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Intersecting the Twin Dragon with rational lines

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Mag.art. Paul Großkopf BSc.
Matrikelnummer: 01221571

ausgeführt am Institut für Diskrete Mathematik und Geometrie
der Fakultät für Mathematik und Geoinformatik
der Technischen Universität Wien

Betreuung
Privatdoz. Dipl.-Ing. Dr.mont. Benoît Loridant

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1 Preface - What is a fractal?

There exists the age old question, whether mathematics is invented by humans or discovered. Mathematical models help us to understand our surrounding, but never describe reality without errors. The assumptions we take are highly generalizing and often not realizable in the real world. Especially in geometry we use perfect polygons, smooth curves or even straight lines, which in reality just exist as concepts. But even in pure mathematics one is often tempted to expect results to follow our experiences in the real world. Fractals were often used as counter examples, like for example the continuous but nowhere differentiable *Weierstrass function* or the space filling *Hilbert curve*. Often described as "mathematical monsters" many mathematicians invented their own shapes with bizarre properties to show the limits of certain mathematical theories. But these monsters themselves never were treated as legitimate objects worthy of studying, until Benoit B. Mandelbrot began investigating them in the 70s of the 20th century. With his Book "The fractal geometry of nature" ([13]) from 1982 he connected these early examples of mathematical monsters, as well as natural phenomea and computer applications. Since then the theory spread and flourished and is now highly used in computer science to generate natural looking images of trees, clouds or mountains. But also in pure mathematics fractal appear quite naturally in dynamical systems and even number theory.

What is a fractal? In the public understanding fractals are often mistaken with self-similar sets, which are geometric object that consist of smaller copies of themselves. In this chapter we will discuss some of the early examples and their counter intuitive properties. However in todays definition not all fractals are self-similar. Random sets of points can be fractal without any underlying structure. Most examples we encounter are in fact self-similar in a broader sense that we will discuss in chapter 3. Self-similarity can often be seen in nature with examples like romanesco broccoli or ice cristals. Nevertheless since the nature of the world is not infinite but rather finite, real strict self-similarity never occurs. As usefull self-similarity is to describe nature it is also just another modell of the world. Also note that not all self-similar sets are fractals, since intervals consist of smaller copies of themselves.

So since self-similarity is not the defining quality of fractals, a broader notion needs to be established. Intuitively speaking we want to develope a measure forthe roughness or the rippling of an object. In Chapter 2 we will develop various concepts, in particular *Hausdorff dimension*, realizing these whishes as well as a general topological notion of dimension that allows us to formalize our intuition of dimension using only topological properties. A fractal is than defined as an object for which these two notions of dimension differ.

Chapter 4 will deal with a number theoretical problem that is strongly conected to fractal geometry. A generalized notion of radix representation leads to various examples of self-similar shapes. These objects can be understood better by using the algebraic structure creating them. One of these shapes is the *Knuth Twin dragon*, which has fractal boundary. We will discuss it in particular since it is the main interest of study in the last chapter.

Finally we want to intersect the Twin dragon with lines of rational slope in Chapter 5. Following a paper by Shigeki Akiyama and Klaus Scheicher [1], who calculated the intersection with the x- and y-axis, we want to use the same technique to determine intersections with other lines. This is the core part of this thesis: it is not based on literature but is self-developed contribution to mathematics.

The rest of this chapter will be regarding three examples of fractals from various viewpoints: the *Cantor set*, the *Sierpinsky triangle* and the Twin dragon itself. On the one hand, they will not be dealt in a rigorous way, since some questions use more general theory from the latter chapters. On the other hand, some properties are proved to give a better understanding the following abstract theory.

Finally I point out that this thesis only gives an introduction to fractal geometry and mainly deals with topological, measure and number theoretical aspects of fractals. There are further approaches to the topic via dynamical systems or computer science. The references cited in this thesis can be a good starting point for interested readers. Please be aware that although all sources are rigorously cited, sometimes definitions, claims or proofs vary to fit our context.

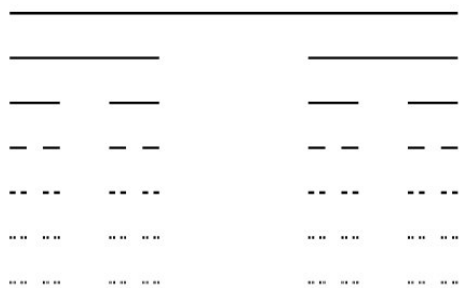

Figure 1: The first few approximations of the Cantor set C_∞

The first object we want to discuss is probably the most elementary fractal ever created. It was first defined by Georg Cantor in 1883 and starts with a real interval, so w.l.o.g $C_0 := [0,1]$. We now obtain $C_1 = [0,1/3] \cup [2/3,1]$ from C_0 by removing its middle third interval. The first approximation of the Cantor set consist of two intervals and we proceed by removing the third of them as well: this generates a decreasing sequence of sets C_n. Now we define the Cantor set as the "limit" of this process.

Definition 1.1. Let $C_0 := [0,1]$ and C_n created by the process described above. Then we define the ***Cantor set*** as

$$C_\infty := \bigcap_{n=0}^{\infty} C_n \tag{1}$$

A direct consequence of this definition is the following proposition.

Proposition 1.2. *Let $C_0, ..., C_n, ..., C_\infty$ be defined as above and let $\mathcal{L}$ note the length of a set. Then*

$$\mathcal{L}(C_\infty) = \lim_{n\to\infty} \mathcal{L}(C_n) = 0 \tag{2}$$

Proof. The proof is straight forward: $\mathcal{L}(C_0) = 1$ and $\mathcal{L}(C_n) = (2/3)\mathcal{L}(C_{n-1}) = (2/3)^n$. This converges to 0 for $n \to \infty$. □

Since $\mathcal{L}$ is not properly defined, this can only be viewed as a heuristic argument, but it holds for the more rigorous notion of *Lebesgue measure*. The next proposition shows that infact most of the original interval is cut out by the process, but the points left over form a self-similar set.

Proposition 1.3. *C_∞ consists of two smaller copies of itself.*

Proof. This can be easily seen by the fact that C_1 consists of two inververvals namely A_0 and B_0. Applying the same process to create the Cantor set on these two starting intervals generates $A_\infty = (1/3)C_\infty, B_\infty = (1/3)(C_\infty + 2)$ which satisfy $C_\infty = A_\infty \cup B_\infty$. □

Another way to see this is by changing the process to approach C_∞. Instead of cutting out intervals we use to functions $f_1(x) := x/3$ and $f_2(x) = x/3 + 2/3$ and define $C_{n+1} = f_1(C_n) \cup f_2(C_n)$. Note that this generates the same approximations as before and the limit object is invariant under $F(C) = f_1(C) \cup f_2(C)$. This is the general approach to self-similar sets, and one can define a notion of limit of these sets. Under pretty weak conditions this sequence allways converges, namely if the generating functions are contractions.

Since most of the unit interval gets lost in the process, Cantor posed the question of how much points are still left. Clearly the rational endpoints remain untouched in every step so countably infinite many points remain in the Cantor set. However the Cantor set is uncountable.

Proposition 1.4. *There is a surjection from the Cantor set to the unit interval.*

Proof. Using ternary representation we can characterize C_∞ as the set of all numbers which can be expressed by only using the digits 0 and 2. Note that instead of 0.1_3 one can express the number $1/3$ as $0.0\overline{2}_3$, so the endpoints are also expressible with only 0 and 2. Now define $f : C_\infty \to [0, 1]$ by replacing all 2's by 1's and interpret the representation in base two. This function is surjective, therefore the Cantor set has uncountable many points. □

This result was quite a shock when first discovered, since it contradicts our natural understanding of quantity. A set with no length but uncoutably many points was something quite hard to accept. Sometimes the Cantor set is refered to as *Cantor dust*, since it is basically nothing but still countains as many points als the unite interval. Furthermore the Cantor set is compact, nowhere dense, totally disconnected.

The next famous example is was proposed by Waclaw Sierpinsky in the year 1915. The Sierpinski triangle can be obtained by starting with a equilateral triangle and in every step diveding the triangle into four subtriangles

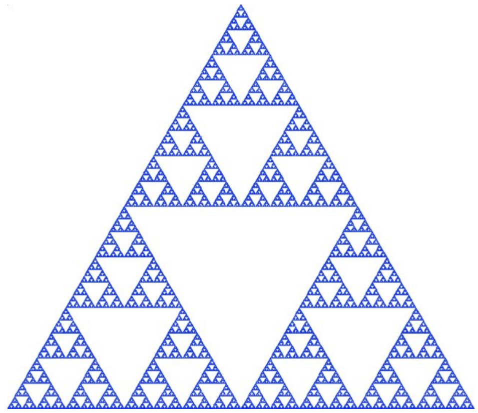

Figure 2: The Sierpinski triangle

and removing the middle one. As with the Cantor set this construction via *initiator* and *generator* is pretty common for fractals and can be mathematically formalized. Another way to generate the Sierpinsky triangle is by only taking the boundary of the triangle and adding more and more triangles. It is not obvious that these two processes determine the same object, but the self-similar property of the set, will guarantee its uniqueness, as we will see in Chapter 3.

Definition 1.5. Let $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{aligned} f_1(x,y) &:= \frac{1}{2}(x,y) \\ f_2(x,y) &:= \frac{1}{2}(x,y) + \left(0, \frac{1}{2}\right) \\ f_3(x,y) &:= \frac{1}{2}(x,y) + \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right) \end{aligned} \tag{3}$$

The set $S \subset \mathbb{R}^2$ fulfilling $S = f_1[S] \cup f_2[S] \cup f_3[S]$ is called the ***Sierpinski triangle*** or ***Sierpinski gasket***.

The existence and uniqueness of this set will be shown later, but still we acknowlegde that different starting objects lead to the same object. In fact the starting object is irrelevant as long as it is not the empty set. Another thing that follows immediately from the two possible approaches is the following.

Proposition 1.6. *The length of the Sierpinsky triangle is infinite and its area is* 0*.*

Proof. In every step the length increases by a half, so the limit object has infinite length, but the area decreases by a quarter and so the limit object has area 0. □

So again we created an object irritating our perception of the world: is S one dimensional since it is a union of lines or is it two dimensional since it still occupies a lot of space? The answer is neither but it makes clear that we need a broader definition of dimension and a finer way to determine it. One heuristic fact one can observe is that hypercubes can be broken up into smaller cubes with the scaling factor 1/2 and depending on the dimension n of the hypercube we obtain 2^n smaller hypercubes. This fact gives us a formula to determine the dimension:

$$\dim = \frac{\log \#\text{self-similar parts}}{\log \text{scaling factor}}. \tag{4}$$

On can apply this to the Sierpinski gasket and obtains $\log 3/\log 2 \approx 1.585$, which makes sense since it lies somewhere between 1 and 2. For the Cantor set we obtain $\log 2/\log 3 \approx 0.63$. This will turn out to be a good measure for dimension in most cases, but will need rigorous mathematical methods.
Astonishingly the Sierpinsky gasket appears in number theory. Taking the *Pascal triangle* and colouring all odd numbers we get a better and better approximation of the Sierpinsky triangle the more we zoom out.
Another occasion where the fractal appears is by playing a so called chaos game. Starting by an equilateral triangle ABC and a point x_0 in the triangle, one plays the game as following:

1. Choose a corner point A, B or C.
2. Construct the line through x_n and the chosen point.
3. Construct the midpoint of the line x_{n+1} and draw it.
4. Repeat from step 1.

This process eventually gives you a point in the Sierpinski triangle and every later point cannot escape the triangle. This process is actually used in generating images of fractals and underlines the bridges to probability theory

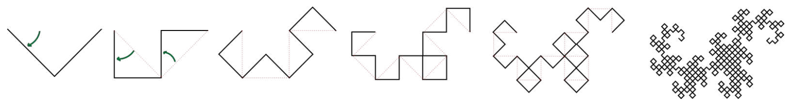

Figure 3: Approximation of the Heighway Dragon

and even more naturally to measure theory.
One close relative of the Sierpinski triangle is the *Sierpinski tetrahedron*, which can be created by stacking up 4 scaled downed versions of an object onto themselfes, where the scaling factor is 1/2. Note that if we apply the dimension calculation from above on the tetrahedron one obtains $\log(4)/\log(2) = 2$. The misscharacterisation that fractals are shapes of non integer dimension is not true. The Sierpinski tetrahedron is exactly two dimensional, but it is still infinitely complex spreaded in 3D-space. Another example is the one dimensional *Takagi function*, which still is nowhere differentiable. For more details consult [19][Chapt. 1].

Last but not least we want to discuss the Knuth Twin dragon, which is named after Donald Knuth, an American mathematician and computer scientist. The Twin dragon can be constructed as a space filling curve and is directely related to the *Heighway dragon curve* which is one of the most popular self-similar shapes since it appeared in the novel *Jurassic Park* by Michael Crichton. The heighway dragon can be approximated by using a single strip of paper and just consequently folding it in half. Unfolding it, such that all angles are 90 degrees we get a curve approximating the limit object. The Twin dragon consists of two Heighway dragons glued together.: therefore the name Twin dragon. Once again the construction is not essential but the underlying self-similarities are.

Definition 1.7. Let $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ with

$$f_1(z) = -\frac{(1+i)z}{2} \qquad f_2(z) = -\frac{(1+i)(z+1)}{2}. \tag{5}$$

The set $\mathcal{K}$ fulfilling $\mathcal{K} = f_1(\mathcal{K}) \cup f_2(\mathcal{K})$ is called the ***Knuth Twin Dragon***.

Geometrically the two functions describe a rotations by 45 degree and downscaling by 1/2, where f_2 also translates. We can also look at the Twin Dragon as an object in $\mathbb{R}^2$. We will often switch between both representations, depending on the context.

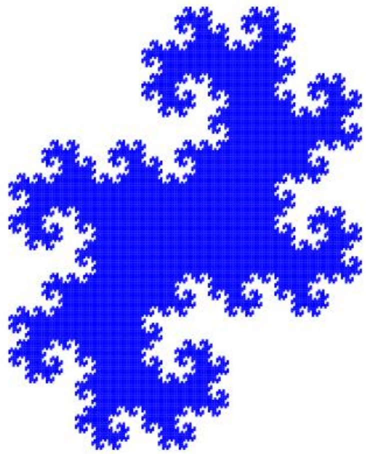

Figure 4: The Knuth Twin Dragon

Proposition 1.8. *The Twin Dragon* $\mathcal{K} \in \mathbb{R}^2$ *is invariant under* $f_1 \cup f_2$ *where* $f_1(x) = B^{-1}x, f_2(x) = B^{-1}(x + (1,0))$ *and*

$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{6}$$

We will in fact see that the Twin Dragon has nonempty interior and dimension 2. This is strange since the approximation can be achieved with a curve. So in every step we get a more and more twisted curve that in the limit fills out a whole area. This was very counterintuitive when first discovered by David Hilbert. These Hilbert space filling curves show that the borders between dimensions are not that strict and one dimensional lines can be injectively and smoothly mapped to two dimensional objects. Since we want to develope a new notion of dimension, we need to be careful claiming invariance of dimension under maps. Depending on the definition we only can achieve invariance under certain maps.
Although the Twin Dragon is not a fractal, its boundary is. We will determine its dimension in the latter and also calculate its intersections with various lines. We will investigate its appearance in number theory and take a tour through measure theory, automata theory and geometry to gather enough techniques to solve the proposed questions. Although this Dragon is at least 50 years old, there is still a lot of mystery surrounding it. This thesis is a small contribution to tame this beautiful mathematical monster.

2 Topological Dimension

The naive understanding of a fractal is a geometric object, which is infinitely curled or has infinitely many holes. In order to understand this aspect of fractals one needs to establish a notion of topological dimension. In linear algebra the dimension of a module is expressed purely through the algebraic relation between its elements. A topological dimension on contrary only uses the topological structure of the space and should coincide in $\mathbb{R}^n$ with the basic understanding of dimensionality. Following Yamaguti, Hata and Kigami in [19] we require for $X \subset \mathbb{R}^n$ that $\dim(X)$ satisfies the following:

(1) $\dim(\{p\}) = 0$, for $p \in \mathbb{R}^n$, $\dim(I^1) = 1$, for I^1 the unit interval and $\dim(I^m) = m$, for I^m the m-dimensional hypercube.

(2) (*Monotonicity*) If $X \subseteq Y$, $\dim(X) \leq \dim(Y)$.

(3) (*Countable Stability*) If $\{X_j\}$ is a sequence of closed subsets of $\mathbb{R}^n$, then

$$\dim\left(\bigcup_{j=1}^{\infty} X_j\right) = \sup_{j\geq 1} \dim(X_j). \tag{7}$$

(4) (*Invariance*) For $\mathcal{F}$ a family of the homeomorphisms of $\mathbb{R}^n$ to $\mathbb{R}^n$, dim shall be invariant, i.e. for $\psi \in \mathcal{F}$

$$\dim(\psi(X)) = \dim(X). \tag{8}$$

The subfamily of homeomorphisms in (4) depends on the definition of the dimension. If a dimension function is invariant under all homeomorphisms, it is called *topologically invariant*. But since this cannot always be achieved, it is sufficient to require invariance for just a specific class, in order to get stronger other properties of the dimension function. A somewhat powerful definition of dimension should at least be invariant under translation, rotation and scaling; so it should at least contain the affine transformations on $\mathbb{R}^n$.

The following chapter works on three different topological dimension-function: the *Lebesgue dimension* or *covering dimension*, the *Hausdorff dimension* and the *box-counting dimension*. Other concepts, which are not included, but should be mentioned, are the *small* and the *large inductive dimensions* as also described in [5].

2.1 Covering dimension

Definition 2.1. Let $n \geq -1$ be an integer. A family $\mathcal{F}$ of sets has ***order*** $\leq n$, iff any distinct $n+2$ sets have empty intersection. For $n \geq 0$ we say $\mathcal{F}$ has order n, iff $\mathcal{F}$ has order $\leq n$, but does not have order $\leq n-1$.

By this definition we see, that the only set families with order -1 are the empty set $\varnothing$ and the set $\{\varnothing\}$. A family of nonempty sets is disjoint iff it is of order 0.

Definition 2.2. Let (X, τ) be a topological space. A family $\mathcal{A} = \{A_i\}_{i \in I}$ of open subsets is called an ***(open) cover*** of X iff $X = \bigcup_{i \in I} A_i$. A ***refinement*** of $\mathcal{A}$ is a cover $\mathcal{B}$ with $\forall B \in \mathcal{B}, \exists A \in \mathcal{A} : B \subseteq A$. We say X has ***covering dimension*** $\leq n$ iff every finite cover of X has a refinement of order $\leq n$. The covering dimension is n iff it is $\leq n$ but not $\leq n-1$, and we write $\mathrm{Cov}(X) = n$. If no such n exists we say $\mathrm{Cov}(X) = \infty$.

Note that if we consider subsets of X and subspace topology we can extend the definition to subsets and consider Cov as a function from all subset of X, $\mathrm{Sub}(X)$, to the set $\mathbb{N} \cup \{\infty\}$.

Proposition 2.3. Cov *is topologically invariant.*

Proof. Let (X, τ), (Y, υ) be two topological spaces and $\psi : X \to Y$ a homeomorphism. Consider the two cases:

Firstly $\mathrm{Cov}(X) = \infty$: Suppose $\mathrm{Cov}(Y) = n \leq \infty$. Let $\mathcal{A} = \{A_i\}_{i=1}^m$ be a finite cover of X. Then $\mathcal{B} := \{\psi(A_i)\}_{i=1}^m$ is a finite open cover of Y and therefore has a refinement of order $\leq n$. Applying ψ^{-1} gives a refinement of $\mathcal{A}$ of order $\leq n$. Since n is independent of the cover, we get a contradiction to the assumption that $\mathrm{Cov}(X) = \infty$. Therefore $\mathrm{Cov}(Y) = \infty$.

Secondly $\mathrm{Cov}(X) = n$. Let $\mathcal{B} = \{B_i\}_{i=1}^m$ be a finite cover of Y, then $\mathcal{A} = \{\psi^{-1}(B_i)\}_{i=1}^m$ forms a cover of X and therefore a has a refinement of order $\leq n$. Applying ψ this times gives us a refinement of $\mathcal{B}$ of order $\leq n$. So $\mathrm{Cov}(Y) \leq n$. To prove that $\mathrm{Cov}(Y) = n$, we use the same method as in the first case to show, that there cannot be a cover with refinement of or der $\leq n-1$. □

We see that this definition is possible in arbitrary topological spaces, but an easy example shows, that if we want to achieve some useful properties, we need to restrict ourselves further. Let $X = \{1, 2, 3, 4\}$ and consider the topology $\{X, \varnothing, \{2,3\}, \{3,4\}, \{3\}, \{2,3,4\}\}$. The set $\{2,4\}$ has dimension 1 but X itself is of dimension 0, since a cover includes X itself, which is of order 0. This violates monotonicity. The problem here is, that there are not enough open sets to cover X, so by restricting ourselves to metric spaces we can work with ϵ-balls and also gain $(T2)$ seperation axiom. In order to prove the other conditions we required for a sufficient dimension function, we need to investigate the properties a little further. A handy lemma in [5, Theorem 3.2.1., p.92] can be put in the more general, not necessarily metric, context.

Lemma 2.4. *Let (X, τ) be a topological space and n a non negative integer. The following are equivalent:*

1. $\mathrm{Cov}(X) \leq n$.
2. $\mathcal{A} = \{A_i\}_{i=1}^k$ *is any finite open cover of X, then there exist a refinement* $\mathcal{B} = \{B_i\}_{i=1}^k$ *of order* $\leq n$ *such that* $B_i \subseteq A_i$ *for* $i \leq k$.
3. $\mathcal{A} = \{A_i\}_{i=1}^{n+2}$ *is any finite open cover of X, then there exist a refinement* $\mathcal{B} = \{B_i\}_{i=1}^{n+2}$ *such that* $B_i \subseteq A_i$ *for* $i \leq n+2$ *and* $\bigcap_{i=1}^{n+2} B_i = \varnothing$.

Proof. 2) $\Rightarrow$ 3) and 2) $\Rightarrow$ 1) are clear.

1) $\Rightarrow$ 2). Suppose $\mathrm{Cov}(X) \leq n$. Then $\mathcal{A}$ admits a refinement $\mathcal{W}$ of order $\leq n$. For each $W \in \mathcal{W}$ there is at least one i such that $W \subseteq U_i$. Choose one of them, and call it $i(W)$. Define

$$B_i := \bigcup \{W \in \mathcal{W} : i(W) = i\}.$$

These sets are open and form a cover. If $x \in X$, then it belongs to at most $n+1$ sets W, because $\mathcal{W}$ has order $\leq n$. So x belongs to at most $n+1$ sets B_i.

3) $\Rightarrow$ 2). Suppose X has property 3) and let $\mathcal{A} = \{A_i\}_{i=1}^k$ be an open cover of X. If $k \leq n+1$, then this cover already is of order $\leq n$. Suppose $k \geq n+2$. Define $W_1 := A_1, W_2 = A_2, \ldots, W_{n+1} = A_{n+1}$ and $W_{n+2} = \bigcup_{i=n+2}^k A_i$. Then there exist open sets $V_i \subseteq W_i$ for $i \leq n+2$ which cover X and have empty intersection. Define $B_1 := V_1, B_2 = V_2, \ldots, B_{n+1} = V_{n+1}$ and $C_{n+2} = V_{n+2} \cap A_{n+2}, \ldots, C_k = V_{n+2} \cap A_k$. These sets form again a cover of X and for $n+2 \leq j \leq k : \bigcap_{i=1}^{n+1} B_i \cap C_j = \varnothing$. Repeat this construction for all subsets of $\{1, \ldots, k\}$ with $n+2$ elements to reach the conclusion that the intersection of all intersections of size $n+2$ are empty. $\square$

Edgar gives a proof of countable stability for metric spaces of dimension 1 [5, Theorem 3.2.11., p.95], which can be extended to arbitrary dimensions. Willi Rinow shows the same in the more abstract $(T4)$ case in [16, 34.19, p.358]. We only state the result and use it to prove the monotonicity of the covering dimension following [5, Theorem 3.2.13., p.96].

Theorem 2.5. *Let (X,d) be a metric space and $F_i, i \in \mathbb{N}$ a family of closed subsets of X. If $\operatorname{Cov}(F_i) \le n$, then $\operatorname{Cov}(\bigcup_{i\in\mathbb{N}} F_i) \le n$.*

Theorem 2.6. *Let (X,d) be a metric space and $T \subset X$, then $\operatorname{Cov}(T) \le \operatorname{Cov}(X)$.*

Proof. If $\operatorname{Cov}(X) = \infty$, then the inequality is true in all cases. For $\operatorname{Cov}(X) = n$ the proof is done in three steps:

1) Assume that T is closed and $\mathcal{A}$ is an open cover of T. We can extend $\mathcal{A}$ by $X \setminus T$ to a cover of X and therefore get a refinement $\mathcal{B}$ of order $\le n$. Filtering out the elements which have nonempty intersection with T gives us a refinement of $\mathcal{A}$ of order $\le n$. So $\operatorname{Cov}(T) \le n$.

2) Assume T is open. T can be written as the union of the countable closed subsets F_i, with

$$F_i := \left\{ x \in X : d(x, X \setminus T) \ge \frac{1}{i} \right\} \tag{9}$$

Since F_i is closed, $\operatorname{Cov}(F_i) \le n$ by 1) and with countable stability it follows that $\operatorname{Cov}(T) \le n$.

3) Finally T is a general subset of X and $\mathcal{U} = \{U_1, ..., U_{n+2}\}$ is an open cover of T. Define $U := \bigcup_{i=1}^{n+2} U_i$, which is an open subset of X, therefore of covering dimension $\le n$ and we get a refinement $\{W_1, ..., W_{n+2}\}$ with empty intersection as in Lemma 2.4. This is also a proper cover of T.

□

In order to check the first condition we are reminding ourselves of a pretty wellknow fact in compact metric spaces.

Definition and Proposition 2.7. Let (X,d) be a compact metric space and $\mathcal{U}$ be an open cover of X. Then there is a positive number r, called the ***Lebesgue number***, such that for any set $A \subseteq X$ with $\operatorname{diam}(A) < r$, there is a set $U \in \mathcal{U}$ with $A \subseteq U$.

Proof. Since X is compact assume $\mathcal{U} = \{U_1, U_2, ..., U_n\}$. Suppose the assertion is false. Then we find sets $A_k, k \in \mathbb{N}$ with $\operatorname{diam}(A_k) < 1/k$, that is not contained in any U_i. So we can take sequences $x_{i,k} \in A_k \setminus U_i$ and again because X is compact, we can assume that these sequences converge to x_i. Now $d(x_{i,k}, x_{j,k}) < 1/k, \forall k \in \mathbb{N}$, hence all x_i are the same point x, which is not contained in any U_i. This is a contradiction to the fact, that $\mathcal{U}$ is a cover of X. □

It is possible to construct a covering of the hypercube with balls of order m. This is not hard, but a bit technical. [15, p.65] shows this for $m = 1, 2$. With the Lebesgue number it is easy to create a refinement of order m of the hypercube $I^m \subseteq \mathbb{R}^m$, since these are compact metric spaces. Therefore the covering dimension is the first dimension which fullfills all four properties for a dimension function. However it is really coarse: the Cantor dust has dimension 0 and the Sierpinski gasket has dimension 1. This is pretty unsatisfying for various reasons, therefore we want to investigate finer notions of dimension. In order to give respect to these special sets, we must give up topological invariance. It is however still possible to define a more powerful dimension, which is still invariant for a sufficiently large class of functions.

2.2 Hausdorff dimension

A notion which is closely connected to dimension, is measure. One-dimensional objects, like Intervals and curves, have length assigned to them, whereas surfaces have area and 3-dimensional objects have volume. On the other hand 2-dimensional objects have no volume and if we divide them into curves, their lengths sum up to infinity. This is the key idea behind the Hausdorff dimension: an object is s-dimensional, if its s-dimensional measure is non trivial. To elaborate this heuristic idea, we recall *Lebesgue-measure* of a subset of $\mathbb{R}^d$.

Definition 2.8. Consider $A = \{(x_1, ..., x_n) | a_i \leq x_i \leq b_i, 0 \leq i \leq d\} \subseteq \mathbb{R}^d$ a axisparallel hyperrectangle, where $a_i \leq b_i \in \mathbb{R}, 0 \leq i \leq d$. Then $\operatorname{Vol}^n(A) := \prod_{i=1}^{d}(b_i - a_i)$. For $A \subseteq \mathbb{R}^d$ an arbitrary subset, we define the ***Lebesgue measure*** of A as

$$\mathcal{L}^n(A) := \inf\left\{\sum_{i=1}^{\infty} \operatorname{Vol}^n(A_i) | A \subseteq \bigcup_{i=1}^{\infty} A_i\right\} \tag{10}$$

where $\{A_i : i \in \mathbb{N}\}$ is a cover of axis-parallel hyper-rectangles.

We want to generalize this concept of approximating a set with hyperrectangles to arbitrary metric spaces. Since we have no distinctive shapes, we allow all covers. Instead of the volume we consider powers of the diameter.

Definition and Proposition 2.9. Let (X,d) be a compact metric space and $\mathcal{U} = \{U_i, i \in I\}$ be an open cover of $A \subseteq X$. $\mathcal{U}$ is called a δ-***cover***, iff $0 \leq \operatorname{diam}(U_i) \leq \delta$. Define

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s | U_i \text{ is a } \delta\text{-cover of } A\right\} \tag{11}$$

and the s-***dimensional Hausdorff measure*** of A as

$$\mathcal{H}^s(A) := \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(A). \tag{12}$$

$\mathcal{H}^s$ is an outer measure on X:

(OM1) $\mathcal{H}^s(\varnothing) = 0$,

(OM2) $A \subseteq B \Rightarrow \mathcal{H}^s(A) \leq \mathcal{H}^s(B)$,

(OM3) $\mathcal{H}^s(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mathcal{H}^s(A_i)$

Proof. We show that the conditions hold for $\mathcal{H}^s_\delta$ and fixed δ and prove the claim by taking limits on both sides.

(OM1) $\mathcal{H}^s_\delta(\varnothing) = 0$, since a singelton already covers $\varnothing$.

(OM2) Let $A \subseteq B$, then every cover of B is a cover of A. It follows that $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s_\delta(B)$.

(OM3) Let $\mathcal{U}_i = \{U_{i,j} : j \in \mathbb{N}\}$ be δ-covers of A_i. Then their union is a δ-cover of $\bigcup_{i=1}^\infty A_i$ and so $\mathcal{H}^s(\bigcup_{i=1}^\infty A_i) \leq \sum_{j=1}^\infty \sum_{i=1}^\infty \operatorname{diam}(U_{i,j})^s$. Since the $\mathcal{U}_i$ where arbitrary we get $\mathcal{H}^s_\delta(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mathcal{H}^s_\delta(A_i)$.

□

On Borel sets the Hausdorff measure even satisfises σ-additivity and therefore is a measure. For $s = n \in \mathbb{N}$ the Hausdorff measure is a multiple of the Lebesgue measure: $\mathcal{H}^n(A) = \pi^{\frac{n}{2}}/2^n\Gamma(\frac{n}{2})\mathcal{L}^n(A)$. The factor comes from the volume of the n-dimensional ball. Sometimes this factor is included into the definition, so that the two measures coincide. Since this thesis focuses on the dimension of object and not their specific measure, it is not necessary to use this factor in the definition; however proofs work for both definitions.

In the following lemma we see that the Hausdorf measure is trivial for allmost all s.

Lemma 2.10. *Let $A \subseteq X$ and $s < t, \delta > 0$, then $\mathcal{H}^t_\delta(A) \leq \delta^{t-s}\mathcal{H}^s_\delta(A)$. Further if $\mathcal{H}^s(A) \leq \infty$, then $\mathcal{H}^t(A) = 0$ and if $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$.*

Proof. Let $\{U_i\}$ be a δ-cover of A, then

$$\sum_{i=1}^{n} \operatorname{diam}(A)^t = \sum_{i=1}^{n} \operatorname{diam}(A)^s \operatorname{diam}(A)^{t-s} \leq \delta^{t-s} \sum_{i=1}^{n} \operatorname{diam}(A)^s.$$

By taking the infimum we get the first claim and by taking limits we easily prove the second two. □

This observation is key to the definition of the Hausdorff dimension. There is a critical point in $[0, \infty]$, where the Hausdorff measure as a function in s jumps from ∞ to 0. Heuristically said: "A set is d-dimensional if its d-dimensional Volume is non trivial."

Definition 2.11. Let (X, d) be a metric space and $A \subseteq X$. The ***Hausdorff dimension*** of A is defined as

$$\dim(A) := \sup\{s : \mathcal{H}^s(A) = \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}. \tag{13}$$

Moreover, A is called an s***-set***, if $0 < \mathcal{H}^s(A) < \infty$.

Note that the Hausdorff measure at this critical point still can be 0 or ∞. We quickly see, that the Hausdorff dimension of the m-dimensional hypercubes is m, since it has non trivial Lebesgue measure.

Proposition 2.12. *a) If $A \subset B$, $\dim(A) \leq \dim(B)$.*

b) If $\{A_i\}_{i=1}^\infty$ is a sequence of sets, then

$$\dim\left(\bigcup_{i=1}^{\infty} A_j\right) = \sup_{i \geq 1} \dim(A_i).$$

Proof. a) Since $\mathcal{H}^s$ is monotone we have $\dim(A) = \inf\{s : \mathcal{H}^s(A) = 0\} \leq \inf\{s : \mathcal{H}^s(B) = 0\} = \dim(B)$.

b) Define $A := \bigcup_{i=1}^{\infty} A_i$. Monotonicity gives us $\dim(A) \geq \sup_{i\geq 1} \dim(A_i)$. Since $\mathcal{H}^s(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(A_i)$ holds, it follows that $\mathcal{H}^s(A) = 0$ for $s > \sup_{i\geq 1} \dim(A_i)$. Hence $\dim(A) \leq \sup_{i\geq 1} \dim(A_i)$.

□

The Hausdorff dimension also satisfies an invariance condition. We cannot achieve topological invariance but a rather big class of functions, which include affine transformations.

Definition 2.13. Let $(X, d), (Y, d')$ be two metric spaces. A function $f : A \subseteq X \to Y$ satisfies the ***Hölder-condition*** or is called ***Hölder continuous***, iff there exist constants $\alpha > 0, c > 0$, such that $\forall x, y \in A$

$$d'(f(x), f(y)) \leq c \cdot d(x, y)^{\alpha}. \tag{14}$$

Iff $\alpha = 1$, we call f a ***Lipschitz transformation***. Iff for $\psi : A \subseteq X \to Y$ there are $0 < c_1 \leq c_2 < \infty$, such that

$$c_1 d(x, y) \leq d'(\psi(x), \psi(y)) \leq c_2 d(x, y) \tag{15}$$

ψ is called ***bi-Lipschitz transformation***.

Note that Hölder continuous functions are actually continuous and bi-Lipschitz transformations are homeomorphisms onto their images.

Proposition 2.14. *Let $f : A \subseteq (X, d) \to (Y, d')$ be Hölder continuous between two metric spaces with constants c, α, $A \subset X$ and $s > 0$. Then*

$$\mathcal{H}^s(f(A)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(A). \tag{16}$$

Further $\dim(f(A)) \leq \alpha^{-1} \dim(A)$. *If ψ is a bi-Lipschitz transformation, then* $\dim(\psi(A)) = \dim(A)$, *hence* $\dim$ *is invariant under bi-Lipschitz transformations. .*

Proof. Let $\{U_i\}$ be a δ-cover of A. Then $\{f(A \cup U_i)\}$ covers $f(A)$ and

$$\operatorname{diam}(f(A \cap U_i)) = \sup_{x,y \in A \cap U_i} d'(f(x), f(y)) \leq c \sup_{x,y \in A \cap U_i} d(x, y)^{\alpha}$$

$$= c \cdot \operatorname{diam}(A \cap U_i)^\alpha \leq c\delta^\alpha =: \epsilon.$$

Therefore $\mathcal{H}_\epsilon^{\frac{s}{\alpha}}(f(A)) \leq c^{\frac{s}{\alpha}} \mathcal{H}_\delta^s(A)$ Taking the limit $\delta \to 0$ we get $\epsilon \to 0$ proves the first claim. For $\alpha = 1$ we get $\dim(f(A)) \leq \dim(A)$ and if we consider $\psi^{-1} : \psi(A) \to A$ of a bi-Lipschitz transformation, we get $\dim(\psi(A)) \leq \dim(A) = \dim(\psi^{-1}(\psi(A))) \leq \dim(\psi(A))$. □

It is possible to show for all metric spaces, that the Hausdorff dimension always exceeds topological dimension. This is shown in [4, p.114, 115], but it uses inductive dimension as the main definition for topological dimension. Since this was not yet covered in this thesis, we restrict ourselves to compact metric spaces following [5, Theorem 6.3.1.11., p. 183]. Note that this will suffices our purposes, because self-similar fractals are compact sets.

Theorem 2.15. *Let* $A \subseteq X$ *with* (X, d) *a metric space.Then* $\operatorname{Cov}(A) \leq \dim(A)$.

Proof. Assume that A is compact. Let $n = \operatorname{Cov}(A)$. Therefore there exist open sets $U_1, ..., U_{n+1}$ covering A, such that for any closed sets $F_i \subseteq U_i$ covering A, we have $\bigcap_{i=1}^{n+1} F_i \neq \varnothing$. Define functions on A as follows:

$$f_i(x) := \operatorname{dist}(x, A \setminus U_i), 1 \leq i \leq n+1 \quad f(x) := f_1(x) + ... + f_{n+1}(x) \tag{17}$$

This functions have Lipschitz constants $L(f_i) = 1, L(f) = n + 1$ and for all $x \in A : f(x) > 0$. Since A is compact, there are positive constants a, b, such that $a \leq f(x) \leq b, x \in A$. Now define

$$h(x) := \left(\frac{f_1(x)}{f(x)}, \frac{f_2(x)}{f(x)}, ..., \frac{f_{n+1}(x)}{f(x)} \right). \tag{18}$$

h is Lipschitz too, since we can deduce

$$\begin{aligned} \left| \frac{f_i(x)}{f(x)} - \frac{f_i(y)}{f(y)} \right| &= \frac{|f(x)f_i(y) - f(y)f_i(x)|}{f(x)f(y)} \\ &\leq \frac{f(x)|f_i(y) - f_i(x)| + f_i(x)|f(x) - f(y)|}{f(x)f(y)} \\ &\leq \frac{b(n+2)}{a^2} d(x, y) \end{aligned}$$

and therefore $L(h) \leq b(n+2)(n+1)/a^2$. We claim that $h[A]$ contains a simplex T and conclude that $\mathrm{Cov}(A) = n = \dim(T) \leq \dim(h[A]) \leq \dim(A)$ with monocity and the change of Hausdorff dimension under Lipschitz functions. Define

$$T := \left\{(t_1, ..., t_{n+1}) \in \mathbb{R}^{n+1} : t_i > 0, \sum_{i=1}^{n+1} t_i = 1\right\} \tag{19}$$

the standard simplex in $\mathbb{R}^{n+1}$ and for $(t_1, ..., t_{n+1}) \in T$ the closed sets F_i satisfying $F_i \subseteq U_i$ and covering A:

$$F_i = \left\{x \in A : \frac{f_i(x)}{f(x)} \geq t_i\right\}.$$

These satisfy the conditions above and we get a point $x \in A$, such that $f_i(x)/f(x) \geq t_i$, but since $\sum_i f_i(x)/f(x) = 1$, we get $f_i(x)/f(x) = t_i$ and therefore $h(x) = (t_1, ..., t_{n+1})$. Hence we conclude that $T \subseteq h[A]$. □

We can now give a formal definition for a fractal.

Definition 2.16. Let $A \subseteq X$ with (X, d) a metric space. Define the ***fractal degree*** as

$$\delta(A) := \dim(A) - \mathrm{Cov}(A). \tag{20}$$

Then A is ***fractal***, iff $\delta(A) > 0$.

This definition is independent from the notion of self-similarity. While most fractals we know of and work on have self-similar charateristics, this is not the defining quality. In applied mathematics and science, Hausdorff dimensions can be calculated for clouds or galaxies which have no underlying mathematical structure. But since self-similar shapes are easier to create, and also appear in various areas of mathematics, they spark great interest in fractal geometry. Before we investigate these further we establish further, properties of the Hausdorff dimension.

2.3 Box-counting dimension

Since the definition of the Hausdorff dimension contains arbitrary covers, it is not that practical to work with. Since we are working in $\mathbb{R}^n$ in most cases, we want to develop a similar notion of dimension, which only uses coverings with similar shapes.
Consider a covering of $F \subseteq \mathbb{R}^n$ by $N_\epsilon(F)$ many balls of radii $\epsilon/2$. Then $\mathcal{H}^s_\epsilon(F) \leq N_\epsilon(F)\epsilon^s$. If $s < \dim(F)$, then $\mathcal{H}^s(F) = \infty$. So for small ϵ we can demand $\mathcal{H}^s_\epsilon(F) > 1$. Taking logarithms gives us

$$0 < \log \mathcal{H}^s_\epsilon(F) \leq \log N_\epsilon(F) + s \log \epsilon.$$

Since we can get arbitrary close with s to $\dim(F)$, we have

$$\dim(F) \leq \frac{\log N_\epsilon(F)}{-\log \epsilon}. \tag{21}$$

In conclusion we started with an arbitrary covering of A with balls and got an upper bound for the Hausdorff dimension. The idea for the box-counting dimension, Box, is now to see how this bound changes in the limit. According to Falconer [9, p. 47] there are multiple ways to define Box.

Definition and Proposition 2.17. For $F \subseteq \mathbb{R}^n$ the ***upper and lower box-counting dimensions*** are defined as

$$\underline{\text{Box}}(F) := \liminf_{\epsilon \to 0^+} \frac{\log N_\epsilon(F)}{-\log \epsilon}, \tag{22}$$

$$\overline{\text{Box}}(F) := \limsup_{\epsilon \to 0^+} \frac{\log N_\epsilon(F)}{-\log \epsilon}, \tag{23}$$

where $N_\epsilon(F)$ notes one of the following:

(i) the minimal number of closed balls with radius ϵ covering F;

(ii) the minimal number of hypercubes with sidelength ϵ covering F;

(iii) the number of ϵ-lattice hypercubes intersecting F;

(iv) the minimal number of set with diam $\leq \epsilon$ covering F;

(v) the maximal number of disjoint balls with radius ϵ with centers in F.

Iff upper and lower box-counting dimension coincide on F, we define ***box-counting dimension*** as $\mathrm{Box}(F) = \underline{\mathrm{Box}}(F) = \overline{\mathrm{Box}}(F)$.

Proof. To see the equivalence of ii) and iv) note that you can cover an arbitrary set with diameter ϵ with an hypercube of sidelength 2ϵ by taking any point as the centre. This cube can be devided into 2^n hypercubes of length ϵ, so $N_\epsilon(F)$ changes mostly by this multiple. Taking logarithms turns it into an additive constant, which disappears in the limit.
Since a covering with closed balls is a 2ϵ-cover, we get factor 2 in the the denominator, which also vanishes in the limit. This shows the equivalence to i).
The equivalence to v) can be achieved by blowing up the balls by the factor 2 so they form a cover, we get an upper bound for the minimal number of covers of 2ϵ-balls. Conversely, the minimal number of a cover is already an upper bound for the max of disjoint balls. Finally the number of ϵ-lattice hypercubes is an upper bound for the minimal number of a cover and lower bound for the maximal number of the number of disjoint ϵ balls. Since these limits meet, we get the equivalence to iii) by sandwich theorem. □

The name box-counting dimension is now self-explanatory since we can calculate it by lying finer and finer mashes on the set and just count how many boxes cover parts of the set. Using (21) we already proved the following corollary.

Corollary 2.18. *For $F \subseteq \mathbb{R}^n$, the following inequality always holds:*

$$\dim(F) \leq \underline{\mathrm{Box}}(F) \leq \overline{\mathrm{Box}}(F). \tag{24}$$

Another notion with which the box-counting dimension can be characterized is by looking at the *Minkowsky content*.

Definition 2.19. Let $F \subseteq \mathbb{R}^n$ and define the ***δ-parallel extension*** of A as

$$F_\delta := \{x \in \mathbb{R}^n : |x - y| \leq \delta, y \in F\} \tag{25}$$

Iff there exist $c > 0, 0 < s \leq n$, such that

$$\lim_{\delta \to 0^+} \frac{\mathcal{L}^n(F_\delta)}{\delta^{n-s}} = c, \tag{26}$$

then c is called the ***δ-Minkowsky content*** of A and s its ***Minkowsky dimension***.

In fact Minkowsky and box-counting dimension coincide.

Theorem 2.20. *Let* $F \subseteq \mathbb{R}^n$, *then*

$$\underline{\text{Box}}(F) = n - \overline{\lim}_{\delta\to 0^+} \frac{\log \mathcal{L}^n(F_\delta)}{\log \delta}, \tag{27}$$

$$\overline{\text{Box}}(F) = n - \underline{\lim}_{\delta\to 0^+} \frac{\log \mathcal{L}^n(F_\delta)}{\log \delta}. \tag{28}$$

Proof. If F is covered by $N_\delta(F)$ balls with radius δ, then we achieve a covering of F_δ by blowing up the balls by a factor of 2. If we denote the volume of the n-dimensional unit ball by c_n, we get the inequality $\text{Vol}^n(F_\delta) \leq N_\epsilon(F)c_n(2\delta)^n$. In the log this transforms to

$$\frac{\log \text{Vol}^n(F_\delta)}{-\log \delta} \leq \frac{\log 2^n c_n + n \log \delta + \log N_\delta(F)}{-\log \delta}.$$

Taking the lim inf yields

$$\underline{\lim}_{\delta\to 0^+} \frac{\log \mathcal{L}^n(F_\delta)}{-\log \delta} \leq -n + \underline{\text{Box}}(F)$$

We prove the second claim analog with lim sup.
Conversely let $N_\delta(F)$ be the maximal number of disjoint balls with radius δ, then we get $\text{Vol}^n(F_\delta) \geq N_\delta(F)c_n(2\delta)^n$ and conclude in the same way as above. □

Not only is the box-counting dimension of practical use, it also appears in various formulas to better understand Hausdorff dimension. In Section 3 we will prove that box-counting dimension coincides with Hausdorff dimension for some self-similar sets and therefore is a strong tool to determine dimensions of self-similar fractals.

2.4 Products and intersections of fractals

In almost all mathematical subjects taking products and intersections are of great interest. Ordinary geometry gives us an idea how dimension should behaves under these operations. So if we take the product of a n-dimensional set E and a m-dimensional set F, the dimension of the carthesian product $E \times F$ is simply the sum $n + m$. In general only $\dim(E \times F) \geq \dim E + \dim F$ is true for Hausdorff dimension, but in specific cases we can prove the full equality.
To investigate how Hausdorff dimension behaves under products we have to establish some methods to calculate Hausdorff dimensions following [9, Chapters 4-7].

Lemma 2.21. *Let $F \subseteq \mathbb{R}^n$ be covered with n_k sets with* diam *at most δ_k for $k \in \mathbb{N}$ and there is a constant $0 < c < 1$ with $\delta_{k+1} \geq c\delta_k$. Then*

$$\dim(F) \leq \underline{\text{Box}}(F) \leq \liminf_{k\to\infty} \frac{\log n_k}{-\log \delta_k}, \tag{29}$$

$$\dim(F) \leq \overline{\text{Box}}(F) \leq \limsup_{k\to\infty} \frac{\log n_k}{-\log \delta_k} \tag{30}$$

hold. If $n_k \delta_k^s$ is bounded in the limit, it yields $\mathcal{H}^s(F) < \infty$

Proof. The inequalities between Hausdorff and box-counting dimension have already been shown. Now let δ be arbitrary, then there is a $k \in \mathbb{N}$ such that $\delta_{k+1} \leq \delta \leq \delta_k$ and it yields

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log n_{k+1}}{-\log \delta_k} \leq \frac{\log n_{k+1}}{-\log \delta_{k+1} + \log(\delta_{k+1}/\delta k)} \leq \frac{\log n_{k+1}}{-\log \delta_{k+1} + \log c}.$$

Taking limits shows the inequalities and for the last claim $\mathcal{H}^s(F) \leq n_k \delta_k^s$ and therefor taking the limit yields $\mathcal{H}^s(F) < \infty$. □

Since it is hard to estimate Hausdorff measure with specific covers, it is often more constructive to work with distributions.

Definition 2.22. A measure μ on $\mathbb{R}^n$ is called a ***distribution***, iff $0 < \mu(\mathbb{R}^n) < \infty$ and μ has bounded support.

Lemma 2.23 (Distribution principal I)**.** *Let* μ *be a distribution on* $F \subseteq \mathbb{R}^n$ *and for* s *there are constants* $c > 0, \delta > 0$*, such that*

$$\mu(U) \leq c \,\mathrm{diam}(U)^s \tag{31}$$

for all U *with* $\mathrm{diam}(U) \leq \delta$*, then* $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$ *and*

$$s \leq \dim(F) \leq \underline{\mathrm{Box}}(F) \leq \overline{\mathrm{Box}}(F) \tag{32}$$

Proof. Let $\{U_i\}$ be a cover of F, then

$$0 < \mu(U) = \mu(\bigcup_i U_i) \leq \sum_i \mu(U_i) \leq c \sum_i \mathrm{diam}(U_i)^s$$

and therefore $\mathcal{H}^s(F) \geq \mu(F)/c > 0$, which shows the last claim □

This principal is important: it gives us a lower bound, while with box-counting dimension we already have an upper bound. We can further sharpen this idea by substituting U with r-balls.

Lemma 2.24 (Distribution principal II)**.** *Let* μ *be a distribution on* $\mathbb{R}^n$*,* $B_r(x)$ *the closed ball with radius* r *and centre* x*,*$F \subseteq \mathbb{R}^n$ *a Borel set and for* s *there is a constant* $c > 0$ *such that*

$$\limsup_{r \to 0} \frac{\mu(B_r(x))}{r^s} \leq c \tag{33}$$

for all $x \in F$*, then* $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$ *and*

$$s \leq \dim(F) \leq \underline{\mathrm{Box}}(F) \leq \overline{\mathrm{Box}}(F). \tag{34}$$

Proof. Define for $\delta > 0$

$$F_\delta := \{x \in F : \mu(B_r(x)) < (c - \epsilon) r^s, \forall 0 < r < \delta, \epsilon > 0\} \tag{35}$$

and let $\{U_i\}$ be a δ-cover for F as well as F_δ. Let $x \in F_\delta \cap U_i$, then the ball B with centre x and radius $\mathrm{diam}(U_i) \leq \delta$ contains U_i. Since $x \in F_\delta$ we get $\mu(U_i) \leq \mu(B) \leq c \,\mathrm{diam}(U_i)^s$ and

$$\mu(F_\delta) \leq \sum_i \{\mu(U_i) : U_i \cap F_\delta \neq 0\} \leq c \sum_i \mathrm{diam}(U_i)^s.$$

Since the cover was arbitrary, we get $\mu(F_\delta) \leq \mathcal{H}^s_\delta \leq \mathcal{H}^s$ and taking the limit $\delta \to 0$ yields $\mu(F) \leq c\mathcal{H}^s(F)$. □

Another concept used in geometry, especially fractal geometry, is *density*. It is used to investigate the structure of fractals further. This thesis limits itself only to definitions and basic results without proofs. For more details the reader is recommended to read [9, chapter 5].

Definition 2.25. Let $F \subset \mathbb{R}^n$ be an s-set. Then the ***upper and lower densities*** of F in $x \in F$ are defined as

$$\underline{D}^s(F,x) := \liminf_{r\to 0} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s} \tag{36}$$

$$\overline{D}^s(F,x) := \limsup_{r\to 0} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s}. \tag{37}$$

A point $x \in F$ is called ***regular*** iff $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 1$, otherwise x is ***irregular***. F is called ***regular/ irregular*** iff $\mathcal{H}^s$-allmost all x are.

Proposition 2.26. *Let $F \subset \mathbb{R}^n$ be an s-set. Then*

a) $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 0$, *for $\mathcal{H}^s$-allmost all $x \notin F$.*

b) $2^{-s} \leq \overline{D}^s(F,x) \leq 1$, *for $\mathcal{H}^s$-allmost all $x \notin F$.*

Theorem 2.27. *Let $F \subset \mathbb{R}^2$ be an s-set, with $s \notin \mathbb{N}$. Then F is irregular.*

Another theorem we will just state, is the existence of s-subsets of higher dimensional sets. For a proof sketch visit [9, p. 73].

Theorem 2.28. *Let $F \subseteq \mathbb{R}^n$ be a Borel set, with $\mathcal{H}^s(F) = \infty$. Then there is a compact Borel subset $E \subset F$, such that $0 < \mathcal{H}^s(E) < \infty$.*

We now have the necessary tools to tackle the product formulas for Hausdorff dimension. The hard work of the proof is done in the following lemma.

Lemma 2.29. *Let $E \subset \mathbb{R}^m, F \subset \mathbb{R}^n$ Borel sets with $\mathcal{H}^s(E), \mathcal{H}^t(F) < \infty$, then there is a constant $c > 0$, which only depends on s and t such that*

$$\mathcal{H}^{s+t}(E \times F) \geq c\mathcal{H}^s(E) \cdot \mathcal{H}^t(F). \tag{38}$$

Proof. If $\mathcal{H}^s(E) = 0$ or $\mathcal{H}^t(F) = 0$ the claim is trivial. So now suppose that E is a s-set and F a t-set. Define a distribution on $\mathbb{R}^m \times \mathbb{R}^n$ as follows:

$$\mu(I \times J) := \mathcal{H}^s(I \cap E) \cdot \mathcal{H}^t(J \cap F) \tag{39}$$

for $I \subseteq \mathbb{R}^m, J \subseteq \mathbb{R}^m$. This is a distribution with $\mu(R^{m+n}) = \mathcal{H}^s(E) \cdot \mathcal{H}^t(F)$. Using Proposition 2.26 b) we get

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s} \leq 1 \text{ and } \limsup_{r \to 0} \frac{\mathcal{H}^t(F \cap B_r(y))}{(2r)^t} \leq 1 \tag{40}$$

for $\mathcal{H}^s$-allmost all $x \in E$ and for or $\mathcal{H}^t$-allmost all $y \in F$. Because we defined μ as the product of $\mathcal{H}^s$ and $\mathcal{H}^t$ we get for μ-allmost all $(x, y) \in E \times F$

$$\mu(B_r(x, y)) \leq \mu(B_r(x) \times B_r(y)) = \mathcal{H}^s(B_r(x) \cap E) \cdot \mathcal{H}^t(B_r(y) \cap F)$$

Using (40) we get for μ-allmost all $(x, y) \in E \times F$

$$\limsup_{r \to 0} \frac{\mu(B_r(x, y))}{(2r)^{s+t}} \leq \limsup_{r \to 0} \frac{\mathcal{H}^s(B_r(x) \cap E)}{(2r)^s} \frac{\mathcal{H}^t(B_r(y) \cap F)}{(2r)^t} \leq 1.$$

Lemma 2.24 garantees us that $\mathcal{H}^{s+t}(E \times F) \geq 2^{-(s+t)} \mu(E \times F) = 2^{-(s+t)} \mathcal{H}^s(E) \cdot \mathcal{H}^t(F)$. □

Theorem 2.30 (Product formula I)**.** *Let $E \subset \mathbb{R}^m, F \subset \mathbb{R}^n$ Borel sets, then*

$$\dim(E \times F) \geq \dim E + \dim F \tag{41}$$

Proof. For $s < \dim(E), t < \dim(F)$ we can use Theorem 2.28 and get Borel sets $E_0 \subset E, F_0 \subset F$ with $0 < \mathcal{H}^s(E_0), \mathcal{F}^t(F_0) < \infty$. Using the previous lemma we get $\mathcal{H}^{s+t}(E \times F) \geq \mathcal{H}^{s+t}(E_0 \times F_0) \geq c\mathcal{H}^s(E_0) \cdot \mathcal{H}^t(F_0) > 0$. Therefore $\dim(E \times F) \geq s + t$ and since s, t are arbitrary close to $\dim(E), \dim(F)$, we proved the claim. □

In general the converse inequality does not hold, but the key to prove the full equality for self-similar set is the second product formula.

Theorem 2.31 (Product formula II)**.** *Let $E \subset \mathbb{R}^m, F \subset \mathbb{R}^n$ arbitrary sets, then*

$$\dim(E \times F) \leq \dim E + \overline{\text{Box}}\, F \tag{42}$$

Proof. For $s > \dim(E), t > \overline{\text{Box}}(F)$ there exists $\delta_0 > 0$ such that F can be covered with $N_\delta(F) \le \delta^{-t}$ many sets of diameter δ for all $\delta \le \delta_0$. Let $\{U_i\}$ be a δ-cover of E with $\sum \text{diam}(U_i)^s < 1$. For all i let $U_{i,j}$ be a δ-cover of F with $N_{diam(U_i)}(F)$ sets of same diameter as U_i. We can cover $U_i \times F$ with $\{U_i \times U_{i,j}\}$ which have diameter $\sqrt{2}\,\text{diam}\, U_i$. Therefore $E \times F$ is covered with $\{U_i \times U_{i,j}\}$ where both i and j vary. We get

$$\begin{aligned}\mathcal{H}^{s+t}_{\sqrt{2}\delta}(E \times F) &\le \sum_i \sum_j \text{diam}(U_i \times U_{i,j})^{s+t} \\ &\le \sum_i \text{diam}(U_i)^{s+t} N_{\text{diam}(U_i)}(F)\sqrt{2}^{s+t} \\ &\le \sqrt{2}^{s+t} \sum_i \text{diam}(U_i)^{s+t}\,\text{diam}(U_i)^{-t} \\ &= \sqrt{2}^{s+t} \sum_i \text{diam}(U_i)^s < \sqrt{2}^{s+t}.\end{aligned}$$

Hence $\mathcal{H}^{s+t}_{\sqrt{2}\delta}(E \times F) \le \infty$ for all $s > \dim(E), t > \overline{\text{Box}}(F)$, which finishes the proof. □

Corollary 2.32. *For* $\overline{\text{Box}}\, F = \dim F$ *it follows that*

$$\dim(E \times F) = \dim E + \dim F \tag{43}$$

The proof of this Corollary is just combining both product formulas. So the goal for the next section about self similarities is to prove that box-counting dimension and Hausdorff dimension coincide. Note that one of the sets E and F has to behave that nicely to have the full equality.

Another thing we are interested in, is how Hausdorff dimension behaves under projection. We state the Projection Theorem:

Theorem 2.33 (Projection Theorem)**.** *Let* $F \subset \mathbb{R}^n$ *be a Borel set.*

a) If $\dim F \le k$, *then* $\dim(proj_\Pi F) = \dim F$, *for allmost all* Π k*-dimensional subspace of* $\mathbb{R}^n$.

b) If $\dim F > k$, *then* $proj_\Pi F$ *has positive* k*-dimensional Lebesgue measure and therefore for almost all* Π *Hausdorff dimension* k.

Since this thesis is interested in the intersection of a line and a fractal, we investigate how Hausdorff dimension behaves under intersection. If we intersect a non fractal shape like a disk with a line we get that in most cases intersection is a line or empty. So dimension reduces by one. If the line is a tangent we get an nonempty intersection, which does not follow that rule. If the shape gets more complicated, even fractal, we hope these exeptional cases do not overrule and the general statement still holds. This is the case for Borel sets. To show this, we start with a powerful lemma.

Lemma 2.34. *Let F be a Borel set in $\mathbb{R}^2$ and L_x the line, which is parallel to the y-axis through $(x,0)$. For $1 \leq s \leq 2$ the following holds:*

$$\int_{-\infty}^{\infty} \mathcal{H}^{s-1}(F \cap L_x) dx \leq \mathcal{H}^s(F) \tag{44}$$

Proof. Let $\epsilon > 0$ and $\{U_i\}$ a δ-cover, such that $\sum \operatorname{diam}(U_i)^s \leq \mathcal{H}^s_\delta + \epsilon$. Let S_i be an axis-parallel square with sidelength $\operatorname{diam}(U_i)$ which contains U_i and χ_i its indicator function. The set $\{S_i \cap L_x\}$ covers $F \cap L_x$ and it follows that

$$\begin{aligned} \mathcal{H}^{s-1}(F \cap L_x) &\leq \sum_i \operatorname{diam}(S_i \cap L_x)^{s-1} = \sum_i \operatorname{diam}(S_i \cap L_x) \operatorname{diam}(U_i)^{s-2} \\ &= \sum_i \operatorname{diam}(U_i)^{s-2} \int \chi_i(x,y) dy \end{aligned}$$

Hence

$$\begin{aligned} \int \mathcal{H}^{s-1}(F \cap L_x) dx &\leq \sum_i \operatorname{diam}(U_i)^{s-2} \int \int \chi_i(x,y)\, dy\, dx \\ &= \sum_i \operatorname{diam}(U_i)^s \leq \mathcal{H}^s_\delta + \epsilon. \end{aligned}$$

Since ϵ was arbitrary small and with the limit $\delta \to 0$ we conclude this proof. □

Corollary 2.35. *Let F be a Borel set in $\mathbb{R}^2$ and L_x the line, which is parallel to the y-axix through $(x,0)$. For (Lebesque-)almost all x, it holds that $\dim(F \cap L_x) \leq \max\{0, \dim F - 1\}$*

Proof. Let $s > \dim(F)$, then $\mathcal{H}^s(F) = 0$. For $s > 1$ (44) yields that $\mathcal{H}^{s-1}(F \cap L_x) = 0$ for allmost all x, therefore $\dim(F \cap L_x) \leq s - 1$. □

Note that since Hausdorff dimension is invariant under rotation, we can generalize this result for arbitrary lines.

Corollary 2.36. *Let F be a Borel set in $\mathbb{R}^2$. For (Lebesque-)allmost all lines L it holds, that* $\dim(F \cap L) \leq \max\{0, \dim F - 1\}$.

Further we can use the same method to proof a higher dimensional analogue to this claim.

Corollary 2.37. *Let F be a Borel set in $\mathbb{R}^n$. For (Lebesque-)almost all m-dimensional subspaces $E \leq \mathbb{R}^n$ it holds, that* $\dim(F \cap E) \leq \max\{0, \dim F + m - n)\}$.

We omit this proof since it basically just varies in the number of integral signs. Another way to state the result is $\dim(F \cap E) \leq \max\{0, \dim F + \dim E - n\}$. Is this true for arbitrary sets E? The answer is no, but we can get an estimate using the dimension of the product, which in some cases actually desolves into the sum (revisit 2.30 and 2.31). This proof is a modified version of Falconer's proof in [9, p.118].

Theorem 2.38. *Let E, F be a Borel sets in $\mathbb{R}^n$. For (Lebesque-)almost all $x \in E$, it holds that*

$$\dim(F \cap (E + x)) \leq \max\{0, \dim(F \times E) - n)\}. \tag{45}$$

Proof. Let $c \in \mathbb{R}^n$ be an arbitrary vector and H_c the n-dimension subspace in $\mathbb{R}^{2n}$ with $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \iff x = y + c$. If $\dim(F \times E) > n$ we can use Corrolary 2.37 and get

$$\dim((F \times E) \cap H_c) \leq \dim(F \times E) + n - 2n \tag{46}$$

for (Lebesque-)almost all $c \in \mathbb{R}^n$. Now $(x, x - c) \in (F \times E) \cap H_c \iff x \in F \cap (E + c)$, so the projection of $((F \times E) \cap H_c)$ on $\mathbb{R}^n \times \{0 \in \mathbb{R}^n\}$ is $F \cap (E + c)$. We use the Projection Theorem 2.33 and get the claim. □

In all statements above we only have ≤, but it is also possible to prove the opposite inequality. A sketch of the proof is outlined in [9, chapt. 8] for a more genereal statement using techniques detailed in previous chapters. This thesis will not make the effort of establishing all of these and therefore the reader is refered to this or other publications. We only state the following result:

Theorem 2.39. *Let E, F be a Borel sets in $\mathbb{R}^n$. It holds, that*

$$\dim(F \cap (E + x)) \geq \dim(E) + \dim(F) - n, \tag{47}$$

for $x \in I$, where $I \subset \mathbb{R}^n$ with positive measure.

3 Self-similarity

Most of the classic examples of fractals as the Cantor set or the Koch curve have *self-similar* properties: parts of the set are just scaled down copies of the whole set. But also non fractal shapes as lines and cubes are self-similar. Cutting a line in half and scaling it by 2 gives us back the original line or splitting a cube in the 8 cubes with half the side length give us $2^3 = 8$ parts which can be scaled up by 2 to the original. This easy example can be shown for n-dimensional hypercubes and show that the dimension is found in the exponent. This idea leads to the definition of *similarity dimension*. We will prove that it coincides with Hausdorff dimension under certain conditions. This will bring us a big step forward in understanding the Twin Dragon curve.
But first we extend the naive definition of self-similarity, since we can prove a lot of theorems in this broad context. Later we will come back to the more stricter definition of *self-affine* set; its theory is connected to linear algebra, classical algebra, number theory and the theory of automata.

Definition 3.1. A function $f : X \to X$, with (X, d) a metric space, is called a ***contraction***, iff it is Lipschitz continuous (def. 2.13) with $c \in (0, 1)$, that is iff $d(x, y) \leq c \cdot d(x, y), \forall x, y \in X$. Iff $d(x, y) = c \cdot d(x, y), \forall x, y \in X$, we call f a ***similar contraction***. We denot the Lipschitz constant of f by $L(f)$.

We state an elementary, but very powerful theorem.

Theorem 3.2 ((Banach) Fixed point theorem)**.** *Let* (X, d) *be a complete metric space and* $f : X \to X$ *a contraction. Then there exists a unique fixed point, that is* $\exists! x \in X : f(x) = x$.

The clue is to put this theorem in the right context to prove the existence of self-similar sets, which we define in the following way:

Definition 3.3. Let $(f_i)_{i=1}^m : X \to X$, be contractions. $\emptyset \neq S \subseteq X$ is called ***self-similar*** iff

$$S = \bigcup_{i=1}^{m} f_i(S). \tag{48}$$

Iff the f_i are similar contractions, S is called a ***self-affine*** set.

3.1 Existence and Uniqueness of self-similar set

Definition and Proposition 3.4. Let (X, d) be a metric space and let $\mathcal{K}(X)$ be the set of all non-empty compact subsets of X. The ***Hausdorff metric*** on $\mathcal{K}(X)$ is defined as

$$d_H(A, B) := \inf_{\delta \geq 0} \{A \subseteq B_\delta \wedge B \subseteq A_\delta\} \tag{49}$$

for $A, B \in \mathcal{K}(X)$ with $A_\delta = \{x \in X | d(x, y) \leq \delta, y \in A\}$ the δ-parallel extension of A (generalisation of Def. 2.19). This is indeed a metric.

Proof. We first prove that the infimum is actually a minimum for compact sets. If we can show that $\bigcap_{\epsilon > \delta} A_\epsilon = A_\delta$, we have $A \subseteq B_\epsilon, \forall \epsilon > \delta \Rightarrow A \subseteq \bigcap_{\epsilon > \delta} B_\epsilon = B_\delta$ and conversely $A \subseteq B_\delta$. Since the A_ϵ are monotone decreasing take a sequence $\epsilon_n \to \delta$. For $x \in \bigcap_n A_{\epsilon_n}$ we get $a_n \in A$ such that, $d(x, a_n) \leq \epsilon_n$. Since A is compact, we can assume, that a_n converges to an $a \in A$. We now have $d(x, a) \leq d(x, a_n) + d(a_n, a) \leq \epsilon_n + d(a_n, a) \to \delta + 0$, therefore $x \in A_\delta$.
Let now $A, B, C \in \mathcal{K}(X)$, then $A_0 = A$, hence $d_H(A, A) = 0$. Conversely if $d(A, B) = 0$, then $A = A_0 \subseteq B \wedge B = B_0 \subseteq A \Rightarrow A = B$. Symmetry is obvious per definitionem. For triangle equality define $\delta_A := d_H(A, C), \delta_B := d_H(B, C)$ and note that $(A_\delta)_\epsilon = A_{\delta+\epsilon}$. We have $A \subseteq C_{\delta_A} \wedge C \subseteq A_{\delta_A}$ and $B \subseteq C_{\delta_B} \wedge C \subseteq B_{\delta_B}$ and therefore $A \subseteq (C_{\delta_A})_{\delta_B} \wedge C_{\delta_B} \subseteq (A_{\delta_A})_{\delta_B} \Rightarrow A \subseteq B_{\delta_A} \wedge B \subseteq A_{\delta_A + \delta_B}$. Hence $d_H(A, B) \leq \delta_A + \delta_B$. □

The space of all compact subsets of X forms a metric space with the Hausdorff metric. We want to use the fixed point theorem on $(\mathcal{K}(\mathbb{R}^n), d_H)$, so we need it to be complete, which is indeed the case. Following [19, Theorem 2.1., p.18] we prove the following theorem.

Theorem 3.5. *Let $\mathcal{K}(\mathbb{R}^n)$ be the set of all non-empty compact subsets of X and d_H the Hausdorff metric. Then $(\mathcal{K}(\mathbb{R}^n), d_H)$ is a complete metric space.*

Proof. Let $\{A_i\} \in \mathcal{K}(\mathbb{R}^n)$ be an arbitrary Cauchy sequence with respect to the Hausdorff metric, that is $\forall \epsilon > 0, \exists m = m(\epsilon), \forall i, j > m : d_H(A_i, A_j) \leq \epsilon$. We define

$$E_k := \overline{\bigcup_{i=k}^{\infty} A_i} \text{ and } E := \bigcap_{k=1}^{\infty} E_k \tag{50}$$

Since $\{A_i\}$ is uniformly bounded, all E_k are closed and bounded and therefore compact. Since $\{E_k\}$ is monotone decreasing, E is compact. We show that

$A_i \to E \in \mathcal{K}(\mathbb{R}^n)$ in the Hausdorff metric to finish the proof.
Let $\epsilon > 0$. We have to prove, that $\forall q \geq m : d_H(E, A_q) < \epsilon$. On the one hand $E \subset E_q \subset (A_q)_\epsilon$ follows directly from the definition. On the other hand let $x \in A_q$, then there exist $y_p \in A_p$, $p > q$, such that $\|x - y_p\| \leq \epsilon$, since $A_q \subseteq (A_p)_\epsilon$. Let z be an accumulation point of $\{y_p\}$. For every $k \in \mathbb{N}$ and $p > k, q$ we have $y_p \in A_p \subseteq E_p \subseteq E_k$. Since E_k is compact, z is in E_k and therefore $z \in E$. Now we calculate $\|x - z\| \leq \|x - y_p\| + \|y_p - z\| \leq \epsilon + \|y_p - z\|$ and since z is an accumulation point, we get $x \in z_\epsilon \subset E_\epsilon$. Thus $d_H(E, A_q) \leq \epsilon$. □

Note that this argument holds for all complete metric spaces with the *Heine-Borel property*, which holds true iff all closed bounded sets are compact in X. Following [5, Theorem 2.5.3., p.72] it is possible to prove the same statement for arbitrary complete metric spaces, where the difficulty only lies in showing that the limit is compact. We can now state and prove the existence and uniqueness theorem.

Theorem 3.6. *Let (X, d) be a complete metric space. Then $(\mathcal{K}(X), d_H)$ is a complete metric space.*

Theorem 3.7 (Existence and Uniqueness of self-similar shape)**.** *Given a family of contractions $(f_i)_{i=1}^m : X \to X, m \geq 2$ there exists a unique self-similar set S.*

Proof. Define the map $F : \mathcal{K}(X) \to \mathcal{K}(X)$ as

$$F(A) := \bigcup_{i=1}^{m} f_i(A). \tag{51}$$

This is a well defined map, since the images of compact sets under continuous maps are compact as well as their finite union. We show, that F is a contraction with respect to the Hausdorff metric.
Let $A, B, C, D \in \mathcal{K}(X)$, then

i) $d_H(f_i(A), f_i(B)) \leq L(f_i) d_H(A, B)$,
ii) $d_H(A \cup B, C \cup D) \leq \max\{d_H(A, C), d_H(B, D)\}$.

To prove i) set $s = d_H(A, B), t = s \cdot L(f_i)$; then $\forall x \in A, \exists y \in B : d(x, y) \leq s$ and

$$d(f_i(x), f_i(y)) \leq L(f_i) d(x, y) \leq t.$$

Hence $\forall f_i(x) \in f_i(A), \exists f_i(y) \in f_i(B) : d(f_i(x), f_i(y)) \leq t$ and the converse claim holds by the same reasoning. Together that shows $d_H(f_i(A), f_i(B)) \leq t$.
For ii) set $s = d_H(A, C), t = d_H(B, D), r = \max\{s, t\}$. We get

$$A \cup B \subseteq C_s \cup D_t \subseteq (C \cup D)_r$$

and conversely $C \cup D \subseteq (A \cup B)_r$.
Now by using i) and ii) repeatedly we get:

$$\begin{aligned} d_H(F(A), F(B)) &= d_H(\bigcup_{i=1}^m f_i(A), \bigcup_{i=1}^m f_i(B)) \\ &\leq \max_{1 \leq i \leq m} d_H(f_i(A), f_i(B)) \\ &\leq \max_{1 \leq i \leq m} L(f_i) d_H(A, B) \end{aligned}$$

So we have a contraction with $L(F) =\leq \max_{1 \leq i \leq m} L(f_i) < 1$. We apply the fixed point theorem (Theorem 3.2) to $\mathcal{K}(X)$ and F. As a result we get a unique set $S \in \mathcal{K}(X)$ with

$$S = F(S) = \bigcup_{i=1}^m f_i(S).$$

□

This theorem not only gives us a proof of the existence of the self-similar set associated to a family of contractions, but the proof of Banach fixed point theorem also shows us how to find it. Starting with an arbitrary compact set $S_0 \neq \emptyset$ one defines a sequence S_n by repeatedly applying F. This converges to the fixed point and self-similar set S. If S is a fractal, the elements of the series are sometimes called *pre-fractals*, but note that they in general do not have fractal dimensions.
The fact that fractals appear as fixed points of contractions shows, its connection to dynamical systems and chaos theory. Therefore we put the previous definitions in a new light.

Definition 3.8. Let S be a self-similar set defined by the contractions $\{f_i : 1 \leq i \leq m\}$. Then the contractions form a so called ***iterated function system***, short ***IFS***, which is the closure of $\{f_i : 1 \leq i \leq m\}$ under composition. S is called an ***attractor***. The map $F(M) = \bigcup_i f_i(M), M \subseteq X$ is called the ***Hutchinson operator***.

It is also possible to allow infinite iterated function system IIFS, as explained in [10]. The main difficulty here is that applying the set function $F(S) = \bigcup_{i \in I} f_i(S)$ does not necessarily map compact set so compact sets, since the countable union of closed sets can be open. For further informations the reader is adviced to read Fernau's book. However in this thesis only finite IFS are considered.

3.2 String models, Mauldin Williams graphs and Büchi Automata

As we recall from our previous work on the Cantor set, it is possible to characterize the points in the Cantor set as numbers with only 0 and 2 as digits. This concept of describing points in a self-similar set with integer strings can be generalized. Moreover self-similar sets are strongly related to discrete mathematics. There are ways to describe them with graphs, which are used in computer science to generate images of them. Self-similar shapes and fractals play a huge role in informatics, for example to create realistic models of natural objects such as trees, mountains or clouds. But also pure mathematics uses this notion to further investigate self-similar shapes. One notion that appears in [1] is the *Büchi automaton*, which wiil be essential to compute intersections with the Twin Dragon in Chapter 5.

First we introduce a little bit of terminology following [5, Chapt. 4], before we establish the *String model theorem*.

Definition 3.9. A ***ratio list*** is a list of positive numbers $(r_1, ..., r_m)$. If $0 < r_i < 1, \forall i$ the ratio list is called ***contracting***. We say a complete metric space (X, d) and $\{f_i : 1 \leq i \leq m\}$ an IFS ***realize*** the ratio list, iff $L(f_i) = r_i$. Let $E = \{1, ..., m\}$ then we call E^ω the ***set of all infinite strings*** over the ***alphabet*** E and $E^* = \{(e_1, ..., e_n) : n \in \mathbb{N}_0, e_i \in E, \forall 1 \leq i \leq n\}$ the set of all finite strings including the empty string Λ. For each $e \in E$ we define the ***right shift*** $\theta_e : E^\omega \to E^\omega$ by $\theta_e(\sigma) := e\sigma$ for $\sigma \in E^\omega$.

The idea now is to create a metric on the strings such that the right shifts realize the given ratio list. Since every string can be listed in a tree diagram we assign to each node a notion of diameter and construct a metric satisfying the claim.

Definition and Proposition 3.10. Given a contracting ratio list $(r_e)_{e\in E}$, there exists a metric on E^ω such that $\{\theta_e, e \in E\}$ is a realization with self-similarities of the ratio list. This is called the ***string model*** of the ratio list.

Proof. Let $\alpha \in E^*$ be a finite string, then we define w_α recursively:

$$w_\Lambda := 1 \qquad\qquad w_{\alpha e} := w_\alpha \cdot r_e, \forall \alpha \in E^*, e \in E. \tag{52}$$

Now define $\delta : E^\omega \times E^\omega \to [0,1]$ as follows: if $\tau, \sigma \in E^\omega$ they have a longest common prefix α. Now we set

$$\delta(\sigma,\tau) := \begin{cases} w_\alpha & ,\tau \neq \sigma \\ 0 & ,\tau = \sigma \end{cases} \tag{53}$$

This definition is clearly symmetrical and $\delta(\tau,\sigma) = 0 \iff \tau = \sigma$. The triangular inequality is satisfied, since if for three strings ρ, σ, τ the first two agree on the prefix α at most, and the last two agree on the prefix β at most then ρ and τ agree on $m := \min\{\alpha, \beta\}$. Therefore $\delta(\rho,\tau) \leq w_m \leq w_\alpha + w_\beta = \delta(\rho,\sigma) + \delta(\sigma,\tau)$.
Now $\delta(\theta_e(\sigma), \theta_e(\tau)) = w_{e\alpha} = r_e w_\alpha = r_e \delta(\sigma,\tau)$. (X,d) clearly forms a complete metric space since elements of Cauchy sequences agree on longer and longer prefixes and therefore have a shared limit. □

The string model of a ratio list is in some sense the biggest realization. On one hand the whole space E^ω is invariant under the IFS, on the other hand for every realization, there is a continuous surjection from the string model onto the self-similar set. This is the string model theorem.

Theorem 3.11 (String model Theorem)**.** *Let (X,d) be a non empty complete metric space and $\{f_e\}_{e\in E}$ an IFS realizing the contracting ratio list $(r_e)_{e\in E}$. Then there is a unique continuous function $h : E^\omega \to S$, such that for $\sigma \in E^\omega, e \in E$*

$$h(e\sigma) = f_e(h(\sigma)) \tag{54}$$

and the image of h is the invariant set of the IFS.

Proof. We will construct a sequence of continuous functions uniformly converging to h. Define $h_0 : E^\omega \to X$ as the constant function with an arbitrary

$x \in X$ and $h_k : E^\omega \to X$ by $h_{k+1}(e\sigma) = f_e(h_k(\sigma))$. h_0 is continuous and suppose the continuity of h_k is already proven. Let $S \subseteq X$, then the preimage of S decomposes into the disjoint subsets $A_e = \{\sigma \in h_{k+1}^{-1}[S] : \exists \sigma' : \sigma = e\sigma'\} = \theta_e[h_k^{-1}[f_e^{-1}[S]]]$. Since f_e, h_k are continuous and θ as a similarity is open, the preimage of S is union of open sets and therefore open itself.
Furthermore the sequence (h_k) converges uniformly and therefore its limit h has the desired properties. Let $r = \max_{e \in E} r_e < 1$. We compute

$$\delta(h_{k+1}(e\sigma), h_k(e\sigma)) = \delta(f_e(h_k(\sigma)), f_e(h_{k-1}(\sigma))) \leq r\delta(h_k(\sigma), h_{k-1}(\sigma)).$$

So $\sup\{\delta(h_{k+1}(\sigma), h_k(\sigma))\} \leq r^k \sup\{\delta(h_1(\sigma), h_0(\sigma))\}$ and since E^ω is compact the later expression is finite. Now for $\sigma \in E^\omega$ arbitrary

$$\delta(h_m(\sigma), h_k(\sigma)) \leq \sum_{j=k}^{m} \delta((h_{j+1}(\sigma)), f_e(h_j(\sigma))) \leq \sum_{j=k}^{m} r^j \sup\{\delta(h_1(\sigma), h_0(\sigma))\}.$$

Since this expression converges to 0, for $m, k \to \infty$, we get uniform convergence for the sequence of functions, proving the continuity of h.
Now we look at the set $h_k[E^\omega] = \bigcup_{e \in E} f_e[h_{k-1}[E^\omega]]$, which converges to $h[E^\omega]$. On the other hand this is exactely the approach by constructing the invariant set of the IFS by prefractals. Finally we prove uniqueness. Let g, h be two functions with the defining properties. Then

$$\delta(h(e\sigma), g(e\sigma)) = \delta(f_e(h(e\sigma)), f_e(g(e\sigma))) \leq r\delta(h(\sigma), g(\sigma)),$$

but since $r < 1$, this means that $\delta(h(\sigma), g(\sigma)) = 0$ for all $\sigma \in E^\omega$ and therefore $h = g$ everywhere. □

This fact gives us a way to describe the points of the invariant set of arbitrary realizations of a ratio list. But note that h in general is not injective.

Definition 3.12. The function h described as above is called the ***addressing function*** and iff $x = h(\sigma)$, we call σ the ***address*** of x.

Another way to approach self-similar sets is via *Mauldin-Williams graphs*. These are graphs with more structure, where we also allow more than one arrow between two vertices. Formally we define this using two function assigning inital vertex and terminal vertex to every edge.

Definition 3.13. A ***directed multigraph*** is the quadruple (V, E, i, t), where V is a (finite) set of vertices and E a (finite) set of edges and $i, t : E \to V$

assigning initial vertex and terminal vertex. A ***path*** is a sequence of edges, such that the terminal vertex of an edge is the initial vertex of the next one. The number of edges is called the ***length*** of the path. A multigraph is called ***strongly connected***, iff for every two vertices there is a path connecting them. A ***Mauldin-Williams graph*** is a multigraph with a function $r : E \to (0, \infty)$. Iff $r(e) < 1, \forall e \in E$, we call the graph ***strictly contracting***.

Mauldin-Williams graph are a good model for self-similar sets and lead to an extended definition of IFS, where the functions don't necessarily map from the same metric space to itself.

Definition 3.14. Given a Mauldin-Williams graph (V, E, i, t, r) we define a ***graph-directed iterated function system*** as a family of complete metric spaces $(X_v, d_v)_{v \in V}$ and Lipschitz functions $f_e : S_v \to S_u$ for $e \in E, i(e) = v, t(e) = u$ and $L(f_e) = r_e$. An ***invariant set list*** is a list of nonempty sets $K_v \subseteq X_v$ such that

$$K_u = \bigcup_{\substack{v \in V \\ e \in E : i(e) = u, t(e) = v}} f_e[K_v]. \tag{55}$$

Anolouge to the previous fixed point arguments we can garantuee an invariant set list for strictly contracting Mauldin-William graphs.

Theorem 3.15. *Let (V, E, i, t, r) be a strictly contracting Mauldin-Williams graph and $(f_e)_{e \in E}$ on $(S_v)_{v \in V}$ realizing the graph. Then there exists a invariant set list.*

Proof. Define $X := \prod_{v \in V} \mathcal{K}(X_v)$ and d the maximum of the Hausdorff metrices on the coordinate spaces. Define

$$F((A_v)_{v \in V}) = \left(\bigcup_{\substack{v \in V \\ e \in E : i(e) = u, t(e) = v}} f_e[K_v] \right)_{u \in V}.$$

This is contracting map on the complete metric space and therefore there exists a fixed point. The coordinates form a invariant set list. □

The last notion discussed in this section is are Büchi automata, which supersede the notion of finite automata. In finite automata only finite paths and finite words are considered, but with Büchi automata we allow these to be infinite, so they are suitable to describe points in self-similar sets.

Definition 3.16. The 5-tuple $\mathcal{A} = (Q, A, E, I, T)$ is called a ***Büchi automaton***, iff $Q = \{q_1, ..., q_N\}$ is a finite set of ***states***, A is a finite ***alphabet***, $E \subset Q \times A \times Q$ is a set of ***edges*** and $I, T \subset Q$ the ***intial*** and ***terminal states***. Let A^* denote the set of all (finite) words and A^ω denote the set of all infinite words. A word $l \in A^*, l = l_1...l_n$ is ***accepted*** by the automaton, iff there are states $q_{i_1}, ..., q_{i_{n+1}}$ such that $q_{i_1} \in I, q_{i_{n+1}} \in F$ and for all k the tripel $(q_{i_k}, l_k, q_{i_{k+1}}) \in E$. We call such a finite path ***successfull*** and we call an infinite path successfull, iff infintley many subpaths are successfull. An infinte word $l \in A^\omega$ is accepted by the automaton, if there exists an infinity successfull paths with the ***label*** l. The set of all $l \in A^\omega$, that are accepted by $\mathcal{A}$ is called the ω-***language*** accepted by the automaton.

Let L_1, L_2 two ω-languages in the same alphabet accepted by $\mathcal{A}$ respectively $\mathcal{B}$. It can be necessary to create automata accepting the union of the languages or their intersection. The union is not difficult: one just uses the union of states and edges, as well as the union of terminal and intial states. The intersection is also possible, but a little bit more elaborate. This will be used in Theorem 5.6 to determine the intersection of the boundary of the Twin Dragon with some lines as it is in [1]

Lemma 3.17. *Let L_1, L_2 be two ω-languages in the same alphabet A accepted by the Büchi automata $\mathcal{A}$ respectively $\mathcal{B}$. Then there is a Büchi automaton accepting $L_1 \cap L_2$.*

Proof. Define $\mathcal{A} \times \mathcal{B} = (Q_\mathcal{A} \times Q_\mathcal{B}, A, E, I_\mathcal{A} \times I_\mathcal{B}, T_\mathcal{A} \times T_\mathcal{A})$, where E consists of the edges $(a, b) \xrightarrow{d} (a', b')$ with $a \xrightarrow{d} a'$ and $b \xrightarrow{d} b'$. Let $l \in A^\omega$ accepted by $\mathcal{A} \times \mathcal{B}$, then there exists an infinite path in the automaton. Projecting to the first coordinate gives a infinite path through $\mathcal{A}$. Therefore $l \in L_1$ and with the same reasoning $l \in L_2$. Now let $l \in L_1 \cap L_2$. There exists a path $a_1 a_2 ...$ through $\mathcal{A}$ and $b_1 b_2 ...$ through $\mathcal{B}$. Therefore $(a_1, b_1), (a_2, b_2), ...$ is accepted by the product automaton. ◻

3.3 Similarity dimension and Hausdorff dimension of self-similar sets

This section will cover the question, what the Hausdorff dimension of a self-similar set is. A goal is to show, that under certain conditions this dimension coincides with box-counting dimension, so that we can use the full versions of the theorems in Chapter 2.
Apart from our previous notions of dimension, self-similar sets have their own measure for their complexity: the *similarity dimension*.

Definition 3.18. Let S be a self-similar set defined by the IFS $(f_i)_{i=1}^m$. The ***similarity dimension*** of S is defined as the root d of the equation

$$\sum_{i=1}^{m} L(f_i)^d = 1. \tag{56}$$

We denote the similarity dimension of S with $\mathrm{Sim}(S)$.

Note that despite the name Sim does not hold the conditions for a dimension function, but it turns out that in certain cases this value agrees with dim and even Box. This is also of very practical use since computing the root of the equation above is a purely analytical and numerical problem, not a geometrical.
Without any restrictions similarity dimension holds as an upper bound for Hausdorff dimension.

Theorem 3.19. *Let S be a self-similar set, then*

$$\dim(S) \leq \mathrm{Sim}(S) \tag{57}$$

Proof. Since $S = F(S)$ with $F(A) := \bigcup_{i=1}^m f_i(A)$, we can split up S by applying F k many times for fixed $k \in \mathbb{N}$ into

$$S = \bigcup_{i_1, i_2, \ldots, i_k} f_{i_1}(f_{i_2}(\ldots(f_{i_k}(S))\ldots)).$$

Denote $v := (i_1, i_2, \ldots, i_k)$ and $S_v := f_{i_1}(f_{i_2}(\ldots(f_{i_k}(S))\ldots))$ and $\lambda := \max\{L(f_i)\} < 1$. We have

$$\mathrm{diam}(S_v) \leq \mathrm{diam}(S) \prod_{j=1}^{k} L(f_{i_j}) \leq \lambda^k \mathrm{diam}(S)$$

and therefore $\{S_v\}$ forms a $\delta-$cover of S with $\delta = \lambda^k \operatorname{diam}(S)$. For $s = \operatorname{Sim}(S)$ we compute

$$\begin{aligned}\mathcal{H}^s_\delta(S) &\leq \sum_{v\in\{1,\dots,m\}^k} \operatorname{diam}(S_v)^s \\ &\leq \operatorname{diam}(S)^s \sum_v \prod_{j=1}^k L(f_{i_j}))^s \\ &\leq \operatorname{diam}(S)^s \left(\sum_{i=1}^m L(f_i)^s\right)^k = \operatorname{diam}(S)^s.\end{aligned}$$

This inequality holds for all small δ, if k is large enough; thus $\mathcal{H}^s(V) \leq \operatorname{diam}(V)^s$. □

The converse inequality does not hold for arbitrary self-similar sets; one needs more conditions. Before directly looking at this problem, we want to answer the question when dim = Box. The *implicit* theorems as in Falconer's book "Techniques in Fractal Geometry" [8, p.42-46] give two conditions for not necessarily self-similar sets. They are called implicit, because they do not explicitly calculate the dimensions, but use the inner structure of the set. In the first condition we are strongly reminded of self-similar sets, since for arbitrary small intersections of the set, we find maps to the whole set with a bounded Lipschitz constant. In the second theorem we look at maps from the whole set to parts of the sets.

Theorem 3.20 (Implicit Theorem I). *Let $E \in \mathcal{K}(\mathbb{R}^n)$, $\dim(E) = s$ and $a, r_0 > 0$. If for every set U intersecting E with $\operatorname{diam}(U) < r_0$ there is a mapping $g : E \cap U \to E$ satisfying*

$$\frac{a}{\operatorname{diam}(U)}\|x - y\| \leq \|g(x) - g(y)\|, \forall x, y \in E \cap U, \tag{58}$$

then $\mathcal{H}^s(E) \geq a^s > 0$ and $\underline{\operatorname{Box}}(E) = \overline{\operatorname{Box}}(E) = s$.

Proof. We remember that $\dim(S) \leq \underline{\operatorname{Box}}(E) \leq \overline{\operatorname{Box}}(E) = s$ always holds (Corollary 2.18). So we prove that for all d with $\mathcal{H}^d(E) < a^d$ it follows that $\overline{\operatorname{Box}}(E) < d$. By letting d converge to s we get the inverse inequality.
Suppose $\mathcal{H}^d(E) < a^d$, then there is a δ-cover $U_1, \dots, U_m$ with $\delta \leq \min\{a/2, r_0\}$

and $\sum \operatorname{diam}(U_i) < a^d$. Since E is compact, we can assume the finiteness of the cover. Let $0 < t < d$ close to d such that

$$a^{-t} \sum_{i=1}^{m} \operatorname{diam}(U_i)^t < 1. \tag{59}$$

By the given conditions we get mappings $g_i : E \cap U_i \to E$ with

$$\|x - y\| \le \frac{\operatorname{diam}(U)}{a} \|g_i(x) - g_i(y)\|, \forall x, y \in E \cap U_i. \tag{60}$$

This garantuees us that the g_i are injective and their are inverses Lipschitz continuous, even contractions. Define $I_k := \{1, .., m\}^k, I := \bigcup_k I_k$ and for each $v = (i_1, ..., i_k)$

$$U_v := g_{i_1}^{-1}(g_{i_2}^{-1}(...(g_{i_k}^{-1}(U))...))$$

analog to the proof of Theorem 3.19. For every k E is covered by the $\{U_v\}, v \in I_k$ and repeated application of (60) gives us

$$\|x-y\| \le a^{-k} \prod_{j=1}^{k} \operatorname{diam}(U_{i_j}) \cdot \|g_{i_1} \circ ... \circ g_{i_k}(x) - g_{i_1} \circ ... \circ g_{i_k}(y)\|, \forall x, y \in E \cap U_v \tag{61}$$

and in particular

$$\operatorname{diam}(U_v) = \max\{\|x - y\|\} \le a^{-k} \prod_{j=1}^{k} \operatorname{diam}(U_{i_j}) \operatorname{diam}(E)$$

Note that the righthand side goes to 0, if we let k get bigger. Let $b = a^{-1} \min_{1 \le i \le m} \operatorname{diam}(U_i)$ and $r < \operatorname{diam}(E)$. Then for every $x \in E$, there is a $v \in I$, such that $x \in U_v$ and

$$br \le a^{-k} \prod_{j=1}^{k} \operatorname{diam}(U_{i_j}) \operatorname{diam}(E) < r$$

. Let $N_r(E)$ denote the minimum number of sets of diameter at most r

covering E (def. 2.17 (iv)), then we calculate

$$\begin{aligned} N_r(E) &\le |\{v \in I : br \le a^{-k} \prod_{j=1}^{k} \operatorname{diam}(U_{i_j}) \operatorname{diam}(E)\}| \\ &\le \sum_{v \in I} (br)^{-t} \left(a^{-k} \prod_{j=1}^{k} \operatorname{diam}(U_{i_j}) \right) \operatorname{diam}(E))^t \\ &\le r^{-t} \frac{\operatorname{diam}(E)}{b^t} \sum_{k=0}^{\infty} a^{-kt} \sum_{v \in I_k} \prod_{j=1}^{k} \operatorname{diam}(U_{i_j})^t \\ &= r^{-t} \frac{\operatorname{diam}(E)}{b^t} \sum_{k=0}^{\infty} \left(a^{-t} \sum_{i=1}^{m} \operatorname{diam}(U_i)^t \right)^k . \end{aligned}$$

The last infinite sum is a geometric series because of (59) and converges to some finite value which does not depend on r. By the definition of box-counting dimension and c_1 the constant calculated above, we conclude

$$\overline{\operatorname{Box}}(E) = \limsup_{r \to 0^+} \frac{\log N_r(E)}{-\log r} \le \lim_{r \to 0^+} \frac{\log(c_1 r^{-t})}{-\log r} = t < d,$$

□

Theorem 3.21 (Implicit Theorem II)**.** *Let $E \in \mathcal{K}(\mathbb{R}^n)$, $\dim(E) = s$ and $a, r_0 > 0$. If for every closed ball B with center in E and radius $r < r_0$ there is a mapping $g : E \to E \cap B$ satisfying*

$$ar\|x - y\| \le \|g(x) - g(y)\|, \forall x, y \in E, \tag{62}$$

then $\mathcal{H}^s(E) \le 4^s a^{-s} < \infty$ and $\underline{\operatorname{Box}}(E) = \overline{\operatorname{Box}}(E) = s$.

Proof. Let $N_r(E)$ be the maximum number of disjoint closed balls of radius r with centres in E (def. 2.17 (v)). Suppose that for some $r < \min\{a^{-1}, r_0\}$. We will derive a contradiction from this assumption.
Let $t > s$, such that $m := N_r(E) > a^{-t} r^{-t}$; then there are disjoint balls $B_1, ..., B_m$ with radius r and centres in E and by the condition of the theorem, we get $g_i : E \to E \cap B_1$ with

$$ar\|x - y\| \le \|g_i(x) - g_i(y)\|, \forall x, y \in E. \tag{63}$$

Let d be the minimal distance between two of the balls. By using (63) $(q-1)$ times we can calculate for $x, y \in E$

$$\|g_{i_1} \circ \dots \circ g_{i_k}(x) - g_{j_1} \circ \dots \circ g_{j_k}(y)\| \geq (ar)^{q-1} d (ar)^q d, \tag{64}$$

with q the least integer such that $i_q \neq j_q$. Let μ the measure on E define by repeated subdivision such that $\mu(g_{i_1} \circ \dots \circ g_{i_k}(E)) = m^{-k}$ for all $(i_1, \dots, i_k)$. Let $U \subseteq \mathbb{R}^n$ that intersects E and $\operatorname{diam}(U) < d$ and k be the least integer such that

$$(ar)^{k+1} d \leq \operatorname{diam}(U) < (ar)^k d.$$

By (64) U intersects $g_{i_1} \circ \dots \circ g_{i_k}(E))$ at most one sequence $(i_1, \dots, i_k)$, so $\mu(U) \leq m^{-k} < (ar)^{kt} \leq (dar)^{-t} \operatorname{diam}(U)^t$. It follow from distribution principal 2.23 that $\dim(E) \geq t > s = \dim(E)$, which is a contradiction.
Therefore $N_r(E) \leq a^{-s} r^{-s}$ for sufficiently small r and therefore $\overline{\operatorname{Box}}(E) \leq s$. By blowing up the balls by the factor 2 we get a cover and $\mathcal{H}^s_{4r}(E) \leq a^{-s} r^{-s} (4r)^s = 4^s a^{-s}$. □

As a direct result of the the two implicit theorems, we can describe the Hausdorff dimension of certain self-affine sets [9, Corr. 3.3, Corr 3.4].

Corollary 3.22. *Let E be a self-affine set with IFS $\{f_i : 1 \leq i \leq m\}$ consisting of similarities. Then $\dim(E) = \operatorname{Box}(E)$ and if $\{f_i(E) : 1 \leq i \leq m\}$ are disjoint sets, then $\operatorname{Sim}(E) = \dim(E)$.*

Proof. Let $r_i := L(f_i)$ be the ratio list of the IFS and $r_0 := \min_i r_i$. Let $x \in E$ and $r \leq \operatorname{diam}(E)$. There is a sequence $(i_1, i_2, \dots.)$ such that $x \in f_{i_1} \circ \dots \circ f_{i_k}(E)$ for all $k \in \mathbb{N}$. We can choose k such that $r_0 r < r_{i_1} \cdots r_{i_k} \operatorname{diam}(E) \leq r$. Then $f_{i_1} \circ \dots \circ f_{i_k} : E \to E \cap B_r(x)$ is a similarity with ratio at least $r_0 \operatorname{diam}(E)^{-1} r$, so by thm. 3.21 we get the $\dim(E) = \operatorname{Box}(E)$ and $\mathcal{H}^s(E) < \infty$. Now suppose $\min_{i \neq j} \operatorname{dist}(f_i(E), f_j(E)) = d > 0$, then it follows, that $\operatorname{dist}(f_{i_1} \circ \dots \circ f_{i_k}(E), f_{j_1} \circ \dots \circ f_{j_k}(E)) \geq r_{i_1} \cdots r_{i_{k-1}} d$ if the index vectors are distinct. Let U intersect E and $\operatorname{diam}(U) \leq d$ and $x \in U \cap E$, then we find $(i_1, \dots, i_k)$ such that $x \in f_{i_1} \circ \dots \circ f_{i_k}(E)$ and $d r_{i_1} \cdots r_{i_k} \leq \operatorname{diam}(U) \leq d r_{i_1} \cdots r_{i_{k-1}}$. Thus U is disjoint from $f_{j_1} \circ \dots \circ f_{j_k}(E)$ for all other index vectors. Hence $E \cap U \subseteq f_{i_1} \circ \dots \circ f_{i_k}(E)$ and therefore $(f_{i_1} \circ \dots \circ f_{i_k})^{-1} : E \cap U \to E$ a similarity with ratio $(r_{i_1} \cdots r_{i_k})^{-1} \geq d \operatorname{diam}(U)^{-1}$. Applying thm. 3.20 we get $0 < \mathcal{H}^s(E)$. Now for disjoint sets it easy to see $\mathcal{H}^s(E) = \sum_{i=1}^m \mathcal{H}^s(f_i(E)) = \sum_{i=1}^m r_i^s \mathcal{H}^s(E)$ and since $0 < \mathcal{H}^s(E) < \infty$, we get $1 = \sum_{i=1}^m r_i^s$ and therefore $s = \operatorname{Sim}(E)$. □

Definition 3.23. Let E be the attractor of the IFS $\{f_i : 1 \le i \le m\}$. If A is a non-empty compact subset of E satisfiying

$$A \supseteq \bigcup_{i=1}^{m} f_i(A) \tag{65}$$

we call it ***super-self-similar***. If

$$A \subseteq \bigcup_{i=1}^{m} f_i(A) \tag{66}$$

it is called ***sub-self-similar***.

Corollary 3.24. *Let E be the attractor of the IFS $\{f_i : 1 \le i \le m\}$ consisting of similarities.*

(a) If A is super-self-similar, then $s := \dim(A) = \mathrm{Box}(A)$ and $\mathcal{H}^s(A) < \infty$

(b) If A is sub-self-similar and $\{f_i(E) : 1 \le i \le m\}$ are disjoint it follows, that $s := \dim(A) = \mathrm{Box}(A)$ and $\mathcal{H}^s(A) > 0$.

The proof to this Corrollary uses the same argument as the one before. Note that for self-affine sets its boundary forms a sub-self-similar set. For self-affine sets like the cantor set the first statement holds, since the similarities are disjoint. But for many self-affine sets this assumption is too strong. In the next subsection we can prove the same result (thm. 3.31) for a broader class of self-affine sets including the Sierpinsky gasket or the Twin Dragon. If the similarities are disjoint we see that in each step of the approximation the set gets split up in more and more disjoint parts, such that the limitset is totally disconnected.

Proposition 3.25. *Let S be a self-affine set with IFS $\{f_i : 1 \le i \le m\}$ consisting of similarities, such that $\{f_i(S) : 1 \le i \le m\}$ are disjoint sets. Then S is totally disconected.*

Proof. Let $x, y \in S, x \ne y$. Then there exists an address $(i_1, i_2, ...)$ of x and an addresse $(j_1, j_2, ...)$ of y. Suppose $i_k \ne j_k$ is the first occation where they differ. Then $(f_{i_k} \circ ... \circ f_{i_1})(S) \subset f_{i_k}(S)$ is disjoint from $(f_{j_k} \circ ... \circ f_{j_1})(S) \subset f_{j_k}(S)$. Since both sets are closed they have distance > 0. Therefore x, y are seperated by two closed sets and cannot be in the same connected component. Since $x \ne y$ where arbitrary, only the singletons are connected. □

Many self-similar sets we want to investigate are not totally disconnected. On the contrary they also can be even connected. The following definition and theorem charactarizes connected self-similar sets. Following [19, p.25] we define:

Definition 3.26. Let $\{A_1, ..., A_m\}$ be m subsets of $\mathbb{R}^n$. We say they form a ***chain***, iff for any $k \neq j$ there is a sequence $(i_1, ..., i_n)$ such that

$$A_k \cap A_{i_1}, A_{i_1} \cap A_{i_2}, ..., A_{i_{n-1}} \cap A_{i_n}, A_{i_n} \cap A_j \tag{67}$$

are all non empty. We say a set $A \subset \mathbb{R}^n$ forms a ϵ-***chain*** if for any $x, y \in A$ there is a sequence $(z_1, ..., z_n)$ such that

$$\|x - z_1\|, \|z_1 - z_2\|, ..., \|z_{n-1} - z_n\|, \|z_n - y\| \tag{68}$$

are all $\leq \epsilon$.

The following proposition follows directly from the definition

Proposition 3.27. *Let $A_1, ..., A_m$ form a chain and each A_i form an ϵ-chain. Then $\bigcup_i A_i$ forms an ϵ-chain. For any contraction ψ on a set A forming an ϵ-chain, $\psi(A)$ form an $L(\psi)\epsilon$-chain.*

This definition is strong enough to fully characterize connected self-similar sets.

Theorem 3.28. *Let S be a self-similar set with IFS $\{f_i : 1 \leq i \leq m\}$. S is connected iff $f_1(S), ..., f_m(S)$ form a chain.*

Proof. Let S be connected, then $\{f_1(S), ..., f_m(S)\}$ must form a chain, otherwise they could be seperated be open sets. Now suppose $\{f_1(S), ..., f_m(S)\}$ form a chain and that S is not connected, so it decomposes in two non-empty disjoint compact sets with distance $> \delta$. For all $\epsilon < \delta$ S does not form an ϵ-chain. Let $\lambda = \max_i L(f_i)$ and since S forms a $\text{diam}(S)$-chain we get by the previous proposition that $f_i(S)$ forms a $\lambda \cdot \text{diam}(S)$-chain. Again by the Prop. and the assumption we get that $S = \bigcup_i f_i(S)$ forms a $\lambda\,\text{diam}(S)$-chain. Be repeating that argument we get that S even forms a $\lambda^l \cdot \text{diam}(S)$-chain and since $\lambda < 1$ we can choose l, such that $\lambda^l \text{diam}(S) \leq \delta$. That is a contradiction. ☐

3.4 Self-affine sets

In this subsection we take a closer look at self-affine sets, which have been well studied, since allmost all of the early examples of fractals are self-affine. But still there are unaswered question regarding the most elementary self-similar sets, like calculating their Hausdorff dimension. We already proved that the Hausdorff dimension coincides with similarity dimension (and box-counting dimension) in the case that the similarities have disjoint imagse. We will sharpen that result and allow disjoint images of one open set. But in general this obviously is not true, since a we can create an IFS realizing an arbitrary ratio list, by just multiplying with the ratios. The attractor is just the origin, which has Hausdorff dimension 0 independend of the similarity dimension of the IFS. Falconer even gives an example of a set that is self-affine, but box-counting and Hausdorff dimension differ [6, Beispiel 9.5]. In general it is very hard to get a complete characterization of self-affine sets that satify some dimension equality.

Definition 3.29. A set of contractions $\{f_i : 1 \le i \le m\}$ satisfies *(Morans)* ***open set condition*** (OSC), iff there exists a nonempty bounded open set $V \in \mathbb{R}^n$, such that

$$\dot{\bigcup} f_i(V) \subseteq V, \tag{69}$$

in other words the contractions restricted $V \to V$ have disjoint images. U is called a ***Moran open set***.

This condition is pretty similar to disjoint images and is allready sufficient for the full equality of all three dimensions. We follow the proof in [6, p.137-139] using the following lemma.

Lemma 3.30. *Let $\{V_i\}_{i\in I}$ be a family of disjoint open subsets of $\mathbb{R}^n$, such that V_i contains a Ball of radius $a_1 r$, $\forall i \in I$ and is contained in a Ball by radius $a_2 r$. Then every Ball B with radius r intersects at most $(1+2a_2)^n a_n^{-n}$ of the closures $\overline{V_i}$.*

Proof. Let q be the number of $\overline{V_i}$ intersecting B. If B intersects $\overline{V_i}$, then V_i is contained in the ball with radius $(1+2a_2)r$ with same centre as B. Summing the volums of the balls contained in V_i we get $q(a_1 r)^n \le (1+2a_2)^n r^n$, which is equivalent to the estimation. □

Theorem 3.31. *Let S be a self-affine set defined by similarities satisfying the open set condition. Then*

$$\mathrm{Sim}(S) = \dim(S) = \mathrm{Box}(S) \tag{70}$$

Proof. Let $(f_i)_{i=1}^m$ be the similarities defining S and denote $s := \mathrm{Sim}\, S$. Define $I := \{1,..,m\}^{\mathbb{N}}$, $J_k := \{1,..,m\}^k$ the set of all infinite series of indices and for $r = (i_1,...,i_k) \in J_k$ let $I_r \subset I$ be the set of all series starting with r. For $A \subseteq \mathbb{R}^n$ we denote by $A_r := f_{i_1} \circ f_{i_2} \circ \circ \circ f_{i_k}(A)$. Therefore we can cover S with $\{S_r\}_{r\in J_k}$. We have

$$\begin{aligned}\sum_{J_k} \mathrm{diam}(S_r) &= \sum_{J_k}\prod_{r} L(f_{i_l})^s \,\mathrm{diam}(S) \\ &= (\sum_{i_1=1}^{m} c_{i_1})^s \cdots (\sum_{i_1=1}^{m} c_{i_k})^s \,\mathrm{diam}(S)^s = \mathrm{diam}(S).\end{aligned}$$

For every $\delta > 0$ one can choose k, such that $S_r \le (\max_i L(f_i))^k \le \delta$, therefore $\mathcal{H}^s_\delta(S) \le \mathrm{diam}(S)^s$. Hence we have an upper bound for the Hausdorff dimension by s, since $\mathcal{H}^s \le \mathrm{diam}(S)^s \le \infty$. For the lower bound, we define a distribution μ on I, with $\mu(I_{(i_1,...,i_k)}) = (L(f_{i_1})\cdots L(f_{i_k}))^s$. Easily we have $\mu(I) = 1$ and

$$\begin{aligned}\mu(\dot{\bigcup_{i=1}^{m}} I_{(i_1,...,i_k,i)}) = \mu(I_{(i_1,...,i_k)}) &= (L(f_{i_1})\cdots L(f_{i_k}))^s \sum_{i=1}^{m} L(f_i)^s \\ &= \sum_{i=1}^{m}(L(f_{i_1})\cdots L(f_{i_k})L(f_i))^s = \sum_{i=1}^{m}\mu(I_{(i_1,...,i_k,i)}).\end{aligned}$$

Therefore we have a distribution on some subsets of I. We cant lift this distribution naturally to a distribution on S with

$$\nu(A) := \mu\left\{(i_1,i_2,...) : \bigcap_{i=1}^{\infty} S_{(i_1,i_2,...i_k)} \in A\right\}. \tag{71}$$

We want to show that ν satisfies the conditions in the distribution principal 2.23. Let V be a Moran open set for S. Since $\overline{V} \supseteq F(\overline{V}) = \bigcup_{i=1}^m f_i(\overline{V})$ the sequence $F^k(\overline{V})$ converges to S monotonely from above. In particular $\overline{V} \supset S$ and $\overline{V}_r \supset S_r$ for all $r \in J_k$. Let B a ball with radius R and estimate $\nu(B)$ by considering the sets V_r intersecting $S \cap B$. For every $(i_1,i_2,...) \in I$ there is a $k \in \mathbb{N}$ such that

$$\min_i L(f_i))R \le \prod_{l=1}^{k} L(f_{i_l}) \le R. \tag{72}$$

Let Q be the set of finite sequences obtained by that process. Since $V_1, ..., V_m$ are disjoint so are $\{V_r\}_{r \in Q}$. Also we get that $S \subseteq \bigcup_Q S_r \subseteq \overline{V}_r$. Since V is bounded and open, there are radii a_1, a_2 like in the lemma above. V_r contains a Ball of radius $a_1 \prod_r L(f_i)$ for $r \in Q$ and therefore one of radius $\min_i L(f_i) a_1 R$ since we defined Q that way. On the other hand V_r is contained in a ball of radius $a_1 \prod_r L(f_i)$ and therefore in a ball of radius R. Let Q_1 be the subset of Q such that $\overline{V}_r$ intersects B for $r \in Q_1$. Using the lemma we get that cardinality of Q_1 is at most $q = (1 + 2a_2)^n a_1^{-n} (\min_i L(f_i))^{-n}$. We calculate

$$\nu(B) = \nu(S \cap B) \leq \mu \left\{ (i_1, i_2, ...) : \bigcap_{i=1}^{\infty} S_{(i_1, i_2, ... i_k)} \in F \cap S \right\}$$
$$\leq \mu \left\{ \bigcup_{Q_1} I_r \right\} = \sum_{Q_1} \mu(I_r) = \sum_{Q_1} \prod_r L(f_i)^s \leq \sum_{Q_1} R^s \leq R^s q.$$

Hence we get for all sets U that $\nu(U) \leq \operatorname{diam}(U) q$. Using the distribution principal 2.23 we get that $\mathcal{H}^s(S) \geq q^{-1} > 0$ and conclude $\dim(S) = s$. Finally we want to show equality with the box-counting dimension. We just need to show $\text{Box} \leq s$. We use the cover of S by $\{V_r\}_{r \in Q}$. One can show inductively that $\sum_Q (\prod_r L(f_i))^s = 1$ and by applying 72 to it we get that the maximal number of elements in Q is $\min_i L(f_i)^{-s} R^{-s}$. We now have a cover with $\operatorname{diam}(\overline{V}_r) \leq R \operatorname{diam}(V)$. Recalling the definition of box-counting dimension we get

$$\text{Box}(S) \leq \lim_{R \to 0} \frac{\log \min_i L(f_i))^{-s} R^{-s}}{-\log(R \operatorname{diam}(V))} = s$$

This concludes the proof. □

We can now calculate the Dimension of the Twin Dragon assuming the open set condition is fulfilled. We will show that in Chapter 4.

Proposition 3.32. *Let $\mathcal{K}$ be the Twin Dragon, that is the attractor of the IFS $\{f_1, f_2\}$ with $f_1(x) = B^{-1}x, f_2(x) = B^{-1}(x + (1, 0))$ and*

$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \tag{73}$$

. *Then* $\dim(\mathcal{K}) = \text{Box}(\mathcal{K}) = \text{Sim}(\mathcal{K}) = 2$.

Proof. If the opens set condition is fulfilled we imediately get the equality of the dimensions and since $L(f_1) = L(f_2) = \sqrt{1/2}$ we easily get $\mathrm{Sim}(\mathcal{K}) = 2$. $\square$

We see in fact, that the Twin Dragon is not(!) fractal. In general it is really hard to calculate Hausdorff dimension of self-similar sets, but there are further classes of fractal with overlaps, where this is possible. For example it is possible, to show that for IFS of *finite type* Hausdorff dimension and box-counting coincide and can be calculated directely, as described in [14].

Since we are also interested in the dimension of the boundary of the Twin Dragon, we state a more general statement about sub-self-similar sets generalizing Corollary 3.24 following [7].

Theorem 3.33. *Let E be the attractor of the IFS $\{f_i : 1 \le i \le m\}$ consisting of similarities satisfying the open set condition. If A is a sub-self-similar subset of E, then $s := \dim(A) = \mathrm{Box}(A)$ and $\mathcal{H}^s(A) > 0$.*

For calculating the dimension of the boundary one needs more structure, but this theorem states that it is sufficient to use box-counting dimension. The Chapter 4 gives a method for boundaries of special self-similar sets.

4 Canonical number systems and tilings

This chapter concentrates on an algebraic, number theoretical problem, where it turns out that fractals appear in a pretty natural way. The Twin Dragon fractal is strongly connected to one specific example of a *canonical number system* (CNS) and to get more knowledge about its geometric, topological properties it is really usefull to investigate this connection. We will also deal with *tilings*, which are ways to cover the plane with copies of a shape, without overlapping. This is a geometric notion, but it also strongly relates to the CNS property. Finite automata show up as well in this chapter, since they are a great tool to calculate Hausdorff dimension and will help us with the main difficulties in the last chapter.

4.1 Radix representation and Canonical number systems

Let us open this subsection with really basic mathematics to have a better understanding for the following definitions and notions. Ever since humans were interested in numbers, there was the need to notate big numbers and have a representation system that lets one easily handle them. Since the roman numerals arguably did not do a good job, we now mostly use the arabic numerals in base 10. But also other bases like 2 or 16 are used to represent numbers. This works because of the following well known statement.

Definition and Proposition 4.1. Let $\gamma, \beta \in \mathbb{N}$. Then there exists a unique representation of γ by

$$\gamma = \sum_{j=0}^{n} d_j \beta^j \tag{74}$$

where $0 \leq d_j \leq \beta, n \in \mathbb{N}$ minimal such that $d_i = 0$ for $i > n$. This is called the ***radix representation*** of γ, β is called the ***radix*** or ***base*** and d_j the respective ***digits***.

If we allow negative exponents and infinite sumation for them, we even can represent all the non negative real numbers. The numbers, which only have negative exponents, form the interval $[0, 1]$.
Now we want to also allow negative basis for example -10 and digits $< |-10|$. It turns out that we have the same properties for negative basis and we can

even expand the area of representable numbers to all reals! The numbers with only non negative exponents shift though to $[-10/11, 1/11]$. Expanding the notion to algebraic and therefore even complex numbers, we have different number fields, that can be represented.
Before we define a CNS, let us define some standard algebraic objects, which will be necessary in the following.

Definition 4.2. A ***number field*** $\mathbb{K}$ is an algebraic field extension of the rational numbers $\mathbb{Q}$. An element $\beta \in \mathbb{K}$ is called an ***algebraic integer***, iff there is a monic polynomial $p \in \mathbb{Z}[x]$, such that $f(\beta) = 0$. The algebraic integers form a ring. Define the ***field norm*** of an element $\alpha \in \mathbb{K}$ as

$$\mathrm{N}(\alpha) = \left(\prod_{j=1}^{n} \sigma_j(\alpha)\right)^{[K:Q(\alpha)]}, \tag{75}$$

where $\sigma_j(\alpha)$ are all of the roots of the minimal polynomial of α.

Following [18] we define the following notion:

Definition 4.3. Let $\mathbb{K}$ be a number field with ring of integers $Z_{\mathbb{K}}$. Let $\beta \in Z_{\mathbb{K}}$ and $\mathcal{N}$ a complete residue system $\mod |N(\beta)|-$. The pair $(\beta, \mathcal{N})$ is called a ***number system*** in K, if each $\gamma \in Z_{\mathbb{K}}$ admits a unique representation of the form (74), where $d_j \in \mathcal{N}, n \in \mathbb{N}$ minimal such that $d_i = 0$ for $i > n$. If $\mathcal{N} = \{0, 1, ..., |N(\beta)| - 1\}$ we call the number system a ***canonical number system*** (CNS). We use the same terminology as in Definition 4.1.

This is a pretty natural extension of the former definition, although in the latter sense positive integers cannot form a base, since $\mathbb{N}$ does not form a ring.
There is an even more general definition used in [1], which only uses polynomials.

Definition 4.4. Let $P(x) = x^m + b_{m-1}x^{m-1} + ... + b_0 \in \mathbb{Z}[x]$ monic polynomial with $m \geq 1$. Let $\mathcal{R} = \mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ and $\mathcal{N}$ a complete residue system $\mod |b_0| - 1$. The pair $(P(x), \mathcal{N})$ is called a ***number system*** in $\mathcal{R}$, if each $\gamma \in \mathcal{R}$ admits a unique representation of the form

$$\gamma = \sum_{j=0}^{n} d_j x^j \tag{76}$$

where $d_j \in \mathcal{N}, n \in \mathbb{N}$ minimal, such that $d_i = 0$ for $i > n$. If $\mathcal{N} = \{0, 1, ..., |b_0| - 1\}$ we call the number system a ***CNS***. If $P(x)$ is irreducible and β is one of its zeros, then $\mathcal{R}$ is isomorphic to $\mathbb{Z}[\beta]$ the ring of integers in $\mathbb{Q}[\beta]$. In this case the definition coincides with Definition 4.3.

These definitions show that the problem of radix representation can be viewed from various angles. There are similar notions in every context. It is important to note that the notions in this thesis may vary from the original sources to suit this context.
We want to define the analogon to the fractional part of radix representation. This will be called the *fundamental domain*. To do so, we need to develope a few notions first following [1, p.558f]. Let $(P(x), \mathcal{N})$ be a CNS and let α_i be the zeros of $P(x)$ ordered such that the first r_1 are the real zeros and the $2r_2$ complex zeros are grouped, such that for alll $j = 1, ..., r_2$:

$$\alpha_{r_1+2j-1} = \overline{\alpha_{r_1+2j}} \qquad \operatorname{Im} \alpha_{r_1+2j-1} = -\operatorname{Im} \alpha_{r_1+2j} > 0.$$

Define the homomorphisms $\Phi_i : \mathcal{R} \to \mathbb{Q}(\alpha_i)$ by

$$\sum_{j=0}^{n-1} d_j x^j \mapsto \sum_{j=0}^{n-1} d_j \alpha_i^j, \tag{77}$$

And $\Phi : \mathcal{R} \to \mathbb{R}^n$ by

$$\Phi := (\Phi_1, ..., \Phi_{r_1}, \operatorname{Re} \Phi_{r_1+1}, \operatorname{Im} \Phi_{r_1+1}, ..., \operatorname{Re} \Phi_{r_1+2r_2-1}, \operatorname{Im} \Phi_{r_1+2r_2-1}.) \tag{78}$$

Further we define the matrix B

$$B := \operatorname{diag}(\alpha_1, ..., \alpha_{r_1}, A_1, ..., A_{r_2}) \text{ where } A_j := \begin{pmatrix} \operatorname{Re} \alpha_{r_1+2j-1} & -\operatorname{Im} \alpha_{r_1+2j-1} \\ \operatorname{Im} \alpha_{r_1+2j-1} & \operatorname{Re} \alpha_{r_1+2j-1} \end{pmatrix}.$$

The matrix B fullfills $\Phi(x\gamma) = B\Phi(\gamma)$ for $\gamma \in \mathcal{R}$. We are now able to define the fundamental domain.

Definition 4.5. Let $(P(x), \mathcal{N})$ be a number system and Φ, B as defined above. The ***fundamental domain*** $\mathcal{F} \subset \mathbb{R}^n$ of the number system is defined by

$$B\mathcal{F} = \bigcup_{d \in \mathcal{N}} (\mathcal{F} + \Phi(d)), \tag{79}$$

which is equivalent to

$$\mathcal{F} = \left\{ \sum_{i=1}^{\infty} B^{-i} \Phi(d_i), d_i \in \mathcal{N} \right\} = \overline{\left\{ \sum_{i=1}^{N} B^{-i} \Phi(d_i), d_i \in \mathcal{N}, N \in \mathbb{N} \right\}}. \tag{80}$$

Once again there is another way to see the problem of number systems. Note that the matrix B looks like a basis in the definition above and the digits therefore are vectors. Therefore we once again define a number system in this context.

Definition 4.6. Let $M \in \mathbb{Z}^{d\times d}$ and $\mathcal{A} = \{A_0 = L, ..., A_{t-1}\}, t = |\det M|$ be a complete residue system of the factor group $L = \mathbb{Z}^d/M\mathbb{Z}^d$.. Let $\mathcal{N} = \{a_0, a_1, ..., a_{t-1}\}$ such that, $a_0 = 0, a_j \in A_j$. The pair $(M, \mathcal{N})$ is called a ***number system*** in $\mathbb{Z}^d$, if each $\gamma \in \mathbb{Z}^d$ admits a unique representation of the form

$$\gamma = \sum_{j=0}^{n} M^j d_j \tag{81}$$

where $d_j \in \mathcal{N}, n \in \mathbb{N}$ minimal, such that $d_i = 0$ for $i > n$.

If we understand these number systems in $\mathbb{Z}^d$, we also understand the number systems, where the base matrix has the specific form that comes from a number system in $\mathbb{Z}[x]$.
The fundamental domain is our main reason to study number systems, since theses can form self-similar shapes. We can express the fundamental domain as follows:

$$\mathcal{F} = \bigcup_{d\in\mathcal{N}} B^{-1}(\mathcal{F} + \Phi(d)). \tag{82}$$

The topological properties of the fundamental domain are studied in [12] under the assumption that all zeros have absolute value bigger than 1. Note that if we look at $P(x) = x^2 + 2x + 2$ with its roots $\alpha_{1,2} = -1 \pm i$ the canonical number system of P has the Twin Dragon as fundamental domain. Therefore studying properties of these general fundamental domains directly translate to the Twin Dragon. Following the proofs of Imre Kátai in [12] in the one dimensional case we adapt them to higher dimensional case, as also proposed in the preprint.

Definition 4.7. A subgroup of $(\mathbb{Z}^d, +)$ is called a ***lattice*** in $\mathbb{Z}^d$. If M is a matrix with integer entrees and eigenvalues $\lambda_1, ..., \lambda_k$ are distinct and $|\lambda_i| > 1$, then $L = M\mathbb{Z}^d$ forms a lattice, called the ***lattice generated by*** M. Let

$\mathcal{A} = \{A_0 = L, ..., A_{t-1}\}, t = |\det M|$ be a complete residue system of the factor group $\mathbb{Z}^d/M\mathbb{Z}^d$. Choose $a_0 = 0, a_j \in A_j$, then define $\mathcal{N} := \{a_0, ..., a_{t-1}\}$. We define a function $J : \mathbb{Z}^d \to \mathbb{Z}^d$ as follows. For $n \in \mathbb{Z}^d$, there is unique $b \in \mathcal{N}$, such that $n - b \in L$. Map n to the unique number $J(n)$ such that $M \cdot J(n) = n - b$.

We want to investigate, if $(M, \mathcal{N})$ form a number system and which properties its fundamental domain has, by analysing the dynamical system $(\mathbb{N}, \mathbb{Z}^d, J)$.

Lemma 4.8. *Let $M, \mathcal{N}$ and J be defined as above. Let $r := \|M^{-1}\| = \sup\{\|M^{-1}x\| : \|x\| = 1\}$, $K := \max_{b \in \mathcal{N}} \|b\|$ and $L = \frac{Kr}{1-r}$. The following holds*

a) If $\|n\| > L$, then $\|J(n)\| < \|n\|$.

b) If $\|n\| \leq L$, then $\|J(n)\| \leq L$.

Since the ball with radius $\|n\|$ contains only finite points of $\mathbb{Z}^d$, the sequence $n, J(n), J^2(n), ...$ ultimately is periodic.

Proof. For $J(n)$ we have

$$\|J(n)\| = \|M^{-1}n - M^{-1}b\| \leq r\|n\| + Kr,$$

and therefore if $\|n\| \leq L$, $\|J(n)\| \leq rL + L(1-r) = L$. Conversely if $\|J\| \geq \|n\|$, we get $\|n\| \leq r\|n\| + L(1-r)$, which is equivalent to $\|n\| \leq L$. By contraposition we get a) and the last statement immediately follows from the two cases. □

Definition and Proposition 4.9. A point $n \in \mathbb{Z}^d$ is called ***periodic*** under J iff there is a $k \in \mathbb{N}$ such that $J^k(n) = n$. Let $\mathcal{P}$ be the set of all periodic elements. The following statements hold.

1. $0 \in \mathcal{P}$,
2. $n \in \mathcal{P} \iff n = b_0 + Mb_1 + M^2b_2 + ... + M^k n, b_j \in \mathcal{N}$,
3. $n \in \mathcal{P} \Rightarrow J(n) \in \mathcal{P}$,
4. $n \in \mathcal{P} \Rightarrow \|n\| \leq L$.

Proof. 1. $0 - 0 \in L$ and $M \cdot 0 = 0$.

2. For $n \in \mathbb{Z}^d$ there is a unique expansion $n = b_0 + MJ(n)$. Further we can use the same argument for $J(n)$ and get $n = b_0 + M(b_1 + MJ^2(n))$. Generally, for $k > 0$, we get

$$n = b_0 + Mb_1 + M^2 b_2 + \dots + M^k J^k(n), b_j \in \mathcal{N}. \tag{83}$$

Therefore the statement is clear.

3. $n = J^k(n) \Rightarrow J(n) = J^k(J(n))$.

4. This follows directly from the previous lemma.

□

The proof gave us a feeling of what $J^k(n)$ does: it gives us the rest term in a digit expension of n. Since we know that this will ultimately be periodic, it can only vanish iff $\mathcal{P} = \{0\}$.

Theorem 4.10. *The system $(M, \mathcal{N})$ is a number system iff $\mathcal{P} = \{0\}$.*

This theorem gives us a way to characterize number system with a simple algorithm. Since $\mathcal{P}$ is in a ball of radius N, we only need to check for finite n, if $J^k(n) = n$. But how long does that take? Luckily there is an upper bound for how long it takes for an arbitrary n to end up in $\mathcal{P}$ by repeatedly applying J.

Lemma 4.11. *Let $l(n)$ be the smallest integer $k \geq 0$, for which $J^k(n) \in \mathcal{P}$. Then there is a constant $c = c(M, \mathcal{N})$, such that $\forall n \in \mathbb{Z}^d \setminus \{0\}$,*

$$\left| l(n) - \frac{\log \|n\|}{\log \|M\|} \right| < c \tag{84}$$

Proof. For the finitely many points $\|n\| \leq 2L$ we can compute the largest c, for which the inequality holds. Let now n be arbitrary: we want estimate when such a n will end up in the ball of radius $4L$ of the first case. Denote by l_1 the smallest number k such that $\|J^k(n)\| \leq 2L$. Then $0 \leq l - l_1 \leq (4L+1)^d$ since this is the number of elements in the cube with side length $2L$ and we know that it contains $\mathcal{P}$. Using (83) for $k = l(n)$ we get: $\|n\| \leq K(1 + \|M\| + \dots + \|M\|^k) + \|M\|^{k+1} 2L \leq (K + 2L)\|M\|^{k+1}$, where the last inequality can be shown elementary using $\|M\| > 1$. This yields

$$\frac{\log \|n\|}{\log \|M\|} - k \leq \log(K + 2L) + 1.$$

Since $J^{k-1}(n)$ is just not yet in the cube we get again by using (83):

$$2L||M||^{k-1} \leq ||J^{k-1}(n)M^{k-1}|| \leq ||n|| + K(1 + ||M|| + \ldots + ||M||^{k-2})$$
$$= ||n|| + \frac{||M||^{k-1} - 1}{||M|| - 1} \leq ||n|| + L||M||^{k-1}.$$

Therefore $\log L + \log ||M||(k-1) \leq \log ||n||$, which yields

$$k - \frac{\log ||n||}{\log ||M||} \leq 1 - \frac{\log(L)}{\log ||M||}.$$

In conclusion we have a constant $c_1 = c_1(M, \mathcal{N})$ for l_1 and get

$$\left| l(n) - \frac{\log ||n||}{\log ||M||} \right| \leq |l(n) - l(n)| + \left| l_1(n) - \frac{\log |n|}{\log ||M||} \right| \leq (4L+1)^d + c_1$$

□

This estimation helps us to get information about the fundamental domain of a number system. Note that the following theorem holds for arbitrary $(M, \mathcal{N})$, especially for number systems.

Theorem 4.12. *Let $\mathcal{F} \subset \mathbb{R}^d$ the set of all $x \in \mathbb{R}^d$ that have at least one expansion of the form:*

$$x = \sum_{i=1}^{\infty} M^{-i} b_i \tag{85}$$

Then $\mathcal{F}$ is compact and, for every $y \in \mathbb{R}^d$, there are $x \in \mathcal{F}, n \in \mathbb{Z}^d : y = n + x$.

Proof. By our characterization of $\mathcal{F}$ by (82) we know, that it is the attractor of an IFS, which is a compact set. But more specifically we can bound $\mathcal{F}$ by $||x|| \leq \sum_i K||M||^{-i} = L$. We now prove the second statement. Let

$$F_m := \left\{ x = \sum_{i=1}^{m} M^{-i} b_i \right\}, F_\infty = \lim_{m \to \infty} F_m, \tag{86}$$

then $\mathcal{F}$ is the closure of F_∞. The previous lemma gives us a constant T, such that for all $m : ||n|| \leq ||M||^m$ we have $l(n) \leq m + T$, for arbitrary $n \in \mathbb{Z}^d$. We express n as

$$n = \sum_{i=0}^{m+T} M^i b_i + pM^{m+T+1}, p \in \mathcal{P}.$$

Now we consider the sets

$$G_m := \left\{M^{-m-T-1}n : \|n\| \leq \|M\|^m\right\}, \tag{87}$$

which satisfy $G_m \subseteq \bigcup_{p\in\mathcal{P}} p + F_m \subseteq \mathcal{P} + \mathcal{F}$. The limit $\lim_{m\to\infty} G_m$ lies dense in the cube of side length $2\|M\|^{T+1}$ with centre in the origin. By rescaling we conclude for the unit cube C that

$$C \subseteq M^{T+1}\mathcal{P} + M^{T+1}\mathcal{F} = M^{T+1}\mathcal{P} + \bigcup_{b_1,\dots,b_{T+1}\in\mathcal{N}} \sum_{i=1}^{T+1} M^{T+1-i} b_i + \mathcal{F}.$$

Therefore every $z \in C$ can be expressed as $z = u + x, u \in Z^d, x \in \mathcal{F}$ and since every $y \in \mathbb{R}^d$ can be expressed by $y = n + z, n \in \mathbb{Z}^d, z \in C$ we have proven the statement. □

Finally we prove some results about the Lebesgue measure of $\mathcal{F}$ and its boundary.

Proposition 4.13. *Let $(M, \mathcal{N})$ be a number system and $\mathcal{F}$ its fundamental domain. Then $\mathcal{L}^d(\mathcal{F}) > 0$ and $\mathcal{L}^d((\mathcal{F}+n) \cap (\mathcal{F}+m)) = 0, \forall n, m \in \mathbb{Z}^d$.*

Proof. Since $\mathbb{R}^d = \mathbb{Z}^d + \mathcal{F}$ we get the contradiction $\mathcal{L}^d(\mathbb{R}^d) = 0$, if $\mathcal{L}^d(\mathcal{F}) = 0$. Since $m, n \in \mathbb{Z}^d$ can be expressed uniquely in base M we get an exponent k, which is the maximum in their expansion. Let Γ_k be the set of all points that have expansion using exponents at most k. Note that $|\Gamma_k| = \det M^{k+1}$ and therefore

$$\mathcal{L}^d(\mathcal{F})(\det M)^k = \mathcal{L}^d(M^k\mathcal{F}) \leq \sum_{n\in\Gamma_{k-1}} \mathcal{L}^d(\mathcal{F}+n) = (\det M)^k \mathcal{L}^d(\mathcal{F}).$$

Therefore $\mathcal{L}^d((\mathcal{F}+n) \cap (\mathcal{F}+m)) = 0$. □

If the last statement is true for $(M, \mathcal{N})$ we call it a *just touching covering system (JTCS)*. We have just shown, that every number system forms a JTCS. It is possible to prove the converse ([12, Chapter 4.]), but we omit this. Further, Kátai gives an algorithm to determine the $n \in \mathbb{Z}^d$, where $\mathcal{F} \cap (\mathcal{F}+n) \neq \varnothing$, but we will develope another algorithm in the next subsection, that is more suitable.
In general we need more conditions on the number system to get more topological properties. Since the Twin Dragon forms the fundamental domain of a canonical number system over $\mathbb{R}^2$ we can show a few properties using [2, p.8/9].

Theorem 4.14. *Let $\mathcal{K}$ be the Twin Dragon as previously defined via the matrix B. Then $\mathcal{K}$ is the closure of its interior and $\mathcal{K}$ satisfies the open set condition.*

Proof. $\mathcal{K}$ is a CNS, since one can check by the algorithm proposed in Theorem 4.10. First we prove that $0 \in \operatorname{int}(\mathcal{K})$. Suppose $0 \in \partial\mathcal{K}$, then there exists a representation

$$0 = n + \sum_{i=1}^{\infty} B^{-i} d_i \in n + \mathcal{K}.$$

Multiplying with B^k yields $0 \in B^k n + \mathcal{K}$ and, since we have a CNS, the vectors $B^k n, k \in \mathbb{N}$ are all different. Therefore 0 is in infinitely many translates of $\mathcal{K}$. Since these get arbitrarily big, it violates the compactness of $\partial\mathcal{K}$. Now we fix $m \in \mathbb{N}$ and F_m as in (86). Then for $g \in F_m$ we get $g \in \operatorname{int}(B^{-m}\mathcal{K} + g)$. Now

$$\mathcal{K} = \bigcup_{g \in F_k} (B^{-m}\mathcal{K} + g)$$

yields $F_k \subseteq \operatorname{int}(\mathcal{K})$ and therefore $F_\infty \subseteq \operatorname{int}(\mathcal{K})$. Taking closure on both sides gives us $\mathcal{K} \subseteq \overline{\operatorname{int}(\mathcal{K})}$ and the reverse relation is trivial, since $\mathcal{K}$ is closed. The open set condition is now satisfied with $V = \operatorname{int}(\mathcal{K})$. □

4.2 Self-similar tiles

Consider the two dimensional plane. We now ask ourselves if there is a way to cover the whole plane with congruent compact sets, such that only the boundaries overlap. Thinking of classic quadratic bathroom tiles or the six sided honey comb we know, that this is possible. More interestingly we can ask for convex tilings of $\mathbb{R}^d$ or if there exists a tiling given a certain set. We will investigate self-similar tiles and their properties. For certain self-similar tiles it is possible to calculate the dimension of its boundary, as it is for the Twin Dragon fractal. Furthermore we will see that these tilings strongly relate to canonical number systems.
Before we define the key notions of this chapter, we introduce a terminology often used in this context.

Definition 4.15. Let (X, τ) be a topological space. We say two sets are ***non-overlapping***, iff their interiors have empty intersection.

Definition 4.16. We call $T \in \mathcal{K}(\mathbb{R}^n)$ a ***tile***, iff there is a countable set I and affine transformations $A_i : \mathbb{R}^n \to \mathbb{R}^n$ for all $i \in I$ such that

$$\mathbb{R}^n = \bigcup_{i \in I} A_i(T), \tag{88}$$

where every two sets are non-overlapping. We call this covering a ***tiling*** of $\mathbb{R}^n$. Tilings of $\mathbb{R}^2$ are sometimes called ***tesselations***.

Now there is a multitude of terminology concerning further structur of the tile, like *convex tilings* or *Euclidean tilings*, where only regular polygons are used. On the other hand one can demand more structure of the affine transformations or even allow different tiles. Finally one can search for tilings of other geometric spaces, like the hyperbolic space or tilings of the torus. This opens the very large theory of tilings.
This thesis is concerned with tiles that are self-similar.

Definition and Proposition 4.17. Let $T \in \mathcal{K}(\mathbb{R}^n)$ be a self-affine set defined by the IFS $\{f_i\}$ with $L(f_i) = 1/c, c < 0, \forall 1 \leq i \leq m$. If $f_i(T)$ and $f_j(T)$ do not overlap for $i \neq j$ and T is the closure of its interior, we call T a ***self-similar tile***. In fact $\mathbb{R}^n$ can be tiled by copies of T.

For specific self-similar tiles it is possible to calculate the Hausdorff dimension of their boundary, including the Twin Dragon. Following [3] we establish the following notions, lemmas and theorems.

Definition 4.18. Let T be self-similar tile such that the IFS $\{f_i\}$ have the form $f_i(x) = A^{-1}(x+d_i), d_i \in D$, where A is a similtude given by an expensive integer matrix and D is a set of coset representatives of $\mathbb{Z}^d/A(\mathbb{Z}^d)$ and $0 \in D$. We call T a ***self-similar digit tile*** and D is the ***digit set***. We use the notation $T = T(A,D)$ to specify the matrix and digit set.

Although we did not prove the last two propositions, note that the Twin Dragon satisfies all needed conditions. Another specification we yet have to show for $\mathcal{K}$ is, that the boundary of the set behaves well, while approximating it by the standard fixed point iteration.

Definition and Proposition 4.19. Let T be a self-similar digit tile, T_0 be the axis parallel unit square at the origin and T_n the n-th step of the approximation by prefractals. T has ***well-behaved boundary***, iff one of the following equivalent conditions is fullfilled

1. $\lim_{n\to\infty} \partial T_n = \partial T$,
2. $\lim_{n\to\infty} \partial T_n$ is not space filling,
3. $\mathcal{L}^d(T) = 1$,
4. $T + \mathbb{Z}^d$ is a tiling of $\mathbb{R}^d$.

These conditions are in fact equivalent

We prove a lemma, that gives us an estimate on how fast the boundary is approximated by the boundary of the prefractals.

Lemma 4.20. *Let T be a self-similar digit tile with well-behaved boundary and c be the expansion factor of A as in the definition above. Let T_0, T_n be the fixed point iteration starting with the unit square. Then there is constant $a > 0$ such that for the Hausdorff metric d_H (3.4) we have the inequality*

$$d_H(\partial T, \partial T_n) < \frac{a}{c^n} \tag{89}$$

Proof. Let F denote the Hutchinson operator and $a = d_H(T, T_0)$. First we see, that $d_H(T, T_n) = d_H(F(T), F(T_{n-1}) \le \frac{1}{c} d_H(T, T_{n-1}) \le \frac{1}{c^n} d_H(T, T_0)$, therefore we know that $d_H(T, T_n) \le a/c^n$. To show that $d_H(T, T_n) \le a/c^n$, we prove, that all $x \in \partial T_n$ have at most a/c^n distance to some point $y \in \partial T$ and otherwise. We use three cases:

First let $x \in \partial T$; then choose $y = x$, so we get distance $0 \leq a/c^n$. Next let $x \notin T$. There is a $z \in T$, such that $d(z,x) \leq a/c^n$, since $d(T,T_n) \leq a/c^n$. Intersecting the line between x and z with the boundary of T gives us a point y with $d(y,x) \leq d(z,x) \leq a/c^n$.
The last case is, if x is in the interior of T. Using the conditions we know that T forms a tiling with the integer lattice and it can be shown that the same is true for T_n. Since $x \in \partial T_n$, there is another copie $y + T_n$, such that x is also in the boundary of it, but $x \notin y + T$, since it is in the interior of T and two copies of T only can intersect on the boundary. Now we use $d_H(y+T, y+T_n) \leq a/c^n$ and find a point $z \in y+T$, such that $d(z,x) \leq a/c^n$ and a point $w \in \partial T \cap \partial(y+T)$ on the segment zx, such that $d(w,x) \leq a/c^n$. □

Another lemma, that happens to be usefull in the proof of the main theorem of this section, is the following.

Lemma 4.21. *Let $T = (A,D)$ be a self-similar digit tile, $\{e_1, ..., e_d\}$ the standard basis of $\mathbb{R}^n$ and $N_0 = \{0\} \cup \{\pm e_1, .., \pm e_d\}$. Then there is a unique smallest finite set $N_0 \subset N \subset \mathbb{Z}^d$, such that $D + N \subseteq AN + D$.*

Proof. We define an algorithm to create N as followed. Start with N_0 and $N_{j+1} = N_j \cup \{x \in \mathbb{Z}^d | Ax + d = y, d \in D, y \in D + N_j\}$. If $N_{n+1} = N_n$ for some n stop and define $N = N_n$. It holds that $AN + D = N + D$ and $N_0 \subseteq N$ by construction. One only has to show, that this algorithm terminates after finitely many steps. Let δ be the maximal norm of a point in D and r_j the maximal norm of a point in N_j. Since $r_{j+1} \leq (1/c)(r_j + 2\delta)$ holds we can estimate

$$r_n \leq 2\delta \sum_{j=1}^{n} \frac{1}{c^j} + \frac{1}{c^n} \leq \frac{2\delta c}{1-c} + 1.$$

Therefore N_n is bounded and the algorithm has to terminate. Since we add in every step the solutions that are necessary and nothing more, this set is the smallest unique set. □

Definition 4.22. We call N the (A,D)-***neighbourhood***. For each $x \in N$ and $d \in D$ there is unique solution $x_d \in N$ to $d + x \in Ax_d + D$, since D is a full coset representative system. Define the ***contact matrix*** as the $(|N|-1) \times (|N|-1)$-matrix $C = (c_{xy})$ with $c_{xy} = |\{d \in D | x_d = y\}|, x, y \in N \setminus \{0\}$.

Using the *Perron-Frobenius* theorem for non-negative matrices we get that the spectral radius of C is an eigenvalue itself. This so called *dominating* eigenvalue λ will appear in the dimension of the boundary of T, but before we can state and prove this result we need one further lemma, regarding the powers of C.

Lemma 4.23. *Let x, y be points in (A, D)-neighbourhood and C the contact matrix. Then c_{xy}^n the entries of C^n count the number of elements $d \in D_n$ such that $d + x \in A^n y + D_n$, where $D_n = D + AD + ... + A^{n-1}D$.*

Proof. This proof is done by induction and for $n = 1$ the statement is just the definition of the contact matrix. Now assume, that the statement is true for $n-1$. Matrix product gives us $c_{xy}^n = \sum_{z \in N \setminus \{0\}} c_{xz} c_{zy}^{n-1}$. Claim that for all $d_1 \in D, d_2 \in D_{n-1}$ with $d_1 + x \in Az + D$ and $d_2 + z \in A^{n-1}y + D_{n-1}$ the element $d = Ad_2 + d_1 \in D$ satisfies $x + d \in A^n y + D_n$, which is easy to check. Secondly we claim the converse statement, meaning every element $d \in D_n$ satisfying $x + d \in A^n y + D_n$ is of the form $d = Ad_2 + d_1$ as above. The decomposition of d, such that $d_1 \in D, d_2 \in D_{n-1}$, is immediate from the definition of D_n. Using the previous lemma we have $d_1 + x = Az + d'$ for some $z \in N, d' \in D$ and $A(d_2 + z) + d' = Ad_2 + Az + d' = d - d_1 + d_1 + x = d + x \in A^n y + D_n = A(A^{n-1}y + D_{n-1}) + D$. Using the induction hypothesis the representation is unique and we get $d_2 + z \in A^{n-1} + D_{n-1}$. Therefore $c_{xy}^n = \sum_{z \in N \setminus \{0\}} c_{xz} c_{zy}^{n-1}$. □

Theorem 4.24. *Let $T = (A, D)$ be a self-similar digit tile with well behaved boundary, where c is the expansion factor of A and C has dominating eigenvalue λ. Then*

$$\dim(\partial T) = \frac{\log \lambda}{\log c}. \tag{90}$$

Proof. Let T_0 be the axis parallel unit square at the origin and $T_n = F^n(T_0)$, where F is the Hutchinson operator creating the self-similar tile. In particular

$$T_n = \bigcup \{A^{-n}(T_0 + d_0 + Ad_1 + ... + A^{n-1}d_{n-1}) | d_i \in D\},$$

in other words T_n is the non-overlapping union of copies of the smaller cubes $A^{-1}T_0$ of edge length $1/c$, since the rest of the terms are translations. Recall the definition of the lattice D_n in Lemma 4.23 and note, that the lattice points are in bijection to the cubes building T_n. Let $N' = N(A, D) \setminus \{0\}$ and for every matrix M let $|M|$ denote the sum of their entries. Lemma 4.23 shows that $|C^n|$ is the number of triples $(x, y, d) \in N' \times N' \times D_n$, that solve the equation $d + x \in A^n y + D_n$. Let B_n be the set of all $d \in D_n$, such that the equation holds for some $x, y \in N'$, B_n' the sets of cubes building T_n and β_n their cardinality. Thus

$$\beta_n \leq |C^n| \leq (|N| - 1)^2 \beta_n. \tag{91}$$

Lemma 4.21 gives us $D + n \subseteq AN + D$ and by induction $D_n + N \subseteq A^n N + D_n$. Let $b = b(A, D)$ be the largest euclidean distance from the origin to a point in N. By Lemma 4.20 the distance between ∂T and the centers of cubes in B_n' is at most $(a + b)/c^n$.
Now consider the tiling of $\mathbb{R}^d$ by cubes of edge length $1/c^n$

$$\{x + A^{-n}(T_0) | x \in A^{-n}(\mathbb{Z}^d)\}$$

The number of tiles within distance $(a + b)/c^n$ is bounded by a constant h only depending on the dimension, not on n. Let α_n be the smallest number of tiles of edge length $1/c^n$ covering ∂T. Thus $\beta_n \leq h\alpha_n$ and $\alpha_n \leq h\beta_n$. Using this inequalities and (91) there are $a', b' > 0$ such that

$$a'|C^n| \leq \alpha_n \leq b'|C_n|.$$

Now we use a formula for calculation the spectrum of a matrix sometimes refrered to as Gelfand theorem: $\lim_{n\to\infty}(|C^n|)^{1/n} = \lambda$. Taking logarithm gives us

$$\lim_{n\to\infty} \frac{1}{n} \log(|C^n|) = \log \lambda$$

Finally we can piece together the calculation for box-counting dimension using the previous inequalities:

$$\text{Box}(\partial T) = \lim_{n\to\infty} \frac{\log \alpha_n}{\log c^n} = \lim_{n\to\infty} \frac{\log |C^n|}{n \log c} = \frac{\log \lambda}{\log c}$$

Since box-counting and Hausdorff dimension coincide on ∂T we get the claimed result. □

This method not only gives us a formula to calculate the dimension, [3] also concluded, that this can be determined to prove, if the boundarys are well behaved.

Theorem 4.25. *Let $T = (T, A)$ be a self-similar digit tile in $\mathbb{R}^d$ and λ the spectral radius of the contact matrix C. Then the boundary of T is well behaved iff $\lambda < |\det A|$.*

Proof. If T has well behaved boundaries Theorem 4.24 holds and $\dim(\partial T) = \log\lambda/\log c$ and $\det(A) = c^d > c^{\dim(\partial T)} = \lambda$. The inequality $d > \dim(\partial T)$ is proven in [11, p.199]. Conversely assume, that T has not well-behaved boundary. Then there are two distinct tiles $T, x+T$ with $x \in \mathbb{Z}^d$ that overlap. Since $\dim_H(T, T_n) \le a/c^n$ still holds, we can show that every point y in the overlap is within distance a/c^n to the boundary of T_n. Firstly we can find $z \in T_n, w \in x + T_n$, such that their distance to y is less than a/c^n and since $T_n \cap (x + T_n) \neq \varnothing$, there is a point $u \in \partial T_n$ on the segment zw, such that $d(y, u) \le a/c^n$.
As in the previous proof T_n decomposes into $|\det(A)|^n$ cubes. Let β_n be the number of cubes contained in the overlap and γ_n the number of cubes in the overlap that intersect ∂T_n. Then there are constants $a_1, a_2 > 0$ such that

$$|\det A|^n \le a_1\beta_n \le a_2\gamma_n \le |C^n|.$$

The last inequality follows from the bijection between D_n and cubes in T_n as in Lemma 4.23. Using Gelfand theorem

$$|\det A| \le \lim_{n\to\infty} |C^n|^{1/n} = \lambda.$$

□

Now we can calculate the Hausdorff dimension of the boundary of the Twin Dragon as it is done in [3].

Proposition 4.26. *Let $\mathcal{K}$ be the Twin Dragon. Then*

$$\dim(\partial\mathcal{K}) = \frac{\log\left(\frac{\sqrt[3]{3\sqrt{87}+28}}{3} + \frac{1}{3\sqrt[3]{3\sqrt{87}+28}} + \frac{1}{3}\right)}{\log\sqrt{2}} = 1.523627... \tag{92}$$

Proof. $\mathcal{K} = \mathcal{K}(B, D)$ is a self-similar digit tile with digit set $D = \{(0,0), (1,0)\}$ and

$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},$$

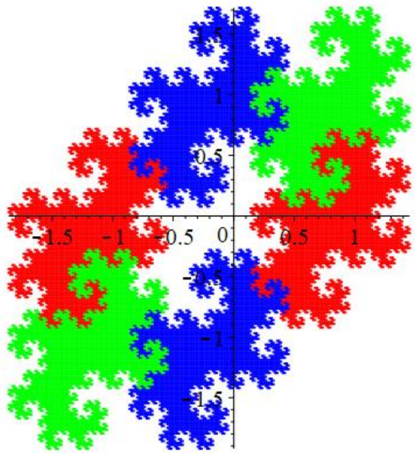

Figure 5: $\mathcal{K}$ and the 6 neighbouring copies

whose expansion factor is $\sqrt{2}$. One can calculate the (B, D)-neighbourhood with the algorithm above and gets $N = N_0 \cup \{(1,1), (-1,-1)\}$ and contact matrix

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The charactristic polynomial is

$$\det(C - \lambda I) = (\lambda + 1)(\lambda^2 - 2\lambda + 2)(\lambda^3 - \lambda^2 - 2),$$

from which the largest root is the real root of the last polynomial $\lambda^3 - \lambda^2 - 2$, which is less than $|\det(B)| = 2$. Using the formula proven in Theorem 4.24, we get the result. □

We also want to characterize the points in the boundary of the Twin Dragon with a Büchi Automaton. We follow a general characterization for self-similar digit tiles with well-behaved boundaries in [17] by Scheicher and Thuswaldner, who developed a faster version of the proposed algorithm.

Proposition 4.27. *Let $T = (T, A)$ be a self-similar digit tile in $\mathbb{R}^d$ such that $T + \mathbb{Z}^d$ is a tiling. Then ∂T can be defined by a graph $G(N)$. N denotes the (A, D)-neighbourhood and holds as the set of vertices in $G(N)$. An edge $x \xrightarrow{d} y, x, y \in N, \in D$ belongs to the graph iff $\exists d' \in D : Ax + d' = y + d$. Let T_0 be the unit cube with center in 0 and T_n the n-th step in the approximation of T. If $B_{x,n} := T_n \cap (T_n + x)$ we have*

$$\partial T = \lim_{n\to\infty} \bigcup_{x\in N\setminus\{0\}} B_{x,n} \tag{93}$$

with

$$B_{x,n} = \bigcup_{x \xrightarrow{d} y} A^{-1}(B_{y,n-1} + d) \tag{94}$$

Proof. Since T has well-behaved boundaries we can approximate its boundary by the boundaries of T_n and $\partial T_n = \bigcup_{x\in\mathbb{R}^d\setminus\{0\}} T_n \cap (T_n + x)$. Further

$$\begin{aligned} B_{x,n} &= T_n \cap (T_n + x) \\ &= A^{-1}(T_{n-1} + D) \cap A^{-1}(T_{n-1} + D + Ax) \\ &= A^{-1} \bigcup_{d,d'\in D} (B_{Ax+d'-d,n-1} + d) \end{aligned}$$

Because the algorithm in 4.21 determined all y such that there is $\exists d, d' \in D$ satisfying$y + d = Ax + d'$ starting with the neighbours of the unit cube. We can express $\partial T_n = \bigcup_{x\in N\setminus\{0\}} B_{x,n}$ and also the second equation holds. □

Since the maps $f_d(x) = A_{-1}(x + d)$ are all contracting, we can interpret this as a Mauldin-Williams graph and Theorem 3.15 gives us an invariant set list. Further we can characterize the points in the boundary of T with a Büchi automaton.

Proposition 4.28. *Let the notation be as in the previous proposition. There exist sets $B_x, x \in N$ such that*

$$\partial T = \bigcup_{x\in N\setminus\{0\}} B_x \tag{95}$$

with

$$B_x = \bigcup_{x \xrightarrow{d} y} A^{-1}(B_y + d). \tag{96}$$

Furthermore a point $p \in T$ belongs to ∂T, iff there exists an infinite path through $G(N)$ labeled with the address of p.

Proof. We only have to prove the last statement. If $p \in \partial T$ then there is an $x \in N \setminus \{0\}$, such that $p \in B_x$. Therefore $p \in A^{-1}(B_y + d)$ for some $x \xrightarrow{d} y \in G(N)$. Following the same argument there is a path labeled with the address of p. On the contrary if p has an address labeled by an infinte path in $G(N)$ starting with x we choose an arbitrary point in B_x and apply $f_d(x) = A_{-1}(x + d)$. This point lies in B_y for $x \xrightarrow{d} y \in G(N)$. If we carry on in the same way converge to the point p but stay in ∂T, which is closed. Therefore $p \in \partial T$. □

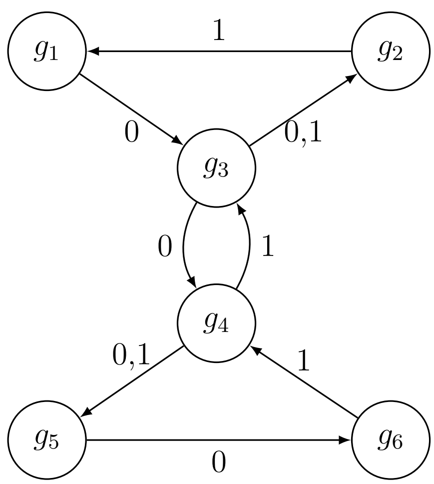

Figure 6: The automaton $\mathcal{G}$ characterizing $\partial\mathcal{K}$

Using these propositions we can now calculate the Büchi automaton $\mathcal{G}$, that accepts the points in the boundary of the Twin Dragon (see Figure 6.).

5 Intersections of the Twin Dragon with rational lines

We finally have gathered enough background information to tackle the main question of this thesis: What is the geometrical structure of intersection between the Twin Dragon and rational lines? Chapter 2 introduced Hausdorff dimension and explicitly states that in almost all cases the dimension is just 1 less than the dimension of the Twin Dragon, respectively its boundary. Therefore we expect a one dimensional shape with a fractal boundary. Chapter 3 gave us the tools to determine these dimension values and introduced further concepts to better understand the self-similar nature of the Twin Dragon. Chapter 4 specifically talked about self-similar digit tiles and CNS. There already a bit of the ground work has been done to understand the results of the following calculations.
We follow the same approach as [1], where Shigeki Akiyama and Klaus Scheicher calculated the intersection with the x-and the y-axis. It turns out that these intersections are just intervalls and the intersection with the boundary are just two points. The points in the intersection are accepted by the Büchi automaton $\mathcal{H}$ (see Figure 7). For the intersection with the x-axis the initial state is h_1, for the y-axis it is h_3.

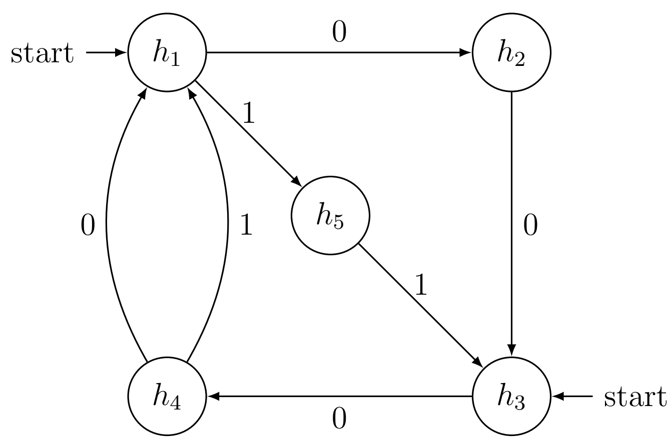

Figure 7: The automaton $\mathcal{H}$ characterizing $\mathcal{K} \cap \{x = 0\}$ (starting with h_3) and $\mathcal{K} \cap \{y = 0\}$ (starting with h_1)

We first analyse the intersection with the identity line and the negative identity line, where we get the same result. In a second round we go through the process again, but using a whole class of lines that are just translations of the result in [1]. Further we look at limits of vertical lines. If we choose the sequence wisely, we can use the approximated results to determine the intersection in the limit. We give two concrete examples of this methode; one intersection achieves fractal boundary.
As before let $\mathcal{K}$ denote the Knuth Twin Dragon throughout this chapter, as well as the notation for CNS in Chapter 4.

5.1 The diagonal and the negative diagonal

Let $\Delta = \{(x,x) \in \mathbb{C}\}$ and $\overline{\Delta} = \{(x,-x) \in \mathbb{C}\}$. Following [1] we state the following result.

Proposition 5.1. *The intersection $\mathcal{K} \cap \Delta$ can be recognized by the Büchi automaton $\mathcal{A}$ (see Figure 8). Let $Q_{\mathcal{A}} = \{q_1, \dots, q_5\}$ be the set of states in $\mathcal{A}$, $I_{\mathcal{A}} = F_{\mathcal{A}} = \{q_1\}$. We have $z \in \mathcal{K} \cap \Delta$, iff*

$$z = \sum_{i=1}^{\infty} B^{-i}\Phi(d_i), \tag{97}$$

where $d = d_1 d_2 \dots$ is accepted by the automaton.

Proof. Since $P(z) = z^2 + 2z + 2$ and $\alpha_{\pm} = -1 \pm i$ we have

$$B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},$$

Since $\Phi(d_i) = (d_i, 0)^T$ for all $d_i \in \mathcal{N} = \{0,1\}$, we can can read (97) as $z = Cd$ with $C \in \mathbb{R}^{2\times\infty}$ and $d = (d_1, d_2, \dots)^T \in \mathbb{R}^\infty$ with

$$C = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & 0 & -\frac{1}{16} & \frac{1}{16} & \dots \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{8} & -\frac{1}{8} & \frac{1}{16} & 0 & \dots \end{pmatrix}. \tag{98}$$

The i-th column of C is given by the first column of B^{-i}. Let c_{ij} denote the entries of C. Then since $B^{-8} = \operatorname{diag}(1/16, 1/16)$ we get $c_{i,j+8} = 1/16 c_{ij}, i = 1,2, j \geq 1$. Let us denote $z = (x,y)$, then we get the digit expansion:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{d_1}{2} + 0 + \frac{d_3}{4} - \frac{d_4}{4} + \frac{d_5}{8} + 0 + -\frac{d_7}{16} + \frac{d_8}{16} + \dots + d_{j+8}\frac{c_{1j}}{16} + \dots \\ -\frac{d_1}{2} + \frac{d_2}{2} - \frac{d_3}{4} + 0 + \frac{d_5}{8} - \frac{d_6}{8} + \frac{d_7}{16} + 0 + \dots + d_{j+8}\frac{c_{2j}}{16} + \dots \end{pmatrix}. \tag{99}$$

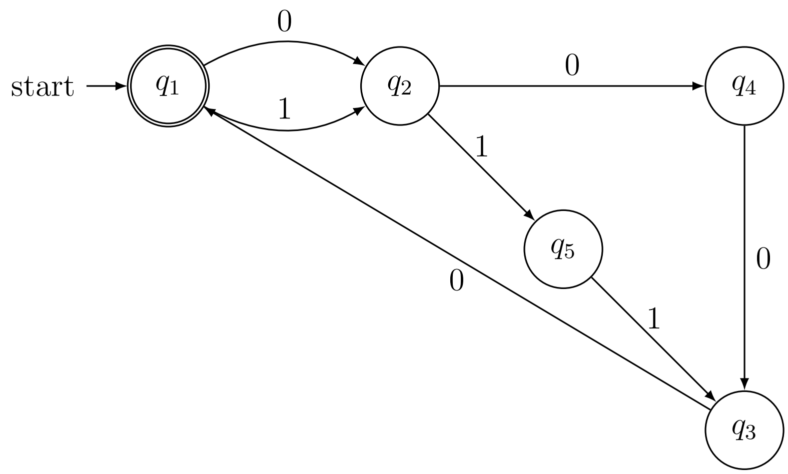

Figure 8: The automaton $\mathcal{A}$ characterizing $\mathcal{K} \cap \Delta$

From here on the proof differs from [1]. First we notice that the digits $d_1, d_5, d_9, ..., d_{4k+1}, ...$ can be choosen arbitrary since these digits keep the relation $x = y$. The approach is now to analyse all the cases for the first eight digits to get $x = y$ or so close, that the other digits can diminish that difference. Luckily the second case does not appear: after eight digits we must already have $x = y$. Summing up all the positive and negativ values for x and y determines how far up or down the remaining of the digits can increase or decrease the values. For example $x_{\min} = \sum_{i=1}^{\infty}(-\frac{1}{4} - \frac{1}{16})\frac{1}{16^i} = -\frac{5}{16} \cdot \frac{1}{15} \approx -0.021$. We get the other value in the same way and have a possible correction possibility of $[-0.021, 0.021]$ in x direction and $[-0.025, 0.038]$ in y-direction.

I. Since d_1 is arbitrary we start with $d_2 = 1$. First case is, that $d_3 = 1$ as well and we see that the current values of x and y are equal.
If $d_3 = 0$, then the difference $x - y = -1/2$. The only useful digit is d_8, since its contribution to the value of x is positive and its contribution to the value of y is non negative. For $d_8 = 1$ we get a difference of -0.4375 which cannot possibly be canceled out within the further digits.
So if $d_2 = 1$, so has to be $d_3 = 1$ and $d_4 = 0$, since this digit just decreases x.

II. Now suppose that $d_2 = 0$ and first let $d_3 = d_4 = 1$. Then the difference $x - y = 1/4$ and the only way to reduce this difference within the first eight digits is $d_7 = 1$. The resulting difference of $-1/8$ can not be compensated. Since that was the best case scenario for $d_2 = 0, d_3 = 1$, we get $d_2 = 0 \Rightarrow d_3 = d_4 = 0$.

We could continue this approach, but because of the structure of the matrix we see that the proof must proceed similarly to the first four digits. That shows that d is a label of an infinite succesfull path through the automaton iff $z \in \mathcal{K} \cap \Delta$. □

The calculation for $\overline{\Delta}$ can be done in a similar way, but there is a more clever observation proving the validity of the proposed automata.

Proposition 5.2. *The intersection $\mathcal{K} \cap \overline{\Delta}$ can be recognized by the Büchi automaton $\mathcal{B}$ (see Figure 9). Let $Q_{\mathcal{B}} = \{q_{-1}, q_0, ..., q_5\}$ be the set of states in $\mathcal{B}$, $I_{\mathcal{B}} = \{q_{-1}\}, F_{\mathcal{B}} = \{q_1\}$. We have $z \in \mathcal{K} \cap \overline{\Delta}$ iff*

$$z = \sum_{i=1}^{\infty} B^{-i}\Phi(d_i)$$

where $d = d_1 d_2 ...$ is accepted by the automaton.

Proof. Observe that if $z = x + ix \in \mathcal{K} \cap \Delta$ then $z(2i)^{-1} = x/2 - x/2i \in \mathcal{K} \cap \overline{\Delta}$. These points correspond to the paths with $d_1 = d_2 = 0$, since then

$$z = \sum_{i=3}^{\infty} B^{-i}\Phi(d_i) = B^{-2}\sum_{i=3}^{\infty} B^{-i+2}\Phi(d_i) = \frac{1}{2i}\sum_{i=1}^{\infty} B^{-i}\Phi(d_{i-2}).$$

Conversely d_1 and d_2 cannot be 1 because the difference $x - (-y)$ cannot be extinguished by the further digits. □

These automata now help us determine the geometric structure of the intersections.

Theorem 5.3. *The intersection $\mathcal{K} \cap \Delta$ consits of the line segment $\{(x, x) : x \in [-\frac{3}{5}, \frac{2}{5}]\}$. The intersection $\mathcal{K} \cap \overline{\Delta}$ consits of the line segment $\{(x, -x) : x \in [-\frac{2}{10}, \frac{3}{10}]\}$*

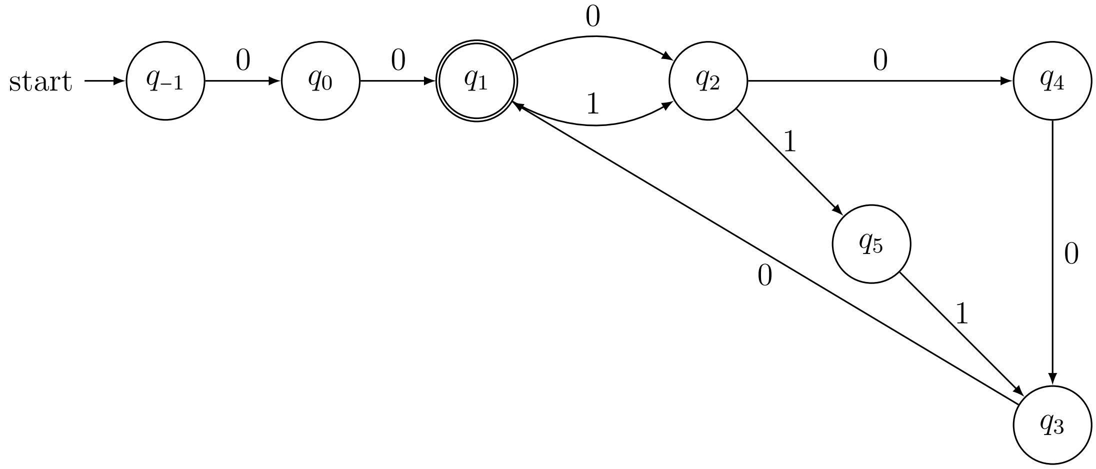

Figure 9: The automaton $\mathcal{B}$ characterizing $\mathcal{K} \cap \overline{\Delta}$

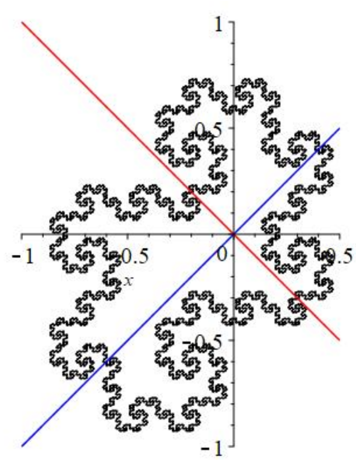

Figure 10: $\mathcal{K}, \Delta$ (blue) and $\overline{\Delta}$ (red)

Proof. Consider an infinite successfull path labeled with $d_1 d_2 d_3$..... accepted by $\mathcal{A}$ described in Proposition 5.1. Then the sequence corresponds to the digit expansion of

$$z = \sum_{i=1}^{\infty} B^{-i}\Phi(d_i) = \sum_{j=1}^{\infty} B^{-4j}\left(\sum_{i=1}^{4} B^{4-k}\Phi(d_{j-1+k})\right).$$

So we divide the digits in blocks of length 4, since B^{-4} corresponds to the number $-\frac{1}{4}$ and regard the blocks as a single digit in base -4. The possible 4-letter words are: $0000, 0110, 1000, 1110$ which correspond to the digit vectors $(0,0), (-1,-1), (2,2), (1,1)$. Since every number in $[-\frac{3}{5}, \frac{2}{5}]$ can be expressed with the digits $-1, 0, 1, 2$ in base -4, we get the first result. The second one follows immediately from multiplication by $\frac{1}{2i}$. □

Theorem 5.4. *The intersection* $\partial\mathcal{K} \cap \Delta$ *consists of the two points* $\{(-\frac{3}{5}, -\frac{3}{5}), (\frac{2}{5}, \frac{2}{5})\}$. *The intersection* $\partial\mathcal{K} \cap \overline{\Delta}$ *consists of the two points* $\{(-\frac{2}{10}, \frac{2}{10}), (\frac{3}{10}, -\frac{3}{10})\}$.

Proof. A point belongs to the intersection $\partial\mathcal{K} \cap \Delta$ if its digit expansion is accepted by the product automaton of $\mathcal{A}$ and $\mathcal{G}$ describing $\Delta \cap \mathcal{K}$ respectively $\partial\mathcal{K}$. Going through all the possible starting points of $\mathcal{G}$, we derive that the only 4 step paths through $\mathcal{G}$, accepted also by $\mathcal{A}$, are:

$$g_1 \overset{0110}{\to} g_3, g_1 \overset{0000}{\to} g_6, g_2 \overset{1000}{\to} g_5, g_4 \overset{1110}{\to} g_3, g_5 \overset{0110}{\to} g_4, g_5 \overset{0110}{\to} g_2, g_5 \overset{0110}{\to} g_6.$$

Since g_3 and g_6 are dead ends, also g_1 and g_4 are not usable for infinite paths. The only possible paths are going back and forth between g_2 and g_5 with the paths $[10000110]^\infty$ and $[01101000]^\infty$. These correspond to the periodic number with digit -9 in base 16 and the periodic number with digit 6 in base 16, which are the proposed points.
For the intersection $\partial\mathcal{K} \cap \overline{\Delta}$ we therefore just have to check on how we can get to g_2 or g_5 with a path labeled 00. This is only possible by $g_1 \overset{00}{\to} g_2, g_3 \overset{00}{\to} g_5$ which correspond to the endpoints of the second segment. □

We see that the two diagonals are counterexamples to the expected result for intersections of fractals and lines, as were the examples of the x- and the y-axis considered in [1].

5.2 Vertical lines and intersections in the limit

We saw in the previous subsection, that it takes a lot of effort to compute an intersection and the outcome cannot be predicted beforehand. To get more interesting results we now take an approach using the data made by Akiyama and Scheicher in [1] regarding the intersection with the line $x = 0$. We can easily derive from this computation the intersection of the Twin Dragon with the line $x = -\frac{1}{2}$. This can only be achieved if the first digit $d_1 = 1$ and the further digits produce a number (x, y) with $x = 0$. So the intersection of $x = -\frac{1}{2}$ and the Twin Dragon is just a translation of the intersection of $\{x = 0\}$ by the vector $(-\frac{1}{2}, -\frac{1}{2})$. Similarly, to reach the intersection with the line $x = -\frac{1}{2} + \frac{1}{8}$ it follows that$d_1 = d_5 = 1$ and the other digits are accepted by the automaton $\mathcal{H}$ characterizing $\mathcal{K} \cap \{x = 0\}$. Following that approach of manipulating the $(4k + 1)$-th digit, we can generalize this result for a great class of lines. The same can be said about horizontal lines and the $(4k+3)$-th digit.

Theorem 5.5. *Let R satisfy*

$$R = \sum_{i=0}^{N} \left(-\frac{a_i}{2} + \frac{b_i}{8} \right) \left(\frac{1}{16} \right)^i, \tag{100}$$

with $a_i, b_i \in \{0, 1\}$. Then the intersection $\mathcal{K} \cap \{x = R\}$ is the translation of $\mathcal{K} \cap \{x = 0\}$ by the vector (R, R) and the intersection $\mathcal{K} \cap \{y = \frac{1}{2}R\}$ is the translation of $\mathcal{K} \cap \{y = 0\}$ by the vector $(-\frac{1}{2}R, -\frac{1}{2}R)$.

Proof. We prove this by induction on N. For $N = 0$ we calculate $x^- = \sum_{i=1}^{\infty} (-\frac{1}{2} - \frac{1}{4} - \frac{1}{16}) \frac{1}{16}^i = -\frac{13}{16} \cdot \frac{1}{15} \approx -0.054 < -\frac{1}{16}$ and $x^+ = \sum_{i=1}^{\infty} (\frac{1}{4} + \frac{1}{8} + \frac{1}{16}) \frac{1}{16}^i = \frac{13}{16} \cdot \frac{1}{15} \approx 0.029 < \frac{1}{16}$. So if we want to have $x = R$ we need to stay within the rage of $[R + x^-, R + x^+]$. There are 16 possible ways for the first eight digits, such that $x = R$, if we require that the remaining digits are accepted by the automaton of $\mathcal{K} \cap \{x = 0\}$:

$$\begin{array}{llll} a_0000b_0000 & a_0000b_0011 & a_0000b_0100 & a_0000b_0111 \\ a_0011b_0000 & a_0011b_0011 & a_0011b_0100 & a_0011b_0111 \\ a_0100b_0000 & a_0100b_0011 & a_0100b_0100 & a_0100b_0111 \\ a_0111b_0000 & a_0111b_0011 & a_0111b_0100 & a_0111b_0111 \end{array}$$

.

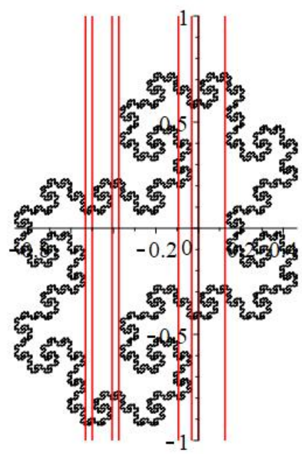

Figure 11: $\mathcal{K}$ and $\{x = R\}$ for all R as described in 5.5 varying a_0, a_1 and b_0

It is now easy to check that if $d_1 \neq a_0$ the remaining of the digits cannot amount to the deficit to stay in the range and similarly if $d_5 \neq b_0$.
Now suppose the only way to approximate a number R_{N-1} within the error interval of the above form is with the prescribed digits for the first $8N$ digits. Let R_N be a number of the proposed form with highest exponent N. We calculate $x_N^- = \sum_{i=N+1}^{\infty}(-\frac{1}{2} - \frac{1}{4} - \frac{1}{16})\frac{1}{16}^i = -\frac{13}{15}\cdot(\frac{1}{16})^{N+1} > -(\frac{1}{16})^{N+1}$ and $x_N^+ = \sum_{i=N}^{\infty}(\frac{1}{4} + \frac{1}{8} + \frac{1}{16})\frac{1}{16}^i = \frac{13}{15}\cdot(\frac{1}{16})^{N+1} < (\frac{1}{16})^{N+1}$. So within the first $8(N+1)$ digits we have to stay in the range of $I = [R_N + x_N^-, R_N + x_N^+]$. Now let R_{N-1} the number such that $R_N - R_{N-1} = (-\frac{a_N}{2} + \frac{b_N}{8})(\frac{1}{16})^N$. We now can extend the error intervall to

$$\begin{aligned} I &\subseteq \left[R_{N-1} + x_N^- - \left(\frac{a_N}{2}\right)\left(\frac{1}{16}\right)^N, R_{N-1} + x_N^+ + \left(\frac{b_N}{8}\right)\left(\frac{1}{16}\right)^N\right] \\ &\subseteq \left[R_{N-1} - \left(\frac{1}{16}\right)^{N+1} - \left(\frac{1}{2}\right)\left(\frac{1}{16}\right)^N, R_{N-1} + \left(\frac{1}{16}\right)^{N+1} + \left(\frac{1}{8}\right)\left(\frac{1}{16}\right)^N\right] \\ &\subset [R_{N-1} + x_{N-1}^-, R_{N-1} + x_{N-1}^+] \end{aligned}$$

Therefore R_N is also an approximation of R_{N-1} within its error intervall. By induction hypothesis the first $8N$ digits are determined and by the same argument as in the induction start the remaining digits are as well.
Now note that the digits of numbers in $\mathcal{K} \cap \{x = R\}$ and $\mathcal{K} \cap \{x = 0\}$ can only vary in finitely many positions, d_{4k+1} for $k = 0, ..., 2N$. Looking at the $(4k+1)$-th column of the matrix in (98) one sees that the result is translation

by the vector (R, R).
The same argument can be used to determine the intersections $\mathcal{K} \cap \{y = \frac{1}{2}R\}$ due to the similar structure of the rows of the matrix in (98). Here the digits d_{4k+3} are determined by the value of R and result in a translation by the negative value, since the x-coordinate is the negative of the y-coordinate in these columns. □

We can characterize points in the intersection with $\{x = R\}$ by going through a pre-automaton $\mathcal{P}$ and connecting to an infinite path in $\mathcal{G}$ characterizing $\partial\mathcal{K} \cap \{x = 0\}$. Fortunately we can also generalize the result for intersection with the boundary of the Twin Dragon.

Theorem 5.6. *The intersection $\partial\mathcal{K} \cap \{x = R\}$ consists of the two points $\{(-\frac{4}{5} + R, R), (\frac{1}{5} + R, R)\}$. The intersection $\partial\mathcal{K} \cap \{y = -\frac{1}{2}R\}$ consists of the two points $\{(-\frac{1}{2}R, \frac{2}{5} - \frac{1}{2}R), (-\frac{1}{2}R, -\frac{3}{5} - \frac{1}{2}R)\}$.*

Proof. Theorem 2.12. in [1] states that the only infinte paths accepted by the product automaton for $\partial\mathcal{K} \cap \{x = 0\}$ are $[00110100]^\infty$ starting in g_4 and $[01000011]^\infty$ stating in g_3. A path accepted by the product automaton for $\partial\mathcal{K} \cap \{x = R\}$ therefore has to pass through g_3 or g_4 at some point. These are all the possibilities for a four step path:

$$g_3 \overset{0100}{\to} g_4, g_3 \overset{1100}{\to} g_4, g_4 \overset{0011}{\to} g_3, g_4 \overset{1011}{\to} g_3$$

Amazingly these are the paths where the first digit is arbitrary, but d_{4k+1} is determined by the value of R till the $8N$-th digit. So there are only two possible paths trough the product automaton corresponding to the translated endpoints.
For the intersection with $\{y = 0\}$ the only paths are $[00001101]^\infty$ starting in g_1 and $[11010000]^\infty$ stating in g_6. Checking all possible four step paths leaves us with:

$$g_1 \overset{0000}{\to} g_6, g_1 \overset{0010}{\to} g_6, g_6 \overset{1101}{\to} g_1, g_6 \overset{1111}{\to} g_1.$$

Again these are the paths where the d_{4k+3} digit is arbitrary, but R determines them till the $8N$-th digit. Therefore the same result holds for the intersection with $\{y = -\frac{1}{2}R\}$ □

What we want to do next, is to look at an arbitrary value $R \in \mathbb{R}$. If we find a sequence $(R_N)_{N \geq 0}$ approximation R and can determine the intersection

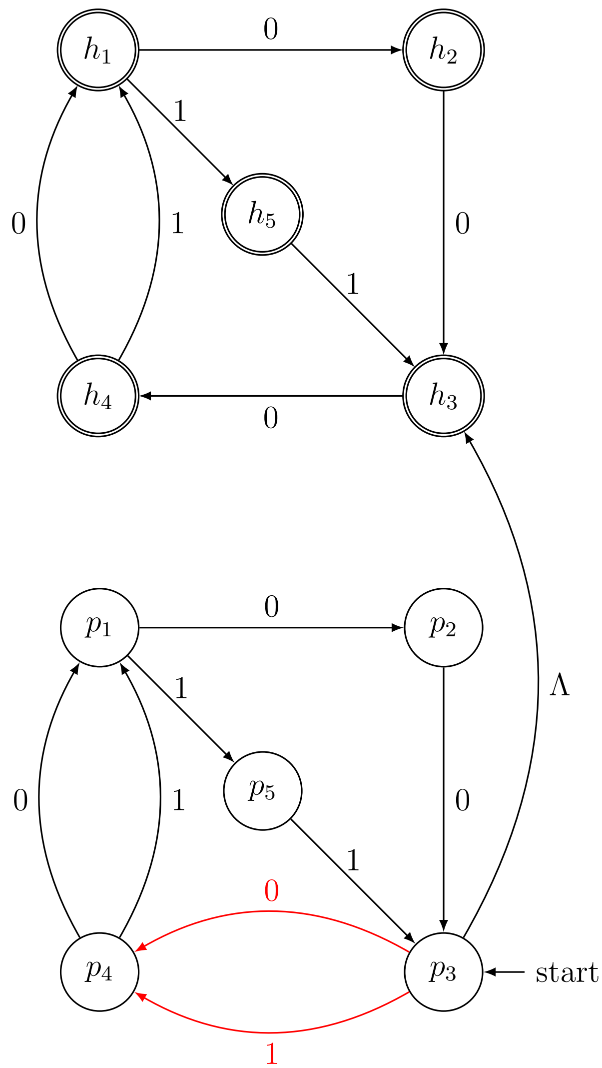

Figure 12: The automaton $\mathcal{P} - \mathcal{H}$ characterizing $\bigcup_R \mathcal{K} \cap \{x = R\}$, where R has the form (100). For a successfull path the labels of the used red edges determine the sequence $(a_0, b_0, a_1, ..., b_N)$.

of the Twin Dragon with $\{x = R_N\}$, is it then possible to calculate the intersection in the limit or do new points appear in the intersection with $\{x = R\}$? We get in fact a lemma, that the second case never appears.

Lemma 5.7. *Let $R \in [-\frac{13}{15}, \frac{7}{15}]$. Then there is a unique sequence $(R_N)_{N\geq 0}$ such that*

- $$R_0 = -\frac{d_{1,0}}{2} + \frac{d_{3,0}}{4} - \frac{d_{4,0}}{4} + \frac{d_{5,0}}{8} - \frac{d_{7,0}}{16} + \frac{d_{8,0}}{16}, \tag{101}$$

  with $d_{i,0} \in \{0,1\}$.

- *For all $N \geq 1$*

  $$R_N = R_{N-1} + \left(-\frac{d_{1,N}}{2} + \frac{d_{3,N}}{4} - \frac{d_{4,N}}{4} + \frac{d_{5,N}}{8} - \frac{d_{7,N}}{16} + \frac{d_{8,N}}{16}\right)\frac{1}{16^N}, \tag{102}$$

  with $d_{i,N} \in \{0,1\}$.

- *For all $N \geq 0$, $R = R_N + \epsilon(N)$ with $\epsilon(N) \in [-\frac{13}{15} \cdot (\frac{1}{16})^{N+1}, \frac{7}{15} \cdot (\frac{1}{16})^{N+1}]$. In particular $\lim_{N\to\infty} R_N = R$.*

Proof. First we show that every number R can be expressed in this way. The end points of the intervall are given by summing all negative values and all positive values in the digit expansion. Now since $-\frac{1}{4} + \frac{1}{8} = -\frac{1}{8}$, we see that all numbers in the fundamental domain of -2 ,$[-\frac{10}{15}, \frac{5}{15}]$, can be expressed since the powers of this base appear in the expansion. Further using the digits $d_{3,i}, d_{7,i}$ we can express every negative of a number in the fundamental domain of -4. Setting the other digits corresponding to a negative number to 1, we get the full intervall $[-\frac{13}{15}, -\frac{8}{15}]$. Conversely setting every other digit corresponding to a positive number to 1 gives us the intervall $[\frac{2}{15}, \frac{7}{15}]$. Since the three intervalls are overlapping, there is a digit expansion for every $R \in [-\frac{13}{15}, \frac{7}{15}]$.
The estimation of $\epsilon(N)$ is clear and has been calculated in Theorem 5.5. Now let $(R_N)_{N\geq 0}, (R'_N)_{N\geq 0}$ approximate the same number R. Suppose they disagree at the N-th term for the first time. Now consider the digits $d_{j,N}, d'_{j,N}$. $d_{1,N}$ and $d'_{1,N}$, as well as $d_{5,N}$ and $d_{5,N}$ cannot vary, since the rest of the expansion cannot correct the resulting error for R. Suppose $d_{3,N} = 1$ and $d'_{3,N} = 0$. If $d_{4,N} = 0$ and $d'_{4,N} = 0$ the resulting impact on R_N, R'_N is the

same. In any other case the difference is too big to be compensated by the remaining digits. Now if $d_{3,N} = d_{3,N}$, we calculate that $d_{4,N} = d'_{4,N}$ in the same way.
The same argument holds for $d_{7,N}, d_{8,N}$. Note that although the digits $d_{i,N}, d'_{i,N}$ can be different, the values R_N, R'_N are the same. Therefore $(R_N)_{N\geq 0}$ is unique. □

We can derive a similar lemma using the second row of the matrix (98) for values for y in $[-\frac{14}{15}, \frac{11}{15}]$, due the similar structure of the row.This lemma has strong consequences: if we visualize the result in Theorem 5.5, we get a pre-automaton only accepting finite paths and the Büchi automaton accepting the remaining infintely many digits. If we get better and better approximations for R the pre-automaton gets bigger and bigger and the automaton $\mathcal{H}$ eventually disappears in the limit, because there is no other way to recieve the value R, than with the sequence $(R_N)_{N\geq 0}$. In general this infinite pre-automaton cannot be written as a Büchi automaton, since it does not have finitely many states. But if we choose the value of R carefully, we get a Büchi automaton.

Theorem 5.8. *Let $R = -\frac{2}{5}$. The intersection $\mathcal{K} \cap \{x = R\}$ is the translation of $\mathcal{K} \cap \{x = 0\}$ by the vector (R, R) and the intersection $\mathcal{K} \cap \{y = \frac{1}{2}R\}$ is the translation of $\mathcal{K} \cap \{y = 0\}$ by the vector $(-\frac{1}{2}R, -\frac{1}{2}R)$. The intersection $\partial\mathcal{K} \cap \{x = R\}$ consists of the two points $\{(-\frac{4}{5} + R, R), (\frac{1}{5} + R, R)\}$. The intersection $\partial\mathcal{K} \cap \{y = -\frac{1}{2}R\}$ consists of the two points $\{(-\frac{1}{2}R, \frac{2}{5} - \frac{1}{2}R), (-\frac{1}{2}R, -\frac{3}{5} - \frac{1}{2}R)\}$.*

Proof. Since R can be expressed as

$$R = \sum_{i=0}^{\infty} \left(-\frac{1}{2} + \frac{1}{8}\right) \frac{1}{16^i},$$

we use the Lemma 5.7 and the results from Theorem 5.5 to get the automaton $\mathcal{P}'$ (see Figure 14) by choosing $a_i = b_i = 1$ and Theorem 5.6 to see, that the intersection with the boundary consists of only the end points. □

We can use this method to find a vertical line with a more interesting intersection. For example, if we look at $\{x = -\frac{1}{4}\}$, we see, that the only acceptable strings of length four in the pre-automaton are: 0001, 0101, 1010, 1110 which correspond to the digit vectors $(1, 0)$, $(1, -2)$, $(1, 3)$, $(1, 1)$ in the base -4. The Büchi automaton for the remaining of the digits accepts

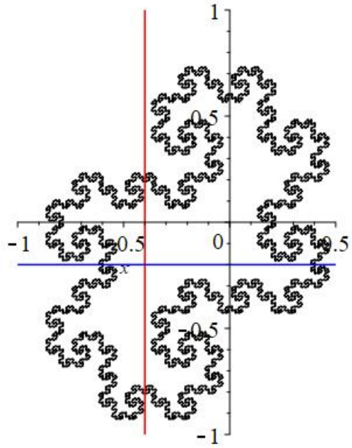

Figure 13: $\mathcal{K}$, $\{x = -0.4\}$ (red) and $\{y = -0.2\}$ (blue)

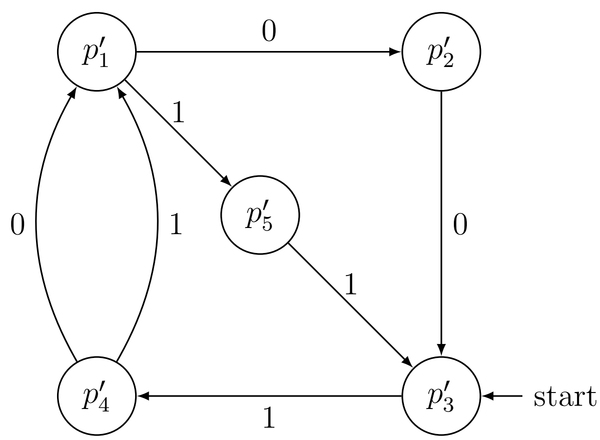

Figure 14: The automaton $\mathcal{P}'$ characterizing $\mathcal{K} \cap \{x = -\frac{2}{5}\}$

the strings $0000, 0011, 0100, 0111$ corresponding to the digit vectors $(0,0)$, $(0,-1)$, $(0,1)$, $(0,2)$. The y-coordinate of points in the intersection has to have a expansion in base -4 starting with $-2, 0, 1, 3$ and followed by digits $-1, 0, 1, 2$. Therefore the intersection splits into four intervalls $[-\frac{3}{20}, \frac{2}{20}] - \frac{d}{4}$, with $d = -2, 0, 1, 3$. So we get

$$\mathcal{K} \cap \left\{x = -\frac{1}{4}\right\} = \left\{(x,y) \in \mathbb{R} : x = -\frac{1}{4}, y \in \left[-\frac{18}{20}, -\frac{13}{20}\right] \cup \left[-\frac{8}{20}, \frac{2}{20}\right] \cup \left[\frac{7}{20}, \frac{12}{20}\right]\right\}.$$

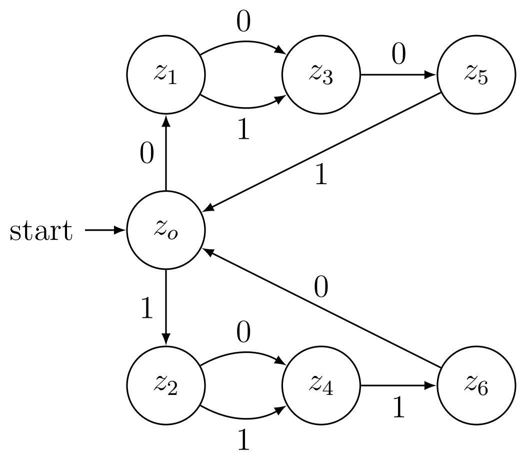

Figure 15: The automaton $\mathcal{Z}$ characterizing $\mathcal{K} \cap \{x = -\frac{1}{5}\}$

We go on with $\{x = -\frac{1}{4} + \frac{1}{16}\}$ and see that points in the intersection have y-coordinate with an expansion in base -4 which starts with two digits out of $\{-2, 0, 1, 3\}$ and end with digits $-1, 0, 1, 2$. By the above lemma we can fully characterize the intersection in the limit with a Büchi-automaton $\mathcal{Z}$.

Theorem 5.9. *Let $R = -\frac{1}{5}$. The intersection $\mathcal{K} \cap \{x = R\}$ can be characterized by the Büchi automaton $\mathcal{Z}$ (see Figure 15) , corresponding to the set*

$$\mathcal{K} \cap \left\{x = -\frac{1}{5}\right\} = \{(x, y) \in \mathbb{R} : x = -\frac{1}{5}, y = 0.[d_1 d_2 d_3 d_4 \ldots]_{-4} : d_i \in \{-2, 0, 1, 3\}\}$$

The intersection with the boundary $\partial\mathcal{K}$ are points with $y = 0.[d_1 d_2 d_3 d_4 \ldots]_{16}$ with either $d_i \in \{-14, -12, -8, -6\}$ for all i or $d_i \in \{-1, 3, 9, 11\}$ for all i.

Proof. We can approximate R by

$$R_N := \sum_{i=0}^{N} \left(-\frac{1}{4} + \frac{1}{16}\right) \frac{1}{16^i}. \tag{103}$$

The values R_N are uniquely determined. Every block of four digits is accepted by the proposed automaton and therefore an infinite successfull path corresponds to a point in $\mathcal{K} \cap \{x = R\}$. By Lemma 5.7 there can be no other point in the intersection. The characterization of the strings 0001, 0101, 1010, 1110 before shows the second claim. The only possible paths with 8 steps accepted by the product automaton are $00011010, 00011110, 01011010, 01011110$ from g_3 to g_3 or $10100001, 10100101, 11100001, 11100101$ from g_4 to g_4. These correspond to the proposed digit expansions. □

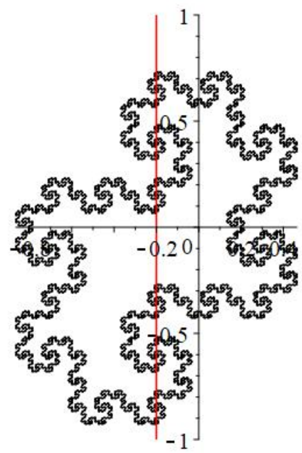

Figure 16: $\mathcal{K}$ and $\{x = -0.2\}$

Theorem 5.10. *The Hausdorff dimension of $\mathcal{K} \cap \{x = -\frac{1}{5}\}$ is 1 and*

$$\dim\left(\partial\mathcal{K} \cap \left\{x = -\frac{1}{5}\right\}\right) = \frac{\log 3}{\log 4} \approx 0.7925. \tag{104}$$

Proof. We can interpret the intersection with $\{x = -\frac{1}{5}\}$ as the self-similar digit tile in $\mathbb{R}$ with $A = -4$ and $D = \{-2, 0, 1, 3\}$. Then $\mathcal{K} \cap \{x = -\frac{1}{5}\}$ has non empty interior and therefor is of dimension 1. As described in Chapter 4 one can calculate the (A, D)-neighbourhood $N = \{0, 1, -1\}$ and the contact matrix C given by:

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

C has the largest eigenvalues $\lambda = 3$ solving $\lambda^2 - 2\lambda - 3 = 0$, A has expansion factor 4 and Theorem 4.24 gives us the proposed formula. □

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