DIPLOMARBEIT

Quantisation of Supersymmetric CP^1 Solitons

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Gewidmet meinen Eltern, Peter und Mathilde
und Geschwistern, Peter, Martin, Mathilde,
Thomas, Markus und Johannes.

Contents

Chapter 1

Introduction

And when I see you
I really see you upside down,
but my brain knows better
it picks you up and turns you around.

Benjamin Gibbard

The investigation of solitons began in the year 1834 when a Scottish engineer, John Scott–Russell, observed a solitary wave in the Union Canal, near Edinburgh. He reproduced the phenomenon in a wave tank, and named it the "Wave of Translation". In 1895, sixty years after this first empirical study, Diederik Korteweg and Hendrik de Vries discovered a nonlinear differential equation describing water waves, the so called KdV–equation, which possesses such a solitary wave solution. However the significance of this discovery was realized only in the 1960s when N. J. Zabusky and M. D. Kruskal [1] did some research on different systems which are subject to the KdV–equation. They figured out that the solutions, which they got by a computational investigation using a finite difference approach, have very special properties (listed in section 2.1) so that Zabusky and Kruskal coined the term *soliton* for this nonlinear waves. These was more or less the starting shot for an intensive investigation of nonlinear differential equations with solitonic solutions.

After this excursion in the "early days" of the soliton research we now turn to (supersymmetric) quantum field theories. If a model, not necessarily supersymmetric, possesses a soliton or instanton, then one aspect of these extended objects is their non–perturbative effects on the theory. Another reason explaining the enormous interest in topological solitons in supersymmetric theories is the existence of a special class of solitons, which are called "critical" or "Bogomol'yni–Prasad–Sommerfield" saturated solitons [2, 3] (in the following abbreviated to BPS saturated solitons). In the seminal paper [4] Witten and Olive noted that in many instances topological charges associated with solitons coincide with the so–called central charge of superalgebras. If the soliton is additionally BPS saturated, one half of the supersymmetry generators vanishes and one is left with a "shortened" multiplet containing only one half of the states (as we will see section 3.1.1).

In this work we are interested in a special limit (the Higgs phase) of an abelian supersymmetric gauge theory with background gauge fields, which reduces to a supersymmetric CP^1 σ–model with twisted mass term [5, 6]. This model exhibits a solitonic solution which classically saturates the Bogomol'yni–bound. From general considerations, which are based on the underlying gauge theory, it is clear that also quantum mechanically the Bogomol'yni–bound should not be violated. But there exist quantum corrections to both, the mass and the central charge of the soliton, as we will see. And the origin of this correction is anomalous in nature.

This thesis is organised as follows. First we will describe the classical properties of solitary waves [7] in chapter 2, including methods to derive the solitons via the Bogomol'yni–Prasad–Sommerfield construction. We then introduce the topological index and the topological charge [8, 7, 9] which later on in the supersymmetric case will be relevant. At the end of chapter 2 we explain in a nutshell the collective coordinates and the moduli space [10].

According to references [11, 12, 4] we show how the central charge of extended supersymmetries can be related to topological objects. Further we investigate the implication of a quantum mechanical BPS saturation on the spectrum of the quantum theory. Although superfields will be important in chapter 5 we only define the representation of the supersymmetry via the Poisson bracket. Thus, we have to refer the reader to the literature (e.g. [11, 13, 12]) for the details on superfields, supernumbers etc..

Chapter 4 presents the basic tools to calculate the quantum fluctuations in a solitonic background. Following reference [14] we derive the Feynman–Kac formula and demonstrate for the harmonic oscillator how one can use it to obtain the vacuum energy. Subsequently we generalise the formulas to field theory and work out some details when the embedding of solitons in higher dimensions is considered.

Then we come to the CP^1 σ–model (a particular nonlinear sigma model) with twisted mass term, which is the model of interest (chapter 5). We will perform a more or less complete classical treatment of the theory. We show that the model can also be obtain from a four dimensional $\mathcal{N} = 1$ theory by Kaluza–Klein reduction[1]. After this classical analysis we will turn to the quantum theory of the nonlinear sigma model. One of the main goals will be to work out the quantum corrections of the mass and the central charge. As a last step of this chapter we investigate the quantisation of the effective Hamiltonian.

With exception of section A.1 which states the conventions used in this work, the appendix A primarily deals with symmetries and their implication on the classical and quantum theory, respectively. Following closely reference [15] we will start from the Hamiltonian formalism after which we state Noether's theorem. Subsequently we write down the "axioms of canonical quantisation" and investigate their effects on the symmetries. And, finally, appendix B summarises the details of the calculations of chapter 5.

[1]This fact is crucial as it allows a consistent supersymmetric treatment of the dimensional regularisation.

Chapter 2

Classical soliton solutions

In the introduction we presented some historical facts on solitary waves and the current interest of research. But now comes an essential question: What is a soliton actually? There are several definitions, one of them given by Drazin and Johnson [16] as follows:

2.1 Definition of a soliton

Drazin and Johnson describe solitons as solutions of nonlinear differential equations which

1. represent waves of permanent form;
2. are localised, so that they decay or approach a constant at infinity;
3. can interact strongly with other solitons, but they emerge from the collision unchanged apart from a phase shift.

The second condition is sometimes a little bit overrestrictive and can be relaxed that not the wave itself decays fast enough at infinity but the energy density $\varepsilon(x)$ so that one can still speak of a localised object.
To get a feeling for this abstract definition let us look at concrete example where we can easily check all these properties.

2.2 The bosonic kink

Let us consider a ϕ^4–theory in $1+1$ dimensional space with the following Lagrange density

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \qquad \text{and the potential} \qquad U(\phi) = \tfrac{\lambda}{4}\left(\phi^2 - \tfrac{\mu^2}{\lambda}\right)^2 \tag{2.1}$$

which is sometimes also called mexican hat or "sombrero" potential (Figure 2.1(a)). Since the mass term is negative there are two distinct vacua[1], one at $\phi = \frac{\mu}{\sqrt{\lambda}} = \phi_{\text{vac}_1}$ and the other at $\phi = -\frac{\mu}{\sqrt{\lambda}} = \phi_{\text{vac}_2}$. The equations of motion (EOM) are derived from the extremum condition of the action. From this follows that if we search for a **static** solution (ϕ_{st}) we have to find a field configuration which minimises the energy, or equivalently, the Hamiltonian $H = \int dx\mathcal{H}$, where the Hamilton density $\mathcal{H}$ is given by

$$\mathcal{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi'^2 + U(\phi)\,, \tag{2.2}$$

[1] The $\mathbb{Z}_2$ symmetry of the Lagrangian is spontaneously broken.

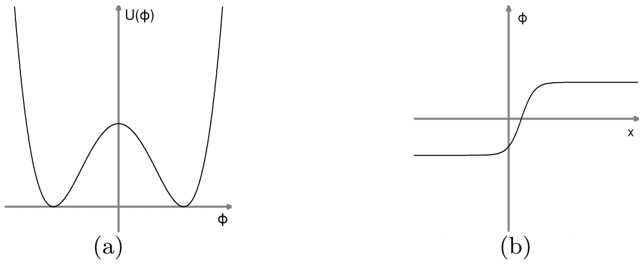


Figure 2.1: (a) The Potential of the ϕ^4-theory; (b) The kink which connects the two vacua.

where the first term vanishes for ϕ_{st}. As a consequence $\mathcal{H}$ has to vanish at least when x goes to infinity. Hence, we get the following boundary conditions:

$$\begin{aligned} \lim_{x\to\pm\infty} \phi_{\text{st}} &= \phi_{\text{vac}_{i_\pm}} \\ \lim_{x\to\pm\infty} \phi'_{\text{st}} &= 0 \end{aligned} \tag{2.3}$$

Thus we can already guess that there may be a solution which interpolates between the two vacua. But to convince ourselves let us have a look at the EOM, given by

$$\partial_\mu \partial^\mu \phi = \frac{\partial U(\phi)}{\partial \phi} \qquad \text{which for } \phi_{\text{st}} \text{ reduces to} \qquad \phi''_{\text{st}} = \frac{\partial U(\phi_{\text{st}})}{\partial \phi_{\text{st}}} . \tag{2.4}$$

The second equation can easily be rewritten as $\phi' d\phi' = dU$ and after integration it follows that

$$\phi'_{\text{st}} = \pm\sqrt{2U(\phi_{\text{st}})} . \tag{2.5}$$

This equation is called Bogomol'yni equation (the nomenclature well become clear in section 2.3), it can be further integrated to

$$x - x_0 = \pm \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{2U(\phi)}} . \tag{2.6}$$

Inserting now the concrete potential from (2.1) and solving for ϕ yields the so–called **kink** and **antikink**, respectively

$$\phi_K(x) = \mp \frac{\mu}{\sqrt{\lambda}} \tanh\left(\tfrac{\mu}{\sqrt{2}}(x - x_0)\right) \tag{2.7}$$

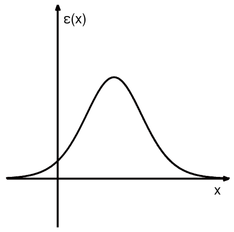
where the + belongs to the kink (Figure 2.1(b)) and the − to the antikink (Figure 2.2(b)).

Let us now go over the definition of a soliton to see if the kink is really one.

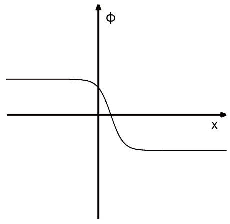
ad 1 So far we have only a stationary field configuration, but since our theory is Lorentz invariant, we just need to boost the coordinate system with velocity u, to get a kink moving in the opposite direction with the same speed.

$$x \to \frac{x - ut}{\sqrt{1-u^2}} \quad \Rightarrow \quad \phi_u(x,t) = \frac{\mu}{\sqrt{\lambda}} \tanh\left(\frac{\mu((x-x_0) - ut)}{\sqrt{2(1-u^2)}}\right), \quad u \in (-1,1) \tag{2.8}$$

Thus the kink is a wave of permanent form.✓



(a)



(b)

Figure 2.2: (a) Energy density of a static kink; (b) The anti–kink

ad 2 To see that the kink is a localised object, we look at the energy density $\varepsilon(x,t)$ of the kink (Figure 2.2(a)), given by

$$\varepsilon(x,t) = \mathcal{H}(\phi_u) = \frac{\mu^4}{2\lambda(1-u^2)} \cosh^{-4}\left(\frac{\mu((x-x_0)-ut)}{\sqrt{2(1-u^2)}}\right) . \tag{2.9}$$

Hence $\lim_{x\to\pm\infty} \varepsilon(x,t) = \mathcal{O}(\left|\frac{1}{x}\right|)$ and consequently also this condition is fulfilled.✓

ad 3 For the third point there exists no analytical solution, at least to my knowledge, so one has to do numerical calculations. But for a similar model, the sine–Gordon theory[2], one can write down the solution explicitly and verify that the solitons emerge from the collision unchanged apart from a phase shift ($x - \frac{u\Delta}{2} \to x + \frac{u\Delta}{2}$ with the phase shift $\Delta := ((1-u^2)/u \ln u)$. For a detailed treatment I refer to the literature [7, 8].✓

Hence the kink is a soliton as we expected.

As a last step let us calculate the energy $E(u)$ of the (moving) kink which is given by the Hamiltonian,

$$E(u) = H[\phi_u] = \frac{1}{\sqrt{1-u^2}} E(u=0) = \frac{1}{\sqrt{1-u^2}} M_{\text{cl}} \qquad \text{with} \qquad M_{\text{cl}} = \frac{2\sqrt{2}\mu^3}{3\lambda} . \tag{2.10}$$

Thus, we can interpret the static (or moving) soliton as a particle with rest mass M_{cl}.

2.3 Bogomol'nyi bound

If the EOM hadn't been that easy to solve, we could have at least derived a lower bound for the mass. This is done by introducing an angular parameter θ into the "static" Hamilton density (2.2),

$$M_{\text{cl}} = \int dx \tfrac{1}{2}\phi'^2 + U(\phi) = \int dx \tfrac{1}{4}[\phi' - \sin\theta\sqrt{4U(\phi)}]^2 + \tfrac{1}{4}[\phi' - \cos\theta\sqrt{4U(\phi)}]^2 + \tag{2.11}$$
$$+ \sin\theta \int dx\, \phi' \sqrt{U(\phi)} + \cos\theta \int dx\, \phi' \sqrt{U(\phi)} \geq (\sin\theta + \cos\theta)\, \Xi$$

[2]One only hast to replace the potential in equation (2.1) by $U(\phi) = \frac{\mu^4}{\lambda}(\cos((\sqrt{\lambda}/\mu)\phi) - 1)$. The kink/soliton solution in this model is given by $\phi_K(x) = 4\frac{\sqrt{\lambda}}{\mu} \arctan[\exp(\mu(x-x_0))] + 4k\pi$, $k \in \mathbb{Z}$, (one only has to redo the calculation with the new potential).

with Ξ given by

$$\Xi = \int dx\, \phi' \sqrt{U(\phi)} = \int d\phi \sqrt{U(\phi)}\,. \tag{2.12}$$

Therefore for all angles θ we have the following bound on the mass:

$$M \geq \sin\theta\, \Xi + \cos\theta\, \Xi\,; \tag{2.13}$$

The sharpest bound occurs when the right hand side is a maximum, which happens for $\cos\theta = \sin\theta$. Thus we find the **Bogomol'nyi bound** for the kink:

$$M \geq \frac{2}{\sqrt{2}} \Xi\,; \tag{2.14}$$

From equation (2.11) and $\cos\theta = \sin\theta$ we see that the bound is saturated if the following first order differential equation, the **Bogomol'nyi equation**, holds:

$$\phi' - \sqrt{2U(\phi)} = 0 \tag{2.15}$$

It was derived for the first time in [2] for the Georgi–Glashow model[3]. We will follow custom and call the solutions to the Bogomol'nyi equation, if they exist, **BPS-solitons**.

2.4 Topological indices

It is often possible to make a topological classification of the solutions of a given system of equations. Specifically, one can define a topological index which is conserved in time. Like other conserved quantities it plays the important role of a 'quantum number' for particle states in the corresponding quantum field theory. It has, however, quite a different origin from that of the other familiar conserved quantities and quantum numbers which are explained in appendix A.

Let us look again at our 2D theory as defined in equation (2.1) but now with a generic potential $U(\phi)$ which has a discrete (not necessarily finite) number of degenerate absolute minima, where it vanishes. Thus we get the following set of vacuum field configurations:

$$\{\phi_{\text{vac}_i}\}_{i\in\mathbb{N}} = \{\phi \in \mathcal{F}(\mathbb{R}) : U(\phi) = 0\}$$

The same conclusions which led us to the boundary conditions (2.3) now lead us to:

$$\lim_{x\to-\infty} \phi(x,t) := \phi(-\infty,t) = \phi_{\text{vac}_i} \qquad \lim_{x\to\infty} \phi(x,t) := \phi(\infty,t) = \phi_{\text{vac}_j}\,, \tag{2.16}$$

where i is not necessarily equal to j. $\phi(-\infty,t)$ and $\phi(\infty,t)$ are time independent because if not we would get contributions to the energy density $\varepsilon(x,t)$ with non–compact support and hence leave the space of finite energy solutions.

Thus we can divide the space of all finite–energy non–singular solutions into sectors, characterised by the values $\phi(-\infty)$ and $\phi(\infty)$. These sectors are topologically unconnected, in the sense that fields from the one sector cannot be distorted continuously into another without violating the requirement of finite energy. In particular, since time evolution is an example of continuous distortion, a field configuration from any one sector stays within that sector as time evolves.

[3] For some details on the Georgi–Glashow model and the calculation of the bound see also [10]

2.4.1 Topological charge and current

Although the conserved topological indices do not come from a continuous symmetry, we can write down the following off–shell conserved (topological) current[4] and its corresponding charge,

$$J^{\mu}_{\text{top}} := \varepsilon^{\mu\nu}\partial_\nu\phi \qquad \text{and} \qquad T = \int dx\, J^0_{\text{top}}\,, \tag{2.17}$$

which has a very strong relation to the topological indices, since $T = [\phi(\infty) - \phi(-\infty)]$. T is the analogue of the topological indices in more complicated systems—such as gauge theories in four dimensions.

To classify the topological sector one needs $\phi(-\infty)$ and $\phi(\infty)$, so that the knowledge of T is not enough, but for quantities which depend only on the difference of the boundary values T is sufficient.
Example: Let us look again at the ϕ^4–theory, the potential (see Figure 2.1(a)) has the two vacua $\phi_{\text{vac}_{1/2}}$. This gives rise to four topological sectors which are summarised in the following index set

$$\{(\phi(\infty), \phi(-\infty))\} = \{(\phi_{\text{vac}_1}, \phi_{\text{vac}_1}), (\phi_{\text{vac}_1}, \phi_{\text{vac}_2}), (\phi_{\text{vac}_2}, \phi_{\text{vac}_1}), (\phi_{\text{vac}_2}, \phi_{\text{vac}_2})\}\ ,$$

but only to three topological indices $T \in \{-1, 0, 1\}$, which we have divided by the factor $2\phi_{\text{vac}_1}$.

Solitary waves are called **topological** if $T \neq 0$, otherwise **non–topological**. Thus the kink and antikink are topological.

2.5 Collective coordinates

The idea behind the **collective coordinates** is to parametrise the **moduli space** of BPS–states, which is the space of physically different field configurations where the energy E attains its minimum. In the case at hand, the collective coordinate is the position of the kink[5] which is parametrised by x_0. Now any motion, however small, increases the kinetic energy of the soliton and makes its total energy strictly greater than the Bogomol'nyi bound. Nevertheless, if we keep the velocity small and if the motion starts off tangent to the space of static BPS–states, energy conservation will prevent the motion from taking the solitons very far away from this space. Much like a point–particle moving slowly near the bottom of a potential well, the motion of slow BPS–solitons may be approximated by motion on the space of static BPS–solitons (*i.e.*, along the flat directions of the potential) and small oscillations in the transverse directions. We can trade the limit of velocities going to zero, for a limit in which the potential well becomes infinitely steep. This suppresses the oscillations in the transverse directions (which become increasingly expensive energetically) and motion is effectively constrained to take place along the flat directions, since this motion costs very little energy. Expanding the action functional around a BPS–state gives rise to an effective theory in terms of collective coordinates. For the kink this is achieved by inserting the kink

[4] $\partial_\mu J^{\mu}_{\text{top}}$ vanishes since we contract a symmetric tensor with an antisymmetric one.
[5] A consequence of the translational symmetry of the theory

solution into the Lagrangian (2.1) and making the moduli parameter (x_0) time dependent

$$L_{\rm eff} = \int dx\, \mathcal{L}(\phi_K) = \int dx \left[\tfrac{1}{2}\phi_K'^2 \dot{x}_0^2 - \tfrac{1}{2}\phi_K'^2 - U(\phi_K)\right] .$$

The obtained effective action is then given by the following very familiar expression:

$$L_{\rm eff} = M_{\rm cl}\frac{\dot{x}_0^2}{2} - M_{\rm cl}\,, \tag{2.18}$$

which is the Lagrangian of a free particle with mass $M_{\rm cl}$ (except for the constant), so we see once more the particle–like character of the kink.

Chapter 3

Supersymmetry

A wise person (Peter van Nieuwenhuizen) *once said that inside every no–go theorem there is a "yes–go" theorem waiting to come out,*

and a wise guy said that we should call it a "go–go" theorem.

unknown

In this chapter, we will briefly review some properties of supersymmetry which are essential for the chapters 5. Originally supersymmetry was investigated to circumvent the restrictions on the most general Lie algebra of symmetries of the S–matrix. Because Coleman and Mandula showed in their celebrated **"no–go" theorem** [1] [18] that the symmetry algebra is a direct sum of the Poincaré algebra and a reductive compact Lie algebra if

1. the S–matrix is based on a local, relativistic quantum field theory in 4 dimensions,
2. there are only a finite number of particles associated with one particle states of a given mass,
3. and there is mass gap between the vacuum and one–particle states.

So Haag, Łopuszánski and Sohnius started to analyse the general structure of *graded* symmetry algebras (they intertwine the fermionic and bosonic part of a theory). Their analysis led to the following most general graded algebra which is compatible with the assumptions of the Coleman–Mandula theorem[2]:

3.1 The supersymmetry algebra

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}_{\dot\beta\, B}\} &= 2\sigma^m_{\alpha\dot\beta} P_m \delta^A_B\,, \qquad \{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta} Z^{AB}\,, \qquad [Q^L_\alpha, T_l] = S_l{}^L{}_M Q^M_\alpha\,, \\ [T_l, T_m] &= i f_{lm}{}^k T_k\,, \qquad \{\bar{Q}_{\dot\alpha\, A}, \bar{Q}_{\dot\beta\, B}\} = \varepsilon_{\dot\alpha\dot\beta} Z^*_{AB}\,, \quad [T^l, \bar{Q}_{\dot\alpha\, L}] = S^{*\,l}{}_L{}^M \bar{Q}_{\dot\alpha\, M}\,; \end{aligned} \tag{3.1}$$

where the Q's are the supercharges, the T_l are the Lie algebra generators of the internal symmetries, α and β are spinor indices, the indices I and J label the spinor representation

[1] For further details and the proof see chapter 24, *Historical Introduction*, of reference [17]

[2] Algebra (3.1) is valid for four dimensions; for SUSY algebras in D dimensions see reference [12].

in case of extended SUSY ($I, J = 1 \ldots \mathcal{N}$) and Z^{IJ}, the central charges[3], which are given by $Z^{IJ} = a^{l\,IJ} T_l$ where the a^l's intertwine the representations S_l and $-S^{*\,l}$.

We are now interested in an integration of this symmetry algebra into a quantum theory. So we will look for its unitary representation. The derivation of the rep. is not difficult but a little bit lengthy, thus we will not go through all the algebraic details. The whole 2nd chapter of reference [11] is devoted to it, but we summarise the results in which we are interested in.

3.1.1 Representations of algebras with central charges

We assume that $P^2 = -M^2$ and study the algebra in the rest frame:

$$\begin{aligned} \{{Q_\alpha}^L, ({Q_\beta}^M)^\dagger\} &= 2M{\delta_\alpha}^\beta \delta^{LM} & \{{Q_\alpha}^L, {Q_\beta}^M\} &= \varepsilon_{\alpha\beta} Z^{LM} \\ \{({Q_\alpha}^L)^\dagger, ({Q_\beta}^M)^\dagger\} &= \varepsilon^{\alpha\beta} Z^{*LM} & Z^{LM} &= -\,Z^{ML} \end{aligned} \tag{3.2}$$

The central charges Z^{LM} commute with all the generators, so we may choose a basis in which the central charges are diagonal with eigenvalues Z^{LM}. These eigenvalues form an antisymmetric $N \times N$ matrix. Any such matrix may be rotated into a standard form by unitary transformation:

$$\tilde{Z}^{LM} = {U^L}_K {U^M}_N Z^{KN} \,. \tag{3.3}$$

The standard form is given by

$$\tilde{Z} = \varepsilon \otimes D \qquad \text{and} \qquad \tilde{Z} = \begin{pmatrix} \varepsilon \otimes D & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4}$$

for N even and N odd, respectively, where D is diagonal with positive real eigenvalues Z_m and ε is the 2×2 antisymmetric matrix with $\varepsilon^{12} = 1$.

We start by decomposing the indices L and M [4] in accord with (3.4), $L = (a, m)$, $M = (b, n)$, where $a, b = 1, 2$ and $n, m = 1, \ldots, \frac{N}{2}$. We then perform a unitary transformation on the ${Q_\alpha}^N$,

$${\tilde{Q}_\alpha}{}^L = {U^L}_K {Q_\alpha}^K \,. \tag{3.5}$$

This allows us to write the algebra (3.1) in the following from:

$$\begin{aligned} \{{\tilde{Q}_\alpha}{}^{am}, ({\tilde{Q}_\beta}{}^{bn})^\dagger\} &= 2M{\delta_\alpha}^\beta {\delta^a}_b {\delta^m}_n \\ \{{\tilde{Q}_\alpha}{}^{am}, {\tilde{Q}_\beta}{}^{bn}\} &= \varepsilon_{\alpha\beta}\varepsilon^{ab}\delta^{mn} Z_n \\ \{({\tilde{Q}_\alpha}{}^{am})^\dagger, ({\tilde{Q}_\beta}{}^{bn})^\dagger\} &= \varepsilon^{\alpha\beta}\varepsilon_{ab}\delta^{mn} Z_n \,. \end{aligned} \tag{3.6}$$

The operators ${\tilde{Q}_\alpha}{}^{am}$ and $({\tilde{Q}_\alpha}{}^{am})^\dagger$ may all be expressed as linear combinations of

$${a_\alpha}^m = \tfrac{1}{\sqrt{2}}[\tilde{Q}_\alpha^{1m} + \varepsilon_{\alpha\rho}({\tilde{Q}_\rho}{}^{2m})^\dagger] \,, \qquad {b_\alpha}^m = \tfrac{1}{\sqrt{2}}[\tilde{Q}_\alpha^{1m} - \varepsilon_{\alpha\rho}({\tilde{Q}_\rho}{}^{2m})^\dagger] \tag{3.7}$$

and their conjugates $({a_\alpha}^m)^\dagger$ and $({b_\alpha}^m)^\dagger$. The operators a and b satisfy the following algebra:

$$\begin{aligned} \{{a_\alpha}^n, {a_\beta}^m\} = \{{b_\alpha}^n, {b_\beta}^m\} = \{{a_\alpha}^n, {b_\beta}^m\} &= 0 & \{{a_\alpha}^n, ({b_\beta}^m)^\dagger\} &= 0 \\ \{{a_\alpha}^n, ({a_\beta}^m)^\dagger\} = \delta_{\alpha\beta}\delta^{mn}(2M + Z_n) & & \{{b_\alpha}^n, ({b_\beta}^m)^\dagger\} &= \delta_{\alpha\beta}\delta^{mn}(2M - Z_n) \,. \end{aligned} \tag{3.8}$$

[3] The central charges are antisymmetric in I and J;

[4] We shall study the case with N even, the case with N odd is analogous.

From these relations we see that $Z_n \leq 2M$ for all n.[5] If a set of $Z_k = 2M$, with $k = 1, \ldots, r$, the corresponding operators b_i must vanish. With the nonvanishing operators we define the following operators:

$$\begin{aligned} \Gamma^{l} &:= \tfrac{1}{\sqrt{2}}[{a_1}^l + ({a_1}^l)^\dagger] & \Gamma^{\frac{N}{2}+l} &:= \tfrac{1}{\sqrt{2}}[{a_2}^l + ({a_2}^l)^\dagger] \\ \Gamma^{N+l} &:= \tfrac{i}{\sqrt{2}}[{a_1}^l - ({a_1}^l)^\dagger] & \Gamma^{\frac{3N}{2}+l} &:= \tfrac{i}{\sqrt{2}}[{a_2}^l - ({a_2}^l)^\dagger] \\ \Gamma^{2N+i} &:= \tfrac{1}{\sqrt{2}}[{b_1}^i + ({b_1}^i)^\dagger] & \Gamma^{\frac{5N}{2}-r+i} &:= \tfrac{1}{\sqrt{2}}[{b_2}^i + ({b_2}^i)^\dagger] \\ \Gamma^{3N-2r+i} &:= \tfrac{i}{\sqrt{2}}[{b_1}^i - ({b_1}^i)^\dagger] & \Gamma^{\frac{7N}{2}-3r+i} &:= \tfrac{i}{\sqrt{2}}[{b_2}^i - ({b_2}^i)^\dagger] \end{aligned} \tag{3.9}$$

where the indices 1 and 2 refer to the $SU(2)$ spinor indices and the indices l and i run from 1 to $\frac{N}{2}$ and $(\frac{N}{2} - r)$, respectively. Thus we get the following Clifford algebra:

$$\{\Gamma^K, \Gamma^M\} = \delta^{KM} \qquad \text{with} \qquad N = 1, \ldots, 4(N-r)\,; \tag{3.10}$$

The fundamental representation of this algebra is spanned by $2^{2(N-r)}$ states. Hence, if we have a set of r central charges which fulfil $2M = Z_k$, the multiplet becomes shortened by a factor of 2^{2r}. Witten and Olive [4] were the first who found an explicit realisation of such a supersymmetry algebra in a theory. They noted that in many instances (supporting topological solitons) topological charges coincide with the central charges of superalgebras. Actually, this seminal paper opened the currently flourishing topic of BPS saturated solitons for investigation. Thus we will briefly review it.

3.2 BPS saturation revisited

Witten and Olive showed in their work [4] that in supersymmetric theories with solitons the usual supersymmetry algebra is not valid. It is modified to include the topological quantum numbers as central charges. Further they used the corrected algebra to show that in the Georgi-Glashow model, quantum corrections preserve the classical equality of the mass and central charge spectrum. We will only summarise some details of the first part of their paper.

The supersymmetric form[6] of a scalar field theory in two dimensions is

$$L = \int d^2x \; \left[\tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{2}\bar{\psi} i \partial\!\!\!/ \psi - \tfrac{1}{2}V^2(\psi) - \tfrac{1}{2}V'(\psi)\bar{\psi}\psi\right] \,, \tag{3.11}$$

where ψ is a Majorana fermion, and $V(\phi)$ an arbitrary function. The conserved symmetry current is

$$J^\mu_{\text{sup}} = (\partial_\nu\phi)\gamma^\nu\gamma^\mu\psi + iV(\phi)\gamma^\mu\psi\,. \tag{3.12}$$

Working with chiral components $\psi^\pm$ of the Fermi field, the chiral components $Q^\pm$ of the supersymmetry charges can be written

$$Q_\pm = \int dx\,[(\partial_0\phi \pm \partial_1\phi)\psi_\pm \mp V(\phi)\psi_\mp] \,. \tag{3.13}$$

[5]For unitary representations it is necessary that $\langle\psi|\{{b_\alpha}^n, ({b_\beta}^m)^\dagger\}|\psi\rangle \geq 0 \quad \Rightarrow \quad Z_n \leq 2M$

[6]So far now we have only presented supersymmetry algebras, but from appendix A it should be clear that a symmetry algebra is derived from the symmetries of the Lagrangian. Thus, for a supersymmetry algebra we need first of all a Lagrangian which is invariant (up to a total derivative) under a certain transformation that intertwines the bosons and the fermions

In this notation, the standard supersymmetry algebra[7] (3.1) gets changed a little

$$Q_+^2 = P_+ \qquad \text{and} \qquad Q_-^2 = P_- \,, \qquad \text{where} \qquad P_\pm = P_0 \pm P_1 \,. \tag{3.14}$$

The central charge Z is then given by

$$Z = \{Q_+ \,,\, Q_-\} = \int dx\, 2V(\phi)\tfrac{\partial\phi}{\partial x} = \int dx\, \tfrac{\partial}{\partial x}(2K(\phi))\,, \tag{3.15}$$

where $K(\phi)$ is a function such that $K'(\phi) = V(\phi)$. Thus $\{Q_+ \,,\, Q_-\}$ is the integral of a total divergence, and naively would vanish. But in a soliton state, the right hand side of equation (3.15) is not necessarily zero[8].

For a typical example, we look at the supersymmetric extended ϕ^4–theory[9] for which we get

$$Z = \int_{-\infty}^{\infty} dx\, \tfrac{\partial}{\partial x}\sqrt{2\lambda}(\tfrac{1}{3}\phi^3 - \tfrac{\mu^2}{\lambda}\phi)\,. \tag{3.16}$$

Z vanishes in a topologically trivial state, and has a positive value in the kink state, a negative value in the antikink state. Although apparently different from the usual topological charge $\int_{-\infty}^{\infty} dx\, \frac{\partial\phi}{\partial x}$, Z actually coincides with it, since both depend only on the topology.

Now let us again treat the algebra from equation (3.14). We see that the mass squared operator $M^2 = P_+P_- = P_-P_+$ can be written

$$M^2 = \tfrac{1}{4}(Z^2 + (\bar{Q}Q)^2)\,, \tag{3.17}$$

where $\bar{Q}Q$ is the Hermitian operator $i(Q_+Q_- - Q_-Q_+)$. Since $(\bar{Q}Q)^2$ is positive, this establishes that $M^2 \geq \frac{1}{4}Z^2$, and saturated only for states $|\alpha\rangle$ such that $\bar{Q}Q\,|\alpha\rangle$ vanishes. In the rest frame $\bar{Q}Q = i(Q_+ - Q_-)(Q_+ + Q_-) = -i(Q_+ + Q_-)(Q_+ - Q_-)$ and so annihilates any state that is annihilated by $(Q_+ + Q_-)$ or $(Q_+ - Q_-)$. This condition may seem rather exeptional, but actually it is satisfied, at least classically, for all the soliton and antisoliton states that satisfy the Bogomol'nyi equation[10]

$$\frac{\partial\phi}{\partial x} = \pm V(\phi) = \pm\sqrt{2U(\phi)}\,. \tag{3.18}$$

If the bound holds also quantum mechanically we get a shortened multiplet structure, as shown above. The corresponding states $|\alpha\rangle$ will be called BPS–states as in the classical regime, see section 2.3.

[7] In equation (3.1) we used the Weyl representation. Now we use the Majorana representation since the real boson under consideration has only one degree of freedom.

[8] In fact, in quantum theory, a matrix element of the operator Z is the difference between the expectation values of $2K(\phi)$ at $x = \infty$ and $x = -\infty$.

[9] To get the Lagrangian for this theory we insert $\sqrt{2U(\phi)}$ of equation (2.1) into equation (3.11).

[10] Implies the saturation of the Bogomol'nyi bound, see section 2.3;

Chapter 4

Perturbation theory in non–trivial backgrounds

4.1 Quantum energy levels for the static solitons

4.1.1 The Feynman–Kac formula

We briefly recall the elementary steps in the derivation of the path integral and apply it for the case of non–trivial background fields. An in–depth discussion is given in [8] and [7].

In a quantum system the time evolution of a state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation[1]

$$\hat{H}\,|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle\ , \tag{4.1}$$

where $\hat{H}$ is the Hamiltonian of the system. In principle it can by solved by the ansatz $|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle$ where $U(t,t_0)$ has to satisfy the composition law

$$U(t'',t) = U(t'',t')U(t',t)$$

and the initial condition $U(t_0,t_0) = \mathbb{1}$.

Of particular interest in the field theoretical context will be the **propagating kernel** K which appears in the present context as the Green function of the Schrödinger equation.

$$\begin{aligned}(\hat{H} - i\hbar\partial_t)K(t,t_0) =& - i\hbar\delta(t-t_0)\mathbb{1}\\ \lim_{t\to t_0^+} K(t,t_0) =& \mathbb{1}\end{aligned} \tag{4.2}$$

The solution to this problem is given by

$$K(t,t_0) = \theta(t-t_0)\mathcal{T}e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}}\ , \tag{4.3}$$

where the $\mathcal{T}$ is time–ordering symbol. For a time–independent Hamiltonian we can omit the time–ordering and get

$$K(t,t_0) = \theta(t-t_0)e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\ , \tag{4.4}$$

[1] We are working in the Schrödinger picture and not in the Heisenberg picture as in appendix A.

which then depends only on $T := t - t_0$. The Fourier transformed kernel $G(E)$ defined by

$$G(E) = \frac{i}{\hbar} \int dT e^{\frac{i}{\hbar}(E+i\varepsilon)T} K = \frac{1}{\hat{H} - E - i\varepsilon} \tag{4.5}$$

satisfies

$$(\hat{H} - E)G(E) = 1 \tag{4.6}$$

where we applied Feynman's pole–prescription in order to guarantee convergence. For a one–dimensional single particle theory with a single degree of freedom

$$\hat{H} = \frac{\hat{p}}{2m} + V(\hat{q}) \tag{4.7}$$

we define the coordinate representation by

$$\hat{q}\,|q\rangle = q\,|q\rangle \qquad q \in \mathbb{R}$$

and normalise $\langle q|\, q'\rangle = \delta(q - q')$. The retarded Feynman propagator or propagating kernel reads

$$K(q'', T|q') = \theta(T) \left\langle q''\right| e^{-\frac{i}{\hbar}\hat{H}T} \left|q'\right\rangle = \theta(T) \left\langle q'', t''\right| q', t'\rangle \qquad \text{with} \quad T = t'' - t' \tag{4.8}$$

where in the last step we have changed to the familiar Heisenberg picture[2],

$$A_H(t) = e^{\frac{i}{\hbar}\hat{H}t} A_S e^{-\frac{i}{\hbar}\hat{H}t} \qquad |\psi\rangle_H = e^{\frac{i}{\hbar}\hat{H}t}\,|\psi\rangle_S\ . \tag{4.9}$$

The basic property of K in this basis is

$$\left\langle q''\right| \psi(t'')\rangle = \int dq'\, K(q'', T|q') \left\langle q'\right| \psi(t')\rangle \tag{4.10}$$

The famous trick of Feynman is to insert a sequence of identities written as completeness relations $\mathbb{1} = \int dq(t_i)\, |q(t_i)\rangle \langle q(t_i)|$ of eigenvectors of the (time–dependent) Heisenberg operator $q_H(T)$ at a sequence of N different times $\{t_i\}$. Absorbing the divergent prefactor in the measure one recovers the useful formula

$$K(q'', T|q') = \int_{(q',t')}^{(q'',t'')} [Dq] e^{\frac{i}{\hbar}S[q,T]} \tag{4.11}$$

which is now easily generalised to field theory.

If we insert $\mathbb{1} = \sum\!\!\!\!\!\!\int |E\rangle \langle E|$ into (4.8) we find

$$K(q'', T|q') = \theta(T) \sum\!\!\!\!\!\!\!\int \left\langle q''\right| E\rangle \langle E|\, e^{-\frac{i}{\hbar}\hat{H}T} \left|q'\right\rangle = \sum\!\!\!\!\!\!\!\int \theta(T)\psi_E(q'')\psi_E^*(q')e^{-\frac{i}{\hbar}ET}\ ,$$

where the $|E\rangle$'s are the eigenstates of the Hamiltonian $\hat{H}$ and the $\sum\!\!\!\!\!\!\int$ stands for integration and summation over the discrete and continuous eigenvalue spectrum, respectively. The so–called **spectral function** or **partition function** is obtained by setting $q'' = q' = q_0$ and integrating over q_0, i.e. we investigate closed paths. Denoting this procedure by Tr one gets

$$K(T) = \mathrm{Tr}\left[e^{-\frac{i}{\hbar}\hat{H}T}\right] = \int dq_0\, K(q_0, T|q_0) = \sum\!\!\!\!\!\!\!\int e^{-\frac{i}{\hbar}ET} \int dq_0\, |\psi_E(q_0)|^2\ . \tag{4.12}$$

[2]In the Heisenberg picture the state vectors become time–independent and all the time information is now carried by the operators, (their time evolution is given in appendix A).

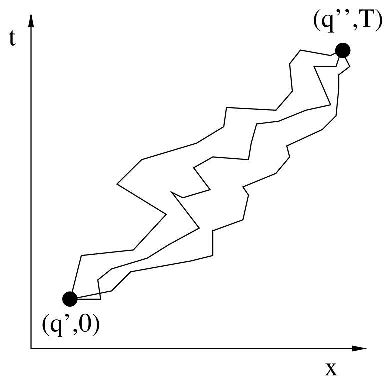


Figure 4.1: All the path with fixed endpoints are considered in the path integral.

A Wick rotation $\tau = \frac{i}{\hbar}T$ and sending $T \to \infty$ picks out $e^{-E_0\tau}$ times the degree k of degeneracy of the ground state:

$$\begin{aligned} E_0 &= -\lim_{\tau\to\infty} \frac{1}{\tau} \ln \int dq_0\, K(q_0, -i\hbar\tau|q_0) = -\lim_{\tau\to\infty} \frac{1}{\tau} \ln \mathrm{Tr}\left[e^{-\hat{H}\tau}\right] \\ k &= \lim_{\tau\to\infty} e^{E_0\tau} \int dq_0\, K(q_0, -i\hbar\tau|q_0) \end{aligned} \tag{4.13}$$

This is the **Feynman–Kac formula**, it allows an evaluation of the ground state energy without detailed knowledge of the propagating kernel. It will be a handy tool when it comes to calculating quantum corrections to masses of topological objects.

4.1.2 Stationary phase approximation in non–trivial backgrounds

The stationary phase approximation when expanding around a non–trivial background is slightly different from the usual vacuum case which is very well known from the ordinary path integral formulation of quantum field theory. We again consider a system with a single degree of freedom.

The classical path $q_{\rm cl}(t)$ between the initial point q' and end point p'' is defined by the extremal principle of the action

$$\delta S[q]\Big|_{q_{\rm cl}} = 0 \qquad \text{with} \qquad S[q] = \int_0^T dt\, \left(\tfrac{1}{2}\dot{q}^2 - V(q)\right) . \tag{4.14}$$

Now we expand the action around this extremum and classical path, respectively, do a partial integration, use the EOM of $q_{\rm cl}$ and get

$$\begin{aligned} S[q] = S[q_{\rm cl} + \eta] = S[q_{\rm cl}] + \dot{q}_{\rm cl}\eta\Big|_0^T + \tfrac{1}{2}\eta\dot{\eta}\Big|_0^T + \tfrac{1}{2}\int_0^T dt\, \eta(-\partial_t^2 - V''(q_{\rm cl}))\eta + \\ + \sum_{k=3}^{N} \int_0^T dt\, \tfrac{1}{k!} V^{(k)}(q_{\rm cl})\eta^3 \, , \end{aligned} \tag{4.15}$$

where the surface terms, $\dot{q}\eta\big|_0^T$ and $\frac{1}{2}\eta\dot{\eta}\big|_0^T$, vanish if the classical path connects the initial and final position[3]. Since this condition is not allways fulfilled one has to be very careful

[3]From Figure 4.1 one can easily see that the fluctuations at the starting $(q', 0)$ and end point (q'', T), respectively, are set to zero, thus $\eta(0) = \eta(T) = 0$ if $q_{\rm cl}$ connects them.

when dropping surface terms and further in the case of a solitonic background, for instance in section 3.2, one may get topological quantum numbers from the surface terms.

We neglect the last sum which only gives corrections to order three or higher. The important terms are the bilinear terms in the fluctuations that yield the determinant of the operator $\hat{O} = -\partial_t^2 - V''(q_{\rm cl})$ and the classical action. With this in mind we get the so–called semi–classical approximate of the kernel

$$K(q'', T|q') = N(T) e^{\frac{i}{\hbar} S[q_{\rm cl}, T]} \frac{1}{\sqrt{\det \hat{O}}} \,. \tag{4.16}$$

Of course the determinate must not vanish so if there are zero modes in the spectrum we have to exclude them from the determinate and treat them in a special manner. In quantum field theory zero modes give scaleless integrals and hence do not contribute in dimensional regularisation. However there is a one to one relation between the zero modes and the collective coordinates and thus the gain importance when we quantise the moduli space (see section 5.3.3).

Before considering quantum field theory let us study this approximation on a simpler example (actually the following example, the harmonic oscillator, is exactly solvable).

4.1.3 The harmonic oscillator

We take the Lagrangian $L = \frac{1}{2}(\dot{q}^2 - \omega^2 q)$ and consider closed paths, $q(0) = q(T) = q_0$. The corresponding differential equation is easily solved and we find

$$q_{\rm cl} = q_0 \left(\cos(\omega t) + \frac{2 \sin^2\left(\frac{\omega T}{2}\right)}{\sin(\omega T)} \sin(\omega t) \right) \,. \tag{4.17}$$

Now we expand the action as before and get

$$S[q] = S[q_{\rm cl}] + \dot{q}\eta\Big|_0^T + \frac{1}{2}\eta\dot{\eta}\Big|_0^T + \frac{1}{2}\int_0^T dt\, \eta(t)(-\partial_t^2 - \omega^2 q^2(t))\eta(t) \,. \tag{4.18}$$

Because we are only interested in the spectral function, for which $q(T) = q(0) = q_0$, one can choose the boundary conditions $\eta(0) = \eta(T) = 0$ since $\forall\ q(T) = q(0) = q_0 \wedge T > 0$ there $\exists\, q_{\rm cl}$. A basis of eigenfunctions of this operator, compatible with the boundary conditions, is

$$\begin{aligned} \psi_n(t) =& \theta(T-t)\sqrt{\frac{2}{T}} \sin(k_n t) \,, \quad \text{with} \quad (-\partial_t^2 - \omega^2)\psi_n = \epsilon_n \psi_n \\ \epsilon =& k_n^2 - \omega^2 \qquad k_n = \frac{n\pi}{T} \qquad n \in \mathbb{N} \end{aligned}$$

Expanding the fluctuations $\eta(t) = \sum_n^\infty a_n \psi_n(t)$ and inserting in (4.18), we find

$$S[q] = S[q_{\rm cl}] + \frac{1}{2}\sum_n \epsilon_n a_n^2 \,, \tag{4.19}$$

where the set $\{a_n\}_{n\in\mathbb{N}}$ parameterises the fluctuations.

For the trace of the time evolution operator we get

$$\mathrm{Tr}\left[e^{-\frac{i}{\hbar}\hat{H}T}\right] = N[T] \int dq_0\, dq_0 e^{\frac{i}{\hbar}S[q_{\rm cl},T]} \int \prod_{n=1}^N da_n e^{\frac{i}{\hbar}\epsilon_n a_n^2} \tag{4.20}$$

A rather tricky calculation, see chapter 1 of reference [15], shows that with a suitable normalisation one finds

$$\mathrm{Tr}\left[e^{-\frac{i}{\hbar}\hat{H}T}\right]=\sum_{n=0}^{\infty}e^{-iT(n+\frac{1}{2})\omega}=\frac{i}{2\sin\left(\frac{T\omega}{2}\right)}\,. \tag{4.21}$$

4.1.4 Generalisation to field theory

We again neglect the zero–mode problem and simply write down the straightforward generalisation of the path integral representation of the propagation kernel (for $T>0$) to bosonic field theory[4]. "ts" indicates that the trace is evaluated in a certain topological sector. This applies to the background of course, but also to the fluctuations.

$$K(T)=\mathrm{Tr}_{\mathrm{ts}}\left[e^{-\frac{i}{\hbar}\hat{H}T}\right]=\int[D\phi_0(x)]_{\mathrm{ts}}\int_{(\phi_0(x),0)}^{(\phi_0(x),T)}[D\phi(x)]_{\mathrm{ts}}e^{\frac{i}{\hbar}S[\phi]} \tag{4.22}$$

The boundary conditions of the one–particle problem translate into

$$\phi(x,0)=\phi(x,T)=\phi_0\rightarrow\eta(x,0)=\eta(x,T)=\eta_0(x) \tag{4.23}$$

when we split $\phi(x)$ into $\phi_{\mathrm{cl}}(x)+\eta(x)$. Expanding the action and imposing the boundary conditions (4.23) leads to

$$S[\phi,T]=S[\phi_{\mathrm{cl}},T]-\frac{1}{2}\int_D d^2x\,\eta\left(-\Box+U''(\phi_{\mathrm{cl}})\right)\eta-\tfrac{1}{2}\int_{\partial D}(2\partial_\mu\phi_{\mathrm{cl}}+\partial_\mu\eta)\eta+\mathcal{O}(\eta^3)\,. \tag{4.24}$$

Reinserting in equation (4.22) gives

$$K(T)=e^{\frac{i}{\hbar}S[\phi_{\mathrm{cl}}]}\int[D\eta_0]\int_{(\eta_0,0)}^{(\eta_0,T)}[D\eta]\exp\left(-\tfrac{i}{2\hbar}\int_D dx\,\eta(-\Box+U''(\phi_{\mathrm{cl}}))\eta+\int_{\partial D}\ldots\right) \tag{4.25}$$

We again expand the fluctuations in eigenfunctions of the spatial part of the operator in the exponent:

$$(-\partial_x^2+U''(\phi_{\mathrm{cl}}))\xi_n=\omega_n\xi_n \qquad\qquad \int_L dx\,\xi_m\xi_n=\delta_{nm} \tag{4.26}$$

with coefficients $\{c_n\}_{n\in\mathbb{N}}$ according to

$$\eta(x,t)=\sum_n c_n(t)\xi_n(x) \qquad \text{and} \qquad \eta_0(x)=\sum_n c_n(0)\xi_n(x)\,. \tag{4.27}$$

This yields for the first term of the exponent

$$\int dx\,\eta\left(-\Box+U''(\phi_{\mathrm{cl}})\right)\eta=\sum_k c_k(t)(\partial_t^2+\omega_k^2)c_k(t)\,. \tag{4.28}$$

[4]More precisely, we look at a two dimensional bosonic field theory for which the action is given by the space–time integral of the Lagrangian density (2.1) (with a generic potential). Generalisations to D dimensional theories can be found in the standard references (e.g. [19, 20]).

Now we investigate the boundary term

$$
\begin{aligned}
-\tfrac{1}{2}\int_D d^2x\,\partial^\mu(2\partial_\mu\phi_{\rm cl}+\partial_\mu\eta)\eta &= \tfrac{1}{2}\int_D dx\,(2\partial_t\phi_{\rm cl}+\partial_t\eta)\eta\Big|_0^T - \tfrac{1}{2}\int_D dt\,(2\partial_x\phi_{\rm cl}+\partial_x\eta)\eta\Big|_{-\infty}^{\infty} \\
&= \tfrac{1}{2}\int_L dx\,\eta\dot\eta\Big|_0^T = \tfrac{1}{2}\sum_{l,k}\int_L dx\,c_l(0)(\dot c_k(T)-\dot c_k(0))\xi_l(x)\xi_k(x) = \\
&= \tfrac{1}{2}\sum_l c_l(0)(\dot c_l(T)-\dot c_l(0)) = \tfrac{1}{2}\sum_l\int_T dt\,\partial_t(c_l(t)\dot c_l(t))\,, \qquad (4.29)
\end{aligned}
$$

where we assumed a static soliton solution and natural boundary conditons. Putting the pieces together

$$
-\tfrac{1}{2}\int_D dx\,\eta(-\Box+U''(\phi_{\rm cl}))\eta - \tfrac{1}{2}\int_{\partial D}\dots \quad = \tfrac{1}{2}\sum_l\int_T dt\,(\dot c_l^2(t)-\omega_l^2c_l^2(t))\,, \qquad (4.30)
$$

the spectral function factorises into harmonic oscillators up to the accuracy of the stationary phase approximation (SPA):

$$
K(T) \overset{SPA}{=} e^{\frac{i}{\hbar}S[\phi_{\rm cl}]}\prod_l\left[\int dc(0)_n\int_{(c(0)_l,0)}^{(c(0),T)}[Dc_l]e^{\frac{i}{2\hbar}\int_T dt\,(\dot c_l^2(t)-\omega_l^2c_l^2(t))}\right]. \qquad (4.31)
$$

We can now calculate the quantum corrections to non–perturbative objects when we plug

$$
K(T) = e^{\frac{i}{\hbar}S[\phi_{\rm cl}]}\prod_l\sum_{n_l=0}^{\infty}e^{-i\omega_lT(n_l+\frac{1}{2})} = e^{\frac{i}{\hbar}S[\phi_{\rm cl}]}\prod_l\frac{1}{2\sinh\left(i\frac{\omega_l}{2}T\right)} \qquad (4.32)
$$

into the Feynman–Kac formula (4.13):

$$
E^{(1)} = E^{(0)} + \hbar\sum_l\frac{\omega_l}{2}\,. \qquad (4.33)
$$

This gives the first order correction to the mass of the ground state in this topological sector. Possible contributions from the counter terms enter if the unrenormalised quantities in $E^{(0)}$ are replaced in this procedure, $E^{(0)} \to E^{(0)} + \delta E$.

As expected, the results of this chapter generalise to the fermionic oscillator if one treats the boundary condition carefully, see [8] and the references given therein.

4.1.5 Embedding soliton solutions in higher dimensions

In the presence of solitons the EOM of the quantum fluctuations around the background differ of course from the trivial case. In a static background, one can separate off the time–dependence just as in the vacuum sector, but in directions where the background is non–trivial one does not find a Helmholtz–type EOM. Typically, the solitons to the resulting eigenvalue can not be given in closed form.

In this section we derive the Fourier decomposition of a quantum field in the presence of a soliton in arbitrary dimensions. To this end we introduce some new notation. Let d be the dimension of space–time and n the number of non–trivial directions $(x^1,\dots,x^n) =: \vec{x}$. The $d-n-1$ trivial directions we denote by $(x^{n+1},\dots,x^{d-1}) =: \vec{y}$. The fluctuation eigenfunction for the n–component momentum $\vec{p}$ is given by $\phi^{(n)}_{\vec{p}}(x)$ and the solution of the free field equation by $\phi^{(d-n-1)}_{\vec{l}}(\vec{y}) = e^{i\vec{l}\vec{y}}$ and they are normalised according to

$$\begin{aligned} \int d^{d-n-1}\vec{y}\, \phi^{(d-n-1)}_{\vec{l}}(\vec{y})\phi^{(d-n-1)^*}_{\vec{l}'}(\vec{y}) =& \frac{1}{(2\pi)^{d-n-1}}\delta^{(d-n-1)}(\vec{l}-\vec{l}') \\ \int d^n\vec{x}\, \phi^{(n)}_{\vec{p}}(\vec{x})\phi^{(n)^*}_{\vec{p}'}(\vec{x}) =& \frac{1}{(2\pi)^n}\delta^{(n)}(\vec{p}-\vec{p}') \end{aligned} \tag{4.34}$$

We get the following expansion of the bosonic fluctuation η

$$\eta(\vec{x},\vec{y},t) = \int \frac{d^{d-n-1}\vec{l}\, d^n\vec{p}}{(2\pi)^{\frac{d-1}{2}}\sqrt{2\omega}} \left[a_{\vec{p},\vec{l}}\, e^{-i(\omega t-\vec{l}\vec{y})}\phi^{(n)}_{\vec{p}}(\vec{x}) + a^\dagger_{\vec{p},\vec{l}}\, e^{i(\omega t-\vec{l}\vec{y})}\phi^{(n)^*}_{\vec{p}}(\vec{x})\right]. \tag{4.35}$$

The same problem for fermions is a bit more involved because one has to take into account the Clifford algebra representation in different dimensions and for example by Schur's lemma we have in odd dimensions ($n \in \mathbb{N}_{\text{odd}}$) $\gamma_0 \cdot \ldots \cdot \gamma_{n-1} \propto \mathbb{1}$. One can therefore not embed the fermionic quantum field in such a general way as the bosonic one. For dimensional regularisation of a $d = 2$, $\mathcal{N} = (2,2)$ SUSY theory featuring a kink background one needs to embed two–dimensional fermions into $2+\varepsilon$ space–time. This is done by solving the $d = 3$ problem and then making the number of the extra dimension continuous. However, before we tackle this problem we need to solve the corresponding $d = 2$ problem. The Dirac equation resulting from dimensional reduction of the former problem has the following structure (see reference [21, 14]) when the trivial direction is put to zero, i.e. $\partial_n \equiv 0$:

$$D\psi_+ - \partial_t\psi_- = 0 \qquad D^\dagger\psi_- - \partial_t\psi_+ = 0\,. \tag{4.36}$$

With the ansatz $\psi_\pm = e^{-i\omega t}\chi^\pm$ we find

$$\begin{pmatrix} D & i\omega_k \\ i\omega_k & D^\dagger \end{pmatrix}\begin{pmatrix} \chi^+_k \\ \chi^-_k \end{pmatrix} = 0 \tag{4.37}$$

and the decomposition

$$\psi(x,t) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \sum\!\!\!\!\!\!\!\int \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2}}\left[a_{\vec{k}}\, e^{-i\omega_k t}\begin{pmatrix} \chi^+_k \\ \chi^-_k \end{pmatrix} + b^\dagger_{\vec{k}}\, e^{i\omega_k t}\begin{pmatrix} \chi^+_k \\ -\chi^-_k \end{pmatrix}\right]. \tag{4.38}$$

For the decomposition in the $2+\varepsilon$ dimensional space we put the soliton into the spatial part of the $1+1$ dimensions and write down the three dimensional Dirac equation

$$D\psi_+ - (\partial_t - \partial_5)\psi_- = 0 \qquad D^\dagger\psi_- - (\partial_t + \partial_5)\psi_+ = 0\,. \tag{4.39}$$

Separating off the trivial momentum l we find

$$\psi(x,y,t) = \int \frac{d^\varepsilon l}{(2\pi)^{\frac{\varepsilon}{2}}} \sum\!\!\!\!\!\!\!\int \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left[a_{\vec{k},l}\, e^{-i(\omega t-yl)}\begin{pmatrix} \alpha\chi^+_k \\ \beta\chi^-_k \end{pmatrix} + b^\dagger_{\vec{k},l}\, e^{i(\omega t-yl)}\begin{pmatrix} \gamma\chi^+_k \\ \delta\chi^-_k \end{pmatrix}\right]. \tag{4.40}$$

From the Dirac equation

$$\begin{pmatrix} D & i(\omega-l) \\ i(\omega+l) & D^\dagger \end{pmatrix}\begin{pmatrix} \alpha\chi^+_k \\ \beta\chi^-_k \end{pmatrix} = 0 \qquad \begin{pmatrix} D & i(-\omega+l) \\ i(-\omega-l) & D^\dagger \end{pmatrix}\begin{pmatrix} \gamma\chi^+_k \\ \delta\chi^-_k \end{pmatrix} = 0 \tag{4.41}$$

we learn that for $\omega^2 = \omega_k^2 + l^2$ the coefficient determinant is zero and α, β, γ and δ are parameterised by

$$\alpha = \omega_k\tilde{\alpha} \qquad \beta = (\omega + l)\tilde{\alpha} \qquad \gamma = \omega_k\tilde{\alpha} \qquad \delta = -(\omega + l)\tilde{\alpha}\,. \tag{4.42}$$

The last input is the anti–commutation relation $\left\{\psi\,,\,\bar{\psi}\right\}$ and the energy of a mode, respectively, it puts $\alpha^2 + \beta^2 = \gamma^2 + \delta^2 = 2\omega$ and thus $\tilde{\alpha} = \frac{1}{\sqrt{\omega_k(\omega+l)}}$. We end up with

$$\psi(x,y,t) = \int \frac{d^\varepsilon l}{(2\pi)^{\frac{\varepsilon}{2}}} \sum\!\!\!\!\!\!\int \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[a_{\vec{k},l}\, e^{-i(\omega t - yl)} \begin{pmatrix} \sqrt{\omega + l}\chi_k^+ \\ \sqrt{\omega - l}\chi_k^- \end{pmatrix} + \right.$$
$$\left. + \, b^\dagger_{\vec{k},l}\, e^{i(\omega t - yl)} \begin{pmatrix} \sqrt{\omega + l}\chi_k^+ \\ -\sqrt{\omega - l}\chi_k^- \end{pmatrix} \right]. \tag{4.43}$$

It is remarkable that this embedding works entirely analogously for the central charge correction[5] and subsequently allows an analogous derivation of quantum corrections from fermions.

[5] And also in the $d = 3 + 1$ monopole case as shown in reference [21, 14];

Chapter 5

The CP^1 σ–model with twisted mass

On ne comprend rien à la vie
tant qu'on n'a pas compris
que tout y est confusion .

Henry de Montherlant

Two dimensional abelian gauge theories with $\mathcal{N} = (2,2)$ supersymmetry exhibit duality, see for instance Witten [22] and Hanany and Hori [6]. One side of this duality can be described by the CP^N σ–model, thus one can get a better understanding of duality, also for more complicated theories, by investigating this toy model. In this work we will not study duality, but we show how one may derive the CP^{N-1} σ–model with twisted mass in the limit of low energy from the former theory and present some results from the dual sector of the theory (for details see [5] and the references therein). Afterwards, in the special case of CP^1, we will perform a classical and quantum analysis of the theory in this phase which is called the Higgs phase of the theory.

5.1 The CP^{N-1} theory as a low energy limit

We start from the Lagrangian density of a superrenormalisable $U(1)$ gauge theory

$$\mathcal{L} = \int d^4\theta \left[\bar{\Phi}_i e^{2V} \Phi_i - \tfrac{1}{4e^2}\bar{\Sigma}\Sigma\right] + \mathcal{L}_F \,, \tag{5.1}$$

where V is the gauge superfield, the Φ_i's, with $i = 1, \ldots, N$, are the chiral superfields, each of charge $+1$, and Σ is the basic gauge invariant field strength of the superspace gauge field.

$$\Sigma = \frac{1}{2\sqrt{2}} \left\{\bar{\mathcal{D}}_+ \,, \mathcal{D}_-\right\} \tag{5.2}$$

is a twisted chiral superfield, a speciality which appears in two dimensions, as it does not exist in four dimensions (for details see reference [23]). Twisted chiral superfields obey the following twisted version of the ordinary chirality condition ($\bar{D}_+\Phi = \bar{D}_-\Phi = 0$):

$$\bar{D}_-\Phi = D_-\Phi = 0 \tag{5.3}$$

where the plus and minus denotes the second and first spinor index, respectively.

The F–term $\mathcal{L}_F$ which contains the Fayet–Iliopoulis term and a topological θ–term[1], is given by

$$\mathcal{L}_F = -rD + \frac{\theta}{2\pi} v_{01} = \int d^2\vartheta \mathcal{W}(\Sigma) + \int d^2\bar{\vartheta} \bar{\mathcal{W}}(\bar{\Sigma}) \tag{5.4}$$

with $\mathcal{W} = i\tau\Sigma/2$, $(\vartheta_1, \vartheta_2) = (\theta_-, \bar{\theta}_+)$, $(\bar{\vartheta}_{\dot{1}}, \bar{\vartheta}_{\dot{2}}) = (\bar{\theta}_-, \theta_+)$ and $\tau = ir + \frac{\theta}{2\pi}$.

For the case $e \gg \Lambda$, where Λ denotes the dynamical scale, we can neglect the kinetic term of the gauge field in equation (5.1) and derive from this new Lagrangian density the following EOM[2]

$$2\bar{\Phi}_i e^{2V} \Phi_i - r = 0\,,$$

which is now an algebraic one. With this, one may integrate out the twisted chiral superfield, and afterwards fix the gauge by gauging one of the nonvanishing fields to unity. Thus the effective superspace Lagrangian with ϕ_j gauged to one becomes

$$\mathcal{L}_{\text{eff}} = r \int d^4\theta \ln\Big(1 + \sum_{\substack{i=1\\ i\neq j}}^{N} \bar{W}_i^{(j)} W_i^{(j)}\Big) + \theta\text{–term}\,,$$

which is the Lagrangian density of a CP^{N-1} σ–model. The bosonic components $w_i^{(j)} = \phi_i/\phi_j$ of the superfield $W_i^{(j)}$, with $i \neq j$, are the coordinates of the coordinate patch $\mathcal{P}_j$ (of the projective space CP^{N-1}) which describe the theory in the vacuum $\mathcal{V}_j$.

5.1.1 Implementation of the twisted mass term

As noticed by Hanany and Hori [6], one may introduce a further relevant parameter for the original gauge theory, namely a twisted mass for the chiral superfields, which corresponds to the expectation value of a background twisted chiral multiplet:

$$\Big\langle \hat{V}_{1\,i} \Big\rangle = \Re(m_i)\,, \quad \Big\langle \hat{V}_{2\,i} \Big\rangle = -\Im(m_i)\,, \quad \Big\langle \hat{V}_{0\,i} \Big\rangle = \Big\langle \hat{V}_{3\,i} \Big\rangle = 0; \tag{5.5}$$

With these background fields the Lagrangian (5.1) becomes

$$\mathcal{L} = \int d^4\theta \left[\bar{\Phi}_i e^{2V + 2\langle \hat{V}_i \rangle} \Phi_i - \frac{1}{4e^2} \bar{\Sigma}\Sigma \right] + \mathcal{L}_F\,. \tag{5.6}$$

For the case $e \gg |m_i - m_j| \gg \Lambda$ we can again integrate out the twisted chiral superfield and use a suitable gauge to get the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = r \int d^4\theta \ln\Big(1 + \sum_{i=1\, i\neq j}^{N} \bar{W}_i^{(j)} \exp(2\langle \hat{V}_i \rangle - 2\langle \hat{V}_j \rangle) W_i^{(j)}\Big) + \theta\text{–term}\,, \tag{5.7}$$

[1]It can be written as total derivative and hence contributes only in a topologically nontrivial sector of the theory.

[2]The topological term does not change the EOM.

for the vacuum $\mathcal{V}_j$. The explicit form of this Lagrangian density for $N = 2$ in terms of the component fields is given by the the following terms

$$\mathcal{L}^{(0)} = -\frac{r}{\rho^2}\left[\partial_\mu \bar{w}\partial^\mu w + |m|^2\,|w|^2 + \frac{\theta}{r2\pi i}\varepsilon^{\mu\nu}\partial_\mu \bar{w}\partial_\nu w\right] \tag{5.8a}$$

$$\mathcal{L}^{(2)} = -\frac{ir}{\rho^2}\Big[\bar{\psi}\gamma^\mu\Big(\frac{\overleftrightarrow{\partial_\mu}}{2} - \frac{1}{\rho}(w^\dagger \overleftrightarrow{\partial_\mu} w)\Big)\psi + i\bar{\psi}\, m_{2\times 2}\,\psi(1 - \frac{2w^\dagger w}{\rho})\Big] \tag{5.8b}$$

$$\mathcal{L}^{(4)} = \frac{r}{\rho^2}\Big[\underbrace{\Big(F - \frac{\bar{w}}{\rho}\psi\psi\Big)}_{=0 \text{ on shell}}\Big(\bar{F} - \frac{w}{\rho}\bar{\psi}\bar{\psi}\Big) - \frac{1}{2\rho^2}\psi\psi\bar{\psi}\bar{\psi}\Big] \tag{5.8c}$$

where $\rho = 1+|w|^2$ and $m_{2\times 2} = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}$. The superscript $^{(i)}$ denotes the number of fermionic fields involved.

5.1.2 The Coulomb phase in a nutshell

In the regime of $\underline{e \ll \Lambda}$ the theory consists of a light gauge multiplet weakly coupled to massive chiral multiplets [22]. In particular the dimensionful gauge coupling is much smaller then the other relevant mass scales and the model can be analysed using ordinary perturbation theory.

A one-loop calculation leads to the following effective twisted superpotential for the gauge field Σ (see equation (110) of [5] and also equation (2.57) of [6])

$$\mathcal{W}_{\text{eff}} = \frac{i}{2}\left(\tau\Sigma - 2\pi i\sum_{i=1}^{N}(\Sigma + m_i)\ln\left(\frac{2}{\mu}(\Sigma + m_i)\right)\right)\,, \tag{5.9}$$

where μ is the renormalisation group subtraction scale. According to chapter 3 the central charges of the supersymmetry algebra are the differences of the vacuum values of the superpotental between which the soliton interpolates. Hence, we find

$$Z_{kl} = 2[\mathcal{W}_{\text{eff}}(e_l) - \mathcal{W}_{\text{eff}}(e_k)] = \frac{1}{2\pi}\left[N(e_l - e_k) - \sum_{i=1}^{N} m_i \ln\left(\frac{e_l - m_i}{e_k + m_i}\right)\right] \tag{5.10}$$

where the e_l for $i = 1, \ldots, N$ stand for the N supersymmetric vacuum values of the bosonic component (σ) of the gauge superfield Σ. By the assumption of BPS saturation the soliton mass is given by $M_{kl} = |Z_{kl}|$

5.2 The classical CP^1 theory

First of all we will deduce the two dimensional $\mathcal{N} = (2,2)$ CP^1 theory from a four dimensional $\mathcal{N} = (1,1)$ CP^1 theory (by dimensional reduction). This will guarantee that later on in the quantum case the dimensional regularisation by embedding can be applied without spoiling supersymmetry. Subsequently we investigate the supersymmetries of the model and look for the Bogomol'yni bound.

5.2.1 Dimensional reduction

Our two dimensional supersymmetric CP^1 model with twisted mass term can be derived by dimensional reduction from a four dimensional one. We will do this in an analogous way as in reference [10], where dimensional reduction is done for super Yang–Mills theories, by making the extra dimensions trivial.

To get the four–dimensional CP^1 theory we start from the Kähler potential

$$K(\bar{\Phi}_i, \Phi_i) = \ln(\bar{\Phi}_i \Phi_i) \qquad \text{with} \quad i = 1, 2\,, \tag{5.11}$$

and use the fact that isometries of a Kähler metric (which are characterised by the Killing potential) can be used to introduce gauge fields (see chapter XX IV of reference [11] for the details of "gauging" a Kähler potential and deriving the component representation of the Lagrangian) but without introducing a kinetic term for the gauge fields. Afterwards one fixes the gauge as in the preceding section[3] and as the final result of this procedure we get the Lagrangian density[4]

$$\mathcal{L} = \int d^4\theta K_m(\Phi, \Phi^\dagger, V) = -g\mathcal{D}_m\phi\mathcal{D}^m\phi^\dagger - ig\psi\sigma^m\mathcal{D}_m\bar{\psi} + \frac{1}{4}R\psi\psi\bar{\psi}\bar{\psi} \tag{5.12}$$

with

$$\mathcal{D}_m\phi = \partial_m\phi - A_m X \qquad \mathcal{D}_m\psi = \partial_m\psi + \Gamma\mathcal{D}_m\phi\psi - A_m\frac{\partial X}{\partial\phi}\psi\,,$$

where $g = \partial_\phi\partial_{\phi^\dagger}K(\phi, \phi^\dagger) = \frac{r}{\rho^2}$ is the Kähler metric, $\Gamma = g^{-1}\partial_\phi g = -2\frac{\phi^\dagger}{\rho}$ the connection, $R = g\partial_{\phi^\dagger}\Gamma = -\frac{2r}{\rho^4}$ the curvature and $X = -i\frac{1}{g}\partial_\phi D = -i\phi$ the Killing vector that follows from the Killing potential ($D = r\frac{\phi^\dagger\phi}{\rho}$). Putting this into equation (5.12) we find

$$\mathcal{L}_{\mathrm{m}} = -\frac{r}{\rho^2}\Big[D_m\phi^\dagger D^m\phi + i\bar{\psi}\gamma^m\big(D_m - 2\frac{\phi^\dagger D_m\phi}{\rho}\big)\psi + \frac{1}{2\rho^2}\psi\psi\bar{\psi}\bar{\psi}\Big] \tag{5.13}$$

where $D_m = \partial_m + iA_m$ and $\gamma^n = \bar{\sigma}^n$.

Now one would first introduce the twisted mass by fixing the values of the static background fields A_m

$$A_0 = 0\,, \qquad A_3\,, \qquad A_x = \Re(m)\,, \qquad A_y = -\Im(m) \tag{5.14}$$

and then dimensional reduce the Lagrangian by taking the fields independent of x and y. But we do it a little bit differently, we introduce the twisted mass and then rotate the coordinate system so that the x'–axis points into the mass direction, see Figure 5.1, and afterwards we make the fields independent of the x'–coordinate but still keep the dependence on the direction where the mass vanishes (y'–direction)[5].

[3]Thus we reduce the target space from $\mathbb{C}^2$ to its physical subspace CP^1 for which we need two patches $\{\mathcal{P}_1\,, \mathcal{P}_2\}$ to cover it.

[4]And as in reference [24] we will use Φ to denote the superfield in CP^1–theory instead of W as in the preceding section.

[5]This will be the ε–dimension when we use dimensional regularisation.

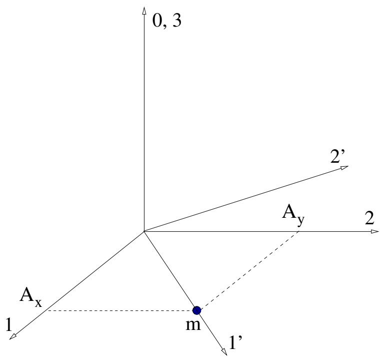


Figure 5.1: We rotate the coordinate system around the z–axe so that the x'–axe points towards the mass direction.

Thus inserting the values of equation (5.14) into (5.13) and applying the following rotation

$$R(\varphi)_n{}^m = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix} \qquad \varphi = \arctan(A_y/A_x)$$

$$\sigma^{2'} = \begin{pmatrix} 0 & \sin(\varphi) - i\cos(\varphi) \\ \sin(\varphi) + i\cos(\varphi) & 0 \end{pmatrix} \qquad \psi \to \psi' = S(\varphi)\psi$$

we get the Lagrangian density

$$\mathcal{L} = -\frac{r}{\rho^2}\Big[\partial_\mu\phi^\dagger\partial^\mu\phi + |m|^2\,\phi^\dagger\phi + i\bar{\psi}'\gamma^\mu\big(\partial_\mu\psi' - \frac{2}{\rho}(\phi^\dagger\partial_\mu\phi)\psi'\big) - \bar{\psi}'\,m_{2\times2}\,\psi'(1 - \frac{2\phi^\dagger\phi}{\rho}) + \frac{1}{2\rho^2}\psi'\psi'\bar{\psi}'\bar{\psi}'\Big] \tag{5.15}$$

where the Greek indices run over the dimensions 0, $2'$ and 3 and the matrix $m_{2\times2}$ is given by

$$m_{2\times2} = \begin{pmatrix} 0 & |m| \\ |m| & 0 \end{pmatrix}. \tag{5.16}$$

If we add a suitable total derivative[6] we finally find the hermitian Lagrangian density

$$\mathcal{L} = -\frac{r}{\rho^2}\Big[\partial_\mu\phi^\dagger\partial^\mu\phi + |m|^2\,\phi^\dagger\phi + i\bar{\psi}'\gamma^\mu\big(\frac{\overleftrightarrow{\partial_\mu}}{2}\psi' - \frac{1}{\rho}(\phi^\dagger\overleftrightarrow{\partial_\mu}\phi)\psi'\big) - \bar{\psi}'m_{2\times2}\psi'(1 - \frac{2\phi^\dagger\phi}{\rho}) + \frac{1}{2\rho^2}\psi'\psi'\bar{\psi}'\bar{\psi}'\Big]. \tag{5.17}$$

5.2.2 Supercharges

We will again start in four dimensions, in order to derive the generators of the supersymmetry transformation, the so–called supercharges. They can be calculated via at least two different ways:

[6] A total derivative neither changes the EOM nor spoils supersymmetry.

1. As a consequence of the algebra (3.1) the supercurrent can be derived by a supersymmetry transformation of the $U(1)$-current:

$$J^\mu = -\frac{r}{\rho^2}\left[i\phi^\dagger \overleftrightarrow{D^\mu}\phi - \bar{\psi}\gamma^\mu\psi(1-2\frac{\phi^\dagger\phi}{\rho})\right] \tag{5.18}$$

2. By making the supersymmetry transformations x-dependent. If an x-independent transformation (not necessarily a supersymmetry transformation) of any Lagrangian density is given by $\delta_\xi\mathcal{L} = \xi\partial_n K^n$ and $\mathcal{L}(\varphi,\partial\varphi)$ is only a function of the fields (φ) and their first derivatives ($\partial\varphi$) then we get for the x-dependent transformations:

$$\delta_\xi\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\xi\Delta\varphi + \frac{\partial\mathcal{L}}{\partial\partial_n\varphi}\partial_n(\xi\Delta\varphi) = \underbrace{\frac{\partial\mathcal{L}}{\partial\varphi}\xi\Delta\varphi + \frac{\partial\mathcal{L}}{\partial\partial_n\varphi}\partial_n(\Delta\varphi)\xi}_{\xi\partial_n K^n} + \frac{\partial\mathcal{L}}{\partial\partial_n\varphi}\partial_n(\xi)\Delta\varphi =$$
$$= \partial_n(\xi K^n) + \left[\frac{\partial\mathcal{L}}{\partial\partial_n\varphi}\Delta\varphi - K^n\right]\partial_n(\xi) = \mathcal{J}^n\partial_n(\xi) + \mathcal{O}(\partial)$$

where $\mathcal{J}^n$ is the conserved current of the symmetry transformation and hence the supercurrent.

We will use the second method[7], so we have to vary the Lagrangian density (5.12) by the supersymmetry transformations of the fields

$$\delta_\xi\mathcal{L} = \delta_\xi\left\{-\frac{r}{\rho^2}\left[D_m\phi^\dagger D^m\phi + i\bar{\psi}\gamma^m(D_m - 2\frac{\phi^\dagger D_m\phi}{\rho})\psi + \frac{1}{2\rho^2}\psi\psi\bar{\psi}\bar{\psi}\right]\right\},$$

where the supersymmetry transformations are given by:

$$\delta_{\xi+\bar{\xi}}\phi = \sqrt{2}\xi\psi \qquad \delta_{\xi+\bar{\xi}}\psi = i\sqrt{2}\sigma^m\bar{\xi}D_m\phi + 2\frac{\phi^\dagger}{\rho}\sqrt{2}(\xi\psi)\psi \tag{5.19a}$$

$$\delta_{\xi+\bar{\xi}}\phi^\dagger = \sqrt{2}\bar{\xi}\bar{\psi} \qquad \delta_{\xi+\bar{\xi}}\bar{\psi} = -i\sqrt{2}\xi\sigma^m D_m\phi^\dagger + 2\frac{\phi}{\rho}\sqrt{2}(\bar{\xi}\bar{\psi})\bar{\psi} \tag{5.19b}$$

After a lengthy calculation, the details of which are given in appendix B, we find

$$\delta_\xi\mathcal{L} = \partial_m\xi\mathcal{J}^m + \mathcal{O}(\partial)$$

with the supercurrent $\mathcal{J}^m$ given by:

$$\mathcal{J}^m = \frac{\sqrt{2}r}{\rho^2}D_n\phi^\dagger\sigma^n\gamma^m\psi \tag{5.20}$$

Finally one obtains the supercharges by integrating the time–component of the supercurrent $\mathcal{J}^0$ over the space dimension(s)

$$Q = \int dv\mathcal{J}^0\,. \tag{5.21}$$

The dimensional reduction will be done in the next subsection together with the central charge.

[7]The second method is preferable since it also allows to check the invariance of the Lagrangian density under the supersymmetry transformations.

5.2.3 Central charges

Looking again at the algebra (3.1) we see that the central charges are given by the anti–commutators of the supercharges. Thus we have to calculate the supersymmetry transformations of the supercurrent to get the current of the central charge from which we find that

$$\delta_\xi \mathcal{J}^m = 0 \tag{5.22a}$$
$$\delta_{\bar\xi} \mathcal{J}^m = \frac{2r}{\rho^2}\bar\xi D_n\bar\psi\sigma^n\gamma^m\psi + \frac{2r}{\rho^2}D_n\phi^\dagger\sigma^n\gamma^m\left(i\sigma^l\bar\xi D_l\phi - \frac{2}{\rho}\bar\xi\bar\psi\phi\psi\right) . \tag{5.22b}$$

The outcome of (5.22a) is clear because we are still in four dimensions. Hence, we have a $\mathcal{N}=1$ supersymmetry[8] which cannot have a central charge. However the RHS of equation (5.22b) does not look like the energy–momentum density $T^n{}_m$. Hence the algebra does not close off shell but on shell (for the details see again appendix B):

$$\delta_{\bar\xi}\mathcal{J}^m = 2i\left\{T'^m{}_n\sigma^n - \sigma^n A_n J^m + \not{\partial}\Lambda^m + \sigma_k \Xi^{km}\right\}\bar\xi \tag{5.23}$$

with

$$\Lambda^m = \frac{ir}{2\rho^2}\bar\psi\gamma^m\psi\,, \qquad \Xi^{km} = \frac{ir}{\rho^2}\epsilon^{nmlk}D_n\phi^\dagger D_l\phi \tag{5.24}$$

and the $U(1)$–current J^m which is given by equation (5.18). Since $[\xi Q,\cdot] = i\delta_\xi$ we only have to multiply (5.23) by i and integrate its time–component over the space dimension(s) to get the following expression of the super algebra:

$$\{Q,\bar Q\} = 2\,\sigma^n\int dv\left(T'^0{}_n - A_n J^0 + \partial_n\Lambda^0 + \eta_{nk}\Xi^{k0}\right) = 2\,\sigma^n P_n$$

Our next step is the dimensional reduction of the previous relation so we rename/redefine the momenta

$$2P_1 =: Z_1 \qquad \text{and} \qquad 2P_2 =: Z_2$$

since they cease to be momenta in two dimensions. Now we go from $3+1$ dimensions (indes m) down to "$1+\varepsilon+1$" dimensions (index μ). Without loss of generality we again choose our coordinate system in such a way that the second dimension is "trivial" and get

$$\begin{aligned} Z^n\sigma_n =& 2\int dv\left(T'^0{}_2\sigma^2 - \sigma^n A_n J^0 + \not{\partial}\Lambda^0 + \sigma_k\Xi^{k0}\right) = \\ =& 2\int dv\Big(T'^0{}_2\sigma^2 - \sigma^1\,|m|\,J^0 + \sigma^k\partial_k\left(\frac{ir}{2\rho^2}\bar\psi\gamma^0\psi\right) + \partial_2\left(\sigma_3\frac{r\,|m|}{\rho}\right) - \partial_3\left(\sigma_2\frac{r\,|m|}{\rho}\right) + \\ & + \sigma^1\frac{ir}{\rho^2}\left(\partial_3\phi^\dagger\partial_2\phi - \partial_2\phi^\dagger\partial_3\phi\right)\Big) \qquad \text{with} \quad n = 1,2 \end{aligned} \tag{5.25}$$

For the classical theory we need only the limit $\varepsilon \to 0$ in which the central charges reduce to

$$Z^n\sigma_n = 2\int dv\Big(-\sigma^1\,|m|\,J^0 - \sigma_2\,\partial_3\frac{r\,|m|}{\rho}\Big) = \epsilon Z' \qquad \text{with} \quad n = 1,2 \tag{5.26}$$

as is expected for a $\mathcal{N}=2$ supersymmetry algebra.

[8] This $\mathcal{N}=1$ supersymmetry in four dimensions will become a $\mathcal{N}=2$ supersymmetry in two dimensions.

5.2.4 Classical BPS saturation

Starting from equation (5.8a) and replacing the fields w and ϕ, respectively, by the real fields φ and α via the coordinate transformation

$$\phi = \tan\left(\frac{\varphi}{2}\right) e^{i\alpha} \tag{5.27}$$

which is a one–to–one mapping of CP^1 onto S^2, we find

$$\mathcal{L}^{(0)} = -\frac{r}{4}\left[\partial_\mu\varphi\,\partial^\mu\varphi + \left(|m|^2 + \partial_\mu\alpha\,\partial^\mu\alpha\right)\sin^2(\varphi)\right] + \frac{\theta}{4\pi}\epsilon^{\mu\nu}\partial_\mu(\cos\varphi)\,\partial_\nu\alpha\,. \tag{5.28}$$

Now we use the methods of section 2.3 to derive the Bogomol'yni bound. First we write down the Hamiltonian for static solutions

$$H[\phi_{\text{st}}] = -L[\phi_{\text{st}}] = \frac{r}{4}\int dz\,\left[(\varphi')^2 + \left(|m|^2 + (\alpha')^2\right)\sin^2(\varphi)\right] = M_{\text{cl}} \tag{5.29}$$

where the last term of (5.28) vanishes for static solutions. Then we reorganise the Lagrangian such that

$$M_{\text{cl}} = \frac{r}{4}\int dz\,\left[\vec{\xi}^2 + \vec{U}^2\right]$$

where $\vec{\xi} = \begin{pmatrix}\alpha'\sin(\varphi)\\ \varphi'\end{pmatrix}$ and $\vec{U} = \begin{pmatrix}0\\ |m|\sin(\varphi)\end{pmatrix}$. Afterwards we go through the remaining steps of section 2.3 to get the following Bogomol'yni equation(s)

$$\vec{\xi} = \pm\vec{U} \qquad \Rightarrow \qquad \frac{\partial\varphi}{\partial z} = \pm\,|m|\sin\varphi \quad \text{and} \quad \frac{\partial\alpha}{\partial z} = 0\,. \tag{5.30}$$

The solution to these equations

$$\varphi_K(z) = 2\arctan\left(e^{|m|(z-z_0)}\right) \qquad \alpha_K = \text{const} \tag{5.31}$$

is a (anti–)kink like soliton that connects the two vacua of the theory, see Figure 5.2.

To get the classical mass of this soliton we have to insert $\phi_K(x)$ into (5.29) and find

$$M_{\text{cl}} = r\,|m| \tag{5.32}$$

Having now the soliton solution and its classical mass we want to end this section by investigating whether this CP^1–kink causes multiplet shortening. As in section 3.2 we have to calculate Z in the presence of a soliton. Thus we put $\phi_K(x)$ into (5.26) and after that compare the result

$$Z = 2r\,|m| \tag{5.33}$$

with M_{cl}. The saturation of the equation $M^2 \geq \frac{1}{4}Z^2$ implies that we get BPS–states and a shortened supersymmetry representation, respectively, at least at classical level[9].

[9]At the end of the next section we will see that this is also true to one loop order.

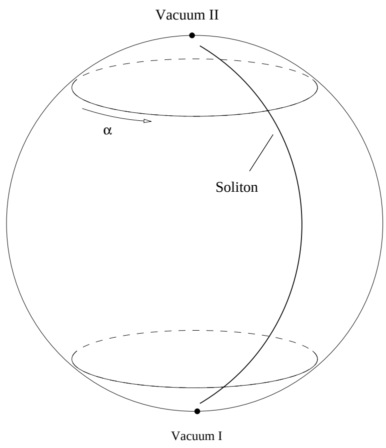


Figure 5.2: The CP^1–soliton mapped to the sphere. (The two different vacua are situated on the south and north pole, respectively)

5.3 Quantum theory of the supersymmetric CP^1 σ–model

In this section we will consider how quantum corrections affect the analysis of the classical theory given above, especially the BPS saturation. To deal with the non-trivial background we must first renormalise the quantum theory in a flat background. (For the details of this procedure we refer to the standard references, e.g. [19].) This will be the task of the following. Then we will apply this theory with all its renormalisation constants to the solitonic sector.

5.3.1 Flat background

Rescaling all the bosonic and fermionic fields in the Lagrangian (5.8), e.g. $\phi = \frac{1}{\sqrt{r}}\tilde{\phi}$ and expanding $\frac{1}{\rho}$ we find

$$
\begin{aligned}
\mathcal{L}^{(0)} &= -\left[\partial_\mu \bar{\bar{\tilde{\phi}}}\partial^\mu \tilde{\phi} + |m|^2 \left|\tilde{\phi}\right|^2 + \frac{g^2\theta}{i2\pi}\epsilon^{\mu\nu}\partial_\mu \bar{\bar{\tilde{\phi}}}\partial_\nu \tilde{\phi}\right] \left(\sum_{n=0}^{\infty}(-1)^n g^{2n}\left|\tilde{\phi}\right|^{2n}\right)^2 \\
\mathcal{L}^{(2)} &= -\, i\left[\bar{\bar{\tilde{\psi}}}\gamma^\mu \frac{\overleftrightarrow{\partial_\mu}}{2}\tilde{\psi} + i\bar{\bar{\tilde{\psi}}}\, m_{2\times 2}\,\tilde{\psi}\right] \left(\sum_{n=0}^{\infty}(-1)^n g^{2n}\left|\tilde{\phi}\right|^{2n}\right)^2 + \\
&\quad + ig^2\left[\bar{\bar{\tilde{\psi}}}\gamma^\mu \tilde{\psi}\,(\bar{\bar{\tilde{\phi}}}\overleftrightarrow{\partial_\mu}\tilde{\phi}) + i2\bar{\bar{\tilde{\psi}}}\, m_{2\times 2}\,\tilde{\psi}\,\bar{\bar{\tilde{\phi}}}\tilde{\phi}\right] \left(\sum_{n=0}^{\infty}(-1)^n g^{2n}\left|\tilde{\phi}\right|^{2n}\right)^3 \\
\mathcal{L}^{(4)} &= -\,\frac{g^2}{2}\tilde{\psi}\tilde{\psi}\bar{\bar{\tilde{\psi}}}\bar{\bar{\tilde{\psi}}} \left(\sum_{n=0}^{\infty}(-1)^n g^{2n}\left|\tilde{\phi}\right|^{2n}\right)^4
\end{aligned}
\tag{5.34}
$$

where we have replaced $\frac{1}{r}$ by g^2. With this redefinition of the fields we can now organise the perturbation theory. We decompose the Lagrangian density into its free part and its perturbation part up to $\mathcal{O}(g^2)$ since we are only interested in its lowest loop corrections. In our case it does not make sense to look at higher corrections in the flat background because with the stationary phase approximation which we will apply in the solitonic background we can only handle one loop order corrections. But in the literatur there are also examples where higher loop calculations have been performed (see e.g. [25] and the references therein).

From equation (5.34) we may read off the **free part** of the Lagrangian

$$\mathcal{L}^{(0)}_{free} = -\left[\partial_\mu \bar{\tilde{\phi}} \partial^\mu \tilde{\phi} + |m|^2 \bar{\tilde{\phi}} \tilde{\phi}\right] \tag{5.35a}$$

$$\mathcal{L}^{(2)}_{free} = -i\left[\tfrac{1}{2}\bar{\tilde{\psi}} \overleftrightarrow{\not\partial}_\mu \tilde{\psi} + i\bar{\tilde{\psi}}\, m_{2\times 2}\, \tilde{\psi}\right] \tag{5.35b}$$

and the **interaction part** up to $\mathcal{O}(\mathbf{g^2})$

$$\mathcal{L}^{(0)}_{\mathcal{O}(g^2)} = 2g^2 \bar{\tilde{\phi}}\tilde{\phi}[\partial_\mu \bar{\tilde{\phi}} \partial^\mu \tilde{\phi} + |m|^2 \bar{\tilde{\phi}}\tilde{\phi}] \tag{5.36a}$$

$$\begin{aligned}\mathcal{L}^{(2)}_{\mathcal{O}(g^2)} =& 2g^2 \bar{\tilde{\phi}}\tilde{\phi} i\left[\tfrac{1}{2}\bar{\tilde{\psi}} \overleftrightarrow{\not\partial}_\mu \tilde{\psi} + i\bar{\tilde{\psi}}\, m_{2\times 2}\, \tilde{\psi}\right] + \\ & + ig^2\left[\bar{\tilde{\psi}}\gamma^\mu \tilde{\psi}\,(\bar{\tilde{\phi}} \overleftrightarrow{\partial_\mu} \tilde{\phi}) + i2\bar{\tilde{\psi}}\, m_{2\times 2}\, \tilde{\psi}\, \bar{\tilde{\phi}}\tilde{\phi}\right]\end{aligned} \tag{5.36b}$$

$$\mathcal{L}^{(4)}_{\mathcal{O}(g^2)} = -\frac{g^2}{2}\tilde{\psi}\tilde{\psi}\bar{\tilde{\psi}}\bar{\tilde{\psi}}\,. \tag{5.36c}$$

Now we are ready to begin with the perturbation theory. In the following we omit the tildes. As usual we derive first the free propagators and the vertices of the interaction and afterwards we calculate the "full" propagator up to one loop order. That means we are working out its loop corrections. As we will see these loop corrections are divergent, so we have to regularise the integrals[10] to handle them systematically and finally get rid of divergences by renormalising the Lagrangian density[11]. So let us start with the propagators.

5.3.1.1 Propagators:

From equation (5.35a) we find the following EOM for the free bosonic field

$$(\partial_\mu \partial^\mu - |m|^2)\phi = 0\,.$$

Using the definition that the propagator is up to a factor i the Green's function[12] of the differential operator that is given by the EOM we can immediately write down the Fourier transform of the **free boson propagator**:

$$D_F(p) = \frac{i}{p^2 + |m|^2 + i\varepsilon}\,. \tag{5.37}$$

We do now the same for the fermions. From equation (5.35b) we get the EOM

$$(-i\not\partial + m_{2\times 2})\psi = 0$$

and the corresponding **free fermion propagator**

$$S_F(p) = \frac{-i}{\not p - m_{2\times 2}} = -i\frac{\sigma^\mu p_\mu - m_{2\times 2}}{p^2 + |m|^2 + i\varepsilon}\,. \tag{5.38}$$

where we used that $\{\sigma^\nu\,,\,\gamma^\mu\} = \{\sigma^\nu\,,\,\bar{\sigma}^\mu\} = -2\eta^{\nu\mu}$. Next we look at the (interaction) vertices.

[10] This done by introducing a regularisation parameter on which the integral continuously depends, e.g. ξ, so that the integral is finite $\forall\ \xi \neq \xi_0$ and in the limit $\xi \to \xi_0$ it becomes the unparametrised one

[11] i.e. one introduces proper counter terms which are dependent on the regularisation parameter so that in thelimit $\xi \to \xi_0$ they cancel the singularities of the loop–integrals

[12] We use the *Feynman contour integral* so that we get the factor $i\varepsilon$ in the denominators of the Fourier transformed Green's functions.

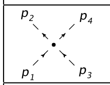
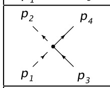
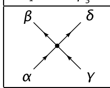
	$= \frac{i}{p^2+\lvert m\rvert^2+i\epsilon}$
	$= -i\frac{\sigma^\nu p_\nu - m_{2\times 2}}{p^2+\lvert m\rvert^2+i\varepsilon}$
	$= i2g^2(4\,\lvert m\rvert^2 + (p_{4\,\mu} + p_{2\,\mu})({p_3}^\mu + {p_1}^\mu))$
	$= -ig^2((\not{p}_2 + \not{p}_1) + 4m_{2\times 2} + (\not{p}_4 + \not{p}_3))$
	$= -2ig^2\varepsilon^{\alpha\gamma}\varepsilon^{\dot\beta\dot\delta}$

Table 5.1: The Feynman rules of the CP^1–theory up to 4–vertex interactions

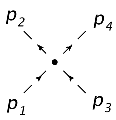
5.3.1.2 Vertices ($\mathcal{O}(\mathbf{g^2})$):

From the interaction part of the Lagrangian one can derive all vertices in momentum space by using the following formula

$$V_{\phi(k_1)\ldots\phi(k_i)\,\bar\phi(k'_1)\ldots\bar\phi(k'_j)\,\psi(p_1)\ldots\psi(p_l)\,\bar\psi(p'_1)\ldots\bar\psi(p'_n)} = (2\pi)^{(i+j+l+n)}\delta(k_1 + \cdots - k'_1 \cdots + p_1 \cdots - p'_n)\cdot$$
$$\cdot\, i\frac{\delta^{(i+j+l+n)}\mathcal{L}_{\text{int}}\big(\phi(x),\bar\phi(x),\psi(x),\bar\psi(x)\big)}{\delta\phi(k_1)\ldots\delta\phi(k_i)\,\delta\bar\phi(k_1)\ldots\delta\bar\phi(k_j)\,\delta\psi(k_1)\ldots\delta\psi(k_l)\,\delta\bar\psi(k_1)\delta\ldots\bar\psi(k_n)} \tag{5.39}$$

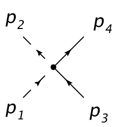
where the $\phi(k)$'s and $\psi(p)$'s are the Fourier transformed fields.

This equation may now easily be applied to (5.36a) and one finds the following vertex

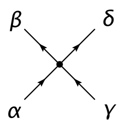


$$= i2g^2(4\,\lvert m\rvert^2 + (p_{4\,\mu} + p_{2\,\mu})({p_3}^\mu + {p_1}^\mu))\,. \tag{5.40}$$

And analogously, from (5.36b) and (5.36c) we get



$$= -ig^2((\not{p}_2 + \not{p}_1) + 4m_{2\times 2} + (\not{p}_4 + \not{p}_3)) \qquad \text{and} \tag{5.41}$$

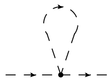
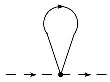


$$= -2ig^2\varepsilon^{\alpha\gamma}\varepsilon^{\dot\beta\dot\delta} \tag{5.42}$$

where we have omitted all δ–functions.

5.3.1.3 Loops and regularisation

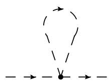
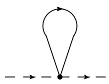
With the Feynman rules, summarised in Table 5.1, we can calculate the one loop corrections of the propagator. For the bosons we find the following:

 + 

$$= \int \frac{d^2p}{(2\pi)^2} \frac{-2g^2(4|m|^2 + (k_\mu + p_\mu)(k^\mu + p^\mu))}{p^2 + |m|^2} - \\ - \int \frac{d^2p}{(2\pi)^2} \mathrm{tr}\left[\frac{\sigma^\nu p_\nu - m_{2\times 2}}{p^2 + |m|^2} g^2 \left(2\not{p} + 4m_{2\times 2} + 2\not{k}\right)\right] =$$

$$= -8g^2|m|^2 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + |m|^2} - 2g^2 \int \frac{d^2p}{(2\pi)^2} \frac{p^2}{p^2 + |m|^2} - 2g^2k^2 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + |m|^2} - \\ - 2g^2 \int \frac{d^2p}{(2\pi)^2} \frac{\mathrm{tr}(\sigma^\nu p_\nu \not{p})}{p^2 + |m|^2} + 4g^2 \int \frac{d^2p}{(2\pi)^2} \frac{\mathrm{tr}(|m|^2_{2\times 2})}{p^2 + |m|^2}.$$

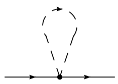
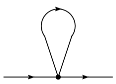
Now we extend the numerator of the second and fourth integral with $|m|^2 - |m|^2$ and apply dimensional regularisation[13]. In the end of this straightforward calculation we obtain

 + 

$$= -i2g^2(k^2 + |m|^2)\left[\frac{\pi^{1+\frac{\epsilon}{2}}}{(2\pi)^{2+\epsilon}}(|m|^2)^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\right] \tag{5.44}$$

where ε is the regularisation parameter. In our case it's an extra dimension since we are using dimensional regularisation. Notice that all the quadratic divergence contributions which one would expect from a naive power counting vanish. This would not have been the case if we had used cutoff regularisation.

For the loop corrections to the fermion propagator we find

 + 

$$= \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + |m|^2} g^2(2\not{k} + 4m_{2\times 2} + 2\not{p}) - \\ - 2g^2 \int \frac{d^2p}{(2\pi)^2} \varepsilon^{\alpha\gamma}\varepsilon^{\dot\beta\dot\delta}\left[\frac{\sigma^\mu p_\mu - m_{2\times 2}}{p^2 + |m|^2}\right]_{\alpha\dot\beta} = -2g^2(\not{k} + m_{2\times 2}) \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + |m|^2} = \\ = -2ig^2(\not{k} + m_{2\times 2})\left[\frac{\pi^{1+\frac{\epsilon}{2}}}{(2\pi)^{2+\epsilon}}(|m|^2)^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\right]. \tag{5.45}$$

Again we get only a logarithmic divergence. The terms which could have led to a linear divergence vanish because the integral of an antisymmetric function over a symmetric interval is zero.

[13] A very useful equation for dimensional regularisation is

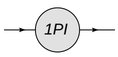
$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \tag{5.43}$$

where l_E is Euclidean momentum. From this equation one immediately sees that scaleless integrals vanish.

5.3.1.4 Renormalisation

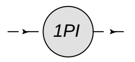
The divergences of the loops can be removed by renormalising the bare constants of the Lagrangian and inserting counter terms into the Lagrangian, respectively. To do this in a consistent way we have to fix the renormalisation conditions.

But first let us define something that we need to write down the renormalisation conditions. A *one–particle irreducible* (1PI) diagram is any diagram that cannot be split in two by removing a single line. Furthermore $-i\Sigma(\not p)$



$$= -\,i\Sigma(\not p) \tag{5.46}$$

denotes the sum of all 1PI diagrams with two external fermion lines and $-i\Xi(p)$ its bosonic counterpart.



$$= -\,i\Xi(p) \tag{5.47}$$

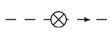
5.3.1.4.1 Renormalisation conditions: We use *on shell* renormalisation. Thus we have the following two renormalisation conditions:

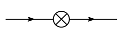
$$\Xi(p^2 = |m|^2) = 0 \tag{5.48a}$$
$$\Sigma(\not p = |m|) = 0 \tag{5.48b}$$

To achieve this we renormalise r_0, the first parameter of the theory.

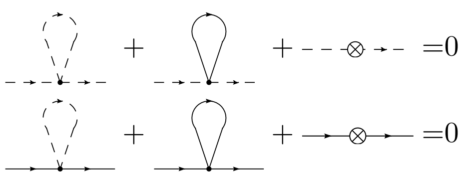
$$r_0 \rightarrow r = r_0 + \delta_{\frac{1}{g_0^2}}$$

Hence we get new terms in the Lagrangian which give two new Feynman rules:


$$= i\delta_{\frac{1}{g_0^2}}(k^2 + m^2) \tag{5.49a}$$


$$= i\delta_{\frac{1}{g_0^2}}(\not k + m_{2\times 2}) \tag{5.49b}$$

Thus the conditions (5.48a) and (5.48b) become up to one loop order:



From this now one can fix the value of $\delta_{\frac{1}{g_0^2}}$ which is given by

$$\delta_{\frac{1}{g_0^2}} = -2\left[\frac{\pi^{1+\frac{\epsilon}{2}}}{(2\pi)^{2+\epsilon}}(|m|^2)^{\frac{\epsilon}{2}}\Gamma\left(-\frac{\epsilon}{2}\right)\right]. \tag{5.50}$$

So we have derived the quantum theory in the flat background up to one loop order.

The result of the renormalisation of the coupling constant is well known in the literature since the (supersymmetric) CP^1 σ–model is a toy model to study asymptotic freedom (see e.g. reference [26] and chapter 13 of reference [19]). From (5.50) it is quite difficult to see that the supersymmetric CP^1 σ–model exhibits asymptotic freedom. But if we had used Pauli–Villars renormalisation and μ as the subtraction point we would have ended up with the following formula for the renormalised coupling constant $g(\mu)$ (see [26])

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{1}{2\pi}\ln\left(\frac{M_{\mathrm{UV}}}{\mu}\right), \tag{5.51}$$

where M_{UV} is the ultra-violet regulator of the Pauli–Villars regularisation. Now looking at (5.51) we see that the model is asymptotically free.

5.3.2 Solitonic background

In this subsection we will derive the quantum corrections to the mass and central charge. We do this by using the stationary phase approximation (SPA) (see also section 4.1.2), that is we replace in the appropriate operators the field operators by their classical values plus the operator valued fluctuations around the non–trivial background. Having the relevant expressions in their expanded form we can use index techniques to calculate their vacuum expectation value (VEV). As we will see these VEV are also divergent thus we use again dimensional regularisation to obtain finite expressions. But to do this in a consistent way we need all expressions in $1 + \varepsilon + 1$ dimension as in section 5.2.1. We begin with the most fundamental, the Lagrangian.

5.3.2.1 The bosonic Lagrangian

Since we use SPA we only treat quadratic fluctuations. The relevant bosonic part of the Lagrangian density (5.17) is thus given by

$$\mathcal{L}^{(0)} = -\frac{r}{\rho^2}(\partial_\mu\bar{\phi}\partial^\mu\phi + |m|^2|\phi|^2) \tag{5.52}$$

where $\rho = 1 + |\phi|^2$ and the Greek indices run over t, y and z. To get a more convenient Lagrangian we replace the field ϕ by $\vec{n}$ by making use of the following transformation (see also [5])

$$\phi = \frac{n_1 + in_2}{1 - n_3} \quad \text{with} \quad \vec{n}\cdot\vec{n} = 1 \quad \Rightarrow \quad \frac{1}{\rho^2} = \frac{1}{4}(1 - n_3)^2$$

which maps the target space of the fields from CP^1 to $O(3)$. Thus the Lagrangian becomes

$$\mathcal{L}^{(0)} = -\frac{r}{4}\big(\partial_\mu n \cdot \partial^\mu n + |m|^2 \underbrace{(n_1^2 + n_2^2)}_{=1-n_3^2}\big)\,.$$

Since we want to expand the fields around their solitonic values we also need the CP^1–kink (5.31) in these new coordinates

$$n_{cl} = (\sin(\varphi_K)\cos(\alpha_K), \sin(\varphi_K)\sin(\alpha_K), -\cos(\varphi_K)) \tag{5.53}$$

where $\varphi_K = 2\arctan\left(e^{|m|(z-z_0)}\right)$ and $\alpha_k = $ const. Now we need a decomposition of the fields into the solitonic parts and the fluctuations which still fulfils the constraint $\vec{n}\cdot\vec{n} = 1$, at least to lowest order. Such a decomposition is given by

$$n = n_{cl} + \delta n = n_{cl} + u_1 \hat{e}_\theta\left(\pi - \varphi_K, \alpha_K\right) + u_2 \hat{e}_\varphi\left(\pi - \varphi_K, \alpha_K\right) \tag{5.54}$$

where $\hat{e}_\theta$ and $\hat{e}_\varphi$ are unit vectors of the spherical coordinate system. With this and equation (4.15) we find the following expansion of the Lagrangian density:

$$\mathcal{L}^{(0)} = \mathcal{L}^{(0)}[\varphi_K, \alpha_K] - \frac{r}{4} \underbrace{(u_1, u_2)}_{=\vec{u}^T} \cdot M_B \cdot \vec{u} + \partial_\mu \mathcal{B} + \mathcal{O}(\vec{u}^3) \tag{5.55}$$

where $\mathcal{B}$ stands for the boundary terms. Without loss of generality we set α_K to zero. To get the concrete form of the matrix M_B we need some relations for the fluctuations.

$$\begin{aligned}
\delta n &= (u_1 \cos(\varphi_K), u_2, u_1 \sin(\varphi_K)) \\
\partial_z \delta n &= (\partial_z u_1 \cos(\varphi) - u_1 |m| \sin^2\varphi, \partial_z u_2, \partial_z u_1 \sin(\varphi) + u_1 |m| \sin\varphi\cos\varphi) \\
-\partial_t \delta n \cdot \partial_t \delta n + \partial_y \delta n \cdot \partial_y \delta n &+ \partial_z \delta n \cdot \partial_z \delta n + |m|^2(\delta n_1^2 + \delta n_2^2) = u_i(-\partial_\mu\partial^\mu + |m|^2)u_i + \\
&+ \partial_\mu(u_i\,\partial^\mu u_i)
\end{aligned}$$

Using these equations we easily find

$$M_B = \begin{pmatrix} -\partial_\mu\partial^\mu + |m|^2 & 0 \\ 0 & -\partial_\mu\partial^\mu + |m|^2 \end{pmatrix},$$

which is quite different to equation (72) of reference [5]. So our result corrects the calculation done in [5].

5.3.2.2 The fermionic Lagrangian

The relevant fermionic part of the Lagrangian density (5.17) is given by

$$\mathcal{L}^{(2)} = -\frac{r}{\rho^2}\left[i\bar{\psi}\gamma^\mu\left(\frac{\overleftrightarrow{\partial_\mu}}{2} - \frac{1}{\rho}(\phi^\dagger \overleftrightarrow{\partial_\mu}\phi)\right)\psi - \bar{\psi} m_{2\times 2}\psi\left(1 - \frac{2\phi^\dagger\phi}{\rho}\right)\right]. \tag{5.56}$$

We again expand the Lagrangian into its classical value and its fluctuation part. With

$$\psi = \psi \qquad\qquad \phi = \phi_k + \delta\phi \tag{5.57}$$

we find

$$\mathcal{L}^{(2)} = -r\bar{\psi}M_F\psi + \mathcal{O}(\hat{o}^3) \tag{5.58}$$

where $\hat{o}$ represents the operators ψ and $\delta\phi$. Using

$$\phi_K^\dagger \overleftrightarrow{\partial_\mu}\phi_K = 0 \qquad \text{and} \qquad 1 - \frac{2\phi^\dagger\phi}{\rho} = \cos(\varphi_K) \tag{5.59}$$

we get

$$M_F = \begin{pmatrix} -i\frac{\overleftrightarrow{\partial_t}}{2\rho_{cl}^2} - i\frac{\overleftrightarrow{\partial}_z}{2\rho_{cl}^2} & -\frac{\overleftrightarrow{\partial_y}}{2\rho_{cl}^2} - \frac{1}{\rho_{cl}^2}|m|\cos(\varphi) \\ \frac{\overleftrightarrow{\partial_y}}{2\rho_{cl}^2} - \frac{1}{\rho_{cl}^2}|m|\cos(\varphi) & -i\frac{\overleftrightarrow{\partial_t}}{2\rho_{cl}^2} + i\frac{\overleftrightarrow{\partial}_z}{2\rho_{cl}^2} \end{pmatrix}.$$

Now we perform two transformations to get a more convenient form of the matrix M_F. First we replace ψ by $\frac{\rho_{cl}}{\sqrt{2}}\psi'$. Hence we get rid of the z-dependent factor $\frac{1}{\rho_{cl}^2}$. Then we rotate the spinors

$$\psi'' = \exp(-i\frac{\pi}{4}\bar{\sigma}^1)\psi' \,. \tag{5.60}$$

Thus the Lagrangian density becomes

$$\mathcal{L}_F = -r\bar{\psi}'' M_F \psi'' \qquad M_F = \frac{1}{2}\begin{pmatrix} -i(\frac{\overleftrightarrow{\partial_t}}{2} + \frac{\overleftrightarrow{\partial_y}}{2}) & \frac{\overleftrightarrow{\partial_z}}{2} - |m|\cos(\varphi_K) \\ -\frac{\overleftrightarrow{\partial_z}}{2} - |m|\cos(\varphi_K) & -i(\frac{\overleftrightarrow{\partial_t}}{2} - \frac{\overleftrightarrow{\partial_y}}{2}) \end{pmatrix} . \tag{5.61}$$

And as a last step we introduce the differential operators

$$D = \partial_z + |m|\cos(\varphi_K) \qquad \text{and} \qquad D^T = -\partial_z + |m|\cos(\varphi_K) \tag{5.62}$$

which fulfil the following relations

$$DD^T = -\partial_z^2 + \cos(2\varphi_K)|m|^2 \qquad \text{and} \qquad D^T D = -\partial_z^2 + |m|^2 \,, \tag{5.63}$$

to get a more compact form for M_F[14]

$$M_F = \frac{1}{2}\begin{pmatrix} -i(\partial_t + \partial_y) & -D^T \\ -D & -i(\partial_t - \partial_y) \end{pmatrix}$$

Having derived the stationary phase approximation of the Lagrangian density we tackle the energy momentum tensor.

5.3.2.3 The energy–momentum tensor

The energy–momentum tensor is the current of the translational symmetry of the Lagrangian. With the methods of appendix A.3 we can get its generic form for a flat space–time

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i}\partial_\nu \varphi^i - \eta^{\mu}{}_{\nu}\mathcal{L} \tag{5.64}$$

where the φ^i's stand for all the fields on which the Lagrangian depends. For the energy density we find the following fluctuation expansion:

$$T_{00} = T_{00}^{\rm cl}(\phi_K) + \frac{r}{4}\left[(\dot{u}_i\dot{u}_i + u_{i,x}u_{i,x} + |m|^2 u_i u_i) - \frac{r}{2}\bar{\psi}\begin{pmatrix} i\partial_y & D^T \\ D & -i\partial_y \end{pmatrix}\psi\right] \tag{5.65}$$

where the term from the fluctuations is denoted as $T_{00}^{(1)}$. If we use the EOM of the fluctuation which can be derived from the Lagrangians (5.55) and (5.58) (see 5.3.2.5) we can rewrite it a bit and find that

$$T_{00}^{(1)} = \frac{r}{2}\left[\dot{u}_i\dot{u}_i + \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial}_t\psi\right] \,. \tag{5.66}$$

To get the energy itself we only have to integrate T_{00} over the whole space.

$$E = \int dv\, T_{00} \tag{5.67}$$

The last quantity which we calculate is the central charge.

[14]Since we are only interested in small fluctuations and not in global objects the final matrix M_F is equal modulo total derivatives to the previous M_F.

5.3.2.4 The central charge

Since the expansion of the central charge is a bit lenghty we will only give the result for each term (the explicit calculations are given in appendix B.1.4.1). We start with the first expression of 5.25 and then give the others:

$$\int dz\, T'^0{}_2 = -\frac{r}{2}\int dz\; \left[\partial_2 u_i \partial^0 u_i + i\bar{\psi}'\gamma^0\partial_2\psi' + \mathcal{O}(\hat{\sigma}^3)\right] \tag{5.68a}$$

$$\int dz\, J^0 = \int dz\; \left[\cos(\phi_K)\left(\epsilon_{3ij}u_i\partial_0 u_j + \bar{\psi}'\gamma^0\psi'\right) + \mathcal{O}(\hat{\sigma}^3)\right] \tag{5.68b}$$

$$\int dz\, \partial_k\left(\frac{ir}{2\rho^2}\bar{\psi}\gamma^0\psi\right) = \int dz\, \partial_k\left(\frac{ir}{4}\bar{\psi}'\gamma^0\psi'\right) \tag{5.68c}$$

$$\int dz\, \partial_2\left(\frac{r\,|m|}{\rho}\right) = -\frac{r\,|m|}{2}\int dz\; \sin(\phi_k)\partial_2 u_1 \tag{5.68d}$$

$$\int dz\, \partial_2\left(\frac{r\,|m|}{\rho}\right) = -\,r\,|m|\,\frac{1}{\rho_{\rm cl}}\Big|_{-\infty}^{\infty} + \frac{r\,|m|}{2}u_i u_i \tag{5.68e}$$

where the u_i and ψ' are the bosonic and fermionic fluctuations from the previous section.

5.3.2.5 Quantum fluctuations with index techniques

In this section we finally want to calculate the quantum corrections of the mass and the central charge. These corrections are nothing more than the VEV of the operators we just derived. Supersymmetry will help us work out these VEV's since we can use techniques of index theorem calculations as we will see.

Using the Lagrangian's (5.55) and (5.58) we get the EOM of the fluctuations u_i and ψ.

$$\begin{array}{cc} \text{Fermions:} & \text{Bosons:} \\ \begin{array}{r} -D^T\psi_- = i(\partial_t + \partial_y)\psi_+ \\ -D\psi_+ = i(\partial_t - \partial_y)\psi_- \end{array} & (\partial_x^2 - |m|^2)u_i = (\partial_t^2 - \partial_y^2)u_i \end{array} \tag{5.69}$$

Since the commutator $[D, \partial_t]$ vanishes, we can iterate the fermionic part and get the following equations:

$$D^T D\psi_+ = -(\partial_t^2 - \partial_y^2)\psi_+ \qquad\qquad D D^T\psi_- = -(\partial_t^2 - \partial_y^2)\psi_-\,; \tag{5.70}$$

From (5.63) we see that the equation for ψ_+ is the same as for u_1 and u_2.
Now we have all ingredients to follow step by step reference [21] to get the quantum fluctuations in the soliton background. But first we briefly summarise the points which we will process afterwards:

1. We make a separation ansatz for $\psi_\pm$ and u_i so that the (iterated) EOM become eigenvalue equations.
2. We write down the the normalised eigen–mode expansion.
3. We insert the eigen–mode representation of ψ and u_i into the operators for the mass and central charge correction and calculate their VEV's. We will find that some of them are functions of the mode density difference.

4. We will use index techniques to get this mode density difference $\Delta\rho$ (see references [10, 27] for details on the calculations and reference [15] for index theorems in general).

ad 1. As in section 4.1.5 we separate off the time and y–dependence. To do this we use the following ansatz:

$$u_i(z,y,t) = \int \frac{d^\varepsilon l}{(2\pi)^{\varepsilon/2}} \sum\!\!\!\!\!\!\int u_{i\,k}(z)\exp(i(\omega t - ly)) \tag{5.71a}$$

$$\psi_\pm(z,y,t) = \int \frac{d^\varepsilon l}{(2\pi)^{\varepsilon/2}} \sum\!\!\!\!\!\!\int \chi_k^\pm(z)\exp(i(\omega t - ly)) \tag{5.71b}$$

Putting this ansatz into (5.69) and (5.70), respectively, we get

$$D^T D\chi_k^+ = (\omega^2 - l^2)\chi_k^+ \quad D^T D u_{i\,k} = (\omega^2 - l^2)u_{i\,k} \quad DD^T\chi_k^- = (\omega^2 - l^2)\chi_k^- \,. \tag{5.72}$$

ad 2. Based on the fact that one may in principle solve these eigenvalue equations[15] we can write down the explicit mode representation for ψ

$$\begin{aligned}\psi(z,y,t) = \int \frac{d^\varepsilon l}{(2\pi)^{\varepsilon/2}} \sum\!\!\!\!\!\!\int \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\Bigg[b_k e^{-i(\omega t - ly)} \begin{pmatrix} \sqrt{\omega + l}\chi_k^- \\ \sqrt{\omega - l}\chi_k^+ \end{pmatrix} + \\ + d_k^\dagger e^{i(\omega t - ly)} \begin{pmatrix} \sqrt{\omega + l}\chi_k^- \\ -\sqrt{\omega - l}\chi_k^+ \end{pmatrix}\Bigg] + \text{zero modes}\end{aligned} \tag{5.73}$$

where the $d_k^\dagger$'s and b_k's are the fermionic creation and annihilation operators ($b_k|0\rangle = d_k|0\rangle = a_k|0\rangle = 0$), respectively. And analogously, we find for u_i

$$u_i(z,y,t) = \int \frac{d^\varepsilon l}{(2\pi)^{\varepsilon/2}} \sum\!\!\!\!\!\!\int \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}(a_{i\,k}e^{-i(\omega t - ly)}\underbrace{u_{i\,k}}_{=\chi_k^+} + h.c.) + \text{zero modes} \tag{5.74}$$

with the bosonic annihilation operators $a_{i\,k}$.

ad 3. First we work out the energy corrections. We insert the results of equation (5.73) and (5.74) into (5.66) and use the following commutator relations[16]

$$\{b_k, b_{k'}^\dagger\} = \tfrac{2}{r}\delta(k-k') \qquad \{d_k, d_{k'}^\dagger\} = \tfrac{2}{r}\delta(k-k') \qquad [a, a_{k'}^\dagger] = \tfrac{2}{r}\delta(k-k') \tag{5.75}$$

we get:

$$\begin{aligned} M^{(1)bulk} &= \langle T_{00}\rangle = M_b^{(1)bulk} + M_f^{(1)bulk} = \\ &= \int dz \int \frac{d^\epsilon l}{(2\pi)^\epsilon}\int \frac{dk}{2\pi}\frac{\omega}{2}|u_{i\,k}|^2 - \int dz \int \frac{d^\epsilon l}{(2\pi)^\epsilon}\int \frac{dk}{2\pi}\frac{\omega}{2}\left[\left|\chi_k^+\right|^2 + \left|\chi_k^-\right|^2\right] = \\ &= \int dz \int \frac{d^\epsilon l}{(2\pi)^\epsilon}\int \frac{dk}{2\pi}\frac{\omega}{2}\left[\left|\chi_k^+\right|^2(z) - \left|\chi_k^-\right|^2(z)\right] \end{aligned}$$

[15] Due to supersymmetry we do not have to solve them.

[16] From the canonical commutator relations follow that the energy operator has to be $\hat{E} = \sum_k \omega_k(\hat{N}_k + \frac{1}{2})$. So we get the normalisation factor $\frac{2}{r}$ for the (anti)commutation relations of the Fourier components.

where we used that $u_{i\,k} = \chi_k^+$. Defining the spectral density of $M^{(1)\,\text{bulk}}$ and mode density difference, respectively, by

$$\Delta\rho(k^2) := \int dz \left[\left|\chi_k^+\right|^2(z) - \left|\chi_k^-\right|^2(z)\right] \tag{5.76}$$

we can write the previous result in a more compact form:

$$M^{(1)\,\text{bulk}} = \int \frac{d^\epsilon l}{(2\pi)^\epsilon} \int \frac{dk}{2\pi}\frac{\omega}{2}\Delta\rho(k^2) \tag{5.77}$$

Now we come to the central charge. The calculations are again much longer than for the mass so we present only the results and refer to appendix B.1.4.1 for the details.

$$\left\langle \int dz\, T'^0{}_2 \right\rangle = \int dz \int \frac{dk}{2\pi} \int \frac{d^\epsilon l}{(2\pi)^\epsilon}\frac{l^2}{2\omega}\left[\left|\chi_k^+\right|^2(z) - \left|\chi_k^-\right|^2(z)\right] \tag{5.78a}$$

$$\left\langle \int dz\, J^0 \right\rangle = 0 \tag{5.78b}$$

$$\left\langle \int dz\, \partial_k \left(\frac{ir}{4}\bar{\psi}'\gamma^0\psi'\right) \right\rangle = 0 \tag{5.78c}$$

$$-\frac{r\,|m|}{2}\left\langle \int dz\, \sin(\phi_k)\partial_2 u_1 \right\rangle = 0 \tag{5.78d}$$

$$\frac{r\,|m|}{2}\langle u_i u_i\rangle = |m|\, I \tag{5.78e}$$

$$\left\langle \int dz\, \frac{ir}{\rho^2}\left(\partial_3\phi^\dagger\partial_2\phi - \partial_2\phi^\dagger\partial_3\phi\right) \right\rangle = 0 \tag{5.78f}$$

where $I = \int \frac{d^{1+\varepsilon}k}{(2\pi)^{1+\varepsilon}} \frac{1}{\sqrt{k^2+|m|^2}}$. Using the spectral density of $M^{(1)\,\text{bulk}}$ we find that

$$\langle Z\rangle = Z_{\text{cl}} + 2\left[\int \frac{dk}{2\pi}\int \frac{d^\epsilon l}{(2\pi)^\epsilon}\frac{l^2}{2\omega}\Delta\rho(k^2) - |m|\, I\right] \tag{5.79}$$

where Z_{cl} is given by equation (5.33).

ad 4. To calculate the mode density difference $\Delta\rho(k^2)$ we define the quantity $\mathcal{J}(M^2)$

$$\mathcal{J}(M^2) := \text{Tr}\left(\frac{M^2}{D^TD + M^2} - \frac{M^2}{DD^T + M^2}\right) \tag{5.80}$$

which in the limit $M^2 \to 0$ gives the index of the operator D. Now we use the fact that D^TD and DD^T exhibit the same non-zero eigenvalues. Hence, we find

$$\mathcal{J}(M^2) - \mathcal{J}(0) = \mathcal{J}_\Delta(M^2) = \int \frac{dk}{2\pi}\frac{M^2}{\omega^2 + M^2}\Delta\rho(k)\,. \tag{5.81}$$

Introducing the operator

$$\not{D} := i\sigma_2 \begin{pmatrix} D & 0 \\ 0 & D^T \end{pmatrix} = \begin{pmatrix} 0 & D^T \\ -D & 0 \end{pmatrix} \qquad \text{with} \qquad \not{D}\not{D} = \begin{pmatrix} -D^TD & 0 \\ 0 & -DD^T \end{pmatrix} \tag{5.82}$$

we can rewrite $\mathcal{J}(M^2)$ as

$$\mathcal{J}(M^2) = -\mathrm{Tr}\left(\frac{M^2}{-\not{D}^2 + M^2}\gamma^5\right) \qquad \text{with} \qquad \gamma^5 = -\sigma^3 \tag{5.83}$$

or, more explicitly, if we write out Tr

$$\mathcal{J}(M^2) = -\int dz\,\mathrm{tr}\left[\left\langle z\left|\frac{M^2}{-\not{D}^2 + M^2}\right|z\right\rangle\gamma^5\right] = \int dz\,J(z,z,M^2) \tag{5.84}$$

where $J(z,z',M^2)$ is the kernel of the operator tr $\left[\frac{M^2}{-\not{D}^2+M^2}\gamma^5\right]$.
Using that $(-\not{D} + M)(\not{D} + M) = -\not{D}^2 + M^2$ and the fact that the trace of an odd number of Pauli matrices vanishes, we can rewrite (5.84) slightly. We find

$$J(x,y,M^2) = -\mathrm{tr}\left[\gamma^5\left\langle x\left|\frac{M}{\not{D} + M}\right|y\right\rangle\right] = -M\mathrm{tr}\big[\gamma^5\Delta(x,y)\big] \tag{5.85}$$

where we have introduced the propagator

$$\Delta(x,y) = \left\langle x\left|\frac{1}{\not{D} + M}\right|y\right\rangle . \tag{5.86}$$

From the following identities

$$\big[\not{D}(x) + M\big]\,\Delta(x,y) = \delta(x-y) \tag{5.87a}$$

$$\Delta(x,y)\left[\overleftarrow{\not{D}}(y) + M\right] = \delta(x-y) \tag{5.87b}$$

one deduces

$$J(x,y,M^2) = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\mathrm{tr}[\gamma^5\sigma^1\Delta(x,y)] - \frac{1}{2}\mathrm{tr}[\gamma^5(K(x) - K(y))\Delta(x,y)]$$

where $K(x) = i\sigma^2\,|m|\cos(\phi_K)$. For the limit $x \to y$ the last term of the previous equation vanishes. Putting this into (5.84) we find

$$\begin{aligned}
\mathcal{J}(M^2) = \int dz\,J(z,z,M^2) &= \tfrac{1}{2}\int dz\,\frac{\partial}{\partial z}\mathrm{tr}\left[\gamma^5\sigma^1\Delta(z,z)\right] = \tfrac{1}{2}\mathrm{tr}\left[\gamma^5\sigma^1\Delta(z,z)\right]\Big|_{-\infty}^{\infty} = \\
&= \tfrac{1}{2}\int\frac{dk}{2\pi}\mathrm{tr}\left[\gamma^5\sigma^1\langle z|k\rangle\langle k|\frac{1}{\not{D} + M}|z\rangle\right]\Bigg|_{-\infty}^{\infty} = \\
&= \tfrac{1}{2}\int\frac{dk}{2\pi}\mathrm{tr}\left[\gamma^5\sigma^1\frac{1}{i\sigma^1 k + i\sigma^2\,|m|\cos(2\varphi) + M}\right]\Bigg|_{-\infty}^{\infty} = \\
&= \tfrac{1}{2}\int\frac{dk}{2\pi}\mathrm{tr}\left[\gamma^5\sigma^1\frac{i\sigma^1 k + i\sigma^2\,|m|\cos(2\varphi) - M}{-k^2 - |m|^2\cos^2(2\varphi) - M^2}\right]\Bigg|_{-\infty}^{\infty} = \\
&= |m|\int\frac{dk}{\pi}\frac{1}{k^2 + |m|^2 + M^2} = \frac{|m|}{\sqrt{|m|^2 + M^2}} .
\end{aligned}$$

From equation (5.81) we immediately see that

$$\frac{|m|}{\sqrt{|m|^2+M^2}}-1=\int\frac{dk}{2\pi}\frac{M^2}{\omega^2+M^2}\Delta\rho(k)\,. \tag{5.88}$$

One can solve this integral equation by a Laplace transform, and the result is

$$\Delta\rho(k^2)=\frac{-2\,|m|}{k^2+|m|^2}\,. \tag{5.89}$$

Having derived the spectral density of the mass operator we can put it back into (5.77) and get

$$M^{(1)\mathrm{bulk}}=\int\frac{dk}{(2\pi)^{1+\epsilon}}\frac{\omega}{2}\Delta\rho=\int\frac{dkd^{\epsilon}l}{(2\pi)^{1+\epsilon}}\frac{\sqrt{k^2+l^2+|m|^2}}{2}\frac{-2\,|m|}{k^2+|m|^2}\,.$$

By twice using (5.43)

$$\begin{aligned}M^{(1)\,\mathrm{bulk}}&=\int\frac{dk}{(2\pi)^{1+\epsilon}}\frac{-\,|m|}{k^2+|m|^2}\pi^{\epsilon/2}(k^2+|m|^2)^{\frac{\epsilon}{2}+\frac{1}{2}}\frac{\Gamma(-\frac{\epsilon}{2}-\frac{1}{2})}{\Gamma(-\frac{1}{2})}=\\&=\frac{-\,|m|\,\pi^{\epsilon/2}}{(2\pi)^{1+\epsilon}}\frac{\Gamma(-\frac{\epsilon}{2}-\frac{1}{2})}{\Gamma(-\frac{1}{2})}\int dk\frac{1}{(k^2+|m|^2)^{\frac{1}{2}-\frac{\epsilon}{2}}}=\\&=\frac{|m|\,\pi^{\epsilon/2}}{(2\pi)^{1+\epsilon}}\frac{2\Gamma(-\frac{\epsilon}{2}+\frac{1}{2})}{(1+\epsilon)\Gamma(-\frac{1}{2})}\pi^{\frac{1}{2}}\left|m^2\right|^{\frac{\epsilon}{2}}\frac{\Gamma(\frac{-\epsilon}{2})}{\Gamma(\frac{1}{2}-\frac{\epsilon}{2})}\end{aligned}$$

we find that

$$M^{(1)\,\mathrm{bulk}}=-\,|m|\,\frac{1}{1+\epsilon}I \tag{5.90}$$

with $I=\int\frac{d^{1+\epsilon}k}{(2\pi)^{1+\epsilon}}\frac{1}{\sqrt{k^2+|m|^2}}$. According to the end of section 4.1.4 and [21, 28] the energy of the ground state and the mass of the CP^1–kink, respectively, is given by:

$$M=M_{\mathrm{cl}}+M^{(1)\,\mathrm{bulk}}=r_0\,|m|+M^{(1)\,\mathrm{bulk}}=(r_0+\delta_{\frac{1}{g_0^2}})\,|m|-\frac{|m|}{\pi}+\mathcal{O}(\varepsilon) \tag{5.91}$$

Thus, the correction of the mass is

$$M_{\mathrm{cor}}=-\frac{|m|}{\pi}+\mathcal{O}(\varepsilon)\,. \tag{5.92}$$

To get the quantum corrections of the central charge we insert (5.89) into (5.79). Working out this equation we find that

$$\langle Z\rangle=2\left(-\frac{|m|}{\pi}+r_0\,|m|-|m|\,I\right)\,. \tag{5.93}$$

As expected we got also in the quantum regime that $M^2=\frac{1}{4}Z^2$. Hence, the Bogomol'yni bound saturation is preserved up to one loop order.

5.3.3 Quantisation of the effective action

So far we have only discussed the nonzero modes of the CP^1–kink but in a semicassical treatment we can also handle the zero modes. As the name suggests these fluctuations cost no energy to excite, hence, these are flat directions of the potential. According to section 2.5 these flat directions parametrise the moduli space. And by making these moduli coordinates time dependent we get an effective Lagrangian as for the kink.

So let us look at the solution of the CP^1–kink (5.31). We see that the bosonic moduli parameter are given by z_0, the kink position, and α which is the azimuth angle if we map the CP^1 onto the sphere (see Figure 5.2). Finding the fermionic moduli parameter is a bit more tricky. We will do it explicit along the lines of reference [10].

We break up the Lagrangian density (5.8) into kinetic minus potential terms $\mathcal{L} = \mathcal{T} - \mathcal{V}$, where

$$\mathcal{T} = \frac{r}{\rho^2}\Big[\partial_0\phi^\dagger\partial_0\phi - \frac{\theta}{r2\pi i}\varepsilon^{01}\partial_{[0}\phi^\dagger\partial_{1]}\phi - i\bar{\psi}\gamma^0\big(\partial_0\psi - \frac{2}{\rho}(\phi^\dagger\partial_0\phi)\psi\big)\Big] \tag{5.94}$$

and

$$\begin{aligned}\mathcal{V} = \frac{r}{\rho^2}\Big[&\partial_3\phi^\dagger\partial_3\phi + |m|^2\,\phi^\dagger\phi + i\bar{\psi}\gamma^3\big(\partial_3\psi - \frac{2}{\rho}(\phi^\dagger\partial_3\phi)\psi\big)\\ &- \bar{\psi}\,m_{2\times2}\,\psi(1 - \frac{2\phi^\dagger\phi}{\rho}) + \frac{1}{2\rho^2}\psi\psi\bar{\psi}\bar{\psi}\Big]\end{aligned} \tag{5.95}$$

The potential is then the integral $V = \int dz\mathcal{V}$. For the BPS–solution (5.31) the potential is given by

$$V = r\,|m| + \int dz\frac{r}{\rho^2}\Big[i\bar{\psi}\gamma^3\big(\partial_3 - \frac{2\,|m|}{\rho}\phi^\dagger\phi\big)\psi - \bar{\psi}\,m_{2\times2}\,\psi(1 - \frac{2\phi^\dagger\phi}{\rho})\Big], \tag{5.96}$$

where we omitted the last term of (5.95). If we now can find a solution to the following Dirac equation

$$\Big[i\gamma^3\big(\partial_3 - \frac{2\,|m|}{\rho}\phi^\dagger\phi\big) - m_{2\times2}(1 - \frac{2\phi^\dagger\phi}{\rho})\Big]\,\psi = 0\,, \tag{5.97}$$

the potential does not get changed, hence, the solution to this equation is a fermionic zero mode. Using the facts that the supercharges commute with the Hamiltonian and that the CP^1–kink is the solution to the bosonic counterpart of (5.97) we only have to apply the supersymmetry transformations (5.19) on (5.31). We obtain

$$\psi = i\sqrt{2}\,|m|\,(\sigma^3 + i\sigma^1)\bar{\xi}\,\phi_K\,, \tag{5.98}$$

where $\bar{\xi}$ is $(1,0)^T$. Having this fermionic zero modes we may introduce the anticommuting coordinates η (which parametrise the fermionic moduli space). We just multiply the ψ with η. Putting now all our results into (5.94) we obtain the following expression for the kinetic term:

$$L = \frac{M_{\rm cl}}{2}\dot{z}_0^2 + \frac{r}{2\,|m|}\dot{\alpha}^2 + \frac{\theta}{2\pi}\dot{\alpha} - ir2\,|m|\,\bar{\eta}\dot{\eta}\; - M_{\rm cl} \tag{5.99}$$

with $M_{\rm cl} = r\,|m|$. To get the Hamiltonian of the system we perform a Legendre transformation of the previous Lagrangian. The result is given by

$$H = \frac{1}{2M_{\rm cl}}P_{z_0}^2 + \frac{|m|}{2r}\left(P_\alpha - \frac{\theta}{2\pi}\right)^2 + M_{\rm cl} \tag{5.100}$$

where $P_{z_0} = M_{\rm cl}\dot{z}_0$ and $P_\alpha = \frac{r}{|m|}\dot{\alpha} + \frac{\theta}{2\pi}$. Since α is angular with period 2π, the eigenvalues n of P_α are quantised in integer units, $n \in \mathbb{Z}$, whence the spectrum looks like

$$E_{k,n} = \frac{1}{2M_{\rm cl}}k^2 + \frac{|m|}{2r}\left(n - \frac{\theta}{2\pi}\right)^2 + M_{\rm cl}\,, \tag{5.101}$$

where $k \in \mathbb{R}$ is the continuous impulse-eigenvalue of P_{z_0}. So we see that the fermionic zero mode does not contribute to the energy spectrum of the CP^1–kink.

Bound states of minimum energy are given by those eigenstates of the Hamiltonian for which $k = 0$. Hence, the states with n not very large should be BPS–states, but now they are dyonic ones. The dyonic character can be seen from (5.18)[17] which does not vanish for states with $n \neq 0$ or $\theta \neq 0$. To check the Bogomol'yni bound saturation we expand the square root $\sqrt{(S + \frac{\theta}{2\pi})^2 + r^2}$ of equation (59) of reference [5] and replace $-S$ by our quantum number n. Comparing the result with (5.101) we find that the two expressions are equivalent, at least up to first order. Thus the energy spectrum obtained from quantising the effective action of the collective coordinates is a small-coupling approximation to the expected BPS energy spectrum. However, even if their energy is only approximately correct, the multiplicity of the bound states can be read of accurately from the effective action.

[17] In equation (5.18) there is no θ–contribution, since we derived the $U(1)$–current by a dimensional reduction. So if one works with a θ–deformed theory one has to add it by hand.

Chapter 6

Conclusion and outlook

Prediction is very difficult,
especially about the future.

Niels Bohr

In this work we investigated classical and quantum mechanical properties of solitons of a supersymmetric CP^1 σ–model with twisted mass term. We started from a gauge theory in which one may implement a twisted mass term via a constant background gauge field. We derived the CP^1 theory as the low energy limit of the gauge theory. In the literature this low energy limit is usually denoted as Higgs phase and is one side of a duality (massive analog of the mirror symmetry [5]). The other side of this duality is called the Higgs phase.

We then rederived our two dimensional CP^1 theory with $\mathcal{N} = 2$ supersymmetry, however in this case not from a gauge theory but from a four dimensional CP^1 theory with $\mathcal{N} = 1$ supersymmetry. The reason is that we want to be sure that dimensional regularisation, which we used in the quantum theory, does not spoil supersymmetry. We calculated the supersymmetry algebra of the theory and found a static BPS saturating soliton, the CP^1–kink. All this was done classically.

In the second part of chapter 5 we performed a quantum mechanical investigation of the CP^1–kink. Starting with a flat background we calculated the counterterm and the renormalised coupling by making use of ordinary quantum field theory techniques. Thereafter we discussed the solitonic sector. Due to supersymmetry and by using index techniques we could derive the quantum corrections of the kink mass and of the central charge. We obtained an anomalous contribution to the spectrum of the mass and the central charge (see also (5.92) and (5.93))

$$M_{\text{anomal}} = -\frac{|m|}{\pi} \qquad\qquad Z_{\text{anomal}} = -2\frac{|m|}{\pi}\,. \tag{6.1}$$

Unlike reference [24], we derived these results by a direct calculation. In the course of the derivation of these two anomalous contributions we also could correct an error in the literature (see the end of section 5.3.2.1). Finally we quantised the moduli space of the CP^1–kink. This semi-classical approach allowed us to extend the analysis of the spectrum to kinks which not only carry topological charge but also Noether charge (called dyons).

But as usual, there are still some open questions which have are left for future research. In our case there are the following three points:

1. *It is not clear how to include the θ–term in a dimensional reduction analysis.* The crucial element in the construction of the θ–term is the two dimensional ϵ–tensor which makes this additional term (in two dimensions) a priori supersymmetric. Hence we have to find a supersymmetric expression in $2+\varepsilon$ dimensions which yields the θ–term in the limit $\varepsilon \to 0$. With a better understanding of the θ–term it also should become clear why we did not obtain non–trivial monodromies in the mass as are predicted by duality considerations.
2. *How to calculate the quantum fluctuations in the presence of a dyonic soliton?* For a soliton carrying only topological charge the techniques of reference [21] are very well suited to calculate the quantum corrections. But for dyonic states it is not yet clear how we have to modify the tools. The primary problem in the dyonic sector is to separate off the time dependence in the iterated fermionic EOM of the fluctuations (the dyonic counterpart to (5.69)).
   Once a solution for the dyonic CP^1–kink is found we think that by analogy we could also derive the quantum fluctuations of dyons in four dimensional super–Yang–Mills theories which still is an open problem in the literature.
3. *According to duality considerations there should be an additional factor $\frac{i}{2}$ in the spectrum of the mass and central charge.* In section 5.1.2 we briefly presented the dual description of the Higgs phase, the Coulomb phase. The spectrum of the mass and central charge (see for example equation (4.2) of reference [24]) are well known in this phase and except of the factor $\frac{i}{2}$ our results match perfectly with them. So if duality is valid it should also appear in the low energy limit.

Recapitulating, we could show once more the power of the techniques of [21] to treat quantum fluctuations in a very clear and consistent manner. By applying these methods to the CP^1 σ–model we were able to present for the first time (at least to our knowledge) a direct calculation to obtain the anomalous corrections to the mass and central charge of the CP^1–kink (see above). In this way we gained an comprehensive understanding of the CP^1–kink and its properties.

Appendix A

Symmetries and quantum theory

A.1 Conventions

Metric: We use the metric $\eta_{nm} = (-, +, \dots, +)$ throughout this work.

Pauli matrices: As a basis for the $SL(2,C)$–matices/group we use the following Pauli matrices:

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

Spinors: With the basis (A.1) we can establish an isomorphism between the "normal" representation of the Lorentz group $SO(1,3)$ and $SL(2,C)/\mathbb{Z}_2$.

$$\begin{aligned} P_m &\rightarrow P'_m = L_m{}^j P_j \\ \sigma^m P_m &\rightarrow \sigma^m P'_m = S(L_i{}^j)\sigma^m P_m S^\dagger(L_i{}^j) \end{aligned}$$

Hence, we get a two dimensional irreducible representation over $\mathbb{C}$ (called Weyl spinors). There is also a second one because $(S^\dagger)^{-1}$ is not equivalent to S ($\nexists A \in SL(2,C)$: $A(S^\dagger)^{-1} = SA$). Thus we have the following two Weyl–representations:

$$\psi'_\alpha = S(L_i{}^j)_\alpha{}^\beta \psi_\beta \qquad \text{and} \qquad \psi'^{\dot\alpha} = S(L_i{}^j)^{-1\dot\alpha}{}_{\dot\beta}\psi^{\dot\beta} \,. \tag{A.2}$$

The representation with the undotted/dotted indices is labeled by $(\frac{1}{2},0)/(0,\frac{1}{2})$. To raise and lower the indices we use the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ ($\varepsilon^{12} = 1$, $\varepsilon_{12} = \varepsilon^{21} = -1$, $\varepsilon_{11} = \varepsilon_{22} = 0$)

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$$

We do this since because of the unimodularity of S and the skew–symmetry we only have to contract the spinor indices to get a Lorentz invariant term, e.g. $\psi^\alpha\psi_\alpha$.

A.2 Hamiltonian formalism

The Lagrangian formalism[1] yields second–order ordinary differential equation (ODE). In contrast, the Hamiltonian formalism gives EOM which are first order in the time derivative[2] and, more importantly, we can make the symplectic structure manifest in the Hamiltonian formalism.

Suppose a Lagrangian L is given. Then the corresponding **Hamiltonian** is introduced via a Legendre transformation of variables as

$$H(q,p) := \sum_k p_k \dot{q}_k - L(q,\dot{q})\,, \tag{A.3}$$

where $\dot{q}$ is eliminated in the left hand side (LHS) in favour of p by making use of the definition of the canonical momentum $p_k := \frac{\partial L(q,\dot{q})}{\partial \dot{q}_k}$. For this transformation to be defined, the Jacobian must satisfy[3]

$$\det\left(\frac{\partial p_i}{\partial \dot{q}_j}\right) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0\,. \tag{A.4}$$

The space with coordinates (q_k, p_k) is called **phase space**.

Let us consider an infinitesimal change in the Hamiltonian induced by δq_k and δp_k,

$$\delta H = \sum_k \left[\delta p_k \dot{q}_k - \frac{\partial L}{\partial q_k}\delta q_k\right]\,. \tag{A.5}$$

It follows from this relation that

$$\frac{\partial H}{\partial p_k} = \dot{q}_k\,, \qquad \frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k} \tag{A.6}$$

which are nothing more than the replacements of independent variables. **Hamilton's equations of motion** are obtained from these equations if the Euler–Lagrange equation ($\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0$) is employed to replace the LHS of the second equation,

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}\,. \tag{A.7}$$

One of the most important tools in the Hamiltonian Formalism is the **Poisson bracket** (which will be replaced by the commutator in the quantum regime, see A.4.1), it is defined as follows:

$$\{A,B\}_{\mathrm{PB}} := \sum_k \left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p_k}\right)\,, \tag{A.8}$$

where $A(q,p)$ and $B(q.p)$ are functions defined on the phase space of the Hamiltonian H. The Poisson bracket is a **Lie bracket**, namely it satisfies

$$\{A, c_1 B_1 + c_2 B_2\}_{\mathrm{PB}} = c_1 \{A, B_1\}_{\mathrm{PB}} + c_2 \{A, B_2\}_{\mathrm{PB}} \quad \text{linearity} \tag{A.9a}$$

$$\{A,B\}_{\mathrm{PB}} = -\{B,A\}_{\mathrm{PB}} \quad \text{skew–symmetry} \tag{A.9b}$$

$$\{\{A,B\}_{\mathrm{PB}},C\}_{\mathrm{PB}} + \{\{B,C\}_{\mathrm{PB}},A\}_{\mathrm{PB}} + \{\{C,A\}_{\mathrm{PB}},B\}_{\mathrm{PB}} = 0 \quad \textbf{Jacobi identity}. \tag{A.9c}$$

[1] We assume that the reader is familiar with it, for details on it see reference [29, 30]

[2] Hence one may introduce flows in the phase space, see [30, 15]

[3] If this condition isn't fulfilled we get constraints, for details on the treatment of constrainted (quantum) systems see e.g. [31]

The fundamental Poisson brackets are

$$\{p_i, p_j\}_{\mathrm{PB}} = \{q_i, q_j\}_{\mathrm{PB}} = 0 \qquad \{q_i, p_j\}_{\mathrm{PB}} = \delta_{ij}\,. \tag{A.10}$$

If a physical quantity depends only on the phase space coordinates, its time evolution is expressed in terms of the Poisson bracket as

$$\frac{dA}{dt} = \{A, H\}_{\mathrm{PB}} \tag{A.11}$$

and consequently also the Hamiltonian EOM themselves are written as

$$\frac{dp_k}{dt} = \{p_k, H\}_{\mathrm{PB}} \qquad \frac{dq_k}{dt} = \{q_k, H\}_{\mathrm{PB}}\,. \tag{A.12}$$

Since we have now derived/written down all the basics we can look for the implications of symmetries

A.3 Symmetries

The most important consequence of a **continuous**[4] symmetry of a physical system[5] is stated in the following theorem:

Theorem A.3.1 (Noether's theorem) *Let $H(q_k, p_k)$ be a Hamiltonian which is invariant under an infinitesimal coordinate transformation*[6] $q_k \to q'_k = T^q_Q(\varepsilon) q_k = q_k + \varepsilon f_k(q)$. *Then*

$$Q = \sum_k p_k f_k(q) \tag{A.13}$$

is conserved.

Proof One has $H(q_k, p_k) = H(q'_k, p'_k)$ by definition. It follows from $q'_k = q_k + \varepsilon f_k(q)$ that the Jacobian associated with the coordinate change is

$$\Lambda_{ij} = \frac{\partial q'_i}{\partial q_j} = \delta_{ij} + \varepsilon \frac{\partial f_i(q)}{\partial q_j} + \mathcal{O}(\varepsilon^2)\,.$$

The momentum transforms under this coordinate change as

$$p_i \to p'_i = T^p_Q(\varepsilon) p_i = \sum_j p_j \Lambda^{-1}_{ji} = p_i - \varepsilon \sum_j p_j \frac{\partial f_j}{\partial q_i}\,.$$

Then, it follows that

$$\begin{aligned} 0 =& H(q'_j, p'_j) - H(q_j, p_j) = \partial_{q_j} H\, \varepsilon f_k(q) - \partial_{p_j} H\, \varepsilon\, p_i \partial_{q_j} f_i = \\ =& \varepsilon \left[\partial_{q_j} H\ f_k(q) - \partial_{p_j} H p_i \partial_{q_j} f_i\right] = \varepsilon\, \{H, Q\}_{\mathrm{PB}} = \varepsilon \frac{dQ}{dt}\,, \end{aligned}$$

which shows that Q is conserved. □

[4]Refers to a continuous parameter/symmetry group in contrast to a discrete symmetry (group);

[5]According to Herman Weyl, we denote a system as symmetric in terms of a transformation T, if we cannot discern after the transformation if it was applied or not.

[6]The superscript q refers to the representation of the transformation group;

This theorem shows that finding a conserved quantity is equivalent to finding a transformation which leaves the Hamiltonian invariant.

A conserved quantity Q is the 'generator' of the transformation under discussion. In fact,

$$\{q_i, Q\}_{\text{PB}} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial Q}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial Q}{\partial q_k} \right) = \sum \delta_{ik} f_k(q) = f_i(q)$$

which shows that

$$\delta q_i = \varepsilon f_i(q) = \varepsilon \{q_i, Q\}_{\text{PB}} \quad \text{and similarly that} \quad \delta p_i = \varepsilon \{p_i, Q\}_{\text{PB}} = -\varepsilon p_j \partial_{q_i} f_j(q)\,. \tag{A.14}$$

Hence, the transformation for a generic function A of the canonical variables p_i and q_i looks like

$$\delta A(q,p) = \delta q_i \frac{\partial A(q,p)}{q_i} + \delta p_i \frac{\partial A(q,p)}{p_i} = \varepsilon \{A, Q\}_{\text{PB}}\ , \tag{A.15}$$

where we have used that $\frac{\partial Q}{p_i} = \{q_i, Q\}_{\text{PB}}$ and $\frac{\partial Q}{q_i} = \{Q, p_i\}_{\text{PB}}$.

Since we now know the effects of symmetries classically we also want to see their consequences in the quantum regime.

A.4 Canonical quantisation

First of all we have to 'define' quantum theory before we can study symmetries in it, so we have to outline here some 'rules' on which quantum theory is based[7].

A.4.1 Axioms of canonical quantisation

Given an isolated dynamical system such as a harmonic oscillator, we can construct a corresponding quantum system following a set of axioms.

A1 There exists a Hilbert space $\mathcal{H}$ for a quantum system and the state of the system is required to be described by a vector $|\psi\rangle \in \mathcal{H}$. In this sense, $|\psi\rangle$ is also called the **state** or a **state vector**. Moreover, two states $|\psi\rangle$ and $c\,|\psi\rangle$ ($c \in \{\mathbb{C} \setminus 0\}$) describe the same state. The state can also be described as a **ray representation**[8] of $\mathcal{H}$.

A2 A physical quantity A in classical mechanics is replaced by a Hermitian operator $\hat{A}$ acting on $\mathcal{H}$. The operator $\hat{A}$ is often called an **observable**. The result obtained when A is measured is one of the eigenvalues of $\hat{A}$.

A3 The Poisson bracket in classical mechanics is replaced by the **commutator**

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \tag{A.16}$$

multiplied by $-i/\hbar$. Units in which $\hbar = 1$ will be employed hereafter. The fundamental commutation relations are

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \qquad [\hat{q}_i, \hat{p}_j] = i\delta_{ij} \tag{A.17}$$

[7] For the a general treatment of quantum mechanics see the standard references [32, 33];

[8] For some details on ray representations see reference [34]

Under this replacement, Hamilton's equations of motion become

$$\frac{d\hat{q}_i}{dt} = \frac{1}{i}[\hat{q}_i, \hat{H}] \qquad \frac{d\hat{p}_i}{dt} = \frac{1}{i}[\hat{p}_i, \hat{H}]\,. \tag{A.18}$$

When a classical quantity A is independent of t explicitly, A satisfies the same equation as Hamilton's equation. By analogy, for $\hat{A}$ which does not depend on t explicitly, one has **Heisenberg's equation of motion**:

$$\frac{d\hat{A}}{dt} = \frac{1}{i}[\hat{A}, \hat{H}] \tag{A.19}$$

A4 Let $|\psi\rangle \in \mathcal{H}$ be an arbitrary state. Suppose one prepares many systems, each of which is in this state. Then, observation of A in these system at time t yields random results in general. Then the expectation value of the results is given by

$$\langle A\rangle_t = \frac{\langle\psi|\hat{A}(t)|\psi\rangle}{\langle\psi|\psi\rangle} \tag{A.20}$$

A5 For any physical state $|\psi\rangle \in \mathcal{H}$, there exists an operator for which $|\psi\rangle$ is one of the eigenstates.

A.4.2 Symmetries in quantum theory

Thus we see that the classical concept by describing particles of their trajectories is gone. Physical systems are now classified by their state vectors and this it where symmetries come in: If we have a transformation which leaves the Hamilton operator invariant[9] we know that its generator commutes with the Hamilton operator ($[H, Q] = 0$) so that we can find a basis which simultaneously diagonalises[10] both operators. Hence we can characterise the physical state of system if we know the eigenvalues of its complete set of commuting observables (CSCO)[11]. The eigenvalues of these operators are normally called 'quantum numbers'.
Example: In particle physics the Poincaré–invariance of free field theories allows one already to do a first classification of the state by its mass (m), spin (s and σ) and three–momentum ($\vec{p}$) for massive quantum field theories (QFT) and by its helicity (λ) and three–momentum ($\vec{p}$) for QFT's[12].

A.4.3 Spontaneous symmetry breaking

If a symmetry of a theory (Lagrangian) is not realised in the ground state[13], which means that there is a transformation, continuously or discrete, that leaves the Lagrangian invariant

[9] From axiom 3 we know that we have to replace the poisson bracket by the commutator in equation A.15 to get the infinitesimal transformation $T_Q(\varepsilon)$ of an observable which depends only on the canonical variables, thus the observable A isn't affected by $T_Q(\varepsilon)$ if $[A, Q]$ vanishes.

[10] Let $\mathcal{B}' = \bigoplus_{i=1}^{k} b'_{a_i}$ be a basis which diagonalises the operator A (the a_i are the eigenvalues of A). If we have now a second operator C which commutes with A ($[A, C] = 0$) its representation concerning the basis $\mathcal{B}'$ will reduce to subspaces ($C = \bigoplus_{i=1}^{k} C_i$) which are spanned by the subbases b'_{a_i} (this is seen from $C\,|a_i\rangle \in b'_{a_i} \Leftrightarrow [A, C] = 0$). Since every suboperator C_i itself is Hermitian we can diagonalise by changing the basis $\mathcal{B}'$ ($\mathcal{B}' \to \mathcal{B}$) without destroying the diagonal structure of A.

[11] These are clearly all the symmetry generators of the system (or observables which can be derived from them); H for instance is the generator of time translations.

[12] For details see ref. [34];

[13] In QFT's this state is called *vacuum state* or short *vacuum* and is denoted by $|0\rangle$;

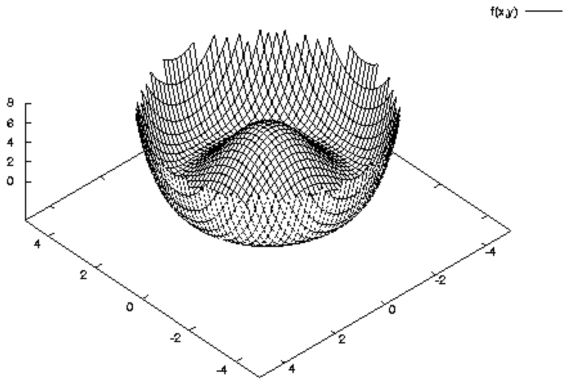


Figure A.1: The mexican hat potential

but not the ground state, one speaks of a spontaneously broken symmetry. Two examples will make this clearer.
Example 1: The Lagrangian of the ϕ^4–theory, see equation (2.1), is invariant under the following discrete $\mathbb{Z}_2$ symmetry

$$\phi \to Z\phi = -\phi \,, \tag{A.21}$$

but not the vacuum. This can be easily seen as follows: Let the vacuum be invariant under Z then[14]

$$\left\langle \hat{\phi} \right\rangle = \langle 0| \, ZZ\hat{\phi}ZZ \, |0\rangle = -\left\langle \hat{\phi} \right\rangle = 0 \,.$$

This is a contradiction to $\left\langle \hat{\phi} \right\rangle = \phi_{\mathrm{vac}_1}$ or ϕ_{vac_1} which are both non vanishing[15]. Hence, the vacuum is not invariant under Z. Thus this symmetry is spontaneously broken.
Example 2: The Lagrangian for the Higgs model is equivalent to the following one

$$L = \int d^n x \left[\partial_\mu \phi_i \partial^\mu \phi_i - \tfrac{\lambda}{4} \left(\phi_i \phi_i - \tfrac{\mu^2}{\lambda} \right)^2 \right] \qquad \text{with} \qquad i = 1,2 \quad \text{and} \quad \lambda > 0 \,.$$

This Lagrangian has a continuous $SO(2)$ symmetry which is again sponaneously broken. Because if

$$\begin{aligned} \underbrace{R(\varepsilon)}_{\mathbb{1}+i\varepsilon Q} |0\rangle = |0\rangle \Rightarrow Q\,|0\rangle = 0 \quad &\Rightarrow \langle \phi_i \rangle = \langle \phi_i \rangle + i\varepsilon \, \langle 0|\, Q\phi_i \,|0\rangle - i\varepsilon \, \langle 0|\, \phi_i Q \,|0\rangle = \\ &= \langle \phi_i + i\varepsilon \, [\phi_i \,, \, Q] \rangle = R_i{}^j(\varepsilon) \, \langle \phi_j \rangle \\ \Rightarrow \langle \phi_i \rangle = 0 \end{aligned}$$

This is a contradiction to the assertion that ϕ_{vac} must not vanish which is enforced by the mexican hat potential, see Figure A.1 and footnote 15. Thus the $SO(2)$ symmetry is spontaneously broken.

[14]Notice that in quantum mechanics the transformation corresponding to A.21 becomes $\hat{\phi} \to Z\hat{\phi}Z = -\phi$

[15]Actually one should be very careful when making the prediction that the vacuum expectation $\langle \hat{\phi} \rangle$ value does not vanish. Since for instance in lattice QCD it happens that $\langle \hat{\phi} \rangle = 0$ although classical it doesn't. But at least for an effective taction on can show that this is true (see chapter 11 of reference [19]).

Appendix B

On the CP^1 σ–model with twisted mass

B.1 Details on some calculations

B.1.1 The equations of motion and canonical momenta

The EOM and the canonical momentum π for ϕ derived from the Lagrangian density (5.64):

$$\frac{2r\partial_k(\phi^\dagger\phi)}{\rho^3}D^k\phi - \frac{r}{\rho^2}D_kD^k\phi - \frac{2r\phi}{\rho^3}\left[D_k\phi^\dagger D^k\phi + i\bar{\psi}\gamma^k(D_k - 2\frac{\phi^\dagger D_k\phi}{\rho})\psi\right] - \frac{2r\phi}{\rho^5}\psi\psi\bar{\psi}\bar{\psi} = 0 \tag{B.1}$$

$$\pi_\phi = -\frac{r}{\rho^2}\left[D^0\phi^\dagger - 2i\bar{\psi}\gamma^0\psi\frac{\phi^\dagger}{\rho}\right] \qquad \pi_{\phi^\dagger} = -\frac{r}{\rho^2}D^0\phi \tag{B.2}$$

And for ψ:

$$\frac{r}{\rho^2}\left[i\gamma^k(D_k - 2\frac{\phi^\dagger D_k\phi}{\rho})\psi + \frac{1}{\rho^2}\psi\psi\bar{\psi}\right] = 0\,, \quad \frac{r}{\rho^2}\left[-i(D_k - 2\frac{\phi D_k\phi^\dagger}{\rho})\bar{\psi}\gamma^k + \frac{1}{\rho^2}\bar{\psi}\bar{\psi}\psi\right] = 0 \tag{B.3}$$

$$\pi_\psi = -i\frac{r}{\rho^2}\bar{\psi}\gamma^0 \tag{B.4}$$

B.1.2 The energy–momentum tensor

The energy–momentum tensor derived from (5.64) and (5.15) is given by

$$\begin{aligned} T^m{}_n = &-\frac{r}{\rho^2}\Big[\partial_n\phi^\dagger D^m\phi + D^m\phi^\dagger\partial_n\phi + i\bar{\psi}\gamma^m\partial_n\psi - 2i\bar{\psi}\gamma^m\psi\frac{\phi^\dagger\partial_n\phi}{\rho} - \\ &- g^m{}_n\left(D_k\phi^\dagger D^k\phi + i\bar{\psi}\gamma^k(D_k - 2\frac{\phi^\dagger D_k\phi}{\rho})\psi + \frac{1}{2\rho^2}\psi\psi\bar{\psi}\bar{\psi}\right)\Big]\,. \end{aligned} \tag{B.5}$$

By making use of the EOM (B.1) we get the following on shell expression for the energy–momentum tensor:

$$T_{\text{on shell}}{}^{m}{}_{n} = -\frac{r}{\rho^2}\Big[\partial_n\phi^\dagger D^m\phi + D^m\phi^\dagger\partial_n\phi + i\bar\psi\gamma^m\partial_n\psi - 2i\bar\psi\gamma^m\psi\frac{\phi^\dagger\partial_n\phi}{\rho} - \\ - g^m{}_n\left(D_k\phi^\dagger D^k\phi - \frac{1}{2\rho^2}\psi\psi\bar\psi\bar\psi\right)\Big] \tag{B.6}$$

Since the Lagrangian from which we started was not Hermitian also the energy–momentum tenser is not, but if we add $\partial_m\left(\frac{ir}{2\rho^2}\bar\psi\gamma^m\psi\right)$ we get a Hermitian one. This modification does not change the supercharges.

From the new Lagrangian we obtain the following hermitian energy–momentum tensor:

$$T'^m{}_n = T^m{}_n + \partial_n\left(\frac{ir}{2\rho^2}\bar\psi\gamma^m\psi\right) - g^m{}_n\partial_l\left(\frac{ir}{2\rho^2}\bar\psi\gamma^l\psi\right) \tag{B.7}$$

$$T'_{\text{on shell}}{}^{m}{}_{n} = T_{\text{on shell}}{}^{m}{}_{n} + \partial_n\left(\frac{ir}{2\rho^2}\bar\psi\gamma^m\psi\right) \tag{B.8}$$

B.1.3 The superinvariance of the Lagrangian and the supercurrent

In the following calculation we show the invariance of (5.15) under the supersymmetry transformations (5.19) and as a by-product we will derive the super charges.

$$\delta_\xi\mathcal{L} = \delta_\xi\left\{-\frac{r}{\rho^2}\Big[D_m\phi^\dagger D^m\phi + i\bar\psi\gamma^m(D_m - 2\frac{\phi^\dagger D_m\phi}{\rho})\psi + \frac{1}{2\rho^2}\psi\psi\bar\psi\bar\psi\Big]\right\} = \\
= \frac{-2\phi^\dagger\sqrt{2}\xi\psi}{\rho}\mathcal{L} - \frac{r}{\rho^2}\Big[D_m\phi^\dagger D^m(\sqrt{2}\xi\psi) + \sqrt{2}\xi\sigma^n D_n\phi^\dagger\gamma^m(D_m - 2\frac{\phi^\dagger D_m\phi}{\rho})\psi + \\
+ i\bar\psi\gamma^m(D_m - 2\frac{\phi^\dagger D_m\phi}{\rho})2\frac{\phi^\dagger}{\rho}\sqrt{2}(\xi\psi)\psi - 2i\bar\psi\gamma^m\psi\Big(\frac{\phi^\dagger D_m(\sqrt{2}\xi\psi)}{\rho} - \frac{\phi^\dagger D_m\phi}{\rho^2}\phi^\dagger\sqrt{2}\xi\psi\Big) - \\
- \frac{i}{\rho^2}\psi\psi\sqrt{2}\xi\sigma^n D_n\phi^\dagger\bar\psi\Big] =$$

$$= \frac{2r}{\rho^3}D_m\phi^\dagger D^m\phi(\phi^\dagger\sqrt{2}\xi\psi) - \frac{r}{\rho^2}\Big[D_m\phi^\dagger D^m(\sqrt{2}\xi\psi) + \sqrt{2}\xi\sigma^n D_n\phi^\dagger\gamma^m(D_m - 2\frac{\phi^\dagger D_m\phi}{\rho})\psi + \\
+ i\bar\psi\gamma^m\psi\partial_m(2\frac{\phi^\dagger}{\rho}\sqrt{2}\xi\psi) - 2i\bar\psi\gamma^m\psi\Big(\frac{\partial_m(\phi^\dagger\sqrt{2}\xi\psi) - D_m\phi^\dagger\sqrt{2}\xi\psi}{\rho} - \\
- \frac{\partial_m(\phi^\dagger\phi) - D_m\phi^\dagger\phi}{\rho^2}\phi^\dagger\sqrt{2}\xi\psi\Big) - \frac{i}{\rho^2}\psi\psi\sqrt{2}\xi\sigma^n D_n\phi^\dagger\bar\psi\Big] =$$

$$= \frac{2r}{\rho^3}D_m\phi^\dagger D^m\phi(\phi^\dagger\sqrt{2}\xi\psi) - \frac{r}{\rho^2}\Big[D_m\phi^\dagger D^m(\sqrt{2}\xi\psi) + \sqrt{2}\xi\sigma^n D_n\phi^\dagger\gamma^m(D_m - 2\overbrace{\frac{\phi^\dagger D_m\phi}{\rho}}^{\partial_m(\phi^\dagger\phi) - D_m(\phi^\dagger)\phi})\psi\Big] = \\
= -D_m\phi^\dagger D^m\left(\sqrt{2}\xi\psi\frac{r}{\rho^2}\right) - \sqrt{2}\xi\sigma^n D_n\phi^\dagger\gamma^m D_m\left(\frac{r}{\rho^2}\psi\right) = \partial_m\xi\sqrt{2}\sigma^n D_n\phi^\dagger\gamma^m\frac{r}{\rho^2}\psi + \mathcal{O}(\partial)$$

B.1.4 The supersymmetry transformation of the supercurrent

The transformation of the supercurrent (5.20) under (5.19).

$$i\frac{2r}{\rho^2}D_n\phi^\dagger\sigma^n\gamma^m\sigma^l\bar{\xi}D_l\phi = i\frac{2r}{\rho^2}\left[\sigma^m D_n\phi^\dagger D^n\phi - \sigma^n D^m\phi^\dagger D_n\phi - \sigma^n D_n\phi^\dagger D^m\phi + \right. \\ \left. + i\epsilon^{nmlk}\sigma_k D_n\phi^\dagger D_l\phi\right]$$

$$\frac{2r}{\rho^2}\bar{\xi}\left(D_n\bar{\psi} - \frac{2}{\rho}\phi D_n\phi^\dagger\bar{\psi}\right)\sigma^n\gamma^m\psi = \\ = \frac{r}{\rho^2}\bar{\xi}\left(\frac{i}{\rho^2}\bar{\psi}\bar{\psi}\psi\sigma_n - \left(D_k\bar{\psi} - \frac{2}{\rho}\phi D_k\phi^\dagger\bar{\psi}\right)\gamma_n\sigma^k\right)\sigma^n\gamma^m\psi = -\frac{ir}{\rho^4}\bar{\psi}\bar{\psi}\psi\psi\sigma^m\bar{\xi} - \\ - \frac{2r}{\rho^2}\sigma^k\bar{\xi}\left(D_k - \frac{2}{\rho}\phi D_k\phi^\dagger\right)\bar{\psi}\gamma^m\psi = -\frac{ir}{\rho^4}\bar{\psi}\bar{\psi}\psi\psi\sigma^m\bar{\xi} + \frac{2r}{\rho^2}\sigma^k\bar{\xi}\bar{\psi}\gamma^m\left(D_k - \frac{2}{\rho}\phi^\dagger D_k\phi\right)\psi - \\ - \sigma^k\bar{\xi}\partial_k\left(\frac{2r}{\rho^2}\bar{\psi}\gamma^m\psi\right)$$

B.1.4.1 Vacuum expectation value of the central charge

From 5.2.3 we see that the central charge is given by :

$$Z' = \int dz\Big(T'^0{}_2\sigma^2 - \sigma^1|m|J^0 + \sigma^k\partial_k\left(\frac{ir}{2\rho^2}\bar{\psi}\gamma^0\psi\right) + \partial_2\left(\sigma_3\frac{r|m|}{\rho}\right) - \partial_3\left(\sigma_2\frac{r|m|}{\rho}\right) + \\ + \sigma^1\frac{ir}{\rho^2}\left(\partial_3\phi^\dagger\partial_2\phi - \partial_2\phi^\dagger\partial_3\phi\right)\Big)$$

Now we derive for each term the vacuum expectation value:

$$\left\langle\int dzT'^0{}_2\right\rangle = -\left\langle\int dz\frac{r}{\rho^2}\left[\partial_2\phi^\dagger\partial^0\phi + \partial^0\phi^\dagger\partial_2\phi + i\bar{\psi}\gamma^0\partial_2\psi - i\bar{\psi}\gamma^0\psi\frac{\phi^\dagger\overleftrightarrow{\partial_2}\phi}{\rho}\right]\right\rangle = \\ = -\frac{r}{2}\left\langle\int dz\Big[\partial_2 n_i\partial^0 n_i + i\bar{\psi}'\gamma^0\partial_2\psi' + 2i\bar{\psi}'\gamma^0\psi'\partial_2\ln(1-n_3)\Big]\right\rangle = \\ = -\frac{r}{2}\int dz\langle\underbrace{\partial_2\delta n_i\partial^0\delta n_i}_{\partial_2 u_i\partial^0 u_i\to 0} + i\bar{\psi}'\gamma^0\partial_2\psi' + \mathcal{O}(\hat{\partial}^3)\rangle = -\frac{r}{2}\int dz\,\langle i\bar{\psi}'\gamma^0\partial_2\psi'\rangle =$$

The integral with the term $\partial_2 u_i\partial^0 u_i$ becomes zero because if we go to Fourier space integrals over momenta with odd power vanish.

$$= \int dz\int\frac{dk}{2\pi}\int\frac{d^\epsilon l}{(2\pi)^\epsilon}\frac{l^2}{2\omega}\left[\left|\chi_k^+\right|^2(z) - \left|\chi_k^-\right|^2(z)\right] = -\int\frac{dk}{2\pi}\int\frac{d^\epsilon l}{(2\pi)^\epsilon}\frac{l^2}{2\omega}\left[\frac{2|m|}{k^2+|m|}\right] = \\ = -\frac{2|m|^{1+\frac{\epsilon}{2}}}{(4\pi)^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{\Gamma\left(1-\frac{\epsilon}{2}\right)}{\sqrt{\pi}(1+\epsilon)} \to -\frac{|m|}{\pi}$$

This is the anomalous contribution to the central charge.

The following three calculations should be self-explanatory.

$$\left\langle \int dz J^0 \right\rangle = \left\langle \int dz \frac{r}{2} \left[\epsilon_{3ij} n_i \partial_0 n_j - n_3 \bar{\psi}' \gamma^0 \psi' \right] \right\rangle = \left\langle \int dz \frac{r}{2} \Big[\epsilon_{3ij} \underbrace{n_{cl\,i} \partial_0 n_{cl\,j}}_{=0} + \right.$$

$$\left. + \epsilon_{3ij} \delta n_i \partial_0 \delta n_j - n_{cl\,3} \bar{\psi}' \gamma^0 \psi' \Big] + \mathcal{O}(\hat{\partial}^3) \right\rangle = - \int dz \frac{r}{2} n_{cl\,3} \left\langle \bar{\psi}' \gamma^0 \psi' \right\rangle =$$

$$= - \int dz \Big(\underbrace{n_{cl\,3}}_{\text{odd func.}} \int \frac{dk}{2\pi} \frac{1}{2} \underbrace{\left[\left| \chi_k^+ \right|^2 (z) + \left| \chi_k^- \right|^2 (z) \right]}_{\text{even func.}} \Big) = 0$$

$$\left\langle \int dz \partial_k \left(\frac{ir}{2\rho^2} \bar{\psi} \gamma^0 \psi \right) \right\rangle = \int dz \partial_k \left\langle \bar{\psi}' \gamma^0 \psi' \right\rangle = \frac{i}{2} \int dz \partial_k \int \frac{dk}{2\pi} \frac{1}{2} \left[\left| \chi_k^+ \right|^2 (z) + \left| \chi_k^- \right|^2 (z) \right] =$$

$$= \frac{i}{2} \delta_{3k} \int dz \partial_3 \int \frac{dk}{2\pi} \frac{1}{2} \left[\left| \chi_k^+ \right|^2 (z) + \left| \chi_k^- \right|^2 (z) \right] =$$

$$= \frac{i}{2} \delta_{3k} \int \frac{dk}{2\pi} \frac{1}{2} \left[\left| \chi_k^+ \right|^2 (z) + \left| \chi_k^- \right|^2 (z) \right] \Bigg|_{-\infty}^{\infty} = 0$$

$$\left\langle \int dz \partial_2 \left(\frac{r\,|m|}{\rho} \right) \right\rangle = - \frac{r\,|m|}{2} \int dz \partial_2 \left\langle n_3 \right\rangle = - \frac{r\,|m|}{2} \int dz \partial_2 \left(n_{cl\,3} + \left\langle \delta n_3 \right\rangle \right) = 0$$

The next integral needs special care since the expansion of $\frac{1}{\rho}$ via the fluctuations u_i is very sensitive around the south pol.

$$\left\langle \int dz \partial_3 \left(\frac{r\,|m|}{\rho} \right) \right\rangle = r\,|m| \left\langle \frac{1}{\rho} \right\rangle \Bigg|_{-\infty}^{\infty} = r\,|m| \left\langle \frac{1}{1 + \frac{n_1^2 + n_2^2}{(1-n_3)^2}} \right\rangle \Bigg|_{-\infty}^{\infty} =$$

$$= -r\,|m| \left\langle \frac{1}{1 + \frac{(n_{cl\,1} + \delta n_1)^2 + (n_{cl\,2} + \delta n_2)^2}{1 - (n_{cl\,3} + \delta n_3)^2}} \right\rangle \Bigg|_{-\infty} = - \left\langle r\,|m| \left(1 - \frac{u_i u_i}{2} + \mathcal{O}(\hat{\partial}^3) \right) \right\rangle \Big|_{-\infty} =$$

$$= -r\,|m| \left(1 - \left\langle u_1 u_1 \right\rangle |_{-\infty} \right) = -r\,|m| + |m|\, I$$

I is given by $I = \int \frac{d^{1+\epsilon}k}{(2\pi)^{1+\epsilon}} \frac{1}{\sqrt{k^2 + |m|^2}}$.

$$\left\langle \int dz \frac{ir}{\rho^2} \left(\partial_3 \phi^\dagger \partial_2 \phi - \partial_2 \phi^\dagger \partial_3 \phi \right) \right\rangle = 0$$

The last integral vanished because the integration over momenta (in Fourier space) with odd power gives zero.

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