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D I P L O M A R B E I T

Approval Voting

A characterization and a compilation of advantages and drawbacks in respect of other voting procedures

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Abstract

This diploma thesis gives a characterization and a compilation of advantages and drawbacks of approval voting in respect of other voting procedures. Approval voting is a voting procedure, where voters can approve of as many candidates as they like, therefore casting a vote for every candidate they approve of. After a short introduction into voting and social choice theory, and the presentation of two discouraging results (Arrow's theorem and the Gibbard-Satterthwaite theorem), the present work evaluates approval voting by some standard social choice criteria. Then, it characterizes approval voting among ballot aggregation functions, it characterizes candidates who can win approval voting elections, and provides advice to voters on what strategies they should employ according to their preference ranking. The main part of this work compiles advantages and disadvantages of approval voting as far as dichotomous, trichotomous and multichotomous preferences, strategy-proofness, election of Pareto and Condorcet candidates, stability of outcomes, Condorcet efficiency, comparison of outcomes to other voting procedures, computational manipulation, vulnerability to majority decisiveness and to the erosion of the majority principle, and subset election outcomes are concerned. Finally it presents some modifications of approval voting to mitigate some of the mentioned drawbacks.

Zusammenfassung

Diese Diplomarbeit enthält eine Charakterisierung und eine Zusammenstellung von Vor- und Nachteilen von "Approval Voting" im Hinblick auf andere Wahlverfahren. Approval Voting ist ein Wahlverfahren, welches den Wählern erlaubt, so vielen Kandidaten ihre Stimme zu geben wie sie möchten, d.h. all jene Kandidaten zu wählen, die sie befürworten. Nach einer kurzen Einführung in die Sozialwahltheorie (engl. social social choice theory bzw. voting theory) und der Präsentation zweier entmutigender Resultate (Arrows Unmöglichkeitssatz und der Satz von Gibbard-Satterthwaite) bewertet die vorliegende Arbeit Approval Voting an Hand einiger bekannter Wahlkriterien. Anschließend charakterisiert sie Approval Voting unter ballot aggregation functions, charakterisiert Kandidaten, die Wahlen mittels Approval Voting gewinnen können und hilft Wählern, sich für die richtige Strategie gemäß ihrer Präferenzen zu entscheiden. Der Hauptteil dieser Arbeit stellt Vor- und Nachteile von Approval Voting zusammen, und zwar im Hinblick auf dichotome, trichotome und multichotome Präferenzen, Manipulierbarkeit, Wahl eines Pareto- und eines Condorcet-Kandidaten, Stabilität der Wahlergebnisse, Condorcet-Effizienz, Vergleich von Wahlergebnissen mit anderen Wahlverfahren, rechnerbetonte Manipulierbarkeit (computational manipulation), Anfälligkeit für Mehrheitsmaßgeblichkeit (majority decisiveness) und für das Aushöhlen des Mehrheitsprinzips, und Wahlergebnisse von Teilmengen von Kandidaten. Schließlich präsentiert sie einige Modifikationen von Approval Voting, um einige der genannten Nachteile zu entschärfen.

Contents

Preface

Democracy must be something more
than two wolves and a sheep
voting on what to have for dinner.

James Bovard

Motivation

Already since grammar school, I have been interested in game theory. In 12^{th} grade I wrote a *Fachbereichsarbeit*, a work for the Austrian school leaving examination (*Matura*), about game theory and analyzed briefly the strategies of the game Master Mind. During my studies of applied mathematics, however, I hardly came into contact with game theory. It was not before Prof. Mehlmann's lecture on game-theoretic modeling (*Spieltheoretische Modellierung*) that I regained interest in this subject and decided to write my diploma thesis in this field. Personally, it was not clear to me from the beginning to study mathematics and computer science. I was also interested in physics, but even more in philosophy, linguistics and sociology. This helps to explain how I decided to write about a topic being right on the intersection point between mathematics, social science and politics: *voting theory.*

Content

This diploma thesis deals with approval voting, a kind of voting procedure where every voter may approve of as many candidates as he or she likes. The first chapter gives a short introduction into voting theory and confronts the reader with two discouraging results: Arrow's theorem and the Gibbard-Satterthwaite theorem. The second chapter presents the most common criteria for voting procedures, defines and evaluates approval voting by these criteria and compares the results with the performance of other voting procedures. Since this approach is not the most satisfactory one, I turn in chapter three to a characterization of approval voting. On the one hand, I present a characterization among *ballot aggregation functions.* This characterization distinguishes approval voting from other voting procedures by three criteria. On the other hand, I present some basic properties of approval voting, including a characterization of candidates who can win approval voting elections. Furthermore, I provide advice to voters on what strategies they should employ according to their preference ranking.

The fourth chapter shows advantages of approval voting in various fields. It also prepares the theoretic background for many issues so that the fifth chapter dealing with the disadvantages of approval voting is a lot shorter thanks to the basics of chapter four.

The sixth chapter gives a little outlook on how approval voting could be modified to mitigate some of the mentioned drawbacks in chapter five.

Acknowledgments

Finally, I want to thank Prof. Mehlmann for the freedom he gave me in determining the field of this work. I also want to thank my friends for their support, especially Natascha Milleschitz, Susanne Huber, Sandra Bichl and Markus Wittberger for proofreading, and Juliane Vymetal for being my "diploma thesis writing partner". A special thanks goes to my mother and the rest of my family who supported me during all these years of my life.

Michael Maurer

1 Voting paradoxes, social choice functions and impossibility theorems

1.1 Voting paradoxes

Suppose a group of nine friends who regularly meet for game evenings and share their games together are planning on what new game to afford. Not everybody, of course, wants to buy the same game, but eventually they can decide on three games to vote for: Poker (P), Monopoly (M) and Twister (T). Everybody has some preferences over the games that group as follows, where for example $P \succ M \succ T$ means that this person strictly prefers Poker to Monopoly to Twister.

Example 1.

4 people: $M \succ P \succ T$
3 people: $T \succ P \succ M$
2 people: $P \succ T \succ M$

As everybody has stated his or her wishes and preferences, what shall our nine friends do now? If everybody just had one vote and therefore voted for their top choice, Monopoly would receive 4 votes, Twister 3 votes and Poker 2 votes. So the overall preference would be $M \succ T \succ P$ and Monopoly would be the winner. This kind of voting is called **plurality voting** and is a very common and popular voting method being used in various single winner elections worldwide, the election for the US president, the US Congress or the UK House of Commons for instance. It incorporates the slogan *one person - one vote*.

Wouldn't it be fairer, however, to employ another form of voting: What about a runoff between the two most preferred games to find out which one really is our friends' darling? If we do that, we will see a runoff between Monopoly and Twister (which got 4 and 3 votes in the first run, respectively), where the first two voter groups will stick to their votes, and the last two people will vote for Twister, since they prefer Twister to Monopoly (regardless of the fact that they would prefer Poker to both of them). So we get an outcome of 4 votes for Monopoly and 5 votes for Twister, so in this case Twister wins! The overall outcome would be $T \succ M \succ P$. This voting method is called **plurality voting with runoff** and is also commonly used, in the election of the French or Austrian president[1], for example.

Thirdly, nobody can prevent us from making pairwise comparisons: If Twister competes directly against Monopoly, we get 5 votes for Twister and 4 votes for

[1] only if none of the candidates gets more than 50% in the first run.

Monopoly[2], so we have $T \succ M$. If Twister directly competes against Poker, we get 3 votes for Twister and 6 for Poker, so $P \succ T$. Finally, a direct competition between Monopoly and Poker will yield an outcome of 4 votes for Monopoly and 5 votes for Poker, so $P \succ M$. Concluding, our friends remark that Poker wins in all pairwise comparisons. Therefore, shouldn't Poker be considered as the most preferred game, since it beats both Monopoly and Twister? The overall outcome of the so called **Condorcet voting** is $P \succ T \succ M$. (This holds because Twister wins once and Monopoly always loses.) Furthermore, since Monopoly loses in all pairwise comparisons, should our nine friends really trust the plurality voting outcome?

The remarkable thing we see here is that the chosen game depends more on the choice of the decision procedure than on the voters' wishes. Even though voter preferences never change, we see that every game can be the winner, only depending on what procedure we use. This can be very dangerous, because if somebody knows about the others' preferences (which is not unlikely in a small group where people know each other well) he or she can intentionally change the voting outcome to his or her favorite outcome only by choosing the appropriate voting procedure.

You can find a similar approach, where Saari lays out the set of an academic department dealing with what beverage to drink at the annual departmental fall banquet [17, pp. 2ff.].

Let us have a look at another example [19, pp. 19-20]. We suppose that a three-body jury has to decide on four different options. Taking the idea from above, the jury makes one-by-one comparisons between the alternatives with the difference that after the first comparison the winner is directly compared with the third alternative and so on. The winner of the last direct comparison will be the overall winner. This kind of voting is called **sequential pairwise voting**.

Example 2. *Suppose the following preferences of our three jury members over the four alternatives we call a, b, c and d:*
Member A: $a \succ b \succ d \succ c$
Member B: $c \succ a \succ b \succ d$
Member C: $b \succ d \succ c \succ a$

To avoid any dispute, the jury is used to voting on the alternatives in alphabetical order. So from the pairing a and b we have a as winner, because both members A and B prefer a to b and only C prefers b to a. Hence, the next pairing is a against c, where c is victorious. The last "battle" will be c against d, and we get the overall winner d with the votes of members A and C, cf. figure 1.1.

What if the jury changed their voting agenda to voting in reversed alphabetical order? We would get the winner d from the pairing d against c, the winner b from d against b and finally the overall winner a from b against a, cf. figure 1.2.

Referring to the figures 1.3 and 1.4 we see that if we used other agendas, we could also get c and b as winners. So it is clear that - as in the previous example - even though none of the jury members changes their mind, any of the four alternatives can win, only depending on the agenda which is employed. Straffin claimed in [19]

[2] since the second and the third group both prefer Twister to Monopoly and only the first group prefers Monopoly to Twister

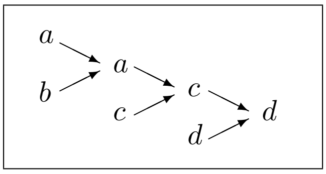

Figure 1.1: Agenda: a, b, c, d - Winner: d

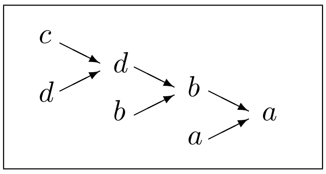

Figure 1.2: Agenda: d, c, b, a - Winner: a

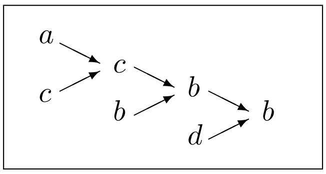

Figure 1.3: Agenda: a, c, b, d - Winner: b

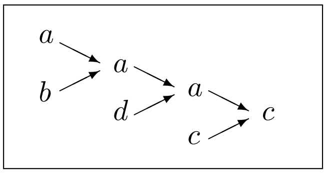

Figure 1.4: Agenda: a, b, d, c - Winner: c

that "chance, or sophisticated manipulation of the agenda, can have as much to do with the outcome as the preferences of the voters". And we have not even changed the voting procedure as in the previous example!

The reason why this is possible is the fact that in this example there is no candidate who would win over all other candidates in pairwise elections, so there is no winner in the Condorcet voting sense as in example 1. Because if there was one, he would win in any election of the pairwise voting, no matter when he is put in for voting, in other words what agenda is used. If such a winner is lacking, we will get quite strange results. Let us state the winners of pairwise elections, which is equal to the overall preference between two candidates: $a \succ b$, $b \succ c$, $d \succ c$, $a \succ d$, $b \succ d$ and $c \succ a$. If we suppose transitivity (from $a \succ b$ and $b \succ c$ follows $a \succ c$ for all a, b, c), we will get $a \succ b \succ d \succ c \succ a$ as an overall outcome, from which both $a \succ c$ and $c \succ a$ follow, which is a contradiction to the definition that $a \succ c$ means that a is strictly preferred to c. Hence, even though the voters have transitive preferences themselves, the overall outcome may not be transitive. What we have seen here is called ***the*** **paradox of voting**, **Condorcet's paradox** or the **paradox of cyclic**

majorities.

1.2 Voting preferences and social choice functions

In this section we want to formalize what we have seen so far in order to provide a basis for the next chapters. In general, we are going to follow the introduction in [6, pp. 180ff]. We presume that there are $n \geq 2$ voters, indexed by $i = 1, 2, \ldots, n$, and a set X of two or more candidates. We will refer to any alternatives as *candidates*, but note that this can mean “people running for office, passage or defeat of a bill, alternative budgets, applicants for a faculty position, or jury verdicts that a judge permits”, not forgetting the different board games from example 1. Also *voters* can have a broad interpretation, meaning “individuals being registered as voters, legislators, trustees, committee members, jurors, or members of some other body whose decisions not only are binding on their members but often a larger community that the body represents”. Voters can also be computer programs which have to decide among different alternatives [13].

We assume that voters have preferences among the candidates. Every voter i is assumed to have a *preference ranking* of the various candidates $\succsim_i$, which is a binary relation on X, so that $\succsim_i$ is transitive ($x \succsim_i y$ and $y \succsim_i z$ imply $x \succsim_i z \ \forall x, y, z \in X$) and complete ($x \succsim_i y$ or $y \succsim_i x \ \forall x, y \in X$). The interpretation of $x \succsim_i y$ is that voter i weakly prefers x to y, which means that he either strictly prefers x to y or that he is indifferent about the two alternatives. We denote the asymmetric (strict) preference part of $\succsim_i$ by $\succ_i$, and the symmetric (indifferent) part of $\succsim_i$ by $\sim_i$:

$$\begin{aligned} &x \succ_i y \text{ if } x \succsim_i y \text{ and not } y \succsim_i x, \\ &x \sim_i y \text{ if } x \succsim_i y \text{ and } y \succsim_i x. \end{aligned}$$

It follows from completeness of $\succsim_i$ that it is reflexive ($x \succsim_i x$, if you set $y = x$) and that exactly one of $x \succ_i y$, $y \succ_i x$ or $x \sim_i y$ holds. The indifference relation $\sim_i$ on X is an equivalence relation which partitions X into r different indifference classes $X_1, X_2, \ldots, X_r$, where $x \sim_i y$ holds for all candidates in the same indifference class and $x \succ_i y$ or $y \succ_i x$ for all candidates in different equivalence classes. Furthermore, an overall ranking $X_1 \succ_i X_2 \succ_i \ldots \succ_i X_r$ can be made, where $X \succ_i Y$ means that $a \succ_i b$ for all pairs $(a, b) \in X \times Y$. When $x \sim_i y \Leftrightarrow x = y$, for all $x, y \in X$, we refer to $\succsim_i$ or its asymmetric part $\succ_i$ as *linear order* or *strict ranking*.

An n-tuple $v = (\succsim_1, \succsim_2, \cdots, \succsim_n)$ of preference rankings on X, one for each voter, is a *voter preference profile*. Let V denote the nonempty set of voter preference profiles in a particular situation.

We will now prepare to define a social choice function, which chooses from a preference profile and the ballots cast by the voters one or more candidates as winner or winners of the election. Then we can define a voting procedure through its social choice function. Three inputs determine the domain of a social choice function. The first is the number $k \geq 1$ of candidates to be chosen by a voting procedure. The second is a nonempty set $\mathcal{X}$ of subsets of X, each of which might arise as the feasible set of candidates or the official set of nominees. We require $|A| \geq k$ for every $A \in \mathcal{X}$.

The third input is the set D_A of ballot response profiles that can occur for each $A \in \mathcal{X}$. Each member of D_A is a ballot response profile $d = (d_1, d_2, \dots, d_n)$, where $d_i \in \mathcal{B}(A)$ for each voter and $\mathcal{B}(A)$ is the set of admissible ballots. The domain of a social choice function is the set

$$\mathcal{D} = \bigcup_{A \in \mathcal{X}} \{(A, d) : d \in D_A\} \tag{1.2.1}$$

of all ordered pairs (A, d) of a set A of feasible candidates and a ballot response profile for that set.

Definition 1. *A* social choice function *is a mapping F from a domain $\mathcal{D}$ into the family $2^{\mathcal{X}}$ of subsets of X such that, for all $(A, d) \in \mathcal{D}$,*

$$F(A, d) \subseteq A \text{ and } |F(A, d)| \geq k.$$

We interpret $F(A, d)$ as the subset of feasible candidates chosen by the voting procedure for situation (A, d). It is assumed to contain at least k candidates. When $|F(A, d)| = k$ for all $(A, d) \in \mathcal{D}$, we say F is *decisive*. By not imposing decisiveness, we allow for the possibility that a choice set contains more candidates than the precise number to be elected, i.e. we admit the possibility of unresolved ties.

So social choice functions give us the winners of an election. The task of receiving a complete social ranking of candidates (from which we can also retrieve the winners, namely the candidates ranked on top) will do a social welfare function for us:

Definition 2.
1. *A* social preference (order) $\succsim$ *on X is a preference ranking of the various candidates of the society as a whole, which is a binary relation on X, so that it is complete.*[3] *We denote the set of all social preference orders as P.*
2. *A* social welfare function *$F : V \to P$ maps a voter preference profile v to a social preference order p.*

1.3 Arrow's impossibility theorem and the Gibbard-Satterthwaite theorem

As we have seen in the first section, there is not really a way to say which would be the best voting procedure whenever we have more than two candidates, especially in cases like example 1. A famous theoretical finding by Kenneth Arrow in 1951, which can probably be considered as important for voting theory as Gödel's incompleteness theorems are for mathematical logic, states that there is no voting procedure (in this version of Arrow's Theorem we will especially examine social welfare functions) which fulfills some quite reasonable criteria if there are more than two alternatives. These criteria have been revised and are not completely the same as in Arrow's original formulation, since some of them were found not necessary. The criteria we examine are:

[3] As we have seen in example 2, it need not be transitive.

Criterion 1 (Unanimity (U)). *A social welfare function respects* unanimity *if $x \succ_i y$ for all $i \in \{1, \ldots, n\}$ implies $x \succ y$.*

Criterion 2 (Independence of irrelevant alternatives (IIA)). *A social welfare function F respects* independence of irrelevant alternatives *if F's ranking of x and y (which can be $x \succ y$, $y \succ x$ or $x \sim y$) only depends on the voters' rankings of x and y and on no other alternatives.*

Criterion 3 (Non-dictatorship (ND)). *A social welfare function F respects* non-dictatorship *if there is no voter i so that F ranks $x \succ y$ whenever voter i ranks $x \succ_i y$ for every pair $x, y \in X$.*

Theorem 1 (Arrow's Impossibility Theorem). *For $|X| \geq 3$, there is no social welfare function F that satisfies transitivity, independence of irrelevant alternatives, unanimity and non-dictatorship. Or, in other words: Every social welfare function F for $|X| \geq 3$ that satisfies transitivity, independence of irrelevant alternatives and unanimity is a dictatorship.*

Proof. To prove the second formulation of Arrow's Theorem we will make use of the first proof in [11]. We suppose a set of candidates $X = \{a, b, c, \ldots\}$. Let us have a closer look at candidate b, who is just one arbitrarily chosen candidate. First we want to show that if every voter ranks b at the top or bottom of his preference order, F must do the same (which implies that even if half of the voters rank b top and the other half ranks it bottom, F can't place it somewhere in the middle). Suppose on the contrary that F ranked $a \succsim b$ and $b \succsim c$ for the given voter preference profile, regardless of any other candidates, but with b not top or bottom ranked. By IIA, if every voter moves c over a, the above ranking must stay the same, since this move would not change the ab and bc rankings of our voters because of the extreme position of b, see figure 1.5. Since every voter i now has $c \succ_i a$, by U we get $c \succ a$, but by transitivity F has to put $a \succsim c$, a contradiction.

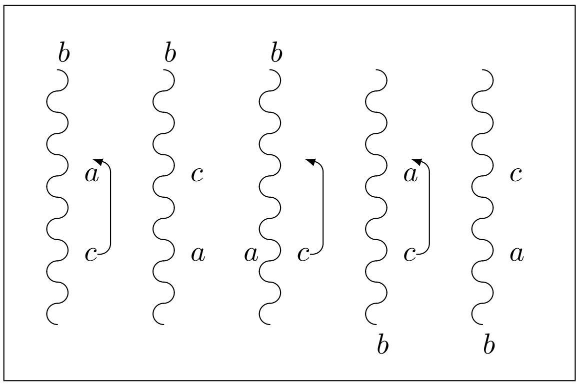

Figure 1.5: Changes between a and c do not affect the ab and bc ratings

Secondly we want to show that there is a voter $\mu^* = \mu(b)$ who is extremely pivotal in the sense that by changing his vote at some profile he can move b from the bottom of the social preference to the top. To see this, consider a voter preference profile where each voter ranks b at the bottom, and arbitrarily ranks the other candidates. By U, F must rank b at the bottom as well. Now, let the voters from 1 to n successively move b to the top without changing anything else. At some voter, F must change its ranking of b, which it has to do at the latest when voter n does the change because of U. Let μ^* be the first voter who causes F to move b. We consider the voter preference profile just before μ^* did the change as profile I and the one right after his change as profile II, see figure 1.6. Since at profile II b is no longer at the bottom of the social ranking, it must be on top because of our first argument. (Also in profile II b is only put either at the top or at the bottom of each voter's order.)

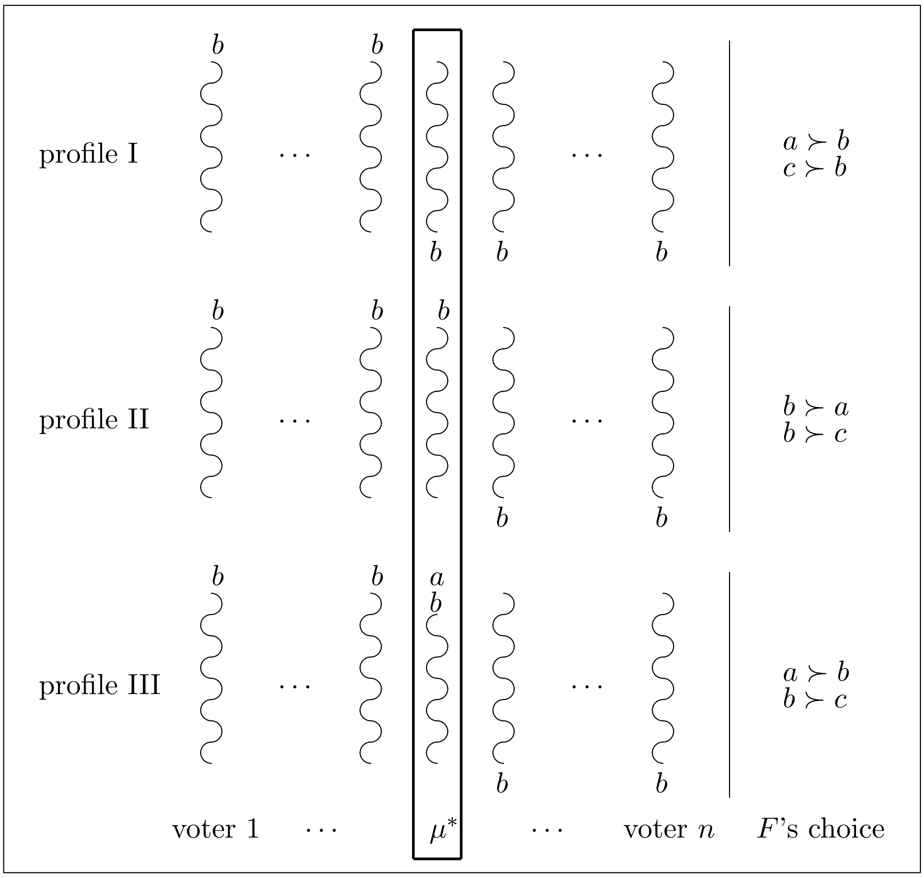

Figure 1.6: Profiles I, II and III with extremely pivotal voter μ^*

Thirdly we argue that μ^* is a dictator over any pair ac not including b, which means that F's preference is always the same as μ^*'s preference between a and c. To see this, we will choose an element of ac, let's say a. We construct profile III from profile II by letting μ^* move a over b such that he has $a \succ_{\mu^*} b \succ_{\mu^*} c$, and the other voters change the position of a and c arbitrarily while leaving b in its extreme position. By IIA, F must put $a \succ b$ (since the rankings of a and b are the same as in profile I) and $b \succ c$ (since the ranking of b and c are the same as in profile II).

So by transitivity we get $a \succ c$, and by IIA $a \succ_{\mu^*} c \Leftrightarrow a \succ c$, because for all voters except μ^* in profile III alternatives a and c are in arbitrary order.

We conclude by showing that μ^* is also a dictator over every pair ab. Take a third distinct alternative c to put at the bottom of each voter's preference order as done in our second argument. Our third argument implies that there must be a voter $\mu(c)$ who is a dictator over every pair $\alpha\beta$ not including c, such as ab. But μ^* can affect the ab ranking in profiles I and II, hence this ab dictator $\mu(c)$ must actually be μ^*. □

This theorem has inspired many other impossibility theorems, from which we will have a look at the Gibbard-Satterthwaite theorem, which says that there is no strategy-proof, non-dictatorial social choice function. The proof can be found in [20] for example. Although we defined a social choice function on $\mathcal{D}$ (see equation 1.2.1), we are looking here at another version of a social choice function, which is defined on $\{X\} \times V$ and which is specified to be decisive with $k = 1$.

Criterion 4 (Strategy-proofness)**.** *A social choice function F is* strategy-proof *if for all $v = (\succsim_1, \dots, \succsim_n)$ and $v' = (\succsim'_1, \dots, \succsim'_n)$ in V and all $i \in \{1, \dots, n\}$,*

$$(\succsim_j = \succsim'_j \text{ for all } j \neq i) \Rightarrow F(X, v) \succsim_i F(X, v').$$

A social choice function which is not strategy-proof is called *manipulable*, which implies that a voter can unilaterally obtain a preferred outcome by voting contrary to his or her true preferences [6, p. 16].

Criterion 5 (Surjective)**.** *A social choice function F is* surjective *if for every $a \in X$ there is voter preference profile $v \in V$ so that $F(X, v) = \{a\}$.*

We still have to define Non-dictatorship for social choice functions, which is quite similar to the definition for a social welfare function. We will do that by defining it as the contrary of dictatorship.

Criterion 6 (Dictatorship)**.** *A social choice function F is* dictatorial *if there exists a voter i so that for every voter preference profile $v = (\succsim_1, \dots, \succsim_n)$ with $A = \{a \in X : a \succsim_i b$ for all $b \in X\}$ we have $F(X, v) = A$.*

Theorem 2 (Gibbard-Satterthwaite Theorem)**.** *For $|X| \geq 3$, a social choice function which is strategy-proof and surjective must be a dictatorship.*

What we can conclude from here is that there is no optimal voting procedure. So we will not be able to find something perfect, but we will still look for those voting procedures which are better than others and specify in which forms they have advantages and disadvantages. So the decision on which procedure to use depends on what criteria are more important than others, amongst others also non-mathematical considerations (not examined here) like simplicity and straightforwardness for people to understand, effort to count the ballots etc.

2 Criteria for voting procedures

2.1 Approval voting evaluated

Now that we know we will not find a perfect voting procedure, we will try to evaluate approval voting by using different criteria which voting procedures should satisfy. Approval voting is a voting procedure, where every voter can approve of as many candidates as they want, therefore casting a vote for every candidate they approve of. More formally, we will define a nonranked voting procedure and give some examples, including approval voting [6, pp. 189f.].

Definition 3. *F on $\{X\} \times \mathcal{B}^n$ with $|X| = m$ is a* nonranked voting procedure *if there is a nonempty subset M of $\{0, 1, \dots, m-1\}$ that includes at least one $j > 0$ such that $\mathcal{B} = \{A \subset X : |A| \in M\}$ and, for all $x \in X$ and $d \in \mathcal{B}^n$,*

$$x \in F(X, d) \Leftrightarrow |\{i : x \in d_i\}| \geq |\{i : y \in d_i\}| \textit{ for all } y \in X. \tag{2.1.1}$$

While the definition of a ballot has been left open in the definition of a social choice procedure in definition 1, for nonranked voting procedures the *set of admissible ballots* $\mathcal{B}$ is now defined as a set of proper subsets of the candidates, $\emptyset \subsetneq B \subsetneq X$ with $|B| \in M$. At least one $j > 0$ is required, so that there cannot be only empty ballots voting for nobody. However, when 0 is included in M, this means that abstention is allowed. Equation 2.1.1 is the condition for winning the election: getting at least as many votes as all the other candidates. We will abbreviate $|\{i : x \in d_i\}|$ with $v(x, d)$, which signifies the number of votes candidate x got at the ballot response profile d.

We will see that all nonranked voting procedures only differ in the set M, which defines the set of admissible ballots:

Plurality voting has $M = \{0, 1\}$, so voters can vote exactly for one candidate if they do not abstain. Brams and Fishburn write in [6]: "It is commonly used in single-winner elections and sometimes in multiple-winner elections. The main criticisms of plurality voting concern its severe limitation on the expression of voter preferences, the dispersion of votes that it produces across ideologically similar candidates - rendering them vulnerable to other candidates, particularly on the ideological extremes, who have no opposition - and the extent to which it encourages voters to vote for candidates other than their favorites when their favorites have no real chance of winning."

Approval voting is defined through $M = \{0, 1, \dots, m-1\}$, so it is the nonranked voting procedure with the largest possible subset of M, the whole set.

Another form of nonranked voting procedures is **Negative voting**, which allows each voter either to vote *for* one candidate or to vote for all but one - which is the same as voting *against* one - and has $M = \{1, m-1\}$, or $M = \{0, 1, m-1\}$ with abstentions allowed.

At this step we are ready to examine our first two voting criteria:

Criterion 7 (Condorcet winner). *If there is a candidate a that wins against every other candidate in pairwise contests, a voting procedure should choose a as the winner.*

Criterion 8 (Condorcet loser). *If there is a candidate a that loses against every other candidate in pairwise contests, a voting procedure should* not *choose a as the winner.*

Lemma 1. *Approval voting fails the Condorcet winner and the Condorcet loser criterion.*

Proof. If you look at example 1 and suppose that every voter only votes for his top preference, the outcome will be the same as plurality voting, where Monopoly was the winner, which by the way was not only *not* the Condorcet winner, but even the Condorcet loser. □

Even though it is true that approval voting need not always elect the Condorcet winner and can sometimes even elect the Condorcet loser, we will show later that approval voting *can* always produce a Condorcet winner as approval voting winner. Suppose in our example 1 that every voter votes for his top two preferences. Hence, we would have the following situation

$$4 \text{ voters: } \{M, P\}$$
$$3 \text{ voters: } \{T, P\}$$
$$2 \text{ voters: } \{P, T\}$$

with Poker - the Condorcet winner - winning with 9 votes, Twister getting 5 votes and Monopoly losing with only 4 votes.

We will examine two more criteria which belong to the family of Condorcet criteria.

Criterion 9 (Majority). *If a majority (= more than half) of voters have a as their first choice, a should win.*

Criterion 10 (Smith's Generalized Condorcet). *If the candidates can be partitioned into two sets A and B such that every alternative in A beats every alternative in B in pairwise contests, then a voting procedure should not elect an alternative in B.*

The explanation of our statement above about the family of Condorcet criteria is that Smith's Generalized Condorcet criterion implies the Condorcet winner and loser criterion, and the Condorcet winner criterion implies the majority criterion. So if a voting procedure satisfies Smith's Generalized criterion, it satisfies all the other three. On the other hand, if it does not satisfy the majority criterion it will neither satisfy the Condorcet winner nor Smith's Generalized Condorcet criterion.

$$\text{Smith's Generalized Condorcet} \Rightarrow \begin{cases} \text{Condorcet Winner} & \Rightarrow \text{Majority} \\ \text{Condorcet Loser} & \end{cases}$$

To see the first implication, just let A be the set which consists only of the Condorcet winner, or B be the set which consists only of the Condorcet loser. To see the second implication consider the following argument: If a majority of voters have a as their first choice, a will also be the Condorcet winner, because with this majority a will win against every other candidate. Now, if a voting procedure picks the Condorcet winner as the winner, the same voting procedure will also elect the majority winner, since he or she is also the Condorcet winner.

Lemma 2. *Approval voting fails Smith's Generalized Condorcet criterion and the majority criterion.*

Proof. To see that approval voting fails Smith's Generalized Condorcet criterion, we just follow the arguments of the implication chain. To see that approval voting fails the majority criterion, have a look at the following example:

$$3 \text{ voters: } a \succ b \succ c \text{ who vote } \{a, b\}$$
$$2 \text{ voters: } b \succ c \succ a \text{ who vote } \{b, c\}$$

Here a is ranked first place by a majority of voters. If the voters vote as suggested, b will win with 5 votes against a with 3 votes. (cf. erosion of the majority principle in section 5.6) □

Criterion 11 (Monotonicity)**.** *If a voting procedure chooses a as the winner, and one or more voters change their preferences in a way favorable for a (without changing anything else), a should still be the winner.*

Lemma 3. *Approval voting satisfies the monotonicity criterion.*

Proof. If a wins, he has got more votes than the other candidates. If now one or more voters rank him higher, some of them might now also vote for a, if they haven't already done so before. So a gets at least all the votes he or she got before, if not more. As far as the other candidates are concerned, they now get at most as many votes as they got before - if not fewer -, because some of those voters might now choose to disapprove of some of the other candidates. So a still gets more votes than the other candidates. □

Criterion 12 (Participation)**.** *Adding one or more voters that prefer a to b should never change the winner from a to b.*

Lemma 4. *Approval voting satisfies the participation criterion.*

Proof. If a gets more votes than any other candidate b, and an additional ballot is cast where at least a is voted for (it could also be voted for b), a still gets more votes. □

If a voting procedure does not meet the participation criterion in a specific situation, we call this situation the *No- Show-Paradox*. In this case it can be better for a voter who prefers a the most to abstain from voting rather than to vote for a. We will see more about this in the next section.

The last criterion (taken from [1]) is one of the most important ones for approval voting, since – as we will show in the next chapter – it will characterize it together with two other criteria (faithfulness and cancellation, cf. criteria 14 and 16) among ballot aggregation functions (for a definition see p. 18).

Criterion 13 (Consistency)**.** *If two societies vote separately over the same set of candidates and there are candidates who win in both societies, then those candidates should also win if the ballots of the two societies are put together.*

If $F(X,d) \cap F(X,d') \neq \emptyset$ *for* $d \in \mathcal{B}^n, d' \in \mathcal{B}^m$ *then* $F(X,d+d') = F(X,d) \cap F(X,d')$,

where $d + d' \in \mathcal{B}^{n+m}$ *signifies the ballots of the two societies put together.*

Lemma 5. *Approval voting satisfies consistency.*

Proof. Let's denote the set of winners in both societies as $W = \{x_1, \ldots, x_k\}$. Now we have,

$$\begin{aligned} \forall x_i \in W: \quad & v(x_i, d) \geq v(x_j, d) & \forall x_j \in X, \\ \forall x_i \in W: \quad & v(x_i, d') \geq v(x_j, d') & \forall x_j \in X, \end{aligned}$$

and therefore we get

$$\forall x_i \in W : v(x_i, d) + v(x_i, d') \geq v(x_j, d) + v(x_j, d') \quad \forall x_j \in X.$$

□

2.2 Voting procedures compared

The difficulty of comparing ranked and nonranked voting procedures is that many criteria can be formulated in a way that is favorable for one or the other system. Take the *Pareto criterion*, for example. If you state it in a way favorable for ranked voting systems like "If all voters prefer a to b, then b should not win", it is obvious that approval voting does not meet this criterion. (It could be that all voters prefer a to b to c and vote for a and b, so a and b both win.) If you state it, however, like "If all voters rank a over b on each ballot, then b should not win", it is clear that approval voting passes this version of the criterion. (If everybody votes for a but not for b, b will certainly lose.) Hence, we will only look at criteria which can be evaluated with both types of voting procedures in a meaningful way.

Besides plurality voting (PV), plurality voting with runoff (PVR) and approval voting (AV), we want to include **Borda count** (BC) and the **Black method** (BM) into our comparison. Borda count is a ranked voting system where each voter gives points to the candidates in order of his preference ranking. The most preferred candidate is given $|X|-1$ points, the second most preferred candidate $|X|-2$ points and so forth leaving the least preferred candidate with 0 points. The points are summed up and the candidate with most points is declared winner. This special case of a ranked voting system is called positional scoring procedure. As we will show, Borda count need not produce a Condorcet winner as the outcome. This is why the Black method has been invented: it is a ranked voting procedure combining Condorcet voting and Borda count: If there is a Condorcet winner, he or she will win the election. If not, regular Borda count is carried out and the Borda count winner wins.

To give a more precise formulation of a positional scoring procedure, we will use the definition in [6]. For a positional scoring procedure the set of admissible ballots $\mathcal{B}$ is the set of all linear orders of X. We denote by s_j the points awarded to a candidate in position j and refer to $s = (s_1, s_2, \dots, s_m)$ as a *positional scoring vector*, where $|X| = m$. It is assumed that $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$. Borda count has $s = (m-1, m-2, \dots, 1, 0)$ and even plurality voting[1] can be considered as a positional scoring procedure with $s = (1, 0, \dots, 0)$. For every $x \in X$, every $j \in \{1, \dots, m\}$ and every ballot response profile $d = (d_1, \dots, d_n)$ in $\mathcal{B}$, let $d(x, j)$ denote the number of voters who rank candidate x in j^{th} position. Clearly, $d(x,1) + d(x,2) + \cdots + d(x,m) = n$. The score of candidate x for ballot response profile d with respect to positional scoring vector s is

$$s(x,d) = \sum_{j=1}^{m} s_j d(x,j)$$

The *positional scoring procedure for* s takes $F(X,d)$ as the subset of candidates that maximize $s(x,d)$ over X for each d in D.

In the following table 2.1 we will summarize our findings about whether the above voting procedures satisfy or fail some chosen criteria. We will prove some of the entries in the table, the rest can be found in Straffin's *Topics in the Theory of Voting* [19].

Table 2.1: Evaluation of voting procedures

	PV	PVR	BC	BM	AV
Majority	YES	YES	NO	YES	NO
Condorcet winner	NO	NO	NO	YES	NO
Condorcet loser	NO	YES	YES	YES	NO
Smith's Generalized Condorcet	NO	NO	NO	NO	NO
Monotonicity	YES	NO	YES	YES	YES
Participation	YES	NO	YES	NO	YES
Consistency	YES	NO	YES	NO	YES

Lemma 6. *Plurality voting fails the Condorcet winner, Condorcet loser and Smith's Generalized Condorcet criterion.*

[1]Approval voting not, since there is *no* fixed positional scoring vector s!

Proof. See example 1 for Condorcet winner and loser, the rest is done by the implication chain. □

Lemma 7. *Plurality voting satisfies the participation and the consistency criterion.*

Proof. Participation: If a gets more votes than any other candidate b, and an additional ballot is cast where a (or neither a nor b) is voted for, a still gets more votes.
Consistency: The proof is exactly the same as for approval voting in lemma 5. □

Lemma 8. *Plurality voting with runoff satisfies the Condorcet loser criterion.*

Proof. If the Condorcet loser comes into the runoff, he will certainly lose against the other candidate in this pairwise comparison. □

Lemma 9. *Plurality voting with runoff fails the Condorcet winner, Smith's Generalized Condorcet, the monotonicity, the participation and the consistency criterion.*

Proof. Condorcet winner and Smith's Generalized Condorcet: See example 1 for Condorcet winner. Failing this criterion implies failing Smith's Generalized Condorcet criterion.
Monotonicity: Have a look at the following example:

$$\begin{aligned} &6 \text{ voters: } a \succ b \succ c \\ &5 \text{ voters: } c \succ a \succ b \\ &4 \text{ voters: } b \succ c \succ a \\ &2 \text{ voters: } b \succ a \succ c \end{aligned}$$

If plurality voting is carried out, candidates a and b will come into the runoff with 6 votes each, where candidate a will win with 11 over 6 votes. If now the last two voters change their minds in favor of alternative a - having $a \succ b \succ c$ - we will have candidates a and c coming into the runoff with 8 and 5 votes respectively, where c will win with 9 against 8 votes. Hence, changes in favor of candidate a have made him lose.
Consistency: Consider the example with the changes made in favor of a, which we will partition into two societies d and d'.

	d	d'
$a \succ b \succ c$	4	4
$c \succ a \succ b$	2	3
$b \succ c \succ a$	4	0

In society d, a and b will be in the runoff with a winning 6 over 4 votes. In society d', a and c will be in the runoff with a winning 4 over 3 votes. However, as we have seen before, if we put the societies together, c will win the election.
Participation: Consider the same example without the partition. If we add two extra voters with $b \succ c \succ a$, we will have a and b in the runoff with a winning 13 over 6 votes. So adding voters that rank c over a makes c lose and a win. □

If you are interested in an amusing story about this issue we suggest reading [5], where the story of a Mr. and Mrs. Smith about “a funny thing happened on the way to the polls” is told. As far as the failure of plurality voting with runoff in satisfying the monotonicity criterion is concerned, following Straffin in [19], we can imagine a news announcement: “Candidate c won today, but if candidate a had received less support, he would have won”. This paradoxical situation is called the *More-Is-Less-Paradox*. Referring to the No-Show-Paradox, we saw here that voting c over a makes c lose and a win. Again, the voters who preferred c to a had rather stayed at home than gone to the polls.

Lemma 10. *Borda count satisfies the participation and the consistency criterion.*

Proof. Participation: If a gets more points than any other candidate b, and an additional ballot is cast where a gets more points than b, a still has more points.
Consistency: This proof goes in line with the proof for approval voting (lemma 5), where $v(x, d)$ is replaced with $s(x, d)$ which signifies the number of points candidate x received. □

Lemma 11. *Borda count fails the majority, the Condorcet winner and Smith's Generalized Condorcet criterion.*

Proof. We only need to show that Borda count fails the majority criterion. The rest will do the implication chain for us. We will make use of the same example used to prove that approval voting fails the majority criterion in lemma 2. In this example the majority winner a gets 6 points, whereas b gets 7 points and wins. □

Lemma 12. *The Black method satisfies the majority, the Condorcet winner and the Condorcet loser criterion.*

Proof. Since the Black method was designed to choose the Condorcet winner, if one exists, it satisfies the Condorcet winner criterion by definition, which implies the majority criterion. If no Condorcet winner exists, Borda count is carried out, which satisfies the Condorcet loser criterion. □

Although the Black method was designed to choose the Condorcet winner, it does not satisfy the generalized Condorcet criterion as we will see in this lemma.

Lemma 13. *The Black method fails Smith's Generalized Condorcet, the participation and the consistency criterion.*

Proof. Smith's Generalized Condorcet: To see this, consider the admittedly constructed example from [19].

$$\begin{aligned}
&1 \text{ voter: } a \succ b \succ x \succ y \succ z \succ w \succ c\\
&1 \text{ voter: } b \succ c \succ x \succ y \succ z \succ w \succ a\\
&1 \text{ voter: } c \succ a \succ x \succ y \succ z \succ w \succ b
\end{aligned}$$

If we partition the alternatives into set $A = \{a, b, c\}$ and $B = \{x, y, z, w\}$, one can easily check that every alternative in A beats every alternative in B. However, there is no Condorcet winner since $a \succ b \succ c \succ a$ (the alternatives beat each other cyclically) and so Borda count is carried out. In points, we will get the following result.

a	b	c	x	y	z	w
6	5	0	4	3	2	1
0	6	5	4	3	2	1
5	0	6	4	3	2	1
11	11	11	12	9	6	3

Candidate x wins although it is in set B.

Consistency: Consider the following example.

	d	d'	$d+d'$
$a \succ b \succ c$	0	4	4
$a \succ c \succ b$	1	2	3
$b \succ a \succ c$	2	0	2
$b \succ c \succ a$	0	5	5
$c \succ a \succ b$	0	1	1

In partition d we clearly get candidate b as the Condorcet winner, since the profile $b \succ a \succ c$ has a majority of votes. In partition d' candidate b beats a 7 to 5 and b beats c 9 to 3, so candidate b is the Condorcet winner and the winner of the Black method. (We do not care about the tie of a against c since those candidates are both beaten by b and therefore cannot win.) Now, in $d + d'$ we have a beating b 8 to 7 and a beating c 9 to 6.

Participation: Consider the following example.

6 voters: $a \succ c \succ b$

5 voters: $b \succ a \succ c$

3 voters: $c \succ b \succ a$

2 voters: $b \succ c \succ a$

1 voter: $a \succ b \succ c$

For the Condorcet voting we get the intransitive outcome $a \succ c \succ b \succ a$. So Borda count is carried out with the following result where a wins.

a	b	c
12	0	6
5	10	0
0	3	6
0	4	2
2	1	0
19	18	14

If we now add two voters who prefer a to b and vote $a \succ b \succ c$, candidate b will beat both a and c 10 to 9 in the Condorcet election. So b is the new winner of the Black method. □

As we have never used the Borda count step in the proof against consistency, we have actually shown that *every* Condorcet method – that is every voting procedure that elects first the Condorcet winner if one exists, and then uses another procedure if not – fails the consistency criterion.

Summing up, we have investigated the following kinds of voting procedures: nonranked singlestage voting procedures (AV, PV), nonranked multistage voting procedures (PVR), ranked Condorcet voting procedures (BM), ranked singlestage positional scoring procedures (BC, PV) and ranked multistage positional scoring procedures (PVR).

3 Characterization and properties of approval voting

3.1 Characterization

In the previous chapter we have shown that approval voting satisfies consistency. As we have announced and will show now, this criterion together with faithfulness and cancellation characterize approval voting in the field of ballot aggregation functions as shown in [1]. A *ballot aggregation function* is more or less the same as a nonranked social choice function, with the difference that there is no confinement on the set of ballots $\mathcal{B}$, allowing fully for $B \in \mathcal{P}(X) \setminus \{\emptyset, X\}$. The ballot aggregation function itself decides whether to count a ballot or not. To do this, we will use a function $\pi : \mathcal{B} \to \mathbb{N}$ called *voter response profile* with the interpretation that $\pi(B)$ is the number of voters who cast ballot B. In this setting, we can define **plurality voting** by

$$f^P(\pi) = \arg\max_{x \in X} \pi(\{x\}),$$

where all non-singleton ballots are discarded; **approval voting** by

$$f^A(\pi) = \arg\max_{x \in X} \sum \{\pi(B) | x \in B \in \mathcal{B}\},$$

where all votes count equally; or the **"split-the-vote-rule"** by

$$f^S(\pi) = \arg\max_{x \in X} \sum \{\frac{1}{|B|}\pi(B) | x \in B \in \mathcal{B}\},$$

where each voter can split their only vote among as many candidates as desired. The advantage of this definition is that it has a variable electorate, i.e. is independent of the number of voters. We define $v(x, \pi)$ as the number of voters who vote for x in π, i.e. $v(x, \pi) = \sum \{\pi(B) | x \in B \in \mathcal{B}\}$. We will redefine consistency for ballot aggregation functions and give the definitions of faithfulness and cancellation. It is immediate to check that approval voting satisfies all of the three criteria, cf. lemma 5 for consistency.

Criterion 14 (Faithfulness)**.** *If a society consists of only one voter (or if only one ballot was cast), a voting procedure should stick to this voter's ballot.*

$$f(B) = B \text{ for all } B \in \mathcal{B}.$$

Criterion 15 (Consistency)**.** *If two societies vote separately over the same set of candidates and there are candidates who win in both societies, then those candidates should also win if the ballots of the two societies are put together.*

$$\text{If } f(\pi) \cap f(\pi') \neq \emptyset \text{ for } \pi, \pi' \in \Pi \text{ then } f(\pi + \pi') = f(\pi) \cap f(\pi'),$$

where $\pi + \pi'$ *defines a new voter response profile putting together the voters and ballots of the two societies.*

Criterion 16 (Cancellation)**.** *If every candidate gets the same number of votes, all of the candidates should be elected.*

$$\text{If } v(x, \pi) = v(y, \pi) \text{ for all } x, y \in X \text{ then } f(\pi) = X.$$

Theorem 3. *Approval voting is the only ballot aggregation function satisfying faithfulness, consistency, and cancellation.*

Proof. Let f be a ballot aggregation function satisfying the above mentioned criteria. We will proceed in three steps and show that this function must be approval voting.

Step 1: For any $\pi \in \Pi$, $B_1, B_2 \in \mathcal{B}$ with $B_1 \cap B_2 = \emptyset$ we want to show that

$$f(\pi + B_1 + B_2) = f(\pi + (B_1 \cup B_2)) \tag{3.1.1}$$

holds. This is the case because by cancellation $f((B_1 \cup B_2) + X \setminus (B_1 \cup B_2)) = X$ (every candidate gets the same number of votes) and therefore by consistency,

$$f(\pi + B_1 + B_2) = f(\pi + B_1 + B_2 + (B_1 \cup B_2) + X \setminus (B_1 \cup B_2))$$

holds, because $f(\pi + B_1 + B_2) \cap \underbrace{f((B_1 \cup B_2) + X \setminus (B_1 \cup B_2))}_{=X} = f(\pi + B_1 + B_2)$.

Through an analog argument we get

$$f(\pi + (B_1 \cup B_2)) = f(\pi + (B_1 \cup B_2) + B_1 + B_2 + X \setminus (B_1 \cup B_2)).$$

As the two right-hand side expressions are the same, the claim is proven.

Step 2: Let $\pi \in \Pi$ be an arbitrary voter response profile. We construct $\pi' \in \Pi$ such that $v(x, \pi) = v(x, \pi')$ for all $x \in X$, but π' only consists of singleton ballots, i.e. $\pi'(B) > 0$ implies $|B| = 1$. Profile π' is constructed from π by "taking apart" each ballot cast under π into separate singleton ballots. This process will result into more ballots and consequently more voters. This does not bother us, since the size of the electorate is not fixed in the definition of a ballot aggregation function[1]. Iteration of 3.1.1 (starting with an empty voter response profile) verifies that the outcome of π and π' will be the same, so $f(\pi) = f(\pi')$.

[1]contrary to the definition of a social choice function, cf. definition 1, or a nonranked voting procedure, cf. definition 3.

Step 3: We define $K = \max_{x \in X} v(x, \pi)$, which exists because X is finite. For each $k = 0, \dots, K$ we define further $X_k = \{x \in X | v(x, \pi) = k\}$, which form a partition of X. Consider the profile

$$\pi^* = X_K + (X_K \cup X_{K-1}) + \cdots + (X_K \cup X_{K-1} \cup \cdots \cup X_1).$$

By faithfulness and consistency we get $f(\pi^*) = X_K$, because

$$\underbrace{f(X_K)}_{=X_K} \cap \underbrace{f(X_K \cup X_{K-1})}_{=X_K \cup X_{K-1}} \cap \cdots \cap \underbrace{f(X_K \cup X_{K-1} \cup \cdots X_1)}_{=X_K \cup X_{K-1} \cup \cdots \cup X_1} = X_K$$

all by faithfulness and partition. So by consistency we get $f(X_K + (X_K \cup X_{K-1}) + \cdots + (X_K \cup X_{K-1} \cup \cdots \cup X_1)) = X_K$. Next we show $f(\pi^*) = f(\pi')$ by iteration of 3.1.1:

$$f(\pi^*) = f(\underbrace{X_K + \cdots + X_K}_{K \text{ times}} + \underbrace{X_{K-1} + \cdots + X_{K-1}}_{K-1 \text{ times}} + \cdots + \cup X_1),$$

which is the same as $f(\pi')$ with all $x \in X_k$ in separate singleton ballots. So from step 2 we get $f(\pi) = f(\pi') = f(\pi^*) = X_K$. As π was arbitrary and results in the set X_K as the set of winners, which are the winners of approval voting (since approval voting is to maximize $v(\cdot, \pi)$), we conclude that f is approval voting. $\square$

Just to give an intuition why this is not true for other voting procedures, take plurality voting for example and suppose $X = \{x, y, z, \dots\}$. Plurality voting violates faithfulness, because if there is only one voter and this voter casts ballot $B = \{x, y\}$, this ballot is ignored (since only singleton ballots are counted) and every candidate has got the maximum of 0 votes. So all candidates are declared winners and the ballot aggregation function does not stick to the voter's ballot. This can occur, because the criterion of faithfulness has to be true for all $B \in \mathcal{P} \setminus \{\emptyset, X\}$, and not only for the ballots permitted by the voting procedure. Further, plurality voting also violates cancellation. If there are only two voters - one votes $\{x, y\}$ and the other one $\{z\}$ - only z wins, even though $v(x, \pi) = v(y, \pi) = v(z, \pi)$.

3.2 Feasible, admissible and sincere strategies for voters

Up to now we have only had a look at the voting procedure alone without giving the voters any clue on what to vote for based on their preferences. This will change in this section. For this purpose we will first define feasible, dominated and admissible strategies and further go into detail for sincere strategies. If not mentioned otherwise, all these definitions will be in the context of nonranked voting procedures.

A *strategy* S for a voter i is a subset of X with the interpretation that i votes for every candidate in S and for none outside S. A *feasible strategy* is a strategy permitted by the voting procedure, i.e. a strategy S is feasible exactly when $|S| \in M$.

Although the terms *strategy* and *ballot* have different motivations and origins, they technically describe the same thing. This is why we will use them equivalently. So, also the terms *strategy profile* and *ballot response profile* are technically the same. Similar to the definition in [4], we will define the strategic complement d_{-i} for the so called *focal voter* i of the ballot response profile d as $d_{-i} := (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$. $F(X, d_{-i} + S)$ will then be the voting outcome if voter i chooses strategy S and the other voters vote as in d_{-i}.

If we want voter i to compare voting outcomes, we will write this as AR_iB meaning that he likes voting outcome A at least as much as voting outcome B, where A and B are sets of candidates. AP_iB means that he strictly prefers voting outcome A to B. We suppose that $\{a\}R_i\{b\} \Leftrightarrow a \succsim_i b$ and $\{a\}P_i\{b\} \Leftrightarrow a \succ_i b$, and that AP_iB and BR_iA cannot hold both. Further, we make the following assumptions for all $i \in \{1, \dots, n\}$, $a, b \in X$ and all $A, B, C \subseteq X$:

Assumption 1 (Assumption P). *If $a \succ_i b$ then $\{a\}P_i\{a, b\}P_i\{b\}$.*

Assumption 2 (Assumption R). *If $A \cup B$ and $B \cup C$ are not empty, and if $a \succsim_i b$, $b \succsim_i c$ and $a \succsim_i c$ for all $a \in A$, $b \in B$ and $c \in C$, then $(A \cup B)R_i(B \cup C)$.*

Assumption P asserts that if a voter prefers candidate a to candidate b he also prefers a voting outcome of $\{a\}$ to a tied outcome of $\{a, b\}$ (whenever both a and b have a positive probability of being elected however a tie is resolved), which in turn he prefers to $\{b\}$. Assumption R asserts that if everything in A is at least as good as everything in B and C, and everything in B is at least as good as everything in C, then outcome $A \cup B$ is at least as good as $B \cup C$ [4, p. 18]. All these assumptions also apply for the social choice, if you just replace $\succsim_i$ by $\succsim$ etc.

Definition 4. *Given P and R for a voter i with $\succsim_i$ on X, strategy S* dominates *strategy T for this voter, or S dom T for $\succsim_i$, if $F(X, d_{-i} + S)R_iF(X, d_{-i} + T)$ for every possible $d_{-i} \in \mathcal{B}^{n-1}$, and $F(X, d_{-i} + S)P_iF(X, d_{-i} + T)$ for at least one d_{-i}.*

This definition does not require T and S to be feasible strategies and is therefore applicable to all nonranked voting systems. It can be interpreted in the way that S dominates T if the focal voter likes every possible outcome (depending on what the others vote) when he votes S at least as much as when he votes T, and strictly prefers the outcome when he votes S to the outcome when he votes T for at least one ballot response profile of the other voters.

Definition 5. *Strategy S is* admissible *for M and $\succsim_i$ if S is feasible for M and there is no strategy T that is also feasible for M and has T dom S for $\succsim_i$. A strategy profile s is admissible, if all the strategies in s are admissible.*

Remember that $\succsim_i$ defines an equivalence order on X and therefore partitions it into $X_1 \succ_i X_2 \succ_i \cdots \succ_i X_r$. A voter is called concerned if $r > 1$ and unconcerned if $r = 1$. For $r = 2$ he is said to have a *dichotomous* preference ranking, for $r = 3$ a *trichotomous* and for $r > 3$ a *multichotomous* preference ranking. Let $H(\succsim_i) = X_1$ denote the subset of the most preferred candidates and $L(\succsim_i) = X_r$ denote the subset of the least preferred candidates for voter i.

Not only because we will need this theorem, which helps us to find admissible strategies, in later proofs, we will present it here (without proof) from [6, p. 22]. To understand the theorem, we need to define the concept of high and low subsets of X [6, p. 18]. A subset $A \subseteq X$ is *high for $\succsim_i$* if

$$x \in A \cap X_k \Rightarrow X_l \subseteq A \text{ for all } l < k,$$

and is *low for* $\succsim_i$ if

$$x \in A \cap X_k \Rightarrow X_l \subseteq A \text{ for all } l > k.$$

Theorem 4 (Admissibility)**.** *Suppose $\succsim_i$ is concerned and assumptions P and R hold. Then strategy S is admissible for a nonranked voting procedure M and preference order $\succsim_i$ if and only if S is feasible for M and either criterion 1 or 2 (or both) holds:*

1. *Every candidate in $H(\succsim_i)$ is in S, and it is impossible to partition S into nonempty subsets S_1 and S_2 such that S_1 is feasible for M and S_2 is low for $\succsim_i$.*
2. *No candidate in $L(\succsim_i)$ is in S, and there is no nonempty $A \subseteq X$ disjoint from S such that $A \cup S$ is feasible for M and A is high for $\succsim_i$.*

Following [6, pp. 21-24], we will prove the following results for a concerned voter with preference ranking $\succsim_i$.

Corollary 1. *For approval voting, S is an admissible strategy if and only if S contains all candidates in $H(\succsim_i)$ and none in $L(\succsim_i)$.*

Proof. "$\Rightarrow$:" If S is admissible, we show via criterion 1 of theorem 4 that the claim holds. Since all candidates of $H(\succsim_i)$ have to be in S, we only have to show that $S \cap L(\succsim_i) = \emptyset$. Suppose on the contrary, that there is a candidate $x \in L(\succsim_i) \cap S$. But now we can partition S into $S_1 = H(\succsim_i)$ and $S_2 = \{x\}$, where S_1 is of course feasible and S_2 is low for $\succsim_i$.

"$\Leftarrow$:" If S contains all candidates in $H(\succsim_i)$ and none in $L(\succsim_i)$ we show again via criterion 1 of theorem 4 that the claim holds. Again, the first part is obviously true. For the second one, assume that there is a subset T of X with $T \subset S, T \cap H(\succsim_i) = \emptyset, T \cap L(\succsim_i) = \emptyset$. No subset of $H(\succsim_i)$ is low, and the only other possibility $S_2 = T$ is not low either, because it would have to contain $L(\succsim_i)$. □

Corollary 2. *For plurality voting, strategy $\{a\}$ is admissible if and only if $\{a\}$ is not in $L(\succsim_i)$.*

Proof. "$\Rightarrow$:" If $\{a\}$ is admissible, we show via criterion 2 of theorem 4 that the claim holds. Since none of the candidates of $L(\succsim_i)$ are allowed to be in S we are done.

"$\Leftarrow$:" If $\{a\}$ is not in $L(\succsim_i)$ we show again via criterion 2 that the claim holds. Again, the first part is obviously true. For the second one, it is clear that we will not find a nonempty set $A \subseteq X$ disjoint from $\{a\}$ such that $A \cup S$ is feasible. □

Corollary 3. *For negative voting,*[2]

- *strategy $\{a\}$ is admissible if and only if the voter strictly prefers a to at least two other candidates,*
- *and strategy $\bar{a}$ (casting a vote against a) is admissible if and only if the voter strictly prefers at least two other candidates to a.*

[2]We assume $|X| \geq 4$, because if $|X| < 4$ negative voting is equivalent to approval voting.

Proof. We start to show the first point:
"$\Rightarrow$:" If $\{a\}$ is admissible, we show via criterion 2 of theorem 4 that the claim holds. So, it is clear that $\{a\}$ cannot be in X_r. We have to show that it cannot be in X_{r-1} either, if $|X_r| = 1$ with $X_r = \{b\}$. Suppose $a \in X_{r-1}$. Then we can define $A = X_1 \cup X_2 \cup \cdots \cup X_{r-2} \cup (X_{r-1} \setminus \{a\}) = X \setminus (\{a\} \cup \{b\})$ with $A \cup S = \bar{b}$ and A is high for $\succsim_i$.

"$\Leftarrow$:" If the voter strictly prefers a to at least two other candidates we show again via criterion 2 that the claim holds. Again, the first part is obviously true (a cannot be in $L(\succsim_i)$). For the second one, the only nonempty set $A \subseteq X$ disjoint from S such that $A \cup S$ is feasible is $A = X \setminus (\{a\} \cup \{b\})$ with $b \in X$. But A is not high, because it contains X_r and so it would have to contain all X_k with $k < r$, which it does not, since a is in one of these X_k.

The second point is quite similar, but we will use criterion 1:
"$\Rightarrow$:" If $\bar{a} = X \setminus \{a\}$ is admissible, we show via criterion 1 that the claim holds. Since every candidate from S has to be in X_1 it is clear that a cannot be in X_1. We have to show that it cannot be in X_2 either, if $|X_1| = 1$ with $X_1 = \{b\}$. Suppose $a \in X_2$. Then we partition S into $S_1 = \{b\}$ and $S_2 = (X_2 \setminus \{a\}) \cup X_3 \cup \cdots \cup X_r = X \setminus (X_1 \cup \{a\})$, where S_1 is feasible and S_2 is low for $\succsim_i$.

"$\Leftarrow$:" If the voter strictly prefers at least two other candidates to a we show via criterion 1 that the claim holds. Again, it is clear for the fist part that a cannot be in X_1, and so all candidates of $\bar{a}$ are in S. For the second point, the only nonempty partition of S such that S_1 is feasible is $S_1 = \{b\}$ with b being any candidate in S. But $S_2 = S \setminus S_1$ is not low, because it contains X_1 and so it would have to contain all X_k with $k > 1$, which it does not, since a is in one of these X_k. □

Furthermore, for a concerned voter, abstention is never an admissible strategy, since it does not satisfy neither criterion 1 nor 2. Likewise, voting for all candidates can also never be an admissible strategy. Although both strategies can be considered feasible, they can be excluded from the analysis for concerned voters [4, p. 25].

With abstentions allowed, approval voting has $2^{|X|} - 1$ different feasible strategies (all possible subsets of X minus the whole set), plurality voting $|X|+1$ (vote for every candidate in X plus abstention) and negative voting $2|X|+1$ for $|X| \geq 3$ (voting for and against every candidate, plus abstention) feasible strategies. Here we see that approval voting offers by far the most possibilities for voters. Speaking of admissible strategies, however, this will radically change, especially under dichotomous preferences, as we will see in the next chapter.

Example 3. *Suppose a voter has the preference order $a \succ b \succ c \succ d$. We will have the following admissible strategies:*

- *Approval voting:* $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$
- *Plurality voting:* $\{a\}$, $\{b\}$, $\{c\}$
- *Negative voting:* $\{a\}$, $\{b\}$, $\bar{c}$, $\bar{d}$

Now we want to set up a concept which decides whether a voter is voting according to his or her true preferences. Referring to the example above, voting $\{a, c\}$ leaves a little awkward impression: Why should a voter approve of one candidate and not approve of a higher ranked one? The concept of sincerity handles this.

Definition 6. *Strategy S is* sincere *for a concerned $\succsim_i$ whenever $x \in S$ implies $y \in S$ for all $y \succ_i x$. A strategy profile s is sincere, if all the strategies in s are sincere. A voting procedure is sincere for $\succsim$ if all admissible strategies are sincere.*

So what are the sincere strategies in the example above? Approval voting has $\{a\}$, $\{a, b\}$, and $\{a, b, c\}$, plurality voting $\{a\}$ and negative voting $\{a\}$ and $\bar{d}$. As we see in this example, approval voting has the highest percentage of admissible strategies which are also sincere. This statement is not true under all circumstances, however. We will examine this more deeply in chapters 4 and 5. If we only allow for linear preference orders (which we will do in the next section), sincere strategies are always admissible if we exclude voting for everybody.

Proposition 1. *For linear preference orders, every feasible and sincere strategy is also admissible.*

Proof. Let us consider criterion 1 of theorem 4. Since we only allow for linear preference orders, $H(\succsim_i)$ only consists of one candidate who we want to call a. Given any sincere strategy S, a has always to be included in S. Now, we cannot find a partition of S in which S_2 is low, because if we found one, this would imply that no matter which candidate is included in S_2 we would always have to include the candidate from $L(\succsim_i)$ into S_2. But this cannot be the case, since S is sincere and we have excluded voting for everybody. □

If not mentioned otherwise, we will always assume strategies to be admissible and sincere.

3.3 Characterization of approval voting outcomes

Our next goal is to characterize election outcomes under approval voting following [7]. "Given a preference profile v, we consider the set of all candidates that *can* be chosen by approval voting when voters use sincere strategies. We call this set *AV outcomes* at v. Clearly, a candidate ranked last by all voters cannot be in this set, because it is inadmissible for any voter to vote for this candidate." We assume linear preferences in this context throughout this work, which only facilitates proofs and does not infringe generality referring to [7]. For every voter preference profile v, we will define a special ballot response profile, the *critical strategy profile* $C_i(v)$ for candidate i as follows: Every voter who has ranked candidate i last only votes for his highest ranked candidate. All the other voters vote for all candidates from the top down to candidate i. It is obvious that all these strategies are sincere and admissible, since no "gap" is allowed between the approved candidates and no lowest ranked candidate is ever approved. To give an example, the critical preference profiles for all candidates in example 1 for the given voter preference profile v are:[3]

[3]For a simpler use of notation we will omit the curly brackets in critical strategy profiles and write only MP instead of $\{M, P\}$.

$C_M(v) = (M, M, M, M, T, T, T, P, P)$, $C_T(v) = (M, M, M, M, T, T, T, PT, PT)$ and $C_P(v) = (MP, MP, MP, MP, TP, TP, TP, P, P)$ with each game winning at its critical strategy profile. This is not by pure chance - the following lemma will show that a candidate cannot do better than at his own critical strategy profile. So if a candidate is to win, he must win at his own critical strategy profile. If he does not, there is no other, better, election outcome where he could possibly win.

Lemma 14. *Assume all voters choose sincere strategies. The AV critical strategy profile for candidate i, $C_i(v)$, maximizes the difference between the number of votes that i receives and the number of votes that every other candidate j receives.*

Proof. If you just look at the definition above, it is obvious that candidate i cannot receive more votes than at his or her critical strategy profile $C_i(v)$. Now consider the number of votes any other candidate j gets at $C_i(v)$. If there are any departures from $C_i(v)$ he or she will get no fewer or sometimes more votes. These departures could be:

1. A voter who ranked candidate i last votes for more than for the highest ranked candidate, possibly for j, if j is not already the highest ranked candidate.

2. All remaining voters can approve of more candidates who are lower ranked than candidate i, possibly including j.

So in either case, candidate j never gets fewer votes and sometimes more. The only other departure from $C_i(v)$ than 1 and 2 would be not to vote for i if he is not ranked last and only to vote for candidates ranked above i. But this would give candidate i fewer votes. So every departure from $C_i(v)$ can only result in the same or fewer votes for i and in the same or more votes for j. So $C_i(v)$ maximizes the difference between number of votes for i and the number of votes for every other candidate j. □

The next theorem will specify what we stated before, i.e. characterize AV outcomes. The second theorem will additionally characterize candidates who cannot be AV outcomes.

Theorem 5. *Candidate i is an AV outcome if and only if i is chosen at his or her critical strategy profile $C_i(v)$.*

Proof. If i is chosen at $C_i(v)$ it obviously is an AV outcome, since all strategies in $C_i(v)$ are sincere. To show the other part, suppose candidate i is not chosen at $C_i(v)$. By lemma 14, candidate i cannot receive more votes at any other ballot response profile consisting of sincere and admissible strategies. So if i is not chosen at $C_i(v)$ it will not be an AV outcome at all. □

Theorem 6. *Given any preference profile v and any candidate i, i cannot be an AV outcome if and only if there exists some other candidate j such that the number of voters who consider j as their best choice and i as their worst choice exceeds the number of voters who prefer i to j.*

Proof. Given two candidates i and j, voters can be partitioned into three (disjoint) classes: (i) those who prefer i to j; (ii) those who consider j as their best choice and i as their worst; and (iii) those who prefer j to i but do not fall in class (ii). At $C_i(v)$ voters of class (i) will only vote for i, those in class (ii) only for j and those in class (iii) will vote for both candidates. Neglecting voters of class (iii) who give i and j the same number of votes, candidate i cannot be elected at $C_i(v)$ if and only if the number of voters in class (ii) exceeds the number of voters in class (i). Hence, by theorem 5 candidate i cannot be an AV outcome. □

To illustrate this, consider the following example:

Example 4. *We define the preference profile v as follows:*

$$\begin{aligned}
&\textit{3 voters: } a \succ b \succ c \succ d\\
&\textit{2 voters: } b \succ c \succ a \succ d\\
&\textit{2 voters: } d \succ b \succ c \succ a
\end{aligned}$$

We have a look at the candidates' critical strategy profiles, which are

$$\begin{aligned}
C_a(v) &= (a, a, a, bca, bca, d, d),\\
C_b(v) &= (ab, ab, ab, b, b, db, db),\\
C_c(v) &= (abc, abc, abc, bc, bc, dbc, dbc) \text{ and}\\
C_d(v) &= (a, a, a, b, b, d, d).
\end{aligned}$$

The winners are, respectively, $\{a\}$, $\{b\}$, $\{b, c\}$ and $\{a\}$, so the set of (all possible) AV outcomes at v is $\{\{a\}, \{b\}, \{b, c\}\}$. Candidate c can only win in a tie with b, and d has no chance of winning at all. We could have seen the latter fact also by applying theorem 6: There are 3 voters who consider a as their best choice and 5 who consider d as their worst choice. This makes 8 voters which exceeds the number of voters who prefer d to a, namely 2.

4 Advantages of approval voting

4.1 On dichotomous preferences

Most advantages of approval voting become obvious when regarded under dichotomous preferences. Dichotomous preferences mean that voters only divide the candidates into two groups: those they approve of and those they disapprove of. This concept of preferences is the one which most closely approaches the idea of approval voting. From what we have stated before, we know that on dichotomous preferences there is a unique admissible strategy for approval voting, namely to vote for all candidates in $H(\succsim_i)$ and none in $L(\succsim_i)$, whereas for plurality voting, for instance, there are $|H(\succsim_i)|$ many admissible strategies. It is obvious that this unique admissible strategy for approval voting is also sincere.

4.1.1 A characterization with strategy-proofness

Why is it of importance that there are few admissible respectively sincere strategies? First, it was often criticized that approval voting offers voters too many choices (as far as feasible strategies are concerned) and makes it too complex for them to understand and to choose one of these many strategies. However, a reasonable voter would only choose an admissible strategy, since all other strategies would be dominated and result in a worse election outcome. Although I suppose that each voter will not analyze their strategies for dominated ones, I still hold the view that it is quite clear to the common public that normally voting for the least preferred candidates is not the best strategy to choose. Second, we bring in another argument: strategy-proofness. A voting procedure is *strategy-proof* if there is only one admissible strategy for $\succsim_i$ for each voter, which then must be sincere. For another definition see criterion 4 on page 8. There, we have already seen what strategy-proof implies: A voter cannot get a better outcome by choosing an admissible strategy which might be insincere even if he knows the strategies of all the other voters. So the fact that approval voting only has a unique admissible strategy also implies that it is strategy-proof on dichotomous preferences.

What about the other voting procedures? Can we really take this as a unique advantage of approval voting or are there other voting procedures which satisfy sincerity and strategy-proofness as well? The next theorem will give us the answer:

Theorem 7. *Suppose every voter has dichotomous preferences, then the following two statements are true:*

(i) Every nonranked voting procedure is sincere.

(ii) Approval voting is the only nonranked voting procedure which is strategy-proof.

Proof. (i) We have to show that all admissible strategies are sincere. To analyze admissible strategies we will make use of theorem 4 and distinguish two cases. Suppose S is an admissible strategy for voter i on dichotomous preferences with $H(\succsim_i) = X_1 \succ_i X_2 = L(\succsim_i)$.
Criterion 1: When every candidate from $H(\succsim_i)$ is in S, S is certainly sincere, because even if candidates from $L(\succsim_i)$ are voted for, sincerity is guaranteed (since all higher ranked candidate are in S), and we do not even have to look at the second half of the condition.
Criterion 2: If there is no candidate from $L(\succsim_i)$ in S, sincerity is guaranteed as well.
(ii) We have to show that all nonranked voting procedures characterized by $M \neq \{0, 1, \ldots, |X|-1\}$ have more than one admissible strategy under appropriate dichotomous preferences $\succsim_i$.[1] Let $\succsim_i$ be such that $|X_1| - 1 \in M \setminus \{0\}$ but $|X_1| \notin M \setminus \{0\}$, which is possible for every M except for approval voting. Let $X_1 = \{x_{k_1}, \ldots, x_{k_l}\}$ with $l = |X_1|$. Now we define two feasible strategies

$$S = \{x_{k_1}, \ldots, x_{k_{l-1}}\} \text{ and}$$
$$S' = \{x_{k_1}, \ldots, x_{k_{l-2}}, x_{k_l}\}.$$

Both strategies fulfill criterion 2 of theorem 4: They include no candidate from X_2 and any nonempty A and A' which is high and disjoint from S and S' would have to contain the one candidate missing from X_1 in either strategy; but $A \cup S$ and $A \cup S'$ are both not feasible. Hence, both S and S' are admissible strategies. □

Hence, there is a "yes" and a "no" answer to the question stated before. "Since the demands of strategy-proofness are more stringent than those for sincere voting, the circumstances that imply strategy-proofness are less likely to obtain than the circumstances that imply sincerity" [4, p. 31]. Furthermore, since every voter has only one sincere and admissible strategy under dichotomous preferences and approval voting, there is only one outcome, and what we will see in corollary 4, this only outcome will always be the Condorcet winner.

4.1.2 A more general characterization with strategy-proofness

Additionally, there is another characterization that can be given for approval voting and strategy-proofness [21]. For this characterization we have to "enhance" our notation a little bit, because this theorem not only permits candidates to be feasible or not as in definition 1, but also allows for a variability of the set of voters. This means that there is a set $A \in \mathcal{X}$ of feasible candidates (or a set of "implementable" alternatives if you wish) as in the original definition of a social choice function on page 5 and a set $\bar{Y} \subseteq \{1, \ldots, n\}$ of voters who do <u>not</u> abstain. The following characterization will also be more general than theorem 7, which characterized approval voting only among nonranked voting procedures. The following theorem gives a characterization among all social choice rules (the term is to be specified later) on dichotomous preferences. It uses another form of a social choice function, namely

[1] We can say "appropriate", because we do not have to show this proposition for all dichotomous preferences. It is enough to show one case, where these voting procedures are not strategy-proof.

$F^{A,\bar{Y}} : V^{\bar{Y}} \to 2^A \setminus \{\emptyset\}$, where $v_{\bar{Y}} \in V^{\bar{Y}}$ specifies the preference profile consisting of the preferences $\succsim_i$ of the non-abstaining voters in $\bar{Y}$. For simpler notation we will write $F^A(v_{\bar{Y}})$ instead of $F^{A,\bar{Y}}(v_{\bar{Y}})$ if no confusion may arise.

We define a *family of social choice functions* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ which takes into account the variability of the two sets. Furthermore, we consider two consistency conditions which try to keep track of the selected candidates while the set of voters and candidates change.

Criterion 17 (Consistency in alternatives)**.** *The family of social choice functions* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ *is* consistent in candidates *if for all sets of feasible candidates* $S \subset T \subseteq X$*, all voters* $\bar{Y}$*, and all preference profiles* $v_{\bar{Y}}$*,*

$$F^S(v_{\bar{Y}}) = F^T(v_{\bar{Y}}) \cap S \textit{ whenever } F^T(v_{\bar{Y}}) \cap S \neq \emptyset.$$

Criterion 18 (Consistency in voters)**.** *Given the set of voters* $B \subset C \subseteq \{1, \dots, n\}$ *and the preference profile* v_C*, let* $v_{C|_B}$ *be the profile obtained by restricting* v_C *to* B*. The family of social choice functions* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ *is* consistent in voters *if for all pairs of candidates* (x, y)*, every electorate* B *and* C*, and all preference profiles* v_B *and* v_C *which are such that* $v_B = v_{C|_B}$ *and* $x \sim_i y$ *for all* $i \in C \setminus B$*, the condition*

$$F^{\{x,y\}}(v_B) = F^{\{x,y\}}(v_C)$$

holds.

Roughly speaking, criterion 17 says that an election outcome of a smaller subset always equals the outcome of the larger subset intersected with the smaller one, and criterion 18 states that unconcerned voters cannot change the outcome of an election with two alternatives. A family of social choice functions which respects the two criteria above, i.e. is consistent in candidates and voters, is called a *social choice rule*. We call $s_x(v_{\bar{Y}}) = \{i \in \bar{Y} | x \in H(\succsim_i)\}$ the *support of candidate* x for voter preference profile $v_{\bar{Y}}$. Generally, this is not equivalent to the number of votes candidate x received – only for approval voting, these two terms coincide.[2] For later use, we define the number of voters who weakly prefer candidate x to y under $v_{\bar{Y}}$ by $s(v_{\bar{Y}}; x, y) = \{i \in \bar{Y} | x \succsim_i y\}$. Under this framework we can define approval voting as the social choice rule where $x \in F^A(v_{\bar{Y}})$ if and only if $s_x(v_{\bar{Y}}) \geq s_y(v_{\bar{Y}})$ for all $y \in A$. The theorem states that a social choice rule is strategy-proof, neutral, anonymous and strictly monotone if and only if it is approval voting. Whereas we want to use definition 4 analogously for strategy-proofness, we still have to define the other three properties.

Criterion 19 (Anonymity)**.** *The social choice rule* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ *is* anonymous *if for all sets of feasible candidates* A*, all voters* $\bar{Y}$*, all preference profiles* $v_{\bar{Y}}$*, and all permutations* σ *of* $\bar{Y}$*,*

$$F^A(v_{\sigma(\bar{Y})}) = F^A(v_{\bar{Y}})$$

holds with $v_{\sigma(\bar{Y})} = (\succsim_{\sigma(i)})_{i \in \bar{Y}}$*.*

[2]Not only these terms coincide, also the voter preference profile and the ballot response profile are equivalent only for approval voting – of course, under dichotomous preferences!

Criterion 20 (Neutrality)**.** *The social choice rule* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ *is* neutral *if for all sets of feasible candidates* A*, all voters* $\bar{Y}$*, all preference profiles* $v_{\bar{Y}}$*, and all permutations* μ *of* A*,*

$$F^{\mu(A)}(\mu(v_{\bar{Y}})) = \mu(F^A(v_{\bar{Y}}))$$

holds, where $\mu(v_{\bar{Y}}) = (\mu(\succsim_i))_{i\in\bar{Y}}$ *and* $\mu(\succsim_i)$ *means that the preference ranking of voter* i *is altered according to the permutation of the candidates so that it represents the original ranking.*

Criterion 21 (Strict monotonicity)**.** *The social choice rule* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ *is* strictly monotone *if for all pairs of candidates* (x, y)*, all voters* $\bar{Y}$*, and all preference profiles* $v_{\bar{Y}}, v'_{\bar{Y}} \in V^{\bar{Y}}$ *which are such that* $x \succ_i y$ *and* $x \sim'_i y$ *for one voter* $i \in \bar{Y}$ *and* $v_{\bar{Y}\setminus\{i\}} = v'_{\bar{Y}\setminus\{i\}}$ *for the rest,*

$$x \in F^{\{x,y\}}(v'_{\bar{Y}}) \Rightarrow F^{\{x,y\}}(v_{\bar{Y}}) = \{x\}$$

holds.

Whereas the criteria of neutrality and anonymity should be taken for granted regarding most voting procedures in democratic elections, the criteria of strategy-proofness and strict monotonicity are quite strong and desired properties that (all four of them) *only* approval voting satisfies under dichotomous preferences. Before we come to the theorem, we need to show the following lemma.

Lemma 15. *Under dichotomous preferences, if the social choice rule* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ *is neutral and strategy-proof, then it satisfies IIA (cf. criterion 2).*

Proof. Suppose IIA is not satisfied. Then there exist voter $\bar{Y}$ and two preference profiles $v_{\bar{Y}}, v'_{\bar{Y}} \in V^{\bar{Y}}$ such that for a pair of candidates (x, y) it satisfies $s(v_{\bar{Y}}; x, y) = s(v'_{\bar{Y}}; x, y)$ and $s(v_{\bar{Y}}; y, x) = s(v'_{\bar{Y}}; y, x)$, but $F^{\{x,y\}}(v_{\bar{Y}}) = \{x\}$ and $F^{\{x,y\}}(v'_{\bar{Y}}) \in \{\{y\}, \{x, y\}\}$.

We define

$$i \in C \subseteq \bar{Y} \Leftrightarrow x \succ_i y \vee y \succ_i x.$$

If $C = \emptyset$, then $x \sim_i y$ must be true for all $i \in \bar{Y}$. This would imply $F^{\{x,y\}}(v_{\bar{Y}}) = \{x, y\}$ because we supposed neutrality and the empty set cannot be selected. This, however, is a contradiction to $F^{\{x,y\}}(v_{\bar{Y}}) = \{x\}$. Hence, $C \neq \emptyset$. Since every voter in $\bar{Y} \setminus C$ is indifferent between x and y we can apply consistency in voters and get $F^{\{x,y\}}(v_{\bar{Y}|C}) = F^{\{x,y\}}(v_{\bar{Y}}) = \{x\}$. As in the mentioned criterion, we will write v_C instead of $v_{\bar{Y}|C}$.

We want to show that for every $j \in C$, $F^{\{x,y\}}(\succsim'_j + v_{C\setminus\{j\}}) = \{x\}$ holds, where $\succsim'_j$ replaces $\succsim_j$ in v_C. We suppose the contrary, having $F^{\{x,y\}}(\succsim'_j + v_{C\setminus\{j\}}) \in \{\{y\}, \{x, y\}\}$. If voter j has $x \succ_j y$, then j can manipulate $F^{\{x,y\},C}$ by pretending $\succsim_j$ would be his true preference instead of $\succsim'_j$, which would yield the preferred outcome $\{x\}$. (To see that j prefers outcome $\{x\}$ to $\{x, y\}$ to $\{y\}$ consider assumption P on page 21.) On the other hand, if voter j has $y \succ_j x$, then j can manipulate $F^{\{x,y\},C}$ by doing the opposite: Pretending that his true preference ranking was $\succsim'_j$

instead of $\succsim_j$. This contradicts strategy-proofness and we have obtained what we wanted.

Now, if we repeat this procedure for each voter $i \in C \setminus \{j\}$ step by step, and change the voter's preferences from $\succsim_i$ to $\succsim_i'$, we will never change the image of $F^{\{x,y\}}$ because of strategy-proofness. Finally, we will have $F^{\{x,y\}}(v_C') = \{x\}$. Again, thanks to consistency of voters we get $F^{\{x,y\}}(v_{\bar{Y}}') = F^{\{x,y\}}(v_C' + v_{\bar{Y}\setminus C}') = F^{\{x,y\}}(v_C') = \{x\}$. This is a contradiction to $y \in F^{\{x,y\}}(v_{\bar{Y}}')$, so IIA must hold. □

Theorem 8. *Under dichotomous preferences, the social choice rule* $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ *is strategy-proof, neutral, anonymous and strictly monotone if and only if it is approval voting.*

Proof. We have already seen that approval voting is strategy-proof, and it is obvious that approval voting satisfies neutrality, anonymity and strict monotonicity. For the other direction let $\{F^{A,\bar{Y}}(v_{\bar{Y}})\}_{A,\bar{Y}}$ be a social choice rule which satisfies the four properties. We will start with $A = \{x, y\}$ and show that for all voters $\bar{Y}$, and for all $v_{\bar{Y}} \in V^{\bar{Y}}$, the subfamily $\left\{F^{\{x,y\},\bar{Y}} : V^{\bar{Y}} \to \{\{x\},\{y\},\{x,y\}\}\right\}_{\bar{Y}}$ orders candidates x and y according to approval voting, which means that (1) $F^{\{x,y\}}(v_{\bar{Y}}) = \{x, y\}$ if $s_x(v_{\bar{Y}}) = s_y(v_{\bar{Y}})$, and that (2) $F^{\{x,y\}}(v_{\bar{Y}}) = \{x\}$ if $s_x(v_{\bar{Y}}) > s_y(v_{\bar{Y}})$.

1. As we have shown in lemma 15, $\{F^{\{x,y\},\bar{Y}}\}_{\bar{Y}}$ satisfies IIA and also via anonymity we get that for all set of voters $\bar{Y}$, the decision of $F^{\{x,y\},\bar{Y}}$ depends on the numbers $s(v_{\bar{Y}}; x, y) = s_x(v_{\bar{Y}}) + |\{i \in \bar{Y} : x, y \in L(\succsim_i)\}|$ and $s(v_{\bar{Y}}; y, x) = s_y(v_{\bar{Y}}) + |\{i \in \bar{Y} : x, y \in L(\succsim_i)\}|$. Neutrality implies that $F^{x,y}(v_{\bar{Y}}) = \{x, y\}$ whenever $s_x(v_{\bar{Y}}) = s_y(v_{\bar{Y}})$.

2. Suppose that for a given set of voters $\bar{Y}$ and a preference profile $v_{\bar{Y}}$, $s_x(v_{\bar{Y}}) - s_y(v_{\bar{Y}}) = 1$. We construct preference profile $v_{\bar{Y}}'$ by picking a voter i who has $x \succ_i y$ and change it to $x \sim_i y$. For all other voters $j \neq i$ we let $\succsim_j' = \succsim_j$. Now we have $s_x(v_{\bar{Y}}') = s_y(v_{\bar{Y}}')$, and thus we are in situation (1) where we had $F^{x,y}(v_{\bar{Y}}) = \{x, y\}$. Now applying strict monotonicity, we can follow that $F^{x,y}(v_{\bar{Y}}) = \{x\}$. It is inductively shown that $F^{x,y}(v_{\bar{Y}}) = \{x\}$ whenever $s_x(v_{\bar{Y}}') > s_y(v_{\bar{Y}}')$.

At this step we want to generalize the situation from the set of candidates $\{x, y\}$ to any set of candidates $A \in \mathcal{X}$. We have to prove that for all sets of feasible candidates A, all sets of non-abstaining voters $\bar{Y}$ and all preference profiles $v_{\bar{Y}}$ the following equivalence must be true:

$$x \in F^A(v_{\bar{Y}}) \Leftrightarrow s_x(v_{\bar{Y}}') \geq s_y(v_{\bar{Y}}') \text{ for all } y \in A.$$

"$\Rightarrow$:" $x \in F^A(v_{\bar{Y}})$ implies with consistency in candidates ($S = \{x, y\}, T = A$) that $x \in F^{\{x,y\}}(v_{\bar{Y}})$ for all $y \in A \setminus \{x\}$, since $F^A(v_{\bar{Y}}) \cap \{x, y\} = F^{\{x,y\}}(v_{\bar{Y}})$ must hold. This, together with the fact from above that $\{F^{\{x,y\},\bar{Y}}\}_{\bar{Y}}$ orders x and y according to approval voting, implies that $s_x(v_{\bar{Y}}') \geq s_y(v_{\bar{Y}}')$ for all $y \in A$.

"$\Leftarrow$:" Suppose that $s_x(v_{\bar{Y}}') \geq s_y(v_{\bar{Y}}')$ for all $y \in A$ holds. Due to the fact we just had, namely that $\{F^{\{x,y\},\bar{Y}}\}_{\bar{Y}}$ orders x and y according to approval voting we know that $x \in F^{\{x,y\}}(v_{\bar{Y}})$ for all $y \in A$. If there is a candidate $z \neq x$ such that $z \in F^A(v_{\bar{Y}})$, then $F^A(v_{\bar{Y}}) \cap \{x,z\} \neq \emptyset$ and consistency in candidates implies with $S = \{x,z\}$ and $T = A$ that $F^{\{x,z\}}(v_{\bar{Y}}) = F^A(v_{\bar{Y}}) \cap \{x,z\}$. But since we know that $x \in F^{\{x,y\}}(v_{\bar{Y}})$ is must also be true that $x \in F^A(v_{\bar{Y}})$. If there is no candidate $z \neq x$ such that $z \in F^A(v_{\bar{Y}})$, then $x \in F^A(v_{\bar{Y}})$ must be true, since $F^A(v_{\bar{Y}})$ is not allowed to choose the empty set by defintion. $\square$

Vorsatz shows in [21] that this theorem is tight, which means that we cannot renounce one criteria so that the theorem would still hold.

4.2 On trichotomous preferences

On trichotomous preferences, conditions for sincereness and strategy-proofness are not so easy to fulfill. Approval voting does worse, but the other nonranked voting procedures do so too. And still, approval voting is the only one still remaining sincere for every trichotomous preference profile.

Theorem 9. *Suppose every voter has trichotomous preferences, then the following two statements are true:*

(i) Approval voting is the only nonranked voting procedure which is sincere.

(ii) There is no nonranked voting procedure which is strategy-proof.

Proof. (i) We will show that approval voting is sincere for all trichotomous preferences profiles, and no other voting procedure can satisfy this condition. Again, we will make use of theorem 4. We show for every voter i with $H(\succsim_i) = X_1 \succ_i X_2 \succ_i X_3 = L(\succsim_i)$ that all admissible strategies are sincere for approval voting. Let S be a feasible strategy.
Criterion 1: This condition can only be fulfilled by approval voting for all voter preferences, so S consists of all candidates from X_1. If now we add a candidate $x \in X_2$, this strategy is admissible and sincere. If we instead add a candidate $x \in X_3$, we can partition S into $S_1 = X_1$ and $S_2 = \{x\}$, but S_2 is low for $\succsim_i$. So this strategy would not be admissible. Hence, all possible strategies arising from this condition are also sincere.
Criterion 2: If only candidates from X_1 are in S we have no problems concerning sincerity. If $S \subseteq X_2$, or if only some candidates from X_1 are in S and $X_2 \cap S \neq \emptyset$, we can choose $|X_1|$ and $|X_2|$ such that except for approval voting there is no nonempty A disjoint from S such that A is high for $\succsim_i$ and $A \cup S$ is feasible. So, only for approval voting, this strategy is not admissible.
(ii) Let $\succsim_i$ be such that

$$\begin{aligned} X_1 &= \{x_{k_1}, \ldots, x_{k_l}\}, \\ X_2 &= \{x_{k_{l+1}}, \ldots, x_{k_j}\}, \\ X_3 &= \{x_{k_{j+1}}\} \cup Y, \end{aligned}$$

with $\emptyset \subseteq Y \subseteq X \setminus (X_1 \cup X_2 \cup \{x_{k_{j+1}}\})$ and $l < j$.

Case (a): $M = \{j\}, j \in \{2, \ldots, |X| - 1\}$: We define

$$S = \{x_{k_1}, \ldots, x_{k_l}, x_{k_{l+1}}, \ldots, x_{k_j}\} \text{ and}$$
$$S' = \{x_{k_1}, \ldots, x_{k_l}, x_{k_{l+1}}, \ldots, x_{k_{j-1}}, x_{k_{j+1}}\}.$$

Both strategies fulfill criterion 1 of theorem 4 because all candidates from X_1 are in S and in S' and there is no partition of S nor S' into nonempty S_1 and S_2 nor S'_1 and S'_2 such that S_1 or S'_1 are feasible. For $j = 1$, we redefine $X_1 = \{x_{k_1}\}$ and $X_2 = \{x_{k_2}\}$. Now, both strategies $S = X_1$ and $S' = X_2$ are admissible, cf. corollary 2.

Case (b): $|M| > 1$: So there exist two integers $j, l \in \{1, \ldots, |X| - 1\}$ with $l < j$. We define

$$S = \{x_{k_1}, \ldots, x_{k_l}\} \text{ and}$$
$$S' = \{x_{k_1}, \ldots, x_{k_l}, x_{k_{l+1}}, \ldots, x_{k_j}\}.$$

Again, both strategies fulfill criterion 1 of theorem 4 because both strategies contain all candidates from X_1 and there is no partition of S nor S' into nonempty S_1 and S_2, nor S'_1 and S'_2, such that S_2 or S'_2 are low, since they both have to contain candidates from $X_1 \cup X_2$, which would imply that S_2 and S'_2 would have to contain all candidates from X_3 as well.

Hence, in both cases we have found two distinct admissible strategies. □

4.3 On multichotomous preferences

If the preference rankings of the electorate are not really restricted, the conclusions we draw are not really strong either. However, at least we can state that assuming multichotomous preferences is no disadvantage for approval voting in comparison with other nonranked voting procedures.

Theorem 10. *Suppose every voter has multichotomous preferences, then the following two statements are true [4, pp. 30-31]:*

(i) There is no nonranked voting procedure which is sincere.

(ii) There is no nonranked voting procedure which is strategy-proof.

Proof. We will prove the second part of the theorem first.

(ii) For $r \geq 4$, let $\succsim_i$ be such that

$$\begin{aligned}
X_1 &= \{x_{k_1}, \ldots, x_{k_l}\}, \\
X_2 &= \{x_{k_{l+1}}, \ldots, x_{k_j}\}, \\
X_3 &= \{x_{k_{j+1}}\}, \\
X_4 &= \{x_{k_{j+2}}\}, \\
&\vdots \\
X_r &= \{x_{k_{j+(r-2)}}\} \cup Y,
\end{aligned}$$

with $\emptyset \subseteq Y \subseteq X \setminus (X_1 \cup X_2 \cup \cdots \cup X_{r-1} \cup \{x_{k_{j+(r-2)}}\})$ and $l < j$. We can again distinguish the same two cases:

Case (a): $M = \{j\}, j \in \{2, \ldots, |X|-1\}$: We again define S and S' as in the proof of theorem 9 above and show in the same manner that they are both admissible strategies. The exception for $j = 1$ works exactly as above, too.

Case (b): $|M| > 1$: So there exist again two integers $j, l \in \{1, \ldots, |X| - 1\}$ with $l < j$, and we define S and S' as in the proof of theorem 9 above and show in the same manner that they are both admissible strategies.

(i) Now for the first part, we consider the strategy

$$S' = \{x_{k_1}, \ldots, x_{k_l}, x_{k_{l+1}}, \ldots, x_{k_{j-1}}, x_{k_{j+1}}\},$$

which is admissible as shown in part (ii) for both cases and obviously insincere. For $j = 1$, strategy $S' = X_3$ is admissible and insincere. □

4.4 Election of a Pareto candidate

A *Pareto candidate* is a candidate who satisfies the Pareto criterion on page 12 in the way that there is no candidate who all voters rank higher. So it seems to be a good idea for a voting procedure to choose such a candidate. We will prove that this is not always the case for approval voting later, but still the following advantage is true [7]:

Proposition 2. *(i) At every preference profile v, there exists a Pareto candidate that is an AV outcome or component of an AV outcome.*
(ii) A non-Pareto candidate may be a component of an AV outcome but never be a unique AV winner.

Proof. (i) Suppose any preference profile v with strict preferences, which is no restriction to this finding referring to [7]. Let all the voters only vote for their top choice. In this way, a Pareto candidate is at least part of an AV outcome, if not the unique winner, because if not, there would have been another candidate who all voters would have ranked higher.

(ii) To show the first part, consider example 4. Here c is part of an AV outcome (together with b), although every voter prefers b to c. To show the second part, suppose non-Pareto candidate i is part of the AV outcome at some preference profile v. Now take any sincere strategy profile s which yields this outcome. Since i is a non-Pareto candidate, there must be a candidate j that all voters prefer to i. And since all voters vote sincerely, all those who vote for i must also vote for j. So, i and j tie for the most votes. Indeed, all candidates j that are ranked higher than i tie for AV winners. Among those candidates j some perhaps Pareto-dominate others, but there must be one who no other candidate is preferred to. Hence, we have shown that this Pareto candidate is part of the AV outcome. □

4.5 Election of a Condorcet winner

As announced on page 10 analyzing the Condorcet winner criterion, we stated that approval voting need not always but always can produce a Condorcet winner as the winner of an election [7]. We will prove this now. Furthermore we will also show a big advantage of approval voting in comparison to *fixed rules*, where voters can cast a vote only for a predetermined number of candidates.

Theorem 11. *(i) Condorcet winners are always AV outcomes.*
(ii) If $|M| = 1$ a unique Condorcet winner need not be a possible outcome for M.

Proof. (i) If candidate i is a Condorcet winner, a majority of voters prefer i to every other candidate j. So fewer voters rank j as their first choice than i as their last choice, which implies by theorem 6 that i is an AV outcome.

(ii) Consider the following example and suppose sincere strategies:

$$
\begin{aligned}
&6 \text{ voters: } a \succ b \succ c \\
&4 \text{ voters: } b \succ a \succ c \\
&4 \text{ voters: } b \succ c \succ a \\
&3 \text{ voters: } c \succ a \succ b
\end{aligned}
$$

If $M = \{1\}$ or $M = \{2\}$, candidate b is elected, whereas candidate a is the Condorcet winner. □

In the example in the proof above we saw that neither fixed rule with $|M| = 1$ is able to elect the Condorcet winner. So plurality voting, for example, is not always able to elect the Condorcet winner, contrary to approval voting. For approval voting there are several strategies that would produce Condorcet winner a as an outcome, e.g. $C_a = (a, ba, b, ca)$ or (a, ba, b, c) with each coordinate representing a whole voter type[3]. Referring to [7] we see that "the flexibility of approval voting may be needed to elect a unique Condorcet winner." On the other hand, dichotomous preferences allow the voter only to have one sincere and admissible strategy, and still the Condorcet winner will be elected.

Corollary 4. *If every voter has dichotomous preferences, approval voting always elects the Condorcet winner.*

Proof. Since approval voting is strategy-proof under dichotomous preferences, every voter has only one sincere and admissible strategy, so the set of AV outcomes will have cardinality one. And, since Condorcet winners are always AV outcomes, this one set must be the Condorcet winner. □

4.6 Stability of Condorcet winners

What about the stability of voting outcomes and what does stability mean? We define two kinds of stability.

[3] A voter type consists of voters having exactly the same preferences.

Definition 7. *Given a preference profile v, a nontied voting outcome is* stable *if and only if there exists a strategy profile s such that no voters of a single type have an incentive to switch their strategy to another sincere strategy in order to produce a preferred outcome.*

A nontied voting outcome is strongly stable *if and only if there exists a strategy profile s such that no types of voters, coordinating their actions, can form a coalition K, all of whose members would have an incentive to switch their strategies to other sincere strategies in order to produce a preferred outcome.*

If a voting outcome is neither stable nor strongly stable it is called *unstable*. Brams and Sanver state in [7] that the strategies associated with a (strongly) stable AV outcome at $C_i(v)$ define a (strong) Nash equilibrium of a voting game in which the voters have complete information about each others' preferences and make simultaneous choices. As with the definition of admissibility there is a proposition that helps us analyzing a nontied outcome for stability: we need only consider critical strategy profiles.

Lemma 16. *(i) A nontied AV outcome i is stable if and only if it is stable at its critical strategy profile $C_i(v)$.*
(ii) A nontied AV outcome i is strongly stable if and only if it is strongly stable at its critical strategy profile $C_i(v)$.

Proof. (i) "$\Leftarrow$:" If i is stable at $C_i(v)$, it is clear that i is stable, because $C_i(v)$ is the strategy profile required for stability.
"$\Rightarrow$:" We have to show that if i is not stable at $C_i(v)$, it cannot be stable at any other strategy profile. So assume that i is unstable at $C_i(v)$. Hence, there exist voters of a single type who have an incentive to switch to another strategy profile s' in order to induce a preferred outcome. By lemma 14 candidate i will receive generally fewer or the same number of votes at s' than at $C_i(v)$. Given another strategy s which elects i, all voters switching from $C_i(v)$ to s' can do so at s too.

(ii) The argument is analogous to part (i). □

Consider again example 1 and $C_M(v)$. Of course, the first type of voters has no incentive to depart from their strategy since M is already their most preferred outcome. The second type, however, does have an incentive to switch from strategy $\{T\}$ to strategy $\{T, P\}$, which would change the winner from M to P. Also, if even only two voters of the second type changed to $\{T, P\}$, this would already result into a tie between M and P with $\{M, P\}P\{M\}$. So M cannot be a stable AV outcome. What about T? At $C_T(v)$ three voters from the first type would prefer voting $\{M, P\}$ rather than $\{M\}$ in order to get a tie between T and P, and if four voters changed they would even change the outcome from T to P. It is obvious that they prefer $\{P\}P\{T, P\}P\{T\}$. Thus, also outcome T is not stable at all, and of course, if an outcome is not stable, it will not be strongly stable either. Our last hope rests on candidate P. Without any doubt voters of the third type do not have any incentive to change their strategy since P is already their most preferred outcome. The voters of the first type do not have any incentive to change their strategy either because even if all four voters voted $\{M\}$ instead of $\{M, P\}$ this would not change a thing.

Neither do voters of the second type, because even if all three voters voted $\{T\}$ instead of $\{T, P\}$ the outcome would not change. So we know that P is a stable AV outcome. Furthermore, the only possible coalition for testing strong stability would consist of first and second type voters. However, even if they both changed their strategies to $\{M\}$ and $\{P\}$ respectively, the outcome would change to M, which is preferred by the first type of voters but not at all by the second type. Thus, we have found a strongly stable AV outcome with candidate P. By the way, P is also the unique Condorcet winner in this example. Is there any relationship between stability and a Condorcet winner? Guess what!

Theorem 12. *A nontied AV outcome is strongly stable if and only if it is a unique Condorcet winner.*

Proof. "$\Leftarrow$:" Suppose candidate i is a unique Condorcet winner at preference profile v. By theorem 11(i) the candidate is surely an AV outcome, and by theorem 5 it is so at least i's critical strategy profile $C_i(v)$. We will show that the outcome is strongly stable at $C_i(v)$, which by lemma 16 implies strong stability. So suppose there is a coalition K consisting of one or more voter types whose members prefer candidate j to candidate i. The members (as far as the voters, not the voter types are concerned) of K are outnumbered by the members of coalition L, whose members prefer i to j, because i is the Condorcet winner. Furthermore, the members of L vote for i, but not for j at $C_i(v)$. Hence, whatever admissible and sincere strategy the members of K switch to, candidate i will always get more votes than candidate j.

"$\Rightarrow$:" Suppose candidate i is not the unique Condorcet winner. Consequently, there exist a candidate j and a majority coalition K consisting of one or more voter types that prefer j to i. We need to show that i is not strongly stable at $C_i(v)$ which then implies by lemma 16 that i is not a strongly stable AV outcome. Suppose that approval voting does not elect i at $C_i(v)$. This means by theorem 5 that i is no AV outcome, and hence, not a strongly stable one. Suppose on the contrary that AV does elect i at $C_i(v)$. Now there is this majority coalition K that can change the winner to j, because all members of K will change their strategy to vote for j and not for i and so j will receive more AV votes than i because of K's majority. $\square$

This strong result becomes even more stringent when we see that no Condorcet method – any voting procedure which is designed to elect the Condorcet winner if one exists – guarantees the Condorcet winner to be in equilibrium. Citing [7] we state that the following proposition casts doubt on the efficacy of Condorcet methods, such as the Black method from chapter 2, to do what they purport to do in equilibrium. By contrast, approval voting ensures that Condorcet winners can be elected as strong Nash equilibria.

Proposition 3. *No Condorcet method which elects a single winner ensures the election of a unique Condorcet winner as a Nash equilibrium.*

Proof. We consider the following example, in which there is no Condorcet winner:

$$\begin{aligned} \text{2 voters: } & a \succ d \succ b \succ c \\ \text{2 voters: } & b \succ d \succ c \succ a \\ \text{1 voter: } & c \succ a \succ b \succ d \end{aligned}$$

If there is no Condorcet winner, we assume that different candidates may be chosen by the voting procedure. From this base example, we will analyze departures for every voter type, one by one. In these departures there exists a unique Condorcet winner, and we show that this unique Condorcet winner is no Nash equilibrium.

Suppose all voters of the first voter type have the true preferences $a \succ c \succ d \succ b$. In this preference profile, there exists a Condorcet winner, namely c. If the winner in the base example was candidate a, the first voter type would vote as in the base example in order to get their preferred outcome a.

Suppose all voters of the second voter type have the true preferences $b \succ d \succ a \succ c$. This would yield the Condorcet winner a. If the winner in the base example was candidate b or d, the second voter type would deviate from their true preferences to the preferences of the base example in order to get their preferred outcome b or d.

Finally, suppose the voter of the third type has the true preferences $c \succ d \succ a \succ b$. In this preference profile we have the Condorcet winner d. If the winner in the base example was candidate c, the third voter type would prefer to vote as in the base example in order to get the preferred outcome c.

Thus, all three cases are not in equilibrium. To prevent the voter types to deviate for all three cases, we would have to prevent candidates a, b, c and d from winning in the base example, which - with no singleton winner sets left - would yield outcomes such as $\{b, c\}$ that are also excluded because the voting system was supposed to elect single winners. Summarizing, there is no way to prevent at least one of the three cases to be not in Nash equilibrium. Hence, we have shown that whatever Condorcet method we use for these examples, there will always be a Condorcet winner not in Nash equilibrium. □

To illustrate this proposition with an example, let's use the Black method with the example in the proof. Since there is no Condorcet winner in the base example, Borda count is carried out with the following results: a: 8 points, b: 9 points, c: 5 points, and d: 8 points. So candidate b would win and we are in the second case of the proof, where the second voter type has $b \succ d \succ a \succ c$ with candidate a being the Condorcet winner. Thus, the Black method fails to ensure the election of a unique Condorcet winner as a Nash equilibrium, since deviating to $b \succ d \succ c \succ a$ would yield the preferred outcome b for the second voter type.

4.7 Condorcet efficiency

So far we have seen that a Condorcet winner always is an approval voting outcome, which means that under a given preference profile a Condorcet winner *can* always, but need not always be elected. What is now the probability of a Condorcet winner really getting elected - of course, only if one exists? This probability is called the Condorcet efficiency for a certain voting procedure. We will assume linear preferences and the following assumption [12]:

Assumption 3 (Impartial Culture)**.** *If a voter is selected at random, he or she will have a linear preference order on the set $X = \{x_1, x_2, \ldots, x_m\}$ of m candidates, and each of the $m!$ possible rankings is equally likely to represent the selected voters' preferences.*

First of all, we will try to compute the probability $P_{CW}(m, n)$ that a Condorcet winner exists, i.e. that we are not in the situation of the Condorcet paradox. For instance, it is easy to compute $P_{CW}(3, 3)$. There are $3! = 6$ different preferences for one voter and $(3!)^3 = 216$ possibilities for the voter preference profile v. If all three or two voters have the same preferences, there will always be a clear Condorcet winner. So we are only interested in the voter preference profiles with three different preference rankings. Their number equals $\frac{6!}{(6-3)!} = 120$, which are permutations without repetition. So 96 preference profiles are so far guaranteed to have a Condorcet winner. Since the order does not matter to us, we will only have $\binom{6}{3} = 20$ combinations (without repetition) to care about. We enumerate all the 6 different voter preferences:

1. $a \succ b \succ c$
2. $a \succ c \succ b$
3. $b \succ a \succ c$
4. $b \succ c \succ a$
5. $c \succ a \succ b$
6. $c \succ b \succ a$

Without enumerating all the 20 combinations, we see at one glance that e.g. all 4 combinations which include 1 and 2 have a Condorcet winner (since they both state $a \succ b$ and $a \succ c$). The same is true for all 4 combinations which include 3 and 4, and all 4 combinations which include 5 and 6. So there remain 8 combinations to be examined:

1. 135: Candidate a is Condorcet winner.
2. 136: Candidate b is Condorcet winner.
3. 145: No Condorcet winner!
4. 146: Candidate b is Condorcet winner.
5. 235: Candidate a is Condorcet winner.
6. 236: No Condorcet winner!

7. 245: Candidate c is Condorcet winner.

8. 246: Candidate c is Condorcet winner.

Thus, there are $2 \times 6 = 12$ out of 216 cases which have no Condorcet winner. This leads to $P_{CW}(3,3) = \frac{204}{216} = \frac{17}{18}$. For values greater than 3 this becomes complex very quickly and we refer to [6, pp. 34-35] and to the references cited therein for further results. Hence, we will confine our interest to a comparison of Condorcet efficiency between approval voting and other fixed rules (nonranked voting procedures with $|M| = 1$).

Second, we find the joint probability $P_{CWAV_m}(m,n,p)$ that candidate $x_m \in X$ is both the Condorcet and the approval voting winner. To do this, we will follow [12] and define a discrete variable x_i^j for every voter $j \leq n$ and for each candidate $i \leq m-1$ by

$$x_i^j = \begin{cases} +1 & \text{if } x_m \succ_j x_i, \\ -1 & \text{if } x_i \succ_j x_m. \end{cases}$$

Let

$$\overline{x}_i = \sum_{j=1}^{n} x_i^j$$

denote the overall value of x_i^j over the n voters. Then candidate x_m will be the Condorcet winner if $\overline{x}_i > 0$ for all $i \leq m-1$.

We also define another discrete variable y_i^j for every voter $j \leq n$, for each candidate $i \leq m-1$, and for a given strategy profile $\mathcal{S} = (S_1, S_2, \ldots, S_n)$ for approval voting by

$$y_i^j = \begin{cases} +1 & \text{if } x_m \in S_j \wedge x_i \notin S_j, \\ -1 & \text{if } x_i \in S_j \wedge x_m \notin S_j, \\ 0 & \text{if } (x_i \in S_j \wedge x_m \in S_j) \vee (x_i \notin S_j \wedge x_m \notin S_j). \end{cases}$$

Let

$$\overline{y}_i = \sum_{j=1}^{n} y_i^j$$

denote the overall value of y_i^j over the n voters. Then candidate x_m will be the approval voting winner if $\overline{y}_i > 0$ for all $i \leq m-1$. Obviously y_i^j depends on the number of candidates voter j chooses to approve. To account for this we define p_q in p as the probability that any voter will choose to approve of q candidates, with $\sum_{q=1}^{m-1} p_q = 1$ and $p_m = 0$.

With all these definitions, $P_{CWAV_m}(m,n,p)$ is given by the joint probability that $\overline{x}_i\sqrt{n} > 0$ and $\overline{y}_i\sqrt{n} > 0$ for all $i \leq m-1$. Let $P_{CWAV}(m,n,p) = mP_{CWAV_m}(m,n,p)$ denote the joint probability that *some* alternative is both the Condorcet and the approval voting winner.

Third, we define the conditional probability that approval voting elects a Condorcet winner if one exists - the Condorcet efficiency - by

$$P_{CEAV}(m,n,p) = \frac{P_{CWAV}(m,n,p)}{P_{CW}(m,n)}.$$

For fixed rules, we will denote the Condorcet efficiency by $P_{CEFR}(m, n, q)$ with one $p_q = 1$

We can now state the following proposition whose proof can be found in [12]. It simply shows that the Condorcet efficiency of approval voting is at least as high as the Condorcet efficiency of plurality voting for an infinite number of voters.

Proposition 4. $P_{CEFR}(m, \infty, 1) \leq P_{CEAV}(m, \infty, p)$

A similar argument can be found concerning the probability that a voting procedure elects the Condorcet loser if one exists. $P_{CLAV}(m, n, p)$ will define the conditional probability that a Condorcet loser is elected under approval voting given that one exists . $P_{CLFR}(m, n, q)$ will do the same for any fixed rule. Gehrlein and Lepelley mention in [12] that the probability $P_{CW}(m, n)$ a Condorcet winner exists is the same as the probability $P_{CL}(m, n)$ a Condorcet loser exists. In the next proposition we will see another advantage of approval voting towards plurality voting whose proof is found in the same paper.

Proposition 5. $P_{CLFR}(m, \infty, 1) \geq P_{CLAV}(m, \infty, p)$

4.8 Comparison of outcomes

We will see in this section that all outcomes that can be achieved by a positional scoring procedure can also be achieved by approval voting, but not vice versa. There are voting outcomes which can only be achieved by approval voting and not by any positional scoring procedure. The same is true for two other voting rules which are not positional scoring procedures. Why is this an advantage? In a sense, approval voting unites all the outcomes of the mentioned voting procedures. So, it could be argued, there is no use in applying Borda count for instance, because all Borda count can achieve, can also be achieved by approval voting, at least as far voting outcomes are concerned. Furthermore, approval voting allows for other candidates that were otherwise neglected. As we will see in the proof of the next theorem, a positional scoring procedure need not elect the Condorcet winner, whereas approval voting always ensures that he or she can be elected. This is a big argument for more inclusiveness and therefore a broader range of possible outcomes. What adds to this argument is that approval voting can better prevent candidates who are not acceptable from winning than any other of the before mentioned voting procedures.

Theorem 13. *At all preference profiles v, a candidate chosen by any positional scoring procedure is an AV outcome. There exist preference profiles v at which a candidate is not chosen by any positional scoring procedure but is, nevertheless, an AV outcome.*

Proof. We begin to show the first statement. Assume candidate i is chosen by any positional scoring procedure at the preference profile v. Let $s = (s_1, s_2, \ldots, s_m)$ be the positional scoring vector as on page 13. Without loss of generality we will normalize $s_1 = 1$ and $s_m = 0$. Let $d(x, k)$ be the number of voters who rank candidate x on k^{th} position as on the same page. Then the following inequality must hold

$$s_1 d(i,1) + s_2 d(i,2) + \cdots + s_m d(i,m) \geq s_1 d(j,1) + s_2 d(j,2) + \cdots + s_m d(j,m) \quad (4.8.1)$$

for all other candidates j.

We will show that i is also chosen by approval voting at i's critical strategy profile $C_i(v)$. For this proof, we will interpret approval voting as a non-classical kind of a positional scoring procedure with the difference that voters give points to the candidates not using a predetermined vector. We abbreviate the left hand side of inequality 4.8.1 by PSP_i for positional scoring procedures and AV_i for approval voting, and the right hand side with PSP_j and AV_j, respectively. We assume that voters vote as suggested by $C_i(v)$ and distinguish two cases:

Case (a): We regard all those voters who rank candidate i last. A positional scoring procedure gives 1 point to the top candidate, 0 points to candidate i, and points between 0 and 1 to all the other candidates. Approval voting gives 1 point to the top candidate, and 0 points to i and to all the other candidates. Now, $AV_i = PSP_i = 0$ and $PSP_j \geq AV_j$ because all candidates ranked between places 2 and $m-1$ will receive points between 0 and 1 under positional scoring procedures, but 0 points under approval voting at $C_i(v)$.

Case (b): We regard all those voters who do not rank candidate i last. A positional scoring procedure gives s_k points to candidate i if he or she is ranked k^{th} best. Approval voting gives candidate i (and all candidates ranked above) 1 point. Hence, it is easy to see that $AV_i \geq PSP_i$ because every $s_k = 1$ in AV_i. If all voters have $j \succ i$ we have $AV_i = AV_j$, if not $AV_i > AV_j$, hence $AV_i \geq AV_j$ holds.

Thus, uniting all the voters of the two cases, we see that if $PSP_i \geq PSP_j$ holds, $AV_i \geq AV_j$ is true.

We still need to show the second statement. Let's consider the following example:

$$\begin{aligned} \text{3 voters: } & a \succ b \succ c \\ \text{2 voters: } & b \succ c \succ a \\ \text{1 voter: } & b \succ a \succ c \\ \text{1 voter: } & c \succ a \succ b \end{aligned}$$

We show that candidate b would be elected by every positional scoring procedure. So the two inequalities

$$d(b,1) + s_2 d(b,2) \geq d(a,1) + s_2 d(a,2)$$
$$d(b,1) + s_2 d(b,2) \geq d(c,1) + s_2 d(c,2)$$

have to hold and form into

$$3 + s_2 \cdot 3 \geq 3 + s_2 \cdot 2$$
$$3 + s_2 \cdot 3 \geq 1 + s_2 \cdot 2,$$

with $0 \leq s_2 \leq 1$. Thus, every positional scoring procedure would elect b, whereas the Condorcet winner a would certainly be elected by approval voting because of theorem 11. $\square$

We will introduce the two other voting procedures which were already mentioned above, **Instant Runoff Voting** (IRV) and **Majoritarian Compromise** (MC), which are two preferential voting systems, which means that they are based on preference rankings. In IRV, after voters haven given their preference rankings, candidates with the fewest first-place votes are eliminated from the preference rankings. So if one voter had the ranking $b \succ a \succ c$ and candidate b was eliminated, the ranking would change into $a \succ c$. This procedure is repeated again until one candidate receives more than half of the votes (or ties with other candidates). If there are only three candidates, this procedure is the same as plurality voting with runoff (PVR). To illustrate IRV, consider the following example.

Example 5.

$$\textit{4 voters: } a \succ b \succ c \succ d$$
$$\textit{3 voters: } c \succ b \succ a \succ d$$
$$\textit{3 voters: } d \succ c \succ b \succ a$$
$$\textit{2 voters: } c \succ d \succ b \succ a$$

In the first round, candidate b has got the fewest first-place votes and is therefore eliminated. This results into:

$$\text{4 voters: } a \succ c \succ d$$
$$\text{3 voters: } c \succ a \succ d$$
$$\text{3 voters: } d \succ c \succ a$$
$$\text{2 voters: } c \succ d \succ a$$

Now in round 2, candidate d has only received 3 first-place votes and is therefore eliminated, which leads to:

$$\text{4 voters: } a \succ c$$
$$\text{3 voters: } c \succ a$$
$$\text{3 voters: } c \succ a$$
$$\text{2 voters: } c \succ a$$

In the last round, candidate c receives more than half of the votes and wins.

Under MC, first-votes, then first- and second-votes, then first-, second- and third-votes, and so on, are summed up for each candidate until one receives more than half of the number of voters. If there is more than one, the candidate with most votes will be declared winner. So referring to example 5, only counting first-votes none of the candidates gets more than 6 votes. So first- and second-votes are summed up for each candidate, which results into candidate a receiving 4 votes, b 7 votes, c 8 votes and d 5 votes. So both candidates b and c receive the required number of votes and therefore c is elected having received most votes.

Theorem 14. *At all preference profiles v, a candidate chosen by IRV or MC is an AV outcome. There exist preference profiles v at which a candidate chosen by approval voting is neither an IRV nor an MC outcome.*

Proof. First we will show that every candidate i chosen by IRV is also an AV outcome. Suppose on the contrary that i is not an AV outcome. By theorem 6 there exists another candidate j such that the number of voters who consider j as their best choice and i as their worst exceeds the numbers of voters who prefer i to j. So candidate i will certainly lose against j in some IRV round.

Second we show that if candidate i is not an AV outcome, it is not chosen by MC either. This holds because j will receive more first place votes than i. If these first-place votes are not more than half of the number of voters, going down to second place votes and lower ranks j will sooner get the required number of votes than i, because at every step there are more voters who have given candidate j first-place votes than there are voters who prefer i to j and so would have given i their vote.

Third we show by considering example 5 that there are AV outcomes which are not IRV or MC outcomes. Whereas both IRV and MC elect candidate c, AV outcomes can also be $\{a\}$, $\{b\}$ and $\{c, d\}$ chosen at the critical strategy profiles $C_a(v)$, $C_b(v)$ and $C_d(v)$ respectively. □

As already mentioned above, approval voting not only allows for a broader range of outcomes, it also makes it easier for voters to prevent the election of certain outcomes they do not approve of.

Theorem 15. *At every preference profile v at which the cardinality of the set of AV outcomes is greater than 1, approval voting can prevent the election of every candidate*[4]*, whereas positional scoring procedures, IRV, and MC cannot prevent the election of all of them.*

Proof. If there is more than one possible winner for a given preference profile v, none of these candidates can be assured of winning, which implies that everybody can be prevented from winning. To show that this is not true for positional scoring procedures, STV and MC consider the following example.

$$\begin{aligned} &1 \text{ voter: } a \succ b \succ c \\ &1 \text{ voter: } b \succ a \succ c \\ &1 \text{ voter: } c \succ b \succ a \end{aligned}$$

Every positional scoring procedure must elect the Condorcet winner b, because – analogous to the proof of theorem 13 –

$$\begin{aligned} 1 + s_2 \cdot 2 &\geq 1 + s_2 \cdot 1 \\ 1 + s_2 \cdot 2 &\geq 1 \end{aligned}$$

holds. Also MC elects b with first-rank and second-rank votes summed up, which give b 3 votes, a 2 votes and c 1 vote. Under IRV candidates a and b can be elected

[4]If there is just one possible outcome there is not much to prevent.

depending on what candidate is eliminated in the first round. If candidates a or c are removed, b wins, and if candidate b is removed, a wins.

Concluding, only candidate c is prevented from winning under these voting procedures, whereas approval voting with $C_a(v) = \{a, ba, c\}$, $C_b(v) = \{ab, b, cb\}$ and $C_c(v) = \{a, b, c\}$ yielding outcomes $\{a\}$, $\{b\}$ and $\{a, b, c\}$ respectively, can prevent the election of all of them. $\square$

We will end this section quoting [7] with a big statement in favor of approval voting:

> We have seen that approval voting allows for outcomes that BC, MC, and [IRV] do not. At the same time, it may preclude outcomes that other systems cannot prohibit. In effect, voters can fine-tune their preferences under approval voting, making outcomes responsive to information that transcends these preferences. (...)
>
> (...) [I]t is worth noting that basing social choice on acceptability rather than on traditional social-choice criteria is a radical departure from the research program initiated by Borda and Condorcet in late 18th-century France. While we do not eschew these criteria, they should not be the be-all and end-all for judging whether outcomes are acceptable or not. Rather, we believe, the pragmatic judgments of voters about who is acceptable and who is not should be decisive. This is information that enriches the standard social-choice framework and should, therefore, be incorporated in it.

4.9 Computational manipulation

In this section we will not focus on strategic behavior of voters who alter their voting strategies in order to obtain a preferred outcome, but move our attention to manipulation exercised by computer programs in the sense that a chair uses computational power to calculate how a possible method could make a certain candidate win (constructive control) or prevent a certain candidate from winning (destructive control). A chair is any kind of person, committee, computer program etc. that has complete information about all voter preferences by knowing the voter preference profile v, and is in charge of conducting the election. We allege that the chair has dishonest motives and wants to manipulate the election by exerting seven different means of control:

1. adding candidates
2. suppressing candidates
3. partition of candidates
4. run-off partition of candidates
5. adding voters

6. suppressing voters

7. partition of voters

Again we want to remind the readers of the fact that an election need not only be any kind of decision-making process among human beings. It can also be a decision-making process among a thousand or millions of computer programs connected via a network, where every program has some information about a certain field and votes in respect of its own parameters. Examples include (web-page) rank aggregation problems or spam filters [13]. So it is easier to imagine that the chair really has complete information about voter preferences. Of course, in certain elections the assumption may not be true. However, even if there are a lot of voters, there are still good ways to estimate voter preferences via more or less precise opinion polls. Furthermore, lower bounds obtained under perfect information presented in these fields are also true if regarded under any natural incomplete information model, because the question whether there is any number of candidates that can be added under incomplete information is generally harder than the same question under perfect information. This is explained in more detail in [13].

So what can be the differences among voting procedures concerning their possibility of being manipulated under these means of control? A voting procedure can be immune or susceptible to a certain type of control, and if it is susceptible it can be vulnerable or resistant.

Definition 8. *A voting procedure is* immune to a type of constructive control *if it is never possible for the chair to change a given candidate from being not the unique winner to being the unique winner by means of this type of constructive control. A voting procedure is* immune to a type of destructive control *if it is never possible for the chair to change a given candidate from being the unique winner to not being the unique winner by means of this type of destructive control.*

A voting procedure which is not immune to a type of control is called susceptible to this type of control.

A voting procedure is vulnerable to a type of control[5] *if it is susceptible to this type of control and the corresponding language problem is solvable in polynomial time.*

A voting procedure is resistant to a type of control *if it is susceptible to this type of control and the corresponding language problem is NP-complete.*

As in [13], we will now start to define the corresponding language problems of the points 1-7 above and analyze them with regard to approval, plurality and Condorcet voting.

Problem 1 (Control by adding candidates)**.** *Given: A set X of qualified candidates and a distinguished candidate $c \in X$, a set $\widehat{X}$ of possible spoiler candidates, and a set Y of voters with voter preference profile v (in the approval case, the ballot response profile d is additionally required; in the other cases it can be derived from v - this*

[5]Contrary to the authors of [13] we will not distinguish between vulnerable and certifiably-vulnerable, since these terms coincide for all the results presented here.

remark is valid for all the following problems and will be henceforth omitted) over $X \cup \widehat{X}$.

Question (constructive): Is there a choice of candidates from $\widehat{X}$ *whose entry into the election would assure that* c *is the unique winner?*

Question (destructive): Is there a choice of candidates from $\widehat{X}$ *whose entry into the election would assure that* c *is not the unique winner?*

Problem 2 (Control by suppressing candidates)**.** *Given: A set* X *of candidates, a distinguished candidate* c*, a set* Y *of voters with voter preference profile* v *and a positive integer* $k < |X|$.

Question (constructive): Is there a set of k *or fewer candidates in* X *whose disqualification would assure that* c *is the unique winner?*

Question (destructive): Is there a set of k *or fewer candidates in* $X \setminus \{c\}$ *whose disqualification would assure that* c *is not the unique winner?*

Problem 3 (Control by partition of candidates)**.** *Given: A set* X *of candidates, a distinguished candidate* c*, a set* Y *of voters with voter preference profile* v.

Question (constructive): Is there a partition of X *into* X_1 *and* X_2 *such that* c *is the unique winner in the sequential two-stage election in which the winners in the subelection* X_1 *who survive the tie-handling rule move forward to face the candidates in* X_2 *(with voter set* Y*)?*

Question (destructive): Is there a partition of X *into* X_1 *and* X_2 *such that* c *is not the unique winner in the sequential two-stage election in which the winners in the subelection* X_1 *who survive the tie-handling rule move forward to face the candidates in* X_2 *(with voter set* Y*)?*

Problem 4 (Control by run-off partition of candidates)**.** *Given: A set* X *of candidates, a distinguished candidate* c*, a set* Y *of voters with voter preference profile* v.

Question (constructive): Is there a partition of X *into* X_1 *and* X_2 *such that* c *is the unique winner of the election in which those candidates surviving (with repect to the tie-handling rule) subelections with candidate set* X_1 *and* X_2*, respectivley, have a run-off with voter set* Y*?*

Question (destructive): Is there a partition of X *into* X_1 *and* X_2 *such that* c *is not the unique winner of the election in which those candidates surviving (with repect to the tie-handling rule) subelections with candidate set* X_1 *and* X_2*, respectivley, have a run-off with voter set* Y*?*

Problem 5 (Control by adding voters)**.** *Given: A set of candidates* X *and a distinguished candidate* $c \in X$*, a set* Y *of registered voters with voter preference profile* v*, an additional set* $\widehat{Y}$ *of yet unregistered voters with voter preference profile* $\hat{v}$*, and a positive integer* $k \leq |\widehat{Y}|$.

Question (constructive): Is there a set of k *or fewer voters from* $\widehat{Y}$ *whose registration would assure that* c *is the unique winner?*

Question (destructive): Is there a set of k *or fewer voters from* $\widehat{Y}$ *whose registration would assure that* c *is not the unique winner?*

Problem 6 (Control by suppressing voters)**.** *Given: A set of candidates X and a distinguished candidate $c \in X$, a set Y of voters with voter preference profile v, and a positive integer $k \leq |Y|$.*

Question (constructive): Is there a set of k or fewer voters from Y whose disenfranchisement would assure that c is the unique winner?

Question (destructive): Is there a set of k or fewer voters from Y whose disenfranchisement would assure that c is not the unique winner?

Problem 7 (Control by partition of voters)**.** *Given: A set of candidates X, a distinguished candidate $c \in X$, and a set Y of voters with voter preference profile v.*

Question (constructive): Is there a partition of Y into Y_1 and Y_2 such that c is the unique winner in the hierarchical two-stage election in which the survivors of the elections with voter set Y_1 and Y_2, respectively, (voting both over the candidate set X) run against each other with voter set Y.

Question (destructive): Is there a partition of Y into Y_1 and Y_2 such that c is not the unique winner in the hierarchical two-stage election in which the survivors of the elections with voter set Y_1 and Y_2, respectively, (voting both over the candidate set X) run against each other with voter set Y.

For the problems with two-stage elections, which are problems 3, 4 and 7, we have to define the mechanism that resolves ties in the first-stage elections. As in [13] we will define the *Ties-Eliminate* mechanism (TE), which does not promote any candidate to the next stage election if there is a tie. Contrary to this, we define the *Ties-Promote* mechanism (TP), which promotes all the candidates tying. All this does not apply to Condorcet voting, since if there is a winner with voters having a strict ranking, he or she will always be unique. We have summarized all the findings of [13] and [14] in table 4.1.

Table 4.1: Computational manipulation of plurality, Condorcet and approval voting

Voting proc.	Plurality		Condorcet		Approval	
Control by	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
Adding candidates	R	R	I	V	I	V
Suppressing candidates	R	R	V	I	V	I
Partition of candidates	TE: R TP: R	TE: R TP: R	V	I	TE: V TP: I	TE: I TP: I
Run-off partition of candidates	TE: R TP: R	TE: R TP: R	V	I	TE: V TP: I	TE: I TP: I
Adding voters	V	V	R	V	R	V
Suppressing voters	V	V	R	V	R	V
Partition of voters	TE: V TP: R	TE: V TP: R	R	V	TE: R TP: R	TE: V TP: V

I = immune, R = resistant, V = vulnerable;
TE=Ties-Eliminate, TP=Ties-Promote.

4.9.1 Approval voting vs. plurality voting

One main advantage of approval voting vs. plurality voting is that for all the problems above, whether in their constructive or destructive form, plurality voting is never immune against any kind of manipulation. Furthermore, there are many cases where approval voting does better than plurality voting. We will now state all these cases. As proving all the propositions would go far beyond the scope of this work, we will only exemplarily show the second of the following nine propositions. All the proofs can be found in [14] and [13].

Proposition 6. *Approval voting is immune to constructive control by adding candidates, whereas plurality voting is only resistant.*

Proposition 7. *Approval voting is immune to destructive control by deleting candidates, whereas plurality voting is only resistant.*

To prove this proposition we will need four lemmas and one construction whose exact use will be seen in the proof.

Lemma 17. *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*

Proof. "$\Rightarrow$:" Suppose the constructive question in problem 1 can be positively answered. Let candidate c not be the unique winner in X, and let $\widetilde{X} \subseteq \widehat{X}$ be the set of candidates added to the candidate set X to make c the unique winner. Define $k = |\widetilde{X}|$. If you now remove $\widetilde{X}$, we have shown that there is a set of k candidates whose removal assures that c is not the unique winner.

"$\Leftarrow$:" Suppose the destructive question in problem 2 can be positively answered. Let candidate c be the unique winner in X. Define $\widehat{X} \subseteq X \setminus \{c\}$ to be set of candidates removed from X to prevent c from winning. If we add now $\widehat{X}$, we have shown that there is a set of candidates whose entry assure that c it the unique winner. □

Unique-WARP is the abbreviation for the "unique" version of the Weak Axiom of Revealed Preference and means that a unique winner among a collection of candidates always remains a unique winner among every subcollection of candidates that includes him or her. Approval voting, for instance, satisfies Unique-WARP, because if a candidate is the unique winner in a collection of candidates, he or she has received most of the votes. If now you subtract any number of candidates except him or her, this candidate will still receive most of the votes, since every voter who voted for him or her before will still do so now. This is not the case for plurality voting for instance, because if some candidates are lacking, voters who had these candidates as first choice will now vote for other candidates. So it can be that another candidate will receive more votes. By the way, also every Condorcet voting method satisfies Unique-Warp: If a candidate does better than every other candidate in pairwise contests, he or she will still do so with fewer candidates. You can find the proof of the next lemma in [14].

Lemma 18. *Any voting system that satisfies Unique-WARP is immune to constructive control by adding candidates.*

For the proof that the destructive part of problem 2 is NP-complete for plurality voting, we will need the following construction. Given the problem of the hitting set, which is NP-complete, this construction transforms it in polynomial time into an election consisting of a set of candidates, a set of voters and a voter preference profile. So the result of the construction will be NP-complete, too.

Problem 8 (Hitting set)**.** *Given: A set $B = \{b_1, b_2, \dots, b_m\}$, a family $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets S_i of B, and a positive integer k.*

Question: Does $\mathcal{S}$ have a hitting set of size at most k? That is, is there a set $B' \subseteq B$ with $|B| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Construction 1 (Construction of an election from a Hitting set instance [13])**.** *Given a triple $(B, \mathcal{S}, k)$ where $B = \{b_1, b_2, \dots, b_m\}$ is a set, $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ is a family of subsets S_i of B, and $k \leq m$ is a positive integer, we construct the following election:*

- *The candidate set is $X = B \cup \{c, w\}$.*
- *The voter set Y and the preference profile v is defined as follows:*
  - *There are $2(m-k) + 2n(k+1) + 4$ voters of the form $c \succ w \succ \cdots$, where "$\cdots$" means that the remaining candidates follow in some arbitrary order.*
  - *There are $2n(k+1) + 5$ voters of the form $w \succ c \succ \cdots$.*
  - *For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form $S_i \succ c \succ \cdots$, where "S_i" denotes the elements of S_i in some arbitrary order.*
  - *Finally, for each j, $1 \leq j \leq m$, there are two voters of the form $b_j \succ w \succ \cdots$.*

Lemma 19. *If B' is a hitting set of size k, then w is the unique plurality voting winner of the election with $X = B' \cup \{c, w\}$, voter set Y and voter preference profile v as in construction 1.*

Proof. Let B' be a hitting set of size k. We get the ballot response profile d by assuming that every voter uses a sincere strategy and therefore votes for his or her top preference available in $B' \cup \{c, w\}$. Then we have

$$\begin{aligned} v(c, d) &= 2(m-k) + 2n(k+1) + 4 \\ v(w, d) &= 2n(k+1) + 5 + 2(m-k) \\ v(b_j, d) &\leq 2(k+1) + 2 \quad \forall j \end{aligned}$$

The $2(m-k)$ votes for candidate w result from the fact that in B', $m-k$ candidates miss from B. Since there are always two voters for each b_j whose second choice is w, there must be $2(m-k)$ additional votes for w. Hence, we see that candidate w receives most of the votes and is the unique plurality winner. $\square$

Lemma 20. *Let $D \subseteq B \cup \{w\}$. If c is not the unique plurality voting winner of the election with candidate set $D \cup \{c\}$, voter set Y and voter preference profile v as in construction 1, then there exists a set $B' \subseteq B$ such that*

1. $D = B' \cup \{w\}$,
2. *w is the unique plurality winner of the election with candidate set $B' \cup \{c, w\}$, voter set Y and voter preference profile v, and*
3. *B' is a hitting set of $\mathcal{S}$ of size less than or equal to k.*

Proof. Let $D \subseteq B \cup \{w\}$ and suppose that c is not the unique plurality voting winner of the election stated above with $X = D \cup \{c\}$. Let d be as in the proof of lemma 19. We begin to note that for all $b \in D \cap B$ we have

$$v(b, d) \leq 2n(k+1) + 2 < 2(m-k) + 4n(k+1) + 9 \leq v(c, d).$$

Since c is not the unique plurality voting winner, there must be a candidate $w \in D$ with $v(w, d) \geq v(c, d)$. Let $B' \subseteq B$ be such that $D = B' \cup \{w\}$. Then $D \cup \{c\} = B' \cup \{c, w\}$. We have

$$\begin{aligned} v(w, d) &= 2n(k+1) + 5 + 2(m - |B'|) \\ v(c, d) &= 2(m-k) + 2n(k+1) + 4 + 2(k+1)l, \end{aligned}$$

where l is the number of sets in S not hit by B', which means that they have an empty intersection with B'. From this we see that $v(c, d)$ is even and $v(w, d)$ is odd, so we have $v(w, d) > v(c, d)$. Hence, w is the unique winner of the plurality election with $X = B' \cup \{c, w\}$ and we have shown points 1 and 2.

For any $B' \subseteq B$ we know that $v(c, d) \leq v(w, d)$ we have

$$2(m-k) + 2(k+1)l \leq 1 + 2(m - |B'|),$$

which implies

$$(k+1)l - k + |B'| \leq \frac{1}{2}.$$

Since the left-hand term of the inequality is a whole number, we can restate it as $(k+1)l - k + |B'| \leq 0$ from what follows $l = 0$ and $|B'| \leq k$. So B' is a hitting set of $\mathcal{S}$ of size at most k. □

Proof of proposition 7. We will show the proposition in three parts: First we will show that approval voting is immune, second that plurality voting is susceptible and third that plurality voting is resistant to destructive control by deleting candidates.

Part 1: From lemma 18 we know that approval voting is immune to constructive control by adding candidates. From lemma 17 we get that it is immune to destructive control by deleting candidates.

Part 2: Let's consider example 4, where a is the unique winner under plurality voting. Now remove for example candidate d. The two voters who had voted for d before will now vote for b. This results in 3 votes for a, but 4 votes for d.

Part 3: We will prove the following claim: $\mathcal{S}$ has a hitting set of size at most k if and only if the election with candidate set X, voter set Y and voter preference profile v as in construction 1 and distinguished candidate c can be destructively controlled by deleting at most $m - k$ candidates.

"$\Rightarrow$:" Let B' be a hitting set of $\mathcal{S}$ of size k. By lemma 19, candidate c is not the unique plurality voting winner with $B' \cup \{c, w\}$. Since $B' \cup \{c, w\} = X \setminus (B \setminus B')$ with $|B \setminus B'| = m - k$, the removal of $B \setminus B'$ changes the winner from c to w.

"$\Leftarrow$:" Let $D \subseteq B \cup \{w\}$ with $|D| \leq m - k$ be so that c is not the unique plurality voting winner of $X \setminus D$. Since $c \in X \setminus D$ it follows from lemma 20 that D there equals $(X \setminus D) \setminus \{c\}$ here, so from (1) we have that $B' \cup \{w\} = (X \setminus D) \setminus \{c\}$ of size at most $(m + 2) - (m - k) + 1 = k + 1$, and so from (3) we get that B' is a hitting set of $\mathcal{S}$ of size at most k.

□

Proposition 8. *Approval voting is immune to constructive control by partition of candidates for tie-breaking mechanism TP, whereas plurality voting is only resistant for TP.*

Proposition 9. *Approval voting is immune to destructive control by partition of candidates for both tie-breaking mechanisms, whereas plurality voting is only resistant.*

Proposition 10. *Approval voting is immune to constructive control by run-off partition of candidates for tie-breaking mechanism TP, whereas plurality voting is only resistant for TP.*

Proposition 11. *Approval voting is immune to destructive control by run-off partition of candidates for both tie-breaking mechanisms, whereas plurality voting is only resistant.*

Proposition 12. *Approval voting is resistant to constructive control by adding voters, whereas plurality voting is vulnerable.*

Proposition 13. *Approval voting is resistant to constructive control by deleting voters, whereas plurality voting is vulnerable.*

Proposition 14. *Approval voting is resistant to constructive control by partition of voters for tie-breaking mechanism TE, whereas plurality voting is vulnerable for TE.*

4.9.2 Approval voting vs. Condorcet voting

As far as approval voting and Condorcet voting are concerned, we can say that in general they share the same results. Approval voting always does at least as well as Condorcet voting, and in two cases it does even better. Doing statistics (and counting twice the Condorcet voting entries for problems which use tie-breaking rules to allow comparison), approval voting is immune 8 times, resistant 4 times and vulnerable 8 times, whereas Condorcet voting is immune 6 times, resistant (also) 4 times and vulnerable 10 times. So those two cases, where the first voting procedure is immune, but the second one is vulnerable, are exactly the two cases where approval voting has an advantage over Condorcet voting.

Proposition 15. *Approval voting is immune to constructive control by partition of candidates for tie-breaking mechanism TP, whereas Condorcet voting is vulnerable for TP.*

Proposition 16. *Approval voting is immune to constructive control by run-off partition of candidates for tie-breaking mechanism TP, whereas Condorcet voting is vulnerable for TP.*

4.10 Conclusion

The main conclusion we have to draw here is that approval voting is unbeatable under dichotomous preferences. No other voting procedure has such a great performance concerning strategy-proofness. However, under all other preferences, especially under multichotomous ones, approval voting does not do so overwhelmingly fine. Still, it is generally speaking not worse than other nonranked voting procedures. Approval voting offers a bunch of advantages concerning the election of a Condorcet winner – in respect of stability it is even better than Condorcet voting methods. It has a higher probability of electing the Condorcet winner than plurality voting, and it offers also many advantages concerning computational manipulability, especially against plurality voting. Approval voting outcomes are more inclusive than Borda count, plurality voting, IRV or MC outcomes for instance, which makes those voting procedures obsolete, one could argue.

Approval voting has also other advantages under other approaches not mentioned here. Take the term "policy space" for instance. Candidates and voters are supposed to have positions on a one, two or even higher dimensional policy space. Then, voters try to elect those candidates that come most closely to their own position. Referring to Dellis and Oak [8], who have adopted a one dimensional model, outcomes, i.e. candidates, under approval voting are more moderate than those under plurality voting under certain assumptions. Thus, approval voting leads to more centrist outcomes compared to plurality voting. Moreover, all feasible alternatives are clustered at no more than two positions, "contrary to the popular intuition that approval voting may lead to a large number and variety of candidates." Furthermore, approval voting is neither subject to the wasting-the-vote nor to the vote-splitting effect. If two candidates run on a similar platform, the former says that voters are discouraged to give their vote to a candidate not having great chances to win, because they fear to "waste their vote", and the latter states that votes will be split among these two candidates. These two effects heavily occur under plurality voting. Hence, plurality voting discourages candidates to run on similar policy platforms, which approval voting does not. If we judge the idea that the best candidate should win as a positive one, it is clearly negative that (possibly better[6]) candidates are already discouraged beforehand to run for office, only because another (possibly worse) candidate has already taken up a similar policy.

[6] The definition of better and worse are clearly no objective ones. However, one could consider whether a candidate is only motivated because of the office he or she could reach, or if there is a real commitment to the policy he or she stands for as a good criterion.

5 Drawbacks of approval voting

5.1 On trichotomous and multichotomous preferences

Under trichotomous and multichotomous preferences we already know that approval voting is no longer strategy-proof; other nonranked voting procedures are not strategy-proof either. In some situations, however, approval voting does even worse as far as the number of admissible strategies is concerned. Consider the following examples in table 5.1 [4, p. 28] and 5.2. In the former example, approval voting does worse than plurality voting in lines 3 and 4, and negative voting in line 3. In the latter example, approval voting does worse in 7 out of 11 cases compared to plurality and in 4 out of 11 cases compared to negative voting. Yet, we can also remark that approval voting does not generally do worse, and that in table 5.1 approval voting is better than negative voting in 3 out of 4 cases, for instance. Still, on trichotomous and multichotomous preferences approval voting has lost its big advantage over other nonranked voting procedures.

Table 5.1: Number of admissible strategies for four candidates and $r \geq 3$

Preferences		AV	PV	NV
trichotomous	$a \sim b \succ c \succ d$	2	3	4
	$a \succ b \succ c \sim d$	2	2	4
	$a \succ b \sim c \succ d$	4	3	2
multichotomous	$a \succ b \succ c \succ d$	4	3	4

Table 5.2: Number of admissible strategies for five candidates and $r \geq 3$

Preferences		AV	PV	NV
trichotomous	$a \sim b \sim c \succ d \succ e$	2	4	5
	$a \sim b \succ c \sim d \succ e$	2	4	5
	$a \sim b \succ c \succ d \sim e$	4	3	6
	$a \succ b \sim c \sim d \succ e$	8	4	2
	$a \succ b \sim c \succ d \sim e$	4	3	5
	$a \succ b \succ c \sim d \sim e$	2	2	5
multichotomous	$a \sim b \succ c \succ d \succ e$	4	4	6
	$a \succ b \sim c \succ d \succ e$	8	4	5
	$a \succ b \succ c \sim d \succ e$	8	4	5
	$a \succ b \succ c \succ d \sim e$	4	3	6
	$a \succ b \succ c \succ d \succ e$	8	4	6

The fact that approval voting offers more admissible strategies in some situations than other nonranked voting procedures makes it more vulnerable to strategic manipulation in these cases. All this has proven the following proposition.

Proposition 17. *Suppose every voter has trichotomous or multichotomous preferences, then approval voting may have more admissible strategies than other voting procedures.*

5.2 Election of a Pareto candidate

We have seen that in the set of AV outcomes there is always a Pareto candidate and that a non-Pareto candidate may be a component of an AV outcome but never a unique outcome. The second fact could also be considered as a disadvantage, because it is at least doubtful that a candidate who is judged inferior to at least one other candidate by all voters is part of an outcome. The disadvantage we state now is that there even can be Pareto candidates not "found" by approval voting.

Proposition 18. *Not every Pareto candidate is necessarily an AV outcome.*

Proof. Consider example 4 where candidate d is not elected at $C_d(v)$ (so not elected at all), but is top ranked by the third voter type. So d is a Pareto candidate, because there is no other candidate that all voters rank higher. □

5.3 Election of a Condorcet loser

Unfortunately for approval voting, a Condorcet loser may be, but need not be an AV outcome. Actually we have already seen this in lemma 1. To show that this need not always be the case, consider example 4 where candidate d is the Condorcet loser, but not an AV outcome. To see that a Pareto candidate may also be a Condorcet loser greatly weakens the disadvantage of approval voting as to proposition 18.

Asking again, what about stability and strong stability? Just to show that there is a real difference between these two terms consider once again example 4. We know that at $C_a(v) = (a, a, a, bca, bca, d, d)$ candidate a wins. The first voter type has no incentive to change their strategy since they cannot obtain any better result. The second voter type could use two other sincere strategies, which are $\{b, c\}$ and $\{b\}$. None of those strategies, though, would change the winner. The third voter type could also use two other sincere strategies, namely $\{d, b\}$ and $\{d, b, c\}$. Again, switching to either of them would not result in a different outcome. Hence, candidate a is a stable AV outcome. However, if both second and third voter types join forces and vote $\{b, c\}$ and $\{d, b\}$ respectively, candidate a would only get 3 votes, candidate c retain his 2 votes, but candidate b would assemble 4 votes who both voter types prefer to candidate a. Thus, candidate a is not a strongly stable AV outcome. Furthermore, this was not the only strategy which could achieve this, also $\{b, c\}$ and $\{d, b, c\}$, $\{b\}$ and $\{b, d\}$, and $\{b\}$ and $\{d, b, c\}$ could achieve this. Whereas we saw in the last chapter that a Condorcet winner always is a strongly stable AV outcome, we will see now that a Condorcet loser can be a stable AV outcome as well.

Proposition 19. *A unique Condorcet loser may be a stable AV outcome, even when there is a different outcome that is a unique Condorcet winner (and therefore strongly stable).*

Proof. Consider the following example from [7]:

$$\begin{aligned} \text{3 voters: } & a \succ b \succ c \succ d \succ e \\ \text{1 voter: } & b \succ c \succ d \succ e \succ a \\ \text{1 voter: } & c \succ d \succ e \succ b \succ a \\ \text{1 voter: } & d \succ e \succ b \succ c \succ a \\ \text{1 voter: } & e \succ b \succ c \succ d \succ a \end{aligned}$$

There exists a Condorcet winner, candidate b, who is therefore a strongly stable AV outcome, and there also exists a Condorcet loser, candidate a. We will show that a is a stable AV outcome. Consider a's critical preference profile $C_a(v) = (a, a, a, b, c, d, e)$. Clearly, first type voters have no incentive to switch their strategy. For all the other voters, they each have three other sincere strategies to apply. However, if only one voter performs a change, none of these strategies would yield another outcome.

To show that this need not always be the case, consider once again example 1. We already know that M is the Condorcet loser. We will show that M is not a stable AV outcome. Consider M's critical preference profile $C_M(v) = (M, M, M, M, T, T, T, P, P)$. Clearly, first type voters have no incentive to switch their strategy. Second type voters have one other sincere strategy which is $\{T, P\}$. This strategy would make P the winner which these voters prefer to M. □

5.4 Condorcet efficiency

Even though approval voting does better than plurality voting as far as the probability of the election of the Condorcet winner and the Condorcet loser is concerned, there still is another voting method with higher chances of electing the first and lower chances of electing the second type of candidate. Quite astonishingly, this is the fixed rule where you have to vote for exactly half of the candidates (if m is odd, then for $\lceil \frac{m}{2} \rceil$ of the candidates). We state the following results with proof in [12].

Proposition 20. $P_{CEAV}(m, \infty, p) \leq P_{CEFR}(m, \infty, \lceil \frac{m}{2} \rceil)$

Proposition 21. $P_{CLAV}(m, \infty, p) \geq P_{CLFR}(m, \infty, \lceil \frac{m}{2} \rceil)$

5.5 Computational manipulation

5.5.1 Approval voting vs. plurality voting

Although approval voting is often better than plurality voting, there also exist five cases where the opposite is true. In all those cases, both systems are always susceptible, but plurality voting is resistant and approval voting vulnerable. As we have already had, table 4.1 summarizes all these results, and the proofs can be found in [13] and [14].

Proposition 22. *Plurality voting is resistant to destructive control by adding candidates, whereas approval voting is vulnerable.*

Proposition 23. *Plurality voting is resistant to constructive control by deleting candidates, whereas approval voting is vulnerable.*

Proposition 24. *Plurality voting is resistant to constructive control by run-off partition of candidates for tie-breaking mechanism TE, whereas approval voting is vulnerable for TE.*

Proposition 25. *Plurality voting is resistant to constructive control by run-off partition of candidates for tie-breaking mechanism TE, whereas approval voting is vulnerable for TE.*

Proposition 26. *Plurality voting is resistant to destructive control by partition of voters for tie-breaking mechanism TP, whereas approval voting is vulnerable for TP.*

5.6 Extreme vulnerability to majority decisiveness and to erosion of the majority principle

In this section we will examine two disadvantages which approval voting is vulnerable[1] to. The two issues can be seen as a dual problem. The first one – *majority decisiveness* – states that even a simple majority can guarantee the election of its most preferred candidate regardless of the other voters' preferences. The second one – *erosion of the majority principle* – states that a candidate preferred by even the largest possible majority is not necessarily elected. These two problems contradict each other in a certain way: The former one is against "majority-dictatorship", the latter one against a "one-person-dictatorship". The former one wants minorities to be heard, the latter one wants majorities not to be overheard. One could say that it is impossible to find a voting procedure that evades both problems. Nevertheless, Baharad and Nitzan present *restricted approval voting*, where they impose a minimum and maximum number of candidates that must be approved, and relax both problems at once [3]. We will consider restricted approval voting in the last chapter in section 6.3, and we will show now that (regular) approval voting is extremely vulnerable to both kinds of problems.

Baharad and Nitzan have already shown before in the context of positional scoring procedures [2] that under sincere voting a majority of size α can guarantee the election of its most preferred candidate when

$$\alpha \geq \frac{s_1 - s_m}{2s_1 - s_2 - s_m}, \tag{5.6.1}$$

where s_i designates the points attributed to a candidate dependent on his or her position in the preference ranking as on page 13, and α is a fraction with denominator

[1] The term "vulnerability" is no longer seen in a computational context as in section 5.5.

n. As in the proof of theorem 13 we will consider approval voting as a kind of a positional scoring procedure with the difference that voters give points to the candidates not using a predetermined vector.

Theorem 16. *Under sincere voting, approval voting is vulnerable to simple majority decisiveness.*

Proof. Suppose that all voters only vote for their most preferred candidate, which is an admissible and sincere strategy. So we get $s_1 = 1$, $s_2 = s_m = 0$, and inequality 5.6.1 is satisfied with $\alpha \geq \frac{1}{2}$. Thus already a simple majority can guarantee its favorite to be elected. □

We could also have proven this by giving the following modification of example 1: Disqualify the last two voters and only allow to vote the 4 voters with $M \succ P \succ T$ and the 3 voters with $T \succ P \succ M$. If now both voter types find their first two choices acceptable, they will vote $\{M, P\}$ and $\{T, P\}$, which will make P the winner. However, the majority – the first type voters – would prefer candidate M to win. So if they now switched to their other sincere and admissible strategy $\{M\}$ they could ensure that M is the winner, regardless of the votes of any minority, i.e. the second voter type.

Theorem 17. *Under sincere voting, approval voting is vulnerable to erosion of the majority principle by the smallest possible minority, that is by a single voter.*

Proof. If every voter approves of at least the first two candidates, we have $s_1 = s_2 = 1$, $s_m = 0$. This implies $\alpha = 1$ in inequality 5.6.1. Thus, there is no majority as large as possible, except unanimity, that could ensure their most preferred candidate to win. Hence, even a single voter could dictate the winner. □

We will also illustrate this (and give an alternative proof) by giving the following example [3]:

Example 6.

$$n-1 \text{ voters: } a \succ b \succ c \succ d$$
$$1 \text{ voter: } b \succ c \succ d \succ a$$

Suppose that all of the $n-1$ voters apply their sincere voting strategy $\{a, b\}$. Now if the last voter votes $\{b, c\}$ for instance, the favorite of a minority - a single voter! - will win, whereas the favorite of this vast majority (all but one) will lose. Here the duality of the problem becomes evident: Of course, the majority could "dictate" their favorite candidate and only vote for a; then the last voter could do anything he or she wanted but his or her voice would not be heard at all.

What about the other voting procedures under sincere voting? Plurality voting, of course, is also vulnerable to majority decisiveness, but not to the erosion of the majority principle. For Borda count we get

$$\alpha \geq \frac{n-1}{n},$$

so we must have $\alpha = \frac{n-1}{n}$ or $\alpha = 1$, which means that only a majority consisting of all but one, or all voters can "dictate" their favorite. The second case is equivalent to the desired feature of unanimity (criterion 1). So Borda count is not vulnerable to majority decisiveness by a simple majority. As to the erosion of the majority principle, a majority smaller than $\frac{n-1}{n}$ can be overruled by a minority, so it is vulnerable to it. Consider example 6 with $n = 3$. A majority of voters prefers a to b, but b will win getting 7 points, whereas candidate a only receives 6 points. Referring to the majority criterion (criterion 9), Condorcet voting is vulnerable to majority decisiveness even by a simple majority. Consider again the above modification of example 1: First type voters simply decide that M is the winner, regardless of what second type voters say. By the majority criterion it is also clear that Condorcet voting is not vulnerable to the erosion of the majority principle, since a favorite of a majority will always win. Concluding we see that the erosion of the majority principle is a drawback to approval voting and Borda count, whereas Condorcet voting and plurality voting always fulfill majority decisiveness. Only Borda count does best here: Only unanimity (but one) can guarantee the election of the group's favorite - a fact that seems quite desirable. Thus, approval voting is the only voting procedure among the ones examined here that is vulnerable to both types of the dual problem. A summary is presented in table 5.3.

Table 5.3: Vulnerability to simple majority decisiveness and to the erosion of the majority principle

	Majority	Erosion
Approval voting	YES	YES
Plurality voting	YES	NO
Borda count	NO	YES
Condorcet voting	YES	NO

Majority = vulnerable to simple majority decisiveness
Erosion = vulnerable to the erosion of the majority principle

5.7 Utilities, Stability and Condorcet winner

In the next two sections we will examine another concept of preferences, from which the old concept can easily be derived. We suppose that there is a *utility function* $u^i : X \to \mathbb{R}$ which assigns every voter i a utility, or payoff, $u^i(x)$ when a candidate x is elected, and $\mathbf{u^i} = (u^i(x))_{x \in X}$. We have not used this concept so far, because I think that it is quite unrealistic to believe that voters can calculate their utilities in terms of money (or any other unity). To my mind, it is far more realistic to assume that voters are able to state preferences over candidates on an ordinal scale than on a cardinal one.[2] Nevertheless, it is surely legitimate to carry out such considerations, where we are able to consider terms like stability in a more game-theoretic way.

[2] An introduction into this argumentation can be found in [19, pp. 47ff.].

Within this framework, we can also consider an election as a game, where each player is a voter who tries to maximize his or her expected payoff. Propositions 27 and 28 present two drawbacks of approval voting within this framework.

Proposition 27. *Given a strategy profile $\mathcal{S}$, a Condorcet winner may receive no vote under approval voting, even if $\mathcal{S}$ survives iterated elimination of (weakly) dominated strategies.*

Proof. Consider the following example with $X = \{a, b, c, d\}$ from [18]:

$$\mathbf{u^1} = (10, 0, 1, 3),$$
$$\mathbf{u^2} = (0, 10, 1, 3),$$
$$\mathbf{u^3} = (1, 0, 10, 3).$$

We already know by corollary 1 that only strategies where voters vote for their most preferred and do not vote for their least preferred candidate are not dominated. So in this example every voter has four admissible strategies. Among these, there is one that is insincere (voting for the first and the third most preferred candidates). We will show that the "plurality voting strategy" $\mathcal{S} = (a, b, c)$, hence the one where voters only vote for their most preferred candidate, survives iterated elimination of (weakly) dominated strategies. These are all admissible strategies:

voter 1: a, ad, acd, ac
voter 2: b, bd, bcd, bc
voter 3: c, cd, acd, ac

If there is a tie in any outcome we suppose that all candidates tying are selected with equal probability. First we verify in table 5.4 - the maximum utility values of each row are in bold face - that the first voter's strategy $\{a\}$ weakly dominates his strategies $\{a, c, d\}$ and $\{a, c\}$. Therefore, we can eliminate these two strategies. Now, in the reduced game the third voter's strategy $\{c\}$ weakly dominates all his other strategies $\{c, d\}$, $\{a, c, d\}$ and $\{a, c\}$. This is verified in table 5.5. Next, with the strategies remaining we see from table 5.6 that the second voter's strategy $\{b\}$ (even strongly) dominates all his other strategies $\{b, d\}$, $\{b, c, d\}$ and $\{b, c\}$. Finally, voter 1 has still two strategies left: $\{a\}$ and $\{a, d\}$. His utility using strategy $\{a\}$ is $\frac{11}{3}$; his utility using strategy $\{a, d\}$ is $\frac{7}{2}$. So strategy $\{a\}$ dominates strategy $\{a, d\}$. Hence, the above defined strategy profile $\mathcal{S}$ survives iterated elimination, where the Condorcet winner d receives no vote.

□

Proposition 28. *A strategy combination which forms a stable set of an approval voting game may contain insincere strategies.*

Proof. Consider the following example:

$$\mathbf{u^1} = (1000, 867, 866, 0),$$
$$\mathbf{u^2} = (115, 1000, 0, 35),$$
$$\mathbf{u^3} = (0, 35, 115, 1000).$$

Table 5.4: Elimination of dominated strategies of voter 1

Strategy of		Scores of candidates				Utilities of voter 1 under strategy		
voter 2	voter 3	a	b	c	d	$\{a\}$	$\{a,c,d\}$	$\{a,c\}$
$\{b\}$	$\{c\}$	0	1	1	0	$\mathbf{\frac{11}{3}}$	1	1
$\{b\}$	$\{c,d\}$	0	1	1	1	$\mathbf{\frac{7}{2}}$	2	1
$\{b\}$	$\{a,c,d\}$	1	1	1	1	**10**	$\frac{14}{3}$	$\frac{14}{2}$
$\{b\}$	$\{a,c\}$	1	1	1	0	**10**	$\frac{14}{3}$	$\frac{14}{3}$
$\{b,d\}$	$\{c\}$	0	1	1	1	$\mathbf{\frac{7}{2}}$	2	1
$\{b,d\}$	$\{c,d\}$	0	1	1	2	**3**	**3**	2
$\{b,d\}$	$\{a,c,d\}$	1	1	1	2	$\mathbf{\frac{13}{2}}$	3	$\frac{14}{3}$
$\{b,d\}$	$\{a,c\}$	1	1	1	1	**10**	$\frac{14}{3}$	$\frac{11}{2}$
$\{b,c,d\}$	$\{c\}$	0	1	2	1	**1**	**1**	**1**
$\{b,c,d\}$	$\{c,d\}$	0	1	2	2	**2**	**2**	1
$\{b,c,d\}$	$\{a,c,d\}$	1	1	2	2	$\mathbf{\frac{14}{3}}$	2	1
$\{b,c,d\}$	$\{a,c\}$	1	1	1	1	$\mathbf{\frac{11}{2}}$	1	1
$\{b,c\}$	$\{c\}$	0	1	2	0	**1**	**1**	**1**
$\{b,c\}$	$\{c,d\}$	0	1	2	1	**1**	**1**	**1**
$\{b,c\}$	$\{a,c,d\}$	1	1	2	1	$\mathbf{\frac{11}{2}}$	1	1
$\{b,c\}$	$\{a,c\}$	1	1	2	0	$\mathbf{\frac{11}{2}}$	1	1

Table 5.5: Elimination of dominated strategies of voter 3

Strategy of		Scores of candidates				Utilities of voter 3 under strategy			
voter 1	voter 2	a	b	c	d	$\{c\}$	$\{c,d\}$	$\{a,c,d\}$	$\{a,c\}$
$\{a\}$	$\{b\}$	1	1	0	0	$\mathbf{\frac{11}{3}}$	$\frac{7}{2}$	1	1
$\{a\}$	$\{b,d\}$	1	1	0	1	$\mathbf{\frac{7}{2}}$	3	2	1
$\{a\}$	$\{b,c,d\}$	1	1	1	1	**10**	$\frac{13}{2}$	$\frac{14}{3}$	$\frac{11}{2}$
$\{a\}$	$\{b,c\}$	1	1	1	0	**10**	**10**	$\frac{11}{2}$	$\frac{11}{2}$
$\{a,d\}$	$\{b\}$	1	1	0	1	$\mathbf{\frac{7}{2}}$	3	2	1
$\{a,d\}$	$\{b,d\}$	1	1	0	2	**3**	**3**	**3**	2
$\{a,d\}$	$\{b,c,d\}$	1	1	1	2	$\mathbf{\frac{13}{2}}$	3	3	$\frac{14}{3}$
$\{a,d\}$	$\{b,c\}$	1	1	1	1	**10**	$\frac{13}{2}$	$\frac{14}{3}$	$\frac{11}{2}$

Table 5.6: Elimination of dominated strategies of voter 2

Strategy of		Scores of candidates				Utilities of voter 2 under strategy			
voter 1	voter 3	a	b	c	d	$\{b\}$	$\{b,d\}$	$\{b,c,d\}$	$\{b,c\}$
$\{a\}$	$\{c\}$	1	0	1	0	$\mathbf{\frac{11}{3}}$	$\frac{7}{2}$	1	1
$\{a,d\}$	$\{c\}$	1	0	1	1	$\mathbf{\frac{7}{2}}$	3	2	1

It is shown in [18] that the strategy profile

$$\mathcal{S} = (ac, \frac{1}{4}b + \frac{3}{4}ab, \frac{1}{4}d + \frac{3}{4}cd)$$

containing mixed strategies is a stable set, where strategy $\{a, c\}$ is insincere. □

5.8 Non-predictability of subset election outcomes

It is desirable for a voting procedure that the outcome of an election of a subset of candidates is not completely independent from the outcome of the original candidate set. Suppose $X = \{a, b, c\}$ with the overall outcome $a \succ b \succ c$. If we now only make pairwise comparisons, an overall outcome of $c \succ a$, $b \succ a$ and $c \succ b$ is not what we would wish for, since this would mean that the Condorcet loser would have won and the Condorcet winner would have lost in the original election. Furthermore, we could assume that if we restrict the strategies voters can use in subset elections dependent on their strategies in the original election, this would ensure the subset election outcome to be dependent on the outcome of the original election. So, before stating the main result of [22], we will define the following restrictions on voters' strategies.

Following [22], let's assume that a voter follows a utility maximizing strategy and only votes for a candidate if the utility derived from the candidate's victory is greater than the average utility over all the candidates running in the election. Let $\widetilde{X} = \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}$ with $1 \le j_1 < j_2 < \cdots < j_r \le m$ be a subset of X. We define the *mean utility function* $\overline{u}_i$ for voter i by

$$\overline{u}_i(\{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}) = \frac{\sum_{k=1}^{r} u_i(x_{j_k})}{r},$$

and the *approval function* $A^i_{j_k}$ for voter i by

$$A^i_{j_k}(\{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}) = \begin{cases} 1 & \text{if } u_i(x_{j_k}) > \overline{u}_i(\{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}) \\ 0 & \text{if } u_i(x_{j_k}) \le \overline{u}_i(\{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}). \end{cases}$$

With strategy profile $\mathcal{S} = (S_1, S_2, \ldots, S_n)$ dependent on a specified $\widetilde{X} \subseteq X$ and the definitions above we have $x_k \in S_i$ for voter $i \Leftrightarrow A^i_{j_k}(\widetilde{X}) = 1$. Facilitating analysis we will only allow for $u_i(x_l) \neq u_i(x_k)$ for all $l \neq k$ and for all voters $i \le n$. The following lemmata show the restrictions implied by this approach.

Lemma 21 (Restriction 1)**.** *For each subset $\widetilde{X} = \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}$ of X (with $r \ge 2$) and any voter i, there exist integers s and t, with $1 \le s, t \le r$, such that $A^i_{j_s}(\widetilde{X}) = 1$ and $A_{j_t}(\widetilde{X}) = 0$.*

Proof. We simply have to show that if there is a set of r pairwise distinct real numbers there will always be one that is below average and one that is above average. If you do not take this for granted, consider the following proof by induction: Without loss of generality, suppose the numbers are in the following order $a_1 < a_2 < a_3 < \cdots$. For $r = 2$, we quickly verify that $a_1 < \frac{a_1+a_2}{2} < a_2$ holds, and the same is done for the inductive step with $a_1 < \frac{a_1+a_2+\cdots+a_r+a_{r+1}}{r+1} < a_{r+1}$. □

So this restriction states that there will always be at least one candidate being approved of and one candidate being disapproved of.

Lemma 22 (Restriction 2). *If $A_s^i(\widehat{X}) = 1$ and $A_t^i(\widehat{X}) = 0$ for any voter i and any $\widehat{X} \subseteq X$, then $A_s^i(\widetilde{X}) \geq A_t^i(\widetilde{X})$ must hold for every $\widetilde{X} \subseteq X$ with $x_s, x_t \in \widetilde{X}$.*

Proof. From the assumption it follows that $u_i(x_s) > \overline{u}_i(\widehat{X}) \geq u_i(x_t)$. Hence we always have $u_i(x_s) > u_i(x_t)$ and so $A_s^i(\widetilde{X}) < A_t^i(\widetilde{X})$ cannot be true. □

This restriction says that if in one subset voter i selects candidate x_s but not x_t, in every other subset containing both candidates, the same voter cannot do the contrary, i.e. vote for x_t but not for x_s.

Lemma 23 (Restriction 3). *If $A_s^i(\widetilde{X}) = 1$ for any $\widetilde{X} \subseteq X$ and any voter i, then $A_t^i(\widetilde{X} \setminus \{x_s\}) \geq A_t^i(\widetilde{X})$ for every candidate $x_t \neq x_s$. Similarly, if $A_s^i(\widetilde{X}) = 0$ for any $\widetilde{X} \subseteq X$ and any voter i, then $A_t^i(\widetilde{X} \setminus \{x_s\}) \leq A_t^i(\widetilde{X})$ for every candidate $x_t \neq x_s$*

Proof. We will show the first part of the lemma, the second one is completely analogous. Since $u_i(x_s) > \overline{u}_i(\widetilde{X})$, we have $\overline{u}_i(\widetilde{X}) > \overline{u}_i(\widetilde{X} \setminus \{x_s\})$. Hence, $u_i(x_t) > \overline{u}_i(\widetilde{X})$ implies $u_i(x_t) > \overline{u}_i(\widetilde{X} \setminus \{x_s\})$. □

The last restriction states that if one removes a candidate a voter votes for, then every other candidate has at least as high chances of getting selected by this voter as in the original set of candidates. Or, in other words, if a better-than-average candidate is removed, the mean utility function of this subset will be lower than the original one. Thus, every other candidate who is better than average in the original set will also be better than average in this subset.

Despite all these restrictions - which we did not impose but which follow alone from the assumption that voters approve of all candidates who have a higher utility than their average utility - that severely infringe voters' strategies in subset elections, unfortunately for approval voting, we come to the following result.

Theorem 18. *Under approval voting, given the overall election results, there are no restrictions on the subset election results, even if every voter follows the recommended utility-maximizing strategy. In other words, information about the results of an election among one group of candidates gives no information about the results of elections among any other groups.*

Proof. We will show this theorem for $m = n = 3$. The general proof of more than two pages can be found in [22]. We want to construct the example stated at the beginning of this section. If the overall outcome is $a \succ b \succ c$ and the subset results are $c \succ a$, $b \succ a$ and $c \succ b$, the theorem follows.

So, let us assume any arbitrary group of voters who yield the outcome $a \succ b \succ c$. We will add three voters to this group with strategies $S_1 = \{a\}$, $S_2 = \{b\}$ and $S_3 = \{c\}$. How are they going to vote in the subset elections? The first voter has to vote $\{a\}$ in subset elections $\{a, b\}$ and $\{a, c\}$ because of lemma 21 (he cannot vote for both) and together with lemma 22 (he must vote for a). As to subset election $\{b, c\}$ we can choose freely whether he should vote $\{b\}$ or $\{c\}$. Since we want the outcome to be $c \succ b$, we let him vote $\{c\}$.

We can follow the same procedure with the two other voters. The second voter will vote $\{b\}$ in $\{a, b\}$ and $\{b, c\}$ and we choose him to vote $\{c\}$ in $\{a, c\}$, since we want to achieve $c \succ a$. The third voter votes $\{c\}$ in $\{a, c\}$ and $\{b, c\}$, and we choose him to vote $\{b\}$ in $\{a, b\}$.

Now, combining the ballots of the three voters in the overall election we obtain $\{a\} \cup \{b\} \cup \{c\} = \{a, b, c\}$, which does not change a thing. In $\{a, b\}$, however, the votes of the first two voters cancel each other out and the third voter gives one vote towards the desired outcome. The same is true for $\{a, c\}$, where we get one additional vote for c, and for $\{b, c\}$ we get an additional vote for c, too. Hence, if we repeat adding the group of these three voters often enough, we will reach the desired outcomes. Of course, we can also achieve any other result with this method, like $a \succ b$, b$\succ$ c and $c \succ a$ only by adjusting the strategies we can choose freely from this group of voters. $\square$

This theorem clearly contradicts Brams' and Fishburn's opinion [4, p. 5] that "[approval voting] is relatively insensitive to the number of candidates running." If we subtract or add candidates we can receive completely different results. However, this is not a drawback solely true for approval voting. Also plurality voting has the same kind of problem as Saari stated in [16]. Since a valid use of approval voting is to vote as in plurality voting, theorem 18 can also be interpreted as a corollary of Saari's finding.

Although the idea of comparing these results with the findings of computational manipulation by adding and deleting candidates sounds alluring, we cannot draw really strong conclusions, because approval voting is immune and vulnerable (two extreme positions!), and plurality voting always resistant (the "middle" category) in these cases.

5.9 Conclusion

Concluding, we repeat that approval voting loses a lot of its glory when considered under trichotomous and multichotomous preferences. In some cases it is even worse compared to plurality or negative voting as far as strategy-proofness is concerned. Being more inclusive, which was seen as an advantage in the previous chapter, also brings the effect that a Condorcet loser may be elected as a stable equilibrium. Approval voting loses against the fixed vote-for-half rule regarding Condorcet efficiency, and it also has some drawbacks with respect to plurality voting concerning computational manipulation. Within the framework of utility functions, other disadvantages become apparent, as the non-predictability of subset election outcomes or the stability of some strategies with unwished attributes.

As a response to the conclusion of chapter 4, we want to mention another finding by Dellis and Oak [9] which says that the advantage of approval over plurality voting concerning the election of more centrist candidates critically hinges upon an assumption. If this assumption shows to be not true in reality, the advantage cannot hold any more either.

6 Modifications of approval voting

6.1 Approval-Condorcet-Elimination

In section 4.1 we have seen that approval voting does excellent on dichotomous preferences. However, what if voters cannot find a partition of the candidates into only two sets? What if they need three categories to classify them, like "favorite", "acceptable" and "disapproved"? In this case, approval voting loses a lot of its glory. We have already seen in section 5.1 that approval voting is no longer strategy-proof under trichotomous (and multichotomous) preferences, nor does it guarantee to *always* elect the Condorcet winner. For the latter we can present a solution called the **Approval-Condorcet-Elimination procedure** (ACE) [23]: Voters are asked to make a trichotomous preference ranking; if they happen to have only a dichotomous one, e.g. $a \sim b \succ c$, ACE provides three different ways of stating them: Either, candidates a and b are both in category "favorite", but candidate c is in category "disapproved". To be aware of this information, we will write this as $a \sim b \succ \emptyset \succ c$. Or, candidates a and b may be only in category "acceptable", and candidate c therefore in category "disapproved", which we denote as $\emptyset \succ a \sim b \succ c$. Finally, it could be that candidates a and b are definitely favorites, and candidate c is still accepted, which would result in $a \sim b \succ c \succ \emptyset$. So this information from ACE is a more powerful one that we cannot obtain in such a way from approval voting.

In [23], ACE is defined as follows: Vote counts are kept for all pairs of candidates so that, in any given pair, the number of votes for a candidate is increased by 1 whenever that alternative is ranked higher in the trichotomous ranking just described. Then, if a candidate is Condorcet winner, this candidate is declared winner. If none exists, any Condorcet loser is eliminated. When there are neither Condorcet winner nor losers, candidate(s) with the largest disapproval vote are eliminated and the Condorcet criteria are applied again. This process is repeated until a winner is found or all remaining candidates are tied. This process reminds a bit of the Black method described on page 12, and has also similarities to IRV as on page 43. Actually it combines two elements from them: it first tries to find a Condorcet winner as in BM, and if there is none, it eliminates candidates round by round as in IRV until a Condorcet winner is found, or the remaining candidates are tied.

The advantages of ACE are (partially) obvious:

Proposition 29. *ACE satisfies the Condorcet winner, the Condorcet loser, the Pareto (the first version on p. 12) and the participation criterion.*

Proof. ACE satisfies the *Condorcet winner* and the *Condorcet loser* criteria by definition.

Pareto: If all voters have $a \succ b$, b will not be the Condorcet winner. Suppose that no other candidate c, $c \neq a, c \neq b$ wins. (If this was the case, we would already be done.) If b is not eliminated before in some round, and a will certainly not be eliminated before b, because a has no (!) disapproval votes and cannot be the Condorcet loser either, candidate b will certainly lose against a in a pairwise competition.

Participation: Suppose a is the winner and x any other alternative. Either a has a majority over x, or x is eliminated in some round due to being the Condorcet loser, or to having the largest disapproval votes. Now we add one or more voters who vote $a \succ x$. (So a and x are different indifferent classes.) For the first case, a now has an even larger majority over x. For the second case, it may be that x receives more votes against another candidate $y \neq a$ and ceases being the Condorcet loser, and for the third case it may be that y receives more disapproval votes than x. So in both cases, candidate y may be eliminated instead of x, but when only candidates a and x are left, a will certainly win, because a has won before and now has one or more additional voters voting $a \succ x$. □

Yet, there are also disadvantages that approval voting does not have. For instance, ACE fails the monotonicity criterion, but it satisfies a weaker form of this criterion.

Criterion 22 (Weak monotonicity)**.** *If a voting procedure chooses a as the winner, and one or more voters move a to a higher indifference class (without changing anything else), a should still be the winner.*

The difference to the original definition of monotonicity in criterion 11 is that the original definition talks about "changing preferences in a way favorable for a (without changing anything else)". So, for example $b \succ c \succ a$ turns into $b \succ a \succ c$. For ACE, these are two changes, because two indifference classes change, and not only one, whereas weak monotonicity would change the same preference ranking into $b \succ c \sim a \succ \emptyset$. However, one could argue that these are as well two changes (the "acceptable"-class changes from cardinality 1 to 2, and the "disapproved"-class from 1 to 0). As we have already mentioned before at the beginning of section 2.2, many criteria can be formulated in a way that is favorable for one or the other system. To my mind, the same phenomenon has occurred here. The following proposition will also consider consistency.

Proposition 30. *ACE fails consistency and monotonicity, but satisfies weak monotonicity.*

Proof. Consistency: See below.

Monotonicity: Consider the following example:

$$\begin{aligned}
4 \text{ voters: } & a \succ c \succ b \\
4 \text{ voters: } & b \succ a \succ c \\
2 \text{ voters: } & b \succ c \succ a \\
2 \text{ voters: } & c \succ b \succ a \\
1 \text{ voter: } & c \succ a \succ b
\end{aligned}$$

Since the overall outcome regarding Condorcet voting is $b \succ a \succ c \succ b$, there is no Condorcet winner nor loser. So candidate b having most disapproval votes is eliminated and a wins over c 8 to 5. Now, if the two voters of the third voter type change their preferences in favor for a from $b \succ c \succ a$ to $b \succ a \succ c$, there still is no Condorcet winner nor loser, but candidate c will now be eliminated with 6 disapproval votes, and b will beat a 8 to 5.

Weak monotonicity: Suppose a is the winner and x any other alternative. Either a has a majority over x, or x is eliminated in some round due to being the Condorcet loser or to having the largest disapproval votes. Now suppose that some voters move a to a higher indifference class, but x stays in his or her indifference class. For the first case, a may get an even larger majority over x. For the second case, x will not get larger support, so still be the Condorcet loser; or a now may receive fewer, but not more disapproval votes, and x gets the same number. Thus, in all cases, a will still win, and x will still lose. □

It may not be surprising that ACE fails consistency, since we already know from lemma 13 that every voting method that first elects a Condorcet winner if one exists, fails consistency. At the same time, though, this is quite astonishing for a modification of approval voting, since approval voting was characterized by consistency among two other criteria in theorem 3. Apparently, these modifications made to approval voting are already very strong and let it drift away from the original definition. On the other hand, there are also some modifications or redefinitions of consistency which ACE satisfies and which may all be considered reasonable [23]. Nevertheless, we will not mention them here, but content ourselves with the proposition above.

6.2 Approval voting combined with disapproval voting

Another way to model trichotomous preferences is the idea to combine approval with disapproval voting. Hence, a voter may not only approve or abstain from approving a candidate, but he or she may also vote against a candidate. Then, the candidate with the largest (net) number of votes (approval votes minus disapproval votes) wins. This modification of approval voting is called **combined approval voting** (CAV) [10].

For CAV we have to apply another definition of strategy, since we have defined a strategy only for nonranked voting procedures, which CAV definitely is not. Thus, a *CAV strategy* for a voter i is a pair $S = (S_1, S_2)$ consisting of two sets $S_1, S_2 \subseteq X, S_1 \cap S_2 = \emptyset$ with the interpretation that i votes for every candidate in S_1 and votes against every candidate in S_2, and abstains from voting for candidates not in $S_1 \cup S_2$.

First of all, we can say that the election outcomes of approval voting and combined approval voting will not differ very much.

Proposition 31. *If all voters vote according to their dominant strategy, the outcomes of approval voting and combined approval voting will be the same.*

Proof. We will sketch a proof for a set $X = \{a, b, c\}$ of three candidates as in [10]. By simple enumeration of all the $3^{|X|} - 2$ (voting for, voting against and abstaining for every candidate in X minus voting for all candidates and voting against all candidates) feasible strategies for a focal voter, plus the situations where he or she can influence the outcome (this is done exhaustively in [10]), table 6.1 presents the result of this process showing all the admissible strategies in comparison to the admissible strategies of approval voting for this case.

Table 6.1: Admissible strategies under CAV and AV for 3 candidates and concerned voters

Preference	Admissible strategies	
rankings	CAV	AV
$a \succ b \succ c$	({a,b},{c}), ({a},{b,c})	{a}, {a,b}
$a \succ c \succ b$	({a,c},{b}), ({a,b},{c})	{a}, {a,c}
$b \succ a \succ c$	({a,b},{c}), ({b},{a,c})	{b}, {a,b}
$b \succ c \succ a$	({b,c},{a}), ({b},{a,c})	{b}, {b,c}
$c \succ a \succ b$	({a,c},{b}), ({c},{a,b})	{c}, {a,c}
$c \succ b \succ a$	({b,c},{a}), ({c},{a,b})	{c}, {b,c}
$c \succ b \sim c$	({a},{b,c})	{c}
$b \succ a \sim c$	({b},{a,c})	{b}
$c \succ a \sim b$	({c},{a,b})	{c}
$a \sim b \succ c$	({a,b},{c})	{a,b}
$a \sim c \succ b$	({a,c},{b})	{a,c}
$b \sim c \succ a$	({b,c},{a})	{b,c}

From the table we conclude that admissible strategies for CAV only include voting for or against a candidate, but never abstaining, and that there are as many admissible strategies for approval voting as for CAV. We show that the outcomes must be the same. Let d be a ballot response profile for approval voting. Candidate a received $v(a, d)$ votes in the approval voting election. In the CAV election, there will also be $v(a, d)$ voters who will vote for a, but since no abstentions are admissible, there will be $(n - v(a, d))$ voters voting against a. So there is a net sum of $v(a, d) - (n - v(a, d)) = 2v(a, d) - n$ for candidate a. Candidates b and c will receive $2v(b, d) - n$ and $2v(c, d) - n$ votes, respectively. Hence, no matter what the result for approval voting was, the same will be true for CAV. $\square$

Yet, there is also a difference between the two systems as far as focal voter efficacy is concerned, which is measured by the probability that a focal voter can change the election outcome.

Definition 9. *Given the ballot of the other $n - 1$ voters, a focal voter n is called*

- weakly decisive*, if the voter is able to create ties between two or more leading candidates.*
- strongly decisive*, if the voter can break a tie between two or more leading candidates, as well as elect a candidate who obtained fewer votes than the leading tied candidates.*

[10] shows that the following proposition holds:

Proposition 32. *The probability that a focal voter is strongly decisive under CAV is greater than under AV.*

The probability that a focal voter is not decisive (so not even weakly decisive) under CAV is greater than under AV.

[10] also shows that these differences are relatively large under a small electorate (committees etc.). So especially under these circumstances voter will prefer one system to the other depending on whether they dislike more being not decisive or they put their attention more on being strongly decisive. Furthermore, trying to be psychologist, an additional point for CAV is the potential psychological effect a voter has by being able to express a distinct disapproval vote against a candidate.

6.3 Restricted approval voting

As announced in section 5.6, **restricted approval voting** (RAV) mitigates the drawbacks of approval voting as far as extreme vulnerability to majority decisiveness and to the erosion of the majority principle is concerned. Let l and u be integers such that $1 \leq l \leq u \leq |X| - 1$, and voters must vote for at least l and at most k candidates [3]. We have the following special cases:

- If $1 < l = u < |X| - 1$, the applied rule equals fixed rules with $|M| = 1$, where voters can only vote for a predetermined number of candidates, cf. p. 35. This is the most inflexible version of RAV.
- If $l = u = 1$, the applied rule is plurality voting.
- If $l = u = |X| - 1$, the applied rule is negative voting.
- If $l = 1$ and $u = |X| - 1$, the applied rule is approval voting.

The following theorem demonstrates how RAV can mitigate the above stated problems [3]:

Theorem 19. 1. *Under sincere voting, if $l > 1$ RAV is not vulnerable to any α-majority decisiveness.*

2. *Under sincere voting, the vulnerability of RAV characterized by $l > 1$ to erosion of the majority principle cannot be eliminated. However, the proportion of preference profiles under which the erosion problem exists can be decreased by reducing u.*

Proof. (1) $l > 1$ implies that $s_1 = s_2 = 1$ as in section 5.6. Consequently, inequality 5.6.1 transforms into

$$\alpha \geq \frac{1 - s_m}{2 - 1 - s_m} = 1.$$

Hence, there is only unanimity that can guarantee its favorite to be elected.

(2) The first claim of part (2) can be directly derived from (1). We prove the second claim by stating that the number of feasible strategies under $s_1 = s_2$ (since $l > 1$) increases when u is increased. In other words, the proportion of preference profiles under which erosion of the majority principle arises can be decreased by reducing u. $\square$

However, this restriction also raises problems. What if $l > 1$ and a voter i has the preference ranking $a \succ_i b \sim_i c$? RAV forces i to vote for a candidate from $L(\succsim_i)$, a strategy not admissible for approval voting, and intuitively not making much sense, either.

6.4 Conclusion

Although all of these restrictions and modifications can weaken or even eliminate some drawbacks, none of them can be judged superior to approval voting, because either not all disadvantages may be eliminated, or the elimination of some flaw brings up another drawback to this modified voting procedure.

Massó and Vorsatz [15] present another modification of approval voting: To allow society to treat unequal alternatives distinctly they propose relaxing the assumption of neutrality. According to this modification, every candidate receives a strictly positive and finite weight already before the election. These weights may differ across candidates. After approval voting was carried out, *weighted approval voting* elects the candidate for which the product of total number of votes times exogenous weight is maximal.

Finally, I want to come back to our nine friends from the introductory example. So, what should they do now? At first, we already know that there is no perfect voting procedure, so all the advice we can give will not be perfect, either. I would suggest that they should try approval voting, because it gives them much more freedom to express their preferences, or perhaps CAV (if we want each voter to have higher chances of being strongly decisive). Also, Borda Count would be an alternative, but to my mind it is too restrictive: Every voter is forced to have a linear ranking (which of course is the case in this example), unless a modification is used which splits the points among equally ranked candidates, and every voter cannot vary his or her strategy without voting insincerely. And this does not imply strategy-proofness as already Borda had to admit [19, p. 29]. The same restriction argument applies for plurality voting, plurality voting with runoff (this at least prevents our friends from electing the Condorcet loser) and all Condorcet methods.

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