Justification Logic for Constructive Modal Logic

Roman Kuznets*
TU Wien, Austria
roma@logic.at

Sonia Marin†
UCL, United Kingdom
s.marin@ucl.ac.uk

Lutz Straßburger
Inria, France
lutz@lix.polytechnique.fr

Abstract

We provide a treatment of the intuitionistic $\Diamond$ modality in the style of justification logic. We introduce a new type of terms, called satisfiers, that justify consistency, obtain justification analogs for the constructive modal logics CK, CD, CT, and $\mathsf{CS4}$, and prove the realization theorem for them.

1 Introduction

Justification logic is a family of modal logics generalizing the Logic of Proofs LP, introduced by Artemov in [6]. The original motivation, which was inspired by works of Kolmogorov and Gödel in the 1930's, was to give a classical provability semantics

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to intuitionistic propositional logic. Gödel [20] made the first steps by translating intuitionistic logic into the modal logic S4, which he rediscovered as a logic of abstract provability. He noted that S4-provability is incompatible with arithmetical reasoning due to the former's acceptance of the reflection principle and outlined, in an unpublished lecture [21], a potential way of overcoming this obstacle by descending to the level of proofs rather than provability. Artemov independently implemented essentially the same idea in the Logic of Proofs by showing that it provides an operational view of the same type of provability as S4 [6, 7].

The language of the Logic of Proofs can be seen as a modal language where occurrences of the $\Box$ modality are replaced with proof terms, also known as *proof polynomials*, *evidence terms*, or *justification terms*, depending on the setting. The intended meaning of the formula '$t : A$' is '*t is a proof of A*' or, more generally, the reason for the validity of A. Thus, the justification language is viewed as a refinement of the modal language, with one provability construct $\Box$ replaced with an infinite family of specific proofs.

It gradually became clear that the applicability of this result goes way beyond the provability interpretation of the modality, and can be equally well considered in other settings, including, notably, epistemic logic [9]. Indeed, the connection between the Logic of Proofs and the modal logic S4 has been extended to other modal logics (based on classical propositional reasoning), including normal modal sublogics of S4 [13], the modal logic S5 [11], all 15 logics of the so-called modal cube between the minimal normal modal logic K and S5 [22], the infinite family of Geach logics [18], to a certain extent to public announcement logic [14], etc. For more information on justification logic, the reader is referred to the entry [3] in the Stanford Encyclopedia of Philosophy, as well as to two recent books [4, 24] on the subject.

The correspondence between a justification logic and a modal logic means that erasing specific reasons in a valid statement about proofs leads to a valid statement about provability and, vice versa, any valid statement about provability can be viewed as a *forgetful projection* of a valid statement about proofs. Moreover, this existential view of $\Box$ as '*there exists a proof*' leads to a first-order provability reading of modal statements and suggests that they can be Skolemized. Such a Skolemization makes negative occurrences of $\Box$ into Skolem variables and positive occurrences into Skolem functions, suggesting a further restriction on the way the $\Box$ modalities are filled in with proof terms—the process called *realization*—negative occurrences should be filled in with distinct proof variables.

The Logic of Proofs was born out of an analysis of intuitionistic logic with the goal of explaining it using *classical reasoning* about proofs. However, other relationships with intuitionistic logic have also been explored. Artemov introduced the first intuitionistic version ILP of the Logic of Proofs in [8] to unify the semantics of modalities and lambda-calculus. Indeed, as simply typed lambda-calculus is in correspondence with intuitionistic proofs, he needed to define an intuitionistic axiomatization of the Logic of Proofs to relate modal logic S4 and λ-calculus. His axiomatization simply changes the propositional base to intuitionistic while keeping the other axioms of Logic of Proofs unchanged. He shows that ILP is in correspondence with the $\Box$-only fragment of the constructive logic CS4 as defined in [12].[1] Recently, Marti and Studer [26] supplied ILP with possible worlds semantics akin to the semantics developed by Fitting for the classical Logic of Proofs [17] and proved internalized disjunction property in its extension [27].

However, this axiomatization is not enough to obtain a proper intuitionistic arithmetical semantics, that is, to interpret '$t : A$' as '*t is a proof of A in Heyting Arithmetic*,' which is the motivation behind another line of work for considering intuitionistic versions of the Logic of Proofs. In order to obtain an intuitionistic Logic of Proofs complete for Heyting arithmetic, Artemov and Iemhoff [5] added to ILP extra axioms that internalize admissible rules of intuitionistic propositional logic. The arithmetical completeness was later shown by Dashkov [16]. Finally, Steren and Bonelli [31] provide an alternative system of terms for ILP based on natural deduction with hypothetical judgments.

What unifies all these versions of intuitionistic justification logics is the exclusive attention to the provability modality. Be the focus on semantics, realization theorem, or arithmetical completeness, the modal language is restricted to the $\Box$ modality. This restriction was quite natural in the classical setting, where $\Diamond$ can simply be viewed as the dual of $\Box$. However, with the freedom of De Morgan shackled comes the responsibility to treat $\Diamond$ as a fully independent modality—a responsibility that we take upon ourselves in this paper. In this first exploration of the kind of terms necessary to represent the operational side of the intuitionistic $\Diamond$ modality, we concentrate on *constructive versions* of several modal logics.[2]

Building on Artemov's treatment of the $\Box$-only fragment, we add a second type of terms, which we call *satisfier terms*, or simply *satisfiers*, and denote by Greek letters. Thus, a formula $\Diamond A$ is to be realized by '$\mu : A$.' The intuitive understanding of these terms is based on the view of $\Diamond$ modality as representing consistency (with $\Box$ still read as provability). A common way of proving consistency of a theory is to provide a model for this theory. Similarly, to prove that a formula is consistent with the theory, it is sufficient to present a model of the theory satisfying this formula. The satisfier μ

[1] Artemov himself called the logic CS4 "the intuitionistic modal logic on the basis of S4" and denoted it IS4.

[2] The reason for this is pragmatic: we discuss here only fragments which can be expressed in ordinary sequent calculus [35, 29, 12]. The more expressive intuitionistic modal logics require more elaborate sequent structures [32, 30]. We come back to this in the conclusion of this paper.

k_1: $\Box(A \supset B) \supset (\Box A \supset \Box B)$	d: $\Box A \supset \Diamond A$
k_2: $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$	t: $(A \supset \Diamond A) \wedge (\Box A \supset A)$
	$\mathsf{4}$: $(\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A)$

Figure 1: Modal axioms used in this paper

justifying the consistency of a formula is, therefore, viewed as an abstract model satisfying the formula. We keep these satisfying models abstract so as not to rely on any specific semantics. All the operations on satisfiers that we employ to ensure the realization theorem for CK, CD, CT, and CS4, as defined in [35, 29, 12], are akin to the operations on proof terms. In particular, the operation $+$ for proof concatenation finds a counterpart in the operation $\sqcup$ for disjoint model union. Similarly, the application operation $\cdot$, which internalizes *modus ponens* reasoning by creating a new proof $t \cdot s$ for B from a given proof t of $A \supset B$ and a given proof s of A, has a counterpart $\star$ that creates a new *satisfier* $t \star \mu$ for B from a given proof t for $A \supset B$ and a given *satisfier* μ for A. The intuition behind this *satisfier propagation* operation $\star$ is that a proof of $A \supset B$, when applied to a satisfier for A provides evidence that the same model is also a satisfier for B. One could, perhaps, call it an internalized *model ponens*.

Outline of the paper: In Sect. 2, we introduce the syntax and proof theory of some constructive modal logics and, in Sect. 3, we give our definition of a justification logic for constructive modal logics. Then, in Sect. 4, we prove the main theorem of this paper, the realization theorem linking the various constructive modal logics to the corresponding justification logic. Finally, in Sect. 5, we point to further questions left as future work, as this paper is only the beginning of the research program consisting in giving justification logic for constructive and intuitionistic versions of modal logics.

2 Constructive modal logic

Let $a \in \mathcal{A}$ for a countable set of propositional variables $\mathcal{A}$. We define

$$A ::= \bot \mid a \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$$

to be formulas in the modal language and use standard conventions regarding parentheses. We denote formulas by A, B, C, ... and define the negation as $\neg A := A \supset \bot$.

In modal logic, the behavior of the $\Box$ modality is determined by the k-axiom $\Box(A \supset B) \supset \Box A \supset \Box B$ and by the *necessitation rule* saying that, if A is valid, then so is $\Box A$, be the logic classical or intuitionistic. In classical modal logic the behavior of the $\Diamond$ modality is then fully determined by the De Morgan duality, which is violated in the intuitionistic case. This means that more axioms are needed to define the behavior of the $\Diamond$.

However, there is no unique way of doing so, and consequently many different variants of "intuitionistic modal logic" do exist. In this paper we consider the variant that is now called *constructive modal logic* [35, 12, 29, 2] and that is defined by adding to intuitionistic propositional logic the two axiom schemes shown in the left column of Fig. 1 together with the necessitation rule mentioned above. We call this logic CK. We also consider (i) the logic CD, which is CK extended with the d-axiom, (ii) the logic CT which is CK extended with the t-axiom, and (iii) the logic CS4 which is CT extended with the 4-axiom; all three axioms in the right column of Fig. 1.

Logics CK and CS4 are among those that have been studied most extensively. They can be given a possible world semantics by combining the interpretation of classical modal operators with that of intuitionistic implication. That is, a model for CK [28] is a tuple $(W, R, \leq, \models)$ where W is a set of worlds, R is a binary relation on W, $<$ is a preorder on W, and $\models$ is a relation between elements of W and formulas. In particular, in constructive modal logic, there can be *fallible* worlds in W such that $w \models \bot$. In a model for CS4 [1], R is additionaly reflexive and transitive (similarly to the case of classical S4) and the interaction of R and $\leq$ is constrained by the following relationship: $(R \circ \leq) \subseteq (\leq \circ R)$. To our knowledge, contrary to the classical case, the correspondence theory of CD and CT has not been investigated.

These logics have simple sequent calculi that can be obtained from any sequent calculus of intuitionistic propositional logic (IPL) by adding the appropriate rules for the modalities. In this paper, a *sequent* is an expression of the shape $B_1, \ldots, B_n \Rightarrow C$ where $B_1, \ldots, B_n$, and C are formulas and the antecedent to the left of $\Rightarrow$ has to be read as a multiset (i.e., the order of formulas is irrelevant, but it matters how often each formula appears). We use Γ, Δ, Σ, ... to denote such multisets of formulas. For a sequent $B_1, \ldots, B_n \Rightarrow C$ we define its *corresponding formula* $fm(B_1, \ldots, B_n \Rightarrow C)$ to be $B_1 \wedge \cdots \wedge B_n \supset C$. Most sequents in this paper consist of modal formulas. Thus, whenever we use the term "sequent" without any qualification, it is assumed that all formulas in it are modal formulas.

We start from the standard sequent calculus G3ip [33] whose rules are shown in Fig. 2. Then, the systems for the logics CK, CD, CT, and CS4, that we call LCK, LCD, LCT, and LCS4 respectively, are obtained by adding the rules in Fig. 3

$$\mathsf{id}\ \frac{}{\Gamma, a \Rightarrow a} \qquad \bot_\mathsf{L}\ \frac{}{\Gamma, \bot \Rightarrow C}$$

$$\vee_\mathsf{L}\ \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \vee_\mathsf{R}\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \vee_\mathsf{R}\ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\wedge_\mathsf{L}\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \wedge_\mathsf{R}\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\supset_\mathsf{L}\ \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \qquad \supset_\mathsf{R}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Figure 2: Sequent calculus G3ip for intuitionistic propositional logic IPL

$$\mathsf{k}_\Box\ \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Box A} \qquad \mathsf{k}_\Diamond\ \frac{\Gamma, B \Rightarrow A}{\Box\Gamma, \Delta, \Diamond B \Rightarrow \Diamond A} \qquad \mathsf{d}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Diamond A}$$

$$\mathsf{4}_\Box\ \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Box A} \qquad \mathsf{4}_\Diamond\ \frac{\Box\Gamma, B \Rightarrow \Diamond A}{\Box\Gamma, \Delta, \Diamond B \Rightarrow \Diamond A} \qquad \mathsf{t}_\Box\ \frac{\Gamma, \Box A, A \Rightarrow B}{\Gamma, \Box A \Rightarrow B} \qquad \mathsf{t}_\Diamond\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}$$

Figure 3: Additional rules for modalities

according to the following table.[3]

$$\begin{array}{rcl} \mathsf{LCK} & = & \mathsf{G3ip} + \mathsf{k}_\Box + \mathsf{k}_\Diamond \\ \mathsf{LCD} & = & \mathsf{G3ip} + \mathsf{k}_\Box + \mathsf{k}_\Diamond + \mathsf{d} \\ \mathsf{LCT} & = & \mathsf{G3ip} + \mathsf{k}_\Box + \mathsf{k}_\Diamond + \mathsf{t}_\Box + \mathsf{t}_\Diamond \\ \mathsf{LCS4} & = & \mathsf{G3ip} + \mathsf{4}_\Box + \mathsf{4}_\Diamond + \mathsf{t}_\Box + \mathsf{t}_\Diamond \end{array} \tag{1}$$

Observe that the axiom rule id is restricted to atomic formulas. We rely on that in the proof of the realization theorem in Sect. 4. However, as expected, using the standard argument by induction on the formula construction, the general form of the axiom rule is derivable.

Lemma 2.1 (Generalized axioms)**.** *For every formula A, the rule* $\mathsf{id}_g\ \frac{}{\Gamma, A \Rightarrow A}$ *is derivable in each of* G3ip*,* LCK*,* LCD*,* LCT*, and* LCS4*.*

[3]For a survey of the classical variants of these systems, see, for example, [34].

Finally, the rule $\mathsf{cut}\,\dfrac{\Gamma \Rightarrow A \qquad \Delta, A \Rightarrow C}{\Gamma, \Delta \Rightarrow C}$ is admissible.

Theorem 2.2 (Cut Admissibility)**.** *Let* $\mathsf{LML} \in \{\mathsf{LCK}, \mathsf{LCD}, \mathsf{LCT}, \mathsf{LCS4}\}$*. If a sequent is provable in* $\mathsf{LML} + \mathsf{cut}$ *then it is also provable in* LML*.*

Proof. For LCK, LCD, and LCT, the proof follows as a special case of [25], and for $\mathsf{CS4}$ the result is stated in [12] as a "routine adaptation of Gentzen's method." □

Using Theorem 2.2, we can easily show the completeness of our system.

Theorem 2.3 (Completeness)**.** *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$ *and* LML *be the corresponding sequent system. If* $\vdash_{\mathsf{ML}} A$*, then* $\vdash_{\mathsf{LML}} \Rightarrow A$*.*

Proof. The axioms of IPL can be proved using $\mathsf{G3ip}$ in Fig. 2; those in Fig. 1 can be proved using the corresponding rules in Fig. 3. Finally, the necessitation rule can be simulated with $\mathsf{k}_\Box$, and *modus ponens* can be simulated using cut. Now completeness of the cut-free systems follows immediately from Theorem 2.2. □

Theorem 2.4 (Soundness)**.** *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$*. If* $B_1, \ldots, B_n \Rightarrow C$ *is a sequent provable in the corresponding sequent system* LML*, then* $B_1 \wedge \cdots \wedge B_n \supset C$ *is a theorem of* ML*.*

Proof. We proceed by induction on the proof π of $B_1, \ldots, B_n \Rightarrow C$ in LML, making a case analysis on the bottom-most rule instance in π. For the rules in $\mathsf{G3ip}$, this is straightforward. Now consider the rule $\mathsf{k}_\Box\,\dfrac{C_1, \ldots, C_n \Rightarrow A}{\Box C_1, \ldots, \Box C_n, D_1, \ldots, D_m \Rightarrow \Box A}$. By induction hypothesis, $\vdash_{\mathsf{ML}} C_1 \wedge \cdots \wedge C_n \supset A$, hence by intuitionistic reasoning, $\vdash_{\mathsf{ML}} C_1 \supset \cdots \supset C_n \supset A$.[4] By necessitation, $\vdash_{\mathsf{ML}} \Box(C_1 \supset \cdots \supset C_n \supset A)$, and, using $\mathsf{k_1}$ and *modus ponens*, we get $\vdash_{\mathsf{ML}} \Box C_1 \supset \cdots \supset \Box C_n \supset \Box A$. Hence, $\vdash_{\mathsf{ML}} \Box C_1 \wedge \cdots \wedge \Box C_n \wedge D_1 \wedge \cdots \wedge D_m \supset \Box A$ follows by intuitionistic reasoning. Other cases are similar. □

3 Justification logic

Justification logic adds *proof terms* directly inside its language using formulas '$t : A$' with the meaning 't *is a proof of* A.' In the constructive version that we propose in this section, we will also add *satisfiers* into the language, using formulas '$\mu : A$' with the underlying intuition that 'μ *is a model of* A.'

[4]Throughout the paper we consider $\supset$ to be right-associative.

Proof terms, intended to replace $\Box$, are denoted $t, s, \ldots$, while *satisfiers*, intended to replace $\Diamond$, are denoted $\mu, \nu, \ldots$ Proof terms are built from a set of *proof variables*, denoted $x, y, \ldots$, and a set of (*proof*) *constants*, denoted $c, d, \ldots$, using the operations *application* $\cdot$, *sum* $+$, and *proof checker* $!$. Satisfiers are built from a set of *satisfier variables*, denoted $\alpha, \beta, \ldots$, using the operations *disjoint union* $\sqcup$ (binary operation on satisfiers) and *propagation* $\star$ (combines a proof term with a satisfier).

$$\begin{array}{rccccccccc} t ::= & c & \mid & x & \mid & (t \cdot t) & \mid & (t + t) & \mid & !\,t \\ \mu ::= & & & \alpha & \mid & (t \star \mu) & \mid & (\mu \sqcup \mu) & & \end{array}$$

While the intuitive meaning of the operations $\cdot$, $+$, and $!$ on proof terms has been well documented in justification logic literature and corresponds to rather well known proof manipulations, it is worth explaining our intuition behind the new operations $\star$ and $\sqcup$ involving satisfiers.

The operation $\star$ is a combination of global and local reasoning. For instance, assume that $\neg\neg A$ is true; therefore, by classical propositional logic, A must be true. Here $\neg\neg A$ being true is a local, contingent fact, whereas the transition is made based on the classical tautology $\neg\neg A \supset A$. The result is the contingent truth of A in the same situation where $\neg\neg A$ is true. We are working in a language with explicit proofs for valid statements and explicit satisfiers representing specific models satisfying a statement. Thus, given a satisfier μ for A and a proof t that generally $A \supset B$, we can conclude B. While B is true whenever A is, the justification used is different in that the former involves a valid transition from A to B justified by t. Hence, instead of using the same satisfier μ, we record our reasoning in the new satisfier $t \star \mu$. For instance, if satisfiers are in principle intended to range over intuitionistic Kripke models, then $x : (\neg\neg A \supset A)$ becomes a non-trivial assumption on whether only classical models are considered. Hence, the truth of A depends not only on the truth of $\neg\neg A$ in a model represented by the satisfier μ but also on the validity of the law of double negation.

The operation $\sqcup$ of disjoint model union is akin to that of disjoint set union. For instance, for sets, one often defines $X \sqcup Y = \big(X \times \{0\}\big) \cup \big(Y \times \{1\}\big)$ in order to avoid potential problems of X overlapping with Y and be able to state facts such as $|X \sqcup Y| = |X| + |Y|$. Intuitively, our disjoint model union works the same way. Whatever the nature of models represented by satisfiers μ and ν, any overlaps among them are resolved before the models are combined and no connection between the μ and ν parts of the satisfier $\mu \sqcup \nu$ exists. For instance, the disjoint union of intuitionistic Kripke models $\mathcal{M}_1 = (W_1, \leq_1, V_1)$ and $\mathcal{M}_2 = (W_2, \leq_2, V_2)$ can be defined as follows: $\mathcal{M}_1 \sqcup \mathcal{M}_2 := (W, \leq, V)$ where $W := \big(W_1 \times \{0\}\big) \cup \big(W_2 \times \{1\}\big)$, $(w, i) \leq (w', j)$ iff $i = j$ and $w \leq_i w'$, and $V\big((w, i)\big) := V_i(w)$.

taut:	Complete finite set of axioms for IPL
$\mathsf{jk}_\Box$:	$t:(A \supset B) \supset (s:A \supset t \cdot s:B)$
$\mathsf{jk}_\Diamond$:	$t:(A \supset B) \supset (\mu:A \supset t \star \mu:B)$
sum:	$s:A \supset (s+t):A$ and $t:A \supset (s+t):A$
union:	$\mu:A \supset (\mu \sqcup \nu):A$ and $\nu:A \supset (\mu \sqcup \nu):A$

$$\mathsf{mp}\,\frac{A \supset B \quad B}{B} \qquad \mathsf{ian}\,\frac{A \text{ is an axiom instance}}{c_1:\ldots c_n:A}$$

Figure 4: Axiomatization of the constructive justification logic JCK

$\mathsf{jd}_\Box$:	$t:\bot \supset \bot$	$\mathsf{jd}_\Diamond$:	$\top \supset \mu:\top$
$\mathsf{jt}_\Box$:	$t:A \supset A$	$\mathsf{jt}_\Diamond$:	$A \supset \mu:A$
$\mathsf{j4}_\Box$:	$t:A \supset !t:t:A$	$\mathsf{j4}_\Diamond$:	$\mu:\nu:A \supset \nu:A$

Figure 5: Additional justification axioms

The formulas of justification logic are obtained from the following grammar:

$$A ::= \bot \mid a \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid t:A \mid \mu:A$$

We propose to extend the formulation of justification logics to realize constructive modal logics. The axiomatization of the basic one is shown in Fig. 4. It is similar to the standard justification counterpart of the classical modal logic K except for the additional axiom $\mathsf{jk}_\Diamond$, which corresponds to the modal axiom k_2. The other axioms taut, $\mathsf{jk}_\Box$, and sum, as well as the rules of *modus ponens* mp and *iterated axiom necessitation* ian are standard, e.g., see [22]. We call this basic logic JCK, and as in the classical setting, we can define extension of JCK using the axioms defined in Fig. 5. The logic JCD is obtained from JCK by adding the axioms $\mathsf{jd}_\Box$ and $\mathsf{jd}_\Diamond$; the logic JCT is obtained from JCK by adding the axioms $\mathsf{jt}_\Box$ and $\mathsf{jt}_\Diamond$; and the logic JCS4 is obtained by adding the axioms $\mathsf{j4}_\Box$ and $\mathsf{j4}_\Diamond$ to JCT. Note that the $\Box$ variant of each axiom corresponds exactly to the one used in the classical setting. Our contribution is the definition of the $\Diamond$ variants operating on the satisfiers instead of the proof terms.

The intuitive reading of these new satisfier axioms is as follows. The axiom $\mathsf{jd}_\Diamond$ states that $\top$ is satisfied in every model. The axiom $\mathsf{jt}_\Diamond$ could be understood as the insistence that the actual model must be part of any other model considered: if A is true, then it is satisfied in every model. Perhaps, the least intuitive is the axiom $\mathsf{j4}_\Diamond$. One way of reading it is to say that truth in models is "context-free." The fact of A being satisfied in a model represented by ν does not depend on ν being considered within the context of another model represented by μ. Put another way, any submodel ν of μ can also be considered in isolation and produces the same truth values.

The logics JCK, JCD, JCT, and $\mathsf{JCS4}$ can be seen as the operational version of the constructive modal logics CK, CD, CT, and $\mathsf{CS4}$ respectively, defined in the previous section. Indeed if one forgets about the proof term and satisfier annotations and considers them as empty $\Box$ and $\Diamond$ respectively, the logics prove the same theorems.

Definition 3.1. We define the operation of *forgetful projection* $(\cdot)^\circ$ that maps justification formulas onto corresponding modal formulas recursively: $\bot^\circ := \bot$, $a^\circ := a$ for all propositional variables a, $(t : A)^\circ := \Box A^\circ$, $(\mu : A)^\circ := \Diamond A^\circ$, and for $* \in \{\wedge, \vee, \supset\}$, finally, $(A * B)^\circ := A^\circ * B^\circ$.

We extend this definition to multisets of formulas: $(A_1, \ldots, A_n)^\circ := A_1^\circ, \ldots, A_n^\circ$.

It is easy to show by induction on the Hilbert derivation in JL that

Lemma 3.2 (Forgetful projection)**.** *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$ *and* ML *be the corresponding modal logic. If* $\vdash_{\mathsf{JL}} F$*, then* $\vdash_{\mathsf{ML}} F^\circ$*.*

The more difficult question however is: can we prove the converse? This result is called realization, namely that every theorem of a certain modal logic can be '*realized*' by a justification theorem. However, it is not such an easy result as it may seem. It is not possible to directly transform a Hilbert proof of a modal theorem into a Hilbert proof of its realization in justification logic as the rule mp in a Hilbert system can create dependencies between modalities. The standard solution to this issue is to consider a proof of the modal theorem in a *cut-free* sequent calculus as the absence of cuts in the proof will prevent the creation of dependencies. The detailed statement and proof of this result can only be presented in the next section, as we have to introduce some basics first.

We state below two lemmas that are crucial for the realization proof: the *Lifting Lemma* and the *Substitution Property.* They are extensions of standard results from the justification logics literature to the constructive case. Repeating verbatim the proof from [7], we obtain the *Lifting Lemma* and its variant showing that necessitation can be internalized within the language of these justification logics.

Lemma 3.3 (Lifting Lemma)**.** *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$*. If*

$$A_1, \ldots, A_n \vdash_{\mathsf{JL}} B,$$

then there exists a proof term $t(x_1, \ldots, x_n)$ *such that for all proof terms* $s_1, \ldots, s_n$

$$s_1 : A_1, \ldots, s_n : A_n \vdash_{\mathsf{JL}} t(s_1, \ldots, s_n) : B.$$

Corollary 3.4. *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$*. If* $\vdash_{\mathsf{JL}} A_1 \wedge \cdots \wedge A_n \supset B$*, then there exists a proof term* $t(x_1, \ldots, x_n)$ *such that for all proof terms* $s_1, \ldots, s_n$ *we have* $\vdash_{\mathsf{JL}} s_1 : A_1 \wedge \cdots \wedge s_n : A_n \supset t(s_1, \ldots, s_n) : B$*.*

In our constructive setting, we also need a $\Diamond$ variant of this statement.

Corollary 3.5. *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$*. If*

$$\vdash_{\mathsf{JL}} A_1 \wedge \cdots \wedge A_n \wedge C \supset B,$$

then there is a satisfier $\mu(x_1, \ldots, x_n, \beta)$ *such that for all proof terms* $s_1, \ldots, s_n$ *and any satisfier* ν

$$\vdash_{\mathsf{JL}} s_1 : A_1 \wedge \cdots \wedge s_n : A_n \wedge \nu : C \supset \mu(s_1, \ldots, s_n, \nu) : B. \tag{2}$$

Proof. By intuitionistic reasoning and Cor. 3.4, we get a proof term $t(x_1, \ldots, x_n)$ such that

$$\vdash_{\mathsf{JL}} s_1 : A_1 \wedge \cdots \wedge s_n : A_n \supset t(s_1, \ldots, s_n) : (C \supset B).$$

Using the instance $t(s_1, \ldots, s_n) : (C \supset B) \supset \nu : C \supset (t(s_1, \ldots, s_n) \star \nu) : B$ of the axiom $\mathsf{jk}_\Diamond$, we can see that (2) holds for $\mu(x_1, \ldots, x_n, \beta) := t(x_1, \ldots, x_n) \star \beta$. $\square$

Finally, we generalize the standard definition of substitution to our setting.

Definition 3.6. A *substitution* σ maps proof variables to proof terms and satisfier variables to satisfiers. The application of a substitution σ to a proof term t or satisfier μ, denoted $t\sigma$ or $\mu\sigma$ respectively, is defined recursively as follows:

$$\begin{aligned} c\sigma &:= c & x\sigma &:= \sigma(x) \\ (t \cdot s)\sigma &:= t\sigma \cdot s\sigma & (t + s)\sigma &:= t\sigma + s\sigma \\ (!\,t)\sigma &:= \,!(t\sigma) & \alpha\sigma &:= \sigma(\alpha) \\ (t \star \mu)\sigma &:= t\sigma \star \mu\sigma & (\mu \sqcup \nu)\sigma &:= \mu\sigma \sqcup \nu\sigma \end{aligned}$$

where c is a proof constant, x is a proof variable, and α is a satisfier variable. The application of σ to a justification formula A yields the formula $A\sigma$, where each proof term t (respectively satisfier μ) appearing in A is replaced with $t\sigma$ (respectively $\mu\sigma$).

The proof of the Substitution Property from [7] is easily adaptable to our case:

Lemma 3.7 (Substitution Property)**.** *Let* $\mathsf{JL} \in \{\mathsf{JCK}, \mathsf{JCD}, \mathsf{JCT}, \mathsf{JCS4}\}$*. If* $\vdash_{\mathsf{JL}} A$*, then* $\vdash_{\mathsf{JL}} A\sigma$ *for any substitution* σ*.*

Remark 3.8. In our formulation, the Substitution Property holds because the rule ian is formulated in its strongest form, with all proof constants being interchangeable. Combined with the schematic formulation of all axioms, this makes derivations impervious to substitutions. A more nuanced formulation would be to restrict ian to a specific set of instances collected in a *constant specification* (our variant corresponds to the *total constant specification*). It is a standard fact in justification logic that the substitution property only holds for *schematic* constant specifications, i.e., those invariant with respect to substitutions. The only difference for our logics is that a schematic constant specification must additionally be schematic with respect to substitutions of satisfiers for satisfier variables.

4 Realization theorem for constructive modal logic

Assume we have a justification formula F and its forgetful projection F°. In that case we call F a *realization* of F°. Similarly, a *justification sequent* $\Gamma \Rightarrow C$, that is, a sequent consisting of justification formulas, can be the *realization* of a modal sequent $\Gamma^\circ \Rightarrow C^\circ$. In order to define the notion of *normal realization* we need the notions of positive and negative occurrences of subformulas.

An occurrence of a subformula A of F is called *positive* if the position of A in the syntactic tree of F is reached from the root by following the left branch of an $\supset$ branching an even number of times; otherwise it is called *negative*. For example, the displayed subformula A is positive in the formula $(A \supset B) \supset C$ but negative in the formula $A \supset (B \supset C)$. The *polarity* of the occurrence of a subformula in a sequent $\Gamma \Rightarrow C$ is given by its polarity in the formula $\mathit{fm}(\Gamma \Rightarrow C)$.

Definition 4.1. A realization $\Gamma \Rightarrow C$ of $\Gamma^\circ \Rightarrow C^\circ$ is called *normal* if the following condition is fulfilled: if $t : A$ (respectively $\mu : A$) is a negative subformula occurrence of $\Gamma \Rightarrow C$, then t is a proof variable (respectively μ is a satisfier variable) that occurs in $\Gamma \Rightarrow C$ exactly once.

We can now state and prove the main theorem of this paper.

Theorem 4.2 (Realization)**.** *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$*,* JL *be the corresponding justification logic, i.e.,* JCK*,* JCD*,* JCT*, or* $\mathsf{JCS4}$ *respectively, and* LML *be the cut-free sequent calculus for* ML*. If* $\vdash_{\mathsf{LML}} \Gamma' \Rightarrow C'$ *for a given modal sequents, then there is a normal realization* $\Gamma \Rightarrow C$ *of* $\Gamma' \Rightarrow C'$ *such that* $\vdash_{\mathsf{JL}} \mathit{fm}(\Gamma \Rightarrow C)$*.*

Corollary 4.3. *Let* $\mathsf{ML} \in \{\mathsf{CK}, \mathsf{CD}, \mathsf{CT}, \mathsf{CS4}\}$ *and* JL *be the corresponding justification logic. If* $\vdash_{\mathsf{ML}} A$*, then* $\vdash_{\mathsf{JL}} F$ *for some justification formula* F *such that* $F^\circ = A$*.*

Proof of Theorem 4.2. The proof goes largely along the lines of that for the $\Box$-only classical fragment (see [7, 13]). The operation $\sqcup$ on satisfiers plays the same role as the operation $+$ on proof terms. Thus, we only show in detail cases for the new rules. As a matter of a shorthand, we say that a justification sequent $\Gamma \Rightarrow C$ is *derivable in* JL if its corresponding formula is, i.e., if $\vdash_{\mathsf{JL}} fm(\Gamma \Rightarrow C)$.

Let π be an LML proof of $\Gamma' \Rightarrow C'$. We assign a unique index $i \in \{1, \dots, n\}$ to each of n occurrences of $\Box$ and $\Diamond$ in its endsequent $\Gamma' \Rightarrow C'$. We define the *modal flow graph* of π, denoted G_π, as follows: its vertices are all occurrences of formulas of the form $\Box A$ and $\Diamond A$ in π. Two such occurrences are connected with an edge iff they are occurrences of the same formula within the same rule instance and

- either one occurs within a side formula in a premise and the other is the same occurrence within the same subformula in the conclusion
- or one occurs within an active formula in a premise and the other is the corresponding occurrence within the principal formula in the conclusion.

Each connected component of G_π has exactly one vertex in the endsequent of π and all vertices in the connected component are assigned the same index as this representative in the endsequent. E.g., in the following instance of $\mathsf{k}_\Box$, modalities connected by edges are vertically aligned and given the same index:

$$\mathsf{k}_\Box \frac{\Box_5 a \lor \Diamond_7 b\ ,\qquad c \supset d \qquad\qquad\qquad \Rightarrow \qquad \Diamond_9 g \supset \Box_8 h}{\Box_2(\Box_5 a \lor \Diamond_7 b), \Box_6(c \supset d), \Box_3 e, \Box_{10}\Diamond_{20} f \Rightarrow \Box_{15}(\Diamond_9 g \supset \Box_8 h)} \tag{3}$$

In the absence of the cut rule, the resulting graph is a forest where each tree has its root in the endsequent and is identified with a unique modality type $\heartsuit$ and unique index i. We denote it a $\heartsuit_i$*-tree*. Branching occurs in the branching rules, as well as in the rules with embedded contraction, e.g., in $\mathsf{t}_\Box$ each modality in A within $\Box A$ in the conclusion of the rule branches to the corresponding occurrence in A and the corresponding occurrence in $\Box A$ in the premise. Each leaf of a $\heartsuit_i$-tree is either in a side formula of an axiom id or $\bot_\mathsf{L}$, in which case it is called an *initial leaf*, or in the conclusion of a modal rule from Fig. 3 that introduced $\heartsuit_i$, in which case it is called a *modal leaf*. For instance, if (3) is used in π, then the $\Box_2$-, $\Box_6$-, $\Box_3$-, $\Box_{10}$-, $\Diamond_{20}$-, and $\Box_{15}$-trees in G_π have modal leaves in the conclusion of (3).

We call the number of modal leaves of a $\heartsuit_i$-tree occurring in the succedents of modal rules the *multiplicity of* i, denoted by m_i, which is a non-negative integer.

From the tree π of modal sequents, we construct another tree π_0 of justification sequents by replacing

each $\Box_i$ for $m_i > 0$ with $z_i := y_{i,1} + \cdots + y_{i,m_i}$ for proof variables $y_{i,1}, \dots, y_{i,m_i}$;

each $\Box_i$ for $m_i = 0$ with $z_i := y_{i,0}$ for a proof variable $y_{i,0}$;

each $\Diamond_i$ for $m_i > 0$ with $\omega_i := \beta_{i,1} \sqcup \cdots \sqcup \beta_{i,m_i}$ for satisfier variables $\beta_{i,1}, \ldots, \beta_{i,m_i}$;

each $\Diamond_i$ for $m_i = 0$ with $\omega_i := \beta_{i,0}$ for a satisfier variable $\beta_{i,0}$.

All proof variables $y_{i,j}$ and all satisfier variables $\beta_{i,j}$ must be pairwise distinct.

Let us call a rule *justificational* if it is one of $\mathsf{k}_\Box$, $\mathsf{k}_\Diamond$, d, or $\mathsf{4}_\Box$. All other rules, including the rules $\mathsf{4}_\Diamond$, $\mathsf{t}_\Box$, and $\mathsf{t}_\Diamond$, as well as the rules in Fig. 2 are *simple*. Let k be the number of instances of justificational rules in π. We will construct a sequence of substitutions $\sigma_1, \ldots, \sigma_k$ that, when applied to π_0, produces a sequence $\pi_1, \ldots, \pi_k$ of trees such that $\pi_{h+1} = \pi_h \sigma_{h+1}$. Note that for any justification sequent in the tree π_h, its forgetful projection is the modal sequent from the corresponding node of the tree π and that every occurrence of $\Box_i$ or $\Diamond_i$ in π is replaced in π_h with $z_i \sigma_1 \ldots \sigma_h$ or $\omega_i \sigma_1 \ldots \sigma_h$ respectfully. For $\tau_h := \sigma_h \circ \cdots \circ \sigma_1$ let us call $z_i \tau_h$ and $\omega_i \tau_h$ the *h-prerealizations* of $\Box_i$, and $\Diamond_i$ respectively.[5] For any sequent occurrence $\Delta \Rightarrow D$ in π, we call $(\Delta \Rightarrow D)\tau_h$ its h-prerealization and denote it $\Delta_h \Rightarrow D_h$.

Let the k justificational rules be ordered linearly in a way consistent with the tree order of π: for arbitrary $k \geq j > i \geq 1$, the jth rule is not inside a subtree rooted at the premise of the ith rule. By induction on $i = 0, \ldots, k$ we will show that,

1. for any subtree of π_i with no occurrences of modal rules $i+1, \ldots, k$ the endsequent $\Delta_i \Rightarrow D_i$ of this subtree is derivable in JL, i.e., h-prerealizations of a sequent occurrence $\Delta \Rightarrow D$ from π become derivable as soon as h overtakes the numbers of all justificational rules used to derive the sequent in π;

2. $y_{i,0}\tau_h = y_{i,0}$ and $\beta_{i,0}\tau_h = \beta_{i,0}$ for all justificational rules above $h = 1, \ldots, k$, i.e., terms prerealizing modalities not contributing to m_i remain fixed points for all substitutions.

In particular, after all justificational rules are processed in π_k, the k-prerealization $\Gamma_k \Rightarrow C_k$ of the endsequent of π will be derivable in JL making it a realization. Moreover, since no negative occurrence of a modality from the endsequent can be traced to a leaf in a succedent of a sequent from π, in this realization all such negative modalities are realized by proof and satisfier variables. We prove it by a secondary induction on the depth of the proof up to the first unprocessed justificational rule.

For a simple rule, the JL-derivability of the i-prerealization of its premise(s) implies the JL-derivability of the i-prerealization of its conclusion. For the rules from Fig. 2 the reasoning is propositional. For rules $\mathsf{t}_\Box$ and $\mathsf{t}_\Diamond$ (applicable only to LCT

[5]The term prerealization is used here in its layman's meaning of an almost but not quite a realization and is unrelated to the use in [22].

and $\mathsf{LCS4}$), it follows by axioms $\mathsf{jt}_\Box$ and $\mathsf{jt}_\Diamond$ of JCT and $\mathsf{JCS4}$. For $\mathsf{LCS4}$, assume that for a rule instance $\mathsf{4}_\Diamond \dfrac{\Box_{k_1}G^1,\dots,\Box_{k_r}G^r,B \Rightarrow \Diamond_j A}{\Box_{k_1}G^1,\dots,\Box_{k_r}G^r,D^1,\dots,D^p,\Diamond_l B \Rightarrow \Diamond_j A}$ from π the i-prerealization of the premise is derivable in JL, i.e.,

$$\vdash_{\mathsf{JL}} z_{k_1} : G^1_i \wedge \dots \wedge z_{k_r} : G^r_i \wedge B_i \supset \omega_j \tau_i : A_i.$$

By Cor. 3.5, there is a satisfier $\mu(x_1,\dots,x_r,\beta)$ such that

$$\vdash_{\mathsf{JL}} !\,z_{k_1} : z_{k_1} : G^1_i \wedge \dots \wedge !\,z_{k_r} : z_{k_r} : G^r_i \wedge \omega_l : B_i \supset \mu(!\,z_{k_1},\dots,!\,z_{k_r},\omega_l) : \omega_j \tau_i : A_i \ .$$

It now follows by $\mathsf{j4}_\Box$ and $\mathsf{j4}_\Diamond$ of $\mathsf{JCS4}$ and propositional reasoning that

$$\vdash_{\mathsf{JL}} z_{k_1} : G^1_i \wedge \dots \wedge z_{k_r} : G^r_i \wedge D^1_i \wedge \dots \wedge D^p_i \wedge \omega_l : B_i \supset \omega_j \tau_i : A_i$$

making the i-prerealization of the conclusion of the rule derivable in JL.

This observation alone establishes the base of the main induction, i.e., that all 0-prerealizations of modal sequents derived without the use of justificational rules are derivable in JL.

For the step of the main induction, consider the premise of the hth justificational rule and assume its $(h-1)$-prerealization is derivable by IH. For each of the justificational rules we will show how to apply an additional substitution to make its conclusion derivable. By the Substitution Property (Lemma 3.7), this substitution preserves the derivability of all h-prerealizations of modal sequents whose $(h-1)$-prerealizations are derivable by the IH, including the premise of the hth justificational rule. Thus, the h-prerealization of its conclusion is also derivable and the argument about simple rules can be applied to extend this result down until the next justificational rule. The cases of the $\mathsf{k}_\Box$ and $\mathsf{4}_\Box$ rules are treated the same way as in [13] by means of Cor. 3.4. It remains to process the two remaining justificational rules.

We start with the case where the hth rule in π is the qth introduction of $\Diamond_j$ by a justificational rule out of m_j: $\mathsf{k}_\Diamond \dfrac{G^1,\dots,G^r,B \Rightarrow A}{\Box_{k_1}G^1,\dots,\Box_{k_r}G^r,D^1,\dots,D^p,\Diamond_l B \Rightarrow \Diamond_j A}$. Assume that the $(h-1)$-prerealization of the premise is derivable in JL, i.e.,

$$\vdash_{\mathsf{JL}} G^1_{h-1} \wedge \dots \wedge G^r_{h-1} \wedge B_{h-1} \supset A_{h-1} \ . \tag{4}$$

By Cor. 3.5 there is a satisfier $\mu(x_1,\dots,x_r,\beta)$ such that

$$\vdash_{\mathsf{JL}} z_{k_1} : G^1_{h-1} \wedge \dots \wedge z_{k_r} : G^r_{h-1} \wedge \omega_l : B_{h-1} \supset \mu(z_{k_1},\dots,z_{k_r},\omega_l) : A_{h-1}.$$

We define $\sigma_h \colon \beta_{j,q} \mapsto \mu(z_{k_1}, \dots, z_{k_r}, \omega_l)$. Note that σ_h affects exactly one satisfier variable, which is neither $y_{i,0}$ nor $\beta_{i,0}$ and which corresponds to the justificational rule being processed. In particular, $\beta_{j,q}\tau_{h-1} = \beta_{j,q}$ and $\beta_{j,q}\tau_h = \beta_{j,q}\sigma_h$. Thus,

$$\vdash_{\mathsf{JL}} z_{k_1} : G^1_{h-1} \wedge \dots \wedge z_{k_r} : G^r_{h-1} \wedge \omega_l : B_{h-1} \supset \beta_{j,q}\tau_h : A_{h-1}.$$

Applying σ_h substitution, we obtain by the Substitution Property,

$$\vdash_{\mathsf{JL}} z_{k_1} : (G^1_{h-1}\sigma_h) \wedge \dots \wedge z_{k_r} : (G^r_{h-1}\sigma_h) \wedge \omega_l : (B_{h-1}\sigma_h) \supset \beta_{j,q}\tau_h : (A_{h-1}\sigma_h)$$

because (a) σ_h does not affect the proof variables $z_{k_1}, \dots, z_{k_r}$, (b) σ_h does not affect the satisfier variable $\omega_l \neq \beta_{j,q}$ because j and l are indices of diamonds of opposite polarity, and (c) σ_h does not affect the satisfier $\beta_{j,q}\tau_h = \mu(z_{k_1}, \dots, z_{k_r}, \omega_l)$ because the only variables occurring in it are $z_{k_1}, \dots, z_{k_r}$, and ω_l. It follows by union that

$$\vdash_{\mathsf{JL}} z_{k_1} : G^1_h \wedge \dots \wedge z_{k_r} : G^r_h \wedge D^1_h \wedge \dots \wedge D^p_h \wedge \omega_l : B_h \supset \omega_j\tau_h : A_h$$

where $\omega_j = \beta_{j,1} \sqcup \dots \sqcup \beta_{j,q} \sqcup \dots \sqcup \beta_{j,m_j}$. Thus, the h-realization of the conclusion is also derivable in JL.

The case of the rule $\mathsf{d}\,\dfrac{G^1, \dots, G^r \Rightarrow A}{\Box_{k_1}G^1, \dots, \Box_{k_r}G^r, D^1, \dots, D^p, \Rightarrow \Diamond_j A}$ for LCD is similar. By the IH, (4) holds for $B_{h-1} = \top$. Repeating all the steps for $\mathsf{k}_\Diamond$ and using a fresh satisfier variable β in place of ω_l for $\Diamond\top$, we obtain

$$\vdash_{\mathsf{JL}} z_{k_1} : G^1_h \wedge \dots \wedge z_{k_r} : G^r_h \wedge D^1_h \wedge \dots \wedge D^p_h \wedge \beta : \top \supset \omega_j\tau_h : A_h \quad .$$

It remains to note that $\vdash_{\mathsf{JL}} \beta : \top$ by axiom $\mathsf{jd}_\Diamond$ of JCD. It follows that

$$\vdash_{\mathsf{JL}} z_{k_1} : G^1_h \wedge \dots \wedge z_{k_r} : G^r_h \wedge D^1_h \wedge \dots \wedge D^p_h \supset \omega_j\tau_h : A_h \quad . \qquad \Box$$

The crucial difference between justificational and simple rules is that, unlike the former, the latter require an additional substitution on top of all the previous ones.

5 Conclusion and future work

In this paper, we proposed justification counterparts for some constructive modal logics, which, for the first time, employ the notion of satisfiers to realize the $\Diamond$-modality. This led us to define an operator combining proof terms and satisfiers, which is crucial to the realization of the constructive modal axiom k_2. However, surprisingly, the only other operation needed on satisfiers is the disjoint union, an equivalent to the sum for proof terms. In particular, while the $\Box$-version of the $\mathsf{4}$-axiom traditionally requires the proof checker operator !, the $\Diamond$-version of axiom $\mathsf{4}$ does not seem to necessitate any additional operation on satisfiers. In the following, we list a handful of directions for future work:

$$\mathsf{k4}_\Box\ \frac{\Box\Gamma, \Gamma \Rightarrow A}{\Delta, \Box\Gamma \Rightarrow \Box A} \qquad \mathsf{k4}_\Diamond\ \frac{\Box\Gamma, \Gamma, B \Rightarrow A}{\Delta, \Box\Gamma, \Diamond B \Rightarrow \Diamond A} \qquad \mathsf{k4}'_\Diamond\ \frac{\Box\Gamma, \Gamma, B \Rightarrow \Diamond A}{\Delta, \Box\Gamma, \Diamond B \Rightarrow \Diamond A}$$

Figure 6: More rules for modalities

- Semantics of our proposed logics. Modular models from [10, 23] should provide a good starting point, but require significant adjustments.

- We have chosen to work with the logics that have simple, known cut-free sequent calculi, a property on which the realization proof strongly relies. The same method can be further extended to $\mathsf{CK4}$ and $\mathsf{CD4}$ that are obtained from CK and CD, respectively, by adding the $\mathsf{4}$-axiom. To our knowledge, these logics have not been independently studied, but it is possible to 'constructivize' the classical rule $\mathsf{k4}_\Box$ in the same way as for the rules in Fig. 3. That is, corresponding sequent systems to $\mathsf{CK4}$ and $\mathsf{CD4}$ may be obtained via the rules in Fig. 6:

$$\begin{aligned} \mathsf{LCK4} &= \mathsf{G3ip} + \mathsf{k4}_\Box + \mathsf{k4}_\Diamond + \mathsf{k4}'_\Diamond \\ \mathsf{LCD4} &= \mathsf{G3ip} + \mathsf{k4}_\Box + \mathsf{k4}_\Diamond + \mathsf{k4}'_\Diamond + \mathsf{d} \end{aligned}$$

  We decided to forgo this extension for pragmatic reasons: without a cut-free calculi for these constructive modal logics in the literature we would need to provide a full cut-elimination proof. Even though it should be possible to directly adapt for example the proof from [25], it would have changed the focus of this paper.

- There exist other, more elaborate realization proofs, e.g., from [19], that provide realizations with additional properties and/or structure. Applying them to modal logics with non-classical propositional basis remains future work.

- We believe that our way of justifying the $\Diamond$ modality would similarly work for the "intuitionistic variant" of modal logic [30], which is obtained from the constructive variant by adding the three axioms $\mathsf{k}_3\colon \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$ and $\mathsf{k}_4\colon (\Diamond A \supset \Box B) \supset \Box(A \supset B)$ and $\mathsf{k}_5\colon \Diamond\bot \supset \bot$. There are no ordinary sequent calculi for such logics, so the proof of realization provided here could not be straightforwardly adapted. However, there are nested sequent calculi for all logics in the *intuitionistic S5-cube* [32], even in a focused variant [15], which means that we might still be able to prove a realization theorem by extending the method used in [22].

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