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DISSERTATION

# Anomalous central charges and quantum masses of BPS states in supersymmetric field theories

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# Kurzfassung

Topologische Solitonen spielen eine entscheidende Rolle im Verständnis nichtstörungstheoretischer Effekte in supersymmetrischen Feldtheorien als auch Superstring-Theorien. Die Masse dieser Solitonen ist umgekehrt proportional zur Kopplungskonstante und gewinnen dadurch im Bereich starker Kopplungen an Bedeutung gegenüber den elementaren Quanten einer Quantenfeldtheorie.

In supersymmetrischen Theorien mit topologischen Solitonzuständen führen diese zu zentralen Ladungen in der Supersymmetriealgebra und zu einer sogenannten "BPS bound", die eine untere Schranke für die Masse der Solitonen darstellen. Im Falle, dass die Schranke abgesättigt wird, spricht man von BPS-Zuständen. Diese haben die besondere Eigenschaft, dass die Dimension der irreduziblen Darstellung der Supersymmetriealgebra gegenüber Nicht-BPS-Zuständen verringert ist. Dies führt zu einer exakten Relation zwischen Masse und zentraler Ladung dieser Zustände, die nicht durch störungstheoretische Korrekturen geändert werden kann.

In der hier vorgelegten Arbeit werden Korrekturen zur Masse sowie zur zentralen Ladung von solitonischen Zuständen in $\mathcal{N} = 1$ supersymmetrischen "kink" und "kink domain wall" Modellen sowie $\mathcal{N} = 2$ Vortizes in drei Dimensionen berechnet. Das Problem der Regularisierung divergenter Ausdrücke in den Quantenkorrekturen wird dabei elegant gelöst, in dem die betrachteten topologischen Objekte in einen höherdimensionalen Raum eingebettet werden. Die so gewonnenen flachen Extradimensionen werden zur dimensionalen Regularisierung benutzt. Die Supersymmetriealgebra wird durch eine die Supersymmetrie respektierende dimensionale Reduktion erhalten. Die zentralen Ladungen sind dann durch die Impulsoperatoren in der Extradimension gegeben. Diese enthalten Terme, die in der ursprünglichen Algebra nicht vorhanden sind und bei Entfernung der Regularisierung verschwinden. Unter Quantenkorrekturen kann jedoch auch nach Entfernung der regularisierenden Extradimension ein endlicher, anomaler, Beitrag bestehen bleiben. So konnten anomale Beiträge zur zentralen Ladung des supersymmetrischen "kink" und der dreidimensionalen supersymmetrischen "domain wall" identifiziert werden. Im Falle des supersymmetrischen Vortex konnte gezeigt werden, dass bisher nicht berücksichtigte Renormierungseffekte, die u.a. für die Eichinvarianz notwendig sind, zu einer Korrektur der Masse und der Ladung führen. In allen untersuchten Fällen wurde die Saturierung der BPS Schranke beobachtet, was vor allem im zweidimensionalen Fall zu einem kuriosen Grundzustand im topologischen Sektor führt, der keine im herkömmlichen Sinne definitive Fermionparität hat. Für diesen Fall konnte ein neuer Fermionparitätsoperator identifiziert werden, bezüglich dem der solitonische Grundzustand bosonisch ist und fermionische Nichtnullmodenanregungen tatsächlich fermionisch sind.

## Acknowledgement

I deeply thank my supervisor Tony Rebhan for a wonderful collaboration during the last two years and for his help to finish this work. I especially thank Peter van Nieuwenhuizen who has become a fatherly friend during our successful collaboration. I thank him very much for the physics he taught me and all the fun we had. The present work would not exist without Peter and Tony.

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# Contents

# 1 Introduction

During the last decade an enormous flurry of activity and also substantial progress has taken place in understanding non-perturbative effects in both supersymmetric field theories and superstring theories [121, 122]. Central to this is the occurrence of extended objects such as solitons and instantons [112], whose masses and actions are inversely proportional to coupling constants so that they gain importance in the strongly coupled regime. As first observed in the two-dimensional sine-Gordon theory [32, 36–39], there is the possibility of an intriguing duality between the ordinary elementary quanta of quantum field theory and bound states of solitons.

In supersymmetric soliton-bearing theories, topological solitons give rise to central charges in the supersymmetry algebra, which commute with all other operators of the algebra, and to a so-called BPS [15,110] bound on the mass spectrum. Classically this bound is saturated by solitons, i.e. exact non-trivial solutions of the non-linear field equations with finite energy or energy density[1] and nontrivial topology in field space. In the quantum theory such states, which saturate the BPS bound, are called *BPS (Bogomol'nyi-Prasad-Sommerfield) states.* This saturation of the bound is closely related to the fact that half (or a quarter, depending on the model under consideration) of the supersymmetry is conserved by BPS states. This is sometimes equally well used as the definition of a BPS state or classical BPS solution, respectively. In the case of BPS saturation the irreducible representations have the same dimension as massless representations rather then massive representations. Thus so called BPS super-multiplets contain less states then massive representations which do not saturate the BPS bound and as a consequence of this "multiplet shortening" there exist exact constraints on the spectrum of quantized solitons as shown by Witten and Olive [146]. If in some approximation equality between the mass of topological states and its topological quantum number (the central charge) is found, this saturation is protected against perturbative quantum corrections, since the number of particle states cannot be changed by perturbative corrections. This is sometimes overstated as implying that there are no quantum corrections to the classical mass spectrum at all, while it rather means that such quantum corrections have to affect mass and central charge by equal amounts. The complete vanishing of quantum corrections to the mass and because of BPS saturation also to the central charge has a different origin than multiplet shortening. This is due to non-renormalization theorems as for example for the superpotential in (generalized) $\mathcal{N} = 1$ Wess-Zumino models in four dimensions [72]. Such non-renormalization theorems are then bequeathed to models obtained by a susy preserving dimensional reduction as for example

[1] We are somewhat cavalier in the notion of solitons since we also consider extended objects like domain walls, which have only finite energy density in a non-compact space.

the two-dimensional $\mathcal{N} = 2$ susy-kink model [104].

In two-dimensional minimal, i.e. $\mathcal{N} = 1$, supersymmetric WZ-models no non-renormalization theorem does apply and nontrivial corrections to the mass and the central charge should be expected. However, the technicalities in these computations are quite involved. This issue was reopened when A. Rebhan and P. v. Nieuwenhuizen found [113] that the simple energy-momentum cutoff used explicitly or implicitly in most of the calculations that obtained a null result was inconsistent with the integrability of the bosonic sine-Gordon model [39]. Careful calculations using mode number cutoff such that boundary energy contributions are avoided subsequently established a nonzero result for the quantum corrections to the mass [61, 63, 98, 104] that agreed in fact with an older result by Schonfeld [120] and which has also been reproduced by different methods [19, 26, 67]. Another involved issue in these two dimensional models is the multiplet shortening mechanism. Irreducible short $\mathcal{N} = 1$ super-multiplets in two dimensions are *one*-dimensional. Thus this single state has to be an eigenstate of one of the two supercharges, which is a fermionic operator in the usual sense of fermion parity. Thus it is impossible to assign a definite fermion parity to such a state in the usual sense, it is half fermionic and half bosonic. Therefore multiplet shortening was discarded at the prize of a reducible two-dimensional representation representation [146]. But the absence of multiplet shortening provides a possible non-saturation of the BPS bound, i.e. a mismatch of the quantum corrections to mass and central charge [113]. A negative mass correction obtained in [104] together with a vanishing correction for the central charge implied not only non-saturation but even violation of the lower BPS bound for the mass. Therefore the authors of [104] conjectured the existence of an anomaly. It was then established by Shifman et al. [123], using a susy-preserving higher-derivative regularization method, that there is an anomalous contribution to the central charge which still leads to BPS saturation at the quantum level, and which was subsequently explained by the possibility of "super-short" single-state super-multiplets in 1+1 dimensions [61, 100] which have no definite fermion number in the usual sense.

Another interesting supersymmetric model providing classical topological soliton solutions is the $\mathcal{N} = 2$ extension of the abelian Higgs model in $2 + 1$ dimensions. The stable topological solitons of this model are the Abrikosov-Nielsen-Oleson [1,42,105,131] vortices, which are an important input in the context of mirror symmetry in (1+1) and (2+1) dimensions [4,80]. In the existing literature [97, 119] a null result for both the mass and the central charge was obtained. However, also this model does not provide a non-renormalization theorem and thus a vanishing result for the quantum corrections called for an explanation. The present author therefore suggested a non-vanishing correction by a renormalization effect which after a private communication was then subsequently obtained in [135]. Also the incomplete null result of ref. [97, 119], where the calculations are lacking of a proper regularization, has been proven in a mathematical rigorous manner in [135].

The present work for the most part is based on research which has been carried out in collaboration with A.Rebhan and P.van Nieuwenhuizen and partly with A. Goldhaber [63, 115–117]. Each chapter focus on different aspects and are almost self-contained. We discuss in separate introductions the details of every particular chapter. Here we give an overview and try to outline the "big picture" which brings all separated parts together.

The renewed activity in this field started, as discussed above, by the discovery of subtleties in the regularization process for quantum correction in the presence of a topological background. In the present work we develop an elegant and susy-preserving variant of dimensional regularization. For supersymmetric theories in the presence of a nontrivial background one has to circumvent two problems: (i) The preservation of susy when changing the dimension and (ii) that the loop-momentum integrals involve nontrivial functions due to the background so that the standard formulas of 't Hooft and Veltman's dimensional regularization are not applicable. Both problems are solved by embedding the nontrivial soliton background in a higher dimensional model with the same field and susy content. At first sight it seems exceptional that such a higher dimensional model exists, but noting that topological states involve central charges in the susy algebra which, except in two dimensions, on representation-theoretical grounds imply extended supersymmetry this is less surprising. In this way four-dimensional $\mathcal{N} = 2$ monopoles are obtained from a six-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory, the three-dimensional $\mathcal{N} = 2$ vortex is obtained from a four-dimensional $\mathcal{N} = 1$ abelian Higgs model and the two-dimensional $\mathcal{N} = 1$ susy kink is obtained from the same Lagrangian in three dimensions. The flat extra-dimension obtained by this embedding is then used for the analytic continuation slightly below the original dimension as usual in dimensional regularization. The susy algebra is obtained by susy-preserving dimensional reduction of the higher dimensional susy algebra. The central charges are then given by the momentum operator in the flat extra-dimension, where the classical central charge, which is also present in the lower-dimensional model, stems from the *antisymmetric* part of the energy momentum tensor while the symmetric part and thus the genuine momentum in the higher dimensional model is only present through the regularization and gives rise to a possible anomalous contribution to the central charge. We always observe the important property that the Dirac operators in a classical BPS background provide a susy-quantum mechanical system, where different components of the fermionic quantum fields are paired to susy-partners. This structure makes it possible to reduce the possible anomalous contributions to surface terms which are thus completely given by the topology of the soliton background. This is quite analogous to chiral anomalies, which are determined by the topology of the gauge field and which can also be related to the index of Dirac operators.

As second general issue is the concept of renormalization. The UV-structure and thus the renormalizability is determined by the vacuum sector and upon specifying one's renormalization conditions, quantum corrections should be calculable also in the topological sector without ambiguity. However, despite of the freedom in specifying one's renormalization conditions, there is need for a complete renormalization of tadpoles, even if they are finite as in odd dimensions in dimensional regularization at one-loop. In the vacuum sector this stabilizes the point of expansion, i.e. the vacuum and in the topological sector this guarantees that the soliton field profile assumes the vacuum value asymptotically. As we will see, the renormalization of tadpoles is also necessary for gauge invariance.

This work is organized as follows: In section 2 we introduce our variant of dimensional regularization by embedding the two-dimensional kink as a domain wall in higher dimensions, whose tensions are obtained in one line with the mass of the (susy) kink. In section 3 we consider once again the mode regularization of the susy-kink and obtain an explanation for the need of averaging over different boundary conditions in terms of the invariance under discrete symmetries. In section 4 we derive the anomalous contribution to the central charge of the susy-kink and susy-kink domain wall. For the susy-kink an anomalous current is identified as the susy transformation of an evanescent counter term in the susy current. For the single-state short susy-multiplet we find a new fermion parity operator, such that the soliton ground state is even and fermionic non-zero-mode excitations are odd. In section 5 we discuss the anomaly multiplet structure obtained by dimensional reduction of supercurrent superfield and compute the non-vanishing corrections to the mass and central charge of the $\mathcal{N} = 2$ vortex.

# 2 Surface tensions of supersymmetric kink domain walls

## 2.1 Introduction

One of the simplest situations where one can study quantum corrections to non-trivial background fields is the calculation of the quantum mass of 1+1-dimensional solitons with exactly known fluctuation spectra [32,37,39, 41, 112, 136]. One-loop corrections can be obtained from computing the difference of the sums (and integrals) of zero-point energies in the soliton background and in the topologically trivial vacuum. The regularization of these sums is a surprisingly delicate matter whose subtleties have been investigated only rather recently, starting with the observation [113] that for example a simple energy-momentum cutoff leads to incorrect results if the same cutoff is used in the topologically distinct sectors. This has been an actual problem in the calculation of the quantum mass of supersymmetric solitons [28,29,84,88,148]. On the other hand, the extension of the mode-number cutoff regularization method introduced by Dashen et al. [37], which begins by discretizing the problem by means of a finite volume, to fermions turns out to lead to new subtleties concerning the choice of boundary conditions which may or may not entail a contamination through energies localized at the boundaries [61,104,123,145].

However, there do exist methods which give correct results that can be formulated a priori in the continuum. In Ref. [104] it has been shown that the derivative of the quantum kink mass with respect to the mass of elementary scalar bosons is less sensitive and can be calculated by energy cutoff regularization, leading to a result for the quantum mass of susy kinks that agrees with S-matrix factorizations [5, 6], validating also previous results obtained by Schonfeld who considered mode-number regularization of the kink-antikink system [120], and by Refs. [21,23,26] using a finite mass formula in terms of only the discrete modes. In Refs. [48,66–68], another viable continuum approach was developed that is based on subtracting successive Born approximations for scattering phase shifts. Ref. [123] introduced susy-preserving higher (space) derivative terms in the action and obtained the correct one-loop results for the energy and the central charge from simple Feynman graphs. Also heat-kernel and zeta-function regularization methods have been applied successfully to this problem [7,19].

In Ref. [107] it has been shown that dimensional regularization through embedding kinks as domain walls in extra dimensions reproduces the known result for the bosonic kink mass, but it was concluded that this method may be difficult to generalize.

The present section relies on [115] and extends the analysis of Ref. [107]. We demonstrate that dimensional regularization also allows one to calculate the surface tensions of kink domain walls in a way that is far simpler than the methods used previously. Moreover, the consideration of domain walls gives insight into where precisely naive cutoff regularization fails, and resolves its ambiguities by observing that finite ambiguities become divergences in higher dimensions. Requiring finiteness in $d+1$ dimensions thus fixes the finite ambiguities in $1+1$ dimensions. In this way we confirm the recent observation in Ref. [98] that the defective energy cutoff method can be repaired by using smooth cutoffs, or sharp cutoffs as limits of smooth ones.

Through dimensional regularization we derive a remarkably compact formula for surface tensions that unifies the diverse results on kink domain walls in 2+1 and 3+1 dimensions, and yields a finite result even in 4+1 dimensions. We discuss the effects of using different renormalization schemes and confirm (most of the) previous one-loop results in the literature on kink domain walls in 2+1 and 3+1 dimensions.

We also show that this way of dimensional regularization works for the supersymmetric case by re-deriving the quantum mass of the 1+1 supersymmetric kink, and find a new result for a 2+1 dimensional supersymmetric kink domain wall with chiral domain-wall fermions, which unlike its 3+1 dimensional analogue has nonzero quantum corrections.

## 2.2 Bosonic kink and dimensional regularization

In 1+1 dimensions, a real $\varphi^4$ theory with spontaneously broken $Z_2$ symmetry $(\varphi \to -\varphi)$

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \varphi_0)^2 - \frac{\lambda_0}{4}(\varphi_0^2 - \mu_0^2/\lambda_0)^2 \tag{1}$$

has topologically non-trivial solutions to the field equations with finite energy: solitons called "kinks", which interpolate between the two degenerate vacuum states $\varphi_0 = \pm\mu_0/\sqrt{\lambda_0} \equiv \pm v_0$. Expressed in terms of renormalized parameters,

$$\begin{aligned} \varphi_0 &= \sqrt{Z}\varphi \quad , \quad Z = 1 + \delta Z \ , \\ \lambda_0 &= Z_\lambda \lambda = \lambda + \delta\lambda \ , \\ v_0^2 &= Z_{v^2} v^2 = v^2 + \delta v^2 \ , \end{aligned} \tag{2}$$

the Lagrangian (1) reads as[2]

$$\begin{aligned} \mathcal{L} = &-\tfrac{1}{2}(\partial\varphi)^2 - \tfrac{\lambda}{4}(\varphi^2 - v^2)^2 \\ &-\tfrac{1}{4}(\delta\lambda + 2\lambda\delta Z)(\varphi^2 - v^2)^2 + \tfrac{\lambda}{2}(\delta v^2 - v^2\delta Z)(\varphi^2 - v^2) \\ &-\tfrac{1}{2}\delta Z(\partial\varphi)^2 + O(\hbar^2), \end{aligned} \tag{3}$$

[2]In a more convenient renormalization scheme, $Z^2\lambda_0 = \lambda + \delta\lambda$ , $\frac{1}{Z}v_0^2 = v^2 + \delta v^2$ the counter term Lagrangian has a more compact form. But for our purposes the scheme (2) leads to simplifications.

where we have omitted higher order counter terms. A kink/anti-kink at rest at $x = x_0$ is classically given by [112]

$$\varphi_{K,\bar{K}} = \pm v \tanh(\tfrac{\mu(x-x_0)}{\sqrt{2}}). \tag{4}$$

This is an exact solution of the classical action, i.e. the first line in (3) but it is not a stationary point of the full Lagrangian (3). This leads to corrections in the field profile through terms linear in the quantum field [62, 123] but does not change the energy at one-loop order (see the remarks at the end of section 5.4 in [112]). Embedding the kink solution in $(d+1)$ dimensions instead of (1+1) gives a domain wall separating the two distinct vacua. This is no longer a finite-energy solution—its energy is proportional to the transverse volume $L^{d-1}$, with classical energy density (surface tension)

$$\mathcal{E}_{DW} = M_0/L^{d-1} = \tfrac{2\sqrt{2}\mu_0^3}{3\lambda_0} = \tfrac{2\sqrt{2}\mu^3}{3\lambda} + O(\hbar), \tag{5}$$

where we have again expressed the classical quantity in terms of the renormalized parameters. In the following both forms will be useful for us.

In $(d+1) \leq 4$ dimensions, (1) is renormalizable or super-renormalizable, and upon specifying one's renormalization conditions, quantum corrections to the energy density should be calculable in perturbation theory without ambiguity. Some authors are somewhat cavalier with regard to fixing the meaning of the parameters of the theory through the renormalization conditions, making their results basically meaningless: since the lowest order involves two parameters, any one- or two-loop result is correct in some renormalization scheme.

In $1+1$ dimensions, where kinks correspond to particles with a calculable quantum mass determined by the parameters of the Lagrangian, the most frequently used renormalization scheme consists of demanding that the tadpole diagrams cancel in their entirety, while $\lambda = \lambda_0$ and $\varphi = \varphi_0$. The cancellation of tadpoles is required by consistency of the perturbation theory in the vacuum sector as well as in the topological sector. In the vacuum sector it guarantees that the starting point of the field expansions remains the vacuum under quantum corrections,

$$\langle \phi \rangle = v \quad \leftrightarrow$$  $$= 0, \tag{6}$$

which in turn guarantees that topological solutions, like the kink (4), terminates in the vacuum, i.e. a zero energy configuration, at spatial infinity. Thus its mass or tension, respectively, is still finite under quantum corrections.

Such a renormalization scheme can still be used in $2+1$ dimensions, whereas in $3+1$ dimensions there is finally the need to renormalize the coupling constant non-trivially in order to absorb all one-loop divergences. In the following we shall concentrate on the particularly natural scheme which fixes the *coupling* constant renormalization such that in addition to the absence of tadpole diagrams the renormalized *mass* of the elementary scalar be equal to the pole of its propagator.

Wave-function renormalization $\varphi_0 = \sqrt{Z}\varphi$ is finite to one-loop order in $3+1$ dimensions and to all orders in lower dimensions and it is therefore not mandatory for the one-loop corrections to the energies of kinks and kink domain walls. Nevertheless we will also consider wave-function renormalizations to compare our results, obtained by embedding in and reduction from an higher dimension, with partial existing results in the literature, to make sure that our method is based on working principles. This will be important in the case of even more delicate calculations as for the central charge corrections considered below.

With (2), the renormalized Lagrangian for elementary excitations $\eta$ around the minimum $v = \frac{\mu}{\sqrt{\lambda}}$, i.e $\varphi = v + \eta$, then reads

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{2}(\partial\eta)^2 - \lambda v^2\eta^2 - \lambda v\eta^3 - \tfrac{\lambda}{4}\eta^4 \\ &\quad +\lambda v(\delta v^2 - v^2\delta Z)\eta + \tfrac{1}{2}(\lambda\delta v^2 - 2v^2\delta\lambda - 5\lambda v^2\delta Z)\eta^2 \\ &\quad -v(\delta\lambda + 2\lambda\delta Z)\eta^3 - \tfrac{1}{4}(\delta\lambda + 2\lambda\delta Z)\eta^4 - \tfrac{1}{2}\delta Z(\partial\eta)^2 + O(\hbar^2) \qquad (7)\end{aligned}$$

which shows that the renormalized mass of the elementary boson at tree-graph level is $m^2 = 2\mu^2$, so that the propagator is given by

$$\Delta(p) = \frac{-i}{p^2 + m^2 - i\varepsilon}. \qquad (8)$$

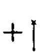
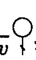
The other *Feynman rules* are:

 $= -i3!\lambda v$ ,  $= -i3!\lambda$ ,  $= i\lambda v(\delta v^2 - v^2\delta Z)$

 $= i(\lambda\delta v^2 - 2\delta\lambda v^2) - i(5\lambda v^2 + p^2)\delta Z.$ (9)

### 2.2.1 Tadpole renormalization with $Z = 1$

For simplicity we choose $Z = 1$, $\delta Z = 0$ for now, postponing the discussion of schemes with nontrivial $Z$ to sect. 2.3.2. The requirement that tadpole graphs are completely canceled by the counter term proportional to $\eta$ fixes $\delta v^2$ at one-loop level:

 $+$  $\overset{!}{=} 0 \quad \Rightarrow \quad \delta v^2 = \frac{i}{\lambda v}$ , (10)

which gives with the rules (9) and including a symmetry factor $\frac{1}{2}$:

$$\delta v^2 = -3i\hbar\int \frac{dk_0 d^d k}{(2\pi)^{d+1}} \frac{1}{k^2 + m^2 - i\varepsilon} = 3\hbar\int \frac{d^d k}{(2\pi)^d} \frac{1}{2[\vec{k}^2 + m^2]^{1/2}}, \qquad (11)$$

where we have re-introduced the $\hbar$-dependence. Using dimensional regularization, where for Euclidean momenta $k_{\mathrm{E}}$

$$\int d^{2\nu} k_{\mathrm{E}} (k_{\mathrm{E}}^2 + M^2)^{-\alpha} = \pi^{\nu} (M^2)^{\nu-\alpha} \Gamma(\alpha - \nu)/\Gamma(\alpha), \tag{12}$$

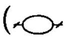
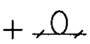
and writing $d = 1 + s$ so that $s$ denotes the number of spatial dimensions orthogonal to the kink axis, we have (setting $\hbar = 1$ henceforth)

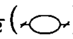
$$\delta v^2 = \frac{3}{2\pi}(1+s) \frac{\Gamma(\frac{-1-s}{2})}{\Gamma(-\frac{1}{2})(4\pi)^{\frac{s}{2}}} \int_0^{\infty} dk (k^2 + m^2)^{\frac{s-1}{2}}, \tag{13}$$

which is written in a form that will turn out to be convenient shortly.

### 2.2.2 On-shell renormalization

Calculating the one-loop correction to the pole mass of the elementary bosons involves local sea-gull diagrams that are exactly canceled by $\delta v^2$ and a non-local diagram with 3-vertices. According to (7) the renormalized mass $m$ will be equal to the pole mass, if the latter diagram evaluated on-shell is canceled by the counter term $\propto \delta\lambda\eta^2$:

(  +  +  $)_{|p^2=-m^2}$ = (  $- 2iv^2\delta\lambda)_{|p^2=-m^2} \stackrel{!}{=} 0$

$\Rightarrow \quad \delta\lambda = \frac{1}{2iv^2}($  $)_{|p^2=-m^2}.$ (14)

Evaluating the graph (see appendix) this determines $\delta\lambda$ as

$$\begin{aligned} \delta\lambda &= 9\lambda^2 \frac{m^{s-2}}{(4\pi)^{\frac{s+2}{2}}} \Gamma(\tfrac{2-s}{2}) \textstyle\int_0^1 dx [1 - x(1-x)]^{\frac{s-2}{2}} \\ &= 9\lambda^2 \frac{m^{s-2}}{(4\pi)^{\frac{s+2}{2}}} \Gamma(\tfrac{2-s}{2}) \left(\tfrac{3}{4}\right)^{\frac{s-2}{2}} {}_2F_1(\tfrac{2-s}{2}, \tfrac{1}{2}; \tfrac{3}{2}; -\tfrac{1}{3}), \end{aligned} \tag{15}$$

where we used ${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt$. For $s \to 2$, i.e. when considering the (3+1)-dimensional theory, $\delta\lambda$ contains a divergence. For $s < 2$, as we have remarked, the choice $\delta\lambda = 0$ is also a possible renormalization scheme, and we shall consider it, too, when applicable.

### 2.2.3 Topological sector

In 1+1 dimensions, the one-loop quantum corrections to the mass of a kink are determined by the functional determinant of the differential operator describing fluctuations around the classical solution (4) compared to that of the trivial vacuum, leading formally to a sum over zero-point energies which contribute according to

$$M^{(1)} = M_0 + \frac{\hbar}{2} \left( \sum \omega - \sum \omega' \right) + O(\lambda) \tag{16}$$

where $\omega$ and $\omega'$ are the eigen frequencies of fluctuations around a kink and the vacuum, respectively. The individual sums as well as their difference are ultraviolet divergent. The latter divergence is removed by the contribution of the counter term Lagrangian (3) to the energy in the kink background,

$$\begin{aligned}\delta M = &\tfrac{1}{2}\delta Z \int dx \; (\partial_x \varphi_K)^2 \\ &-\tfrac{\lambda}{2}(\delta v^2 - v^2 \delta Z) \int dx \; (\varphi_K^2 - v^2) + \tfrac{1}{4}(\delta\lambda + 2\lambda\delta Z) \int dx \; (\varphi_K^2 - v^2)^2 \\ = &m \delta v^2 + \frac{m^3}{6\lambda^2}\delta\lambda. \end{aligned} \tag{17}$$

Note that the terms proportional $\delta Z$ vanish because of the *Bogomol'nyi* equation for the kink,

$$\partial_x \varphi_K = U(\varphi_K) = \sqrt{\tfrac{\lambda}{2}}(\varphi_k^2 - v^2). \tag{18}$$

For a simple two dimensional scalar field model this is equivalent to the equipartition theorem, but as will we see later this is an important structure especially in the supersymmetric case. Since the field strength renormalization is a variation of the field and thus proportional to the classical e.o.m. the one loop contribution proportional $\delta Z$ obviously vanishes for a classical background. Nevertheless the form as given in the first line of (17) will be useful for us. Equivalently the counterterms are obtained by rewriting the bare kink mass $M_0$ in terms of renormalized parameters

$$M_0 = \frac{2\sqrt{2}}{3}(v_0^2)^{3/2}\lambda_0^{1/2} = \frac{m_0^3}{3\lambda_0} = \frac{m^3}{3\lambda} + \delta M \tag{19}$$

with

$$\delta M = m\delta v^2 + \frac{m^3}{6\lambda^2}\delta\lambda \equiv \delta_v M + \delta_\lambda M \tag{20}$$

However, as reviewed in the introduction, the regularization of the sums over zero-point energies is a highly delicate matter, and for instance a simple cutoff regularization fails [113]. Using the same sharp cutoff in energy or, equivalently, momentum in both the trivial and soliton sector, gives a finite result where the cutoff can be removed, but this differs from other regularization procedures by a finite amount. In fact, it has been shown that cutoff regularization can be repaired by using smooth cutoffs [98] which are in fact also required in the calculation of Casimir energies in order that sums over zero-point energies there can be evaluated by means of the Euler-McLaurin formula [86]. The limit of a sharp cutoff differs from a straightforward sharp cutoff by a delta-function peak in the spectral density at the integration boundary which must not be omitted. A completely different procedure using sharp cutoffs which depend on the coordinate $x$ has recently been proposed in Ref. [62] and independently in Ref. [145]. This "local mode regularization" has been used in Ref. [62] to calculate the local distribution of the quantum energies of 1+1 dimensional solitons.

In the following, we shall however employ dimensional regularization, which has been shown in Ref. [107] to reproduce correctly the quantum mass of the bosonic 1+1 dimensional kink, and also consider the higher-dimensional kink domain walls.[3] By analytic continuation of the number $s$ of extra transverse dimensions of a kink domain wall, no further regularization is needed. In the vacuum this is indeed consistent with standard (isotropic) dimensional regularization over $s+1$ spatial dimensions, as its formulae continue to apply if one first integrates over a subset of dimensions.

Denoting the momenta pertaining to the $s$ transverse dimensions by $\ell$ and reserving $k$ for the momentum along the kink, i.e. perpendicular to the kink domain wall, the energy of the latter per transverse volume $L^s$ follows from (16)

$$\begin{aligned}\frac{M^{(1)}}{L^s} &= \frac{m^3}{3\lambda}+\frac{1}{2}\sum_B\int_{-\infty}^{\infty}\frac{d^s\ell}{(2\pi)^s}\sqrt{\omega_B^2+\ell^2}\\ &+\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk\,d^s\ell}{(2\pi)^{s+1}}\sqrt{k^2+\ell^2+m^2}\,\delta_K'(k)+\delta M \qquad (21)\end{aligned}$$

where the discrete sum is over the normalizable states $B$ of the 1+1-dimensional kink with energy $\omega_B$, and the integral is over the continuum part of the spectrum. We give a more detailed derivation of formulae like this one in section 4 below. For the moment we focus on the regularization of such expressions.

The spectrum of fluctuations for the 1+1-dimensional kink is known exactly [112]. It consists of a zero-mode, a bound state with energy $\omega_B^2/m^2 = 3/4$, and scattering states $\phi_k$ in a reflectionless potential for which the phase shift $\delta_K(k) = -2\arctan(3mk/(m^2-2k^2))$ in the kink background provides the difference in the spectral density between kink and trivial vacuum

$$\int_{-\infty}^{\infty}dx\,(|\phi_k(x)|^2-1)=\delta_K'(k)=-\frac{3m}{2}\frac{2k^2+m^2}{(k^2+m^2)(k^2+m^2/4)}, \qquad (22)$$

The zero mode ($\omega_B = 0$), which trivially does not contribute to the mass of a kink because of its vanishing energy, corresponds to a massless mode with energy $\sqrt{\ell^2}$ for $s \neq 0$, but does also not contribute to the energy densities of kink domain walls in dimensional regularization, because in the latter integrals without a mass scale vanish. However, it does contribute in cut-off regularization, as we shall discuss further below.

[3]Dimensional regularization adapted to domain wall configurations has in fact been discussed already long ago in Ref. [16], however without giving concrete results for the surface tension.

The nontrivial contribution comes from the integral in (21) containing the phase shift. It is this part which led to a lot of confusion during the last decade. The remarkable thing of the dimensional regularization by embedding the nontrivial background is now as follows. The $\ell$ integral in (21) can be treated with standard methods of dimensional regularization as an $s$-dimensional integral, i.e. by analytical continuation in $s$ of the resulting integral (see appendix)

$$\int \frac{d^s \ell}{(2\pi)^s} (\ell^2 + k^2 + m^2)^{\frac{1}{2}} = \frac{1}{(4\pi)^{s/2}} \frac{\Gamma(\frac{-1-s}{2})}{\Gamma(\frac{1}{2})} (k^2 + m^2)^{(s-1)/2}. \tag{23}$$

Thus the continuous dimension of the $\ell$ integration has become a continuous exponent for the remaining strict one dimensional $k$-integration. The $k$-integration is, because of the nontrivial background reflected by the phase shift (22), originally not of the form of a standard integral in dimensional regularization. Now this integral can again be evaluated by using the 't Hooft-Veltman integration formulas, but now for an integer dimension and an arbitrary, and finally analytical continued exponent $s$.

The leading divergence in the last integral of (21) matches the divergence in $\delta_v M$ and can be combined with it using (13) to give (with $x \equiv k/m$)

$$\begin{aligned} \frac{M^{(1)}}{L^s} &= \frac{m^3}{3\lambda} + \frac{\Gamma(\frac{-1-s}{2}) m^{s+1}}{\Gamma(-\frac{1}{2})(4\pi)^{\frac{s}{2}}} \Bigg\{ \frac{1}{2}\left(\frac{3}{4}\right)^{\frac{s+1}{2}} \\ &\quad + \frac{3}{4\pi} \int_{-\infty}^{\infty} dx (x^2+1)^{\frac{s-1}{2}} \left[ \frac{-1}{4x^2+1} + s \right] \Bigg\} + \delta_\lambda M. \end{aligned} \tag{24}$$

Here the first term inside the braces is the contribution from the bound state with nonzero energy.

In the limit $s \to 0$, which corresponds to the 1+1 dimensional kink, where one may renormalize "minimally" by putting $\delta_\lambda M = 0$, one obtains

$$\Delta M^{(1)}_{s=0} \equiv M^{(1)}_{s=0} - \frac{m^3}{3\lambda} = \frac{m}{4\sqrt{3}} - \frac{3m}{2\pi}, \tag{25}$$

reproducing the well-known DHN result [37]. It is interesting to note that it is the last term in (25) that would be missed in a sharp-cutoff calculation (see Ref. [113]) and that it now arises from the last term in the square brackets of (24). The latter arises because the counter term due to $\delta v^2$ does no longer match all of the divergences of the integral involving $\delta'_K$ for $s > 0$, but dimensional regularization gives a finite result as $s \to 0$.

In energy cutoff regularization this term can be recovered by implementing the cutoff as $\delta(k) \to \delta(k)\theta(\Lambda - k)$ which gives a Dirac-delta in the spectral density by differentiating $\theta$ [98] and a finite contribution because the scattering phase $\delta(k)$ decays only like $1/k$ at large momenta. The need for such subtle corrections is nicely avoided by dimensional regularization:

for sufficiently negative transverse dimensionality $s$ the ultraviolet behavior of the scattering phases in the longitudinal direction is made harmless.

For $s = 1, 2, 3$, the integral in (24) is divergent and gives poles in dimensional regularization, but as the final results will show, these divergences are canceled by the other terms in (24): for $s = 1, 3$, they come from the bound state contribution, whereas for $s = 2$, they are provided by $\delta_\lambda M$.

However, naive cutoff regularization would give rise to problems which in fact point to the necessity of its modification as in Ref. [98]. In contrast to dimensional regularization, cutoff regularization leads to singularities for linear and quadratic divergences. Let us consider as an example the 2+1 case, i.e. $s = 1$. Using a sharp cutoff in the $k$-integral of (21) and $\delta M = \delta_v M$, one can combine these integrals yielding

$$\frac{M^{(1)}_{s=1}}{L} = \frac{m^3}{3\lambda} + \int_{-\infty}^{\infty} \frac{d\ell}{2\pi} \Big\{ \frac{1}{2}\sqrt{\ell^2} + \frac{1}{2}\sqrt{\ell^2 + 3m^2/4}$$
$$-\frac{1}{\pi}\left[\sqrt{\ell^2}\arctan\sqrt{\ell^2/m^2} + \sqrt{\ell^2 + 3m^2/4}\arctan\sqrt{3 + 4\ell^2/m^2}\right]\Big\}. \quad (26)$$

In this expression, the quadratic divergences cancel (for which it is necessary that the kink zero-mode is not omitted!), but because $\arctan(x) = \pi/2 - 1/x + O(1/x^2)$ for large $x$ the terms in the square bracket also contain linear divergences that do not cancel. However, if the $k$-integral in (21) is evaluated with a cutoff that is obtained from a smooth cutoff through a limiting procedure, the Dirac-delta peak in the spectral density [98] contributes the additional term

$$\lim_{\Lambda_k \to \infty} \int_{-\infty}^{\infty} \frac{d\ell}{2\pi} \frac{-3m\sqrt{\Lambda_k^2 + \ell^2 + m^2}}{2\pi\Lambda_k} \quad (27)$$

where we have used $\delta(\Lambda_k) \sim 3m/\Lambda_k$. This renders the complete result finite, and equal to that obtained in dimensional regularization.

Our study of domain walls thus resolves the ambiguities previously found in the calculation of the kink mass. Finite ambiguities in 1+1 dimensions become divergences in $d + 1$ dimensions with $d > 1$. Requiring finiteness in $d + 1$ dimensions fixes the finite ambiguities in 1+1 dimensions.

## 2.3 Surface tension of bosonic kink domain walls

For $d > 1$, it is straightforward to extract the finite answers for the one-loop surface tensions of the bosonic kink domain walls by expanding $s$ around integer values, which leads to elementary integrals. But instead of giving these individual results, some of which have been obtained previously, we shall aim at covering them all together.

| $s$ | ${}_2F_1(\frac{2-s}{2},\frac{1}{2};\frac{3}{2};-\kappa)$ |
|---|---|
| 0 | $\arctan(\sqrt{\kappa})/\sqrt{\kappa}$ |
| 1 | $\mathrm{Arsinh}(\sqrt{\kappa})/\sqrt{\kappa}$ |
| 2 | 1 |
| 3 | $\frac{1}{2}\left[\sqrt{1+\kappa}+\mathrm{Arsinh}(\sqrt{\kappa})/\sqrt{\kappa}\right]$ |

Table 1: *Special cases of ${}_2F_1$ in (28) for the values $s$ of physical interest.*

#### 2.3.1 Renormalization schemes with $Z = 1$

First we shall consider renormalization schemes where the wave-function renormalization constant is kept at $Z = 1$ so that $\varphi = \varphi_0$, which is a valid and convenient choice at all loop orders for $s < 2$ and to one-loop order for $s = 2$.

For general non-integer $s$, the integral in (24) can be expressed in terms of the same hypergeometric function that appeared in the counter term $\delta\lambda$, eq. (15), which was chosen so as to let $m$ coincide with the physical pole mass of the elementary scalar bosons.[4] This leads to the following remarkably compact formula for the energy densities of $s$-dimensional bosonic kink domain walls

$$\frac{\Delta M^{(1)}_{\rm OS}}{L^s} = \frac{m^{s+1}}{(4\pi)^{\frac{s+2}{2}}}\frac{2\Gamma(\frac{2-s}{2})}{s+1}\left\{(s+2)\left(\frac{3}{4}\right)^{\frac{s}{2}}{}_2F_1(\frac{2-s}{2},\frac{1}{2};\frac{3}{2};-\frac{1}{3})-3\right\}, \tag{28}$$

where $m$ is the physical (pole) mass of the elementary scalar, and the term proportional to $-3$ is produced by the term proportional to $s$ in (24). This is a finite expression for $-1 < s < 4$. (The more minimal renormalization scheme where $Z_\lambda = 1$, which is possible for $s < 2$ only, is obtained by replacing $(2 + s)$ in the first term by 1.)

For the integer values of $s$ of physical interest, the hypergeometric function in (28) can be reduced to elementary functions given in Table 1.

In the 3+1 dimensional case, one has ${}_2F_1(0,\dots) \equiv 1$, giving a zero for the content of the braces in eq. (28), but multiplying a pole of the Gamma function. Here one has to expand around $s = 2$, for which one needs the

[4]Using for example formula (3.259.3) of Ref. [64] together with the linear transformation formulas (9.131), (9.132).

| s | $\frac{\Delta M^{(1)}}{L^s}/m^{s+1}$ (OS) | $\frac{\Delta M^{(1)}}{L^s}/m^{s+1}$ (MR) |
|---|---|---|
| 0 | $\frac{1}{2\sqrt{3}} - \frac{3}{2\pi} \approx -0.189$ | $\frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \approx -0.333$ |
| 1 | $\frac{3}{32\pi}(3\ln 3 - 4) \approx -0.0210$ | $\frac{3}{32\pi}(\ln 3 - 4) \approx -0.0866$ |
| 2 | $\frac{3}{16\pi^2} - \frac{1}{8\pi\sqrt{3}} \approx -0.00397$ | - |
| 3 | $\frac{9(4-5\ln 3)}{(32\pi)^2} \approx -0.00133$ | - |

Table 2: *One-loop contributions to the quantum mass of the bosonic kink ($s = 0$) and to the surface tension of $s$-dimensional domain walls for the on-shell (OS), and minimal renormalization (MR) schemes, both with wave-function renormalization $Z = 1$.*

| s | $\frac{\Delta M^{(1)}}{L^s}/m^{s+1}$ (OSR) | $\frac{\Delta M^{(1)}}{L^s}/m^{s+1}$ (ZM) |
|---|---|---|
| 0 | $\frac{2}{3\sqrt{3}} - \frac{2}{\pi} \approx -0.252$ | $\frac{1}{4\sqrt{3}} - \frac{19}{16\pi} \approx -0.234$ |
| 1 | $\frac{5}{32\pi}(3\ln 3 - 4) \approx -0.0350$ | $\frac{3}{32\pi}\left(\ln 3 - \frac{13}{6}\right) \approx -0.0319$ |
| 2 | $\frac{3}{8\pi^2} - \frac{1}{4\pi\sqrt{3}} \approx -0.00795$ | $-\frac{1}{64\pi^2} - \frac{1}{32\pi\sqrt{3}} \approx -0.00733$ |
| 3 | $\frac{21(4-5\ln 3)}{(32\pi)^2} \approx -0.00310$ | $\frac{-20-9\ln 3}{(32\pi)^2} \approx -0.00296$ |

Table 3: *One-loop contributions to the quantum mass of the bosonic kink ($s = 0$) and to the surface tension of $s$-dimensional domain walls for the on-shell scheme with normalized residue (OSR) and the zero-momentum (ZM) scheme.*

following, easily derivable relation

$$\begin{aligned}
&\lim_{\epsilon\to 0}\Gamma(\epsilon)\left[{}_2F_1(\epsilon,\frac{1}{2};\frac{3}{2};-\kappa)-1\right] \\
&= \sum_{n=1}^{\infty}\frac{(-\kappa)^n}{n(2n+1)} = -\ln(1+\kappa) - \frac{2}{\sqrt{\kappa}}\arctan(\sqrt{\kappa}) + 2\,. \qquad (29)
\end{aligned}$$

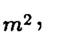
The numerical results for $s = 0, 1, 2, 3$ following from (28) are given in Table 2 for both the physical on-shell renormalization scheme (OS) and, where applicable, the minimal one with $\delta\lambda = 0$ (MR).

### 2.3.2 Renormalization schemes with nontrivial $Z$

In the OS scheme with a nontrivial $Z = 1 + \delta_Z$, the equation defining $\delta v^2$ is obtained by replacing $\delta v^2 \to \delta v^2 - v^2 \delta_Z$ in the left-hand side of (11), since with nonzero $\delta Z$ the 1-point counter term vertex in (9) gives

 $+$  $\stackrel{!}{=} 0 \quad \Rightarrow \quad \delta v^2 - v^2 \delta Z = \frac{i}{\lambda v}$  . (30)

The equation defining $\delta\lambda$ by the substitution $\delta\lambda \to \delta\lambda + \lambda\delta Z$ in the left-hand side of (15), since

$\delta\lambda + \lambda\delta Z \;=\; \frac{1}{2iv^2}($  $)_{|p^2=-m^2}$, (31)

where we have inserted $p^2 = -m^2 = -2\lambda v^2$ also for the 2-point counter term vertex in (9).

Using the *Bogomol'nyi* equation, $\varphi'_K = U(\varphi_K)$, the contribution from the counter term Lagrangian (17) reduces thus to

$$
\begin{aligned}
\delta M &= -\tfrac{\lambda}{2}(\delta v^2 - v^2\delta Z)\int dx\ (\varphi_K^2 - v^2) + \tfrac{\delta\lambda + 3\lambda\delta Z}{4}\int dx\ (\varphi_K^2 - v^2)^2 \\
&= \delta M^{OS}_{Z=1} + \delta Z\ M_{cl}.
\end{aligned} \tag{32}
$$

For the last equality we have used the replacement determined by (31, 30) and the classical mass

$$
M_{cl} = \frac{\lambda}{2}\int dx\ (\varphi_K^2 - v^2)^2 = \frac{m^3}{3\lambda}. \tag{33}
$$

For any $\delta Z$ the above replacements (30, 31) in the OS scheme preserve the relation $\lambda = m^2/(2v^2)$, but with the definition of $m$ fixed, that changes the coupling appearing in the classical expression $M_{cl.} = m^3/(3\lambda)$ according to $\lambda = \lambda|_{Z=1}(1 - \delta Z)$. Thus the extra contribution to $\Delta M^{(1)}$ can therefor again alternatively obtained from the classical expression of the mass $M_{cl}$ as $+M_{cl.}\,\delta Z$.

A natural refinement of the OS scheme, where $m$ is given by the physical (pole) mass, is to require that the residue of this pole be unity. This leads to

$$
\begin{aligned}
\delta Z &= -9\lambda\frac{m^{s-2}}{(4\pi)^{\frac{s+2}{2}}}\Gamma(\tfrac{4-s}{2})\int_0^1 dx\, x(1-x)[1 - x(1-x)]^{\frac{s-4}{2}} \\
&= 9\lambda\frac{m^{s-2}}{(4\pi)^{\frac{s+2}{2}}}\Gamma(\tfrac{4-s}{2})(\tfrac{3}{4})^{\frac{s-2}{2}}\left[{}_2F_1(\tfrac{2-s}{2},\tfrac{1}{2};\tfrac{3}{2};-\tfrac{1}{3}) - \tfrac{4}{3}\,{}_2F_1(\tfrac{4-s}{2},\tfrac{1}{2};\tfrac{3}{2};-\tfrac{1}{3})\right].
\end{aligned} \tag{34}
$$

Curiously enough, with the help of Gauss' recursion relations [64] the particular combination of hypergeometric functions in this expression can be recast in a form proportional to (28),

$$
\delta Z = 2\lambda\frac{m^{s-2}}{(4\pi)^{\frac{s+2}{2}}}\Gamma(\frac{2-s}{2})\left\{(s+2)\left(\frac{3}{4}\right)^{\frac{s}{2}}{}_2F_1(\frac{2-s}{2},\frac{1}{2};\frac{3}{2};-\frac{1}{3}) - 3\right\}. \tag{35}
$$

The energy densities of kink domain walls in an on-shell renormalization scheme with physical pole mass and unit residue (OSR) is thus given by the simple conversion formula

$$\frac{\Delta M^{(1)}_{\rm OSR}}{L^s} = \frac{\Delta M^{(1)}_{\rm OS}}{L^s} + \frac{m^3}{3\lambda}\delta Z = \frac{s+4}{3}\frac{\Delta M^{(1)}_{\rm OS}}{L^s} \tag{36}$$

and the particular results for the values $s$ of interest are listed in Table 3.

For the sake of comparison with previous results in the literature, Table 3 also includes another widely used renormalization scheme [101], where the mass is renormalized at zero momentum (ZM) according to $m^2_{ZM} = \Gamma^{(1)}(0)$ with $\Gamma^{(1)}(k^2)$ the inverse propagator to one-loop order and $\delta Z$ is chosen such that $[\partial\Gamma^{(1)}/\partial k^2](0) = 1$. In this scheme, formula (28) gets replaced by

$$\begin{aligned}\frac{\Delta M^{(1)}_{\rm ZM}}{L^s} \quad = \quad & \frac{m^{s+1}}{(4\pi)^{\frac{s+2}{2}}}\frac{2\Gamma(\frac{2-s}{2})}{s+1}\left\{\left(\frac{3}{4}\right)^{\frac{s}{2}} {}_2F_1(\frac{2-s}{2},\frac{1}{2};\frac{3}{2};-\frac{1}{3})\right.\\ &\left.+\frac{3}{4}(s-3) - \frac{1}{16}(s+1)(2-s)\right\},\end{aligned} \tag{37}$$

where the very last term within the braces is the contribution of $\delta Z$.

The surface tension of $\varphi^4$ domain walls has been calculated in the ZM scheme to one-loop order in 3+1 dimensions in Ref. [102] by considering the energy splitting of the two lowest states in a finite volume using zeta-function techniques, and our result completely agrees with that. Our result is also consistent with the older Ref. [22] using $\epsilon$-expansion (in the limit $\epsilon \to 0$), which employed yet another renormalization scheme that is closer (but not identical) to an $\overline{\rm MS}$-scheme. We do not, however, agree with the ZM-scheme result reported in Ref. [8] nor with its correction in Ref. [40] [5].

In 2+1 dimensions, the surface tension of the kink domain wall has been calculated in Ref. [103], and in Ref. [79] to two-loop order in the ZM scheme. Our one-loop ZM result reproduces that given in Ref. [79], while the one-loop result of Ref. [103] cannot be directly compared with ours as it re-expresses the ZM result in terms of the physical pole mass without using the coupling of either our OS or OSR scheme. We also agree with the most recent work [65], where the 2+1 dimensional kink domain wall energy density was calculated using the Born approximation methodology of Refs. [48,69] in the MR scheme. Compared to Ref. [65], the present calculation in dimensional regularization turns out to be considerably simpler and more straightforward, as the former has to exert some care in identifying "half-bound" states and to employ certain non-trivial sum rules for phase shifts. On the other hand, the methods of [48,69] will be useful also in cases where one can determine phase shifts only numerically.

[5]The latter reports the same result as that contained in Ref. [22] (for $\epsilon \to 0$), while formulating different renormalization conditions amounting to the ZM scheme at one-loop order.

Comparing finally the size of the one-loop corrections in the four different renormalization schemes considered in Tables 2 and 3, one notices that the corrections are largest in the MR scheme and significantly smaller in the other schemes, with the ZM and OSR results being rather close, but with noticeable differences.

These issues are of relevance in practical applications, and, indeed, the surface tension of the $\varphi^4$ kink domain wall can be related to universal quantities that can be investigated by lattice simulations of the Ising model and experimentally in binary mixtures [79]. In a comparison of the field-theoretic results with lattice studies, the different definitions of mass in the OS and in the ZM scheme correspond to the true (exponential) correlation length and to the second moment of the correlation function, respectively, both of which can be found in the literature (see e.g. [76] and references therein).

Of perhaps mere academic interest is the case of kink domain walls in 5 dimensions ($s = 3$) where our formulae still give finite results. In 5 dimensions, $\varphi^4$ theory is of course no longer renormalizable, though it may still be of interest as an effective theory.

### 2.4 The susy kink and domain string

In 1+1 and 2+1 dimensions ($s = 0$ and $s = 1$), the model (1) has the supersymmetric extension [44, 82]

$$\mathcal{L} = -\tfrac{1}{2}\left[(\partial_\mu\varphi)^2 + U(\varphi)^2 + \bar{\psi}\gamma^\mu\partial_\mu\psi + U'(\varphi)\bar{\psi}\psi\right], \tag{38}$$

where $\psi$ is a Majorana spinor, $\bar{\psi} = \psi^{\mathrm{T}}C$ and

$$U(\varphi) = \sqrt{\frac{\lambda_0}{2}}\left(\varphi^2 - v_0^2\right), \qquad v_0^2 \equiv \mu_0^2/\lambda_0. \tag{39}$$

(In 1+1 dimensions, $U \propto \sin(\sqrt{\gamma}\varphi/2)$ gives the sine-Gordon model, which is however not renormalizable in 2+1 dimensions.)

Embedding the susy kink in 2+1 dimensions gives a domain wall centered about a one-dimensional string on which the fermion mass vanishes (since $U'(\varphi_K) \propto \varphi_K$ vanishes at the center of the kink). In the following we shall succinctly refer to this particular domain wall as "domain string", postponing a brief discussion of higher-dimensional kink domain walls to the next subsection.

Going from 1+1 to 2+1 dimensions, the discrete symmetry content of (38) in fact changes. In 1+1 dimensions, (38) has the $Z_2$ symmetry,

$$\varphi \to -\varphi \ , \ \ \psi \to \gamma^5\psi, \tag{40}$$

with $\gamma^5 = \gamma^0\gamma^1$. In 2+1 dimensions, on the other hand, $\gamma^5 = \gamma^0\gamma^1\gamma^2 \propto \pm 1$, and the sign of the fermion mass term can no longer be reversed by $\psi \to \gamma^5\psi$. By the same token, (38) breaks parity, because a sign change of one of the spatial $\gamma$ matrices cannot be effected by an equivalence transformation, but leads to the other of the two inequivalent irreducible representations of a Clifford algebra in odd space-time dimensions.

### 2.4.1 Vacuum sector and renormalization

In what follows we shall consider the quantum corrections to both, the mass of the susy kink and the tension of the domain string, together. In both cases we shall continue to use a renormalization scheme where we put $Z_\varphi = 1 = Z_\psi$ at one-loop order. For this reason we have already dropped a subscript 0 for the unrenormalized fields in (38). We shall however consider the possibility of (finite) coupling constant renormalization, again by requiring that the renormalized mass of elementary scalars and fermions be given by the physical pole mass, together with the requirement of vanishing tadpoles, which fixes $\delta v^2$.

The Yukawa coupling in (38) introduces a mass term and vertices for the fermions. Renormalization of the coupling according (2) gives for the Yukawa term

$$\mathcal{L}_Y = \sqrt{2\lambda_0}\ \varphi\bar{\psi}\psi = \sqrt{2\lambda}\ \varphi\bar{\psi}\psi + \tfrac{\delta\lambda}{\sqrt{2\lambda}}\ \varphi\bar{\psi}\psi, \tag{41}$$

and expansion around the vacuum, i.e. $\varphi = v + \eta$ gives

$$\mathcal{L}_Y = \sqrt{2\lambda}\ v\bar{\psi}\psi + \sqrt{2\lambda}\ \eta\bar{\psi}\psi + \tfrac{\delta\lambda}{\sqrt{2\lambda}}\ (v+\eta)\bar{\psi}\psi. \tag{42}$$

From this we can see that the tree-level mass for the fermion is $m_F = \sqrt{2\lambda}v = m$ and thus the same as for the elementary bosonic excitation $\eta$. The bose-fermi interaction in the vacuum sector is given by the $\bar{\psi}\psi\eta$-vertex in (42). At order $\hbar$ this gives rise to a fermionic tadpole as well to a non-local diagram with 3-vertices which are now both divergent in two dimensions (see appendix).

Inclusion of the fermionic tadpole loop replaces 3 by $(3-2)$ in (11) so that compared to the bosonic result we have

$$\delta v^2|_{\rm susy} \equiv \delta\tilde{v}^2 = \frac{1}{3}\delta v^2|_{\rm bos.}$$

(When useful we distinguish quantities in the susy case by twiddles.)

In the OS scheme, the supersymmetric version of (15) is obtained by the replacement[6]

$$9m^2 \to 9m^2 - 2(2m^2 + \frac{1}{2}q^2)|_{q^2=-m^2} = 6m^2,$$

[6]The counter term $\frac{1}{2}\lambda\delta v^2\eta^2$ induced by the tadpole with a fermionic loop cancels only those contributions to the bosonic selfenergy due to a fermionic loop which contain one propagator. The remaining contributions have two propagators and are proportional to the bosonic contribution to the selfenergy.

and thus

$$\delta\tilde{\lambda} = \frac{2}{3}\delta\lambda|_{\text{bos.}}.$$

### 2.4.2 Topological sector

In a Majorana representation of the Dirac matrices in terms of the usual Pauli matrices $\sigma^k$ with $\gamma^0 = -i\tau^2$, $\gamma^1 = \tau^3$, $\gamma^2 = \sigma^1$ (added for $s = 1$), and $C = \tau^2$ so that $\psi = \binom{\psi^+}{\psi^-}$ with real $\psi^+(x,t)$ and $\psi^-(x,t)$, the equations for the bosonic and fermionic normal modes with frequency $\omega$ and longitudinal momentum $\ell$ (nonzero only when $s = 1$) in the kink background $\varphi = \varphi_K$ read

$$[-\partial_x^2 + U'^2 + UU'']\eta = (\omega^2 - \ell^2)\eta, \tag{43}$$
$$(\partial_x + U')\psi^+ + i(\omega + \ell)\psi^- = 0, \tag{44}$$
$$(\partial_x - U')\psi^- + i(\omega - \ell)\psi^+ = 0. \tag{45}$$

We can now eliminate the $\psi^-$ component in (45) by algebraically relating it to $\psi^+$ through (44)

$$\psi^- = \frac{i}{\sqrt{\omega + \ell}}(\partial_x + U')\psi^+, \tag{46}$$

*iff* $\omega + \ell \neq 0$. Inserting in this (45) shows that $\psi^+$ satisfies the same equation as the bosonic fluctuation $\eta$. Compared to $\psi^+$, the component $\psi^-$ has a continuous spectrum whose modes differ by an additional phase shift $\theta = -2\arctan(m/k)$ when traversing the kink from $x_1 = -\infty$ to $x_1 = +\infty$, which is determined only by $U'(\varphi_K(x_1 = \pm\infty)) = \pm m$. This follows from the algebraic relation (46) since asymptotically $\psi^+$ becomes a plane wave. Correspondingly, the difference of the spectral densities of the $\psi^+$-fluctuations in the kink and in the trivial vacuum equals that of the $\eta$-fluctuations, given in (22), whereas that of $\psi^-$-fluctuations is obtained by replacing $\delta'_K \to \delta'_K + \theta'$.

In the sum over zero-point energies for the one-loop quantum mass of the kink (when $s = 0$),

$$\tilde{M} = \tilde{M}_{cl.} + \frac{1}{2}\left(\sum \omega_B - \sum \omega'_B\right) - \frac{1}{2}\left(\sum \omega_F - \sum \omega'_F\right) + \delta\tilde{M}\ , \tag{47}$$

one thus finds that the bosonic contributions from the continuous spectrum are canceled by the fermionic contributions except for the additional contribution involving $\theta'(k)$ in the spectral density of the $\psi^-$ modes.

**Zero modes**. The discrete bound states cancel exactly, apart from the subtlety that the fermionic zero mode should be counted as half a fermionic mode [61]. In strictly 1+1 dimensions, the zero modes do not contribute simply because they carry zero energy, and for $s > 0$, where they become massless modes, they do not contribute in dimensional regularization.

In a cutoff regularization in $s = 1$, as we already discussed and shall further discuss below, they in fact do play a role. Remarkably, the half-counting of the fermionic zero mode for $s = 0$ has an analog for $s = 1$ where the bosonic and fermionic zero modes of the kink correspond to massless modes with energy $|\omega| = |\ell|$. From (44) and (45) one finds that the fermionic kink-zero mode $\psi^+ \propto \varphi_K'$, $\psi^- = 0$ is a solution only for $\omega = +\ell$. It therefore cancels only half of the contributions from the bosonic kink-zero mode which for $s = 1$ have $\omega = \pm\ell$. For $s = 1$ one thus finds that the fermionic zero mode of the kink corresponds to a chiral (Majorana-Weyl) fermion on the ($s$=1)-dimensional domain string [24, 57, 78].[7]

In dimensional regularization, however, the kink zero modes and their massless counterparts for $s > 0$ can be dropped, and the energy density of the susy domain wall reads

$$\frac{\tilde{M}^{(1)}}{L^s} = \frac{m^3}{3\lambda} - \frac{1}{4}\int \frac{dk\, d^s\ell}{(2\pi)^{s+1}} \sqrt{k^2 + \ell^2 + m^2}\, \theta'(k) + \delta\tilde{M}, \tag{48}$$

where

$$\theta'(k) = \frac{2m}{k^2 + m^2}. \tag{49}$$

With $\delta_v \tilde{M} = \frac{1}{3}\delta_v M$ the logarithmic divergence in the integral in (48) as $s \to 0$ gets canceled. A naive cut-off regularization at $s = 0$ would actually lead to a total cancellation of the $k$-integral with the counter term $\delta_v \tilde{M}$, giving a vanishing quantum correction in renormalization schemes with $\lambda = \lambda_0$. Note that the spectral density (49) has in principle the form of an usual integrand in dimensional regularization. So in principle (48) could be evaluated for $s = 0$ and thus $\ell = 0$ by performing the $k$-integration in $d = 1 - \epsilon$ dimensions. But this would also lead to a vanishing result. Below we will see that dimensional regularization by dimensional reduction to $d = 1 - \epsilon$ leads to evanescent counter terms. In dimensional regularization there is now however a mismatch for $s \neq 0$ and a finite remainder in the limit $s \to 0$ proportional to $s\Gamma(-s/2)$. Including the optional $\lambda$-renormalization the final result reads

$$\frac{\tilde{M}^{(1)}}{L^s} = \frac{m^3}{3\lambda} - \frac{m^{s+1}}{(4\pi)^{\frac{s+2}{2}}} \frac{2\Gamma(\frac{2-s}{2})}{s+1} + \delta_\lambda \tilde{M}. \tag{50}$$

In the minimal renormalization (MR) scheme one has $\delta_\lambda \tilde{M} = 0$, whereas in the more physical OS scheme, where $m$ is the pole mass of the elementary

[7] Choosing a different sign for $\gamma_1$ reverses the allowed sign of $\ell$ for these fermionic modes and thus their chirality (with respect to the domain string world sheet). This corresponds to the other, inequivalent representation of the Clifford algebra in 2+1 dimensions.

| s | $\frac{\Delta\tilde{M}^{(1)}}{L^s}/m^{s+1}$ (OS) | $\frac{\Delta\tilde{M}^{(1)}}{L^s}/m^{s+1}$ (MR) |
|---|---|---|
| 0 | $\frac{1}{6\sqrt{3}}-\frac{1}{2\pi}\approx -0.063$ | $-\frac{1}{2\pi}\approx -0.159$ |
| 1 | $\frac{1}{8\pi}(\ln 3-1)\approx +0.004$ | $-\frac{1}{8\pi}\approx -0.040$ |

Table 4: *One-loop contributions to the quantum mass of the susy kink ($s=0$) and to the surface tension of the (s=1)-dimensional susy kink domain "wall" for the on-shell (OS) and minimal renormalization (MR) schemes.*

bosons as well as fermions, one has $\delta_\lambda \tilde{M} = \frac{2}{3}\delta_\lambda M$, yielding

$$\frac{\Delta\tilde{M}^{(1)}}{L^s} = \frac{m^{s+1}\Gamma(\frac{2-s}{2})}{(4\pi)^{\frac{s+2}{2}}}\left\{\left(\frac{3}{4}\right)^{\frac{s-2}{2}} {}_2F_1(\frac{2-s}{2},\frac{1}{2};\frac{3}{2};-\frac{1}{3}) - \frac{2}{s+1}\right\}. \tag{51}$$

The respective results for the 1+1 dimensional susy kink ($s=0$) and for the (s=1)-dimensional susy kink domain "wall" (domain string) are given in Table 4. Again we find that there is much faster apparent convergence in the OS scheme compared to the MR one where only the tadpoles are subtracted. Results for renormalization schemes with a nontrivial $Z$ are quoted in [115].

In the literature, to the best of our knowledge, only the case of a supersymmetric kink ($s=0$) in the MR scheme[8] has been considered and dimensional regularization reproduces the result obtained before by Refs. [21, 67, 104, 120, 123]. However, a (larger) number of papers have missed the contribution $-m/(2\pi)$ because of the (mostly implicit) use of the inconsistent energy cutoff scheme [28, 29, 84, 88, 148] or have obtained different answers because of the use of boundary conditions that accumulate a finite amount of energy at the boundaries [113, 133]. The former result is however now generally accepted and, in the case of the super-sine-Gordon model (where the same issues arise with the same results) in agreement with S-matrix factorization [6].

In Ref. [98] the correct susy kink mass has also been obtained by employing a smooth energy (momentum) cutoff, the necessity of which becomes apparent, as in the purely bosonic case, by considering the 2+1 dimensional domain wall. Using a naive cutoff for $s=1$ one finds quadratic divergences which cancel only upon inclusion of the zero modes (which become massless modes in 2+1 dimensions). As we have discussed above, unlike the other

[8]In Refs. [88,113] the respective results have also been expressed in terms of the physical pole mass, but keeping $\lambda$ as in the MR scheme. Such a renormalization scheme yields a tadpole contribution proportional to $\delta\lambda$ and should not be confused with the OS scheme considered here, where both the mass and the coupling is renormalized such as to have both vanishing tadpoles and a physical pole mass for the elementary bosons.

bound states, these do not cancel because the fermionic zero mode becomes a chiral fermion on the domain-string world-sheet and thus cancels only half of the bosonic zero (massless) mode contribution, yielding

$$\int_0^\infty \frac{d\ell}{2\pi}\left\{\frac{1}{2}\sqrt{\ell^2} - \int_{-\Lambda_k}^{\Lambda_k} \frac{dk}{2\pi}\left[\sqrt{k^2+\ell^2+m^2}\frac{m}{k^2+m^2} - \frac{1}{\sqrt{k^2+\ell^2+m^2}}\right]\right\}$$
$$\stackrel{\Lambda_k\to\infty}{\longrightarrow} \int_0^\infty \frac{d\ell}{2\pi}\left\{\frac{\ell}{2} - \frac{\ell}{\pi}\arctan\frac{\ell}{m}\right\} \sim \int_0^\infty \frac{d\ell}{\pi}\frac{m}{2\pi} \tag{52}$$

which is however still linearly divergent. Smoothing out the cutoff in the $k$-integral does pick an additional (and for $s=0$ the only) contribution $-m/(2\pi)$, which is now necessary to have a finite result for $s=1$. This finite result then reads

$$\frac{\tilde{M}^{(1)}_{s=1}}{L} = -\frac{1}{\pi}\int_0^\infty \frac{d\ell}{2\pi}\left(m - \ell\arctan\frac{m}{\ell}\right) = -\frac{m^2}{8\pi} \tag{53}$$

in agreement with the result obtained above in dimensional regularization.

### 2.5 Susy kink domain walls in 3+1 dimensions

For completeness we shall also discuss kink domain walls in the 3+1-dimensional $\mathcal{N}=1$ Wess-Zumino-model [142]. In accordance with Ref. [30, 45] we shall demonstrate that in this model there is no nontrivial quantum correction to the surface tension.

A Wess-Zumino model with a spontaneously broken $Z_2$ symmetry now requires two real scalar fields to pair up with the now four-component Majorana spinor. For the classical Lagrangian we choose

$$\mathcal{L} = -\frac{1}{2}(\partial A)^2 - \frac{1}{2}(\partial B)^2 - V(A,B) - \frac{1}{2}\bar{\psi}[\partial\!\!\!/ + \sqrt{2\lambda}(A + i\gamma_5 B)]\psi$$
$$V(A,B) = \frac{\lambda}{4}(A^2 - B^2 - v^2)^2 + \lambda A^2 B^2, \tag{54}$$

where $A$ is a real scalar with non-vanishing vacuum expectation value, while $B$ is a real pseudo-scalar without one. For $B \equiv 0$ the potential coincides with that of the kink model (1), and correspondingly a classical domain wall solution is given by $A_K(x) = \phi_K(x_1)$ and all other fields zero.

As is well known [72, 83], in the 3+1-dimensional Wess-Zumino-model there is only one non-trivial renormalization constant $Z$ for the kinetic term. The superpotential

$$\mathcal{W} = \frac{\sqrt{\lambda_0}}{2}\left(\frac{\mu_0^2}{\lambda_0}\phi_0 + \frac{1}{3}\phi_0^3\right) \quad \text{with} \quad V(A,B) = |\mathcal{W}'|^2, \tag{55}$$

where $\phi = A + iB$, does not renormalize. Thus the renormalization of the field strength due to the renormalization of the kinetic term, $\phi_0 = \sqrt{Z}\phi$, must be compensated by the renormalization of the parameters $\mu_0$ and $\lambda_0$. This implies

$$\lambda_0 = Z^{-3}\lambda \quad , \quad \mu_0 = Z^{-1}\mu, \tag{56}$$

and thus the classical mass (5) is unchanged and one obtains a vanishing counter-term $\delta M$ for the kink wall energy density:

$$\delta M = \delta\left(\frac{2^{\frac{3}{2}}\mu_0^3}{3\lambda_0}\right) = \delta\left(\tfrac{Z^{-3}}{Z^{-3}}\right) M_{cl} = 0. \tag{57}$$

### 2.5.1 Topological sector

The fluctuation equations for $\eta = A - A_K$, $B$, and $\psi$ read

$$\begin{aligned} &\partial^2\eta - (U'^2 + UU'')\eta = 0 \\ &\partial^2 B - (\lambda A_K^2 + \mu^2)B = 0 \\ &[\not{\partial} + U']\psi = 0, \end{aligned} \tag{58}$$

with $U$ as in (39). $A_K$ satisfies the Bogomol'nyi equation $A'_K = -U(A_K)$, and the $x$-dependent parts of the $\eta$ and $B$ field equations factorize as $-(\partial_x - U')(\partial_x + U')$ and $-(\partial_x + U')(\partial_x - U')$, respectively.

Both the $\eta$ and $B$ fluctuation equations involve reflectionless potentials of the form

$$-\partial_z^2 - \frac{n(n+1)}{\cosh^2 z} + n^2, \tag{59}$$

where $z := \frac{mx}{2}$, $m := \sqrt{2}\mu$.

The kink fluctuation modes $\phi_k(x)$ correspond to $n = 2$, and the $\eta$ fluctuations are given by the former multiplied by plane waves with momentum $\vec{\ell} = (\ell_2, \ell_3)$ in the trivial directions. Their spectrum thus consists of one massless mode and one massive mode localized on the domain wall with $\omega_0^2(\ell) = \ell^2$ and $\omega_B^2(\ell) = \frac{3}{4}m^2 + \ell^2$ and delocalized ones with $\omega_k^2(\ell) = k^2 + \ell^2 + m^2$.

The $x$-dependence of the $B$-fluctuations on the other hand involves the potential (59) with $n = 1$, like the fluctuation equations for the sine-Gordon soliton, but with different energies according to

$$\left(-\partial_z^2 - \frac{2}{\cosh^2 z} + 1\right) s(z) = [\frac{4}{m^2}(\omega^2 - \ell^2) - 3]s(z). \tag{60}$$

The spectrum of the sine-Gordon system is now shifted by $\ell^2 + \frac{3}{4}m^2$ so that the sine-Gordon zero-mode matches the bound state of the kink, and the continuous part of the spectrum also coincide. The spectrum of the $B$-fluctuations thus equals that of the $\eta$-fluctuations apart from the absence of the massless (zero) mode. The spectral densities for the delocalized modes are, however, different and the bosonic contribution to the one-loop surface tension reads

$$\frac{\Delta^{\mathrm{b}}\tilde{M}^{(1)}}{L^s} = \frac{1}{2}\int \frac{d^s\ell}{(2\pi)^s}\left(\omega_0(\ell) + 2\omega_B(\ell) + \int \frac{dk}{2\pi}\omega_k(\ell)[\delta'_K(k) + \delta'_{SG}(k)]\right) \tag{61}$$

where $s = 2 - \epsilon$. The phase shifts are given by the spectral densities w.r.t. the vacuum:

$$\begin{aligned} \delta'_K(k) &= \int dx(|\phi_k(x)|^2 - 1) \\ \delta'_{SG}(k) &= \int dx(|s_k(x)|^2 - 1). \end{aligned} \tag{62}$$

Choosing the Majorana representation for the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \tag{63}$$

and writing $\psi$ in terms of two 2-component spinors

$$\psi^{(\pm)} = e^{\mp i(\omega t - \ell x_{||})}\begin{pmatrix} \psi_A^{(\pm)} \\ \psi_B^{(\pm)} \end{pmatrix}, \tag{64}$$

for positive and negative energy modes the fermionic fluctuation equation of (58) becomes:

$$(\partial_x + U')\psi_A^{\pm} \pm i[\omega + \not{\ell}]\psi_B^{\pm} = 0 \tag{65}$$

$$i[\omega - \not{\ell}]\psi_A^{\pm} \pm (\partial_x - U')\psi_B^{\pm} = 0, \tag{66}$$

where $\not{\ell} = \sigma_1\ell_2 + \sigma_3\ell_3$. Through (65) the bi-spinor $\psi_B^{\pm}$ is algebraically related to $\psi_A^{\pm}$, and inserted into (66) gives for $\psi_A^{\pm}$ the same equation as for the bosonic fluctuations $\eta$. Thus $\psi_A^{\pm}$ and the energies $\omega$ are given by

$$\begin{aligned} &\psi_A^+ = \phi_k(x)\sqrt{\omega + \not{\ell}}\,\xi_s^+,\ \psi_A^- = \phi_k^*\sqrt{\omega + \not{\ell}}\,\xi_s^-, \\ &\omega = \sqrt{\omega_{Kink}^2 + \vec{\ell}^2}, \end{aligned} \tag{67}$$

where $\omega_{Kink}^2 = \frac{3}{4}m^2$ for the bound state and $k^2 + m^2\}$ for the continuum modes. The bi-spinors $\xi_{s=1,2}^{\pm}$ are the two linear independent polarizations for each, the positive and negative energy solutions, and they are chosen such that they form a complete orthonormal set. From (65) one gets for $\psi_B^{\pm}$:

$$\psi_B^{\pm} = \pm i\frac{1}{\sqrt{\omega^2 - \ell^2}}(\partial_x + U')\psi_A^{\pm}, \tag{68}$$

which gives with (67)

$$\psi_B^+ = is_k(x)\sqrt{\omega - \not{\ell}}\,\xi_s^+,\ \psi_B^- = -is_k^*(x)\sqrt{\omega - \not{\ell}}\,\xi_s^-, \tag{69}$$

where we have used that and $\frac{1}{\omega_{Kink}}(\partial_x + U')\phi_k = s_k$. Thus the operator in (68) does not exist for the zero (massless) mode $\omega_{Kink} = 0$, so that this case has to be treated separately. Again we have also for the fermionic modes themselves the algebraic structure of a susy-quantum system (see appendix).

For the massless (zero) mode ($\omega_{Kink} = 0$) only $(\partial_x + U')\psi_A = 0$ in (65) has a normalizable solution, which is located at the domain wall. The other equation, $(\partial_x - U')\psi_B = 0$, has normalizable solutions only if boundaries for the $x$-direction were introduced, and would be localized there.

#### 2.5.2 Fermionic correction

The fermionic quantum field is given by

$$\psi = \int d^d\ell \sum_B\!\!\!\!\!\!\!\int\ dk \sum_s \left(a_k(\ell)\ \psi_{k,s}^{(+)}(\ell) + a_k^\dagger(\ell)\ \psi_{k,s}^{(-)}(\ell)\right), \tag{70}$$

and contains now an additional sum over spins $s$. The fermionic correction to the domain wall tension is now easily obtained as ($s = 2 - \epsilon$):

$$\Delta\mathcal{E}_{DW}^{ferm} = -\int \frac{d^s\ell}{(2\pi)^s}\left(\frac{\omega_0(\ell)}{2} + \omega_B(\ell) + \int dk \frac{\omega_k(\ell)}{2}[\rho_\phi(k) + \rho_s(k)]\right), \tag{71}$$

where we have used that for a complete set $\{\xi_s\}$ is $\sum_s \xi_s^\dagger\ \not{\ell}\xi_s = Tr\ \not{\ell} = 0$. The functions $\rho_\phi$ and $\rho_s$ are the normal ordered spectral densities,

$$\begin{aligned}\rho_\phi(k) &= \int dx(|\phi_k(x)|^2 - 1)\\ \rho_s(k) &= \int dx(|s_k(x)|^2 - 1),\end{aligned} \tag{72}$$

which are, because of the additional sum over spin-degrees of freedom in (70), exactly the same as in the bosonic sector given by (62).

As a result, the fermionic contribution to the one-loop correction of the domain wall tension becomes identical to the bosonic one, but with a negative sign,

$$\frac{\Delta^{\mathrm{f}}\tilde{M}^{(1)}}{L^s} = -\frac{\Delta^{\mathrm{b}}\tilde{M}^{(1)}}{L^s}. \tag{73}$$

In perfect agreement with the non-renormalization theorem of the superpotential (which does not apply at the lower dimensions considered above), there is no quantum correction to the classical value of the surface tension of the susy kink domain wall in 3+1 dimensions.

This cancellation of the quantum corrections can also be linked to the cancellation of quantum corrections to the $N = 2$ susy kink mass [104,123]. Such a cancellation is also to be expected for 4+1 dimensional supersymmetric theories with domain walls. In contrast to 2+1 dimensions, in 4+1 dimensions there are no Majorana fermions, so one needs to extend the supersymmetry algebra to involve a Dirac fermion. From the point of view of the 1+1 dimensional kink, this will imply $N = 4$ supersymmetry. On the then 4-dimensional domain wall one may have chiral fermions, but as pointed out in Ref. [57], these domain-wall fermions necessarily come in pairs containing both chiralities.

## 2.6 Conclusion

In this section we have shown that dimensional regularization allows one to compute the one-loop contributions to the quantum energies of bosonic and supersymmetric kinks and kink domain walls in a very simple manner. The ambiguities associated with ultraviolet regularization observed in the 1+1 dimensional kinks has been shown to be eliminated by considering their extension to kink domain walls in higher dimensions.

For the bosonic kink domain walls, which are of interest also in the context of condensed matter physics, we have derived a compact $d$-dimensional formula, which reproduces and (mostly) confirms existing results in the literature, and we have also discussed in detail the dependence on particular renormalization schemes.

In the supersymmetric case, we confirmed previous results in 1+1 and 3+1 dimensions. While in the latter case quantum corrections to the surface tension vanish, we have obtained a nontrivial one-loop correction for a 2+1 dimensional $N = 1$ susy kink domain wall with chiral domain wall fermions.

# 3 Clash of discrete symmetries for the supersymmetric kink on a circle

## 3.1 Introduction

The so called mode regularization was successfully introduced to compute the order-$\hbar$ quantum correction to the mass of the bosonic kink [37]. However, in the minimal supersymmetric case, which in addition to the real scalar field includes a Majorana fermion, the implementation of fermionic boundary conditions in the topological background is highly delicate matter [120] [9], [61,104]. This section is based on the article [63] and explains the need for averaging over different boundary conditions by the requirement of invariance under discrete symmetries. In this way spurious boundary energy, which may be present in individual sectors, does not contaminate the mass correction of the susy kink.

Subtleties in the application of the discrete symmetries charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, and time reversal $\mathcal{T}$ to Majorana fermions have long been a topic of interest [106,111,149]. Past discussions generally have dealt with local processes and properties, but the main aim of the present work is to study an anomalous global behavior of these discrete symmetries in a model with a topological structure. For this we consider the simplest possible system: the supersymmetric (susy) kink with what would seem to be natural boundary conditions.

Some time ago the concept of locally invisible boundary conditions was introduced [61,104]: for a two component Majorana fermion $\psi = (\psi_1, \psi_2)$ in a kink background in a box of length $L$, the twisted periodic (TP) and twisted antiperiodic (TAP) boundary conditions

$$\text{TP}: \qquad \psi_1(-L/2) = \psi_2(L/2), \quad \psi_2(-L/2) = \psi_1(L/2) \tag{74}$$

$$\text{TAP}: \qquad \psi_1(-L/2) = -\psi_2(L/2), \quad \psi_2(-L/2) = -\psi_1(L/2) \tag{75}$$

amount to putting the system on a circle without introducing a point where a boundary is present: The kink solution $\varphi_K(x) = v \tanh \frac{mx}{2}$ is invariant under the simultaneous transformation $x = L/2 \to x = -L/2$ and $\varphi_K \to -\varphi_K$. Thus the points $x = \pm L/2$ may be identified. The action for the susy kink

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}U^2(\varphi) - \frac{1}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi - \frac{1}{2}U'(\varphi)\bar{\psi}\psi \tag{76}$$

with $U(\varphi) = U(-\varphi)$ is invariant under the transformation

$$\varphi \to -\varphi \ , \quad \psi \to e^{i\alpha}\gamma^5 \psi \ , \tag{77}$$

[9] "the familiar sum of frequencies ... is unacceptably sensitive to the definition of the infinite volume limit"

which is compatible with the Majorana condition for $\alpha = 0$ or $\pi$ whereas for Dirac fermions an arbitrary phase would be allowed. Here we use a Majorana representation of the Dirac matrices with $\gamma^0 = -i\sigma_2$, $\gamma^1 = \sigma_3$, $\gamma^5 = \sigma_1$. In these terms, the TP and TAP boundary conditions in (74), (75) are simply $\psi \to \pm\gamma^5\psi$, clearly satisfying (77). As a consequence there is no visible boundary (meaning no locally observable discontinuity or cusp) at $x = \pm L/2$. Note that it is not necessary in these considerations to assume that the center of the kink is located at the point $x = 0$. That will be helpful later on in defining the parity operation in a simple manner, but for any other purpose the matching point for the transition or jump conditions (77) may be chosen arbitrarily, as befits a locally invisible boundary.

The TP and TAP boundary conditions arise naturally if one begins with a kink-antikink system with periodic (P) boundary conditions, and looks at the values of $\psi_1$, $\psi_2$ between the kink and antikink. One finds then that for P conditions for the kink-antikink system, the fermions satisfy either TP or TAP conditions. Here we also consider a natural extension of these ideas: we begin with antiperiodic (AP) boundary conditions for the kink-antikink system, and find then that if the fermionic modes are written as plane waves $e^{-i(\omega t - kx)}$ far away from the kink-antikink system, then in between the kink and antikink they satisfy imaginary twisted periodic and antiperiodic (iTP and iTAP) boundary conditions

$$\text{iTP}: \qquad \psi_1(-L/2) = i\psi_2(L/2), \quad \psi_2(-L/2) = i\psi_1(L/2) \tag{78}$$

$$\text{iTAP}: \qquad \psi_1(-L/2) = -i\psi_2(L/2), \quad \psi_2(-L/2) = -i\psi_1(L/2)\ , \tag{79}$$

where $\psi_{1,2}$ now refer to the fermionic mode functions as opposed to the complete field. If one prefers to avoid working with complex boundary conditions for the Majorana fermions, one may take the real and imaginary parts of the distorted plane waves, but this then leads to the *nonlocal* boundary conditions $(-\partial_x^2 + m^2)^{1/2}\psi_1(-L/2) = \pm(\partial_x - m)\psi_2(L/2)$ and similar conditions for $\psi_2(-L/2)$. In the following we consider only the algebraic boundary conditions (78) and (79). For periodic boundary conditions on the kink-antikink system, one finds only the real boundary conditions for a single kink given in (74) and (75), whether one uses complex or real mode functions.

In the trivial sector, P and AP boundary conditions are invisible boundary conditions, and, having introduced iTP/iTAP it seems only natural to also include iP and iAP boundary conditions

$$\begin{aligned} \text{iP}: &\qquad \psi_1(-L/2) = i\psi_1(L/2), \quad \psi_2(-L/2) = i\psi_2(L/2) \\ \text{iAP}: &\qquad \psi_1(-L/2) = -i\psi_1(L/2), \quad \psi_2(-L/2) = -i\psi_2(L/2). \end{aligned} \tag{80}$$

With imaginary boundary conditions, one finds a generalized Majorana identity, in which the adjoint of the field for one of the two boundary conditions is equal to the field for the other boundary condition, so that only if one averages over both conditions is it meaningful to describe the fermions as Majorana particles.

In Ref. [61], it was found that for a single kink one has to consider suitable averages over subsets of the mentioned boundary conditions to obtain the correct susy kink mass, because for particular individual cases one encounters localized boundary energy. This localized energy is due to boundary conditions which distort the field at the boundary and may be called visible boundary conditions. In the kink sector, the P/AP and iP/iAP boundary conditions are visible, whereas in the trivial sector, the twisted versions are visible.

To cancel out localized boundary energy, one needs to average over the results of a twisted and an untwisted boundary condition. In this paper, we shall show that there is a reason to average also over the two twisted boundary conditions, because a single (real) twisted boundary condition breaks parity $\mathcal{P}$ (as well as $\mathcal{T}$), giving rise to delocalized momentum proportional to the ultraviolet cutoff, which cancels only in the average. (In the case of imaginary boundary conditions, a similar phenomenon arises with iP/iAP in the trivial sector.) This was overlooked in Ref. [61], which had assumed parity-invariance for the spectrum and incorrectly claimed the appearance of delocalized energy.

One might expect that one can find other boundary conditions in the kink sector which preserve parity. Indeed, the invisible boundary conditions iTP, and iTAP have a $\mathcal{P}$ and $\mathcal{T}$ invariant spectrum, but instead violate $\mathcal{C}$ (and thus $\mathcal{CPT}$), so that these mode functions do not allow one to build a local quantum field theory with Majorana fields. Because $\mathcal{C}$ selects different locally invisible boundary condition from $\mathcal{P}$ and $\mathcal{T}$, it follows that there is no choice which preserves all three symmetries simultaneously. This obstruction occurs despite the fact that the action as a local expression in Bose and Fermi fields is invariant under all the symmetries. Hence, one encounters here a phenomenon which we call with some hesitation a discrete symmetry anomaly, induced by the kink. There is no local counter term which can remove this anomaly. One can, of course, choose as boundary conditions $\psi = 0$ in which case there are no problems with the discrete symmetries, but then one has localized boundary energy, and our aim here is to study the discrete symmetries in the presence of invisible boundary conditions, which means with the kink put on a circle.

The possibility that a nontrivial structure of spacetime can lead to anomalies in discrete symmetries has been studied before. For example, in Ref. [89, 90] a $\mathcal{CPT}$ anomaly was claimed to arise by compactification of some dimensions of (3+1) spacetime.

In our example, both a nontrivial space-time and a nontrivial field topology is present. In Ref. [115], it was found that in 2+1 dimensions there arise chiral fermions living on a susy kink domain wall (see also section 2); these fermions are massless in 2+1 dimensions (their energy is equal to the momentum along the domain wall) and they correspond to fermionic zero modes of the susy kink in 1+1 dimensions. In this case the spectrum is again parity-nonsymmetric (the massless fermions on the domain wall move in one direction but not in the other) but now this is not due to boundary conditions but rather due to the presence of the kink, in accordance with the general results of Refs. [24, 57]. In Ref. [3] the connection between instantons and the breaking of supersymmetry and the discrete symmetries $\mathcal{C}, \mathcal{P}, \mathcal{T}$ was considered.

In the following we discuss how the symmetries $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ act on the boundary conditions in the kink and in the trivial sector and we work out the fermionic spectra for the 16 sets of boundary conditions (8 sets in the kink sector, and 8 sets in the trivial sector). We also determine how the total mass and momentum of the kink depend on the choice of boundary conditions. We regulate by mode regularization, i.e., requiring equal numbers of modes in the trivial and kink sector, counting fermionic zero modes according to the rules derived in Ref. [61].

## 3.2 Discrete symmetries and their implementation

For the single-particle Dirac Hamiltonian

$$H = i\hbar\sigma_1\partial_x + \hbar\sigma_2 m\varphi_K(x)/v \ , \tag{81}$$

one has simple and unique representations of the three symmetry operations, charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, and time reversal $\mathcal{T}$, which leave this differential operator invariant. $\mathcal{C}$ at the single-particle level is an antiunitary operation which reverses the sign of $H$, and because in this representation $H$ is purely imaginary the transformation is accomplished by simple complex conjugation of fermion wave functions: $\mathcal{C} = \mathcal{K}$. For $\mathcal{P}$, which must include the transformation $x \to -x$, a subtlety arises because this operation by itself turns the kink into an antikink. Therefore, in the kink sector, one must require for the action of parity on the classical bosonic field $\varphi_K(x) \to -\varphi_K(-x) = \varphi_K(x)$. In the kink background the combined transformation reverses the derivative term but not the mass or Yukawa term in $H$, and we find for the action on fermion wave functions $\mathcal{P} = (x \to -x) \times i\sigma_2$. For the antiunitary $\mathcal{T}$ one needs an operation including $\mathcal{K}$, but it must leave $H$ invariant. To do this requires a matrix factor anticommuting with $H$, yielding $\mathcal{T} = \sigma_3\mathcal{K}$. Note that each of these discrete operations on fermion wavefunctions is the same in the trivial sector as it is in the kink sector. Of course, in the trivial sector, the action of parity on the (constant) classical background field $\varphi = v$ is simply to preserve it. Thus, to keep the background invariant one treats the background field as scalar in the trivial sector but pseudo scalar in the kink sector.

While the discrete transformations can be defined consistently for the differential operator, we still need to look at their effects on the matching or boundary conditions. Let us write these conditions in a general form which covers all the choices described above:

$$\psi(x = -L/2) = \Gamma e^{i\alpha}\psi(x = +L/2) \quad . \tag{82}$$

The twisted boundary conditions which we now analyse correspond to $\Gamma = \gamma^5 = \sigma_1$. The conditions could be applied at any point (see [104] for the details of the precise procedure), but let us choose symmetric placement around the center of the kink to make the action of the parity symmetry as simple as possible. Evidently we obtain the four different possibilities mentioned above by choosing $\alpha = 0, \pi, \pi/2, -\pi/2$, respectively. The action of $\mathcal{C}$ takes $e^{i\alpha}$ to $(e^{i\alpha})^*$, so that only $\alpha = 0, \pi$ (TP and TAP) are left unchanged. For parity, because of the interchange of left and right boundaries along with the presence of the matrix $\sigma_2$, one has $e^{i\alpha} \to -(e^{i\alpha})^{-1}$, so that only $\alpha = \pm\pi/2$ (iTP and iTAP) are left unchanged. For $\mathcal{T}$, the matrix $\sigma_3$ implies $e^{i\alpha} \to -(e^{i\alpha})^*$, and again $\alpha = \pm\pi/2$ (iTP and iTAP) are left unchanged.[10]

The purely real TP and TAP conditions commute with $\mathcal{C}$, but $\mathcal{T}$ and $\mathcal{P}$ each interchange TP with TAP. Consequently, with one of these conditions by itself only $\mathcal{C}$ holds: It is possible to choose wave functions which are real, and a fermion field operator which is Hermitean, but (positive-energy) waves of positive and negative wavenumber $k$ are not degenerate with each other. In fact, it is easy to check that for each solution with $k$ of one sign for TP there is a degenerate solution with $k$ of the opposite sign for TAP. As will be shown below there is a delocalized quantity, a net momentum proportional to the ultraviolet cutoff energy $\Lambda$.

On the other hand, with iTP and iTAP conditions, $\mathcal{P}$ and $\mathcal{T}$ symmetries leave the conditions invariant, but $\mathcal{C}$ interchanges them. Once again, to have all three symmetries one must use an average over the two boundary conditions. This time, if one just chooses one of these boundary condition there is a difference in energy spectrum from the other boundary condition (but the spectra are each parity-symmetric). Now a new difficulty arises, that it is impossible to write a Hermitean Majorana field, because a positive energy state with positive momentum does not have an equal negative energy partner with negative momentum. A different way to reach the same conclusion is to consider the operation $\mathcal{CPT}$, which is a well-accepted symmetry for local quantum field theory.[11]

Evidently this symmetry leaves the field Hamiltonian density invariant only for the TP and TAP conditions, which therefore are the ones uniquely

[10] We use the passive point of view according to which we equate (82) to $\psi'(-L/2) = M\psi'(L/2)$ and solve for $M$.

[11] There has been recent interest in anomalous $\mathcal{CPT}$ violation in chiral theories in 4 dimensions [54, 91] and in 2 dimensions [89, 90]. We consider the present work (which does not include chiral gauge couplings) complementary to those studies, but the chiral nature of the twisted boundary conditions suggests that there may be a connection to the anomaly in explicitly chiral theories.

allowed as consistent conditions in quantum field theory. For these conditions to achieve vanishing delocalized momentum one must average over TP and TAP, while for iTP and iTAP implementing $\mathcal{CPT}$ symmetry forces averaging over the two sets.

For completeness, we should examine the effects of the discrete symmetries in the trivial sector. Now, invisible boundary conditions have the unit matrix in place of the matrix $\sigma_1$. One sees immediately that the P and AP conditions satisfy all three discrete symmetries, while iP and iAP do not satisfy any. This means that one could implement the discrete symmetries with either P or AP, but implementation for imaginary conditions would require both iP and iAP.

To describe the discrete symmetries as transformations on the Majorana field we need a dictionary relating these transformations to those already discussed for the the single particle wave functions. For charge conjugation this is

$$U_C \psi(x,t) U_C^{-1} = \psi^\dagger(x,t) \quad , \tag{83}$$

so that the Majorana condition becomes simply the hermiticity or self adjointness of the field $\psi$. Note that what had been an antiunitary operation taking $H$ into its negative for the single-particle description now is a unitary operation leaving the Hamiltonian density $\mathcal{H}(x.t)$ invariant. This result depends critically on the fact that $\mathcal{H}$ includes a commutator of $\psi$ with $\psi^\dagger$, which reverses sign under charge conjugation. For parity we have[12]

$$U_P \psi(x,t) U_P^{-1} = i\sigma_2 \psi(-x,t) \quad , \tag{84}$$

identical with the single-particle rule. For time reversal one finds the greatest subtlety, because this operation remains antiunitary:

$$V_T \psi(x,t) V_T^{-1} = \sigma_3 \psi^*(x,-t) \quad . \tag{85}$$

The subtlety has to do with defining complex conjugation for the raising and lowering operators $a^\dagger$ and $a$ appearing in the mode expansion of the field. The simplest assumption is that this operation leaves the operators invariant, but instead each one could be multiplied by a different phase factor. In that case, the phase factor would have to be explicitly compensated in the action of $V_T$ on each raising or lowering operator. It is easy to verify that these new definitions are consistent with the earlier analysis of the relation between discrete symmetries and boundary conditions, with the obvious proviso that the boundary conditions now are applied to the field exactly as they previously were applied to the wave functions.

The issues discussed here all arise because we are dealing with Majorana fermions. How would the discussion change if one considered instead an

[12] As is the case for Majorana fermions in 4 dimensions [111,149], $\mathcal{P}^2 = -1$.

$N = 2$ theory, with Dirac fermions? Now the field $\psi$ no longer need be equivalent to its charge conjugate, so it might seem that one could choose just one boundary condition instead of averaging over a pair. It is enticing to imagine that the Dirac fermion charge could be coupled to a U(1) gauge field, so that the phase $\alpha$ in (82) would reflect a magnetic flux threading the circle. However, for no choice of $\alpha$ would the spectrum obey all three discrete symmetries, just as we found already; that deduction holds regardless of the assumption $N = 1$ or $N = 2$. Thus we still require a pair of boundary conditions if the symmetries all are to be obeyed simultaneously. In the $N = 2$ theory however, continuous values of $\alpha$ are allowed, and except for the values considered before any other would break all three symmetries, as one would expect for arbitrary irrational flux through the circle. The $N = 2$ theory exhibits the Jackiw-Rebbi half-fermion charge localized at the kink [87], and it is amusing that this is consistent with the possibility of tunneling between kink and antikink [14], as the latter also would possess charge one-half. The physical interpretation of this analysis, when combined with what we saw earlier, seems to be that the problem of the kink on a circle 'knows' that it really is half of the kink-antikink problem on a doubled circle. Thus the discrete symmetries which are obeyed for half an Aharonov-Bohm quantum of flux through the large circle also are obeyed for one-quarter flux through the small circle, but only when one averages over a suitable pair (iTP and iTAP) of boundary conditions.

### 3.3 Mode number regularization of fermionic contributions to the one-loop susy kink mass

We now turn to the explicit calculation of the fermionic contributions to the susy kink mass at one-loop order in mode number regularization, extending and partially correcting the results presented in Ref. [61].

The $\phi^4$-kink model corresponds to using $U(\varphi) = \sqrt{\lambda/2}(\varphi^2 - v^2)$ in the Lagrangian (76), but the following discussion applies (mutatis mutandis) to other models such as sine-Gordon, where $U \propto \sin(\gamma\phi/2)$.

In the trivial vacuum, one has $U(v) = 0$ and $U'(v) = \sqrt{2\lambda}\ v = m$, whereas with the nontrivial kink background field $\varphi_K(x) = v\tanh(m(x - x_0)/2)$ one has the Bogomol'nyi equation $U(\varphi_K) = -\partial_x\varphi_K$ and $U'(\varphi_K) = m\varphi_K/v$, leading to a fluctuation equation for the fermionic mode functions governed by the differential operator (81).

The fermionic mode functions will be written

$$\psi(x,t) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} e^{-i\omega t} \tag{86}$$

so that the Dirac equation becomes

$$-i\omega\psi_1 = (\partial_x - U')\psi_2, \quad -i\omega\psi_2 = (\partial_x + U')\psi_1. \tag{87}$$

The fermionic contribution to the one-loop quantum mass of a kink is given by sums over zero-point energies according to

$$M_f^{(1)} = -\frac{\hbar}{2}\left[\sum \omega_K - \sum \omega_V\right] + \Delta M_f \tag{88}$$

where the indices $K$ and $V$ refer to kink and trivial vacuum, respectively, and $\Delta M_f$ is the fermionic contribution to the counter-term due to renormalizing the theory in the trivial vacuum. A minimal renormalization scheme that can be chosen is to require that tadpoles vanish and all other renormalization constants are trivial. This gives [61]

$$\Delta M_f = -\frac{2}{3}\Delta M_b = -\frac{m\hbar}{2\pi}\int_{-\Lambda}^{\Lambda}\frac{dk}{\sqrt{k^2+m^2}}. \tag{89}$$

In (global)[13] mode regularization the spectrum of fluctuations about a kink (and in the trivial vacuum) is discretized by considering an interval of (large) length $L$ and choosing boundary conditions. The sums in (88) are then cut off at a given large value $N$ of the number of modes, which according to the principle of mode regularization is chosen to be the same in the trivial and in the kink sector.

As argued in Ref. [61], this requires *fixed* boundary conditions, meaning that they are identical for the trivial and the kink sector. But because invisible boundary conditions in one sector are visible ones in the other, it becomes necessary to average over boundary conditions such that boundary energies cancel in the average.

The correct answer this average has to give is, as has been established by a variety of methods [19–21, 61, 62, 67, 98, 104, 115, 120, 123],

$$M_f^{(1)} = -M_b^{(1)} - \frac{\hbar m}{2\pi} \tag{90}$$

where $M_b^{(1)}$ is the bosonic contribution, so that there is in total a non-vanishing negative correction for the susy kink mass $M^{(1)} = M_f^{(1)} + M_b^{(1)}$ which is in fact entirely due to an interesting anomalous contribution to the central charge operator [100, 116, 123] (see also section 4 below).

### 3.3.1 Quantization conditions

To explicitly compute the difference of the sums in Eq. (88) for the various boundary conditions discussed in Sect. 1, we have to derive the quantization conditions on an interval of length $L$. For ease of comparison with Ref. [61], Sect. VB, we let the spatial coordinate run from 0 to $L$ and put the center of the kink at $x = L/2$. We shall have to consider carefully both the discrete

[13] See Refs. [62, 145] for a local variant which avoids the subtleties discussed here as well as allowing one to calculate the local energy distribution.

and continuous[14] spectrum.

### I. Trivial sector

If one puts

$$\psi_1 = e^{ikx} + ae^{-ikx} \tag{91}$$

then it follows from the Dirac equation (87) with $U' \equiv m$ that

$$\psi_2 = -[e^{i(kx+\frac{\theta}{2})} - ae^{-i(kx+\frac{\theta}{2})}] \tag{92}$$

where we define $\theta$ such that

$$e^{i\frac{\theta}{2}} = \frac{k - im}{\omega}, \quad \omega = \pm\sqrt{k^2 + m^2}. \tag{93}$$

So $\theta = -2\arctan(m/k)$, but the branch of the arctan is fixed such that (for positive frequencies $\omega$) $\theta$ goes from $-2\pi$ to 0 as $k$ runs from $-\infty$ to $+\infty$. This conforms with the definition adopted in [104] but deviates from Ref. [61]. The definition (93) has the advantage of avoiding explicit sign functions $\mathrm{sgn}(k)$ in the quantization conditions.

The quantization conditions for untwisted P and AP boundary conditions are simply $kL = 2\pi n$ and $kL = 2\pi n + \pi$; iP and iAP have $kL = 2\pi n - \pi/2$ and $kL = 2\pi n + \pi/2$, respectively. Notice that iP and iAP in the trivial sector each have a set of solutions which is not symmetrical under $k \to -k$.

The twisted boundary conditions read $\psi_1(0) = \rho\psi_2(L)$ and $\psi_2(0) = \rho\psi_1(L)$, where $\rho = e^{i\alpha} = (+1, -1, +i, -i)$ for TP, TAP, iTP, and iTAP, respectively. Plugging these conditions into (91) and (92) and solving for $a$ gives

$$\frac{-\rho e^{i(kL+\frac{\theta}{2})} - 1}{-\rho e^{-i(kL+\frac{\theta}{2})} + 1} = a = \frac{-e^{i\frac{\theta}{2}} - \rho e^{ikL}}{\rho e^{-ikL} - e^{-i\frac{\theta}{2}}} \,. \tag{94}$$

Multiplying out, this gives

$$(\rho^2 - 1)(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}) = 2\rho(e^{ikL} - e^{-ikL}). \tag{95}$$

For $\rho^2 = 1$, (TP and TAP), this is equivalent to $\sin kL = 0$, i.e. $kL = \pi n$, with $n \neq 0$, because $n = 0$ corresponds to the trivial solution $\psi_1 = \psi_2 = 0$.

Imaginary twisted periodic/antiperiodic boundary conditions (iTP/iTAP) have $\rho^2 = -1$, and one finds for $\rho = \pm i$ the two sets of solutions a) $kL = 2\pi n - \frac{\theta}{2} \pm \frac{\pi}{2}$, b) $kL = 2\pi n + \frac{\theta}{2} \pm \frac{\pi}{2}$. (For these conditions the numerator

[14]More precisely the part of the discretized spectrum that becomes continuous in the limit $mL \to \infty$.

and denominator on one side of (94) vanish, but not on the other side.) To every solution with $k$ there is one with $-k$, but the two correspond to the same solution (up to normalization) so it suffices to consider $k \geq 0$; $k = 0$ has again $a = -1$ such that $\psi_1 = \psi_2 = 0$ everywhere and therefore must not be counted.

There are also potentially *zero modes*, $\omega = 0$, and *almost-zero modes*, $\omega \approx 0$, which have to be treated separately. For $\omega = 0$, the solutions to the Dirac equation read

$$\psi_I = \begin{pmatrix} a_1 e^{-mx} \\ a_2 e^{mx} \end{pmatrix}, \tag{96}$$

where $a_1$ and $a_2$ are determined by the boundary conditions.

Only TP and TAP give nontrivial solutions for $a_1$ and $a_2$ and thus are compatible with these solutions. There is one such zero mode for each of these boundary conditions.

The imaginary twisted boundary conditions iTP/iTAP on the other hand have *almost-zero* modes with energy $\omega^2 \to 4m^2 e^{-2mL}$ for $mL \to \infty$, with the positive-frequency solution satisfying iTP, and the negative-frequency one satisfying iTAP. To verify this, one can use the ansatz

$$\psi_1 = e^{-\kappa x} + a e^{\kappa x}, \quad -i\omega\psi_2 = (m - \kappa)e^{-\kappa x} + a(m + \kappa)e^{\kappa x} \tag{97}$$

with $\omega^2 = m^2 - \kappa^2$ and make the approximation $\kappa \approx m$ which becomes valid in the limit $mL \to \infty$.

The untwisted boundary conditions P, AP, iP, and iAP have neither zero nor almost-zero modes in the trivial sector.

### II. Kink sector

In the kink sector, one has asymptotic expressions

$$\psi_1 = \begin{cases} e^{i(kx-\frac{\delta}{2})} + a e^{-i(kx-\frac{\delta}{2})}, & x \approx 0 \\ e^{i(kx+\frac{\delta}{2})} + a e^{-i(kx+\frac{\delta}{2})}, & x \approx L \end{cases} \tag{98}$$

$$\psi_2 = -\begin{cases} e^{i(kx-\frac{\delta}{2}-\frac{\theta}{2})} - a e^{-i(kx-\frac{\delta}{2}-\frac{\theta}{2})}, & x \approx 0 \\ e^{i(kx+\frac{\delta}{2}+\frac{\theta}{2})} - a e^{-i(kx+\frac{\delta}{2}+\frac{\theta}{2})}, & x \approx L \end{cases} \tag{99}$$

where $\delta = -2\arctan(3mk/(m^2 - k^2))$ is the phase shift function also appearing for bosonic fluctuations. So $\psi_1$ behaves as the latter, while $\psi_2$ has a modified phase shift $\delta + \theta$.

For $\delta(k)$ we adopt the convention that $\delta(k \to \pm\infty) \to 0$ so that there is a discontinuity at $k = 0$ which in accordance with Levinson's theorem is $2\pi$ times the number of bound states. For $\theta$ we however keep the definition

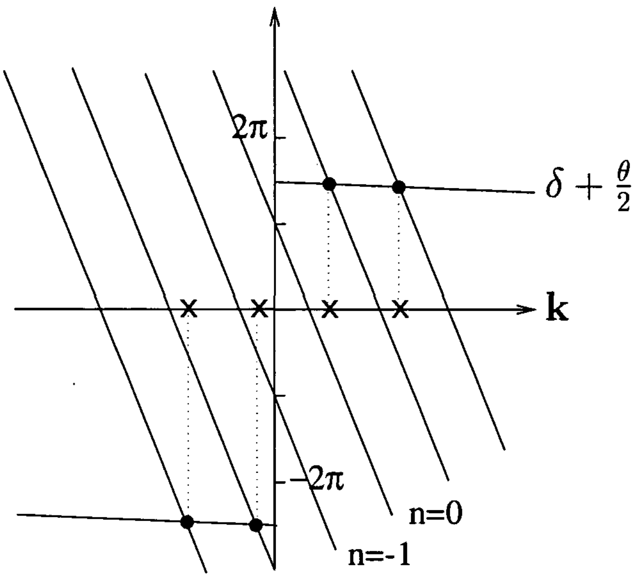



Figure 1: The quantization conditions for the fermionic modes in the case of TP boundary conditions obtained from solving $\delta + \frac{\theta}{2} = 2\pi n + \pi - kL$ for positive $\omega$. The spectrum is clearly not invariant under $k \to -k$.

of Eq. (93), which has the advantage of avoiding a separate treatment of positive and negative values of $k$.

We begin with discussing the untwisted boundary conditions. The (real) P and AP conditions can be satisfied either for a) $a = 1$ and $kL = 2\pi n + \pi - \delta - \theta$ or b) $a = -1$ and $kL = 2\pi n + \pi - \delta$, where only positive $n$ need to be considered to obtain a complete set of solutions and solutions with $k = 0$ have to be excluded, for they correspond to $\psi_1 = \psi_2 = 0$. Because these quantization conditions involve only $e^{i\theta}$ rather than $e^{i\theta/2}$, in this (and only in this) case it would make no difference to define $\theta$ such as to vanish for $k \to \pm\infty$, as done for example in Ref. [113] (which obtained an incorrect result for the susy kink mass only because there is a localized boundary energy contribution [61], as we shall see shortly).

The imaginary untwisted boundary conditions iP and iAP on the other hand have identical quantization conditions, which are given by the two sets a) $kL = 2\pi n + \frac{\pi}{2} - \delta - \frac{\theta}{2}$, $n \geq 1$, b) $kL = 2\pi n - \frac{\pi}{2} - \delta - \frac{\theta}{2}$, $n \geq 2$. Again, only positive $n$ need to be considered since (in contrast to iP/iAP in the trivial sector) $k \to -k$ does not lead to further independent solutions.

Turning now to the twisted boundary conditions, the TP ones lead to $kL = 2\pi n + \pi - \delta - \theta/2$. As shown in Fig. 1, this has solutions for all $n$ except $n = 0, -1$, and the set of these solutions is not symmetric under $k \to -k$. The solutions generated by the latter transformation instead obey TAP boundary conditions, which require $kL = 2\pi n - \delta - \theta/2$.

Table 5: Summary of fermionic quantization conditions, numbered in conformity with Ref. [61] where applicable, and the number of (almost-)zero modes ($n_z$) in each case. An upper index $\pm$ to the number $n_z$ indicates that these modes are only almost-zero modes; an index + or − indicates that only the positive or negative frequency mode, respectively, is compatible with the given boundary condition (b.c.).

| *i*) | b.c. | sector | $k_{i)}L$ | $n_z$ |
|---|---|---|---|---|
| 1) | P | trivial | $2\pi n$, all $n$ | 0 |
| 2) | AP | trivial | $2\pi n + \pi$, all $n$ | 0 |
| 3) | P | kink | a) $2\pi n - \delta - \theta$, $n \geq 1$ | 2 |
| | | | b) $2\pi n - \delta$, $n \geq 2$ | |
| 4) | AP | kink | a) $2\pi n + \pi - \delta - \theta$, $n \geq 1$ | $2^{\pm}$ |
| | | | b) $2\pi n + \pi - \delta$, $n \geq 1$ | |
| 1') | iP | trivial | $2\pi n - \pi/2$, all $n$ | 0 |
| 2') | iAP | trivial | $2\pi n + \pi/2$, all $n$ | 0 |
| 3')=4') | iP/iAP | kink | a) $2\pi n + \pi/2 - \delta - \theta/2$, $n \geq 1$ | $2^{\pm}$ |
| | | | b) $2\pi n + \pi/2 - \delta - \theta/2$, $n \geq 2$ | |
| 5)=6) | TP/TAP | trivial | a) $2\pi n$, $n \geq 1$ | 1 |
| | | | b) $2\pi n + \pi$, $n \geq 0$ | |
| 7) | TP | kink | $2\pi n + \pi - \delta - \theta/2$, all $n$, $n \neq 0, -1$ | 1 |
| 8) | TAP | kink | $2\pi n - \delta - \theta/2$, all $n$, $n \neq 0, -1$ | 1 |
| 5') | iTP | trivial | a) $2\pi n + \pi/2 - \theta/2$, $n \geq 0$ | $1^{+}$ |
| | | | b) $2\pi n + \pi/2 + \theta/2$, $n \geq 1$ | |
| 6') | iTAP | trivial | a) $2\pi n - \pi/2 - \theta/2$, $n \geq 1$ | $1^{-}$ |
| | | | b) $2\pi n - \pi/2 + \theta/2$, $n \geq 1$ | |
| 7') | iTP | kink | $2\pi n + \pi/2 - \delta - \theta/2$, all $n$, $n \neq 0, -1$ | $1^{+}$ |
| 8') | iTAP | kink | $2\pi n - \pi/2 - \delta - \theta/2$, all $n$, $n \neq 0, +1$ | $1^{-}$ |

The imaginary twisted boundary conditions iTP/iTAP differ from TP/TAP simply by an additional term $-\pi/2$ on the r.h.s. of the quantization conditions (for positive-frequency solutions). For iTP the exemptions are $n = 0, -1$ as with TP. For iTAP, $n = 0$ has to be excluded, while $n = \pm 1$ corresponds to the threshold mode $k = 0$, $\omega = m$, which is proportional to $(\psi_1, \psi_2) = (1 - 3\tanh^2(mx/2), -2i\tanh(mx/2))$, and thus consistent with iTAP boundary conditions (it does not appear in any of the other boundary conditions). Thus $n = \pm 1$ has to be counted only once.

In contrast to TP/TAP, the sets of allowed $k$-values for iTP and iTAP are each symmetric under $k \to -k$ (while the corresponding solutions are linearly independent), but a positive-frequency solution with momentum $k$ for iTP or iTAP has a negative-frequency partner only for the other of the two imaginary twisted boundary conditions.

For the counting of modes in the next subsection we also need to know how many zero modes there are for each boundary condition in the kink sector. For real boundary conditions these have been discussed in Ref. [61] and are recapitulated in Table 5, which summarizes the results of this subsection. The imaginary boundary conditions iP and iAP each have a pair of approximately-zero modes; however, for iTP there is only one approximately-zero mode with positive frequency, while the complex conjugated negative-frequency mode satisfies iTAP boundary conditions. (For iTP and iTAP boundary conditions, one can take $\psi_1$ real and $\psi_2$ purely imaginary as this is consistent with the Dirac equation, while for iP and iAP both $\psi_1$ and $\psi_2$ are complex combinations of two real solutions.)

Finally, in the kink sector there is one bound state with energy squared $\omega_B^2 = \frac{3}{4}m^2$. One can verify that on a finite interval it is possible to satisfy any of the boundary conditions considered by slightly increasing or decreasing the value of $\kappa_B$ in $\omega_B^2 = m^2 - \kappa_B^2$. This is easy to see for P, AP, iTP, and iTAP boundary conditions where the mode functions $\psi_1$ and $\psi_2$ are antisymmetric and symmetric around the kink center, respectively; for TP, TAP, iP, and iAP, we have verified the compatibility of the boundary conditions numerically. By contrast, the situation is more complicated for the zero modes, because there $\kappa_0$ can only be decreased from its maximal value $\kappa_0 = m$. Increasing $\kappa_0$ would turn $\omega^2$ negative, but the Hamiltonian (81) is self-adjoint with a Hermitean inner product.

### 3.3.2 Mode sums

**I. Real boundary conditions**

Evaluating (88) with an equal number of modes in the trivial and in the kink sector, one thus obtains for P and AP boundary conditions [61,113,133]

$$\begin{aligned} M_f^{(1)}(\mathrm{P}) &= \frac{\hbar}{2}\sum_{n=-N}^{N}\omega_{1)} - \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{3a)} - \frac{\hbar}{2}\sum_{n=2}^{N}\omega_{3b)} - 0 - \frac{\hbar\omega_B}{2} + \Delta M_f \\ &= -\frac{\hbar\omega_B}{2} + \hbar m + \hbar\int_0^{\Lambda}\frac{dk}{2\pi}\omega'\left(\delta + \frac{\theta}{2}\right) + \Delta M_f \end{aligned} \tag{100}$$

and

$$\begin{aligned} M_f^{(1)}(\mathrm{AP}) &= \hbar\sum_{n=0}^{N}\omega_{2)} - \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{4a)} - \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{4b)} - 0 - \frac{\hbar\omega_B}{2} + \Delta M_f \\ &= M_f^{(1)}(\mathrm{P}), \end{aligned} \tag{101}$$

where the sums for the trivial sectors are written first, with $\omega_{i)} = \sqrt{k_{i)}^2 + m^2}$ according to Table 5; explicit zeros indicate the presence of (almost-)zero modes. This leads to

$$M_f^{(1)}(\mathrm{P}) = M_f^{(1)}(\mathrm{AP}) = M_f^{(1)} + \frac{\hbar m}{4}\,, \tag{102}$$

implying that there is a finite amount of boundary energy equivalent to the contribution of one half of that of a low-lying continuum mode. Since P and AP are invisible boundary conditions in the trivial sector, this must be attributed to the kink sector.

For fixed TP and TAP boundary conditions, we find (correcting Ref. [61])

$$\begin{aligned} M_f^{(1)}(\mathrm{TP}) &= \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{5a)} + \frac{\hbar}{2}\sum_{n=0}^{N}\omega_{5b)} - \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{7)} - \frac{\hbar}{2}\sum_{n=-2}^{-(N+1)}\omega_{7)} \\ &\quad -\frac{\hbar\omega_B}{2} + \Delta M_f \\ &= -\frac{\hbar\omega_B}{2} + \frac{\hbar m}{2} + \hbar\int_0^\Lambda \frac{dk}{2\pi}\omega'\left(\delta + \frac{\theta}{2}\right) + \Delta M_f \\ &= M_f^{(1)} - \frac{\hbar m}{4} \end{aligned} \tag{103}$$

and

$$\begin{aligned} M_f^{(1)}(\mathrm{TAP}) &= \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{6a)} + \frac{\hbar}{2}\sum_{n=0}^{N}\omega_{6b)} - \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{8)} - \frac{\hbar}{2}\sum_{n=-2}^{-(N+1)}\omega_{8)} \\ &\quad -\frac{\hbar\omega_B}{2} + \Delta M_f \\ &= -\frac{\hbar\omega_B}{2} + \frac{\hbar m}{2} + \hbar\int_0^\Lambda \frac{dk}{2\pi}\omega'\left(\delta + \frac{\theta}{2}\right) + \Delta M_f \\ &= M_f^{(1)}(\mathrm{TP}). \end{aligned} \tag{104}$$

TP/TAP are invisible boundary conditions in the kink sector, so that any boundary energy must now be attributed to the trivial sector. As one can see, it has equal magnitude but opposite sign than in the results for P/AP, in agreement with the discussion in Ref. [61]. (Twisting the fermions from P in the trivial sector to TP in the kink sector, the localized boundary energy does not change.) However, because $M_f^{(1)}(\mathrm{TAP}) = M_f^{(1)}(\mathrm{TP})$, there is no delocalized boundary energy in the sense of Ref. [61].

Taking the average of the results of one of the untwisted and one of the twisted boundary conditions eliminates the localized boundary energy and yields the correct result (90).

In Ref. [104] it was found that mode number regularization with the completely invisible "topological" boundary conditions of P in the trivial sector and TP in the kink sector produces the correct finite part, but leaves an infinite (but $m$-independent) term corresponding to the contribution of one half of that of a continuum mode with $k = \Lambda$. The latter is removed by the derivative regularization method proposed in Ref. [104]. For mode regularization to give finite results it is crucial to have fixed boundary conditions. The localized boundary energies that this produces has then to be eliminated by averaging over one twisted and one untwisted boundary condition.

However, taking either TP or TAP for the twisted boundary condition, parity $\mathcal{P}$ is not a symmetry and thus the one-loop correction to the momentum in the kink sector need not be zero.

The momentum operator is diagonal asymptotically far away from the kink, and one obtains for TP

$$
\begin{aligned}
P_f^{(1)}(\mathrm{TP}) &= \frac{\hbar}{2L}\left(\sum_{n=1}^{N}+\sum_{n=-2}^{-(N+1)}\right)[2\pi n+\pi-\delta-\theta/2] \\
&= \frac{\hbar}{2}\int_0^\Lambda \frac{dk}{2\pi}[-\delta-\theta/2]+\frac{\hbar}{2}\int_{-\Lambda}^{0}\frac{dk}{2\pi}[-\delta-\theta/2] \\
&= +\frac{\hbar}{4}\Lambda \qquad (105)
\end{aligned}
$$

and, for TAP,

$$
\begin{aligned}
P_f^{(1)}(\mathrm{TAP}) &= \frac{\hbar}{2L}\left(\sum_{n=1}^{N}+\sum_{n=-2}^{-(N+1)}\right)[2\pi n-\delta-\theta/2] \\
&= \frac{\hbar}{2}\int_0^\Lambda \frac{dk}{2\pi}[-\delta-\theta/2]+\frac{\hbar}{2}\int_{-\Lambda}^{0}\frac{dk}{2\pi}[-2\pi-\delta-\theta/2] \\
&= -\frac{\hbar}{4}\Lambda\,. \qquad (106)
\end{aligned}
$$

Both results correspond to the contribution of one-half of a high-energy mode $|k|=\Lambda$, but with opposite sign. So there is an infinite amount of "delocalized momentum", which cancels only in the average over TP and TAP.

**II. Imaginary boundary conditions**

As discussed in Sect. 3.2, the imaginary versions of the above boundary conditions have the problem that each of iP, iAP, iTP, and iTAP separately break $\mathcal{C}$ and make it impossible to define Majorana quantum fields. In fact, $\mathcal{CPT}$ is equally violated.

Nevertheless, it may make sense to consider these boundary conditions in an averaged sense. Summing over positive frequencies only one has for iP

$$
\begin{aligned}
M_f^{(1)}(\mathrm{iP}) &= \frac{\hbar}{2}\sum_{n=-N}^{N}\omega_{1')}-\frac{\hbar}{2}\sum_{n=1}^{N}\omega_{3a')}-\frac{\hbar}{2}\sum_{n=2}^{N}\omega_{3b')}-0-\frac{\hbar\omega_B}{2}+\Delta M_f \\
&= -\frac{\hbar\omega_B}{2}+2\times\frac{\hbar m}{2}+\hbar\int_0^\Lambda\frac{dk}{2\pi}\omega'\left(\delta+\frac{\theta}{2}\right)+\Delta M_f \\
&= M_f^{(1)}+\frac{\hbar m}{4}, \qquad (107)
\end{aligned}
$$

and the same for $M_f^{(1)}(\text{iAP})$ because $\sum_{-N}^{N}\omega_{1')} = \sum_{-N}^{N}\omega_{2')}$ and (3')=(4') according to Table 5. The iP/iAP results for the one-loop energies thus coincide with the corresponding results for P/AP.

Analogously, for iTP one obtains

$$\begin{aligned}
M_f^{(1)}(\text{iTP}) &= 0 + \frac{\hbar}{2}\sum_{n=0}^{N}\omega_{5a')} + \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{5b')} - 2\times\frac{\hbar}{2}\sum_{n=1}^{N}\omega_{7')} - 0 \\
&\quad -\frac{\hbar\omega_B}{2} + \Delta M_f \\
&= -\frac{\hbar\omega_B}{2} + \frac{\hbar m}{2} + \hbar\int_0^\Lambda \frac{dk}{2\pi}\omega'\left(\delta + \frac{\theta}{2}\right) + \Delta M_f \\
&= M_f^{(1)} - \frac{\hbar m}{4}
\end{aligned} \tag{108}$$

and for iTAP

$$\begin{aligned}
M_f^{(1)}(\text{iTAP}) &= \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{6a')} + \frac{\hbar}{2}\sum_{n=1}^{N}\omega_{6b')} - \frac{\hbar m}{2} - 2\times\frac{\hbar}{2}\sum_{n=2}^{N}\omega_{8')} \\
&\quad -\frac{\hbar\omega_B}{2} + \Delta M_f \\
&= -\frac{\hbar\omega_B}{2} + \frac{\hbar m}{2} + \hbar\int_0^\Lambda \frac{dk}{2\pi}\omega'\left(\delta + \frac{\theta}{2}\right) + \Delta M_f \\
&= M_f^{(1)}(\text{iTP}).
\end{aligned} \tag{109}$$

Although $\mathcal{C}$ is broken, the two results coincide, so there is still no delocalized boundary energy in the sense of Ref. [61].

Because $\mathcal{P}$ is intact with either iTP or iTAP, there is also no delocalized momentum as with real twisted boundary conditions. However, iP/iAP in the trivial sector now break $\mathcal{P}$ (whereas the kink sector is symmetric under $k \to -k$), and one finds that there is delocalized momentum associated with the trivial sector,

$$P_f^{(1)}(\text{iP}) = \frac{\hbar}{2L}\sum_{n=-N}^{N}(2\pi n - \frac{\pi}{2}) = -\frac{\hbar}{4}\Lambda \tag{110}$$

and

$$P_f^{(1)}(\text{iAP}) = \frac{\hbar}{2L}\sum_{n=-N}^{N}(2\pi n + \frac{\pi}{2}) = +\frac{\hbar}{4}\Lambda, \tag{111}$$

which again corresponds to the contribution of one-half of a high-energy mode $|k| = \Lambda$ for iP and iAP separately, but with opposite sign.

Thus, averaging over the results of the mode sums for all four imaginary boundary conditions removes both localized boundary energies and delocalized momentum. In fact, only in such an average one effectively removes also the obstruction to the Majorana condition (and $\mathcal{CPT}$) that positive and negative frequency modes have different spectra.

Curiously enough, the necessity to consider iTP and iTAP together in order to have at least effectively no violation of $\mathcal{C}$ and $\mathcal{CPT}$ means that the threshold mode $k = 0$, which only appears under iTAP boundary conditions, is in the average counted like half a mode. In Ref. [67], in a different regularization method, threshold modes had to be treated explicitly as modes to be counted only half.

### 3.4 Discussion

We found that no single set of locally invisible boundary conditions preserved all three discrete symmetries. The real boundary conditions TP and TAP preserved $\mathcal{CPT}$, but break both $\mathcal{P}$ and $\mathcal{T}$. The imaginary variants iTP and iTAP on the other hand respect $\mathcal{P}$ and $\mathcal{T}$, but violate $\mathcal{C}$ and therefore even $\mathcal{CPT}$, so that these boundary conditions cannot be used for local quantum field theory, although this obstruction is effectively removed by averaging over iP and iAP, or iTP and iTAP. The cancellation of local boundary energy in the mode regularization scheme requires averaging over the results obtained with one twisted and one untwisted boundary condition, where these conditions have to be used both in the trivial and in the kink sector.

For compatibility with the Euler-Lagrange variational principle, one should require that boundary terms due to partial integrations cancel. In our case these "boundary field equations" read

$$\begin{aligned}&\psi_1(-L/2)\delta\psi_2(-L/2) + \psi_2(-L/2)\delta\psi_1(-L/2)\\ &\quad=\psi_1(L/2)\delta\psi_2(L/2) + \psi_2(L/2)\delta\psi_1(L/2). \qquad (112)\end{aligned}$$

It is easy to see that the real boundary conditions P, AP, TP, and TAP all satisfy this requirement, but the imaginary versions iP, iAP, iTP, and iTAP each violate it.

This means that none of the imaginary boundary conditions can be used in a Lagrangian formulation with Majorana fermions, although the Hamiltonian (81) with a Hermitean inner product is still self-adjoint. The same conclusion was reached by looking at the spectrum (derived from bulk field equations and imposing the boundary conditions). The problem with imaginary boundary conditions then turned out to be that for a given momentum $k$ and positive frequency $\omega$ there is no corresponding mode in the spectrum with $-k$ and $-\omega$, and no Majorana field can be built.

To avoid this problem, one would have to switch to complex fermions by giving up supersymmetry, as in the original Jackiw-Rebbi model [87], or go to $N=2$ susy models. We summarize our assertions about averaging over invisible boundary conditions to restore all three discrete symmetries. In the trivial sector, one may average over P and AP or iP and iAP, or both sets. However, because P and AP separately obey all symmetries, there is no need to average if one chooses one of these real periodic boundary conditions. In the kink sector, one may average over TP and TAP or iTP and iTAP, or both sets. Any of these is an acceptable method to restore the symmetries, but this time there is no single boundary condition which simultaneously satisfies all three, so that averaging over at least one pair is necessary. That fact is the main point of this section.

The idea that one must average over a set of boundary conditions to restore a symmetry is known in string theory, where the spinning string maintains modular invariance (large general coordinate transformations) and unitarity and supersymmetry only if one sums over all spin structures (the requirement that fermions on a closed surface are periodic or antiperiodic in spacelike or timelike directions) (see [70] pp. 279–281).

We close with some speculative remarks. The fact that no *locally invisible* boundary condition for the fermionic quantum fluctuations satisfies all three symmetries $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ simultaneously, whereas the classical action is $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ invariant, suggests that we are dealing with a discrete anomaly. The origin of this effect is the global structure (analogous to a Möbius strip in our case [104]), whereas the usual chiral anomaly is a local effect. Clearly, one should not confuse this with the anomalies due to instantons, where the effective action contains terms of the form $\psi^4 + \bar{\psi}^4$; these preserve parity but break chiral invariance.

Whether or not the striking loss of simultaneous $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ invariance should be called an anomaly in the sense of the chiral anomaly, it certainly satisfies the definition of an anomaly as a 'clash of quantum consistency conditions'.

# 4 The anomaly in the central charge of the supersymmetric kink

## 4.1 Introduction

The calculation of quantum corrections to the mass of a supersymmetric (susy) kink and to its central charge has proved to be a highly nontrivial task fraught with subtleties and pitfalls.

Initially it was thought that supersymmetry would lead to a complete cancellation of quantum corrections [34,35,81] and thereby guarantee Bogomol'nyi-Prasad-Sommerfield (BPS) saturation at the quantum level. Then, by considering a kink-antikink system in a finite box and regularizing the ultraviolet divergences by a cutoff in the number of the discretized modes, Schonfeld [120] found that there is a nonzero, *negative* quantum correction at one-loop level, $\Delta M^{(1)} = -m/(2\pi)$, but remarked that "the familiar sum of frequencies ... is unacceptably sensitive to the definition of the infinite volume limit". Most of the subsequent literature [28, 29, 88, 148] considered instead a single kink directly, using an energy-momentum cutoff which gave again a null result. A direct calculation of the central charge [84] also gave a null result, apparently confirming a conjecture[15] of Witten and Olive [146] that BPS saturation in the minimally susy 1+1 dimensional case would hold although arguments on multiplet shortening did not seem to apply.

Using a mode regularization scheme and periodic boundary conditions in a finite box Ref. [113] obtained a susy kink mass correction $\Delta M^{(1)} = +m(1/4 - 1/2\pi) > 0$ (obtained previously also in Ref. [133]) which together with the null result for the central charge appeared to be consistent with the BPS bound, but implying non-saturation. But as we saw in section 3 it turns out that one has to average over sets of boundary conditions to cancel both localized boundary energy and delocalized momentum in mode regularization [61, 63]. This indeed leads to a different result, namely that originally obtained by Schonfeld [120], which however appeared to be in conflict with the BPS inequality for a central charge without quantum corrections.

Since this appeared to be a pure one-loop effect, Ref. [104] proposed " ... the interesting conjecture that it may be formulated in terms of a topological quantum anomaly." It was then shown by Shifman et al. [123], using a susy-preserving higher-derivative regularization method, that there is an anomalous contribution to the central charge balancing the quantum corrections to the mass so that BPS saturation remains intact. In fact, it was later understood that multiplet shortening can occur even in minimally susy 1+1 dimensional theories, giving rise to single-state supermultiplets [99,100], as we will discuss at the end of this section.

[15] "While we suspect that this is true we have no proof." [146]

Both results, the non-vanishing mass correction and thus the necessity of a non-vanishing correction to the central charge, have been confirmed by a number of different methods [19,61–63,67,98,145] validating also the finite mass formula in terms of only the discrete modes derived in Refs. [20, 26] based on the method of [23]. However, some authors claimed a nontrivial quantum correction to the central charge [27, 67] apparently without the need of the anomalous term proposed in Ref. [123].

In the previous section 2, we have introduced a particularly simple and elegant regularization scheme that yields the correct quantum mass of the susy kink is dimensional regularization, if the kink is embedded in higher dimensions as a domain wall Such a scheme was not considered before for the susy kink because both susy and the existence of finite-energy solutions seemed to tie one to one spatial dimension.

In fact, the 1+1 dimensional susy kink can be embedded in 2+1 dimensions with the same field content while keeping susy invariance. For the corresponding classically BPS saturated domain wall (a 1+1 dimensional object by itself), we have found a nontrivial negative correction to the surface (i.e. string) tension [115]. In order to have BPS saturation at the quantum level, there has to be a matching correction to the momentum in the extra dimension which is the analog of the central charge of the 1+1 dimensional case.

In this section we show that in dimensional regularization by means of dimensional reduction from 2+1 dimensions, which preserves susy, one finds the required correction to the extra momentum to have a BPS saturated domain wall at the quantum level. This nontrivial correction is made possible by the fact that the 2+1 dimensional theory spontaneously breaks parity, which also allows the appearance of domain wall fermions of only one chirality.

By dimensionally reducing to 1+1 dimensions, the parity-violating contributions to the extra momentum turn out to provide an anomalous contribution to the central charge as postulated in Ref. [123], thereby giving a novel physical explanation of the latter. This is in line with the well-known fact that central charges of susy theories can be reinterpreted as momenta in higher dimensions.

Here an important observation takes place. The classical central charge stems entirely from the *antisymmetric* part of the energy momentum tensor of the $2+1$ dimensional theory and thus would missed in dimensional reduction if starting with the improved energy momentum tensor in $2+1$ dimensions. Whereas the symmetric part of the $2+1$ dimensional EM-tensor gives the anomalous contribution to the central charge. This anomalous contribution contribution can be reduced to a surface term and is thus completely determined by the topology of the soliton background, independent of the field profile in the bulk. This is quite analogous to the "geometric" chiral anomalies, which are determined by the topology of the gauge field background. Therefore when we refer to the $\varphi^4$ kink in the following, this is just a special realization of this general situation.

In the case of the susy kink, dimensional regularization is seen to be compatible with susy invariance only at the expense of a spontaneous parity violation, which in turn allows nonvanishing quantum corrections to the extra momentum in one higher spatial dimension. On the other hand, the surface term that usually exclusively provides the central charge does not receive quantum corrections in dimensional regularization, by the same reason that led to null results previously in other schemes [84, 104, 113]. The nontrivial anomalous quantum correction to the central charge operator is thus seen to be entirely the remnant of the spontaneous parity violation in the higher-dimensional theory in which a susy kink can be embedded by preserving minimal susy.

We also (in Sect. 4.3.3) pinpoint what we believe to be the error in Ref. [67] who arrived at the conclusion of BPS saturation apparently without the need for an anomalous additional term in the central charge operator.

In addition we consider dimensional regularization by dimensional reduction from 1 to 1-$\epsilon$ spatial dimensions, which also preserves supersymmetry. In this case we show that an anomalous contribution to the central charge arises from the necessity to add an evanescent counterterm to the susy current. This counterterm preserves susy but produces an anomaly in the conformal-susy current. We also construct the conformal central-charge current whose divergence is proportional to the ordinary central-charge current and thus contains the central-charge anomaly as superpartner of the conformal-susy anomaly.

## 4.2 Minimally supersymmetric kink and kink domain wall

### 4.2.1 The model

A real scalar field model in 1+1 dimensions with spontaneously broken $Z_2$ symmetry ($\varphi \to -\varphi$) has topologically nontrivial finite-energy solutions called "kinks" which interpolate between the two neighboring degenerate vacuum states, as for example $\varphi = \pm v$. If the potential is of the form $V(\varphi) = \frac{1}{2}U^2(\varphi)$ it has a minimally[16] , $\mathcal{N} = 1$ supersymmetric extension [44]

$$\mathcal{L} = -\frac{1}{2}\left[(\partial_\mu\varphi)^2 + U(\varphi)^2 + \bar{\psi}\gamma^\mu\partial_\mu\psi + U'(\varphi)\bar{\psi}\psi\right] \tag{113}$$

where $\psi$ is a Majorana spinor, $\bar{\psi} = \psi^{\mathrm{T}}C$ with $C\gamma^\mu = -(\gamma^\mu)^T C$. We again use a Majorana representation of the Dirac matrices with $\gamma^0 = -i\sigma^2$, $\gamma^1 = \sigma^3$, and $C = \sigma^2$ in terms of the standard Pauli matrices $\sigma^k$ so that $\psi = \binom{\psi^+}{\psi^-}$ with real $\psi^+(x,t)$ and $\psi^-(x,t)$.

[16] In the literature this is often noted by $\mathcal{N} = (1,1)$ supersymmetry, but we are having also three dimensions in mind.

The $\varphi^4$ model is defined as the special case

$$U(\varphi) = \sqrt{\frac{\lambda}{2}}\left(\varphi^2 - v_0^2\right), \qquad v_0^2 \equiv \mu_0^2/\lambda \tag{114}$$

where the $Z_2$ symmetry of the susy action as mentioned already (40) also involves the fermions according to $\varphi \to -\varphi, \psi \to \gamma^5\psi$ with $\gamma^5 = \gamma^0\gamma^1$. For the classical kink at rest at $x = 0$ which interpolates between the two vacua see (4).

At the quantum level we have to renormalize, and we shall employ the simplest possible scheme discussed in section 2 which consists of putting all renormalization constants to unity except for a mass counter term chosen such that tadpole diagrams cancel completely in the trivial vacuum. At the one-loop level and using dimensional regularization this gives(see (11))

$$\delta\mu^2 = \lambda\,\delta v^2 = \lambda \int \frac{dk_0 d^dk}{(2\pi)^{d+1}} \frac{-i}{k^2 + m^2 - i\epsilon} = \lambda \int \frac{d^dk}{(2\pi)^d} \frac{1}{2[\vec{k}^2 + m^2]^{1/2}}, \tag{115}$$

where $m = U'(v) = \sqrt{2}\mu$ is the mass of elementary bosons and fermions and $k^2 = \vec{k}^2 - k_0^2$.

The susy invariance of the model (113) under

$$\delta\varphi = \bar{\epsilon}\psi \quad , \quad \delta\psi = (\partial\!\!\!/\varphi - U)\epsilon, \tag{116}$$

(with $\mu_0^2$ replaced by $\mu^2 + \delta\mu^2$) leads to the on-shell conserved Noether current

$$j_\mu = -(\partial\!\!\!/\varphi + U(\varphi))\gamma_\mu\psi \tag{117}$$

and two conserved charges $Q^\pm = \int dx\, j_0^\pm$.

The model (113) is equally supersymmetric in 2+1 dimensions, where we use $\gamma^2 = \sigma^1$. The same renormalization scheme can be used, only the renormalization constant (115) has to be evaluated for $d = 2 - \epsilon$ in place of $d = 1 - \epsilon$ spatial dimensions.

While classical kinks in 1+1 dimensions have finite energy (rest mass) $M = m^3/\lambda$, in (non-compact) 2+1 dimensions there exist no longer solitons of finite-energy. Instead one can have (one-dimensional) domain walls with a profile given by (4) which have finite surface (string) tension $M/L = m^3/\lambda$. With a compact extra dimension one can of course use these configurations to form "domain strings" of finite total energy proportional to the length $L$ of the string when wrapped around the extra dimension.

The 2+1 dimensional case is different also with respect to the discrete symmetries of (113). In 2+1 dimensions, $\gamma^5 = \gamma^0\gamma^1\gamma^2 = \pm\mathbf{1}$ corresponding to the two inequivalent choices available for $\gamma^2 = \pm\sigma^1$ (in odd space-time dimensions the Clifford algebra has two inequivalent irreducible representations). Therefore, the sign of the fermion mass (Yukawa) term can no longer be reversed by $\psi \to \gamma^5\psi$ and there is no longer the $Z_2$ symmetry $\varphi \to -\varphi, \psi \to \gamma^5\psi$.

What the 2+1 dimensional model does break spontaneously is instead *parity*, which corresponds to changing the sign of one of the spatial coordinates. The Lagrangian is invariant under $x^m \to -x^m$ for a given spatial index $m = 1$ or $m = 2$ together with $\varphi \to -\varphi$ (which thus is a pseudo scalar) and $\psi \to \gamma^m\psi$. Each of the trivial vacua breaks these invariances spontaneously, whereas a kink background in the $x^1$-direction with $\varphi_K(-x^1) = -\varphi_K(x^1)$ transforms correctly with respect to $x^1$-reflections, but breaks $x^2 = y$ reflection invariance.

This is to be contrasted with the 1+1 dimensional case, where parity ($x^1 \to -x^1$) can be represented either by $\psi \to \gamma^0\psi$ and a true scalar $\varphi \to \varphi$ or by $\psi \to \gamma^1\psi$ and a pseudoscalar $\varphi \to -\varphi$. The former leaves the trivial vacuum invariant, and the latter the ground state of the kink sector.

### 4.2.2 Susy algebra

The susy algebra for the 1+1 and the 2+1 dimensional cases can both be covered by starting from 2+1 dimensions, the 1+1 dimensional case following from reduction by one spatial dimension.

In 2+1 dimensions one obtains classically [58]

$$\begin{aligned} \{Q^\alpha, \bar{Q}_\beta\} &= 2i(\gamma^M)^\alpha{}_\beta P_M \ , \quad (M = 0,1,2) \\ &= 2i(\gamma^0 H + \gamma^1(\tilde{P}_x + \tilde{Z}_y) + \gamma^2(\tilde{P}_y - \tilde{Z}_x))^\alpha{}_\beta, \end{aligned} \tag{118}$$

where we separated off two surface terms $\tilde{Z}_m$ in defining

$$\tilde{P}_m = \int d^dx \tilde{\mathcal{P}}_m, \qquad \tilde{\mathcal{P}}_m = \dot{\varphi}\, \partial_m\varphi - \frac{1}{2}(\bar{\psi}\gamma^0\partial_m\psi), \tag{119}$$

$$\tilde{Z}_m = \int d^dx \tilde{\mathcal{Z}}_m, \qquad \tilde{\mathcal{Z}}_m = U(\varphi)\partial_m\varphi = \partial_m\mathcal{W}(\varphi) \tag{120}$$

with $\mathcal{W}(\varphi) \equiv \int d\varphi\, U(\varphi)$. Note that the usual central charge density of the two-dimensional model, $\tilde{\mathcal{Z}}_m$, is obtained by dimensional reduction of the *antisymmetric* part of the three dimensional energy momentum tensor. The local version of the susy algebra (118) is obtained by a susy variation (116) of the supercurrent (117) as follows

$$\begin{aligned} T^{MN} &\sim Tr(\gamma^M \delta j^N) = Tr(\gamma^M\gamma^N\gamma^P)\ \partial_P\varphi U(\varphi) + \text{symm. part} \\ &\sim \varepsilon^{MNP}\partial_P\varphi U(\varphi) + \text{symm. part}, \end{aligned} \tag{121}$$

and the central charge density is then the momentum density $T_{03}$ in the reduced extra dimension.

Having a kink profile in the $x$-direction, which satisfies the Bogomol'nyi equation $\partial_x \varphi_K = -U(\varphi_K)$, one finds that with our choice of Dirac matrices

$$Q_\pm = \int d^2x[(\dot{\varphi} \mp \partial_y \varphi)\psi^\pm + (\partial_x \varphi \pm U(\varphi))\psi^\mp], \tag{122}$$

$$\{Q_\pm, Q_\pm\} = 2(H \pm (\tilde{Z}_x - \tilde{P}_y)), \quad \{Q_+, Q_-\} = 2P_x \tag{123}$$

and the charge $Q_+$ leaves the classical topological (domain-wall) vacuum ($\varphi = \varphi_K,\ \psi = 0$) invariant. This corresponds to classical BPS saturation, since with $P_x = 0$ and $\tilde{P}_y = 0$ one has $\{Q_+, Q_+\} = 2(H + \tilde{Z}_x)$ and, indeed, with a kink domain wall $\tilde{Z}_x/L^{d-1} = W(+v) - W(-v) = -M/L^{d-1}$.

**BPS bound** At the quantum level, Hermiticity of $Q_\pm$ and the positivity of the Hilbert space norm implies a lower bound for the energy(density):

$$\langle \Sigma | H | \Sigma \rangle \geq |\, \langle \Sigma | P_y | \Sigma \rangle \,| \equiv |\left\langle \Sigma | (\tilde{P}_y - \tilde{Z}_x) | \Sigma \right\rangle|. \tag{124}$$

This inequality is saturated when

$$Q_+ |\Sigma\rangle = 0. \tag{125}$$

By virtue of (123) also $\langle P_x \rangle = 0$ in this case, so that BPS states correspond to massless states, since with $[H, P_m] = 0$ one has

$$\langle P_M P^M \rangle = -\tfrac{1}{4}\langle (Q_+^2 Q_-^2 - \{Q_+, Q_-\}\rangle = 0 \tag{126}$$

for BPS saturated states (125) with $\langle P_y \rangle = M$ for a kink domain wall in the $x$-direction [100]. However with infinite momentum and energy unless the $y$-direction is compact with finite length $L$. An antikink domain wall has instead $Q_- |\Sigma\rangle = 0$. In both cases, half of the supersymmetry is spontaneously broken.

Omitting regularization the susy algebra in 1+1 dimensions is obtained from (118) simply by dropping $\tilde{P}_y$ as well as $\tilde{Z}_y$ so that $P_x \equiv \tilde{P}_x$. The term $\gamma^2 \tilde{Z}_x$ remains, however, with $\gamma^2$ being the nontrivial $\gamma^5$ of 1+1 dimensions. The susy algebra simplifies to

$$\{Q_\pm, Q_\pm\} = 2(H \pm Z), \quad \{Q_+, Q_-\} = 2P_x \tag{127}$$

and one obtains the quantum BPS bound

$$\langle \psi | H | \psi \rangle \geq |\, \langle \psi | Z | \psi \rangle \,| \tag{128}$$

for any state $|\psi\rangle$. BPS saturated states have $Q_+|\psi\rangle = 0$ or $Q_-|\psi\rangle = 0$, corresponding to kink and antikink, respectively, and break half of the supersymmetry.

The above considerations involve Heisenberg operators and Heisenberg states, which are both not known in perturbation theory. How and if an observed BPS saturation in perturbation theory can be extended to such general statements as made above will be discussed at the end of this section.

## 4.3 Quantum corrections to the susy algebra in dimensional regularization

### 4.3.1 Fluctuations

In a kink (or kink domain wall) background one spatial direction is singled out and we choose this to be along $x$. The direction orthogonal to the kink direction (parallel to the domain wall) will be denoted by $y$.

The quantum fields can then be expanded in the eigenfunctions, which are known analytically for the $\varphi^4$ and sine-Gordon-kink [112], times plane waves in the extra dimensions. For the bosonic fluctuations we have $[-\Box + (U'^2 + UU'')]\eta = 0$ which is solved by

$$\eta = \int \frac{d^{d-1}\ell}{(2\pi)^{\frac{d-1}{2}}} \sum\!\!\!\!\!\!\int \frac{dk}{\sqrt{4\pi\omega}} \left( a_{k,\ell}\, e^{-i(\omega t - \ell y)} \phi_k(x) + a^\dagger_{k,\ell}\, e^{i(\omega t-\ell y)} \phi^*_k(x) \right). \tag{129}$$

The kink eigenfunctions $\phi_k$ are normalized according to $\int dx|\phi|^2 = 1$ for the discrete states and to Dirac distributions for the continuum states according to $\int dx\, \phi^*_k \phi_{k'} = 2\pi\delta(k-k')$. The mode energies are $\omega = \sqrt{\omega_k^2 + \ell^2}$ where $\omega_k$ is the energy in the 1+1-dimensional case.

The canonical equal-time commutation relations $[\eta(\vec{x}), \dot{\eta}(\vec{x}')] = i\delta(\vec{x}-\vec{x}')$ are fulfilled with

$$[a_{k,\ell}, a^\dagger_{k',\ell'}] = \delta_{kk'}\delta(\ell - \ell'), \tag{130}$$

where for the continuum states $\delta_{k,k'}$ becomes a Dirac delta.

For the fermionic modes which satisfy the Dirac equation $[\partial\!\!\!/ + U']\psi = 0$ one finds

$$\begin{aligned}
\psi &= \psi_0 + \int \frac{d^{d-1}\ell}{(2\pi)^{\frac{d-1}{2}}} \sum\!\!\!\!\!\!\int{}' \frac{dk}{\sqrt{4\pi\omega}} \left[ b_{k,\ell}\, e^{-i(\omega t-\ell y)} \begin{pmatrix} \sqrt{\omega+\ell}\ \phi_k(x) \\ \sqrt{\omega-\ell}\ i s_k(x) \end{pmatrix} + b^\dagger_{k,\ell}\ (\text{c.c.}) \right], \\
\psi_0 &= \int \frac{d^{d-1}\ell}{(2\pi)^{\frac{d-1}{2}}} b_{0,\ell}\, e^{-i\ell(t-y)} \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix}, \quad b_0^\dagger(\ell) = b_0(-\ell).
\end{aligned} \tag{131}$$

As seen in section 2 the fermionic zero mode[17] of the susy kink turns into massless modes located on the domain wall, which have only one chirality, forming a Majorana-Weyl domain wall fermion [24, 57, 78, 115].[18]

[17] By a slight abuse of notation we shall always label this by a subscript 0, but this should not be confused with the threshold mode $k = 0$ (which does not appear explicitly anywhere below).

[18] The mode with $\ell = 0$ corresponds in 1+1 dimensions to the zero mode of the susy kink. It has to be counted as half a degree of freedom in mode regularization [61]. For dimensional regularization such subtleties do not play a role because the zero mode only gives scaleless integrals and these vanish.

For the massive modes the Dirac equation relates the eigenfunctions appearing in the upper and the lower components of the spinors as follows:

$$s_k = \frac{1}{\omega_k}(\partial_x + U')\phi_k = \frac{1}{\sqrt{\omega^2 - \ell^2}}(\partial_x + U')\phi_k, \tag{132}$$

so that the function $s_k$ is the SUSY-quantum mechanical (B) [147] partner state of $\phi_k$ and thus coincides with the eigen modes of the sine-Gordon model if $\phi_k$ belongs to the $\varphi^4$-kink (hence the notation) [33]. With (132), their normalization is the same as that of the $\phi_k$. It is the relation (132) and, the fact that it relates bosonic to fermionic modes, as well as the different components of fermionic to each other, which makes it possible to compute the one loop-correction to the energy and the anomaly in the central charge, without explicit knowledge of the mode functions.

The canonical equal-time anti-commutation relations $\{\psi^\alpha(\vec{x}), \psi^\beta(\vec{x}')\} = \delta^{\alpha\beta}\delta(\vec{x} - \vec{x}')$ are satisfied if

$$\begin{aligned} \{b_0(\ell), b_0^\dagger(\ell')\} &= \{b_0(\ell), b_0(-\ell')\} = \delta(\ell - \ell'), \\ \{b_{k,\ell}, b_{k',\ell'}^\dagger\} &= \delta_{k,k'}\delta(\ell - \ell'), \end{aligned} \tag{133}$$

and again the $\delta_{k,k'}$ becomes a Dirac delta for the continuum states. The algebra (133) and the solution for the massless mode (131) show that the operator $b_0(\ell)$ creates right-moving massless states on the wall when $\ell$ is negative and annihilates them for positive momentum $\ell$. Thus only massless states with momentum in the positive $y$-direction can be created. Changing the representation of the gamma matrices by $\gamma^2 \to -\gamma^2$, which is inequivalent to the original one, reverses the situation. Now only massless states with momenta in the positive $y$-direction exist. Thus depending on the representation of the Clifford algebra one chirality of the domain wall fermions is singled out. This is a reflection of the spontaneous violation of parity when embedding the susy kink as a domain wall in 2+1 dimensions.

Notice that in (131) $d$ can be only 2 or 1, for which $\ell$ has 1 or 0 components, so for strictly $d = 1$ $\ell \equiv 0$. In order to have a susy-preserving dimensional regularization scheme by dimensional reduction, we shall start from $d = 2$ spatial dimensions, and then make $d$ continuous and smaller than 2.

### 4.3.2 Energy corrections

Before turning to a direct calculation of the anomalous contributions to central charge and momentum, we derive, as promised in section 2, the one-loop energy density of the susy kink (domain wall) in dimensional regularization.

Expanding the Hamiltonian density of the model (113),

$$\mathcal{H} = \frac{1}{2}[\dot{\varphi} + (\nabla\varphi)^2 + U^2(\varphi)] + \frac{1}{2}\psi^\dagger i\gamma^0[\vec{\gamma}\,\vec{\nabla} + U'(\varphi)]\psi, \tag{134}$$

around the kink/domain wall, using $\varphi = \varphi_K + \eta$, one obtains

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}[(\partial_x \varphi_K)^2 + U^2] - \frac{\delta\mu^2}{\sqrt{2\lambda}} U - \partial_x(U\eta) + \\ &+ \frac{1}{2}[\dot{\eta}^2 + (\nabla\eta)^2 + \frac{1}{2}(U^2)''\eta^2] + \frac{1}{2}\psi^\dagger i\gamma^0[\vec{\gamma}\,\vec{\nabla} + U']\psi + O(\hbar^2), \end{aligned} \tag{135}$$

where $U$ without an explicit argument implies evaluation at $\varphi = \varphi_K$ and use of the renormalized $\mu^2$. The first two terms on the r.h.s. are the classical energy density and the counterterm contribution. The terms quadratic in the fluctuations are the only ones contributing to the total energy.[19] They give

$$\begin{aligned} \int dx\, d^{d-1}y\, \langle \mathcal{H}^{(2)} \rangle &= \frac{L^{d-1}}{2} \int dx \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \Big[ \frac{\omega}{2} |\phi_k|^2 \\ &\quad + \frac{1}{2\omega}(\ell^2 |\phi_k|^2 + |\phi_k{}'|^2 + \frac{1}{2}(U^2)''|\phi_k|^2) \Big] \\ &- \frac{L^{d-1}}{2} \int dx \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \frac{\omega}{2}(|\phi_k|^2 + |s_k|^2), \end{aligned} \tag{136}$$

where the two sum-integrals are the bosonic and fermionic contributions, respectively.

Using $\frac{1}{2}(U^2)'' = U'^2 - \partial_x U'$ which follows from the Bogomol'nyi equation $\partial_x \varphi_K = -U$ and partially integrating (or alternatively from the equipartition theorem for the energy of the bosonic fluctuations in (135)), one obtains the expected sum-integrals over zero-point energies,

$$\int dx\, d^{d-1}y\, \langle \mathcal{H}^{(2)} \rangle = \frac{L^{d-1}}{2} \int dx \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \frac{\omega}{2}(|\phi_k|^2 - |s_k|^2). \tag{137}$$

In these expressions, the massless modes (which correspond to the zero mode of the 1+1 dimensional kink) can be dropped in dimensional regularization as scaleless and thus vanishing contributions, and the massive discrete modes cancel between bosons and fermions.[20] Carrying out the $x$-integration over the continuous mode functions gives a difference of spectral densities, namely

$$\int dx(|\phi_k(x)|^2 - |s_k(x)|^2) = -\theta'(k) = -\frac{2m}{k^2 + m^2}, \tag{138}$$

where $\theta(k)$ is the additional phase shift of the mode functions $s_k$ compared to $\phi_k$.

So one can see that the entire one-loop correction to the energy, and as we will see also the anomaly in the central charge, is determined by

[19] The third term in (135) is of relevance when calculating the energy profile [62, 123].

[20] The zero mode contributions in fact do not cancel by themselves between bosons and fermions, because the latter are chiral. This non-cancellation is in fact crucial in energy cutoff regularization (see Ref. [115]).

the difference in the spectral densities of the fermionic modes. With the help of the susy-quantum mechanical relation (132) for the fermionic modes in the BPS background the integral (138) can also be computed without detailed knowledge of the mode functions: Denoting the operator in (132) by $A = \partial_x + U'$ the fluctuation equation above (129) of the "bosonic" modes $\phi_k$ factorizes as

$$A^\dagger A \phi_k = \omega_k^2 \phi_k, \tag{139}$$

where $A^\dagger = -\partial_x + U'$ is the adjoint operator. Using (132) the spectral density (138) can be written as

$$\begin{aligned} -\theta'(k) &= \int dx\; [|\phi_k(x)|^2 - \tfrac{1}{\omega_k^2}(A\phi_k)^*(A\phi_k)] \\ &= \int dx\; [|\phi_k(x)|^2 - \tfrac{1}{\omega_k^2}\phi_k^* A^\dagger A \phi_k] + \text{surface term}. \end{aligned} \tag{140}$$

The first term simply vanishes because of (139). The surface term results from the fact, that the operator $A^\dagger$ is Hermitian conjugated of $A$ only on the space of mode functions and their derivatives, but in (140) the factor $(A\phi_k)$ contains the derivative of the potential $U'$ in the kink background, which is a function of nontrivial topology. Thus we obtain for the spectral density difference

$$-\theta'(k) = -\tfrac{1}{\omega_k^2}\int dx\; \partial_x(U'|\phi_k(x)|^2) = -\frac{2m}{k^2+m^2}, \tag{141}$$

where we have used only the asymptotic values $U'[\varphi_K(x=\pm\infty)] = U'(\pm v) = \pm m$ and that the mode functions are plane waves asymptotically, i.e. $|\phi_k|^2 \to 1$. The result coincides with the explicitly integrated mode functions (138).

Combining (137) and (138), and adding in the counterterm contribution from (115) leads to a simple integral

$$\begin{aligned} \frac{\Delta M^{(1)}}{L^{d-1}} &= -\frac{1}{4}\int \frac{dk\, d^{d-1}\ell}{(2\pi)^d}\omega\, \theta'(k) + m\delta v^2 \\ &= -\frac{1}{4}\int \frac{dk\, d^{d-1}\ell}{(2\pi)^d}\frac{\ell^2}{\omega}\theta'(k) = -\frac{2}{d}\frac{\Gamma(\frac{3-d}{2})}{(4\pi)^{\frac{d+1}{2}}}m^d. \end{aligned} \tag{142}$$

This reproduces the correct known result for the susy kink mass correction $\Delta M^{(1)} = -m/(2\pi)$ (for $d=1$) and the surface (string) tension of the 2+1 dimensional susy kink domain wall $\Delta M^{(1)}/L = -m^2/(8\pi)$ (for $d=2$) [115].

Notice that the entire result is produced by an integrand proportional to the extra momentum component $\ell^2$, which for strictly $d=1$ would not exist. This can also be observed by recasting $\langle \mathcal{H}^{(2)} \rangle$ in (136) with the help of (132) in the form

$$\begin{aligned}\langle \mathcal{H}^{(2)} \rangle &= -\partial_x \left( \frac{1}{2} \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} U' \frac{|\phi_k|^2}{2\omega} \right) + \\ &+ \frac{1}{2} \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \frac{\ell^2}{2\omega} (|\phi_k^2| - |s_k|^2). \end{aligned} \tag{143}$$

When integrated, the first term, which is a pure surface term, cancels exactly the counterterm (see (115)), because

$$\int dx \langle \frac{1}{2} \partial_x (U' \eta^2) \rangle = \frac{1}{2} U' \langle \eta^2 \rangle |_{-\infty}^{\infty} = m \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \int \frac{dk}{2\pi} \frac{1}{2\omega} \equiv m \delta v^2, \tag{144}$$

where we have used that $U'(x = \pm\infty) = \pm m$. The second contribution in (143), on the other hand, is precisely the r.h.s. of (142). The first term in (143) was first given by Yamagishi [148] but because of the lack of proper regularization he missed the second term which gives the nontrivial correction. Anyhow, fourteen years later the same mistake was done by Graham and Jaffe [67] but now in the opposite direction when writing the central charge in the form of the Hamiltonian, as we explain below. Therefore they missed the anomaly in the central charge.

### 4.3.3 Anomalous contributions to the central charge and extra momentum

In a kink (domain wall) background with only nontrivial $x$ dependence, the central charge density $\tilde{\mathcal{Z}}_x$ receives nontrivial contributions. Expanding $\tilde{\mathcal{Z}}_x$ around the kink background gives

$$\tilde{\mathcal{Z}}_x = U \partial_x \varphi_K - \frac{\delta\mu^2}{\sqrt{2\lambda}} \partial_x \varphi_K + \partial_x (U\eta) + \frac{1}{2} \partial_x (U' \eta^2) + O(\eta^3). \tag{145}$$

Again only the part quadratic in the fluctuations contributes to the integrated quantity at one-loop order[21]. However, this leads just to the contribution shown in (144), which matches precisely the counterterm $m\delta v^2$ from requiring vanishing tadpoles. Straightforward application of the rules of dimensional regularization thus leads to a null result for the net one-loop correction to $\left\langle \tilde{Z}_x \right\rangle$ in the same way as found in Refs. [84, 104, 113] in other schemes.

On the other hand, by considering the less singular combination $\left\langle H + \tilde{Z}_x \right\rangle$ and showing that it vanishes exactly, it was concluded in Ref. [67] that $\left\langle \tilde{Z}_x \right\rangle$ has to compensate any nontrivial result for $\langle H \rangle$, which in Ref. [67] was obtained by subtracting successive Born approximations for scattering phase

[21] Again, this does not hold for the central charge density locally [62, 123].

shifts. In fact, Ref. [67] explicitly demonstrates how to rewrite $\left\langle \tilde{Z}_x \right\rangle$ into $-\langle H\rangle$, apparently without the need for the anomalous terms in the quantum central charge operator postulated in Ref. [123].

The resolution of this discrepancy is that Ref. [67] did not regularize $\left\langle \tilde{Z}_x \right\rangle$ and the manipulations needed to rewrite it as $-\langle H\rangle$ (which eventually is regularized and renormalized) are ill-defined. Using dimensional regularization one in fact obtains a nonzero result for $\left\langle H + \tilde{Z}_x \right\rangle$, apparently in violation of susy.

However, dimensional regularization by embedding the kink as a domain wall in (up to) one higher dimension, which preserves susy, instead leads to

$$\left\langle H + \tilde{Z}_x - \tilde{P}_y \right\rangle = 0, \tag{146}$$

i.e. the saturation of (124), as we shall now verify.

The bosonic contribution to $\left\langle \tilde{P}_y \right\rangle$ involves

$$\frac{1}{2}\langle \dot{\eta}\partial_y \eta + \partial_y \eta \dot{\eta}\rangle = -\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\, \frac{\ell}{2} |\phi_k(x)|^2. \tag{147}$$

The $\ell$-integral factorizes and gives zero both because it is a scale-less integral and because the integrand is odd in $\ell$. Only the fermions turn out to give interesting contributions:

$$\begin{aligned}
\langle \tilde{\mathcal{P}}_y\rangle &= \frac{i}{2}\langle \psi^\dagger \partial_y \psi\rangle \\
&= \frac{1}{2}\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\frac{\ell}{2\omega}\left[(\omega+\ell)|\phi_k|^2 + (\omega-\ell)|s_k|^2\right] \\
&= \frac{1}{2}\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}}\, \ell\, \theta(-\ell)\; |\phi_0|^2 + \\
&\quad + \frac{1}{2}\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int{}' \frac{dk}{2\pi}\left(\frac{\ell}{2}(|\phi_k|^2 + |s_k|^2) + \frac{\ell^2}{2\omega}(|\phi_k|^2 - |s_k|^2)\right)
\end{aligned} \tag{148}$$

From the last sum-integral we have separated off the contribution of the zero mode of the kink, which turns into chiral domain wall fermions for $d > 1$. The contribution of the latter no longer vanishes by symmetry, but the $\ell$-integral is still scale-less and therefore put to zero in dimensional regularization. The first sum-integral on the right-hand side is again zero by both symmetry and scalelessness, but the final term is not. The $\ell$-integration no longer factorizes because $\omega = \sqrt{k^2 + \ell^2 + m^2}$, and is in fact identical to the finite contribution in $\langle \mathcal{H}\rangle$ obtained already in (143):

$$-\Delta Z = \int dx\; \langle \tilde{\mathcal{P}}_y\rangle = \frac{1}{4}\int \frac{dk\, d^{d-1}\ell\, \ell^2}{(2\pi)^d\;\;\omega}\theta'(k) = \frac{2}{d}\frac{\Gamma(\frac{3-d}{2})}{(4\pi)^{\frac{d+1}{2}}}\, m^d. \tag{149}$$

So for all $d \leq 2$ we have BPS saturation, $\langle H \rangle = |\left\langle \tilde{Z}_x - \tilde{P}_y \right\rangle|$, which in the limit $d \to 1$, the susy kink, is made possible by a nonvanishing $\left\langle \tilde{P}_y \right\rangle$. The anomaly in the central charge is seen to arise from a parity-violating contribution in $d = 1 + \epsilon$ dimensions which is the price to be paid for preserving supersymmetry when going up in dimensions to embed the susy kink as a domain wall.

It is again the difference in the spectral densities, $\theta'$, which determines the one-loop corrections, which thus depend only on the derivative of the pre-potential (or equivalently the second derivative of super potential $\mathcal{W} = \int d\varphi U(\varphi)$) at the critical points $\pm v$. In general the spectral density difference for a model with spontaneously broken $\mathbb{Z}_2$ symmetry is given by

$$\theta'(k) = \frac{U'(v) - U'(-v)}{k^2 + U'(v)^2}, \tag{150}$$

which has an obvious generalization for $\mathbb{Z}_N$ symmetric models like the the sine-Gordon model. From (139,140) one can see that this quantity is closely related to the index of the operator $AA^\dagger$. For the here considered simple models, where only one spatial direction is nontrivial, $\theta'(k)$ is easily obtained from the Dirac equation in the asymptotic regions $x \to \pm\infty$, far away from the kink [113]. But as we will see in section 5, in case of a less trivial background like the vortex, the formulation as surface term as given in (140,141) will provide essential simplifications.

### 4.4 Dimensional reduction and evanescent counterterms

In the above, we have effectively used the 't Hooft-Veltman version of dimensional regularization [130] in which the space-time dimensionality $n$ is made larger than the dimension of interest. In general this breaks susy because the numbers of bosons and fermions are not the same anymore when one moves up in dimensions. But in our particular model the number of states are the same in 1+1 and 2+1 dimensions, so that we could preserve susy, though this led to new physics like spontaneous parity violation and chiral domain wall fermions.

In 2+1 dimensions, we have $P_y = \tilde{P}_y - \tilde{Z}_x$ and $|\langle P_y \rangle| = \langle H \rangle$, where $\tilde{P}$ and $\tilde{Z}$ were defined in (118). Classically, this BPS saturation is guaranteed by $\tilde{Z}_x$ alone. At the quantum level, however, the quantum corrections to the latter are cancelled completely by the counterterm from renormalizing tadpoles to zero. All nontrivial corrections come from the "genuine" momentum operator $\tilde{P}_y$, and are due to having a spontaneous breaking of parity.

In the limit of 1+1 dimensions, because $\gamma^2|_{D=2+1} = \gamma^5|_{D=1+1}$, one has to make the identification $Z = \tilde{Z}_x - \tilde{P}_y$. For $\tilde{Z}_x$, one again does not obtain net quantum corrections. However, the expectation value $\left\langle \tilde{P}_y \right\rangle$ does not vanish in the limit $d \to 1$, although there is no longer an extra dimension. The spontaneous parity violation in the 2+1 dimensional theory, which had to be considered in order to preserve susy, leaves a finite imprint upon dimensional reduction to 1+1 dimensions by providing an anomalous additional contribution to $\left\langle \tilde{Z}_x \right\rangle$ balancing the nontrivial quantum correction $\langle H \rangle$.

We now comment on how the central charge anomaly can be recovered from Siegel's version of dimensional regularization [25, 124, 127] where $n$ is smaller than the dimension of space-time and where one keeps the number of field components fixed, but lowers the number of coordinates and momenta from 2 to $n < 2$. At the one-loop level one encounters 2-dimensional $\delta_\mu^\nu$ coming from Dirac matrices, and $n$-dimensional $\hat{\delta}_\mu^\nu$ from loop momenta. An important concept which is going to play a role are the evanescent counterterms [17, 18] involving the factor $\frac{1}{\epsilon}\hat{\hat{\delta}}_\mu^\nu \gamma_\nu \psi$, where $\hat{\hat{\delta}}_\mu^\nu \equiv \delta_\mu^\nu - \hat{\delta}_\mu^\nu$ has only $\epsilon = 2 - n$ nonvanishing components.

For the chiral anomaly in 3+1 dimensions due to a massless Dirac fermion coupled to on-shell photons one finds from dimensional reduction the following expression for the regularized but not yet renormalized chiral current [71, 73, 74]

$$j_\mu = \frac{1}{2}\partial_\mu \frac{1}{\Box} F^{\rho\sigma}\tilde{F}_{\rho\sigma} - \frac{2}{\epsilon}\tilde{F}_{\mu\nu}\hat{\hat{A}}^\nu \tag{151}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and $\hat{\hat{A}}^\nu \equiv \hat{\hat{\delta}}_\mu^\nu A^\mu$. This current is gauge invariant because $\delta\hat{\hat{A}}_\nu = \hat{\hat{\partial}}_\nu \lambda = 0$ as coordinates only lie in the $n$-dimensional subspace. It is conserved since total antisymmetrization of five indices in 4 dimensions yields

$$\tilde{F}^{\mu\nu}\partial_\mu \hat{\hat{A}}_\nu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\partial_\mu \hat{\hat{\delta}}_\nu^\lambda A_\lambda = -\tilde{F}^{\rho\nu}\partial_\rho \hat{\hat{A}}_\nu + \epsilon\frac{1}{2}F^{\mu\nu}\tilde{F}_{\mu\nu}. \tag{152}$$

Clearly, $\partial^\mu j_\mu = \frac{1}{2}F^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{2}{\epsilon}(\frac{1}{4}\epsilon F^{\mu\nu}\tilde{F}_{\mu\nu}) = 0$. The composite operator $j_\mu$ is renormalized by subtracting the divergence $-\frac{2}{\epsilon}\tilde{F}_{\mu\nu}\hat{\hat{A}}^\nu$ (operator mixing), and thus in dimensional reduction the chiral anomaly is produced by the (evanescent) counterterm, and not by the loop graph.

Consider now the supercurrent $j_\mu = -(\partial\!\!\!/\varphi + U(\varphi))\gamma_\mu\psi$. In the trivial vacuum, expanding into quantum fields yields

$$j_\mu = -\left(\partial\!\!\!/\eta + U'(v)\,\eta + \frac{1}{2}U''(v)\,\eta^2\right)\gamma_\mu\psi + \frac{1}{\sqrt{2\lambda}}\delta\mu^2\gamma_\mu\psi. \tag{153}$$

Only matrix elements with one external fermion are divergent. The term involving $U''(v)\eta^2$ in (153) gives rise to a divergent scalar tadpole that is cancelled completely by the counterterm $\delta\mu^2$ (which itself is due to an $\eta$ and a $\psi$ loop). The only other divergent diagram is due to the term involving $\partial\!\!\!/\eta$ in (153) and has the form a $\psi$-selfenergy. Its singular part reads

$$\langle 0|j_\mu|p\rangle^{\mathrm{div}} = iU''(v)\int_0^1 dx \int \frac{d^n\kappa}{(2\pi)^n}\frac{\kappa\!\!\!/\gamma_\mu\,\kappa\!\!\!/}{[\kappa^2 + p^2x(1-x) + m^2]^2}u(p). \tag{154}$$

Using $\hat{\delta}_\mu^\nu \equiv \delta_\mu^\nu - \hat{\hat{\delta}}_\mu^\nu$ we find that under the integral

$$\not{\kappa}\gamma_\mu\not{\kappa} = -\kappa^2(\delta_\mu^\lambda - \frac{2}{n}\hat{\delta}_\mu^\lambda)\gamma_\lambda = \frac{\epsilon}{n}\kappa^2\gamma_\mu - \frac{2}{n}\kappa^2\hat{\hat{\delta}}_\mu^\lambda\gamma_\lambda$$

so that

$$\langle 0|j_\mu|p\rangle^{\text{div}} = \frac{U''(v)}{2\pi}\frac{\hat{\hat{\delta}}_\mu^\lambda}{\epsilon}\gamma_\lambda u(p). \tag{155}$$

Hence, the regularized one-loop contribution to the susy current contains the evanescent operator

$$j_\mu^{\text{div}} = \frac{U''(\varphi)}{2\pi}\frac{\hat{\hat{\delta}}_\mu^\lambda}{\epsilon}\gamma_\lambda\psi. \tag{156}$$

This is by itself a conserved quantity, because all fields depend only on the $n$-dimensional coordinates, but it has a nonvanishing contraction with $\gamma^\mu$. The latter gives rise to an anomalous contribution to the renormalized conformal-susy current $\not{x}j_\mu^{\text{ren.}}$ where $j_\mu^{\text{ren.}} = j_\mu - j_\mu^{div}$,

$$\partial^\mu(\not{x}j_\mu^{\text{ren.}})_{\text{anom.}} = -\gamma^\mu j_\mu^{\text{div}} = -\frac{U''}{2\pi}\psi. \tag{157}$$

(There are also nonvanishing nonanomalous contributions to $\partial^\mu(\not{x}j_\mu)$ because our model is not conformal-susy invariant at the classical level.)

Ordinary susy on the other hand is unbroken; there is no anomaly in the divergence of $j_\mu^{\text{ren.}}$. A susy variation of $j_\mu$ involves the energy-momentum tensor and the topological central-charge current $\zeta_\mu$ according to

$$\delta j_\mu = -2T_\mu{}^\nu\gamma_\nu\epsilon - 2\zeta_\mu\gamma^5\epsilon, \tag{158}$$

where classically $\zeta_\mu = \epsilon_{\mu\nu}U\partial^\nu\varphi$.

At the quantum level, the counter-term $j_\mu^{\text{ct}} = -j_\mu^{\text{div.}}$ induces an additional contribution to the central charge current

$$\zeta_\mu^{\text{anom}} = \frac{1}{4\pi}\frac{\hat{\hat{\delta}}_\mu^\nu}{\epsilon}\epsilon_{\nu\rho}\partial^\rho U' \tag{159}$$

which despite appearances is a *finite* quantity: using that total antisymmetrization of the three lower indices has to vanish in two dimensions gives

$$\hat{\hat{\delta}}_\mu^\nu\epsilon_{\nu\rho} = \epsilon\epsilon_{\mu\rho} + \hat{\hat{\delta}}_\rho^\nu\epsilon_{\nu\mu} \tag{160}$$

and together with the fact the $U'$ only depends on $n$-dimensional coordinates this finally yields

$$\zeta_\mu^{\text{anom}} = \frac{1}{4\pi}\epsilon_{\mu\rho}\partial^\rho U' \tag{161}$$

in agreement with the anomaly in the central charge as obtained previously.

We emphasize that $\zeta_\mu$ itself does not require the subtraction of an evanescent counterterm. The latter only appears in the susy current $j_\mu$, which gives rise to a conformal-susy anomaly in $\not{x} j_\mu$. A susy variation of the latter shows that it forms a conformal current multiplet involving besides the dilatation current $T_{\mu\nu}x^\nu$ and the Lorentz current $T_\mu{}^\nu x^\rho \epsilon_{\nu\rho}$ also a current $j^{(\zeta)\mu}_{(\nu)} = x^\rho \epsilon_{\rho\nu}\zeta^\mu$. We identify this with the conformal central-charge current, which is to be distinguished from the ordinary central-charge current $\zeta_\mu$.

Since $\partial_\mu j^{(\zeta)\mu}_{(\nu)} = \epsilon_{\mu\nu}\zeta^\mu$, and $\epsilon_{\mu\nu}$ is invertible, the entire central-charge current $\zeta^\mu$ enters in the divergence of the conformal central-charge current, whereas in the case of the conformal-susy current it was the contraction $\gamma_\mu j^\mu$.

The current $j^{(\zeta)}$ thus has the curious property of being completely determined by its own divergence. For this reason it is in fact not associated with any continuous symmetry (as is also the case for the ordinary central-charge current, which is of topological origin). In the absence of classical breaking of conformal invariance it is conserved trivially by its complete disappearance and then there is no symmetry generating charge operator. Nevertheless, in the conformally noninvariant susy kink model this current arises and has in addition to its nonanomalous divergence an anomalous one, namely the anomalous contribution to the central charge current inherited from the evanescent counterterm in the renormalized susy current.

### 4.5 Zero modes, multiplet shortening and BPS saturation

For the following we consider strictly two dimensions, since multiplet shorting is somewhat peculiar for N=1 susy in two dimensions. The classical kink background ($\varphi = \varphi_K$ , $\psi = 0$) spontaneously breaks symmetries of the system (113). Then the background transforms nontrivial under infinitesimal symmetry transformation. Now at the one hand a variation of a classical solution fulfills the fluctuation equation, i.e. the linearized e.o.m. On the other hand, a symmetry transformation of a classical solution is again an exact classical solution. Therefore a spontaneously broken symmetry gives rise to zero modes. In the case of the kink background the broken symmetries are the translational symmetry and half of the supersymmetry (116): [22]

$$\delta\varphi_K = \Delta x \partial_x \varphi_K \quad , \quad \delta\psi_+ = 4\partial_x \varphi_K \epsilon. \tag{162}$$

where we have used the Bogomol'nyi (BPS) equation $\partial_x \varphi_K = -U(\varphi_K)$ for the kink, which also implies that the second supersymmetry ($\epsilon = \epsilon_-$) in (116) is still unbroken. This in turn can be used as a definition for BPS solutions and equations.

[22] Boost do not lead to an independent zero mode

### 4.5.1 Abstract representations

We construct representations of the strictly two dimensional algebra

$$Q_+^2 = H + Z \quad , \quad Q_-^2 = H - Z \quad , \quad \{Q_+, Q_-\} = 2P, \tag{163}$$

where $Q_\pm^\dagger = Q_\pm$ are Hermitian, by the method of induced representations [144].Going to the rest frame in the topological sector, i.e. with non-vanishing central charge (a hat indicates that the operator refers to a certain representation), one obtains:

$$\hat{P}_\mu|\Sigma\rangle = \begin{pmatrix} M \\ 0 \end{pmatrix} |\Sigma\rangle \quad , \quad \hat{Z}|\Sigma\rangle = Z|\Sigma\rangle, \tag{164}$$

where $M, Z$ are ordinary numbers now. Since the central charge commutes with all other generators it is of course possible to diagonalize $P$ and $Z$ simultaneously. With the rescaling $Q_\pm = \sqrt{M \pm Z}\, q_\pm$ the algebra between states $|\Sigma\rangle$ becomes

$$q_+^2 = 1 \quad , \quad q_-^2 = 1 \quad , \quad \{q_+, q_-\} = 0. \tag{165}$$

This is a two dimensional Clifford algebra and the irreducible representations, which are all equivalent, are also two dimensional $\{|\Sigma_-\rangle|\Sigma_+\rangle\}$. The super-charges are then obtained as[23]

$$\hat{Q}_+ = \sqrt{M + Z}\, \sigma_1 \quad , \quad \hat{Q}_- = \sqrt{M - Z}\, \sigma_2. \tag{166}$$

**Short multiplet, BPS states** For BPS states per definition the absolute value of the central charge is equal to the energy of this state, i.e. for the eigen values in (163) we have

$$M - |Z| = 0. \tag{167}$$

We choose $Z = -M$ which corresponds to our convention for the kink. The algebra (163) in such a BPS representation becomes now

$$\hat{Q}_+^2 = 0 \quad , \quad \hat{Q}_-^2 = 2M \quad , \quad \{\hat{Q}_+, \hat{Q}_-\} = 0. \tag{168}$$

Because of the Hermiticity of $Q_+$, (168) implies $||Q_+|\Sigma\rangle||^2 = 0$ and thus

$$Q_+|\Sigma\rangle = 0. \tag{169}$$

This equation is equivalent to (167) for the definition of BPS states and means that the BPS states are left invariant by half of the supersymmetry, namely $Q_+$ in our case. Operators and states can be characterized by the cohomology of the operator $Q_+$. Analogous to BRST exact operators which

[23] We choose $a = \frac{1}{2}(q_+ - iq_-)$ so that $\{a, a^\dagger\} = 1$ and $a|\Sigma_-\rangle = 0$.

have vanishing matrix elements for physical states we can say that each operator which is $Q_+$–exact has vanishing expectation value for BPS states:

$$\mathcal{O} = \{Q_+, \mathcal{O}'\} \Rightarrow \langle BPS|\mathcal{O}|BPS\rangle = 0. \tag{170}$$

Linear representation spaces for a nilpotent operator decompose into singlets $Q_+|\phi\rangle = 0$ with $|\phi\rangle \neq Q_+|\phi'\rangle$ and doublets $(|\psi\rangle, Q_+|\psi\rangle)$. But as already mentioned the Hermiticity of $Q_+$ implies that BPS states are singlets. Thus $\hat{Q}_+$ is represented trivial, i.e. identical zero. Then $\hat{Q}_-$ in (168) forms a one dimensional Clifford algebra and there exists two inequivalent irreducible representations,

$$\hat{Q}_-|\Sigma\rangle = \pm\sqrt{2M}|\Sigma\rangle, \tag{171}$$

which are connected by a $\mathbb{Z}_2$ transformation $\psi \to -\psi$ for all fermions, which is clearly a symmetry for each fermionic action. Thus the irreducible representation is one-dimensional and the *fermionic* operator is diagonal [100]. This is the reason why it was originally thought that multiplet shortening does not occur in two dimensions [113, 123, 146]. Therefore a reducible two dimensional representation for the soliton states was assumed such that the fermion parity operator $(-1)^F$ is still defined. For a reducible two dimensional representation $\{|\Sigma_{\rm b}\rangle,\ |\Sigma_{\rm f}\rangle\}$ we may choose:

$$\hat{Q}_- = \sqrt{2M}\,\sigma_1 \quad , \quad (-1)^F = \sigma_3, \tag{172}$$

so that $(-1)^F$ is diagonal in this representation and $Q_-$ has fermion parity $-1$, i.e. $\{\hat{Q}, (-1)^F\} = 0$. Note that this is the direct sum of the two inequivalent irreducible representations (171), which are obtained as $\frac{1}{2}(|\Sigma_{\rm b}\rangle \pm |\Sigma_{\rm f}\rangle)$.

Witten and Olive [146] observed, that in four dimensional susy gauge theories the number of particle states is not changed in the Higgs phase, although massive representations have $2^{\mathcal{N}}$ times as many states than massless one. Thus, they concluded, that the Higgs phase corresponds to a BPS saturated representation which has the same dimension as the massless representation. Because of this *multiplet shortening* the BPS saturation should be protected against perturbative corrections since they cannot change the number of particle states.

The counting of susy soliton states in two dimensions is somewhat peculiar (see below) and the loss of fermion parity (171) suggested a two dimensional representation, as for the non-susy soliton [87], and thus no multiplet shortening would occur. In [123], nevertheless BPS-saturation was assumed, to match the central charge correction to the mass correction obtained in [113]. The crucial relation for BPS saturation is the annihilation by one super charge (169). It was stated that this relation is protected without multiplet shortening, by analogous arguments that constraint supersymmetry breaking [147]. A simple argument shows that this is not sufficient.

Assume that in some *approximation* a reducible multiplet is BPS saturated, i.e. $\hat{Q}_+|\Sigma_i\rangle = 0$. Since the operator $Q_+$ is Hermitian its representation is of the form (because of the reducibility the $\mathbb{Z}_2$–grading through $(-1)^F$ exists)

$$\hat{Q}_+ = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}, \tag{173}$$

and the BPS states can be separated in zero-eigên states of $M$ and $M^\dagger$. To answer the question if the multiplet remains BPS saturated under perturbations (corrections) we consider the quantity[24]

$$\lim_{\beta\to\infty} \mathrm{Tr}\left[(-1)^F\, e^{-\beta\, Q_+^2}\right] = n_\mathrm{b}^0 - n_\mathrm{f}^0 = \mathrm{Ind}(M), \tag{174}$$

which is the difference between the number of zero-eigenvalue eigen states (singlets) of $M$ and $M^\dagger$. That this index is invariant under perturbative corrections can be seen quite analogously to the arguments of [147]. If a state is no longer annihilated by $Q_+$, say for example one with fermionic parity, then also the bosonic super-partner state is no longer annihilated:

$$0 \neq Q_+|\mathrm{f}\,\rangle \sim Q_+Q_-|\mathrm{b}\rangle = -Q_-Q_+|\mathrm{b}\rangle \to Q_+|\mathrm{b}\rangle \neq 0. \tag{175}$$

So the difference in the number of singlet states is unchanged under perturbative corrections, and thus it can be calculated in a semi-classical approximation. So what can this index tell us? In case that it would be nonzero, there would exist, at least one, BPS saturated state, which is then protected against quantum corrections. But in this case $(-1)^F$ is no longer defined as we will see immediately. If the index vanishes in an approximation, $n_\mathrm{b}^0 - n_\mathrm{f}^0 = 0$, the number of fermionic and bosonic singlets coincides. The trivial case is of course that they both vanish already in the approximation and there are no BPS states. In the nontrivial case there exist susy pairs of BPS singlets in the approximation, but susy does not protect them from being lifted pairwise above the BPS bound as described in (175). But this is exactly the case of the $\mathbb{Z}_2$ symmetric two-dimensional multiplet (172). So the equality between the mass correction and the anomalous contribution to the central charge needs a different explanation, and in fact it was found that one has to give up the usual fermion parity for the topological soliton state which is then a single-state short super-multiplet (171) [100]. If we look now again on the BPS saturation equation (169), we see immediately that a lift above the BPS bound would give a twice as long irreducible multiplet (166) which cannot be caused by perturbative corrections. So, in the absence of other mechanism as for example a difference in a conserved quantum number, multiplet shortening is a necessary condition for BPS saturation being protected.

[24]Note that BPS states ($Q_+^2 = 0$) contribute $\mathrm{Tr}|_{BPS}(-1)^F = n_\mathrm{b}^0 - n_\mathrm{f}^0$, whereas for non-BPS states is $Q_+^2 = \langle H + Z\rangle > 0$ such that their contribution vanish for $\beta \to \infty$.

### 4.5.2 Realization in a quantum field theory

Up to now we have only discussed abstract representations of the susy algebra (163). In a quantum field theory the operators in the algebra (163) and their representations correspond to Heisenberg operators of conserved, i.e. time-invariant, charges and Heisenberg states. In general neither operators nor states in the Heisenberg picture are known explicitly. But instead one quantizes the field operators in the interaction picture in terms of creation/annihilation operators which are defined w.r.t. a perturbative ground state. The canonical commutation relations imply an algebra for the mode coefficients which usually has to be represented in an irreducible manner. This determines which kind of the above representations is realized in the (perturbative) quantum field theory.

In our case this is the algebra (133) (for $\ell = 0$ now)

$$\{b_k, b_{k'}^\dagger\} = \delta_{k,k'} \quad , \quad b_0^2 = 1. \tag{176}$$

The representation of the bosonic algebra is less involved, except that the collective coordinate governed by the translational zero mode is represented with vanishing momentum $\dot{q}_0 = 0$.

Let us first consider the quasi-classical case [100], i.e. the quantum mechanics of the soliton moduli. For this one includes only the classical background $\varphi_K$ and the bosonic and fermionic zero modes (moduli), with operators

$$b_0^2 = 1 \ \text{ and } \ [\dot{q}, q] = -i. \tag{177}$$

With the BPS equation $\partial_x \varphi_K + U = 0$ and $\dot{q}_0 = 0$ one obtains for the supercharges (122)

$$Q_+ = 0 \quad , \quad Q_- = 2\int dx \partial_x \varphi_K \psi_0 = \sqrt{2M_{cl}}\, b_0, \tag{178}$$

where for $b_0$ there exist two inequivalent irreducible representations $b_0|s_p m\rangle = \pm|s_\pm\rangle$, which corresponds to the two short BPS multiplets (171).

Instead of discussing how to represent the quasi-classical quantum mechanics let us consider quantum field theory. In the semi-classical approximation, one has to include all modes of the quantum field, which differs from the quasi-classical case qualitatively, since the number of mode degrees of freedom is changed from 2 to infinity. Thus we are now looking for a representation of (176). This is an infinite dimensional Clifford algebra of pairs of generators $\gamma_k^+ = (b_k + b_k^\dagger)$ and $\gamma_k^- = i(b_k - b_k^\dagger)$ and the single generator $b_0$. This is quite analogous to an odd dimensional Clifford algebra, and the operator $b_0$ plays the role of the $\gamma_5$ of the even-dimensional algebra $\gamma_k^\pm$. So there are two inequivalent representations of the full algebra governed by the sign of the "gamma five" operator $b_0$. Because $b_0$ has to anti-commute with the $b_k$'s this time it cannot be represented as a number. The $b_k$-algebra can be represented as usual on a Fock space, constructed from the Clifford vacuum $|\Omega\rangle$ with $b_k|\Omega\rangle = 0$. The whole algebra, including $b_0$ can then be realized in two inequivalent irreducible representations [61]:

$$|s_\pm\rangle = \tfrac{1}{2}(1 \pm b_0)|\Omega\rangle \quad , \quad b_0|s_\pm\rangle = \pm|s_\pm\rangle \quad , \quad b_k|s_\pm\rangle = 0. \tag{179}$$

In the usual fermion parity counting $b_0$ is an odd, i.e. fermionic operator, and thus the ground states $|s_\pm\rangle$ are half fermionic and half bosonic. But in two dimensions there is less distinction between fermions and bosons. In fact, since there are no rotations in two dimensions, the definition of fermion parity is more abstract. Especially the statistics of different vacua depends on the sign of the eigenvalue of fermion mass matrix at the considered vacuum [147]. In the case of the kink this means that if the vacuum located at $+v$ is defined to be bosonic the vacuum at $-v$ is automatically fermionic. Now a topological state like the kink connects these two vacua with opposite statistics which explains that this state cannot have a definite fermion parity, as it is usually defined, namely to be even for bosonic fields $\varphi$ and odd for fermionic fields $\psi$.

Let us now check if also semi-classically the BPS saturation condition is satisfied. With the regularized mode expansion for the quantum fields (129,131) and the BPS equation $\partial_x\varphi_K + U = 0$ one obtains

$$\begin{aligned} Q_+|s_\pm\rangle &= \int dx\, [\dot{\eta}\psi^+ + (\partial_x\eta + U'\eta)\psi^-]|s_\pm\rangle + O(\hbar) \\ &= i\int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}(\sqrt{\omega-\ell} - \sqrt{\omega+\ell})a_k^\dagger b_k^\dagger|s_\pm\rangle = 0. \end{aligned} \tag{180}$$

For higher order calculations also the motion of the perturbative ground state $|s_\pm\rangle$ has to be taken into account. So both states are BPS saturated semi-classically. The question is now whether the field operator is represented reducibly or irreducibly, i.e. by a single state $|s_+\rangle$ or $|s_-\rangle$. First we note that any state generated from $|s_+\rangle$ is orthogonal to any state generated from $|s_-\rangle$. Since $b_0$ is diagonal for this states a general state is of the form

$$|A_\pm\rangle = (A^e_\pm + A^o_\pm)|s_\pm\rangle, \tag{181}$$

where $A^e, A^o$ contains an even and odd number of non-zero-mode oscillators. The scalar product $\langle A_-|A_+\rangle$ vanishes since an odd number of oscillators has vanishing expectation value. The same argument shows that operators with an even number of fermionic fields have vanishing matrix elements between $|s_+\rangle$ and $|s_-\rangle$. Thus except for the usual fermion parity, there is no reason to implement a two dimensional reducible representation. But as we discussed already, in two dimensions the boson fermion distinction is somewhat arbitrary. The BPS saturation under quantum corrections, obtained in the previous subsection is of course strong evidence for multiplet shortening. The multiplet shortening can also clearly be seen by a soft breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ supersymmetry, where one of the soliton states disappears by the phenomenon of delocalization [118]. Also the strange effective multiplicity of $\sqrt{2}$ found by [50] can be explained by multiplet shortening.

As a last thing we show that by changing the interpretation of $b_0$ it is possible to define a definite fermion parity also in the topological sector built on a single-state multiplet. Since $b_0^2 = 1$, $b_0$ is an involution and it generates a $\mathbb{Z}_2$ grading on the Hilbert space. Each state built on $|s_+\rangle$ for example can be decomposed as

$$|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle \quad \text{with} \quad |\psi_\pm\rangle = P_\pm|\psi\rangle \quad , \quad P_\pm = \tfrac{1}{2}(1 \pm b_0). \tag{182}$$

The operator $b_0$ is then of the form

$$b_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{183}$$

Consider now physical observables $\mathcal{O}$ which contain an even number of fermionic fields, and thus are of the form

$$\mathcal{O} = \alpha + \beta b_0 B + \gamma B^2, \tag{184}$$

where $B$ and $B^2$ stand symbolically for an odd and even number of nonzero-mode oscillators and $\alpha, \beta, \gamma$ are ordinary numbers. From this follows immediately

$$[\mathcal{O}, b] = 0 \quad \Rightarrow \quad \mathcal{O} = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{D} \end{pmatrix}. \tag{185}$$

So the $\mathbb{Z}_2$ graduation is respected by physical observables. The form (183) of $b_0$ suggests the identification

$$b_0 = (-1)^F \tag{186}$$

in the soliton sector. The choice between the two inequivalent representations $|s_\pm\rangle$ is then analogous to the two different parities of the vacua $\pm v$, where here $|s_+\rangle$ would be bosonic.

# 5 Non-vanishing quantum corrections to the mass and central charge of the $N = 2$ vortex and BPS saturation

## 5.1 Introduction

So far we have developed on the basis of rather simple models a very elegant method of dimensional regularization for computations of quantum corrections in the presence of a nontrivial, topological background. By employing dimensional regularization through susy-preserving dimensional reduction from a higher-dimensional model. The correct quantum corrections to the mass of the susy kink are obtained [115] without having to deal with energy located at boundaries introduced in other methods, and the anomalous contribution to the central charge can be obtained from corrections to the momentum operator in the extra dimensions, which in the case of a kink background leave a finite remainder in the limit of 2 dimensions [116].

In this section we consider the Abrikosov-Nielsen-Oleson [1,42,105,131] vortex solution of the abelian Higgs model in 2+1 dimensions which has a supersymmetric extension [46,119] such that classically the Bogomol'nyi bound [15] is saturated. We employ our variant of dimensional regularization to the $\mathcal{N} = 2$ vortex by dimensionally reducing the $\mathcal{N} = 1$ abelian Higgs model in $3 + 1$ dimensions. This model includes a Fayet-Iliopoulos term which mediates the spontaneous breakdown of gauge-symmetry and gives rise to topological distinct vacua. Because of the absence of a superpotential stays supersymmetry, despite of the Fayet-Iliopoulos term, unbroken. To obtain the same field content for the three-dimensional $\mathcal{N} = 2$ descendant model as for the original $\mathcal{N} = 2$ three-dimensional vortex model [46,119], one has to start with one chiral multiplet only for the matter fields in four dimension. This model is inconsistent in strict four dimensions because of a chiral anomaly. To achieve anomaly cancellation one needs at least a second chiral multiplet with opposite charged.

We confirm the results of [97,119,135] that in a particular gauge (background-covariant Feynman-'t Hooft) the sums over zero-point energies of fluctuations in the vortex background cancel completely, but contrary to [97,119] we find a non-vanishing quantum correction to the vortex mass coming from a finite renormalization of the expectation value of the Higgs field in this gauge. The renormalization of tadpoles in the vacuum sector is necessary for a consistent perturbative expansion, since otherwise the point of expansion, the might be vacuum, starts move when corrections are included. In the topological sector the renormalization of vacuum-tadpoles guarantees that the vortex field-profile asymptotically takes the real vacuum value. As we will see, the renormalization of the expectation value of the Higgs field depends the gauge parameter and thus the remaining contributions to mass and central charge must be gauge dependent too, so the final result is gauge independent. This in addition demonstrates the need for renormalizing tadpoles although they are finite in dimensional regularization. This also shows that the so far obtained null results for the zero-point energy sums [97,119] are gauge dependent statements.

In contrast to [119], where a null result for the quantum corrections to the central charge was stated, we show that the central charge receives also a net non-vanishing quantum correction, namely from a nontrivial phase in the fluctuations of the Higgs field in the vortex background, which contributes to the central charge even though the latter is a surface term that can be evaluated far away from the vortex. This reflects the long range force of the vortex-gauge field. The correction to the central charge exactly matches the correction to the mass of the vortex.

In Ref. [97], it was claimed that the usual multiplet shortening arguments in favor of BPS saturation would not be applicable to the $N = 2$ vortex since in the vortex background there would be two rather than one fermionic zero modes [96], leading to two short multiplets which have the same number of states as one long multiplet.[25] We show however that the extra zero mode postulated in [97] has to be discarded because its gaugino component is singular, and that only after doing so there is agreement with the results from index theorems [80,96,139]. For this reason, standard multiplet shortening arguments do apply, explaining the BPS saturation at the quantum level that we observe in our explicit one-loop calculations.

## 5.2 Embedding the vortex

The three dimensional $\mathcal{N} = 2$ abelian vortex can be embedded as a string in a four dimensional $\mathcal{N} = 1$ abelian gauge theory, consisting of a vector multiplet $V \sim (A, \lambda, D)$ and chiral scalar multiplets $\Phi_k \sim (\phi_k, \psi_k, F_k)$, with a vanishing superpotential but including a Fayet-Iliopoulos (FI)-term [49]:

$$\mathcal{L} = \int \left( \frac{-i\tau}{16\pi} d\theta^2 W^\alpha W_\alpha + h.c. \right) + \sum_i \int d\theta^4 \bar{\Phi}_i e^{q_i V} \Phi_i + 2\kappa \int d\theta^4 V. \quad (187)$$

The FI-term, which is gauge invariant for abelian (sub)groups only, induces spontaneous symmetry breaking of the gauge symmetry and masses for the fields. Note, that in the absence of a superpotential supersymmetry stays unbroken. The Lagrangian (187) is invariant under complexified $U_{\mathbb{C}}(1)$ gauge transformations

$$V \to V + i(\Lambda - \bar{\Lambda}) \quad , \quad \Phi_k \to e^{-iq_k \Lambda} \Phi_k \ , \quad (188)$$

[25]Incidentally, Refs. [96,97] considered supersymmetric Maxwell-Chern-Simons theory, which contains the supersymmetric abelian Higgs model as a special case.

where $\Lambda$ is a chiral superfield and $q_k$ is the charge of the field $\Phi_k$. The theta angle in the gauge coupling

$$\tau = i\frac{4\pi}{g^2} + \frac{\vartheta}{2\pi} \tag{189}$$

parameterizes the topological gauge field term. This term is unimportant for our purposes and we will set $\vartheta = 0$ henceforth. Because of the Bianchi identity $D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$ only the real part of the FI-coupling $\kappa$ enters in the Lagrangian which we thus choose to be real.

In (187) chiral currents are coupled to the gauge field, which are, as is well known, anomalous [2, 12]. For a consistent coupling of the currents to the gauge field these anomalies have to cancel, which is achieved if the trace of the $U(1)$ charges vanishes [9], i.e. if

$$\sum_k q_k = 0. \tag{190}$$

Eliminating the auxiliary fields $F_k, D$ through their *algebraic* equations of motions one obtains the (positive) potential and thus the (classical) vacua as follows:

$$F_k = 0 = \bar{F}_k \quad , \quad D = -g^2(\kappa + \tfrac{1}{2}\Sigma q_k|\phi_k|^2) \tag{191}$$

$$V = \tfrac{1}{2}U^2 = \frac{1}{2g^2}D^2 = \frac{g^2}{2}(\kappa + \tfrac{1}{2}\Sigma q_k|\phi_k|^2)^2. \tag{192}$$

In the case of only one chiral multiplet ($q = -1$) the vacuum manifold is $S^1$ and consists of gauge-equivalent configurations

$$|v_1|^2 = 2\kappa \quad \to v_1 = 2\kappa e^{i\alpha}. \tag{193}$$

In the case of two opposite charged chiral multiplets ($q_{1,2} = \mp 1$) the vacuum manifold is more complicated:

$$|v_1|^2 - |v_2|^2 = 2\kappa \quad \to v_1 = \sqrt{r^2 + 2\kappa}\; e^{i\alpha_1} \; , \; v_2 = re^{i\alpha_2}. \tag{194}$$

The vacuum manifold parametrized by $(\alpha_1, \alpha_2, r)$ is $S^1 \times S^1 \times \mathbb{R}_+$. To divide out gauge equivalent configurations we write (194) as

$$v_1 = \tfrac{\sqrt{r^2+2\kappa}}{r}\; e^{i\beta} v_2 \quad , \quad \beta = (\alpha_1 - \alpha_2). \tag{195}$$

By a gauge transformation with $\frac{\beta}{2}$ or $\frac{\beta}{2} + \pi$ the phase in (195) is gone. The remaining configurations solving the vacuum constraint (194) given by

$$\mathcal{M} = v_2/\{v_2 \to -v_2\} = \{r \geq 0, \alpha \in [0, \pi]\}, \tag{196}$$

are physical inequivalent and called moduli space of the model. The modulus $r$, however, in lower dimensions ($D = 2, 3$) is dynamical selected to be zero by vortex configurations [108, 128]. The existence of moduli spaces, i.e. inequivalent vacua called flat directions, are typical for supersymmetric theories, where the accidental degeneracy of vacua is protected against quantum corrections.

The two cases considered above are the most interesting one for us. The case of only one charged multiplet gives exactly the field content of the $\mathcal{N}=2$ vortex in three dimensions [119]. But this model is anomalous in four dimensions, since the trace of $U(1)$ charges (190) does not vanish. Adding a second multiplet with opposite charge is the minimal extension of the $3D$-abelian vortex model to a non-anomalous theory in four dimensions. This case will be discussed in more detail elsewhere [114].

### 5.2.1 Dimensional reduction

We now dimensional reduce the Lagrangian (187) with one charged multiplet to three dimensions to obtain the minimal three dimensional $\mathcal{N}=2$ vortex Lagrangian. For this we write (187) in terms of the component fields in the *Wess-Zumino (WZ) gauge* [143] (see appendix C)[26]:

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}F_{mn}^2 - i\lambda\sigma^m\partial_m\bar{\lambda} + \frac{1}{2}D^2 + ev^2 D \\ & -|D_m\phi|^2 - i\bar{\psi}\bar{\sigma}^m D_m\psi + |F|^2 + i\sqrt{2}e(\bar{\lambda}\bar{\psi}\phi - \lambda\psi\bar{\phi}) - e|\phi|^2 D, \end{aligned} \tag{197}$$

where $D_m = \partial_m - ieA_m$. The first line in (197) is the gauge field Lagrangian including the FI-term, where the second line is the gauge-matter field Lagrangian in (187) in the WZ-gauge. The bosonic three dimensional Lagrangian is easily obtained. The fermionic one is treated in more detail in appendix D. We reduce trivially[27] the $z$-direction and introduce the real scalar field $N := A_3$:

$$\begin{aligned}\mathcal{L}_B &= -\tfrac{1}{4}F_{\mu\nu}^2 - \tfrac{1}{2}(\partial N)^2 - |D_\mu\phi|^2 - \tfrac{1}{2}e^2(|\phi|^2 - v^2)^2 - e^2|\phi|^2 N, & (198)\\ \mathcal{L}_F &= -\bar{\lambda}\partial\!\!\!/\lambda - \bar{\psi}D\!\!\!\!/\psi + eN\bar{\psi}\psi - i\sqrt{2}e(\bar{\psi}\lambda\phi - \bar{\lambda}\psi\bar{\phi}), & (199)\end{aligned}$$

where we have set the auxiliary fields on-shell (191). The Lagrangian (198) is of the same form as in [96]. Since we now know how the three dimensional fields are related to their four-dimensional origin we will keep the four dimensional language henceforth.

## 5.3 Supercurrent superfield and anomaly multiplet

Before we finally switch to the component Lagrangian and the WZ-gauge (see below) we discuss some (classical) properties of the model (187) in

[26]We rescale the vector multiplet $(A,\lambda,D) \to \frac{1}{g}(A,\lambda,D)$ and define $e = -\frac{1}{2}qg$ as well $v^2 = -\frac{2\kappa}{q}$.

[27]With trivially we mean that all fields are independent of $x_3 = z$ so that $\partial_3 \equiv 0$. Later we will keep derivatives w.r.t. the extra dimension for the purpose of regularization.

terms of superfields. The Lagrangian (187) has several (global) symmetries, $R$-symmetry, supersymmetry and translational symmetry, and associated currents which form a super multiplet, the supercurrent superfield [52]. The starting point is the so-called $R$-symmetry.

### 5.3.1 $R$-symmetry

The $R$-symmetry can be represented as $U(1)$ rotations of the fermionic superspace coordinates

$$\theta \to e^{i\alpha}\theta \quad , \quad \bar{\theta} \to e^{-i\alpha}\bar{\theta}, \tag{200}$$

which thus have per construction charge $r_{\theta,\bar{\theta}} = \pm 1$, and also a superfield has a definite $R$-charge $r_F$ under such transformations

$$F'(\theta', \bar{\theta}') = e^{ir_F\alpha} F(\theta, \bar{\theta}). \tag{201}$$

The $R$-charge of the different fields in (187) is determined by the requirement of invariance under the above defined transformations:

$$\begin{aligned} \delta\mathcal{L}_W &= e^{2i\alpha(r_W - 1)}\mathcal{L}_W \quad , \quad \delta\mathcal{L}_{\bar{W}} = e^{2i\alpha(r_{\bar{W}}+1)}\mathcal{L}_{\bar{W}} \\ \delta\mathcal{L}_M &= e^{i\alpha(r_{\bar{\Phi}} + r_\Phi)}\mathcal{L}_M \quad , \quad \delta\mathcal{L}_{FI} = e^{i\alpha r_V}\mathcal{L}_{FI}, \end{aligned} \tag{202}$$

where $\mathcal{L}_W$ (and its complex conjugated), $\mathcal{L}_M$ and $\mathcal{L}_{FI}$ are the terms in (187). From the transformation of the gauge and matter-gauge coupling in (202) one obtains the weights

$$r_W = 1 \ , \ r_{\bar{W}} = -1 \to r_V = 0 \quad \text{and} \quad r_\Phi = -r_{\bar{\Phi}} = r. \tag{203}$$

With (203) the FI-Lagrangian does not give any further constraint and the $R$-weight $r$ of the chiral superfield is arbitrary so far. Associated to the $R$-symmetry, as a continuous symmetry, is a conserved conserved current, the $R$-current. From the component expansion (appendix C) one read off the $R$-charges of the component fields

$$r_\lambda = 1 \ , \ r_\phi = r \ , \ r_\psi = r - 1 \ , \ r_F = r - 2, \tag{204}$$

where we have used that the Grassmann coordinates have unit charge, $r_\theta = -r_{\bar{\theta}} = 1$. The complex conjugated fields are opposite charged and all other fields are invariant under the $R$-symmetry. The Lagrangian transforms modulo e.o.m. into the divergence of the Noether current, $\delta\mathcal{L} = \partial^m R_m + e.o.m.$, where in the WZ-gauge

$$R_m = -\tfrac{1}{g^2}\lambda\sigma_m\bar{\lambda} + ir(\bar{\phi}D_m\phi - \phi\bar{D}_m\bar{\phi}) - (r-1)\psi\sigma_m\bar{\psi}, \tag{205}$$

and which is conserved on-shell since $\delta\mathcal{L} = 0$. Because the FI-term is invariant and also linear ($\sim D$) it does not contribute to the current. This current is per definition the lowest component of the supercurrent superfield and this will be our starting point for the construction of this superfield. The principles which we impose for the "engineering" process are: (i) supersymmetric covariance, since it should be a super field. (ii) gauge invariance, so that it is meaningful to start with the lowest component in a certain gauge. First we write the $R$-current in spinorial coordinates[28]:

$$\begin{aligned} R_{\alpha\dot{\alpha}} &= \sigma^m_{\alpha\dot{\alpha}} R_m \\ &= \tfrac{2}{g^2}\lambda_\alpha \bar{\lambda}_{\dot{\alpha}} + 2(r-1)\psi_\alpha \bar{\psi}_{\dot{\alpha}} + ir(\bar{\phi} D_{\alpha\dot{\alpha}}\phi - \phi D_{\alpha\dot{\alpha}}\bar{\phi}). \end{aligned} \tag{206}$$

The superfield expression for the first term is easily guessed to be $\frac{2}{g^2} W_\alpha \bar{W}_{\dot{\alpha}}$ since[29] $\lambda_\alpha = W_\alpha|$ and $W_\alpha$ is super covariant and gauge invariant. For the other terms we introduce the super- and gauge-covariant derivative

$$\nabla_\alpha \Phi = (D_\alpha + qD_\alpha V)\Phi \quad , \quad \bar{\nabla}_{\dot{\alpha}}\bar{\Phi} = (\bar{D}_{\dot{\alpha}} + q\bar{D}_{\dot{\alpha}} V)\bar{\Phi}, \tag{207}$$

which transforms covariant under gauge transformations and in the abelian case the projection on the lowest component gives the same as for the super derivative [30], i.e.

$$\nabla_\alpha \Phi \to e^{-i\Lambda}\nabla_\alpha \Phi \quad , \quad \nabla_\alpha \Phi| = D_\alpha \Phi|. \tag{208}$$

With this we see that $\sqrt{2}\psi_\alpha = \nabla_\alpha \Phi|$. So the gauge *in*variant expression for the second term in (206) is $(r-1)e^{qV}\nabla_\alpha \Phi \nabla_{\dot{\alpha}}\bar{\Phi}$. More educated but analogous "guessing" gives for the third term in (206) $\frac{r}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}](\Phi e^{qV}\bar{\Phi}) - re^{qV}\nabla_\alpha \Phi \nabla_{\dot{\alpha}}\bar{\Phi}$. So the supercurrent superfield reads as

$$V_{\alpha\dot{\alpha}} = \frac{2}{g^2} W_\alpha \bar{W}_{\dot{\alpha}} - e^{qV}\nabla_\alpha \Phi \nabla_{\dot{\alpha}}\bar{\Phi} + \frac{r}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}](\Phi e^{qV}\bar{\Phi}). \tag{209}$$

Note that the lowest component uniquely defines a superfield and so does the $R$-current (206) for the supercurrent superfield. An alternative method to construct the supercurrent-SF is to vary the action with respect to external supergravity [125]. The supercurrent-SF (209) has exactly the same form as in the case of massless, i.e. super-conformal QED[31] if one chooses the $R$-charge of the chiral super field to be $r = \frac{2}{3}$ [94]. As we will see below it is exactly the conformal structure which will fix $r = \frac{2}{3}$ [109]. As for the $R$-current the FI-term does not contribute to the expression for the supercurrent SF but it enters through the equations of motions.

[28]Note that $D_{\alpha\dot{\alpha}}\bar{\phi} = \sigma^m_{\alpha\dot{\alpha}} \bar{D}_m \bar{\phi}$.

[29]The bar | means projection on the lowest component.

[30]In the non-abelian case this is only true under the trace

[31]We have treated here only one charge $q$. For different charges one just have to sum over the matter contribution to the supercurrent-SF (209)

### 5.3.2 Trace equation

The supercurrent-SF (209) is conserved on-shell

$$\partial_m V^m = \partial_m(-\tfrac{1}{2}\bar{\sigma}^{m\alpha\dot{\alpha}} V_{\alpha\dot{\alpha}}) = 0, \tag{210}$$

which is clear from the fact that the lowest component, the $R$-current is conserved:

$$\partial_m V^m| = \partial_m R^m = 0 \ \Rightarrow \ \partial_m V^m = 0, \tag{211}$$

since $\partial_m V^m$ is also a superfield. Thus the supercurrent-SF contains a multiplet of conserved currents, which emerge as coefficients in a theta expansion

$$V_m = R_m - i(\theta J_m - \bar{\theta}\bar{J}_m) - 2\theta\sigma^n\theta(\Theta_{nm} + \zeta_{mn}) + \dots\,, \tag{212}$$

where we have written only the most relevant terms. The components in (212) are the $R$-current, supercurrents and the energy-momentum tensor, which we have split into its symmetric and antisymmetric part. These currents are improved currents (the symmetric part of the energy momentum tensor only) such that their moments

$$S_m = \not{x} J_m \ , \ D_m = x^m \Theta_{mn} \tag{213}$$

are the currents of the super-conformal symmetry [52]. Our model is conformal invariant only in the limit $\kappa = 0$, so that the super-conformal currents (213) will not be conserved but will contain explicit breaking terms proportional to $\kappa$. The conformal structure is summarized in the so-called "trace equation" of the supercurrent-SF [109]. This equation gives the (non)conservation laws for the conformal currents. First we need the **superspace equations of motion**. Following [51] we obtain[32]:

$$\nabla^2\Phi = 0 = \bar{\nabla}^2\bar{\Phi}, \tag{214}$$

$$\tfrac{1}{g^2} D^\alpha W_\alpha = q\bar{\Phi}e^{qV}\Phi + 2\kappa. \tag{215}$$

From (215) one can see how the FI-term enters the supercurrent-SF. Using the equations of motions and the identity
$\bar{D}^{\dot{\alpha}} D_\alpha \bar{D}_{\dot{\alpha}} = \frac{1}{2}(\bar{D}^2 D_\alpha + D_\alpha \bar{D}^2)$ one obtains for the trace equation:

$$\begin{aligned}\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = {} & 2q(1 - \tfrac{3}{2}r) W_\alpha \bar{\Phi} e^{qV}\Phi + (\tfrac{3}{2}r - 1) e^{qV}(\bar{D}^{\dot{\alpha}}\nabla_\alpha \Phi)\bar{\nabla}_{\dot{\alpha}}\bar{\Phi} \\ & + 4\kappa W_\alpha. \end{aligned} \tag{216}$$

Thus the correct $R$-weight for the chiral superfield, which is consistent with the conformal structure, is $r = \frac{2}{3}$. With this choice only the explicit breaking by the FI-term survives in the trace equation. Because of the Bianchi

[32] In our conventions $\frac{1}{4}D^2 = d\theta^2$ up to surface terms

identity for the field strength $W_\alpha$ this corresponds to the "$B_\alpha \neq 0, S = 0$"-case [13,109]:

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = 4\kappa W_\alpha \quad , \quad D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}. \tag{217}$$

In even dimensions chiral and conformal currents are in general anomalous. Thus in a supersymmetric theory the anomalies sit in a superfield (217) and form a multiplet, the so called anomaly multiplet. Concretely the following anomalies are related by the trace equation (217):

$$\partial_m R^m \quad , \quad \partial_m D^m = \Theta^m_m \quad , \quad \partial_m S^m = \Gamma^m J_m. \tag{218}$$

It is now interesting what the trace equation (217) gives if one dimensionally reduces the theory. For this we switch to four-spinor notation [140] so that the supercurrent superfield and the conformal trace equation reads as

$$V_m = R_m + i\bar{\theta}\Gamma_5 J_m - \frac{i}{2}\bar{\theta}\Gamma_5\Gamma^n\theta(\Theta_{mn} + \zeta_{mn}) + \dots ; \tag{219}$$

$$D_\beta V_{\alpha\beta} = -(\bar{\partial}_\beta + \Gamma^m_{\beta\rho}\,\theta\rho\,\partial_m)V_{\alpha\beta} = 0. \tag{220}$$

We start in four (or any higher even dimension $D$) and reduce step by step to two dimensions less, so that the indices take the values $m = (0, \dots, i = D-2, \mu = D-1, D)$. For the first and second term in (219) nothing important happens, so that the trace equation (220) gives essentially

$$i\Gamma_5\Gamma^m J_m + \theta\partial^m R_m \quad , \quad i\Gamma_5\Gamma^\mu J_\mu + \theta\partial^\mu R_\mu \quad , \quad i\Gamma_5\Gamma^i J_i + \theta\partial^i R_i \dots , \tag{221}$$

in the different dimensions. The interesting thing happens in the third term of (219), where the essential terms are ($T_{mn} = \Theta_{mn} + \zeta_{mn}$)

$$D: \quad -i\Gamma_5\Gamma^m\Gamma^n T_{mn}\theta = -i\Gamma_5\Theta^m_{\ m}\theta + \dots , \tag{222}$$

$$\begin{aligned} D-1: \quad & -i\Gamma_5(\Gamma^\mu\Gamma^\nu T_{\mu\nu} + \Gamma^D\Gamma^\mu T_{\mu D} + \Gamma^\nu\Gamma^D T_{D\nu})\theta \\ & = -i\Gamma_5(\Theta^\mu_{\ \mu} + 2\Gamma^D\Gamma^\mu\zeta_{\mu D})\theta \dots , \end{aligned} \tag{223}$$

$$\begin{aligned} D-2: \quad & -i\Gamma_5(\Gamma^i\Gamma^j T_{ij} + \Gamma^{D-1}\Gamma^i T_{i\ D-1} + \Gamma^j\Gamma^{D-1} T_{D-1\ j})\theta \\ & = -i\Gamma_5(\Theta^i\ i + 2\Gamma^{D-1}\Gamma^i\zeta_{iD-1})\theta \dots . \end{aligned} \tag{224}$$

Thus through the dimensional reduction the conformal projection (contraction with a gamma matrix) of the *antisymmetric* part of the energy momentum tensor enters the anomaly multiplet at the same level as the trace anomaly of the energy momentum tensor. So in reduced dimensions the multiplet of moments (213) and anomalies (218) become ($a = \mu$ or $i$)

$$S_a = \not{x} J_a \ , \ D_a = x^b\Theta_{ba} \ , \ Z_a = \not{x}\zeta_a. \tag{225}$$

$$\partial_a S^a = \gamma^a J_a \ , \ \partial_a D^a = \Theta^a_{\ a} \ , \ \partial_a Z^a = \Gamma^a\zeta_a, \tag{226}$$

where $\zeta_a = \zeta_{Da}$ is the central charge current and stems from the antisymmetric part of the energy momentum only. The asymmetric EM-tensor can differ from the symmetric (improved) one only by an improvement term. Thus the antisymmetric part is a surface term and does not act on localized states. But in the dimensional reduced theory extended finite energy states may exist, so that the surface term can no longer be neglected. In the higher dimension the antisymmetric part can be important for domain-walls or -strings where they become central charge densities of this objects, which are extended but not finite energy states [58, 115]. The momentum of the central charge current, $Z_a$, gives for the two-dimensional $\mathcal{N}=1$ kink the anomalous correction to the central charge [116]. Although the dimensional reduced theory is $\mathcal{N}=2$ symmetric this analysis has merit also for the $\mathcal{N}=1$ kink, since softly breaking the $\mathcal{N}=2$ symmetry and sending the masses of the "regulator" multiplet to infinity gives the anomaly in the central charge of the $\mathcal{N}=1$ kink [123]. The above analysis is also valid if one starts in six dimensions and reduces to four dimensions. Starting with a $\mathcal{N}=1$ Yang-Mills theory in $D=6$ one obtains the $\mathcal{N}=2$ monopole model [85]. Concluding this analysis, the possibility of a central charge anomaly is directly connected to the existence of super-conformal anomalies, which depends on the supersymmetry content as well as on the dimension. Since supersymmetry connects the conformal anomalies with the chiral $R$-current anomaly one cannot expect a conformal anomaly in odd dimensions, but there are nevertheless anomalous contributions to the central charge possible, i.e. finite reminder from the momentum in the extra dimension, as we have seen in the case of the susy-kink domain wall (2). A last subtlety remains. The above analysis treats the central charge *current*. It is of course possible that anomalies of local currents do not contribute to integrated quantities. In $D=4$, $\mathcal{N}=1$ super QED the Konishi anomaly [31,92] of the supercurrent-SF induces a central charge density [30]

$$\{Q_\alpha, J^0_\beta\} = \mathcal{T}_{\alpha\beta}, \tag{227}$$

where $\mathcal{T}_{\alpha\beta}$ is proportional to the Konishi anomaly. Because of the absence of topological distinct vacua this integrates to zero. Also in the dimensional reduced ($D=3$) theory this gives no contributions, since there the anomaly vanishes.

## 5.4 The BPS-vortex solutions

First we consider BPS solution from its purely bosonic origin [15]. For this we set the fermionic fields as well as the bosonic field $N$, coming from the extra dimension, in (197) to zero. This gives the abelian Higgs- or Landau-Ginzburg model for a superconducter [59], respectively:

$$\mathcal{L}_{LG} = -\tfrac{1}{4}F^2_{\mu\nu} - |D_\mu\phi|^2 - \tfrac{1}{2}\, U^2(|\phi|), \tag{228}$$

where we have introduced the pre-potential

$$U(|\phi|) = \lambda\, (|\phi|^2 - v^2). \tag{229}$$

Here occurs an important point for the structure of the bosonic as well as supersymmetric theory. To emphasize this fact we have introduced a new coupling $\lambda$ in (229). It is only for the case that the Higgs coupling in (229) equals the gauge coupling, i.e.

$$\lambda \stackrel{!}{=} e, \tag{230}$$

that the bosonic theory provides BPS solutions [15]. Otherwise the Bogomol'nyi trick of completing squares (see below) does not work. In the Landau-Ginzburg theory this is the transition point between type I and type II superconductors. The $\mathcal{N} = 1$ supersymmetric extension of (228) at this coincidence of couplings becomes $\mathcal{N} = 2$ supersymmetric, as shown classically in [46].[33] Thus there is a deep connection between the existence of a topological charge, i.e. a BPS solution for the bosonic model and the supersymmetry content of the supersymmetric extended model [77].

The vacuum manifold is give by $\phi_0 = ve^{i\alpha}$ and is thus topological an $S^1$. We are looking for static finite energy solutions or strings of finite tension in the four dimensional space, i.e. all fields are independent of the extra dimension. Thus we start with the ansatz

$$\dot{\phi} = 0 = \dot{A}_{i=1,2} \quad , \quad A_0 \equiv 0, \tag{231}$$

where we have also set the unphysical mode $A_0$, which is just a Lagrange multiplier for the Gauß-constraint, to zero. For finite energy each term in (228) must separately vanish asymptotically, thus

$$\phi \to ve^{i\alpha(x)}, \tag{232}$$

$$(\partial_i - ieA_i)\phi \to 0 \Rightarrow A_i \to \frac{1}{ie}\phi^{-1}\partial_i\phi, \tag{233}$$

fast enough to be integrable in two dimensions. (233) implies that the gauge field is asymptotically pure gauge and thus also the field strength vanishes:

$$A_i \to \frac{1}{ie}\partial_i \log\phi \Rightarrow F_{ij} \to 0. \tag{234}$$

### 5.4.1 Homotopy classification

The two dimensional spatial boundary at infinity is topological an $S^1$ as well as the vacuum manifold. Thus the scalar field at spatial infinity is the map

$$\phi|_{\partial\mathcal{M}} : S^1 \to S^1 = \pi_1(S^1) = \mathbb{Z}, \tag{235}$$

[33] To be more specific, the two couplings really have to be identical, i.e. also renormalize in the same way.

where $\pi_1$ stands for the first homotopy group. Thus there are infinitely many different and disconnected topological sectors in the space of solutions. The continuity in the angular variable $\theta$ of the asymptotic solutions (232) imply

$$\alpha(\theta + 2\pi) = \alpha + 2\pi n_T \Rightarrow \alpha = n_T\theta, \tag{236}$$

where we have assumed that the radial dependence has been separated already. With (233) we can now classify all finite energy solutions by their asymptotic behavior and topological index, the *winding number* $n_T$ as follows [34]

$$\phi \to v\ e^{in_T\theta} \quad , \quad A_\theta \to \frac{n_T}{e} \quad , \quad n_T \in \mathbb{Z}. \tag{237}$$

**Flux-quantization:** To the above discussed gauge fields $A = A_i dx^i$ one can associate a magnetic field $B = dA$ [35] with a magnetic flux in the direction of the extra dimension through the $xy$-plane

$$\Phi = \int_{\mathcal{M}} B = \int_{\partial\mathcal{M}} A_\theta d\theta = \frac{2\pi n_T}{e}. \tag{238}$$

This quantization does not include the fundamental constant of quantum theory, $\hbar$. It enters through the quantization of the electrical charge of the Cooper pairs [112]

$$Q_{el} = e\langle \int (\dot{\phi}\bar{\phi} - \phi\dot{\bar{\phi}})\rangle = \frac{e\hbar}{2} \Rightarrow \Phi = \frac{2\pi n_T \hbar}{2Q_{el}}. \tag{239}$$

### 5.4.2 Bogomol'nyi bound, central charge and BPS equations

With the Bogomol'nyi trick (completing squares) the energy (density) of the above discussed classical solutions can be written as

$$\begin{aligned} E &= \int dx^2 \left( |D_i\phi|^2 + \tfrac{1}{2}F_{12}^2 + \tfrac{1}{2}U^2 \right) \\ &= \int dx^2 \left\{ \tfrac{1}{2}|(D_i \pm i\varepsilon_{ij}D_j)\phi|^2 + \tfrac{1}{2}(F_{12} \pm U)^2 \right\} \pm Z. \end{aligned} \tag{240}$$

Since the first two terms in (240) are positive definite one obtains a lower bound for the energy in terms of the *central charge* $Z$:

$$E \geq |Z| \quad , \quad Z = \int dx^2 \partial_i \varepsilon_{ij}[ev^2 A_j - i\bar{\phi}D_j\phi]. \tag{241}$$

The bound is saturated if the *BPS equations*

$$(D_i \pm i\varepsilon_{ij}D_j)\phi = 0, \tag{242}$$

$$F_{12} \pm U = 0 \tag{243}$$

[34] In a normalized frame $e^\theta = \frac{1}{r}d\theta$ with $g(e^\theta, e^\theta) = 1$ the gauge field coordinate would read $\tilde{A}_\theta = \frac{1}{r}\frac{n_T}{e}$

[35] Note that $B_3 = (rotA)_3 = \varepsilon_{3jk}\partial_j A_k = dA|_{\mathbb{R}^2}$, where $B_1 = B_3 = 0$.

are fulfilled. in contrast to the equations of motions these equations are of first order and should thus be easier to solve than the e.o.m. Unfortunately no solution in terms of elementary functions is known [42, 131]. The important thing is now that the BPS equations (242) imply the equations of motions. Acting with $D_i$ on (242) and using (243) gives:

$$D_i{}^2\phi \pm eF_{12}\phi = D_i{}^2 - e\phi U = 0, \tag{244}$$

which is exactly the e.o.m. for the scalar field with the ansatz (231). The e.o.m. for the gauge field is obtained by acting with $\varepsilon_{ij}\partial_j$ on (243) and with the help of (242):

$$\varepsilon_{ij}\partial_j F_{12} \pm \varepsilon_{ij}\partial_j U = \varepsilon_{ij}\partial_j F_{12} + ie(\bar{\phi}D_i\phi - \phi D_i\bar{\phi}), \tag{245}$$

which is by noting $(\partial_\mu F^{\mu\nu})_i = -\varepsilon_{ij}\partial_j F_{12}$, the e.o.m. for the gauge field.

### 5.4.3 Vortices

The solutions of the BPS-equations (242,243) are called *vortices* and since they are finite energy solutions they fall in the classification (237). In $2+1$ dimensions they are really finite energy solutions, in $3+1$ dimensions they are strings of finite string tension and in $1+1$ dimensions they occur as instantons. The upper/lower sign in (242,243) refer to vortices and anti-vortices, respectively. We will always refer to vortices except stated otherwise. The BPS equation (242) can be written as

$$(D_1 + iD_2)\phi = D_+\phi = 0\ . \tag{246}$$

For an angular symmetric vortex with winding number $n_\tau$ we make the following ansatz:

$$\varphi = f(r)\ e^{in_\tau\theta}\quad ,\quad \mathcal{A}_\theta = \frac{n_\tau - a(r)}{e}. \tag{247}$$

Further we express $F = dA = -\partial_r a(r) dr d\theta$ through $F_{12}$ by (see appendix E)

$$F_{12} = \partial_1 x_{i'}\partial_2 x_{j'} F_{i'j'} = \frac{1}{r}F_{r\theta} = -\frac{\partial_r a(r)}{r}. \tag{248}$$

The equations (242,243) thus become

$$r\frac{d}{dr}\ln f(r) {=} a(r) \tag{249}$$

$$\frac{1}{r}\frac{d}{dr}a(r) {=} e^2\ (f^2(r) - v^2), \tag{250}$$

and the boundary conditions are [42, 131]:

$$f(\infty) = v\quad ,\quad f(r\to 0) \sim r^n + O(r^{n+2}) \tag{251}$$

$$a(\infty) = 0\quad ,\quad a(r\to 0) \sim n + O(r^2), \tag{252}$$

where the asymptotic values are reached exponentially fast. The central charge and thus the energy of this solutions are

$$E = Z = \int_{\mathcal{M}} d\xi = \int_{\partial\mathcal{M}} \xi = \int_0^{2\pi} d\theta \xi_\theta \tag{253}$$

$$= \int_0^{2\pi} d\theta \left(ev^2 \mathcal{A}_\theta - i\bar{\varphi} D_\theta \varphi\right) = 2\pi n_\tau v^2. \tag{254}$$

The above (247) considered configuration is a single $n_\tau$-vortex, i.e. a vortex with winding number $n_\tau$. We would have obtained the same result for the energy and central charge of a configuration of $n_\tau$ one-vortices distributed in the plane. Configurations of $k$ $n_i$-vortices are similar to one $\sum_1^k n_i$-vortex in many respects, especially topologically and according the number of zero modes [137]. A $n$-vortex can be seen as a limiting process of approaching centers of $n$ one-vortices. From the $3+1$ dimensional point of view $n$ one-vortices are $n$ flux tubes located at the centers of the vortices.

## 5.5 Supersymmetry and WZ-gauge

The superspace Lagrangian (187) is manifestly supersymmetric. The transformation for the different superfields $F = (V, \Phi_i)$ are certain translations in superspace:

$$\begin{aligned} \delta_\epsilon F &= (\epsilon Q + \bar{\epsilon}\bar{Q})F \\ Q_\alpha &= \partial_\alpha - i(\sigma^m \bar{\theta})_\alpha \partial_m \\ \bar{Q}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^m)_{\dot{\alpha}} \partial_m. \end{aligned} \tag{255}$$

By a component expansion of the lh.- and rh.-side of (255) one can read off the transformation rules for the component fields. The Lagrangian (187) is also invariant under the super-gauge transformations (188). Since the vector superfield $V$ transforms inhomogeneously one can eliminate component fields of $V$ by a field dependent super-gauge transformation. The maximal elimination leads to the so called *WZ-gauge*, where only the gauge field, the gaugino and the real auxiliary field are left, $V_{WZ} \sim (A_m, \lambda, D)$. As always in gauge theories, any general configuration for the vector superfield $V$ can be obtained from the gauge fixed $V_{WZ}$ by a field dependent super-gauge transformation, i.e.

$$V = V_{WZ} + i(\Lambda - \bar{\Lambda}) \quad , \quad \Lambda = \Lambda(B, \chi, C), \tag{256}$$

where $B, \chi, C$ are the component fields of $V$ eliminated in the $WZ$-gauge. The elimination of the $B, \chi$- and $C$-component of $V$ fixes the super-gauge transformation $\Lambda$ completely in terms of $B, \chi$- and $C$ except the real part of the scalar field component $L$ of $\Lambda$. This residual gauge freedom in the $WZ$-gauge is the ordinary gauge symmetry

$$\delta_g A_m = \partial_m \mathrm{Re} L. \tag{257}$$

The problem of the $WZ$-gauge is that it is not invariant under the supersymmetry transformations (255), i.e. it is a non-super-covariant gauge. The susy variations of the $C$ and $\chi$ components are proportional to $\lambda$ and $A_m$, respectively, and thus in the nontrivial case a susy transformation leads out of the $WZ$-gauge. The $WZ$-gauge can be restored, if the susy transformations are followed by a suitable super-gauge transformation. Adopting the $WZ$-gauge the Lagrangian is no longer invariant under general super-gauge transformations nor under susy transformations (255) separately, but under a combination of susy and super-gauge transformations, which takes us back to the $WZ$-gauge. Of course the residual, i.e. ordinary gauge transformations, are still a symmetry. The Lagrangian in $WZ$-gauge (197) is invariant under the following *combined* transformations:

$$\begin{aligned}\delta A_m = (\epsilon\sigma_m\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_m\lambda) \quad &, \quad \delta D = i(\epsilon\sigma^m\partial_m\lambda + \bar{\epsilon}\bar{\sigma}^m\partial_m\lambda) \\ \delta\lambda_\alpha = -iF_{mn}(\sigma^{mn}\epsilon)_\alpha + D\epsilon_\alpha \quad &, \quad \delta\bar{\lambda}^{\dot{\alpha}} = iF_{mn}(\bar{\sigma}^{mn}\bar{\epsilon})^{\dot{\alpha}} + D\bar{\epsilon}^{\dot{\beta}}, \end{aligned} \qquad (258)$$

for the gauge multiplet, and the matter multiplet transforms as

$$\begin{aligned}\delta\phi &= -i\sqrt{2}\epsilon\psi \ , \quad \delta\psi_\alpha = -\sqrt{2}D_m\phi(\sigma^m\bar{\epsilon})_\alpha + \sqrt{2}F\epsilon_\alpha \\ \delta\bar{\phi} &= i\sqrt{2}\bar{\epsilon}\bar{\psi} \ , \quad \delta\bar{\psi}_{\dot{\alpha}} = -\sqrt{2}(\epsilon\sigma^m)_{\dot{\alpha}}\bar{D}_m\bar{\phi} + \sqrt{2}\bar{\epsilon}_{\dot{\alpha}}\bar{F} \\ \delta F &= i\sqrt{2}\bar{\epsilon}\bar{\sigma}^m D_m\psi - 2ie\phi\bar{\epsilon}\bar{\lambda} \ , \ \delta\bar{F} = i\sqrt{2}\epsilon\sigma^m\bar{D}_m\bar{\psi} + 2ie\bar{\phi}\epsilon\lambda \end{aligned} \qquad (259)$$

These transformations have the nice property that they involve covariant derivatives, instead of ordinary derivatives as the transformations (255). In fact this transformation rules were found by de Wit and Freedman when they where looking for covariant susy transformations which close on bosonic symmetries of the Lagrangian [43]. Thus the above given transformation rules are the *de Wit-Freedman* transformations and are of the form [140]

$$\delta_{WF}\Phi = \delta_s\Phi + \delta_g\Phi, \qquad (260)$$

where $\delta_s$ and $\delta_g$ are a susy and a super-gauge- transformation, respectively. The commutator of two combined transformations (260) closes on translations and ordinary gauge transformations (257) and is thus still a symmetry of the Lagrangian in the $WZ$-gauge, as it should be.

The Noether currents, associated with the symmetry transformations (258,259), are most easily obtained by making the susy parameters $\epsilon_\alpha$, $\bar{\epsilon}_{\dot{\alpha}}$ local [134]. The Lagrangian then transforms as

$$\delta\mathcal{L} = \partial_m\epsilon^\alpha J^m_\alpha + \partial_m\bar{\epsilon}_{\dot{\alpha}}J^{m\dot{\alpha}} + \text{tot. diverg.} \qquad (261)$$

For each of the two parameters one gets a Noether current which are obtained as:

$$J^m_\alpha = (iU\eta^{mn} - F^{mn}_+)(\sigma_n\bar{\lambda})_\alpha - i\sqrt{2}\bar{D}_n\bar{\phi}(\sigma^n\bar{\sigma}^m\psi)_\alpha \ , \qquad (262)$$

$$\bar{J}^{m\dot{\alpha}} = (iU\eta^{mn} + F^{mn}_-)(\bar{\sigma}_n\lambda)^{\dot{\alpha}} + i\sqrt{2}D_n\phi(\bar{\sigma}^n\sigma^m\bar{\psi})^{\dot{\alpha}}. \qquad (263)$$

Here we have introduced the (anti)self-dual field strength and the pre-potential:

$$F_{\pm}^{mn} = F^{mn} \pm i\tilde{F}^{mn} \ , \ \tilde{F}^{mn} = \tfrac{1}{2}\varepsilon^{mnpq}F_{pq} \ , \ U = e(|\phi|^2 - v^2). \tag{264}$$

These two currents are invariant under super-gauge transformations.

### 5.5.1 Supersymmetry algebra

Having the Noether currents (262) one can in principle calculate the algebra for the associated conserved charges. But in a gauge theory things are not so straight forward. Because of the residual ordinary gauge symmetry (257) the canonical structure is plagued by first class constraints and the supersymmetry is generally broken by gauge fixing (see below). Thus the susy charges become time-dependent. For certain gauges it is possible to define the gauge fixing Lagrangian in superspace which is then manifestly supersymmetric [56]. However, because of our vortex background we are not able to implement the the gauge fixing in a supersymmetric way. Anyhow, we can derive the classical susy algebra from the currents (262) using the transformations (258,259), without need to refer to a canonical structure. The implications of the gauge fixing and the involved supersymmetry breaking will be discussed below using BRST symmetry.

The susy algebra is now obtained as follows. The variation of the supercurrent (262) governed by the transformations (258,259)

$$\delta J_\alpha^m = i[\epsilon Q + \bar{Q}\bar{\epsilon} \ , \ J_\alpha^m] = i\epsilon^\beta\{Q_\beta, J_\alpha^m\} - i\{J_\alpha^m, \bar{Q}_{\dot\alpha}\}\epsilon^{\dot\alpha}, \tag{265}$$

reproduces the non-integrated susy algebra, closing on translations without gauge transformations since the Noether currents (262) are super-gauge invariant. So we expect for the two independent transformations with parameter $\epsilon_\alpha$ and $\epsilon_{\dot\alpha}$:

$$\{Q_\beta, J_\alpha^m\} = -i\delta_\beta J_\alpha^m = 0 \tag{266}$$

$$\{\bar{Q}_{\dot\alpha}, J_\alpha^m\} = i\bar{\delta}_{\dot\alpha} J_\alpha^m = 2\sigma_{\alpha\dot\alpha}^n T^m{}_n, \tag{267}$$

where we have factored out the transformation parameters. The trivial relation (266) may be changed in four dimensions by central charge densities due to the Konishi anomaly [30], but in three dimensions the anomaly vanishes. Anyhow this might be of importance for the embedded cosmic string.[36] To obtain the trivial relation (266) by transforming the supercurrent one has to use the algebraic e.o.m $F = 0$ as well as the non-algebraic e.o.m. for the

[36] On representation theoretical grounds there are no central charges allowed for the $\mathcal{N} = 1$ algebra in four dimensions . But a cosmic string obtained by embedding the vortex in four dimensions, like a domain wall, is not a finite energy state and is thus not part of the usual spectrum or representation, respectively.

gaugino $\lambda$. To project out the energy momentum tensor of the nontrivial relation (267) we use the trace-relation $Tr\bar{\sigma}^m\sigma^n = -2\eta^{mn}$ and thus obtain:

$$\begin{aligned}\frac{1}{4i}\, Tr[\bar{\sigma}_n\bar{\delta}J_m] &= \Theta_{mn} + \zeta_{mn} \\ &= F_m{}^k F_{nk} + D_{(m}\phi\bar{D}_{n)}\bar{\phi} + \tfrac{i}{2}\bar{\lambda}\bar{\sigma}_{(m}\partial_{n)}\lambda + \tfrac{i}{2}\bar{\psi}\bar{\sigma}_{(m}D_{n)}\psi + \eta_{mn}\mathcal{L} \\ &\quad + \varepsilon_{mnpq}\left( \tfrac{1}{2}UF^{pq} - iD^p\phi\bar{D}^q\bar{\phi} \right). \end{aligned} \tag{268}$$

Here we have again used the trivial e.o.m. $D = U$ but also the nontrivial e.o.m. for the matter fermion $\psi$ as well as the gaugino $\lambda$. The use of the nontrivial equations of motions is justifiable since the Noether currents are conserved only on-shell. On the other hand this is of course a source for a potential anomaly, i.e. if an e.o.m., although dealing with Heisenberg operators at the quantum level, is not satisfied for some matrix element (insertion). Besides that we are considering only classical quantities in this section we are on the safe side, since there are no poles in three (odd) dimensions at the one-loop level in dimensional regularization.

From (268) we see that the energy momentum tensor consists of the improved symmetric part $\Theta_{mn}$, which is given in the second line, and an antisymmetric part $\zeta_{mn}$, given in the third line of (268). The antisymmetric part, and thus the difference to the improved energy momentum tensor is the surface term

$$\begin{aligned}\zeta_{mn} &= \varepsilon_{mnpq}\left( \tfrac{1}{2}UF^{pq} - iD^p\phi\bar{D}^q\bar{\phi} \right) \\ &= -\varepsilon_{mnpg}\partial^p[ev^2A^q - i\bar{\phi}D^p\phi],\end{aligned} \tag{269}$$

where we have used that $\frac{1}{2}[D_q, D_p]\phi = i\frac{e}{2}F_{pq}$. The four-dimensional theory does not provide extended finite energy states and thus this surface term does not act on (transform) any state of the regular spectrum. But for infinite energy strings with finite energy density, obtained by embedding a three dimensional vortex, this surface term becomes important. As promised in (222), the antisymmetric part of the energy momentum tensor gives the classical central charge in the dimensional reduced theory$(i, j = 1, 2)$:

$$\zeta := \zeta_{03} = \partial_i\varepsilon_{ij}[ev^2A_j - i\bar{\phi}D_j\phi], \tag{270}$$

where we have used $\varepsilon_{0123} = -1$. This is exactly the central charge density obtained by the Bogomol'nyi trick (241). The symmetric contribution $\Theta_{03}$ to three momentum $P_{03} = T_{03}$ gives rise to possible anomalous[37] central charge contributions as has been seen for the kink and kink-domain walls [116]. This will be investigated below.

[37]This terminology is not precise since this possible contribution cannot come from an anomaly in three dimensions. We just want to emphasize that this possible contributions are finite reminders of the finally reduced extra dimension which are missed without this regularization. See for example the charge of the kink domain wall in three dimensions (2, [116]).

### 5.5.2 Partial susy breaking and BPS solution

BPS equations and their solutions have a number of curious properties. Although they are obtained from the purely bosonic model by the Bogomol'nyi trick (240, [15]) they can also be defined by their supersymmetry properties. BPS solutions conserve (or break) half (or a quarter, depending on the model) of the supersymmetry and are thus left invariant by half of the transformations (258,259). Also does the existence of the bosonic BPS solution imply relations between coupling constants for which the supersymmetry is extended [77], as already mentioned in (230). We look now for transformations which leave the (anti)vortex invariant (or equivalently show that the invariance under some restricted susy transformations leads to the BPS equations). Since the fermionic fields are assumed to be zero for classical solutions one only has to require that the fermions stay zero under the susy variations. One obtains for the gaugino, iff the only non-vanishing field strength component is given by $F_{12} = \mp U$:

$$\delta\lambda_\alpha = U[\mathbb{1} \pm \sigma^3] \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad , \quad \delta\bar{\lambda}^{\dot\alpha} = U[\mathbb{1} \mp \sigma^3] \begin{pmatrix} \bar{\epsilon}_{\dot 2} \\ -\bar{\epsilon}_{\dot 1} \end{pmatrix} . \tag{271}$$

The fermions of the matter multiplet transform as follows, iff also the scalar field depends only on $x_{1,2}$:

$$\delta\psi_\alpha = -\sqrt{2} \begin{pmatrix} -D_-\varphi\bar{\epsilon}_{\dot 1} \\ D_+\varphi\bar{\epsilon}_{\dot 2} \end{pmatrix} \quad , \quad \delta\bar{\psi}_{\dot\alpha} = -\sqrt{2} \begin{pmatrix} (D_-\varphi)^*\epsilon_1 \\ (D_+\varphi)^*\epsilon_2 \end{pmatrix} . \tag{272}$$

So we can read off the susy parameters $\epsilon, \bar\epsilon$ that leave the (anti) vortex invariant and vice versa:

$$\text{vortex}: \ D_+\varphi = 0 \ , \ F_{12} = -U \ \Leftrightarrow \ \epsilon_1 = 0 = \bar{\epsilon}_{\dot 1} \ , \tag{273}$$

$$\text{antivortex}: \ D_-\varphi = 0 \ , \ F_{12} = U \ \Leftrightarrow \ \epsilon_2 = 0 = \bar{\epsilon}_{\dot 2} \ . \tag{274}$$

The broken susy transformations, on the other hand, lead to fermionic zero modes. Since the susy-transformations (258,259) are symmetries of the Lagrangian and thus of the equations of motion the transformed field configurations are again classical solutions, especially since we do not require any boundary conditions, which might break susy. These fermionic zero modes will be investigated in detail below, since they are the essential input for the (non) multiplet shortening mechanism (see below).

So far we have considered only the classical action, i.e. without gauge fixing. In the next section we investigate the issue of susy breaking gauge fixing. For this we need the structure of BRST symmetry.

## 5.6 Gauge fixing and BRST symmetry

Gauge symmetry is one of the most important concepts for theories of fundamental interactions. This local symmetry makes it is possible to quantize spin-one-, i.e. vector-, fields without loosing the probability interpretation of quantum theory. Because of gauge invariance it possible to choose a certain gauge such that physical amplitudes are manifestly unitary. This comes at a prize of technical problems. In the classical theory the general solution of the equations of motions contain arbitrary functions of time. This is due to the local symmetry. In the case of a global symmetry at a certain (initial) time some degrees of freedom (DOF) (depending on the number of independent symmetries) can be fixed by the use of this global symmetry, but the time evolution of these DOF is then determined by the e.o.m. In the case of a local symmetry, the DOF can be fixed at *every* time at all *positions.* Thus these DOF can be completely eliminated and they are undetermined by the e.o.m. In the canonical approach to quantum theory one has to implement constraints in the canonical quantization procedure. Using path integral quantization one has to separate the over-counting of gauge equivalent configurations to absorb them in normalization factors. Perturbation theory can be implemented elegantly by introducing additional (unphysical) degrees of freedom, "ghosts", which compensates the dynamic of the unphysical DOF of gauge invariant models. Underlying to this procedure is a surprising structure which is observed as the so called BRST symmetry [10, 11, 132]. The precursor of the work of [10, 11, 132] is the Faddeev-Popov method [47] for the path integral quantization of gauge theories. Thereby the gauge fixing condition

$$F(A_m, \phi, \dots) - f(x) = 0 \tag{275}$$

is implemented as a functional delta-function in the path integral. This induces the Faddeev-Popov determinant

$$\Delta_{FP} = \det \frac{\delta F}{\delta \omega} \tag{276}$$

in the integration measure, which can be treated perturbatively when written as integration over Grassmannian (ghost) fields. Averaging over different gauges, mediated by the function $f$ in (275), with a quadratic weight function[38] results in an additional contribution to the exponent in the path integral. The "effective" or quantum Lagrangian for the path integral reads then

$$\mathcal{L}_Q = \mathcal{L}_{inv} + \mathcal{L}_{GF} + \mathcal{L}_{FP} \quad , \quad \mathcal{L}_{GF} = \frac{1}{2\xi} F^2 \quad , \quad \mathcal{L}_{FP} = b(\Delta_{FP})c, \tag{277}$$

where $\xi$ is a gauge parameter and $b, c$ are the Faddeev-Popov ghosts. The structure of this Lagrangian, independent of the path integral formalism, will be investigated with the help of the BRST symmetry. At the first look this might be an excess of formalism for abelian gauge fields. But as we will see, both the nontrivial background and especially supersymmetry will be related to this structure.

[38]A non quadratic weight function induces further, so-called Nielsen-Kallosh ghosts [126]

### 5.6.1 BRST symmetry

First we outline some generalities of the BRST symmetry which we then apply to our specific case. BRST symmetry is defined as a gauge symmetry, where the local gauge parameter is factored into a global Grassmannian parameter and a Grassmannian field, called *ghost* field:

$$\omega(x) = \lambda c(x). \tag{278}$$

The BRST transformations of the various fields $\varphi_i$ are then defined as $\delta_{gauge}\varphi_i = \lambda\ \delta_{BRS}\varphi_i$. For the component fields one thus obtains

$$\delta_{BRS}A_m = \partial_m c \quad , \quad \delta_{BRS}\phi = iec\phi \quad , \quad \delta_{BRS}\psi = iec\psi. \tag{279}$$

To fix the transformation of the ghost field $c$ one requires nilpotency for the BRST transformation. The reason for this will become clear immediately. Applied to the scalar field this gives (279)

$$\delta^2_{BRS}\phi = ie(\delta_{BRS}c - iec^2) \stackrel{!}{=} 0 \rightarrow \delta_{BRS}c = 0, \tag{280}$$

where we have used that in the abelian case $c^2 = 0$. One can easily convince oneself that with this transformation of the ghost,

$$\delta^2_{BRS} = 0 \tag{281}$$

for all fields. In addition to the ghost $c$ one introduces the anti-ghost $b$ and associates a so-called ghost number $N_{FP} = \pm 1$ for each ghost/anti-ghost, which is additive for products of ghost fields. As we will see below these are "good" quantum numbers, i.e. eigenvalues of a conserved charge. The anti ghost per definition transforms into a Lagrange multiplier field

$$\delta_{BRS}b = B \quad , \quad \delta_{BRS}B = 0. \tag{282}$$

The real field $B$ is an auxiliary (Nakanishi-Lautrup) field, and was originally introduced to generalize the Gupta-Bleuler subsidiary condition [93]. With this transformation rules the BRST transformation is still nilpotent for all fields. The gauge-fixing Lagrangian is introduced as follows:

$$\mathcal{L}_{GF+FP} = \delta_{BRS}\Psi \quad , \quad \Psi = b\mathcal{F}(A_m, \phi, \dots). \tag{283}$$

Here $\Psi$ is the gauge fixing "fermion" and $\mathcal{F}$ is related to the gauge fixing function in (275) as follows:

$$\mathcal{F} = F + \tfrac{\xi}{2}B. \tag{284}$$

If $F$ does not contain ghost fields, which we will always assume, the gauge fixing Lagrangian (283) separates as

$$\mathcal{L}_{GF+FP} = \mathcal{L}_{GF} + \mathcal{L}_{FP}. \tag{285}$$

There are some comments necessary:(i) The BRST transformation $\delta_{BRS}$ is independent of the gauge fixing function $F(A_m, \phi, \dots)$. (ii) The gauge-fixing Lagrangian $\mathcal{L}_{GF+FP}$ is BRST exact and thus BRST invariant,

$$\delta_{BRS}\mathcal{L}_{GF+FP} = \delta_{BRS}^2\Psi = 0, \tag{286}$$

because of the *nilpotency* of the BRST transformation. Since the BRST transformation is a gauge transformation for the classical (invariant) Lagrangian the whole quantum Lagrangian is BRST invariant

$$\delta_{BRS}\mathcal{L}_Q = \delta_{BRS}\mathcal{L}_{inv} + \delta_{BRS}\mathcal{L}_{GF+FP} = 0. \tag{287}$$

From the point of view of the quantum Lagrangian $\mathcal{L}_Q$ the BRST transformation is a *global* transformation among all physical and ghost fields with constant Grassmannian parameter $\lambda$. (iii) The Hermiticity (complex conjugation) properties of the ghosts are defined by the requirement that the quantum Lagrangian $\mathcal{L}_Q$ is hermitian and therefore induces a unitary time evolution. Thus we have [39]

$$c^\dagger = c \ \ , \ \ b^\dagger = b \ \ \rightarrow \ \ \lambda^* = -\lambda, \tag{288}$$

That the BRST parameter is purely imaginary follows from (278).

The quantum Lagrangian (287) is no longer gauge invariant and thus first class constraints have become at most second class constraints, which can be solved. The canonical quantization replaces now Dirac brackets by (anti)commutators. A question remains in the canonical quantization procedure: What are the physical states? To answer this one needs the BRST charge.

### 5.6.2 BRST charge and physical states

For the BRST symmetry (287) of the quantum Lagrangian (286) exists an associated conserved current:

$$\delta\mathcal{L}_Q = 0 = \partial_m\left(\frac{\delta\mathcal{L}_Q}{\delta\partial_m\Phi_i}\delta\Phi_i\right) + e.o.m. = \partial_m(\lambda J^m_{BRST}) + e.o.m. \tag{289}$$

The conserved charge, which then induces the BRST transformations, is:

$$Q_{BRST} = \int dx^3 J^0_{BRST} \ \ , \ \ [i\lambda Q_{BRST}, \Phi_i] = \lambda\delta_{BRS}\Phi_i. \tag{290}$$

The Faddeev-Popov Lagrangian in (285) is bilinear in the ghosts, so that the quantum Lagrangian $\mathcal{L}_Q$ is invariant under the transformations

$$c \rightarrow e^\rho c \quad b \rightarrow e^{-\rho} b. \tag{291}$$

[39]Originally it was thought that the anti-ghost is the hermitian conjugate ghost.

Because of the Hermiticity of the ghost fields (288) the parameter $\rho$ has to be real. This symmetry leads to another conserved charge, the ghost charge $Q_{gh}$, which together with the BRST charge form the BRST algebra [93]

$$\{Q_{BRST}, Q_{BRST}\} = 0 \quad , \quad [Q_{gh}, Q_{gh}] = 0 \quad , \quad [iQ_{gh}, Q_{BRST}] = Q_{BRST}. \tag{292}$$

The ghost charge $Q_{gh}$ defines the ghost number operator $N_{FP} = iQ_{gh}$, mentioned already below (281), and the last commutator in (292) shows that the BRST charge has ghost number $N_{FP}(Q_{BRST}) = 1$.

Because of the indefinite space time metric $\eta_{mn}$ the Fock-space for vector fields contains negative norm states in a covariant approach. This is inconsistent with the probability interpretation of quantum theory and thus one needs a selection rule, which is consistent with the time evolution to identify physical states. Such a selection rule was given by Kugo-Ojima [93]:

$$Q_{BRST} \, |phys\rangle = 0. \tag{293}$$

Because $Q_{BRST}$ is conserved this condition is consistent with the time evolution. The space of states (293) still contain zero-norm (BRST-exact) states, which have to be factored out to obtain the physical Hilbert space.

### 5.6.3 Background $R_\xi$-gauge

In a spontaneously broken gauge theory or/and for a nontrivial background one has to expand the classical Lagrangian around the non-vanishing vacuum expectation values or the background, respectively. Expanding (197) around the vortex solution (247)

$$\phi \to \varphi + \eta \quad , \quad A_m \to \mathcal{A}_m + \alpha_m, \tag{294}$$

the Lagrangian (197) has still a gauge invariance acting on the quantum fields:

$$\delta\alpha_m = \partial_m \omega \quad , \quad \delta\eta = ie\omega(\varphi + \eta) \quad \delta\psi = ie\omega\psi. \tag{295}$$

It is this gauge invariance we have to deal with to obtain the correct Faddeev-Popov determinant in (276). In the BRST approach this can easily be formulated in a somewhat more background independent way, which will be useful for the supersymmetry investigations. But let us first reproduce the results used in the literature concerning the vortex so far [97]. In the spontaneously broken vacuum sector the background in (294) becomes $\varphi = v$ and $\mathcal{A}_m = 0$. The BRST transformations, as described in (278), are obtained by setting $\omega(x) = \lambda c(x)$ in (295). Next we define the $R_\xi$-gauge fixing function:

$$F = \partial_m \alpha^m - ie\xi(\varphi\bar{\eta} - \bar{\varphi}\eta), \tag{296}$$

where the choice $\xi = 1$ is the Feynman-'t Hooft gauge. The gauge fixing Lagrangian is now easily obtained as

$$\begin{aligned}\mathcal{L}_{GF} + \mathcal{L}_{FP} &= \delta_{BRS}\Psi \\ &= BF + \tfrac{\xi}{2}B^2 + b[\Box - e^2\xi(2|\varphi|^2 + \varphi\bar{\eta} + \bar{\varphi}\eta)]c, \end{aligned} \tag{297}$$

where we have used the gauge fixing fermion $\Psi$ from (283). Eliminating the auxiliary field $B$ by its algebraic equation of motion, one obtains for the first two terms in (297) the usual background $R_\xi$ [53, 129] gauge fixing Lagrangian:

$$\mathcal{L}_{GF} = -\tfrac{1}{2\xi}[\partial_m \alpha^m - ie\xi(\varphi\bar{\eta} - \bar{\varphi}\eta)]^2. \tag{298}$$

The second term in (297) is the ghost Lagrangian. But note that for a linear BRST symmetry the auxiliary Nakanishi-Lautrup field $B$ is essential.

The Feynman-'t Hooft gauge ($\xi = 1$) has, per construction, the nice property that the off-diagonal terms in the spontaneously broken vacuum of (197), which mixes scalar- and gauge fields in propagators, are canceled by the mixed scalar-gauge field terms of $\mathcal{L}_{GF}$. Also in the non-trivial background this gauge leads to simplifications, as we will see. The quantum fields, variations of the background, are quadratic only in $\mathcal{L}_{GF}$, there are no linear terms. Thus the classical e.o.m. are not modified by the gauge fixing procedure, since they are proportional to the linear variations of the fields.

Let us now rewrite the above gauge fixing and BRST transformations in a more background independent way. For this we rewrite the gauge fixing function $F$ (296) in terms of the full quantum fields $\phi = \varphi + \eta$ and $A_m = \mathcal{A}_m + \alpha_m$ instead of the fluctuations:

$$F = \partial_m A^m - ie\xi(\varphi\bar{\phi} - \bar{\varphi}\phi), \tag{299}$$

where we have assumed $\partial_m \mathcal{A}^m = 0$ for the gauge field background, as it is true for the vortex background (247). This assumption is not needed in what follows, it just simplifies the notation. With (299) we do not have to refer to a certain decomposition into background and fluctuation or a expansion for the classical Lagrangian $\mathcal{L}_{inv}$ in (277). The functions $\varphi, \bar{\varphi}$ in (299) can be considered as arbitrary but fixed background functions. Since we do not expand the classical Lagrangian as described in (294) the gauge symmetry and thus the BRST symmetry acts now on the whole fields

$$\delta_g A_m = \partial_m \omega \rightarrow \delta_{BRST} A_m = \partial_m c \tag{300}$$

$$\delta_g \phi = ie\omega\phi \rightarrow \delta_{BRST}\phi = iec\phi. \tag{301}$$

The transformation for $\psi$ (295) is of course unchanged. The ghost Lagrangian

$$\mathcal{L}_{FP} = b\ \delta_{BRST} F = b[\Box - e^2\xi(\varphi\bar{\phi} + \bar{\varphi}\phi)]c, \tag{302}$$

does not change by this redefinitions, after expanding $\phi$ and $A_m$ again according (294). Before considering the supersymmetry we just note again that the background function $\varphi$ is fixed and invariant under gauge- or BRST transformations.

### 5.6.4 SUSY and BRST symmetry

Now we come back to the issue of supersymmetry in the presence of a non-supersymmetric gauge fixing Lagrangian as for the $R_\xi$ background gauge (297,298) for the vortex. The naive (classical) susy charges associated with the Noether currents (262) are no longer conserved and are thus *time-dependent* when the gauge fixing Lagrangian is included in the quantum Lagrangian $\mathcal{L}_Q$ (277). At first sight this looks catastrophic but as we will see the *broken* charge algebra closes on the physical Hilbert space. The cancellation of gauge artifacts in the supersymmetry algebra is controlled by the BRST symmetry. This relies on the following two important results [55]:

If the gauge fixing Lagrangian is given by $\mathcal{L}_g = \delta_{BRST}\Psi$, where $\Psi$ is a gauge fixing fermion, the susy charge density $J^0$ can be written as

$$J^0 = J^0_{(\text{naive})} + BRST - \text{exact piece}, \tag{303}$$

*if* for a localized susy transformation (local parameters)

$$[\delta_S, \delta_B]\Psi = 0. \tag{304}$$

Here the naive charge density is the one obtained before gauge fixing, i.e. the one we obtained in (262). The transformations $\delta_S$ and $\delta_B$ are susy- and BRST-transformations, respectively, including the Grassmannian transformation parameters (otherwise the anti-commutator has to vanish on $\Psi$). Thus if the gauge fixing procedure is "BRST-covariant" the time dependent part of the integrated charges in (303) manifestly vanishes on the physical Hilbert space. To obtain the algebra we have to go one step further:

*If* two independent localized susy transformations commute with the BRST-transformation on the gauge fixing fermion, i.e.

$$[\delta_{S_2}\delta_{S_1}, \delta_B]\Psi = 0, \tag{305}$$

then the equal-time anti-commutator relation for the time dependent super charges reads as ($\alpha, \beta$ can take any values depending on the model):

$$\{Q_\alpha(x^0), Q_\beta(y^0)\}_{x^0=y^0} = \{Q_\alpha, Q_\beta\}_{(\text{physical})} + BRST - \text{exact piece}. \tag{306}$$

The physical as well as the BRST-exact piece are obtained by a susy transformation of the susy charge density in the path integral formulation (that is the reason why localized susy transformations are needed). Thus the physical contribution coincides with the naive expression as obtained from the classical Lagrangian in (262) up to anomalies. But as we have already argued below (268) we do not expect any anomalies here. The unphysical fields (ghosts and auxiliary fields, this does not include gauge modes (ghost) of the vector field) coming from the gauge fixing fermion $\Psi$ are defined to be invariant under susy transformations. For a proof of this statements we refer to [55].

What remains to show is that the above theorems apply to our case with the background gauge fixing. If this is true, we can simply take the naive expressions from (268) to calculate quantum corrections to the energy and central charge of the vortex.

For a general gauge fixing fermion of the form (283,284) the commutator of one or two susy transformations with a BRST-transformation vanishes if

$$[\delta_B, \delta_S]F = 0, \tag{307}$$

where $\delta_S = \delta_{S_1}$ or $\delta_{S_2}\delta_{S_1}$. This can be seen as follows:

$$\begin{aligned}\delta_B\delta_S\Psi &= \delta_B(b\delta_S F) = B\delta_S F + \delta_B\delta_S F \\ &= \delta_S(BF + b\delta_B F) = \delta_S\delta_B[b(F + \tfrac{\xi}{2}B)] \\ &= \delta_S\delta_B\Psi.\end{aligned} \tag{308}$$

In the second equality we have used the assumed relation (307). In all other equalities we used the invariance of the unphysical fields under susy.

To show (307) we use the background "independent" formulation (299). Then the susy transformations are the one given in (258,259). The background function $\varphi$ is fixed and invariant also under this susy transformation. This is also consistent with the classical vortex solution, where all fermions vanish, and thus the bosonic fields $\varphi, \mathcal{A}_m$ do not transform under (258,259). So (307) is true for the background independent gauge fixing function (299) if the transformations commute on the gauge- and scalar fields, i.e.

$$[\delta_S, \delta_B]A_m = [\delta_S, \delta_B]\phi = [\delta_S, \delta_B]\bar{\phi} = 0, \tag{309}$$

where $\delta_S$ is again a single or a product of two localized susy transformation. This is easily shown to be true, so that finally we have the desired relation for our case

$$[\delta_{S_2}\delta_{S_1}, \delta_B]\Psi = 0, \tag{310}$$

and thus on the physical subspace the susy algebra is simply given by (268).

## 5.7 Vacuum sector and renormalization

At the classical level, the energy and central charge of vortices are multiples of $2\pi v^2$. Renormalization of tadpoles, even when only by finite amounts, will therefore contribute directly to the quantum mass and central charge of the $\mathcal{N}=2$ vortex, a fact that has been overlooked in the original literature [97, 119] on quantum corrections to the $N=2$ vortex.[40]

Adopting a "minimal" renormalization scheme where the scalar wave function renormalization constant $Z_\phi = 1$, the renormalization of $v^2$ is fixed by the requirement of vanishing tadpoles in the trivial sector of the 2+1 dimensional model. The calculation can be conveniently performed by using dimensional regularization of the 3+1 dimensional $\mathcal{N}=1$ model. For the calculation of the tadpoles we decompose, after the renormalization $v^2 \to v^2 + \delta v^2$ in (197), $\phi = v + \eta \equiv v + (\sigma + i\rho)/\sqrt{2}$, where $\sigma$ is the Higgs field and $\rho$ the would-be Goldstone boson. The gauge fixing term (298) avoids mixed $a_\mu$-$\rho$ propagators (we denote the gauge field fluctuations in the vacuum sector by $a_\mu$ in contrast to $\alpha_\mu$ in the vortex background), but there are mixed $\lambda$-$\psi$ propagators, which can be diagonalized by introducing new spinors

$$s = \tfrac{1}{\sqrt{2}}(\psi + \lambda) \quad , \quad d = \tfrac{1}{\sqrt{2}}(\psi - \lambda). \tag{311}$$

The quadratic fermionic Lagrangian in terms of four-component Majorana spinors $s_M = (s_\alpha, \bar{s}^{\dot\alpha})$ is then obtained as

$$\mathcal{L}_F^{(2)} = -\tfrac{1}{2}\bar{s}_M(\partial\!\!\!/ + m)s_M - \tfrac{1}{2}\bar{d}_M(\partial\!\!\!/ - m)d_M \ , \tag{312}$$

where the mass $m = \sqrt{2}ev$.

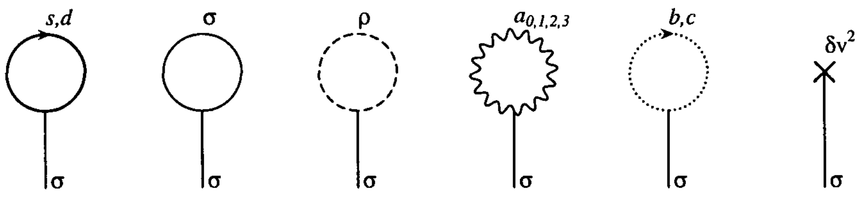
The part of the interaction Lagrangian which is relevant for $\sigma$ tadpoles to one-loop order is given by

$$\mathcal{L}^{\rm int}_{\sigma-\rm tadpoles} = -e\sigma(\bar{s}_M s_M - \bar{d}_M d_M) - \frac{em}{2}(\sigma^2 + \rho^2)\,\sigma \tag{313}$$

$$-em(a_\mu^2 + \xi b\, c - \delta v^2)\,\sigma, \tag{314}$$

where $b$ and $c$ are the Faddeev-Popov fields.

The one-loop contributions to the $\sigma$ tadpole thus read




$$= (-em) \times \tag{315}$$

$$\{-\mathrm{tr}\mathbf{1}_4 I(m) + \frac{3}{2}I(m) + \frac{1}{2}I(\xi^{\frac{1}{2}}m) + [3I(m) + \xi I(\xi^{\frac{1}{2}}m)] - \xi I(\xi^{\frac{1}{2}}m) - \delta v^2\},$$

[40] The nontrivial renormalization of $v^2$ has however been included in the recent work of Ref. [135].

where

$$I(m) = \int \frac{d^{3+\epsilon}k}{(2\pi)^{3+\epsilon}} \frac{-i}{k^2+m^2} = -\frac{m^{1+\epsilon}}{(4\pi)^{1+\epsilon/2}} \frac{\Gamma(-\frac{1}{2}-\frac{\epsilon}{2})}{\Gamma(-\frac{1}{2})} = -\frac{m}{4\pi} + O(\epsilon). \tag{316}$$

Requiring that the sum of tadpole diagrams (315) vanishes fixes $\delta v^2$,

$$\delta v^2 = \frac{1}{2}\left(I(m) + I(\xi^{\frac{1}{2}}m)\right)\Big|_{D=3} = -\frac{1+\xi^{\frac{1}{2}}}{8\pi}m. \tag{317}$$

Because in dimensional regularization there are no poles in odd dimensions at the one-loop level, the result for $\delta v^2$ is finite, but it is non-vanishing. Because the classical mass of the vortex is $M_{\mathrm{V}} = 2\pi v^2 = \pi m^2/e^2$, the counter term $\delta v^2$ is the only one that is of importance to the one-loop corrections to $M_{\mathrm{V}}$. Since $\delta v^2$ is gauge-parameter dependent, the remaining contributions to mass and central charge must be gauge dependent, too, so that the final result is gauge independent.

## 5.8 Propagators in the vortex background

In this section we discuss the fluctuation equations and asymptotic propagators in the vortex background. This results will be needed for the computation of the quantum corrections to the mass as well as to the central charge. Throughout this section we use the Feynman-'t Hooft gauge $\xi = 1$.

### 5.8.1 Fermionic fluctuation

We start with the fermionic fluctuations, and as a consequence of supersymmetry this results will be very useful also for the bosonic sector. Shifting the fields by the vortex background (247)

$$\phi \to \varphi + \eta \quad A_m \to \mathcal{A}_m + \alpha_m, \tag{318}$$

and grouping spinor components as $U = \begin{pmatrix}\psi_1 \\ \bar{\lambda}^{\dot{2}}\end{pmatrix}$, $V = \begin{pmatrix}\psi_2 \\ \bar{\lambda}^{\dot{1}}\end{pmatrix}$, the fermionic part of the classical Lagrangian (197) bilinear in quantum fields can be written as follows ($\partial_\tau := -\partial_0 + \partial_3$ and $\bar{\partial}_\tau := -\partial_0 - \partial_3$):

$$\mathcal{L}_F^{(2)} = -iU^*(\bar{\partial}_\tau U - L^\dagger V) - iV^*(\partial_\tau V + LU), \tag{319}$$

where the operators $L, L^\dagger$ are given by ($D_\pm = D_1 \pm iD_2$, see appendix E)

$$L = \begin{pmatrix} -D_+ & -\sqrt{2}e\varphi \\ \sqrt{2}e\varphi^* & \partial_- \end{pmatrix} \quad , \quad L^\dagger = \begin{pmatrix} D_- & \sqrt{2}e\varphi \\ -\sqrt{2}e\varphi^* & -\partial_+ \end{pmatrix}. \tag{320}$$

The covariant derivatives are defined with respect to the background, i.e. $D_i = \partial_i - ie\mathcal{A}_i$. In the BPS-vortex background, with $D_+\varphi = 0$, the two hermitian conjugated operators $L, L^\dagger$ have following important properties:

$$LL^\dagger = \begin{pmatrix} -D_+D_- + 2e^2|\varphi|^2 & 0 \\ 0 & -\partial_+\partial_- + 2e^2|\varphi|^2 \end{pmatrix}, \tag{321}$$

and

$$L^\dagger L = \begin{pmatrix} -D_-D_+ + 2e^2|\varphi|^2 & -\sqrt{2}eD_-\varphi \\ -\sqrt{2}e(D_-\varphi)^* & -\partial_-\partial_+ + 2e^2|\varphi|^2 \end{pmatrix}. \tag{322}$$

For the anti-vortex ($D_-\varphi = 0$) $L^\dagger L$ is diagonal, but not the other combination $LL^\dagger$. These two operators form a quantum mechanical susy system (see appendix B). Now the free generating functional, and thus the propagator, are defined by adding (Grassmann-valued) sources to (319)

$$S_0(J) = \int dx^D(\mathcal{L}_F^{(2)} + W^*J + J^*W), \tag{323}$$

where we have grouped $W = (U, V)$. The generating functional is the obtained as

$$Z_0[J, J^*] = \int \mathcal{D}WW^* \; e^{\,iS_0(J)} = \; e^{-\int J^*\Delta J} \;, \text{ with} \tag{324}$$

$$\begin{pmatrix} \bar{\partial}_t & -L^\dagger \\ L & \partial_\tau \end{pmatrix} \Delta = \mathbb{1}\delta^D(x - x'). \tag{325}$$

Finding a solution for the propagator $\Delta$ in (325) is equivalent to solving the fluctuation equations, i.e. the linearized e.o.m. for the operators.[41] From (319) we can read off the fluctuation equations to be

$$LU = -\partial_\tau V \quad , \quad L^\dagger V = \bar{\partial}_\tau U. \tag{326}$$

Separating of the trivial motion, $U_n^{(\pm)} = e^{\pm i(E_nt-\ell z)}u_n^{(\pm)}$ and analogous for $V$, and iterating (326) one gets

$$LL^\dagger v_n^{(\pm)} = \omega_n^2 v_n^{(\pm)} \quad , \quad E_n^2 =: \omega_n^2 + \ell^2 \tag{327}$$

$$u_n^{(\pm)} = \tfrac{\pm i}{E_n-\ell} L^\dagger v_n^{(\pm)}. \tag{328}$$

The equation for the $v$-modes (327) is diagonal because of (321) and can thus be decomposed into "spin"-states $v_{n,s=1,2}$ (since positive and negative frequency modes fulfill the same equation we omit this indication for the $v$-modes in the following). The $u$-modes are algebraically related to the $v$-modes by (328) and up to the correct normalization the two equations represent the structure of quantum mechanical susy system (428-430). Iterating (326) in opposite sequence one obtains the quantum mechanical susy-partner Hamiltonian:

$$L^\dagger L u_n = \omega_n^2 u_n. \tag{329}$$

[41]This corresponds to the interaction picture in the vortex background.

### 5.8.2 Asymptotic solutions, spectral density

First we look for asymptotic solutions of the diagonal equation for the $v$-modes (327). Since $|\varphi|^2 \to v^2$ exponentially fast (251) the fluctuation operator becomes ($i = 1, 2$)

$$LL^\dagger \to \begin{pmatrix} -D^2_{i|\mathrm{asym}} + M^2 & 0 \\ 0 & -\partial_i^2 + M^2 \end{pmatrix}, \tag{330}$$

where the mass $M = \sqrt{2}ev$, so that the "spin"-decomposed $v$-modes fulfill

$$(-D_i^2 + M^2)v_{n,1} = \omega_n^2 v_{n,1} \quad , \quad (-\partial_i^2 + M^2)v_{n,2} = \omega_n^2 v_{n,2}. \tag{331}$$

The square of the covariant derivative in the background (247) is given by

$$\begin{aligned} D_i^2 &= \partial_i^2 - 2i\tfrac{n_\tau - a(r)}{r^2}\partial_\theta - \tfrac{(n_\tau - a(r))^2}{r^2} \\ &\to \tfrac{1}{r}\partial_r(r\partial_r) + \tfrac{1}{r^2}\partial_\theta^2 - \tfrac{1}{r^2}(2in_\tau\partial_\theta + n_\tau^2), \end{aligned} \tag{332}$$

where we have used that asymptotically $a(r) \to 0$ exponentially fast. The first two terms in (332) are simply the two-dimensional free (vacuum) Laplacian in polar coordinates $(r, \theta)$. The third term reflects the long-range force of the non-trivial gauge field background, which remains also asymptotically in the case of nontrivial topology (winding) $n_\tau \neq 0$. To solve the free equation in (331) we separate off the angular dependence

$$s = 2: \quad v_{msk} = e^{im\theta} R_{mk}(r), \quad m = 0, \pm 1, \pm 2, \ldots \tag{333}$$

so that $e^{im\theta}$ form a complete set. With $\omega_m^2 = k_m^2 + M^2$ and $\rho = k_m r$ (331) becomes

$$\rho^2 R'' + \rho R' + (\rho^2 - m^2)R = 0. \tag{334}$$

This is *Bessel's equation of order* $m$, and a general solution is given by a linear combination the Bessel function of first and second kind (Neumann function) of order $|m|$ [95]. Because of the symmetry $m \to -m$ of (334) we consider only the positive $m$-values below.

$$R_m(kr) \sim \; a_m(k) J_m(kr) + b_m(k) N_m(kr) \; , \; m = 0, 1, 2, \ldots \tag{335}$$

For the threshold mode $k = 0$ the coefficient $b_m$ has to vanish since only $J_m$ is regular at the origin. The normalizations will be discussed below. The spectral density of the continuous momentum quantum numbers $|k_m| \in \mathbb{R}_+$ can be obtained by introducing temporarily boundary conditions. We require that all mode functions, and thus the $V$-spinor field, vanish at the radius $r = L$. This selects the Bessel functions of first kind as the eigen system of (334) [75]:

$$R_m(k_m L) \sim J_m(k_m r) = 0 \Rightarrow k_m(n) = \frac{x_n^{(m)}}{L}, \tag{336}$$

where $x_n^{(m)}$ is the $n$'th zero of the order $m$ Bessel function $J_m(\rho)$. These are simple zeros symmetrically distributes w.r.t. $\rho = 0$, which for $m > 0$ is itself a zero (of order $m$), on the real line [95]. The spectral density is then given as

$$\rho_m(k) = \left(\frac{dk_m}{dn}\right)^{-1} = L\left(\frac{dx_n^{(m)}}{dn}\right)^{-1}. \tag{337}$$

In the continuum limit $L \to \infty$ the momenta $k_m(n)$ become a continuous quantum number for all $m$, and so the energies are degenerated as

$$\omega_m^2(k) = \omega^2(k) = k^2 + M^2 \quad , \quad k \in \mathbb{R}. \tag{338}$$

What happens now with the charged, "spin" $s = 1$ modes which feel the gauge field background? Again we separate the angular dependence as

$$s = 1: \quad v_{m'sk} = e^{im'\theta} R_{m'k}(r). \tag{339}$$

With the the same substitutions, $\omega_{m'}^2 = k_{m'}^2 + M^2$ and $\rho = k_{m'}r$, because of the nontrivial operator (332) equation (331) becomes now

$$\rho^2 R'' + \rho R' + (\rho^2 - (m' - n_\tau)^2)R = 0. \tag{340}$$

This is again a Bessel equation, but now its order $m' - n_\tau$, is shifted by the winding number. The rest is quite analogous as before. To obtain the same momenta, and thus mode energies (336), as well as the same spectral density (337) in the continuum limit, as for the "spin" $s = 2$ mode we have to identify

$$m' = m + n_\tau, \tag{341}$$

so that the angular momentum of the charged mode is increased by the winding number $n_\tau$ compared to the uncharged mode of the same energy. Thus for the positive frequency modes and their spectral densities we obtain

$$v_{msk} \sim e^{im\theta} J_{|m|}(kr)\varepsilon_s \quad , \quad \varepsilon_1 = \begin{pmatrix} e^{in_\tau\theta} \\ 0 \end{pmatrix} \quad , \quad \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{342}$$

$$\omega^2(k) = k^2 + M^2 \quad , \quad \rho_m(k) = L(\tfrac{d}{dn} x_n^{(m)})^{-1}. \tag{343}$$

The negative frequency modes are just the complex conjugated, so that the whole quantum field $V$ is written as:

$$V = \sum\!\!\!\!\!\!\int_{msk} dk \int \frac{dp^d}{(2\pi)^d} (a_{msk} e^{-i[E_{mk}t - \ell z]} v_{msk} + b_{msk}^\dagger e^{i[E_{mk}t - \ell z]} v_{msk}^*) \tag{344}$$

### 5.8.3 Completeness and propagator

Now we turn to the exact problem again. To get the right normalization to identify the equations (327,328) with the quantum mechanical susy-system (428-430) we write the modes as follows:

$$v_{mks} = \sqrt{E-\ell}\;\tilde{v}_{mks} \quad , \quad u^{(\pm)}_{mks} = \mp i\sqrt{E+\ell}\;\tilde{u}_{mks}. \tag{345}$$

The fluctuation equations (327,328) become then

$$LL^{\dagger}\tilde{v}_{mks} = \omega^2(k)\tilde{v}_{mks} \quad , \quad \tilde{u}_{mks} = \frac{1}{\omega(k)}L^{\dagger}\tilde{v}_{mks}. \tag{346}$$

We again omit the positive/negative frequency label $(\pm)$ since both fulfill the same equations. We have now the correct normalizations, so that if $\tilde{v}_{mks}$ are normalized, and thus as eigen functions of the self-adjoint operator $LL^{\dagger}$ form a complete set[42].

$$\sum\!\!\!\!\!\!\!\int_{mks} dk\; \tilde{v}_{mks}(x)\tilde{v}^{\dagger}_{mks}(x') = \mathbb{1}_2\delta^2(x-x'), \tag{347}$$

the $u$-modes are also ortho-normalized. They also form a complete set, since *each* eigenfunction, except zero-modes, of the self-adjoint susy partner operator $L^{\dagger}L$ is algebraically related to the eigen-functions $v_{mks}$ of $LL^{\dagger}$ as given in (346). To be precise, in the case of degenerated eigenvalues this only true for the eigen spaces and not for each eigenfunction separately, but this does of course not change the completeness of the whole set of eigen-functions. Therefore, including zero-modes, the $u$'s of (346) form a complete set

$$\sum\!\!\!\!\!\!\!\int_{msk} \tilde{u}_{mks}(x)\tilde{u}^{\dagger}_{mks}(x') = \mathbb{1}_2\delta^2(x-x'). \tag{348}$$

We can now write down the propagator which solves (325) $(p_a = (p_0, \ell))$:

$$\Delta_{\alpha\beta}(x-x') = \sum\!\!\!\!\!\!\!\int_{msk} dk \int \frac{dp^d}{(2\pi)^d}\; \frac{i\; w_{\alpha}(x)\; w^{\dagger}_{\beta}(x')\; e^{ip_a(x-x')^a}}{p^2+\omega^2(k)-i\varepsilon} \tag{349}$$

with the spinors grouped as

$$w_{\alpha} = \begin{pmatrix} u^{(+)}_{mks} \\ v^{(+)}_{mks} \end{pmatrix} = \begin{pmatrix} -i\sqrt{p_0+\ell}\;\tilde{u}_{mks} \\ \sqrt{p_0-\ell}\;\tilde{v}_{mks} \end{pmatrix}. \tag{350}$$

The energies $E(k)$ are off-shell now, as usual for the propagator. Note that because of the "funny" normalization factors in (350) the propagator (349)

[42]The sum-integral is a sum for discrete (bound) modes and an integral for continuum modes

is asymmetric in the flat extra momentum $\ell$, even for $x' = x$. This is not the case for the bosonic propagators (see below). The need for this normalization factors is a direct consequence of the regularization by a flat extra dimension. This $\ell$-asymmetry is responsible for the anomalous contribution to the central charge for the susy kink and susy kink-domain wall. In checking equation (325) one can see the importance for the normalization factors required in (345) and the relations (346). Note that because of our dimensional regularization by an extra momentum $\ell$, possible zero modes of the nontrivial operators $L^\dagger L, LL^\dagger$ are included in the propagator (350) as massless part of the discrete spectrum. But in one-loop calculations they contribute only scaleless and thus in dimensional regularization vanishing integrals.

### 5.8.4 Bosonic fluctuations

For the bosonic propagators we also have to take into account the gauge-fixing Lagrangian (298). Again, expanding the fields as described in (319), one obtains for the bosonic part of the quantum Lagrangian $\mathcal{L}_Q$, bilinear in quantum fields:

$$\begin{aligned}\mathcal{L}^{(2)}_{Q,B} &= \mathcal{L}^{(2)}_B + \mathcal{L}_{GF} \\ &= \tfrac{1}{2}\alpha^m(\Box - 2e^2|\varphi|^2)\alpha_m + \bar{\eta}(D_m^2 - e^2(3|\varphi|^2 - v^2))\eta \\ &\quad - 2ie\alpha^m(D_m\varphi\bar{\eta} - \eta\bar{D}_m\bar{\varphi}).\end{aligned} \tag{351}$$

The $m = 0,3$ components of the vortex gauge field are zero (247) and the background fields do not depend on $x_{0,3}$. So these two components of the gauge quantum field $\alpha_m$ decouple from the scalar field. The fluctuation equations can be read off (351) to be $(a = 0,3)$

$$(\Box - 2e^2|\varphi|^2)\alpha_a = 0\ , \quad \alpha_a = \varepsilon(s)_a e^{-i(Et-\ell z)}S(x,k), \tag{352}$$

and analogously for the negative frequency part. As one can see this gives for $S$ the same equation as for the $v_{s=2}$-modes (321,330) which we thus identify, $S(x,k) \equiv \tilde{v}_{mk2} := \tilde{v}_{mk}(x)$. To get correct normalization of the propagator we have to identify $S$ with the normalized, i.e. tilded modes (345). The propagator is therefore given by $(p_a = (p_0, \ell)$:

$$\Delta_{ab}(x-x') = \eta_{ab} \sum\!\!\!\!\!\!\int_{mk} dk \int \frac{dp^d}{(2\pi)^d} \frac{-i\ \tilde{v}_{mk}(x)\tilde{v}^*_{mk}(x')e^{ip_a(x-x')^a}}{p^2 + \omega_m(k)^2 - i\varepsilon} \tag{353}$$

$$\eta^{ab}(\Box - 2e^2|\varphi|^2)\Delta_{dc} = i\delta^a_c\delta^D(x-x'), \tag{354}$$

Here we have used the completeness $\sum_s \varepsilon(s)_a\varepsilon^*(s)_b = \eta_{ab}$ of the spin vectors. The free generating functional reads as

$$Z_0[J_a] = \int \mathcal{D}\alpha_{0,3}\ e^{i\int(\mathcal{L}^{(2)}_{QB} + J^a\alpha_a)} = e^{-\frac{1}{2}\int J^a\Delta_{ab}J^b}. \tag{355}$$

Asymptotically we again obtain the second equation of (331), i.e. free fields of mass $M = \sqrt{2}ev$.

The $\alpha_{m=1,2}$ components mix with the scalar field fluctuations $\eta$ through the gauge fixing term in (351). Introducing holomorphic coordinates $\alpha_\pm = \alpha_1 \pm i\alpha_2$ (see appendix E) and using the BPS equation $D_+\varphi = 0$ (246) for the *vortex* background, the quadratic Lagrangian (351) coupled to sources for this fields writes as

$$\mathcal{L}_B(\alpha_i, \eta, J) = \bar{Z}_k M_{kj} Z_j + Z_k \bar{J}_k + \bar{Z}_k J_k \quad , \tag{356}$$

$$Z_k = \begin{pmatrix} \eta \\ \frac{-i}{\sqrt{2}}\alpha_+ \end{pmatrix} \quad , \quad M_{kj} = \begin{pmatrix} D_n^2 - e^2(3|\varphi|^2 - v^2) & \sqrt{2}eD_-\varphi \\ \sqrt{2}e(D_-\varphi)^* & \Box - 2e^2|\varphi|^2 \end{pmatrix} . \tag{357}$$

Using the BPS equations (242,243) the differential operators in holomorphic coordinates are $(a = 0, 3)$

$$D_n^2 = \partial_a^2 + D_-D_+ + e^2(|\varphi|^2 - v^2) \quad , \quad \Box = \partial_a^2 + \partial_-\partial_+ . \tag{358}$$

So the operator in (356) becomes

$$M = \partial_a^2 - L^\dagger L \tag{359}$$

Therefore the doublet $\vec{Z} = e^{\pm i(Et - \ell z)}\vec{z}$ fulfills the same mode equation as the fermionic $u$-modes (329)

$$L^\dagger L \vec{z}_n = \omega_n^2 \vec{z}_n \quad , \quad E_n^2 = \omega^2 + \ell^2, \tag{360}$$

which we thus identify, $\vec{z}_m \equiv \tilde{u}_{mks}$. Again we have to take the normalized (tilded) modes, so that the propagator and the free generating functional read

$$\Delta(x - x') = \sum\!\!\!\!\!\!\int_{mks} dk \int \frac{dp^d}{(2\pi)^d} \, \frac{-i\ \tilde{u}_{mks}(x)\tilde{u}^\dagger_{mks}(x')\ e^{ip_a(x-x')^a}}{p^2 + \omega_{ms}^2(k) - i\varepsilon} \tag{361}$$

$$Z_0[\bar{J}_k, J_i] = e^{-\int J_i \Delta_{ik} \bar{J}_k} \quad , \quad M_{ik}\Delta_{kj}(x - x') = i\delta_{ij}\delta(x - x'). \tag{362}$$

### 5.8.5 Asymptotic propagator

We now consider the asymptotic propagator $\mathbf{\Delta}$ (361) and thus asymptotic solutions for the $u$ modes, which are algebraically related to the $v$-modes (346). We know already that compared to the asymptotically trivial $v_{mks=2}$ modes the charged $v_{mks=1}$ modes have asymptotically an additional phase (342), i.e.

$$\tilde{v}_{mk1} = e^{in_\tau\theta}\tilde{v}_{mk2} \Rightarrow D_-\tilde{v}_{mk1} \to e^{in_\tau\theta}\partial_-\tilde{v}_{mk2} . \tag{363}$$

So through the algebraic relation (346) we can asymptotically express the propagator (361) in terms of the vacuum modes $v_{mk2} = v_{mk}$. Since the frequencies $\omega$ are independent of the "spin" $s$ we can carry out the $s$-sum and obtain for the asymptotic propagator:

$$\Delta(x-x') \to$$
$$\sum\!\!\!\!\!\!\int_{mk} dk \int \frac{dp^d}{(2\pi)^d} \frac{-i\, \tilde{v}_{mk}(x)\tilde{v}^*_{mk}(x')\; e^{ip_a(x-x')^a}}{p^2+\omega^2_{ms}(k)-i\varepsilon} \begin{pmatrix} e^{in_\tau(\theta-\theta')} & 0 \\ 0 & 1 \end{pmatrix}. \tag{364}$$

Here we have used that for the vacuum modes asymptotically $\partial_\pm \tilde{v}_{mk2} \to ik_\pm v_{mk2}$. Up to the extra phase factor for the $\eta-\bar{\eta}$ component this propagator coincides with the the $\alpha_{a=0,3}$ propagator (353) This also reflects the result, that the fluctuation operator (322) becomes asymptotically diagonal

$$L^\dagger L \to \begin{pmatrix} -D^2_{i|\mathrm{asym}}+M^2 & 0 \\ 0 & -\partial_i^2+M^2 \end{pmatrix}, \tag{365}$$

and is thus exactly the same operator as the asymptotic $LL^\dagger$ operator (330).

### 5.8.6 Ghost propagator

The last but trivial thing to do is to write down the ghost propagator. The ghost Lagrangian (297) quadratic in quantum fields is given by

$$\mathcal{L}^{(2)}_{gh} = b(\Box - 2e^2|\varphi|^2)c, \tag{366}$$

and thus the fluctuations equation are the same as for the $\alpha_{a=0,3}$ components (352). The propagator and the free generating functional is given by

$$Z_0[\beta,\kappa] = \int \mathcal{D}c\; b\; e^{i\int(\mathcal{L}^{(2)}_{gh}+\beta b+\kappa c)} = e^{\int \kappa\Delta\beta} \tag{367}$$
$$(\Box - 2e^2|\varphi|^2)\Delta(x-x') = i\delta(x-x') \tag{368}$$
$$\Delta(x-x') = \Delta_{1,1}(x-x'), \tag{369}$$

where $\Delta_{1,1}(x-x')$ is the 11-component of the propagator (353).

### 5.8.7 Bound states and zero modes

The discrete spectrum of the exact (non-asymptotic) fluctuation operators (321,322) cannot be obtained by the asymptotic equations. Only the continuous (scattered) spectrum is accessible in this approximation. The discrete spectrum is non-perturbative in this respect. For the computation of one-loop corrections in susy-preserving dimensional regularization this makes no problems. Zero-modes are massless modes due to the extra momentum and give only scale-less contributions, which vanish in dimensional regularization. Contributions of discrete bosonic and fermionic bound states are matched exactly by supersymmetry to cancel each other, since our regularization procedure respects supersymmetry. It is only in the continuous spectrum that fermionic and bosonic contributions may have different spectral densities due to the nontrivial background.

Zero modes will be of special interest in the context of multiplet shortening which we will discuss at the end of this section.

### 5.9 Interaction Lagrangian in the vortex background

For completeness we write down the interaction Lagrangian in the background. The bosonic Lagrangian including the gauge fixing term contains the following interaction Lagrangian:

$$\begin{aligned}\mathcal{L}_B^{(I)} + \mathcal{L}_{GF}^{(I)} &= \frac{e^2}{4}\,|\eta|^4 + e^2\alpha^2|\eta|^2 \\ &\quad + e^2\varphi(\bar\eta|\eta|^2 + \bar\eta\alpha^2 + c.c.) + ie\alpha^m(\bar\eta D_m\eta - \eta\bar D_m\bar\eta). \quad (370)\end{aligned}$$

The first line gives background independent four-vertices, with four scalars and two scalars- two gauge fields interactions. The second line gives background dependent three-vertices. The first term, a three scalar interaction, depends on the scalar field background where the second term, a two scalar-one gauge field vertex, depends on the gauge field background. The ghost Lagrangian (297) contributes the following interactions:

$$\mathcal{L}_{gh}^{(I)} = -e^2(\varphi\; bc\eta + \bar\varphi\; bc\bar\eta). \quad (371)$$

These are scalar-ghost-ghost three-vertices depending on the scalar field background. And finally the fermionic interaction Lagrangian reads as

$$\mathcal{L}_F^{(I)} = -e\alpha_m\bar\psi\bar\sigma^m\psi + i\sqrt{2}e(\lambda\psi\bar\eta - \bar\lambda\bar\psi\eta), \quad (372)$$

which is the usual gauge field and Yukawa interaction, completely independent of the background fields.

### 5.10 Mass and central charge corrections

The expressions for the central charge and stress tensor can be constructed from the classical action without any gauge artifacts. However, when one evaluates one-loop corrections, one uses the gauge-fixing term to obtain propagators and well-defined fluctuation equations, and one has to expand the classical Hamiltonian according the fluctuations depending on the gauge-fixing term. Alternatively we can consider trace of the time-evolution operator (the spectral function, see e.q. [145]) by integrating all fields in the path-integral with the action of the gauge-fixed quantum Lagrangian $\mathcal{L}_Q$ (277). This gives the sums over zero-point energies including unphysical degrees of freedom and Faddeev-Popov ghosts. This can be done in a well-defined manner by using dimensional regularization by dimensional reduction from the 3+1 dimensional model. Using this method, the central charge contains the standard 2+1 dimensional terms and, as a potential anomalous contribution, a remainder from the momentum operator in the extra spatial dimension.

### 5.10.1 Mass

In the $\xi = 1$ gauge the sum over zero-point energies is formally

$$\begin{aligned}\frac{1}{2}\sum \omega_{\rm bos} - \frac{1}{2}\sum \omega_{\rm ferm} &= \sum \omega_\eta + \sum \omega_{\alpha+} - \sum \omega_U - \sum \omega_V \\ &= \sum \omega_U - \sum \omega_V, \end{aligned} \tag{373}$$

where the quartet $(\alpha_3, \alpha_0, b, c)$ cancels separately. (Note that in (373) all frequencies appear twice because all fields are complex).

Using dimensional regularization as developed in section 4 the difference of mode sums (373)can be written in terms of the difference of the densities of $u$- and $v$-modes which are governed by the operators $L^\dagger L$ and $LL^\dagger$. The modes are related to each other up to zero modes by the susy-quantum mechanical relation (327-330). In dimensional regularization, where the zero-mode contributions continue to give zero because scaleless integrals vanish. Analogously to section 4 we thus obtain for the mode contribution to the energy

$$\frac{1}{2}\sum \omega_{\rm bos} - \frac{1}{2}\sum \omega_{\rm ferm} =$$
$$\sum\!\!\!\!\!\!\int_{mks} dk \int \frac{d^{d-1}\ell}{(2\pi)^{d-1}} E_n \int dx^2 \, \{|\tilde{u}_{mks}|^2 - |\tilde{v}_{mks}|^2\}, \tag{374}$$

where $E_n$ are the mode energies given in (327). The spatial integral again gives the difference in the spectral densities. We show in the next subsection, when computing the central charge correction, that this time the difference in the spectral density vanishes. This can be done in a very elegant manner by reducing the spatial integral to a surface term. The vanishing result confirms the existing literature that proves that $L^\dagger L$ and $LL^\dagger$, which govern $U$ and $V$, respectively, are isospectral up to zero modes [97,119], but in these investigations a proper regularization is lacking. A rigorous treatment was carried out just recently using the heat-kernel approach [135] confirming the cancellation of mode contributions.

We can therefore conclude that in the $\xi = 1$ gauge there is a complete cancellation of the sums over zero-point energies. All that remains is the finite renormalization of $\delta v^2$:

$$E = 2\pi|n|v_0^2 = 2\pi|n|(v^2 + \delta v^2|_{\xi=1}) = 2\pi|n|(v^2 - \frac{m}{4\pi}) \equiv |n| \left(\frac{\pi m^2}{e^2} - \frac{m}{2}\right) \tag{375}$$

(In gauges other than $\xi = 1$ the fluctuation equations for the fields $\eta, \alpha_+$, no longer match those of the $U$ fermions.)

This result agrees with [135], where a careful analysis of boundary conditions in the heat-kernel approach was carried out because the vortex had to be put in a box to discretize the spectrum. In dimensional regularization one does not need to put the system in a box, and as a consequence there is no need to study the contributions from artificial boundaries.

### 5.10.2 Central charge

We now calculate the central charge by starting from the susy algebra in 3+1 dimensions (268) and dimensionally reduce to 2+1 dimensions. The in this way susy preserving regulated central charge is then obtained from the 03-component of the four dimensional energy momentum tensor $T_{mn}$, i.e. the momentum operator for the flat extra dimension. The *antisymmetric* part (270) of $T_{mn}$ gives the standard expression for the central charge density (253), while the symmetric part is a genuine momentum in the extra dimension:

$$\langle Z \rangle = \int d^2x \langle T_{03} \rangle = \langle \tilde{Z} + \tilde{P}_3 \rangle. \tag{376}$$

The naive/non anomalous central charge density (253,270),

$$\zeta = \partial_i \varepsilon_{ij} [e v_0^2 A_j - i\bar{\phi} D_j \phi], \tag{377}$$

is a total divergence and thus the quantum corrections to $\tilde{Z}$ can be evaluated at spatial infinity. To do so, we renormalize $\zeta$ ($v_0^2 = v^2 + \delta v^2$) and expand the fields in (377) around the vortex background (332), ($\mathrm{d}_j := \partial_i \varepsilon_{ij}$) and obtain:

$$\begin{aligned} \zeta =\ & \mathrm{d}_j [e v^2 \mathcal{A}_j - i\bar{\varphi} D_j \varphi] + e\delta v^2 \mathrm{d}\mathcal{A} - i\mathrm{d}_j(\bar{\eta} D_j \eta) && (378)\\ & -\mathrm{d}_j(U\alpha_j) - i\mathrm{d}_j(\bar{\varphi} D_j \eta + \bar{\eta} D_j \varphi) - e\mathrm{d}_j(\bar{\varphi}\eta\alpha_j + \varphi\bar{\eta}\alpha_j), && (379) \end{aligned}$$

where we have omitted the terms $\sim \delta v^2 \alpha_j$ and $\sim \alpha_j \eta\bar{\eta}$ which contribute foremost to order $O(\hbar^2)$. The covariant derivative is now again defined w.r.t. the background, $D_j = \partial_j - ie\mathcal{A}_j$, and $U = e(|\varphi|^2 - v^2)$ is the prepotential. Using the asymptotic properties of the vortex solution (237), the three terms in the second line vanish at the boundary and may contribute only to the local central charge density. For the first term this is obvious, since $U \to 0$. The second term can be written as

$$-i\mathrm{d}_j[\partial_j(\bar{\varphi}\eta) - \bar{D}_j\bar{\varphi}\eta + D_j\varphi\bar{\eta}] \to -i\varepsilon_{ij}\partial_i\partial_j(\bar{\varphi}\eta) = 0, \tag{380}$$

since $D_j\varphi \to 0$. From the interaction Lagrangian (370) one can see that one-loop contributions from the third term can come only from mixed $\alpha -$

$\eta$ tadpoles of the propagator (361). But asymptotically this propagator becomes diagonal (364), so that there are no mixed tadpoles at the boundary. So for the naive central charge to one loop order we obtain

$$
\begin{aligned}
\langle \tilde{Z} \rangle &= \int dx^2 \, \langle \zeta \rangle \\
&= 2\pi n_\tau \{ v_0^2 - \langle \bar{\eta}\eta \rangle |_{r\to\infty} \} - i \int_0^{2\pi} d\theta \, \{ \langle \bar{\eta} \partial_\theta \eta \rangle \} \, |_{r\to\infty} \\
&\equiv Z_a + Z_b.
\end{aligned} \tag{381}
$$

The first contribution, $Z_a$, can be easily evaluated for arbitrary gauge parameter $\xi$, yielding

$$
\begin{aligned}
Z_a &= 2\pi n \{ v_0^2 - \frac{1}{2} ( \langle \sigma\sigma \rangle + \langle \rho\rho \rangle ) |_{r\to\infty} \} \\
&= 2\pi n \{ v_0^2 - \frac{1}{2} [ I(m) + I(\xi^{\frac{1}{2}} m) ] \} \\
&= 2\pi n ( v_0^2 - \delta v^2 ) = 2\pi n v^2 .
\end{aligned} \tag{382}
$$

If this was everything, this would correspond to a cancellation of all quantum corrections to $Z$, just as in the naive calculation of $Z$ in the susy kink [84, 113]. The second contribution in (381), however, does not vanish when taking the limit $r \to \infty$. In the trivial vacuum such a term would vanish by symmetric integration, but due to the long range force of the vortex gauge field background, the charged field $\eta$ does even at the boundary feel the vortex. This results in a additional $\theta$-dependence compared to vacuum modes and propagator, respectively, as discussed in (341,364), also asymptotically. This contribution is simplest in the $\xi = 1$ gauge, where the asymptotic $\eta$-propagator is given by (364). We thus have, in the $\xi = 1$ gauge,

$$
Z_b = -i \int_0^{2\pi} d\theta \, \langle \bar{\eta} \partial_\theta \eta \rangle_{\xi=1} = 2\pi n \, \langle \bar{\eta}\eta \rangle_{\xi=1, r\to\infty} = 2\pi n \, \delta v^2 \Big|_{\xi=1}, \tag{383}
$$

where we have used, that the term where $\theta$-derivative does not act on the extra phase in (364) gives the same as in the trivial vacuum and is thus zero by symmetric (momentum)-integration. This is exactly the result for the one-loop correction to the mass of the vortex in eq.(375), implying saturation of the BPS bound provided that there are now no anomalous contributions to the central charge operator as there are in the case in the $\mathcal{N} = 1$ susy kink [116].

In dimensional regularization by dimensional reduction from a higher-dimensional model such anomalous contributions to the central charge operator come from a finite remainder of the genuine extra momentum operator, contained in the symmetric energy momentum tensor $\Theta_{mn}$ (268):[43]

$$
\begin{aligned}
Z_c = \langle \tilde{P}_3 \rangle &= \int d^2x \, \langle \Theta_{03} \rangle \\
&= \int d^2x \, \langle F_{0i}F_{3i} + D_{(0}\phi \bar{D}_{3)}\bar{\phi} + i\lambda\sigma_{(0}\partial_{3)}\bar{\lambda} + i\bar{\psi}\bar{\sigma}_{(0}\partial_{3)}\psi \rangle \, . \qquad (384)
\end{aligned}
$$

Expanding the fields around the vortex solution (247) one obtains for the bosonic part of (384)

$$
\begin{aligned}
\langle \tilde{P}_3^{\text{bos}} \rangle = \int d^2x \, \langle &\partial_0\alpha_i\partial_3\alpha_i + 2e^2|\varphi|^2\alpha_0\alpha_3 + \\
&\partial_0\eta\partial_3\bar{\eta} + ie[\bar{\varphi}\alpha_3\partial_0\eta - \varphi\alpha_0\partial_3\bar{\eta} + (0 \leftrightarrow 3)] \rangle + O(\hbar^2). \qquad (385)
\end{aligned}
$$

Because of the symmetry in the extra momentum of the bosonic propagators, as discussed below (268), the terms containing derivatives in (385) vanish. At one-loop order the second term could contribute only a $\alpha_0 - \alpha_3$-tadpole, but this propagator is even in the bulk of the background diagonal (353). Thus we have $\langle \tilde{P}_3^{\text{bos}} \rangle = 0$. However, the fermionic contribution does not obviously vanish. Grouping $W = \binom{U}{V}$, where $U, V$ are given below (318), the fermionic part of (385) can be written as ($\partial_\tau := -\partial_0 + \partial_3$ and $\bar{\partial}_\tau := -\partial_0 - \partial_3$):

$$
\langle \tilde{P}_3^{\text{ferm}} \rangle = \frac{i}{2} \int d^2x \; Tr \left\langle W^\dagger \begin{pmatrix} \bar{\partial}_\tau & 0 \\ 0 & -\partial_\tau \end{pmatrix} W \right\rangle . \qquad (386)
$$

Using $\bar{\partial}_\tau \; e^{i(p_a x^a)} = i(p_0 - \ell) \; e^{i(p_a x^a)}$ and $-\partial_\tau \; e^{i(p_a x^a)} = -i(p_0 + \ell) \; e^{i(p_a x^a)}$ and the propagator (349) one obtains

$$
\langle \tilde{P}_3^{\text{ferm}} \rangle = \frac{i}{2} \sum\!\!\!\!\!\!\int_{mks} dk \int \frac{dp^d}{(2\pi)^2} \frac{p^2}{p^2 + \omega^2(k) - i\varepsilon} \int dx^2 \; \{|\tilde{u}_{mks}|^2 - |\tilde{v}_{mks}|^2\}. \qquad (387)
$$

Here again is $d = 2 - \epsilon$ and $p^2 = -p_0^2 + \ell^2$. The mode energies $\omega^2 = k^2 + M^2$ (343) refer to the nontrivial fluctuation equations. Note that without the normalization factors due to dimensional regularization in (350) the numerator in (387) would be linear in $p_0$ and $\ell$, and thus this momentum integration would give zero due to symmetric integration. For zero modes, i.e $\omega^2 = 0$ the momentum integration of the trivial dimensions ($p_a$) is scale-less and thus zero in dimensional regularization. Also would this term not be present without the momentum operator for this extra dimension. In the case for the kink this would mean the there is no anomalous contribution to the central charge. Starting already in the lower dimension a possible anomaly

[43]For later convenience we have, compared to (268), re-written the $\lambda$-terms in a form which differs by a total derivative in the trivial directions $x_0, x_3$.This is valid because the nontrivial propagator (349) does not depend on these two coordinates for $x' = x$, i.e $\partial_{0,3}\Delta(x, x) = 0$.

comes from an evanescent counter term in the vacuum sector. Anyhow, even in that case the dimensional reduction should take place only in spatial directions, so that the charges associated with currents containing evanescent counter terms do not change (see footnote 6 on page 14 of [60]).

Now it remains to calculate (387). The spatial integral gives the difference in the spectral densities of the $u$- and $v$-modes. At first sight this looks rather hard to calculate, since the background and thus also the modes are not explicitly known. In [97] it has been shown that it is possible to carry out a phase shift analysis by studying the asymptotics for the mode function. However, because of the susy-quantum mechanics (346) inherent in fermionic fluctuation operators in a BPS background we are able to convert the $x^2$-integral into a surface term. With $\tilde{u}_{mks} = \frac{1}{\omega(k)} L^\dagger \tilde{v}_{mks}$ we get for the spectral density in (387):

$$\Delta\rho_m(k) := \sum_{s=1,2} \int dx^2 \, \left\{ \frac{1}{\omega^2(k)} (L^\dagger \tilde{v}_{mks})^\dagger (L^\dagger \tilde{v}_{mks}) - |\tilde{v}_{mks}|^2 \right\}$$

$$= \sum_{s=1,2} \int dx^2 \, \left\{ \frac{1}{\omega^2(k)} \tilde{v}^\dagger_{mks} L L^\dagger \tilde{v}_{mks} - |\tilde{v}_{mks}|^2 \right\} + \text{surface term.} \quad (388)$$

The integral vanishes because $\tilde{v}_{mks}$ are the eigen modes of the operator $LL^\dagger$ (346). The emergence of the surface term seems surprising, since $L^\dagger$ was said to be the hermitian conjugate of the operator $L$. This is true on the space of mode (eigen) functions $\tilde{u}_{mks}, \tilde{v}_{mks}$, which are asymptotically oscillating functions of trivial topology. But in (388) one has to bring by conjugation $L^\dagger$ from $\tilde{v}_{mks}$ to $L$ acting on functions $(L^\dagger \tilde{v}_{mks})$, and the latter contains functions of nontrivial topology due to the vortex background which contribute non-vanishing surface terms. Since the $v$-modes (331) are diagonal we obtain with (320):

$$L^\dagger \tilde{v}_{mk1} = \begin{pmatrix} D_- \tilde{v}_{mk1} \\ -\sqrt{2} e \bar{\varphi} \tilde{v}_{mk1} \end{pmatrix} \quad , \quad L^\dagger \tilde{v}_{mk1} = \begin{pmatrix} \sqrt{2} e \varphi \tilde{v}_{mk2} \\ -\partial_+ \tilde{v}_{mk2} \end{pmatrix} . \quad (389)$$

Only derivative terms can contribute to the surface term and only the covariant derivative in the first component for $s = 1$ contains a function of nontrivial topology. So we obtain for (388) (see appendix E for holomorphic coordinates)

$$\Delta\rho_m(k) = -ie \int dx^2 \, \partial_+ (\mathcal{A}_- |v_{mk1}|^2) \quad (390)$$

$$= \int_0^{2\pi} d\theta \, \left[ (n_\tau - a(r)) |v_{mk1}|^2 \right]_{r=0}^{\infty}$$

$$+ i \int_0^\infty dr \, \tfrac{n_\tau - a(r)}{r} \int_0^{2\pi} d\theta \, \partial_\theta |v_{mk1}|^2 . \quad (391)$$

The first integral in the second equality vanishes, since for $r \to 0$ one has $a(r) \to n$ and $|v_{mk1}|^2$ is regular, while $a(r)$ vanishes for $r \to \infty$ but all cylinder functions behave like $|v_{mk1}|^2 \sim \frac{1}{r}$ [95]. The second integral vanishes since for all modes $|v_{mk1}|^2$ is either $2\pi$-periodic or independent of $\theta$. So the difference in the spectral density cancels up to zero-modes which contribute only scale-less integrals and thus vanish in dimensional regularization. This confirms the phase shift analysis by Lee and Min [97]. Hence there is no anomalous contribution to the central charge,

$$Z_c = \langle \tilde{P}_3^{\mathrm{ferm}} \rangle \sim \Delta\rho_m(k) = 0. \tag{392}$$

The BPS bound is indeed saturated at the (one-loop) quantum level:

$$|Z| = |Z_a + Z_b| = E. \tag{393}$$

Thus contrary to the susy kink the contribution from the momentum in the extra dimension vanishes, although in both cases this contribution is proportional to a surface term due to the topological background. The difference between these two cases is that the spatial boundary of the nontrivial dimension(s) $\partial\mathcal{M}$ is in the kink and kink-domain wall case non-compact, namely two discrete points $\partial\mathcal{M} = \{x = \pm\infty\}$, but for the vortex it is the compact manifold $\partial\mathcal{M} = S^1$. Future investigations will show whether this compactness of the spatial boundary manifold will also lead to a cancellation of anomalous contributions for the four dimensional monopole.

### 5.11 Zero modes and multiplet shortening

Massive representations of the Poincaré supersymmetry algebra for which the absolute value of the central charge equals the energy, i.e. when the BPS bound is saturated, contain as many states as massless representations, which in 2+1 dimensions for the $\mathcal{N} = 2$ super-Poincaré algebra [97]is half of that of massive representations for which the BPS bound is not saturated.

In subsection 5.5.2 we have seen that the vortex background is invariant under susy transformations with parameters $\epsilon_1 = 0 = \bar{\epsilon}_{\dot{1}}$. The corresponding supercharges are obtained from (261) (note that $\epsilon_1 = -\epsilon^2$):

$$Q_1 = \int dx^2\ J_1^0 \ , \quad \bar{Q}_{\dot{1}} = Q_1^\dagger = -\int dx\ \bar{J}_{\dot{1}}^0 \ , \tag{394}$$

where the currents are given in (262) and we have used the fact that $(\bar{J}_{\dot{1}}^m)^* = -J_1^m$. The invariance of the vortex solution under these transformations implies that the perturbative ground state in the topological sector is annihilated, at least semi-classical, by this supercharges

$$Q_1|v\rangle = 0 = \bar{Q}_{\dot{1}}|v\rangle. \tag{395}$$

From the susy algebra (267) follows with the definition (270) for the central charge

$$\{\bar{Q}_{\dot{1}}, Q_1\} = 2(H - Z) \quad , \quad \{\bar{Q}_{\dot{2}}, Q_2\} = 2(H + Z), \tag{396}$$

so that the invariance (395) implies BPS saturation, i.e. $\langle H\rangle = \langle Z\rangle$.

The residual susy-transformations of the vortex solution ($D_+\varphi = 0$, $F_{12} = -U$) with parameters $\epsilon_1$, $\bar{\epsilon}_{\dot{1}} \neq 0$ lead to fermionic zero-modes (271, 272):

$$\lambda^0_\alpha = 2U\begin{pmatrix}\epsilon_1\\0\end{pmatrix} \quad , \quad \bar{\lambda}^0_{\dot{\alpha}} = 2U\begin{pmatrix}\bar{\epsilon}_{\dot{1}}\\0\end{pmatrix} \tag{397}$$

$$\psi^0_\alpha = \sqrt{2}D_-\varphi\begin{pmatrix}\bar{\epsilon}_{\dot{1}}\\0\end{pmatrix} \quad , \quad \bar{\psi}^0_{\dot{\alpha}} = \sqrt{2}(D_-\varphi)^*\begin{pmatrix}\epsilon_1\\0\end{pmatrix}, \tag{398}$$

which thus appear as complex conjugated pairs. The pair $(\psi_1, \bar{\lambda}^{\dot{2}})$ and it is complex conjugated form a $U$-zero-mode of the operator $L^\dagger L$. The vortex ground state is thus degenerated and the nontrivial susy charges in (396) form a two dimensional representation in the space spanned by

$$|v_{\mathrm{b}}\rangle \quad , \quad |v_{\mathrm{f}}\rangle = a^\dagger|v_{\mathrm{b}}\rangle, \tag{399}$$

where $a^\dagger$ is the creation operator of the fermionic $U$-zero-mode. This is a short multiplet, its half as long as the massive irreducible representation, and therefore the BPS saturation condition (395) is protected against perturbative corrections, as discussed in section 4.

However, if there indeed is a second fermionic zero mode in the model as postulated in [96, 97], in second quantization it would be present in the mode expansion of the fermionic quartet $U$ and $V$,

$$\begin{pmatrix}U\\V\end{pmatrix} = \left[a_{\mathrm{I}}\begin{pmatrix}u_{\mathrm{I}}\\0\end{pmatrix} + a_{\mathrm{II}}\begin{pmatrix}u_{\mathrm{II}}\\0\end{pmatrix} + c.c\right] + \text{non-zero modes}. \tag{400}$$

As a result, there would then be a further degeneracy, namely a quartet of BPS states

$$|v\rangle,\ a^\dagger_{\mathrm{I}}|v\rangle,\ a^\dagger_{\mathrm{II}}|v\rangle,\ a^\dagger_{\mathrm{I}}a^\dagger_{\mathrm{II}}|v\rangle \tag{401}$$

comprising two short multiplets of $\mathcal{N} = 2$ susy, which together have as many states as one long multiplet without BPS saturation. As stressed in [97], the standard argument for stability of BPS saturation under quantum corrections from multiplet shortening [146] thus would not be applicable.

However, we shall now show that there is in fact only a single fermionic zero mode in a vortex background with winding number $n = 1$. To this end, we first observe that the zero modes must lie in $U$, because $V$ is governed by the operator $LL^\dagger$ of Eq. (321), whose only zero mode solution is $V_0 \equiv 0$. A zero mode for $U$ must satisfy $LU = 0$, and to analyze this equation we follow [97] and set $\psi_1(x,y) = e^{i(j-\frac{1}{2}+n)\theta}u(r)$ and $\bar{\lambda}^{\dot{2}} = e^{i(j+\frac{1}{2})\theta}d(r)$. The equation $LU = 0$ reduces then to

$$\begin{pmatrix} \partial_r - \frac{a+j-\frac{1}{2}}{r} & \sqrt{2}ef \\ \sqrt{2}ef & \partial_r + \frac{j+\frac{1}{2}}{r} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = 0, \tag{402}$$

where $f = f(r)$ and $a = a(r)$ satisfy $f' = \frac{a}{r}f$ and $a' = re^2(f^2 - e^2)$. Iterating this equation yields

$$\left( \partial_r^2 + \frac{1}{r}\partial_r - \frac{(j-\frac{1}{2})^2}{r^2} - 2e^2f^2 \right) \frac{u}{f} = 0. \tag{403}$$

Given a solution for $u$, the corresponding solution for $d$ follows from $LU = 0$.

For given $j$, this equation has two independent solutions, a linear combination of which yields solutions which decrease exponentially fast as $r \to \infty$. Hence, *both* solutions should be regular at $r = 0$. For $j \neq \frac{1}{2}$, one has, using $f(r \to 0) \sim r^n$,

$$\psi^1 \sim u \sim r^n (C_1 r^{j-\frac{1}{2}} + C_2 r^{-(j-\frac{1}{2})}) \quad \text{for } r \to 0 \tag{404}$$

which selects for $n = 1$ only $j = -\frac{1}{2}$. This solution is the zero mode that is obtained by transforming the background solutions (397,398). For $j = \frac{1}{2}$, one finds for $n = 1$ near$r = 0$

$$\psi^1 \sim C_1\,(x+iy) + C_2\,(x+iy)\ln r\,. \tag{405}$$

For large $r$, $\psi_1 \sim e^{-mr}e^{i\theta}$, as follows from (403). This solution corresponds to the second fermionic zero mode postulated in Ref. [97].

However, while (405) is regular at the origin, the associated gaugino component is not: (402) implies that

$$\bar{\chi}^{\dot{1}} \sim C_2 \frac{e^{i\theta}}{r}, \tag{406}$$

so this solution has to be discarded when $C_2 \neq 0$.

Similarly, one can show that for winding number $n > 1$ regularity of the gaugino component generically requires that $j \leq -\frac{1}{2}$ so that the correct quantization condition for normalizable fermionic zero modes is $-n + \frac{1}{2} \leq j \leq -\frac{1}{2}$. Hence, there are $n$ independent fermionic zero modes, not $2n$ as concluded in [97]. It is in fact only the former value that agrees with the results [80, 96] obtained from the index theorem [139]. While it is true, as remarked in [96], that in a particular gauge the bosonic zero modes, of which there are $2n$, satisfy a set of equations equivalent to those for the fermionic zero modes, it also has to be noted that the linearly dependent solutions $\binom{U}{0}$ and $i\binom{U}{0}$ correspond to linearly independent solutions for the bosonic zero modes $\delta A$ and $\delta\phi$ (for an analogous case see eq. (3.8) of Ref. [138]). There are therefore only half as many fermionic zero modes than there are bosonic ones.

We thus conclude that for the basic vortex (winding number $n = 1$) there is exactly one fermionic zero mode and this gives rise to a single short multiplet at the quantum level. Standard multiplet shortening arguments therefore do apply and explain the preservation of BPS saturation that we verified at one-loop order.

# 6 Conclusions

Topological static classical solutions represent an important part of the spectrum of a quantum field theory, especially concerning dualities as for example mirror symmetry. The existence of a topological conservation law guarantees the stability of these states in the quantum theory. We have seen that BPS states have the special property that there exists an relation between the mass and the central charge of these states, which can be protected against higher order corrections. Nevertheless the mass and the central charge both may receive quantum corrections. Only for models providing a non-renormalization theorem the individual operators may be protected against nontrivial corrections.

We have introduced a rather simple and elegant variant of dimensional regularization by embedding the nontrivial BPS background in a higher dimensional model, which allows us to compute quantum corrections without the need of discussing artificial boundary conditions and spurious boundary energies. We observe two different sources for nontrivial corrections to the central charge of the investigated BPS states. For $\mathcal{N} = 1$ kink and kink-domain wall states the central charge corrections are obtained as an anomalous contribution from the momentum operator in the extra-dimension, whereas for the $\mathcal{N} = 2$ vortex the anomalous contribution to the central charge vanishes. However, a nontrivial renormalization in the vacuum sector, which is needed for consistency reasons and gauge invariance, leads to a nontrivial correction of the mass and central charge. In all considered cases we are able to formulate the anomalous contribution as a surface term, because the Dirac operators form a susy-quantum mechanical system in the BPS background. In the case of a non-vanishing anomalous contribution the correction is related to an infrared property of the theory, namely the topology of the BPS background. It is typical for anomalies that they emerge in UV-calculation but at the same time have an IR origin. The second source for the correction is in contrast to the anomalous contribution a purely UV-effect. An important difference between the two cases is that the boundary of the nontrivial spatial directions is in the anomalous case non-compact, namely the two points at spatial infinity, where for latter the spatial boundary is the compact manifold $S^1$ and surface term vanish. Therefore we conjecture that also for the four-dimensional monopole, where the spatial boundary is also compact, the anomalous contribution vanishes.

For all considered cases we observe BPS saturation at the one-loop level which is understood as a result of multiplet shortening. In the two dimensional case the multiplet shortening results in a single-state super-multiplet which forms a curious ground state in the topological sector, which does not have a definite fermion parity in the usual sense. But as we have shown also in this case exists a well defined parity operator.

# A One-loop integrals in dimensional regularization

## A.1 Feynman parameters

With the Feynman parameterization

$$\frac{1}{AB} = \int_0^1 \frac{1}{[xA + (1-x)B]^2} \tag{407}$$

typical denominators of one-loop integrands containing two propagators are rewritten as

$$\begin{aligned}
\frac{1}{k^2+m^2-i\varepsilon}\,\frac{1}{(p+k)^2+m^2-i\varepsilon} &= \\
&= \int_0^1 \frac{dx}{[(k+(1-x)p)^2-(1-x)^2p^2+(1-x)p^2+m^2-i\varepsilon]^2}, \\
&= \int_0^1 \frac{dx}{[k'^2+x(1-x)p^2+m^2-i\varepsilon]^2},
\end{aligned} \tag{408}$$

where in the last equality we have substituted $k' = k+(1-x)p \to \int d^Dk' = \int d^Dk$.

## A.2 Wick rotation

To evaluate the $k_0$-integration along the contour determined by the pole prescription $i\varepsilon$ we close the contour in the first and third quadrant, so that no pole is enclosed. Assuming that the integrand vanishes on the auxiliary contour, i.e. for $|k_0| \to \infty$, one gets for integrands regular in first and second quadrant ($i\varepsilon$-pole prescription). For $F|_{\mathcal{C}_{aux}} \to 0$:

$$\int_{-\infty}^{\infty} dk^0 F(\cdots + i\varepsilon) = \int_{-i\infty}^{i\infty} dk^0 F(\dots) = i\int_{-\infty}^{\infty} dk_E^0 F(ik_E^0 \dots), \tag{409}$$

where in the last equality we substituted $k^0 = ik_E^0$.

## A.3 't Hooft-Veltman-Euclidean integrals

The magic of dimensional regularization is to evaluate the integrals in $D$ dimensions and analytical continue the result as a function of $D$:

$$\begin{aligned}
\int \frac{d^D\ell_E}{(2\pi)^D}\frac{1}{(\ell_E^2+\Delta)^n} &= \frac{1}{(4\pi)^{D/2}}\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}(\Delta)^{\frac{D-2n}{2}} \\
\int \frac{d^D\ell_E}{(2\pi)^D}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} &= \frac{1}{(4\pi)^{D/2}}\frac{D}{2}\frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)}(\Delta)^{\frac{D-2n+2}{2}}
\end{aligned} \tag{410}$$

**Gamma function** $\Gamma(z)$ The Gamma function is uniquely defined over the complex plane $\mathbb{C}$ and has *simple* poles at $z = 0, -1, -2, \ldots$. It is a solution of the functional equation (recursion)

$$\Gamma(z+1) = z\Gamma(z). \tag{411}$$

Special values and expansions are

$$\begin{aligned} \Gamma(1) = 1\,, \quad \Gamma(\tfrac{1}{2}) &= \sqrt{\pi}\,, \quad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi} \\ \Gamma(\frac{\epsilon}{2})|_{\epsilon\to 0} &= \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \\ \Gamma(-1+\frac{\epsilon}{2})|_{\epsilon\to 0} &= -\frac{2}{\epsilon} - 1 + \gamma + \mathcal{O}(\epsilon), \end{aligned} \tag{412}$$

where $\gamma$ is the Euler-Mascheroni constant.

## A.4 One loop integrals for Majorana fermions

We give here only the fermionic one loop contributions. The needed bosonic one loop integrals are contained in the fermionic ones.

### A.4.1 Tadpole

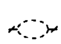
Using the Feynman rules section 2 the fermionic tadpole-graph is given by:

$$\begin{aligned} &= \underbrace{\frac{1}{2}}_{S} (-i\sqrt{2\lambda})(-1)\; Tr \int \frac{d^D p}{(2\pi)^D} \frac{(\not{p} - im)}{p^2 + m^2 - i\varepsilon} \\ &= im\sqrt{2\lambda} \int \frac{d^D p_E}{(2\pi)^D} \frac{1}{p_E^2 + m^2} = i2\lambda v m^{D-2} \frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{D/2}}, \end{aligned} \tag{413}$$

where we have used that $Tr(\gamma^\mu) = 0$ and $Tr(\mathbb{1}m) = 2m$. It is important to include a symmetry factor $\frac{1}{2}$ for Majorana fermions.

### A.4.2 3-vertex loop

For the in two dimensions finite two point correction one obtains:



$$\begin{aligned} &= \underbrace{\frac{1}{2}}_{S} (-i\sqrt{2\lambda})^2(-1)\; Tr \int \frac{d^D k}{(2\pi)^D} \frac{(\not{k} - im)}{k^2 + m^2} \frac{(\not{k} + \not{p} - im)}{(k+p)^2 + m^2} \\ &= \lambda \int \frac{d^D k}{(2\pi)^D} \frac{2(k^2 + kp - m^2)}{[k^2 + m^2][(k+p)^2 + m^2]}, \end{aligned} \tag{414}$$

where we have again used $Tr(\gamma_\mu) = 0$ and $Tr(\mathbb{1}) = 1$. Using the Feynman parameterization (407) and the Wick rotation (409) one gets for(414):

$$\begin{aligned} -2\lambda \int_0^1 dx \int \frac{d^Dk}{(2\pi)^D} \frac{k^2 + \Delta_p(x)}{[k^2 - \Delta_p(x) + i\varepsilon]^2} &= \\ -2\lambda \int_0^1 dx \int \frac{d^Dk_E}{(2\pi)^D} \frac{i[-k_E^2 + \Delta_p(x)]}{[k_E^2 + \Delta_p(x)]^2}, & \end{aligned} \tag{415}$$

where we have used that the numerator under the variable transformation used in (408) writes as

$$\begin{aligned} k^2 + kp + m^2 \rightarrow & [k - (1-x)p]^2 + [k - (1-x)p]p + m^2 \\ &= k^2 + x(x-1)p^2 + m^2 + \text{terms linear in } k \\ &= k^2 + \Delta_p(x) + \text{terms linear in } k, \end{aligned} \tag{416}$$

and that the terms linear in $k$ do not contribute. Performing the $k$-integration with the help of (410) we get for (415):

$$2i\lambda \frac{1}{(4\pi)^{D/2}} \left[\tfrac{D}{2}\Gamma(1 - \tfrac{D}{2}) - \Gamma(2 - \tfrac{D}{2})\right] \int_0^1 dx [\Delta_p(x)]^{\frac{D}{2}-1}, \tag{417}$$

with the recursion formula (411) we get finally for the 3-vertex fermion loop:

$$= 2i\lambda \frac{\Gamma(1 - \frac{D}{2})}{(4\pi)^{D/2}} (D-1) \int_0^1 dx \, [x(x-1)p^2 + m^2]^{\frac{D-2}{2}} \tag{418}$$

# B Quantum mechanical susy systems

Here we collect briefly some facts about factorization and isospectral operators, which will be extensively needed for solving fluctuation equations. We follow [33]. Assume an given operator can be factorized as follows

$$H_1 = A^\dagger A. \tag{419}$$

This operator is obviously hermitian and has an complete set of normalized eigen functions

$$H_1 \psi_m^{(1)} = E_m^{(1)} \psi_m^{(1)}, \quad \sum_m \psi_m^{(1)}(x) \psi_m^{(1)\dagger}(x') = \delta(x - x'). \tag{420}$$

Here $m$ stands for a set of quantum numbers and also the symbolic sum in (420) means a suitable summation/integration over this set. Also $\delta(x - x')$ is symbolic for the unit in vector and functional space. There exists an isospectral operator

$$H_2 = AA^\dagger. \tag{421}$$

Which is also obviously Hermitian and thus has an normalized complete set of eigen functions

$$H_2\psi_m^{(2)} = E_m^{(2)}\psi_m^{(2)}, \quad \sum_m \psi_m^{(2)}(x)\psi_m^{(2)\dagger}(x') = \delta(x-x'). \tag{422}$$

The spectra are related as follows: First of all the operators are positive semidefinite:

$$\langle\psi_m^{(1)}|H_1|\psi_m^{(1)}\rangle = E_m^{(1)} = ||A\psi_m^{(1)}||^2 \geq 0, \tag{423}$$

and exactly analogously one obtains the positivity for $H_2$. Now because of the special form of $H_{1,2}$ for *non-zero eigenstates* the eigen- functions and values are related as follows: For each eigen function $\psi_m^{(1)}$ of $H_1$ with non-zero eigenvalue

$$H_1\psi_m^{(1)} = A^\dagger A\psi_m^{(1)} = E_m^{(1)}\psi_m^{(1)}, \tag{424}$$

follows that $A\psi_m^{(1)}$ is an eigen function of the susy-partner Hamiltonian $H_2$

$$H_2(A\psi_m^{(1)}) = AA^\dagger A\psi_m^{(1)} = E_m^{(1)}(A\psi_m^{(1)}). \tag{425}$$

On the other Hand is for each eigen function $\psi_m^{(2)}$ of $H_2$ with non-zero eigenvalue

$$H_2\psi_m^{(2)} = AA^\dagger\psi_m^{(2)} = E_m^{(2)}\psi_m^{(2)}, \tag{426}$$

follows that $A\dagger\psi_m^{(1)}$ is an eigen function of the susy-partner Hamiltonian $H_1$

$$H_2(A^\dagger\psi_m^{(2)}) = A^\dagger AA^\dagger\psi_m^{(2)} = E_m^{(2)}(A^\dagger\psi_m^{(2)}). \tag{427}$$

Thus up to zero-modes *all* eigen functions of $H_1 = A^\dagger A$ are algebraically related to the eigen functions $H_2 = AA^\dagger$ and vice versa. The relations are as follows:

$$E_m^{(1)} = E_m^{(2)} \tag{428}$$

$$\psi_m^{(2)} = \frac{1}{\sqrt{E_m^{(1)}}}\, A\psi_m^{(1)} \tag{429}$$

$$\psi_m^{(1)} = \frac{1}{\sqrt{E_m^{(2)}}}\, A^\dagger\psi_m^{(2)}. \tag{430}$$

The left hand side of (429,430) is normalized if the $\psi$'s on the right hand side are normalized.

# C 4D-superfields and WZ-gauge

Here we set up our spinor and superspace conventions in four dimensions. We closely follow [141] and [9]:
**Space:**

$$\eta_{mn} = (-,+,+,+) \quad , \quad \varepsilon^{0123} = 1 = -\varepsilon_{0123} \tag{431}$$

**Spinors:**

$$\begin{aligned}
&\psi_{\dot\alpha} = (\psi_\alpha)^* \quad , \quad \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \quad , \quad \psi\chi = \psi^\alpha\chi_\alpha \quad , \quad \bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} \\
&\varepsilon^{12} = 1 \quad , \quad \varepsilon^{\alpha\beta}\varepsilon_{\beta\rho} = \delta^\alpha_\rho \quad , \quad \varepsilon_{12} = -1 \\
&\sigma^m_{\alpha\dot\alpha} = (-\mathbb{1}, \vec\sigma)_{\alpha\dot\alpha} \quad , \quad \bar\sigma^{m\dot\alpha\alpha} = (-\mathbb{1}, -\vec\sigma)^{\dot\alpha\alpha}.
\end{aligned} \tag{432}$$

**Grassmann derivatives:**

$$\begin{aligned}
&\partial_\alpha(\theta\chi) = \partial_\alpha\theta\chi - \theta\partial_\alpha\chi \quad , \quad \partial_\alpha\theta^\beta = \delta^\beta_\alpha \quad , \quad \partial^\alpha\theta_\beta = \delta^\alpha_\beta \quad , \\
&\partial^\alpha = -\varepsilon^{\alpha\beta}\partial_\beta \quad , \quad \partial_\alpha = -\varepsilon_{\alpha\beta}\partial^\beta, \\
&D_\alpha = \partial_\alpha + i(\sigma^m\bar\theta)_\alpha\partial_m \quad , \quad \bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i(\theta\sigma^m)_{\dot\alpha}\partial_m
\end{aligned} \tag{433}$$

and the same for $\bar\partial_{\dot\alpha}$ acting on $\bar\theta^{\dot\beta}$ a.s.o.
**Grassmann integration:**

$$d\theta_\alpha\theta^\beta = \delta^\beta_\alpha \quad d\theta_\alpha = -\varepsilon_{\alpha\beta}d\theta^\beta \tag{434}$$

$$d\theta^2 = \tfrac{1}{4}d\theta^\alpha d\theta_\alpha \quad , \quad d\bar\theta^2 = \tfrac{1}{4}d\bar\theta_{\dot\alpha}d\bar\theta^{\dot\alpha} \quad , \quad d\theta^4 = d\theta^2 d\bar\theta^2 \tag{435}$$

$$\int d\theta^2\, \theta^2 = 1 = \int d\bar\theta^2\, \bar\theta^2. \tag{436}$$

**Chiral SF** The chiral SF is defined by $\bar D_{\dot\alpha}\Phi = 0$ an can be written as

$$\Phi = \varphi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \quad , \quad y^m = x^m + i\theta\sigma^m\bar\theta \tag{437}$$

**Vector SF** The real vector SF $\bar V = V$ is given by

$$\begin{aligned}
V = &\ B + \theta\chi + \bar\theta\bar\chi + \theta^2 C + \bar\theta^2\bar C \\
&-\theta\sigma^m\bar\theta A_m + i\theta^2\bar\theta(\bar\lambda + \tfrac{1}{2}\bar\sigma^m\partial_m\chi) - i\bar\theta^2\theta(\lambda - \tfrac{1}{2}\sigma^m\partial_m\bar\chi) \\
&+\tfrac{1}{2}\theta^2\bar\theta^2(D + \Box B),
\end{aligned} \tag{438}$$

where $A_m, B, D$ are real. Super gauge transformations leave the reality condition invariant, i.e.

$$V \to V' = V + i(\Lambda - \bar\Lambda) \quad , \quad \Lambda \sim (L, \psi_\Lambda, F_\Lambda)\ldots\text{chiral}. \tag{439}$$

In components this writes as

$$\delta B = i(L - L^*) \ , \ \delta C = iF_\Lambda \ , \ \delta\lambda = 0 \ ; \ \delta D = 0 \tag{440}$$
$$\delta\chi = i\sqrt{2}\psi_\Lambda \ , \ \delta\bar{\chi} = -i\sqrt{2}\bar{\psi}_\Lambda \ , \ \delta A_m = \partial_m(L + L^*). \tag{441}$$

In the WZ-gauge one chooses $\Lambda$ such that

$$B = \chi = C = 0. \tag{442}$$

For the vector $V$ field one obtains

$$\begin{aligned} V_{WZ} &= -\theta\sigma^m\bar{\theta}A_m + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 D \\ V_{WZ}^2 &= -\tfrac{1}{2}\theta^2\bar{\theta}^2 A^2 \\ V_{WZ}^3 &= 0, \end{aligned} \tag{443}$$

so that $e^V$ becomes polynomial in the WZ-gauge. The gauge invariant field strength is defined as

$$W_\alpha = -\tfrac{1}{4}\bar{D}^2 D_\alpha V \ , \ W_{\dot{\alpha}} = -\tfrac{1}{4}D^2\bar{D}_{\dot{\alpha}}V. \tag{444}$$

They are chiral and anti-chiral, respectively and fulfills the Bianchi identity $D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}W^{\dot{\alpha}}$.

# D Dimensional reduction

Here we discuss some details of the dimensional reduction of (chiral) spinors from four to three dimensions. The main constraint comes from the fact that the spinors have definite transformation properties from their four-dimensional origin. Identifying two-component chiral $4D$-spinors with three-dimensional two component Dirac spinors the restricted $4D$ transformations have to realize a Lorenz transformation of $3D$ Dirac spinors. The four-dimensional $SL(2,\mathbb{C})$ transformations are restricted to three dimensional transformations as follows:

$$\begin{aligned} \delta\psi_\alpha &= -\frac{i}{2}(\vec{\varphi} - i\vec{\nu})\vec{\sigma}\ \psi_\beta \ , \ \delta\bar{\lambda}^{\dot{\alpha}} = -\frac{i}{2}(\vec{\varphi} + i\vec{\nu})\vec{\sigma}\ \bar{\lambda}^{\dot{\beta}} \\ \vec{\varphi}_{|3D} &= (0,0,\varphi) \ \ldots \ \text{rotations in } xy - \text{plane} \\ \vec{\nu}_{|3D} &= (\nu_1,\nu_2,0) \ \ldots \ \text{boosts in } x,y \text{ directions,} \end{aligned} \tag{445}$$

where as three dimensional Dirac spinors transform as

$$\delta\psi_D = \tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\psi_D \ , \ \Sigma^{\mu\nu} = \tfrac{1}{4}[\gamma^\mu,\gamma^\nu]. \tag{446}$$

We now make the following identifications:

$$\psi_D = \psi_\alpha \ , \ \lambda_D = -i\gamma^0\bar{\lambda}^{\dot{\alpha}} \ , \ \gamma^\mu = (i\sigma_3, -\sigma_2, \sigma_1). \tag{447}$$

With this choices the transformations in (445) and (446) coincide. The choice for $\lambda_D$ is such that the Yukawa coupling has a simple form in three dimensions. The $3D$ fermionic Lagrangian reads: [44]

$$\mathcal{L}_{F,3D} = -\bar{\lambda}\partial\!\!\!/\lambda - \bar{\psi}D\!\!\!\!/\psi + eN\bar{\psi}\psi - i\sqrt{2}e(\bar{\psi}\lambda\phi - \bar{\lambda}\psi\bar{\phi}). \tag{448}$$

This Lagrangian has the same form as in [97]. But note that the representation of the gamma matrices (447), which were determined by the form of the kinetic term of the $\psi$-field, differ by a relative sign from [97]. This corresponds to the two inequivalent representations in odd (three) dimensions for the Clifford algebra.

A different possibility of the dimensional reduction is to work with Majorana spinors and Majorana representations in four and three dimensions:

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \quad , \quad \Gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{449}$$

Choosing a Majorana representation for the three dimensional gamma matrices $\gamma^\mu$ also the four dimensional representation is real. The crucial point is the transformation property. With (449) the restricted four dimensional Lorentz generators are block diagonal:

$$\frac{1}{4}[\Gamma^\mu, \Gamma^\nu] = \begin{pmatrix} \Sigma^{\mu\nu} & 0 \\ 0 & \Sigma^{\mu\nu} \end{pmatrix}. \tag{450}$$

Thus the bi-spinor components of the real four spinor,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{451}$$

transform correctly as three dimensional two-component spinors. The dimensional reduction in terms of $\psi_1, \psi_2$ and analogously for $\lambda_1, \lambda_2$ leads to the Lagrangian obtained by [119]. The three dimensional spinors in (448) a related to the $3D$ Majorana spinors as follows:

$$\psi = \psi_1 - i\psi_2 \quad , \quad \lambda = \lambda_1 + i\lambda_2. \tag{452}$$

# E Polar and holomorphic coordinates

**Holomorphic** coordinates we define an technical grounds through the derivatives as follows:

$$\partial_\pm = \partial_1 \pm i\partial_2 \Rightarrow D_\pm = D_1 \pm iD_2 \tag{453}$$

$$X_\pm = X_1 \pm iX_2 \Rightarrow X_iY_i = \tfrac{1}{2}(X_+Y_- + X_-Y_+). \tag{454}$$

[44] Here we dimensionally reduce trivially, i.e $\partial_3 \equiv 0$.

(453) implies for the spatial coordinates $x^\pm = \frac{1}{2}(x^1 \mp ix^2)$, and (454) refers to a one form.
**Polar** coordinates are defined as usual:

$$x_1 = r\cos\theta \ , \ x_2 = r\sin\theta \ \Rightarrow \ \partial_i r = \frac{x_i}{r} \ , \ \partial_i\theta = -\varepsilon_{ij}\frac{x_j}{r}. \tag{455}$$

From the transformation rules

$$X_{i'} = \frac{\partial x_k}{\partial x_{i'}} X_k \quad , \quad X_k = \frac{\partial x_{i'}}{\partial x_k} X_{i'}, \tag{456}$$

one obtains

$$\begin{pmatrix} X_r \\ X_\theta \end{pmatrix} = \begin{pmatrix} \cos X_1 + \sin X_2 \\ -r\sin X_1 + r\cos X_2 \end{pmatrix} \quad , \quad X_i = \frac{x_i}{r} X_r - \varepsilon_{ij}\frac{x_j}{r} X_\theta. \tag{457}$$

Thus the holomorphic field components and covariant derivatives expressed in polar coordinates read

$$A_\pm = e^{\pm i\theta}(A_r \pm \frac{i}{r} A_\theta) \quad , \quad D_\pm = e^{\pm i\theta}(D_r \pm \frac{i}{r} D_\theta). \tag{458}$$

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