**Lemma 6.1.3.** *Suppose $R$ and $S$ are two relations in an allegory $\mathcal{A}$ that allow factorisations $R = f^\circ \circ E$ and $S = g^\circ \circ E$. Suppose furthermore that the division $\frac{E}{E}$ exists. Then*

$$\frac{R}{S} = f^\circ \circ \frac{E}{E} \circ g.$$

*Moreover, in that case, $\frac{E}{R} \subseteq \frac{E}{S}$ iff $S \subseteq R$.*

**Proof.** We apply Lemma 3.1.2 repeatedly. $T \subseteq f^\circ \circ \frac{E}{E} \circ g \Leftrightarrow f \circ T \subseteq \frac{E}{E} \circ g \Leftrightarrow f \circ T \circ g^\circ \subseteq \frac{E}{E} \Leftrightarrow f \circ TS = f \circ T \circ g^\circ \circ E \subseteq E \Leftrightarrow TS \subseteq f^\circ \circ E = R$.
For any $T$ we know that $T \subseteq \frac{E}{R} \Rightarrow T \subseteq \frac{E}{S}$. If we set $T = f$ we get $f \subseteq \frac{E}{S}$ and hence $S \subseteq f^\circ E = R$. ∎

One might be tempted to define the union of two relations $R$ and $S$ to be the supremum of $R$ and $S$ with respect to the order $\subseteq$ straight-away, but mere supremum is not necessarily preserved by composition, this condition has to be included in the definition.

**Definition 6.1.4.** *Let $R$ and $S$ be two morphisms in an allegory $\mathcal{A}$. The* union *of $R$ and $S$, if it exists, is the unique relation that fulfils*

$$R \cup S \subseteq T \Leftrightarrow R \subseteq T \text{ and } S \subseteq T \text{ as well as}$$

$$(R \cup S)T = RT \cup ST.$$

*Likewise, the* zero relation $0_{BA}$ *is the smallest element in the semilattices on the Homset $(A, B)_{\mathcal{A}}$ that satisfy $R \circ 0_{BA} = 0_{CA}$. A* distributive allegory *is an allegory equipped with a union operation $\cup$ and a zero operation $0_{BA}$.*

With presence of a zero relation, disjointness of maps can be expressed by the condition $u_2{}^\circ \circ u_1 = 0$. Therefore, for any two *objects* $X$ and $Y$, the *disjoint union* $(X + Y, u_1, u_2)$ can be introduced, $u_1, u_2$ have to be disjoint, monic and jointly epi maps in this case. Such a disjoint union is a coproduct in $\mathcal{A}$ as well as in Map($\mathcal{A}$). Together with the morphisms $P_1{}^\circ$ and $P_2{}^\circ$ as projections it is also the product of $X$ and $Y$, the product in $\mathcal{A}$, only, of course.

## 6.2 Allegory Axioms for Algebraic Set Theory

In this section, the aim is to single out a class (subcategory) of relations $\mathcal{S}$ from an allegory $\mathcal{A}$, just as in Gödel-Bernays set theory a class of sets is singled out from the entity of classes.

Intuitively, in our approach to Algebraic Set Theory, a relation $R : X \to I$ should be considered as a collection $(X_i)_{i\in I} = (\{x : iRx\})_{i\in I}$ of subsets of $X$, not necessarily disjoint nor non-empty nor covering $X$. Whenever we will feature axioms or statements in the following, an informal reformulation in terms of collections will be included, enclosed in brackets $\{\}$

Those axioms will postulate that collections of components with at most one element ought to be small, the collection of small relations should be closed with respect to composition and subset, every small relation can be internally represented and there is also a formulation of a powerset axiom.

So in short, we propose in this chapter a set of axioms for *families of sets indexed by classes*. This should be put in contrast to prior approaches to Algebraic Set Theory by axiomising either *disjoint families of sets indexed by classes* ([JM95], [Sim99], [ABSS03], etc.) or *families of subsets of a given class $X$ indexed by classes* ([But03]).

As presented in section 2.3 the canonical indexed category $\mathbb{C}_{\mathcal{C}}$ over a category $\mathcal{C}$ describes, in some sense, the category $\mathcal{C}$ itself from the viewpoint of $\mathcal{C}$. But the component categories $\mathcal{C}^I = \mathcal{C}/I$ of $\mathbb{C}_{\mathcal{C}}$ model disjoint families of sets. This fact clashes with our premise to include potentially intersecting sets in some sense. Working with a different indexed category over $\mathcal{C}$, arbitrary families of sets can be integrated in the setting of indexed categories.

In the relational indexed category $\mathbb{R}_{\mathcal{C}}$ over $\mathcal{C}$, the component $\mathcal{R}^I$ is no longer the conventional slice category $\mathcal{C}/I$, but formulation of slice category of relations. The objects of $\mathcal{R}^I$ are relations $R : X \rightsquigarrow I$ for some $X$, a morphism between $R : X \rightsquigarrow I$ and $S : Y \rightsquigarrow I$ is a relation $T : X \rightsquigarrow Y$ satisfying $S \circ T \subseteq R$.

$$\begin{tikzcd}
X \arrow[rr, rightsquigarrow, "T"] \arrow[dr, rightsquigarrow, "R"'] & \subseteq & Y \arrow[dl, rightsquigarrow, "S"] \\
& I &
\end{tikzcd}$$

If all involved morphisms are maps, then the diagram commutes, thus retaining the slice category $\mathcal{C}/I$ as a subcategory.

The functor $u^*$ is defined by $u^*(R) = u^\circ R$ and $u^*(T) = T$ for morphisms, which is indeed a morphism from $u^*(R)$ to $u^*(S)$ as $u^\circ S \circ T \subseteq u^\circ R$. The relational indexed category $\mathbb{R}$ is no indexed subcategory of $\mathbb{C}$ (unless in trivial cases), because $u^*(f) = u^\circ f$ need not be a map again. Yet in the other direction, there is an indexed functor $\mathbb{R} \to \mathbb{C}$, the disjointification functor, mapping each relation $(l, k) : R \subseteq I \times X$ to the map $l$.

The construction of a relational indexed category works also for allegories, i.e. with the same definition as above, we see that for every allegory $\mathcal{A}$, there is an indexed category $\mathbb{R}_{\mathcal{A}}$ over the category $\mathrm{Map}(\mathcal{A})$.

From a meta-level perspective, there are the following demands on a structure of smallness:

- $\mathcal{S}$ should be an indexed subcategory of $\mathbb{R}_A$
- Suppose that $\mathcal{A}$ is an allegory coming from a regular category $\mathcal{C}$. The structure of smallness should be preserved when applying the disjointification functor $\mathbb{R}_\mathcal{C} \to \mathbb{C}_\mathcal{C}$. Conversely, given a structure of smallness in $\mathbb{C}_\mathcal{C}$, the preimage should give a small structure in $\mathbb{R}_\mathcal{C}$

In the latter condition, with structure of smallness in a category $\mathcal{C}$, a subcategory subject to the axioms of [Sim99] is meant. In Theorem 6.2.7 the comparison to the mainstream approach of axiomising disjoint families of sets indexed by classes is established, taking into account the considerations on indexed categories from above.

It is not overbold to argue that the axioms in this chapter presents itself in a more tidy form than prevalent sets of axioms for Algebraic Set Theory. In fact, especially the Representability and Power Axiom have an inherent relation-like character which is perceptible in other treatises, too. Some statements that else would require a different treatment are included in a way within the Union axiom. Moreover, the following section is a witness that these axioms constitute a successful solution to the challenge to take arbitrary sets that potentially intersect as a basis for a formulation of Algebraic Set Theory. From a set theory viewpoint, renouncing the additional requirement to have disjoint families appears much more natural.

Opponents would allege that this comes with the cost that these axioms are formulated in an allegory setting. As such, the theory is weaker, only within a regular allegory imposing the condition that small relations come from small maps, it gets equivalent to prevalent axiomisations of small maps in a regular category. Although it comes not most direct, an integration into an indexed category setting is well feasible as show above.

Furthering Algebraic Set Theory, a next step is to study universes, that are objects $U$ together with a bijection $U \to \mathsf{P}(U)$, so that there is a global element relation on $U$. This chapter ends with a brief examination on universes in our relational approach to Algebraic Set Theory.

## The Axioms

**Definition 6.2.1.** *Let $\mathcal{A}$ be an allegory. A class $\mathcal{S}$ of relations is called class of* small relations *if it fulfils the following axioms A1-A5:*

- **A1 Subsingleton** *One-to-one relations are small.*
- **A2 Union** *Smallness is closed with respect to composition:*
  $\{$*For small families* $(S_i)$ *and* $(R_y)$, $\bigcup_{y \in S_i} R_y$ *is small.*$\}$
- **A3 Separation** *Smallness is closed with respect to* $\subseteq$.
- **A4 Representability** *For every object $X$ there exists an object $\mathsf{P}_\mathcal{S}(X)$ and a small relation $\ni_X$: $X \rightarrow \mathsf{P}_\mathcal{S}(X)$ such that for every small relation $R$ there exactly one map $f$ allowing the factorisation $R = f^\circ \ni_X$.*

$$\begin{tikzcd}
X \arrow[rr, squiggly, "\ni"] \arrow[dr, "R"'] & & \mathsf{P}_\mathcal{S}(X) \arrow[dl, squiggly, "f^\circ"'] \\
& I \arrow[ur, bend right, "f"'] &
\end{tikzcd}$$

  $\{$*The family* $(B)_{B \in \mathsf{P}_\mathcal{S} X}$ *is small and for every small family* $(R_i)_{i \in I}$ *there is a morphism* $f : I \rightarrow \mathsf{P}_\mathcal{S}(X)$ *which represents the family, i.e.* $R_i = \{x : x \in f(i)\}\}$

- **A5 Power** *For every $X$ the relation $\frac{\ni_X}{\ni_X}$ exists and is small.*
  $\{(\mathsf{P}(B))_{B \in \mathsf{P}_\mathcal{S} X}$ *exists and is small*$\}$

We will drop the index from $\ni_X$ whenever it is unambiguous, $\in$ is an abbreviation for $(\ni)^\circ$. The relation $\frac{\ni_X}{\ni_X}$ corresponds to an internal superset relation $\supseteq_X$, but we will keep the notation $\frac{\ni}{\ni}$ to avoid mix-up with with (external) superset in the allegory $\mathcal{A}$. Recall from the definition of the division that $T \subseteq \frac{\ni}{\ni} \Leftrightarrow T \circ \ni \subseteq \ni$. Applying Lemma 6.1.3 to A4 and A5 tells us that the division $\frac{R}{S}$ of small relations $R$ and $S$ exists as soon as $\frac{\ni}{\ni}$ exists because $\frac{R}{S} = f_R{}^\circ \circ \frac{\ni}{\ni} \circ g_S$ for maps $f_R$ and $f_S$ representing $R$ and $S$ respectively. If $S$ is small, then $\frac{R}{\ni}$ is small, too.

The axioms A1-A2 imply the existence of some additional structure on the subcategory $\mathcal{S}$ of small relations. The map $X \mapsto \mathsf{P}_\mathcal{S}(X)$ extends to a contravariant functor on $\mathcal{S}$, for example. That is because if $R$ is small then there is a map $\mathsf{P}_\mathcal{S}(R)$ such that $\ni \circ R = \mathsf{P}_\mathcal{S}(R)^\circ \ni$. As $f^\circ$ is always small, $\mathsf{P}_\mathcal{S}$ also extends to a covariant functor on Map($\mathcal{A}$). We denote this *"image map"* by $f_!$, so that $\ni f^\circ = (f_!)^\circ \ni$.

For a small map $g : X \longrightarrow A$ the *"fibre map"* $g^{-1} : A \longrightarrow \mathsf{P}_S(X)$ is simply the unique representing map of $g$ as $g = g^{-1\circ} \ni$.

The *singleton* $\{\}_X$ or $\sigma_X$ is the representing map of the identity, *internal union* $\bigcup^{(X)} : \mathsf{P}_S\mathsf{P}_S(X) \longrightarrow \mathsf{P}_S(X)$ is defined as the representing map of $\ni\ni$ (or in greater detail $\ni_{\mathsf{P}_S(X)}\ni_X$).

**Lemma 6.2.2.** $(\mathsf{P}_S, \sigma, \bigcup)$ *is a monad in the category of maps* $\mathrm{Map}(\mathcal{A})$.
*This means that* $\sigma\colon Id \to \mathsf{P}_S$ *and* $\bigcup\colon \mathsf{P}_S\mathsf{P}_S \to \mathsf{P}_S$ *are natural transformations which fulfil both the unit law* $\bigcup^{(X)} \sigma_{\mathsf{P}_S X} = \Delta = \bigcup^{(X)}(\sigma_X)_!$ *and the associativity law* $\bigcup^{(X)}(\bigcup^{(X)}_!) = \bigcup^{(X)}(\bigcup^{(\mathsf{P}_S X)})$.

**Proof.** Let us expound only the proof of naturality of the singleton map, as the other statements are similar: Apply converse to $\sigma_Y f = f_!\sigma_X$. Because representables are unique, the equation follows from $f^\circ\sigma_Y{}^\circ \ni_Y = \sigma_X{}^\circ f_!{}^\circ \ni_Y = \sigma_X{}^\circ \ni_X f^\circ$, as $\sigma^\circ \ni = \Delta$ by definition of the singleton. ■

The *internal intersection* $\bigcap$ would be the representing map of $(\frac{\in_X}{\ni_{P(X)}})^\circ$, but $\in_X$ is not small in general, so $(\frac{\in_X}{\ni_{P(X)}})^\circ$ need not exist nor be small. The composition $\in\ni$ is a sort of all relation on symmetric small objects. We note that $R^\circ R = \in ff^\circ \ni \subseteq \in\ni$.

In *Set*, for example, the relation $\ni\in$ consist of all disjoint pairs of subsets; the pair of empty subsets is not disjoint, so $\Delta$ is not in $\ni\in$. But, for example an easy consequence from the axioms is that $h^\circ \subseteq h^\circ \ni\in$ iff $h^\circ \ni$ is full.

## Discussing Axioms

There are two immediate, but conceptually and meta-mathematically important implications from the axioms A1 and A2.
First, recall that in the motivation in Chapter 1, there was the promise that in Algebraic Set Theory there will be a natural reformulation of the replacement axiom in categorical, algebraic terms. To reveal the answer to this challenge, the replacement axiom takes the following form in our setting:

> **B2.1 Replacement** $Sf^\circ$ is small for a small relation $S$.
> {for a small family $(S_i)_{i\in I}$ the image $f(S_i)_{i\in I}$ is small}

We see that this statement indeed holds as $f^\circ$ is one-to-one. On the other hand, if we multiply with $h^\circ$ from the right for a map $h : I \longrightarrow J$, we obtain

**B2.2 Index Stability** $f^\circ R$ is small for a small relation $R$.
{for a small family $(R_i)_{j\in J}$, the family $(R_{h(i)})_{i\in I}$ is small}

So from this corollary, we learn that the category of all small relations is indeed an indexed subcategory in $\mathbb{R}_{\mathcal{A}}$.

We might state A3 Separation in an equivalent way, that is to require $R \cap S$ to be small for a small relation $S$. This formulation might be closer to the intuition to separate something small out of large family.

Comparing to [Sim99] there are two implications from the axioms that follow much more direct in our setting:

**D1 Cancellation** If $gR$ is a small relation, then also $R$.
{If the disjoint union $\left(\bigcup_{j\in f^{-1}(i)} (R_j)\right)_{i\in I}$ is small, then $(R_j)_{j\in J}$ is small itself}

**D2 Quotients** $Re$ small where $e$ is epi implies that also $R$ small.
{For $e : W \twoheadrightarrow X$ if $\{w : e(w) \in R_i\}_{i\in I}$ is small then $(R_i)_{i\in I}$ is small, too.}

Both properties follow very direct from $R \subseteq (g^\circ g)R = g^\circ \circ (gR)$ and $R \subseteq (Re)e^\circ$. For some related other statements, though, we need that $\mathcal{A}$ comes from an nearly regular category, i.e. that every relation $R$ has a tabulation $R = l \circ k^\circ$.

**Lemma 6.2.3.** *Let $\mathcal{A}$ be an allegory that comes from a nearly regular category. Then the following statements are valid:*

**D3 Relational Descent** *Suppose $e$ epi and $e^\circ R$ small, then $R$ is small, too.*
*{Suppose $e : I \twoheadrightarrow J$ is epi and $(R_{e(i)})_{i\in I}$ is small then $(R_j)_{j\in J}$ is small, too.}*

**D4 Descent** *Assume a pullback situation $fe' = eg$ with $e$ epi, $g, e'$ jointly monic. Then $g$ is small $\Rightarrow f$ is small.*
*{For disjoint families we have: Suppose $e : I \twoheadrightarrow J$ is epi and $(R_{e(i)})_{i\in I}$ is small then $(R_j)_{j\in J}$ is small, too.}*

**Proof.** As $e^\circ R$ is small, there is a map $d$ such that $d^\circ \ni = e^\circ R$. We show, that $ed^\circ$ is a map. It is total as $e$ is epi ($\Delta = ee^\circ \subseteq ed^\circ de^\circ$).

For univalentness we need that for every $R$ there is a tabulation $R = lk^\circ$. From $ex = ey$ follows $dx = dy$ because of unique part in (Power)

$$x^\circ d^\circ \ni = x^\circ e^\circ R = y^\circ e^\circ R = y^\circ d^\circ \ni \Rightarrow dx = dy$$

With Lemma 3.1.2 this result can be stated as $e^\circ e = xy^\circ \Rightarrow d^\circ d = xy^\circ$. Note that there need not exist any such $x, y$ if $\mathcal{A}$ is not tabular. But when there is a tabulation for every relation, we can follow $e^\circ e \subseteq d^\circ d$ and so $de^\circ$ is univalent. So we have found a representing map $de^\circ$ for $R = ee^\circ R = (de^\circ)^\circ \ni$. For the descent axiom for maps we set $R = ge'^\circ$. ■

## Comparing to other Approaches to Algebraic Set Theory

Next we explore the relation of the axioms given in this text to other versions of Algebraic Set Theory. We chose the formulation of ALEX SIMPSON [Sim99] for a comparison. In that paper, the following axioms for Algebraic Set Theory (there called classic structure) were proposed:

**Definition 6.2.4.** *Let $C$ be a regular category. $\mathcal{S} \subseteq C$ is a class of* S-small *maps if it fulfils the axioms of [Sim99, Def 2.4.], i.e. if*

- **S1 Subsingleton** *monomorphisms are S-small*
- **S2 Union** *$\mathcal{S}$ is a subcategory of $C$, i.e. S-smallness is closed with respect to composition.*
- **S3 Pullback stability** *$\mathcal{S}$ is stable under pullback.*
- **S4 Representability** *(Axiom 1 in [Sim99])*
  *For every object $X$ there is an object $\mathsf{P}_S(X)$ and a small relation $\ni_X \subseteq \mathsf{P}_S(X) \times X$ such that, for any small relation $R \subseteq A \times X$ there exists a unique morphism $h : A \to \mathsf{P}_S(X)$ fitting into a pullback diagram of the form*

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[d, "{(l,k)}"'] & \ni_X \arrow[d, "{(t,s)}"] \\ A \times X \arrow[r, "(h \times id)"'] & \mathsf{P}_S(X) \times X \end{tikzcd} \tag{6.1}$$

*where a relation $R \subseteq A \times X$ is S-small if its first component $R \to A$ is S-small.*

**S5 Power** *(Axiom 2 in [Sim99])*
*For every object $X$, the so called superset relation $\supseteq_X$ is small, where $\supseteq_X \hookrightarrow \mathsf{P}_S(X) \times \mathsf{P}_S(X)$, is defined as the relation satisfying: any morphism $(h, g) : A \to \mathsf{P}_S(X) \times \mathsf{P}_S(X)$ factors through the subobject $\supseteq_X \hookrightarrow \mathsf{P}_S(X) \times \mathsf{P}_S(X)$ if and only if, in the pullback diagram below, $Q \hookrightarrow A \times X$ factors through $P \hookrightarrow A \times X$.*

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, hook, "{(l,k)}"'] & \ni_X \arrow[d, hook, "{(t,s)}"'] & Q \arrow[l] \arrow[d, hook] \\ A \times X \arrow[r, "(h \times id)"'] & \mathsf{P}_S(X) \times X & A \times X \arrow[l, "(g \times id)"'] \end{tikzcd} \tag{6.2}$$

Both Representability and Power Axiom call for a reformulation in a relational setting. To that aim the following lemma will be helpful:

**Lemma 6.2.5.** *Let $R = lk^\circ : X \rightsquigarrow A$ and $S = ts^\circ : X \rightsquigarrow B$ be relations, $h : A \to B$ be a map. Then:*

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[d, hook, "{(l,k)}"'] & S \arrow[d, hook, "{(t,s)}"'] \\ A \times X \arrow[r, "(h \times id)"'] & B \times X \end{tikzcd} \quad \textit{is a pullback} \quad \textit{iff} \quad R = h^\circ S$$

**Proof.** The diagram commutes iff it commutes projected to the first and to the second coordinate. It is a pullback iff it is a pullback when it is projected to the first coordinate. In other words, with tabulations $R = lk^\circ$ and $S = ts^\circ$, the diagram just expresses that $su = k$, $lu^\circ = h^\circ t$ with $u, l$ jointly monic. So $h^\circ S = h^\circ ts^\circ = lu^\circ s^\circ = lk^\circ = R$.

$$\begin{tikzcd} R \arrow[rr, "u"] \arrow[dr, "k"] \arrow[dd, "l"'] & & S \arrow[dr, "s"] \arrow[dd, "t"] & \\ & X \arrow[rr, "id"] & & X \\ A \arrow[rr, "h"'] & & B & \end{tikzcd}$$

Figure 6.2: Splitting up the diagram for lemma 6.2.5

On the other hand if $R = h^\circ S$ we can define $u$ by $u := s^\circ k \cap t^\circ hl$ and check that it is indeed a map that is jointly monic with $l$. ■

**Lemma 6.2.6.** *With allegory notation, the axiom S4 Representability (Axiom 1 in [Sim99]) is equivalent to A4 Representability for the category of relations of C, i.e. it holds iff*
*there exists a S-small relation $\ni_X$ such that for any S-small relation R there exists a unique morphism h such that $R = h^\circ \ni$.*

*Moreover, the superset relation $\supseteq_X$ of [Sim99] is given by $\frac{\ni_X}{\ni_X}$. For categories of relations, the axiom S5 Power (Axiom 2 in [Sim99]) is thus equivalent to A5 Power.*

**Proof.** The first statement gets trivial in presence of Lemma 6.2.5. For the second, split the morphism $(h, g)$ into $(id \times g) \circ (h \times id) \circ \delta$. We know that a morphism $f$ factors through a monomorphism $m$ iff the pullback $f^*(m)$ of $m$ along $f$ is an isomorphism (Lemma A.1.3). So $(h, g)$ factors through a subobject $\varphi : \Phi \hookrightarrow P_S(X) \times P_S(X)$ iff

$$((id \times g) \circ (h \times id) \circ \delta)^*(\varphi) = \delta^*(h \times id^*(id \times g^*(\varphi)))$$

is an isomorphism. Applying Lemma 6.2.5 twice, we see that this is equivalent to the statement that $\delta^*(h^\circ \Phi g)$ is an isomorphism, or, applying Lemma A.1.3 again, to $\Delta \subseteq h^\circ \Phi g$. Allegory calculus yields the formulation $hg^\circ \subseteq \Phi$, finally.

The relations $P$ and $Q$ have the representations $P = h^\circ \ni$ and $Q = g^\circ \ni$, respectively. Therefore $Q$ factors through $P$ iff $g^\circ \ni \subseteq h^\circ \ni$ or equivalently iff $hg^\circ \ni \subseteq \ni$.

Putting these reformulations together, superset relation takes the following form: $\Phi$ is the $\supseteq$-relation iff

$$hg^\circ \subseteq \Phi \Leftrightarrow hg^\circ \ni \subseteq \ni \qquad \text{for every pair } (h, g).$$

But, in fact, as every relation $R$ arises as $hg^\circ$ for some pair $(h, g)$, this is nothing else than the definition of the division $\frac{\ni}{\ni}$. ■

**Theorem 6.2.7.** *Let C be a regular category equipped with a subcategory $S_C$ of S-small maps. Then the S-small relations fulfil the axioms of smallness A1-A5, if we define a relation $R \hookrightarrow I \times X$ to be S-small if $R \hookrightarrow I \times X \to I$ is S-small.*

*Conversely, let $\mathcal{A}$ be a regular allegory equipped with a subcategory $S_\mathcal{A}$ of small relations. Then the small maps in* Map($\mathcal{A}$) *fulfil the axioms of S-smallness S1-S5 provided that a relation $R : X \rightsquigarrow I, R = lk^\circ$ is small iff its left tabulation map l is small.*

**Proof.** It remains to show A3 Separation and A2 Union for S-small relations in a category of relations and S3 Pullback stability for small maps in a regular allegory.

If $R \subseteq S$ holds for two relations $(l, k) : R \hookrightarrow I \times X$ and $(t, s) : S \hookrightarrow I \times X$, then there exists a monomorphism $i : R \hookrightarrow S$ such that $ti = l \wedge si = k$. So S1 and S2 imply that the separation axiom A3 holds for S-small relations. In fact, a similar argument using the last statement of Lemma 3.1.3 shows that in any *regular* allegory, A4 is redundant.

$$\begin{tikzcd}
 & & RS \arrow[ddll, bend right, "l_{RS}"'] \arrow[ddrr, bend left, "k_{RS}"] & & \\
 & & P \arrow[u, "e"] \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[d, phantom, "\text{pullb.}"] & & \\
 & R \arrow[dl, "l_R"] \arrow[dr, "k_R"'] & & S \arrow[dl, "l_S"] \arrow[dr, "k_S"'] & \\
Z & & Y & & X
\end{tikzcd}$$

Figure 6.3: Composition of relations in a regular category

For the composition of relations consider the diagram in figure 6.2. The map $t$ is S-small by pullback stability, hence $l_R \circ t$ is S-small. Now apply Theorem 1.2 (Quotients) of [Sim99]: if $f \circ e$ is S-small and $e$ epi then $f$ is S-small. Since $l_{RS} \circ e = (l_R \circ t)$ the left tabulation $l_{RS}$ is S-small as desired.

Finally, consider a pullback situation $hg = fh'$ with $g, h'$ jointly monic and $f$ small. From $gh'^\circ \subseteq h^\circ f$ follows that $gh'^\circ$ is small and hence $g$. ∎

We did not yet consider the symmetric division in context of the setting of small relations. In fact for small relations, the symmetric division always exists – it can be calculated as a composition involving only the representing arrows.

**Lemma 6.2.8.** *Suppose in an allegory $\mathcal{A}$ the symmetric division $\frac{\ni}{\ni}$ has a factorisation $\frac{\ni}{\ni} = k \circ l^\circ$, not necessarily jointly monic. Then $\frac{\ni}{\ni} = \Delta$ and for $R = f^\circ \ni$ and $S = g^\circ \ni$ the symmetric division can be expressed as $\frac{R}{S} = f^\circ g$.*

**Proof.** For $T = \frac{\ni}{\ni}$ we unfold the definition of the symmetric division.

$$T \subseteq \frac{\ni}{\ni} = \frac{\ni}{\ni} \cap \left(\frac{\ni}{\ni}\right)^\circ \Leftrightarrow \begin{array}{l} k \circ l^\circ \ni = T \circ \ni \subseteq \ni \\ l \circ k^\circ \ni = T^\circ \circ \ni \subseteq \ni \end{array} \text{ and}$$

The map $k$ is total, so $\Delta \subseteq k^\circ k$ and therefore $l^\circ \ni \subseteq k^\circ k l^\circ \ni \subseteq k^\circ \ni$. The same derivation can be done with $l$ and $k$ reversed, so $k^\circ \ni = l^\circ \ni$. The representability axiom A4 implies that $l = k$, as there is no more than one map representing a relation. So $\frac{\ni}{\ni} = T = kl^\circ = kk^\circ \subseteq \Delta$ because $k$ is univalent. On the other hand $\Delta$ is in $\frac{\ni}{\ni}$ and therefore in $\frac{\ni}{\ni}$. The general formula for symmetric division follows from Lemma 6.1.3 and 3.1.2. ■

In the final remarks closing the thesis, we present some of the standard axioms of set theory in a very condensed, not to say in a cryptical form.

For that aim, consider in a formulation of a Gödel-Bernays set theory the allegory of classes and class-relations, the class of all sets $U$ and the global element relation $\ni: U \rightsquigarrow U$. The division $\frac{\ni}{S}$ – if it exists – is then the class

$$\Big\{(y,x) : \{z : (x,z) \in S\} = \{z : (y,z) \in \ni\}\Big\} \quad = \quad \Big\{(y,x) : \{z : (x,z) \in S\} = y\Big\}.$$

So given a class relation $S$, the condition that

$$\frac{\ni}{S} \text{ exists and is a map}$$

is just a reformulation of the statement that for every $x$ the class $\{z : (x,z) \in S\}$ is a set. This way, (some of) the axioms of a Gödel-Bernays set theory can be equivalently reformulated as:

**Extensionality** $\frac{\ni}{\ni} = \Delta$.

**Replacement** $\frac{\ni}{\ni f^\circ}$ is a map where $f$ is a map.

**Union** $\frac{\ni}{\ni\ni}$ is a map.

**Powerset** $\frac{\ni}{\frac{\ni}{\ni}}$ is a map.

In fact, these axioms can not only be interpreted in the allegory of classes in a Gödel-Bernays set theory, but also in the context of the allegory version of Algebraic Set Theory that has been introduced in this chapter. Define for an allegory $\mathcal{A}$ equipped with a structure $\mathcal{S}$ of small relations a *universe* to be an object $U$ equipped with an isomorphism map $d : U \to \mathrm{P}_{\mathcal{S}}(U)$. For such a universe $(U, d)$ in $\mathcal{A}$, the relation $d^\circ\ni$ serves as a global element relation $\ni_{\mathcal{U}}$ on $U$.

Lemma 6.2.8 exhibits that a relation of the form $\frac{\ni_{\mathcal{U}}}{S}$ is a map iff $S$ has a factorisation $f^\circ \ni$, that is iff $S$ is small. Thus, the validity of these five statements follows from the axioms of smallness.

# Appendix A

# Tables, Sheets, Formulas

This appendix collects properties, formulas and theorems in general category theory in a condensed form. Parts of the material below can be found in good accounts on general category theory, e.g. [AHS90], the remainder is supposed to be general *folklore* in category theory. In the scientific community, these basic facts are assumed to be commonly known and are applied, in some variants, throughout in the literature. Proofs are mostly exercise-level or can otherwise be found in either [Bor94, AHS90] or [PPT04] (for Table A.3.3)

## A.1 General

**Table A.1.1.** *Suppose* $U : \mathcal{A} \to \mathcal{C}$ *is a functor,* $\mathcal{D} \leq \mathcal{C}$ *its image. Then:*

| | | | | |
|---|---|---|---|---|
| *U is faithful* | $\Leftrightarrow$ | $\mathcal{F} \leq \mathcal{D}$ *is a subcategory* | $\Rightarrow$ | *U reflects monos and epis* |
| *full and faithful* | $\Leftrightarrow$ | *full subcategory* | $\Rightarrow$ | *monos, epis, limits and colimits* |
| *additionally preserves limits* | $\Leftrightarrow$ | *additionally closed under limits (in* $\mathcal{C}$*!)* | | |

**Table A.1.2.** *Suppose $C$ is a category. A morphism $e$ is split epi if there is an $m$ with $e \circ m = id$. In that case*

$$R \underset{m}{\overset{e}{\leftrightarrows}} A \circlearrowright^{\alpha}, \quad \alpha^2 = \alpha = m \circ e,$$

$$R = equ(id, \alpha) = coeq(id, \alpha) = \varprojlim A \circlearrowright^{\alpha} = \varinjlim A \circlearrowright^{\alpha}$$

*The following implications of variants of epi morphisms $f$ hold:*

$$\begin{array}{ccccccccc} iso & \Rightarrow & split & \Rightarrow & stably\ regular & \Rightarrow & stably\ strong & \Rightarrow & stably\ extremal \\ & & & & \Downarrow & & \Downarrow & & \Downarrow \\ & & & & regular & \Rightarrow & strong & \Rightarrow & extremal \\ & & & & \Downarrow & & & & \\ & & & & epi & & & & \end{array}$$

*In a topos, this table reduces to iso $\Rightarrow$ split $\Rightarrow$ epi.*

**Lemma A.1.3.** *Suppose $f$ is morphism in a category $C$ and $m$ a monomorphism. Consider the following diagram:*

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d] & {} \arrow[d] \arrow[dl, dotted, "\exists\kappa"'] \\ {} \arrow[r, hook, "m"'] & {} \end{tikzcd}$$

*Then: $f$ is an isomorphism iff both the square above is a pullback and there is such a $\kappa$ which fits into the diagram.*

## A.2 Adjunctions

**Definition A.2.1.** *An adjunction* $\vee : F \underset{\eta,\varepsilon}{\dashv} U : \wedge$, *where* $\mathcal{X} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{A}$, *is defined by one of the items below. They are related by the following equations:*

$$
\begin{array}{lcl@{\qquad}c@{\qquad}lcl}
\eta_X & = & \mathit{id}^{\vee}_{FX} & \textit{and} & \varepsilon_A & = & \mathit{id}^{\wedge}_{UA} \\[1ex]
f^{\vee} & = & Uf \circ \eta_X & \textit{and} & g^{\wedge} & = & \varepsilon_A \circ Fg \\
\textit{for} & f: & FX \to A & & \textit{for} & g: & X \to UA \\[1ex]
Fh & = & (\eta_{X_2} \circ h)^{\wedge} & \textit{and} & Uk & = & (k \circ \varepsilon_{A_1})^{\vee} \\
\textit{for} & h: & X_1 \to X_2 & & \textit{for} & k: & A_1 \to A_2
\end{array}
$$

- $F$ and $U$ are two functors and there are two natural transformations: The *unit* $\eta : \mathit{Id} \Rightarrow UF$ and the *counit* $\varepsilon : FU \Rightarrow \mathit{Id}$ with

$$\mathit{id} = U\varepsilon \circ \eta_U \text{ and } \mathit{id} = \varepsilon_F \circ F\eta$$

- $U$ is a functor and for each object $X$ there is a *free object* $FX$ and an embedding of generators $\eta_X : X \to U(FX)$ such that: For every object $A$ and map $g : X \to UA$ there is an unique extension $g^{\wedge}$ making the diagram

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & UFX \arrow[d, dashed, "U(g^{\wedge})"] & FX \arrow[d, dashed, "g^{\wedge}"] \\
& UA & A
\end{tikzcd}
$$

commute.

- $F$ is a functor and for each object $A$ there is a *cofree object* $UA$ and a morphism $\varepsilon_A : F(UA) \to A$ such that: For every object $X$ and map $f : FX \to A$ there is an unique extension $f^{\vee}$ making the diagram

$$
\begin{tikzcd}
UA & FUA \arrow[r, "\varepsilon_A"'] & A \\
X \arrow[u, dashed, "f^{\vee}"] & FX \arrow[u, dashed, "F(f^{\vee})"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

commute.

- $F$ and $U$ are two functors and there is a natural bijection $-^{\vee} : (FX, A) \cong (X, UA) : -^{\wedge}$

- $F$ and $U$ are two functors and there is a bijection $-^{\vee} : (FX, A) \cong (X, UA) : -^{\wedge}$ with

$$\begin{tikzcd} & UA_1 \arrow[dd, "U(h)"] \\ X \arrow[ur, "f_1^{\vee}"] \arrow[dr, "f_2^{\vee}"'] & \\ & UA_2 \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} & A_1 \arrow[dd, "h"] \\ FX \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & A_2 \end{tikzcd}$$

$$\begin{tikzcd} X_1 \arrow[dr, "g_1"] & \\ & UA \\ X_2 \arrow[uu, "k"] \arrow[ur, "g_2"'] & \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} FX_1 \arrow[dr, "g_1^{\wedge}"] & \\ & A \\ FX_2 \arrow[uu, "F(k)"] \arrow[ur, "g_2^{\wedge}"'] & \end{tikzcd}$$

- $F$ and $U$ are two functors and there is an isomorphism $-^{\vee} : F \downarrow \mathcal{A} \cong \mathcal{X} \downarrow U : -^{\wedge}$
- $U$ is a functor and for every $X$ there is an initial object $X \xrightarrow{\eta} UFX$ in $X \downarrow U$.
- $U$ is a functor that preserves limits and $FX = \varprojlim(X \downarrow U \to \mathcal{A})$ exists.

**Existence and Properties of Adjoints**

**Theorem A.2.2** (Adjoint Functor Theorem). *Suppose $\mathcal{C}$ has small homsets, lower bounds on small diagrams, idempotents split.*
*Then: $\mathcal{C}$ has an initial object $\Leftrightarrow$ there is a set of jointly weakly initial objects in $\mathcal{C}$.*

*Suppose $\mathcal{C}$ has small homsets, limits on small diagrams, $U$ preserves them.*
*Then: There is a $U$-free object $(FX, \eta_X)$ $\Leftrightarrow$ there is a small set of solutions $(S, s)$ such that for every $X \to UA$ there is a $(s, S)$ and $h$ with*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "g"'] & US \arrow[d, dashed, "U(h)"] \\ & UA \end{tikzcd} \qquad \begin{tikzcd} S \arrow[d, dashed, "h"] \\ A \end{tikzcd}$$

**Theorem A.2.3** (Special Adjoint Functor Theorem). *Suppose $\mathcal{C}$ has small homsets, subsets and limits. Then:*
*$\mathcal{C}$ has an initial object $\Leftarrow$ there is a cogenerating set.*

*Suppose $\mathcal{C}, \mathcal{D}$ have small homsets, subsets and limits; $U$ preserves them.*
*Then: $F \dashv U \Leftarrow$ there is a cogenerating set.*

**Lemma A.2.4.** *Let $F \underset{\eta,\varepsilon}{\dashv} U$ be an adjunction between $\mathcal{X}$ and $\mathcal{A}$, and $T = UF$, $G = FU$, $\mu = U\varepsilon_F$, $\delta = F\eta_U$ and $K$ is the comparison functor for the monad $(T, \eta, \mu)$ (see A.2.10).*

Then these properties always hold

$\varepsilon_F$ *split epi* $\qquad U\varepsilon$ *split epi* $\qquad \eta_U$ *split mono* $\qquad F\eta$ *split mono*

and the following equivalences can be recorded:

| | | | | | |
|---|---|---|---|---|---|
| $\varepsilon$ *epi* | iff | $U$ *faithful* | $\eta$ *mono* | iff | $F$ *faithful* |
| | iff | $^\wedge$ *preserves epis* | | | |
| $\varepsilon$ *extremal epi* | iff | $U$ *conservative* | $\eta$ *extr. mono* | iff | $F$ *conservative* |
| $\varepsilon$ *regular epi* | iff | $K$ *full faithful* | $\eta$ *split mono* | iff | |
| $\varepsilon$ *split epi* | iff | $U$ *monadic*[1] | $\eta_A$ *split mono* | iff | $\exists B, A \hookrightarrow UB$ *split mono* |
| $\varepsilon$ *split mono* | iff | $U$ *full* | $\eta$ *split epi* | iff | $F$ *full* |
| $\varepsilon$ *iso* | iff | $U$ *full faithful* | $\eta$ *iso* | iff | $F$ *full faithful* |
| | iff | $G \underset{\overline{\eta},\varepsilon}{\dashv} I,\ U\overline{\eta} = \eta$ | | | |

Moreover,

$\varepsilon_F$ *mono* $\quad$ iff $\quad U\varepsilon$ *mono* $\quad$ iff $\quad \eta_U$ *epi* $\quad$ iff $\quad F\eta$ *epi*

iff the following pairs of morphisms are isos

$$FUF \underset{F\eta}{\overset{\varepsilon_F}{\rightleftarrows}} F \qquad U \underset{U\varepsilon}{\overset{\eta_U}{\rightleftarrows}} UFU$$

iff $\quad G\varepsilon = \varepsilon_G \quad$ iff $\quad \eta_T = T\eta$

iff $\quad U|_{im(F)}$ *equivalence* $\quad$ iff $\quad F|_{im(U)}$ *equivalence*

[1] if $\mathcal{X}$ has pullbacks.

## Reflections and Sheaves

**Definition A.2.5.** *Suppose $\mathcal{C}$ is a category. A* reflection

$$\mathrm{a} \underset{\eta,\varepsilon}{\dashv} \iota \colon \mathcal{L} \underset{\iota}{\overset{\mathrm{a}}{\leftrightarrows}} \mathcal{E}; \quad L := \iota\mathrm{a}$$

*is given by either*

- *an adjunction where $\varepsilon$ is iso, i.e. $\iota$ is full and faithful*
- *a full subcategory $\mathcal{L} \leq \mathcal{X}$ and for every $X$ there is a best approximation $\mathrm{a}X$ and an inclusion $\eta_X : X \to \mathrm{a}X$ such that*
  *Every morphism into $\mathcal{L}$ factors uniquely through the best approximation*
- *an idempotent monad $(L, \eta, \mu)$, where $\mu = U\varepsilon_F$ iso, and $\mathcal{L}$ is the category of $L$-algebras (see A.2.10).*

**Definition A.2.6.** *Suppose $\mathcal{E}$ is a topos. An inclusion $m : \mathcal{L} \to \mathcal{E}$ is given by one of the items below. They are related by the following equations:*

$$\begin{array}{lcllcl}
m^* & = & \mathrm{a} & m_* & = & \iota \\
\overline{U}^j & = & \eta^*(\mathrm{a}U) = \{x | j(x \in U)\} & \overline{U}(C) & = & \{x : \exists M \text{ covers } C\ \forall f \in M : x{\cdot}f \in U\} \\
J & = & \overline{\top} = \{\omega | j\omega = \omega\} & j & \text{is} & \text{the characteristic map for } J \subseteq \Omega \\
\mathcal{L} & = & \mathcal{E}_j = \mathrm{Sh}(\mathcal{C}) & jR & = & \overline{R} \text{ for } R \in \mathrm{Sub}\,\mathrm{y}C = \Omega(C) \\
R & \text{is} & \text{a cover of } C \quad \Leftrightarrow & R & \in & J(C) \subseteq \Omega(C) = \mathrm{Sub}\,\mathrm{y}C
\end{array}$$

- $\mathrm{a} \dashv \iota$ *is a reflection where* a *preserves finite limits*
- *$j : \Omega \to \Omega$ is a map which satisfies $j \circ j = j$, $j0 = j$ and $j \circ \wedge = \wedge(j \times j)$*
- *$U \mapsto \overline{U}$ is a closure operator on* $\mathrm{Sub}(A)$ *and $f^*\overline{U} = \overline{f^*U}$*
- *{ if $\mathcal{E} = [\mathcal{C}^\circ, \mathcal{S}]$: } a coverage on $\mathcal{C}$*

*$\mathcal{L} = \mathcal{E}_j = \mathrm{Sh}(\mathcal{C})$ is called the category of sheaves or s for $j$ (or for the notion of a coverage on $\mathcal{C}$).*

**Lemma A.2.7.** *The formula*

$$\mathrm{Sub}_{\mathsf{Sh}(C)}(\mathsf{a}X) = j\text{-closed-Sub}(X)$$

*is a short cut for the following fact:*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, hook] & A' \arrow[d, hook] \\ X \arrow[r, "\eta_X"] & \mathsf{a}X \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[dr, hook] \arrow[rr, bend left, "\eta_U"] & jU \arrow[r, "\eta_{jU}"'] \arrow[d, hook] & \mathsf{a}U = \mathsf{a}jU \arrow[d, hook] \\ & X \arrow[r, "\eta_X"] & \mathsf{a}X \end{tikzcd}$$

*The first square is a pullback iff $A$ is closed; $A' = \mathsf{a}A$ and $\varphi = \eta_A$ in this case. We thus obtain $jU$ by pulling $\mathsf{a}U$ back along $\eta$.*

**Lemma A.2.8.** *Let $C$ be a category equipped with a notion of a coverage and suppose $R \subseteq \mathsf{y}C$ and $f^*(R) = \{s : fs \in R\}$. Then the following statements are equivalent:*

*(1) There is an $S \subseteq \mathsf{y}C$ which covers and $f^*(R)$ covers for all $f \in S$*

*(2) $\{f : f^*R$ covers$\}$ covers*

*(3) $\{f : f^*R$ covers$\} = \mathsf{y}C$*

*(4) $R$ covers*

## Left and Right Adjoints

**Table A.2.9.** *The following functors appear as right and left adjoints:*

| *$F$ is the left adjoint* | $F \dashv U$ | *$U$ is the right adjoint* |
|---|---|---|
| *free* | | *forgetful* |
| $(FA, B)$ | | $(A, UB)$ |
| $FA \to B$ | | $A \to UB$ |
| | | |
| $f^*$ | | $f_*$ |
| $f^{-1}$ | | |
| | | |
| is and $F$ preserves: | | is and $U$ preserves: |
| *colimits* | | *limits* |
| $\begin{tikzcd} L \arrow[r, dotted] & {} \\ D_1 \arrow[u] \arrow[r] & D_2 \arrow[ul] \arrow[r] & D_3 \arrow[ull, bend right] \arrow[r] & \cdots \end{tikzcd}$ | | $\begin{tikzcd} L \arrow[d] \arrow[dr] \arrow[drr, bend left] & {} \arrow[l, dotted] \\ D_1 & D_2 \arrow[l] & D_3 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$ |
| *direct limit* | | *inverse limit* |
| $\varinjlim$ | | $\varprojlim$ |
| $\int^C$ | | $\int_C$ |
| | | |
| $0$ | | $1$ |
| *sum* $\Sigma$ | | |
| *coproduct* $\coprod$ | | *product* $\prod$ |
| $A + B$ | | $A \times B$ |
| | | |
| *pushout* | | *pullback* |
| *coequaliser* | | *equaliser* |
| *quotient* | | *subobject* |
| *epi* | | *mono* |
| | | |
| *Supremum* $\bigvee$ | | *Infinum* $\bigwedge$ |
| $\exists$ | | $\forall$ |
| | | |
| in a po: | | |
| $= \bigwedge \{x : y \leq Ux\}$ | | $= \bigvee \{y : Fy \leq x\}$ |
| | | |
| $A \times -$ | | $(\ )^A$ *exponential* |

## Monads and Kan Extensions

**Definition A.2.10.** *A* monad $\mathbf{T}$ *on a category $\mathcal{C}$ is given by a triple $(T, \eta, \mu)$ with*

$$T : \mathcal{C} \to \mathcal{C} \qquad \eta : Id \Rightarrow T \qquad \mu : T^2 \Rightarrow T$$

$$\begin{tikzcd} T^3 \arrow[r, shift left, "\mu_T"] \arrow[r, shift right, "T\mu"'] & T^2 \arrow[r, "\mu"] & T \arrow[l, bend right=50, "T\eta"'] \arrow[l, bend left=50, "\eta_T"] \end{tikzcd}$$

$$\mu \circ T\eta = id = \mu \circ \eta_T$$
$$\mu \circ T\mu = \mu \circ \mu_T$$

*An* algebra *for a monad $(T, \eta, \mu)$ is given by*

$$A = (X, h) \qquad h : TX \to X$$

$$\begin{tikzcd} T^2X \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "Th"'] & TX \arrow[r, "h"] & X \arrow[l, bend left=50, "\eta_X"] \end{tikzcd}$$

$$h \circ \eta_X = id$$
$$h \circ Th = h \circ \mu_X$$

*where $U^T(A) = X$ is the underlying object of $A$. For two algebras $A = (X, h)$ and $B = (Y, k)$, a homomorphism $A \to B$ is determined by a morphism $f$ between the underlying objects satisfying*

$$\begin{tikzcd} TX \arrow[r, "h"'] \arrow[d, "Tf"'] & X \arrow[d, "f"'] \\ TY \arrow[r, "k"'] & Y \end{tikzcd}$$

*The* free algebra $F^T(X)$ *is given by $(TX, \mu_X)$; it satisfies indeed the universal property:*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & TX \arrow[d, dashed, "k \circ Tg"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, dashed, "T(k \circ Tg)"] & TX \arrow[d, dashed, "k \circ Tg"] \\ TY \arrow[r, "k"'] & Y \end{tikzcd}$$

*in other words $F^T \underset{\eta^T, \varepsilon^T}{\dashv} U^T$ where $\eta^T = \eta$; $\varepsilon^T = h$: $(TX, \mu_X) \to (X, h)$.*

*The* Eilenberg-Moore comparison functor $K$ *is given by*

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, bend right, "F^T"'] \arrow[dr, bend left, "F"] & \\ \mathcal{C}^T \arrow[ur, "U^T"'] \arrow[rr, bend right, "\Phi"] & & \mathcal{B} \arrow[ul, "U"] \end{tikzcd} \qquad \begin{tikzcd} B_1 \arrow[d, "\varphi"] & \mapsto & (UB_2, U\varepsilon B_2) \arrow[d, "Uf"] \\ B_2 & & (UB_2, U\varepsilon B_2) \end{tikzcd}$$

Every algebra has a representation as coequalisers of free algebras given by

$$\begin{tikzcd} F^TU^TF^T|A| \arrow[r, shift left, "Th"] \arrow[r, shift right, "\mu_{|A|}"'] & F^T|A| \arrow[r, "h"'] \arrow[l, dotted, bend left=60, "F^T\eta^T_{U^TA}"] & A \end{tikzcd}$$

**Definition A.2.11** (Kan Extension). *Suppose $F\colon \mathcal{B} \to \mathcal{C}$ and $T\colon \mathcal{B} \to \mathcal{S}$ are functors.*
*The* left Kan extension *is given by a functor $L = \varinjlim_F\colon \mathcal{C} \to \mathcal{S}$ and a natural transformation $l\colon T \to L \circ F$ such that for any other functor $S\colon \mathcal{C} \to \mathcal{S}$ and natural transformation $\varepsilon\colon T \to S \circ F$ there is a unique $\nu\colon L \to S$ such that $\nu F \circ l = \varepsilon$ as in the diagram*

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "F"] \arrow[dd, "T"'] & & \mathcal{C} \arrow[ddll, "L"] \arrow[ddll, bend left=40, "S"] \\ & \arrow[phantom, "l\ \ \varepsilon\ \ \nu F"] & \\ \mathcal{S} & & \end{tikzcd}$$

*that is $(\varinjlim_F T, S) \cong (T, S \circ F)$ for short.*

*In other words, the functor assigning $T$ to the left Kan extension $\varinjlim_F T$ is left adjoint to the functor $S \mapsto S \circ F$; the dual concept, a right Kan extension, is then the right adjoint to $- \circ F$:*

$$\varinjlim_F \dashv - \circ F \dashv \varprojlim_F.$$

*Kan extensions are special (co)limits:*

$$(\varprojlim_F T)(C) = \varprojlim(C{\downarrow}F \to \mathcal{B} \xrightarrow{T} \mathcal{S}) \quad \textit{and} \quad (\varinjlim_F T)(C) = \varinjlim(F{\downarrow}C \to \mathcal{B} \xrightarrow{T} \mathcal{S})$$

*and conversely limits and adjoints can be expressed as Kan extensions:*

$$F \dashv U \Leftrightarrow F = \varprojlim_U Id \quad \textit{and} \quad \varprojlim T = \varprojlim_{!_\mathcal{J}} T$$

## A.3 Logic and Point-free Topology

**Table A.3.1** (External Semantics – Generalised Elements). *Suppose $X, Y, S, T$ are objects in a topos $\mathcal{E}$ and generalised elements are defined by*

$$a \in_T^X A \text{ iff } a : T \to X \text{ factors through } A \hookrightarrow X.$$

*Then the $T$-elements of a subobject $\{x|\varphi\} \hookrightarrow X$ are determined by induction on the complexity of the formula with the following equivalences as building blocks:*

$$
\begin{array}{lcl}
a, b \in_T^X \{x, y | f(x) = g(y)\} & \Leftrightarrow & f(a) = g(b) \\
a \in_T^X \{x | \mathit{true}\} & \Leftrightarrow & a \in_T X \\
a \in_T^X \{x | \mathit{false}\} & \Leftrightarrow & T = 0 \\
a \in_T^X \{x | \varphi \wedge \psi\} & \Leftrightarrow & a \in_T^X \{x|\varphi\} \wedge a \in_T^X \{x|\psi\} \\
b \in_T^Y \{y | \exists y\ \varphi\} & \Leftrightarrow & \exists\ e \colon S \twoheadrightarrow T,\ a \in_S X : \\
 & & (a, b \circ e) \in_S^{X \times Y} \{(x, y) | \varphi(x, y)\} \\
a \in_T^X \{x | \varphi \vee \psi\} & \Leftrightarrow & \exists \begin{array}{c} S_1 \xrightarrow{e} \\ S_2 \xrightarrow[d]{} \end{array} T \quad \textit{covering such that} \\
 & & a \circ e \in_{S_1}^X \{x|\varphi\}, a \circ d \in_{S_2}^X \{x|\psi\} \\
a \in_T^X \{x | \varphi \Rightarrow \psi\} & \Leftrightarrow & \forall\ e \colon S \to T : \\
 & & a \circ e \in_S^X \{x|\varphi(x)\} \Rightarrow a \circ e \in_S^X \{x|\psi(x)\} \\
a \in_T^X \{x | \neg\varphi\} & \Leftrightarrow & \forall\ e \colon S \to T : a \circ e \notin_S^X \{x|\varphi(x)\} \\
b \in_T^Y \{y | \forall y\ \varphi\} & \Leftrightarrow & \forall\ e \colon S \to T,\ a \in_S X : \\
 & & (a, b \circ e) \in_S^{X \times Y} \{(x, y) | \varphi(x, y)\}
\end{array}
$$

**Table A.3.2** (Analogy Continuous Sets, Names and Rings). *Suppose $\mathcal{C}$ is a category, $P$ a partial order and $K$ an integral domain.*

| *Category* | *Forcing* | *Rings* |
|---|---|---|
| $\mathrm{Sh}_{\neg\neg}(\mathcal{C}/C)$ | $C$-*names below* $C$ | *polynoms* $K[X]$*(as formal expressions)* |
| $\mathrm{Sh}_{\neg\neg}(P)$ | $P$-*names* | |
| *constant sets* $\alpha = \gamma^*(I)$ | *names* $\check{x}$ | *scalars* $K$ |
| *internal language in* $\mathrm{Sh}(\mathcal{C})$ | *working in the model* $\mathsf{M}[G]$ | *operations defined on* $K[X]$ |
| *generic filtered functor* $F$ *with induced geometric morphism* $f$ | *generic filter* $\mathcal{G}$ | *element* $a$ *of* $K$ |
| $f^*X$ | $x[\mathcal{G}]$ | $p(a)$ |

**Table A.3.3** (Point-free Topology). *Suppose $X$ is a locale.*

| | | | |
|---|---|---|---|
| *Adjunction:* | $\mathcal{O}(-) \dashv pt$ | | |
| *Spaces* | $Top \underset{pt}{\overset{\mathcal{O}(-)}{\rightleftarrows}} Loc$ | *Locales* | |
| *Sober Spaces* | $Sob \cong sLoc$ | *Spatial Locales* | |

| Sublocale | | $\wedge$ | $\bigvee$ | open<br>closed<br>dense | |
|---|---|---|---|---|---|
| subobject<br>$A \hookrightarrow X$ | $X \hookrightarrow X$<br>$\uparrow$<br>$0 \hookrightarrow X$ | pullback | | $m^* \dashv m_* =$<br>$(- \wedge R) \dashv (R \to -)$<br>$(- \vee R) \dashv id$<br>$m^*U = 0 \Rightarrow U = 0$ | $A \overset{j}{\leftrightarrows} X$<br>$X/\vartheta \underset{\bigvee}{\overset{\kappa}{\leftrightarrows}} X$ |
| nucleus<br>$j : X \to X$ | $id$<br>$\downarrow$<br>$\equiv X$ | | $\bigcap j_i$ | $R \to -$<br>$- \vee R$<br>$j0 = 0$ | $j = m_* m^*$<br>$jU = \bigvee\{W : W \,\vartheta\, U\}$ |
| congruence | $\Delta_X$<br>$\downarrow$<br>$X \times X$ | $\langle \bigcup \vartheta_i \rangle_{trans}$ | $\bigcap \vartheta_i$ | $U \,\vartheta\, V \Leftrightarrow$<br>$U \wedge R = V \wedge R$<br>$U \vee R = V \vee R$<br>$0 \,\vartheta\, U \Rightarrow U = 0$ | $U \,\vartheta\, V$<br>$\Leftrightarrow m^*U = m^*V$<br>$\Leftrightarrow jU = jV$ |
| fixset | $X$<br>$\uparrow$<br>$0$ | $\bigcap$ | $\{\bigwedge a_i \mid a_i \in A_i\}$ | $-$<br>$\uparrow R$<br>$-$ | $A = \{U : jU = U\}$ |

# Bibliography


[ABSS03] AWODEY, STEVE, BUTZ, CARSTEN, SIMPSON, ALEX K. & STREICHER, THOMAS. *Relating topos theory and set theory via categories of classes*. Carnegie Mellon University Department of Philosophy Technical Report, 146, 2003
URL http://www.phil.cmu.edu/projects/ast/Papers/sets.pdf

[AHS90] ADÁMEK, JIŘÍ, HERRLICH, HORST & STRECKER, GEORGE E. *Abstract and concrete categories - The joy of cats*. Nr. 17 in Reprints in Theory and Applications of Categories. 1990
Open Access http://www.tac.mta.ca/tac/reprints/articles/17/tr17.pdf

[Awo00] AWODEY, STEVE. *Continuity and Logical Completeness: an application of sheaf theory and topoi*. PhilSci Archive ID 175 , 2000
Open Access http://philsci-archive.pitt.edu/archive/00000175/

[Bel88] BELL, JOHN L. *Toposes and local set theories*, Vol. 14 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, 1988

[Bor94] BORCEUX, FRANCIS. *Handbook of categorical algebra. 1-3*, Vol. 50-52 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1994

[Bru90] BRUNNER, NORBERT. *The Fraenkel-Mostowski method, revisited*. Notre Dame J. Formal Logic, 31(1):pp. 64–75, 1990

[Bru96] BRUNNER, NORBERT. *75 years of independence proofs by Fraenkel-Mostowski permutation models*. Math. Japon., 43(1):pp. 177–199, 1996

[BS89] BLASS, ANDREAS & SCEDROV, ANDRE. *Freyd's models for the independence of the axiom of choice*. Mem. Amer. Math. Soc., 79(404):pp. viii+134, 1989

[Bun74] BUNGE, MARTA C. *Topos theory and Souslin's hypothesis.* J. Pure Appl. Algebra, 4:pp. 159–187, 1974

[But03] BUTZ, CARSTEN. *Bernays-Gödel type theory.* J. Pure Appl. Algebra, 178(1):pp. 1–23, 2003

[BW85] BARR, MICHAEL & WELLS, CHARLES. *Toposes, triples and theories.* Nr. 12 in Reprints in Theory and Applications of Categories. 1985
Open Access `http://www.tac.mta.ca/tac/reprints/articles/12/tr12.pdf`

[Cho01] CHOW, TIM. *Forcing for dummies.* posted on sci.math.research 10.03.01 , 2001
Open Access `http://www-math.mit.edu/~tchow/mathstuff/forcingdum`

[Eas05] EASWARAN, KENNY. *A Cheerful Introduction to Forcing.*, 2005
URL `http://www.ocf.berkeley.edu/~easwaran/papers/forcing.pdf`

[Epp02] EPPENSCHWANDTNER, WOLFGANG E. *Tensorprodukte Topologischer Vektorräume – ein Kategorientheoretischer Zugang.* Technische Universität Wien, May 2002
Open Access `http://dmg.tuwien.ac.at/eppenschwandtner/tensorkat.pdf`

[Epp04] EPPENSCHWANDTNER, WOLFGANG E. *Der Raum in der Mathematik.* Texte zur Sommerakademie 2004 zum Thema Raum, PRO SCIENTIA, Wien, 2004
Open Access `http://dmg.tuwien.ac.at/eppenschwandtner/raummath.pdf`

[Epp05] EPPENSCHWANDTNER, WOLFGANG E. *Graphen, Nichtklassische Logik und Unabhängigkeitsbeweise.* Wissenschaftliche Nachrichten, 2005
Open Access `http://dmg.tuwien.ac.at/eppenschwandtner/grapheni.pdf`

[Fou80] FOURMAN, MICHAEL P. *Sheaf models for set theory.* J. Pure Appl. Algebra, 19:pp. 91–101, 1980

[Fre80] FREYD, PETER. *The axiom of choice.* J. Pure Appl. Algebra, 19:pp. 103–125, 1980

[Fre87] FREYD, PETER. *All topoi are localic or why permutation models prevail.* J. Pure Appl. Algebra, 46(1):pp. 49–58, 1987

[FS90] FREYD, PETER J. & SCEDROV, ANDRE. *Categories, allegories,* Vol. 39 of *North-Holland Mathematical Library.* North-Holland Publishing Co., 1990

[GJ95] GOLDSTERN, MARTIN & JUDAH, HAIM. *The incompleteness phenomenon.* A K Peters Ltd., 1995

[Hay81] HAYASHI, SUSUMU. *On set theories in toposes.* In *Logic Symposia, Hakone 1979, 1980 (Hakone, 1979/1980)*, Vol. 891 of *Lecture Notes in Math.*, 23–29. Springer, 1981

[Jac99] JACOBS, BART. *Categorical logic and type theory*, Vol. 141 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., 1999

[Jec03] JECH, THOMAS. *Set theory.* Springer Monographs in Mathematics. Springer, 2003

[JM95] JOYAL, ANDRÉ & MOERDIJK, IEKE. *Algebraic set theory*, Vol. 220 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1995

[Joh91] JOHNSTONE, PETER T. *The art of pointless thinking: a student's guide to the category of locales.* In *Category theory at work (Bremen, 1990)*, Vol. 18 of *Res. Exp. Math.*, 85–107. Heldermann, 1991

[Joh02] JOHNSTONE, PETER T. *Sketches of an Elephant: A Topos Theory Compendium 1-2*. Oxford University Press, 2002

[Kaw95] KAWAHARA, YASUO. *Relational set theory.* Nr. 953 in Lecture Notes in Computer Science. 1995

[Kel76] KELLY, GREGORY M. *MatSciNet Review on: Carboni, Aurelio, Categorie di relazioni.*. Mathematical Reviews, MR0506555 (58 #22228), 1976

[Kun83] KUNEN, KENNETH. *Set theory*, Vol. 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1983

[McL90] MCLARTY, COLIN. *The uses and abuses of the history of topos theory.* British J. Philos. Sci., 41(3):pp. 351–375, 1990

[ML98] MAC LANE, SAUNDERS. *Categories for the working mathematician*, Vol. 5 of *Graduate Texts in Mathematics*. Springer, 2. Ed., 1998

[MLM92] MAC LANE, SAUNDERS & MOERDIJK, IEKE. *Reals and forcing with an elementary topos.* In *Logic from computer science (Berkeley, CA, 1989)*, Vol. 21 of *Math. Sci. Res. Inst. Publ.*, 373–385. Springer, 1992

[MLM94] MAC LANE, SAUNDERS & MOERDIJK, IEKE. *Sheaves in geometry and logic*. Universitext. Springer, 1994

[Pho92] PHOA, WESLEY. *Categorical logic and type theory*. LFCS report ECS-LFCS-92-208, 1992
Open Access `http://www.lfcs.inf.ed.ac.uk/reports/92/ECS-LFCS-92-208/`

[PPT04] PICADO, JORGE, PULTR, ALEŠ & TOZZI, ANNA. *Locales*. In *Categorical foundations*, Vol. 97 of *Encyclopedia Math. Appl.*, 49–101. Cambridge University Press, 2004

[Sce84] SCEDROV, ANDRE. *Forcing and classifying topoi*. Mem. Amer. Math. Soc., 48(295):pp. x+93, 1984

[Sim99] SIMPSON, ALEX K. *Elementary axioms for categories of classes (extended abstract)*. In *14th Symposium on Logic in Computer Science (Trento, 1999)*, 77–85. IEEE Computer Soc., Los Alamitos, CA, 1999
URL `http://homepages.inf.ed.ac.uk/als/Research/classes.ps.gz`

[Tie72] TIERNEY, MYLES. *Sheaf theory and the continuum hypothesis*. In *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N.S., 1971)*, 13–42. Lecture Notes in Math., Vol. 274. Springer, 1972

The tag Open Access indicates that the article is published in an open access journal or archive. The other web links point to resources which are only temporary free or of unknown status with respect to access policy. All web addresses cited here have been checked on 22.10.2006.

# Index

# Curriculum Vitae

Wolfgang E. Eppenschwandtner
geb. 13. April 1979, Wien

Promenadegasse 57/D/5, 1170 Wien

weppens@dmg.tuwien.ac.at

**Ausbildung**

| | |
|---|---|
| 1989–1997 | Bundesrealgymnasium Wien 16, Maroltingergasse |
| 1997–2002 | Studium der Technischen Mathematik, Studienzweig: Mathematische Computerwissenschaften) und Informatik an der Technischen Universität Wien |
| 2002 | Diplom in Technischer Mathematik (ausgez. Erfolg) Diplomarbeit: *"Tensorprodukte topologischer Vektorräume - ein kategorientheoretischer Zugang"* [Epp02] |

**Beruf und Erfahrungen**

| | |
|---|---|
| seit 2004 | Projektassistent (FWF) am Institut für Diskrete Mathematik und Geometrie, TU Wien |
| 2005 | Forschungsaufenthalt an der *Université Catholique de Louvain* in Louvain-La-Neuve, Belgien |
| 2003-2005 | Studienrichtungsvertreter für Doktorat TU Wien |
| seit 2005 | Mitglied im Council von EURODOC, der europäischen Vereinigung für Doktoranden und Nachwuchsforscher |
| seit 2000 | Insgesamt 7 Semester Leiter einer Übungsgruppe Mathematik als Tutor/Studienassistent, TU Wien und Boku Wien. |

**Sonstiges**

| | |
|---|---|
| 2003-2005 | Förderung durch das Österreichische Studienförderungswerk PRO SCIENTIA |