TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Diplomarbeit

Stability of a Closed-Loop Control System – Applied to a Gantry Crane with Heavy Chains

Ausgeführt am Institut für

Analysis und Scientific Computing
der Technischen Universität Wien

unter der Anleitung von

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15. Oktober 2009

Unterschrift

Acknowledgements

I wish to thank Professor Dr. Arnold, who has aroused my interest in partial differential equations in many lectures and seminaires, and who gave me the opportunity to write my diploma thesis in this very interesting field. I am very grateful that he took so much time for questions, discussions, and to give me advice.

Many thanks also to MSc. Maja Miletic, who provided some additional references which were very useful.

Furthermore, I want to express my gratitude to Dipl.-Ing. Claudia Scheimbauer and Dipl.-Ing. Clemens Koppensteiner for their very helpful comments and suggestions, and furthermore for the very enriching collaboration during the past five years.

Finally, I wish to thank my parents, Mag. Otto and Dr. Claire Stürzer, who have always encouraged me and supported my interests, and who made it possible for me to write these lines.

Contents

Chapter 1

Introduction

In this thesis we discuss the stability of a linear control system, which is a slight generalization of a controller applied to an infinite-dimensional model of a gantry crane with heavy chains. In this introduction we describe the underlying situation and briefly sketch the derivation of the equations for the controller, as discussed in [11], see also [10].

The model consists of a cart of mass m_w, which moves horizontally along a rail, a chain of length L with mass per length ρ, attached to the cart[1], and a load of mass m_p at its end, which is considered to be point-like. The chain is assumed to be non-elastic. Furthermore, any friction occuring in this system is assumed to be negligible, and is therefore not taken into account.

The objective of applying a controller to this system is to move the cart in such a way that the chain follows a desired trajectory. In most cases this will be to damp out vibrations of the chain, e.g. oscillations arising from an abrupt stop of the cart. The controller will be realized as an appropriate, time-dependent force F acting upon the cart. The situation is sketched in Figure 1.1.

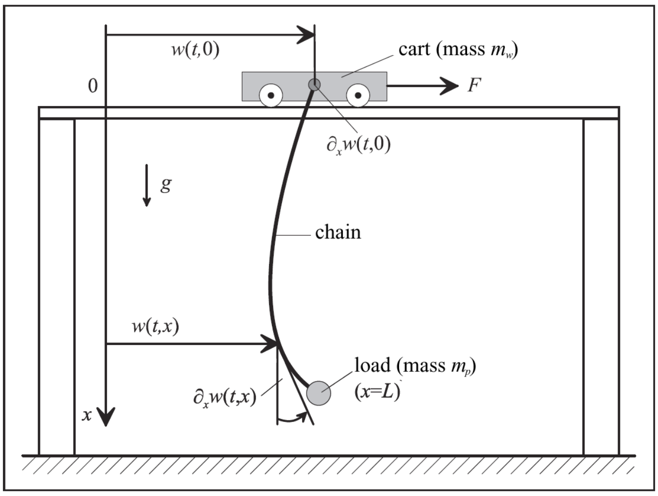

Figure 1.1: Model of a Gantry Crane (from [11]).

Before specifying the control force F, we have to find the equations of motion of the system. The horizontal deflection of the chain at height x (vertical distance from the cart) and at time t is denoted by $w(t, x)$. In the model considered here we assume that the occuring oscillations, especially the function $\partial_x w(t, x)$, remain sufficiently small for all $t > 0$, such that we can assume $\arctan \partial_x w(t, x) \approx \partial_x w(t, x)$. With this, the equations describing the dynamics of the system, already taking into account an exterior force F acting upon the cart, can be derived via Hamilton's principle. The obtained equations of motion are the following:

$$\rho \partial_t^2 w(t, x) - \partial_x (P(x) \partial_x w(t, x)) = 0, \quad (1.1a)$$

$$m_w \partial_t^2 w(t, 0) - P(0) \partial_x w(t, 0) = F(t), \quad (1.1b)$$

$$m_p \partial_t^2 w(t, L) + P(L) \partial_x w(t, L) = 0. \quad (1.1c)$$

The function $P(x)$ represents the gravitational pull on the chain at height x, given by $P(x) = g[\rho(L - x) + m_p]$, where g denotes the gravitational acceleration. We note that $P \geq m_p > 0$

[1] Please note that here only a single chain is considered, unlike the pair of parallel chains as used in [11]. This change corresponds to the substitution $\rho \to \rho/2$.

holds uniformly on $[0, L]$. In this thesis we expand this model developped in [11], as described above, by considering more general functions P : We only require that $P \in H^2(0, L)$ and that $P > 0$ holds uniformly on $[0, L]$.
Let w_d denote a given trajectory, satisfying system (1.1) with a certain force F_d acting upon the cart. The desired property of the controller is to determine a force F such that the trajectory w approaches the desired trajectory w_d. Defining the error function $w_e := w - w_d$, this is equivalent to the requirement that $w_e \to 0$ in an appropriate way. We refer to this as *asymptotic stability* of the controller. We find, due to linearity of the equations (1.1), that w_e also satisfies

$$\rho \partial_t^2 w_e(t, x) - \partial_x(P(x)\partial_x w_e(t, x)) = 0, \tag{1.2a}$$

$$m_p \partial_t^2 w_e(t, L) + P(L)\partial_x w_e(t, L) = 0. \tag{1.2b}$$

In order to determine the controlling force F, we first introduce a function F_e such that

$$F(t) = -P(0)\partial_x w(t, 0) + m_w \partial_t^2 w_d(t, 0) + F_e(t).$$

From (1.1b) we find that

$$F_e(t) = m_w \partial_t v_e(t, 0), \tag{1.3}$$

where $v_e(t, 0) := \partial_t w_e(t, 0)$. In order to design the controller, a first approach is to consider a functional V_e depending on w_e, v_e, describing the energy of the system, and find conditions on F_e such that the derivative of V_e with respect to the time satisfies $\frac{\mathrm{d}}{\mathrm{d}t} V_e(t) \leq 0$. This condition implies that the controller extracts energy from the error system (1.2), (1.3), and that therefore the error function w_e is expected to be non-increasing (in an appropriate norm). As discussed in [11] it is reasonable to assume that F_e is of the form

$$F_e(t) = \vartheta_1 v_e(t, 0) + \vartheta_2 \partial_x v_e(t, 0) + \vartheta_3 w_e(t, 0) + \vartheta_4 \partial_x w_e(t, 0),$$

with real coefficients ϑ_i. In this thesis we explicitly define an inner product in an appropriate Hilbert space, such that the corresponding norm serves as the function V_e itself. As discussed in Chapter 3, this allows us to make use of the theory of C_0-semigroups of contractions to derive conditions on the ϑ_i such that the norm of the error function w_e is non-increasing along every trajectory. In practice, this is a very important property of a controller which clearly must be fulfilled. However, in general, this is not sufficient for the proof of asymptotic stability since the error function w_e may remain uniformly bounded but not necessarily tends to zero. Therefore, in Chapter 4, we explicitly prove asymptotic stability. Finally, in Chapter 5 we show that the controller not only is asymptotically stable, but even exponentially stable, which means that the norm of the error function w_e tends to zero exponentially.
The conclusion of these statements is that, under certain conditions on the ϑ_i, the discussed controller satisfies the requirements of stabilizing the system, i.e. the chain will follow the desired trajectory with decreasing error. For example, if the state desired to attain is the passive state with non-oscillating chain, the controller will proceed to damp any initial vibrations of the chain, and in a finite time, any vibrations remaining will be below a certain threshold, e.g. the accuracy of the sensors attached to the chain.

Chapter 2

Preliminaries

To begin with we present the problem considered in this thesis in an appropriate mathematical framework, which is needed for further investigations.

In this thesis we analyse the stability of the following system satisfied by the error function w_e, obtained from (1.2) and (1.3)[1]:

$$\begin{aligned} \rho\partial_t^2 w(t,x) - \partial_x(P(x)\partial_x w(t,x)) &= 0, \quad x \in (0,L),\, t>0, && (2.1a)\\ m_p\partial_t^2 w(t,L) + P(L)\partial_x w(t,L) &= 0, && (2.1b)\\ m_w\partial_t^2 w(t,0) &= F(t), && (2.1c)\end{aligned}$$

where $F(t) := \vartheta_1\partial_t w(t,0) + \vartheta_2\partial_x\partial_t w(t,0) + \vartheta_3 w(t,0) + \vartheta_4\partial_x w(t,0)$, with $\vartheta_i \in \mathbb{R}$ to be specified later. The function $P \in H^2(0,L) \hookrightarrow C^1[0,1]$ is real-valued, satisfying $\min_{x\in[0,L]} P(x) > 0$, and ρ, m_p, and m_w are positive constants. Before mathematically analyzing the above problem, we eliminate most numerical coefficients. To this end, we rescale length and time, i.e. we introduce new parameters $(\tilde{x},\tilde{t})$, corresponding to the linear transformation $x = s_1\tilde{x}$, and $t = s_2\tilde{t}$, and $s_i > 0$. Setting $s_1 = \frac{m_p}{\rho P(L)}$ and $s_2 = s_1\sqrt{\rho}$, equations (2.1a) and (2.1b) become

$$\begin{aligned} \partial_{\tilde{t}}^2 w(\tilde{t},\tilde{x}) - \partial_{\tilde{x}}(P(\tilde{x})\partial_{\tilde{x}} w(\tilde{t},\tilde{x})) &= 0, \quad \tilde{x} \in (0,\tilde{L}), \tilde{t}>0, && (2.1a')\\ \partial_{\tilde{t}}^2 w(\tilde{t},\tilde{L}) + \partial_{\tilde{x}} w(\tilde{t},\tilde{L}) &= 0 && (2.1b')\end{aligned}$$

in new coordinates. In (2.1c), m_w and all additional factors arising from the change of coordinates can be absorbed in the coefficients ϑ_i appearing in F, which yields new coefficients $\tilde{\vartheta}_i$. In the following, we will only consider the equations in the new coordinates $\tilde{x}$ and $\tilde{t}$. Thus we can omit the symbol $\tilde{\cdot}$ and simply write x, t and ϑ_i.

Before we analyse the stability of this system, we re-write it in terms of an initial value problem in an appropriate Hilbert space: for this we define the complex Hilbert space

$$\mathcal{H} = \{z = (w,v,\xi,\psi) : w \in H^2(0,L),\, v \in H^1(0,L),\, \xi = v(L),\, \psi = v(0)\}, \qquad (2.2)$$

which is a closed subspace of $H^2 \times H^1 \times \mathbb{C} \times \mathbb{C}$. The auxiliary scalar variables ξ, ψ are introduced in order to include the dynamical boundary conditions (2.1b), (2.1c) into the initial value problem. $\mathcal{H}$ is equipped with the natural inner product given by

$$\langle z_1, z_2\rangle_* = \langle w_1, w_2\rangle_{H^2} + \langle v_1, v_2\rangle_{H^1} + \xi_1\bar{\xi}_2 + \psi_1\bar{\psi}_2,$$

where $\bar{\xi}$ denotes the complex conjugate of ξ. Let the linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be defined as

$$A : \begin{bmatrix} w \\ v \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} v \\ (Pw')' \\ -w'(L) \\ \vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0) \end{bmatrix}, \qquad (2.3)$$

[1] Since we only consider the error system in the following, we write $w = w_e$ and $F = F_e$ for simplicity.

where w' denotes the spatial derivative of w. The (dense) domain of A is defined as

$$D(A) := \big\{z = (w, v, \xi, \psi) : \quad w \in H^3(0, L),\ v \in H^2(0, L),\ \xi = v(L),\ \psi = v(0), \qquad (Pw')'(L) = -w'(L),\ (Pw')'(0) = F\big\}, \tag{2.4}$$

where $F := \vartheta_1 v(0) + \vartheta_2 \partial_x v(0) + \vartheta_3 w(0) + \vartheta_4 \partial_x w(0)$. The boundary conditions stated in $D(A)$ arise naturally from the requirement that $\mathcal{R}(A) \subset \mathcal{H}$. With these definitions, we can rewrite system (2.1) as the following initial value problem in $\mathcal{H}$:

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0 \in D(A). \end{cases} \tag{2.5}$$

In this thesis we prove that, under certain conditions on the ϑ_i, A generates a C_0-semigroup of contractions, and that every solution of $\dot{z} = Az$ exponentially converges to 0. For the proofs, the natural inner product $\langle \cdot, \cdot \rangle_*$ on $\mathcal{H}$ is not always useful to work with. Therefore, we define an equivalent inner product, which is more suitable for the considered problem:

$$\begin{aligned} \langle z_1, z_2 \rangle_{\mathcal{H}} \quad := \quad & \alpha_1 \int_0^L \left[\gamma (Pw_1')'(P\bar{w}_2')' + Pw_1'\bar{w}_2'\right] \mathrm{d}x + \alpha_1 \gamma P(L) w_1'(L) \bar{w}_2'(L) \\ & + \alpha_2 w_1(0)\bar{w}_2(0) + \alpha_1 \int_0^L \left(\gamma P v_1' \bar{v}_2' + v_1 \bar{v}_2\right) \mathrm{d}x + \alpha_1 P(L) \xi_1 \bar{\xi}_2 + \alpha_2 \gamma \psi_1 \bar{\psi}_2 \\ & + \frac{1}{2}\big(\psi_1 - 2\alpha_1 P(0) w_1'(0) + 2\alpha_2 w_1(0)\big)\big(\bar{\psi}_2 - 2\alpha_1 P(0)\bar{w}_2'(0) + 2\alpha_2 \bar{w}_2(0)\big), \end{aligned}$$

where α_1, α_2, and γ for now are arbitrary positive constants and are specified later in the proof of Lemma 3.6. We have the following lemma:

Lemma 2.1. *The norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the natural norm $\|\cdot\|_*$ on $\mathcal{H}$.*

Proof. We have to prove the existence of constants $c_1, c_2 > 0$ such that $c_1\|z\|_* \le \|z\|_{\mathcal{H}} \le c_2\|z\|_*$ holds for all $z \in \mathcal{H}$. To verify the first inequality, it remains to show the existence of $\tilde{c}_1$ such that

$$\int_0^L \gamma((Pw')')^2 + P(w')^2 \,\mathrm{d}x \ge \tilde{c}_1 \int_0^L (w'')^2 + (w')^2 \,\mathrm{d}x \tag{2.6}$$

holds for all real-valued $w \in H^2(0, L)$. Using the properties of P mentioned above, Lemma A.1 can be applied pointwise in x with $a = \sqrt{\gamma}\, P'(x)$, $b = \sqrt{\gamma}\, P(x)$, $\varepsilon = P(x)$, $x_1 = w'(x)$, and $x_2 = w''(x)$, which directly yields the desired inequality (2.6).
To verify the second inequality, it suffices to apply Cauchy's inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, $a, b \in \mathbb{R}$, to all non-quadratic terms in $\|z\|^2_{\mathcal{H}}$. □

Chapter 3

Generation of a Semigroup of Contractions

In the following we characterize the conditions on the ϑ_i such that the operator A defined in (2.3) generates a C_0-semigroup of contractions. The main statement is the following theorem:

Theorem 3.1. *Let A be the operator defined in (2.3) and let there be constants $a, b > 0$ satisfying $(a+b-1)^2 < 4ab$, such that*

$$\vartheta_1 = \frac{\vartheta_3}{b} - a, \quad \vartheta_2 = \frac{\vartheta_4}{b}, \tag{3.1}$$

and $\vartheta_1, \vartheta_3 < 0$ and $\vartheta_2, \vartheta_4 > 0$. Then A is the infinitesimal generator of a C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$ on $\mathcal{H}$.

This result follows readily from the following theorem (cf. [9], Theorem 4.3, Chapter I):

Theorem 3.2 (Lumer-Phillips)**.** *Let X be a Banach space. A linear operator $\mathcal{A}$ with dense domain $D(\mathcal{A}) \subset X$ is the infinitesimal generator of a C_0-semigroup of contractions if and only if $\mathcal{A}$ is dissipative and* $\operatorname{ran}(\mathcal{A} - \lambda_0) = X$ *for a $\lambda_0 > 0$.*

Remark 3.3. If X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, the dissipativity of a densely defined, linear operator $\mathcal{A}$ is equivalent to the condition (cf. [9], [13])

$$\operatorname{Re}\langle z, \mathcal{A}z \rangle \leq 0, \quad \forall z \in D(\mathcal{A}).$$

In the case considered here, it is sufficient to use the following corollary of the above theorem (cf. [6], Theorem 1.2.4):

Theorem 3.4. *Let X be a Banach space, and $\mathcal{A}$ a linear operator with dense domain $D(\mathcal{A}) \subset X$. Then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions if $\mathcal{A}$ is dissipative and* $0 \in \rho(\mathcal{A})$, *the resolvent set of $\mathcal{A}$.*

A proof can be found in [6]. Here we give an alternative proof:

Proof. From the fact, that $\rho(\mathcal{A})$ is an open set, and $0 \in \rho(\mathcal{A})$, we conclude that $\operatorname{ran}(\mathcal{A} - \lambda_0) = X$ for all $|\lambda_0|$ sufficiently small. The theorem now follows from the Lumer-Phillips theorem. □

In order to prove Theorem 3.1, we need to verify the conditions stated in Theorem 3.4.

Lemma 3.5. *The domain $D(A)$ of the operator A defined in (2.3) is dense in $\mathcal{H}$.*

Proof. Let $z_0 = (w_0, v_0, \xi_0, \psi_0) \in \mathcal{H}$. Since the inclusions $H^3(0, L) \subset H^2(0, L) \subset H^1(0, L)$ are dense, there exists a sequence $z_n = (w_n, v_n, \xi_n, \psi_n) \in H^3(0, L) \times H^2(0, L) \times \mathbb{C}^2 \cap \mathcal{H}$ such that $z_n \to z_0$ in $\mathcal{H}$. Now, in general, the second derivatives $\partial^2 w_n(0), \partial^2 w_n(L)$ will not satisfy the boundary conditions necessary for $z_n \in D(A)$.

The fact that $H_0^1(0,L) \subset L^2(0,L)$ dense ensures the existence of a sequence $\{u_n\} \subset H^1(0,L)$ satisfying $u_n(0) = a$ for all $n \in \mathbb{N}$ and any fixed $a \in \mathbb{C}$, with $\|u_n\|_{L^2} \to 0$. The sequence $\{y_n\}$ defined by

$$y_n := \int_0^x \int_0^\xi u_n(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}\xi$$

satisfies $\partial^2 y_n(0) = a$ for all $n \in \mathbb{N}$, and $\|y_n\|_{H^2} \to 0$.
This shows that for the sequence $\{w_n\}$ the values $\partial^2 w_n(0)$, $\partial^2 w_n(L)$ can be modified such that $\{z_n\} \subset D(A)$, but still $z_n \to z_0$ in $\mathcal{H}$. □

Lemma 3.6. *Let the assumptions of Theorem 3.1 hold. Then the operator A is dissipative.*

Proof. For all $z \in D(A)$ we have:

$$\begin{aligned}
\operatorname{Re}\langle z, Az\rangle_{\mathcal{H}} &= \operatorname{Re}\Bigg[\alpha_1\gamma \int_0^L (Pw')'(P\bar{v}')'\,\mathrm{d}x + \alpha_1 \int_0^L Pw'\bar{v}'\,\mathrm{d}x \\
&\quad + \alpha_1\gamma P(L)w'(L)\bar{v}'(L) + \alpha_2 w(0)\bar{v}(0) \\
&\quad + \alpha_1\gamma \int_0^L Pv'(P\bar{w}')''\,\mathrm{d}x + \alpha_1 \int_0^L v(P\bar{w}')'\,\mathrm{d}x \\
&\quad - \alpha_1 P(L)v(L)\bar{w}'(L) + \alpha_2\gamma v(0)\bar{F} \\
&\quad + \frac{1}{2}\big[v(0) - 2\alpha_1 P(0)w'(0) + 2\alpha_2 w(0)\big]\big[\bar{F} - 2\alpha_1 P(0)\bar{v}'(0) + 2\alpha_2\bar{v}(0)\big]\Bigg] \\
&= \operatorname{Re}\Bigg[\alpha_1\gamma \int_0^L \big[Pv'(P\bar{w}')'\big]'\,\mathrm{d}x + \alpha_1 \int_0^L \big[Pv\bar{w}'\big]'\,\mathrm{d}x \qquad (3.2) \\
&\quad + \alpha_1\gamma P(L)w'(L)\bar{v}'(L) + \alpha_2 w(0)\bar{v}(0) \\
&\quad - \alpha_1 P(L)v(L)\bar{w}'(L) + \alpha_2\gamma v(0)\bar{F} \\
&\quad + \frac{1}{2}\big[v(0) - 2\alpha_1 P(0)w'(0) + 2\alpha_2 w(0)\big]\big[\bar{F} - 2\alpha_1 P(0)\bar{v}'(0) + 2\alpha_2\bar{v}(0)\big]\Bigg].
\end{aligned}$$

Using the boundary conditions in $D(A)$ to evaluate the term $Pv'(P\bar{w}')'|_0^L$, we find that the real parts of all terms at $x = L$ cancel against the real part of the third term of (3.2). The remaining terms are

$$\begin{aligned}
\operatorname{Re}\langle z, Az\rangle_{\mathcal{H}} &= \operatorname{Re}\Big[\bar{v}(0)[-\alpha_1 P(0)w'(0) + \alpha_2 w(0)] + \gamma\bar{F}[-\alpha_1 P(0)v'(0) + \alpha_2 v(0)] + \\
&\quad + \frac{1}{2}\big[v(0) - 2\alpha_1 P(0)w'(0) + 2\alpha_2 w(0)\big]\big[\bar{F} - 2\alpha_1 P(0)\bar{v}'(0) + 2\alpha_2\bar{v}(0)\big]\Big].
\end{aligned}$$

By introducing the functional $J : w \mapsto -2\alpha_1 P(0)w'(0) + 2\alpha_2 w(0)$, we simplify the expression:

$$\operatorname{Re}\langle z, Az\rangle_{\mathcal{H}} = \frac{1}{2}\operatorname{Re}\big\{\bar{v}(0)J(w) + \gamma\bar{F}J(v) + \big[v(0) + J(w)\big]\big[\bar{F} + J(\bar{v})\big]\big\}. \qquad (3.3)$$

Using (3.1) we can write $F = -av(0) - bJ(w) - J(v)$ with $a, b > 0$. This fixes

$$\alpha_1 := \frac{\vartheta_2}{2P(0)} > 0, \qquad \alpha_2 := -\frac{\vartheta_2\vartheta_3}{2\vartheta_4} > 0.$$

Then the right hand side of (3.3) only depends on the three independent values $v(0)$, $J(w)$ and $J(v)$. Introducing the new variables $y_1 = \sqrt{a}\,v(0)$, $y_2 = \sqrt{b}\,J(w)$ and $y_3 = \sqrt{\gamma}\,J(v)$ yields:

$$\operatorname{Re}\langle z, Az\rangle_{\mathcal{H}} = \frac{1}{2}\big(P_3(\operatorname{Re} y_1, \operatorname{Re} y_2, \operatorname{Re} y_3) + P_3(\operatorname{Im} y_1, \operatorname{Im} y_2, \operatorname{Im} y_3)\big), \qquad (3.4)$$

where P_3 is the polynomial defined by

$$P_3(x_1, x_2, x_3) := -x_1^2 - x_2^2 - x_3^2 - 2x_1x_2\Big(\frac{a+b-1}{2\sqrt{ab}}\Big) - 2x_2x_3\Big(\frac{\sqrt{b\gamma}}{2}\Big) - 2x_1x_3\Big(\frac{\sqrt{a\gamma}}{2}\Big). \qquad (3.5)$$

Hence, the dissipativity of A is equivalent to

$$x_1^2+x_2^2+x_3^2+2x_1x_2\Big(\frac{a+b-1}{2\sqrt{ab}}\Big)+2x_2x_3\Big(\frac{\sqrt{b\gamma}}{2}\Big)+2x_1x_3\Big(\frac{\sqrt{a\gamma}}{2}\Big)\geq 0,\quad \forall x_1,x_2,x_3\in\mathbb{R}. \tag{3.6}$$

Applying Lemma A.2 yields the following conditions:

$$a,b\leq\frac{4}{\gamma},\qquad \frac{(a+b-1)^2}{4ab}\leq 1,\qquad \frac{(a+b-1)^2}{4ab}\leq 1-\frac{\gamma}{4}.$$

Since $\gamma>0$ has not yet been specified, we can choose γ arbitrary small, so that the above conditions reduce to the single condition

$$\frac{(a+b-1)^2}{4ab}<1. \tag{3.7}$$

Matching the coefficients in $F=\vartheta_1 v(0)+\vartheta_2 v'(0)+\vartheta_3 w(0)+\vartheta_4 w'(0)=-av(0)-bJ(w)-J(v)$ yields the desired relations for $\vartheta_1,\ldots,\vartheta_4$. □

Remark 3.7. Condition (3.7) can be simplified by making an coordinate transformation, $\hat{a}=a+b$, $\hat{b}=a-b$. The condition then reduces to $\hat{b}^2<2\hat{a}-1$, which is satisfied exactly by the set of points inside the parabola touching the coordinate-axes at the points $(a,b)=(1,0)$ and $(a,b)=(0,1)$.

Lemma 3.8. *Let A be the operator defined in (2.3), and $\vartheta_3\neq 0$. Then A^{-1} exists and $D(A^{-1})=\mathcal{H}$.*

Proof. We will prove this Lemma by showing that the equation $Az=(f,g,g(L),g(0))$ has a unique solution $z\in D(A)$ for every $(f,g,g(L),g(0))\in\mathcal{H}$:

$$\begin{bmatrix} v \\ (Pw')' \\ -w'(L) \\ \vartheta_1 v(0)+\vartheta_2 v'(0)+\vartheta_3 w(0)+\vartheta_4 w'(0)\end{bmatrix}=\begin{bmatrix} f \\ g \\ g(L) \\ g(0)\end{bmatrix}. \tag{3.8}$$

From the first line we immediately get $v=f\in H^2(0,L)$, which also fixes the values $v(0)$ and $v'(0)$. The third line yields $w'(L)=-g(L)$. After integration of the second line we get

$$w'(x)=\frac{P(L)}{P(x)}w'(L)+\frac{1}{P(x)}\int_L^x g(y)\,\mathrm{d}y. \tag{3.9}$$

The occuring integral exists, because $g\in H^1(0,L)$. Since $w'(L)$ is already known, we can compute the value $w'(0)$ from this equation. Together with the already known values $v(0)$ and $v'(0)$, we obtain $w(0)$ from the fourth line in (3.8), since $\vartheta_3\neq 0$. With this, $w(x)$ can uniquely be determined by integration of equation (3.9):

$$w(x)=w(0)+\int_0^x\frac{P(L)}{P(y)}w'(L)\,\mathrm{d}y+\int_0^x\frac{1}{P(y)}\int_L^y g(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}y. \tag{3.10}$$

All integrals exist, since $P(x)>0$ holds uniformly, and $\int_L^x g(y)\,\mathrm{d}y$ is continuous. It remains to check that $w\in H^3(0,L)$ is satisfied. This follows from (3.9) with $g\in H^1(0,L)$ and $P^{-1}\in H^2(0,L)$. Thus, the inverse A^{-1} exists and is defined on $\mathcal{H}$. □

Lemma 3.9. *The operator A^{-1} is compact.*

Proof. We show that for $(f,g,g(L),g(0))\in\mathcal{H}$ and $z=A^{-1}(f,g,g(L),g(0))$ the norm of z in $\mathcal{J}=H^3(0,L)\times H^2(0,L)\times\mathbb{C}^2$ is uniformly bounded by $\|(f,g,g(L),g(0))\|_{\mathcal{H}}$.
Due to the continuous embedding $H^1(0,L)\hookrightarrow C[0,L]$ in one dimension (see e.g. [1]), we have the estimates $|g(L)|$, $|g(0)|\leq C\|g\|_{H^1}$. From the third line in (3.8) we therefore get $|w'(L)|\leq C\|g\|_{H^1}$. With this and (3.9) we find the estimate

$$\|w'\|_{L^2}\leq C\|g\|_{H^1}. \tag{3.11}$$

Applying this result to the identity $Pw'' = g - P'w'$, which is obtained from the second line in (3.8), and using the fact that $P' \in L^2(0, L)$, yields

$$\|w''\|_{L^2} \leq C\|g\|_{H^1}. \tag{3.12}$$

Similarly, from $(Pw')'' = g'$ we obtain the estimate

$$\|w'''\|_{L^2} \leq C\|g\|_{H^1}. \tag{3.13}$$

For v we immediately get $\|v\|_{H^2} = \|f\|_{H^2}$ using the first line in (3.8). Due to the continuous embedding $H^k(0, L) \hookrightarrow C^{k-1}[0, L]$ in one dimension (cf. [1]), we find the following estimates

$$|v(0)| \leq C\|f\|_{H^2}, \tag{3.14}$$
$$|v'(0)| \leq C\|f\|_{H^2}. \tag{3.15}$$

Using the above estimate for $w'(L)$ and (3.9) we obtain

$$|w'(0)| \leq C\|g\|_{H^1}. \tag{3.16}$$

Applying the above estimate to $g(0)$ and (3.14), (3.15), (3.16) to the fourth line of (3.8) yields

$$|w(0)|^2 \leq C(\|f\|_{H^2}^2 + \|g\|_{H^1}^2). \tag{3.17}$$

Altogether, we get

$$\|w\|_{H^3}^2 \leq C(\|f\|_{H^2}^2 + \|g\|_{H^1}^2). \tag{3.18}$$

Thus, we have $\|w\|_{H^3}^2 + \|v\|_{H^2}^2 \leq C(\|f\|_{H^2}^2 + \|g\|_{H^1}^2)$, which shows that A^{-1} maps bounded sets in $\mathcal{H}$ to bounded sets in $\mathcal{J}$. Since the embeddings $H^3(0, L) \subset\subset H^2(0, L) \subset\subset H^1(0, L)$ are compact, A^{-1} is a compact operator. □

Now, Theorem 3.1 follows directly from the above results:

Proof of Theorem 3.1. Due to the results of Lemmata 3.6, 3.8 and 3.9, all assumptions of Theorem 3.4 are satisfied. Hence, we have proved Theorem 3.1. □

The following corollary follows as a consequence of Theorem 3.1, due to elementary properties of generators of C_0-semigroups of operators (see [9] for more details):

Corollary 3.10. *Under the assumptions of Theorem 3.1, the initial value problem*

$$\begin{cases} \dot{z}(t) = Az(t) \\ z(0) = z_0 \end{cases} \tag{3.19}$$

has a unique mild solution $z(t) := T(t)z_0$ for all $z_0 \in \mathcal{H}$, where $\{T(t)\}_{t\geq 0}$ is the C_0-semigroup generated by A. If $z_0 \in D(A)$, then $z(t)$ is continuously differentiable on $[0, \infty)$ and $z(t) \in D(A)$ for all $t \geq 0$, and therefore is a classical solution. Furthermore, the norm $\|z(t)\|_{\mathcal{H}}$ is non-increasing.

Remark 3.11. The particular controller developed in [11] satisfies the conditions for $\vartheta_1, \dots, \vartheta_4$ in Theorem 3.1 with $b = a + 1$. With this identity, the condition $(a + b - 1)^2 < 4ab$ clearly holds, and therefore the controller generates a C_0-semigroup of contractions.

Chapter 4

Asymptotic Stability

After having shown that the norm of every solution of the initial value problem (3.19) is non-increasing in the previous chapter, we now prove that the norm even tends to zero as $t \to \infty$. We refer to this as asymptotic stability of the C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$ generated by the dissipative operator A defined in (2.3).

Definition 4.1 (Asymptotic Stability)**.** A C_0-semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space X is said to be *asymptotically stable* if for every $z \in X : \|S(t)z\| \to 0$ as $t \to \infty$.

To show asymptotic stability of the semigroup $\{T(t)\}_{t\geq 0}$ from Chapter 3, we apply the following theorem (see [7], Theorem 3.26):

Theorem 4.2. *Let $\{S(t)\}_{t\geq 0}$ be a uniformly bounded C_0-semigroup in a Banach space X with generator $\mathcal{A}$, and assume that the resolvent $R(\lambda, \mathcal{A})$ is compact for some $\lambda \in \rho(\mathcal{A})$. Then $\{S(t)\}_{t\geq 0}$ is asymptotically stable if and only if* $\operatorname{Re}\lambda < 0$ *for all $\lambda \in \sigma(\mathcal{A})$.*

Remark 4.3. Lemma B.1 shows that it is sufficient to show that the resolvent $R(\lambda, \mathcal{A})$ is compact for one $\lambda = \lambda_0 \in \rho(\mathcal{A})$, as this already implies the compactness of $R(\lambda, \mathcal{A})$ for all $\lambda \in \rho(\mathcal{A})$.

Before we apply Theorem 4.2, we first need to show the following lemma:

Lemma 4.4. *Under the assumptions of Theorem 3.1, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subset \rho(A)$ holds.*

Proof. As a consequence of the Hille-Yoshida theorem (see [9], Corollary 1.3.6), for any infinitesimal generator of a C_0-semigroup of contractions we have the inclusion $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(A)$. Hence, it remains to prove that $\mathrm{i}\mathbb{R} \subset \rho(A)$.
We already showed in Lemma 3.9 that the operator A^{-1} is compact. Therefore, its spectrum $\sigma(A^{-1})$ is at most countable and consists only of the point spectrum, i.e. $\sigma(A^{-1}) = \sigma_p(A^{-1})$. From this, we immediately get $\sigma(A) = \sigma_p(A) = \sigma_p(A^{-1})^{-1}$. Thus, it is sufficient to show that $A - \mathrm{i}\tau$ is injective for all $\tau \in \mathbb{R}$, that is to show that the system

$$\begin{bmatrix} v - \mathrm{i}\tau w \\ (Pw')' - \mathrm{i}\tau v \\ -w'(L) - \mathrm{i}\tau v(L) \\ \vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0) - \mathrm{i}\tau v(0) \end{bmatrix} = 0 \tag{4.1}$$

only has the trivial solution in $D(A)$. We can rewrite this system in terms of the following equivalent boundary value problem (BVP) for $w \in H^3(0, L) \hookrightarrow C^2[0, L]$:

$$(Pw')' + \tau^2 w = 0, \qquad x \in (0, L), \tag{4.2a}$$
$$w'(0) = c_0 w(0), \tag{4.2b}$$
$$w'(L) = c_L w(L), \tag{4.2c}$$

where $c_0 := -\frac{\vartheta_3+\tau^2+\mathrm{i}\tau\vartheta_1}{\vartheta_4+\mathrm{i}\tau\vartheta_2}$ and $c_L := \tau^2$. It is important to note that the conditions (3.1) on the ϑ_i imply that $c_0 \notin \mathbb{R}$ for all $\tau \in \mathbb{R}$. We now multiply equation (4.2a) by the complex conjugate $\bar{w}$ and integrate by parts, which yields

$$-\int_0^L P|w'|^2\,\mathrm{d}x + \tau^2\|w\|_{L^2}^2 + P(L)w'(L)\bar{w}(L) = P(0)w'(0)\bar{w}(0).$$

Due to the boundary conditions (4.2b) and (4.2c) the left hand side of above identity is real, but the right hand side is either non-real or zero. Thus $w'(0)\bar{w}(0) = 0$, and (4.2b) implies that $w(0) = w'(0) = 0$. Therefore, every solution of the boundary value problem (4.2) also satisfies the initial value problem

$$\begin{aligned} (Pw')' + \tau^2 w &= 0, \qquad x \in (0, L), \\ w(0) &= 0, \\ w'(0) &= 0. \end{aligned}$$

Hence, $w \equiv 0$, and this shows that $A - \mathrm{i}\tau$ is injective for all $\tau \in \mathbb{R}$. □

Now we can prove the main statement of this chapter:

Theorem 4.5. *Let the assumptions of Theorem 3.1 hold. Then the C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$ generated by A is asymptotically stable.*

Proof. In Lemma 3.9 we showed that $0 \in \rho(A)$ and that $A^{-1} = R(0, A)$ is compact. By Lemma 4.4 we have $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$. Therefore, our assertion follows from Theorem 4.2. □

Chapter 5

Exponential Stability

Here we show an even stronger result than asymptotic stability, namely exponential stability of the semigroup $\{T(t)\}_{t\geq 0}$, i.e. we prove that every solution of the initial value problem (3.19) tends to zero exponentially. We follow a strategy similar to the one applied in [8].

Definition 5.1 (Exponential stability). A C_0-semigroup $\{S(t)\}_{t\geq 0}$ is said to be *exponentially stable* if there exist constants $M \geq 1$ and $\omega > 0$ such that $\|S(t)\| \leq M \exp(-\omega t)$ for all $t \geq 0$.

To investigate exponential stability of a C_0-semigroup, we use the following theorem (see [7], Corollary 3.36, and [4]):

Theorem 5.2 (Huang). *Let $\{S(t)\}_{t\geq 0}$ be a uniformly bounded C_0-semigroup in a Hilbert space, and let $\mathcal{A}$ be its generator. Then $\{S(t)\}_{t\geq 0}$ is exponentially stable if and only if* $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$ *and*

$$\sup_{\tau\in\mathbb{R}} \|R(\mathrm{i}\tau, \mathcal{A})\| < \infty. \tag{5.1}$$

Remark 5.3. We know from the previous chapter that $\mathrm{i}\mathbb{R} \subset \rho(A)$. The map $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$ (cf. [13]), so, in particular, $\lambda \mapsto \|R(\lambda, A)\|$ is continuous on $\mathrm{i}\mathbb{R}$. Therefore it remains to prove that $\|R(\mathrm{i}\tau, A)\|$ is uniformly bounded as $|\tau| \to \infty$.

Now we are able to prove the main theorem of this chapter:

Theorem 5.4. *Assume that the conditions in Theorem 3.1 are satisfied. Then the C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$ generated by A is exponentially stable.*

Proof. In order to prove that $\|R(\mathrm{i}\tau, A)\|$ is uniformly bounded for $|\tau| \to \infty$, we need to find an appropriate estimate for the solution $z = (w, v, v(L), v(0))$ of the equation

$$(A - i\tau)z = (f, g, g(L), g(0)) \in \mathcal{H} \tag{5.2}$$

in terms of the right hand side. We show that the unique solution[1] (w, v) of the BVP

$$\begin{aligned}
v - \mathrm{i}\tau w &= f, \qquad x \in (0, L), && (5.3a)\\
(Pw')' - \mathrm{i}\tau v &= g, \qquad x \in (0, L), && (5.3b)\\
-w'(L) - \mathrm{i}\tau v(L) &= g(L), && (5.3c)\\
\vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0) - \mathrm{i}\tau v(0) &= g(0) && (5.3d)
\end{aligned}$$

satisfies the estimate

$$\|w\|_{H^2} + \|v\|_{H^1} \leq C(\|f\|_{H^2} + \|g\|_{H^1}) \tag{5.4}$$

uniformly for all $f \in H^2(0, L)$, $g \in H^1(0, L)$ and for all $|\tau|$ sufficiently large.
Since v and w are directly related via equation (5.3a), we replace v in (5.3b)-(5.3d) by $v = f + \mathrm{i}\tau w$

[1] See Lemma 4.4.

to obtain the following BVP for w :

$$\begin{aligned}
(Pw')' + \tau^2 w &= g + \mathrm{i}\tau f, \qquad x \in (0,L), && (5.5a)\\
-w'(L) + \tau^2 w(L) &= (g+\mathrm{i}\tau f)(L), && (5.5b)\\
\underbrace{(\vartheta_4 + \mathrm{i}\tau\vartheta_2)}_{=:\gamma_1} w'(0) + \underbrace{(\vartheta_3 + \tau^2 + \mathrm{i}\tau\vartheta_1)}_{=:\gamma_2} w(0) &= (g+\mathrm{i}\tau f)(0) - \vartheta_1 f(0) - \vartheta_2 f'(0). && (5.5c)
\end{aligned}$$

With this, we first show the desired estimate for w.

Step 1: Homogeneous boundary conditions.
To begin with, we shall transform (5.5) into a BVP with homogeneous boundary conditions. To this end, we use (5.5a) to eliminate the terms $w(0)$ and $w(L)$. This yields, after differentiating (5.5a), the following BVP for $\tilde{y} := Pw'$:

$$\begin{aligned}
\tilde{y}'' + \frac{\tau^2}{P}\tilde{y} &= g' + \mathrm{i}\tau f', \qquad x \in (0,L), && (5.6a)\\
\tilde{y}(L) + P(L)\tilde{y}'(L) &= 0, && (5.6b)\\
\frac{\gamma_1}{P(0)}\tilde{y}(0) - \frac{\gamma_2}{\tau^2}\tilde{y}'(0) &= \underbrace{-\frac{g(0)}{\tau^2}(\vartheta_3 + \mathrm{i}\tau\vartheta_1) - \frac{\mathrm{i}\vartheta_3}{\tau} f(0) - \vartheta_2 f'(0)}_{=:R_1}. && (5.6c)
\end{aligned}$$

In order to make the second boundary condition homogeneous, we determine a first order polynomial $h(x) = a_1 x + a_0$, such that $h(x)$ satisfies the boundary conditions (5.6b),(5.6c). The coefficients can be determined uniquely:

$$a_1 = -\frac{\tau^2 P(0) R_1}{\gamma_1 \tau^2 (L + P(L)) + P(0)\gamma_2}, \qquad a_0 = -(L + P(L))a_1. \tag{5.7}$$

We note that, as already mentioned in the previous section, $\gamma_1/\gamma_2 \notin \mathbb{R}$, and so a_1 is always well defined. For $|\tau| > 1$ we find the estimate

$$|a_i| \le \frac{C}{\tau^2}(\|g\|_{H^1} + |\tau|\|f\|_{H^2}), \tag{5.8}$$

by using the continuous embedding $H^k(0,L) \hookrightarrow C^{k-1}[0,L]$ in one dimension (cf. [1]) to estimate the terms occuring in R_1. Now, the function $y := \tilde{y} - h$ satisfies the following problem with homogeneous boundary conditions:

$$\begin{aligned}
y'' + \frac{\tau^2}{P} y &= H := g' + \mathrm{i}\tau f' - \frac{\tau^2}{P} h, \qquad x \in (0,L), && (5.9a)\\
y(L) + P(L) y'(L) &= 0, && (5.9b)\\
\frac{\gamma_1}{P(0)} y(0) - \frac{\gamma_2}{\tau^2} y'(0) &= 0. && (5.9c)
\end{aligned}$$

We note that the function H can be split into two parts $H = H_g + H_f$, such that H_g only depends on g, and H_f only on f.

Step 2: Solution estimate.
Now we determine the solution of (5.9). Let $\{\varphi_1, \varphi_2\}$ be a basis of solutions of the homogeneous equation $y'' + \frac{\tau^2}{P} y = 0$. Then, the general solution of the inhomogeneous equation (5.9a) can be obtained by variation of constants:

$$\begin{aligned}
y(x) &= c_1\varphi_1(x) + c_2\varphi_2(x) + \int_0^x H(t) \frac{\varphi_1(t)\varphi_2(x) - \varphi_2(t)\varphi_1(x)}{\varphi_1(t)\varphi_2'(t) - \varphi_1'(t)\varphi_2(t)}\,\mathrm{d}t && (5.10)\\
&= c_1\varphi_1(x) + c_2\varphi_2(x) + \int_0^x H(t) J(x,t)\,\mathrm{d}t, && (5.11)
\end{aligned}$$

where $J(x,t)$ is the Green's function introduced in Lemma C.1, and $c_i \in \mathbb{C}$ are arbitrary constants. The derivative $y'(x)$ satisfies

$$y'(x) = c_1\varphi_1'(x) + c_2\varphi_2'(x) + \int_0^x H(t)\partial_x J(x,t)\,\mathrm{d}t. \tag{5.12}$$

In order to determine the constants c_i we now specify the initial conditions of the solutions φ_1, φ_2 :

$$\begin{aligned} \varphi_1(0) &= 0, & \varphi_2(0) &= 1,\\ \varphi_1'(0) &= \tau, & \varphi_2'(0) &= 0. \end{aligned}$$

We point out that these conditions imply that the functions φ_i are real-valued. From the boundary conditions (5.9b),-(5.9c) we then find

$$c_1 = \frac{-\int_0^L H(t)J(L,t)\,\mathrm{d}t - P(L)\int_0^L H(t)\partial_x J(L,t)\,\mathrm{d}t}{\varphi_1(L) + P(L)\varphi_1'(L) + \frac{\gamma_2 P(0)}{\gamma_1\tau}[\varphi_2(L) + P(L)\varphi_2'(L)]}, \qquad c_2 = \frac{\gamma_2 P(0)}{\gamma_1\tau}c_1. \tag{5.13}$$

Again, since $\gamma_2/\gamma_1 \notin \mathbb{R}$ and φ_1, φ_2 are linearly independent, the coefficients c_1, c_2 are well defined. Next we estimate these coefficients. First, we find that for $|\tau| \to \infty$ we have

$$\frac{\gamma_2 P(0)}{\gamma_1\tau} \to -\frac{\mathrm{i}P(0)}{\vartheta_2}.$$

Therefore, we can find some constant $C > 0$, independent of τ, such that the denominator N of c_1 can be estimated as follows:

$$\begin{aligned} |N|^2 &:= \left|\varphi_1(L) + P(L)\varphi_1'(L) + \frac{\gamma_2 P(0)}{\gamma_1\tau}[\varphi_2(L) + P(L)\varphi_2'(L)]\right|^2 \\ &\geq C\left(|\varphi_1(L) + P(L)\varphi_1'(L)|^2 + |\varphi_2(L) + P(L)\varphi_2'(L)|^2\right). \end{aligned}$$

From the initial conditions of φ_1, φ_2 and Lemma C.1 we find that the Wronskian satisfies $\varphi_1'(L)\varphi_2(L) - \varphi_1(L)\varphi_2'(L) = \tau$. Since $\|\varphi_i\|_{L^\infty}$ is uniformly bounded for all τ sufficiently large by Lemma C.2, we conclude that $|\varphi_1'(L)| + |\varphi_2'(L)| \geq C\tau$, for some constant $C > 0$ independent of τ. With this result, we obtain the estimate

$$|N| \geq C|\tau|, \tag{5.14}$$

for all $|\tau| > 1$, and C independent of τ.
Now it remains to estimate the integrals occuring in c_1 and those in (5.11) and (5.12). Due to the estimates for h found in (5.8), by applying Theorem C.3 on H_f and the Cauchy-Schwarz inequality on the integral containing H_g, we find that

$$\left\|\int_0^x H(t)J(x,t)\,\mathrm{d}t\right\|_{L^\infty} \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.15}$$

$$\left\|\int_0^x H(t)\partial_x J(x,t)\,\mathrm{d}t\right\|_{L^\infty} \leq C(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.16}$$

with $C > 0$ independent of τ, and for all $|\tau| > 1$. Therefore we conclude that the estimate $|c_i| \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2})$ holds uniformly in τ. Applying these results and the estimates found in Lemma C.2 to the basis-functions φ_1, φ_2, we find that the following estimates hold uniformly for $|\tau| > 1$:

$$\|y\|_{L^2} \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.17}$$

$$\|y\|_{H^1} \leq C(\|g\|_{H^1} + \|f\|_{H^2}). \tag{5.18}$$

Using (5.8), we see that the same estimates hold for $\tilde{y}$. Furthermore, by using $\tilde{y} = Pw'$ and the equation (5.5a), we find

$$\|w\|_{H^1} \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.19}$$

$$\|w\|_{H^2} \leq C(\|g\|_{H^1} + \|f\|_{H^2}). \tag{5.20}$$

For the L^2-estimate we used here (5.5b):

$$|w(L)| \leq \frac{1}{\tau^2}\left|\frac{\tilde{y}(L)}{P(L)} + g(L) + \mathrm{i}\tau f(L)\right| \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}).$$

Finally, from equation (5.3a) and by using (5.19) we get the desired estimate

$$\|v\|_{H^1} \leq C(\|g\|_{H^1} + \|f\|_{H^2}),$$

which completes the proof. □

Alternative Version of the Proof

Here we present an alternative proof of Theorem 5.4. In the above proof, a "direct" estimate of w in the H^2-norm was avoided by intruducing a new function y, for which only the H^1-norm was needed. With the estimates from Appendix C.1 applied in this proof, a direct estimate of the H^2-norm would not have been uniformly bounded. In the following, alternative, proof of Theorem 5.4, the desired uniform H^2-estimate of w is obtained in a more direct way. But for this a more careful analysis of the occuring terms and more precise estimates are necessary.

Proof of Theorem 5.4 (alternative version). In order to show that $\|R(\mathrm{i}\tau, A)\|$ is uniformly bounded for $|\tau| \to \infty$, we need to prove the uniform estimate

$$\|w\|_{H^2} + \|v\|_{H^1} \leq C(\|f\|_{H^2} + \|g\|_{H^1}) \tag{5.4}$$

for the solutions of the system (5.3). Analogously to the the above proof, we replace $v = f + \mathrm{i}\tau w$ in (5.3b)-(5.3d) to obtain the following BVP for w :

$$(Pw')' + \tau^2 w = g + \mathrm{i}\tau f, \qquad x \in (0, L), \tag{5.21a}$$
$$-w'(L) + \tau^2 w(L) = R_L := (g + \mathrm{i}\tau f)(L), \tag{5.21b}$$
$$\underbrace{(\vartheta_4 + \mathrm{i}\tau\vartheta_2)}_{=:\gamma_1} w'(0) + \underbrace{(\vartheta_3 + \tau^2 + \mathrm{i}\tau\vartheta_1)}_{=:\gamma_2} w(0) = R_0 := (g + \mathrm{i}\tau f)(0) - \vartheta_1 f(0) - \vartheta_2 f'(0). \tag{5.21c}$$

With this, we first show the desired estimate for w.

Step 1: Homogeneous boundary conditions.
To begin with, we transform (5.21) into a BVP with homogeneous boundary conditions. For this, we determine a function

$$h(x) = \int_0^x \frac{a_1}{P(t)}\,\mathrm{d}t + a_0, \qquad a_i \in \mathbb{C},$$

such that $h(x)$ satifies the boundary conditions (5.21b)-(5.21c). The coefficients are

$$a_0 = \frac{\frac{\gamma_1 R_L}{P(0)} + R_0(\frac{1}{P(L)} - \tau^2 I_P)}{\frac{\gamma_1\tau^2}{P(0)} + \gamma_2(\frac{1}{P(L)} - \tau^2 I_P)}, \qquad a_1 = \frac{R_0\tau^2 - R_L\gamma_2}{\frac{\gamma_1\tau^2}{P(0)} + \gamma_2(\frac{1}{P(L)} - \tau^2 I_P)}, \tag{5.22}$$

where $I_P := \int_0^L \frac{1}{P}\,\mathrm{d}t$. Since $\gamma_1/\gamma_2 \notin \mathbb{R}$, these coefficients are well-defined. For $|\tau| > 1$ the following estimate holds uniformly:

$$|a_i| \leq \frac{C}{\tau^2}(\|g\|_{H^1} + |\tau|\|f\|_{H^2}). \tag{5.23}$$

Setting $y = w - h$ yields the following BVP with homogeneous boundary conditions:

$$(Py')' + \tau^2 y = H := g + \mathrm{i}\tau f - \tau^2 h, \qquad x \in (0, L), \tag{5.24a}$$
$$-y'(L) + \tau^2 y(L) = 0, \tag{5.24b}$$
$$\gamma_1 y'(0) + \gamma_2 y(0) = 0. \tag{5.24c}$$

We note that the function H can be split into two parts $H = H_g + H_f$, such that H_g only depends on g, and H_f only on f. We obtain the following estimates for those components:

$$\|H_g\|_{H^1} \leq C\|g\|_{H^1}, \qquad \|H_f\|_{H^2} \leq C|\tau|\|f\|_{H^2}, \tag{5.25}$$

$$|H_g(x)| \leq \tfrac{C}{|\tau|}\|g\|_{H^1}, \qquad |H_f(x)| \leq C\|f\|_{H^2}, \qquad x \in \{0, L\}, \tag{5.26}$$

which hold uniformly for $|\tau| > 1$. For (5.26), the expression $g + \mathrm{i}\tau f - \tau^2 h$ is evaluated explicitly (without using the estimate (5.23)).

Step 2: Solution estimate.
Now we to determine the solution of (5.24). Let $\{\varphi_1, \varphi_2\}$ be a basis of solutions of the homogeneous equation $(Py')' + \tau^2 y = 0$. Then the general solution of the inhomogeneous equation (5.24a) is obtained by variation of constants:

$$y(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \int_0^x H(t)J(x,t)\,\mathrm{d}t, \tag{5.27}$$

where $J(x,t)$ is the Green's function explicitly given in (C.9). The first derivative satisfies

$$y'(x) = c_1\varphi_1'(x) + c_2\varphi_2'(x) + \int_0^x H(t)\partial_x J(x,t)\,\mathrm{d}t. \tag{5.28}$$

In order to determine the coefficients c_1, c_2, we specify the initial conditions of the basis functions φ_1, φ_2 :

$$\begin{array}{rclcrcl} \varphi_1(0) & = & 0, & \qquad & \varphi_2(0) & = & 1, \\ \varphi_1'(0) & = & \tau, & & \varphi_2'(0) & = & 0. \end{array} \tag{5.29}$$

These conditions imply that the solutions φ_i are real-valued. From the boundary conditions (5.24b)-(5.24c) we get

$$c_1 = \frac{\int_0^L H(t)\partial_x J(L,t)\,\mathrm{d}t - \tau^2\int_0^L H(t)J(L,t)\,\mathrm{d}t}{-\varphi_1'(L) + \tau^2\varphi_1(L) - \frac{\gamma_1\tau}{\gamma_2}[-\varphi_2'(L) + \tau^2\varphi_2(L)]}, \qquad c_2 = -\frac{\gamma_1\tau}{\gamma_2}c_1. \tag{5.30}$$

Again, since $\frac{\gamma_1}{\gamma_2} \notin \mathbb{R}$, and the φ_i are linearly independent, the coefficients are well-defined for all $\tau \in \mathbb{R}$. In order to find an estimate for the c_i, we first note that $\frac{\gamma_1\tau}{\gamma_2} \to \mathrm{i}\vartheta_2$ for $|\tau| \to \infty$. Therefore, we obtain the following estimate of the denominator N of c_1 :

$$|N|^2 \geq C\left([-\varphi_1'(L) + \tau^2\varphi_1(L)]^2 + [-\varphi_2'(L) + \tau^2\varphi_2(L)]^2\right),$$

uniformly for $|\tau| > 1$. Due to Lemma C.4 and the initial conditions specified above, we find that the Wronskian $W(t) := \varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t)$ satisfies $P(t)W(t) \equiv P(0)\tau$. From this, it follows that $|\varphi_1(L)| + |\varphi_2(L)| \geq C > 0$ holds uniformly, since the derivatives occuring in $W(t)$ satisfy $|\varphi_i'(L)| \leq C|\tau|$ due to Lemma C.5. Therefore, we get the estimate $|N| \geq C\tau^2$ for the denominator N.
By seperating $H = H_g + H_f$, we find the following estimates from Theorem C.6 due to the inequalities (5.25)-(5.26):

$$\left|\int_0^L H(t)J(L,t)\,\mathrm{d}t\right| \leq \frac{C}{\tau^2}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.31}$$

$$\left|\int_0^L H(t)\partial_x J(L,t)\,\mathrm{d}t\right| \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}). \tag{5.32}$$

With this, we can estimate the numerator in the c_i. Together with the estimate $|N| \geq C\tau^2$ found above, we obtain, uniformly in $|\tau| > 1$:

$$|c_i| \leq \frac{C}{\tau^2}(\|g\|_{H^1} + \|f\|_{H^2}). \tag{5.33}$$

For the integrals in (5.27), (5.28), Theorem C.6 yields

$$\left\| \int_0^x H(t)J(x,t)\,\mathrm{d}t \right\|_{L^2} \le \frac{C}{\tau^2}(\|g\|_{H^1} + |\tau|\|f\|_{H^2}), \tag{5.34}$$

$$\left\| \int_0^x H(t)\partial_x J(x,t)\,\mathrm{d}t \right\|_{L^2} \le \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}). \tag{5.35}$$

Combining the estimates (5.33)-(5.35) and the estimates (C.6) for φ_i, we get

$$\|y\|_{H^1} \le \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.36}$$

uniformly for $|\tau| > 1$.

To estimate y'', we first consider $(Py')'$. By (C.8) we obtain

$$\partial_x \left(P(x)\partial_x \int_0^x H(t)J(x,t)\,\mathrm{d}t \right) = -\tau^2 \int_0^x H(t)J(x,t)\,\mathrm{d}t + H(x).$$

Hence, Theorem C.6 implies that

$$\left\| \partial_x \left(P(x)\partial_x \int_0^x H(t)J(x,t)\,\mathrm{d}t \right) \right\|_{L^2} \le C(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.37}$$

uniformly for $|\tau| > 1$. Using the estimates (C.6) for $(P\varphi_i')' = -\tau^2\varphi_i$ then yields

$$\|(Py')'\|_{L^2} \le C(\|g\|_{H^1} + \|f\|_{H^2}),$$

uniformly for $|\tau| > 1$. Using $y'' = \frac{1}{P}[(Py')' - P'y']$ and estimate (5.36) we obtain

$$\|y\|_{H^2} \le C(\|g\|_{H^1} + \|f\|_{H^2}), \tag{5.38}$$

uniformly for $|\tau| > 1$. Due to (5.23), the same estimates (5.36), (5.38) also hold for $w = y + h$. From the estimate for $\|w\|_{H^1}$, by using (5.3a) we find that

$$\|v\|_{H^1} \le C(\|g\|_{H^1} + \|f\|_{H^2}),$$

which proves the theorem. □

Appendix A

Useful Inequalities

Lemma A.1. *Let $a_0, b_0, \varepsilon_0 > 0$ be given. Then there exist constants $c, d > 0$ such that*

$$(ax_1 + bx_2)^2 + \varepsilon x_1^2 \geq cx_1^2 + dx_2^2 \tag{A.1}$$

holds uniformly for all $x_1, x_2 \in \mathbb{R}$ and $|a| \leq a_0$, $b \geq b_0$ and $\varepsilon \geq \varepsilon_0$.

Proof. Inequality (A.1) can be re-written in the equivalent form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} a^2 + \varepsilon - c & ab \\ ab & b^2 - d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0,$$

where the occuring 2×2-matrix will be denoted as M. Since this inequality has to hold for all $x_1, x_2 \in \mathbb{R}$, it is equivalent to M being positive semi-definite. Applying the Sylvester criterion yields the following conditions:

$$\begin{aligned} b^2 - d &\geq 0, & \text{(A.2)} \\ (\varepsilon - c)(b^2 - d) &\geq a^2 d. & \text{(A.3)} \end{aligned}$$

If $a = 0$, we can take $c = \varepsilon_0$ and $d = b_0^2$. Otherwise, we see from condition (A.3) that $d \neq b^2$, so that it can be written as:

$$c \leq \varepsilon - a^2 \frac{d}{b^2 - d}. \tag{A.4}$$

Because of monotonicity of the right hand side we find the estimate

$$\varepsilon - a^2 \frac{d}{b^2 - d} \geq \varepsilon_0 - a_0^2 \frac{d}{b_0^2 - d}.$$

So, for (A.4) to hold, it is sufficient that c, d satisfy the stricter inequality

$$c \leq \varepsilon_0 - a_0^2 \frac{d}{b_0^2 - d}. \tag{A.5}$$

For d sufficiently small, the right hand side becomes positive, and therefore a $c > 0$ satisfying (A.5) exists. □

Lemma A.2. *Let $\alpha, \beta, \delta \in \mathbb{R}$ and*

$$P_3(x_1, x_2, x_3) := x_1^2 + x_2^2 + x_3^2 + 2\alpha x_1 x_2 + 2\beta x_2 x_3 + 2\delta x_1 x_3$$

be a polynomial. Then the inequality $P_3(x_1, x_2, x_3) \geq 0$ holds for all $x_1, x_2, x_3 \in \mathbb{R}$ if and only if the coefficients satisfy the conditions

$$\begin{gathered} \alpha^2 \leq 1, \quad \beta^2 \leq 1, \quad \delta^2 \leq 1, \\ \alpha^2 + \beta^2 + \delta^2 \leq 1 + 2\alpha\beta\delta. \end{gathered}$$

Proof. The polynomial can be written as

$$P_3(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 1 & \alpha & \delta \\ \alpha & 1 & \beta \\ \delta & \beta & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

with M denoting the 3×3 matrix. Now the property $P_3(x_1, x_2, x_3) \geq 0$, $\forall x_1, x_2, x_3 \in \mathbb{R}$ is equivalent to M being positive semi-definite. Applying the Sylvester criterion to M yields exactly the desired conditions. □

Appendix B

Compact Resolvents

Here we present a useful result concerning compact resolvents of linear operators. A proof of the following lemma can be found in [5]. Here we give an alternative, shorter proof.

Lemma B.1. *Let $\mathcal{A}$ be a closed, densely defined linear operator in a Banach space X. If there exists a $\lambda_0 \in \rho(\mathcal{A})$ such that the resolvent $R(\lambda_0, \mathcal{A})$ is a compact operator, then $R(\lambda, \mathcal{A})$ is compact for all $\lambda \in \rho(\mathcal{A})$.*

Proof. Let $\lambda \in \rho(\mathcal{A})$. Then we can write

$$R(\lambda, \mathcal{A}) = R(\lambda_0, \mathcal{A})(\mathcal{A} - \lambda_0)R(\lambda, \mathcal{A}). \tag{B.1}$$

Since $R(\lambda, \mathcal{A})$ is continuous, and $(\mathcal{A} - \lambda_0)$ is closed, we find that $(\mathcal{A} - \lambda_0)R(\lambda, \mathcal{A}) : X \to X$ is closed and also continuous, since its domain is the entire space X. Therefore, (B.1) is the decomposition of $R(\lambda, \mathcal{A})$ into the product of a compact and a continuous operator, and hence it is also compact (cf. [13]). □

Appendix C

ODEs with a Parameter: Uniform Estimates

Before we present the results of this section, we briefly discuss an introductory example in order to motivate the following analysis. Consider the homogeneous linear ordinary differential equation

$$y'' + \tau^2 y = 0, \quad x \in (0, L),\ \tau > 0. \tag{C.1}$$

The functions $\sin \tau x$, $\cos \tau x$ form a basis of the solutions of (C.1). From the theory of Fourier-Integrals it is known that for any $f \in L^1(0, L)$ the following properties hold (for more details, see [12]):

$$\lim_{\tau\to\infty} \int_0^L \cos \tau x f(x)\, \mathrm{d}x = 0, \quad \lim_{\tau\to\infty} \int_0^L \sin \tau x f(x)\, \mathrm{d}x = 0. \tag{C.2}$$

If, for example, the stronger requirement $f \in W^{1,1}(0, L)$ is assumed, it can even be shown that the above integrals are $O(\tau^{-1})$.

In the following we consider appropriate generalizations of (C.1) and show results analogous to (C.2), and also give information about the order of convergence.

C.1 Estimates – Part 1

In this section we discuss the behaviour of the classical solutions $y \in C^2[0, L]$ of the equation

$$y'' + \frac{\tau^2}{P(x)} y = 0, \qquad x \in (0, L), \tag{C.3}$$

where $\tau \in \mathbb{R}$ and $P \in C^1[0, L]$ is a real-valued function satisfying $P^0 \le P(x) \le P^1$ uniformly for $x \in [0, L]$ for some positive constants P^0, P^1. Since τ only occours squared, we can assume that $\tau \ge 0$ holds in the following.

Lemma C.1 ([3])**.** *Let (φ_1, φ_2) be an arbitrary pair of linearly independent solutions of (C.3). Then the Green's function of the equation is given by*

$$J(x, t) := \frac{\varphi_1(x)\varphi_2(t) - \varphi_2(x)\varphi_1(t)}{\varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t)}. \tag{C.4}$$

Furthermore, the Wronskian $W(t) := \varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t)$ is constant for $t \in [0, L]$. Hence, (C.4) *simplifies to $J(x, t) = C[\varphi_1(x)\varphi_2(t) - \varphi_2(x)\varphi_1(t)]$.*

With the prescribed initial data $\varphi(0)$ and $\varphi'(0)$, we shall denote the unique classical solution of (C.3) by φ_τ. The behaviour of solutions of (C.3) is stated in the following Lemma. For the proof, see Prop. 2.1 in [2].

Lemma C.2. *There exists a constant $C > 0$ such that for any family of solutions $\{\varphi_\tau\}_{\tau>1}$ of (C.3) the following estimates hold uniformly for $\tau > 1$:*

$$\|\varphi_\tau\|_{L^\infty} \le \frac{C}{\tau}\left(\tau|\varphi_\tau(0)| + |\varphi_\tau'(0)|\right), \qquad \|\varphi_\tau'\|_{L^\infty} \le C\left(\tau|\varphi_\tau(0)| + |\varphi_\tau'(0)|\right).$$

Now we are able to prove the following theorem:

Theorem C.3. *Let $\{J_\tau\}_{\tau>1}$ be the family of Green's functions defined in Lemma C.1. Then there exists a constant $C > 0$ such that the following estimates hold uniformly for all $f \in H^1(0,L)$ and $\tau > 1$:*

$$\left\| \int_0^x f(t) J_\tau(x,t)\,\mathrm{d}t \right\|_{L^\infty} \le \frac{C}{\tau^2}\|f\|_1, \tag{C.5}$$

$$\left\| \int_0^x f(t) \partial_x J_\tau(x,t)\,\mathrm{d}t \right\|_{L^\infty} \le \frac{C}{\tau}\|f\|_1. \tag{C.6}$$

Proof. We are going to show (C.5), the proof of (C.6) can be done analogously. The index τ is omitted for sake of simplicity. First, we make the substitution $t = x-\xi$ in the left hand integral, and define the family of functions $\psi_x : \xi \mapsto J(x, x-\xi)$ with parameter x. These functions are solutions of the equation

$$\psi_x'' + \frac{\tau^2}{P(x-\xi)}\psi_x = 0, \tag{C.7}$$

with $'$ denoting here derivatives with respect to ξ. ψ_x takes the initial values $\psi_x(\xi = 0) = 0$ and $\psi_x'(\xi = 0) = 1$. Now, integrating by parts yields

$$\begin{aligned}
\left| \int_0^x f(x-\xi)\psi_x(\xi)\,\mathrm{d}\xi \right| &= \left| -\int_0^x \partial_\xi(fP)(x-\xi) \int_0^\xi \frac{\psi_x(\zeta)}{P(x-\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi + f(0)P(0)\int_0^x \frac{\psi_x(\zeta)}{P(x-\zeta)}\,\mathrm{d}\zeta \right| \\
&\le 2\frac{\|\psi_x'\|_{L^\infty}}{\tau^2}\left(\int_0^x |\partial_\xi(fP)(x-\xi)|\,\mathrm{d}\xi + |f(0)P(0)| \right) \\
&\le C\frac{\|\psi_x'\|_{L^\infty}\|f\|_1}{\tau^2},
\end{aligned}$$

where we used (C.7) in the second step. And in the last step we used the continuous embedding $H^1(0,L) \hookrightarrow C[0,L]$. From Lemma C.2 and the known initial conditions of ψ_x we find that $\|\psi_x'\|_{L^\infty}$ is uniformly bounded for all $\tau > 1$, which proves (C.5). □

C.2 Estimates – Part 2

In this section we discuss the behaviour of the classical solutions $y \in C^2[0,L]$ of the equation

$$(Py')' + \tau^2 y = 0, \tag{C.8}$$

where $\tau \in \mathbb{R}$ and $P \in C^1[0,L]$ is a real-valued function satisfying $P^0 \le P(x) \le P^1$ uniformly for $x \in [0,L]$ for some positive constants P^0, P^1. Since τ only occours squared, we can assume that $\tau \ge 0$ holds in the following.

Lemma C.4 ([3])**.** *Let (φ_1, φ_2) be a arbitrary pair of linearly independent solutions of (C.8). Then the Green's function of the equation is given by*

$$J(x,t) := \frac{\varphi_1(x)\varphi_2(t) - \varphi_2(x)\varphi_1(t)}{P(t)[\varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t)]} \tag{C.9}$$

Furthermore, the Wronskian $W(t) = \varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t)$ satisfies: $P(t)W(t)$ is constant on $[0,L]$. Hence, (C.9) *simplifies to $J(x,t) = C[\varphi_1(x)\varphi_2(t) - \varphi_2(x)\varphi_1(t)]$.*

The behaviour of solutions of (C.8) is stated in the following lemma:

Lemma C.5. *There exists some constant $C > 0$ such that for any family of solutions $\{\varphi_\tau\}_{\tau>1}$ of (C.8) the following estimates hold uniformly for all $\tau > 1$:*

$$\|\varphi_\tau\|_{L^\infty} \leq \frac{C}{\tau}\left(\tau|\varphi_\tau(0)| + |\varphi_\tau'(0)|\right), \qquad \|\varphi_\tau'\|_{L^\infty} \leq C\left(\tau|\varphi_\tau(0)| + |\varphi_\tau'(0)|\right). \tag{C.10}$$

Proof. The functions of the family $u = u_\tau := P\varphi_\tau'$ satisfy the equation

$$u'' + \frac{\tau^2}{P(x)}u = 0, \qquad x \in (0, L). \tag{C.11}$$

From Lemma C.2 above we know that u satisfies the estimates:

$$\|u\|_{L^\infty} \leq \frac{C}{\tau}\left(\tau|u(0)| + |u'(0)|\right), \qquad \|u'\|_{L^\infty} \leq C\left(\tau|u(0)| + |u'(0)|\right). \tag{C.12}$$

From the definition of u and (C.8), we find that $|u(0)| \leq P^1|\varphi'(0)|$ and $|u'(0)| = \tau^2|\varphi(0)|$. On the other hand, we have $\|u\|_{L^\infty} \geq P^0\|\varphi'\|_{L^\infty}$ and $\|u'\|_{L^\infty} = \tau^2\|\varphi\|_{L^\infty}$. With this, we find the desired estimates for φ from the estimates (C.12). □

Now we are able to prove the following

Theorem C.6. *Let $\{J_\tau\}_{\tau>1}$ be the family of Green's functions defined in Lemma C.4. Let $M > 0$ be a constant, and $\{H_\tau\}_{\tau>1}$ a family of functions, satisfying one of the following conditions uniformly in τ :*

(a) $H_\tau \in W^{1,1}(0, L) : \|H_\tau\|_{W^{1,1}} \leq M$,

(b) $H_\tau \in W^{2,1}(0, L) : \|H_\tau\|_{W^{2,1}} \leq \tau M$ *and* $|H_\tau(0)|, |H_\tau(L)| \leq M$.

Then there exists some constant $C > 0$, independent of H_τ and M, such that the following estimates hold uniformly for all $\tau > 1$:

$$\left\|\int_0^x H_\tau(t)J_\tau(x,t)\,\mathrm{d}t - \frac{H(x)}{\tau^2}\right\|_{L^\infty} \leq \frac{CM}{\tau^2}, \tag{C.13}$$

$$\left\|\int_0^x H_\tau(t)\partial_x J_\tau(x,t)\,\mathrm{d}t\right\|_{L^\infty} \leq \frac{CM}{\tau}. \tag{C.14}$$

Proof. For the sake of simplicity we omit the index τ as its role should be clear from the context. We begin by showing (C.13). First, we note that for all $x \in [0, L]$ fixed, the function $\psi : t \mapsto J(x, t)$ is a solution of the homogeneous equation (C.8) with initial values $\psi(x) = 0$, $\partial_t\psi(x) = -1/P(x)$. Thus the following estimates hold uniformly for $\tau > 1$ due to Lemma C.5:

$$\|\psi\|_{L^\infty} \leq \frac{C}{\tau}, \quad \|\partial_t\psi(x)\|_{L^\infty} \leq C. \tag{C.15}$$

Now we can integrate by parts:

$$\int_0^x H(t)J(x,t)\,\mathrm{d}t = -\int_0^x H'\Psi\,\mathrm{d}t + H\Psi\big|_0^x, \tag{C.16}$$

where $\Psi(t) := -\frac{1}{\tau^2}P(t)\psi'(t)$ is a primitive of ψ (use (C.8)), and $'$ denotes here the derivative with respect to t. With this, we get for the case $H \in W^{1,1}(0, L)$:

$$\begin{aligned}\left\|\int_0^x H(t)J(x,t)\,\mathrm{d}t\right\|_{L^\infty} &= \frac{1}{\tau^2}\left\|\int_0^x PH'\psi'\,\mathrm{d}t - PH\psi'\big|_0^x\right\|_{L^\infty} \\ &\leq \frac{C\|H\|_{W^{1,1}}\|\psi'\|_{L^\infty}}{\tau^2} \\ &\leq \frac{CM}{\tau^2},\end{aligned}$$

where we used the continuous embedding $H \in W^{1,1}(0,L) \hookrightarrow C[0,L]$ (see [1]). For the other case, $H \in W^{2,1}(0,L)$, we need to integrate by parts a second time in (C.16):

$$
\begin{aligned}
\int_0^x H(t)J(x,t)\,\mathrm{d}t &= \frac{1}{\tau^2}\left(\int_0^x PH'\psi'\,\mathrm{d}t - PH\psi'\big|_0^x\right) \\
&= \frac{1}{\tau^2}\left(-\int_0^x (PH')'\psi\,\mathrm{d}t - PH\psi'\big|_0^x + PH'\psi\big|_0^x\right) \\
&= \frac{1}{\tau^2}\left(-\int_0^x (PH')'\psi\,\mathrm{d}t + P(0)H(0)\psi'(0) - P(0)H'(0)\psi(0)\right) + \frac{1}{\tau^2}H(x),
\end{aligned}
$$

where the initial conditions of ψ at $t = x$ were used. Due to the assumed estimates on $H(0)$, $H'(0)$, and with (C.15), we find that the terms in the brackets are uniformly bounded by M. The remaining term outside the brackets is the only one not having the right order in τ, but it cancels in (C.13), and we obtain the desired estimate.
The second estimate can be obtained completely analogously. The reduced order in τ is due to the fact that the estimates for $\partial_x J(x,t)$ are one factor of τ "weaker" than those for $J(x,t)$. □

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