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On optimal and threshold dividend strategies for four different risk models

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durch

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Contents

Chapter 1

Introduction

This diploma theses deals with an optimization problem of control theory, that has been discussed and worked at in the past from different point of views.

In 1957, Bruno De Finetti was the first one, who published a paper, which discussed the optimal dividend problem. Afterwards several people addressed this problem, for instance Asmussen & Taksar, Gerber & Shiu and Schmidli. Although the mathematical models were different, the underlying question remained the same.

How can the dividend payments of a company be paid to the shareholders in an optimal way?

To understand what is meant with the word optimal, we seize a suggestion, which was made by Bruno De Finetti and afterwards was taken over, namely, that a company would seek to maximize the expectation of the present value of all dividend payments before ruin. Of course, this approach differs from the aim to calculate and minimize the ruin probability, but it seems to be relevant, as we can make us clear in the following way.

Imagine an insurance company, which receives premiums from the policy holders and in exchange charges for a certain amount of insurance coverage. Besides the insurance company has shareholders, which are not only interested in the profit of the company in which they invest, but especial in high dividends. As too high dividend payments could result in a fast ruin, the insurance company has to think about a tactical strategy for the dividend payments.

In order to find the optimal dividend strategy we first of all need a mathematical model, which describes how the surplus of the insurance company evolves over time. Simply seen, we have to model the incoming premiums, the outgoing claims and the outgoing dividend payments. There are several possibilities for it. But a very common approach to model the outgoing claims is the use of the compound Poisson process.

To get an insight of different models, which build the fundament of the resulting considerations, four different models are presented. Whereas the compound

Poisson model and the diffusion model go hand in hand with the paper by Gerber & Shiu and the book by Schmidli, the other models were developed self-contained by applying the proceeding of the compound Poisson and the diffusion model. One of them, the dual model, shows that the problem of optimal dividend payments can also be discussed for companies, which do not act as insurer. As the compound Poisson and the diffusion model base on different stochastic processes, namely the compound Poisson process respectively the Brownian motion, the link between these two models is explained in chapter 6. For the other two models, the dual model and the perturbed model, the amount of total claims is described by the compound Poisson process, too.

Chapter 2 to 5 are organized after the same scheme to make it easy to compare the different models.

At the beginning the mathematical model and the problem are formulated and it is argued that we are not only interested in the optimal dividend payments but also in the expected value of the present value of all this optimal dividend payments before ruin. This expected value, a function of the initial capital of the company, is called value function.

To find the optimal dividend strategy and the value function, the so-called Hamilton-Jacobi-Bellman equation is motivated in the second section. Therefore one has to make some assumptions to the unknown value function. Out of the Hamilton-Jacobi-Bellman equation, a candidate for the optimal dividend strategy can be found. At that time we can not be sure whether the candidate really is the optimal strategy.

By solving the Hamilton-Jacobi-Bellman equation, a candidate for the value function can be calculated. Except for the perturbed model explicit solutions can be obtained.

Afterwards a sort of verification theorem has to be done, which proves that the candidates really are the optimal dividend strategy and the value function.

Finally it is discussed that the optimal dividend strategy is a so-called threshold strategy, which means that the dividend payments only depend on whether the current surplus is higher or lower than the threshold. Again, except for the perturbed model, the optimal threshold can be calculated explicit.

By way of illustration numerical examples are discussed at the end of chapters 2, 3 and 4.

Chapter 2

Compound Poisson model

The following considerations and the solution of the problem of optimal dividend payments in the compound Poisson model base on a paper by Gerber and Shiu [5] and on a book by Schmidli [8].

2.1 Problem formulation

Without dividend payments, the surplus of an insurance company depends on the initial capital x, the constant premium income c and the random occurring claims.

In the compound Poisson model the claims are modeled by the compound Poisson process $\{S_t\}$, which consists of the number of occurred claims and of the size of the individual claims. The number of claims is modeled by a Poisson process $\{N_t\}$ with rate λ, i.e., $\mathbb{E}[N_t] = \lambda \cdot t$. Further, the individual claims Y_i are independent and identically distributed (i.i.d.), positive and independent of N with probability density $p(y), y \geq 0$. By $0 < T_1 < T_2 < ...$ we denote the claim occurrence times. Then

$$S_t = \sum_{i=1}^{N_t} Y_i$$

is the sum of claims, which have to be paid by the insurance company from time 0 until t and the surplus without dividend payments is

$$X_t = x + ct - S_t.$$

The dividend payments at time $t \geq 0$ are described by a process $\{U_t\}$, whereas we only allow dividend rate processes, which are adapted and cadlag. Further, we restrict the dividend payments by

$$0 \leq U_t \leq \alpha < \infty \qquad \forall t \geq 0,$$

i.e., the several dividend payments must not exceed some given barrier called $\alpha < c$. Overall, the surplus of an insurance company in the compound Poisson model is given by

$$X_t^U = x + ct - S_t - \int_0^t U_s \, ds.$$

As dividends can only be paid until time of ruin $\tau^U = \inf\{t : X_t^U < 0\}$, the value of the dividends is

$$V^U(x) = \mathbb{E}\left[\int_0^{\tau^U} e^{-\delta t} U_t \, dt\right].$$

Discounting the dividend payments with $\delta > 0$ describes, that money today is better than money tomorrow.

We search for dividend rates, which yield the maximal value of dividends. Therefore we aim at the function

$$V(x) = \sup_U V^U(x) = V^{U^*}(x),$$

which is called value function and describes the maximal expected value of dividend payments, which is possible. The dividend rate process $\{U_t^*\}$ is the optimal dividend strategy.

We suppose that the individual claims Y_i are exponentially distributed,

$$p(y) = \beta e^{-\beta y}, \qquad \beta > 0, y > 0.$$

2.2 Motivation of the Hamilton-Jacobi-Bellman equation

To find the Hamilton-Jacobi-Bellman (HJB) equation we use Itôs formula

$$\begin{aligned} f(t, X_t^U) - f(0, x) &= \int_0^t \left[\frac{\partial f}{\partial s}(s, X_{s-}^U) + (c - U_s)\frac{\partial f}{\partial x}(s, X_{s-}^U)\right] ds \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left[f(s, X_{s-}^U - y) - f(s, X_{s-}^U)\right] J_X(\,ds\,dy), \end{aligned}$$

which is an adaption to Cont & Tankov's proposition 8.13 in [4]. J_X is a random measure associated to the jumps and f is an arbitrary measurable function. We pay no attention to this random measure, as we do not need it to find the HJB equation. Define $f(t, X_t^U)$ as the discounted value function,

$$f(t, X_t^U) := e^{-\delta t} V(X_t^U).$$

Further, define

$$Z_t := f(t, X_t^U) + \int_0^t e^{-\delta s} U_s \, ds - f(0, X_0). \tag{2.1}$$

Equating the drift in Z_t to zero and using that $p(y) = 0$ for $y \leq 0$ gives

$$-\delta V(X_t^U) + (c - U_t) V'(X_t^U) + \lambda \int_0^\infty V(X_t^U - y) p(y) \, dy - \lambda V(X_t^U) + U_t = 0.$$

If there is a claim y, which exceeds the surplus X_t^U of the insurance company, the company is ruined. Therefore $V(X_t^U - y) = 0$ for $y > X_t^U$ and the integral in the former equation goes only up to X_t^U.

Taking the supremum over all admissible dividend strategies yields the Hamilton-Jacobi-Bellman equation

$$\max_{0 \leq u \leq \alpha} \{u(1 - V'(x))\} - V(x)(\lambda + \delta) + cV'(x) + \lambda \int_0^x V(x - y) p(y) \, dy = 0. \tag{2.2}$$

$u(1 - V'(x))$ is maximized, if

$$u = \alpha \text{ for } V'(x) < 1, \qquad u = 0 \text{ for } V'(x) > 1. \tag{2.3}$$

If $V'(x) = 1$, any value $\in [0, \alpha]$ is possible for u.

2.3 Threshold strategies

A so-called threshold strategy is a dividend strategy, which pays dividends according to a threshold. Whenever the surplus is below the threshold, no dividends are paid. But as soon as the surplus exceeds the threshold, the maximal dividend rate α is paid. According to the considerations above, we do have such a threshold strategy with threshold b, if the value function satisfies

$$V'(x) = \begin{cases} > 1 & \text{for } x < b \\ < 1 & \text{for } x > b. \end{cases}$$

A priori we do not know whether there is a threshold b, which fulfills the given property. And we also do not know, which value such a threshold has. But for the following considerations we assume, that there is such a threshold. How the surplus is affected by such a threshold, is illustrated in the figures below. Figure 2.1 shows the surplus X_t without dividend payments for one sample path. The same sample path, but with dividend payments according to a threshold strategy with threshold $b > x$, is displayed in figure 2.2. In the third figure the threshold b is assumed to be less than the initial capital x, which clarifies, that the numerical value of the threshold has a big influence on the development of the surplus X_t^U.

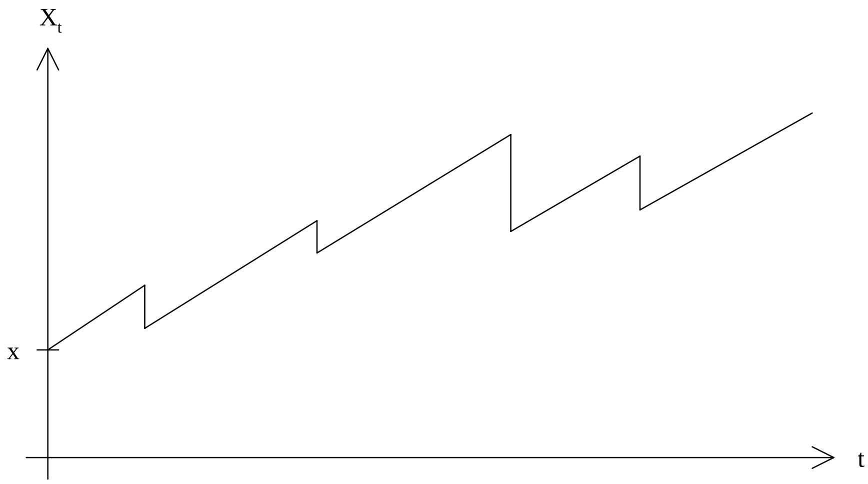

Figure 2.1: Sample path for the surplus without dividend payments.

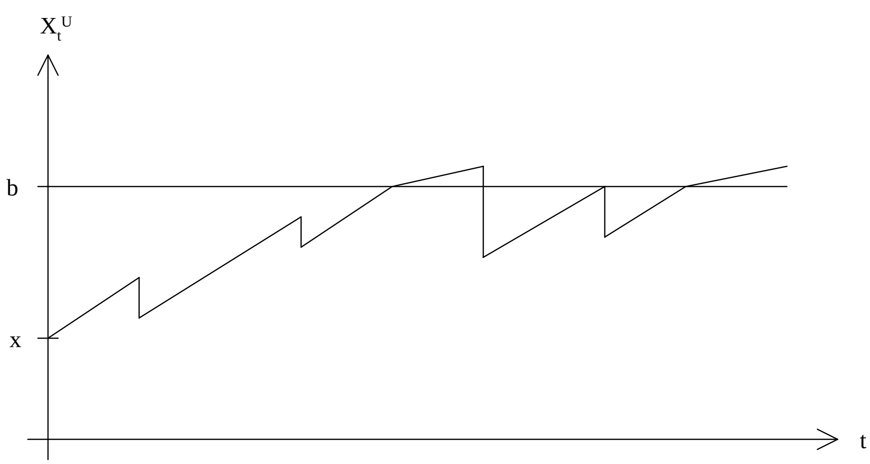

Figure 2.2: Sample path for the surplus with threshold $b > x$.

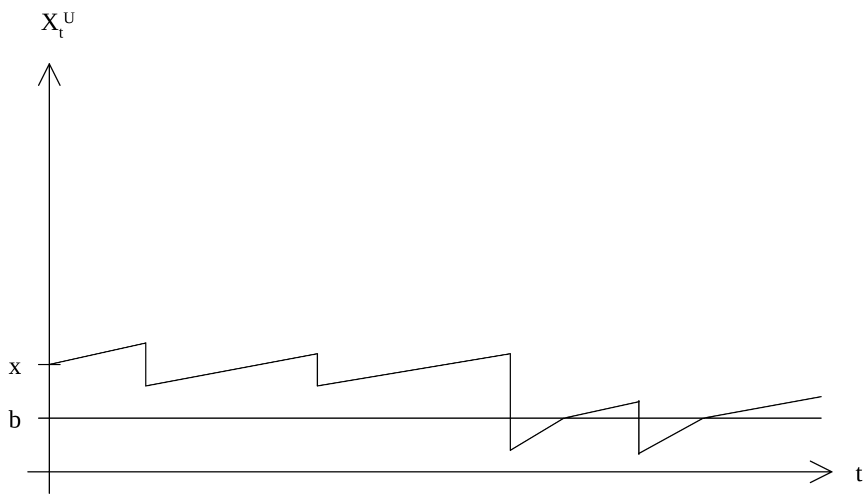

Figure 2.3: Sample path for the surplus with threshold $b < x$.

Let $V(x;b)$ denote the expectation of the present value of all dividends until ruin, where x is the initial capital and b is the threshold. Then $V(x;b)$ satisfies the following integro-differential equation:

$$cV'(x;b) - (\lambda+\delta)V(x;b) + \lambda \int_0^x V(x-y;b)p(y)dy = 0 \qquad 0 \leq x < b \tag{2.4}$$

$$\alpha + (c-\alpha)V'(x;b) - (\lambda+\delta)V(x;b) + \lambda \int_0^x V(x-y;b)p(y)dy = 0 \qquad x > b \tag{2.5}$$

We do not know yet, how the value function looks like, but we aim at a value function, that is continuous at $x = b$. Therefore we suppose that

$$V(b-;b) = V(b+;b).$$

So we construct a solution, which fulfills this continuity condition and afterwards we show that this found solution is the value function. Supposing this, we see that

$$cV'(b-;b) = (c-\alpha)V'(b+;b) + \alpha, \tag{2.6}$$

which follows from (2.4) and (2.5). This relation will be used later.

As the integro-differential equation

$$ch'(x) - (\lambda+\delta)h(x) + \lambda \int_0^x h(x-y)p(y)dy = 0 \tag{2.7}$$

for $x > 0$ has a unique solution $h(x)$ (apart from a constant factor),

$$V(x; b) = \gamma h(x)$$

where γ does not depend on x.

2.4 Solution of the HJB equation

We assume now, that there exists a b as in the previous subsection, so that there is a threshold strategy. Then we look at the case $0 < x < b$, where the optimal strategy is to pay no dividends ($u = 0$).

- $u = 0$

So we have the integro-differential equation (2.4), respectively (2.7). We take the derivative of (2.7) in order to find a solution h.

$$ch''(x) - (\lambda + \delta)h'(x) + \lambda \frac{d}{dx}\left(\int_0^x h(x-y)p(y)dy\right) = 0 \tag{2.8}$$

If h is continuously differentiable the Leibniz integral rule can be used to determine the derivation of the integral. As we now only search for a solution and will verify it afterwards, we can use the Leibniz integral rule without knowing whether h is continuously differentiable or not.

$$\begin{aligned}
\frac{d}{dx}\left(\int_0^x h(x-y)p(y)dy\right) &= \frac{d}{dx}\left(\int_0^x h(y)p(x-y)dy\right) \\
&= \int_0^x h(y)\frac{d}{dx}p(x-y)dy + \beta h(x) \\
&= \beta\left(h(x) - \int_0^x h(y)p(x-y)dy\right) \\
&= \beta\left(h(x) - \int_0^x h(x-y)p(y)dy\right)
\end{aligned} \tag{2.9}$$

Using (2.9) in (2.8) yields

$$ch''(x) - (\lambda + \delta)h'(x) + \lambda\beta\left(h(x) - \int_0^x h(x-y)p(y)\,dy\right) = 0 \tag{2.10}$$

and as (2.7) is fulfilled, (2.8) becomes

$$ch''(x) + (c\beta - \lambda - \delta)h'(x) - \beta\delta h(x) = 0. \tag{2.11}$$

This is a second order homogenous linear differential equation with constant coefficients, which can be solved by an exponential ansatz

$$h(x) = C_0 e^{rx} + C_1 e^{sx}. \tag{2.12}$$

Setting the ansatz into the differential equation yields that $r > 0$ and $s < 0$ are the roots of the quadratic equation

$$c\xi^2 + (c\beta - \lambda - \delta)\xi - \beta\delta = 0. \tag{2.13}$$

The left-hand side of (2.13) is $\beta\lambda$ (> 0) for $\xi = -\beta$ and $-\beta\delta$ (< 0) for $\xi = 0$. As the premium rate c is positive, the negative root of the quadratic equation lies between $-\beta$ and 0: $-\beta < s < 0$. Therefore $\beta + s$ is positive.

To get an information about the constants C_0 and C_1 we substitute $h(x)$ in (2.7) for (2.12). Therefore we calculate $\int_0^x h(x-y)p(y)\,dy$ for the given probability density and equate the coefficient of $e^{-\beta x}$ with 0. Hence we get

$$\frac{C_0}{r+\beta} + \frac{C_1}{s+\beta} = 0. \tag{2.14}$$

This yields

$$h(x) = -\frac{C_1}{s+\beta} \cdot \left((r+\beta)e^{rx} - (s+\beta)e^{sx}\right). \tag{2.15}$$

Now we know $V(x;b)$ except for a constant factor γ, which does not depend on x:

$$V(x;b) = \gamma\left[(r+\beta)e^{rx} - (s+\beta)e^{sx}\right] \qquad 0 \le x < b. \tag{2.16}$$

In order to determine this constant γ we use the continuity condition $V(b-;b) = V(b+;b)$ (see section 2.3). But as we do not know $V(x;b)$ for $x > b$ yet, we first have to do the same approach as for the determination of $V(x;b)$ for $0 \le x \le b$.

- $u = \alpha$

Differentiating (2.5) yields the second order inhomogeneous linear differential equation with constant coefficients

$$(c-\alpha)V''(x;b) + [\beta(c-\alpha) - \lambda - \delta]V'(x;b) - \beta\delta V(x;b) + \beta\alpha = 0 \qquad x > b. \tag{2.17}$$

The homogeneous part of this differential equation is again solved by an exponential ansatz and delivers

$$V(x;b)_{hom} = D_0 e^{vx} + D_1 e^{wx},$$

where $v > 0$ and $w < 0$ are the roots of

$$\xi^2(c-\alpha) + \xi(\beta c - \alpha\beta - \lambda - \delta) - \beta\delta = 0. \tag{2.18}$$

It is easy to see that the particular solution is

$$V(x;b)_{part} = \frac{\alpha}{\delta}.$$

Therefore the general solution is

$$V(x;b) = \frac{\alpha}{\delta} + D_0 e^{vx} + D_1 e^{wx}.$$

To get an information about the constants, we need the following two propositions.

Proposition 2.4.1 *$V(x;b)$ is bounded by $\frac{\alpha}{\delta}$.*

Proof. To get an estimate up for $V(x;b)$ we assume that the company always pays the maximal dividend rate α and that the company is never ruined, i.e. $\tau = \infty$:

$$V(x;b) \leq \int_0^\infty e^{-\delta t} \alpha \, dt = \frac{\alpha}{\delta}.$$

□

Proposition 2.4.2 *$V(x;b)$ converges to $\frac{\alpha}{\delta}$ as $x \to \infty$.*

Proof. Consider the strategy $U_t = \alpha$. As $x \to \infty$, the ruin time τ^α converges to infinity. Therefore $\lim_{x\to\infty} e^{-\delta\tau^\alpha} = 0$ and $\mathbb{E}\big[e^{-\delta\tau^\alpha}\big]$ converges to zero by dominated convergence. With this strategy the value of the dividends is

$$V^\alpha(x;b) = \mathbb{E}\left[\int_0^{\tau^\alpha} e^{-\delta t} \alpha \, dt\right] = \frac{\alpha}{\delta}\left(1 - \mathbb{E}\big[e^{-\delta\tau^\alpha}\big]\right)$$

and converges to $\frac{\alpha}{\delta}$ as $x \to \infty$. On the one hand $V(x;b) \geq V^\alpha(x;b)$ and it follows that $\lim_{x\to\infty} V(x;b) \geq \frac{\alpha}{\delta}$. But on the other hand $V(x;b)$ is bounded by $\frac{\alpha}{\delta}$ and therefore $\lim_{x\to\infty} V(x;b) = \frac{\alpha}{\delta}$. □

Proposition 2.4.2 gives

$$\begin{aligned} \lim_{x\to\infty} V(x;b) &= \lim_{x\to\infty} \left(\frac{\alpha}{\delta} + D_0 e^{vx} + D_1 e^{wx}\right) && (2.19) \\ &= \frac{\alpha}{\delta} + D_0 \underbrace{\lim_{x\to\infty} e^{vx}}_{\to\infty} + D_1 \underbrace{\lim_{x\to\infty} e^{wx}}_{\to 0} = \frac{\alpha}{\delta}. && (2.20) \end{aligned}$$

From the last equation it already follows that

$$D_0 = 0.$$

This yields

$$D_1 < 0$$

and altogether

$$V(x;b) = \frac{\alpha}{\delta} + D_1 e^{wx} \tag{2.21}$$

with

$$w < 0 \text{ and } D_1 < 0.$$

We now want to quantify the constants C_1 and D_1. Therefore we need two equations, which contain C_1 and D_1. The first one we get from the continuity condition, as already mentioned,

$$V(b-;b) = V(b+;b), \tag{2.22}$$

which leads to

$$-\frac{C_1}{s+\beta}\left((r+\beta)e^{rb} - (s+\beta)e^{sb}\right) = \frac{\alpha}{\delta} + D_1 e^{wb}. \tag{2.23}$$

For the second equation we use (2.5) and split the integral in it in two parts:

$$\begin{aligned}\int_0^x V(x-y;b)p(y)dy &= \int_0^x V(y;b)p(x-y)dy \\ &= \int_0^b V(x-y;b)p(y)dy + \int_b^x V(x-y;b)p(y)dy\end{aligned}$$

Using (2.15), (2.21) and calculating the integrals results in

$$\begin{aligned}\int_0^x V(x-y;b)p(y)dy &= e^{-\beta x}\left[\frac{C_1\beta}{s+\beta}\left(e^{b(s+\beta)} - e^{b(r+\beta)}\right) - \frac{\alpha}{\delta}e^{\beta b} - \frac{D_1\beta}{w+\beta}e^{b(w+\beta)}\right] \\ &\quad + \frac{\alpha}{\delta} + D_1\frac{\beta}{w+\beta}e^{wx}.\end{aligned}$$

If we look at (2.5) again, we see, that the other components do not evolve like $e^{-\beta x}$. Because of this the coefficient of $e^{-\beta x}$ has to be zero. As $e^{\beta b}$ is included in every term of the coefficient, we finally get as the two conditions for C_1 and D_1:

$$\begin{aligned}\frac{C_1\beta}{s+\beta}(e^{bs} - e^{br}) - \frac{\alpha}{\delta} - \frac{D_1\beta}{w+\beta}e^{wb} &= 0 \\ \frac{C_1}{s+\beta}\left[(s+\beta)e^{sb} - (r+\beta)e^{rb}\right] - \frac{\alpha}{\delta} &\overset{(2.23)}{=} D_1 e^{wb}\end{aligned}$$

This two-dimensional system can be solved by putting the right-hand side of the second equation into the first one.

$$C_1 = \frac{\alpha}{\delta}\frac{w}{\beta}\frac{s+\beta}{(r-w)e^{br} - (s-w)e^{bs}} \tag{2.24}$$

$$\begin{aligned}D_1 &= e^{-wb}\left[\frac{\alpha}{\delta}\frac{w}{\beta}\frac{(s+\beta)e^{sb} - (r+\beta)e^{rb}}{(r-w)e^{br} - (s-w)e^{bs}} - \frac{\alpha}{\delta}\right] \\ &= e^{-wb}\left[V(b;b) - \frac{\alpha}{\delta}\right]\end{aligned} \tag{2.25}$$

Putting this into (2.15) and (2.21) yields

$$V(x;b) = -\frac{\alpha}{\delta}\frac{w}{\beta}\frac{(r+\beta)e^{rx}-(s+\beta)e^{sx}}{(r-w)e^{br}-(s-w)e^{bs}} \qquad 0 \le x \le b \tag{2.26}$$

$$V(x;b) = \frac{\alpha}{\delta}\left(1-e^{w(x-b)}\right)+V(b;b)e^{w(x-b)} \qquad x \ge b \tag{2.27}$$

If both $x = 0$ and $b = 0$ then

$$V(0;0) = -\frac{w}{\beta}\frac{\alpha}{\delta}. \tag{2.28}$$

As we know that w is the negative root of (2.18), it is easy to see that $0 < -\frac{w}{\beta} < 1$ and thus $V(0;0) < \frac{\alpha}{\delta}$:

$$\begin{aligned} \xi &= 0 \quad \Rightarrow \text{left-hand side of (2.18)} < 0 \\ \xi &= -\beta \Rightarrow \text{left-hand side of (2.18)} > 0 \\ &\Rightarrow -\beta < w < 0 \end{aligned}$$

2.5 Verification of the solution

As we are interested in the optimal dividend strategy, we want to figure out, whether the solution we found really is the value function. Therefore we show that an increasing, bounded and positive function, which solves the HJB equation, equals the value function. That the solution $V(x;b)$ found in the prior section fulfills these conditions, is discussed at the end of this section.

Theorem 2.5.1 *Suppose that $f(x)$ is an increasing, bounded and positive solution to (2.2) and $f(x) = 0$ for $x < 0$. Then $f(x) = V(x)$ and the optimal dividend strategy is given by (2.3), i.e. $U_t^* = \alpha 1_{V'(X_t^*)<1}$*

Proof. The theorem is proven in two parts.

- The function $f(x)$ majorizes $V(x)$.

Let U be an arbitrary strategy. Then the process M with

$$\begin{aligned} M_t = &\sum_{i=1}^{N_{\tau^U \wedge t}} \left(f(X_{T_i}^U) - f(X_{T_i-}^U)\right) e^{-\delta T_i} \\ &-\lambda \int_0^{\tau^U \wedge t} e^{-\delta s} \left(\int_0^{X_s^U} f(X_s^U - y)\, dG(y) - f(X_s^U) \right) ds \end{aligned} \tag{2.29}$$

is a martingale. This can be shown by using the integration theorem in Brémaud [3] [p. 235]. Using the fundamental theorem of calculus for the function $e^{-\delta t} f(X_t^U)$

and its derivative and taking into consideration that no claims occur between T_{i-1} and T_i-, we get that

$$f(X^U_{T_i-})e^{-\delta T_i} - f(X^U_{T_{i-1}})e^{-\delta T_{i-1}} = \int_{T_{i-1}}^{T_i-} \left[(c-U_s)f'(X^U_s) - \delta f(X^U_s)\right] e^{-\delta s}\, ds. \tag{2.30}$$

Then,

$$\begin{aligned} Z_t \;:=\; & f(X^U_{\tau^U \wedge t})e^{-\delta(\tau^U \wedge t)} - \\ & \int_0^{\tau^U \wedge t} \left[(c-U_s)f'(X^U_s) + \lambda \int_0^{X^U_s} f(X^U_s - y)\, dG(y) - (\lambda+\delta)f(X^U_s)\right] e^{-\delta s}\, ds \end{aligned}$$

is a martingale, too. This can easily be seen by using (2.29) and (2.30) to find that

$$Z_t = M_t + f(x) - \sum_{i=1}^{N_{\tau^U \wedge t}} \underbrace{\int_{T_i-}^{T_i} \left[(c-U_s)f'(X^U_s) - f(X^U_s)\delta\right] e^{-\delta s}\, ds}_{=0}.$$

As $f(x)$ is deterministic, Z_t is a martingale and $\mathbb{E}[Z_t] = f(x)$. According to the assumption, f solves the HJB equation (2.2) and because U is an arbitrary dividend strategy, we have

$$(c-U_s)f'(X^U_s) + \lambda \int_0^{X^U_s} f(X^U_s - y)\, dG(y) - (\lambda+\delta)f(X^U_s) \leq -U_s. \tag{2.31}$$

Consequently, we get a lower bound for Z_t,

$$Z_t \geq f(X^U_{\tau^U \wedge t})e^{-\delta(\tau^U \wedge t)} + \int_0^{\tau^U \wedge t} U_s e^{-\delta s}\, ds. \tag{2.32}$$

Taking the expectation and using that f is positive, gives

$$f(x) \geq \mathbb{E}\left[\int_0^{\tau^U \wedge t} U_s e^{-\delta s}\, ds\right].$$

As for $t \to \infty$ the limes and expectation may be interchanged by dominated convergence,

$$\begin{aligned} f(x) \;&\geq\; \mathbb{E}\left[\lim_{t\to\infty} \int_0^\infty 1_{s\leq\tau^U} 1_{s\leq t} U_s e^{-\delta s}\, ds\right] \\ &=\; \mathbb{E}\left[\int_0^{\tau^U} U_s e^{-\delta s}\, ds\right] \\ &=\; V^U(x). \end{aligned}$$

Since the reasoning is valid for all dividend strategies, it is also valid for the optimal dividend strategy U^* and therefore $f(x)$ majorizes the value function,

$$f(x) \geq \mathbb{E}\left[\int_0^{\tau^*} U_s^* e^{-\delta s}\, ds\right] = V(x).$$

- $f(x) = V(x)$

For $U = U^*$, we have equality in (2.31). Taking the expectation gives

$$f(x) = \mathbb{E}\left[f(X_{\tau^*\wedge t}^*) e^{-\delta(\tau^*\wedge t)} + \int_0^{\tau^*\wedge t} U_s^* e^{-\delta s}\, ds\right].$$

Letting $t \to \infty$ yields

$$\begin{aligned} f(x) &= \mathbb{E}\left[\lim_{t\to\infty}\left(\underbrace{f(X_t^*)}_{bounded}\underbrace{e^{-\delta t}}_{\to 0} 1_{t\leq\tau^*} + f(\underbrace{X_{\tau^*}^*}_{<0})\underbrace{e^{-\delta\tau^*}}_{\leq 1} 1_{t>\tau^*}\right)\right] + V(x) \\ &= V(x). \end{aligned}$$

□

$V(x;b)$ given by (2.26) and (2.27) fulfills the assumptions in theorem 2.5.1:

- $V(x;b)$ increasing

  As $V(x;b)$ is the difference of an increasing and decreasing function on $[0,b]$, $V(x;b)$ is increasing on $[0,b]$. For $x > b$, $V(x;b)$ is the sum of an increasing and a decreasing function. But because $\frac{\alpha}{\delta} > V(b;b)$ (as D_1 in (2.25) is negative), the increasing part outweighs the decreasing part and $V(x;b)$ is increasing on $[b,\infty)$:

  $$V(x;b) = \frac{\alpha}{\delta}\underbrace{(1 - e^{w(x-b)})}_{increasing} + V(b;b)\underbrace{e^{w(x-b)}}_{decreasing}$$

  Because $V(x;b)$ is continuous at $x = b$, $V(x;b)$ is increasing on $[0,\infty)$.

- $V(x;b)$ bounded

  $V(x;b)$ is bounded for $x \geq b$ by $\frac{\alpha}{\delta}$ by construction (see (2.21)).
  As $V(x;b)$ is increasing on $[0,b]$ and $V(b-;b) = V(b+;b) < \frac{\alpha}{\delta}$, $V(x;b)$ is bounded by $\frac{\alpha}{\delta}$ on $[0;b]$, too.

- $V(x;b)$ positive

$0 \le x \le b$
$V(x;b)$ is positive, because $r > 0 > s$ and $w < 0$.

$x \ge b$
$V(x;b)$ is positive, because $V(b;b)$ is positive and therefore both summands are positive.

- $V(x;b) = 0$ for $x < 0$

  Fulfilled by construction.

2.6 Optimal threshold strategies

In the previous section we have not ready thought about whether a threshold b really exists. We chose a strategy, where we made the dividend payments dependent of some point b. Was the surplus under the level b, we didn't pay anything, but if the surplus was above b we paid the maximum α. From this strategy we got two functions $V(x;b)$ for $x < b$ and $x > b$. Now we consider whether there exists a point b^* for which $V'(x;b^*)$ is > 1 for $x < b^*$ and $V'(x;b^*) < 1$ for $x > b^*$. If such a point exists, we have found the optimal threshold b^*.

According to (2.6)

$$V'(b-;b) = \left(1 - \frac{\alpha}{c}\right) V'(b+;b) + \frac{\alpha}{c} \cdot 1$$

is a weighted average of $V'(b+;b)$ and 1. Therefore $V'(b-;b)$ lies in the closed interval $[V'(b+;b);1]$ or $[1;V'(b+;b)]$. So $V'(b+;b)$ and $V'(b-;b)$ are both less than 1, greater than 1 or both equal to 1. Because of this

$$V'(x;b^*) > 1 \quad \text{for } x < b^* \tag{2.33}$$
$$V'(x;b^*) < 1 \quad \text{for } x > b^* \tag{2.34}$$

becomes

$$\begin{array}{l} V'(b^*-;b^*) > 1 \\ \\ V'(b^*+;b^*) < 1 \end{array} \quad \Rightarrow V'(b^*-;b^*) = V'(b^*+;b^*) = 1. \tag{2.35}$$

Theorem 2.6.1 *For exponential claims a threshold b^* exists. If*

$$(-w)\frac{\alpha}{\delta}\left[1 + \frac{w}{\beta}\right] \le 1, \tag{2.36}$$

$b^* = 0$. *Otherwise the optimal threshold* b^* *is given by*

$$b^* = \frac{1}{r-s} \ln\left(\frac{s^2 - ws}{r^2 - wr}\right). \tag{2.37}$$

Proof. First, consider the case $b^* = 0$, where dividends are always paid at rate α, i.e. $V'(x;0) < 1 \ \forall x > 0$. We will find a condition, which ensures that $b^* = 0$. Then we consider the case $b^* > 0$, calculate b^* and then we prove that b^* really is the optimal threshold we searched for.

- $b^* = 0$
  From (2.27) it follows that

$$V'(x;b) = (-w)\left[\frac{\alpha}{\delta} - V(b;b)\right] e^{w(x-b)} \quad \forall x > b. \tag{2.38}$$

  Using $V'(x;0) < 1 \ \forall x > 0$ yields $(-w)\left[\frac{\alpha}{\delta} - V(0;0)\right] e^{wx} < 1$ and as this must be valid $\forall x > 0$,

$$(-w)\left[\frac{\alpha}{\delta} - V(0;0)\right] \leq 1.$$

  As $V(0;0) = \frac{w}{\beta}\frac{\alpha}{\delta}$, we get

$$(-w)\frac{\alpha}{\delta}\left[1 + \frac{w}{\beta}\right] \leq 1$$

  as condition for $b^* = 0$.

- $b^* > 0$
  If the left hand side is > 1, then $b^* > 0$. For an optimal threshold b^*, $V'(b^*+;b^*) = 1$ (see (2.35)). Together with (2.38) we have

$$V'(b^*+;b^*) = (-w)\left[\frac{\alpha}{\delta} - V(b^*+;b^*)\right] = 1,$$

  i.e. b^* is the solution of $V(b^*+;b^*) = \frac{\alpha}{\delta} + \frac{1}{w}$. We use this equation to find b^*. As V is continuous at $x = b$, $V(b^*-;b) = V(b^*+;b)$ and we can use (2.26) in $V(b^*+;b^*)$ to calculate b^*. Rearranging terms yields

$$b^* = \frac{1}{r-s} \ln\left(\frac{s^2 - ws}{r^2 - wr}\right) \tag{2.39}$$

  Now we prove that the optimal strategy is really a threshold strategy with threshold b^*.

  - $V'(x;b^*) < 1 \ \forall x > b^*$
    Using (2.27) and $V(b^*, b^*) = \frac{\alpha}{\delta} + \frac{1}{w}$ gives $V'(x;b^*) = e^{w(x-b^*)} < 1$ for all $x > b^*$, as w is negative. Further, $V'(b^*, b^*) = 1$.

– $V'(x;b^*) > 1 \; \forall x < b^*$
As $V'(b^*;b^*) = 1$, it is sufficient to show that V' is decreasing, i.e. $V''(x;b^*) < 0 \; \forall x < b^*$. As $V''(x;b)$ is increasing for $0 < x < b$, $V''(x;b^*) < 0$ is equivalent to

$$V''(b^*-;b^*) < 0.$$

Considering (2.11), (2.17), the continuity condition (2.22) and $V'(b^*+;b^*) = 1$, we get

$$cV''(b^*-;b^*)+(\beta c-\lambda-\delta)V'(b^*-;b^*) = (c-\alpha)V''(b^*+;b^*)+\beta c-\lambda-\delta.$$

Furthermore, setting (2.37) into $V'(b^*-;b^*)$ and using (2.26) yields $V'(b^*-;b^*) = 1$. Hence,

$$V''(b^*-;b^*) < 0$$

is equivalent to

$$V''(b^*+;b^*) < 0$$

which is certainly true as $V''(b^*+;b^*) = -w^2 \left[\frac{\alpha}{\delta} - \underbrace{V(b^*;b^*)}_{\frac{\alpha}{\delta}+\frac{1}{w}} \right] = w.$ □

2.7 Numerical example

In the following take $c = 4$, $\beta = 1$, $\lambda = 3.5$, $\delta = 0.02$ and $\alpha = 2$.

Then $V(x;b)$ can be calculated for different values of b by using (2.13), (2.18), (2.26) and (2.27). Figure 2.4 shows $V(x;b)$ as a function of b for four different values of the surplus x. Each of these four functions reaches its maximum at $b = 14.3$. Calculating the optimal threshold b^* by (2.37) yields $b^* \approx 14.3$.

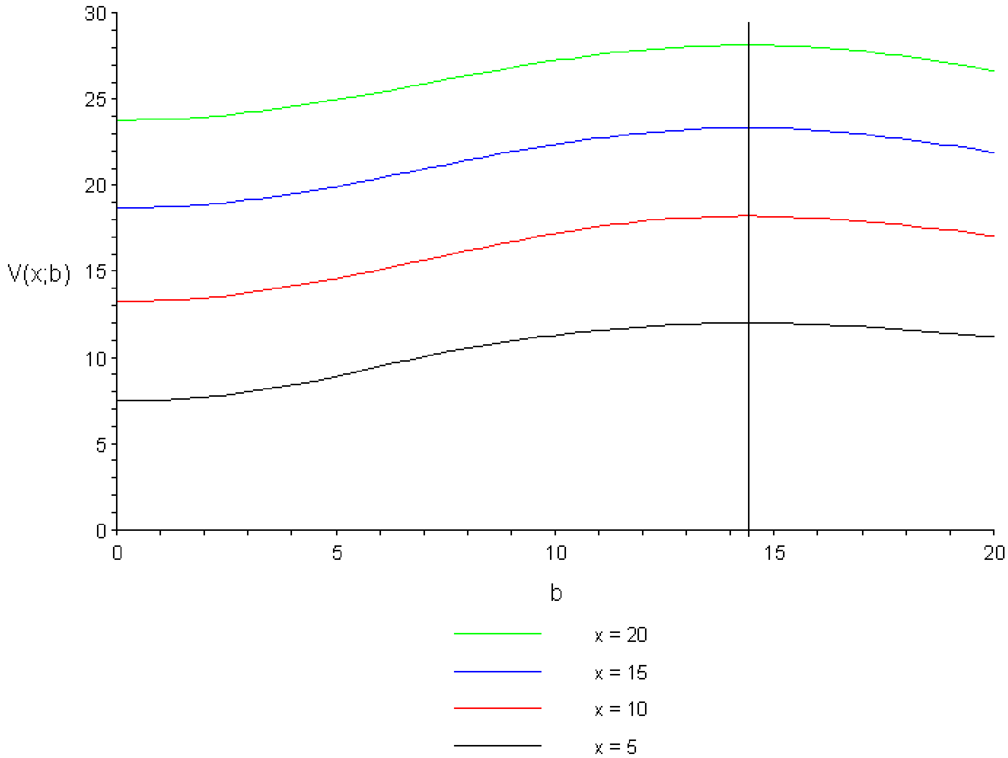

Figure 2.4: $V(x;b)$ as a function of b for different values of x.

Figure 2.5 illustrates $V(x;b)$ as a function of the surplus x for different values of b. It can be seen, that $V(x;b^*) > V(x;b)$ for $b \neq b^*$ and that the demand on the first derivative, see (2.33) and (2.34), need not be fulfilled necessarily for $b \neq b^*$.

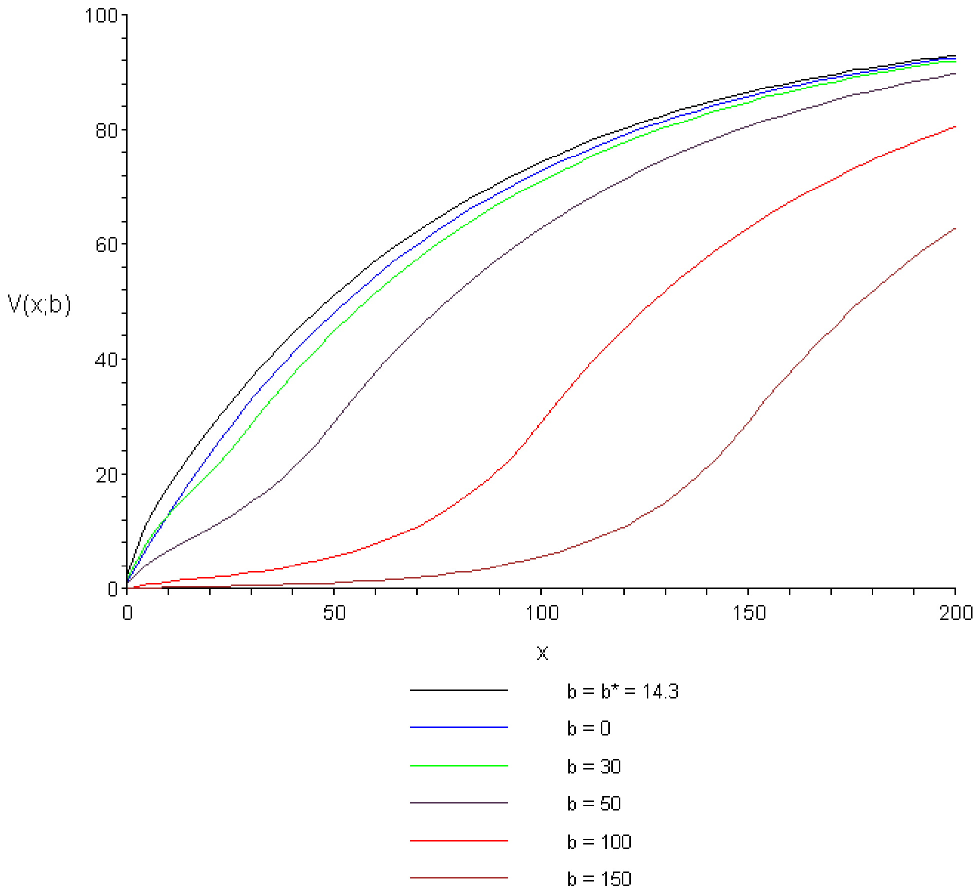

Figure 2.5: $V(x;b)$ as a function of x for different values of b.

Chapter 3

Diffusion model

The considerations in this chapter are based on a paper of Asmussen and Taksar [1] and also on a book written by Schmidli [8][chapter 2.5].

3.1 Problem formulation

In the diffusion model the surplus of an insurance company evolves like a Brownian motion with drift parameter μ and variance parameter σ^2. Additionally the insurance company pays restricted dividends U_s, with $0 \leq U_s \leq \alpha$, where $\alpha < \infty$ is some constant. Therefore the surplus process is given by

$$X_t^U = x + \mu t + \sigma W_t - \int_0^t U_s \, ds,$$

where x is the initial surplus. The corresponding stochastic differential equation is

$$dX_t^U = (\mu - U_t)dt + \sigma dW_t.$$

The filtration $\{\mathfrak{F}_t\}$ is the filtration generated by the Brownian motion. Only dividend rate processes $\{U_t\}$, which are adapted, are allowed. If the surplus becomes negative, the company is ruined and the surplus process is stopped at the ruin time $\tau^U = \inf\{t \geq 0 : X_t^U < 0\}$. The purpose of the following sections is to maximize the expected value of the discounted dividend payments $V^U(x)$:

$$V^U(x) = \mathbb{E}\left[\int_0^{\tau^U} U_s e^{-\delta s} \, ds\right].$$

So we are searching for a dividend strategy U_s^*, such that

$$V(x) = \sup_U V^U(x) = \mathbb{E}\left[\int_0^{\tau^*} U_s^* e^{-\delta s} \, ds\right],$$

where we call $V(x)$ the value function. We have the boundary condition $V(0) = 0$ as, if the initial capital is zero, ruin will happen immediately by the fluctuations of the Brownian motion.

3.2 Motivation of the Hamilton-Jacobi-Bellman equation

The aim of this section is to motivate but not to prove the HJB equation. To find it, we make several assumptions on the value function. Therefore, after solving the found equation in section 3.4, we then have to show, that this solution is really the value function. This will be done in a sort of verification theorem in section 3.5. Now, we start with motivating the Hamilton-Jacobi-Bellman equation.

Let $\epsilon > 0$. Then for each $x > 0$ consider a dividend strategy U^x, which is $\epsilon-$optimal in the sense that

$$V^{U^x}(x) \geq V(x) - \epsilon$$

Fix $0 \leq u \leq \alpha$ and $h > 0$ and consider the dividend strategy

$$U_t = \begin{cases} u & \text{if } 0 \leq t \leq \tau \wedge h, \\ U_{t-h}^{X_h} & \text{if } t > h \text{ and } \tau > h, \end{cases}$$

which means, we pay dividends at rate u until time h or the time of ruin τ, whichever has occurred first, and then we switch to the $\epsilon-$ optimal dividend strategy. We do not worry about measurability here. Using this dividend strategy and splitting the integral brings

$$\begin{aligned} V(x) &\geq V^U(x) \\ &= \mathbb{E}\left[\int_0^{\tau\wedge h} ue^{-\delta s}\,ds\right] + \mathbb{E}\left[1_{\tau>h}\int_h^{\tau} U_{s-h}^{X_h} e^{-\delta(s+h)}\,ds\right] \\ &= u\mathbb{E}\left[\frac{1-e^{-\delta(\tau\wedge h)}}{\delta}\right] + e^{-\delta h}\mathbb{E}\left[1_{\tau>h}\underbrace{V^{U^{X_h}}(X_h)}_{\geq V(X_h)-\epsilon}\right] \\ &\geq u\mathbb{E}\left[\frac{1-e^{-\delta(\tau\wedge h)}}{\delta}\right] + e^{-\delta h}\mathbb{E}\left[1_{\tau>h}V(X_h)\underbrace{-1_{\tau>h}\epsilon}_{\geq-\epsilon}\right]. \end{aligned}$$

Because $X_\tau < 0$, $V(X_\tau) = 0$ and $1_{\tau>h}V(X_h)$ can be replaced by $V(X_{\tau\wedge h})$. It follows that

$$V(x) \geq u\mathbb{E}\left[\frac{1-e^{-\delta(\tau\wedge h)}}{\delta}\right] + e^{-\delta h}\mathbb{E}\left[V(X_{\tau\wedge h})\right] - \epsilon e^{-\delta h}.$$

As ϵ is arbitrary, we find that

$$V(x) \geq u\mathbb{E}\left[\frac{1-e^{-\delta(\tau\wedge h)}}{\delta}\right] + e^{-\delta h}\mathbb{E}\left[V(X_{\tau\wedge h})\right]. \tag{3.1}$$

If we assume that $V(x)$ is twice continuously differentiable, we can use Itô's formula, which can be found for example in [6]. Applying Itô's formula to our notation, we get

$$V(X_{\tau\wedge h}) - V(x) = \int_0^{\tau\wedge h} \sigma V'(X_s)\, dW_s + \int_0^{\tau\wedge h} \left((\mu - u)V'(X_s) + \frac{\sigma^2}{2} V''(X_s) \right) ds. \tag{3.2}$$

Now assume that $\{\int_0^t V'(X_s)\, dW_s\}$ is a martingale. Then the stochastic integral disappears by taking the expectation. Combining (3.1) and (3.2) and dividing everything by h brings

$$\begin{aligned}\frac{1}{h}V(x) \geq \frac{u}{h}\,\mathbb{E}\left[\frac{1 - e^{-\delta(\tau\wedge h)}}{\delta}\right] \\ + e^{-\delta h}\frac{1}{h}\,\mathbb{E}\left[V(x) + \int_0^{\tau\wedge h} ((\mu - u)V'(X_s) + \frac{\sigma^2}{2}V''(X_s))\, ds\right].\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}u\mathbb{E}\left[\frac{1 - e^{-\delta(\tau\wedge h)}}{\delta h}\right] - \frac{1 - e^{-\delta h}}{h}V(x) \\ + e^{-\delta h}\mathbb{E}\left[\frac{1}{h}\int_0^{\tau\wedge h} \left((\mu - u)V'(X_s) + \frac{\sigma^2}{2}V''(X_s)\right) ds\right] \leq 0.\end{aligned}$$

Letting $h \to 0$, assuming that limit and expectation may be interchanged and using l'Hôpital's rule shows

$$\frac{1}{2}\sigma^2 V''(x) + (\mu - u)V'(x) - \delta V(x) + u \leq 0.$$

As this inequality must be true for all $0 \leq u \leq \alpha$,

$$\sup_{0\leq u\leq\alpha} \left\{ \frac{1}{2}\sigma^2 V''(x) + (\mu - u)V'(x) - \delta V(x) + u \right\} \leq 0. \tag{3.3}$$

It can be shown that the inequality is tight for at least one $u \in [0, \alpha]$, see [1][p. 15]. But we will use a different approach. We solve the Hamilton-Jacobi-Bellman equation with the accompanying boundary condition

$$\sup_{0\leq u\leq\alpha} [u(1 - V'(x))] + \frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) = 0 \tag{3.4}$$

$$V(0) = 0 \tag{3.5}$$

and after verifying that the solution found is the value function, we see that the value function satisfies (3.3) with equality.

3.3 Threshold strategies

Suppose that $f(x)$ is a solution of (3.4) and (3.5). Then the optimal dividend strategy u is

$$u = \begin{cases} 0 & \text{if } f'(x) > 1 \\ \alpha & \text{if } f'(x) \leq 1 \end{cases}$$

If there exists a point $b \leq 0$ such that $f'(x) > 1$ for $x < b$ and $f'(x) \leq 1$ for $x \geq b$, we have a strategy called threshold strategy, i.e. dividend payments depend on whether the surplus x is above or under this level b and the dividend rates are either zero or maximal.
Assume that the solution $f(x)$ is concave, which will be shown later.

Proposition 3.3.1 *If the solution $f(x)$ is concave, there exists a threshold strategy $b \geq 0$.*

Proof.

- If $f'(x)$ ever falls below 1, there exists a point $b \in \mathbb{R}$ such that $f'(x) > 1$ for $x < b$ and $f'(x) \leq 1$ for $x \geq b$, i.e. the threshold b exists. If $b < 0$, we set $b = 0$, because the company is ruined if x becomes negative.

- If $f'(x) < 1\ \forall x \in \mathbb{R}$, set $b = 0$ again. Then for all $x \geq 0$ the dividend rate α is paid.

- The case $f'(x) > 1\ \forall x \in \mathbb{R}$ is not possible, because $V(x)$ is bounded by $\frac{\alpha}{\delta}$, as we will see in (3.12).

□

Hence the differential equation can be split into two differential equations, a homogeneous one with no dividend payments ($u = 0$) and an inhomogeneous one with maximal dividend payments ($u = \alpha$). In the following we write $f(x; b)$ instead of $f(x)$ to denote the dependence on the threshold.

$$\frac{1}{2}\sigma^2 f''(x;b) + \mu f'(x;b) - \delta f(x;b) = 0, \qquad 0 \leq x \leq b \tag{3.6}$$

$$\frac{1}{2}\sigma^2 f''(x;b) + (\mu - \alpha) f'(x;b) - \delta f(x;b) + \alpha = 0, \qquad x \geq b. \tag{3.7}$$

As we will see later in (3.13) to (3.15), we can write "$x \leq b$" in (3.6) instead of "$x < b$".

3.4 Solution of the HJB equation

Both differential equations can be solved separately by the exponential ansatz.

$$f(x;b) = C_0 e^{rx} + C_1 e^{sx}, \qquad 0 \le x \le b \tag{3.8}$$

where $r > 0$ and $s < 0$ are the roots of

$$\frac{1}{2}\sigma^2\xi^2 + \mu\xi - \delta = 0. \tag{3.9}$$

The homogenous solution of the second differential equation is

$$f_{hom}(x;b) = D_0 e^{vx} + D_1 e^{wx}, \qquad x \ge b$$

where $v > 0$ and $w < 0$ are the roots of

$$\frac{1}{2}\sigma^2\xi^2 + (\mu - \alpha)\xi - \delta = 0. \tag{3.10}$$

The particular solution is

$$f_{part}(x;b) = \frac{\alpha}{\delta}, \qquad x \ge b$$

and therefore the general solution is

$$f(x;b) = \frac{\alpha}{\delta} + D_0 e^{vx} + D_1 e^{wx}, \qquad x \ge b. \tag{3.11}$$

To get an information about the constants, we first consider the definition of the value function $V(x)$:

$$\begin{aligned} V(x) &= \sup_{0 \le u \le \alpha} \mathbb{E}\left[\int_0^\tau e^{-\delta t} U_t \, dt\right] \\ &\le \mathbb{E}\left[\int_0^\tau e^{-\delta t} \alpha \, dt\right] \\ &\le \int_0^\infty e^{-\delta t} \alpha \, dt \\ &= \frac{\alpha}{\delta}. \end{aligned} \tag{3.12}$$

Hence our candidate $f(x;b)$ for the value function $V(x)$ has to be bounded by $\frac{\alpha}{\delta}$, which gives us an information about D_0 and D_1 in (3.11). As $v > 0$, $D_0 = 0$ and as $w < 0$, $D_1 < 0$. Furthermore the boundary condition $V(0) = 0$ in (3.5) yields $C_0 = -C_1$ and as we aim for a solution $f(x;b)$, which is increasing, and as $r > 0$, we get $C_0 > 0$ in (3.8). To find C_0 and D_1, we use the principle of smooth fit. We do this, because we aim at a solution $f(x;b)$, which is twice continuously differentiable.

$$f(b-;b) = f(b+;b) \tag{3.13}$$
$$f'(b-;b) = 1 \tag{3.14}$$
$$f'(b+;b) = 1 \tag{3.15}$$

Under these conditions the second derivative $f''(x;b)$ is automatically continuous at $x = b$, which can be checked by setting (3.14) in (3.6) and (3.15) in (3.7). Using the information about the constants in these conditions, we get the following system of equations:

$$C_0\left(e^{rb} - e^{sb}\right) = \frac{\alpha}{\delta} + D_1 e^{wb} \tag{3.16}$$
$$C_0\left(re^{rb} - se^{sb}\right) = 1 \tag{3.17}$$
$$D_1 w e^{wb} = 1 \tag{3.18}$$

When solving (3.18) and using it in (3.16), we come across a necessary condition for α and δ:

$$\underbrace{C_0}_{>0}\underbrace{\left(e^{rb} - e^{sb}\right)}_{>0 \text{ for } x>b} = \frac{\alpha}{\delta} + \frac{1}{w} \Rightarrow \frac{\alpha}{\delta} + \frac{1}{w} > 0, \tag{3.19}$$

with $w < 0$ negative root of (3.10). Therefore the solution of the equation system (3.16) to (3.18) can only exist if

$$M := \frac{\alpha}{\delta} + \frac{1}{w} > 0. \tag{3.20}$$

Due to (3.17) and (3.19)

$$C_0\left(e^{rb} - e^{sb}\right) = MC_0\left(re^{rb} - se^{sb}\right),$$

which yields

$$\underbrace{e^{(r-s)b}}_{>0} = \frac{Ms-1}{Mr-1}. \tag{3.21}$$

As the nominator is negative ($M > 0$, $s < 0$), the denominator also has to be negative, i.e., $Mr < 1$ has to be fulfilled. To show that this is true for all α, δ, μ and σ, we make use of the following proposition which is proven easily.

Proposition 3.4.1 $\forall a, b > 0 : \sqrt{a^2 + b} - a < \frac{b}{2a}$

Proof.

$$a^2 + b < (a + \frac{b}{2a})^2$$
$$a^2 + b < a^2 + b + \frac{b^2}{4a^2}$$
$$0 < \frac{b^2}{4a^2}$$

□

Proposition 3.4.2 $Mr < 1$.

Proof.
We do a case differentiation, first consider $\mu \geq \alpha$. As

$$r = \frac{1}{\sigma^2}\left(-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}\right),$$

we can use proposition 3.4.1 to get $r < \frac{\delta}{\mu}$. Therefore

$$Mr < M\frac{\delta}{\mu} = \left(\frac{\alpha}{\delta} + \underbrace{\frac{1}{w}}_{<0}\right)\frac{\delta}{\mu} < \frac{\alpha}{\mu} \leq 1.$$

Now, consider $\mu < \alpha$. $Mr < 1$ is equivalent to $\frac{\alpha}{\delta} < \frac{1}{r} - \frac{1}{w}$ and

$$\begin{aligned} -w &= \frac{1}{\sigma^2}\left((\mu - \alpha) + \sqrt{(\mu - \alpha)^2 + 2\delta\sigma^2}\right) \\ &= \frac{1}{\sigma^2}\left(-(\alpha - \mu) + \sqrt{(\alpha - \mu)^2 + 2\delta\sigma^2}\right) \\ &< \frac{\delta}{\alpha - \mu}, \end{aligned}$$

where we used proposition 3.4.1 again. So

$$\frac{1}{r} - \frac{1}{w} > \frac{1}{r} + \frac{\alpha - \mu}{\delta} > \frac{\mu}{\delta} + \frac{\alpha - \mu}{\delta} = \frac{\alpha}{\delta}.$$

□

As we know now, that the necessary condition $Mr < 1$ holds, we can go on with calculating the constants C_0 and D_1 to get the desired solution $f(x)$. From (3.18),

$$D_1 = \frac{1}{w} \cdot e^{-wb}. \tag{3.22}$$

From (3.17),

$$C_0 = \frac{1}{re^{rb} - se^{sb}}. \tag{3.23}$$

Theorem 3.4.1 *There exists a twice continuously differentiable concave solution to (3.4) and (3.5). If* $M := \frac{\alpha}{\delta} + \frac{1}{w} > 0$*, then*

$$f(x; b) = \begin{cases} \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}} & \text{for } 0 \leq x \leq b \\ \frac{\alpha}{\delta} + \frac{1}{w}e^{w(x-b)} & \text{for } x \geq b. \end{cases}$$

Otherwise,

$$f(x) = \frac{\alpha}{\delta}\left(1 - e^{wx}\right) \qquad \text{for } x \geq 0.$$

Proof. First consider $M := \frac{\alpha}{\delta} + \frac{1}{w} > 0$.

- Concavity: $f(x;b)$ is concave, if $f'(x;b)$ is decreasing, i.e., $f''(x;b) < 0$. On $[b,\infty)$, $f''(x;b) = we^{w(x-b)} < 0$ and therefore $f(x;b)$ is concave on $[b,\infty)$. To show concavity on [0,b], we look at $f'''(x;b) = \frac{r^3e^{rx}-s^3e^{sx}}{re^{rb}-se^{sb}}$, which is ≥ 0, as $r > 0$ and $s < 0$. Because of this $f''(x;b)$ is increasing on $[0,b]$. As discussed above, $f''(b-;b) = f''(b+;b)$ by construction. Therefore $f''(b-;b) < 0$ and since $f''(x;b)$ is increasing, $f''(x;b) < 0$ on $[0,b]$. Overall, $f(x;b)$ is concave on $[0,\infty)$.

- Now we check, that $f(x;b)$ is twice continuously differentiable in x. On $[0,b]$ $f(x;b)$ and its derivatives are continuous, as the denominator is only zero for the trivial case $\mu = \sigma = \delta = 0$. $f(x;b)$ and its derivatives are continuous on $[b,\infty]$, too, as the exponential function is continuous. By construction f, f' and f'' are continuous at $x = b$.

- The function $f(x;b)$ solves the HJB equation: For $0 \leq x \leq b$, $f'(x) \geq 1$ and $r = 0$, and as f solves (3.6) and $f(0;b) = 0$, f solves (3.4) and (3.5). For $x \geq b$, $f'(x) \leq 1$ and $r = \alpha$ and as f solves (3.7), f solves (3.4), too.

Now, consider $M := \frac{\alpha}{\delta} + \frac{1}{w} \leq 0$. First, $f(x) = \frac{\alpha}{\delta}(1 - e^{wx})$ is concave, because $f''(x;b) < 0$. Furthermore, $f'(0) \leq 1$ and therefore $f'(x) \leq 1$ $\forall x \geq 0$ and $(\alpha - u)(f'(x) - 1) \leq 0$ for $0 \leq u \leq \alpha$. As additionally $f(x)$ solves (3.7) and $f(0) = 0$, f solves (3.4) and (3.5). □

Remark. There can be found a second form for the solution $f(x;b)$ for $0 \leq x \leq b$. Therefore (3.16) is multiplied by w and subtracted from (3.17) to get

$$-w\frac{\alpha}{\delta}\frac{1}{e^{rb}(w-r) - e^{sb}(w-s)} = \frac{1}{re^{rb} - se^{sb}}. \tag{3.24}$$

This yields

$$f(x;b) = -w\frac{\alpha}{\delta}\frac{e^{rx} - e^{sx}}{e^{rb}(r-w) + e^{sb}(w-s)}, \qquad 0 \leq x \leq b \tag{3.25}$$

3.5 Verification of the solution

In the previous section we found a function f, which solves the Hamilton-Jacobi-Bellman equation and the corresponding boundary condition. The dividend strategy was given by $U_t = \alpha \cdot 1_{\{X_t \geq b\}}$. In this section it is our aim to show that the function f we found maximizes the expected value of the discounted dividend payments and hence the given dividend strategy is the optimal one. We show this in two parts.

Corollary 3.5.1 *The function f in theorem 3.4.1 majorizes $V(x)$.*

Proof. To simplify the notation the subscription with the threshold b is omitted in this proof. Using Itô's formula, see for example [6], for $f(X_t^U, t) = e^{-\delta t} f(X_t^U)$, we get

$$e^{-\delta(t\wedge\tau)} f(X_{t\wedge\tau}^U) - f(x) = \int_0^{t\wedge\tau} \left(\frac{1}{2}\sigma^2 f''(X_s^U) + (\mu - U_s) f'(X_s^U) - \delta f(X_s^U)\right) e^{-\delta s}\, ds + \int_0^{t\wedge\tau} e^{-\delta s} f'(X_s^U)\sigma\, dW_s \quad (3.26)$$

As f is concave, $f'(x)$ is bounded by $f'(0)$. Hence the second integral on the right hand side of the equation above is a square integrable martingale with mean zero, see for example [6][p. 97]. Using that f fulfills the HJB equation (3.4) and taking the expectation yields

$$\mathbb{E}\left[e^{-\delta(t\wedge\tau)} f(X_{t\wedge\tau}^U)\right] + \mathbb{E}\left[\int_0^{t\wedge\tau} U_s e^{-\delta s}\, ds\right] \leq f(x) \quad (3.27)$$

As $f \geq 0$, the first expectation on the left hand side is also $\geq$ and

$$f(x) \geq \mathbb{E}\left[\int_0^{t\wedge\tau} U_s e^{-\delta s}\, ds\right]. \quad (3.28)$$

Letting $t \to \infty$ yields

$$f(x) \geq \mathbb{E}\left[\int_0^{\tau} U_s e^{-\delta s}\, ds\right], \quad (3.29)$$

as, by dominated convergence, the limit and expectation may be interchanged. As this inequality is valid for every admissible control $\{U_s\}$,

$$f(x) \geq \sup_U \mathbb{E}\left[\int_0^{\tau} U_s e^{-\delta s}\, ds\right] = V(x). \quad (3.30)$$

□

Corollary 3.5.2 *The function $f(x)$ is identical with $V(x)$ and an optimal dividend strategy U^* is given by $U_t^* = \alpha \cdot 1_{\{X_t^* \geq b\}}$.*

Proof. In the case $U = U^*$ equality holds in (3.27). If $\tau^* \leq t$, then $f(X^*_{t\wedge\tau^*}) = 0$. By the boundedness of $f(x)$ and dominated convergence,

$$\lim_{t\to\infty} \mathbb{E}\big[e^{-\delta(t\wedge\tau^*)} f(X^*_{t\wedge\tau})\big] = 0$$

and

$$f(x) = \mathbb{E}\left[\int_0^{\tau^*} U^*_s e^{-\delta s}\, ds\right] = V(x).$$

□

Consequently we have not only found a solution to the HJB equation, but also proved, that this solution we found is the value function and furthermore we know the optimal dividend strategy, which maximizes the expected discounted value of the dividend payments until ruin.

As we have already shown in proposition 3.3.1 and section 3.4, that this optimal strategy is a threshold strategy , it just remains to calculate the optimal threshold b^*. This is done in the next section.

3.6 Optimal threshold strategies

For $M = \frac{\alpha}{\delta} + \frac{1}{w} > 0$, the optimal threshold b^* can be obtained from equation (3.21) and we get

$$b^* = \frac{1}{r-s} \ln \frac{1-Ms}{1-Mr}. \tag{3.31}$$

If $M \leq 0$, the optimal threshold is

$$b^* = 0.$$

This can be seen by considering the solution for $M \leq 0$ in theorem 3.4.1, computing the first derivative and using $\frac{\alpha}{\delta} + \frac{1}{w} \leq 0$ and $w < 0$ to see that

$$f'(x) = \frac{\alpha}{\delta}(-we^{wx}) \leq -\frac{1}{w}(-we^{wx}) = e^{wx} < 1 \qquad \forall x > 0.$$

3.7 Numerical example

In the following take $\mu = 1$, $\delta = 0.02$ and $\alpha = 0.8$.

Considering different values for the variance parameter $\sigma > 0$ the optimal threshold b^* can be calculated using (3.31). Therefore r, s and M can be calculated using (3.9), (3.10) and (3.20). The results of table 3.1 are illustrated in figure 3.1.

Table 3.1: Optimal threshold for different values of σ.

σ	0.1	0.2	0.3	0.4	0.5	0.8	1.0	2.0	5.0	8.0
b^*	0.05	0.18	0.38	0.62	0.91	1.98	2.82	7.38	12.81	4.39

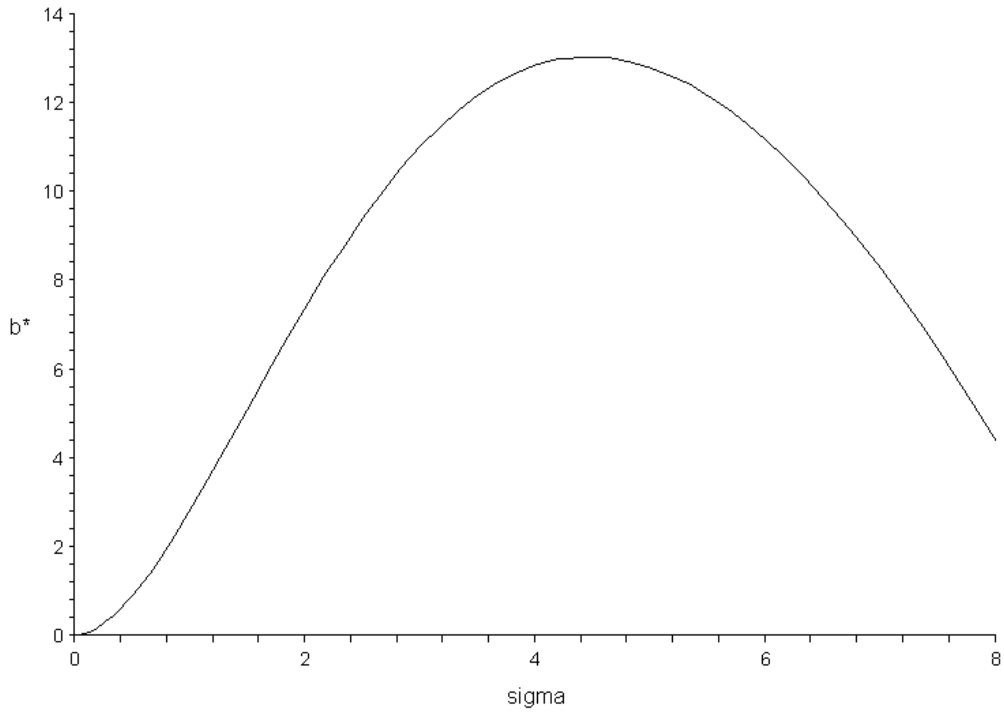

Figure 3.1: Optimal threshold b^* for different values of σ.

Figure 3.2 shows the value function $V(x; b^*)$ for different values of σ.

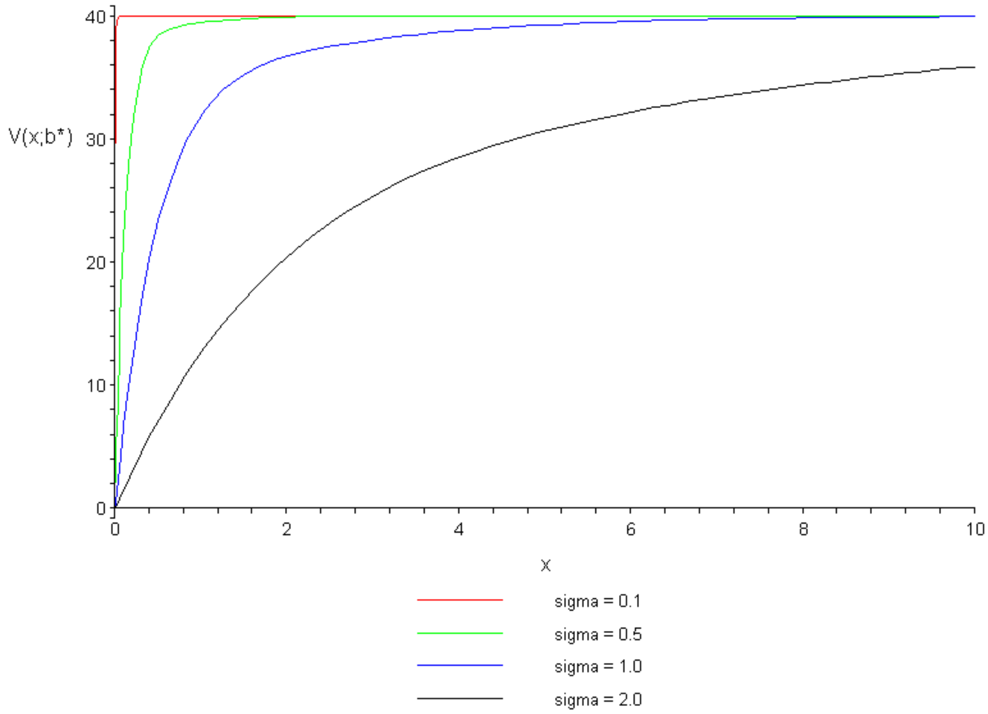

Figure 3.2: Value function $V(x; b^*)$ for different values of σ.

3.8 How the surplus evolves under the optimal strategy

The following analysis is based on [8][p. 102] and estimates the probability of ruin. We investigate two cases, which differ in the size of the drift parameter μ.

case 1: $\mu > \alpha$

For $\mu > \alpha$, $\mu - \alpha$ is positive and the surplus process is described by a Brownian motion with positive drift. Consider the strategy $U_t = \alpha\ \forall t$, which always pays the maximal dividend rate to the shareholders, independent of the surplus process. The surplus, which is reached with this strategy, is at all times less than the surplus which is reached with the optimal dividend strategy. The probability of ruin ψ, in which we are interested, is well known for a Brownian motion with positive drift μ and volatility σ and can be found for instance in [8] [p. 206]:

$$\begin{aligned} \psi(x) &= \mathbb{P}[\tau < \infty] = e^{\frac{-2\mu x}{\sigma^2}} \\ \Rightarrow \psi^\alpha(x) &= e^{\frac{-2(\mu-\alpha)x}{\sigma^2}} \\ \Rightarrow \psi^*(x) &= e^{\frac{-2(\mu-u^*)x}{\sigma^2}} \leq \psi^\alpha(x) < 1 \ \forall x > 0. \end{aligned}$$

As we have seen now, the ruin probability is less than 1 and therefore it is possible, that the company is never ruined.

case 2: $\mu \leq \alpha$

Suppose, that the company starts with the initial capital $x = b + 1$. Then the surplus reaches the level b almost surely as a Brownian motion with negative drift diverges to $-\infty$ for $t \to \infty$. Hence, $\liminf_{t\to\infty} X_t^U \leq b$. We define

$$\begin{aligned} T_0 &:= \inf\left\{t : X_t^U \leq b+1\right\} \\ T_n &:= \inf\left\{t \geq T_{n-1}+1 : X_t^U \leq b+1\right\} \\ A_n &:= \left\{\inf\left\{\mu t + \sigma(W_{T_n+t} - W_{T_n}) : 0 \leq t \leq 1\right\} < -b-1\right\} \end{aligned}$$

A_n describes the event, that the surplus, which starts with a value less than $b+1$, falls more than $b+1$ within one time unit or even faster. If A_n happens, ruin occurs. The probability for the event A_n is independent of n, as

$$\mu t + \sigma(W_{T_n+t} - W_{T_n}) \sim N(\mu t, \sigma^2 t)$$

and μt and $\sigma^2 t$ are independent of T_n.

$$\Rightarrow \mathbb{P}[A_n] = \gamma > 0, \ \ \{A_n\} \text{ independent}$$

As the events $\{A_n\}$ are independent and $\sum_{n\geq 1} \mathbb{P}[A_n] = \sum_{n\geq 1} \gamma = \infty$, the Borel-Cantelli lemma can be used, which says, that

$$\mathbb{P}\left[\limsup_{n\to\infty} A_n\right] = 1.$$

This means that the event A_n happens infinitely often with probability 1 . Hence, ruin occurs almost surely under the optimal strategy, if $\mu \leq \alpha$. Consequently, sooner or later the surplus becomes negative.

With this knowledge we can ask whether both cases are even possible for $M > 0$ and $M \leq 0$ or whether they are mutually exclusive. In section 3.4 M has been defined as $M := \frac{1}{w} + \frac{\alpha}{\delta}$, whereas $w < 0$ and $\frac{\alpha}{\delta} > 0$.

- $M \leq 0$: $M \leq 0$ led to $b^* = 0$. In this case the maximal dividend rate is paid all the time. We are interested in whether $M \leq 0$ makes one of the two cases ($\mu > \alpha$ or $\mu \leq \alpha$) impossible. To check this, we use (3.10) to get

$$u = -\frac{\mu - \alpha}{\sigma^2} - \frac{\sqrt{(\mu - \alpha)^2 + 2c\sigma^2}}{\sigma^2}$$

and obtain

$$M \leq 0 \Leftrightarrow \frac{\sigma^2}{-(\mu - \alpha) - \sqrt{(\mu - \alpha)^2 + 2\delta\sigma^2}} + \frac{\alpha}{\delta} \leq 0.$$

Trying to find numerical values for σ, μ, α and δ, so that both $M \leq 0$ and $\mu > \alpha$, respectively $\mu \leq \alpha$, are fulfilled, is successful. For both cases numerical values can be found, even though they are not really practical as can be seen in the table below.

- $M > 0$: Also $M > 0$ does not a priori exclude one of the two cases for the drift parameter μ, which is stated in following table. The case $M > 0$ seems to be more relevant, because here the numerical values we found lie in an interval, which is imaginable for practical application. Besides, in the case $M \leq 0$, the optimal threshold is zero and therefore this case is trivial.

Table 3.2: Examples for parameters.

	$M \leq 0$		$M > 0$	
	$\mu > \alpha$	$\mu \leq \alpha$	$\mu > \alpha$	$\mu \leq \alpha$
σ	1	2	0.1	0.1
μ	2	1	2	1
α	0.1	1.05	1	2
δ	0.45	1.0	0.05	0.05

Chapter 4

The dual model

The dual model was introduced by [2], where a barrier strategy was considered.

4.1 Problem formulation

In the dual model the surplus of a company is given by

$$X_t = x - ct + S_t, \tag{4.1}$$

where x is again the initial surplus. In this model, c does not represent premiums, which are paid continuously to the company, but deterministic and fixed expenses of the company. The process

$$S_t = \sum_{i=1}^{N_t} Y_i$$

is, as in the compound Poisson model, a compound Poisson process with parameter λ and probability density function

$$p(y) = \beta e^{-\beta y}, \qquad \beta > 0, y > 0.$$

This compound Poisson process does not model claims, but positive gains. Whereas the compound Poisson model described above seems to be natural for an insurance company, the dual model seems to be more convenient for a company, which has occasional gains and fixed costs. Until the company is ruined, dividends are paid and the surplus including the dividend payments is of the form

$$X_t^U = x - ct + S_t - \int_0^t U_s \, ds, \tag{4.2}$$

where $\int_0^t U_s \, ds$ denotes the aggregate dividends paid between time 0 and t. We assume that dividends are paid according to a ceiling, i.e. the dividend payments are restricted by a constant α:

$$0 \leq U_s \leq \alpha < \infty.$$

Again, we aim for a dividend strategy, which maximizes the discounted dividend payments until ruin and the corresponding value function

$$V(x) = \sup_U \mathbb{E}\left[\int_0^{\tau^U} e^{-\delta t} U_t \, dt\right] = \mathbb{E}\left[\int_0^{\tau^*} e^{-\delta t} U_t^* \, dt\right]. \tag{4.3}$$

4.2 Motivation of the Hamilton-Jacobi-Bellman equation

To find the HJB equation for the dual model we proceed as in the compound Poisson model and use Itôs formula

$$\begin{aligned} f(t, X_t^U) - f(0,x) &= \int_0^t \left[\frac{\partial f}{\partial s}(s, X_{s-}^U) - (c + U_s)\frac{\partial f}{\partial x}(s, X_{s-}^U)\right] ds \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left[f(s, X_{s-}^U + y) - f(s, X_{s-}^U)\right] J_X(\, ds\, dy), \end{aligned}$$

see, for example, Cont & Tankov's proposition 8.13 in [4]. J_X is a random measure associated to the jumps and f is an arbitrary measurable function. We pay no attention to this random measure, as we do not need it to find the HJB equation. Define

$$f(t, X_t^U) := e^{-\delta t} V(X_t^U)$$

and

$$Z_t := f(t, X_t^U) + \int_0^t e^{-\delta s} U_s \, ds - f(0, X_0). \tag{4.4}$$

Equating the drift in Z_t to zero and using that $p(y) = 0$ for $y \le 0$ gives

$$-\delta V(X_t^U) - (c + U_t)V'(X_t^U) + \lambda \int_0^\infty V(X_t^U + y)p(y)\, dy - \lambda V(X_t^U) + U_t = 0.$$

Finally, take the supremum over all admissible dividend strategies to get the Hamilton-Jacobi-Bellman equation,

$$\max_{0 \le u \le \alpha} \{u(1 - V'(x))\} - V(x)(\lambda + \delta) - cV'(x) + \int_0^\infty V(x+y)\lambda p(y)\, dy = 0. \tag{4.5}$$

This HJB equation of the dual model looks a little bit different to the HJB equation in the compound Poisson model. Anyhow, the solution of this equation can be done similar to the compound Poisson model.

$u(1 - V'(x))$ is maximized, if

$$u = \alpha \text{ for } V'(x) < 1, \qquad u = 0 \text{ for } V'(x) > 1. \tag{4.6}$$

If $V'(x) = 1$, any value $\in [0, \alpha]$ is possible for u.

4.3 Threshold strategies

If there is a point b, for which

$$V'(x) > 1 \quad \text{for} \ \ x < b \tag{4.7}$$

and

$$V'(x) < 1 \quad \text{for} \ \ x > b, \tag{4.8}$$

we have a threshold strategy as in the compound Poisson model. As we do not know yet, whether such a point b exists, we define our dividend strategy u in the following way. We choose an arbitrary point b and define

$$u = \begin{cases} 0 & \text{if } x < b \\ \alpha & \text{if } x > b. \end{cases}$$

This threshold strategy means that no dividends are paid as long as the surplus is less than the threshold b. As soon as the surplus exceeds the threshold dividends are paid at the maximal rate α. So there are just two cases, either paying no dividends or paying the maximal dividend rate.

Of course, such a threshold strategy has an influence on the development of the surplus. How this threshold strategy affects the surplus, is shown in figure 4.1 and figure 4.2 for one sample path. Figure 4.1 demonstrates the temporal fluctuation of the surplus X_t due to the stochastic gains and the constant and deterministic costs without dividend payments. The figure beneath describes the effect of the threshold strategy on the surplus X_t^U with dividend payments.

In order to show that it makes a big difference, whether the threshold b is high or low, figure 4.3 illustrates the same sample path as figure 4.2, but with a lower threshold.

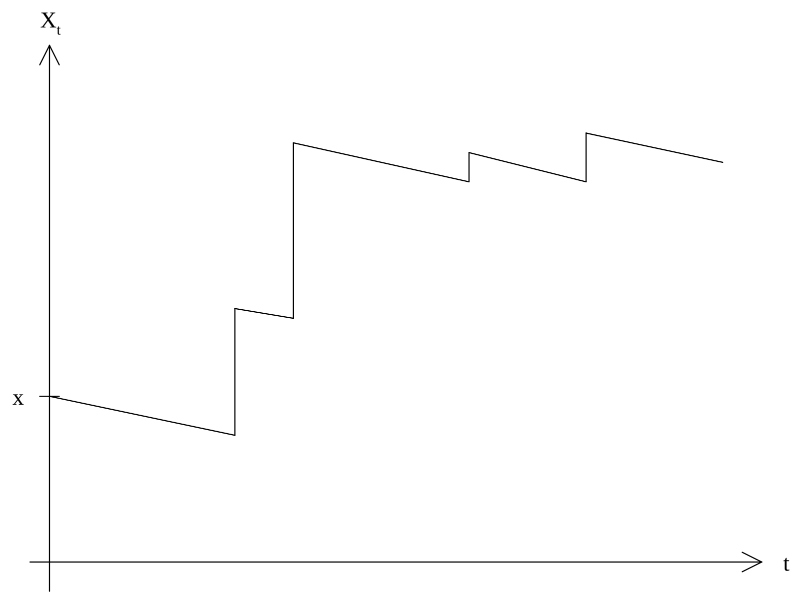

Figure 4.1: Sample path for the surplus without dividend payments.

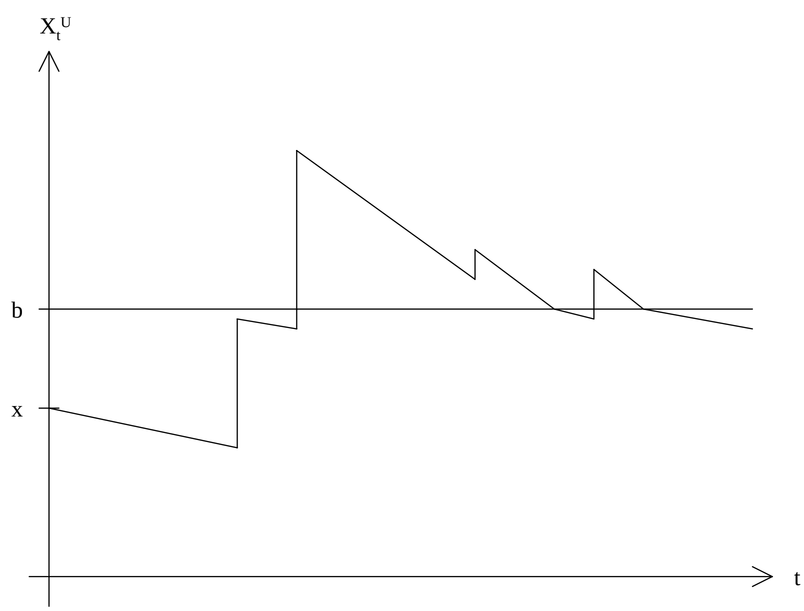

Figure 4.2: Sample path for the surplus with threshold $b > x$.

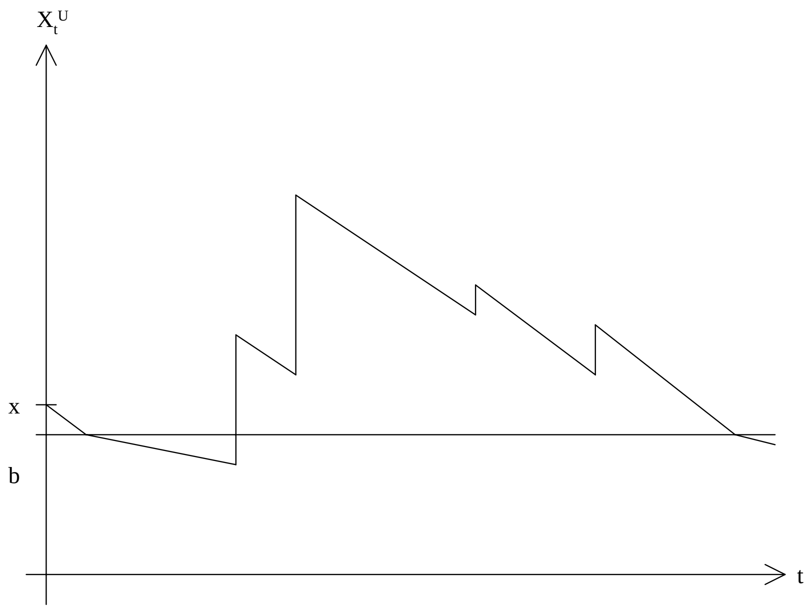

Figure 4.3: Sample path for the surplus with threshold $b < x$.

Now, let $V(x;b)$ denote the expectation of the present value of all dividends until ruin, where x is the initial capital and b is the threshold. Then $V(x;b)$ satisfies the following integro-differential equation:

$$-V(x;b)(\lambda+\delta) - cV'(x;b) + \int_0^\infty V(x+y;b)\lambda p(y)\,dy \qquad 0 < x < b \tag{4.9}$$

$$\alpha - V(x;b)(\lambda+\delta) - (c+\alpha)V'(x;b) + \int_0^\infty V(x+y;b)\lambda p(y)\,dy \qquad x > b. \tag{4.10}$$

4.4 Solution of the HJB equation

First, consider $x < b$. Then there are no dividend payments, which means that $u = 0$.

- $u = 0$

To find a solution we differentiate the integro-differential equation (4.9) with respect to x and find that

$$\frac{d}{dx}\left(\int_0^\infty V(x+y;b)p(y)dy\right) = \beta\left(-V(x;b) + \int_0^\infty V(x+y;b)p(y)dy\right).$$

This yields the second order homogenous linear differential equation

$$V''(x;b)c + V'(x;b)(\lambda+\delta-\beta c) - V(x;b)\beta\delta = 0. \tag{4.11}$$

Again, as in the compound Poisson model, this differential equation with constant coefficients can be solved by an exponential ansatz to find that

$$V(x;b) = C_0 e^{rx} + C_1 e^{sx},$$

where $r > 0$ and $s < 0$ are the roots of

$$c\xi^2 + \xi(\lambda + \delta - \beta c) - \beta\delta = 0. \tag{4.12}$$

As $r > 0$ and $s < 0$,

$$e^{rx} > e^{sx}.$$

Because the value function (4.3) is nonnegative, the constant C_0 has to be positive. To get more information about the constants we consider the value function for the case $x = 0$. Then the initial capital is zero. Considering (4.2), we see that ruin will occur immediately if $x = 0$, as $N_0 = 0$. This yields

$$V(0;b) = C_0 + C_1 = 0 \Rightarrow C_1 = -C_0.$$

The left-hand side of (4.12) is positive for $\xi = \beta$ and negative for $\xi = 0$. As the premium rate c is positive, the positive solution of the quadratic equation lies between 0 and β and

$$\beta - r > 0. \tag{4.13}$$

Hence,

$$V(x;b) = C_0 \left(e^{rx} - e^{sx}\right), \qquad 0 < x < b, \tag{4.14}$$

where $C_0 > 0$, $r > 0$ and $s < 0$. Before the constant C_0 can be determined, the case $u = \alpha$ has to be considered to find the necessary condition.

- $u = \alpha$

Differentiating the integro-differential equation (4.10) yields the second order inhomogeneous linear differential equation

$$(c+\alpha)V''(x;b) + V'(x;b)\left[\lambda + \delta - \beta(c+\alpha)\right] - V(x;b)\beta\delta + \alpha\beta = 0. \tag{4.15}$$

The solution

$$V(x;b) = \frac{\alpha}{\delta} + D_0 e^{vx} + D_1 e^{wx}$$

is again found by an exponential ansatz. The constants $v > 0$ and $w < 0$ are the roots of the quadratic equation

$$(c+\alpha)\xi^2 + \left[\lambda + \delta - \beta(c+\alpha)\right]\xi - \beta\delta = 0. \tag{4.16}$$

Analogous to the compound Poisson model, $\lim_{x\to\infty} V(x) = \frac{\alpha}{\delta}$ and $V(x) \le \frac{\alpha}{\delta}$ can be verified. Because of this, $D_0 = 0$ and $D_1 < 0$ have to be fulfilled. Therefore the solution of the inhomogeneous differential equation is

$$V(x) = \frac{\alpha}{\delta} + D_1 e^{wx}, \qquad x > b \tag{4.17}$$

with $D_1 < 0$ and $w < 0$. To determine C_0 and D_1, we use on the one hand the continuity condition and on the other hand we split the integral in (4.10) into two parts. The continuity condition

$$V(b-;b) = V(b+;b)$$

gives

$$C_0\left(e^{rb} - e^{sb}\right) = \frac{\alpha}{\delta} + D_1 e^{wb}. \tag{4.18}$$

Splitting the integral and using (4.14) and (4.17) leads to

$$\begin{aligned}
\int_0^\infty V(x+y)p(y)\,dy &= \int_x^\infty V(y)p(y-x)\,dy \\
&= \int_x^b V(y)p(y-x)\,dy + \int_b^\infty V(y)p(y-x)\,dy \\
&= e^{\beta x}e^{-\beta b}\left[C_0\left(\frac{\beta}{r-\beta}e^{br} - \frac{\beta}{s-\beta}e^{bs}\right) + \frac{\alpha}{\delta} - D_1\frac{\beta}{w-\beta}e^{bw}\right] \\
&\quad -C_0\left(\frac{\beta}{r-\beta}e^{xr} - \frac{\beta}{s-\beta}e^{xs}\right)
\end{aligned}$$

As all other terms in (4.10) do not involve the exponential function $e^{\beta x}$, the coefficient of $e^{\beta x}$ is set to zero and it follows that

$$C_0\left(\frac{\beta}{r-\beta}e^{br} - \frac{\beta}{s-\beta}e^{bs}\right) = -\frac{\alpha}{\delta} + D_1\frac{\beta}{w-\beta}e^{bw} \tag{4.19}$$

Solving this system of equations ((4.18),(4.19)) yields

$$C_0 = \frac{\alpha}{\delta}\frac{w}{\beta}\frac{1}{e^{rb}\frac{r-w}{r-\beta} - e^{sb}\frac{s-w}{s-\beta}} \tag{4.20}$$

and

$$\begin{aligned}
D_1 &= e^{-wb}\left[\frac{\alpha}{\delta}\frac{w}{\beta}\frac{e^{rb}-e^{sb}}{e^{rb}\frac{r-w}{r-\beta} - e^{sb}\frac{s-w}{s-\beta}} - \frac{\alpha}{\delta}\right] \\
&= e^{-wb}\left[V(b;b) - \frac{\alpha}{\delta}\right].
\end{aligned} \tag{4.21}$$

Putting (4.20) in (4.14) and (4.21) in (4.17) gives

$$V(x;b) = \frac{\alpha}{\delta}\frac{w}{\beta}\frac{e^{rx}-e^{sx}}{e^{rb}\frac{r-w}{r-\beta} - e^{sb}\frac{s-w}{s-\beta}} \qquad 0 < x \le b \tag{4.22}$$

and

$$V(x;b) = \frac{\alpha}{\delta}\left(1 - e^{w(x-b)}\right) + e^{w(x-b)}V(b;b) \qquad x \ge b. \tag{4.23}$$

4.5 Verification of the solution

Theorem 4.5.1 *Suppose that $f(x)$ is an increasing, bounded and positive solution to (4.5) and $f(x) = 0$ for $x < 0$. Then $f(x) = V(x)$ and the optimal dividend strategy is given by $U_t^* = \alpha \cdot 1_{V'(X_t^*)<1}$.*

As in the compound Poisson model, the theorem is proven in two steps.

- $f(x) \geq V(x)$

Let U be an arbitrary dividend strategy. To verify that the solution f is the desired value function, we consider the process M with

$$\begin{aligned} M_t \;=\; & \sum_{i=1}^{N_{\tau^U \wedge t}} \left(f(X_{T_i}^U) - f(X_{T_i-}^U) \right) e^{-\delta T_i} \\ & -\lambda \int_0^{\tau^U \wedge t} e^{-\delta s} \left(\int_0^\infty f(X_s^U + y)\, dG(y) - f(X_s^U) \right] ds. \end{aligned} \tag{4.24}$$

This process is a martingale, which can be proven analogue to the compound Poisson model by Brémaud's integration theorem ([3], p.235) with the difference that we do have upward jumps instead of downward jumps. Applying the fundamental theorem of calculus for

$$g(t, X_t^U) := e^{-\delta t} f(X_t^U)$$

gives

$$f(X_{T_i-}^U) e^{-\delta T_i} - f(X_{T_{i-1}}^U) e^{-\delta T_{i-1}} = \int_{T_{i-1}}^{T_i-} \left[(-c - U_s) f'(X_s^U) - \delta f(X_s^U) \right] e^{-\delta s}\, ds. \tag{4.25}$$

Then,

$$\begin{aligned} Z_t \;:=\; & f(X_{\tau^U \wedge t}^U) e^{-\delta(\tau^U \wedge t)} - \\ & \int_0^{\tau^U \wedge t} \left[(-c - U_s) f'(X_s^U) + \lambda \int_0^\infty f(X_s^U + y)\, dG(y) - (\lambda + \delta) f(X_s^U) \right] e^{-\delta s}\, ds \end{aligned}$$

is a martingale, too. This can easily be seen by using (4.24) and (4.25) to find that

$$Z_t = M_t + f(x) - \sum_{i=1}^{N_{\tau^U \wedge t}} \underbrace{\int_{T_i-}^{T_i} \left[(-c - U_s) f'(X_s^U) - f(X_s^U)\delta \right] e^{-\delta s}\, ds}_{=0}.$$

As $f(x)$ is deterministic, Z_t is a martingale and $\mathbb{E}[Z_t] = f(x)$. According to the assumption f solves the HJB equation (4.5) and because U is an arbitrary dividend strategy, we have

$$(-c - U_s)f'(X_s^U) + \lambda \int_0^\infty f(X_s^U + y)\, dG(y) - (\lambda + \delta) f(X_s^U) \leq -U_s.$$

Consequently, we get a lower bound for Z_t,

$$Z_t \geq f(X_{\tau^U \wedge t}^U) e^{-\delta(\tau^U \wedge t)} + \int_0^{\tau^U \wedge t} U_s e^{-\delta s}\, ds \tag{4.26}$$

and after taking the expectation and using that f is positive, we get a lower bound for our solution $f(x)$,

$$f(x) \geq \mathbb{E}\left[\int_0^{\tau^U \wedge t} U_s e^{-\delta s}\, ds\right].$$

As for $t \to \infty$ the limes and expectation may be interchanged by dominated convergence,

$$f(x) \geq V^U(x),$$

analogue to the compound Poisson model. Since the reasoning is valid for all dividend strategies, it is also valid for the optimal dividend strategy U^* and therefore $f(x)$ majorizes the value function

$$f(x) \geq \mathbb{E}\left[\int_0^{\tau^*} U^* e^{-\delta s}\, ds\right] = V(x).$$

Now we are going to prove that the solution f does not only majorize the value function, but it is the value function.

- $f(x) = V(x)$

Therefore consider the optimal dividend strategy U^* (4.6), which we used in section 4.4 to find the solution f. Then, we have equality in (4.26) and taking the expectation gives

$$f(x) = \mathbb{E}\big[f(X_{\tau^* \wedge t}^*) e^{-\delta(\tau^* \wedge t)}\big] + \mathbb{E}\left[\int_0^{\tau^* \wedge t} U_s^* e^{-\delta s}\, ds\right].$$

Letting $t \to \infty$, we are allowed to interchange the expectation and the limes, as the conditions for dominated convergence are fulfilled. Because being ruined means that the surplus is negative and as by assumption $f(x) = 0$ for $x < 0$, the term above can be simplified to

$$\begin{aligned} f(x) &= \mathbb{E}\left[\lim_{t\to\infty}\left(1_{\tau^*<\infty}\cdot 0\cdot e^{-\delta\tau^*} + 1_{\tau^*=\infty}\underbrace{f(X_t)}_{\le\frac{\alpha}{\delta}}e^{-\delta t}\right)\right] + \mathbb{E}\left[\int_0^{\tau^*} U_s^* e^{-\delta s}\,ds\right] \\ &= \mathbb{E}\left[\int_0^{\tau^*} U_s^* e^{-\delta s}\,ds\right] \\ &= V(x). \end{aligned}$$

□

In section 4.4 a solution of the HJB was constructed. Showing that this solution given by (4.22) and (4.23) satisfies the conditions of theorem 4.5.1 yields that the function $V(x;b)$ we have found is really the desired value function.

- $V(x;b)$ increasing

  To show that $V(x;b)$ is increasing, we argue in almost the same manner as in the compound Poisson model. Additionally, the factor

$$\frac{\alpha}{\delta}\frac{w}{\beta}\frac{1}{e^{rb}\frac{r-w}{r-\beta} - e^{sb}\frac{s-w}{s-\beta}}$$

  is positive by construction.

- $V(x;b)$ bounded

  This condition can be shown just like in the compound Poisson model.

- $V(x;b)$ positive

  This is proven in the same way is in the compound Poisson model.

- $V(x;b) = 0$ for $x < 0$

  Fulfilled by construction.

4.6 Optimal threshold strategies

Up to now we have supposed, that there is a threshold, which fulfills (4.7) and (4.8). As in the compound Poisson model we are going to show that such a b really exists by calculating it. If we find such a point, we have found the optimal threshold b^* as the chosen strategy is optimal. Therefore we search for a useful condition, which will allow us to find b^*. For $x = b^*-$ and $x = b^*+$ the integro-differential equations (4.9) and (4.10) become

$$\begin{aligned} -V(b^*-)(\lambda+\delta) - cV'(b^*-) + \int_0^\infty V(b^* - +y)\lambda p(y)\,dy &= 0 \\ \alpha - (c+\alpha)V'(b^*+) - V(b^*+)(\lambda+\delta) + \int_0^\infty V(b^* + +y)\lambda p(y)\,dy &= 0. \end{aligned}$$

Using the continuity condition $V(b^*-;b^*) = V(b^*+;b^*)$ and subtracting the two equations gives

$$V'(b^*+) = \frac{c}{c+\alpha}V'(b^*-) + \left(1 - \frac{c}{c+\alpha}\right)\cdot 1,$$

which means that $V'(b^*+;b^*)$ is a weighted average of $V'(b^*-;b^*)$ and 1. Hence, either $V'(b^*+;b^*)$ and $V'(b^*-;b^*)$ are both greater than 1, less than 1 or both equal 1. Thus, if there is a b^*, which fulfills (4.7) and (4.8), both derivatives have to be 1,

$$V'(b^*-;b^*) = V'(b^*+;b^*) = 1.$$

This condition is necessary, but we do not know whether it is sufficient to show that b^* is the optimal threshold. However, with this condition we will find a candidate for the threshold and afterwards we will prove that this candidate really fulfills (4.7) and (4.8).

Theorem 4.6.1 *For exponential claims a threshold b^* exists. If*

$$-\frac{\alpha}{\delta}w \le 1, \tag{4.27}$$

$b^* = 0$. *Otherwise the optimal threshold b^* is given by*

$$b^* = \frac{1}{r-s}\ln\left(\frac{\left(\frac{1}{w}+\frac{\alpha}{\delta}\right)\frac{s-w}{s-\beta} - \frac{\alpha}{\delta}\frac{w}{\beta}}{\left(\frac{1}{w}+\frac{\alpha}{\delta}\right)\frac{r-w}{r-\beta} - \frac{\alpha}{\delta}\frac{w}{\beta}}\right). \tag{4.28}$$

Proof. First, consider the case $b^* = 0$, where dividends are always paid at rate α, i.e. $V'(x;0) < 1\ \forall x > 0$. We will find a condition, which ensures that $b^* = 0$. Then we consider the case $b^* > 0$, calculate b^* and then we prove that b^* really is the optimal threshold we searched for.

- $b^* = 0$
  $b^* = 0$ means that the first derivative is less than 1 $\forall x > 0$ and

$$V(x;0) = \frac{\alpha}{\delta}\left(1 - e^{wx}\right) + e^{wx}V(0;0)$$

because of (4.23). Computing the first derivative gives

$$V'(x;0) = -w\frac{\alpha}{\delta}e^{wx} + we^{wx}V(0;0) < 1.$$

As in the dual model having no capital immediately leads to ruin, $V(0;0) = 0$ and it follows that

$$-\frac{\alpha}{\delta}we^{wx} < 1.$$

Considering that w is negative, e^{wx} can be replaced by 1 and we get the condition stated in the theorem.

- $b^* > 0$
  To find the optimal threshold b^* we use that b^* has to fulfill $V'(b^*-;b^*) = V'(b^*+;b^*) = 1$. Using (4.23) for $x = b^*+$ and computing the first derivative we find the condition

  $$-w\frac{\alpha}{\delta} + wV(b^*;b^*) = 1$$

  Hence b^* has to be the solution of

  $$V(b^*;b^*) = \frac{1}{w} + \frac{\alpha}{\delta}. \tag{4.29}$$

  As V is continuous at $x = b^*$ because of the continuity condition and using (4.22) we get

  $$\frac{1}{w} + \frac{\alpha}{\delta} = \frac{\alpha}{\delta}\frac{w}{\beta}\frac{e^{rb} - e^{sb}}{e^{rb}\frac{r-w}{r-\beta} - e^{sb}\frac{s-w}{s-\beta}}.$$

  Rearranging terms gives

  $$b^* = \frac{1}{r-s}\ln\left(\frac{\left(\frac{1}{w} + \frac{\alpha}{\delta}\right)\frac{s-w}{s-\beta} - \frac{\alpha}{\delta}\frac{w}{\beta}}{\left(\frac{1}{w} + \frac{\alpha}{\delta}\right)\frac{r-w}{r-\beta} - \frac{\alpha}{\delta}\frac{w}{\beta}}\right). \tag{4.30}$$

  Now we prove that the optimal strategy is really a threshold strategy with threshold b^*.

  - $V'(x;b^*) < 1 \ \forall x > b^*$:
    For $x > b^*$

    $$V(x;b^*) = \frac{\alpha}{\delta}\left(1 - e^{w(x-b^*)}\right) + e^{w(x-b^*)}V(b^*;b^*).$$

    Computing the first derivative and using (4.29) gives

    $$V'(x;b^*) = e^{w(x-b^*)}.$$

    As w is negative and $x > b^*$, $V'(x;b^*) < 1$ is fulfilled. For $x = b^*$ $V' = 1$.For $x = b^*+$ we have $V'(b^*+;b^*) = 1$.

– $V'(x; b^*) > 1 \; \forall x < b^*$:
  As $V'(b^*; b^*) = 1$, it is sufficient to show that V' is decreasing, i.e. $V''(x; b^*) < 0 \; \forall x < b^*$. Because of (4.14) $V''(x; b)$ is increasing in x for $0 < x < b$, $V''(x; b^*) < 0$ is equivalent to

$$V''(b^*-; b^*) < 0.$$

  Considering (4.11) for $x = b^*-$, (4.15) for $x = b^*+$, the continuity condition and $V'(b^*+; b^*) = 1$, we get

$$cV''(b^*-; b^*) + (\lambda + \delta - \beta c)V'(b^*-; b^*) + \beta c - \lambda - \delta = (c + \alpha)V''(b^*+; b^*).$$

  If $V'(b^*-; b^*) = 1$ can be proven, the equation simplifies to

$$cV''(b^*-; b^*) = (c + \alpha)V''(b^*+; b^*).$$

  Then

$$V''(b^*-; b^*) < 0$$

  is equivalent to

$$V''(b^*+; b^*) < 0$$

  which is certainly true as

$$V''(b^*+; b^*) = w^2 e^{w(x-b^*)} \left(-\frac{\alpha}{\delta} + \underbrace{V(b^*; b^*)}_{\leq \frac{\alpha}{\delta}} \right) < 0.$$

  Therefore it just remains to show $V'(b^*-; b^*) = 1$. One alternative is to set (4.30) into $V'(b^*-; b^*)$ calculated from (4.22), equating $V'(b^*-; b^*)$ with 1 and recalculating that this is true. Indeed this method is tedious but successful. □

4.7 Numerical example

In the following take $c = 3.5$, $\beta = 1$, $\lambda = 4$, $\delta = 0.02$ and $\alpha = 2$.

As in the compound Poisson model, $V(x; b)$ can be calculated for different values of b by using (4.12), (4.16), (4.22) and (4.23). Figure 4.4 shows $V(x; b)$ as a function of b for four different values of the surplus x. Each of these four functions reaches its maximum at $b \approx 13.5$. Calculating the optimal threshold b^* by (4.28) yields $b^* \approx 13.5$.

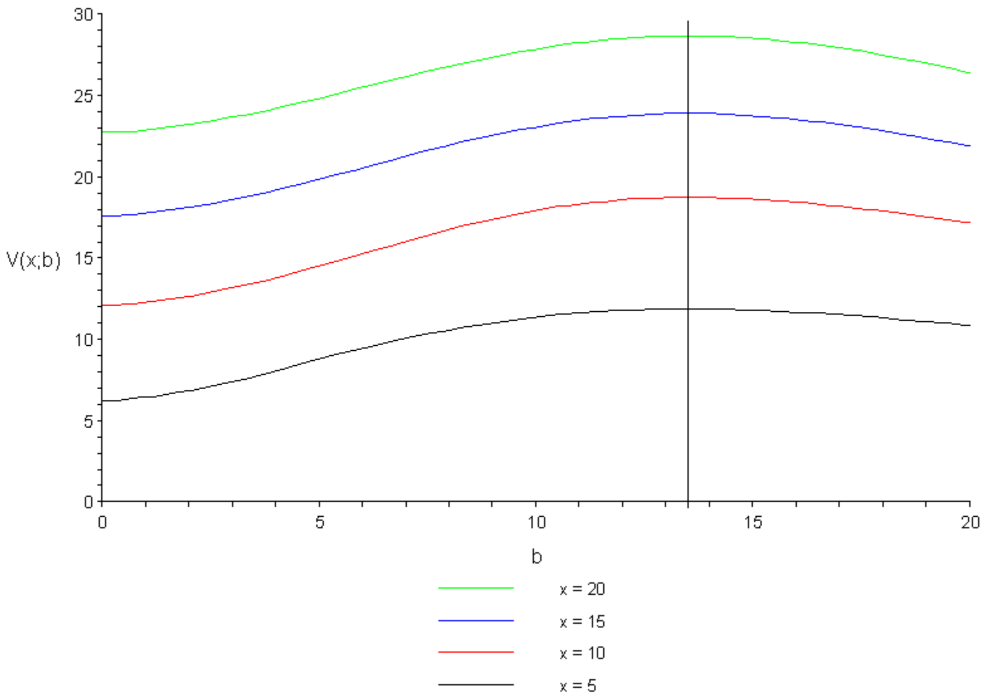

Figure 4.4: $V(x;b)$ as a function of b for different values of x.

Figure 2.5 illustrates $V(x;b)$ as a function of the surplus x for different values of b. It can be seen, that $V(x;b^*) > V(x;b)$ for $b \neq b^*$ and that the demand on the first derivative, see (4.7) and (4.8), need not be fulfilled necessarily for $b \neq b^*$.

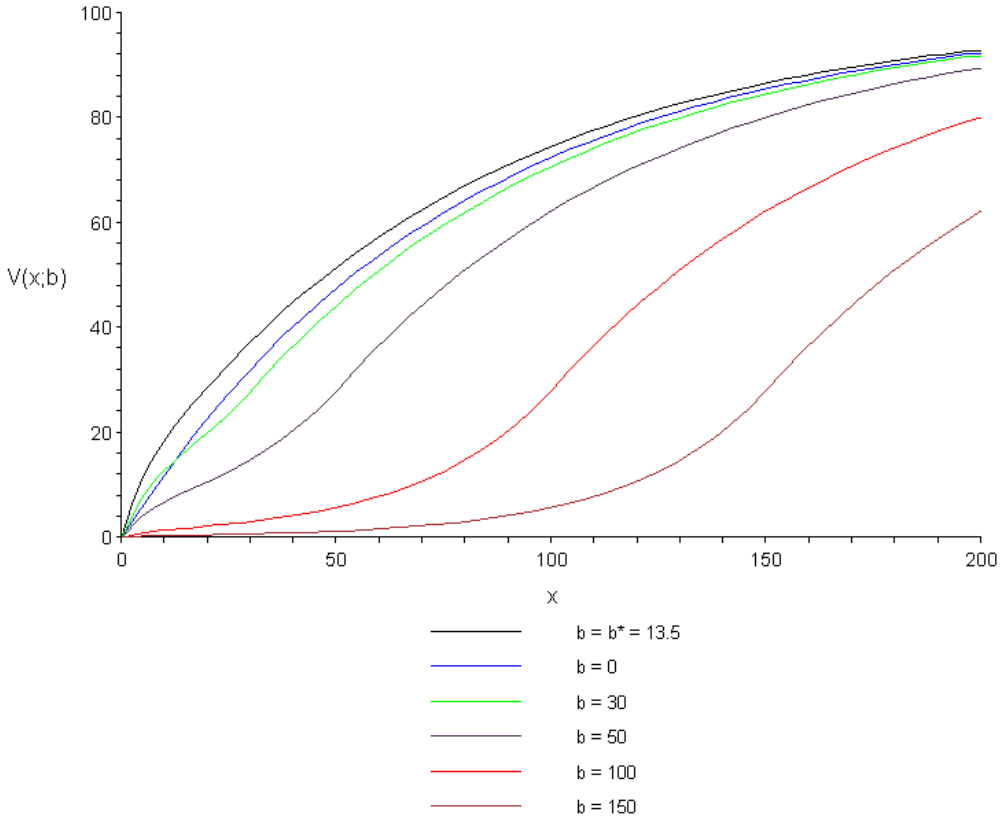

Figure 4.5: $V(x;b)$ as a function of x for different values of b.

Chapter 5

Compound poisson model perturbed by a diffusion

The compound Poisson model perturbed by diffusion (see [7] and the references therein) is an extension of the compound Poisson model described in chapter 2. To some extent, asking for the optimal dividend strategy leads to more difficult calculations than in the model without perturbation.

5.1 Problem formulation

The surplus of an insurance company is modeled as in the compound Poisson model with the exception that the development of the surplus depends not only on the claims, premiums and dividends, but also on the development of a Brownian motion $\{W_t\}$. How much this random process affects the surplus is described by the dispersion parameter $\sigma > 0$.

$$X_t^U = x + ct - \sum_{i=1}^{N_t} Y_i + \sigma W_t - \int_0^t U_s \, ds. \tag{5.1}$$

As in the compound Poisson model, $\beta > 0$ is the parameter of the exponential distribution, which describes the claim sizes Y_i and $\lambda > 0$ is the parameter of the homogenous Poisson process $\{N_t\}$, which describes the number of claims. Again, we are searching for the dividend strategy $\{U_t^*\}$, which maximizes the present value of all dividend payments until ruin time τ. Only dividend rate processes $\{U_t\}$ which are adapted, cadlag and bounded by a constant $\alpha < c$ are allowed.

5.2 Motivation of the Hamilton-Jacobi-Bellman equation

To find the HJB equation, we proceed similar to the compound Poisson model and the dual model. Define $f(t, X_t^U)$ as the discounted value function,

$$f(t, X_t^U) := e^{-\delta t} V(X_t^U).$$

Further, define

$$Z_t := f(t, X_t^U) + \int_0^t e^{-\delta s} U_s \, ds - f(0, X_0). \tag{5.2}$$

As the surplus X_t^U evolves as described in (5.1), we can use Itô's formula for jump-diffusion processes, see [4][prop.8.14], to get $f(t, X_t^U) - f(0, X_0)$ in (5.2). For it f has to be twice continuously differentiable in x and continuously differentiable in t.

$$\begin{aligned} f(t, X_t^U) - f(0, X_0) \quad = \quad & \int_0^t \left[f_s(s, X_s^U) + f_x(s, X_s^U) b_s \right] \, ds \\ & + \frac{1}{2} \int_0^t \sigma^2 f_{xx}(s, X_s^U) \, ds + \int_0^t f_x(s, X_s^U) \sigma \, dW_s \\ & + \int_0^t \lambda \int_{-\infty}^{\infty} \left[f(s, X_{s-}^U - y) - f(s, X_{s-}^U) \right] p(y) \, dy \, ds \end{aligned}$$

where

$$\begin{aligned} b_s &= c - U_s, \\ p(y) &= \beta e^{-\beta y} \end{aligned}$$

and f_x denotes the first, f_{xx} the second derivative with respect to x and f_s denotes the first derivative with respect to time. Then, equating the drift in Z_t to zero and using that $p(y) = 0$ for $y \leq 0$ gives

$$\begin{aligned} -\delta V(X_t^U) + (c - U_t) V'(X_t^U) + \frac{1}{2} \sigma^2 V''(X_t^U) & \\ + \lambda \int_0^{\infty} V(X_t^U - y) p(y) \, dy - \lambda V(X_t^U) + U_t = 0. & \end{aligned}$$

If there is a claim y, which exceeds the surplus X_t^U of the insurance company, the company is ruined. Therefore $V(X_t^U - y) = 0$ for $y > X_t^U$ and the integral in the former equation goes only up to X_t^U.

Hence, the Hamilton-Jacobi-Bellman equation is

$$\max_{0 \leq u \leq \alpha} \{ u(1 - V'(x)) \} - V(x)(\lambda + \delta) + cV'(x) + \frac{1}{2} \sigma^2 V''(x) + \lambda \int_0^x V(x - y) p(y) \, dy = 0. \tag{5.3}$$

5.3 Solution of the HJB equation

Suppose that $f(x)$ is a solution of the HJB equation. Then the dividend strategy, which maximizes $u(1-f'(x))$ is

$$u = \alpha \text{ for } f'(x) < 1, \qquad u = 0 \text{ for } f'(x) > 1.$$

If $f'(x) = 1$, any value $\in [0, \alpha]$ is possible for u.

If f has the property that

$$f'(x) = \begin{cases} > 1 & \text{for } x < b \\ < 1 & \text{for } x > b \end{cases}$$

for some b, then this dividend strategy is again a threshold-strategy. In the following we assume that there is such a b.

To find the solution f, we proceed as in the compound Poisson model and compute the derivative of the integro-differential equation.

- $u = 0$

For $x < b$ the dividend rate is zero and the solution f satisfies

$$-f(x)(\lambda + \delta) + cf'(x) + \frac{1}{2}\sigma^2 f''(x) + \lambda \int_0^x f(x-y)p(y)\,dy = 0. \tag{5.4}$$

Computing the derivative, using $p(y) = \beta e^{-\beta y}$ and resetting the integral in (5.4) gives

$$\frac{1}{2}\sigma^2 f'''(x) + \left(c + \frac{1}{2}\sigma^2\beta\right) f''(x) + (\beta c - \lambda - \delta) f'(x) - \delta\beta f(x) = 0.$$

In contrast to the compound Poisson model this is a third order homogenous linear differential equation. To find f, we use the exponential ansatz, analogue to the compound Poisson model,

$$f(x) = Ae^{\xi_1 x} + Be^{\xi_2 x} + Ce^{\xi_3 x}.$$

Setting the ansatz into the differential equation yields that ξ solves the following cubic equation.

$$\xi^3 + \left(\frac{2c}{\sigma^2} + \beta\right)\xi^2 + \frac{2(\beta c - \lambda - \delta)}{\sigma^2}\xi - 2\frac{\beta\delta}{\sigma^2} = 0. \tag{5.5}$$

To analyze, whether the solutions of this cubic equation are positive, negative or even complex, we calculate two function values of the cubic function

$$h(\xi) = \xi^3 + \left(\frac{2c}{\sigma^2} + \beta\right)\xi^2 + \frac{2(\beta c - \lambda - \delta)}{\sigma^2}\xi - 2\frac{\beta\delta}{\sigma^2}$$

to see that

$$h(-\beta) = 2\frac{\beta\lambda}{\sigma^2} > 0$$

and

$$h(0) = -2\frac{\beta\delta}{\sigma^2} < 0.$$

This gives that the function $h(\xi)$ has one positive and two negative roots as visualized in the following figure.

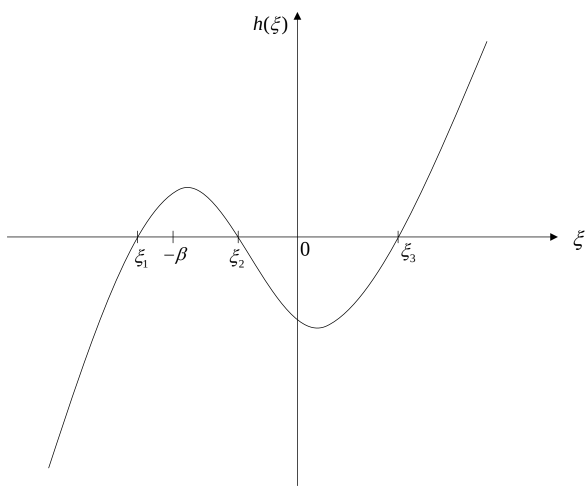

Figure 5.1: Roots of $h(\xi)$.

Therefore,

$$\xi_1 < -\beta < \xi_2 < 0 < \xi_3. \tag{5.6}$$

The determination of the constants A,B and C is described in the next subsection.

- $u = \alpha$

For $x > b$, the dividend rate is maximal, namely $u = \alpha$ and the solution f satisfies

$$\alpha(1 - f'(x)) - f(x)(\lambda + \delta) + cf'(x) + \frac{1}{2}\sigma^2 f''(x) + \lambda \int_0^x f(x-y)p(y)\,dy = 0. \tag{5.7}$$

Analogue to the case $x < b$, computing the derivative gives

$$\frac{1}{2}\sigma^2 f'''(x) + (c - \alpha + \frac{1}{2}\sigma^2\beta)f''(x) + (\beta(c-\alpha) - \lambda - \delta)f'(x) - \delta\beta f(x) + \beta\alpha = 0$$

and using the exponential ansatz yields, that the solution of this inhomogeneous differential equation of third order is

$$f(x) = \frac{\alpha}{\delta} + De^{\chi_1 x} + Ee^{\chi_2 x} + Fe^{\chi_3 x},$$

where χ_1,χ_2 and χ_3 are the solutions of

$$\chi^3 + \left(\frac{2(c-\alpha)}{\sigma^2} + \beta\right)\chi^2 + \frac{2(\beta(c-\alpha) - \lambda - \delta)}{\sigma^2}\chi - \frac{2\beta\delta}{\sigma^2} = 0.$$

Just as in the case $u = 0$ we find that setting $\xi = -\beta$ gives a negative value and setting $\xi = 0$ gives a positive value and therefore we get

$$\chi_1 < -\beta < \chi_2 < 0 < \chi_3.$$

The constants D, E and F are determined in the next subsection.

5.3.1 Determination of the constants

To determine the constants A,B,C,D,E and F we need six conditions. Beginning with the solution f for $x < b$, we find the condition

- $f(0) = 0 \Leftrightarrow A + B + C = 0$,

as having no capital leads to ruin immediately because of the Brownian motion. Furthermore, setting the solution f in the integro-differential equation (5.4) and equating the coefficient of $e^{-\beta x}$ with zero gives

- $\frac{A}{\xi_1+\beta} + \frac{B}{\xi_2+\beta} + \frac{C}{\xi_3+\beta} = 0.$

Looking at the case $x > b$ gives two more conditions, namely

- $F = 0$,

as otherwise f, which is the candidate for the value function, would develop exponentially. Splitting the integral in (5.7) in two parts and equating the coefficient of $e^{-\beta x}$ with zero again, gives

- $\frac{A}{\xi_1+\beta}\left(e^{b(\xi_1+\beta)} - 1\right) + \frac{B}{\xi_2+\beta}\left(e^{b(\xi_2+\beta)} - 1\right) + \frac{C}{\xi_3+\beta}\left(e^{b(\xi_3+\beta)} - 1\right) - \frac{\alpha}{\delta}\frac{e^{\beta b}}{\beta}$
  $- \frac{D}{\chi_1+\beta}e^{b(\chi_1+\beta)} - \frac{E}{\chi_2+\beta}e^{b(\chi_2+\beta)} = 0.$

For the last two conditions we use the principle of smooth fit, which means that we demand that f and f' are continuous at $x = b$,

- $Ae^{\xi_1 b} + Be^{\xi_2 b} + Ce^{\xi_3 b} = \frac{\alpha}{\delta} + De^{\chi_1 b} + Ee^{\chi_2 b}$,
- $A\xi_1 e^{\xi_1 b} + B\xi_2 e^{\xi_2 b} + C\xi_3 e^{\xi_3 b} = D\chi_1 e^{\chi_1 b} + E\chi_2 e^{\chi_2 b}$.

With these six conditions the constants can be determined numerically for some b. To find a candidate for the optimal threshold b^*, we use the condition

- $f'(b-) = f'(b+) = 1$.

From this equation b can be calculated numerically.

But we cannot be sure, whether b fulfills the conditions for a threshold, namely

$$f'(x) > 1 \qquad \forall x < b \tag{5.8}$$

and

$$f'(x) < 1 \qquad \forall x > b. \tag{5.9}$$

Furthermore it has to be verified that the solution of (5.3) is really the value function. But this is more complicated and left to future research.

Chapter 6

About the coherence of the compound Poisson and diffusion model

In this chapter the connection of the compound Poisson model and the diffusion model is demonstrated. This helps to motivate the diffusion assumption in chapter 3.

Before the link between these two models can be formulated mathematical, we have to recall two facts from probability theory first.

Proposition 6.0.1 *Let X^n be a sequence of Lévy-processes and X a Lévy-process. Then*

$$X^n \xrightarrow{d} X \Leftrightarrow X_1^n \xrightarrow{d} X_1,$$

where $\xrightarrow{d}$ on the left-hand side denotes convergence in distribution of processes on the Skorokhod space and $\xrightarrow{d}$ on the right-hand side denotes convergence in distribution of $\mathbb{R}$-valued random variables. X_1 denotes the Lévy-process X at time 1*.*

This proposition means that a sequence of Lévy-processes converges in distribution towards a second Lévy-process if and only if the sequence of Lévy-processes at time 1 converges in distribution towards the second Lévy-process at time 1.

Further, we take advantage of the Lévy continuity theorem, which says that a sequence of random variables converges in distribution if and only if the sequence of the corresponding characteristic functions converges.

Proposition 6.0.2 *If Y^n is a sequence of random variables, Y a random variable and $\phi_{Y^n}(z)$, $\phi_Y(z)$ the corresponding characteristic functions, the following equivalence holds.*

$$Y^n \xrightarrow{d} Y \Leftrightarrow \lim_{n\to\infty} \phi_{Y^n}(z) = \phi_Y(z).$$

Now we can prove the following corollary.

Corollary 6.0.1 *Let X_t^{CP} describe the surplus without dividends in the compound Poisson model*

$$X_t^{CP} = x + ct - S_t,$$

where $S_t = \sum_{i=1}^{N_t} Y_i$ is a compound Poisson process with intensity λ and the claim amount Y_i is exponentially distributed with parameter β.

$$X_t^D = x + \mu t + \sigma W_t$$

describes the surplus without dividend payments in the diffusion model. Define the sequence $X^{CP(n)}$ of surplus processes with premium rate c_n, intensity λ_n and parameter β_n, set $\lambda_n := n$ and fit the first two moments for $t = 1$,

$$\mathbb{E}\left[X_1^{CP(n)}\right] = \mathbb{E}\left[X_1^D\right]$$

$$\mathrm{Var}\left(X_1^{CP(n)}\right) = \mathrm{Var}\left(X_1^D\right).$$

Then,

$$X^{CP(n)} \xrightarrow{d} X^D \ \textit{for} \ n \to \infty.$$

Proof. Because the Brownian motion and the compound Poisson process are Lévy-processes, it suffices to show $X_1^{CP(n)} \xrightarrow{d} X_1^D$ for $n \to \infty$, whereby the processes become random variables. Therefore the Lévy continuity theorem can be used, which says that it suffices to prove

$$\lim_{n\to\infty} \phi_{X_1^{CP(n)}}(z) = \phi_{X_1^D}(z).$$

The characteristic function of the sequence of processes is known as we know the characteristic function of a compound Poisson process at time $t = 1$,

$$\phi_{X_1^{CP(n)}}(z) = \mathbb{E}\left[e^{izX_1^{CP(n)}}\right] = e^{iz(x+c_n)}\,\mathbb{E}\left[e^{i(-z)S_1^{(n)}}\right] = e^{iz(x+c_n)}e^{\lambda_n\left(\frac{\beta_n}{\beta_n+iz}-1\right)}.$$

As $W_1 \sim \mathcal{N}(0,1)$, the characteristic function of the surplus process in the diffusion model is

$$\phi_{X_1^D}(z) = \mathbb{E}\left[e^{izX_1^D}\right] = e^{iz(x+\mu)}\,\mathbb{E}\left[e^{iz\sigma W_1}\right] = e^{iz(x+\mu)}e^{-\frac{\sigma^2}{2}z^2}.$$

Fitting the moments gives

$$c_n - \mathbb{E}\left[S_1^{(n)}\right] = \mu \Rightarrow c_n = \mu + \frac{\lambda_n}{\beta_n} \tag{6.1}$$

and

$$\frac{2\lambda_n}{\beta_n^2} = \sigma^2 \Rightarrow \beta_n = \frac{\sqrt{2\lambda_n}}{\sigma}. \tag{6.2}$$

Therefore, letting $n \to \infty$ causes $c_n \to \infty$ and $\beta_n \to \infty$, as $\lambda_n = n$. Now, the convergence of the characteristic functions can be shown easily, whereas it suffices to show that the exponents converge. By rearranging terms and using (6.1), (6.2) and $\lambda_n = n$, we see that

$$iz(x + c_n) + \lambda_n \left(\frac{\beta_n}{\beta_n + iz} - 1 \right) = iz \left(x + \mu + \frac{nz^2}{\frac{\sqrt{2n}}{\sigma} \left(\frac{2n}{\sigma^2} + z^2 \right)} \right) - \frac{nz^2}{\frac{2n}{\sigma^2} + z^2}.$$

Dividing by n and letting n tend to infinity yields

$$\lim_{n \to \infty} iz(x + c_n) + \lambda_n \left(\frac{\beta_n}{\beta_n + iz} - 1 \right) = iz(x + \mu) - \frac{\sigma^2}{2} z^2.$$

□

Consequently, the value function of the compound Poisson model, calculated in section 2.4, converges to the value function of the diffusion model, calculated in section 3.4.
In the compound Poisson model the roots $r > 0$, $s < 0$ and $w < 0$ were given by the quadratic equations (2.13) and (2.18). Letting $\beta \to \infty$, $c \to \infty$, $\lambda \to \infty$ and using $c = \mu + \frac{\lambda}{\beta}$ and $\beta = \frac{\sqrt{2\lambda}}{\sigma}$ as discussed above, the quadratic equations converge to

$$\frac{\sigma^2}{2} \xi^2 + \mu \xi - \delta = 0$$

and

$$\frac{\sigma^2}{2} \xi^2 + (\mu - \alpha) \xi + \delta = 0,$$

which are the quadratic equations (3.9) and (3.10) in the diffusion model. Considering the value functions in the compound Poisson and the diffusion model, one can easily see, that the value function of the compound Poisson model converges to the value function of the diffusion model for $\beta \to \infty$.

For $0 \leq x \leq b$, the value function of the compound Poisson model was

$$V(x; b) = -w \frac{\alpha}{\delta} \frac{\left(1 + \frac{r}{\beta}\right) e^{rx} - \left(1 + \frac{s}{\beta}\right) e^{sx}}{(r - w)e^{rb} - (s - w)e^{sb}},$$

see (2.26). Letting $\beta \to \infty$, this value function converges to the value function in the diffusion model for $0 \leq x \leq b$,

$$\lim_{\beta \to \infty} V(x; b) = -w \frac{\alpha}{\delta} \frac{e^{rx} - e^{sx}}{(r - w)e^{rb} - (s - w)e^{sb}},$$

see (3.25).

For $x \geq b$, the value function (2.27) of the compound Poisson model remains valid in the limit, as can be seen in the following way. Using $V'(b;b) = 1$ in (2.38) to get

$$V(b;b) = \frac{1}{w} + \frac{\alpha}{\delta}$$

and setting it into (2.27) results in

$$V(x;b) = \frac{\alpha}{\delta} + \frac{1}{w}e^{w(x-b)},$$

which is the value function for $x \geq b$ in the diffusion model, see theorem 3.4.1.

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