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Colombeau Generalized Functions on Manifolds

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Gratias Ago

Hoc opus studiae doctrinae mathematicae per multos annos durans frux est. His in annis parentes me adiuvabant et operibus omnibus sustinebant itaque vitam ab omnibus curis vacuam efficiebant. De itinere meo semper sollicitudinem gaudiumque exhibebant praeterea auxilium pecuniarium ferebant. Eis ideo primo gratias ago. Secundo maritae meae qui sempiternus animum meum animamque amore suo nutrit gratias ago. Tertio gratias patriae et professoribus nominatim consuasori meo ago, ut studia doctrinae mathematicae in libertate formare et ad meum arbitrium omnia facere possum.

Contents

Kurzfassung

Diese Arbeit gibt einen Überblick über jüngere Entwicklungen der Theorie der Algebren verallgemeinerter Funktionen im Sinne von Colombeau auf differenzierbaren Mannigfaltigkeiten. Ursprünglich von J.-F. Colombeau in den achtziger Jahren entwickelt, enthalten Colombeau-Algebren glatte Funktionen als Unteralgebra sowie Distributionen als einen linearen Unterrraum und ermöglichen es somit, nichtlineare Operationen auf Distributionen zu erklären. Dies spielt vor allem bei der Modellierung von nichtlinearen physikalischen Phänomenen – etwa in der allgemeinen Relativitätstheorie – eine wichtige Rolle. Es zeigt sich, dass wohlbekannte Konzepte wie der Fluss von Vektorfeldern und pseudo-Riemannsche Geometrie sinnvoll in diesem Kontext erklärt werden können. Als physikalische Anwendung dieser Theorie werden die Geodäten einer durch eine impulsive Gravitationswelle beschriebenen Raum-Zeit bestimmt.

Abstract

This work gives a survey on recent developments in the field of algebras of generalized functions in the sense of Colombeau on differentiable manifolds. Developed by J.-F. Colombeau in the 1980s, Colombeau Algebras contain smooth functions as a subalgebra and distributions as a linear subspace and thus provide a way to define nonlinear operations on distributions. This plays an important role in the modelling of nonlinear physical phenomena, for instance in the theory of general relativity. It turns out that well-known concepts like the flow of vector fields and pseudo-Riemannian geometry can be extended to this new setting. As a practical application in physics we consider a space-time given by an impulsive gravitational wave and determine its geodesics.

Chapter 1

Notation and Terminology

This section introduces notational conventions and tools used throughout the rest of this work. Readers with no more than a basic background in differential geometry will find [Die72, Abr67, GHV72] to be valuable resources to deepen their knowledge sufficiently to follow the presentation. Care was taken not to overwhelm those only partly adept in the language of differential geometry but to explain in more detail than would be strictly necessary.

Real Analysis

In general, the letters denoting dimensions will not be explicitly introduced and should be taken from the context. $\mathbb{N}$ is the set of all positive integers, $\mathbb{N}_0$ of all nonnegative ones. $\mathbb{R}_+$ is the set of nonnegative real numbers. Partial derivatives with respect to a multi-index $\alpha = (\alpha^1, \ldots, \alpha^k) \in \mathbb{N}_0^k$ are denoted by ∂^α, D is the total derivative. We define $A \subset\subset B$ to mean that A is a compact subset of B° (the interior of B). $\|f\|_\infty$ is the uniform norm of a real-valued function f. The expression *smooth* means differentiable of infinite order. $B_r(x)$ is the open Euclidean ball of radius r around $x \in \mathbb{R}^n$. While $|\cdot|$ is used for the absolute value in $\mathbb{R}$, $\|\cdot\|$ shall be the Euclidean norm on $\mathbb{R}^n$.

Sheaves

We will also emplay the notion of a sheaf; as giving the whole background about sheaves would certainly cause acute mental strain in both the author and the reader, the following will suffice for our purposes. With $(\Omega_\lambda)_{\lambda \in \Lambda}$ an open covering of $U \subseteq \mathbb{R}^n$ a sheaf of differential algebras (or modules, rings, ...) on U is a mapping which assigns to each $\Omega \subseteq U$ a differential algebra (module, ring, ...) $\mathcal{G}(U)$ such that the following conditions are satisfied:

(S0) For $u \in \mathcal{G}(\Omega)$ and $\Omega'' \subseteq \Omega' \subseteq \Omega$, $(u|_{\Omega'})|_{\Omega''} = u|_{\Omega''}$.

(S1) If $u, v \in \mathcal{G}(\Omega)$ and $u|_{\Omega_\lambda} = v|_{\Omega_\lambda}$ for all $\lambda \in \Lambda$ then $u = v$.

(S2) If there is given a family of $u_\lambda \in \mathcal{G}(\Omega_\lambda)$ satisfying $u_\lambda|_{\Omega_\lambda \cap \Omega_\mu} = u_\mu|_{\Omega_\lambda \cap \Omega_\mu}$ for all $\lambda, \mu \in \Lambda$ with $\Omega_\lambda \cap \Omega_\mu \neq \emptyset$, then there exists a $u \in \mathcal{G}(\Omega)$ such that $u|_{\Omega_\lambda} = u_\lambda$ for all $\lambda \in \Lambda$.

Nets

A net is defined as a mapping on a directed set into any other set ([Kel55]). There are various ways of denoting nets; as the main objects of Colombeau algebras are nets of functions (or points) we strive for a presentation which is short and readable but still explicit enough to be intuitively understood. We will only consider such nets where also the dependence on the index $\varepsilon \in I := (0, 1]$ is smooth; therefore a net will always be an element of some $C^\infty(I \times \Omega, \Omega')$ for some sets Ω, Ω' such that we can speak of smoothness. We will denote such a net by a symbol indexed by ε, e.g., $u_\varepsilon \in C^\infty(I \times \Omega, \Omega')$. Note that a priori the symbol u_ε has no relationship at all with the symbol u (though often it will be defined to have such). The class of a net u_ε with respect to some factor space will be written as $[u_\varepsilon]$.

Differential Geometry

We will find ourselves situated on manifolds which are without exception smooth, paracompact, Hausdorff, connected, and of finite dimension; the letters X, Y and Z shall denote such. A chart in X where the domain U is an open set in X and φ is a homeomorphism of U onto an open set in $\mathbb{R}^n$ is written as (U, φ) and an atlas of X is given as $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ without further specification of the index set A.

$\mathfrak{X}(X)$ and $\mathfrak{X}^*(X)$ shall be the space of vector fields and one forms on a manifold X, respectively, $\mathfrak{X}(\alpha)$ the space of vector fields along a curve α in X.

A vector bundle E over X with projection $\pi : E \to X$ is written as (E, X, π). Vector bundle charts (*fibered charts* in [Die72]) will be given as mappings of the form

$$\begin{aligned}\Psi : \pi^{-1}(U) &\to \varphi(U) \times \mathbb{R}^{n'} \\ x &\mapsto (\varphi(\pi(x)), \boldsymbol{\varphi}(x))\end{aligned}$$

where (U, φ) is a chart in X and $\boldsymbol{\psi}$ is a linear isomorphism of each fiber $\pi^{-1}(x), x \in U$ to $\mathbb{R}^{n'}$. Vector bundle charts shall be denoted by uppercase Greek letters as in (U, Φ) with $U \subseteq X$; the corresponding lowercase Greek letter φ will stand for the chart in the base manifold and, in bold script like $\boldsymbol{\varphi}$, for the linear isomorphism.

The space of smooth sections of a vector bundle (E, X, π) is denoted by $\Gamma(X, E)$. The local expression s_α of a section $s \in \Gamma(X, E)$ relative to a vector bundle chart (U_α, Φ_α), given by $s_\alpha := \Phi_\alpha \circ s \circ \varphi_\alpha^{-1}$, is a mapping of the form

$$\begin{aligned}\varphi(U_\alpha) &\to \varphi(U_\alpha) \times \mathbb{R}^{n'} \\ x &\to (x, s_\alpha^1, \ldots, s_\alpha^{n'}).\end{aligned}$$

The s_α^i are called *components* of s.

If (E, X, π_X) and (F, Y, π_Y) are vector bundles, a vector bundle homomorphism from E to F is a pair (f, g) where $f : E \to F$ and $g : X \to Y$ are smooth mappings satisfying $g \circ \pi_X = \pi_Y \circ f$ and such that the restriction $f|_{\pi_X^{-1}(x)} : \pi_X^{-1}(x) \to \pi_Y^{-1}(g(x))$ is linear on each fiber $\pi_X^{-1}(x), x \in E$ of E. We will simply write $\underline{f}$ in place of g and f in place of $(f, \underline{f})$. The space of vector bundle homomorphisms from E to F is denoted by $\mathrm{Hom}(E, F)$.

The local expression of $f \in \mathrm{Hom}(E, F)$ with respect to vector bundle charts (U, Φ) in E and (V, Ψ) in F is of the form

$$\begin{aligned}\varphi(U) \times \mathbb{R}^{n'} &\to \psi(V) \times \mathbb{R}^{m'} \\ (x, \xi) &\mapsto (f_{\Psi\Phi}^{(1)}(x), f_{\Psi\Phi}^{(2)}(x) \cdot \xi)\end{aligned} \tag{1.1}$$

where $f_{\Psi\Phi}^{(1)} = \psi \circ \underline{f} \circ \varphi^{-1}$ is a smooth mapping of $\varphi(U)$ into $\psi(V)$ and $f_{\Psi\Phi}^{(2)}$ a smooth mapping of $\varphi(U)$ into the set of linear mappings from $\mathbb{R}^{n'}$ to $\mathbb{R}^{m'}$ (identified with the set of $n' \times m'$-matrices over $\mathbb{R}$).

A Riemannian metric on a vector bundle (E, X, π) is defined as a section $g \in \Gamma(X, E_2^0)$ such that $g(p)$ is symmetric and positive definite on every fiber ([GHV72], Ch. II, 2.17). A norm $\|\cdot\|_g$ on the fibers of E is then induced by $\|e\|_g := g(\pi(e))(e, e)\ \forall e \in E$.

The following Lemma will be used frequently in order to derive estimates chart-wise on compact sets.

Lemma 1.1. *Let $K \subset\subset X$ and $(U_\alpha, \varphi_\alpha)$ be an atlas of X. Then there are $r \in \mathbb{N}, \alpha_i \in A$ and $K_i \subset\subset U_{\alpha_i}$ for $i = 1, \ldots, r$ such that $K = \bigcup_{i=1}^r K_i$.*

Proof. For each $p \in K$ there is an $\alpha_p \in A$ with $p \in U_{\alpha_p}$. As X is regular as a topological space we can choose open neighborhoods U_p of p satisfying $\overline{U_p} \subset\subset U_{\alpha_p}$ for all $p \in K$. Because $K \subseteq \bigcup_{p \in K} U_p$ and K is compact there are $r \in \mathbb{N}$ and $p_1, \ldots, p_n \in K$ such that $K \subseteq \bigcup_{i=1}^r U_{p_i}$. We set $K_i := K \cap \overline{U_{p_i}}$. □

Differential Operators

If E and F are complex vector bundles over X, a *(C^∞ linear) differential operator from E to F* is a continuous linear mapping $P : \Gamma(X, E) \to \Gamma(X, F)$ for which $\operatorname{supp} Pu \subseteq \operatorname{supp} u$ holds for all sections u ([Die72]). Operators satisfying this latter condition are called *local*. If on E and F we have vector bundle charts (U, Φ) and (U, Ψ), respectively, the maps

$$f \mapsto \boldsymbol{\varphi} \circ f \circ \varphi^{-1} : \Gamma(U, E) \to C^\infty(\varphi(U))^{n'} \quad \text{and}$$
$$f \mapsto \boldsymbol{\psi} \circ f \circ \varphi^{-1} : \Gamma(U, F) \to C^\infty(\varphi(U))^{n''}$$

are isomorphisms. For any differential operator P from E to F, the value $(P \cdot f)|_U$ depends only on $f|_U$. Therefore there exists a linear mapping $Q : C^\infty(\varphi(U))^{n'} \to C^\infty(\varphi(U))^{m'}$ such that

$$\begin{array}{ccc} \Gamma(U, E) & \xrightarrow{P} & \Gamma(U, F) \\ {\scriptstyle f \mapsto \boldsymbol{\varphi} \circ f \circ \varphi^{-1}} \big\downarrow & & \big\downarrow {\scriptstyle f \mapsto \boldsymbol{\psi} \circ f \circ \varphi^{-1}} \\ C^\infty(\varphi(U))^{n'} & \xrightarrow[Q]{} & C^\infty(\varphi(U))^{m'} \end{array}$$

is commutative, i.e.

$$\boldsymbol{\theta} \circ (P \cdot f)|_U \circ \varphi^{-1} = Q \cdot (\boldsymbol{\varphi} \circ (f|_U) \circ \varphi^{-1}). \tag{1.2}$$

Q is then called the *local expression* of the operator P corresponding to the charts φ, Φ and Ψ.

Peetre's theorem states that such P can be characterized locally in terms of linear mappings and partial derivatives.

Theorem 1.2 (Peetre)**.** *In order that a linear mapping P of $\Gamma(X, E)$ into $\Gamma(X, F)$ is a differential operator, it is necessary and sufficient that for each*

$x \in X$ there exist vector bundle charts of E and F at the point x such that the corresponding local expression of P is of the form

$$g \mapsto \sum_{|\alpha| \leq p} A_\alpha \cdot \partial^\alpha g,$$

where, for each multi-index α such that $|\alpha| \leq p$, the mapping $y \mapsto A_\alpha(y)$ is a C^∞-mapping of $\varphi(U)$ into the vector space $Hom_{\mathbb{K}}(\mathbb{K}^{n'}, \mathbb{K}^{m'})$ (the space of $\mathbb{K}$-linear mappings which can be identified with the space of $m' \times n'$ matrices over $\mathbb{K}$).

Chapter 2

Introduction to Colombeau Algebras

It is assumed that the reader already is familiar with the special variant of Colombeau algebras on $\mathbb{R}^n$. The most important definitions and results will be given in this chapter in order to fix notation and terminology and to serve as reference points for the extension of concepts to manifolds. For proofs and further background, consult [GOKS01].

Generalized functions on $\mathbb{R}^n$

The nets of functions we consider are indexed by $I = (0, 1]$. The constituting parts of the Colombeau algebra on a subset Ω of $\mathbb{R}^n$ are then defined by

$$
\begin{aligned}
\mathcal{E}(\Omega) :=& \big\{u_\varepsilon \in C^\infty(I \times \Omega)\big\},\\
\mathcal{E}_M(\Omega) :=& \big\{u_\varepsilon \in \mathcal{E}(\Omega) \bigm| \forall K \subset\subset \Omega \ \forall \alpha \in \mathbb{N}_0^n\\
&\quad \exists N \in \mathbb{N} : \sup_{x \in K} \|\partial^\alpha u_\varepsilon(x)\| = O(\varepsilon^{-N})\big\},\\
\mathcal{N}(\Omega) :=& \big\{u_\varepsilon \in \mathcal{E}_M(\Omega) \bigm| \forall K \subset\subset \Omega \ \forall \alpha \in \mathbb{N}_0^n\\
&\quad \forall m \in \mathbb{N} : \sup_{x \in K} \|\partial^\alpha u_\varepsilon(x)\| = O(\varepsilon^{m})\big\},
\end{aligned}
$$

and the algebra itself by

$$\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

$\mathcal{E}(\Omega)$ is called *base space*, nets in $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ *moderate* and *negligible* functions, respectively. $\mathcal{G}(\Omega)$ is the *Colombeau algebra of generalized functions* on Ω, its elements are written as $u_\varepsilon + \mathcal{N}(\Omega)$ or $[u_\varepsilon]$. $\mathcal{G}(\Omega)$ is an associative commutative algebra with unit $[(\varepsilon, x) \mapsto 1]$ where operations are defined component-wise, i.e. $[u_\varepsilon] + [v_\varepsilon] := [u_\varepsilon + v_\varepsilon]$ and $[u_\varepsilon] \cdot [v_\varepsilon] := [u_\varepsilon \cdot v_\varepsilon]$. It is also a differential algebra with respect to $\partial_i[u_\varepsilon] := [\partial_i u_\varepsilon]$. Important properties of $\mathcal{G}(\Omega)$ are the existence of injective embeddings

$$\iota : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega) \text{ and } \sigma : C^\infty(\Omega) \to \mathcal{G}(\Omega) \tag{2.1}$$

and derivatives ∂_i which extend the derivations of $\mathcal{D}'$ onto $\mathcal{G}(\Omega)$; furthermore, multiplication extends the pointwise multiplication of C^∞-functions. The mapping $\mathcal{G}(\_) : \Omega \mapsto \mathcal{G}(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$ is a fine sheaf of differential algebras on $\mathbb{R}^n$.

Negligible functions can be characterized in a simpler way by

$$\mathcal{N}(\Omega) := \left\{ u_\varepsilon \in \mathcal{E}_M(\Omega) \;\middle|\; \forall K \subset\subset \Omega \; \forall m \in \mathbb{N} : \sup_{x \in K} |u_\varepsilon(x)| = O(\varepsilon^m) \right\}. \tag{2.2}$$

Generalized Numbers

Generalized numbers are defined analogously to generalized functions by moderate and negligible objects

$$\begin{aligned}
\mathcal{E} &:= \left\{ v_\varepsilon \in C^\infty(I, \mathbb{K}) \mid \exists N \in \mathbb{N} : |v_\varepsilon| = O(\varepsilon^{-N}) \right\}, \\
\mathcal{N} &:= \left\{ v_\varepsilon \in C^\infty(I, \mathbb{K}) \mid \forall m \in \mathbb{N} : |v_\varepsilon| = O(\varepsilon^{m}) \right\}, \\
\mathcal{K} &:= \mathcal{E} / \mathcal{N}.
\end{aligned}$$

$\mathcal{K}$ is a ring. For $\mathcal{K} = \mathbb{R}$ we will also write $\mathcal{R}$ in place of $\mathcal{K}$. The *point value* of a generalized function $u = [u_\varepsilon] \in \mathcal{G}(\Omega)$ at a point $x \in \Omega$ given by $u(x) := u_\varepsilon(x) + \mathcal{N}$ is a well-defined element of $\mathcal{K}$. The point values at all points of Ω are, however, not enough to uniquely determine the function - we need to extend our concept of points to generalized points. On a subset $\Omega \subseteq \mathbb{R}^n$, we therefore consider the set

$$\Omega_M := \left\{ x_\varepsilon \in C^\infty(I, \Omega) \mid \exists N \in \mathbb{N} : \|x_\varepsilon\| = O(\varepsilon^{-N}) \right\}.$$

Elements of Ω_M are called equivalent, written $x_\varepsilon \sim y_\varepsilon$, if $\|x_\varepsilon - y_\varepsilon\| = O(\varepsilon^m)$ for all $m \in \mathbb{N}$. We then factor Ω_M by this equivalence relation, and of the resulting set we only consider those elements $[x_\varepsilon]$ for which there is a compact set K with $x_\varepsilon \in K$ for small ε. Those are called *compactly supported generalized points* and are denoted by $\widetilde{\Omega}_c$.

Point Values

Proposition 2.1. *For $u = [u_\varepsilon] \in \mathcal{G}(\Omega)$, the evaluation at the generalized point $x = [x_\varepsilon] \in \widetilde{\Omega}_c$ given by $u(x) := [u_\varepsilon(x_\varepsilon)]$ is a well-defined element of $\mathcal{K}$ called the generalized point value of u at x.*

Proposition 2.2. *Let $u \in \mathcal{G}(\Omega)$. Then $u = 0$ if and only if $u(x) = 0$ for all $x \in \widetilde{\Omega}_c$.*

Invertibility

Lemma 2.3. *Let $A \in \mathcal{K}^{n^2}$ be a square matrix. Then the following statements are equivalent.*

(i) A *is non-degenerate, i.e.,* $\xi \in \mathcal{K}^n$, $\xi^t A\eta = 0\ \forall \eta \in \mathcal{K}^n$ *implies* $\xi = 0$.

(ii) $A : \mathcal{K}^n \to \mathcal{K}^n$ *is injective.*

(iii) $A : \mathcal{K}^n \to \mathcal{K}^n$ *is bijective.*

(iv) $\det(A)$ *is invertible.*

Proposition 2.4. *$u \in \mathcal{G}(\Omega)$ is invertible if and only if $u(x)$ is invertible in $\mathcal{K}$ for each $x \in \widetilde{\Omega}_c$.*

Theorem 2.5. *A generalized function $u \in \mathcal{G}(\Omega)$ is invertible if and only if for each representative u_ε of u and each $K \subset\subset \Omega$ there exist $\varepsilon_0 > 0$ and $m \in \mathbb{N}$ such that $\inf_{x\in K}|u_\varepsilon(x)| \geq \varepsilon^m$ for all $\varepsilon < \varepsilon_0$.*

Chapter 3

Colombeau Generalized Functions on Manifolds

3.1 Generalized Functions

Basic Definition

In the following, charts shall be taken from an atlas $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ if not stated otherwise.

In order to construct Colombeau algebras on manifolds, we need notions of moderate and negligible functions there. One natural starting point is to use the framework already established on $\mathbb{R}^n$, enabling us to define moderate functions on a manifold X as nets $u_\varepsilon \in C^\infty(I \times X)$ whose local expression on charts is moderate, i.e., $u_\varepsilon \circ \varphi_\alpha^{-1} \in \mathcal{E}_M(\varphi_\alpha(U_\alpha))$. This makes sense, as moderateness then is invariant under change of charts.

On the other hand, on $\mathbb{R}^n$ growth conditions are imposed on the partial derivatives $\partial^\alpha u_\varepsilon$. In the manifold setting, one can do the same with Lie derivatives, applying the same growth conditions to $L_{\xi_1} \dots L_{\xi_k} u_\varepsilon$ with each $\xi_i \in \mathfrak{X}(X)$ being a vector field.

A third door is opened by Peetre's theorem, relating differential operators on a manifold to local expressions made up of partial derivatives. We denote by $\mathcal{P}(X)$ the space of differential operators as defined above with vector bundles $E = F = X \times \mathbb{R}$. In this case the sections of E and F are the real valued smooth functions on X. For each chart (U, φ) in X a vector bundle chart is then given by $(U, \varphi \times \mathrm{id}_\mathbb{R})$, so the corresponding local expression of an operator $P \in \mathcal{P}(X)$ is given by P itself (as can be verified by replacing

$\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ in (1.2) by $\mathrm{id}_{\mathbb{R}^m}$) and by Peetre's theorem we obtain

$$(Pf)(x) = \left(\sum A_\alpha \partial^\alpha (f \circ \varphi^{-1})\right) \circ \varphi(x) \tag{3.1}$$

for $x \in U$ and $f \in C^\infty(X)$.

For the construction of $\mathcal{G}(X)$ we define $\mathcal{E}(X) := C^\infty(I \times X)$ as *base space*, consisting of the nets of functions which will model our generalized functions. Note that the smooth dependence on $\varepsilon \in I$ is not necessary for the construction of Colombeau algebras.

Theorem 3.1. *The following conditions for a net $u_\varepsilon \in \mathcal{E}(X)$ of smooth functions on the manifold X are equivalent:*

(i) $\forall K \subset\subset X\ \forall P \in \mathcal{P}(X)\ \exists N \in \mathbb{N} : \sup_{p \in K} |Pu_\varepsilon(p)| = O(\varepsilon^{-N})$.

(ii) $\forall K \subset\subset X\ \forall l \in \mathbb{N}_0\ \forall \xi_1, \ldots, \xi_l \in \mathfrak{X}(X)\ \exists N \in \mathbb{N}$:

$$\sup_{p \in K} |L_{\xi_1} \ldots L_{\xi_l} u_\varepsilon(p)| = O(\varepsilon^{-N}).$$

(iii) $u_\varepsilon \circ \varphi^{-1} \in \mathcal{E}_M(\varphi(U))$ *for all charts* (U, φ) *in* X.

Proof. *(i)* $\Rightarrow$ *(ii)*: For $u_\varepsilon \in C^\infty(X)$ and $\xi \in \mathfrak{X}(X)$, the Lie derivative $L_\xi u_\varepsilon$ as well as the iterated Lie derivatives are easily seen to be local and therefore elements of $\mathcal{P}(X)$ by the coordinate description

$$L_\xi u_\varepsilon(x) = \sum_{i=1}^{n} \xi^i \frac{\partial u_\varepsilon}{\partial x^i}.$$

(ii) $\Rightarrow$ (iii) follows as well from the local form of the Lie derivative and (iii) $\Rightarrow$ (i) is a direct consequence of Peetre's theorem, taking into account the local form (3.1) of P. $\square$

Definition 3.2. The space $\mathcal{E}_M(X)$ of moderate functions on X is defined as the set of all elements of $\mathcal{E}(X)$ satisfying one of the equivalent conditions of Theorem 3.1.

Replacing "$\exists N$", "ε^{-N}" and $\mathcal{E}_M$ by "$\forall m$", "ε^m" and "$\mathcal{N}$" in (i)-(iii), respectively, yields the following characterization of negligible nets.

Theorem 3.3. *The following conditions for a net $u_\varepsilon \in \mathcal{E}_M(X)$ are equivalent:*

(i) $\forall K \subset\subset X \; \forall P \in \mathcal{P}(X) \; \forall m \in \mathbb{N} : \sup_{p\in K} |Pu_\varepsilon(p)| = O(\varepsilon^m)$.

(ii) $\forall K \subset\subset X \; \forall l \in \mathbb{N}_0 \; \forall \xi_1, \dots, \xi_l \in \mathfrak{X}(X) \; \forall m \in \mathbb{N}$:

$$\sup_{p\in K} |L_{\xi_1} \dots L_{\xi_l} u_\varepsilon(p)| = O(\varepsilon^m).$$

(iii) $u_\varepsilon \circ \varphi^{-1} \in \mathcal{N}(\varphi(U))$ *for all charts* (U, φ) *in* X.

The equivalence of these conditions can be seen in total analogy to the corresponding statement for moderateness. A simplification can be made by a glance at (2.2) and the local description (iii); one does not need to take into account the derivatives at all.

Definition 3.4. The space $\mathcal{N}(X)$ of negligible functions on X is defined as the set of nets $u_\varepsilon \in \mathcal{E}_M(X)$ satisfying

$$\forall K \subset\subset X \; \forall m \in \mathbb{N} : \sup_{p\in K} |u_\varepsilon(p)| = O(\varepsilon^m).$$

Definition 3.5. The Colombeau algebra of generalized functions on X is defined as

$$\mathcal{G}(X) := \mathcal{E}_M(X)/\mathcal{N}(X).$$

Elements of $\mathcal{G}(X)$ are again denoted by $[u_\varepsilon] = u_\varepsilon + \mathcal{N}(X)$. Naturally, the zero element $0 + \mathcal{N}(X)$ in $\mathcal{G}(X)$ is denoted by 0 and the multiplicative unit $[(\varepsilon, p) \mapsto 1]$ by 1. Smooth functions $f \in C^\infty(X)$ can be embedded into $\mathcal{G}(X)$ by the constant embedding $\sigma(f) := [\varepsilon \mapsto f]$.

Proposition 3.6. $\mathcal{G}(X)$ *is an associative commutative algebra with unit and a differential algebra with respect to Lie derivatives, where the operations* $+$, $\cdot$ *and* L_ξ *are defined component-wise.*

Proof. That $\mathcal{E}_M(X)$ is a subalgebra of $\mathcal{E}(X)$ follows from the linearity of P (or L_ξ) and the fact that L_ξ satisfies the Leibniz rules for derivatives. From the latter we also conclude that $\mathcal{N}(X)$ is an ideal in $\mathcal{E}_M(X)$, so multiplication is well-defined. For L_ξ to be well-defined we need $L_\xi(\mathcal{E}_M(X)) \subseteq \mathcal{E}_M(X)$ and $L_\xi(\mathcal{N}(X)) \subseteq \mathcal{N}(X)$ for all $\xi \in \mathfrak{X}(X)$, which immediately result from the corresponding definitions. □

As seen above, moderateness and negligibility on manifolds can be described locally by the respective conditions on $\mathbb{R}^n$. In fact, we even have the following identification.

Proposition 3.7. *A generalized function* $u = [u_\varepsilon] \in \mathcal{G}(X)$ *can be identified with the family* $(u_\alpha)_\alpha$ *of generalized functions defined by*

$$u_\alpha := [u_\varepsilon \circ \varphi_\alpha^{-1}] \in \mathcal{G}(\varphi_\alpha(U_\alpha))$$

satisfying the transformation law

$$u_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} = u_\beta|_{\varphi_\beta(U_\alpha \cap U_\beta)} \circ \varphi_\beta \circ \varphi_\alpha^{-1}$$

for $\alpha, \beta \in A$ *and* $U_\alpha \cap U_\beta \neq \emptyset$*, where* $\big\{(U_\alpha, \varphi_\alpha)_\alpha \mid \alpha \in A\big\}$ *is an atlas of* X*.*

Proof. The one-to-one correspondence is evident. □

Theorem 3.8. $\mathcal{G}(\_) : \Omega \to \mathcal{G}(\Omega)$ *is a sheaf of differential algebras on* X*.*

Proof. (S0) is obvious.

For the rest suppose $(\Omega_\lambda)_\lambda$ to be an open covering of Ω.

(S1) For $u, v \in \mathcal{G}(\Omega)$ and $u|_{\Omega_\lambda} = v|_{\Omega_\lambda}$ $\forall\lambda$, we have to show that $u = v$. This follows using the local descriptions $(u_\alpha)_\alpha$ and $(v_\alpha)_\alpha$ and the sheaf property already established for Colombeau algebras on $\mathbb{R}^n$, as $\varphi_\alpha(U_\alpha)$ is covered by the open sets $\varphi_\alpha(U_\alpha \cap \Omega_\lambda)$ on each of which $u_\alpha = v_\alpha$ holds; therefore $u_\alpha = v_\alpha$ on $\varphi_\alpha(U_\alpha)$ $\forall\alpha$ which is equivalent to $u = v$.

(S2) Given $u_\lambda \in \mathcal{G}(\Omega_\lambda)$ such that $u_\lambda|_{\Omega_\lambda \cap \Omega_\mu} = u_\mu|_{\Omega_\lambda \cap \Omega_\mu}$ $\forall\lambda, \mu$, we first fix a chart $(U_\alpha, \varphi_\alpha)$ and note that $u_\lambda|_{U_\alpha}$ is in $\mathcal{G}(U_\alpha \cap \Omega_\lambda)$ and satisfies $u_\lambda = u_\mu$ on $U_\alpha \cap \Omega_\lambda \cap \Omega_\mu$. Hence we have the local expression $(u_\lambda)_\alpha \in \mathcal{G}(\varphi_\alpha(U_\alpha \cap \Omega_\lambda))$ with $(u_\lambda)_\alpha = (u_\mu)_\alpha$ on $\varphi_\alpha(U_\alpha \cap \Omega_\lambda \cap \Omega_\mu)$. As $(\varphi_\alpha(U_\alpha \cap \Omega_\lambda))_\lambda$ is an open covering of $\varphi_\alpha(U_\alpha)$, there is a $\tilde{u}_\alpha \in \mathcal{G}(\varphi_\alpha(U_\alpha))$ by the sheaf property on $\mathbb{R}^n$ such that $\tilde{u}_\alpha|_{\varphi_\alpha(U_\alpha \cap \Omega_\lambda)} = (u_\lambda)_\alpha$. This can be done on each chart, giving a family of functions $(\tilde{u}_\alpha)_\alpha$ satisfying the transformation law

$$\begin{aligned}\tilde{u}_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta \cap \Omega_\lambda)} = (u_\lambda)_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} &= (u_\lambda)_\beta|_{\varphi_\alpha(U_\alpha \cap U_\beta)} \circ \varphi_\beta \circ \varphi_\alpha^{-1} \\ &= \tilde{u}_\beta|_{\varphi_\alpha(U_\alpha \cap U_\beta \cap \Omega_\lambda)} \circ \varphi_\beta \circ \varphi_\alpha^{-1}\end{aligned}$$

for each λ and, consequently,

$$\tilde{u}_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} = \tilde{u}_\beta|_{\varphi_\alpha(U_\alpha \cap U_\beta)} \circ \varphi_\beta \circ \varphi_\alpha^{-1}$$

resulting – via the identity of proposition 3.7 – in a generalized function $u \in \mathcal{G}(\Omega)$ for which $u|_{\Omega_\lambda} = u_\lambda$ holds because the respective local expressions

$$(u|_{\Omega_\lambda})_\alpha = \tilde{u}_\alpha|_{\varphi_\alpha(\Omega_\lambda \cap U_\alpha)} = (u_\lambda)_\alpha$$

are equal. Uniqueness follows from (S1). □

Point Values

Generalized functions on manifolds can be characterized by their point values, but evaluation on all points $x \in X$ is not enough for that; we need generalized points. To this end, we define compactly supported points on X as

$$X_c := \left\{p_\varepsilon \in C^\infty(I, X) \mid \exists K \subset\subset X\ \exists \varepsilon_0 \in I:\ p_\varepsilon \in K\ \forall \varepsilon \leq \varepsilon_0\right\}$$

and their equivalence by

$$p_\varepsilon \sim q_\varepsilon \iff d_h(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m) \quad \forall m \in \mathbb{N},$$

where d_h is the Riemannian distance induced by a Riemannian metric h on X. The equivalence classes with respect to $\sim$ are called *compactly supported generalized points on* X. This set will be denoted by $\widetilde{X}_c$. For $X = \mathbb{R}$ we will write $\mathcal{R}_c$ instead of $\widetilde{\mathbb{R}}_c$. Lemma 3.13 will establish that $\widetilde{X}_c$ in fact does not depend on the metric h, but first we need some preparations.

Lemma 3.9. *Let h_1 and h_2 be Riemannian metrics on X. Then for all $K \subset\subset X$ there exist $C_1, C_2 > 0$ such that*

$$C_1 \|v\|_{h_2(p)} \leq \|v\|_{h_1(p)} \leq C_2 \|v\|_{h_2(p)} \quad \forall p \in K\ \forall v \in T_pX.$$

Proof. By virtue of Lemma 1.1 we may assume without loss of generality that $K \subset\subset U_\alpha$ for some chart $(U_\alpha, \varphi_\alpha)$. If we denote by h_i^α the local expression of h_i $(i = 1, 2)$, we can define the function

$$f(x, v) := \frac{h_1^\alpha(x)(v, v)}{h_2^\alpha(x)(v, v)}$$

which is continuous on $\varphi_\alpha(K) \times \mathbb{R}^n \setminus \{0\}$. Therefore, the supremum of f on this set, which is equal to the supremum on $\varphi_\alpha(K) \times \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, is finite, giving a constant C_2 such that $\|v\|_{h_1} \leq C_2 \|v\|_{h_2}$. By the same procedure we obtain C_1 as required. □

We will also need the same result for Riemannian metrics on vector bundles instead of manifolds.

Lemma 3.10. *Let $K \subset\subset X$ and g_1, g_2 be Riemannian metrics on a vector bundle (E, X, π) inducing the norms $\|\cdot\|_{g_1}$ and $\|\cdot\|_{g_2}$ on the fibers of E. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \|p\|_{g_1} \leq \|p\|_{g_2} \leq C_2 \|p\|_{g_1} \quad \forall p \in \pi^{-1}(K).$$

Proof. Without loss of generality we may assume that there is a vector bundle chart (U, Φ) such that $K \subset\subset U$. The local expressions of g_1 and g_2 with respect to this chart are then given by

$$(\Phi_* g_i)(x)(\xi_1, \xi_2) := g_i\big(\varphi^{-1}(x)\big)\big((\boldsymbol{\varphi}|_{E_{\varphi^{-1}(x)}})^{-1}(\xi_1), (\boldsymbol{\varphi}|_{E_{\varphi^{-1}(x)}})^{-1}(\xi_2)\big).$$

By the continuity of

$$f(x, v) := \frac{(\Phi_* g_1)(x)(v, v)}{(\Phi_* g_2)(x)(v, v)}$$

on $\varphi(K) \times \mathbb{R}^{n'} \setminus \{0\}$, there exists $C > 0$ such that $\sup_{(x,v)\in\varphi(K)\times\mathbb{R}^{n'}} f(x, v) = \sup_{(x,v)\in\varphi(K)\times B_1(0)} f(x, v) = C$. For all $e \in \pi^{-1}(K)$ we obtain

$$\begin{aligned}
\|e\|_{g_1} &= g_1(\pi(e))(e, e) \\
&= (\Phi_* g_1)\big(\varphi(\pi(e))\big)\big(\boldsymbol{\varphi}(e), \boldsymbol{\varphi}(e)\big) \\
&\leq C(\Phi_* g_2)\big(\varphi(\pi(e))\big)\big(\boldsymbol{\varphi}(e), \boldsymbol{\varphi}(e)\big) \\
&= C g_2(\pi(e))(e, e) = C\|e\|_{g_2}.
\end{aligned}$$ □

Corollary 3.11. *If g is a Riemannian metric on E, (U, Φ) a vector bundle chart in E and $K \subset\subset U$ there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|e\|_g \leq \|\boldsymbol{\varphi}(e)\| \leq C_2\|e\|_g \quad \forall e \in \pi^{-1}(K).$$

Proof. Immediate by extending the inner product on $\mathbb{R}^{n'}$ to a Riemannian metric h on E and applying the previous lemma. □

Lemma 3.12. *Let h_1 and h_2 be Riemannian metrics on X. Then for all $K \subset\subset X$ there exist constants $\varepsilon_0, C > 0$ such that $\forall p \in K$ $\forall \varepsilon \leq \varepsilon_0$:*

$$B_\varepsilon^{(2)}(p) \subseteq B_{C\varepsilon}^{(1)}(p)$$

where $B_\varepsilon^{(i)}(p) = \{q \in X \mid d_i(p, q) < \varepsilon\}$ and d_i denotes the Riemannian distance with respect to h_i.

Proof. For any $p \in X$ we can choose (by virtue of [O'N83], Chapter 5, Proposition 7) a geodesically convex (with respect to h_2) and relatively compact neighborhood U_p of p. Applying Lemma 3.9 to $\overline{U}_p$ we obtain a constant $C_p > 0$ such that $\|v\|_{h_1(q)} \leq C_p\|v\|_{h_2(q)}$ $\forall q \in \overline{U}_p$ $\forall v \in T_qX$. For any $q, q' \in U_p$, let α be the connecting h_2-geodesic. Denoting by L_i the length of α with respect to h_i we then have $d_1(q, q') = L_1(\alpha) \leq C_p L_2(\alpha) = C_p d_2(q, q')$.

For each $p \in K$ we choose ε_p such that $B^{(2)}_{\varepsilon_p}(p) \subseteq U_p$. As K is compact there exist finitely many p_i $(i = 1, \dots, m)$ such that K is contained in $U := \bigcup_{i=1}^m B^{(2)}_{\varepsilon_{p_i}/2}(p_i)$. Then, set $\varepsilon_0 := \min(d_2(K, \partial U), \varepsilon_{p_1}/2, \dots, \varepsilon_{p_m}/2)$ and $C := \max_{1\leq i\leq m} C_{p_i}$.

Finally, take $p \in K$, $\varepsilon \leq \varepsilon_0$ and $q \in B^{(2)}_\varepsilon(p)$. As K was covered by d_2-balls with radius $\varepsilon_{p_i}/2$, there exists i such that $d_2(p, p_i) \leq \varepsilon_{p_i}/2$. By the choice of ε, $d_2(p, q) \leq \varepsilon_{p_i}/2$ and therefore $p, q \in B^{(2)}_{\varepsilon_{p_i}}(p_i) \subseteq U_{p_i}$. This results in $d_1(p, q) \leq C_{p_i} d_2(p, q) \leq C\varepsilon$ or $q \in B^{(1)}_{C\varepsilon}(p)$. □

Lemma 3.13. *Let h_i be Riemannian metrics inducing the Riemannian distances d_i on X $(i = 1, 2)$. Then for $K, K' \subset\subset X$ there exists $C > 0$ such that $d_2(p, q) \leq C d_1(p, q)$ for all $p \in K$ and $q \in K'$.*

Proof. Assume to the contrary that there exist subsequences p_m, q_m such that $d_2(p_m, q_m) > m d_1(p_m, q_m)$ $\forall m \in \mathbb{N}$. While m tends to ∞, $d_1(p_m, q_m)$ converges to zero. As both sequences will stay inside a fixed compact set for small ε, we may additionally choose suitable subsequences and suppose that both p_m and q_m converge to some $p \in K$. Let V be a relatively compact neighborhood of p. By virtue of Lemma 3.12 there exist $r_0, \alpha > 0$ such that $B^{(1)}_r(q) \subseteq B^{(2)}_{\alpha r}(q)$ for all $q \in V$ and $r \leq r_0$. For $m > \alpha$ sufficiently large, p_m and q_m are in V, $d_1(p_m, q_m) \leq r_0$ and we arrive at the contradiction $d_2(p_m, q_m) \leq \alpha d_1(p_m, q_m)$. □

The following two lemmas will be needed in order to introduce point value evaluation for functions in $\mathcal{G}(X)$.

Lemma 3.14. *If $K \subset\subset U$ with (U, φ) a chart in X and h any Riemannian metric on X there exists $C > 0$ (depending on h) such that $\|\xi\| \leq C\|T_{\varphi(p)}\varphi^{-1}\xi\|_h$ $\forall p \in K$ $\forall \xi \in \mathbb{R}^n$.*

Proof. The inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ can be extended to a Riemannian metric g on a neighborhood of K. For a point p in this neighborhood we define g at p as

$$\begin{aligned} g_p : T_pX \times T_pX &\to \mathbb{R} \\ (u, v) &\to \chi(p) \cdot \langle T_p\varphi\, u, T_p\varphi\, v\rangle \end{aligned}$$

where $\chi \in \mathcal{D}(X)$ is a cut-off function satisfying $\chi \equiv 1$ on some open neighborhood W of K with $\operatorname{supp}\chi \subseteq V$ and $\overline{W} \subset\subset V$. Lemma 3.9 then gives a constant $C > 0$ such that for $\xi \in \mathbb{R}^n$ and $p \in K$ we have

$$\|\xi\| = \|T_{\varphi(p)}\varphi^{-1}\xi\|_g \leq C\|T_{\varphi(p)}\varphi^{-1}\xi\|_h.$$

□

Lemma 3.15. *For nets $p_\varepsilon, q_\varepsilon \in C^\infty(I, X)$ compactly supported in some W_α which is geodesically convex with respect to a Riemannian metric h on X and satisfies $\overline{W}_\alpha \subset\subset U_\alpha$ for some chart $(U_\alpha, \varphi_\alpha)$, the following equivalence holds true.*

$$p_\varepsilon \sim q_\varepsilon \Longleftrightarrow \|\varphi_\alpha(p_\varepsilon) - \varphi_\alpha(q_\varepsilon)\| = O(\varepsilon^m) \quad \forall m \in \mathbb{N}.$$

Proof. ($\Rightarrow$): Denoting by $\gamma_\varepsilon : [\alpha_\varepsilon, \beta_\varepsilon] \to W_\alpha$ the unique geodesic in W_α joining p_ε and q_ε, we have

$$d_h(p_\varepsilon, q_\varepsilon) = \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \|\gamma_\varepsilon'(s)\|_h \mathrm{d}s = O(\varepsilon^m) \quad \forall m > 0.$$

By Lemma 3.14 there exists $C > 0$ such that

$$\|\xi\| \le C \|T_{\varphi_\alpha(p)}\varphi_\alpha^{-1}\xi\|_h \quad \forall p \in \overline{W}_\alpha \; \forall \xi \in \mathbb{R}^n$$

and therefore

$$\|\varphi_\alpha(p_\varepsilon) - \varphi_\alpha(q_\varepsilon)\| \le \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \|(\varphi_\alpha \circ \gamma_\varepsilon)'(s)\| \mathrm{d}s \le C \int_{\alpha_\varepsilon}^{\beta_\varepsilon} \|\gamma_\varepsilon'(s)\|_h \,\mathrm{d}s = O(\varepsilon^m).$$

($\Leftarrow$): If for some Riemannian metric g on X we establish $d_g(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m)$, the claim immediately follows from Lemma 3.13. Choosing $K \subset\subset W_\alpha$ such that $p_\varepsilon, q_\varepsilon \in K$ for small ε, we construct g as in the proof of Lemma 3.14 employing a cut-off function supported in U_α equal to 1 in a neighborhood W' of W_α with $K \subset\subset W_\alpha$. For small ε the line connecting p_ε and q_ε is contained in W', hence we can write (for small ε again)

$$d_g(p_\varepsilon, q_\varepsilon) = d_{g|_{W'}}(p_\varepsilon, q_\varepsilon) = \|\varphi_\alpha(p_\varepsilon) - \varphi_\alpha(q_\varepsilon)\| = O(\varepsilon^m). \qquad \square$$

Proposition 3.16. *For $u = [u_\varepsilon] \in \mathcal{G}(X)$ and $p = [p_\varepsilon] \in \widetilde{X}_c$, the point value of u at p defined by $u(p) := [u_\varepsilon(p_\varepsilon)]$ is a well-defined element of $\mathcal{K}$.*

Proof. Because p is compactly supported, $u_\varepsilon(p_\varepsilon)$ is moderate or negligible if u_ε is. Therefore different representatives of u_ε give the same result. Choosing any other representative q_ε of p, we have to establish that $u_\varepsilon(p_\varepsilon) \sim u_\varepsilon(q_\varepsilon)$. For small ε, p_ε and q_ε will stay inside a fixed $K \subset\subset X$. Cover K by finitely many geodesically convex sets W_α with respect to some Riemannian metric h having the property $\overline{W}_\alpha \subset\subset U_\alpha$ ([O'N83]). For all ε small enough there exists i_ε such that the whole line connecting $\varphi_{\alpha_{i_\varepsilon}}(p_\varepsilon)$ and $\varphi_{\alpha_{i_\varepsilon}}(q_\varepsilon)$ is contained in $\varphi_{\alpha_{i_\varepsilon}}(W_{\alpha_{i_\varepsilon}})$. In order to obtain moderateness estimates we recall that for each differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\frac{\partial}{\partial\sigma} f\big(a - \sigma(a-b)\big) = (Df)\big(a - \sigma(a-b)\big) \cdot (b-a)$$

and by integration

$$|f(b) - f(a)| \le \int_0^1 \big|(Df)\big(a - \sigma(a-b)\big)(b-a)\big| \,\mathrm{d}\sigma.$$

Substituting $u_\varepsilon \circ \varphi_\alpha^{-1}$ for f and $\varphi_\alpha(p_\varepsilon) := p_{\varepsilon,\alpha}$ and $\varphi_\alpha(q_\varepsilon) := q_{\varepsilon,\alpha}$ for a and b, respectively, we get

$$|u_\varepsilon(q_\varepsilon) - u_\varepsilon(p_\varepsilon)| \le \int_0^1 \big|D(u_\varepsilon \circ \varphi_\alpha^{-1})\big(p_{\varepsilon,\alpha} - \sigma(q_{\varepsilon,\alpha} - p_{\varepsilon,\alpha})\big)(p_{\varepsilon,\alpha} - q_{\varepsilon,\alpha}))\big| \,\mathrm{d}\sigma$$

where $\alpha = \alpha_{i_\varepsilon}$. For sufficiently small ε, the line connecting $p_{\varepsilon,\alpha}$ and $q_{\varepsilon,\alpha}$ is contained in $\varphi_\alpha(W_\alpha)$ and the claim follows from Lemma 3.15 and the fact that $u_\varepsilon \circ \varphi_\alpha^{-1}$ is moderate. □

We finally arrive at the following point value characterization of generalized functions on a manifold.

Theorem 3.17. *Let $u \in \mathcal{G}(X)$. Then $u = 0$ in $\mathcal{G}(X)$ if and only if $u(p) = 0$ in $\mathcal{K}$ for all $p \in \widetilde{X}_c$.*

Proof. Necessity is clear from Proposition 3.16. Conversely, let u_ε be a representative of u. Take a Riemannian metric h and cover X by geodesically convex sets W_α with $\overline{W}_\alpha \subset\subset U_\alpha$ as above. For $x = [x_\varepsilon] \in \varphi_\alpha(W_\alpha)^\sim_c$, the class of $p_\varepsilon := \varphi_\alpha^{-1}(x_\varepsilon)$ is a well-defined element of $\widetilde{X}_c$. By assumption, $u_\varepsilon(p_\varepsilon) = u_\varepsilon \circ \varphi_\alpha^{-1}(x_\varepsilon)$ is negligible, so we have $u_\varepsilon \circ \varphi_\alpha^{-1} = 0$ in $\mathcal{G}(\varphi_\alpha(W_\alpha))$ for all α and therefore $u = 0$. □

3.2 Generalized Sections of Vector Bundles

Sections of vector bundles can be generalized in the sense of Colombeau as well. There are two equivalent ways to define moderate and negligible sections; the first is via the components.

Definition 3.18. Let (E, X, π) be a vector bundle. Then the base space $\Gamma_{\mathcal{E}}(X, E)$, the space of moderate sections $\Gamma_{\mathcal{E}_M}(X, E)$ and the space of negligible sections $\Gamma_{\mathcal{N}}(X, E)$ are defined as follows.

$$\begin{aligned}
\Gamma_{\mathcal{E}}(X, E) &:= \big\{s_\varepsilon \in C^\infty(I \times X, E) \mid s_\varepsilon \in \Gamma(X, E)\ \forall \varepsilon \in I\big\}, \\
\Gamma_{\mathcal{E}_M}(X, E) &:= \{s_\varepsilon \in \Gamma_{\mathcal{E}}(X, E) \mid \forall \alpha,\ \forall i:\ (s_\varepsilon)^i_\alpha \in \mathcal{E}_M(\varphi_\alpha(U_\alpha))\}, \\
\Gamma_{\mathcal{N}}(X, E) &:= \{s_\varepsilon \in \Gamma_{\mathcal{E}_M}(X, E) \mid \forall \alpha,\ \forall i:\ (s_\varepsilon)^i_\alpha \in \mathcal{N}(\varphi_\alpha(U_\alpha))\}.
\end{aligned}$$

For those spaces to be well-defined, moderateness and negligibility should be preserved under change of charts. This, however, is immediately clear as a change of vector bundle charts has the form

$$\Psi_\beta \circ \Psi_\alpha^{-1}(y, w) = \big(\varphi_{\beta\alpha}, \boldsymbol{\varphi}_{\beta\alpha}(y)w\big)$$

with $\boldsymbol{\varphi}_{\beta\alpha} \in GL(n', \mathbb{R})$, so the components of a section change linearly.

The second way to introduce generalized sections is a characterization by Peetre's theorem. We denote by $\mathcal{P}(X, E)$ the space of linear differential operators from E into itself.

Proposition 3.19. *For $s_\varepsilon \in \Gamma_{\mathcal{E}}(X, E)$ and $\|\cdot\|$ a norm on the fibers of E induced by any Riemannian metric on E, the following equivalences hold true.*

$$\begin{aligned}
&(i) \quad s_\varepsilon \in \Gamma_{\mathcal{E}_M}(X, E) \iff \forall P \in \mathcal{P}(X, E)\ \forall K \subset\subset X \\
&\qquad\qquad\qquad\qquad\qquad \exists N \in \mathbb{N} : \sup_{p \in K} \|P u_\varepsilon(p)\| = O(\varepsilon^{-N}). \\
&(ii) \quad s_\varepsilon \in \Gamma_N(X, E) \iff \forall P \in \mathcal{P}(X, E)\ \forall K \subset\subset X \\
&\qquad\qquad\qquad\qquad\qquad \forall m \in \mathbb{N} : \sup_{p \in K} \|P u_\varepsilon(p)\| = O(\varepsilon^{m}).
\end{aligned}$$

Proof. For both directions it suffices to show the estimates for compact sets fully contained in the domain of some chart. The general case then follows by application of Lemma 1.1.

($\Rightarrow$): For each vector bundle chart (U_α, Φ_α) on E, every differential operator $P \in \mathcal{P}(X, E)$ gives rise to its corresponding local expression Q such that

$$(P \cdot u_\varepsilon)|_{U_\alpha} = \boldsymbol{\varphi}_\alpha^{-1} \circ Q \cdot (\boldsymbol{\varphi}_\alpha \circ u_\varepsilon|_{U_\alpha} \circ \varphi_\alpha^{-1}) \circ \varphi_\alpha$$

holds. Because of $\boldsymbol{\varphi}_\alpha \circ u_\varepsilon|_{U_\alpha} \circ \varphi_\alpha^{-1} = \big((u_\varepsilon)^1_\alpha, \ldots, (u_\varepsilon)^{n'}_\alpha\big)$ we can make use of the fact that by Peetre's theorem Q consists only of a combination of linear mappings and partial derivatives. Hence it follows that the components of $Q \cdot (\boldsymbol{\varphi}_\alpha \circ u_\varepsilon|_{U_\alpha} \circ \varphi_\alpha^{-1}) \circ \varphi_\alpha$ satisfy the necessary estimates. The mapping $\boldsymbol{\varphi}_\alpha^{-1}$ then only adds a constant (Lemma 3.11).

($\Leftarrow$): To establish growth estimates on derivatives of $(u_\varepsilon)_\alpha^i$ we construct a differential operator $P \in \mathcal{P}(X, E)$ by

$$(P \cdot u_\varepsilon)|_{U_\alpha} := \boldsymbol{\varphi}_\alpha^{-1} \circ Q \cdot (\boldsymbol{\varphi}_\alpha \circ u_\varepsilon|_{U_\alpha} \circ \varphi_\alpha^{-1}) \circ \varphi_\alpha,$$

where Q is chosen such that the components $(u_\varepsilon)_\alpha^i$ are mapped to their partial derivatives of the required degree, immediately resulting in the claim. □

Definition 3.20. The space of generalized sections of (E, X, π) is defined as

$$\Gamma_{\mathcal{G}}(X, E) := \Gamma_{\mathcal{E}_M}(X, E)/\Gamma_{\mathcal{N}}(X, E).$$

Proposition 3.21. *With operations defined component-wise,* $\Gamma_{\mathcal{G}}(X, E)$ *is a* $\mathcal{G}(X)$*-module.*

Proof. We need to establish that the product of any given $[s_\varepsilon] \in \Gamma_{\mathcal{G}}(X, E)$ and $[u_\varepsilon] \in \mathcal{G}(X)$ is well defined. First, to show that $[s_\varepsilon u_\varepsilon]$ is in $\Gamma_{\mathcal{G}}(X, E)$ we need to have $(s_\varepsilon u_\varepsilon)_\alpha^i \in \mathcal{E}_M(\varphi_\alpha(U_\alpha))$. We set $p = \varphi_\alpha^{-1}(x)$ and observe that

$$\begin{gathered}(s_\varepsilon u_\varepsilon)_\alpha(x) = \Phi_\alpha \circ s_\varepsilon u_\varepsilon \circ \varphi_\alpha^{-1}(x) = \\ \big(\varphi_\alpha(\pi(s_\varepsilon(p)u_\varepsilon(p))), \boldsymbol{\varphi}_\alpha(s_\varepsilon(p)u_\varepsilon(p))\big) = \\ (\varphi_\alpha(\pi(s_\varepsilon(p)u_\varepsilon(p)), u_\varepsilon(p) \cdot \boldsymbol{\varphi}_\alpha(s_\varepsilon(p)))\end{gathered}$$

and from $\varphi_\alpha^i \circ s_\varepsilon \circ \varphi_\alpha^{-1} \in \mathcal{E}_M(\varphi_\alpha(U_\alpha))$ and $u_\varepsilon \circ \varphi_\alpha^{-1} \in \mathcal{E}_M(\varphi_\alpha(U_\alpha))$ we also have a moderate product $\varphi_\alpha^i \circ s_\varepsilon \circ \varphi_\alpha^{-1} \cdot (u_\varepsilon \circ \varphi_\alpha^{-1}) \in \mathcal{E}_M(\varphi_\alpha(U_\alpha))$, therefore $(s_\varepsilon u_\varepsilon)_\alpha^i \in \mathcal{E}_M(\varphi_\alpha(U_\alpha))$.

If we take different representatives

$$\begin{aligned}[s_\varepsilon] &= [s_\varepsilon + m_\varepsilon], m \in \Gamma_{\mathcal{N}}(X, E) \\ \text{and } [u_\varepsilon] &= [u_\varepsilon + n_\varepsilon], n \in \mathcal{N}(X),\end{aligned}$$

we also have $[(s_\varepsilon + m_\varepsilon)(u_\varepsilon + n_\varepsilon)] = [(s_\varepsilon u_\varepsilon)]$ because $s_\varepsilon n_\varepsilon$, $m_\varepsilon u_\varepsilon$, and $m_\varepsilon n_\varepsilon$ are seen to be in $\Gamma_{\mathcal{N}}(X, E)$ in exactly the same manner as above. □

For a generalized section $s = [s_\varepsilon] \in \Gamma_{\mathcal{G}}(X, E)$ and a chart $(U_\alpha, \varphi_\alpha)$ the functions $(s_\varepsilon)_\alpha^i$ are in $\mathcal{E}_M(\varphi_\alpha(U_\alpha))$ or $\mathcal{N}(\varphi_\alpha(U_\alpha))$ if s_ε is moderate or negligible, respectively. Therefore we may define the *components* s_α^i of s and its *vector part* s_α relative to this chart as

$$\begin{aligned}s_\alpha^i &:= [(s_\varepsilon)_\alpha^i] \in \mathcal{G}(\varphi_\alpha(U_\alpha)), \\ s_\alpha &:= (s_\alpha^1, \ldots, s_\alpha^{n'}) \in \mathcal{G}(\varphi_\alpha(U_\alpha))^{n'}.\end{aligned}$$

Proposition 3.22. *$\Gamma_{\mathcal{G}}(X,E)$ can be identified with the set of all families $(s_\alpha)_\alpha$ with $s_\alpha \in \mathcal{G}(\varphi_\alpha(U_\alpha))^{n'}$ satisfying the transformation law*

$$s_\alpha = \boldsymbol{\varphi}_\alpha \circ \boldsymbol{\varphi}_\beta^{-1} \circ s_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1} \text{ on } \varphi_\alpha(U_\alpha \cap U_\beta) \tag{3.2}$$

or written in components

$$s_\alpha^i(x) = (\boldsymbol{\varphi}_{\alpha\beta})^i_j(\varphi_\beta \circ \varphi_\alpha^{-1}(x)) s^j_\beta(\varphi_\beta \circ \varphi_\alpha^{-1}(x))$$

where $\boldsymbol{\varphi}_\alpha \circ \boldsymbol{\varphi}_\beta^{-1} =: \boldsymbol{\varphi}_{\alpha\beta} = (\boldsymbol{\varphi}_{\alpha\beta})^i_j$.

Proof. Evident by definition. □

Proposition 3.23. *$\Gamma_{\mathcal{G}}(X,E)$ is a fine sheaf of $\mathcal{G}(X)$-modules.*

Proof. (S0) is clear. Now take an open covering $(\Omega_\lambda)_\lambda$ of X. Equality of two sections on each set of the covering implies equality of the components on $\varphi_\alpha(U_\alpha \cap \Omega_\lambda)$ $\forall \lambda, \alpha$ and therefore on $\varphi_\alpha(U_\alpha)$, giving (S1). For (S2), restriction to U_α gives components of generalized sections locally which in course satisfy the transformation law (3.2). □

Smooth sections $s \in \Gamma(X,E)$ can be embedded into $\Gamma_{\mathcal{G}}(X,E)$ by the constant embedding $\Sigma(s) := [\varepsilon \mapsto s]$.

Generalized Tensor Fields

If E is the tensor bundle $T^r_s(X)$ over X, the space $\mathcal{G}^r_s(X) := \Gamma_{\mathcal{G}}(X, T^r_s(X))$ is called the *space of generalized (r,s) tensor fields on X*. We will also write $\mathfrak{X}_{\mathcal{G}}(X)$ in place of $\mathcal{G}^1_0(X)$ and $\mathfrak{X}^*_{\mathcal{G}}(X)$ in place of $\mathcal{G}^0_1(X)$.

Theorem 3.24. *As $\mathcal{G}(X)$-module, $\mathcal{G}^r_s(X)$ is isomorphic to the space of multilinear maps $L_{\mathcal{G}(X)}(\mathfrak{X}^*_{\mathcal{G}}(X)^r, \mathfrak{X}_{\mathcal{G}}(X)^s; \mathcal{G}(X))$.*

Proof. Let $t = [t_\varepsilon] \in \mathcal{G}^r_s(X)$ be a generalized tensor field, $\omega^k = [\omega^k_\varepsilon] \in \mathfrak{X}^*_{\mathcal{G}}(X)$ generalized one-forms for $1 \le k \le r$ and $\xi_l = [\xi_{l\varepsilon}] \in \mathfrak{X}_{\mathcal{G}}(X)$ generalized vector-fields on X for $1 \le l \le s$. We then can define an isomorphism $\iota : \mathcal{G}^r_s(X) \to L_{\mathcal{G}(X)}(\mathfrak{X}^*_{\mathcal{G}}(X)^r, \mathfrak{X}_{\mathcal{G}}(X)^s; \mathcal{G}(X))$ by

$$\iota(t)(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_s) := [t_\varepsilon(\omega^1_\varepsilon, \dots, \omega^r_\varepsilon, \xi_{1\varepsilon}, \dots, \xi_{s\varepsilon})]$$

in which the right-hand side is well-defined by classical theory. Using the local description it is straightforward to obtain well-definedness of ι. For showing injectivity assume $\iota(t) = 0$, i.e., $t_\varepsilon(\omega^1_\varepsilon, \dots, \omega^r_\varepsilon, \xi_{1\varepsilon}, \dots, \xi_{s\varepsilon}) \in \mathcal{N}(X)$. If we can show $t = 0$ locally it holds globally, as $\mathcal{G}^r_s(X)$ is a sheaf. For a chart $(U_\alpha, \varphi_\alpha)$ in X we therefore choose $K \subset\subset U_\alpha$ and one-forms ω^k as well as vector fields ξ_l whose compact support is contained in U_α such that $\omega = \Sigma(dx^i)$ and $\xi = \Sigma(\partial_j)$ on an open neighborhood U of K in U_α. Then $t^{i_1\dots i_r}_{j_1\dots j_s}|_U = t_\varepsilon(\omega^1_\varepsilon, \dots, \omega^r_\varepsilon, \xi_{1\varepsilon}, \dots, \xi_{s\varepsilon}) \in \mathcal{N}(U)$, hence $t = 0$ is established.

For surjectivity let $\tilde{t} \in L_{\mathcal{G}(X)}(\mathfrak{X}^*_{\mathcal{G}}(X)^r, \mathfrak{X}_{\mathcal{G}}(X)^s; \mathcal{G}(X)$ be given. We define the n^{r+s} components of a section t with respect to a chart $(U_\alpha, \varphi_\alpha)$ as

$$(t_\alpha)^{i_1\dots i_r}_{j_1\dots j_s} := \tilde{t}|_{U_\alpha}(dx^{i_1}, \dots, dx^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) \circ \varphi_\alpha^{-1} \in \mathcal{G}(\varphi_\alpha(U_\alpha)),$$

which are well-defined by Lemma 3.25 below and constitute a coherent family satisfying (3.2). The $(t_\alpha)^{i_1\dots i_r}_{j_1\dots j_s}$ thus are the local coordinates of a unique section $t \in \mathcal{G}^r_s(X)$ which by construction is mapped onto $\tilde{t}$ by ι. □

Lemma 3.25. *Take some $t \in L_{\mathcal{G}(X)}(\mathfrak{X}^*_{\mathcal{G}}(X)^r, \mathfrak{X}_{\mathcal{G}}(X)^s; \mathcal{G}(X))$, $\omega^k \in \mathfrak{X}^*_{\mathcal{G}}(X)$ for $1 \le k \le r$ and $\xi_l \in \mathfrak{X}_{\mathcal{G}}(X)$ for $1 \le l \le s$. Suppose that the restriction of some ω^k or ξ_l onto some open set $U \subset\subset X$ is the zero element. Then $t(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_s)|_U = 0$.*

Proof. Assume that, say, $\xi_s|_U = 0$. For each point $p \in U$ there is an open neighborhood U_p of p such that $\overline{U}_p \subset\subset V_\alpha$ for some chart $(U_\alpha, \varphi_\alpha)$. We may therefore assume without loss of generality that $\overline{U} \subset\subset U_\alpha$ for some chart U_α, as the general case then follows by the sheaf property of $\mathcal{G}(X)$. In local coordinates we can write $\xi|_{U_\alpha} = \xi^i \partial_i$ with $\xi^i \in \mathcal{G}(U_\alpha)$ and $\xi_i|_U = 0$. Let $f \in \mathcal{D}(U_\alpha)$ such that $f|_{\overline{U}} = 1$. We use the embedding $F := \sigma(f) \in \mathcal{G}(X)$ of f and obtain

$$\begin{aligned} t(\omega^1, \dots, \xi_s)|_U &= (F|_U)^2 t(\omega^1, \dots, \xi_s)|_U \\ &= (F^2 t(\omega^1, \dots, \xi_s))|_U \\ &= t(\omega^1, \dots, F\xi^i_s F\partial_i)|_U \\ &= (F\xi^i_s t(\omega^1, \dots, F\partial_i))|_U \\ &= (F\xi^i_s)|_U t(\omega^1, \dots, F\partial_i)|_U \\ &= 0. \end{aligned}$$ □

A consequence of Lemma 3.25 is that for $V \subseteq X$, $\omega^k \in \mathfrak{X}^*_{\mathcal{G}}(V)$ $(1 \le k \le r)$ and $\xi_l \in \mathfrak{X}_{\mathcal{G}}(V)$ $(1 \le l \le s)$ the restriction $t|_V(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_s)$ is well-defined.

Remark 3.26. For a generalized tensor field $t \in \mathcal{G}^r_s(X)$ the *components* of t relative to a chart $(U_\alpha, \varphi_\alpha)$ are the functions

$$t^{i_1 \dots i_r}_{j_1 \dots j_s} := t|_{U_\alpha}(dx^{i_1}, \dots, dx^{i_r}, \partial_{j_1}, \dots, \partial_{j_s}) \in \mathcal{G}(U_\alpha).$$

Now let $(U_\alpha, \varphi_\alpha)$ be charts covering X. Then the natural vector bundle charts $(T^r_s U_\alpha, (T\varphi)^r_s|_{T^r_s U_\alpha})$ constitute an atlas for $T^r_s X$ ([Abr67], Definition 6.10). The vector part t_α of t at a point $x \in \varphi_\alpha(U_\alpha)$ then is an element of $T^r_s(\mathbb{R}^n)$ and has tensor components (where e_j is the canonical basis of $\mathbb{R}^n$ and ε^i its dual basis)

$$(t_\alpha)^{i_1 \dots i_r}_{j_1 \dots j_s}(x) = t_\alpha(x)(\varepsilon^{i_1}, \dots, \varepsilon^{i_r}, e_{j_1}, \dots, e_{j_s})$$

which in fact are the coordinates of t_α in the n^{r+s}-dimensional vector space $T^r_s(\mathbb{R}^n)$. Because of the identity

$$t^{i_1 \dots i_r}_{j_1 \dots j_s}(p) = (t_\alpha)^{i_1 \dots i_r}_{j_1 \dots j_s}(\varphi_\alpha(p)) \quad \forall p \in U_\alpha,$$

moderateness and negligibility estimates of a generalized tensor can be determined in terms of its components $t^{i_1 \dots i_r}_{j_1 \dots j_s}$.
Inserting a generalized point $\tilde{x} \in \varphi_\alpha(U_\alpha)^\sim_c$ into the vector part of t naturally gives a multilinear mapping $t_\alpha(\tilde{x}) : \mathcal{K}^{n^{r+s}} \to \mathcal{K}$.

Proposition 3.27. *Let $t \in \mathcal{G}^r_s(X)$ and $p = [p_\varepsilon] \in \widetilde{X}_c$. Let $\omega^1, \dots, \omega^r$ and $\omega^{1'}, \dots, \omega^{r'}$ be generalized one-forms with $\omega^i(p) = \omega^{i'}(p)$ for $1 \le i \le r$. Let $\xi_1, \dots, \xi_s$ and $\xi'_1, \dots, \xi'_s$ be generalized vector fields with $\xi_j(p) = \xi'_j(p)$ for $1 \le j \le s$. Then*

$$t(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_s)(p) = s(\omega^{1'}, \dots, \omega^{r'}, \xi'_1, \dots, \xi'_s)(p).$$

Proof. The first step of our proof is to show that if any ω^i or ξ_j is zero at p, then $s(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_s)(p) = 0$. Suppose that $\xi_s(p) = 0$. Also, assume that there exists a chart (U, φ) and $K \subset\subset U$ such that $p_\varepsilon \in K$ for small ε. Choose a smooth bump function f with support in U such that $f \equiv 1$ in a neighborhood of K. Then $F := [f] \in \mathcal{G}(U)$ and $F\partial_j \in \mathfrak{X}_\mathcal{G}(X)$. Hence

$$\begin{aligned} &F^2 t(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_s) \\ &= t(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_{s-1}, F^2\xi_s) \\ &= t(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_{s-1}, \sum_j F\xi^j_s F\partial_j) \\ &= \sum_j F\xi^j_s t(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_{s-1}, F\partial_j). \end{aligned}$$

Since $\xi_s(p) = 0$, the coordinates ξ_s^j are zero and inserting p into the above equation yields the claim of the first step.

In the second step, we remit the above assumption on K, which now does not have to be wholly contained in a chart. Now suppose that

$$t(\omega^1, \dots, \omega^r, \xi_1, \dots, \xi_s)(p) \neq 0. \tag{3.3}$$

Then there exist representatives t_ε, w_ε^i, $\xi_{j\varepsilon}$ and p_ε, a sequence $\varepsilon_k \to 0$ and $m_0 \in \mathbb{N}$ such that

$$|t_\varepsilon(\omega^1_{\varepsilon_k}, \dots, \omega^r_{\varepsilon_k}, \xi_{1\varepsilon_k}, \dots, \xi_{s\varepsilon_k})(p_{\varepsilon_k})| > \varepsilon_k^{m_0}$$

for all $k \in \mathbb{N}$. By restricting p_{ε_k} to a subsequence of itself we may assume without loss of generality that p_{ε_k} converges to some $p \in X$. This enables us to choose representatives such that the argument of step one gives a contradiction to equation (3.3). $\square$

Definition 3.28. For $\xi \in \mathfrak{X}_\mathcal{G}(X)$ and $t \in \mathcal{G}^r_s(X)$, the generalized Lie-derivative of t with respect to ξ is defined as

$$L_\xi t := [L_{\xi_\varepsilon} t_\varepsilon],$$

where for $u \in \mathcal{G}(X)$ we also write $\xi(u)$ in place of $L_\xi(u)$.

Well-definedness of the generalized Lie-derivative follows at once from the local form

$$(L_{\xi_\varepsilon} t_\varepsilon)^{i_1 \dots i_r}_{j_1 \dots j_s} = \frac{\partial t^{i_1 \dots i_r}_{j_1 \dots j_s}}{\partial x_k} \xi^k_\varepsilon + \sum_{\mu=1}^{s} t^{i_1 \dots i_r}_{j_1 \dots l \dots j_s} \frac{\partial \xi^l_\varepsilon}{\partial x_{j_\mu}} - \sum_{\nu=1}^{r} t^{i_1 \dots l \dots i_r}_{j_1 \dots j_s} \frac{\partial \xi^{i_\nu}}{\partial x_l}.$$

All properties of the classical Lie derivative are valid for the generalized case as well, as they carry over component-wise.

3.3 Generalized Functions Valued in a Manifold

In order to introduce generalized functions on a manifold X which take values not in $\mathbb{R}^n$ but in another manifold Y we build on [AB91], where the space $\mathcal{G}[\Omega, \Omega']$ of generalized functions on $\Omega \subseteq \mathbb{R}^n$ taking values in $\Omega' \subseteq \mathbb{R}^m$ is introduced, called $\mathcal{G}_*(\Omega; \Omega')$ therein.

We will first give the definition of $\mathcal{G}[\Omega,\Omega']$: we denote by $\mathcal{E}[\Omega,\Omega']$ the set of all $u_\varepsilon \in C^\infty(I \times \Omega, \Omega')$ satisfying

$$\forall K \subset\subset \Omega\ \exists \varepsilon_0 \in I\ \exists K' \subset\subset \Omega'\ \forall \varepsilon < \varepsilon_0 :\ u_\varepsilon(K) \subseteq K' \tag{3.4}$$

and by $\mathcal{E}_M[\Omega,\Omega'] := \mathcal{E}[\Omega,\Omega'] \cap \mathcal{E}_M(\Omega)^m$ the subset of moderate elements. Functions satisfying condition (3.4) will also be called *compactly bounded* or *c-bounded*. Then the space of *compactly bounded generalized functions from* Ω *into* Ω' is defined as $\mathcal{G}[\Omega,\Omega'] := \mathcal{E}_M[\Omega,\Omega']/(\mathcal{N}(\Omega)^m)$.

There turns out to be a way to characterize those functions by charts which can be transferred to manifolds easily.

Proposition 3.29. *For $u_\varepsilon \in \mathcal{E}[\Omega,\Omega']$, the following statements are equivalent:*

(a) $u_\varepsilon \in \mathcal{E}_M[\Omega,\Omega']$.

(b) *(i)* $\forall K \subset\subset \Omega\ \exists \varepsilon_0 \in I\ \exists K' \subset\subset \Omega'\ \forall \varepsilon < \varepsilon_0 :\ u_\varepsilon(K) \subseteq K'$.

(ii) $\forall k \in \mathbb{N}_0$*, for all charts (U,φ) in Ω and (V,φ) in Ω', each $L \subset\subset U$ and $L' \subset\subset V$ there exists $N \in \mathbb{N}$ with*

$$\sup_{x \in L \cap u_\varepsilon^{-1}(L')} \|D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1})(\varphi(x))\| = O(\varepsilon^{-N})$$

where $\|\cdot\|$ is any norm on the respective space of multilinear maps.

Proof. First, note that condition (ii) is independent of the choice of the norm.

Regarding (a)⇒(b), (i) holds by definition. For (ii), fix k, (U,φ), (V,ψ), L, and L' as required. Using the chain rule we then are able to estimate $\|D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1})(\varphi(x))\|$ for $x \in L \cap u_\varepsilon^{-1}(L')$ by expressions consisting of sums of products of $\sup_{y\in\varphi(L)}\|D^{(j_1)}(\varphi^{-1})(y)\|$, $\sup_{z\in L'}\|D^{(j_2)}(\psi)(z)\|$ (which clearly are finite) and $\sup_{x\in L}\|D^{(j_3)}(u_\varepsilon)(x)\|$ (which is $O(\varepsilon^{-N_{j_3}})$) with $1 \le j_1, j_2, j_3 \le k$. In total this gives an estimate of $O(\varepsilon^{-N})$ with $N := \max_{1\le j_3 \le k} N_{j_3}$.

(b)⇒(a): To establish moderateness on $K \subset\subset \Omega$, we set $(U,\varphi) = (\Omega, \mathrm{id}_\Omega)$, $(V,\psi) = (\Omega', \mathrm{id}_{\Omega'})$, $L = K$, and $L' = K'$ where K' is as in (i). □

Moving on, we define moderate mappings between manifolds X and Y the same way. We set $\mathcal{E}[X,Y] := C^\infty(I \times X, Y)$ for the base space.

Definition 3.30. The space $\mathcal{E}_M[X,Y]$ of *compactly bounded moderate maps* from X to Y is defined as the set of all nets $u_\varepsilon \in \mathcal{E}[X,Y]$ satisfying

(i) $\forall K \subset\subset X\ \exists \varepsilon_0 \in I\ \exists K' \subset\subset Y\ \forall \varepsilon < \varepsilon_0 :\ u_\varepsilon(K) \subseteq K'$.

(ii) $\forall k \in \mathbb{N}_0$, for all charts (U,φ) in X and (V,ψ) in Y, each $L \subset\subset U$ and $L' \subset\subset V$ there exists $N \in \mathbb{N}$ with

$$\sup_{p \in L \cap u_\varepsilon^{-1}(L')} \|D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^{-N})$$

where $\|\cdot\|$ is any norm on the respective space of multilinear maps.

Proposition 3.31. *In Definition (3.30) it suffices to require condition (ii) merely for charts from given atlases of X and Y.*

Proof. Suppose that condition (ii) is satisfied for charts from given atlases $\mathcal{A}_X = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ and $\mathcal{A}_Y = \{(V_\beta, \psi_\beta) \mid \beta \in B\}$ of X and Y, respectively, and fix k, (U,φ), (V,ψ), L, and L' as required. Using Lemma 1.1 we write

$$L = \bigcup_{i=1}^{r} L_i \text{ and } L' = \bigcup_{j=1}^{s} L'_j \text{ with } L_i \subset\subset U_{\alpha_i},\ L'_j \subset\subset V_{\beta_j}$$

for some $r, s \in \mathbb{N}$ and $\alpha_i \in A$, $\beta_j \in B$. For each i and j we obtain estimates

$$\sup_{p \in L_i \cap u_\varepsilon^{-1}(L'_j)} \|D^{(k)}(\psi_{\beta_j} \circ u_\varepsilon \circ \varphi_{\alpha_i}^{-1})(\varphi_{\alpha_i}(p))\| \le C_{ij}\varepsilon^{-N_{ij}} \text{ for } \varepsilon \le \varepsilon_{ij}.$$

For each $p \in L \cap u_\varepsilon^{-1}(L')$ we have i,j such that $p \in L_i \cap u_\varepsilon^{-1}(L'_j)$. Let $N := \max_{i,j} N_{ij}$ and $\varepsilon_0 := \min_{i,j} \varepsilon_{ij}$. In a neighborhood of p we can write

$$\psi \circ u_\varepsilon \circ \varphi^{-1} = \psi \circ \psi_{\beta_j}^{-1} \circ (\psi_{\beta_j} \circ u_\varepsilon \circ \varphi_{\alpha_i}^{-1}) \circ \varphi_{\alpha_i} \circ \varphi^{-1} \tag{3.5}$$

and by the chain rule and the boundedness of derivatives of $\psi \circ \psi_{\beta_j}^{-1}$ and $\varphi_{\alpha_i} \circ \varphi^{-1}$ on $\psi_{\beta_j}(L'_j)$ and $\varphi(L_i)$, respectively, there is a constant C such that

$$\sup_{p \in L \cap u_\varepsilon^{-1}(L')} \|D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| \le C\varepsilon^{-N} \text{ for } \varepsilon \le \varepsilon_0. \qquad \square$$

In $\mathbb{R}^n$ the concept of negligible elements is an essential ingredient in the construction of Colombeau algebras. There, the ideal $\mathcal{N}$ gives rise to an equivalence relation (via $x \sim y \Leftrightarrow x - y \in \mathcal{N}$) by which the space of moderate functions is factored. In the absence of a linear structure, as is the case with $\mathcal{E}[X,Y]$, equivalence has to be defined directly.

Definition 3.32. Two nets $u_\varepsilon, v_\varepsilon \in \mathcal{E}_M[X,Y]$ are called equivalent, written $u_\varepsilon \sim v_\varepsilon$, if the following conditions are satisfied:

(i) $\forall K \subset\subset X$: $\sup_{p\in K} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) \to 0$ for $\varepsilon \to 0$, where h is some Riemannian metric on Y.

(ii) $\forall k \in \mathbb{N}_0$, $\forall m \in \mathbb{N}$, for each chart (U,φ) in X and (V,ψ) in Y, each $L \subset\subset U$ and $L' \subset\subset V$:

$$\sup_{p\in L\cap u_\varepsilon^{-1}(L')\cap v_\varepsilon^{-1}(L')} \|D^{(k)}(\psi\circ u_\varepsilon\circ\varphi^{-1} - \psi\circ v_\varepsilon\circ\varphi^{-1})(\varphi(p))\| = O(\varepsilon^m).$$

An auxiliary result will be needed in order to show that equivalence can be determined as well by verifying the requirements of Definition 3.32 (ii) merely on given atlases.

Lemma 3.33. *For all $f : \Omega \subseteq \mathbb{R}^n \to \Omega' \subseteq \mathbb{R}^m$ which are continuously differentiable and all $K \subset\subset \Omega$ there exists a constant $C \geq 0$ such that $\|f(x) - f(y)\| \leq C\|x-y\|\ \forall x,y \in K$.*

Proof. Choose $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\chi \subseteq \Omega$ and equal to 1 in a neighborhood of K. Then χf can be extended to a smooth function $\tilde{f}$ on $\mathbb{R}^n$. Because $\tilde{f}(x) - \tilde{f}(y) = (x-y)\cdot D\tilde{f}(\xi)$ for some ξ between x and y, the supremum of $\|D\tilde{f}\|$ on the convex hull of K is the needed constant C. □

Remark 3.34. As the mean value theorem ([Die60], 8.5.2) can be stated for any function taking values in a Banach space, the previous Lemma is in particular also valid for f taking values in the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, without modification of the proof.

Proposition 3.35. *Definition 3.32 (i) is independent of the choice of h. Furthermore, it suffices to require (ii) merely for charts from given atlases.*

Proof. The first assertion follows immediately from Definition 3.30 (i) and Lemma 3.13. For the second assertion fix atlases $\mathcal{A}_X = \{(U_\alpha,\varphi_\alpha) \mid \alpha \in A\}$ and $\mathcal{A}_Y = \{(V_\beta,\psi_\beta) \mid \beta \in B\}$ of X and Y, respectively, $k \in \mathbb{N}_0$, $m \in \mathbb{N}$, (U,φ) and (V,ψ) charts in X and Y, respectively, $L \subset\subset U$, and $L' \subset\subset V$. Using Lemma 1.1 we write

$$L = \bigcup_{i=1}^r L_i \text{ and } L' = \bigcup_{j=1}^s L'_j \text{ with } L_i \subset\subset U_{\alpha_i},\ L'_j \subset\subset V_{\beta_j}$$

for some r, s, α_i, β_j. We require that both $u_\varepsilon(p)$ and $v_\varepsilon(p)$ are in a compact subset of the domain of the same chart. For this we choose neighborhoods

V_j' of L_j' with $\overline{V_j'} \subset\subset V_{\beta_j}$ for $j = 1, \ldots, s$ and infer from Definition 3.32 (i) the existence of $\varepsilon_1 > 0$ such that

$$\sup_{p \in L} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) < \min_{j=1,\ldots,s} d_h(L_j', \partial V_j') \quad \forall \varepsilon < \varepsilon_1.$$

Then for any $p \in L \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')$ there are i and j such that $p \in L_i$, $u_\varepsilon(p) \in V_j'$ and $v_\varepsilon(p) \in V_j'$. Because Definition 3.32 (ii) holds on the given atlases we obtain constants C_{ij} and ε_{ij} for which

$$\|D^{(k)}(\psi_{\beta_j} \circ u_\varepsilon \circ \varphi_{\alpha_i}^{-1} - \psi_{\beta_j} \circ v_\varepsilon \circ \varphi_{\alpha_i}^{-1})(\varphi(p))\| \leq C_{ij}\varepsilon^m$$

holds for all $\varepsilon < \varepsilon_{ij}$ and $p \in L_i \cap u_\varepsilon^{-1}(V_j') \cap v_\varepsilon^{-1}(V_j')$. In a neighborhood of such p we have

$$\begin{aligned}\psi \circ u_\varepsilon \circ \varphi^{-1} - \psi \circ v_\varepsilon \circ \varphi^{-1} =& (\psi \circ \psi_{\beta_j}^{-1}) \circ (\psi_{\beta_j} \circ u_\varepsilon \circ \varphi_{\alpha_i}^{-1}) \circ (\varphi_{\alpha_i} \circ \varphi^{-1}) \\ &- (\psi \circ \psi_{\beta_j}^{-1}) \circ (\psi_{\beta_j} \circ v_\varepsilon \circ \varphi_{\alpha_i}^{-1}) \circ (\varphi_{\alpha_i} \circ \varphi^{-1}).\end{aligned} \tag{3.6}$$

Lemma 3.33 together with the chain rule gives the needed estimates. □

Proposition 3.36. *The relation $\sim$ of Definition 3.32 is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious. For transitivity suppose $u_\varepsilon, v_\varepsilon, w_\varepsilon \in \mathcal{E}_M[X, Y]$ with $u \sim v$ and $v \sim w$. (i) of Definition 3.32 is obvious from the triangle inequality. Regarding (ii) for any given atlases $\mathcal{A}_X = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of X and $\mathcal{A}_Y = \{(V_\beta, \psi_\beta) \mid \beta \in B\}$ of Y, fix $k \in \mathbb{N}_0$, $m \in \mathbb{N}$, charts $(U, \varphi) \in \mathcal{A}_X$, $(V, \psi) \in \mathcal{A}_Y$, $L \subset\subset U$ and $L' \subset\subset V$. We then can write $L = \bigcup_{i=1}^r L_i$ and $L' = \bigcup_{j=1}^s L_j'$ with $L_i \subset\subset U_{\alpha_i}, L_j' \subset\subset V_{\beta_j}$ for some r, s, α_i, β_j and choose neighborhoods V_j' of L_j' with $\overline{V_j'} \subset\subset V_{\beta_j}$ $\forall j \in \{1, \ldots, s\}$ as well as ε_0 such that each of $\sup_{p\in L} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big)$, $\sup_{p\in L} d_h\big(v_\varepsilon(p), w_\varepsilon(p)\big)$ and $\sup_{p\in L} d_h\big(u_\varepsilon(p), w_\varepsilon(p)\big)$ in course is smaller than $\min_{j=1,\ldots,s} d_h(L_j', \partial V_j')/2$ for $\varepsilon < \varepsilon_0$. From this we gain that for each $p \in L \cap u_\varepsilon^{-1}(L') \cap w_\varepsilon^{-1}(L')$ and $\varepsilon < \varepsilon_0$ there are i, j such that $p \in L_i \cap u_\varepsilon^{-1}(V_j') \cap v_\varepsilon^{-1}(V_j')$ as well as $p \in L_i \cap v_\varepsilon^{-1}(V_j') \cap w_\varepsilon^{-1}(V_j')$. Equivalence of u_ε and w_ε then follows directly by writing

$$\begin{aligned}\psi \circ u_\varepsilon \circ \varphi^{-1} - \psi \circ w_\varepsilon \circ \varphi^{-1} =& (\psi \circ u_\varepsilon \circ \varphi^{-1} - \psi \circ v_\varepsilon \circ \varphi^{-1}) \\ &+ (\psi \circ v_\varepsilon \circ \varphi^{-1} - \psi \circ w_\varepsilon \circ \varphi^{-1})\end{aligned}$$

and estimating. □

We arrive at the definition of manifold-valued generalized functions.

Definition 3.37. The quotient $\mathcal{G}[X,Y] := \mathcal{E}_M[X,Y]/\sim$ is called the space of *compactly bounded Colombeau generalized functions* from X to Y.

In fact, $\mathcal{G}[\_,Y]$ is a sheaf of sets (for proof see [KSV06], Theorem 2.3).

Points in $\widetilde{X}_c$ can be embedded into $\mathcal{G}[X,X]$ injectively (i.e., equivalence preserving) in the following way. Define the embedding

$$\begin{aligned} \iota_X : \widetilde{X}_c &\to \mathcal{G}[X,X] \\ p = [p_\varepsilon] &\mapsto [(\varepsilon,x) \mapsto p_\varepsilon]. \end{aligned}$$

Then moderateness of $\iota_X(p)$ is obvious from the corresponding definitions, as is the property

$$[p_\varepsilon] = [q_\varepsilon] \text{ in } \widetilde{X}_c \Longleftrightarrow [\iota_X p_\varepsilon] \sim [\iota_X q_\varepsilon] \text{ in } \mathcal{G}[X,X], \tag{3.7}$$

which ensures well-definedness and injectivity.

3.4 Generalized Vector Bundle Homomorphisms

We will now introduce generalized vector bundle homomorphisms in order to treat tangent mappings of elements of $\mathcal{G}[X,Y]$ later on. As base space we define

$$\mathcal{E}^{VB}[E,F] := \left\{ u_\varepsilon \in C^\infty(I \times E, F) \;\middle|\; u_\varepsilon \in \mathrm{Hom}(E,F) \;\forall \varepsilon \in I \right\}.$$

Definition 3.38. The space $\mathcal{E}_M^{VB}[E,F]$ is defined as the set consisting of all nets $u_\varepsilon \in \mathcal{E}^{VB}[E,F]$ satisfying

(i) $\underline{u_\varepsilon} \in \mathcal{E}_M[X,Y]$.

(ii) $\forall k \in \mathbb{N}_0$, for all vector bundle charts (U,Φ) in E and (V,Ψ) in F, each $L \subset\subset U$ and $L' \subset\subset V$ there exists $N \in \mathbb{N}$ with

$$\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L')} \| D^{(k)}(u^{(2)}_{\varepsilon\Psi\Phi})(\varphi(p)) \| = O(\varepsilon^{-N})$$

where $\|\cdot\|$ is any norm on the respective space of multilinear maps.

It suffices to require (ii) merely for charts from given vector bundle atlases of E and F. The proof is very similar to that of Proposition 3.31; the only significant change is in equation (3.5). If any k, (U_0, Φ_0), (V_0, Ψ_0), $L \subset\subset U$, and $L' \subset\subset V$ are given as required in (ii), we write the change of charts on the base as $\varphi_{\alpha_i 0}$ and $\psi_{0\beta_j}$ and the transition functions as $\boldsymbol{\varphi}_{\alpha_i 0}$ and $\boldsymbol{\psi}_{0\beta_j}$. Then we replace equation (3.5) by

$$\begin{aligned} u^{(2)}_{\varepsilon\Psi_0\Phi_0}(\varphi_0(p)) = \boldsymbol{\psi}_{0\beta_j}(u^{(1)}_{\varepsilon\Psi_{\beta_j}\Phi_{\alpha_i}}(\varphi_{\alpha_i 0}(\varphi(p)))) \\ \cdot u^{(2)}_{\varepsilon\Psi_{\beta_j}\Phi_{\alpha_i}}(\varphi_{\alpha_i 0}(\varphi(p))) \cdot \boldsymbol{\varphi}_{\alpha_i 0}(\varphi(p)) \end{aligned} \tag{3.8}$$

and estimate by the chain rule.

Definition 3.39. Two elements $u_\varepsilon, v_\varepsilon \in \mathcal{E}^{VB}_M[E, F]$ are called *vb-equivalent*, written $u_\varepsilon \sim_{vb} v_\varepsilon$, if the following conditions are satisfied:

(i) $\underline{u_\varepsilon} \sim \underline{v_\varepsilon}$ in $\mathcal{E}_M[X, Y]$.

(ii) $\forall k \in \mathbb{N}_0$, $\forall m \in \mathbb{N}$, for each vector bundle chart (U, Φ) in E and (V, Ψ) in F, each $L \subset\subset U$ and $L' \subset\subset V$:

$$\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L') \cap \underline{v_\varepsilon}^{-1}(L')} \|D^{(k)}(u^{(2)}_{\varepsilon\Psi\Phi} - v^{(2)}_{\varepsilon\Psi\Phi})(\varphi(p))\| = O(\varepsilon^m).$$

Furthermore, u_ε and v_ε are called vb0-equivalent, written $u_\varepsilon \sim_{vb0} v_\varepsilon$, if $\underline{u_\varepsilon} \sim_0 \underline{v_\varepsilon}$ holds and (ii) is satisfied for $k = 0$.

As before, it suffices to require (ii) in Definition 3.39 only for charts of given vector bundle atlases of E and F. The proof is very similar to that of Proposition 3.35; with notation as above, the only significant change is in equation (3.6), which is replaced in the same manner as shown in (3.8). Furthermore, $\sim_{vb}$ is seen to be an equivalence relation the same way as in Proposition 3.36.

Definition 3.40. The quotient $\mathrm{Hom}_{\mathcal{G}}[E, F] := \mathcal{E}^{VB}_M[E, F]/{\sim_{vb}}$ is called the space of *Colombeau generalized vector bundle homomorphisms.*

$\mathrm{Hom}_{\mathcal{G}}(\pi_X^{-1}(\_), F)$ is a sheaf of sets on X (for proof see [KSV06], Theorem 2.5).

For $u = [u_\varepsilon] \in \mathrm{Hom}_{\mathcal{G}}[E, F]$ we set $\underline{u} := [\underline{u_\varepsilon}] \in \mathcal{G}[X, Y]$.

Later on, we will give an injective embedding of generalized vector bundle points of E (still to be defined) into $\mathrm{Hom}_{\mathcal{G}}[E, E]$.

Definition 3.41. For $u = [u_\varepsilon] \in \mathcal{G}[X, Y]$ we define the tangent map

$$Tu := [Tu_\varepsilon] \in \mathrm{Hom}_{\mathcal{G}}(TX, TY).$$

Well-definedness of Tu follows immediately from the definitions of $\sim$ and $\sim_{vb}$.

3.5 Hybrid Generalized Functions

We will now define the space $\mathcal{G}^h[X, F]$ of hybrid generalized functions on X taking values in a vector bundle F over Y. As base space we will use $\mathcal{E}^h[X, F] := C^\infty(I \times X, F)$ and write $\underline{u_\varepsilon} := \pi_Y \circ u_\varepsilon$ for any net $u_\varepsilon \in \mathcal{E}^h[X, F]$.

Definition 3.42. The space $\mathcal{E}^h_M[X, F]$ of hybrid moderate generalized functions from X to F is defined as the set of all nets $u_\varepsilon \in \mathcal{E}^h[X, F]$ satisfying

(i) $\underline{u_\varepsilon}$ is c-bounded.

(ii) $\forall k \in \mathbb{N}_0$, for each chart (U, φ) in X and each vector bundle chart (V, Ψ) in F, each $L \subset\subset U$ and $L' \subset\subset V$ there exists $N \in \mathbb{N}$ such that

$$\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L')} \|D^{(k)}(\Psi \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^{-N})$$

where $\|\cdot\|$ is any norm on the respective space of multilinear maps.

Definition 3.43. Two elements $u_\varepsilon, v_\varepsilon \in \mathcal{E}^h_M[X, F]$ are called equivalent, written $u_\varepsilon \sim_h v_\varepsilon$, if the following conditions are satisfied:

(i) $\forall K \subset\subset X : \sup_{p \in K} d_h\big(\underline{u_\varepsilon}(p), \underline{v_\varepsilon}(p)\big) \to 0$ for $\varepsilon \to 0$, where h is some Riemannian metric on Y.

(ii) $\forall k \in \mathbb{N}_0$, $\forall m \in \mathbb{N}$, for each chart (U, φ) in X and each vector bundle chart (V, Ψ) in Y, each $L \subset\subset U$ and $L' \subset\subset V$:

$$\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L') \cap \underline{v_\varepsilon}^{-1}(L')} \|D^{(k)}(\Psi \circ u_\varepsilon \circ \varphi^{-1} - \Psi \circ v_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^m).$$

Furthermore, u_ε and v_ε are called equivalent of order zero, written $u_\varepsilon \sim_{h0} v_\varepsilon$, if (ii) is satisfied for $k = 0$.

In Definitions 3.42 and 3.43 it suffices to require condition (ii) merely for charts from given atlases of X and F. The proof is almost the same as the proof of Propositions 3.31 and 3.35.

The resulting space of generalized maps is, as usual, the quotient of moderate functions by $\sim_h$, where the latter again is easily seen to be an equivalence relation.

Definition 3.44. The quotient $\mathcal{G}^h[X,F] := \mathcal{E}^h_M[X,F]/\sim_h$ is called the space of *hybrid Colombeau generalized functions* from X to F.

For $\xi = [\xi_\varepsilon] \in \mathcal{G}^h[X,F]$ we set $\underline{\xi} := [\underline{\xi_\varepsilon}] \in \mathcal{G}[X,Y]$.

Definition 3.45. For $u \in \mathcal{G}[X,Y]$, the space of generalized sections along u is defined as $\mathcal{G}[X,F](u) := \left\{\xi \in \mathcal{G}^h[X,F] \mid \underline{\xi} = u\right\}$. The space of generalized vector fields along u is defined as

$$\mathfrak{X}_{\mathcal{G}}(u) := \left\{\xi \in \mathcal{G}^h[X,TY] \mid \underline{\xi} = u\right\}.$$

Chapter 4

Characterization results

In the preceding chapter several kinds of generalized functions were defined, distinguished by the structure of their domain of definition and their range space. While for generalized functions on a manifold and generalized sections a characterization was given which does not depend on local expressions on charts (Theorems 3.1, 3.3 and Proposition 3.19), the definition of generalized functions valued in a manifold, generalized vector bundle homomorphisms, and hybrid generalized functions so far rests solely on such a description. We will therefore aim to supplement those spaces with intrinsic characterizations of moderate and negligible elements, i.e., one without the use of charts, and will also give point value characterizations. Furthermore, we will show that equivalence can be defined without imposing restrictions on the derivatives but only on the functions themselves, as is the case in practically all other variants of Colombeau algebras.

The central idea of the following results is to replace the local expression on charts – in other words, local composition with smooth functions – by composition with globally defined smooth functions on the manifold.

4.1 Manifold-Valued Generalized Functions

Intrinsic Characterization

Proposition 4.1. *Let $u_\varepsilon \in \mathcal{E}[X, Y]$. Then the following conditions are equivalent.*

(i) u_ε is c-bounded.

(ii) $f \circ u_\varepsilon$ *is c-bounded for all* $f \in C^\infty(Y)$.

(iii) $f \circ u_\varepsilon$ *is moderate of order zero for all* $f \in C^\infty(Y)$, *i.e.,*

$$\forall K \subset\subset X \ \exists N \in \mathbb{N} : \sup_{p \in K} |f \circ u_\varepsilon(p)| = O(\varepsilon^{-N}) \quad \forall f \in C^\infty(Y).$$

(iv) $u_\varepsilon(x_\varepsilon) \in Y_c \quad \forall x_\varepsilon \in X_c$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ is easy to see as well as $(i) \Leftrightarrow (iv)$. For $(iii) \Rightarrow (i)$, any non-compact Y can be covered by compact sets L_n with $L_n \subseteq L_{n+1}$ and $L^\circ_{n+1} \setminus L_n \neq \emptyset \ \forall n \in \mathbb{N}$. Assuming (i) to be false, we have

$$\exists K \subset\subset X \ \forall n \in \mathbb{N} \ \exists \varepsilon_n \leq \frac{1}{n} \ \exists p_n \in K : u_{\varepsilon_n}(p_n) \notin L_n.$$

Without loss of generality we may write $u_{\varepsilon_n}(p_n) \in L^\circ_{n+1} \setminus L_n$. Now choose $f_n \in \mathcal{D}(L^\circ_{n+1} \setminus L_n)$ with $f_n(u_{\varepsilon_n}(p_n)) = e^{1/\varepsilon_n}$ and set $f := \sum_{n=1}^\infty f_n \in C^\infty(Y)$. Taking into account (iii) there exists $N \in \mathbb{N} : \sup_{p \in K} |f \circ u_\varepsilon(p)| \leq C\varepsilon^{-N}$ for small ε, giving $e^{1/\varepsilon_n} \leq C\varepsilon_n^{-N}$ which is a contradiction for large n. □

Remark 4.2. Note that for (iii) ⇒ (i) in 4.1, the growth does not have to be like $O(\varepsilon^{-N})$. In fact, any estimate $\sup_{p \in K} |f \circ u_\varepsilon(p)| = h(\varepsilon)$ with arbitrary h would be sufficient to give the implication, as we can set $f_n(u_{\varepsilon_n}(p_n)) = g(\varepsilon_n)$ with g chosen such that $g(\varepsilon_n) > h(\varepsilon_n)$, which then is a contradiction to (iii) stating $g(\varepsilon_n) \leq h(\varepsilon_n)$.

Proposition 4.3. *Let* $u_\varepsilon \in \mathcal{E}[X, Y]$. *Then the following statements are equivalent.*

(a) $u_\varepsilon \in \mathcal{E}_M[X, Y]$.

(b) *(i)* u_ε *is c-bounded and*

(ii) $f \circ u_\varepsilon \in \mathcal{E}_M(X) \quad \forall f \in \mathcal{D}(Y)$.

(c) $f \circ u_\varepsilon \in \mathcal{E}_M(X) \quad \forall f \in C^\infty(Y)$.

Proof. (c) ⇒ (b) follows from Proposition 4.1.

Regarding (b) ⇒ (a), (i) of Definition 3.30 equals (i) of (b). For (ii) thereof, fix $k \in \mathbb{N}_0$, charts (U, φ) in X and (V, ψ) in Y, $L \subset\subset U$, and $L' \subset\subset V$. We may establish the claim by locally replacing the coordinates of the chart in Y by functions in $\mathcal{D}(Y)$: if the dimension of Y equals m, we can choose $f \in \mathcal{D}(W)^m$ such that $f \equiv \psi$ in a neighborhood of L'. Setting $f_j := \mathrm{pr}_j \circ f$

(the projection onto the j-th coordinate) we have $\psi^j \circ u_\varepsilon = f_j \circ u_\varepsilon$ in a neighborhood of each $p \in L \cap u_\varepsilon^{-1}(L')$, where ψ^j is the j-th coordinate of ψ. Hence $\|D^{(k)}(\psi^j \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = \|D^{(k)}(f_j \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^{-N_j})$ for some N_j, resulting in moderateness of u.

Concerning (a) $\Rightarrow$ (c), fix $f \in C^\infty(Y)$ and $K \subset\subset X$. Without loss of generality we may assume $K \subset\subset U$ for some chart (U, φ) in X (in the general case, the following procedure would be applied to each component of a suitable partitioning of K). As u is c-bounded there exist $K' \subset\subset Y$ and $\varepsilon_0 > 0$ such that $u_\varepsilon(K) \subseteq K'$ for all $\varepsilon < \varepsilon_0$. Covering K' by finitely many charts (V_l, ψ_l) in Y we can, by virtue of Lemma 1.1, write $K' = \bigcup_{l=1}^r K'_l$ with $K'_l \subset\subset V_l$ $(1 \le l \le r)$. Applying (ii) of Definition 3.30 on $L := K$ for the chart (U, φ) and $L' := K'_l$ for the chart (V_l, ψ_l), we obtain (for fixed k) constants C_l, N_l and ε_l such that

$$\sup_{p \in K \cap u_\varepsilon^{-1}(K'_l)} \|D^{(k)}(\psi_l \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| \le C_l \varepsilon^{-N_l} \quad \text{for } \varepsilon < \varepsilon_l,$$

and because each $p \in K$ is mapped into some K'_l by u_ε there is $N \in \mathbb{N}$ with

$$\sup_{p \in K} \|D^{(k)}(\psi_l \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p)\| = O(\varepsilon^{-N}) \quad (\varepsilon \to 0).$$

Furthermore, $f \circ u_\varepsilon = (f \circ \psi_l^{-1}) \circ (\psi_l \circ u_\varepsilon)$ in a neighborhood of p. By the chain rule we arrive at moderateness of $f \circ u$. $\square$

The same methods are now used to obtain similar characterizations of equivalence; additionally, we will refer to equivalence of order zero in the following sense.

Definition 4.4. $u_\varepsilon, v_\varepsilon \in \mathcal{E}_M[X, Y]$ are said to be equivalent of order zero, written $u_\varepsilon \sim_0 v_\varepsilon$, if Definition 3.32 holds with $k = 0$ in (ii), which can equally be rewritten as

(i) $\forall K \subset\subset X$ and for some (hence every) Riemannian metric h on Y, $\sup_{p \in K} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) \to 0$ for $\varepsilon \to 0$.

(ii) $\forall K \subset\subset X$, $\forall m \in \mathbb{N}$, each chart (V, ψ) in Y and each $L' \subset\subset V$:

$$\sup_{p \in K \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')} \|\psi \circ u_\varepsilon(p) - \psi \circ v_\varepsilon(p)\| = O(\varepsilon^m).$$

That (ii) in Definition 4.4 is indeed equivalent to Definition 3.32 (ii) with $k = 0$ can be inferred by the same methods as used in the proof of Proposition 3.31.

Theorem 4.5. *Let $u_\varepsilon, v_\varepsilon \in \mathcal{E}_M[X,Y]$. Then the following statements are equivalent.*

(a) $u_\varepsilon \sim v_\varepsilon$.

(b) $u_\varepsilon \sim_0 v_\varepsilon$.

(c) $f \circ u_\varepsilon - f \circ v_\varepsilon \in \mathcal{N}(X) \quad \forall f \in \mathcal{D}(Y)$.

(d) $f \circ u_\varepsilon - f \circ v_\varepsilon \in \mathcal{N}(X) \quad \forall f \in C^\infty(Y)$.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c): Fix $f \in \mathcal{D}(Y)$ and $K \subset\subset X$. In order to establish

$$\sup_{p\in K} |f \circ u_\varepsilon(p) - f \circ v_\varepsilon(p)| = O(\varepsilon^m) \quad \forall m \in \mathbb{N}$$

we choose (by c-boundedness of u_ε and v_ε) $K' \subset\subset Y$ and $\varepsilon_0 > 0$ such that $u_\varepsilon(K) \cup v_\varepsilon(K) \subseteq K' \; \forall \varepsilon < \varepsilon_0$. We cover K' by finitely many open sets V_l' with $\overline{V_l'} \subset\subset V_l$ for charts (V_l, ψ_l) in Y. For each l there exists by virtue of Lemma 3.33 a constant $C \geq 0$ independent of ε such that

$$\begin{aligned} |(f \circ u_\varepsilon - f \circ v_\varepsilon)(p)| &= |(f \circ \psi_l^{-1}) \circ (\psi_l \circ u_\varepsilon)(p) - (f \circ \psi_l^{-1}) \circ (\psi_l \circ v_\varepsilon)(p)| \\ &\leq C \|(\psi_l \circ u_\varepsilon - \psi_l \circ v_\varepsilon)(p)\| = O(\varepsilon^m) \end{aligned}$$

for all $p \in K \cap u_\varepsilon^{-1}(V_l') \cap v_\varepsilon^{-1}(V_l')$ and all $m \in \mathbb{N}$ by assumption. As for small ε (as in the proof of Proposition 3.35) each $p \in K$ gets mapped into a certain V_l' by both u_ε and v_ε, the claim follows from the definition of equivalence.

(c) $\Rightarrow$ (a) First we need to establish that for all $K \subset\subset X$ and for each Riemannian metric h on Y $\sup_{p\in X} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) \to 0$. Assuming the contrary we have

$$\exists K \subset\subset X \; \exists \delta > 0 \; \forall k \in \mathbb{N} \; \exists \varepsilon_k < \frac{1}{k} \; \exists p_k \in K : \; d_h\big(u_{\varepsilon_k}(p_k), v_{\varepsilon_k}(p_k)\big) \geq \delta \tag{4.1}$$

with some Riemannian metric h. Because $u_\varepsilon, v_\varepsilon \in \mathcal{E}_M[X,Y]$ there exists $K' \subset\subset Y$ with $u_\varepsilon(K) \cup v_\varepsilon(K) \subseteq K'$ for small ε. By choosing suitable subsequences of ε_k and p_k we may assume without loss of generality that $u_{\varepsilon_k}(p_k) \to q_1 \in K'$ and $v_{\varepsilon_k}(p_k) \to q_2 \in K'$ with (4.1) implying $q_1 \neq q_2$. Take $f \in \mathcal{D}(Y)$ such that $f(q_1) = 1$ and $f(q_2) = 0$. Then the first part of implication (a) is a direct consequence of the contradiction arising from

$1 = \lim_{k\to\infty}|f(u_{\varepsilon_k}(p_k)) - f(v_{\varepsilon_k}(p_k))|$ by construction on the one hand and $\sup_{p\in K}|f(u_{\varepsilon_k}(p)) - f(v_{\varepsilon_k}(p))| = O(\varepsilon_k^m)$ for all $m \in \mathbb{N}$ on the other hand.

Second, take $L \subset\subset U$ and $L' \subset\subset V$ for charts (U,φ) in X and (V,ψ) in Y. Again we work with coordinates ψ^j of ψ. For each j we may choose $f_j \in \mathcal{D}(V)$ such that $f_j \equiv \psi^j$ in a neighborhood of L'. For each point $p \in L \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')$, $\psi^j \circ u_\varepsilon = f_j \circ u_\varepsilon$ and $\psi^j \circ v_\varepsilon = f_j \circ v_\varepsilon$ in some neighborhood of p. The final conclusion establishing the needed estimates reads

$$\|D^{(k)}(\psi^j \circ u_\varepsilon \circ \varphi^{-1} - \psi^j \circ v_\varepsilon \circ \varphi^{-1})(\varphi(p))\| \leq \\ \sup_{p'\in L}\|D^{(k)}(f_j \circ u_\varepsilon \circ \varphi^{-1} - f_j \circ v_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^m)$$

for the coordinates of ψ and therefore for ψ itself.

(d) $\Rightarrow$ (c) is obvious; for the converse fix $f \in C^\infty(Y)$ and $K \subset\subset X$. Set K' such that $u_\varepsilon(K) \cup v_\varepsilon(K) \subseteq K'$. Choosing $\tilde{f} \in \mathcal{D}(Y)$ with $f \equiv \tilde{f}$ on K', (c) gives

$$\sup_{x\in K}|(f \circ u_\varepsilon - f \circ v_\varepsilon)(x)| = \sup_{x\in K}|(\tilde{f} \circ u_\varepsilon - \tilde{f} \circ v_\varepsilon)(x)| = O(\varepsilon^m).$$

As this can be done for each K, we are finished. □

There is an intrinsic characterization of equivalence which will be useful when introducing point values.

Theorem 4.6. *For $u_\varepsilon, v_\varepsilon \in \mathcal{E}_M[X,Y]$ to be equivalent it is necessary and sufficient that for some (hence every) Riemannian metric h on Y, every $m \in \mathbb{N}$ and each $K \subset\subset X$ we have*

$$\sup_{p\in K} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) = O(\varepsilon^m) \text{ for } \varepsilon \to 0.$$

Proof. We establish necessity indirectly. Assume u_ε and v_ε to be equivalent and

$$\exists K \subset\subset X\ \exists m_0 \in \mathbb{N}\ \forall k \in \mathbb{N}\ \exists \varepsilon_k < \frac{1}{k}\ \exists p_k \in K : d_h\big(u_{\varepsilon_k}(p_k), v_{\varepsilon_k}(p_k)\big) \geq \varepsilon_k^{m_0}.$$

By choosing a suitable subsequence of p_k we may assume $p_k \to p \in K$, $u_{\varepsilon_k}(p_k) \to q \in Y$ and $v_{\varepsilon_k}(p_k) \to q$. The latter have the same limit because $u_\varepsilon \sim v_\varepsilon$. Taking a chart (V,ψ) in Y containing q we set $L' = \psi^{-1}(\overline{B_r(\psi(q))})$ for suitable r such that $u_{\varepsilon_k}(p_k), v_{\varepsilon_k}(p_k) \in L' \subset\subset V$ for all k larger than some k_0. Employing a smooth cut-off function equal to 1 on L' and supported in V we extend the inner product on $\mathbb{R}^m$ to a Riemannian metric g on a neighborhood of L' like in the proof of Lemma 3.14. For $k > k_0$ we have

$$d_g\big(u_{\varepsilon_k}(p_k), v_{\varepsilon_k}(p_k)\big) = \|\psi \circ u_{\varepsilon_k}(p_k) - \psi \circ v_{\varepsilon_k}(p_k)\| = O(\varepsilon^m) \quad \forall m \in \mathbb{N}$$

by Definition 4.4 (ii). Lemma 3.13, however, gives the contradiction

$$d_h\big(u_{\varepsilon_k}(p_k), v_{\varepsilon_k}(p_k)\big) \le C d_g\big(u_{\varepsilon_k}(p_k), v_{\varepsilon_k}(p_k)\big)$$

for some $C > 0$ and all k.

Conversely, 4.4 (i) is obvious. Concerning (ii), fix $K \subset\subset X$ and take a chart (V, ψ) in Y and $L' \subset\subset V$. First suppose that L' is contained in a geodesically convex set V' with $\overline{V'} \subset\subset V$. Any $m \in \mathbb{N}$ gives rise to constants $\varepsilon' > 0$ and $C' > 0$ such that

$$\sup_{p \in K} d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) \le C'\varepsilon^m \text{ for } \varepsilon < \varepsilon'.$$

Denoting by $\gamma_\varepsilon : [a_\varepsilon, b_\varepsilon] \to V'$ the unique geodesic joining $u_\varepsilon(p)$ and $v_\varepsilon(p)$ for some $p \in K \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')$ we obtain

$$d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) = \int_{a_\varepsilon}^{b_\varepsilon} \|\gamma_\varepsilon'(s)\| \,\mathrm{d}s. \tag{4.2}$$

Glancing at Lemma 3.14 we use

$$\exists C'' > 0 : \|\xi\| \le C''\|T_{\psi(p)}\psi^{-1}(\xi)\|_h \quad \forall p \in V' \ \forall \xi \in \mathbb{R}^n$$

and (4.2) to estimate

$$\begin{aligned} \|\psi(u_\varepsilon(p)) - \psi(v_\varepsilon(p))\| &\le \int_{a_\varepsilon}^{b_\varepsilon} \|(\psi \circ \gamma_\varepsilon)'(s)\| \,\mathrm{d}s \\ &\le C'' \int_{a_\varepsilon}^{b_\varepsilon} \|\gamma_\varepsilon'(s)\|_h \,\mathrm{d}s = C'' d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) = O(\varepsilon^m), \end{aligned}$$

giving the claim for L' as above.

Arbitrary $L' \subset\subset V$ can be covered by finitely many geodesically convex open sets V_i ([O'N83], Proposition 5.7) whose closure is compact and contained in V. Then there are $L_i' \subset\subset V_i$ ($1 \le i \le k$) such that $L' = \bigcup_{i=1}^k L_i'$. We may choose $\varepsilon'' > 0$ with $d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) < \min_{i=1,\dots,k} d_h(L_i', \partial V_i)$ for all $\varepsilon < \varepsilon''$ and all $p \in K$.

For such ε and $p \in u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')$ there exists i with $u_\varepsilon(p), v_\varepsilon(p) \in V_i$. By what was proven above there are, for each $i = 1, \dots, k$, positive constants ε_i and C_i such that $\|\psi(u_\varepsilon(p)) - \psi(v_\varepsilon(p))\| \le C_i \varepsilon^m$ for all ε smaller than ε_i and p in $K \cap u_\varepsilon^{-1}(V_i) \cap v_\varepsilon^{-1}(V_i)$. Setting $\varepsilon_1 = \min(\varepsilon', \varepsilon_1, \dots, \varepsilon_k)$ and $C = \max_{i=1,\dots,k} C_i$ establishes the claim. $\square$

Point Values

In order to introduce point values for elements of $\mathcal{G}[X,Y]$ we need the following terminology. Let $f:(X,g)\to(Y,h)$ be a smooth map between Riemannian manifolds and $p\in X$. We denote by $\|T_pf\|_{g,h}$ the norm of the linear map $T_pf:(T_pX,\|\ \|_g)\to(T_{f(p)}Y,\|\ \|_h)$. First, we establish an auxiliary result.

Lemma 4.7. *Let $u_\varepsilon\in\mathcal{E}_M[X,Y]$ and take Riemannian metrics g and h on X and Y, respectively. Then for any $K\subset\subset X$ there exists $N\in\mathbb{N}$ such that*

$$\sup_{p\in K}\|T_pu_\varepsilon\|_{g,h}=O(\varepsilon^{-N}).$$

Proof. As the local expression of T_pu_ε is the Jacobian of the local expression of f, the claim follows directly from the definition of moderateness. $\square$

Proposition 4.8. *Let $u=[u_\varepsilon]\in\mathcal{G}[X,Y]$, $p=[p_\varepsilon]\in\widetilde{X}_c$. Then the point value of u at p defined by $u(p):=[u_\varepsilon(p_\varepsilon)]$ is a well-defined element of $\widetilde{Y}_c$.*

Proof. Membership in $\widetilde{Y}_c$ follows from c-boundedness of u.

Next we have to show that different representatives of u and p still give the same element of $\widetilde{Y}_c$. Let h be any Riemannian metric on Y.

First, let $u'_\varepsilon\in\mathcal{E}_M[X,Y]$ with $u_\varepsilon\sim u'_\varepsilon$. Choosing $K\subset\subset X$ such that $p_\varepsilon\in K$ for small ε, by Theorem 4.6 we have

$$d_h\big(u_\varepsilon(p_\varepsilon),u'_\varepsilon(p_\varepsilon)\big)\le\sup_{p\in K}d_h\big(u_\varepsilon(p),u'_\varepsilon(p)\big)=O(\varepsilon^m)$$

for all $m\in\mathbb{N}$, which establishes that $u_\varepsilon(p_\varepsilon)\sim u'_\varepsilon(p_\varepsilon)$.

Second, let $p'_\varepsilon\in X_c$ with $p_\varepsilon\sim p'_\varepsilon$. As X is assumed to be paracompact and connected, there exists a geodesically complete Riemannian metric g on X ([NO61]). For all ε there exists a g-geodesic $\gamma_\varepsilon:[a_\varepsilon,b_\varepsilon]\to X$ connecting p_ε and p'_ε such that

$$d_g(p_\varepsilon,p'_\varepsilon)=\int_{a_\varepsilon}^{b_\varepsilon}\|\gamma'_\varepsilon(s)\|_g\mathrm{d}s.$$

In order to apply Lemma 4.7 we choose any $q\in K$ and a suitable $r\in\mathbb{R}_+$ such that the ball $\{p\in X\mid d_g(p,q)\le r\}$ covers K and subsequently set $K':=\{p\in X\mid d_g(p,q)\le 2r\}$. For any $s\in[a_\varepsilon,b_\varepsilon]$, the g-distance of $\gamma_\varepsilon(s)$ to either p_ε or p'_ε is no more than r because $d_g(p_\varepsilon,p'_\varepsilon)\le 2r$. As $d_g(p_\varepsilon,q)\le r$ and $d_g(p'_\varepsilon,q)\le r$ we get $d_g(\gamma_\varepsilon(s),q)\le 2r$, i.e., $\gamma_\varepsilon(s)\in K'$. Employing the

Hopf-Rinow theorem, which states that K' is compact, there are by Lemma 4.7 positive constants C and ε_0 such that $\sup_{p\in K'}\|T_p u_\varepsilon\|_{g,h} \le C\varepsilon^{-N}$ for all $\varepsilon < \varepsilon_0$. We finally infer $u_\varepsilon(p_\varepsilon) \sim u_\varepsilon(p'_\varepsilon)$ from

$$\begin{aligned} d_h\big(u_\varepsilon(p_\varepsilon), u_\varepsilon(p'_\varepsilon)\big) \le \int_{a_\varepsilon}^{b_\varepsilon} \|(u_\varepsilon \circ \gamma_\varepsilon)'(s)\|_h \mathrm{d}s &\le \int_{a_\varepsilon}^{b_\varepsilon} \|T_{\gamma_\varepsilon(s)} u_\varepsilon\|_{g,h} \|\gamma'_\varepsilon(s)\|_g \mathrm{d}s \\ &\le C\varepsilon^{-N} d_g(p_\varepsilon, p'_\varepsilon) \quad \forall \varepsilon < \varepsilon_0 \end{aligned}$$

and the fact that $d_g(p_\varepsilon, p'_\varepsilon) = O(\varepsilon^m)\ \forall m > 0$ by assumption. Summing up, for any choice of u'_ε and p'_ε such that $u_\varepsilon \sim u'_\varepsilon$ and $p_\varepsilon \sim p'_\varepsilon$ it follows that $u_\varepsilon(p_\varepsilon) \sim u_\varepsilon(p'_\varepsilon) \sim u'_\varepsilon(p'_\varepsilon)$, i.e., $[u_\varepsilon(p_\varepsilon)] = [u'_\varepsilon(p'_\varepsilon)]$. □

We can thus give a point value characterization of elements of $\mathcal{G}[X, Y]$.

Proposition 4.9. *Let u and v be in $\mathcal{G}[X, Y]$. Then $u = v$ if and only if $u(p) = v(p)$ for all $p \in \widetilde{X}_c$.*

Proof. Necessity was already shown in the proof of Proposition 4.8. For sufficiency we employ Theorem 4.6 and suppose indirectly that u and v are not equivalent. Let h be any Riemannian metric on X and choose u_ε and v_ε to be representatives of u and v, respectively. In this way we obtain

$$\exists K \subset\subset X\ \exists m \in \mathbb{N}\ \forall k \in \mathbb{N}\ \exists \varepsilon_k < \frac{1}{k}\ \exists p_k \in K:\ d_h\big(u_\varepsilon(p), v_\varepsilon(p)\big) > \varepsilon_k^m.$$

We choose a $\tilde{p} = [\tilde{p}_\varepsilon] \in \widetilde{X}_c$ with $\tilde{p}_{1/k} = p_k$ and obtain $u(\tilde{p}) \neq v(\tilde{p})$. □

4.2 Generalized Vector Bundle Homomorphisms

Generalized Vector Bundle Points

Before we attempt to characterize elements of $\mathrm{Hom}_{\mathcal{G}}[E, F]$ by point values we shall introduce an appropriate concept of generalized vector bundle points.

Definition 4.10. Let (E, X, π) be a vector bundle. We consider the set of all nets $e_\varepsilon \in C^\infty(I, E)$ satisfying

(i) $\pi(e_\varepsilon) \in X_c$.

(ii) For each Riemannian metric h on E inducing the norm $\| \ \|_h$ on the fibers of E, there exists $N \in \mathbb{N}$ such that

$$\|e_\varepsilon\|_h = O(\varepsilon^{-N}).$$

These are called vb-moderate generalized points.

On this set we define an equivalence relation $\sim_{vb}$ and call two elements $e_\varepsilon, e'_\varepsilon$ equivalent if

(iii) $\pi(e_\varepsilon) \sim \pi(e'_\varepsilon)$ in X_c.

(iv) For all $m \in \mathbb{N}$, vector bundle charts (U, Φ) in E, and $L \subset\subset U$ there exist $\varepsilon_1 > 0$ and $C \geq 0$ such that

$$\|\boldsymbol{\varphi} e_\varepsilon - \boldsymbol{\varphi} e'_\varepsilon\| \leq C\varepsilon^m$$

for all $\varepsilon < \varepsilon_1$ whenever both $\pi(e_\varepsilon)$ and $\pi(e'_\varepsilon)$ lie in L.

This set of equivalence classes is denoted by $E_c^{\sim_{vb}}$.

Again, it suffices to require (iv) merely for charts of a given vector bundle atlas.

There is an injective (i.e., equivalence preserving) embedding of $E_c^{\sim_{vb}}$ into $\mathrm{Hom}_{\mathcal{G}}[E, E]$ given by

$$\begin{aligned} \iota_E : E_c^{\sim_{vb}} &\to \mathrm{Hom}_{\mathcal{G}}[E, E] \\ e = [e_\varepsilon] &\mapsto [(\varepsilon, f) \mapsto e_\varepsilon]. \end{aligned}$$

It is easy to see moderateness of $\iota_E(e)$ as well as the property

$$e_\varepsilon \sim_{vb} e'_\varepsilon \text{ in } E_c^{\sim_{vb}} \iff \iota_E e_\varepsilon \sim_{vb} \iota_E e'_\varepsilon \text{ in } \mathrm{Hom}_{\mathcal{G}}[E, E]. \tag{4.3}$$

For $e = [e_\varepsilon] \in E_c^{\sim_{vb}}$ we call $\pi(e) := [\pi(e_\varepsilon)]$ the *base point* of e. For any $p \in \widetilde{X}_c$ we set $(E_c^{\sim_{vb}})_p := \{e \in E_c^{\sim_{vb}} \mid \pi(e) = p\}$.

Lemma 4.11. *Let $e = [e_\varepsilon] \in (E_c^{\sim_{vb}})_p$, $p = [p_\varepsilon] \in \widetilde{X}_c$. Then there exists a representative e'_ε of e such that $\pi(e'_\varepsilon) = p_\varepsilon$ for small ε.*

Proof. There exists $K \subset\subset X$ such that $p_\varepsilon \in K$ for small ε. Then there are vector bundle charts (U_i, Φ_i) covering K such that (employing Lemma 1.1) we can write $K = \bigcup_{i=1}^k K_i$ with $K_i \subset\subset U_i$. Let U'_i be a neighborhood of K_i whose closure is compact and contained in U_i. For small ε all $\pi(e_\varepsilon)$ and p_ε are contained in the same $\pi(U'_i)$. For such ε we consequently set $e'_\varepsilon = \Phi_i^{-1}(\varphi_i(p_\varepsilon), \boldsymbol{\varphi}_i(e_\varepsilon))$. By construction, $e_\varepsilon \sim_{vb} e'_\varepsilon$ and $\pi(e'_\varepsilon) = p_\varepsilon$. $\square$

Intrinsic Characterization

In order to derive characterizations of generalized vector bundle homomorphisms similar to those for $\mathcal{G}[X,Y]$ we replace the chart-wise description in F by composition with vector bundle homomorphisms from F to $\mathbb{R}\times\mathbb{R}^{m'}$. For $u_\varepsilon \in \mathcal{E}^{VB}[E,\mathbb{R}\times\mathbb{R}^{m'}]$ and (U,Φ) a vector bundle chart in E we can write the local expression of u_ε like in (1.1) as

$$u_\varepsilon \circ \Phi^{-1}(x,\xi) = (u^{(1)}_{\varepsilon\,\mathrm{id}\,\Phi}(x), u^{(2)}_{\varepsilon\,\mathrm{id}\,\Phi}(x)\cdot\xi).$$

Proposition 4.12. *(a) Let $u_\varepsilon \in \mathcal{E}^{VB}[E,\mathbb{R}\times\mathbb{R}^{m'}]$. Then u_ε is moderate if and only if*

(i) $\underline{u_\varepsilon}$ is c-bounded and

(ii) $u^{(1)}_{\varepsilon\,\mathrm{id}\,\Phi} \in \mathcal{E}_M(\varphi(U))$ and $u^{(2)}_{\varepsilon\,\mathrm{id}\,\Phi} \in \mathcal{E}_M(\varphi(U))^{n'\cdot m'}$ for all vector bundle charts (U,Φ) in E.

(b) Let $u_\varepsilon, v_\varepsilon \in \mathcal{E}^{VB}_M[E,\mathbb{R}\times\mathbb{R}^{m'}]$. Then $u_\varepsilon \sim_{vb} v_\varepsilon$ holds if and only if $u^{(1)}_{\varepsilon\,\mathrm{id}\,\Phi} - v^{(1)}_{\varepsilon\,\mathrm{id}\,\Phi} \in \mathcal{N}(\varphi(U))$ and $u^{(2)}_{\varepsilon\,\mathrm{id}\,\Phi} - v^{(2)}_{\varepsilon\,\mathrm{id}\,\Phi} \in \mathcal{N}(\varphi(U))^{n'\cdot m'}$ for all vector bundle charts (U,Φ) in E.

Proof. This is a straightforward calculation from the definitions. □

In the following we denote by $\mathrm{Hom}_c(E,\mathbb{R}\times\mathbb{R}^{m'})$ the set of all vector bundle homomorphisms $f \in \mathrm{Hom}(E,\mathbb{R}\times\mathbb{R}^{m'})$ such that $\underline{f} : X \to \mathbb{R}$ has compact support.

Proposition 4.13. *Let $u_\varepsilon \in \mathcal{E}^{VB}[E,F]$. Then the following conditions are equivalent.*

(a) $u_\varepsilon \in \mathcal{E}^{VB}_M[E,F]$.

(b) *(i)* $\underline{u_\varepsilon}$ *is c-bounded and*

(ii) $\hat f \circ u_\varepsilon \in \mathcal{E}^{VB}_M[E,\mathbb{R}\times\mathbb{R}^{m'}] \quad \forall \hat f \in \mathrm{Hom}_c(F,\mathbb{R}\times\mathbb{R}^{m'})$.

(c) $\hat f \circ u_\varepsilon \in \mathcal{E}^{VB}_M[E,\mathbb{R}\times\mathbb{R}^{m'}] \quad \forall \hat f \in \mathrm{Hom}(F,\mathbb{R}\times\mathbb{R}^{m'})$.

Proof. (a) ⇒ (c): Let $\hat f \in \mathrm{Hom}(F,\mathbb{R}\times\mathbb{R}^{m'})$ and (U,Φ) be a vector bundle chart in E. By assumption, $\underline{u_\varepsilon}$ is c-bounded and therefore $\underline{\hat f} \circ \underline{u_\varepsilon}$ also is; $(\hat f \circ u_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi} = \underline{\hat f} \circ \underline{u_\varepsilon} \circ \varphi^{-1} \in \mathcal{E}_M(\varphi(U))$ by Proposition 4.3. We then need that $g := (\hat f \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi} \in \mathcal{E}_M(\varphi(U))^{n'\cdot m'}$ or, in coordinates g_{ij} of g:

$$\forall K \subset\subset \varphi(U)\ \forall \alpha \in \mathbb{N}_0^n\ \exists N_{ij} \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha g_{ij}(x)| = O(\varepsilon^{-N_{ij}})$$

for each i and j. This requirement is equivalent to

$$\forall L \subset\subset U \; \forall k \in \mathbb{N}_0 \; \exists N \in \mathbb{N} : \sup_{p \in L} \| D^{(k)} g(\varphi(p)) \| = O(\varepsilon^{-N}),$$

which is what we will show.

Fix $\varepsilon_0 > 0$ and $L' \subset\subset Y$ such that $\underline{u_\varepsilon}(L) \subseteq L' \; \forall \varepsilon < \varepsilon_0$. Covering L' by finitely many vector bundle charts (V_j, Ψ_j) in F we subsequently choose $L'_j \subset\subset V_j$ $(j = 1, \dots, s)$ such that $L' = \bigcup_{j=1}^s L'_j$. As each $p \in L$ gets mapped into some L'_j by $\underline{u_\varepsilon}$ we can write for all p' in a neighborhood of such p:

$$(\hat{f} \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}(\varphi(p')) = (\hat{f} \circ \Psi_j^{-1})^{(2)}(\psi_j \circ \underline{u_\varepsilon}(p')) \cdot u^{(2)}_{\varepsilon \Psi_j \Phi}(\varphi(p'))$$

and thus $\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L'_j)} \| D^{(k)} g(\varphi(p)) \|$ can be estimated with the help of the chain rule by terms consisting of

$$\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L'_j)} \| D^{(k')}((\hat{f} \circ \Psi_j^{-1})^{(2)}(\psi_j \circ \underline{u_\varepsilon}(p)) \|,$$

which is finite, and

$$\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L'_j)} \| D^{(k'')}(u^{(2)}_{\varepsilon \Psi_j \Phi})(\varphi(p)) \|,$$

which satisfies moderateness estimates by assumption.

(c) $\Rightarrow$ (b): (i) follows directly from Proposition 4.1, (ii) is evident.

(b) $\Rightarrow$ (a): In order to establish (i) of Definition 3.38 it suffices by Proposition 4.3 to show that $f \circ \underline{u_\varepsilon} \in \mathcal{E}_M(X)$ for all $f \in C^\infty(Y)$. Since for all $\hat{f} \in \mathrm{Hom}(F, \mathbb{R} \times \mathbb{R}^{m'})$ and every vector bundle chart (U, Φ) in E we have $(\hat{f} \circ u_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi} = \underline{\hat{f}} \circ \underline{u_\varepsilon} \circ \varphi^{-1} \in \mathcal{E}_M(\varphi(U))$ we only need to choose, for every given $f \in C^\infty(Y)$, any $\hat{f} \in \mathrm{Hom}(F, \mathbb{R} \times \mathbb{R}^{m'})$ with $\underline{\hat{f}} = f$ to reach $\underline{u_\varepsilon} \in \mathcal{E}_M[X, Y]$. Then, fix vector bundle charts (U, Φ) in E, (V, Ψ) in F, $L \subset\subset U$, and $L' \subset\subset V$. As for all $x \in \varphi(L \cap \underline{u_\varepsilon}^{-1}(L'))$ and $\xi \in \mathbb{R}^{n'}$ we have

$$(\Psi \circ u_\varepsilon \circ \Phi^{-1})(x, \xi) = (u^{(1)}_{\varepsilon \Psi \Phi}(x), u^{(2)}_{\varepsilon \Psi \Phi}(x) \cdot \xi)$$

we choose an open neighborhood V' of L' with $\overline{V'} \subset\subset V$ and for any $l \in \{1, \dots, m\}$ define $\hat{f}_l \in \mathrm{Hom}_c(F, \mathbb{R} \times \mathbb{R}^{m'})$ such that

$$\hat{f}_l|_{\pi_Y^{-1}(V')} = (\mathrm{pr}_l \times \mathrm{id}_{\mathbb{R}^{m'}}) \circ \Psi|_{\pi_Y^{-1}(V')}$$

(where pr_l is the projection on the j-th coordinate) and have

$$u^{(2)}_{\varepsilon \Psi \Phi}(\varphi(p)) = (\hat{f}_l \circ u_\varepsilon \circ \Phi^{-1})^{(2)}(\varphi(p)) \quad \forall p \in L \cap \underline{u_\varepsilon}^{-1}(L'), \tag{4.4}$$

giving the desired estimates for $\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L')} \| D^{(k)}(u^{(2)}_{\varepsilon \Psi \Phi})(\varphi(p)) \|$. $\square$

Theorem 4.14. *Let $u_\varepsilon, v_\varepsilon \in \mathcal{E}_M^{VB}[E,F]$. Then the following statements are equivalent.*

(i) $u_\varepsilon \sim_{vb} v_\varepsilon$.

(ii) $u_\varepsilon \sim_{vb0} v_\varepsilon$.

(iii) $\hat{f} \circ u_\varepsilon \sim_{vb} \hat{f} \circ v_\varepsilon$ *in* $\mathcal{E}_M^{VB}[E, \mathbb{R} \times \mathbb{R}^{m'}] \quad \forall \hat{f} \in \mathrm{Hom}_c(F, \mathbb{R} \times \mathbb{R}^{m'})$.

(iv) $\hat{f} \circ u_\varepsilon \sim_{vb} \hat{f} \circ v_\varepsilon$ *in* $\mathcal{E}_M^{VB}[E, \mathbb{R} \times \mathbb{R}^{m'}] \quad \forall \hat{f} \in \mathrm{Hom}(F, \mathbb{R} \times \mathbb{R}^{m'})$.

Proof. (i) $\Rightarrow$ (ii) is evident.

(ii) $\Rightarrow$ (iv): Take $\hat{f} \in \mathrm{Hom}[F, \mathbb{R} \times \mathbb{R}^{m'}]$. We now need to establish that

$$\begin{aligned} (\hat{f} \circ u_\varepsilon - \hat{f} \circ v_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi} &\in \mathcal{N}(\varphi(U)) \\ \text{and } (\hat{f} \circ u_\varepsilon - \hat{f} \circ v_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi} &\in \mathcal{N}(\varphi(U))^{n' \cdot m'} \end{aligned} \tag{4.5}$$

for every vector bundle chart (U, Φ) in E.

First, $\underline{u_\varepsilon} \sim \underline{v_\varepsilon}$ in $\mathcal{E}_M[X, \mathbb{R}]$ by assumption and therefore $f \circ \underline{u_\varepsilon} - f \circ \underline{v_\varepsilon} \in \mathcal{N}(X)$ for all $f \in C^\infty(Y)$ by the characterization in Theorem 4.5. This gives

$$(\hat{f} \circ u_\varepsilon - \hat{f} \circ v_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi}(\varphi(p)) = (\underline{\hat{f}} \circ \underline{u_\varepsilon} - \underline{\hat{f}} \circ \underline{v_\varepsilon}) \circ \varphi^{-1} \in \mathcal{N}(\varphi(U)).$$

Second, condition (4.5) is equivalent to

$$\forall L \subset\subset U \ \forall m \in \mathbb{N} : \sup_{p \in L} \|(\hat{f} \circ u_\varepsilon - \hat{f} \circ v_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}(\varphi(p))\| = O(\varepsilon^m). \tag{4.6}$$

For $L \subset\subset U$ choose $\varepsilon_1 > 0$ and $L' \subset\subset Y$ such that $\underline{u_\varepsilon}(L) \cup \underline{v_\varepsilon}(L) \subseteq L'$ for all $\varepsilon < \varepsilon_1$. With (V_l, Ψ_l) vector bundle charts in F, cover L' by finitely many open sets V_l' satisfying $\overline{V_l'} \subset\subset V_l$. As $u \sim_{vb0} v$ implies $\underline{u_\varepsilon} \sim \underline{v_\varepsilon}$ and therefore $\sup_{p \in L} d_h\big(\underline{u_\varepsilon}(p), \underline{v_\varepsilon}(p)\big) \to 0$ ($\varepsilon \to 0$), both $\underline{u_\varepsilon}(p)$ and $\underline{v_\varepsilon}(p)$ are contained in some V_l' for $p \in L$ and small ε as in the proof of Proposition 3.35.

We finally write for $p \in L \cap \underline{u_\varepsilon}^{-1}(V_l') \cap \underline{v_\varepsilon}^{-1}(V_l')$

$$\begin{aligned} &(\hat{f} \circ u_\varepsilon - \hat{f} \circ v_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}(\varphi(p)) = \\ &(\hat{f} \circ \Psi_l^{-1})^{(2)}(\psi_l \circ \underline{u_\varepsilon}(p)) \cdot u^{(2)}_{\varepsilon \Psi_l \Phi}(\varphi(p)) - (\hat{f} \circ \Psi_l^{-1})^{(2)}(\psi_l \circ \underline{v_\varepsilon}(p)) \cdot v^{(2)}_{\varepsilon \Psi_l \Phi}(\varphi(p)) = \\ &\quad \big((\hat{f} \circ \Psi_l^{-1})^{(2)}(\psi_l \circ \underline{u_\varepsilon}(p)) - (\hat{f} \circ \Psi_l^{-1})^{(2)}(\psi_l \circ \underline{v_\varepsilon}(p))\big) \cdot u^{(2)}_{\varepsilon \Psi_l \Phi}(\varphi(p)) + \\ &\quad\quad (\hat{f} \circ \Psi_l^{-1})^{(2)}(\psi_l \circ \underline{v_\varepsilon}(p))\big(u^{(2)}_{\varepsilon \Psi_l \Phi}(\varphi(p)) - v^{(2)}_{\varepsilon \Psi_l \Phi}(\varphi(p))\big) \end{aligned}$$

and obtain the needed estimates for (4.6) by

$$\sup_{p\in L\cap\underline{u_\varepsilon}^{-1}(\overline{V_l'})}\|u^{(2)}_{\varepsilon\Psi_l\Phi}(\varphi(p))\| = O(\varepsilon^{-N_l}) \text{ for some } N_l\in\mathbb{N},$$

$$\sup_{p\in L\cap\underline{u_\varepsilon}^{-1}(\overline{V_l'})\cap\underline{v_\varepsilon}^{-1}(\overline{V_l'})}\|u^{(2)}_{\varepsilon\Psi_l\Phi}(\varphi(p)) - v^{(2)}_{\varepsilon\Psi_l\Phi}(\varphi(p))\| = O(\varepsilon^m) \quad \forall m\in\mathbb{N}$$

and the help of Lemma 3.33 and the remark following it: as there is a compact set $\Omega\subset\subset\mathbb{R}^m$ such that

$$\psi_l\circ\underline{u_\varepsilon}(L\cap\underline{u_\varepsilon}^{-1}(\overline{V_l'}))\cup\psi_l\circ\underline{v_\varepsilon}(L\cap\underline{v_\varepsilon}^{-1}(\overline{V_l'}))\subseteq\Omega$$

we get for all $p\in L\cap\underline{u_\varepsilon}^{-1}(\overline{V_l'})\cap\underline{v_\varepsilon}^{-1}(\overline{V_l'})$

$$\begin{aligned}&\|(\hat f\circ\Psi_l^{-1})^{(2)}(\psi_l\circ\underline{u_\varepsilon}(p)) - (\hat f\circ\Psi_l^{-1})^{(2)}(\psi_l\circ\underline{v_\varepsilon}(p))\|\\ &\qquad\le C\|\psi_l\circ\underline{u_\varepsilon}(p)-\psi_l\circ\underline{v_\varepsilon}(p)\|\end{aligned}$$

and conclude the argument by applying Definition 4.4 (ii) and the assumption $\underline{u_\varepsilon}\sim\underline{v_\varepsilon}$.

(iv) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i): As $\underline{u_\varepsilon}\sim\underline{v_\varepsilon}$ in $\mathcal{E}_M[X,Y]$ is equivalent to $f\circ\underline{u_\varepsilon}-f\circ\underline{v_\varepsilon}\in\mathcal{N}(X)$ for all $f\in\mathcal{D}(Y)$ by Theorem 4.5, we choose any $\hat f\in\mathrm{Hom}_c(F,\mathbb{R}\times\mathbb{R}^{m'})$ such that $\underline{\hat f}=f$ and obtain by assumption (iii) and Proposition 4.12 that

$$(\hat f\circ u_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi} - (\hat f\circ v_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi} = (f\circ\underline{u_\varepsilon}-f\circ\underline{v_\varepsilon})\circ\varphi^{-1}\in\mathcal{N}(\varphi(U))$$

for each vector bundle chart (U,Φ), which gives equivalence of $\underline{u_\varepsilon}$ and $\underline{v_\varepsilon}$. Part (ii) of Definition 3.39 follows easily from a representation as in (4.4) of both $u^{(2)}_{\varepsilon\Psi\Phi}$ and $v^{(2)}_{\varepsilon\Psi\Phi}$ and an $\hat f_l$ chosen as in the Proof of 4.13, (b) $\Rightarrow$ (a). $\square$

Point Values

Now we can state a corollary of Theorem 4.14 which transfers its conclusions to generalized vector bundle points.

Corollary 4.15. *Let $e=[e_\varepsilon], e'=[e'_\varepsilon]\in E_c^{\sim_{vb}}$. Then the following statements are equivalent.*

(i) $e=e'$.

(ii) $\hat{f}(e) = \hat{f}(e')$ *in* $(\mathbb{R} \times \mathbb{R}^{n'})_c^{\sim vb}$ *for all* $\hat{f} \in \mathrm{Hom}_c(E, \mathbb{R} \times \mathbb{R}^{n'})$ *(or for all* $\hat{f} \in \mathrm{Hom}(E, \mathbb{R} \times \mathbb{R}^{n'})$*).*

(iii) $\|\hat{f}(e_\varepsilon) - \hat{f}(e'_\varepsilon)\| = O(\varepsilon^m)$ $(\varepsilon \to 0)$ $\quad \forall m \in \mathbb{N}, \forall \hat{f} \in \mathrm{Hom}_c(E, \mathbb{R} \times \mathbb{R}^{n'})$ *(or* $\forall \hat{f} \in \mathrm{Hom}(E, \mathbb{R} \times \mathbb{R}^{n'})$*).*

Proof. (i) $\Leftrightarrow$ (ii) follows from Theorem 4.14, property (4.3) and the fact that $\hat{f}(\iota_E e_\varepsilon) = \iota_{\mathbb{R}\times\mathbb{R}^{n'}} \hat{f}(e_\varepsilon)$. (ii) $\Leftrightarrow$ (iii) is clear from the definitions. $\square$

Proposition 4.16. *Let* $u = [u_\varepsilon] \in \mathrm{Hom}_\mathcal{G}[E, F]$ *and* $e = [e_\varepsilon] \in E_c^{\sim vb}$*. Then the point value of* u *at* e *defined by* $u(e) := [u_\varepsilon(e_\varepsilon)]$ *is a well-defined element of* $F_c^{\sim vb}$*.*

Proof. (i) of Definition 4.10 follows from $\pi_Y(u_\varepsilon(e_\varepsilon)) = \underline{u_\varepsilon}(\pi_X(e_\varepsilon)) \in Y_c$.

For (ii), we assume without loss of generality that there are vector bundle charts (U, Φ) in E and (V, Ψ) in F, $L \subset\subset U$, $L' \subset\subset V$, and $\varepsilon_0 > 0$ such that $\pi_X(e_\varepsilon) \in L$ and $\pi_Y(u_\varepsilon(e_\varepsilon)) \in L'$ for all $\varepsilon < \varepsilon_0$; in the general case we would employ Lemma 1.1 as in previous proofs. Let h and $\tilde{h}$ be any Riemannian metrics on E and F, respectively. By Corollary 3.11 we can write for all $\varepsilon < \varepsilon_0$

$$\begin{aligned}\|u_\varepsilon(e_\varepsilon)\|_{\tilde{h}} \leq& C\|\boldsymbol{\psi} \circ u_\varepsilon \circ \Phi^{-1} \circ \Phi(e_\varepsilon)\| \\ =& C\big\|u^{(2)}_{\varepsilon\Psi\Phi}\big(\varphi(\pi_X(e_\varepsilon))\big) \cdot \boldsymbol{\varphi}(e_\varepsilon)\big\|,\end{aligned}$$

and because for all $\varepsilon < \varepsilon_0$ we have

$$\exists N \in \mathbb{N}: \ \big\|u^{(2)}_{\varepsilon\Psi\Phi}\big(\varphi(\pi_X(e_\varepsilon))\big)\big\| \leq \sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L')} \big\|u^{(2)}_{\varepsilon\Psi\Phi}\big(\varphi(p)\big)\big\| = O(\varepsilon^{-N}) \text{ and}$$
$$\exists N' \in \mathbb{N},\ C' > 0: \ \|\boldsymbol{\varphi}(e_\varepsilon)\| \leq C'\|e_\varepsilon\|_h = O(\varepsilon^{-N}),$$

vb-moderateness of $u_\varepsilon(e_\varepsilon)$ is ensured.

Now suppose that $u_\varepsilon \sim u'_\varepsilon$ and $e_\varepsilon \sim e'_\varepsilon$ for some $u'_\varepsilon \in \mathcal{E}_M^{VB}[E, F]$ and vb-moderate e'_ε. In both cases we easily get (iii) of Definition 4.10 from $\pi_Y(u_\varepsilon(e_\varepsilon)) = \underline{u_\varepsilon}(\pi_X(e_\varepsilon)) \sim \underline{u'_\varepsilon}(\pi_X(e'_\varepsilon)) = \pi_Y(u'_\varepsilon(e'_\varepsilon))$. Then, writing

$$\begin{aligned}&\|\boldsymbol{\psi}(u_\varepsilon(e_\varepsilon)) - \boldsymbol{\psi}(u'_\varepsilon(e_\varepsilon))\| \\ &\quad = \|\boldsymbol{\psi} \circ u_\varepsilon \circ \Phi^{-1} \circ \Phi(e_\varepsilon) - \boldsymbol{\psi} \circ u'_\varepsilon \circ \Phi^{-1} \circ \Phi(e_\varepsilon)\| \\ &\quad = \big\|u^{(2)}_{\varepsilon\Psi\Phi}\big(\varphi(\pi_X(e_\varepsilon))\big) \cdot \boldsymbol{\varphi}(e_\varepsilon) - (u'_\varepsilon)^{(2)}_{\Psi\Phi}\big(\varphi(\pi_X(e_\varepsilon))\big) \cdot \boldsymbol{\varphi}(e_\varepsilon)\big\| \\ &\quad \leq \big\|u^{(2)}_{\varepsilon\Psi\Phi}\big(\varphi(\pi_X(e_\varepsilon))\big) - (u'_\varepsilon)^{(2)}_{\Psi\Phi}\big(\varphi(\pi_X(e_\varepsilon))\big)\big\| \cdot \|\boldsymbol{\varphi}(e_\varepsilon)\|\end{aligned}$$

gives (iv) because the first factor is $O(\varepsilon^m)$ and the second is $O(\varepsilon^{-N})$ by assumption.

Finally, we know from Corollary (4.15) that $u_\varepsilon(e_\varepsilon) \sim u_\varepsilon(e'_\varepsilon)$ is equivalent to $(\hat{f} \circ u_\varepsilon)(e_\varepsilon) \sim (\hat{f} \circ u_\varepsilon)(e'_\varepsilon)$ in $(\mathbb{R} \times \mathbb{R}^{m'})_c$ for all $\hat{f} \in \mathrm{Hom}_c(F, \mathbb{R} \times \mathbb{R}^{m'})$. A consequence of

$$\begin{aligned} \|(\hat{f} \circ u_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi}(\varphi(\pi(e_\varepsilon))) - (\hat{f} \circ u_\varepsilon)^{(1)}_{\mathrm{id}\,\Phi}(\varphi(\pi(e'_\varepsilon)))\| = \\ \|\underline{\hat{f}} \circ \underline{u_\varepsilon}(\pi(e_\varepsilon)) - \underline{\hat{f}} \circ \underline{u_\varepsilon}(\pi(e'_\varepsilon))\|, \end{aligned} \tag{4.7}$$

$\underline{\hat{f}} \circ \underline{u_\varepsilon} \in \mathcal{E}_M[X, \mathbb{R}]$ and $\pi(e_\varepsilon) \sim \pi(e'_\varepsilon)$ in X_c is that (4.7) is $O(\varepsilon^m)$ for all $m \in \mathbb{N}$. All that remains to be shown now is that

$$\Big\|(\hat{f} \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}\big(\varphi(\pi_X(e_\varepsilon))\big) \cdot \boldsymbol{\varphi}(e_\varepsilon) - (\hat{f} \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}\big(\varphi(\pi_X(e'_\varepsilon))\big) \cdot \boldsymbol{\varphi}(e'_\varepsilon)\Big\|$$

is $O(\varepsilon^m)$, which follows easily from $\pi_X(e_\varepsilon) \sim \pi_X(e'_\varepsilon)$, moderateness of u_ε and the fact that $\|\boldsymbol{\varphi}(e_\varepsilon) - \boldsymbol{\varphi}(e'_\varepsilon)\| = O(\varepsilon^m)$. □

Theorem 4.17. *Let $u, v \in \mathrm{Hom}_{\mathcal{G}}[E, F]$. Then $u = v$ holds if and only if $u(e) = v(e)$ in $F_c^{\sim vb}$ for all $e \in E_c^{\sim vb}$.*

Proof. Necessity follows from the proof of Corollary 4.16. Conversely, let u_ε and v_ε be representatives of u and v. Suppose that $u_\varepsilon \not\sim v_\varepsilon$ holds, which either means $\underline{u_\varepsilon} \not\sim \underline{v_\varepsilon}$ or that (ii) of Definition 3.39 is violated for $k = 0$ – remember that vb-equivalence equals vb0-equivalence. The first option implies the existence of $p = [p_\varepsilon] \in \widetilde{X}_c$ such that $\underline{u}(p) \neq \underline{v}(p)$, so with any $e = [e_\varepsilon] \in (E_c^{\sim vb})_p$ such that $\pi_X(e_\varepsilon) = p_\varepsilon$ (Lemma 4.11), $\pi_Y(u_\varepsilon(e_\varepsilon)) = \underline{u_\varepsilon}(\pi_X(e_\varepsilon)) = \underline{u_\varepsilon}(p_\varepsilon) \not\sim \underline{v_\varepsilon}(p_\varepsilon) = \underline{v_\varepsilon}(\pi_X(e_\varepsilon)) = \pi_Y(v_\varepsilon(e_\varepsilon))$ subsequently implies that $u(e) \neq v(e)$.

The second option translates into the existence of $m \in \mathbb{N}$, vector bundle charts (U, Φ) in E and (V, Ψ) in F, $L \subset\subset U$, and $L' \subset\subset V$ satisfying

$$\begin{aligned} \forall C > 0\ \forall k \in \mathbb{N}\ \exists \varepsilon_k < \frac{1}{k}\ \exists p_k \in L \cap \underline{u_{\varepsilon_k}}^{-1}(L') \cap \underline{v_{\varepsilon_k}}^{-1}(L') : \\ \|(u^{(2)}_{\varepsilon_k \Psi\Phi} - v^{(2)}_{\varepsilon_k \Psi\Phi})(\varphi(p_k))\| > C\varepsilon_k^m. \end{aligned}$$

This means there is a bounded sequence of vectors $v_k \in \mathbb{R}^{n'}$ satisfying

$$\|(u^{(2)}_{\varepsilon_k \Psi\Phi} - v^{(2)}_{\varepsilon_k \Psi\Phi})(\varphi(p_k)) \cdot v_k\| > C\varepsilon_k^m.$$

Now choose an element $e = [e_\varepsilon]$ of $E_c^{\sim vb}$ which satisfies

$$e_{1/k} = \Phi^{-1}(\varphi(p_k), v_k).$$

Then $\pi_Y(u_{\varepsilon_k}(e_{\varepsilon_k})) = \underline{u_{\varepsilon_k}}(\pi_X(e_{\varepsilon_k})) = \underline{u_{\varepsilon_k}}(p_k) \in L'$ and $\pi_Y(v_{\varepsilon_k}(e_{\varepsilon_k})) \in L'$ but

$$
\begin{aligned}
&\|\boldsymbol{\psi}(u_{\varepsilon_k}(e_{\varepsilon_k})) - \boldsymbol{\psi}(v_{\varepsilon_k}(e_{\varepsilon_k}))\| \\
&= \|u^{(2)}_{\varepsilon_k \Psi\Phi}(\varphi(\pi_X(e_{\varepsilon_k}))) \cdot \boldsymbol{\varphi}(e_{\varepsilon_k}) - v^{(2)}_{\varepsilon_k \Psi\Phi}(\varphi(\pi_X(e_{\varepsilon_k}))) \cdot \boldsymbol{\varphi}(e_{\varepsilon_k})\| \\
&= \|(u^{(2)}_{\varepsilon_k \Psi\Phi} - v^{(2)}_{\varepsilon_k \Psi\Phi})(\varphi(p_k)) \cdot v_k\| > C\varepsilon_k^m \quad \forall k \in \mathbb{N},
\end{aligned}
$$

which is a contradiction to requirement (iv) of Definition 4.10. □

4.3 Hybrid Generalized Functions

In the following, we will write $f = (f^{(1)}, f^{(2)})$ for any function f which maps into $\mathbb{R} \times \mathbb{R}^n$.

Intrinsic Characterization

Lemma 4.18. *Let $u_\varepsilon \in \mathcal{E}^h[X, \mathbb{R} \times \mathbb{R}^{m'}]$. Then u_ε is moderate if and only if*

(i) $\underline{u_\varepsilon}$ is c-bounded,

(ii) $\underline{u_\varepsilon} \in \mathcal{E}_M(X)$, and

(iii) $u^{(2)}_\varepsilon \in \mathcal{E}_M(X)^{m'}$.

Furthermore, for $u_\varepsilon, v_\varepsilon \in \mathcal{E}^h_M[X, \mathbb{R} \times \mathbb{R}^{m'}]$, $u_\varepsilon \sim_h v_\varepsilon$ holds if and only if

(iv) $\underline{u_\varepsilon} \sim \underline{v_\varepsilon}$ in $\mathcal{E}_M(X)$ and

(v) $u^{(2)}_\varepsilon - v^{(2)}_\varepsilon \in \mathcal{N}(X)^{m'}$.

Proof. Taking the single chart $(\mathbb{R} \times \mathbb{R}^{m'}, \mathrm{id}_{\mathbb{R}\times\mathbb{R}^{m'}})$ as atlas for F in Definition 3.42, the condition "$\forall L' \subset\subset W$" is redundant and "$p \in L \cap \underline{u_\varepsilon}^{-1}(L')$" can be replaced by "$p \in L$". The claim then is evident as growth conditions can be applied coordinate-wise. □

Remark 4.19. Note that by Definition 3.4, for (v) in Lemma 4.18 we only need to establish growth estimates of order zero.

Proposition 4.20. *Let $u_\varepsilon \in \mathcal{E}^h[X, F]$. Then the following conditions are equivalent.*

(a) $u_\varepsilon \in \mathcal{E}^h_M[X, F]$.

(b) *(i)* $\underline{u_\varepsilon}$ *is c-bounded and*

(ii) $\hat{f} \circ u_\varepsilon \in \mathcal{E}^h_M[X, \mathbb{R} \times \mathbb{R}^{m'}] \quad \forall \hat{f} \in \mathrm{Hom}_c(F, \mathbb{R} \times \mathbb{R}^{m'})$.

(c) $\hat{f} \circ u_\varepsilon \in \mathcal{E}^h_M[X, \mathbb{R} \times \mathbb{R}^{m'}] \quad \forall \hat{f} \in \mathrm{Hom}(F, \mathbb{R} \times \mathbb{R}^{m'})$.

Proof. (a) $\Rightarrow$ (c): Take $\hat{f} \in \mathrm{Hom}(F, \mathbb{R} \times \mathbb{R}^{m'})$. Then $\underline{\hat{f}} \circ \underline{u_\varepsilon}$ is c-bounded because $u_\varepsilon \in \mathcal{E}_M[X, Y]$ is. Furthermore, Proposition 4.3 tells us that $\underline{\hat{f}} \circ \underline{u_\varepsilon}$ is in $\mathcal{E}_M(X)$. (i) and (ii) of Lemma 4.18 are thus established.

Now given any chart (U, φ) in X and $L \subset\subset U$ for (iii), we need to show the existence of some $N \in \mathbb{N}$ such that

$$\sup_{p \in L} \| D^{(k)}(\hat{f}^{(2)} \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p)) \|. \tag{4.8}$$

Fix $L' \subset\subset Y$ such that $\underline{u_\varepsilon}(L) \subseteq L'$ for small ε and decompose L' into $L' = \bigcup_{j=1}^s L'_j$ with $L'_j \subset\subset V_j$ and (V_j, Ψ_j) vector bundle charts in F. Then we can write for $\underline{u_\varepsilon}(p) \in L'_j$:

$$D^{(k)}(\hat{f}^{(2)} \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p)) = D^{(k)}\big((\hat{f}^{(2)} \circ \Psi_j^{-1}) \circ (\Psi_j \circ u_\varepsilon \circ \varphi^{-1})\big)(\varphi(p)).$$

For each j we have for any $k' \in \mathbb{N}_0$, by moderateness of u_ε, the existence of $N_j \in \mathbb{N}$ such that

$$\sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L'_j)} \| D^{(k')}(\Psi_j \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p)) \| = O(\varepsilon^{-N_j}).$$

As $\Psi_j(L'_j)$ is compact and $\hat{f}^{(2)} \circ \Psi_j^{-1} : \mathbb{R}^{m'} \to \mathbb{R}^{m'}$ is smooth, every derivative of the latter on the former is bounded and by the chain rule we obtain (4.8).

(c) $\Rightarrow$ (b): (ii) is clear, (i) follows directly from Proposition 4.1.

(b) $\Rightarrow$ (a): For (ii) of Definition 3.42, given any chart (U, φ) in X, any vector bundle chart (V, Ψ) in F, $L \subset\subset U$, and $L' \subset\subset V$, choose an open neighborhood V' of L' whose closure is contained in V. For each coordinate ψ^j of ψ $(1 \le j \le m)$ take $\hat{f}_j \in \mathrm{Hom}_c(F, \mathbb{R} \times \mathbb{R}^{m'})$ such that

$$\hat{f}_j|_{\pi_Y^{-1}(V')} = (\mathrm{pr}_j \times \mathrm{id}_{\mathbb{R}^{m'}}) \circ \Psi|_{\pi_Y^{-1}(V')}.$$

For each $p \in L \cap \underline{u_\varepsilon}^{-1}(L')$ we then have (in some neighborhood of p)

$$\hat{f}_j \circ u_\varepsilon = (\psi^j \circ \underline{u_\varepsilon}, \boldsymbol{\psi} \circ u_\varepsilon).$$

In order to estimate

$$\sup_{p\in L\cap \underline{u_\varepsilon}^{-1}(L')} \|D^{(k)}(\Psi\circ u_\varepsilon\circ\varphi^{-1})(\varphi(p))\|$$
$$= \sup_{p\in L\cap \underline{u_\varepsilon}^{-1}} \left\|D^{(k)}\left((\psi\circ\underline{u_\varepsilon}\circ\varphi^{-1}, \boldsymbol{\psi}\circ u_\varepsilon\circ\varphi^{-1})\right)\left(\varphi(p)\right)\right\|$$

we resort to the coordinates of ψ and obtain

$$\sup_{p\in L\cap \underline{u_\varepsilon}^{-1}(L')} \left\|D^{(k)}\left((\psi^j\circ\underline{u_\varepsilon}\circ\varphi^{-1}, \boldsymbol{\psi}\circ u_\varepsilon\circ\varphi^{-1})\right)\left(\varphi(p)\right)\right\| =$$
$$\sup_{p\in L\cap \underline{u_\varepsilon}^{-1}(L')} \|D^{(k)}(\hat{f}_j\circ u_\varepsilon\circ\varphi^{-1})(\varphi(p))\|,$$

which satisfies moderateness estimates by assumption. □

Theorem 4.21. *Let $u_\varepsilon, v_\varepsilon \in \mathcal{E}^h_M[X,F]$. Then the following statements are equivalent.*

(i) $u_\varepsilon \sim_h v_\varepsilon$.

(ii) $u_\varepsilon \sim_{h0} v_\varepsilon$.

(iii) $\hat{f}\circ u_\varepsilon \sim_h \hat{f}\circ v_\varepsilon$ *in* $\mathcal{E}^h_M[X,\mathbb{R}\times\mathbb{R}^{m'}]$ $\quad\forall \hat{f}\in \mathrm{Hom}_c(F,\mathbb{R}\times\mathbb{R}^{m'})$.

(iv) $\hat{f}\circ u_\varepsilon \sim_h \hat{f}\circ v_\varepsilon$ *in* $\mathcal{E}^h_M[X,\mathbb{R}\times\mathbb{R}^{m'}]$ $\quad\forall \hat{f}\in \mathrm{Hom}(F,\mathbb{R}\times\mathbb{R}^{m'})$.

Proof. (i) ⇒ (ii) is evident.
(ii) ⇒ (iv): Let $\hat{f}\in\mathrm{Hom}(F,\mathbb{R}\times\mathbb{R}^{m'})$. As $\underline{u_\varepsilon}\in\mathcal{E}_M[X,Y]$ and $\underline{\hat{f}}\in C^\infty(Y)$, $\underline{\hat{f}}\circ\underline{u_\varepsilon}\sim\underline{\hat{f}}\circ\underline{v_\varepsilon}$ by Theorem 4.5, giving (iv) of Lemma 4.18. For (v), choose a chart (U,φ) in X and $L\subset\subset U$. By Remark 4.19 we now only need to estimate

$$(\hat{f}\circ u_\varepsilon)^{(2)} - (\hat{f}\circ v_\varepsilon)^{(2)}. \tag{4.9}$$

For this we choose $L'\subset\subset Y$ such that $\underline{u_\varepsilon}(L)\subseteq L'$ for small ε. As in the proof of Proposition 3.35, we may choose finitely many vector bundle charts (V_j,Ψ_j) in F, $L'_j\subset\subset V_j$ such that $L'=\bigcup_{j=1}^s L'_j$ and open neighborhoods V'_j of L'_j with $\overline{V'_j}\subset\subset V_j$ for $j=1,\dots,s$ such that for each p contained in $L\cap\underline{u_\varepsilon}^{-1}(L')\cap\underline{v_\varepsilon}^{-1}(L')$ there is a $j\in\{1,\dots,s\}$ such that both $\underline{u_\varepsilon}(p)$ and $\underline{v_\varepsilon}(p)$ are in V'_j. Expression (4.9) can then be rewritten as

$$(\hat{f}^{(2)}\circ\Psi_l^{-1})\circ(\Psi_l\circ u_\varepsilon) - (\hat{f}^{(2)}\circ\Psi_l^{-1})\circ(\Psi_l\circ v_\varepsilon). \tag{4.10}$$

As $\Psi_l(\overline{V_j'})$ is compact we can happily employ Lemma 3.33 to obtain a positive constant C such that the supremum of the norm of (4.10) on $K := L \cap \underline{u_\varepsilon}^{-1}(V_j') \cap \underline{v_\varepsilon}^{-1}(L')$ can be estimated by

$$C \cdot \sup_{p \in K} \|(\Psi_l \circ u_\varepsilon - \Psi_l \circ v_\varepsilon)(p)\| = O(\varepsilon^m)$$

for all $m \in \mathbb{N}$ by assumption.

(iv) $\Rightarrow$ (iii) is clear like sunlight.

(iii) $\Rightarrow$ (i): Definition 3.43 (i) follows from Theorem 4.5. For (ii) fix k, m, (U, φ), (V, Ψ), L, and L' as required. We choose an open neighborhood V' of L' satisfying $\overline{V'} \subset\subset V$ and for each $j \in \{1, \ldots, m\}$ we take some $\hat{f}_j \in \mathrm{Hom}_c(F, \mathbb{R} \times \mathbb{R}^{m'})$ with

$$\hat{f}_j|_{\pi_Y^{-1}(V')} = (\mathrm{pr}_j \times \mathrm{id}_{\mathbb{R}^{m'}}) \circ \Psi|_{\pi_Y^{-1}(V')}.$$

For $p \in L \cap \underline{u_\varepsilon}^{-1}(L') \cap \underline{v_\varepsilon}^{-1}(L')$ we can then write

$$\hat{f}_j \circ u_\varepsilon - \hat{f}_j \circ v_\varepsilon = (\psi_j \circ \underline{u_\varepsilon} - \psi_j \circ \underline{v_\varepsilon}, \boldsymbol{\psi} \circ u_\varepsilon - \boldsymbol{\psi} \circ v_\varepsilon),$$

which enables us to estimate via coordinates ψ^j of ψ, after abbreviating $L'' := L \cap \underline{u_\varepsilon}^{-1}(L') \cap \underline{v_\varepsilon}^{-1}(L')$:

$$\sup_{p \in L''} \|D^{(k)}(\psi^j \circ \underline{u_\varepsilon} \circ \varphi^{-1} - \psi^j \circ \underline{v_\varepsilon} \circ \varphi^{-1}, \boldsymbol{\psi} \circ u_\varepsilon \circ \varphi^{-1} - \boldsymbol{\psi} \circ v_\varepsilon \circ \varphi^{-1})(\varphi(p))\|$$
$$= \sup_{p \in L''} \|D^{(k)}(\hat{f}_j \circ u_\varepsilon - \hat{f}_j \circ v_\varepsilon)(\varphi(p))\|.$$

By assumption, the last expression satisfies negligibility estimates. □

Point Values

We end up at the following point value characterization.

Proposition 4.22. *Let $u = [u_\varepsilon] \in \mathcal{G}^h[X, F]$, $p = [p_\varepsilon] \in \widetilde{X}_c$. Then the point value of u at p, defined by $u(p) := [u_\varepsilon(p_\varepsilon))]$, is a well-defined element of $F_c^{\sim vb}$.*

Proof. First, $\pi_Y(u_\varepsilon(p_\varepsilon))$ is compactly supported because of Definition 3.42 (i).

Second, we need to estimate $\|u_\varepsilon(p_\varepsilon)\|_h$ for some Riemannian metric h on F. As $p_\varepsilon \in \widetilde{X}_c$ and $\underline{u_\varepsilon}(p_\varepsilon) \in \widetilde{Y}_c$ there exist $\varepsilon_0 > 0$ as well as compact sets

$L \subset\subset X$ and $L' \subset\subset Y$ such that $p_\varepsilon \in L \cap \underline{u_\varepsilon}^{-1}(L')$ for all $\varepsilon < \varepsilon_0$. By Lemma 1.1 we can write $L = \bigcup_{i=1}^r L_i$ and $L' = \bigcup_{j=1}^s L'_j$ with $L_i \subset\subset U_i$ and $L'_j \subset\subset V_j$ where (U_i, φ_i) and (V_j, Ψ_j) are charts in X and vector bundle charts in F, respectively ($1 \le i \le r$, $1 \le j \le s$).

For each $\varepsilon < \varepsilon_0$ there are i, j such that $p_\varepsilon \in L_i \cap \underline{u_\varepsilon}^{-1}(L'_j)$. By Corollary 3.11 there exists $C > 0$ such that

$$\sup_{q \in \pi_Y^{-1}(L'_j)} \|q\|_h \le C \|\boldsymbol{\psi}_j q\| \quad \forall j \in \{1, \dots, s\}\ \forall \varepsilon < \varepsilon_0,$$

so we can estimate

$$\begin{aligned}
\|u_\varepsilon(p_\varepsilon)\|_h &\le C \|(\boldsymbol{\psi}_j \circ u_\varepsilon \circ \varphi_i^{-1})(\varphi_i(p_\varepsilon)\| \\
&\le C \sup_{p \in L_i \cap \underline{u_\varepsilon}^{-1}(L'_j)} \|\boldsymbol{\psi}_j \circ u_\varepsilon \circ \varphi_i^{-1})(\varphi_i(p))\| = O(\varepsilon^{-N_{ij}})
\end{aligned}$$

for some $N_{ij} \in \mathbb{N}$ by moderateness of u_ε and after setting $N := \max_{i,j} N_{ij}$ we subsequently conclude that $\|u_\varepsilon(p_\varepsilon)\|_h = O(\varepsilon^{-N})$. So $u_\varepsilon(p_\varepsilon)$ indeed is an element of $F_c^{\sim_{vb}}$.

What remains to be shown is independence of the choice of representatives of u and p. So let $p'_\varepsilon \in X_c$ be given with $p_\varepsilon \sim p'_\varepsilon$. Corollary 4.15 states that $[u_\varepsilon(p_\varepsilon)] = [u_\varepsilon(p'_\varepsilon)]$ in $F_c^{\sim_{vb}}$ is equivalent to $[\hat{f}(u_\varepsilon(p_\varepsilon))] = [\hat{f}(u_\varepsilon(p'_\varepsilon))]$ in $(\mathbb{R} \times \mathbb{R}^{m'})_c^{\sim_{vb}}$ for all $\hat{f} \in \mathrm{Hom}_c(F, \mathbb{R} \times \mathbb{R}^{m'})$. There exists $K \subset\subset X$ such that $p_\varepsilon, p'_\varepsilon \in K$ for small ε. We cover K by charts (U_i, φ_i) and as in previous proofs choose sets $L_i \subset\subset U_i$ such that for all small ε there exists i with $p_\varepsilon, p'_\varepsilon \in L_i$. We then can write

$$\begin{aligned}
\|\hat{f} \circ u_\varepsilon(p_\varepsilon) - \hat{f} \circ u_\varepsilon(p'_\varepsilon)\| &= \|\hat{f} \circ u_\varepsilon \circ \varphi_i^{-1} \circ \varphi_i(p_\varepsilon) - \hat{f} \circ u_\varepsilon \circ \varphi_i^{-1} \circ \varphi_i(p'_\varepsilon)\| \\
&\le \sup_{q \in L_i} \|D(\hat{f} \circ u_\varepsilon \circ \varphi_i^{-1})(\varphi_i(q))\| \cdot \|\varphi(p_\varepsilon) - \varphi(q_\varepsilon)\|
\end{aligned}$$

and obtain vb-equivalence of $u_\varepsilon(p_\varepsilon)$ and $u_\varepsilon(p'_\varepsilon)$ by moderateness of $\hat{f} \circ u_\varepsilon$ and the assumption $p_\varepsilon \sim p'_\varepsilon$.

Now let $u_\varepsilon \sim_h u'_\varepsilon$. Then $u_\varepsilon(p_\varepsilon) \sim_{vb} u'_\varepsilon(p_\varepsilon)$ follows at once from the definition of equivalence in $\mathcal{G}^h[X, F]$. □

Proposition 4.23. *Let $u = [u_\varepsilon], v = [v_\varepsilon] \in \mathcal{G}^h[X, F]$. Then $u = v$ if and only if $u(p) = v(p)$ for all $p \in \widetilde{X}_c$.*

Proof. Necessity has already been established in Proposition 4.22. For the converse suppose that $u_\varepsilon \not\sim_h v_\varepsilon$, which by Theorem 4.21 is equivalent to $u_\varepsilon \not\sim_{h0} v_\varepsilon$. Then there are two possibilities:

First, $\underline{u_\varepsilon} \not\sim_0 \underline{v_\varepsilon}$ implies by Theorem 4.9 the existence of a $p \in \widetilde{X}_c$ with $\underline{u}(p) \neq \underline{v}(p)$ and therefore $u(p) \neq v(p)$.

Second, if (ii) of Definition 3.43 is violated for $k = 0$ there exist $m \in \mathbb{N}$, a chart (U, φ) in X, a vector bundle chart (W, Ψ) in F, sets $L \subset\subset U$ and $L' \subset\subset V$ and for each $n \in \mathbb{N}$ an $\varepsilon_n < 1/n$ and $p_n \in L \cap \underline{u_{\varepsilon_n}}^{-1}(L') \cap \underline{v_{\varepsilon_n}}^{-1}(L')$ satisfying

$$\|\Psi \circ u_\varepsilon(p_n) - \Psi \circ v_\varepsilon(p_n)\| > n\varepsilon_n^m.$$

This allows us to construct a point $[p_\varepsilon] \in \widetilde{X}_c$ by $p_{\varepsilon_n} := p_n$ for which then $u(p) \neq v(p)$ holds. $\square$

Chapter 5

Composition

We can now state results about composition of generalized functions of various kinds.

Between Manifolds

Theorem 5.1. *Let* $u = [u_\varepsilon] \in \mathcal{G}[X,Y]$ *and* $v = [v_\varepsilon] \in \mathcal{G}[Y,Z]$. *Then* $v \circ u := [v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}[X,Z]$.

Proof. For c-boundedness of $v \circ u$ we note that for each $K \subset\subset X$ we get

$$\begin{aligned} &\exists K' \subset\subset Y,\ \varepsilon_0' > 0 : u_\varepsilon(K) \subseteq K' \quad \forall \varepsilon < \varepsilon_0' \text{ and} \\ &\exists K'' \subset\subset Z,\ \varepsilon_0'' > 0 : v_\varepsilon(K') \subseteq K'' \quad \forall \varepsilon < \varepsilon_0'' \end{aligned}$$

which gives $(v_\varepsilon \circ u_\varepsilon)(K) \subseteq K''\ \forall \varepsilon < \min(\varepsilon_0', \varepsilon_0'')$.
Regarding moderateness we fix $k \in \mathbb{N}_0$, charts (U, φ) in X and (W, ζ) in Z, $L \subset\subset U$, and $L'' \subset\subset W$. As u_ε is c-bounded there is $\varepsilon_0 > 0$ and $L' \subset\subset Y$ such that $u_\varepsilon(L) \subseteq L'\ \forall \varepsilon < \varepsilon_0$. We cover L' by finitely many charts (V_i, ψ_i) in Y and decompose it into $L' = \bigcup_{i=1}^r L_i'$ with $L_i' \subset\subset V_i$ and $r \in \mathbb{N}$. By Definition 3.30 there are $N_i, N_i' \in \mathbb{N}$ $(i = 1, \ldots, r)$ such that

$$\begin{aligned} &\sup_{p \in L \cap u_\varepsilon^{-1}(L_i')} \| D^{(k)}(\psi_i \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p)) \| = O(\varepsilon^{-N_i}) \text{ and} \\ &\sup_{p \in L_i' \cap v_\varepsilon^{-1}(L'')} \| D^{(k)}(\zeta \circ v_\varepsilon \circ \psi_i^{-1})(\psi_i(p)) \| = O(\varepsilon^{-N_i'}). \end{aligned} \tag{5.1}$$

For every $p \in L \cap (v_\varepsilon \circ u_\varepsilon)^{-1}(L'')$ and $\varepsilon < \varepsilon_0$ there is an $i \in \{1, \ldots, r\}$ such that $u_\varepsilon(p) \in L_i'$. We thus write for all p' in a neighborhood of p

$$D^{(k)}(\zeta \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p')) = D^{(k)}((\zeta \circ v_\varepsilon \circ \psi_i^{-1}) \circ (\psi_i \circ u_\varepsilon \circ \varphi^{-1}))(\varphi(p'))$$

and obtain moderateness by estimates (5.1) and the chain rule.

In order to show that $v \circ u$ is indeed well-defined we assume some u'_ε and v'_ε in $\mathcal{E}_M^{VB}[X,Y]$ to satisfy $u_\varepsilon \sim u'_\varepsilon$ and $v_\varepsilon \sim v'_\varepsilon$.

Proposition 4.9 states that for all $p_\varepsilon \in X_c$, $[u_\varepsilon(p_\varepsilon)] = [u'_\varepsilon(p_\varepsilon)]$ holds in $\widetilde{Y}_c$. By Proposition 4.8, point value evaluation is independent of the specific representatives of both the point and the function, therefore we know that $[v_\varepsilon(u_\varepsilon(p_\varepsilon))] = [v'_\varepsilon(u'_\varepsilon(p_\varepsilon))]$ holds in $\widetilde{Z}_c$. Again by Proposition 4.9, composition of v and u is well-defined. □

Corollary 5.2. *(i) Let $u \in C^\infty(X,Y)$ and $v = [v_\varepsilon] \in G[Y,Z]$. Then $v \circ u := v_\varepsilon \circ u \in \mathcal{G}[X,Z]$ is well-defined.*

(ii) Let $u = [u_\varepsilon] \in \mathcal{G}[X,Y]$ and $v \in C^\infty(Y,Z)$. Then $v \circ u := [v \circ u_\varepsilon]$ is a well-defined element of $\mathcal{G}[X,Z]$.

Proof. The proof of Theorem 5.1 is easily adapted to these cases. □

Theorem 5.3. *Let $u = [u_\varepsilon] \in \mathcal{G}[X,Y]$ and $v = [v_\varepsilon] \in \mathcal{G}(Y)$ be given. Then $v \circ u := [v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}(X)$ is well-defined.*

Proof. For moderateness of $v_\varepsilon \circ u_\varepsilon$ we need to estimate

$$\sup_{p \in L} \| D^{(k)}(v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p)) \|$$

for each $k \in \mathbb{N}_0$ and L compactly contained in the domain of a chart (U,φ) in X. As u_ε is c-bounded, we have

$$\exists \varepsilon_0 > 0 \; \exists L' \subset\subset Y : u_\varepsilon(L) \subseteq L' \; \forall \varepsilon < \varepsilon_0,$$

and write $L' = \bigcup_{j=1}^s L'_j$ with $L'_j \subset\subset V_j$ and (V_j, ψ_j) charts in Y. Moderateness estimates are then established by the chain rule applied to $v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1} = (v_\varepsilon \circ \psi_j^{-1}) \circ (\psi_j \circ u_\varepsilon \circ \varphi^{-1})$.

We now take any $u' = [u'_\varepsilon] \in \mathcal{G}[X,Y]$ and $v' = [v'_\varepsilon] \in \mathcal{G}(Y)$ with $u_\varepsilon \sim u'_\varepsilon$ and $v_\varepsilon \sim v'_\varepsilon$. By Proposition 4.9, $u(x) = u'(x)$ holds in $\widetilde{Y}_c$ for all $x \in \widetilde{X}_c$ and $v(u(x)) = v'(u'(x))$ as well. By the same Proposition, $v \circ u = v' \circ u'$ follows and therefore composition is well-defined. □

Between Vector Bundles

Theorem 5.4. *Let* $u = [u_\varepsilon] \in \mathrm{Hom}_{\mathcal{G}}[E, F]$, $v = [v_\varepsilon] \in \mathrm{Hom}_{\mathcal{G}}[F, G]$. *Then* $v \circ u := [v_\varepsilon \circ u_\varepsilon]$ *is a well-defined element of* $\mathrm{Hom}_{\mathcal{G}}[E, G]$.

Proof. For $v_\varepsilon \circ u_\varepsilon \in \mathcal{E}_M^{VB}[E, G]$ it suffices to show $\hat{f} \circ v_\varepsilon \circ u_\varepsilon \in \mathcal{E}_M^{VB}[E, \mathbb{R} \times \mathbb{R}^{k'}]$ for each $\hat{f} \in \mathrm{Hom}_c(G, \mathbb{R} \times \mathbb{R}^{k'})$. Because Proposition 4.13 implies that $\hat{f} \circ v_\varepsilon$ is in $\mathcal{E}_M^{VB}[F, \mathbb{R} \times \mathbb{R}^{k'}]$, we may assume without loss of generality that $G = \mathbb{R} \times \mathbb{R}^{k'}$. (i) of Definition 3.38 is clear by Theorem 5.1 from $\underline{v_\varepsilon \circ u_\varepsilon} = \underline{v_\varepsilon} \circ \underline{u_\varepsilon} \in \mathcal{E}_M[X, Z]$. Now regarding (ii), it suffices to show (by Proposition 4.12) that for any vector bundle chart (U, Φ) in E, $(v_\varepsilon \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}$ is an element of $\mathcal{E}_M(\varphi(U))^{n' \cdot m'}$. Again, this is equivalent to

$$\forall L \subset\subset U \ \forall k \in \mathbb{N}_0 \ \exists N \in \mathbb{N} : \sup_{p \in L} \| D^{(k)} (v_\varepsilon \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}(\varphi(p)) \| = O(\varepsilon^{-N}).$$

Let $L \subset\subset U$ and choose $\varepsilon_0 > 0$ and $L' \subset\subset Y$ such that $\underline{u_\varepsilon}(L) \subseteq L'$ for all $\varepsilon < \varepsilon_0$. Then cover L' by finitely many vector bundle charts (V_i, Ψ_i) in F and choose $L'_i \subset\subset V_i$ such that $L' = \bigcup_{i=1}^{l} L'_i$. For each $p \in L$ and $\varepsilon < \varepsilon_0$ there is an $i \in \{1, \dots, l\}$ such that $u_\varepsilon(p) \in L'_i$. Then for p' in a neighborhood of p we have

$$(v_\varepsilon \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}\big(\varphi(p')\big) = v^{(2)}_{\varepsilon\,\mathrm{id}\,\Psi_i}\big(\psi_i \circ \underline{u_\varepsilon}(p')\big) \cdot u^{(2)}_{\varepsilon \Psi_i \Phi}\big(\varphi(p')\big).$$

By Definition 3.38 (ii), we have

$$\forall k' \in \mathbb{N}_0 \ \exists N' \in \mathbb{N} : \sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L'_i)} \| D^{(k')} (u^{(2)}_{\varepsilon \Psi_i \Phi})\big(\varphi(p)\big) \| = O(\varepsilon^{-N'}),$$

while Proposition 4.12 tells us that

$$\forall k'' \in \mathbb{N}_0 \ \exists N'' \in \mathbb{N} : \sup_{p \in L \cap \underline{u_\varepsilon}^{-1}(L'_i)} \| D^{(k'')} (v^{(2)}_{\varepsilon\,\mathrm{id}\,\Psi_i})(\psi_i \circ \underline{u_\varepsilon}(p)) \| \leq \\ \sup_{p \in L'_i} \| D^{(k)} (v^{(2)}_{\varepsilon\,\mathrm{id}\,\Psi_i})(\psi_i(p)) \| = O(\varepsilon^{-N''}).$$

The estimate for $D^{(k)}\big((v_\varepsilon \circ u_\varepsilon)^{(2)}_{\mathrm{id}\,\Phi}\big)$ thus follows by the chain rule.

Finally let $u'_\varepsilon \in \mathcal{E}_M^{VB}[E, F]$ and $v'_\varepsilon \in \mathcal{E}_M^{VB}[F, G]$ be given with $u_\varepsilon \sim u'_\varepsilon$ and $v_\varepsilon \sim v'_\varepsilon$. Then for $e_\varepsilon \in E_c^{\sim_{vb}}$ the equivalences $u_\varepsilon(e_\varepsilon) \sim_{vb} u'_\varepsilon(e_\varepsilon)$ as well as $v_\varepsilon \circ u_\varepsilon(e_\varepsilon) \sim_{vb} v'_\varepsilon \circ u'_\varepsilon(e_\varepsilon)$ imply well-definedness of $v \circ u$. $\square$

Hybrids

Theorem 5.5. *For any $u = [u_\varepsilon] \in \mathcal{G}[X,Y]$, $v = [v_\varepsilon] \in \mathcal{G}^h[Y,G]$ and $w = [w_\varepsilon] \in \mathrm{Hom}_\mathcal{G}[G,H]$ the compositions $v \circ u := [v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}^h[X,G]$ and $w \circ v = [w_\varepsilon \circ v_\varepsilon] \in \mathcal{G}^h[Y,H]$ are well-defined.*

Proof. Instead of repeating parts of previous proofs ad nauseam, it should suffice to say at this point that moderateness follows in the obvious way and well-definedness is an immediate consequence of Proposition 4.9, Theorem 4.17 and Proposition 4.23. □

Theorem 5.6. *(i) Let $u = [u_\varepsilon] \in \mathcal{G}[X,Y]$ and $v = [v_\varepsilon] \in \Gamma_\mathcal{G}(Y,F)$. Then $v \circ u := [v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}^h[X,F]$ is well-defined.*

(ii) Let $u = [u_\varepsilon] \in \Gamma_\mathcal{G}(X,E)$ and $v = [v_\varepsilon] \in \mathrm{Hom}_\mathcal{G}[E,F]$. Then the composition $v \circ u := [v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}^h[X,F]$ is well-defined.

Proof. (i) For moderateness of $v_\varepsilon \circ u_\varepsilon$ note that $\underline{v_\varepsilon \circ u_\varepsilon} = (\pi_Y \circ v_\varepsilon) \circ u_\varepsilon = u_\varepsilon$ immediately gives (i) of Definition 3.42. Next, choose $k \in \mathbb{N}$, a chart (U,φ) in X, a vector bundle chart (V,Ψ) in Y, $L \subset\subset U$, and $L' \subset\subset V$. Then

$$\sup_{p \in L \cap u_\varepsilon^{-1}(L')} \|D^{(k)}(\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\|$$

can be estimated to be $O(\varepsilon^{-N})$ for some $N \in \mathbb{N}$ by writing

$$\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1} = (\Psi \circ v_\varepsilon \circ \psi^{-1}) \circ (\psi \circ u_\varepsilon \circ \varphi^{-1})$$

and incorporating moderateness of u_ε and v_ε.

In order to be well-defined the composition has to be independent of the specific choice of representatives. Let $u'_\varepsilon \in \mathcal{E}_M[X,Y]$ satisfy $u_\varepsilon \sim u'_\varepsilon$. Then (i) of Definition 3.43 for $K \subset\subset X$ follows from

$$\sup_{p \in K} d_h\big((\underline{v_\varepsilon \circ u_\varepsilon})(p), (\underline{v_\varepsilon \circ u'_\varepsilon})(p)\big) = \sup_{p \in K} d_h\big(u_\varepsilon(p), u'_\varepsilon(p)\big) \to 0 \quad (\varepsilon \to 0)$$

by Theorem 4.6. For (ii), choose L, L', (U,φ) and (V,Ψ) as required. Then, write $L' = \bigcup_{j=1}^r L'_j$ with $L'_j \subset\subset \overline{V'_j} \subset\subset V_j$ where V'_j is an open neighborhood of L'_j and (V_j, Ψ_j) are vector bundle charts in F. Then for each $p \in L \cap u_\varepsilon^{-1}(L') \cap (u'_\varepsilon)^{-1}(L')$ and small ε there exists a j such that $u_\varepsilon(p)$ and $u'_\varepsilon(p)$ are in V_j. Subsequently, derivatives of

$$\begin{aligned}
&(\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1} - \Psi \circ v_\varepsilon \circ u'_\varepsilon \circ \varphi^{-1})(\varphi(p)) \\
&\quad = \big((\Psi \circ \Psi_j^{-1}) \circ (\Psi_j \circ v_\varepsilon \circ \psi_j^{-1}) \circ (\psi_j \circ u_\varepsilon \circ \varphi^{-1}) \\
&\qquad - (\Psi \circ \Psi_j^{-1}) \circ (\Psi_j \circ v_\varepsilon \circ \psi_j^{-1}) \circ (\psi_j \circ u'_\varepsilon \circ \varphi^{-1})\big)(\varphi(p))
\end{aligned}$$

can be estimated with the help of Lemma 3.33, using moderateness of v_ε and equivalence of u_ε and u'_ε.

Now let $v_\varepsilon \in \Gamma_{\mathcal{E}_M}(Y,F)$ such that $v_\varepsilon - v'_\varepsilon \in \Gamma_{\mathcal{N}}(Y,F)$. For establishing $v_\varepsilon \circ u_\varepsilon \sim_h v'_\varepsilon \circ u_\varepsilon$, (i) of 3.43 is clear from $\underline{v_\varepsilon \circ u_\varepsilon} = \underline{v'_\varepsilon \circ u_\varepsilon} = u_\varepsilon$. For (ii), with $L \cap \underline{v_\varepsilon \circ u_\varepsilon}^{-1}(L') \cap \underline{v'_\varepsilon \circ u_\varepsilon}(L') = L \cap u_\varepsilon^{-1}(L')$ we can easily ascertain that

$$\begin{aligned}
&\sup_{p \in L \cap u_\varepsilon^{-1}(L')} \|D^{(k)}(\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1} - \Psi \circ v'_\varepsilon \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| \\
&\qquad = \sup_{p \in L \cap u_\varepsilon^{-1}(L')} \left\| D^{(k)}\big(\Psi \circ (v_\varepsilon - v'_\varepsilon) \circ \psi^{-1} \circ (\psi \circ u_\varepsilon \circ \varphi^{-1})\big)\big(\varphi(p)\big) \right\|
\end{aligned}$$

satisfies negligibility estimates, as derivatives of $\Psi \circ (v_\varepsilon - v'_\varepsilon) \circ \psi^{-1}$ are on $\psi(L')$ for all $m \in \mathbb{N}$ and each derivative of $\psi \circ u_\varepsilon \circ \varphi^{-1}$ is $O(\varepsilon^{-N})$ on $L \cap u_\varepsilon^{-1}(L')$ for some $N \in \mathbb{N}$. In total, the composition of v and u is well-defined.

(ii) C-boundedness of $\underline{v_\varepsilon \circ u_\varepsilon} = \underline{v_\varepsilon}$ means (i) of 3.42. For (U,φ) a chart in X, (V,Ψ) a vector bundle chart in F, $L \subset\subset U$, and $L' \subset\subset V$ there should exist an $N \in \mathbb{N}$ such that

$$\sup_{p \in L \cap \underline{v_\varepsilon}^{-1}(L')} \|D^{(k)}(\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^{-N}). \tag{5.2}$$

Writing $\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1} = (\Psi \circ v_\varepsilon \circ \Phi^{-1}) \circ (\Phi \circ u_\varepsilon \circ \varphi^{-1})$, where (U,Φ) is a vector bundle chart in E over (U,φ), we obtain (5.2) by moderateness of u and v.

For well-definedness, assume that $u_\varepsilon - u'_\varepsilon$ is in $\Gamma_{\mathcal{N}}(X,E)$ for some $u'_\varepsilon \in \Gamma_{\mathcal{E}_M}(X,E)$. Part (i) of Definition 3.43 is clear from $\underline{v_\varepsilon \circ u_\varepsilon} = \underline{v_\varepsilon \circ u'_\varepsilon} = \underline{v_\varepsilon}$. For (ii) we use the notation

$$\Phi \circ u_\varepsilon \circ \varphi^{-1}(x) = (x, u^{(2)}_{\varepsilon\Phi}(x))$$

and proceed with

$$\begin{aligned}
\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1}(x) &= (\Psi \circ v_\varepsilon \circ \Phi^{-1}) \circ (\Phi \circ u_\varepsilon \circ \varphi^{-1})(x) \\
&= \big(v^{(1)}_{\varepsilon\Psi\Phi}, v^{(2)}_{\varepsilon\Psi\Phi}(x) \cdot u^{(2)}_{\varepsilon\Phi}(x)\big),
\end{aligned}$$

giving in turn

$$\begin{aligned}
&\Psi \circ v_\varepsilon \circ u_\varepsilon \circ \varphi^{-1}(x) - \Psi \circ v_\varepsilon \circ u'_\varepsilon \circ \varphi^{-1}(x) \\
&\qquad\qquad = \big(0, v^{(2)}_{\varepsilon\Psi\Phi}(x) \cdot (u^{(2)}_{\varepsilon\Phi} - (u'_{\varepsilon\Phi})^{(2)})(x)\big).
\end{aligned}$$

Because $\|v^{(2)}_{\varepsilon\Psi\Phi}\|$ is $O(\varepsilon^{-N})$ and $\|u^{(2)}_{\varepsilon\Phi} - (u'_{\varepsilon\Phi})^{(2)}\|$ is $O(\varepsilon^m)$, from this we obtain $v_\varepsilon \circ u_\varepsilon \sim v_\varepsilon \circ u'_\varepsilon$.

Last but not least, let $v'_\varepsilon \in \mathcal{E}^{VB}_M[E, F]$ with $v_\varepsilon \sim_{vb} v'_\varepsilon$. Similarly as before, we now need to estimate

$$\big\|\big(v^{(1)}_{\varepsilon\Psi\Phi} - v^{(1)}_{\varepsilon\Psi\Phi}, (v^{(2)}_{\varepsilon\Psi\Phi} - v'^{(2)}_{\varepsilon\Psi\Phi}) \cdot u^{(2)}_{\varepsilon\Phi}(x)\big)\big\|,$$

which is straightforward from the assumptions. $\square$

Chapter 6

Generalized ODEs and Flow

In the previous chapters a theory of generalized functions on $\mathbb{R}^n$ as well as on manifolds was presented. Our aim now is to compare certain aspects of the classical and the generalized setting in the field of differential equations, namely the flow properties of autonomous ODE systems. A certain degree of consistency - even with some restrictions - would be desirable in order to ascertain the validity of the approach taken.

Auxiliary Results

We first review some results of classical ODE theory, taken from [Ama83] and [Aul04].

First, we will need the Lemma of Gronwall in the following form ([Ama83], Theorem 6.2).

Lemma 6.1. *Let J be an interval in $\mathbb{R}$, $t_0 \in J$ and let $\alpha, \beta, u \in C(J, \mathbb{R}_+)$ satisfy*

$$u(t) \leq \alpha(t) + \Big|\int_{t_0}^{t} \beta(s)u(s)\mathrm{d}s\Big| \quad \forall t \in J.$$

Furthermore, assume that $\alpha(t) = \alpha_0(|t - t_0|)$ for a nondecreasing function $\alpha_0 \in C(\mathbb{R}_+, \mathbb{R}_+)$. Then

$$u(t) \leq \alpha(t) \exp\Big(\Big|\int_{t_0}^{t} \beta(s)\mathrm{d}s\Big|\Big) \quad \forall t \in J.$$

The following theorem ([Aul04], Theorem 2.5.6) assures existence as well as uniqueness of a global solution for a nonlinear ODE system.

Theorem 6.2. *For $D := (a,b) \times \mathbb{R}^n$ with $-\infty \leq a < b \leq \infty$, let the function $F : D \to \mathbb{R}^n$ be continuous, Lipschitz-continuous in x and linearly bounded, i.e.,*

$$\|F(t,x)\| \leq L(t)\|x\| + M(t) \quad \forall t \in (a,b)\ \forall x \in \mathbb{R}^n$$

with $M, T : (a,b) \to \mathbb{R}_+$ continuous. Then the initial value problem

$$\begin{aligned} \dot{x}(t) &= F(t,x) \\ x(t_0) &= x_0 \\ (t_0, x_0) &\in (a,b) \times \mathbb{R}^n \end{aligned}$$

has a unique solution on (a,b).

As we will have to deal with ODE systems depending on a parameter $\varepsilon \in I$, we will use the following theorem to assure that the solution is not only smooth in t but also depends smoothly on initial values and parameters ([Ama83], Theorem 10.3).

Theorem 6.3. *For $m \in \mathbb{N} \cup \{\infty\}$, $I \subseteq \mathbb{R}$ open, $J \subseteq \mathbb{R}$ an interval and $D \subseteq \mathbb{R}^n$ open, let $F \in C^m(J \times D \times I, \mathbb{R}^n)$. Then the unique maximal solution $u = u(t; t_0, x_0, \varepsilon_0)$ of*

$$\begin{aligned} \dot{x}(t) &= F(t,x,\varepsilon_0) \\ x(t_0) &= x_0 \\ (t_0, x_0, \varepsilon_0) &\in J \times D \times I \end{aligned}$$

is of class C^m on its domain of definition.

The next theorem ([Ama83], Theorem 10.3) ensures smoothness for the flow of an autonomous system. As a system with parameters can be transformed into an equivalent system without parameters ([Aul04], Theorem 7.1.2), the flow then even depends smoothly on the parameter if the right-hand side does so.

Theorem 6.4. *If $F \in C^m(\mathbb{R}^n, \mathbb{R}^n)$, its corresponding flow is of class C^m on its domain of definition.*

6.1 Euclidean Space Setting

Classical case

For completeness and clarity of presentation we first consider the classical, non-generalized case. The object of our studies is a system of autonomous nonlinear ODEs on $\mathbb{R}^n$ given by

$$\begin{aligned} \dot{x}(t) &= F(x(t)) \\ x(t_0) &= x_0 \\ (t_0, x_0) &\in \mathbb{R}^{1+n} \end{aligned} \tag{6.1}$$

with $x : \mathbb{R} \to \mathbb{R}^n$ and $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Furthermore, we assume that F is linearly bounded, i.e., there exist constants $M, L > 0$ such that

$$\|F(x)\| \leq M + L\|x\| \quad \forall x \in \mathbb{R}^n.$$

Then by Theorems 6.2 and 6.3 there exists a unique globally defined smooth solution of system (6.1) for each choice of (t_0, x_0). Theorem 6.4 assures that the corresponding flow, which is defined globally, also is smooth. These two statements will now be generalized to the Colombeau setting.

Generalized case

If we state system (6.1) with $F = [F_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)^n$, $x_0 = [x_{0\varepsilon}] \in \mathcal{R}_c^n$ and $t_0 \in \mathbb{R}$, on the level of representatives it takes the form

$$\begin{aligned} \dot{x}_\varepsilon(t) &= F_\varepsilon(x_\varepsilon(t)) \\ x_\varepsilon(t_0) &= x_{0\varepsilon} \end{aligned} \tag{6.2}$$

where the solution x should be an element of $\mathcal{G}(\mathbb{R}, \mathbb{R}^n)$. We now present a basic theorem about existence and uniqueness of solutions.

Theorem 6.5. *Let the system given by* (6.2) *satisfy*

(i) $\exists M, L, \varepsilon_0 > 0 : \|F_\varepsilon(x)\| \leq M + L\|x\| \quad \forall x \in \mathbb{R}^n \ \forall \varepsilon < \varepsilon_0$ *and*

(ii) $\forall K \subset\subset \mathbb{R}^n : \sup_{x \in K} \|DF_\varepsilon(x)\| = O(|\log \varepsilon|)$.

Then the initial value problem (6.2) *has a unique solution in* $\mathcal{G}[\mathbb{R}, \mathbb{R}^n]$.

Proof. By Theorems 6.2 and 6.3 there exists a net $x_\varepsilon \in C^\infty(I \times \mathbb{R}, \mathbb{R}^n)$ such that

$$\begin{aligned} \dot{x}_\varepsilon(t) &= F_\varepsilon(x_\varepsilon(t)) \\ x_\varepsilon(t_0) &= x_{0\varepsilon} \end{aligned}$$

for all $\varepsilon \in I$. For x_ε to be in $\mathcal{G}[\mathbb{R}, \mathbb{R}^n]$ we first need to establish c-boundedness. By our assumptions on F_ε we have for all $\varepsilon < \varepsilon_0$ and all $t \in \mathbb{R}$

$$\begin{aligned}\|x_\varepsilon(t)\| &\leq \|x_{0\varepsilon}\| + \Big|\int_{t_0}^{t} \big\|F_\varepsilon(x_\varepsilon(s))\big\| \mathrm{d}s\Big| \\ &\leq \|x_{0\varepsilon}\| + |t - t_0|M + \Big|\int_{t_0}^{t} L\big\|x_\varepsilon(s)\big\| \mathrm{d}s\Big|\end{aligned}$$

and are in the position to apply Gronwall's Lemma to obtain

$$\|x_\varepsilon(t)\| \leq (\|x_{0\varepsilon}\| + |t - t_0|M)e^{|t-t_0|L} \quad \forall \varepsilon < \varepsilon_0$$

which implies c-boundedness of x_ε, as $x_{0\varepsilon}$ is moderate and the remainder of the right hand side is continuous and independent of ε. By assumption (i), $\dot{x}_\varepsilon$ is c-bounded as well. Next we are going to show moderateness of x_ε, which requires

$$\forall K \subset\subset \mathbb{R}\ \forall k \in \mathbb{N}_0\ \exists N \in \mathbb{N} : \sup_{t \in K} \|x_\varepsilon^{(k)}(t)\| = O(\varepsilon^{-N}). \tag{6.3}$$

This condition is clearly satisfied for $k \in \{0, 1\}$ by the statements about c-boundedness. For $k = 2$ we have

$$\|\ddot{x}_\varepsilon(t)\| \leq \|(DF_\varepsilon)(x_\varepsilon(t))\| \cdot \|\dot{x}_\varepsilon(t)\| \tag{6.4}$$

which satisfies (6.3) by moderateness of F and c-boundedness of x_ε and $\dot{x}_\varepsilon$. Derivatives of higher order are estimated inductively by differentiating equation 6.4, giving a term consisting of derivatives of F_ε and x_ε up to order $k - 1$. So x_ε is indeed moderate. We finally need to show that the solution is independent of the choice of representatives of F and x_0. To this end we consider the system

$$\begin{aligned}\dot{y}_\varepsilon(t) &= G_\varepsilon(y_\varepsilon(t)) \\ y_\varepsilon(t_0) &= y_{0\varepsilon}\end{aligned} \tag{6.5}$$

with $G_\varepsilon = F_\varepsilon + N_\varepsilon$, $N_\varepsilon \in \mathcal{N}(\mathbb{R}^n)^n$, $y_{0\varepsilon} = x_{0\varepsilon} + n_\varepsilon$, $n_\varepsilon \in \mathcal{N}$ and with G_ε satisfying (i) and (ii). We will now provide all ingredients necessary for showing negligibility estimates of order zero, i.e.,

$$\forall K \subset\subset \mathbb{R}\ \forall m \in \mathbb{N} : \sup_{t \in K} \|(x_\varepsilon - y_\varepsilon)(t)\| = O(\varepsilon^m).$$

For the following, fix K and m. It is evident that

$$\begin{aligned}(x_\varepsilon - y_\varepsilon)(t) &= x_{0\varepsilon} - y_{0\varepsilon} + \int_{t_0}^{t} \big(F_\varepsilon(x_\varepsilon(s)) - G_\varepsilon(y_\varepsilon(s))\big)\mathrm{d}s \\ &= -n_\varepsilon + \int_{t_0}^{t} \big(F_\varepsilon(x_\varepsilon(s)) - F_\varepsilon(y_\varepsilon(s)) - N_\varepsilon(y_\varepsilon(s))\big)\mathrm{d}s. \qquad (6.6)\end{aligned}$$

As y_ε is c-bounded there exists $C > 0$ such that

$$\sup_{t\in K}\Big|\int_{t_0}^{t} \big\|N_\varepsilon(y_\varepsilon(s))\big\|\mathrm{d}s\Big| \le C\varepsilon^m$$

for small ε. Choosing $K' \subset\subset \mathbb{R}$ such that $x_\varepsilon(s) \in K'$ and $y_\varepsilon(s) \in K'$ for all s with $\min(t_0, \min K) \le s \le \max(t_0, \max K)$ and small ε, we obtain by the mean value theorem for the remainder of the integral in (6.6)

$$\|F_\varepsilon(x_\varepsilon(s)) - F_\varepsilon(y_\varepsilon(s))\| \le \sup_{\sigma\in K'} \|(DF_\varepsilon)(\sigma)\| \cdot \|(x_\varepsilon - y_\varepsilon)(s)\|.$$

By assumption (ii) there exists $C' > 0$ such that

$$\sup_{\sigma\in K'} \|(DF_\varepsilon)(\sigma)\| \le C'|\log\varepsilon|$$

for small ε. Putting all pieces together we can now apply Gronwall's inequality to

$$\|(x_\varepsilon - y_\varepsilon)(s)\| \le \|n_\varepsilon\| + C\varepsilon^m + \Big|\int_{t_0}^{t} C'|\log\varepsilon| \cdot \|(x_\varepsilon - y_\varepsilon)(s)\|\mathrm{d}s\Big|$$

to obtain, for small ε,

$$\|(x_\varepsilon - y_\varepsilon)(s)\| \le (\|n_\varepsilon\| + C\varepsilon^m) \cdot e^{|t-t_0|C'\log\varepsilon}.$$

As m was chosen arbitrarily, $\|(x_\varepsilon - y_\varepsilon)(s)\| = O(\varepsilon^m)$ holds for all $m \in \mathbb{N}$, implying uniqueness of the solution. $\square$

We will now give the corresponding flow theorem for our generalized system.

Theorem 6.6. *For $F \in \mathcal{G}(\mathbb{N}^n)^n$ satisfying (i) and (ii) of Theorem 6.5 there exists a unique generalized function $\Phi \in \mathcal{G}[\mathbb{R}^{n+1}, \mathbb{R}^n]$, the generalized flow of F, such that*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, x) = F(\Phi(t, x))$ *in* $\mathcal{G}[\mathbb{R}^{n+1}, \mathbb{R}^n]$, *(ii)* $\Phi(0,\cdot) = \mathrm{id}_{\mathbb{R}^n}$ *in* $\mathcal{G}[\mathbb{R}^n,\mathbb{R}^n]$*, and*

(iii) $\Phi(t+s,\cdot) = \Phi(t,\Phi(s,\cdot))$ *in* $\mathcal{G}[\mathbb{R}^{2+n},\mathbb{R}^n]$.

Proof. From classical theory (Theorem 6.4) we infer the existence of a globally defined smooth flow Φ^ε satisfying (i)-(iii) on the level of representatives. For establishing c-boundedness of Φ^ε we consider

$$\begin{aligned}\|\Phi^\varepsilon(t,x)\| &\le \|\Phi^\varepsilon(0,x)\| + \left|\int_0^t \big\|F_\varepsilon(\Phi^\varepsilon(s,x))\big\| \mathrm{d}s\right| \\ &\le \|x\| + tM + \left|\int_0^t L\|\Phi^\varepsilon(s,x)\|\mathrm{d}s\right|\end{aligned}$$

and apply the Gronwall inequality to get

$$\|\Phi^\varepsilon(t,x)\| \le (\|x\| + |t|M)\cdot e^{|t|L}.$$

Then, $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^\varepsilon$ is c-bounded as well because of

$$\left\|\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Phi^\varepsilon\Big)(t,x)\right\| = \|F_\varepsilon(\Phi^\varepsilon(t,x))\| \le M + L\|\Phi^\varepsilon(t,x)\|. \tag{6.7}$$

Higher order derivatives of Φ^ε with respect to t are estimated inductively, as $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Phi^\varepsilon$ consists of derivatives of F_ε and Φ^ε up to order $k-1$.
For moderateness of Φ^ε we need to show that

$$\forall K = K_1\times K_2 \subset\subset \mathbb{R}\times\mathbb{R}^n\ \forall k\in\mathbb{N}_0\ \exists N\in\mathbb{N}: \qquad \sup_{(t,x)\in K}\|(D^{(k)}\Phi^\varepsilon)(t,x)\| = O(\varepsilon^{-N}).$$

Starting with

$$\Phi^\varepsilon(t,x) = \Phi^\varepsilon(0,x) + \int_0^t F_\varepsilon(\Phi^\varepsilon(s,x))\mathrm{d}s$$

we form the partial derivative with respect to x,

$$(D_x\Phi^\varepsilon)(t,x) = \mathrm{id}_{\mathbb{R}^n} + \int_0^t (DF_\varepsilon)(\Phi^\varepsilon(s,x))\cdot(D_x\Phi^\varepsilon)(s,x)\mathrm{d}s. \tag{6.8}$$

By c-boundedness of Φ^ε there exist $K'\subset\subset\mathbb{R}^n$ containing $\Phi^\varepsilon([0,t]\times K_2)$ for all $t\in K_1$ and $C>0$ with $\sup_{x\in K'}\|(DF_\varepsilon)(x)\| \le C|\log\varepsilon|$, both for small ε. We can now apply Gronwall's inequality to

$$\|(D_x\Phi^\varepsilon)(t,x)\| \le 1 + \left|\int_0^t C|\log\varepsilon|\cdot\|(D_x\Phi^\varepsilon)(s,x)\|\mathrm{d}s\right| \quad \forall (t,x)\in K$$

to obtain

$$\|(D_x\Phi^\varepsilon)(t,x)\| \le \varepsilon^{-|t|C} \quad \forall (t,x) \in K \text{ and small } \varepsilon$$

from which the moderateness estimate is obvious.

Estimates for higher order derivatives of Φ^ε with respect to x are assured in the same manner by differentiating equation (6.8), employing moderateness of F_ε, assumption (ii) and the estimates already established for lower orders.

Derivatives with respect to t are obviously moderate for orders zero and one by c-boundedness of Φ^ε and $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^\varepsilon$. Higher orders are obtained inductively by differentiating equation (6.7).

Mixed derivatives follow suit, first differentiating equation (6.8) with respect to x and then with respect to t as needed.

In order to substantiate uniqueness, assume that $\Psi \in \mathcal{G}[\mathbb{R}^{n+1}, \mathbb{R}]$ is another generalized function satisfying (i)-(iii). As for any t_0 in $\mathbb{R}$ and $x_0 = [x_{0\varepsilon}]$ in $\mathcal{R}_c^n$ the functions $x_\varepsilon(t) := \Phi^\varepsilon(t, x_0)$ and $y_\varepsilon(t) := \Psi^\varepsilon(t, x_0)$ solve the initial value problem (6.2), $x_\varepsilon \sim y_\varepsilon$ follows from Theorem 6.5. Therefore, the point value characterization of Proposition 4.9 first gives $\Phi(t,x) = \Psi(t,x)$ $\forall (t,x) \in \mathcal{R}_c^{1+n}$ and then $\Phi = \Psi$ in $\mathcal{G}[\mathbb{R}^{n+1}, \mathbb{R}^n]$. As properties (i)-(iii) hold on the level of representatives, we are finished. $\square$

6.2 Manifold Setting

Classical case

The notion of an autonomous ODE system as in (6.1) can be transferred to a manifold. When x is a mapping from $\mathbb{R}$ into X, the derivative $\dot{x}(t)$ is a tangent vector in $T_{x(t)}X$ and the right hand side has to be a vector field on X. We thus get the system

$$\begin{aligned} \dot{x}(t) &= \xi(x(t)) \\ x(t_0) &= x_0 \\ (t_0, x_0) &\in (\mathbb{R} \times X) \end{aligned} \tag{6.9}$$

with $\xi \in \mathfrak{X}(X)$ and $x : \mathbb{R} \to X$. If ξ is complete there exists a globally defined smooth solution of system (6.9) for each choice of the initial condition. Furthermore, there is a globally defined smooth flow ([Lan95], Chapter IV, Theorem 2.6).

Generalized case

If for the left-hand side we consider a generalized function $x \in \mathcal{G}[\mathbb{R}, X]$, the derivative $\dot{x}(t)$ is an element of $\mathfrak{X}_{\mathcal{G}}(x) = \{v \in \mathcal{G}^h[\mathbb{R}, TX] \mid \pi_X \circ v = x\}$. For a a generalized vector field $\xi \in \mathfrak{X}_{\mathcal{G}}(X)$, on the right-hand side the composition $\xi \circ x$ is an element of $\mathfrak{X}_{\mathcal{G}}(x)$ by virtue of Theorem 5.6. This allows us to formulate a generalized differential equation of first order on a manifold, for which the next theorem ensures the existence of a solution. But first we need to introduce a way of describing certain properties of vector fields on a manifold.

Definition 6.7. Let $\xi \in \mathfrak{X}_{\mathcal{G}}(X)$ and h be a Riemannian metric on X. Then we call ξ

(i) locally bounded if for all $K \subset\subset X$ there exists for one (hence every) representative ξ_ε of ξ a constant $C > 0$ such that

$$\sup_{p \in K} \|\xi_\varepsilon(p)\|_h \leq C \quad \forall \varepsilon \in I,$$

(ii) locally of L^∞-log-type if for all $K \subset\subset X$ there exists for one (hence every) representative ξ_ε of ξ a constant $C > 0$ such that

$$\sup_{p \in K} \|\xi_\varepsilon(p)\|_h \leq C|\log \varepsilon| \quad \forall \varepsilon \in I,$$

(iii) globally bounded with respect to h if for one (hence every) representative ξ_ε of ξ there exists a constant $C > 0$ such that

$$\sup_{p \in X} \|\xi_\varepsilon(p)\|_h \leq C \quad \forall \varepsilon \in I.$$

If (i) or (ii) is satisfied for a Riemannian metric, then by Lemma 3.9 it also holds for any other Riemannian metric on X.

Theorem 6.8. *Consider the differential equation system*

$$\begin{aligned} \dot{x}(t) &= \xi(x(t)) \\ x(t_0) &= x_0 \\ (t_0, x_0) &\in \mathbb{R} \times \widetilde{X}_c \end{aligned} \tag{6.10}$$

with $\xi \in \mathfrak{X}_{\mathcal{G}}(X)$. Furthermore, let ξ be globally bounded with respect to a complete Riemannian metric h on TX. Then there exists a unique solution $x \in \mathcal{G}[\mathbb{R}, X]$ of system (6.10).

Proof. For each $\varepsilon \in I$, on the level of representatives we have, because of completeness of h and the assumption on ξ, the existence of a global solution $x_\varepsilon \in C^\infty(\mathbb{R} \times I, X)$ of

$$\begin{aligned} \dot{x}_\varepsilon(t) &= \xi_\varepsilon(x_\varepsilon(t)) \\ x_\varepsilon(t_0) &= x_{0\varepsilon}. \end{aligned}$$

For c-boundedness of x_ε we will show that for all $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and ε_0 small enough the set $\bigcup_{\varepsilon<\varepsilon_0} x_\varepsilon([t_1, t_2])$ is compact. In fact we can write

$$L(x_\varepsilon|_{[t_1,t_2]}) = \int_{t_1}^{t_2} \|\dot{x}_\varepsilon(s)\|_h \mathrm{d}s = \int_{t_1}^{t_2} \|\xi_\varepsilon(x_\varepsilon(s))\|_h \mathrm{d}s \le C|t_2 - t_1| \qquad (6.11)$$

by assumption and thus, choosing $K \subset\subset X$ and ε_0 such that $x_\varepsilon(t_1) \in K$ for all $\varepsilon < \varepsilon_0$, we obtain

$$\bigcup_{\varepsilon<\varepsilon_0} x_\varepsilon([t_1, t_2]) \subseteq \big\{p \in X \mid d_h(p, K) \le C|t_2 - t_1|\big\}.$$

By the Hopf-Rinow theorem the latter set is compact, so x_ε is c-bounded. Concerning moderateness we need to estimate $D^{(k)}(\varphi \circ x_\varepsilon)(t)$ for some chart (U, φ). For $k = 0$ the estimates are an obvious consequence of c-boundedness of x_ε; for $k \ge 1$ the equality

$$D^{(k)}(\varphi \circ x_\varepsilon)(t) = D^{(k-1)}(T\varphi \circ \xi_\varepsilon \circ x_\varepsilon)(t) \quad \forall k \ge 1$$

together with c-boundedness of x_ε and moderateness of ξ_ε entails moderateness of x_ε.

In order to establish uniqueness of the solution, choose $a > 0$, $K \subset\subset X$ and $\varepsilon_0 > 0$ such that $x_\varepsilon([-a-1, a+1]) \cup y_\varepsilon([-a-1, a+1]) \subseteq K$ for all $\varepsilon < \varepsilon_0$. We take different representatives in system (6.10), i.e.,

$$\begin{aligned} \dot{y}(t) &= \eta(x(t)) \\ y(t_0) &= y_0 \end{aligned}$$

with $\xi \sim \eta$ in $\mathfrak{X}_\mathcal{G}(X)$ and $x_0 \sim y_0$ in $\widetilde{X}_c$. Let $t_0 \in (-a, a)$. By [Aub82], Theorem 1.36, there exists $r > 0$ such that we can cover K by finitely many metric balls $B_r(p_i)$, $p_i \in K$ with $B_{4r}(p_i)$ a geodesically convex domain for the chart $\psi_i := \exp_{p_i}^{-1}$. Then we choose $\varepsilon_1 < \varepsilon_0$ such that

$$d_h\big(x_\varepsilon(t_0), y_\varepsilon(t_0)\big) = d_h(x_{0\varepsilon}, y_{0\varepsilon}) < r \quad \forall \varepsilon < \varepsilon_1.$$

With C being the constant of equation 6.11 we choose $0 < d < r/C$. For each $\varepsilon < \varepsilon_1$ there exists an i such that $x_{0\varepsilon} \in B_r(p_i)$ and therefore

$y_{0\varepsilon} \in B_{2r}(p_i)$. Then for each t with $|t - t_0| < d$ the entire line connecting $x_\varepsilon(t)$ and $y_\varepsilon(t)$ is contained in $B_{3r}(p_i)$ and by convexity of the charts, the line connecting $\psi_i(x_\varepsilon(t))$ and $\psi_i(y_\varepsilon(t))$ is contained in $\psi_i(B_{3r}(p_i))$. Applying a Gronwall argument as in 6.5 then shows that there exist $\varepsilon_2 < \varepsilon_1$ and $C' > 0$ such that

$$\|\psi_i \circ x_\varepsilon(t) - \psi_i \circ y_\varepsilon(t)\| \le C'\varepsilon^m \quad \forall \varepsilon < \varepsilon_2$$

where ε_2 and C' depend only on K, $x_{0\varepsilon}$, $y_{0\varepsilon}$ and ψ_i and thus can be chosen uniformly for $i \in \{1, \dots, k\}$ and $t \in [t_0 - d, t_0 + d]$. This gives

$$\sup_{t \in [t_0 - d, t_0 + d]} d_h\big(x_\varepsilon(t), y_\varepsilon(t)\big) \le C''\varepsilon^m \quad \forall \varepsilon < \varepsilon_2$$

which implies $x_\varepsilon \sim y_\varepsilon$ on $(t_0 - d, t_0 + d)$. It follows that if x and y coincide at any point of $(-a, a)$ then they agree on the whole interval. As a was arbitrary, x and y agree globally. □

Theorem 6.9. *In system* (6.10)*, let ξ_ε be globally bounded with respect to a complete Riemannian metric h on TX and such that for each differential operator $P \in \mathcal{P}(X, TX)$ of first order $P\xi$ is locally of L^∞-log-type. Then there exists a globally defined function $\Phi \in \mathcal{G}[\mathbb{R} \times X, X]$, the generalized flow of ξ, satisfying*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, x) = \xi(\Phi(t, x))$ *in* $\mathcal{G}^h[\mathbb{R} \times X, TX]$,

(ii) $\Phi(0, \cdot) = \mathrm{id}_X$ *in* $\mathcal{G}[X, X]$, *and*

(iii) $\Phi(s + t, \cdot) = \Phi(s, \Phi(t, \cdot))$ *in* $\mathcal{G}[\mathbb{R}^2 \times X, TX]$.

Proof. For c-boundedness, let $K_1 \times K_2 \subset\subset \mathbb{R} \times X$, where $K_1 \subseteq [t_1, t_2]$ with $t_1, t_2 \in \mathbb{R}$. Then the assumption that ξ is globally bounded tells us

$$L\big(\Phi^\varepsilon([t_1, t_2], x)\big) \le C|t_2 - t_1| \quad \forall x \in X, \tag{6.12}$$

so $\Phi^\varepsilon(K_1 \times K_2)$ is a subset of $\{p \in X \mid d_h(p, K) \le C|t_2 - t_1|\}$, which is compact by the Hopf-Rinow theorem.

Moderateness of Φ^ε requires that for all charts (U, φ) and (V, ψ) in X, $L \subset\subset \mathbb{R} \times U$ and $L' \subset\subset V$ there exists an $N \in \mathbb{N}$ such that

$$\sup_{(t,p) \in L \cap (\Phi^\varepsilon)^{-1}(L')} \big\| D^{(k)}\big(\psi \circ \Phi^\varepsilon \circ (\mathrm{id} \times \varphi^{-1})\big)\big(t, \varphi(p)\big)\big\| = O(\varepsilon^{-N}). \tag{6.13}$$

We may assume without loss of generality that $L = T \times K$ with $T := [0, a]$, $a \in \mathbb{R}_+$ and $K \subset\subset U$. By c-boundedness of Φ^ε there exists a compact subset of X containing $\Phi^\varepsilon(L)$ for all small ε, which by Lemma 1.1 can be written as the union of compact sets K_j $(1 \le j \le n \in \mathbb{R})$ with $K_j \subset\subset U_j$ where the (U_j, φ_j) are charts taken from an atlas of X. Then we choose open neighborhoods U'_j of K_j such that $\overline{U'_j} \subset\subset U_j$, which implies that $d_j := d_h(K_j, \partial U'_j)$ is positive.

Now take $p \in K$ contained (after a possible renumbering of the K_j) in K_1. As $t \in T$ grows, we can follow the integral curve $\Phi^\varepsilon(t,p)$ through one or more of the U'_j. As soon as it leaves U'_j, it will have travelled at least the distance d_j and will enter some K_l, so that we may repeat the above procedure with K_l and U'_l. Because of inequality (6.12), the length of each such integral curve is bounded by aC. This means that we can write the time interval as $T = \bigcup_{i=1}^n [t_{i-1}, t_i]$ with $n \in \mathbb{N}$, $t_0 = 0$ and $t_n = a$ such that $\Phi^\varepsilon([t_{i-1}, t_i], p) \subset\subset U_i$, where n is by construction necessarily smaller than $n_{\max} := aC/\min_j d_j + 1$.

For shorter notation we set

$$\begin{aligned} &T_i = [t_{i-1}, t_i],\\ &f_i = \varphi_i \circ \Phi^\varepsilon \circ (\mathrm{id} \times \varphi_1^{-1}), \text{ and}\\ &g_i = T\varphi_i \circ \xi_\varepsilon \circ \varphi_i^{-1} \end{aligned}$$

and note that $\frac{\mathrm{d}}{\mathrm{d}t} f_i = g_i \circ f_i$.

We will establish the moderateness estimate for the first derivative with respect to x; higher x-derivatives and mixed x,t-derivatives are obtained by differentiating the equations and following the same process. We set $x := \varphi_1(p)$ and begin with

$$f_1(t,x) = f_1(0,x) + \int_0^t f'_1(s,x)\mathrm{d}s = x + \int_0^t g_1\big(f_1(s,x)\big)\mathrm{d}s \quad \forall t \in T_1$$

and differentiate with respect to x to get

$$D_x f_1(t,x) = \mathrm{id} + \int_0^t Dg_1\big(f_1(s,x)\big) \cdot D_x f_1(s,x)\mathrm{d}s. \tag{6.14}$$

We now need to estimate $\|Dg_1(f_{11}(s,x))\|$. For this we define a differential operator P of first order on each K_j by

$$\begin{gathered} T\varphi_j \circ (Pf) \circ \varphi_j^{-1} = D(T\varphi_j \circ f \circ \varphi_j^{-1}) \text{ or}\\ Pf = T\varphi_j^{-1} \circ D(T\varphi_j \circ f \circ \varphi_j^{-1}) \circ \varphi_j \quad \forall f \in \Gamma(K_j, TK_j) \end{gathered}$$

and because $P\xi$ is locally of L^∞-log-type we obtain

$$\sup_{p \in K_j} \|D(T\varphi_j \circ \xi_\varepsilon|_{K_j} \circ \varphi_j^{-1})(\varphi_j(p))\| \le C'_j \sup_{p \in K_j} \|(P\xi_\varepsilon|_{K_j})(p)\| \le C_j |\log\varepsilon|$$

with some constants $C'_j, C_j > 0$. As each $(s,p) \in L$ gets mapped into a certain K_j by Φ^ε, there exists a constant C such that

$$\left\| D(T\varphi_j \circ \xi_\varepsilon \circ \varphi_j^{-1})\big(\varphi_j \circ \Phi^\varepsilon \circ (\mathrm{id} \times \varphi_i^{-1})(s, \varphi_i(p))\big) \right\| \leq C|\log \varepsilon|$$

uniformly for each choice of $(s,p) \in L$ and corresponding j and i. So this gives us

$$\left\| Dg_j\big(f_j(s,x)\big) \right\| \leq C|\log \varepsilon| \quad \forall s \in T.$$

Taking the norm in equation (6.14) we can apply the Gronwall inequality on

$$\|D_x f_1(t,x)\| \leq 1 + \int_0^t C|\log \varepsilon| \cdot \|D_x f_1(s,x)\| \mathrm{d}s \quad \forall t \in T_1$$

and obtain

$$\|D_x f_1(t,x)\| \leq \varepsilon^{-Ct} \quad \forall t \in T_1 \ \forall \varepsilon < \varepsilon_0.$$

Note that this estimate does not depend on the starting point x, but only on t.

Now we demonstrate the inductive step which will incorporate the change to the next chart and continue the Gronwall estimate. The inductive assumption for i is

$$\|D_x f_i(t,x)\| \leq C_i \varepsilon^{-N_i t} \quad \forall \varepsilon < \varepsilon_i \ \forall t \in T_i$$

with $C_1 := 1$ and $N_1 := C$. We continue with

$$\begin{aligned} f_{i+1}(t_i + t, x) &= f_{i+1}(t_i, x) + \int_0^t f'_{i+1}(t_i + s, x) \mathrm{d}s \\ &= (\varphi_{i+1} \circ \varphi_i^{-1}) \circ f_i(t_i, x) + \int_0^t g_{i+1}\big(f_{i+1}(t_i + s, x)\big) \mathrm{d}s \end{aligned}$$

and, differentiating,

$$\begin{aligned} D_x f_{i+1}(t_i + t, x) = {}& D(\varphi_{i+1} \circ \varphi_i^{-1})\big(f_i(t_i, x)\big) \cdot D_x f_i(t_i, x) \\ &+ \int_0^t Dg_{i+1}\big(f_{i+1}(t_i + s, x)\big) \cdot D_x f_{i+1}(t_i + s, x) \mathrm{d}s \end{aligned} \tag{6.15}$$

for all $t \in [0, t_{i+1} - t_i]$. There is a uniform constant E such that

$$\|D(\varphi_{i+1} \circ \varphi_i^{-1})(f_i(t,x))\| \leq E$$

for all (t, x) appearing in the estimate. We then take the norm in equation (6.15) and achieve

$$\|D_x f_{i+1}(t_i + t, x)\| \leq EC_i \varepsilon^{-N_i t_i} + \int_0^t C|\log \varepsilon| \cdot \|D_x f_{i+1}(t_i + s, x)\| \mathrm{d}s$$

for all $t \in [0, t_{i+1} - t_i]$. Applying the Gronwall inequality again gives

$$\|D_x f_{i+1}(t_i + t, x)\| \leq EC_i \varepsilon^{-N_i t_i} \varepsilon^{-Ct} \quad \forall t \in [0, t_{i+1} - t_i]$$

from which we derive $C_{i+1} := EC_i = E^i$ and $N_{i+1} = N_i = C$ $(i > 1)$. We finally obtain estimate (6.13) from

$$\|D_x f_n(t, \varphi(p))\| \leq E^{n-1} \varepsilon^{-Ct} \leq E^{n_{\max}-1} \varepsilon^{-Ct} \quad \forall (t, p) \in L.$$

Uniqueness of the flow follows as in Theorem 6.6. □

Chapter 7

Generalized Pseudo-Riemannian Geometry

We shall recall some statements of classical pseudo-Riemannian Geometry (see [O'N83], Chapter 3).

(G1) A symmetric bilinear form is non-degenerate if and only if its matrix relative to one (hence every) basis is invertible.

(G2) The *index* of a symmetric bilinear form b on a vector space V is the largest integer that is the dimension of a subspace $W \subseteq V$ on which $b|_W$ is negative definite.

(G3) A metric tensor g on a smooth manifold M is a symmetric non-degenerate (0,2) tensor field on M of constant index.

(G4) A pseudo-Riemannian manifold is a smooth manifold M furnished with a metric tensor g. If the index of g is zero, we speak of a Riemannian manifold.

We will now step by step establish corresponding statements in the context of generalized functions. The object of our interest will naturally be a generalized (0,2) tensor field $\hat{g} \in \mathcal{G}^0_2(X)$.

Non-degeneracy

A tensor field $g \in \mathcal{T}^0_2(X)$ is by definition non-degenerate if the function $g(x) : T_xX \times T_xX \to \mathbb{R}$ is non-degenerate in every point $x \in X$. With statement (G1) this means that for each chart $(U_\alpha, \varphi_\alpha)$ and each $x \in \varphi_\alpha(U_\alpha)$ the matrix consisting of the coordinates of the tensor $g_\alpha(x) \in T^0_2(\mathbb{R}^n)$ is invertible. Now by Remark 3.26, $\hat{g}_\alpha(\tilde{x})$ is a map from $\mathcal{K}^n \times \mathcal{K}^n$ into $\mathcal{K}$ for each $\tilde{x} \in \varphi_\alpha(U_\alpha)^\sim_c$. As non-degeneracy is well-defined for such maps (Lemma 2.3), we can thus define non-degeneracy for generalized (0,2) tensor fields in the next theorem, which will also establish a connection to classical pseudo-Riemannian metrics.

Theorem 7.1. *Let $\hat{g} \in \mathcal{G}^0_2(X)$. Then the following statements are equivalent.*

(i) For each chart $(U_\alpha, \varphi_\alpha)$ and each choice of $x \in (\varphi_\alpha(U_\alpha))^\sim_c$ the mapping $\hat{g}_\alpha(x) : \mathcal{K}^n \times \mathcal{K}^n \to \mathcal{K}$ is symmetric and non-degenerate.

(ii) For each chart $(U_\alpha, \varphi_\alpha)$, the map $\hat{g} : \mathfrak{X}_\mathcal{G}(X) \times \mathfrak{X}_\mathcal{G}(X) \to \mathcal{G}(X)$ is symmetric and $\det \hat{g}_\alpha$ *is invertible in $\mathcal{G}(\varphi_\alpha(U_\alpha))$.*

(iii) For each chart $(U_\alpha, \varphi_\alpha)$, $\det \hat{g}_\alpha$ *is invertible in $\mathcal{G}(\varphi_\alpha(U_\alpha))$ and for each relatively compact open set $V \subseteq X$ there exist a representative $\hat{g}_\varepsilon$ of $\hat{g}$ and $\varepsilon_0 > 0$ such that $\hat{g}_\varepsilon|_V$ is a smooth pseudo-Riemannian metric for all $\varepsilon < \varepsilon_0$.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 2.3 and Proposition 2.4, $\det \hat{g}_\alpha$ is invertible. For symmetry of $\hat{g} \in L_\mathcal{G}(\mathfrak{X}_\mathcal{G}(X)^2; \mathcal{G}(X))$ note that this map is given by the assignment $(\xi, \eta) \mapsto [g_\varepsilon(\xi_\varepsilon, \eta_\varepsilon)]$ (cf. the Proof of Theorem 3.24). By Proposition 3.7 we can identify $\hat{g}(\xi, \eta)$ with the family of local expressions given by

$$\hat{g}(\xi, \eta)_\alpha = [g_\varepsilon(\xi_\varepsilon, \eta_\varepsilon) \circ \psi_\alpha^{-1}] \in \mathcal{G}(\varphi_\alpha(U_\alpha))$$

whence by inserting a point $x \in \varphi_\alpha(U_\alpha)^\sim_c$ and using local coordinates and Proposition 3.27 we obtain

$$\begin{aligned} \hat{g}(\xi, \eta)_\alpha(x) =& \hat{g}_\alpha(x)\big(\xi(x), \eta(x)\big) \\ =& \hat{g}_\alpha(x)\big(\eta(x), \xi(x)\big) = \hat{g}(\eta, \xi)_\alpha(x). \end{aligned}$$

Therefore $\hat{g}(\xi, \eta) = \hat{g}(\eta, \xi)$ holds in $\mathcal{G}(X)$ for every choice of ξ and η and $\hat{g}$ is symmetric.

(ii) $\Rightarrow$ (iii): As $\hat{g}$ is symmetric we have

$$\hat{g}(\xi, \eta) = \frac{1}{2}\big(\hat{g}(\xi, \eta) + \hat{g}(\eta, \xi)\big).$$

If for a representative $\hat{f}_\varepsilon$ of $\hat{g}$ we denote the mapping $(\xi, \eta) \mapsto \hat{f}_\varepsilon(\eta_\varepsilon, \xi_\varepsilon)$ by $\check{f}_\varepsilon$, we can write

$$\hat{g}(\xi, \eta) = \big[1/2(\hat{f}_\varepsilon + \check{f}_\varepsilon)(\xi_\varepsilon, \eta_\varepsilon)\big] \text{ in } L_{\mathcal{G}}(\mathfrak{X}_{\mathcal{G}}(X)^2; \mathcal{G}(X))$$

and therefore $\hat{g}_\varepsilon := 1/2(\hat{f}_\varepsilon + \check{f}_\varepsilon)$ is a representative of $\hat{g}$ such that each $g_\varepsilon : \mathfrak{X}(X) \times \mathfrak{X}(X) \to C^\infty(X)$ is symmetric. As $\overline{V}$ is compact in X we can write it as the union of compact sets each of which is contained in the domain of one of finitely many charts. For each of these compact sets – denote one by K – there exists by Theorem 2.5 an $\varepsilon_0 > 0$ and $m \in \mathbb{N}$ such that $\inf_{p \in K} |\det(\hat{g}_\varepsilon(p))| > \varepsilon^m$ for $\varepsilon < \varepsilon_0$, so each $\hat{g}_\varepsilon$ is non-degenerate and therefore a pseudo-Riemannian metric on K for such ε. We conclude that $\hat{g}_\varepsilon$ restricted to V also is a pseudo-Riemannian metric.

(iii) $\Rightarrow$ (i): Let $x \in \varphi_\alpha(U_\alpha))^\sim_c$ be supported in $K \subset\subset \varphi_\alpha(U_\alpha)$ and $\hat{g}_\varepsilon$ be a representative of $\hat{g}$ such that each $\hat{g}_\varepsilon$ is a pseudo-Riemannian metric on a neighborhood of $\varphi_\alpha^{-1}(K)$ for small ε. Symmetry of $\hat{g}_\alpha(x)$ then follows directly from symmetry of $\hat{g}_\varepsilon$ and non-degeneracy is a consequence of Lemma 2.3. □

We thus have a notion of non-degeneracy for generalized (0,2) tensor fields and can state a 'generalized' version of statement (G1).

The Index

Definition 7.2. A generalized (0,2)-tensor field $\hat{g}$ is called non-degenerate if it satisfies one of the equivalent conditions in Theorem 7.1.

Concerning a generalization of (G2), somehow incorporating the index of the representatives in Theorem 7.1 (iii) apparently is the most straightforward approach. Let $\hat{g}$ be a non-degenerate (0,2) tensor field, V a relatively compact subset of X and $\hat{g}_\varepsilon$ a representative of $\hat{g}$ as in Theorem 7.1 (iii). We require that the index of $\hat{g}_\varepsilon$ is the same for all (small) ε and all choices of V, but we also need independence of the specific representative of $\hat{g}$ in order to properly define an index for $\hat{g}$. So fix V for the moment and let $\tilde{g}_\varepsilon$ be another representative of $\hat{g}$ as in Theorem 7.1 (iii). We need to show that for each ε the index of $\hat{g}_\varepsilon|_V$ equals the index of $\tilde{g}_\varepsilon|_V$.

As a side-note, it is not surprising that while in the classical case the index can be determined pointwise, in the generalized case we have to examine a relatively compact subset of the manifold.

In order to incorporate invertibility of $\det \hat{g}_\alpha$ we split $\overline{V}$ into finitely many sets compactly contained in some charts $(U_\alpha, \varphi_\alpha)$. Let $K \subset\subset U_\alpha$ be one of these sets. Denote by $\hat{\lambda}^1_\varepsilon(x) \geq \cdots \geq \hat{\lambda}^n_\varepsilon(x)$ for $x \in \varphi_\alpha(K)$ the eigenvalues of $(\hat{g}_\alpha)_\varepsilon(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Because the operator norm of a matrix is an upper bound for its absolute eigenvalues, all $\hat{\lambda}^i_\varepsilon$ are moderate of order zero, i.e., grow like $O(\varepsilon^{-N})$ for some N on compact subsets of $\varphi_\alpha(K)$. Because $\det \hat{g}_\alpha$ is invertible in $\mathcal{G}(\varphi_\alpha(K))$ it satisfies

$$\inf_{x \in L} |\det(\hat{g}_\alpha)_\varepsilon(x)| = \inf_{x \in L} |\hat{\lambda}^1_\varepsilon(x) \cdots \hat{\lambda}^n_\varepsilon(x)| \geq \varepsilon^k \tag{7.1}$$

for each $L \subset\subset \psi_\alpha(K)$ and some $k \in \mathbb{N}$. Therefore there has to exist an $m \in \mathbb{N}$ such that $\inf_{x \in L} |\hat{\lambda}^i_\varepsilon(x)| \geq \varepsilon^m$ for all small ε and $1 \leq i \leq n$, as otherwise (7.1) would not hold. Denoting the eigenvalues of $(\tilde{g}_\alpha)_\varepsilon$ by $\tilde{\lambda}^1_\varepsilon \geq \ldots \geq \tilde{\lambda}^n_\varepsilon$, we can estimate

$$\max_i |\tilde{\lambda}^i_\varepsilon - \hat{\lambda}^i_\varepsilon| \leq \|(\tilde{g}_\alpha)_\varepsilon - (\hat{g}_\alpha)_\varepsilon\| = O(\varepsilon^m) \quad \forall m \in \mathbb{N}.$$

Therefore each $\tilde{\lambda}^i_\varepsilon$ has the same sign as $\hat{\lambda}^i_\varepsilon$ for small ε. This enables us to unambiguously define the index for symmetric non-degenerate generalized (0,2) tensor fields.

Definition 7.3. Let $\hat{g} \in \mathcal{G}^0_2(X)$ be non-degenerate. If there exists some $j \in \mathbb{N}_0$ with the property that for each relatively compact open subset V of X there exists a representative $\hat{g}_\varepsilon$ of $\hat{g}$ as in Theorem 7.1 (iii) such that the index of each $\hat{g}_\varepsilon|_V$ equals j, we call j the index of $\hat{g}$.

With this result we may state the generalized counterparts of (G3) and (G4).

Definition 7.4. A generalized metric tensor on a manifold X is a symmetric non-degenerate generalized (0,2) tensor field possessing an index.

Definition 7.5. A pseudo-Riemannian manifold $(X, \hat{g})$ is a manifold X furnished with a metric tensor $\hat{g}$. If the index of $\hat{g}$ is zero, $(X, \hat{g})$ is called a generalized Riemannian manifold.

We will also write $\langle \xi, \eta \rangle$ in place of $\hat{g}(\xi, \eta)$.

Inverse Metric

Proposition 7.6. *Let $(X, \hat{g})$ be a generalized pseudo-Riemannian manifold and $\hat{g}_\varepsilon$ a representative of $\hat{g}$. Then the inverse metric $\hat{g}^{-1} := [\hat{g}_\varepsilon^{-1}]$ is a well-defined element of $\mathcal{G}^2_0(X)$.*

Proof. Cover X by open sets W_α such that $\overline{W}_\alpha \subset\subset U_\alpha$ where $(U_\alpha, \varphi_\alpha)$ are charts on X. As $\det \hat{g}_\alpha$ is invertible in $\mathcal{G}(\varphi_\alpha(W_\alpha))$, Theorem 2.5 implies the existence of $\varepsilon_0 > 0$ and $m \in \mathbb{N}$ such that

$$\inf_{x\in\overline{W}_\alpha} |\det(\hat{g}_\varepsilon)_{ij}(x)| = \inf_{x\in\overline{W}_\alpha} |\det(\hat{g}_\varepsilon)_\alpha(\psi_\alpha(x))| = \\ \inf_{x\in\overline{W}_\alpha} |\det(\hat{g}_\alpha)_\varepsilon(\psi_\alpha(x))| = \inf_{x\in\psi_\alpha(\overline{W}_\alpha)} |(\det \hat{g}_\alpha)_\varepsilon(x)| \geq \varepsilon^m \quad \forall\varepsilon < \varepsilon_0$$

and therefore $(\hat{g}_\varepsilon)_{ij}(x)$ is invertible for all $x \in \overline{W}_\alpha$ and $\varepsilon < \varepsilon_0$. In turn we obtain the inverse matrix $(\hat{g}_\varepsilon)^{ij}(x)$ for all such x and ε. We can now define a tensor $(\hat{g}_\varepsilon^{-1})_\alpha \in \Gamma(W_\alpha, T_0^2(W_\alpha))$ in the obvious way by

$$(\hat{g}_\varepsilon^{-1})_\alpha(x)(a_i dx_i, b_j dx_j) := (\hat{g}_\varepsilon)^{ij}(x) a_i b_j \quad \forall x \in W_\alpha \ \forall\varepsilon < \varepsilon_0.$$

This net is moderate on W_α by the cofactor formula of matrix inversion,

$$(\hat{g}_\varepsilon)^{ij} = \frac{\operatorname{cof}(\hat{g}_\varepsilon)_{ij}}{\det(\hat{g}_\varepsilon)_{ij}},$$

and the $(\hat{g}_\varepsilon^{-1})_\alpha$ form a coherent family of sections which, by Proposition 3.22, defines a unique element $\hat{g}^{-1} \in \mathcal{G}_0^2(X)$. By the same reasoning, this definition is independent of the specific representative $\hat{g}_\varepsilon$ of $\hat{g}$. $\square$

From now on we denote the components of the inverse metric by $\hat{g}^{ij}$. The following Lemma will prove to be useful for certain calculations.

Lemma 7.7. *On a generalized pseudo-Riemannian manifold $(X, \hat{g})$, let $\hat{g}_\varepsilon$ be a representative of $\hat{g}$. Then for every particular representative $\hat{g}_\varepsilon^{-1}$ of the inverse metric $\hat{g}^{-1}$ there exists an element $\hat{n}_\varepsilon \in \Gamma_\mathcal{N}(X, T_0^2(X)$ such that $(\hat{g}_\varepsilon^{-1})^{ij} = ((\hat{g}_\varepsilon)_{ij})^{-1} + (\hat{n}_\varepsilon)^{ij}$.*

Proof. The statement follows at once from the cofactor formula of matrix inversion above. $\square$

Remark 7.8. The previous Lemma in fact assures that in component-wise calculation with a generalized metric and its inverse we need not worry too much about different representatives and calculate as habitual in classical theory - the error will be negligible. The proof of the next Proposition will give an example of this.

Proposition 7.9. *Let $(X, \hat{g})$ be a generalized pseudo-Riemannian manifold. Then the following statements hold.*

(i) If for some $\xi \in \mathfrak{X}_{\mathcal{G}}(X)$ the equality $\hat{g}(\xi, \eta) = 0$ holds for all $\eta \in \mathfrak{X}_{\mathcal{G}}(X)$ then $\xi = 0$ follows.

(ii) The mapping $\xi \mapsto \hat{g}(\xi, \cdot)$ is a $\mathcal{G}(X)$-linear isomorphism from $\mathfrak{X}_{\mathcal{G}}(X)$ into $\mathfrak{X}^*_{\mathcal{G}}(X)$.

Proof. (i) Let the representatives of $\hat{g}$ and ξ be $\hat{g}_\varepsilon$ and ξ_ε, respectively. As $\mathfrak{X}_{\mathcal{G}}(X)$ is a sheaf it suffices to show $\xi \in \Gamma_{\mathcal{N}}(U_\alpha, T^1_0(U_\alpha))$ for charts $(U_\alpha, \varphi_\alpha)$ covering X. In local coordinates with respect to such a chart our assumption reads

$$\hat{g}(\xi, \eta) = [\hat{g}_\varepsilon(\xi_\varepsilon, \eta_\varepsilon)] = [(\hat{g}_\varepsilon)_{ij}\xi^i_\varepsilon \eta^j_\varepsilon] = 0 \quad \text{in } \mathcal{G}(U_\alpha).$$

If we set $\eta^j_\varepsilon = \sum_l \hat{g}^{jl}_\varepsilon \xi^l_\varepsilon$ this gives $\sum_i (\xi^i_\varepsilon)^2 \in \mathcal{N}(U_\alpha)$ (Remark 7.8) and therefore $\xi^i_\varepsilon \in \mathcal{N}(U_\alpha)$ (Remark 3.26), which means $\xi = 0$.

(ii) By Theorem 3.24, $\hat{g}(\xi, \cdot)$ is an element of $\mathfrak{X}^*_{\mathcal{G}}(X)$ and the assignment is linear. Injectivity follows from (i), surjectivity remains to be shown. Let thus $a = [a_\varepsilon] \in \mathfrak{X}^*_{\mathcal{G}}(X)$ with $a_\varepsilon = a_{\varepsilon i} dx^i$ in local coordinates. With $\xi_\varepsilon = (\hat{g}_\varepsilon)^{ki} a_{\varepsilon k} \partial_i$ we obtain for any $\eta \in \mathfrak{X}_{\mathcal{G}}(X)$

$$\begin{aligned} \hat{g}(\xi, \eta) &= [\hat{g}_\varepsilon(\xi_\varepsilon, \eta_\varepsilon)] = [(\hat{g}_\varepsilon)_{ij}\xi^i_\varepsilon \eta^j_\varepsilon] \\ &= [(\hat{g}_\varepsilon)_{ij}(\hat{g}_\varepsilon)^{ki} a_{\varepsilon k} \eta^j_\varepsilon] = [a_{\varepsilon i}\eta^i_\varepsilon] = [a_\varepsilon(\eta_\varepsilon)] = a(\eta) \end{aligned}$$

which concludes the proof by the sheaf property of $\mathfrak{X}^*_{\mathcal{G}}(X)$. □

A generalized vector field is said to be *metrically equivalent* to a generalized one-form if they correspond via the isomorphism between $\mathfrak{X}_{\mathcal{G}}(X)$ and $\mathfrak{X}^*_{\mathcal{G}}(X)$ introduced in Proposition 7.9.

The Levi-Civita Connection

Definition 7.10. A generalized connection on a manifold X is a mapping $\hat{D} : \mathfrak{X}_{\mathcal{G}}(X) \times \mathfrak{X}_{\mathcal{G}}(X) \to \mathfrak{X}_{\mathcal{G}}(X)$ satisfying

(D1) $\hat{D}_\xi \eta$ is $\mathcal{R}$-linear in η,

(D2) $\hat{D}_\xi \eta$ is $\mathcal{G}(X)$-linear in ξ, and

(D3) $\hat{D}_\xi(u\eta) = u\hat{D}_\xi \eta + \xi(u)\eta$ for all $u \in \mathcal{G}(X)$.

$\hat{D}_\xi \eta$ is called the covariant derivative of η with respect to ξ for the connection $\hat{D}$.

Theorem 7.11. *Any generalized pseudo-Riemannian manifold $(X, \hat{g})$ admits a unique generalized connection $\hat{D}$, called generalized Levi-Civita connection of X, satisfying*

(D4) $[\xi, \eta] = \hat{D}_\xi \eta - \hat{D}_\eta \xi$ *and*

(D5) $\xi\langle \eta, \zeta\rangle = \langle \hat{D}_\xi \eta, \zeta\rangle + \langle \eta, \hat{D}_\xi \eta\rangle$

for all $\xi, \eta, \zeta \in \mathfrak{X}_\mathcal{G}(X)$. $\hat{D}$ *is characterized by the Koszul formula*

(D6)

$$\begin{aligned} 2\langle \hat{D}_\xi \eta, \zeta\rangle = \xi\langle \eta, \zeta\rangle + \eta\langle \zeta, \xi\rangle - \zeta\langle \xi, \eta\rangle \\ -\langle \xi, [\eta, \zeta]\rangle + \langle \eta, [\zeta, \xi]\rangle + \langle \zeta, [\xi, \eta]\rangle. \end{aligned} \tag{7.2}$$

Proof. We commence with any generalized connection $\hat{D}$ on X satisfying (D4) and (D5). To show uniqueness we proceed precisely as in the classical case and write

$$\begin{aligned} \xi\langle \eta, \zeta\rangle &= \langle \hat{D}_\zeta \eta, \zeta\rangle + \langle \eta, \hat{D}_\xi \zeta\rangle \\ \eta\langle \zeta, \xi\rangle &= \langle \hat{D}_\eta \zeta, \xi\rangle + \langle \zeta, \hat{D}\eta \xi\rangle \\ \zeta\langle \xi, \eta\rangle &= \langle \hat{D}_\zeta \xi, \eta\rangle + \langle \zeta, \hat{D}\zeta \eta\rangle \end{aligned}$$

which gives (D6) by adding the first two relations, subtracting the third and using (D4). Therefore, if another connection $\hat{D}'$ is given,

$$\langle \hat{D}_\xi, \eta - \hat{D}'_\xi, \eta, \zeta\rangle = 0 \quad \forall \xi, \eta, \zeta \in \mathfrak{X}_\mathcal{G}(X)$$

implies $\hat{D}_\xi \eta = \hat{D}'_\xi \eta \ \forall \xi, \eta \in \mathfrak{X}_\mathcal{G}(X)$ by Proposition 7.9 (i).

Now for existence, define $F(\xi, \eta, \zeta)$ to be 1/2 times the right hand side of (7.2). For fixed ξ, ζ the mapping $\xi \mapsto F(\xi, \eta, \zeta)$ is $\mathcal{G}(X)$-linear and therefore defines a generalized one-form by Theorem 3.24 which by Proposition 7.9 is metrically equivalent to a unique generalized vector field we denote by $\hat{D}_\xi \eta$. (D1)-(D5) are then routinely verified as in the classical proof. □

Definition 7.12. Let $(U_\alpha, \varphi_\alpha)$ be a chart and $\hat{D}$ a generalized connection on X. The *Christoffel symbols* of $\hat{D}$ for this chart are the functions $\hat{\Gamma}^k_{ij} \in \mathcal{G}(U_\alpha)$ such that

$$\hat{D}_{\partial_i} \partial_j = \sum_k \hat{\Gamma}^k_{ij} \partial_k \quad (1 \leq i, j \leq n).$$

Proposition 7.13. *With $\hat{D}$ the generalized Levi-Civita connection on a generalized pseudo-Riemannian manifold $(X, \hat{g})$, $(U_\alpha, \varphi_\alpha)$ a chart on X with coordinates x^i and any vector field $\xi \in \mathfrak{X}_{\mathcal{G}}(X)$ we have*

$$\hat{D}_\xi \eta = \Big(\xi^i \frac{\partial \eta^k}{\partial x_i} + \xi^i \eta^j \Gamma^k_{ij}\Big)\partial_k.$$

The generalized Christoffel symbols of $\hat{D}$ are symmetric in the lower pair of indices and are given by

$$\hat{\Gamma}^k_{ij} = \frac{1}{2}\hat{g}^{km}\Big(\frac{\partial \hat{g}_{jm}}{\partial x^i} + \frac{\partial \hat{g}_{im}}{\partial x^j} - \frac{\partial \hat{g}_{ij}}{\partial x^m}\Big). \tag{7.3}$$

Proof. Immediate from (D3), (D4) and (D6) as in the classical case. □

Remark 7.14. Let $(X, \hat{g})$ be a generalized pseudo-Riemannian manifold with a representative $\hat{g}_\varepsilon$ of $\hat{g}$ as in Theorem 7.1 (iii). We may ask what the relationship between the generalized Christoffel symbols $\hat{\Gamma}^k_{ij}$ of the generalized Levi-Civita connection of $(X, \hat{g})$ and the Christoffel symbols $(\hat{\Gamma}_\varepsilon)^k_{ij}$ of each $\hat{g}_\varepsilon$ is. The latter clearly are moderate as nets, as the classical equivalent of equation (7.3) shows. Now let $(\hat{\Gamma}^k_{ij})_\varepsilon$ be a representative of $\hat{\Gamma}^k_{ij}$. We then get

$$\big[(\hat{\Gamma}^k_{ij})_\varepsilon\big] = \big[(\hat{\Gamma}_\varepsilon)^k_{ij}\big] \quad \text{in } \mathcal{G}(U_\alpha)$$

on each chart U_α as a direct consequence of Remark 7.8 and equation (7.3).

Covariant derivative

In order to be able to define geodesics in generalized pseudo-Riemannian manifolds we need the notion of the induced covariant derivative of a generalized metric along a generalized curve.

Let $\gamma : I \to M$ be a curve in a semi-Riemannian manifold X. The *induced covariant derivative* is the unique function $\xi \mapsto \xi'$ from $\mathfrak{X}(\alpha)$ to $\mathfrak{X}(\alpha)$ such that

(i) $(r\xi_1 + s\xi_2)' = r\xi_1' + s\xi_2' \quad \forall r, s \in \mathbb{R},\ \xi_1, \xi_2 \in \mathfrak{X}(\gamma)$,

(ii) $(u\xi)' = \frac{du}{dt}\xi + u\xi' \quad \forall u \in C^\infty(J),\ \xi \in \mathfrak{X}(\alpha)$, and

(iii) $(\xi \circ \gamma)'(t) = \hat{D}_{\gamma'(t)}\xi$ in $\mathfrak{X}_{\mathcal{G}}(\gamma) \quad \forall t \in J,\ \forall \xi \in \mathfrak{X}(X)$.

Moreover, it satisfies

(iv) $\frac{d}{dt}\langle\xi_1,\xi_2\rangle = \langle\xi_1',\xi_2\rangle + \langle\xi_1,\xi_2'\rangle \quad \forall\xi_1,\xi_2 \in \mathfrak{X}(\gamma)$.

Proposition 7.15. *Let $J \subseteq \mathbb{R}$ be an interval and $\gamma \in \mathcal{G}[J,X]$ a curve in a generalized pseudo-Riemannian manifold $(X,\hat{g})$. Then there exists a unique mapping $\xi \mapsto \xi'$ from $\mathfrak{X}_{\mathcal{G}}(\gamma)$ to $\mathfrak{X}_{\mathcal{G}}(\gamma)$ such that*

(i) $(r\xi_1 + s\xi_2)' = r\xi_1' + s\xi_2' \quad \forall r,s \in \mathcal{K},\ \xi_1,\xi_2 \in \mathfrak{X}_{\mathcal{G}}(\gamma)$,

(ii) $(u\xi)' = \frac{du}{dt}\xi + u\xi' \quad \forall u \in \mathcal{G}(J),\ \xi \in \mathfrak{X}_{\mathcal{G}}(\alpha)$, *and*

(iii) $(\eta\circ\gamma)' = \hat{D}_{\gamma'}\xi$ *in* $\mathfrak{X}_{\mathcal{G}}(\gamma) \quad \forall\eta \in \mathfrak{X}_{\mathcal{G}}(X)$.

Proof. For each $K \subset\subset J$ we can choose $\varepsilon_0 > 0$ and $K' \subset\subset X$ such that $\gamma_\varepsilon(K) \subseteq K'$ for small ε and therefore by Theorem 7.1 (iii) there exists a representative $\hat{g}_\varepsilon$ of $\hat{g}$ such that g_ε is a pseudo-Riemannian metric in a neighborhood of K' for small ε. For fixed ε, let ξ_ε' denote the induced covariant derivative of ξ_ε along $\gamma_\varepsilon|_K$. In local coordinates it is given by

$$\xi_\varepsilon' = \sum_k \Big(\frac{d\xi_\varepsilon^k}{dt} + \sum_{i,j}(\hat{\Gamma}_\varepsilon)_{ij}^k \frac{d\gamma_\varepsilon^i}{dt}\xi_\varepsilon^j\Big)\partial_k \tag{7.4}$$

where the $(\hat{\Gamma}_\varepsilon)_{ij}^k$ denote the Christoffel symbols of $\hat{g}_\varepsilon$. Moderateness of ξ_ε' follows easily from moderateness of ξ_ε, $(\hat{\Gamma}_\varepsilon)_{ij}^k$ and γ_ε, independence of representatives is seen in the same way.

We thus have a moderate net $\xi_\varepsilon'|_K \in C^\infty(I\times K, TX)$ with $\pi_X \circ \xi_\varepsilon' = \gamma_\varepsilon$ on K, i.e., its class is an element of $\mathfrak{X}_{\mathcal{G}}(\gamma|_K)$. If we cover J by relatively compact (in J) subintervals we can patch together the nets obtained for different choices of K and obtain a generalized vector field $\xi_\varepsilon' \in \mathfrak{X}_{\mathcal{G}}(\gamma)$ along γ. Although $\xi' = [\xi_\varepsilon']$ is well-defined then, we still have to assure that this map is characterized uniquely by (i)-(iii). This, however, follows as in the classical case by employing these properties in order to show that the induced covariant derivative is completely determined by the connection $\hat{D}$. □

Definition 7.16. The induced covariant derivative of $\xi \in \mathfrak{X}_{\mathcal{G}}(\gamma)$ along a generalized curve $\gamma \in \mathcal{G}[J,X]$, $J \subseteq \mathbb{R}$, is defined as $\xi' := [\xi_\varepsilon']$.

We now have all ingredients ready to define generalized geodesics. For an interval $J \subseteq \mathbb{R}$, the tangent bundle of J is $J\times\mathbb{R}$ and there exists a canonical section ι such that $\iota(t) = 1$ for all $t \in J$. For a curve $\alpha \in C^\infty(J,X)$ the velocity field of α then is defined as $\alpha'(t) = T\alpha \circ \iota(t)$.

In the generalized case we start with a generalized curve $\gamma \in \mathcal{G}[J, X]$ and obtain a well-defined $\gamma' := T\gamma \circ \iota \in \mathfrak{X}_{\mathcal{G}}(\gamma)$ by Definition 3.41 and Theorem 5.6 (ii). Therefore, γ'' is defined as the induced covariant derivative of γ' along γ.

Definition 7.17. A geodesic in a generalized pseudo-Riemannian manifold is a generalized curve $\gamma \in \mathcal{G}[J, X]$ satisfying $\gamma'' = 0$ or, in local coordinates,

$$\left[\frac{d^2\gamma_\varepsilon^k}{dt^2} + \sum_{i,j} (\hat{\Gamma}_{ij}^k)_\varepsilon \frac{\gamma_\varepsilon^i}{dt} \frac{\gamma_\varepsilon^j}{dt}\right] = 0 \quad \text{in } \mathfrak{X}_{\mathcal{G}}(\gamma).$$

Chapter 8

Geodesics for impulsive gravitational waves

8.1 Lyrical Introduction

Now with the theory laid out in front of our eyes we see
some applications in physics, namely this will be
a gravitational wave[1], but not the general kind
plane fronted it shall be, consisting of parallel rays
but lest this physical stuff will clutter the reader's mind
we'll leave all physics behind and save it for future days.

$$\begin{aligned} \mathrm{d}s^2 &= H(u,x,y)\mathrm{d}u^2 - \mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 \\ u &= t - z, v = t + z \end{aligned} \tag{8.1}$$

Here's a Minkowski space-time, or rather its line element
u and v are special: "null coordinates" they are called;
x and y are "transverse". With H we then have meant
some kind of profile function; this is the form it takes:

$$H(u,x,y) = f(x,y)\delta(u), \quad f \in C^\infty(\mathbb{R}^2)$$

We shall find geodesics in this Minkowski space
as they tell a story on what exactly takes place.
First the equations we need contain the Christoffel symbols

[1] See [Pen72] for more about *impulsive plane fronted gravitational waves with parallel rays.*

$$\begin{aligned}
\Gamma^v_{uu} &= -f(x,y)\dot{\delta}(u) \\
\Gamma^x_{uu} &= -1/2\partial_x f(x,y)\delta, \quad \Gamma^y_{uu} = -1/2\partial_y f(x,y)\delta \\
\Gamma^v_{ux} &= -\partial_x f\delta, \quad \Gamma^v_{uy} = -\partial_y f(x,y)\delta/u)
\end{aligned}$$

and having them in our hand we list the equations then.

$$\begin{aligned}
u''(r) &= 0 \\
v''(r) &= f(x(r),y(r))\dot{\delta}(u(r)) + 2\big(\partial_x f(x(r),y(r))x'(r) \\
&\qquad + \partial_y f(x(r),y(r))y'(r)\big)u'(r)\delta(u(r)) \\
x''(r) &= \frac{1}{2}\partial_x f\big(x(r),y(r)\big)(u'(r))^2\delta(u(r)) \\
y''(r) &= \frac{1}{2}\partial_y f\big(x(r),y(r)\big)(u'(r))^2\delta(u(r))
\end{aligned}$$

u is a linear function of r so we cunningly may
– after we rule out the case of u being constant in $\mathbb{R}$
which is the trivial case with nothing exciting to say –
use it to parametrize our curve and end up this far:

$$\begin{aligned}
\ddot{v}(u) &= f(x(u),y(u))\dot{\delta}(u) + 2\big(\partial_x f(x(u),y(u))\dot{x}(u) \\
&\qquad + \partial_y f(x(u),y(u))\dot{y}(u)\big)\delta(u) \\
\ddot{x}(u) &= \frac{1}{2}\partial_x f(x(u),y(u))\delta(u) \\
\ddot{y}(u) &= \frac{1}{2}\partial_y f(x(u),y(u))\delta(u)
\end{aligned} \tag{8.2}$$

Alas! Solving these ODEs in $\mathcal{D}'$ is bound to fail:
if we integrate twice the latter pair of equations

$$\begin{aligned}
x(u) &= x(0) + \dot{x}(0) + \frac{1}{2}\partial_x f(x(0),y(0))u_+ \\
y(u) &= y(0) + \dot{y}(0) + \frac{1}{2}\partial_y f(x(0),y(0))u_+
\end{aligned}$$

inserting this in the first shows our struggle to have no avail:
Heaviside is multiplied by delta. Congratulations!

8.2 The Way to go

So there is a problem with ill-defined products of distributions. A rule of thumb like $\theta\delta := \frac{1}{2}\delta$ might somehow bring valid results in some situations, but defies mathematical reasoning. Luckily the theory of Colombeau generalized functions gives us the possibility to define products of distributions, even if the result is, in general, no distribution anymore but lies in the space of generalized functions. However, the notion of distributional convergence is a valuable tool for interpreting the solution obtained in the space of generalized Colombeau functions. We will therefore embed the singularities in system (8.2) above into $\mathcal{G}(\mathbb{R})$ and try to find solutions there which can then be examined for association with distributions.

So how do we embed δ? Of course we strive for the most general way to do this, as taking just any regularization of δ might leave us worrying if the result was the same if a different regularization was used. Therefore we consider all nets $\rho \in C^\infty(I \times \mathbb{R}^n)$ satisfying

(i) $\operatorname{supp} \rho_\varepsilon \subseteq [-\varepsilon, \varepsilon] \quad \forall \varepsilon \in (0, 1]$,

(ii) $\int_{\operatorname{supp} \rho_\varepsilon} \rho_\varepsilon(x) \mathrm{d}x \to 1 \quad (\varepsilon \to 0)$, and

(iii) $\exists \eta > 0 \; \exists C \geq 0 : \int_{\operatorname{supp} \rho_\varepsilon} |\rho_\varepsilon(x)| \mathrm{d}x \leq C \quad \forall \varepsilon \in (0, \eta)$.

Any such net converges to δ distributionally. We call all elements of $\mathcal{G}(\mathbb{R}^n)$ satisfying (i)-(iii) above *generalized delta functions*. With ι the embedding (2.1) of $\mathcal{D}'(\mathbb{R}^n)$ into $\mathcal{G}(\mathbb{R}^n)$, $\iota(\delta)$ is such. Furthermore, every generalized delta function is associated to δ.

8.3 Calculations

With δ replaced by any generalized delta function $\rho = [\rho_\varepsilon]$ we can now state (8.2) as

$$
\begin{aligned}
\ddot{v}_\varepsilon(u) &= f\big(x_\varepsilon(u), y_\varepsilon(u)\big)\dot{\rho}_\varepsilon(u) + \\
&\qquad 2\big(\partial_x f(x_\varepsilon(u), y_\varepsilon(u))\dot{x}_\varepsilon(u) + \partial_y f(x_\varepsilon(u), y_\varepsilon(u))\dot{y}_\varepsilon(u)\big)\rho_\varepsilon(u) \\
\ddot{x}_\varepsilon(u) &= \frac{1}{2}\partial_x f\big(x_\varepsilon(u), y_\varepsilon(u)\big)\rho_\varepsilon(u) \\
\ddot{y}_\varepsilon(u) &= \frac{1}{2}\partial_y f\big(x_\varepsilon(u), y_\varepsilon(u)\big)\rho_\varepsilon(u)
\end{aligned}
$$

with initial conditions at $u=-1$:

$$\begin{aligned} v_\varepsilon(-1) &= v_0, \quad & x_\varepsilon(-1) &= x_0, \quad & y_\varepsilon(-1) &= y_0,\\ \dot v_\varepsilon(-1) &= \dot v_0, \quad & \dot x_\varepsilon(-1) &= \dot x_0, \quad & \dot y_\varepsilon(-1) &= \dot y_0. \end{aligned}$$

For the following calculations set

$$\begin{aligned} w_\varepsilon(u) &= \begin{pmatrix} x_\varepsilon(u)\\ y_\varepsilon(u)\end{pmatrix}, \quad w_0 = \begin{pmatrix} x_0\\ y_0\end{pmatrix}, \quad \dot w_0 = \begin{pmatrix} \dot x_0\\ \dot y_0\end{pmatrix}, \text{ and}\\ g(x,y) &= \frac{1}{2}\begin{pmatrix} \partial_x f(x,y)\\ \partial_y f(x,y)\end{pmatrix} \end{aligned}$$

for shorter notation. The system then reads

$$\ddot w_\varepsilon(u) = g(w_\varepsilon(u))\rho_\varepsilon(u), \quad w_\varepsilon(-1) = w_0, \quad \dot w_\varepsilon(-1) = \dot w_0 \tag{8.3}$$

for all $\varepsilon \in I$ which is equivalent to

$$w_\varepsilon(u) = (Aw_\varepsilon)(u) := w_0 + \dot w_0(u+1) + \int_{-\varepsilon}^{u}\int_{-\varepsilon}^{s} g(w_\varepsilon(r))\rho_\varepsilon(r)\mathrm{d}r\mathrm{d}s. \tag{8.4}$$

Solving this equation is trivial on $(-\infty,-\varepsilon)$ and (ε,∞) because ρ_ε is zero there. It will thus suffice to show existence of a solution in $J_\varepsilon := [-1,-\varepsilon+\alpha]$ for any $\alpha > 0$ and all $\varepsilon < \varepsilon_0$ with some constant $\varepsilon_0 > 0$. For all ε smaller than $\min(\alpha,\varepsilon_0)$ we will then have assured the existence of a solution of (8.3) in the interval $[-\varepsilon,\varepsilon]$. The approach to do this is to show that the operator A has a fixed point in some suitable space. Let ε remain fixed for the moment. The space of functions we consider shall be

$$F_\varepsilon := \left\{ f \in C^\infty(J_\varepsilon,\mathbb{R}^2) \;\middle|\; \|f(u)-w_0\|_\infty \le c \right\}$$

where $c,\alpha \in \mathbb{R}_+$ still need to be defined such that A is a contraction on F_ε, i.e.,

(i) $Aw_\varepsilon \in F_\varepsilon \quad \forall w_\varepsilon \in F_\varepsilon$ and

(ii) $\exists K \in [0,1) : \|Aw_\varepsilon - A\tilde w_\varepsilon\|_\infty \le K\|w_\varepsilon - \tilde w_\varepsilon\|_\infty \quad \forall w_\varepsilon, \tilde w_\varepsilon \in F_\varepsilon$.

Regarding condition (i) we set $M := \overline{B}_c(w_0)$ and estimate

$$\begin{aligned} \|(Aw_\varepsilon)(u) - w_0\| &\le \|\dot w_0\|(1+\alpha-\varepsilon) + \int_{-\varepsilon}^{\alpha-\varepsilon}\int_{-\varepsilon}^{s} \|g|_M\|_\infty \|\rho_\varepsilon(r)\|\mathrm{d}r\mathrm{d}s\\ &\le \|\dot w_0\|(1+\alpha) + \alpha\|g|_M\|_\infty\|\rho_\varepsilon\|_1 \quad \forall u \in J_\varepsilon. \end{aligned} \tag{8.5}$$

Because c will be somewhere near $\|\dot{w}_0\|$, as a look at equation (8.4) suggests, we assume that $c = \|\dot{w}_0\| + b$ for some $b > 0$. Estimate (8.5) then results in

$$\|(Aw_\varepsilon)(u) - w_0\|_\infty \leq \alpha(\|\dot{w}_0\| + \|g|_M\|_\infty C) + \|\dot{w}_0\| \leq b + \|\dot{w}_0\|$$

where for the last inequality to hold we must set

$$\alpha \leq \frac{b}{\|\dot{w}_0\| + \|g|_M\|_\infty \|\rho_\varepsilon\|_1}.$$

This ensures that A maps F_ε into itself.

For condition (ii) we assume g to be Lipschitz-continuous with constant L on M in order to calculate

$$\begin{aligned}
\|(Aw_\varepsilon - A\tilde{w}_\varepsilon)(u)\| &\leq \int_{-\varepsilon}^{u}\int_{-\varepsilon}^{s} \|g(w_\varepsilon(r)) - g(\tilde{w}_\varepsilon(r))\| \cdot \|\rho_\varepsilon(r)\| \mathrm{d}r\mathrm{d}s \\
&\leq (\varepsilon + u)L\|w_\varepsilon - \tilde{w}_\varepsilon\|_\infty \|\rho_\varepsilon\|_1 \\
&\leq \alpha L \|w_\varepsilon - \tilde{w}_\varepsilon\|_\infty C \quad \forall u \in J_\varepsilon
\end{aligned}$$

and infer that α also has to satisfy

$$\alpha < \frac{1}{LC}.$$

As the right-hand side of (8.3) vanishes for $t \notin [-\varepsilon, \varepsilon]$ it suffices to restrict α to be smaller than 1. Furthermore, a solution of (8.3) (which is smooth by smoothness of g) exists for all $t \in \mathbb{R}$, being linear outside of $[-\varepsilon, \varepsilon]$ and satisfying $\|w_\varepsilon(t) - w_0\| \leq b + \|\dot{w}_0\|$ on J_ε. Thus w_ε clearly is bounded uniformly in ε on compact sets for $\varepsilon < \min(\varepsilon_0, \alpha)$ and the same holds for $\dot{w}_\varepsilon$:

$$\|\dot{w}_\varepsilon(t)\| \leq \|\dot{w}_0\| + \int_{-\varepsilon}^{t} \|g(w_\varepsilon(s))\| \cdot \|\rho_\varepsilon(t)\| \mathrm{d}s \leq \|\dot{w}_0\| \cdot \|g|_M\|_\infty C \quad \forall t \in [-\varepsilon, \varepsilon].$$

Summing up we have proved the following theorem (the generalization from $\mathbb{R}^2$ to $\mathbb{R}^n$ is straightforward).

Theorem 8.1. *Consider the system*

$$\ddot{w}_\varepsilon(u) = g(w_\varepsilon(u))\rho_\varepsilon(u), \quad w_\varepsilon(-1) = w_0, \quad \dot{w}_\varepsilon(-1) = \dot{w}_0 \tag{8.6}$$

for $\varepsilon \in I$, $w_0, \dot{w}_0 \in \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ a smooth function which is Lipschitz-continuous with constant L on $\{x \in \mathbb{R}^n : \|x - w_0\| \leq \|\dot{w}_0\| + b\}$ with arbitrary $b > 0$, and ρ_ε a net of smooth functions satisfying (i) and (iii) above. Furthermore, let $W = \{u \in \mathbb{R}^n \mid \|u - u_0\| \leq b + \|\dot{w}_0\|\}$ and $\alpha = \min\{b/(C\|g|_W\|_\infty + \|\dot{w}_\varepsilon\|), 1/(2LC), 1\}$. Then (8.6) has a unique solution on $J_\varepsilon = [-1, \alpha - \varepsilon]$. For $\varepsilon < \alpha$, w_ε is globally defined and both w_ε and $\dot{w}_\varepsilon$ are bounded uniformly in ε on compact sets.

By differentiating (8.3) one inductively obtains moderateness of w_ε, which requires g to be infinitely differentiable. By the same reasoning moderateness of v_ε follows. Now in order to assess what we reached so far, note that a triple $(V, X^1, X^2) \in \mathcal{G}(\mathbb{R})^3$ can only be said to be a solution of the system

$$\begin{aligned}
&\ddot{V} = (f \circ X^i)\dot{\rho} + 2(\partial_i f \circ X^i)\dot{X}^i \rho, \\
&\ddot{X}^i = \frac{1}{2}(\partial_i f \circ X^i)\rho \quad \text{in } \mathcal{G}(\mathbb{R}), \\
&V(-1) = v_0, \quad X^i(-1) = x_0^i, \\
&\dot{V}(-1) = \dot{v}_0, \quad \dot{X}^i(-1) = \dot{x}_0^i \quad \text{in } \mathcal{R},
\end{aligned} \tag{8.7}$$

if these equations hold for all representatives $V_\varepsilon, X^i_\varepsilon, \rho_\varepsilon$, i.e.,

$$\ddot{V}_\varepsilon - (f \circ X^i_\varepsilon)\dot{\rho}_\varepsilon - 2(\partial_i f \circ X^i_\varepsilon)\dot{X}^i_\varepsilon \rho_\varepsilon \in \mathcal{N}(\mathbb{R}), \tag{8.8}$$

$$\ddot{X}^i_\varepsilon - \frac{1}{2}(\partial_i f \circ X^i_\varepsilon)\rho_\varepsilon \in \mathcal{N}(\mathbb{R}), \tag{8.9}$$

$$\begin{aligned}
&V_\varepsilon(-1) - v_0 \in \mathcal{N}, \quad X^i_\varepsilon(-1) - x_0^i \in \mathcal{N}, \\
&\dot{V}_\varepsilon(-1) - \dot{v}_0 \in \mathcal{N}, \quad \dot{X}^i_\varepsilon(-1) - \dot{x}_0^i \in \mathcal{N}.
\end{aligned} \tag{8.10}$$

Now take the moderate solution $(v_\varepsilon, x_\varepsilon, y_\varepsilon)$ we constructed for each ε and set $V = [v_\varepsilon]$, $X^1 = [x_\varepsilon]$ and $X^2 = [y_\varepsilon]$. In order to show that (V, X^1, X^2) is a proper solution of (8.8) we take any representatives V_ε, X^i_ε of V, X^i, i.e.,

$$V_\varepsilon = v_\varepsilon + M_\varepsilon, \quad X^i_\varepsilon = x^i_\varepsilon + N^i_\varepsilon \tag{8.11}$$

with $M_\varepsilon, N^i_\varepsilon \in \mathcal{N}(\mathbb{R})$ and let $\tilde{\rho}_\varepsilon$ be a representative of ρ. We then have to show that conditions (8.8) and (8.10) hold. Inserting (8.11) we get

$$\begin{aligned}
\ddot{X}^i_\varepsilon - \frac{1}{2}(\partial_i f \circ X^i_\varepsilon)\tilde{\rho}_\varepsilon &= \ddot{x}^i_\varepsilon + \ddot{N}^i_\varepsilon - \frac{1}{2}(\partial_i f \circ X^i_\varepsilon)\tilde{\rho}_\varepsilon \\
&= \frac{1}{2}(\partial_i f \circ x^i_\varepsilon)\rho_\varepsilon - \frac{1}{2}(\partial_i f \circ X^i_\varepsilon)\tilde{\rho}_\varepsilon + \ddot{N}^i_\varepsilon \\
&= \frac{1}{2}(\partial_i f \circ x^i_\varepsilon)(\rho_\varepsilon - \tilde{\rho}_\varepsilon) + \frac{1}{2}\big(\partial_i f \circ x^i_\varepsilon - \partial_i f \circ X^i_\varepsilon\big)\tilde{\rho}_\varepsilon + \ddot{N}^i_\varepsilon.
\end{aligned}$$

By assumption $\partial_i f \circ x^i_\varepsilon$ is moderate, $\rho_\varepsilon - \tilde{\rho}_\varepsilon$ and $\ddot{N}^i_\varepsilon$ are negligible, and x^i_ε is bounded uniformly in ε on compact sets. Consequently, we can estimate $|(\partial_i f \circ x^i_\varepsilon - \partial_i f \circ X^i_\varepsilon)(u)| \le C|(x^i_\varepsilon - X^i_\varepsilon)(u)|$ for all u in a compact set for some $C \ge 0$ by Lemma 3.33. Therefore, $\tilde{\rho}_\varepsilon$ is moderate and X^i_ε satisfies (8.9). For $\ddot{V}_\varepsilon$ we get the same result and the triple $(V, X^1, X^2) \in \mathcal{G}(\mathbb{R})^3$ is a solution of (8.7).

Of course we ask if this solution is unique. Suppose that $W = [W_\varepsilon]$ and $Y^i = [Y^i_\varepsilon]$ are solutions of (8.7), too. In order to show $X^i_\varepsilon - Y^i_\varepsilon \in \mathcal{N}(\mathbb{R})$ we set

$$\begin{aligned}
(n_{x^i})_\varepsilon &:= X^i_\varepsilon(-1) - x^i_0 \in \mathcal{N} \\
(n_{y^i})_\varepsilon &:= Y^i_\varepsilon(-1) - y^i_0 \in \mathcal{N} \\
(n_{\dot{x}^i})_\varepsilon &:= \dot{X}^i_\varepsilon(-1) - \dot{x}^i_0 \in \mathcal{N} \\
(n_{\dot{y}^i})_\varepsilon &:= \dot{Y}^i_\varepsilon(-1) - \dot{y}^i_0 \in \mathcal{N} \\
N^i_\varepsilon &:= \ddot{X}^i_\varepsilon - \ddot{Y}^i_\varepsilon - \frac{1}{2}\rho_\varepsilon(\partial_i f \circ X^i_\varepsilon - \partial_i f \circ Y^i_\varepsilon) \in \mathcal{N}(\mathbb{R})
\end{aligned}$$

as in conditions (8.9),(8.10) and write the difference $X^i_\varepsilon - Y^i_\varepsilon$ (for any representative ρ_ε of ρ) as

$$\begin{aligned}
(X^i_\varepsilon - Y^i_\varepsilon)(u) =& (n_{x^i})_\varepsilon - (n_{y^i})_\varepsilon + (u+1)(n_{\dot{x}^i})_\varepsilon - (n_{\dot{y}^i})_\varepsilon \\
&+ \frac{1}{2}\int_{-1}^{u}\int_{-1}^{s} \rho_\varepsilon(r)\big((\partial_i f)(X^i_\varepsilon(r)) - (\partial_i f)(Y^i_\varepsilon(r))\big)\mathrm{d}r\mathrm{d}s \\
&- \int_{-1}^{u}\int_{-1}^{s} N^i_\varepsilon(r)\mathrm{d}r\mathrm{d}s
\end{aligned} \tag{8.12}$$

It suffices to show the negligibility estimate on $[-T, T]$ for each $T > 0$. First, $(n_{x^i})_\varepsilon$, $(n_{y^i})_\varepsilon$, $(n_{\dot{x}^i})_\varepsilon$ and $(n_{\dot{y}^i})_\varepsilon$ grow like $O(\varepsilon^m)$ for all $m \in \mathbb{R}$. Then we employ

$$\begin{aligned}
&(\partial_i f)(X^i_\varepsilon(r)) - (\partial_i f)(Y^i_\varepsilon(r)) = \\
&\qquad \int_0^1 (D\partial_i f)\big(\sigma X^i_\varepsilon(r) + (1-\sigma)Y^i_\varepsilon(r)\big) \cdot \big|(X^i_\varepsilon - Y^i_\varepsilon)(r)\big|\mathrm{d}\sigma
\end{aligned}$$

and can easily apply the Lemma of Gronwall on (8.12) which yields negligible growth of $X_\varepsilon - Y_\varepsilon$. The same procedure gives an identical result for $\dot{X}_\varepsilon - \dot{Y}_\varepsilon$, and as higher derivatives are seen to inductively satisfy the same growth estimates, $X^i_\varepsilon - Y^i_\varepsilon \in \mathcal{N}(\mathbb{R})$ follows and consequently $V_\varepsilon - W_\varepsilon \in \mathcal{N}(\mathbb{R})$ is verified by inserting into the integral equation for V. We thus have proved the following theorem.

Theorem 8.2. *Let $D \in \mathcal{G}(\mathbb{R})$ be a generalized delta-function, $f \in C^\infty(\mathbb{R}^2)$*

and $v_0, \dot{v}_0, x_0^i, \dot{x}_0^i \in \mathbb{R}$*. Then the initial value problem*

$$\begin{aligned}
\ddot{V} &= (f \circ X^i)\dot{\rho} + 2\partial_i(f \circ X^i)\dot{X}^i\rho, \\
\ddot{X} &= \frac{1}{2}(\partial_i f \circ X^i)\rho, \\
V(-1) &= v_0, \quad X^i(-1) = x_0^i, \\
\dot{V}(-1) &= \dot{v}_0, \quad \dot{X}^i(-1) = \dot{x}_0^i
\end{aligned} \tag{8.13}$$

has a unique solution (V, X^1, X^2) *in* $\mathcal{G}(\mathbb{R})^3$*.*

8.4 Distributional Limits

It can be shown (cf. [Ste98]) that the distributional limits of the unique solutions to the initial value problem 8.13 are given by

$$\begin{aligned}
v_\varepsilon(u) &\to v_0 + \dot{v}_0(1+u) + f(0)\theta(u) + \partial_i f(0)(\dot{x}_0^i + \frac{1}{4}\partial_i f(0))u_+, \\
x_\varepsilon^i(u) &\to x_0^i + \dot{x}_0^i(1+u) + \frac{1}{2}\partial_i f(0)u_+,
\end{aligned}$$

which means that the geodesics of the space-time (8.1) are given by refracted, broken straight lines. It is worthy of observation that these distributional limits are independent of the regularization chosen. [KS99] treats this problem in the context of full Colombeau algebras, as opposed to the special Colombeau algebras used here.

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