DISSERTATION

zum Thema

ausgeführt zum Zwecke der Erlangung des akademischen Grades
Doktor der technischen Wissenschaften

eingereicht an der Technischen Universität Wien
Technisch Naturwissenschaftliche Fakultät

von

Dipl.–Ing. Mahmoud Nikbakht–Tehrani

Neilreichgasse 23/14-15, A-1100 Wien
geboren am 3. Dezember 1964 in Babol

Wien, im März 1997   ....

The classical solutions of perturbative string theory with unbroken $N = 1$ spacetime supersymmetry provide us with the only known consistent string vacua in four dimensions. As is well-known, the $(0, 2)$ superconformal invariance on the string worldsheet together with an integrality condition on the $U(1)$ charges in the superconformal algebra are equivalent to $N = 1$ spacetime supersymmetry [7, 33, 95]. Therefore, the $(0, 2)$ superconformal field theories (particularly the $(0, 2)$ Calabi-Yau sigma models) seem to be the natural candidates for (geometric) string compactification. Yet another source of interest in $(0, 2)$ models is due to the fact that the phenomenological prospects of such models are much more promising than those of, e.g. $(2, 2)$ theories, because they lead to more realistic gauge groups [70, 109] like $SO(10)$ and $SU(5)$.

Using various exact conformal field theory methods many models of this type were constructed already several years ago [47, 93]. In the geometrical setting, on the other hand, these models received less attention. This was in part because of the assertion made in [33] that the generic $(0, 2)$ Calabi-Yau sigma models suffer from destabilization by worldsheet instantons. Recent work [100], however, shows that large classes of these models are not destabilized by such nonperturbative sigma model effects. In spite of the early works [25, 35, 36, 57] the technical difficulty in constructing $(0, 2)$ models remained

another big obstacle in the study of these models. Witten's gauged linear sigma model approach [111] has dramatically changed the state of affairs. It provided a powerful tool in constructing $(2,2)$ and $(0,2)$ models and in analyzing their 'phase structures'. Using this framework the authors of [37] have constructed and analyzed plenty of $(0,2)$ models in their 'Landau-Ginzburg phase'. The subsequent works [14, 15] proposed an identification between an exact $(0,2)$ superconformal field theory and a certain model of [37]. The involved exact $(0,2)$ superconformal field theory was constructed using the simple current methods. Inspired by this proposal the authors of [16, 75] have tried to extend this identification to a larger class of $(0,2)$ models. The starting point was a general solution of anomaly cancellation condition yielding a large set of consistent sigma model data. In these works attention has been paid primarily to the connection of the exact superconformal field theories and Landau-Ginzburg models.

The geometric data defining a $(0,2)$ supersymmetric $U(1)$ gauge theory in general result in a singular Calabi-Yau variety with some stable vector bundle, the so-called gauge bundle, whereas its corresponding physical theory is well-behaved. Therefore, the naive phase picture of the moduli space of such models is not complete. In order to fill this important and obvious gap, we are going to resolve these singularities and consider the moduli space of this desingularized model into which the moduli space of the original model will be embedded in the present work. In the framework of gauged linear sigma models in the process of desingularization we first embed the original model into a new one which has $U(1) \times \ldots \times U(1)$ (N copies) as its gauge group and has $N - 1$ new chiral scalar superfields. Next, we determine the charges of these fields with respect to the full gauge group. By varying the N Kähler moduli parameters of the resulting model and finding the minima of the scalar potential one can recover the phase structure of the moduli space.

As is known [3, 5], there is an equivalent formulation of the whole story in terms of toric geometry. In this context, the relevant information describing a

model is encoded in the combinatorial data of a reflexive polytope in a lattice. The phase structure of the theory is then determined by the possible triangulations of this polytope. The 'Calabi-Yau phases' correspond to the possible maximal triangulations . Resolving the singularities of the base variety determines at the same time the charges of the right-handed fermions, which couple to the tangent bundle. In $(0, 2)$ models we still have to deal with other independent degrees of freedom, namely the left-handed fermions, which couple to the gauge bundle. As discussed in [39], the gauge bundle on the desingularized variety is taken to be defined through an 'exact sequence' which reduces to the original one if one removes the exceptional divisors. The gauge bundles defined in this way are supposed to satisfy the condition of vanishing first Chern class and of anomaly cancellation.

In this work we develop a powerful machinery for resolving the singularities of a huge class of $(0, 2)$ Calabi-Yau sigma models whose defining data come from the solution of the anomaly cancellation equation found in [16, 75]. We will show that with each singular $(0, 2)$ Calabi-Yau sigma model there is associated a large set of nonsingular ones which are possible candidates for perturbative $(0, 2)$ string vacua. Using the intersection theory in toric varieties and the methods of computational commutative algebra we have constructed a new algorithmic way for the calculation of the net number of generations of matter fields by applying the Riemann–Roch theorem. However, in order to be admissible candidates for perturbative string vacua, the gauge bundles of the resolved models still have to satisfy the constraint of stability. Checking this condition is a hard technical problem for which no efficient tools are known. All what can be done at the moment in the cases of interest to us is to check a necessary condition which states that the lowest and the highest cohomology groups of the gauge bundles must vanish [36, 88]. This, in turn, requires some sophisticated cohomology calculations. In order to handle these extensive calculations we develop a comprehensive framework for approaching such calculations [86].
What can be learned on the physical side from this high-browse mathematics?

The ambiguity arising in the process of desingularization indicates that there should be some sort of physical processes that allow transitions among different $(0, 2)$ string vacua through a point in the moduli space representing the original singular model. Up to now very little is known about such phenomena which are an active area of current research. Apart from the fact that the study of $(0, 2)$ string vacua is per se of great interest, it is also desirable in other respects. Recent years have witnessed some exciting progress in our understanding of the structure of the space of string vacua in which dualities play an important role. The relevant duality in our context is the one between heterotic strings and F-theory compactifications [13, 42, 49]. The insight gained from this work contributes to our knowledge on the side of heterotic strings and will therefore be very useful in analyzing this duality. Finally we stress that our work is also relevant for the study of '$(0, 2)$ dualities' [37] in the resolved moduli space.

The organization of this work is as follows. In chapter 2 we briefly review some general aspects of heterotic string compactification in the geometrical setting of nonlinear sigma models. Chapter 3 deals with the basic ingredients of the gauged linear sigma model approach. After discussing the necessary technicalities for the formulation of $(0, 2)$ supersymmetric $U(1)$ gauge theories, we analyze in the simplest nontrivial example the 'phase structure' which is exhibited by such models. At the end of this chapter we comment on the relation of toric geometry to gauged linear sigma models and its relevance for analyzing the phase structure in more complicated situations. Chapter 4 begins with reviewing the basics of toric geometry which puts the necessary machinary for constructing the models of interest and for addressing the issue of resolution of singularities at our disposal. In the next section we discuss the intersection ring and the Riemann-Roch theorem which provide us with the main tools for the calculations of the last section. After that we outline a general framework for the calculation of cohomology groups of twisting sheaves on toric varieties which plays an important role for future works in this direction. In the last section we present our examples. We conclude this work with a mathematical appendix on the Gröbner basis.

In this chapter we are going to briefly review some standard issues on string compactification in the geometrical context. Our starting point will be the heterotic string. From the two possible gauge symmetry groups $SO(32)$ and $E_8 \times E_8$ of the heterotic string we choose the latter one which is phenomenologically more attractive. As is standard practice, one of these E_8 will be relegated to a hidden sector, leaving the observable sector with a single E_8 gauge symmetry.

The propagation of a heterotic string in a nontrivial background spacetime M can be described by a two-dimensional nonlinear sigma model with target space M. This nonlinear sigma model has $(0,1)$ worldsheet supersymmetry and is coupled to the two-dimensional $(0,1)$ supergravity. A critical dimension of 10 emerges in this case. We then assume that the ten-dimensional target space is of the form $M = \mathbb{M} \times X$ where $\mathbb{M}$ is the four-dimensional Minkowski space and X is a six-dimensional compact manifold. We will see below that the specific choice of the so-called internal manifold X is far from arbitrary. There is a close link between the topology of X and the observable low energy physics. The string propagation on $\mathbb{M}$ corresponds to a free field theory.

Of particular interest, for phenomenology, are compactifications which lead to an unbroken $N = 1$ supersymmetry in four dimensions [21, 107, 30]. As is well-known [7, 40, 95], the requirement of unbroken $N = 1$ spacetime supersymmetry is equivalent to $(0, 2)$ superconformal symmetry on the worldsheet, provided a certain integrality condition on the $U(1)$ charges in the superconformal algebra holds. This condition is required so that we may define a chiral GSO projection.

The field content of the nonlinear sigma model describing the propagation of a string on the internal manifold X consists of the following:

- $\phi : \Sigma \to X$ .... the sigma model map which describes the embedding of the worldsheet Σ in X,

- $\psi \in ?\,(\Sigma, S_+ \otimes \phi^*(TX))$ .... a right-handed worldsheet spinor which is coupled to the pullback of the tangent bundle TX of X via the sigma model map ϕ,

- $\lambda \in ?\,(\Sigma, S_- \otimes \phi^*(E))$ .... a left-handed worldsheet spinor which is coupled to the pullback of a vector bundle E on X via the sigma model map ϕ.

Now the existence of $(0, 2)$ supersymmetry in the classical sigma model requires that the internal manifold X be Kähler[1] and the vector bundle E be a holomorphic vector bundle whose Yang-Mills connection is holomorphic [70]. In a complex coordinate system, the sigma model action can be written as

$$S = \frac{i}{2\pi} \quad d^2 z \quad \frac{1}{2}\, g_{i\bar{j}}\, (\partial\phi^i \bar{\partial}\phi^{\bar{j}} + \partial\phi^{\bar{j}}\bar{\partial}\phi^i) - \frac{1}{2}\, i\, b_{i\bar{j}}\, (\partial\phi^i \bar{\partial}\phi^{\bar{j}} - \partial\phi^{\bar{j}}\bar{\partial}\phi^i)$$

$$+ i\,(\psi_{\bar{i}} D\psi^{\bar{i}} + \lambda_a \bar{D}\lambda^a) + \frac{1}{4}\, F^a{}_{b\,i\bar{j}}\, \lambda_a \lambda^b \psi^i \psi^{\bar{j}} \quad , \qquad (2.1)$$

where $g_{i\bar{j}}$ is the Kähler metric of X, $b_{i\bar{j}}$ is a closed two-form on X and $F^a{}_{b\,i\bar{j}}$ is the curvature of the holomorphic connection $A^a{}_{b\,i}$ on E. The covariant derivatives

---

[1] This is the case if the connection on X is torsionfree, otherwise one only has a Hermitian manifold with torsion [70].

D and $\bar{D}$ in (2.1) are given by

$$D\psi^i = \partial\psi^i + \partial\phi^j \, ?^i_{jk} \, \psi^k \qquad \text{and} \qquad \bar{D}\lambda^a = \bar{\partial}\lambda^a + \bar{\partial}\phi^i \, A^a_{bi} \, \lambda^b \, ,$$

where $?^i_{jk}\, (= \overline{?^i_{jk}}\,)$ is the antiholomorphic part of the torsionfree Hermitian connection on the Kähler manifold X. We note that the holomorphicity of the connection A^a_{bi} on E is equivalent to the fact that its curvature form is of $(1,1)$ type, i.e. $F_{ij} \equiv F_{\bar{i}\bar{j}} \equiv 0$.

In order to be a candidate for a perturbative string vacuum with an unbroken $N = 1$ supersymmetry, the sigma model still has to satisfy two sets of conditions imposed on its defining data. The first set of these conditions, which are of topological nature, has been imposed by the cancellation of (global) worldsheet diffeomorphism anomalies [48, 108]

$$c_1(X) \equiv c_1(E) \equiv 0 \pmod 2 \, , \qquad (2.2)$$

$$\mathrm{ch}_2(X) = \mathrm{ch}_2(E) \, , \qquad (2.3)$$

where $c_i(E)$ and $c_i(X)$ $(:= c_i(TX)\,)$ denote the i-th Chern classes of E and X, respectively. $\mathrm{ch}_2 = \frac{1}{2}c_1^2 - c_2$ is the second Chern character. Chern classes c_i are elements of the integer cohomology of X, i.e. $c_i \in H^{2i}(X, \mathbb{Z})$, whereas Chern characters ch_i belong to the rational cohomology of X, i.e. $\mathrm{ch}_i \in H^{2i}(X, \mathbb{Z})\otimes\mathbb{Q}$. Despite of this latter fact, (2.3) holds due to (2.2) in the integer cohomology. As pointed out in [48], (2.2) and (2.3) are exactly the conditions in [105] for modular invariance. The mod 2 reduction of the first Chern class of a complex vector bundle is its second Stiefel-Whitney class which is the obstruction to the existence of spinors. Therefore, condition (2.2) states that the manifold X and the vector bundle E admit spinors. Due to the left-right asymmetric construction of the sigma model in our case, there are sigma model anomalies present. The conditions (2.2) and (2.3) are also precisely what are needed to cancel such anomalies [48, 70, 83].

The second set of conditions, which are of geometrical nature, comes from the vanishing of the lowest order beta function. It requires that the Kähler metric

be Ricci-flat, i.e.

$$\text{Ric}(X) = 0 \tag{2.4}$$

and that the holomorphic connection on E satisfy

$$g^{i\bar{j}} \, F_{i\bar{j}} = 0 \; . \tag{2.5}$$

The Ricci-flatness of the Kähler metric on the compact Kähler manifold X is equivalent to the vanishing of the first Chern class of X. That it is a necessary condition is easy to verify. Let $\rho = i \, R_{i\bar{j}} \, d\phi^i \wedge d\phi^{\bar{j}}$ be the Ricci two-form which is associated with the Ricci tensor of the Kähler metric. Locally, it is given by $\rho = i\partial\bar{\partial} \, \log \left(\det(g_{i\bar{j}}) \right)^{1/2}$. A representative of the first Chern class $c_1(X)$ is $\frac{1}{2\pi}\rho$. It is a well-known fact that the Chern classes do not depend on the choice of the Kähler metric[2], i.e. $\rho(\tilde{g}) = \rho(g) + d\alpha$. If now the Ricci tensor of $g_{i\bar{j}}$ vanishes, then $c_1(X)$ has to be trivial. The sufficiency of this condition was conjectured by Calabi and proved by Yau [112]. A compact Kähler manifold with trivial first Chern class is called a Calabi-Yau manifold [3].

A consequence of Ricci-flatness of the Calabi-Yau manifold X is that its holonomy group is contained in $SU(3)$. The proof is simple, hence we sketch it now. Take a vector $\boldsymbol{v} = v^i \, \partial_i$ from the holomorphic tangent space at p and parallel transport it along a parallelogram with 'sides' ϵ^i and $\eta^{\bar{j}}$ back to p. Then, we have $\delta\boldsymbol{v}^k = v^\ell \, R^k{}_{\ell i\bar{j}}\epsilon^i \, \eta^{\bar{j}}$, where $R^k{}_{\ell i\bar{j}}\epsilon^i \, \eta^{\bar{j}}$ belongs to the holonomy algebra. Due to $U(3) = SU(3) \times U(1)$ we have the decomposition $u(3) = su(3) \oplus u(1)$, where $su(3)$ is the traceless part of $u(3)$ while $u(1)$ is its trace part. Because of the Ricci-flatness the trace part, i.e. the $u(1)$ part, of $R^k{}_{\ell i\bar{j}}\epsilon^i \, \eta^{\bar{j}}$ vanishes.

If the holonomy group is $SU(3)$, then we precisely have $N = 1$ spacetime supersymmetry. Another consequence of $c_1(X) = 0$ is that the canonical bundle of X, i.e. the highest exterior power of the holomorphic cotangent bundle, is

---

[2]The variation of the Ricci two-form under $g_{i\bar{j}} \to g_{i\bar{j}} + \delta g_{i\bar{j}}$ is $\delta\rho = -\frac{i}{2}d \; (\partial - \bar{\partial}) \, g^{i\bar{j}} \, \delta g_{i\bar{j}} \; .$

[3]Calabi-Yau manifolds of interest here have complex dimension three.

trivial.

Equations of type (2.5) have been studied by Donaldson [43] for the case of X having complex dimension two and by Uhlenbeck and Yau [104] for the higher dimensional case. Before giving the result of [104], we spend a moment explaining a few mathematical concepts which appear in the statement of their theorem.

Let E be a holomorphic vector bundle of rank r with a Hermitian fiber metric $h_{a\bar{b}}$ on an n-dimensional compact Kähler manifold X with a Kähler metric $g_{i\bar{j}}$. Further, let $A\,(\in\,?\,(\mathrm{End}\,E\otimes T^*X))$ be the holomorphic Hermitian connection on E and F be its curvature. E is called Hermitian-Yang-Mills if

$$g^{i\bar{j}}\,F^a_{\ b\,i\bar{j}} = \mu\,\delta^a_{\ b}\,, \tag{2.6}$$

for all $0 \leq a, b \leq r$, where μ is a constant called the Hermitian-Einstein factor. μ can be expressed in terms of invariants of E and X as follows. The first Chern class of E is given by $c_1(E) = \frac{i}{2\pi}h^{a\bar{b}}\,F_{a\bar{b}\,i\bar{j}}\,d\phi^i\wedge d\phi^{\bar{j}}$. Denoting the Kähler form on X by $\omega\,(:=\frac{i}{2\pi}g_{i\bar{j}}\,d\phi^i\wedge d\phi^{\bar{j}})$, we have

$$c_1(E)\wedge *\omega = \frac{1}{(n-1)!}\,c_1(E)\wedge\omega^{n-1} = \ h^{a\bar{b}}\,g^{i\bar{j}}\,F_{a\bar{b}\,i\bar{j}}\ \frac{\omega^n}{n!} = \mu\cdot\mathrm{rank}(E)\,\frac{\omega^n}{n!}\,.$$

Integrating the above relation on X, we obtain (with an appropriate normalization)

$$\mu = \mu(E) = \frac{1}{\mathrm{rank}(E)}\ \int_X c_1(E)\wedge *\omega\,.$$

Now let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{E}$ (= locally free sheaf of local sections of E) [4]. We define

$$\mu(\mathcal{F}) := \frac{1}{\mathrm{rank}(\mathcal{F})}\ \int_X c_1(\mathcal{F})\wedge *\omega\,,$$

---

[4]The category of locally free sheaves and vector bundles on a variety $(X, \mathcal{O}_X)$ are equivalent. To each vector bundle E one can associate the locally free sheaf of its local sections. To recover a vector bundle from a locally free sheaf $\mathcal{E}$ we first take a covering $X = \cup_\alpha V_\alpha$ and consider the isomorphisms $\varphi_\alpha : \mathcal{E}_{|V_\alpha} \to \mathcal{O}^n_{V_\alpha}$. Then $C_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \mathcal{O}^n_{V_\alpha \cap V_\beta} \to \mathcal{O}^n_{V_\alpha \cap V_\beta}$

where the first Chern class of $\mathcal{F}$ is defined through one of its locally free resolutions: If

$$0 \to \mathcal{F}_k \to \mathcal{F}_{k-1} \to \ldots \to \mathcal{F}_0 \to \mathcal{F} \to 0$$

is a locally free resolution of $\mathcal{F}$, then $c_1(\mathcal{F}) = \sum_{i=0}^{k} (-1)^i\, c_1(\mathcal{F}_i)$. The rank of $\mathcal{F}$ is defined as the rank of $\mathcal{F}_{|_{X \setminus S(\mathcal{F})}}$, where $S(\mathcal{F})$ is the singularity set of $\mathcal{F}$ [5].

Definition: A holomorphic vector bundle E on a compact Kähler manifold X with the Kähler form ω is called stable if $\mu(\mathcal{F}) < \mu(E)$, for every coherent subsheaf $\mathcal{F}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(E)$.

The Uhlenbeck-Yau theorem is the following:

Theorem: A stable holomorphic vector bundle on a compact Kähler manifold admits a unique holomorphic Hermitian-Yang-Mills connection.

In the case of interest to us, the Hermitian-Einstein factor vanishes, implying

$$\int_X c_1(E) \wedge * \omega = 0 \ . \tag{2.7}$$

If the vector bundle E satisfies this integrability condition, then the above theorem guarantees the existence of a unique solution of equation (2.5). Imposing the stronger condition $c_1(E) = 0$ on E than (2.2), condition (2.7) is obviously satisfied.

The above conditions suffice to provide us with a solution to the string equations of motion to lowest nontrivial order in sigma model perturbation theory.

---

is an isomorphism. (We have assumed that X is connected.) The matrices $C_{\alpha\beta}$ obviously satisfy the cocyle condition. Hence they define some vector bundle E. Note that the fiber of E over x, E_x, is $\mathcal{E}_x/m_x\,\mathcal{E}_x$, where $\mathcal{E}_x$ is the stalk of $\mathcal{E}$ at x and m_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$.

[5] $S(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \text{ is not free over } \mathcal{O}_{X,x} \}$ is a closed subvariety of X of codimension at least one.

At higher loop order of sigma model perturbation theory the beta function is no longer nonzero for the original configuration and needs to be readjusted. A 'nonrenormalization theorem' which has been proved in [109] for spacetime superpotential at tree level in string theory guarantees that such a readjustment is possible. This has been checked explicitly in low orders in [110].

We sketch the basic ideas behind the proof of this nonrenormalization theorem. For the sake of simplicity, we assume that the Kähler form ω of X is the only nontrivial element of $H_{\bar{\partial}}^{1,1}(X)$ [6] and that $b_{i\bar{j}}$ are the components of ω . To the harmonic form $b_{i\bar{j}}$ there corresponds a massless particle a in spacetime. The vertex operator for this particle is

$$V_a = \int_{\Sigma} d^2z\, b_{i\bar{j}}\; \left[(\partial\phi^i + ik_\mu\psi^\mu\psi^i)\bar{\partial}\phi^{\bar{j}} - (\partial\phi^{\bar{j}} + ik_\mu\psi^\mu\psi^{\bar{j}})\bar{\partial}\phi^i \right]\; e^{ik_\mu\phi^\mu}\, , \quad (2.8)$$

where the index μ refers to spacetime. At zero momentum V_a represents a topological quantity (it is the pullback of a closed two-form via the sigma model map ϕ integrated over the two-cycle Σ (= worldsheet)) and vanishes in the perturbative sector. Another massless particle of interest to us in four dimensions is the dilatonic mode r of the Kähler metric. Its vertex operator is given by

$$V_r = \int_{\Sigma} d^2z\, g_{i\bar{j}}\; \left[(\partial\phi^i + ik_\mu\psi^\mu\psi^i)\bar{\partial}\phi^{\bar{j}} + (\partial\phi^{\bar{j}} + ik_\mu\psi^\mu\psi^{\bar{j}})\bar{\partial}\phi^i \right]\; e^{ik_\mu\phi^\mu}\, . \quad (2.9)$$

r describes the 'size' of the internal manifold X and $r^{-1/2}$ is the coupling constant of the sigma model describing the propagation of the string on X. In fact, a and r in the combination $r + ia$ give the scalar component of a chiral superfield R in spacetime. Note that these two fields have the same internal wave function (up to a trivial symmetry change).

The spacetime superpotential is a holomorphic function of various chiral superfields. Particularly, it must be a holomorphic function of R. But we have

---

[6]For the Calabi-Yau manifold X, the cohomology groups $H_{\bar{\partial}}^{0,2}(X)$ and $H_{\bar{\partial}}^{2,0}(X)$ are trivial.

just seen that in sigma model perturbation theory, a decouples at zero momentum, so the superpotential must be independent of a and hence of r. Thus the superpotential receives no correction in any finite order of sigma model perturbation theory.

At the level of worldsheet instantons, i.e. topologically nontrivial maps ϕ, this nonrenormalization theorem can break down. The reason is that for topologically nontrivial maps (2.8) does not vanish at zero momentum and so in general the spacetime superpotential gets renormalized by the worldsheet instantons.

In [33] it has been claimed that in a generic $(0, 2)$ model the superpotential does receive corrections from worldsheet instantons destabilizing the vacuum configuration. The basic assumption underlying their arguments was that there are no zero modes of the left-handed fermions λ in the instanton background.

However, if there are zero modes of λ in the instanton background, then the instanton contribution must vanish [35, 36]. Regarding this observation, we now look for conditions under which the existence of zero modes of the left-handed fermions is guaranteed [36]. To begin with, we first explain how the instantons look like in our sigma models. The instanton action

$$\tilde{S} = i \int d^2z \, g_{i\bar{j}} \, (\partial\phi^i \bar{\partial}\phi^{\bar{j}} + \partial\phi^{\bar{j}} \bar{\partial}\phi^i)$$

is positive. The absolute value of the 'topological charge'

$$\tilde{Q} = \int d^2z \, b_{i\bar{j}} \, (\partial\phi^i \bar{\partial}\phi^{\bar{j}} - \partial\phi^{\bar{j}} \bar{\partial}\phi^i)$$

is a lower bound for the instanton action, i.e. $\tilde{S} \geq \mid \tilde{Q} \mid$. The equality is achieved only for the fields ϕ satisfying $\bar{\partial}\phi = 0$ ($\leftrightarrow$ holomorphic instantons) or $\partial\phi = 0$ ($\leftrightarrow$ antiholomorphic instantons) [89].

An argument similar to that used in the proof of the nonrenormalization theorem shows that only the holomorphic instantons can renormalize the space-

time superpotential [33]. Moreover, because of a nonrenormalization theorem[7], proved in [31, 81], we only need to consider the holomorphic instantons of genus zero.

The number of zero modes of the left-handed fermions λ in the instanton background ϕ is related to the Euler characteristic of the twisted spin bundle $S_- \otimes \phi^*(E)$ on $\Sigma \simeq \mathbb{P}^1$. Before going further, we pause a moment recalling some well-known facts about $\mathbb{P}^1$ which we will need in the following.

The Picard group of $\mathbb{P}^1$, $\mathrm{Pic}\,(\mathbb{P}^1)$, is isomorphic to $\mathbb{Z}$. It follows that the isomorphism classes of invertible sheaves on $\mathbb{P}^1$ are of the form $\mathcal{O}_{\mathbb{P}^1}(m) = \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes m}$, where $\mathcal{O}_{\mathbb{P}^1}(1)$ is the Serre twisting sheaf on $\mathbb{P}^1$. (For $m < 0$, $\mathcal{O}_{\mathbb{P}^1}(1)^{\otimes m} := \mathcal{O}_{\mathbb{P}^1}(-1)^{\otimes |m|}$, where $\mathcal{O}_{\mathbb{P}^1}(-1)$ is the dual of $\mathcal{O}_{\mathbb{P}^1}(1)$.) For the cohomology of $\mathcal{O}_{\mathbb{P}^1}(m)$ the following formulae hold

$$
\dim H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) =
\begin{cases}
\begin{matrix} m+1 \\ m \end{matrix} & \text{for} \quad q = 0\,,\ m \geq 0 \\[2em]
\begin{matrix} -m-1 \\ -m-2 \end{matrix} & \text{for} \quad q = 1\,,\ m \leq -2 \\[2em]
0 & \text{otherwise .}
\end{cases}
\tag{2.10}
$$

On $\mathbb{P}^1$, the dual of the Serre twisting sheaf, i.e. $\mathcal{O}_{\mathbb{P}^1}(-1)$, corresponds to S_-. With $\mathcal{E}_\phi(-1)$ we denote the sheaf $\phi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ which is associated to the twisted spin bundle.

There exists a splitting theorem due to Grothendieck [60] which states that every rank r locally free sheaf $\mathcal{F}$ on $\mathbb{P}^1$ has the form

$$
\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)
$$

---

[7]The statement of this theorem is the following: With the assumption that spacetime supersymmetry is not anomalous, the spacetime superpotential is not renormalized by string loops.

with uniquely determined $a_1, \ldots, a_r \in \mathbb{Z}$ obeying $a_1 \geq \ldots \geq a_r$. According to this theorem, $\phi^* \mathcal{E}$ will split on $\mathbb{P}^1$ as

$$\phi^* \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r) . \tag{2.11}$$

The triviality of the first Chern class of E, which we have supposed in order to satisfy (2.7), leads to

$$\sum_{i=1}^{r} a_i = 0 . \tag{2.12}$$

As mentioned above, we are actually interested in knowing whether

$$\dim H^0(\mathbb{P}^1, \mathcal{E}_\phi(-1)) - \dim H^1(\mathbb{P}^1, \mathcal{E}_\phi(-1)) \tag{2.13}$$

vanishes or not. Using (2.10), (2.11) and taking (2.12) into account, it can be easily seen that (2.13) is nonvanishing if $\phi^* \mathcal{E}$ splits nontrivially, i.e. if not all a_i vanish simultaneously.

Summarizing the above discussion, we have the following criterion for the stability of a $(0, 2)$ compactification [36]: If $\phi^* \mathcal{E}$ splits nontrivially for every genus zero holomorphic instanton $\phi : \Sigma(\simeq \mathbb{P}^1) \to X$, then the $(0, 2)$ model is a solution of the string equations of motion to all orders in the semiclassical expansion[8].

In order to verify that a given $(0, 2)$ compactification evades destabilization by worldsheet instantons, one should, therefore, first determine all genus zero holomorphic instantons $\phi : \Sigma(\simeq \mathbb{P}^1) \to X$ and then check whether $\phi^* \mathcal{E}$ splits nontrivially. It goes without saying that this is in general a very hard task!

In view of the technical difficulties in working with nonlinear sigma models, one makes use of another more fruitful approach based on the ideas of the

---

[8]It may happen that X does not contain holomorphic curves of genus zero. In such cases the instanton mechanism which destabilizes the $(0, 2)$ compactification would not operate.

renormalization group. The starting point in this approach is a theory which is not conformally invariant but flows in the infrared limit to a fixed point of the renormalization group action. This fixed point is then the desired conformally invariant theory. Since all quantum field theories which lie in the same universality class flow to the same infrared fixed point, one looks for a family of quantum field theories as the starting point which is simpler to analyze. This will be the subject of the next chapter.

As pointed out in the previous chapter, the $(0, 2)$ Calabi-Yau compactifications are rather difficult to construct and to analyze. Because of this fact the $(0, 2)$ Calabi-Yau compactifications have received less attention in the early days of string theory than their $(2, 2)$ counterparts.

The state of affairs has changed dramatically when Witten introduced his gauged linear sigma model approach [111]. It provided a powerful tool in constructing $(0, 2)$ (and $(2, 2)$) models and in analyzing their 'phase structure'. In this chapter we will review the basic ideas of this approach without going into details.

The starting point in this approach is a $(0, 2)$ (resp. $(2, 2)$ if one wants to study $(2, 2)$ models) supersymmetric $U(1)$ gauge theory. Such a theory represents a nonconformal member of the universality class of a $(0, 2)$ (resp. $(2, 2)$) superconformal field theory. Therefore, we now begin to work out the necessary tools which we need to deal with such theories.

We will work in $(0,2)$ rigid superspace parametrized by $(z \; ; \; \bar{z}, \theta^+, \bar{\theta}^+)$, where the fermionic coordinate is complex. The canonical basis of the tangent space is defined by $(\partial \; ; \; \bar{\partial}, D_+, \bar{D}_+)$ with

$$\partial = \frac{\partial}{\partial z} \; , \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}$$

$$D_+ = \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \bar{\partial} \tag{3.1}$$

$$\bar{D}_+ = \frac{\partial}{\partial \bar{\theta}^+} + \theta^+ \bar{\partial} \; .$$

In the matter sector we have two types of $(0,2)$ multiplets, namely the chiral scalar multiplet and the chiral spinor multiplet. The $(0,2)$ chiral scalar multiplet is described by a complex scalar superfield Φ that satisfies the condition $\bar{D}_+ \Phi = 0$. Explicit θ-expansion of Φ yields

$$\Phi(z \; ; \; \bar{z}, \theta^+, \bar{\theta}^+) = \phi(z, \bar{z}) + \theta^+ \psi(z, \bar{z}) - \bar{\theta}^+ \theta^+ \bar{\partial} \phi(z, \bar{z}) \; , \tag{3.2}$$

where ϕ is a complex scalar field and ψ is a right-handed spinor field. The next matter multiplet, i.e. the $(0,2)$ chiral spinor multiplet, is described by a left-handed complex spinor superfield Λ that satisfies $\bar{D}_+ \Lambda = 0$. Its θ-expansion is given by

$$\Lambda(z \; ; \; \bar{z}, \theta^+, \bar{\theta}^+) = \lambda(z, \bar{z}) + \theta^+ G(z, \bar{z}) - \bar{\theta}^+ \theta^+ \bar{\partial} \lambda(z, \bar{z}) \; , \tag{3.3}$$

where λ is a left-handed spinor field and G is an auxiliary field. We now come to the $(0,2)$ gauge multiplet. Introducing a $U(1)$ gauge connection $\boldsymbol{A} = (A_z \; ; \; A_{\bar{z}}, A_+, \bar{A}_+)$ in superspace, one can define the covariant derivatives $\nabla, \overline{\nabla}, \nabla_+$ and $\overline{\nabla}_+$. Assuming that these covariant derivatives satisfy the following set of constraints

$$\{\nabla_+, \nabla_+\} = 0 \qquad \{\overline{\nabla}_+, \overline{\nabla}_+\} = 0 \; ,$$
$$[\, \nabla_+ \, , \, \overline{\nabla} \,] = 0 \qquad [\, \overline{\nabla}_+ \, , \, \overline{\nabla} \,] = 0 \; , \tag{3.4}$$
$$\{\nabla_+, \overline{\nabla}_+\} = 2\overline{\nabla} \; ,$$

it can be shown [20, 111] that the A_+, $\bar{A}_+$ and $A_{\bar{z}}$ components of $\boldsymbol{A}$ can be expressed in terms of a real scalar superfield V in the following way

$$A_+ = e^{-V} D_+ e^V \quad,\quad \bar{A}_+ = e^V \bar{D}_+ e^{-V} \quad,\quad A_{\bar{z}} = \frac{1}{2}(D_+ \bar{A}_+ + \bar{D}_+ A_+)\,. \quad (3.5)$$

Under a supergauge transformation V and A_z transform as $\delta V = i(\chi - \bar{\chi})$ and $\delta A_z = i(\chi + \bar{\chi})$, respectively, where χ is a chiral scalar superfield. Making use of such a supergauge transformation one can go to the so-called Wess-Zumino gauge in which

$$V = -\bar{\theta}^+ \theta^+ \bar{a} \quad,\quad A_z = a + \bar{\theta}^+ \alpha - \theta^+ \bar{\alpha} - \bar{\theta}^+ \theta^+ D\,, \quad (3.6)$$

where $a, \bar{a}$ are z- and $\bar{z}$-components of the gauge field, $\alpha, \bar{\alpha}$ are the left-handed gauginos and D is a real auxiliary field. In the Wess-Zumino gauge we still have the residual gauge symmetry under a supergauge transformation by $\chi = \eta - \bar{\theta}^+ \theta^+ \partial\eta$, where η is a real scalar field.

Let Φ and Λ carry $U(1)$ charges q and $\bar{q}$, respectively. Under a supergauge transformation Φ and Λ transform as

$$\begin{aligned}
\Phi &\to e^{2iq\chi}\Phi & \Lambda &\to e^{2i\bar{q}\chi}\Lambda\,, \\
\bar{\Phi} &\to e^{-2iq\bar{\chi}}\bar{\Phi} & \bar{\Lambda} &\to e^{-2i\bar{q}\bar{\chi}}\bar{\Lambda}\,.
\end{aligned}$$

A gauge invariant kinetic term for the bosonic matter field Φ is

$$\begin{aligned}
S_\Phi &= \frac{1}{2} \int d^2 z\, d^2\theta \quad (\partial - qA_z)e^{qV}\bar{\Phi} \cdot e^{qV}\Phi - e^{qV}\bar{\Phi} \cdot (\partial + qA_z)e^{qV}\Phi \\
&= \int d^2 z \quad (\partial - qa)\bar{\phi}\,(\bar{\partial} + q\bar{a})\phi + (\bar{\partial} - q\bar{a})\bar{\phi}\,(\partial + qa)\phi \\
&\quad\quad + \bar{\psi}(\partial + qa)\psi + q\bar{\alpha}\bar{\psi}\phi - q\alpha\psi\bar{\phi} - q\bar{\phi}\phi\,D \quad. \quad (3.7)
\end{aligned}$$

A gauge invariant kinetic term for the fermionic matter field Λ is

$$\begin{aligned}
S_\Lambda &= \frac{1}{2} \int d^2 z\, d^2\theta \ \bar{\Lambda}\,e^{\bar{q}V}\,\Lambda \\
&= \int d^2 z \quad \bar{\lambda}\,(\bar{\partial} + \bar{q}\,\bar{a})\,\lambda + \frac{1}{2}\bar{G}\,G \quad. \quad (3.8)
\end{aligned}$$

Introducing the gauge invariant field strengths

$$
\begin{aligned}
W & := [\nabla, \nabla_+] = \bar{\alpha} + \bar{\theta}^+ \, (\partial\bar{a} - \bar{\partial}a - D) \, + \, \bar{\theta}^+\theta^+ \, \bar{\partial}\bar{\alpha} \ , \\
\overline{W} & := [\nabla, \overline{\nabla}_+] = -\alpha + \theta^+ \, (\partial\bar{a} - \bar{\partial}a + D) \, + \, \bar{\theta}^+\theta^+ \, \bar{\partial}\alpha \ ,
\end{aligned}
\tag{3.9}
$$

the kinetic term for the gauge field can be written as

$$
\begin{aligned}
S_{\text{gauge}} & = -\frac{1}{2e^2} \quad d^2z \, d^2\theta \ \overline{W}\, W \\
& = -\frac{1}{2e^2} \quad d^2z \quad (\partial\bar{a} - \bar{\partial}a)^2 \, - \, D^2 \, + \, 2\,\alpha\,\bar{\partial}\bar{\alpha} \quad .
\end{aligned}
\tag{3.10}
$$

The Fayet-Iliopoulos D-term and the θ-angle term are

$$
\begin{aligned}
S_{D,\theta} & = -\frac{t}{2} \quad d^2z \, d\theta^+ \ W \, + \, \frac{\bar{t}}{2} \quad d^2z \, d\bar{\theta}^+ \ \overline{W} \\
& = r \quad d^2z \, D \, + \, \frac{i\theta}{\pi} \quad d^2z \, (\partial\bar{a} - \bar{\partial}a) \ ,
\end{aligned}
\tag{3.11}
$$

where $t = r - \frac{i\theta}{\pi}$. The $(0,2)$ superpotential is of the following form

$$
\begin{aligned}
S_{\text{w}} & = \quad d^2z \, d\theta^+ \, \Lambda \, F(\Phi) \, + \quad d^2z \, d\bar{\theta}^+ \, \bar{\Lambda} \, \overline{F(\Phi)} \\
& = \quad d^2z \quad (\, G\, F(\phi) - \frac{\partial F}{\partial \phi}\lambda\psi \,) + (\, \bar{G}\, \overline{F(\phi)} - \overline{\frac{\partial F}{\partial \phi}} \, \bar{\psi}\bar{\lambda} \,) \quad ,
\end{aligned}
\tag{3.12}
$$

where F is a homogeneous polynomial of appropriate degree so that (3.12) is gauge invariant.

We now apply these tools to construct $(0,2)$ supersymmetric $U(1)$ gauge theories. Let Φ_i $(i = 1, \ldots, n+1)$ and P be chiral scalar superfields, and Λ^a $(a = 1, \ldots, \ell+1)$ and ? be chiral spinor superfields. They represent the matter fields of the model, whose $U(1)$ charge assignment is as follows

field	Φ_i	P	Λ^a	?
charge	w_i	$-m$	q_a	$-d$

where d and m are supposed to be equal to the sum of w_i and q_a, respectively. Furthermore, w_i and q_a are taken to be positive. The action S which describes the model is

$$S = S_{\text{gauge}} + S_{\text{matter}} + S_{\mathcal{W}} + S_{D,\theta} \; ,$$

where S_{matter} denotes the sum of the kinetic terms for the bosonic and the fermionic matter fields. The simplest $(0,2)$ superpotential that one can write down has the following form

$$S_{\mathcal{W}} = \quad d^2 z \, d\theta^+ \; (?\, W(\Phi_i) + P\Lambda^a F_a(\Phi_i)) \; + \; h.c. \; , \qquad (3.13)$$

where W and F_a are homogeneous polynomials in Φ_i of degree d and $m - q_a$, respectively. For the purposes of this chapter we assume that W is a transversal polynomial[1] and that the F_a do not vanish simultaneously on $W(\phi) = 0$. Integrating out the auxiliary D field in the gauge multiplet (3.6) and the auxiliary fields in the chiral spinor superfields (cf. (3.3)), we get the scalar potential

$$U = |W(\phi_i)|^2 + |p|^2 \quad |F_a(\phi_i)|^2 + \frac{e^2}{2} \quad w_i|\phi_i|^2 - m|p|^2 - r \Bigg)^{\!2} , \quad (3.14)$$
$$ {}_a \phantom{|F_a(\phi_i)|^2 + \frac{e^2}{2} \quad} {}_i$$

where the parameter r is the coefficient in the Fayet-Iliopoulos D-term (3.11) and ϕ_i, p denote the lowest terms of the chiral scalar superfields Φ_i and P, respectively (cf. (3.2)).

Now varying the parameter r in (3.14) this model exhibits different 'phases'. By minimizing the scalar potential (3.14) for large positive r we obtain

$$w_i|\phi_i|^2 = r, \quad W(\phi_i) = 0, \quad p = 0 \; . \qquad (3.15)$$
$${}_i$$

---

[1]This means that $\phi_1 = 0, \dots, \phi_{n+1} = 0$ is the only common solution of $W(\phi) = 0$ and $\partial_i W(\phi) = 0$ $(i = 1, \dots, n+1)$.

Taking the quotient by the action of the gauge group $U(1)$ these equations describe a Calabi-Yau hypersurface X as the zero locus of the homogeneous polynomial $W(\phi_i)$ in the weighted projective space $\mathbb{P}(w_1, \dots, w_{n+1})$ with the Kähler class proportional to r. (Note that $c_1(X) = \sum_i w_i - d = 0$!)

It is noteworthy that such varieties are in general singular. We assume that the Calabi-Yau varieties that arise in this way have at worst canonical singularities. We recall that a (normal) variety X is said to have canonical singularities if rK_X is a Cartier divisor for some integer $r \geq 1$ and if $f : \tilde{X} \to X$ is a local resolution of singularities then

$$rK_{\tilde{X}} = f^*(rK_X) + \sum_j a_j E_j \;, \tag{3.16}$$

where K_X and $K_{\tilde{X}}$ are the canonical divisors of X and $\tilde{X}$, respectively, E_j are the exceptional prime divisors of f and a_j are nonnegative integers. For the time being, we ignore all issues associated with singularities in what follows. We will come back to this subject later.

There are two sets of Yukawa couplings in the action which can give mass to the spinors. One of these sets comes from the kinetic terms (3.7) for the bosonic matter fields and contains the Yukawa couplings between the gauginos and the superpartners of the scalar fields. The other set comes from the superpotential part (3.12) of the action and contains the Yukawa couplings between the right- and left-handed spinor matter fields.

The right-handed spinor π, the superpartner of p, and the left-handed spinor γ, the lowest component of the chiral spinor superfield ?, become massive through their Yukawa couplings

$$F_a(\phi)\,\lambda^a \pi \quad \text{and} \quad \frac{\partial W}{\partial \phi_i}\,\gamma \psi_i \;,$$

and drop out of the low energy theory. The right-handed massless spinors are

those which are in the kernel of the map

$$g : (\psi_1, \dots, \psi_{n+1}) \longmapsto \sum_{i=1}^{n+1} \psi_i \, \frac{\partial W}{\partial \phi_i}$$

but not in the image of the map

$$f : \ \psi \longmapsto (w_1 \phi_1 \psi, \dots, w_{n+1} \phi_{n+1} \psi) \ .$$

In other words, the right-handed massless spinors arise from the cohomology of the sequence

$$0 \to \mathcal{O} \xrightarrow{f} \bigoplus_{i=1}^{n+1} \mathcal{O}(w_i) \xrightarrow{g} \mathcal{O}(d) \to 0 \ . \tag{3.17}$$

The cohomology of this sequence defines the tangent sheaf T_X of the Calabi-Yau hypersurface X in $\mathbb{P}(w_1, \dots, w_{n+1})$.

The situation for the left-handed spinors is slightly different. There are no Yukawa couplings of λ^a with the gauginos. The left-handed massless spinors are those which lie in the kernel of the map

$$F : (\lambda^1, \dots, \lambda^{\ell+1}) \longmapsto \sum_{a=1}^{\ell+1} \lambda^a \, F_a(\phi) \ ,$$

They are, therefore, sections of the sheaf $\mathcal{E}$ defined by the exact sequence

$$0 \to \mathcal{E} \to \bigoplus_{a=1}^{\ell+1} \mathcal{O}(q_a) \xrightarrow{F} \mathcal{O}(m) \to 0 \ . \tag{3.18}$$

So we find that our gauged linear sigma model reduces for large positive r in the infrared limit to a $(0,2)$ Calabi-Yau sigma model with the target space X, a hypersurface in the weighted projective space $\mathbb{P}(w_1, \dots, w_{n+1})$, and a rank ℓ vector bundle E on X which is defined by the exact sequence (3.18). With the above choice of $U(1)$ charges, the first Chern classes of E and X vanish

$$c_1(X) = \sum_{i=1}^{n+1} w_i - d = 0 \ , \quad c_1(E) = \sum_{a=1}^{\ell+1} q_a - m = 0 \ . \tag{3.19}$$

The above geometric data still have to satisfy an important condition that comes from the cancellation of the $U(1)$ gauge anomaly. Imposing the condition $c_2(E) = c_2(X)$ guarantees this cancellation [38]. This leads, in turn, to the following quadratic Diophantine equation

$$m^2 - \sum_{a=1}^{\ell+1} q_a^2 = d^2 - \sum_{i=1}^{n+1} w_i^2 \ . \tag{3.20}$$

The conditions (3.19) and (3.20) are exactly those which have been discussed in chapter 1 in the context of nonlinear sigma models.

For a generic map F in the exact sequence (3.18), the choice of $\ell = 4$ corresponds to the unbroken gauge group $SO(10)$ in the heterotic string compactification. The massless spectrum is [36]

rep. of $SO(10)$	*cohomology group*
45	$H^\bullet(X, \mathcal{O}_X)$
16	$H^\bullet(X, \mathcal{E})$
10	$H^\bullet(X, {}^2\,\mathcal{E})$
1	$H^\bullet(X, End\,\mathcal{E})$

The net number of generations of matter fields, i.e. **16**'s of $SO(10)$, is given by the index of the twisted Dolbeault operator, $ind\,\bar\partial_\varepsilon$. According to the Riemann-Roch theorem

$$ind\,\bar\partial_\varepsilon = \int_X ch\,(\mathcal{E})\,Td\,(X)\,, \tag{3.21}$$

where $Td(X)$ denotes the Todd class of X (cf. chapter 4 for more details). Because of (3.19) the formula (3.21) reduces in our case to

$$ind\,\bar\partial_\varepsilon = \frac{1}{2}\int_X c_3(\mathcal{E})\,. \tag{3.22}$$

Now we turn to the Landau-Ginzburg phase. For large negative values of r the vanishing of the scalar potential yields

$$\phi_i = 0 \quad (\text{ for all } i \text{ }), \quad |p|^2 = \frac{-r}{m} \quad .$$

In this case p and its superpartner become massive and drop out of the low energy theory. Since the charge of p is m the gauge group $U(1)$ breaks down to the subgroup $\mathbb{Z}_m$.

We are therefore left with a $(0,2)$ Landau-Ginzburg orbifold in the infrared limit. Absorbing the vacuum expectation value of p by a trivial rescaling of the fields we get the superpotential

$$S_{\mathcal{W}} = \quad d^2 z \, d\theta \quad (\, ? \, W(\Phi_i) + \Lambda^a F_a(\Phi_i) \,) \quad . \tag{3.23}$$

Summarizing the above discussion, we have found that our simple model exhibits two different 'phases', namely the Calabi-Yau phase and the Landau-Ginzburg phase. In other words, we have found that the Calabi-Yau σ models and the Landau-Ginzburg models can be interpreted as two different phases of the same underlying theory.

It should be noted that the parameter r, which has been used to label the infrared fixed points of the renormalization group action, is actually not a renormalization group invariant. It gets additively renormalized at one loop. The beta function is proportional to the sum of $U(1)$ charges of the scalar fields [38, 111], i.e. $d - m$ in our case, which can in general be nonvanishing. In order to cancel this one loop contribution, one can add to the superpotential $S_{\mathcal{W}}$ the term

$$d^2 z \, d\theta^+ \, \Omega \, \Upsilon \, + \, h.c. \, ,$$

in which Υ is a chiral scalar superfield of $U(1)$ charge $m - d$ and Ω is a chiral spinor superfield of the opposite charge.

Obviously, the scalar potential U is modified by introducing these additional fields. However, it can be shown that by minimizing U one always obtains $v = 0$ (v is the lowest component of Υ) and that the fluctuations of Ω and Υ are massive and drop out of the low energy theory. Therefore, they do not affect the low energy physics (cf. [38] for more details on this point).

The construction that has been reviewed here can be easily generalized in many different respects [37]. We give two examples which will be of interest to us. Suppose that we take as our starting point a $(0, 2)$ supersymmetric $U(1)$ gauge theory with the following superpotential

$$S_{\mathrm{w}} = \int d^2z \, d\theta^+ \left[\Lambda^j W_j(\Phi_i) + P\Lambda^a F_a(\Phi_i) \right] + h.c. , \qquad (3.24)$$

where W_j are homogeneous polynomials of degrees d_j, Λ^j are chiral spinor superfields of $U(1)$ charges $-d_j$ and $\sum_j d_j = \sum_i w_i$.

An analysis similar to what we have done above shows that in the Calabi-Yau phase this model describes a $(0, 2)$ nonlinear sigma model whose target space X is now the complete intersection of the hypersurfaces $W_j(\phi_i) = 0$ in the weighted projective space $\mathbb{P}(w_1, \ldots, w_{n+1})$. The vector bundle E is defined as before by (3.18). The condition $c_2(E) = c_2(X)$ reads as follows

$$m^2 - \sum_a q_a^2 = \sum_j d_j^2 - \sum_i w_i^2 . \qquad (3.25)$$

As another example we consider a model in which the superpotential is given by

$$S_{\mathrm{w}} = \int d^2z \, d\theta^+ \left[\Lambda W(\Phi_i) + P_j\Lambda^a F_a^j(\Phi_i) \right] + h.c. , \qquad (3.26)$$

where F_a^j are homogeneous polynomials of degrees $m_j - q_a$, P_j are chiral scalar superfields of $U(1)$ charges $-m_j$. We further assume that $\sum_j m_j = \sum_a q_a$.

The Calabi-Yau phase of this model describes a $(0, 2)$ nonlinear sigma model whose target space is, as before, the Calabi-Yau hypersurface X defined as the

zero locus of $W(\phi_i) = 0$ in $\mathbb{P}(w_1, \dots, w_{n+1})$. The vector bundle E is defined in this case by the following exact sequence

$$0 \to \mathcal{E} \to \bigoplus_a \mathcal{O}(q_a) \xrightarrow{F} \bigoplus_j \mathcal{O}(m_j) \to 0 . \tag{3.27}$$

The condition $c_2(E) = c_2(X)$ leads to

$$\sum_j m_j^2 - \sum_a q_a^2 = d^2 - \sum_i w_i^2 . \tag{3.28}$$

In concluding this chapter we comment on some general points about the phase structure of generic gauged linear sigma models.

If we rely no more on our naive arguments in the above discussion and take the issues related to the singularities into account, then the phase picture resulting from such a model will become more complicated. As we will see below, this is mainly due to the fact that the study of a generic singular model in the framework of the gauged linear sigma models leads us to consider models with gauge group $U(1) \times \dots \times U(1)$ (N copies).

The analysis of the phase structure of such models goes exactly along the same lines as in our simple case above. Since instead of one parameter r in the scalar potential U we have now N different parameters $r_1, \dots, r_N$ which can be independently varied, this analysis becomes increasingly difficult.

As is known [3, 4, 5], there exists an 'equivalent' formulation of the whole story in terms of toric geometry[2]. It provides us with some efficient computational tools that remain tractable in more complicated examples. We discuss the relevant concepts from the toric geometry in chapter 4.

---

[2]The mathematical construction underlying the gauged linear sigma model approach is the symplectic quotient [111]. As is well-known, this can be rephrased as a holomorphic quotient which is, in turn, closely related to the toric geometry [23].

In resolving the (0,2) singularities we make intensive use of the methods of toric geometry. Hence we begin this chapter with reviewing the basic concepts of toric geometry which are necessary for understanding the geometrical setting of our discussion in later sections. They also provide us with the necessary tools for the calculations of section 4.4. For the most part we have omitted the proofs of theorems which can be found in the standard works [26, 52, 73, 87].

Toric geometry is a part of algebraic geometry in which many algebraic geometrical problems such as the construction of a resolution of singularities, calculation of intersection indices, fundamental groups and other invariants are reduced to purely combinatorial questions on certain lattices and convex polyhedral cones in them. It is noteworthy that the class of toric varieties, the basic objects of study in toric geometry, is relatively small. It suffices to mention that all toric varieties are rational!

Fans and toric varieties

Let $\mathbf{N}$ be a free Abelian group of rank d ($=$ a lattice of rank d) and $\mathbf{M} = \mathrm{Hom}(\mathbf{N}, \mathbb{Z})$ be its dual lattice. Further, let

$$\langle \cdot, \cdot \rangle : \mathbf{M} \times \mathbf{N} \to \mathbb{Z}$$

be the canonical pairing. By $\mathbf{N}_{\mathbb{R}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbf{M}_{\mathbb{R}} = \mathbf{M} \otimes_{\mathbb{Z}} \mathbb{R}$ we denote the $\mathbb{R}$-scalar extensions of $\mathbf{N}$ and $\mathbf{M}$, respectively. The $\mathbb{R}$-extension of the canonical pairing will be denoted by the same symbol $\langle \cdot, \cdot \rangle$.

Definition: A convex polyhedral cone (or simply a cone) σ in $\mathbf{N}_{\mathbb{R}}$ is a subset of the form $\sigma := \{ a_1 \boldsymbol{v}_1 + \ldots + a_k \boldsymbol{v}_k \,|\, \boldsymbol{v}_i \in \mathbf{N}_{\mathbb{R}} \,,\, a_i \in \mathbb{R}_+ \text{ for } i = 1, \ldots, k \}$.

The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are called the generators of σ. To underline this fact we write $\sigma = \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \rangle$. For cones σ and τ we denote by $\sigma \pm \tau$ the cones $\{ \boldsymbol{v} \pm \boldsymbol{w} \,|\, \boldsymbol{v} \in \sigma \,,\, \boldsymbol{w} \in \tau \}$, respectively. $\sigma + (-\sigma)$ is the smallest subspace of $\mathbf{N}_{\mathbb{R}}$ containing σ. The dimension of σ is defined as the dimension of the subspace $\sigma + (-\sigma)$. We call σ a 'big cone' if $\dim \sigma = d$.

Definition: The dual $\sigma^{\vee}$ of a cone σ is the subset $\sigma^{\vee} := \{ \boldsymbol{w} \in \mathbf{M}_{\mathbb{R}} \,|\, \langle \boldsymbol{w}, \boldsymbol{v} \rangle \geq 0 \text{ for all } \boldsymbol{v} \in \sigma \}$ [1].

A face τ of σ is the set $\{ \boldsymbol{v} \in \sigma \,|\, \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0 \text{ for some } \boldsymbol{u} \in \sigma^{\vee} \}$ which is the intersection of σ with a supporting hyperplane of it defined by $\boldsymbol{u}$. τ is also a convex polyhedral cone. It is generated by all $\boldsymbol{v}_i$ that satisfy $\langle \boldsymbol{u}, \boldsymbol{v}_i \rangle = 0$. For the dual of τ we have

$$\tau^{\vee} = \sigma^{\vee} + \mathbb{R}_+ \left(-\boldsymbol{u} \right) . \tag{4.1}$$

It is obvious that a cone has finitely many faces. The intersection of a number of faces is again a face. 'τ is a face of σ' is abbreviated by $\tau \prec \sigma$.

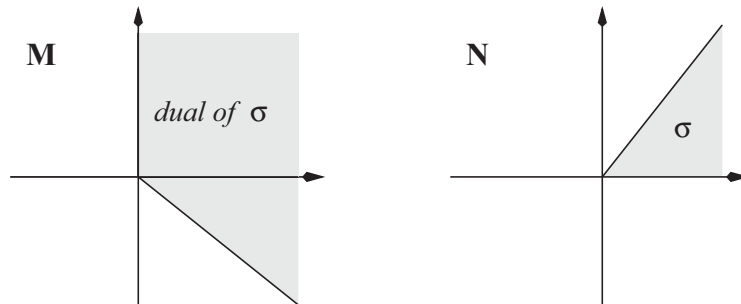[1] $\sigma^{\vee}$ is, in turn, a convex polyhedral cone in $\mathbf{M}_{\mathbb{R}}$ (Farkas' theorem).

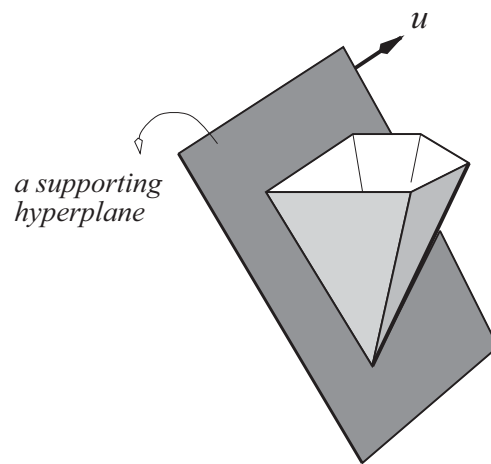Figure 4.1: A cone σ and its dual $\sigma^\vee$

Figure 4.2: A supporting hyperplane of σ and its corresponding $\boldsymbol{u}$

A cone σ is called rational if its generators are lattice vectors. The dual of such a cone is also rational. A simplicial cone is a cone whose generators are linearly independent over $\mathbb{R}$. If the generators of a simplicial cone σ can be extended to a basis of $\mathbf{N}$ then σ is said to be a basic cone.

The cospan, $\operatorname{cosp}\sigma\ (:= \sigma \cap (-\sigma))$, of σ is the greatest subspace of $\mathbf{N}_\mathbb{R}$ contained in σ. If $\operatorname{cosp}\sigma = \{0\}$, then σ is said to be strongly convex. Such a cone has an apex at the origin. It should be clear from the above definitions what one means by a strongly convex rational polyhedral cone! Note that the dual of such a cone is strongly convex if σ is a big cone.
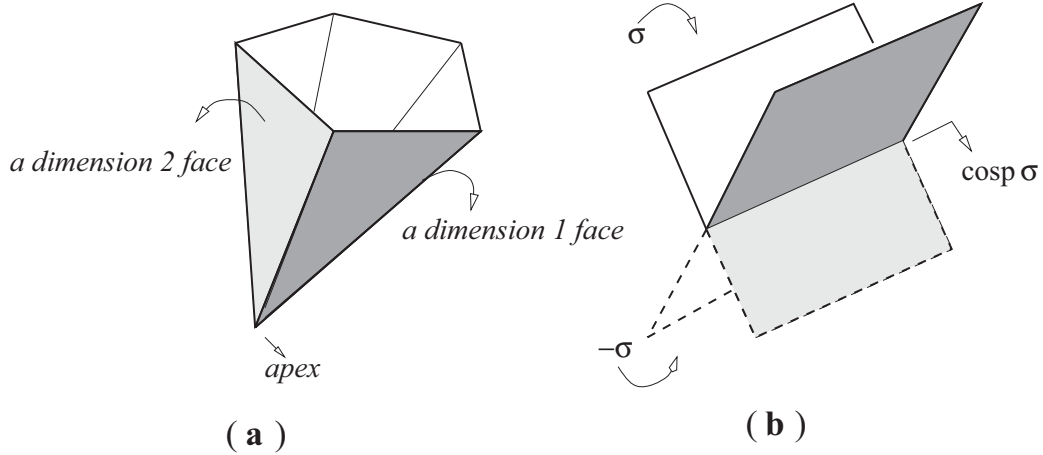
a dimension 2 face

a dimension 1 face

apex

(a)

σ

$\mathrm{cosp}\,\sigma$

$-\sigma$

(b)

Figure 4.3: (**a**) A strongly convex polyhedral cone in $\mathbb{R}^3$, (**b**) A convex polyhedral cone in $\mathbb{R}^3$

Gordan's lemma: Let σ be a convex rational polyhedral cone in $\mathbf{N}_{\mathbb{R}}$. Then $\sigma \cap \mathbf{N}$ is a finitely generated (commutative) monoid.

We sketch the simple proof of this lemma. That $\sigma \cap \mathbf{N}$ is a monoid is obvious. Without loss of generality, we assume that $\sigma = \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_k \rangle$ is simplicial. It can be easily seen that each point $\boldsymbol{v}$ of $\sigma \cap \mathbf{N}$ can be uniquely written as $\boldsymbol{v} = \boldsymbol{p} + \sum_i n_i \boldsymbol{v}_i$, where n_i are nonnegative integers and $\boldsymbol{p}$ lies in the parallelepiped $P = \{ a_1 \boldsymbol{v}_1 + \dots + a_k \boldsymbol{v}_k \,|\, 0 \leq a_i < 1 \ \text{ for } i = 1, \dots, k \}$. Now, the lemma follows from the observation that $\#(P \cap \mathbf{N})$ is finite.

Example: Take the cone σ in $\mathbb{R}^2$ generated by $\boldsymbol{v}_1 = (7, 2)$ and $\boldsymbol{v}_2 = (2, 5)$. It can be easily checked that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ together with $(1, 1), (2, 1), (3, 1), (1, 2)$ yield a (minimal) generating set for $\sigma \cap \mathbb{Z}^2$.

Let σ be a strongly convex rational cone in $\mathbf{N}_{\mathbb{R}}$. Let $A_{\sigma} := \mathbb{C}\,[\,\sigma^{\vee} \cap \mathbf{M}\,]$ be the monoid $\mathbb{C}$-algebra constructed from $\sigma^{\vee} \cap \mathbf{M}$. It consists of all expressions

$$\sum_{m \in \sigma^\vee \cap \mathbf{M}} \alpha_m \chi^m \,,$$ where $\alpha_m \in \mathbb{C}$ and almost all $\alpha_m = 0$, with multiplication determined by the addition in $\sigma^\vee \cap \mathbf{M}$: $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. It follows from the above lemma that A_σ is a finitely generated commutative $\mathbb{C}$-algebra. Moreover, A_σ has no zero divisors. As is well-known, we can associate with A_σ the affine variety

$$U_\sigma = \operatorname{Spec} A_\sigma \,.$$

We call U_σ the affine toric variety associated to the cone σ. A_σ will then be the ring of regular functions on U_σ, i.e. $A_\sigma = ?\,(U_\sigma, \mathcal{O}_{U_\sigma})$, where $\mathcal{O}_{U_\sigma}$ denotes the structure sheaf of U_σ. A_σ is integrally closed which means that the affine toric variety U_σ is normal. U_σ can be represented as a closed subvariety of some affine space $\mathbb{C}^m$. To see this, we first choose a generating set $\{\xi_1, \dots, \xi_m\}$ for A_σ and then take a (finite) presentation

$$0 \to I \to \mathbb{C}[x_1, \dots, x_m] \to \mathbb{C}[\xi_1, \dots, \xi_m] \to 0 \,,$$

where $\mathbb{C}[x_1, \dots, x_m]$ is the polynomial ring in the variables $x_1, \dots, x_m$ over $\mathbb{C}$ and I denotes the ideal of relations. Now, U_σ can be identified by the subvariety $\boldsymbol{V}(I)$ of common zeros of the polynomials in I. The points of $\boldsymbol{V}(I)$ are in one to one correspondence to the maximal ideals (= closed points) in $\operatorname{Spec} A_\sigma$.

Let $\{e_i\}_{i=1}^d$ be a basis in $\mathbf{M}$ ($\cong \mathbb{Z}^d$). A generating set for the monoid $\mathbf{M}$ is given by $e_1, \dots, e_d, -e_1, \dots, -e_d$. Associating with each $(m_1, \dots, m_d) \in \mathbf{M}$ the Laurent monomial $X_1^{m_1} \dots X_d^{m_d}$, the monoid algebra $\mathbb{C}[\mathbf{M}]$ can be identified by the $\mathbb{C}$-algebra

$$\mathbb{C}[X_1, X_1^{-1}, \dots, X_d, X_d^{-1}]$$

of Laurent polynomials in the variables $X_1, \dots, X_d$. The corresponding affine toric variety is

$$T = \operatorname{Spec} \mathbb{C}[X_1, X_1^{-1}, \dots, X_d, X_d^{-1}] \cong (\mathbb{C}^*)^d \tag{4.2}$$

which is the d-dimensional complex algebraic torus. Since each cone $\sigma^\vee$ is contained in $\mathbf{M}_\mathbb{R}$, the monoid $\sigma^\vee \cap \mathbf{M}$ is a submonoid of $\mathbf{M}$: $\sigma^\vee \cap \mathbf{M} \hookrightarrow \mathbf{M}$.

This homomorphism can be extended to a homomorphism of the $\mathbb{C}$-algebra $\mathbb{C}[\sigma^\vee \cap \mathbf{M}]$ and $\mathbb{C}[\mathbf{M}]$, which, in turn, yields an embedding of the torus T as a dense open subset in U_σ.

Note that A_σ, the $\mathbb{C}$-algebra of regular functions on U_σ, is a monoidal algebra, i.e. it is a subalgebra of $\mathbb{C}[X_1, X_1^{-1}, \ldots, X_d, X_d^{-1}]$ which is generated by finitely many Laurent monomials. It is not hard to see that the ideal of relations in a presentation of A_σ is generated by binomials[2]

$$x_1^{a_1} \ldots x_m^{a_m} - x_1^{b_1} \ldots x_m^{b_m} \ ,$$

where $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ determine a (positive) linear relation of the generating set $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ of $\sigma^\vee \cap \mathbf{M}$, i.e.

$$\sum_i a_i \, \boldsymbol{u}_i = \sum_j b_j \, \boldsymbol{u}_j \ .$$

Example: Let σ be a cone in $\mathbb{R}^2$ generated by $\boldsymbol{v}_1 = 2\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{v}_2 = \boldsymbol{e}_2$. Then the monoid $\sigma^\vee \cap \mathbf{M}$ is generated by $\boldsymbol{u}_1 = \boldsymbol{e}_1$, $\boldsymbol{u}_2 = \boldsymbol{e}_1 + \boldsymbol{e}_2$ and $\boldsymbol{u}_3 = \boldsymbol{e}_1 + 2\boldsymbol{e}_2$.
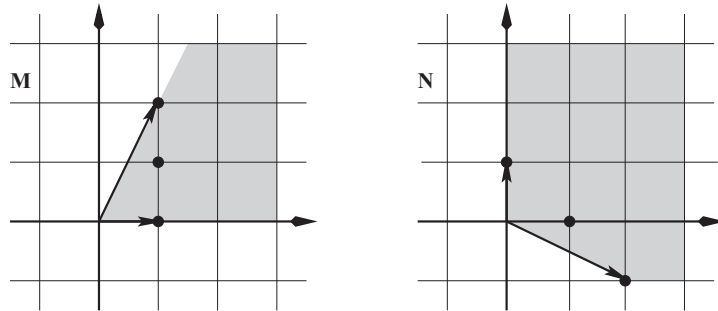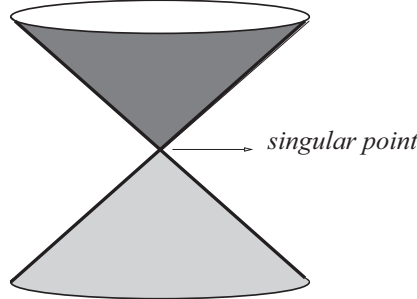

Figure 4.4: σ and its dual

Therefore, $A_\sigma = \mathbb{C}[X_1, X_1 X_2, X_1 X_2^2]$. There is a linear relation $\boldsymbol{u}_1 + \boldsymbol{u}_3 = 2\boldsymbol{u}_2$. Hence $I = \langle x_1 x_3 - x_2^2 \rangle$. U_σ is a quadratic cone with a singularity at the origin.

---

[2]For a subvariety of $\mathbb{C}^m$ to be toric, it is necessary and sufficient that it be defined by equations of the type $x_1^{a_1} \ldots x_m^{a_m} - x_1^{b_1} \ldots x_m^{b_m} = 0$.

Figure 4.5: The quadratic cone $x_1 x_3 = x_2^2$

Gluing together affine toric varieties, we obtain the toric varieties. The gluing information is encoded in a certain complex of cones in $\mathbf{N}_\mathbb{R}$ which is called a fan.

Definition: A fan Σ in $\mathbf{N}_\mathbb{R}$ is a collection of strongly convex rational polyhedral cones[3] satisfying the following conditions:

- if $\tau \prec \sigma$ and $\sigma \in \Sigma$, then $\tau \in \Sigma$,
- for any $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face of both σ and σ' .

The support $|\Sigma|$ of a fan Σ is the set $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. A fan Σ is called simplicial if it consists of simplicial cones. Σ is said to be complete if $|\Sigma| = \mathbf{N}_\mathbb{R}$. By $\Sigma^{(k)}$ we denote the set of all k-dimensional cones in Σ .

Let $\tau \prec \sigma$. Then it follows from (4.1) that $\tau^\vee \cap \mathbf{M} = \sigma^\vee \cap \mathbf{M} + \mathbb{Z}_+ (-\boldsymbol{u})$. It is obvious from this relation that A_τ is the localization of A_σ at an element in the monoid algebra associated to $\boldsymbol{u}$, i.e $\chi^{\boldsymbol{u}}$. Consequently, the corresponding affine toric variety U_τ represents a principal open subset of U_σ. Now, let τ be a common face of σ and σ'. According to the above discussion, we have an isomorphism, called a gluing map, between the principal open subsets in U_σ and $U_{\sigma'}$ defined by τ.

---

[3]The condition of strong convexity guarantees that the duals of the cones in the fan are all big cones in $\mathbf{M}_\mathbb{R}$.
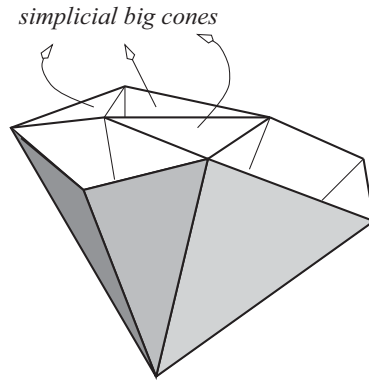
Figure 4.6: A fan in $\mathbb{R}^3$

Using these gluing data, the toric variety $\mathbb{P}_\Sigma$ can be constructed as follows. First, we take the disjoint union of the affine toric varieties U_σ, $\bigsqcup_{\sigma \in \Sigma} U_\sigma$, and then identify the points which are mapped to each other by the gluing maps. Note that the toric varieties are separated.

Example (The complex projective plane $\mathbb{P}^2$): The fan Σ consists of the following cones: $\sigma_1 = \langle e_1, e_2 \rangle$, $\sigma_2 = \langle e_2, -(e_1 + e_2) \rangle$, $\sigma_3 = \langle e_1, -(e_1 + e_2) \rangle$, $\tau_1 = \sigma_2 \cap \sigma_3 = \langle -(e_1 + e_2) \rangle$, $\tau_2 = \sigma_3 \cap \sigma_1 = \langle e_1 \rangle$, $\tau_3 = \sigma_1 \cap \sigma_2 = \langle e_2 \rangle$ and the zero-dimensional cone.
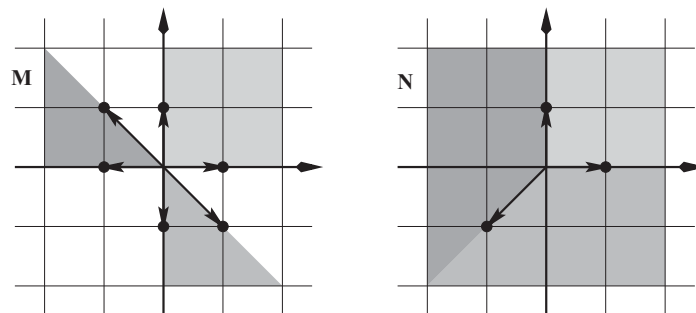

Figure 4.7: The big cones of the fan Σ and their duals

Gluing the affine pieces $U_{\sigma_1} = \operatorname{Spec} \mathbb{C}\,[\,X_1, X_2\,] \cong \mathbb{C}^2$, $U_{\sigma_2} = \operatorname{Spec} \mathbb{C}\,[\,X_1^{-1}, X_1^{-1} X_2\,] \cong \mathbb{C}^2$, $U_{\sigma_3} = \operatorname{Spec} \mathbb{C}\,[\,X_1 X_2^{-1}, X_2^{-1}\,] \cong \mathbb{C}^2$ along the principal open

subsets defined by τ_1, τ_2 and τ_3, we obtain the toric variety $\mathbb{P}^2$. Let (x_i, y_i) be the 'coordinate functions' on the affine piece U_{σ_i}. Then the transition functions are

$$x_2 = x_1^{-1} \qquad x_3 = x_2 \, y_2^{-1} \qquad x_1 = y_3 \, x_3^{-1}$$

$$y_2 = y_1 \, x_1^{-1} \qquad y_3 = y_2^{-1} \qquad y_1 = x_3^{-1} \ .$$

Generalizing this example we see that $\mathbb{P}^n$ is also a toric variety defined by the fan whose big cones are given by $\sigma_i = \langle \, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{i-1}, \boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_n, -(\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n) \rangle \rangle$ for $i = 1, \ldots, n$, and $\sigma_0 = \langle \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \rangle$. As in the above example, the $n+1$ affine toric varieties U_{σ_i} are the copies of the affine space $\mathbb{C}^n$.

Example (The weighted projective space $\mathbb{P}(w_1, \ldots, w_{n+1})$): Let $w_1, \ldots, w_{n+1}$ be relatively prime positive integers. Further, let $\{\, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1} \,\}$ be a spanning set of an n-dimensional real vector space V satisfying the linear relation $w_1 \boldsymbol{v}_1 + \ldots + w_{n+1} \boldsymbol{v}_{n+1} = 0$. Let the integer span of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1}$ define the lattice $\mathbf{N}$ whose $\mathbb{R}$-extension is obviously V. The fan Σ consists of all simplicial cones generated by proper subsets of $\{\, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1} \,\}$. The corresponding toric variety will be the weighted projective space $\mathbb{P}(w_1, \ldots, w_{n+1})$ [4].

Example (Hirzebruch surface): Let (η_0, η_1) and $(\zeta_0, \zeta_1, \zeta_2)$ be the homogeneous coordinates of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. The Hirzebruch surface $\mathbb{F}_a$, $a \in \mathbb{Z}$, is the rational ruled surface in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the equation $\eta_0^a \zeta_0 = \eta_1^a \zeta_1$. $\mathbb{F}_a$ is a toric variety whose defining fan Σ consists of the following big cones: $\sigma_1 = \langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$, $\sigma_2 = \langle -\boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$, $\sigma_3 = \langle -\boldsymbol{e}_1, a\boldsymbol{e}_1 - \boldsymbol{e}_2 \rangle$, $\sigma_4 = \langle \boldsymbol{e}_1, a\boldsymbol{e}_1 - \boldsymbol{e}_2 \rangle$.

---

[4]The weighted projective space is usually defined as a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the diagonal action of the multiplicative group $\mathbb{C}^*$:

$$(\lambda, (z_1, \ldots, z_{n+1})) \mapsto (\lambda^{w_1} z_1, \ldots, \lambda^{w_{n+1}} z_{n+1}) \quad (\lambda \in \mathbb{C}^*) \, .$$
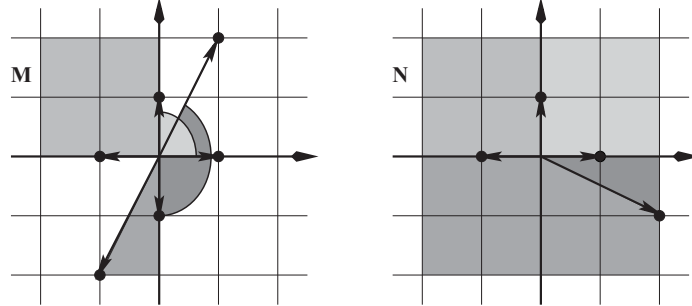
Figure 4.8: The big cones of the fan Σ and their duals

The corresponding affine toric varieties are

$$U_{\sigma_1} = \operatorname{Spec} \mathbb{C}\,[\,X_1, X_2\,] \qquad U_{\sigma_3} = \operatorname{Spec} \mathbb{C}\,[\,X_1^{-1} X_2^{-a}, X_2^{-1}\,]$$
$$U_{\sigma_2} = \operatorname{Spec} \mathbb{C}\,[\,X_1^{-1}, X_2\,] \qquad U_{\sigma_4} = \operatorname{Spec} \mathbb{C}\,[\,X_1 X_2^a, X_2^{-1}\,]\,.$$

The affine toric varieties U_{σ_1} and U_{σ_2} patch together to yield $\mathbb{P} \times \mathbb{C}$. The same holds for the other two pieces. All together one obtains a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

Divisors and invertible sheaves

We have seen that the complex algebraic torus T is contained as a dense open subset in each toric variety $\mathbb{P}_\Sigma$. The natural action of T as a group on itself can be extended to an action on $\mathbb{P}_\Sigma$. Let U_σ be an affine piece of $\mathbb{P}_\Sigma$. The (closed) points of U_σ are in one to one correspondence with the semigroup homomorphism from $\sigma^\vee \cap \mathbf{M}$ to the multiplicative semigroup $\mathbb{C}$. Let $t : \mathbf{M} \to \mathbb{C}$ and $p : \sigma^\vee \cap \mathbf{M} \to \mathbb{C}$ be two points lying in T and U_σ, respectively. Then, the action of T on U_σ is defined by

$$(t, p) \mapsto t \cdot p \quad , \quad t \cdot p\,(\boldsymbol{u}) := t\,(\boldsymbol{u})\, p\,(\boldsymbol{u})\,.$$

It can be shown that the action defined in this way is compatible with the gluing maps and therefore can be extended to an action of T on the whole $\mathbb{P}_\Sigma$.[5]

---

[5]This is the characterizing property of toric varieties, i.e. if a normal variety X contains T as a dense open subvariety and the natural action of T on itself can be extended to an

For each $\sigma \in \Sigma^{(k)}$, there is an orbit $\mathcal{O}_\sigma$ of the action of T on $\mathbb{P}_\Sigma$ which is isomorphic to $(\mathbb{C}^*)^{d-k}$. $\mathcal{O}_\sigma$ is an open subvariety of its closure $\mathcal{F}_\sigma$, which is, in turn, a closed subvariety of U_σ. In fact, $\mathcal{F}_\sigma = \operatorname{Spec} \mathbb{C}[\operatorname{cosp} \sigma^\vee \cap \mathbf{M}]$. The closed embedding $\mathcal{F}_\sigma \hookrightarrow U_\sigma$ is defined by the surjective homomorphism $\mathbb{C}[\sigma^\vee \cap \mathbf{M}] \to \mathbb{C}[\operatorname{cosp} \sigma^\vee \cap \mathbf{M}]$ coming from the extension of the map $\chi^m \mapsto \boldsymbol{cf}(m) \cdot \chi^m$, where $\boldsymbol{cf}$ is the characteristic function of the set $\operatorname{cosp} \sigma^\vee \cap \mathbf{M}$.

$\mathcal{F}_\sigma$ is again a toric variety. Its defining fan is the projection of the star of σ ($= \{\sigma' \in \Sigma \mid \sigma \prec \sigma'\}$) onto $\mathbf{N}_\mathbb{R}/\sigma + (-\sigma)$.
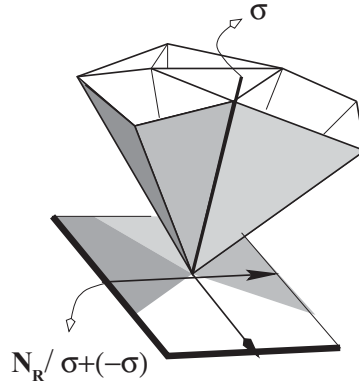
Figure 4.9: The fan describing the closed T-orbit associated to σ

Let $\Sigma^{(1)} = \{\rho_1, \ldots, \rho_n\}$ and e_i be the primitive lattice vectors on ρ_i. The (irreducible) closed codimension one T-orbits D_i associated to ρ_i freely generate the group of T-invariant Weil divisors $\operatorname{WDiv}(\mathbb{P}_\Sigma)$ in $\mathbb{P}_\Sigma$. Therefore, each T-invariant Weil divisor D can be written as

$$D = \sum_{i=1}^{n} a_i D_i,$$

where $a_1, \ldots, a_n \in \mathbb{Z}$. A T-invariant Cartier divisor D in $\mathbb{P}_\Sigma$ is defined by a family $\{m_\sigma\}_{\sigma \in \Sigma}$, where $m_\sigma \in \mathbf{M}/\mathbf{M} \cap \operatorname{cosp} \sigma^\vee$, such that if $\tau \prec \sigma$ then

action on the whole variety X, then X is of the form $\mathbb{P}_\Sigma$.

m_σ must map to m_τ under the canonical projection from $\mathbf{M} \,/\, \mathbf{M} \cap \operatorname{cosp} \sigma^\vee$ to $\mathbf{M} \,/\, \mathbf{M} \cap \operatorname{cosp} \tau^\vee$. On each open affine subset U_σ, $\sigma \in \Sigma$, D is given by

$$- \sum_{i=1}^{n} \langle\, m_\sigma, e_i \,\rangle D_i \ .$$

The group of T-invariant Cartier divisors will be denoted by $\operatorname{CDiv}(\mathbb{P}_\Sigma)$. The family $\{m_\sigma\}_{\sigma \in \Sigma}$ described above determines a continuous real function ψ_D on $|\Sigma|$ with the following properties:

- ψ_D is integral, i.e. $\psi_D(|\Sigma| \cap \mathbf{N}) \subset \mathbb{Z}$,

- ψ_D is Σ-piecewise linear, i.e. the restriction of ψ_D to each cone σ in Σ is an $\mathbb{R}$-linear function.

Conversely, any such function comes from a unique T-invariant Cartier divisor. Therefore, a T-invariant Cartier divisor D can be equivalently given by

$$D = - \sum_{i=1}^{n} \psi_D(e_i) \, D_i \ .$$

This shows that each T-invariant Cartier divisor defines a T-invariant Weil divisor.

Each $m \in \mathbf{M}$ gives a character $\chi^m : T \to \mathbb{C}^*$ and hence χ^m defines a rational function on $\mathbb{P}_\Sigma$, which, in turn, determines a principal T-invariant Cartier divisor

$$\operatorname{div}(\chi^m) = - \sum_{i=1}^{n} \langle\, m, e_i \,\rangle D_i \ .$$

Let Σ be a complete fan. Then the following theorem shows the relations between the Picard group $\operatorname{Pic}(\mathbb{P}_\Sigma)$, the Chow group $A_{d-1}(\mathbb{P}_\Sigma)$, $\operatorname{CDiv}(\mathbb{P}_\Sigma)$, $\operatorname{WDiv}(\mathbb{P}_\Sigma)$ and $\mathbf{M}$.

Theorem: There exists a commutative diagram with exact rows[6]

---

[6] The main point in this theorem is that one can compute the Picard group $\operatorname{Pic}(\mathbb{P}_\Sigma)$ (the Chow group $A_{d-1}(\mathbb{P}_\Sigma)$) by using only the T-invariant Cartier (Weil) and principal divisors.

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathrm{CDiv}\,(\mathbb{P}_\Sigma) \longrightarrow \mathrm{Pic}\,(\mathbb{P}_\Sigma) \longrightarrow 0$$

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathrm{WDiv}\,(\mathbb{P}_\Sigma) \longrightarrow A_{d-1}(\mathbb{P}_\Sigma) \longrightarrow 0$$

Moreover, the Picard group is a finitely generated free Abelian group[7].

A T-invariant Cartier divisor $D = \sum_i a_i\, D_i$, where $a_i = -\psi_D(e_i)$, determines a rational convex polyhedron Δ_D in $\mathbf{M}_\mathbb{R}$:

$$\Delta_D := \left\{\, \boldsymbol{u} \in \mathbf{M}_\mathbb{R} \mid \langle\, \boldsymbol{u}, e_i\,\rangle \geq -a_i \ \text{ for all } i = 1, \dots, n \,\right\}.$$

The points of $\Delta_D \cap \mathbf{M}$, correspond to the global sections of the (T-linearized) invertible sheaf $\mathcal{O}_{\mathbb{P}_\Sigma}(D)$ associated to D, i.e.

$$?\,(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(D)) = \bigoplus_{m \in \Delta_D \cap \mathbf{M}} \mathbb{C} \cdot \chi^m\,.$$

If the fan Σ is complete, then Δ_D is an integral (convex) polytope. In the following we assume that Σ is complete. An invertible sheaf $\mathcal{O}_{\mathbb{P}_\Sigma}(D)$ is said to be semi-ample if it is generated by its global sections. The following theorem characterizes the semi-ample invertible sheaves.

Theorem: $\mathcal{O}_{\mathbb{P}_\Sigma}(D)$ is semi-ample if and only if ψ_D is (upper) convex[8]. Moreover, ψ_D is uniquely determined by Δ_D : $\psi_D(e_i) = \min\limits_{m \in \Delta_D \cap \mathbf{M}} \langle\, m, e_i\,\rangle\,.$

Theorem: A T-invariant Cartier divisor D is very ample if and only if ψ_D is strictly convex and for every big cone σ the monoid $\sigma^\vee \cap \mathbf{M}$ is generated by

---

[7] $\mathrm{rank}\,(\mathrm{Pic}\,(\mathbb{P}_\Sigma)) \leq \mathrm{rank}\,(A_{d-1}\,(\mathbb{P}_\Sigma))$.
[8] ψ_D is upper convex $:\Longleftrightarrow \psi_D(t\boldsymbol{v} + (1-t)\boldsymbol{w}) \geq t\psi_D(\boldsymbol{v}) + (1-t)\psi_D(\boldsymbol{w})$ for $\boldsymbol{v}, \boldsymbol{w} \in |\Sigma|$ and $t \in [0,1]$.

$$\{\, m - m_\sigma \mid m \in \Delta_D \cap \mathbf{M}\,\}.$$

$D = -\sum_i \psi_D(e_i)\, D_i$ is said to be ample if some positive multiple of D is very ample. The ampleness of D is equivalent to the strict convexity of ψ_D.

Toric varieties from polytopes

Let Δ be a lattice polytope in $\mathbf{M}_\mathbb{R}$. We define a monomial subalgebra S_Δ of $\mathbb{C}[\, X_0, X_1, X_1^{-1}, \ldots, X_d, X_d^{-1}\,]$ as follows. S_Δ is spanned as a complex vector space by monomials $X_0^k X_1^{m_1} \ldots X_d^{m_d}$ such that the lattice point $m = (m_1, \ldots, m_d) \in \mathbf{M}$ belongs to $k\,\Delta$. This ring can be graded by setting

$$deg\,(\, X_0^k X_1^{m_1} \ldots X_d^{m_d}\,) = k\,.$$

The (projective) variety $\mathbb{P}_\Delta$ defined by $\mathbb{P}_\Delta = \mathrm{Proj}\, S_\Delta$ is the toric variety associated to Δ. The toric variety $\mathbb{P}_\Delta$ constructed in this way is isomorphic to a toric variety $\mathbb{P}_\Sigma$ whose defining fan Σ is the normal fan of Δ[9].

Note that the construction of a toric variety $\mathbb{P}_\Delta$ from a lattice polytope Δ also provides us with an ample T-invariant Cartier divisor corresponding to Δ. Conversely, if we want to construct a polytope Δ from a fan Σ such that $\mathbb{P}_\Delta \simeq \mathbb{P}_\Sigma$, then we will need an ample T-invariant Cartier divisor D in $\mathbb{P}_\Sigma$. Δ will be then the lattice polytope associated to D.

Singularities

First, we recall some general definitions. Let X be a normal variety. X is said to have only $\mathbb{Q}$-Gorenstein singularities if the canonical divisor K_X is $\mathbb{Q}$-Cartier, i.e. if $r K_X$ is Cartier for some integer $r \geq 1$. If K_X is Cartier, then X is said to have only Gorenstein singularities. The variety X is said to have

---

[9]The normal fan Σ of Δ is defined as follows. Let Θ be a k-dimensional face of Δ and $\boldsymbol{u} \in \mathbf{M}_\mathbb{R}$ a point lying strictly inside Θ. We set $\sigma_\Theta^\vee = \sum_{\lambda \in \mathbb{R}_+} \lambda \cdot (\Delta - \boldsymbol{u})$. Then $\sigma_\Theta \in \Sigma^{(d-k)}$. The complete fan Σ will be the set of all σ_Θ, where Θ ranges over the faces of Δ.

only canonical singularities if it is $\mathbb{Q}$-Gorenstein and if $f : \tilde{X} \to X$ is a local resolution of singularities then $rK_{\tilde{X}} = f^*(rK_X) + \sum_j a_j E_j$, where E_j are the exceptional prime divisors of f and a_j are nonnegative integers. If $a_j > 0$ for all j, then the singularities of X are said to be terminal.

We now turn to the toric varieties. Since the issue of singularity is a local one, we can restrict ourselves to the case of affine toric varieties.

Theorem: An affine toric variety U_σ, $\sigma \in \Sigma^{(k)}$, is nonsingular if and only if the cone σ is basic, in which case $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{d-k}$.

For a simplicial cone $\sigma \in \Sigma^{(k)}$ which is not a basic one, the affine toric variety U_σ is of the form $U_\sigma \cong \mathbb{C}^k / G \times (\mathbb{C}^*)^{d-k}$, where the group G is the quotient of sublattices N' and N'' of $\mathbf{N}$ which are generated by $\sigma \cap \mathbf{N}$ and $\{e_i \mid \rho_i \prec \sigma\}$, respectively. This means that U_σ for a simplicial cone σ has only quotient singularities.

The following theorem gives us a combinatorial characterization of canonical singularities in the toric case.

Theorem: The affine toric variety U_σ, $\sigma \in \Sigma^{(k)}$, has at worst canonical singularities if and only if

- there is a $\boldsymbol{u} \in \mathbf{M}_\mathbb{Q}$ such that $\langle \boldsymbol{u}, e_i \rangle = 1$ for all $\rho_i \prec \sigma$, [10]

- for each nonzero $\boldsymbol{v} \in \sigma \cap \mathbf{N}$, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq 1$.

If for each nonzero $\boldsymbol{v} \in \sigma \cap \mathbf{N}$ in the above theorem holds $\langle \boldsymbol{u}, \boldsymbol{v} \rangle > 1$, then the singularities are terminal.

As mentioned in section 3.2, we are interested in the Calabi-Yau varieties which have at worst canonical singularities. We now describe how such varieties can

---

[10]This condition states when U_σ is $\mathbb{Q}$-Gorenstein. In particular, if $\boldsymbol{u} \in \mathbf{M}$ then U_σ is Gorenstein.

be realized as hypersurfaces in toric varieties [8].

Let Δ be an integral (convex) polytope in $\mathbf{M}_{\mathbb{R}}$. Δ is reflexive if it contains the origin as an interior point and if its dual $\Delta^* := \{\, \boldsymbol{v} \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq -1$ for all $\boldsymbol{u} \in \Delta \,\}$ is also an integral polytope in $\mathbf{N}_{\mathbb{R}}$. Starting from a reflexive polytope Δ, we construct the toric variety $\mathbb{P}_{\Delta}$. The ample divisor in $\mathbb{P}_{\Delta}$ which is associated to Δ is the anticanonical divisor $-K = \sum_i D_i$. Note also that $-K$ is obviously a T-invariant Cartier divisor. Hence, $\mathbb{P}_{\Delta}$ is a Gorenstein Fano toric variety. Conversely, it can be shown that each Gorenstein Fano toric variety arises from a reflexive polytope. It is known that the Gorenstein toric varieties are canonical [90]. Consequently, the toric varieties coming from reflexive polytopes have at worst canonical singularities. Now, it follows from Bertini theorems and the adjunction formula that a generic section of the anticanonical sheaf realizes our Calabi-Yau variety.

Homogeneous coordinate ring approach

In this subsection we briefly discuss the homogeneous coordinate ring approach which is more appropriate for our field theoretical considerations. The original motivation for its development was, however, the desire to have a construction of toric varieties and related objects similar to those of $\mathbb{P}^n$ in the classical algebraic geometry [23].

As discussed previously, each one-dimensional cone ρ_i defines a T-invariant Weil divisor D_i. Assuming that the fan Σ is complete, we have seen that there is an injective map from $\mathbf{M}$ into the free Abelian group of Weil divisors $\mathrm{WDiv}\,(\mathbb{P}_{\Sigma}) = \bigoplus_{i=1}^{n} \mathbb{Z} \cdot D_i$:

$$\boldsymbol{\alpha} : \mathbf{M} \longrightarrow \bigoplus_{i=1}^{n} \mathbb{Z} \cdot D_i \;,\;\; m \mapsto \sum_{i=1}^{n} \langle m, e_i \rangle \, D_i \;. \tag{4.3}$$

The cokernel of this map coincides with the Chow group $A_{d-1}(\mathbb{P}_{\Sigma})$ which is a finitely generated Abelian group of rank $n - d$. Therefore, we have the

following exact sequence

$$0 \longrightarrow \mathbf{M} \xrightarrow{\ \boldsymbol{\alpha}\ } \bigoplus_{i=1}^{n} \mathbb{Z} \cdot D_i \xrightarrow{\ deg\ } A_{d-1}(\mathbb{P}_\Sigma) \longrightarrow 0 \ , \tag{4.4}$$

where deg denotes the canonical projection. Now consider

$$G = \mathrm{Hom}_{\mathbb{Z}}(A_{d-1}(\mathbb{P}_\Sigma), \mathbb{C}^*)$$

which is in general isomorphic to a product of $(\mathbb{C}^*)^{n-d}$ and a finite group [23]. By applying $\mathrm{Hom}_{\mathbb{Z}}(\ \cdot\ , \mathbb{C}^*)$ to (4.4) we get

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 1 \ ,$$

which defines the action of G on $\mathbb{C}^n$:

$$g \cdot (x_1, \dots, x_n) = (\, g(deg\, D_1)\, x_1, \dots, g(deg\, D_n)\, x_n \,)$$

for $g \in G$ and $(x_1, \dots, x_n) \in \mathbb{C}^n$.

Let $S = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring over $\mathbb{C}$ in the variables $x_1, \dots, x_n$, where the x_i correspond to the one-dimensional cones ρ_i in Σ. Each monomial $x_1^{a_1} \dots x_n^{a_n}$ in S determines a divisor $\sum_{i=1}^{n} a_i\, D_i$. This ring is graded in a natural way by $deg(x_i) := deg\, D_i$:

$$S = \bigoplus_{q \in A_{d-1}(\mathbb{P}_\Sigma)} S_q \ ,$$

where S_q is generated by all monomials $x_1^{a_1} \dots x_n^{a_n}$ such that $deg\, (\sum_{i=1}^{n} a_i D_i) = q$. Let B denote the monomial ideal in S generated by $\boldsymbol{x}^\sigma = \prod_{\rho_i \nless \sigma} x_i$ for all $\sigma \in \Sigma$. Note that the set of monomials $\{\boldsymbol{x}^\sigma \mid \sigma \in \Sigma^{(d)}\}$, is the unique minimal basis of this ideal. The ring S defines the n-dimensional affine space $\mathbb{A}^n = \mathrm{Spec}\, S$. The ideal B gives the variety

$$\boldsymbol{Z}_\Sigma = \boldsymbol{V}(B) \ ,$$

which is denoted as the exceptional set. Removing the exceptional set $\boldsymbol{Z}_\Sigma$ we obtain the Zariski open set

$$\boldsymbol{U}_\Sigma = \mathbb{A}^n \setminus \boldsymbol{Z}_\Sigma \ ,$$

which is invariant under the action of G. For the case of a complete simplicial fan the geometric quotient of $\boldsymbol{U}_\Sigma$ by G exists and gives rise to $\mathbb{P}_\Sigma$ [23]!

Using this construction of a toric variety $\mathbb{P}_\Sigma$, the sheaves of $\mathcal{O}_{\mathbb{P}_\Sigma}$-modules can be studied in a way similar to that of $\mathbb{P}^n$. For example, the twisting sheaves $\mathcal{O}_{\mathbb{P}_\Sigma}(\boldsymbol{p})$ are sheaves which are associated to the graded S-module $S(\boldsymbol{p})$, i.e. $\mathcal{O}_{\mathbb{P}_\Sigma}(\boldsymbol{p}) = S(\boldsymbol{p})^\sim$, where $S(\boldsymbol{p})_{\boldsymbol{q}} = S_{\boldsymbol{p}+\boldsymbol{q}}$ for all $\boldsymbol{q}$. It can be shown that $\mathcal{O}_{\mathbb{P}_\Sigma}(\boldsymbol{p}) \simeq \mathcal{O}_{\mathbb{P}_\Sigma}(D)$, where D is a T-invariant divisor with $deg\, D = \boldsymbol{p}$. In particular, $\mathcal{O}_{\mathbb{P}_\Sigma} = S^\sim$. Furthermore, we have $S \simeq \underset{\boldsymbol{p}}{} H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\boldsymbol{p}))$. We conclude this subsection with the following theorem.

Theorem: Let Σ be a complete simplicial fan. Then every coherent sheaf $\mathcal{F}$ on $\mathbb{P}_\Sigma$ is of the form $\mathcal{F} = M^\sim$, where M is a finitely generated graded S-module.

We first recall some general concepts. Let X be a nonsingular variety. An (algebraic) k-cycle ξ on X is a finite sum

$$\xi = \sum_i a_i\, S_i\ ,$$

where $a_i \in \mathbb{Z}$ and S_i are irreducible k-dimensional subvarieties of X. ξ is called effective if $a_i \geq 0$ for all i. The support $\mathrm{supp}\,\xi$ of ξ is the union of all S_i for which a_i is nonzero. The set $\mathcal{Z}_k(X)$ of all k-cycles with the addition defined in an obvious way is an Abelian group. A family of k-cycles on X with base Y, a nonsingular irreducible curve, is a $(k+1)$-cycle ξ on $X \times Y$ such that the image of $\mathrm{supp}\,\xi$ under the projection to Y is dense in Y. ξ can be considered as $\{\xi_s\}_{s\in S}$, where ξ_s are k-cycles on X.

Two cycles are said to be algebraically equivalent if there exists a family of cycles containing both of them. If the base Y of this family is $\mathbb{P}^1$, then we speak of rational equivalence of cycles. Factorizing $\mathcal{Z}_k(X)$ with respect to rational equivalence of cycles, we obtain the Chow group $A_k(X)$ of k-cycle classes on X.

We have already seen that with each $(d - k)$-dimensional cone σ there is associated the closed T-orbit $\mathcal{F}_\sigma$ which defines a k-cycle on $\mathbb{P}_\Sigma$. The following theorem describes the role of such toric k-cycles.

Theorem: $A_k(\mathbb{P}_\Sigma)$ is generated by $\mathcal{F}_\sigma$, where $\sigma \in \Sigma^{(d-k)}$.

Now we turn our attention to the discussion of the intersection ring. Let

$$A_\bullet(\mathbb{P}_\Sigma) = \bigoplus_k A_k(\mathbb{P}_\Sigma)$$

be the Chow ring of $\mathbb{P}_\Sigma$. Note that the Abelian groups $A_k(\mathbb{P}_\Sigma)$ are finitely generated. The multiplicative structure of $A_\bullet(\mathbb{P}_\Sigma)$ is given by the intersection of cycles. It is determined by the combinatorial data encoded in the fan Σ. If σ_1 and σ_2 are two cones in Σ that are faces of at least one other cone τ in the fan, then their corresponding cycles do intersect, otherwise the intersection is empty. The intersection cycle in the former case is the one that is associated to τ. It is convenient to consider $A^k(\mathbb{P}_\Sigma) := A_{d-k}(\mathbb{P}_\Sigma)$ because by intersecting cycles the codimensions add up. Therefore, we obtain the graded commutative ring

$$A^\bullet(\mathbb{P}_\Sigma) = \bigoplus_k A^k(\mathbb{P}_\Sigma) \,,$$

which is called the intersection ring of $\mathbb{P}_\Sigma$. Let $\mathbb{Q}[z_1, \dots, z_n]$ be the polynomial ring in the variables $z_1, \dots, z_n$ over $\mathbb{Q}$, where the z_i correspond to the one-dimensional cones ρ_i in the fan Σ. Further, let

$$I = \left\langle \sum_{i=1}^n \langle m, e_i \rangle \, z_i \; : \; m \in \mathbf{M} \right\rangle \quad \text{and} \quad J = \left\langle \prod z_i \; : \; \text{for all } \mathcal{P} \right\rangle \qquad (4.5)$$

be ideals in $\mathbb{Q}[z_1, \ldots, z_n]$, where $\mathcal{P}$ stands for a primitive collection. It is a subset $\{\rho_{i_1}, \ldots, \rho_{i_k}\}$ of $\Sigma^{(1)}$ which does not generate a k-dimensional cone, whereas any proper subset of it generates a cone in Σ. The ideal J is the so-called Stanley–Reisner ideal.

It can be shown that for a complete simplicial toric variety $\mathbb{P}_\Sigma$ one has the following isomorphisms for the intersection ring $A^\bullet(\mathbb{P}_\Sigma)_\mathbb{Q}$ $(= A^\bullet(\mathbb{P}_\Sigma) \otimes_\mathbb{Z} \mathbb{Q})$

$$A^\bullet(\mathbb{P}_\Sigma)_\mathbb{Q} \simeq \mathbb{Q}[z_1, \ldots, z_n] / (I + J) \simeq H^\bullet(\mathbb{P}_\Sigma, \mathbb{Q}) ,$$

where the isomorphism in the direction of the rational cohomology ring doubles the degree.

We are actually interested in the intersection ring of desingularized Calabi-Yau varieties $\tilde{X}$ which, as we will discuss below, are generic sections of the anticanonical sheaf on $\mathbb{P}_\Sigma$ ($\simeq \mathbb{P}_\Delta$). The ring $A^\bullet(\tilde{X})_\mathbb{Q}$ is isomorphic to the quotient of $A^\bullet(\mathbb{P}_\Sigma)_\mathbb{Q}$ by the annihilator of the canonical divisor [9], i.e.

$$A^\bullet(\tilde{X})_\mathbb{Q} \simeq A^\bullet(\mathbb{P}_\Sigma)_\mathbb{Q} / \operatorname{ann}(z_1 + \ldots + z_n) .$$

Let $\mathcal{F}$ be a coherent sheaf on the (complete) variety X. As is known, the Euler characteristic $\chi(X, \mathcal{F})$ of $\mathcal{F}$ is defined by

$$\chi(\mathcal{F}) = \chi(X, \mathcal{F}) = \sum_{p \geq 0} (-1)^p \, H^p(X, \mathcal{F}) .$$

The main property of the Euler characteristic is its additivity. It means that for an exact sequence of coherent sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ it holds that $\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G})$.

The Riemann-Roch theorem allows us to express the Euler characteristic of a coherent sheaf $\mathcal{F}$ on the (nonsingular) variety X in terms of the intersection of algebraic cycles on X. We now define the intersection form $(\cdot, \cdot)$ on $A^\bullet(X)$ which we need in the statement of the Riemann-Roch theorem. It is a map

$$A^\bullet(X) \times A^\bullet(X) \to \mathbb{Q}$$

that is the composition of the multiplication in $A^\bullet(X)$ and the linear functional

$$\deg\,:\,A^\bullet(X) \to \mathbb{Q}$$

defined as follows. deg associates to each cycle $\xi \in A^\bullet(X)$ the degree of its 0-cycle[11] part $\xi_0 = \sum_i a_i[P_i]$:

$$\deg(\xi_0) := \sum_i a_i\,.$$

For a nonsingular projective variety X the Riemann-Roch theorem reads

$$\chi(\mathcal{F}) = (\,\mathrm{ch}\,(\mathcal{F}), \mathrm{Td}\,(X)\,)\,,$$

where $\mathrm{ch}\,(\mathcal{F})$ is the Chern character of $\mathcal{F}$ and $\mathrm{Td}\,(X) = \mathrm{td}\,(T_X)$ is the Todd class of the tangent sheaf. $\mathrm{ch}\,(\mathcal{F})$ and $\mathrm{Td}\,(X)$ are elements of $A^\bullet(X)$! We recall that the Chern character of a coherent sheaf $\mathcal{F}$ can be defined through its locally free resolutions. For a locally free sheaf $\mathcal{E}$ of rank r we have:

$$
\begin{aligned}
\mathrm{ch}\,(\mathcal{E}) \;=\;& r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1 c_2 + 3c_3) \\
& + \frac{1}{24}(c_1^4 - 4c_1^2 c_2 + 4c_1 c_3 + 2c_2^2 - 4c_4) + \ldots \qquad (4.6) \\
\mathrm{td}\,(\mathcal{E}) \;=\;& 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1 c_2 \\
& - \frac{1}{720}(c_1^4 - 4c_1^2 c_2 - 3c_2^2 - c_1 c_3 + c_4) + \ldots \,, \qquad (4.7)
\end{aligned}
$$

where $c_i = c_i(\mathcal{E})$ is the i-th Chern class of $\mathcal{E}$.

In this section we outline a general framework for the calculation of the co-homology groups of twisting sheaves on toric varieties. To begin with, we

---

[11]On a complete variety the degree of a 0-cycle is preserved by rational equivalence.

recall some definitions and facts from the cohomology of sheaves. Let $\mathcal{F}$ be an Abelian sheaf on a topological space X. Consider the sheaf $C^0(\mathcal{F})$ defined by

$$C^0(\mathcal{F})\,(U) = \prod_{x \in U} \mathcal{F}_x \,.$$

The sheaf $\mathcal{F}$ is canonically embedded in $C^0(\mathcal{F})$ by associating to $s \in \mathcal{F}(U)$ the family $(s(x)) \in \prod_{x \in U} \mathcal{F}_x$. The sheaf $C^0(\mathcal{F})$ is always flabby. Now consider the family $\{C^n(\mathcal{F})\}_{n \geq 1}$ of sheaves that is recursively defined by

$$C^1(\mathcal{F}) := C^0(C^0(\mathcal{F})\,/\,\mathcal{F})\,,$$
$$C^{n+1}(\mathcal{F}) := C^0(C^n(\mathcal{F})\,/\,d^{n-1}\,C^{n-1}(\mathcal{F}))\,,$$

where $d^n : C^n(\mathcal{F}) \to C^{n+1}(\mathcal{F})$ is defined as a composition

$$C^n(\mathcal{F}) \longrightarrow (\,C^n(\mathcal{F})\,/\,d^{n-1}\,C^{n-1}(\mathcal{F})\,) \longrightarrow C^0(\,C^n(\mathcal{F})\,/\,d^{n-1}\,C^{n-1}(\mathcal{F})\,)\,.$$

In this way we obtain the Godement canonical flabby resolution $0 \to \mathcal{F} \to C^\bullet(\mathcal{F})$. The cohomology of the complex $C^\bullet(\mathcal{F})\,(X)$ of Abelian groups is said to be the cohomology of the sheaf $\mathcal{F}$ and is denoted by $H^\bullet(X, \mathcal{F})$. Clearly $?(X, \mathcal{F}) = H^0(X, \mathcal{F})$ and $H^p(X, \mathcal{F}) = 0$ for $p < 0$. If $\mathcal{F}$ is flabby, then $H^p(X, \mathcal{F}) = 0$ for $p > 0$. The definition of the cohomology group given here is not very enlightening. For this reason we now discuss some other methods which provide us with more useful tools for the cohomology computations.

First, we recall the definition of the Čech cohomology group. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X, where the index set is well-ordered. For an Abelian sheaf $\mathcal{F}$ on X we set

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \ldots < i_p} ?(U_{i_0} \cap \ldots \cap U_{i_p}, \mathcal{F})$$

and define $d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d^p\,\alpha)_{i_0 \ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k\,\alpha_{i_0 \ldots \hat{i}_k \ldots i_{p+1}}\big|_{U_{i_0} \cap \ldots \cap U_{i_{p+1}}}\,,$$

where the $\hat{\ }$ over i_k means that this index is to be omitted. $C^\bullet(\mathcal{U}, \mathcal{F})$ is called the Čech complex for the open covering $\mathcal{U}$ with values in $\mathcal{F}$. The cohomology groups $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$ of this complex are called the Čech cohomology groups of $\mathcal{F}$ for the open covering $\mathcal{U}$.

We are primarily concerned with the coherent sheaves (of $\mathcal{O}_X$-modules) on an algebraic variety $(X, \mathcal{O}_X)$. The following theorem of Serre plays a fundamental role in the cohomology theory of such sheaves.

Theorem (Serre): Let $\mathcal{F}$ be a quasi-coherent sheaf on an affine variety X. Then $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

If the variety X is separated, then the following theorem gives us a useful tool for the computation of cohomology groups.

Theorem: Let $\mathcal{F}$ be a quasi-coherent sheaf on a separated variety X and $\mathcal{U} = \{U_i\}_{i=0}^n$ be an open covering of X by open affine subvarieties. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$. [12]

Having reduced the calculation of cohomology to that of Čech cohomology for a finite covering, we now come to the question how these cohomology groups can be effectively calculated. The Koszul complex method provides us with the necessary tools. Before going further, we mention the following vanishing theorem of Grothendieck.

Theorem: Let X be an n-dimensional Noetherian topological space. Then, $H^p(X, \mathcal{F}) = 0$ for all $p > n$ and all Abelian sheaves $\mathcal{F}$ on X.

---

[12] The key point in the proof of this theorem is to show that $H^q(U_{i_0} \cap \ldots \cap U_{i_{p+1}}, \mathcal{F}) = 0$ for all $(p+1)$-tuples $i_0 < \ldots < i_{p+1}$ and $q > 0$. This follows already from Serre's theorem because the separability of X implies that $U_{i_0} \cap \ldots \cap U_{i_{p+1}}$ are affine.

Now, let R be a commutative ring with unit. Consider the free R-module R^d with the canonical basis $\{e_i\}_{i=1}^d$. Further, let

$$d_1 : R^d \longrightarrow R \ , \quad e_i \mapsto a_i := d_1(e_i)$$

be a homomorphism of R-modules. We are going to construct a complex of R-modules. Let $K_p(\boldsymbol{a}) = K_p(a_1, \dots, a_d) = {}^p R^d = \sum\limits_{i_1 < \dots < i_p} R\, e_{i_1} \wedge \dots \wedge e_{i_p}$.
Define $d_p : K_p(\boldsymbol{a}) \to K_{p-1}(\boldsymbol{a})$ by

$$d_p\left(e_{i_1} \wedge \dots \wedge e_{i_p}\right) = \sum_{k=1}^{p} (-1)^k d_1(e_{i_k})\, e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \wedge \dots \wedge e_{i_p} \ .$$

It can be easily seen that $d_{p-1}\, d_p = 0$. Therefore, we get the following complex

$$0 \to {}^d R^d \to {}^{d-1} R^d \to \dots \to R^d \xrightarrow{d_1} R \to 0 \ ,$$

which is called a (homological) Koszul complex. Let M be a finitely generated R-module. Then $K_\bullet(\boldsymbol{a}, M) := K_\bullet(\boldsymbol{a}) \otimes_R M$ $(K^\bullet(\boldsymbol{a}, M) := \mathrm{Hom}_R(K_\bullet(\boldsymbol{a}), M))$ is called the homological (cohomological) Koszul complex of M with respect to $(a_1, \dots, a_d)$. Its corresponding homology (cohomology) groups are denoted by $H_\bullet(\boldsymbol{a}, M)$ $(H^\bullet(\boldsymbol{a}, M))$. Note that $K^\bullet(\boldsymbol{a}, M) = \sum\limits_{i_1 < \dots < i_p} M$. Therefore, $\alpha \in K^\bullet(\boldsymbol{a}, M)$ can be considered as a sequence $\alpha = \left(m_{i_1 \dots i_p}\right)_{i_1 < \dots < i_p}$ of elements of M. The coboundary operator $d^p : K^p(\boldsymbol{a}, M) \to K^{p+1}(\boldsymbol{a}, M)$ is given by

$$\left(d^p \alpha\right)_{i_1 \dots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^k d_1\left(e_{i_k}\right) m_{i_1 \dots \hat{i}_k \dots i_{p+1}} \ .$$

It is not hard to see that $H_0(\boldsymbol{a}, M) = M / \langle a_1, \dots, a_d \rangle M$ and $H_d(\boldsymbol{a}, M) = \{ m \in M \mid \langle a_1, \dots, a_d \rangle m = 0 \}$.

Theorem: If $\langle a_1, \dots, a_d \rangle = R$, then $K_\bullet(\boldsymbol{a}, M)$ is acyclic.

A sequence $(b_1, \dots, b_r)$ in R is said to be M-regular if $\langle b_1, \dots, b_r \rangle \neq R$ and the image of b_i in $M / \langle b_1, \dots, b_{i-1} \rangle M$ is no zerodivisor for $i = 1, \dots, r$.

Theorem: If $H_p(\boldsymbol{a}, M) = 0$ for $p > r$ while $H_r(\boldsymbol{a}, M) \neq 0$, then every maximal M-sequence in $\langle a_1, \dots, a_d \rangle$ has length r. In particular, if $(a_1, \dots, a_d)$ is an M-regular sequence in R, then $H_0(\boldsymbol{a}, M) = M / \langle a_1, \dots, a_d \rangle M$ and $H_p(\boldsymbol{a}, M) = 0$ for $p = 1, \dots, d$.

Theorem: $H_p(\boldsymbol{a}, M) \simeq H^{d-p}(\boldsymbol{a}, M)$.

After these preliminaries, we now turn our attention to the relation between Koszul and Čech complexes. We will show that the Čech complex for a given covering is related to the limit of an inductive system of Koszul complexes. Let U_i be the principal open subsets in $\operatorname{Spec} R$ associated to a_i and $\mathcal{U} = \{U_i\}_{i=1}^d$. Let $U = \bigcup_{i=1}^d U_i$ and $\mathcal{F} = M^\sim$. Then $?(U_{i_1} \cap \dots \cap U_{i_p}, \mathcal{F})$ is equal to the localization of M at $a_{i_1} \dots a_{i_p}$, i.e. $M_{a_{i_1} \dots a_{i_p}}$. Now, consider the family $\{K^\bullet(\boldsymbol{a}^m, M)\}_{m \in \mathbb{N}}$ of Koszul complexes of M with respect to $\boldsymbol{a}^m := (a_1^m, \dots, a_d^m)$. Note that $d_m^p : K^p(\boldsymbol{a}^m, M) \to K^{p+1}(\boldsymbol{a}^m, M)$ is given by

$$(d_m^p \alpha)_{i_1 \dots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^k a_{i_k}^m \, m_{i_1 \dots \hat{i}_k \dots i_{p+1}}.$$

Let $f_{mn}^p : K^p(\boldsymbol{a}^m, M) \to K^p(\boldsymbol{a}^n, M)$ be defined by

$$f_{mn}^p : (m_{i_1 \dots i_p})_{i_1 < \dots < i_p} \mapsto (a_{i_1} \dots a_{i_p})^{n-m} (m_{i_1 \dots i_p})_{i_1 < \dots < i_p}$$

for $n \geq m$. It can be easily checked that $f_{mm}^p = id$ and $f_{mn}^p f_{\ell m}^p = f_{\ell n}^p$. Using the definitions of coboundary operators and f_{mn}^p one can show that the following diagram commutes

$$
\begin{array}{ccc}
K^p(\boldsymbol{a}^m, M) & \longrightarrow & K^{p+1}(\boldsymbol{a}^m, M) \\
\downarrow & & \downarrow \\
K^p(\boldsymbol{a}^n, M) & \longrightarrow & K^{p+1}(\boldsymbol{a}^n, M)
\end{array}
$$

Therefore, the family $\{K^\bullet(\boldsymbol{a}^m, M)\}_{m \in \mathbb{N}}$ together with $\{f^\bullet_{mn}\}_{m \leq n}$ build an inductive system of Koszul complexes. Let $K^\bullet(\boldsymbol{a}^\infty, M) = \varinjlim K^\bullet(\boldsymbol{a}^m, M)$.

We now define the map $\varphi^p_m : K^p(\boldsymbol{a}^m, M) \to C^{p-1}(\mathcal{U}, \mathcal{F})$ by

$$(m_{i_1 \ldots i_p}) \mapsto \frac{(m_{i_1 \ldots i_p})}{(a_{i_1} \ldots a_{i_p})^m} \quad .$$

Obviously, the following diagram

$$
\begin{array}{ccc}
K^p(\boldsymbol{a}^m, M) & & \\
 & \searrow^{\varphi^p_m} & \\
{\scriptstyle f^p_{mn}}\Big\downarrow & & C^{p-1}(\mathcal{U}, \mathcal{F}) \\
 & \nearrow^{\varphi^p_n} & \\
K^p(\boldsymbol{a}^n, M) & &
\end{array}
$$

commutes. Therefore, $C^{p-1}(\mathcal{U}, \mathcal{F})$ is a target object for the inductive system $\{K^p(\boldsymbol{a}^m, M)\}_{m \in \mathbb{N}}$. Because of the universal property of the inductive limit there exists a map $\psi^p : K^p(\boldsymbol{a}^\infty, M) \to C^{p-1}(\mathcal{U}, \mathcal{F})$ such that the diagram

$$
\begin{array}{ccccc}
K^p(\boldsymbol{a}^m, M) & & & & \\
 & \searrow^{f^p_m} & & \searrow^{\varphi^p_m} & \\
{\scriptstyle f^p_{mn}}\Big\downarrow & & K^p(\boldsymbol{a}^\infty, M) \xrightarrow{\ \psi^p\ } & & C^{p-1}(\mathcal{U}, \mathcal{F}) \\
 & \nearrow^{f^p_n} & & \nearrow^{\varphi^p_n} & \\
K^p(\boldsymbol{a}^n, M) & & & &
\end{array}
$$

commutes. We now prove that ψ^p is actually an isomorphism. Let $\beta \in C^{p-1}(\mathcal{U}, \mathcal{F})$. Then, for $i_1 < \ldots < i_p$, $\beta_{i_1 \ldots i_p} = (a_{i_1} \ldots a_{i_p})^{-m} \cdot m_{i_1 \ldots i_p}$, where $m_{i_1 \ldots i_p} \in M$. Therefore, $\alpha = (m_{i_1 \ldots i_p})_{i_1 < \ldots < i_p} \in K^p(\boldsymbol{a}^m, M)$ and $\beta = \psi^p(f^p_m(\alpha))$.

This shows the surjectivity of ψ^p. Injectivity of ψ^p : Let $f_m^p(\alpha) \in \ker \psi^p$, i.e. $\psi^p(f_m^p(\alpha)) = 0$. It follows from the commutativity of the above diagram that $\psi^p(f_m^p(\alpha)) = (\psi^p \circ f_m^p)(\alpha) = \varphi_m^p(\alpha) = 0$, i.e. $(a_{i_1} \ldots a_{i_p})^{-m} \cdot m_{i_1 \ldots i_p} = 0$ for each $i_1 < \ldots < i_p$. Consequently, there exists some k such that $(a_{i_1} \ldots a_{i_p})^k \cdot m_{i_1 \ldots i_p} = 0$. But $(a_{i_1} \ldots a_{i_p})^k \cdot m_{i_1 \ldots i_p} = (f_{m,m+k}^p(\alpha))_{i_1 \ldots i_p}$. That is $f_{m,m+k}^p(\alpha) = 0$. On the other hand, $f_m^p(\alpha) = (f_{m+k}^p \circ f_{m,m+k}^p)(\alpha) = f_{m+k}^p(f_{m,m+k}^p(\alpha)) = 0$. Hence $\ker \psi^p = \{0\}$.

Summarizing the above discussion, we can write down the following exact sequence of complexes

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\tilde{C}^\bullet(\mathcal{U},\mathcal{F}) & : \quad 0 \longrightarrow & 0 & \longrightarrow & C^0(\mathcal{U},\mathcal{F}) & \longrightarrow & C^1(\mathcal{U},\mathcal{F}) & \rightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
K^\bullet(\boldsymbol{a}^\infty,M) : \quad 0 \longrightarrow & K^0(\boldsymbol{a}^\infty,M) & \longrightarrow & K^1(\boldsymbol{a}^\infty,M) & \rightarrow & K^2(\boldsymbol{a}^\infty,M) & \rightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
M[0] & : \quad 0 \longrightarrow & K^0(\boldsymbol{a}^\infty,M) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

which yields

$$0 \to H^0(\boldsymbol{a}^\infty,M) \to M \to \check{H}^0(\mathcal{U},\mathcal{F}_{|U}) \to H^1(\boldsymbol{a}^\infty,M) \to 0$$

$$\tag{4.8}$$

$$\check{H}^{p-1}(\mathcal{U},\mathcal{F}_{|U}) \simeq H^p(\boldsymbol{a}^\infty,M) \qquad \text{for } p > 1.$$

It is noteworthy that there is a close relation between $H^p(\boldsymbol{a}^\infty,M)$ and Ext_R^p :

$$H^p(\boldsymbol{a}^\infty,M) \simeq \varinjlim \mathrm{Ext}_R^p(R/I^m,M),$$

where $I^m = \langle a_1^m, \ldots, a_d^m \rangle$.

Now, the calculation of the cohomology of twisting sheaves on $\mathbb{P}_\Sigma$ can be approached in the following way. As mentioned before, in the homogeneous

coordinate ring construction the toric variety $\mathbb{P}_\Sigma$ is realized as the geometric quotient $\boldsymbol{U}_\Sigma \,/\, G$ provided that Σ is simplicial. An open covering of $\boldsymbol{U}_\Sigma$ is given by the principal open subsets $U_i := D(\boldsymbol{x}^{\sigma_i})$, where the $\boldsymbol{x}^{\sigma_i}$ belong to the minimal basis of the ideal B :

$$\boldsymbol{U}_\Sigma = \bigcup_{\sigma_i \in \Sigma^{(d)}} U_i \ .$$

Note that the projection map $\pi : \boldsymbol{U}_\Sigma \to \mathbb{P}_\Sigma$ is an affine morphism. For such morphisms we have the following lemma.

Lemma: Let $f : X \to Y$ be an affine morphism of separated varieties X and Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then, $H^p(X, \mathcal{F}) \simeq H^p(Y, f_*\mathcal{F})$.

Let $\mathcal{U} = \{U_i\}_{i=1}^m$ be a covering of Y by open affine subsets and $\tilde{\mathcal{U}} = \{f^{-1}(U_i)\}_{i=1}^m$. It follows from the definition of $f_*\mathcal{F}$ that $C^0(\tilde{\mathcal{U}}, \mathcal{F}) = C^0(\mathcal{U}, f_*\mathcal{F})$. It is also obvious that $C^p(\tilde{\mathcal{U}}, \mathcal{F}) = C^p(\mathcal{U}, f_*\mathcal{F})$ for $p > 0$. Therefore, $\check{H}^p(\tilde{\mathcal{U}}, \mathcal{F}) = \check{H}^p(\mathcal{U}, f_*\mathcal{F})$. Since the Čech cohomology groups with respect to an affine covering coincide with the cohomology groups, we have $H^p(X, \mathcal{F}) \simeq H^p(Y, f_*\mathcal{F})$.

It follows from our discussion in the subsection on the homogeneous coordinate ring approach that $\pi_*\mathcal{O}_{\boldsymbol{U}_\Sigma} = \bigoplus_p \mathcal{O}_{\mathbb{P}_\Sigma}(\boldsymbol{p})$. Consequently, if we calculate $H^p(\boldsymbol{U}_\Sigma, \mathcal{O}_{\boldsymbol{U}_\Sigma})$ using (4.8), then the above lemma yields the desired result!

For the special case of (weighted) projective space $\mathbb{P}^d$ ($\mathbb{P}(w_1, \dots, w_{d+1})$) we find that $B = \langle x_1, \dots, x_{d+1} \rangle$. Since the sequence $(x_1, \dots, x_{d+1})$ is regular in $S = \mathbb{C}[x_1, \dots, x_{d+1}]$ we obtain

$$H^p(\boldsymbol{U}_\Sigma, \mathcal{O}_{\boldsymbol{U}_\Sigma}) = \begin{cases} S & \text{for} \quad p = 0 \\ 0 & \text{for} \quad 0 < p < d \\ \bigoplus_{i=1}^{d+1}\left(S_{x_i} \,/\, S\right) & \text{for} \quad p = d \end{cases}$$

where $\boldsymbol{U}_\Sigma = \mathbb{A}^{d+1} \setminus \{0\} = \bigcup_{i=1}^{d+1} U_i$. We have also made use of the fact that the inductive limit commutes with the cohomology.

First we give a general solution of anomaly cancellation condition found in [16, 75]. Take the following geometric data that define a rank four stable vector bundle E on a Calabi-Yau hypersurface X in $\mathbb{P}(w_1, \ldots, w_5)$:

$$ 0 \to \mathcal{E} \to \bigoplus_{a=1}^{5} \mathcal{O}(q_a) \xrightarrow{F} \mathcal{O}(m) \to 0 \ . \tag{4.9} $$

By setting $m = d$ and $\{q_1, \ldots, q_5\} = \{w_1, \ldots, w_5\}$ equation (3.20) is trivially satisfied. Assume that for one of the weights, say w_5, we have $d/w_5 \in 2\mathbb{Z} + 1$. Replace w_5 by $2w_5$ and define $w_6 := (m - w_5)/2$. Furthermore, take instead of m the new integers $m_1 := m - w_5$ and $m_2 := (m + 3w_5)/2$. One can easily check that $\{w_1, w_2, w_3, w_4, w_5; m\}$ and $\{w_1, w_2, w_3, w_4, 2w_5, w_6; m_1, m_2\}$ satisfy

$$ m^2 - \sum_{i=1}^{5} w_i^2 = m_1^2 + m_2^2 - \sum_{i=1}^{4} w_i - (2w_5)^2 - w_6^2 \ . $$

In [75] this equation has been interpreted as the anomaly cancellation condition for the defining data of a $(0,2)$ Calabi-Yau σ model with the same gauge bundle as in (4.9), defined now on a Calabi-Yau complete intersection in $\mathbb{P}(w_1, w_2, w_3, w_4, 2w_5, w_6)$ (cf. (3.25)). Contrary to this interpretation we assume that these data describe a $(0,2)$ supersymmetric $U(1)$ gauge theory whose 'Calabi-Yau phase' is determined by the following exact sequence

$$ 0 \to \mathcal{E} \to \bigoplus_{i=1}^{4} \mathcal{O}(w_i) \oplus \mathcal{O}(2w_5) \oplus \mathcal{O}(w_6) \xrightarrow{\boldsymbol{F}} \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \to 0 \tag{4.10} $$

on the same Calabi-Yau variety X in $\mathbb{P}(w_1, \ldots, w_5)$ as before (cf. (3.28)).

As pointed out in section (3.2), the Calabi-Yau varieties which have been defined as hypersurfaces in weighted projective spaces are in general singular,

whereas their corresponding physical theories are well-behaved. Therefore, our phase picture of the moduli space is not complete. To remedy this we have to desingularize our model and consider the moduli space of this new model into which the moduli space of the original model will be embedded.

In the framework of gauged linear sigma models the process of desingularization amounts above all to embedding the original theory in a new one with gauge group $U(1) \times \ldots \times U(1)$ (N copies) and $N-1$ new chiral scalar superfields $\Upsilon_1, \ldots, \Upsilon_{N-1}$ and then determining the charges of these fields with respect to the full gauge group. This new model then has N Kähler moduli parameters $r_1, \ldots, r_N$, one for each Fayet-Iliopoulos D-term of the $U(1)$ factors of the new gauge group. As mentioned before, by varying these parameters and finding the minima of the scalar potential one can recover the phase structure of the moduli space.

In section (3.2) we have already noted that there is an equivalent formulation of the whole story in terms of toric geometry [3, 4, 5]. It provides us with some efficient computational tools for analyzing the phase structure of the moduli space. The basic idea here is that the relevant information describing a theory is encoded in the combinatorial data of some reflexive polytope Δ in a lattice $\mathbf{N}$ and the phase structure of the theory is then determined by the possible triangulations of this polytope. By the Calabi-Yau phase we now mean a phase which corresponds to a maximal triangulation of Δ .

We will now describe the process of resolving the singularities. At first we have to resolve the singularities of the base variety. Our starting point is a reflexive polytope Δ in $\mathbf{N}$ which corresponds to the weighted projective space $\mathbb{P}(w_1, \ldots, w_5)$. Let Σ be the fan in $\mathbf{N}_{\mathbb{R}}$ associated to Δ. The toric variety $\mathbb{P}_\Sigma$ then has an ample anticanonical sheaf, whose generic section realizes our (canonical) Calabi-Yau variety (cf. p.43).
Taking a maximal triangulation of Δ leads in our case , i.e. $d = 4$, to a fully resolved Calabi-Yau variety $\tilde{X}$ [8]. A maximal triangulation of Δ amounts

above all to adding new one-dimensional cones to $\Sigma^{(1)}$ which are associated with the points on the faces of Δ. In the context of gauged linear sigma models these correspond to the additional chiral scalar superfields. As mentioned above, we also need to determine the charges of the fields with respect to the full gauge group. Translated into the geometric language this means that we have to determine the grading of the variables in the homogeneous coordinate ring S. Using (4.3) this can be done by solving

$$\sum_{i=1}^{n} w_i^{(k)} \boldsymbol{\alpha}_{ij} = 0$$

for all $k = 1, \ldots, N$ and $j = 1, \ldots, d$, which gives the charges $\boldsymbol{w}_i$ of the x_i : $\boldsymbol{w}_i = deg\,(x_i) = (w_i^{(1)}, \ldots, w_i^{(N)})$. Note that the desingularization of the base variety simultaneously resolves the tangent sheaf to which the right-handed fermions couple. Therefore, these fermions have the same charges as their superpartners.

What about the left-handed fermions? The geometric data of the gauge bundle E in a $(0, 2)$ model are the additional degrees of freedom which we still have to deal with. After resolving the singularities of the base variety we should pull the exact sequence (4.10) back to the desingularized base variety. We do this by taking the sequence

$$0 \to \tilde{\mathcal{E}} \to \bigoplus_{i=1}^{6} \mathcal{O}(\boldsymbol{q}_i) \xrightarrow{\tilde{\boldsymbol{F}}} \bigoplus_{j=1}^{2} \mathcal{O}(\boldsymbol{p}_j) \to 0 \,, \tag{4.11}$$

which reduces to (4.10) if we remove the exceptional divisors that arise in the process of desingularization. We impose the same conditions as before on these geometric data guaranteeing the existence of spinors and the cancellation of gauge anomalies. The conditions $c_1(\tilde{T}_X) = c_1(\tilde{\mathcal{E}}) = 0$ result in

$$\boldsymbol{d} = \sum_{i=1}^{5} \boldsymbol{w}_i \quad \text{and} \quad \boldsymbol{p}_1 + \boldsymbol{p}_2 = \sum_{i=1}^{6} \boldsymbol{q}_i \,, \tag{4.12}$$

where $\tilde{T}_X$ denotes the resolved tangent sheaf. The anomaly cancellation con-

dition leads to the following Diophantine equations:

$$d^{(l)}d^{(k)} - \sum_{i=1}^{5} w_i^{(l)} w_i^{(k)} = \sum_{j=1}^{2} p_j^{(l)} p_j^{(k)} - \sum_{i=1}^{6} q_i^{(l)} q_i^{(k)} \tag{4.13}$$

for $l, k = 1, \ldots, N$. Each solution of these equations gives possible gauge bundle data for the desingularized theory. One should be careful about the exactness of (4.11). It may happen that one can not choose the polynomials in $\tilde{F}$ such that not all of them vanish simultaneously on the base variety. If this is the case, then one has to deal with a sheaf which is no longer locally free. The examples that will be considered here avoid this problem.

We now come to the discussion of a few examples which belong to the class of models given by the geometric data of (4.10) on a Calabi-Yau variety in the weighted projective space $\mathbb{P}(w_1, \ldots, w_5)$.

Example 1 : $\mathbb{P}(1, 1, 1, 3, 3)$

The gauge bundle E is given by

$$0 \to \mathcal{E} \to \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(3) \oplus \mathcal{O}(6) \oplus \mathcal{O}(3) \to \mathcal{O}(6) \oplus \mathcal{O}(9) \to 0 \ .$$

The reflexive polytope Δ corresponding to $\mathbb{P}(1, 1, 1, 3, 3)$ is defined by the vertices

$$e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0)$$
$$e_4 = (0, 0, 0, 1), e_5 = (-1, -1, -3, -3)$$

with respect to the canonical basis in the lattice $\mathbf{N}$. Apart from the unique inner point, Δ still has one additional point e_6 which lies on the codimension 2 face of Δ generated as the convex hull of the points e_1, e_2 and e_5 : $e_6 = (0, 0, -1, -1)$. Therefore, the desingularization in this case gives rise to an extra $U(1)$ factor. We now proceed to find the charges of the fields. As before

we denote by x_i and D_i the variables in S and the divisors associated to e_i. In the canonical basis of $\mathbf{M}$, the map $\boldsymbol{\alpha}$ is represented by

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & -1 & -3 & -3 \\
0 & 0 & -1 & -1
\end{pmatrix} ,
$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	$(1,0)$	$(1,0)$	$(3,1)$	$(3,1)$	$(1,0)$	$(0,1)$

Using this table we determine in the next step the data of the resolved gauge bundle $\tilde{E}$. From (4.12) and (4.13) we obtain

$$q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 = 0 \ ,$$
$$q_1 + q_2 + q_3 + 3q_4 + 6q_5 + 3q_6 - 6p_1 - 9p_2 = -12 \ ,$$
$$q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 = -2 \ ,$$

where we have dropped the upper index '2' on q's and p's. Here is a set of solutions of the above Diophantine equations

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
0	-1	1	1	4	0	2	3
0	0	0	1	0	1	0	2
1	1	0	0	0	0	2	0
0	0	0	1	1	1	1	2

which results in

(a)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,0) \oplus \mathcal{O}(1,-1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(6,4)$ $\oplus \mathcal{O}(3,1) \oplus \mathcal{O}(3,0) \to \mathcal{O}(6,2) \oplus \mathcal{O}(9,3) \to 0$
(b)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,0) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(3,1)$ $\oplus \mathcal{O}(6,0) \oplus \mathcal{O}(3,1) \to \mathcal{O}(6,0) \oplus \mathcal{O}(9,2) \to 0$
(c)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(3,0)$ $\oplus \mathcal{O}(6,0) \oplus \mathcal{O}(3,0) \to \mathcal{O}(6,2) \oplus \mathcal{O}(9,0) \to 0$
(d)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,0) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(3,1)$ $\oplus \mathcal{O}(6,1) \oplus \mathcal{O}(3,1) \to \mathcal{O}(6,1) \oplus \mathcal{O}(9,2) \to 0$

We now consider a fan Σ corresponding to the maximal triangulation of Δ given by the 'big cones'

$$\langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_6 \rangle, \langle e_1 e_2 e_4 e_6 \rangle,$$
$$\langle e_1 e_3 e_4 e_5 \rangle, \langle e_1 e_3 e_5 e_6 \rangle, \langle e_1 e_4 e_5 e_6 \rangle,$$
$$\langle e_2 e_3 e_4 e_5 \rangle, \langle e_2 e_3 e_5 e_6 \rangle, \langle e_2 e_4 e_5 e_6 \rangle,$$

where $\langle e_i e_j e_k e_l \rangle$ denotes the cone generated by e_i, e_j, e_k and e_l. The primitive collections of Σ are $\{e_3, e_4, e_6\}$ and $\{e_1, e_2, e_5\}$. With these combinatorial data at hand we can write down the ideals I and J:

$$I = \langle z_1 - z_5, z_2 - z_5, z_3 - 3z_5 - z_6, z_4 - 3z_5 - z_6 \rangle \ , \quad J = \langle z_1 z_2 z_5, z_3 z_4 z_6 \rangle \ .$$

Because we are going to make calculations in a polynomial ring, it is convenient to use the Gröbner basis method [1, 22, 44]. A Gröbner basis of $I + J$ with respect to the lex order $z_1 > \ldots > z_6$ is given by

$$I + J \ = \langle \ \ z_1 - z_5 \ , \ z_2 - z_5 \ , \ z_3 - 3z_5 - z_6 \ , \ z_4 - 3z_5 - z_6,$$
$$z_5^3 \ , \ 9z_5^2 z_6 + 6z_5 z_6^2 + z_6^3 \ , \ 9z_5 z_6^3 + 2z_6^4 \ , \ z_6^5 \ \rangle \ .$$

Let K be the ideal in the polynomial ring $\mathbb{Q}[z_1, \ldots, z_6]$ generated by $z_1 + \ldots + z_6$, which is a representative of the canonical class in the intersection ring $A^\bullet(\mathbb{P}_\Sigma)$. Then, the annihilator of $z_1 + \ldots + z_6$ in $A^\bullet(\mathbb{P}_\Sigma)$ is given by

$$\mathrm{ann}\,(z_1 + \ldots + z_6) = (I + J) : K \;,$$

where ':' denotes the quotient of ideals. $\mathrm{ann}\,(z_1 + \ldots + z_6)$ in this example is

$$\mathrm{ann}\,(z_1 + \ldots + z_6) \;=\; \big\langle \;\; z_1 + z_2 + z_3 + z_4 - 8z_5 - 2z_6\,,\; z_2 - z_5,$$
$$z_3 - 3z_5 - z_6\,,\; z_4 - 3z_5 - z_6\,,\; z_5^3\,,\; 3z_5 z_6 + z_6^2\,,\; z_6^4 \;\big\rangle\;.$$

A look at (4.6) and (4.7) shows that in the product of $\mathrm{ch}\,(\tilde{\mathcal{E}})$ and $\mathrm{Td}\,(\tilde{X})$ in the intersection ring $A^\bullet(\tilde{X})$ the only 0-cycle part is given by $1/2\; c_3(\tilde{\mathcal{E}})$. Applying the degree functional to this term gives us the Euler characteristic of the respective gauge bundle. One should be careful about the normalization of the product of cycles. For the big cones of Σ the normalization is fixed by

$$\langle D_{i_1} \ldots D_{i_4} \rangle = \frac{1}{\mathrm{mult}(e_{i_1}, \ldots, e_{i_4})} \;, \tag{4.14}$$

where $\mathrm{mult}\,(e_{i_1}, \ldots, e_{i_4})$ denotes the index in $\mathbf{N}$ of the lattice spanned by these vectors. Multiplying the top terms in the intersection ring of the desingularized Calabi-Yau variety by the representative of the canonical divisor and using (4.14) together with the 'algebraic moving lemma' [26, 52] yields the normalization in $A^\bullet(\tilde{X})$.

All big cones in this example have volume one. Therefore, $\langle D_{i_1} \ldots D_{i_4} \rangle = 1$ for all big cones in Σ. As we will see below, the third Chern class of the gauge bundle is represented by a degree three monomial in z_6. Its normalization in $A^\bullet(\tilde{X})$ is given by $\langle z_6^3 \rangle = 27$. We have summarized the result of the calculations for the resolved bundles found above in the following table

$\tilde{\mathcal{E}}$	$c_3(\tilde{\mathcal{E}})$	$\chi(\tilde{\mathcal{E}})$
(a)	$-\frac{20}{3}z_6^3$	-90
(b)	$-8z_6^3$	-108
(c)	$-\frac{22}{9}z_6^3$	-33
(d)	$-8z_6^3$	-108

Example 2 : $\mathbb{P}(1,1,2,2,3)$

The gauge bundle E is given by

$$0 \to \mathcal{E} \to \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(6) \oplus \mathcal{O}(3) \to \mathcal{O}(6) \oplus \mathcal{O}(9) \to 0 \ .$$

The reflexive polytope Δ corresponding to $\mathbb{P}(1,1,2,2,3)$ is defined by the vertices

$$e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0)$$
$$e_4 = (0,0,0,1), e_5 = (-1,-2,-2,-3), e_6 = (0,-1,-1,-1) \ .$$

This exhausts the set of boundary points of Δ. The desingularization gives rise, as before, to an extra $U(1)$ factor. The map $\boldsymbol{\alpha}$ is represented by

$$\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & -2 & -2 & -3 \\
0 & -1 & -1 & -1
\end{array} \ ,$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	$(1,0)$	$(2,1)$	$(2,1)$	$(3,1)$	$(1,0)$	$(0,1)$

Using this table we obtain the following equations for the data of the resolved gauge bundle $\tilde{E}$

$$q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 = 0 \ ,$$
$$q_1 + q_2 + 2q_3 + 2q_4 + 6q_5 + 3q_6 - 6p_1 - 9p_2 = -20 \ ,$$
$$q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 = -6 \ .$$

Two solutions of these equations are

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
1	1	1	0	2	0	3	2
2	0	1	0	3	0	4	2

which lead to

(a)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(2,1) \oplus \mathcal{O}(2,0)$
	$\oplus \mathcal{O}(6,2) \oplus \mathcal{O}(3,0) \to \mathcal{O}(6,3) \oplus \mathcal{O}(9,2) \to 0$
(b)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,2) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(2,1) \oplus \mathcal{O}(2,0)$
	$\oplus \mathcal{O}(6,3) \oplus \mathcal{O}(3,0) \to \mathcal{O}(6,4) \oplus \mathcal{O}(9,2) \to 0$

The big cones of the fan Σ corresponding to the maximal triangulation of Δ are

$$\langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_5 \rangle, \langle e_1 e_2 e_4 e_6 \rangle,$$
$$\langle e_1 e_2 e_5 e_6 \rangle, \langle e_1 e_3 e_4 e_6 \rangle, \langle e_1 e_3 e_5 e_6 \rangle,$$
$$\langle e_2 e_3 e_4 e_5 \rangle, \langle e_2 e_4 e_5 e_6 \rangle, \langle e_3 e_4 e_5 e_6 \rangle \ .$$

The primitive collections of Σ are $\{e_2, e_3, e_6\}$ and $\{e_1, e_4, e_5\}$. Using these combinatorial data we find $I = \langle\ z_1 - z_5\ ,\ z_2 - 2z_5 - z_6\ ,\ z_3 - 2z_5 - z_6\ ,\ z_4 - 3z_5 - z_6\ \rangle$ and $J = \langle\ z_1 z_4 z_5\ ,\ z_2 z_3 z_6\ \rangle$. With respect to the lex order $z_1 > \ldots > z_6$ the Gröbner bases of $I + J$ and $\mathrm{ann}\,(z_1 + \ldots + z_6)$ are given by

$$
\begin{aligned}
I + J\ &=\ \langle\quad z_1 - z_5\ ,\ z_2 - 2z_5 - z_6\ ,\ z_3 - 2z_5 - z_6\ , \\
&\qquad z_4 - 3z_5 - z_6\ ,\ 3z_5^3 + z_5^2 z_6\ ,\ 4z_5^2 z_6 + \\
&\qquad 4z_5 z_6^2 + z_6^3\ ,\ 5z_5 z_6^3 + 2z_6^4\ ,\ z_6^5\ \rangle\ , \\
\mathrm{ann}\,(z_1 + \ldots + z_6)\ &=\ \langle\quad z_1 + z_2 + z_3 + z_4 - 8z_5 - 3z_6\ ,\ z_2 - 2z_5 - z_6\ , \\
&\qquad z_3 - 2z_5 - z_6\ ,\ z_4 - 3z_5 - z_6\ ,\ z_5^2 + 7z_5 z_6 + 4z_6^2\ , \\
&\qquad 8z_5 z_6^2 + 5z_6^3\ ,\ z_6^4\ \rangle\ .
\end{aligned}
$$

The normalization in this case is as follows: $\langle D_1 D_2 D_3 D_5 \rangle = \frac{1}{3}$ and all other big cones have unit volume. This leads to the normalization $\langle z_6^3 \rangle = 8$ in the intersection ring $A^\bullet(\tilde{X})$. Therefore, we obtain

$\tilde{\mathcal{E}}$	$c_3(\tilde{\mathcal{E}})$	$\chi(\tilde{\mathcal{E}})$
(a)	$-\frac{51}{4} z_6^3$	-51
(b)	$-\frac{21}{2} z_6^3$	-42

Example 3 : $\mathbb{P}(1, 2, 2, 3, 4)$

The gauge bundle E is given by

$$
0 \to \mathcal{E} \to \mathcal{O}(1) \oplus \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(3) \oplus \mathcal{O}(8) \oplus \mathcal{O}(4) \to \mathcal{O}(8) \oplus \mathcal{O}(12) \to 0\ .
$$

The reflexive polytope Δ corresponding to $\mathbb{P}(1, 2, 2, 3, 4)$ is defined by the vertices

$$
\begin{aligned}
e_1 &= (1, 0, 0, 0),\ e_2 = (0, 1, 0, 0),\ e_3 = (0, 0, 1, 0) \\
e_4 &= (0, 0, 0, 1),\ e_5 = (-2, -2, -3, -4)\ .
\end{aligned}
$$

Δ still has one other boundary point $e_6 = (-1, -1, -1, -2)$ which lies on the codimension three face generated by e_3 and e_5. The desingularization gives rise as before to an extra $U(1)$ factor. The map $\boldsymbol{\alpha}$ is represented by

$$
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-2 & -2 & -3 & -4 \\
-1 & -1 & -1 & -2
\end{array} \quad ,
$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	$(2,1)$	$(2,1)$	$(3,1)$	$(4,2)$	$(1,0)$	$(0,1)$

Therefore we obtain the following equations for the data of the resolved gauge bundle $\tilde{E}$

$$
q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 = 0 \ ,
$$
$$
q_1 + 2q_2 + 2q_3 + 3q_4 + 8q_5 + 4q_6 - 8p_1 - 12p_2 = -45 \ ,
$$
$$
q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 = -18 \ .
$$

Two solutions of these equations are

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
1	1	1	2	0	2	5	2
2	0	2	3	0	0	6	1

which lead to

(a)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,1) \oplus \mathcal{O}(2,1) \oplus \mathcal{O}(2,1) \oplus \mathcal{O}(3,2)$
	$\oplus \mathcal{O}(8,0) \oplus \mathcal{O}(4,2) \to \mathcal{O}(8,5) \oplus \mathcal{O}(12,2) \to 0$
(b)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,2) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(2,2) \oplus \mathcal{O}(3,3)$
	$\oplus \mathcal{O}(8,0) \oplus \mathcal{O}(4,0) \to \mathcal{O}(8,6) \oplus \mathcal{O}(12,1) \to 0$

The big cones of the fan Σ corresponding to the maximal triangulation of Δ are

$$\langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_6 \rangle, \langle e_1 e_2 e_4 e_5 \rangle,$$
$$\langle e_1 e_2 e_5 e_6 \rangle, \langle e_1 e_3 e_4 e_6 \rangle, \langle e_1 e_4 e_5 e_6 \rangle,$$
$$\langle e_2 e_3 e_4 e_6 \rangle, \langle e_2 e_4 e_5 e_6 \rangle .$$

The primitive collections of Σ are $\{e_3, e_5\}$ and $\{e_1, e_2, e_4, e_6\}$. Using these combinatorial data we find $I = \langle\ z_1 - 2z_5 - z_6\ ,\ z_2 - 2z_5 - z_6\ ,\ z_3 - 3z_5 - z_6\ ,\ z_4 - 4z_5 - 2z_6\ \rangle$ and $J = \langle\ z_3 z_5\ ,\ z_1 z_2 z_4 z_6\ \rangle$. With respect to the lex order $z_1 > \ldots > z_6$ the Gröbner bases of $I + J$ and $\mathrm{ann}\,(z_1 + \ldots + z_6)$ are given by

$$
\begin{aligned}
I + J \ &= \langle\ z_1 - 2z_5 - z_6\ ,\ z_2 - 2z_5 - z_6\ ,\ z_3 - 3z_5 - z_6\ ,\\
&\qquad z_4 - 4z_5 - 2z_6\ ,\ 3z_5^2 + z_5 z_6\ ,\ 26 z_5 z_6^3 + 9 z_6^4\ ,\ z_6^5\ \rangle\ ,\\
\mathrm{ann}\,(z_1 + \ldots + z_6) \ &= \langle\ z_1 + z_2 + z_3 + z_4 - 11 z_5 - 5 z_6\ ,\ z_2 - 2z_5 - z_6\ ,\\
&\qquad z_3 - 3z_5 - z_6\ ,\ z_4 - 4z_5 - 2z_6\ ,\ 3z_5^2 + z_5 z_6\ ,\\
&\qquad 8 z_5 z_6^2 + 3 z_6^3\ ,\ z_6^4\ \rangle\ .
\end{aligned}
$$

The normalization in this case is as follows:

$$\langle D_1 D_2 D_3 D_6 \rangle = \langle D_1 D_2 D_5 D_6 \rangle = \frac{1}{2}\quad,\quad \langle D_1 D_2 D_4 D_5 \rangle = \frac{1}{3}$$

and all other big cones have unit volume. This results in the normalization $\langle z_6^3 \rangle = -24$ in the intersection ring $A^\bullet(\tilde{X})$. Therefore, we obtain

$\tilde{\mathcal{E}}$	$c_3(\tilde{\mathcal{E}})$	$\chi(\tilde{\mathcal{E}})$
(a)	$-\frac{7}{2}z_6^3$	**42**
(b)	z_6^3	**-12**

Example 4 : $\mathbb{P}(1,1,3,3,4)$

The gauge bundle E is given by

$$0 \to \mathcal{E} \to \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(3)^{\oplus 2} \oplus \mathcal{O}(8) \oplus \mathcal{O}(4) \to \mathcal{O}(8) \oplus \mathcal{O}(12) \to 0 ,$$

and the vertices

$$e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0)$$
$$e_4 = (0,0,0,1), e_5 = (-1,-3,-3,-4) .$$

define the reflexive polytope Δ corresponding to $\mathbb{P}(1,1,3,3,4)$. There still exists one other boundary point $e_6 = (0,-1,-1,-1)$ of Δ which lies on the codimension two face spanned by e_1, e_4 and e_5. An extra $U(1)$ factor arises from the desingularization. The map $\boldsymbol{\alpha}$ is represented by

$$\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & -3 & -3 & -4 \\
0 & -1 & -1 & -1
\end{array} ,$$

which yields

field	x_1	x_2	x_3	x_4	x_5	x_6
charge	$(1,0)$	$(3,1)$	$(3,1)$	$(4,1)$	$(1,0)$	$(0,1)$

This leads to the following equations for the data of the resolved gauge bundle $\tilde{E}$

$$q_1 + q_2 + q_3 + q_4 + q_5 + q_6 - p_1 - p_2 = 0 \ ,$$
$$q_1 + q_2 + 3q_3 + 3q_4 + 8q_5 + 4q_6 - 8p_1 - 12p_2 = -26 \ ,$$
$$q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - p_1^2 - p_2^2 = -6 \ .$$

Two solutions of these equations are

q_1	q_2	q_3	q_4	q_5	q_6	p_1	p_2
1	1	0	0	2	1	3	2
2	0	0	0	3	1	4	2

which result in

(a)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(3,0) \oplus \mathcal{O}(3,0)$
	$\oplus \mathcal{O}(8,2) \oplus \mathcal{O}(4,1) \to \mathcal{O}(8,3) \oplus \mathcal{O}(12,2) \to 0$
(b)	$0 \to \tilde{\mathcal{E}} \to \mathcal{O}(1,2) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(3,0) \oplus \mathcal{O}(3,0)$
	$\oplus \mathcal{O}(8,3) \oplus \mathcal{O}(4,1) \to \mathcal{O}(8,4) \oplus \mathcal{O}(12,2) \to 0$

The big cones of the fan Σ corresponding to the maximal triangulation of Δ are

$$\langle e_1 e_2 e_3 e_4 \rangle, \langle e_1 e_2 e_3 e_5 \rangle, \langle e_1 e_2 e_4 e_6 \rangle,$$
$$\langle e_1 e_2 e_5 e_6 \rangle, \langle e_1 e_3 e_4 e_6 \rangle, \langle e_1 e_3 e_5 e_6 \rangle,$$
$$\langle e_2 e_3 e_4 e_5 \rangle, \langle e_2 e_4 e_5 e_6 \rangle, \langle e_3 e_4 e_5 e_6 \rangle \ .$$

The primitive collections of Σ are $\{e_2, e_3, e_6\}$ and $\{e_1, e_4, e_5\}$. From these combinatorial data we find $I = \langle \ z_1 - z_5 \ , \ z_2 - 3z_5 - z_6 \ , \ z_3 - 3z_5 - z_6 \ , \ z_4 - 4z_5 - z_6 \ \rangle$ and $J = \langle \ z_1 z_4 z_5 \ , \ z_2 z_3 z_6 \ \rangle$. With respect to the lex order $z_1 > \ldots > z_6$

the Gröbner bases of $I + J$ and $\text{ann}\,(z_1 + \ldots + z_6)$ are given by

$$
\begin{aligned}
I + J \ =\ \langle\ & z_1 - z_5\ ,\ z_2 - 3z_5 - z_6\ ,\ z_3 - 3z_5 - z_6\ , \\
& z_4 - 4z_5 - z_6\ ,\ 4z_5^3 + z_5^2 z_6\ ,\ 9z_5^2 z_6 + \\
& 6z_5 z_6^2 + z_6^3\ ,\ 18z_5 z_6^3 + 5z_6^4\ ,\ z_6^5\ \rangle\ , \\
\text{ann}\,(z_1 + \ldots + z_6) \ =\ \langle\ & z_1 + z_2 + z_3 + z_4 - 11z_5 - 3z_6\ ,\ z_2 - 3z_5 - z_6\ , \\
& z_3 - 3z_5 - z_6\ ,\ z_4 - 4z_5 - z_6\ ,\ 4z_5^3 + z_5^2 z_6\ , \\
& 3z_5 z_6 + z_6^2\ ,\ z_6^4\ \rangle\ .
\end{aligned}
$$

The normalization in this case is as follows: $\langle D_1 D_2 D_3 D_5 \rangle = \tfrac{1}{4}$ and all other big cones have unit volume. This leads to the normalization $\langle z_6^3 \rangle = 36$ in the intersection ring $A^\bullet(\tilde{X})$. Therefore, we obtain

$\tilde{\mathcal{E}}$	$c_3(\tilde{\mathcal{E}})$	$\chi(\tilde{\mathcal{E}})$
(a)	$-\frac{26}{9} z_6^3$	-52
(b)	$-\frac{20}{9} z_6^3$	-40

Let $k[x_1, \ldots, x_m]$ denote a polynomial ring in m variables $x_1, \ldots, x_m$ over the field k. To each monomial $x^\alpha = x_1^{\alpha_1} \ldots x_m^{\alpha_m}$ we associate the element $(\alpha_1, \ldots, \alpha_m)$ in the semigroup $(\mathbb{N}^m, +)$.

Definition: A *monomial ordering* in $k[x_1, \ldots, x_m]$ is an order relation on the set of monomials induced by a total ordering $>$ on $\mathbb{N}^m$ which is consistent with its semigroup structure and such that $>$ is a well-ordering on $\mathbb{N}^m$.

Here are two examples:

- *lex*(icographic) order: $x^\alpha >_{lex} x^\beta :\Leftrightarrow$ (the left-most nonzero entry in $\alpha - \beta$ is positive),

- g(raded) *lex* order: $x^\alpha >_{glex} x^\beta :\Leftrightarrow (\sum_i \alpha_i > \sum_i \beta_i)$ or $(\alpha = \beta$ and $x^\alpha >_{lex} x^\beta)$.

Given a nonzero polynomial $f = \sum_\alpha a_\alpha x^\alpha$ in $k[x_1, \ldots, x_m]$ with a monomial order we define: $\deg(f) = \alpha_{max} = \max(\alpha \in \mathbb{N}^m : a_\alpha \neq 0)$, leading term of $f = lt(f) = a_{\alpha_{max}} x^{\alpha_{max}}$, leading monomial of $f = lm(f) = x^{\alpha_{max}}$. We now come to the division algorithm.

ivision algorithm in $k[x_1, \ldots, x_m]$

    input: an s-tuple of polynomials $F = (f_1, \ldots, f_s)$ and
        a nonzero polynomial f ,
    output: the remainder r $(= \bar{f}^F)$ of dividing f by
        F and the quotients $q_1, \ldots, q_s$,
    algorithm: $p := f$, $r := 0$, $q_i := 0$ for all $i = 1, \ldots, s$
        repeat
            $i := 1$, dividing:=true
            while $(i \leq s)$ and (dividing) **do**
            if $lt(f_i)$ divides $lt(p)$ **then**
                $u := lt(p)/lt(f_i)$, $q_i := q_i + u$,
                $p := p - u\, f_i$, dividing:=false
            else $i := i + 1$
            if dividing **then**
                $r := r + lt(p)$, $p := p - lt(p)$
        until $p = 0$

It should be noted that, for a given monomial order, r and q_i depend on the order of f_i in F. Given $f, g \in k[x_1, \ldots, x_m]$ with h being the least common multiple of $lm(f)$ and $lm(g)$ we define the S-polynomial of f and g as $S(f, g) = h \cdot (f/lt(f) - g/lt(g))$. Now let I be an ideal in $k[x_1, \ldots, x_m]$.

Definition: A *Gröbner basis* of I is a generating set $G = \{f_1, \ldots, f_s\}$ such that $\overline{S(f_i, f_j)}^G = 0$ for all i and j.

The remainder of dividing a polynomial by G is unique! Using the *Buchberger's algorithm* one can find a Gröbner basis of a given ideal.

Buchberger's algorithm

    input: an s-tuple of polynomials $F = (f_1, \ldots, f_r)$ generating I ,
    output: a Gröbner basis $G = (g_1, \ldots, g_s)$ of I,

algorithm: $G := F$
      repeat
          $G' := G$
          for each i , j with $i \neq j$ in G' **do**
          $S := \overline{S(f_i, f_j)}^{\,G'}$
          if $S \neq 0$ **then** $G := G \cup \{S\}$
      until $G = G'$

Using the Gröbner basis we can do algorithmic calculations in a polynomial ring. As an example we give the algorithm for the calculation of $I : J = \{f \in k[x_1, \ldots , x_m] \mid fJ \subset I\}$. First, we determine a Gröbner basis of $I \cap J$. It is given as the intersection of $k[x_1, \ldots , x_m]$ with a Gröbner basis of the ideal $tI - (1-t)J$ in $k[t, x_1, \ldots , x_m]$ with respect to a lex order in which t is greater than x_i. Let $J = \langle f_1, \ldots , f_n \rangle$. Taking $I : J = I : \langle f_1, \ldots , f_n \rangle = \bigcap_{i=1}^{n} I : \langle f_i \rangle$ into account we only need to calculate a Gröbner basis of $I : \langle f_i \rangle$. Because of $I : \langle f_i \rangle = 1/f(I \cap \langle f_i \rangle)$ it reduces to the case just discussed above (cf. [1, 22, 44] for more details).

[1] W. W. Adams, P. Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Math., Vol. 3, AMS, (1994)

[2] P.S. Aspinwall, B.R. Greene, D.R. Morrison, *Multiple mirror manifolds and topology change in string theory*, Phys. Lett. **303B** (1993) 249, hep-th/9301043

[3] P.S. Aspinwall, B.R. Greene, D.R. Morrison, *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory*, Nucl. Phys. **B416** (1994) 414, hep-th/9309097

[4] P.S. Aspinwall, B.R. Greene, D.R. Morrison, *Space-time topology change and stringy geometry*, J. Math. Phys. **35** (1994) 5321

[5] P.S. Aspinwall, B.R. Greene, *On the geometric interpretation of $N = 2$ superconformal theories*, Nucl. Phys. **B437** (1995) 205, hep-th/9409110

[6] T. Banks, N. Seiberg, *Non–perturbative infinities*, Nucl. Phys. **B273** (1986) 157

[7] T. Banks, L.J. Dixon, D. Friedan, E. Martinec, *Phenomenology and conformal field theory or can string theory predict the weak mixing angle?*, Nucl. Phys. **B299** (1988) 613

[8] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Alg. Geom. **3** (1994) 493, alg-geom/9310003

[9] V.V. Batyrev, *Quantum cohomology rings of toric manifolds*, Journées de Géométrie Algébrique d'Orsay (Juillet 1992), Astérisque, Vol. 218, Société Mathématique de France, (1993), p. 9, alg-geom/9310004

[10] V.V. Batyrev, D.A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, alg-geom/9306011

[11] V.V. Batyrev, L.A. Borisov, *Dual cones and mirror symmetry for generalized Calabi–Yau manifolds*, alg-geom/9402002; *On Calabi-Yau complete intersections in toric varieties*, alg-geom/9412017

[12] M. Beltrametti, L. Robbiano, *Introduction to the theory of weighted projective spaces*, Expo. Math. **4** (1986) 111

[13] M. Bershadski, A. Johansen, T. Pantev, V. Sadov, *Four-dimensional compactifications of F-theory*, hep-th/9701165

[14] R. Blumenhagen, A. Wißkirchen, *Exactly solvable (0,2) supersymmetric string vacua with GUT gauge groups*, Nucl. Phys. **B454** (1995) 561, hep-th/9506104; *Exploring the moduli space of (0,2) strings*, Nucl. Phys. **B475** (1996) 225, hep-th/9604140

[15] R. Blumenhagen, R. Schimmrigk, A. Wißkirchen, *The (0,2) exactly solvable structure of chiral rings, Landau–Ginzburg theories and Calabi–Yau manifolds*, Nucl. Phys. **B461** (1996) 460, hep-th/9510055

[16] R. Blumenhagen, R. Schimmrigk, A. Wißkirchen, *(0,2) mirror symmetry*, hep-th/9609167

[17] G. Bohnhorst, H. Spindler, *The stability of certain vector bundles on $\mathbb{P}^n$*, Lecture Notes in Math. **1507** (1992) 39

[18] L.A. Borisov, *Towards the mirror symmetry for Calabi-Yau complete intersections in Gorenstein toric Fano varieties*, alg-geom/9310001

[19] V. Brînzănescu, *Holomorphic vector bundles over compact complex surfaces*, Lecture Notes in Math. **1624** (1996)

[20] R. Brooks, F. Muhammad, S.J. Gates, *Unidexterous D=2 supersymmetry in superspace*, Nucl. Phys. **B268** (1986) 599

[21] P. Candelas, G. Horowitz, A. Strominger, E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. **B258** (1985) 46

[22] D. Cox, J. Little, D. O'Shea, *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra*, UTM, Springer Verlag, (1992)

[23] D.A. Cox, *The homogeneous coordinate ring of a toric variety*, alg-geom/9210008

[24] D.A. Cox, *Recent developments in toric geometry*, alg-geom/9606016

[25] M. Cvetič, *Exact construction of (0,2) Calabi–Yau manifolds*, Phys. Rev. Lett. **59** (1987) 2829

[26] V.I. Danilov, *The geometry of toric varieties*, Russian Math. Survey **33**, n.2 (1978) 97

[27] V.I. Danilov, *Algebraic varieties and schemes*, Encycl. Math. Sci. **23**, Springer Verlag (1994) 167

[28] V.I. Danilov, *Cohomology of algebraic varieties*, Encycl. Math. Sci. **35**, Springer Verlag (1996) 1

[29] F. Delduc, F. Gieres, S. Gourmelen, *d=2, N=2 superconformal symmetries and models*, hep-th/9609182

[30] M. Dine, R. Rohm, N. Seiberg, E. Witten, *Gluino condensation in superstring models*, Phys. Lett. **156B** (1985) 55

[31] M. Dine, N. Seiberg, *Nonrenormalization theorems in superstring theory*, Phys. Rev. Lett. **57** (1986) 2625

[32] M. Dine, N. Seiberg, *(2,0) superspace*, Phys. Lett. **180B** (1986) 364

[33] M. Dine, N. Seiberg, X.G. Wen, E.Witten, *Nonperturbative effects on the string worldsheet I and II*, Nucl. Phys. **B278** (1986) 769, **B289** (1987) 319

[34] M. Dine, N. Seiberg, *Are (0,2) models string miracles?*, Nucl. Phys. **B306** (1988) 137

[35] J. Distler, *Resurrecting (2,0) compactifications*, Phys. Lett. **188B** (1987) 431

[36] J. Distler, B. Greene, *Aspects of (2,0) string compactifications*, Nucl. Phys. **B304** (1988) 1

[37] J. Distler, S. Kachru, *(0,2) Landau–Ginzburg theory*, Nucl. Phys. **B413** (1994) 213, hep-th/9309110

[38] J. Distler, *Notes on (0,2) superconformal field theories*, Proceedings of the 1994 Trieste Summer School, hep-th/9502012

[39] J. Distler, B.R. Green, D.R. Morrison, *Resolving singularities in (0,2) models*, hep-th/9605222

[40] L. Dixon, *Some world-sheet properties of superstring compactifications, on orbifolds and otherwise*, in: Proceedings of the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology, eds. G. Furlan et al.

[41] I. Dolgachev, *Weighted projective varieties*, Lecture Notes in Math. **956** (1981) 34

[42] R.Y. Donagi, *Principal bundles on elliptic fibrations*, alg-geom/9702002

[43] S. Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985) 1

[44] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, GTM 150, Springer Verlag, (1995)

[45] G. Ewald, *Combinatorial convexity and algebraic geometry*, Springer Verlag, (1996)

[46] A.R. Fletcher, *Working with complete intersections*, Bonn preprint MPI/89–35 (1989)

[47] A. Font, L.E. Ibáñez, M. Mondragon, F. Quevedo, G.G. Ross, *(0,2) heterotic string compactifications from N=2 superconformal theories*, Phys. Lett. **227B** (1989) 34 ;
A. Font, L.E. Ibáñez, F. Quevedo, A. Sierra, *Twisted N=2 coset models, discrete torsion and asymmetric heterotic string compactification*, Nucl. Phys. **B337** (1990) 119

[48] D.S. Freed, *Determinants, torsions and strings*, Commun. Math. Phys. **107** (1986) 483

[49] R. Friedman, J. Morgan, E. Witten, *Vector bundles and F-theory*, hep-th/97011162

[50] W. Fulton, *Introduction to intersection theory in algebraic geometry*, CBMS Lecture Notes **54** (1983)

[51] W. Fulton, *Intersection theory,* Springer Verlag, (1984)

[52] W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, Princeton, (1993)

[53] W. Fulton, B. Sturmfels, *Intersection theory on toric varieties*, alg-geom/9403022

[54] R. Godement, *Topologie algébrique et théorie des faisceaux*, Herman, Paris (1964)

[55] M. Green, *Koszul cohomology and geometry*, in: Lectures on Riemann Surfaces, Trieste, Italy, 1987, Proceedings of the College on Riemann Surfaces, eds. M. Cornalba et al. , World Scientific, (1989)

[56] M.B. Green, J.H. Schwarz, E. Witten, *Superstring theory*, Vols. 1&2, Cambridge Univ. Press, Cambridge (1987)

[57] B.R. Greene, *Superconformal compactifications in weighted projective space,* Commun. Math. Phys. **130** (1990) 335

[58] B.R. Greene, S.-S. Roan, S.-T. Yau, *Geometric singularities and spectra of Landau–Ginzburg models,* Commun. Math. Phys. **142** (1991) 245

[59] P. Griffith, J. Harris, *Principles of algebraic geometry*, John Wiley and Sons, New York, (1978)

[60] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann,* Amer. J. Math. **79** (1956) 137

[61] A. Grothendieck, *Local cohomology,* (Notes by R. Hartshorne), Lecture Notes in Math. **41** (1967)

[62] A. Grothendieck, J. Dieudonné, *Eléments de géométrie algébrique,* Publ. Math. IHES **4** (1960), **11** (1961), **17** (1963)

[63] R. Hartshorne, *Algebraic geometry,* GTM 52, Springer Verlag, (1977)

[64] M. Herrman, R. Schmidt, W. Vogel, *Theorie der normalen Flachheit,* Teubner–Texte zur Mathematik, Teubner Verlagsgesellschaft, Leipzig, (1977)

[65] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie,* Springer Verlag, (1956)

[66] F. Hirzebruch, T. Berger, R. Jung, *Manifolds and modular forms,* A Publication of the Max–Planck–Institut für Mathematik, Bonn, Vieweg, (1992)

[67] S. Hosono, A. Klemm, S. Theisen, *Lectures on mirror symmetry,* Lecture Notes in Phys. **436** (1994) 235, hep-th/9403096

[68] P.S. Howe, G. Papadopoulos, *Ultra–violet behaviour of two–dimensional supersymmetric non–linear σ–models,* Nucl. Phys. **B289** (1987) 264

[69] K. Hulek, *Der Satz von Riemann–Roch,* preprint pr276, Univ. Hannover, (1996)

[70] C.M. Hull, E. Witten, *Supersymmetric sigma models and the heterotic string,* Phys. Lett. **160B** (1985) 398

[71] T. Hübsch, *Calabi-Yau manifolds: A bestiary for physicists,* World Scientific, (1992)

[72] B. Iverson, *Cohomology of sheaves,* Springer Verlag, (1986)

[73] G. Kempf, F. Knudsen, D. Mumford, B. Saint–Donat, *Toroidal embeddings I,* Lecture Notes in Math. **339** (1973)

[74] M. Kreuzer, H. Skarke, *On the classification of reflexive polyhedra,* hep-th/9512204

[75] M. Kreuzer, M. Nikbakht–Tehrani, *(0,2) string compactifications,* to be published in Nucl. Phys. Proc., hep-th/9611130

[76] J. Kollár, *Flips, flops, minimal models, etc.,* Surveys in Diff. Geom. **1** (1991) 113

[77] E. Kunz, *Einführung in die kommutative Algebra und algebraische Geometrie,* Vieweg, (1980)

[78] E. Kunz, *Residuen und Dualität auf projektiven algebraischen Varietäten,* Der Regensburger Trichter, Band **19** (1986)

[79] D. Lüst, S. Theisen, *Lectures on string theory,* Lecture Notes in Phys. **346** (1989)

[80] D.G. Markushevich, M.A. Olshanetsky, A.M. Perelomov, *Description of a class of superstring compactifications related to semi-simple Lie algebras*, Commun. Math. Phys. **111** (1987) 247

[81] E. Martinec, *Nonrenormalization theorems and fermionic string finiteness*, Phys. Lett. **171B** (1986) 189

[82] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Adv. Math. **8**, Cambridge Univ. Press, Cambridge (1986)

[83] G. Moore, P. Nelson, *Aetiology of sigma model anomalies*, Commun. Math. Phys. **100** (1985) 83

[84] D.R. Morrison, M.R. Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, Nucl. Phys. **B440** (1995) 279, hep-th/9412236

[85] M. Nikbakht-Tehrani, *On the Calabi-Yau phase of* $(0, 2)$ *models*, to be published in Nucl. Phys. B, hep-th/9612067

[86] M. Nikbakht-Tehrani, *On the cohomology of twisting sheaves on toric varieties*, in preparation

[87] T. Oda, *Convex bodies and algebraic geometry*, Springer Verlag, (1988)

[88] C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*, Birkhäuser, (1980)

[89] A.M. Perelomov, *Instantons and Kähler manifolds*, Commun. Math. Phys. **63** (1978) 137

[90] M. Reid, *Canonical 3–folds*, Journées de Géometrié Algébrique d'Angers, ed. A. Beauville, Sijthoff & Noordhoff, Alphen ann den Rijn, (1980) 273

[91] M. Reid, *Young person's guide to canonical singularities*, Algebraic Geometry, Bowdoin 1985, ed. S. Bloch, Proc. Sym. Pure Math. **46**, Part 1, AMS, Providence, RI, (1987) 345

[92] S.-S. Roan, S.-T. Yau, *On Ricci flat 3–fold*, Acta Math. Sinicia, New Series, Vol. 3, No. **3** (1987) 256

[93] A.N. Schellekens, S. Yankielowicz, *New modular invariants for N=2 tensor products and four-dimensional strings*, Nucl. Phys. **B330** (1990) 103 ;
A.N. Schellekens, S. Yankielowicz, *Tables supplements*, CERN–TH. 5440S/89 and CERN–TH. 5440T/89 (unpublished);
A.N. Schellekens, *Electric charge quantization in string theory*, Phys. Lett. **237B** (1990) 363

[94] H. Schmitt, *Grundtatsachen über lokale Kohomologiemoduln*, Lecture Notes in Math. **238** (1971) 33

[95] A. Sen, *(2,0) supersymmetry and space-time supersymmetry in the heterotic string theory*, Nucl. Phys. **B278** (1986) 189

[96] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. **61** (1955) 197

[97] J.-P. Serre, *Algébre locale. Multiplicités*, Lecture Notes in Math. **11** (1965)

[98] I.R. Shafarevich, *Basic algebraic geometry*, Vols. 1&2, Springer Verlag, (1994)

[99] P. Shanahan, *The Atiyah–Singer index theorem: An introduction*, Lecture Notes in Math. **638** (1978)

[100] E. Silverstein, E. Witten, *Criteria for conformal invariance of (0,2) models*, Nucl. Phys. **B444** (1995) 161, hep-th/9503212

[101] H. Skarke, *Weight systems for toric Calabi-Yau varieties and reflexivity of Newton polyhedra*, alg-geom/9603007

[102] A. Strominger, *Superstrings with torsions*, Nucl. Phys. **B274** (1986) 253

[103] B. Sturmfels, *Equations defining toric varieties*, alg-geom/9610018

[104] K. Uhlenbeck, S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Commun. Pure and App. Math. XXXIX (1986) S257

[105] C. Vafa, *Modular invariance and discrete torsion on orbifolds*, Nucl. Phys. **B273** (1986) 592

[106] X.G. Wen, E. Witten, *World sheet instantons and the Peccei-Quinn symmetry*, Phys. Lett. **166B** (1986) 397

[107] E. Witten, *Symmetry breaking patterns in superstring models,* Nucl. Phys. **B258** (1985) 75

[108] E. Witten, *Global anomalies in string theory,* in: Anomalies, geometry and topology, eds. W.A. Bardeen et al. , World Scientific, Singapore, 1985

[109] E. Witten, *New issues in manifolds of SU(3) holonomy,* Nucl. Phys. **B268** (1986) 79

[110] E. Witten, L. Witten, *Large radius expansion of superstring compactifications,* Nucl. Phys. **B281** (1987) 109

[111] E. Witten, *Phases of N=2 theories in two dimensions,* Nucl. Phys. **B403** (1993) 159, hep-th/9301042

[112] S.-T. Yau, *Compact three dimensional Kähler manifolds with zero Ricci curvature,* in: Anomalies, geometry and topology, eds. W.A. Bardeen et al. , World Scientific, Singapore, (1985)