TECHNISCHE UNIVERSITÄT WIEN

JOHANNES KEPLER UNIVERSITÄT LINZ

On equationally additive clones

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Background

Topic: Universal algebraic geometry

- 1997: B. Plotkin: *Some concepts of algebraic geometry in univ. alg.*
- 1999: G. Baumslag, A. Mjasnikov, V. Remeslennikov: *Algebraic geometry over groups. I. Algebraic sets and ideal theory*
- 2011/12: È. Danijarova, A. Mjasnikov, V. Remeslennikov: *Algebraic geometry over algebraic structures. II. Foundations*
- 2010: È. Danijarova, A. Mjasnikov, V. Remeslennikov: *Algebraic geometry over algebraic structures. IV. Equational domains and codomains*
- 2017: A. Pinus: *Algebraic sets of universal algebras and algebraic closure operator*
- 2016: A. Pinus: *On algebraically equivalent clones*
- 2020: E. Aichinger, B. Rossi: *A clonoid based approach to some finiteness results in universal algebra*

Basic concepts

Algebraic sets over clone $F \leq \mathcal{O}_A$
($=$ solution sets of systems of equations over F)

$\varrho \subseteq A^n$ algebraic $\iff$ $\varrho = \{\mathrm{x} \in A^n \mid \forall i \in I\colon \quad f_i(\mathrm{x}) = g_i(\mathrm{x})\}$

for some $f_i, g_i \in F^{(n)}$ ($i \in I$, I any set).

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$\varrho_{12} = \{(x_1, x_2, x_3, x_4) \in A^4 \mid x_1 = x_2\}$

algebraic over any clone; solution set of 1 equation:

$$e_1^{(4)}(x_1, x_2, x_3, x_4) = e_2^{(4)}(x_1, x_2, x_3, x_4).$$

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$\varrho_{34} = \{(x_1, x_2, x_3, x_4) \in A^4 \mid x_3 = x_4\}$

algebraic over any clone; solution set of 1 equation:

$$e_3^{(4)}(x_1, x_2, x_3, x_4) = e_4^{(4)}(x_1, x_2, x_3, x_4).$$

Equationally additive clones

$\mathrm{Alg}^{(n)} F := \{\, \varrho \subseteq A^n \mid \varrho \text{ algebraic over } F \}$ $\quad \mathrm{Alg}\, F := \bigcup_{n\in\mathbb{N}_+} \mathrm{Alg}^{(n)} F$

Algebraic equivalence of clones $F, G \leq \mathcal{O}_A$

$F \equiv_{\mathrm{alg}} G$ algebraically equivalent $\iff$ $\mathrm{Alg}\, F = \mathrm{Alg}\, G$
(same algebraic geometry)

Theorem: for finite A: Pinus, 2016

$|\{\, F \leq \mathcal{O}_A \mid F \text{ 'equationally additive'} \}/\equiv_{\mathrm{alg}}| < \aleph_0$.

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Clone $F \leq \mathcal{O}_A$ equationally additive

$\iff \forall n \in \mathbb{N}_+ \, \forall \varrho, \sigma \in \mathrm{Alg}^{(n)} F: \quad \varrho \cup \sigma \in \mathrm{Alg}^{(n)} F$
(algebraic sets closed under finite unions)

Easy consequence

For a clone $F \leq \mathcal{O}_A$

F equationally additive

$$\Longrightarrow \Delta_A^{(4)} = \{(x_1, x_2, x_3, x_4) \in A^4 \mid x_1 = x_2 \text{ or } x_3 = x_4\} \in \mathrm{Alg}^{(4)} F$$

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- $\implies \Delta_A^{(4)} = \varrho_{12} \cup \varrho_{34} \in \mathrm{Alg}^{(4)} F$

since F is equationally additive

A characterisation of equational additivity

Theorem — Danijarova, Mjasnikov, Remeslennikov, 2010

A clone $F \leq \mathcal{O}_A$ is equationally additive $\iff$ $\Delta_A^{(4)} \in \operatorname{Alg} F$

A characterisation of equational additivity

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In a field

- $\varrho = \{ a \in A^n \mid \forall i \in I: \quad f_i(a) = 0\} \in \operatorname{Alg} F$
- $\sigma = \{ a \in A^n \mid \forall j \in J: \quad g_j(a) = 0\} \in \operatorname{Alg} F$
- $\implies \varrho \cup \sigma = \{ a \in A^n \mid \forall i \in I \, \forall j \in J: \quad f_i(a) \cdot g_j(a) = 0\}$

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In general

- $\varrho = \{a \in A^n \mid \forall i \in I: \quad f_i(a) = f_i'(a)\} \in \operatorname{Alg} F$
- $\sigma = \left\{a \in A^n \,\middle|\, \forall j \in J: \quad g_j(a) = g_j'(a)\right\} \in \operatorname{Alg} F$
- $\Delta_A^{(4)} = \{a \in A^4 \mid \forall k \in K: \quad h_k(a) = h_k'(a)\} \in \operatorname{Alg} F$
- $\implies \varrho \cup \sigma = \Big\{a \in A^n \,\Big|\, \forall k \in K\, \forall i \in I\, \forall j \in J:$
  $h_k(f_i(a), f_i'(a), g_j(a), g_j'(a)) = h_k'(f_i(a), f_i'(a), g_j(a), g_j'(a))\Big\}$

Goal

Which clones in Post's lattice are equationally additive?

Exploiting the Boolean domain

On finite sets A

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Theorem Tóth, Waldhauser, 2017

…study solution sets of finitely many equations on finite sets

$\forall F \leq \mathcal{O}_{\{0,1\}}\colon \quad \mathrm{Alg}\, F = \mathrm{Inv}_A\, F^*$ $\qquad F^*$…centraliser of F

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With characterisation of eqn. additivity:

$F \leq \mathcal{O}_2$ equationally additive $\iff \Delta_2^{(4)} \in \operatorname{Alg} F = \operatorname{Inv}_2 F^*$

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With characterisation of eqn. additivity:

$F \leq \mathcal{O}_2$ equationally additive $\iff \Delta_2^{(4)} \in \operatorname{Alg} F = \operatorname{Inv}_2 F^*$

Consequence: $\operatorname{Alg} F = \operatorname{Alg} F^{**}$

$F \leq \mathcal{O}_2$ equationally additive $\iff F^{**}$ equationally additive

Characterisation for Boolean clones part 1

For $F \leq \mathcal{O}_2$ Aichinger, Rossi, MB

F eqn. additive $\iff \Delta_2^{(4)} \in \mathrm{Inv}_2\, F^*$

Characterisation for Boolean clones part 1

For $F \leq \mathcal{O}_2$ — Aichinger, Rossi, MB

$$F \text{ eqn. additive} \iff \Delta_2^{(4)} \in \operatorname{Inv}_2 F^* \iff F^* \subseteq \operatorname{Pol}_2\left\{\Delta_2^{(4)}\right\}$$

Characterisation for Boolean clones part 1

Known fact — e.g. Pöschel/Kalužnin, 1.3.1

$$\operatorname{Pol}_A\left\{\Delta_A^{(4)}\right\} = \operatorname{Pol}_A\left\{\Delta_A^{(3)}\right\} = \left\langle \mathcal{O}_A^{(1)} \right\rangle_{\mathcal{O}_A}$$

$$\Delta_A^{(3)} = \{(x_1, x_2, x_3) \in A^3 \mid x_1 = x_2 \text{ or } x_2 = x_3\}$$

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$$= (S \cap T_0 \cap T_1)^*$$

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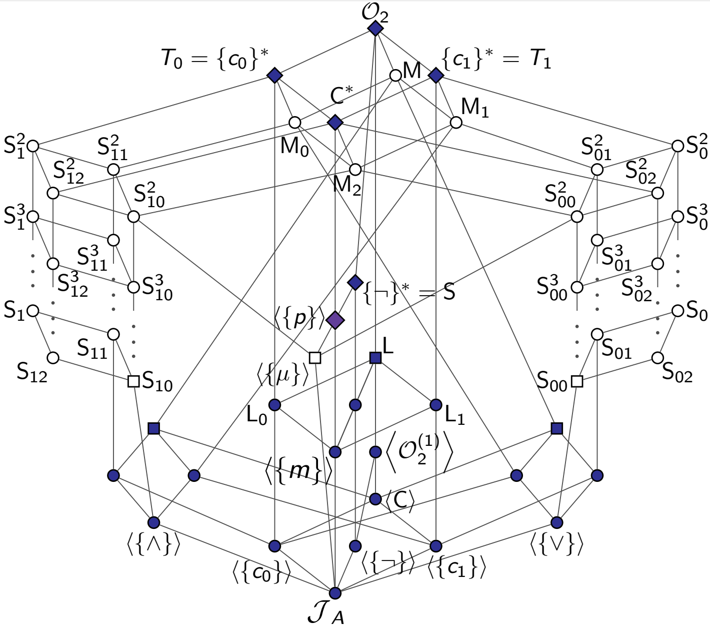
&\qquad\qquad\qquad\qquad = (S \cap T_0 \cap T_1)^* \\
&\iff \langle\{p\}\rangle_{\mathcal{O}_2} = \langle\{\mu, m\}\rangle_{\mathcal{O}_2} = S \cap T_0 \cap T_1 \subseteq F^{**} \\
&\iff F^{**} \text{ has majority } \mu \text{ \& minority op. } m \\
&\iff F^{**} \text{ has Pixley op. } p
\end{aligned}$$

Boolean centraliser clones



Kuznecov, 1979
Hermann, 2008

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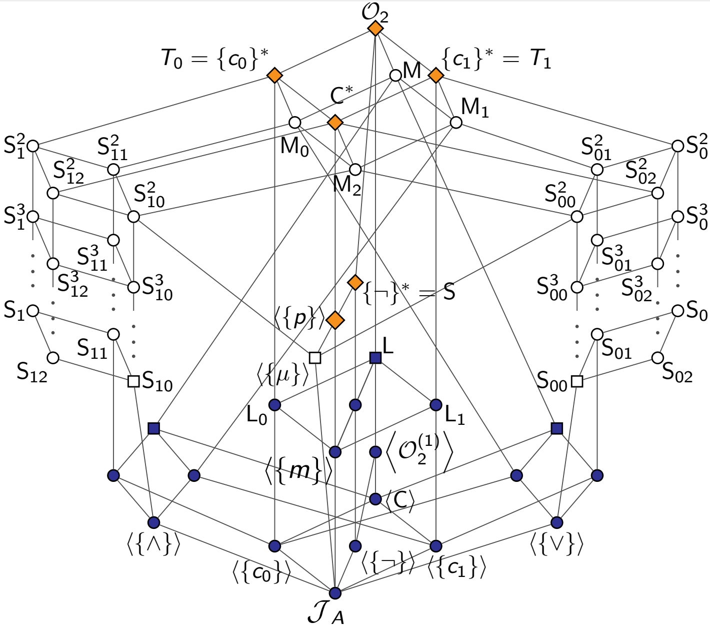

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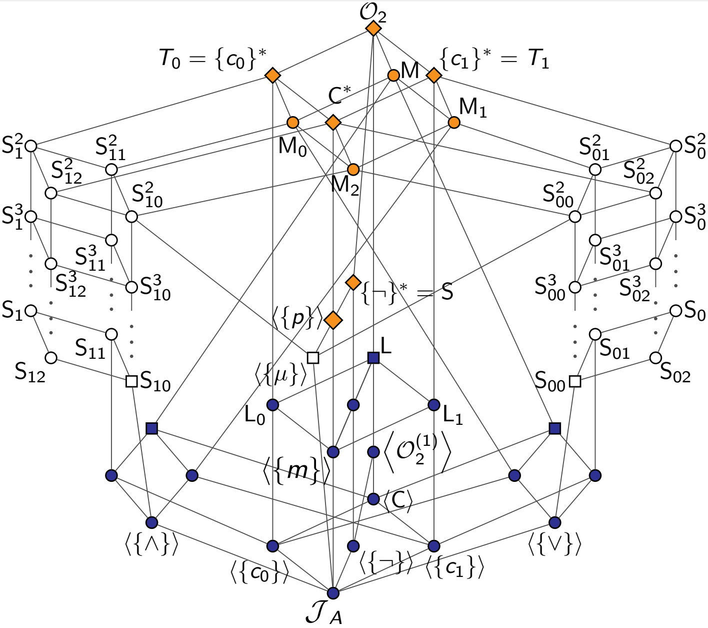
$F \leq \mathcal{O}_2$ eqn. add. $\iff$ $\langle\{p\}\rangle \subseteq F^{**}$

- centraliser
- no centraliser
- eqn. additive

Boolean centraliser clones



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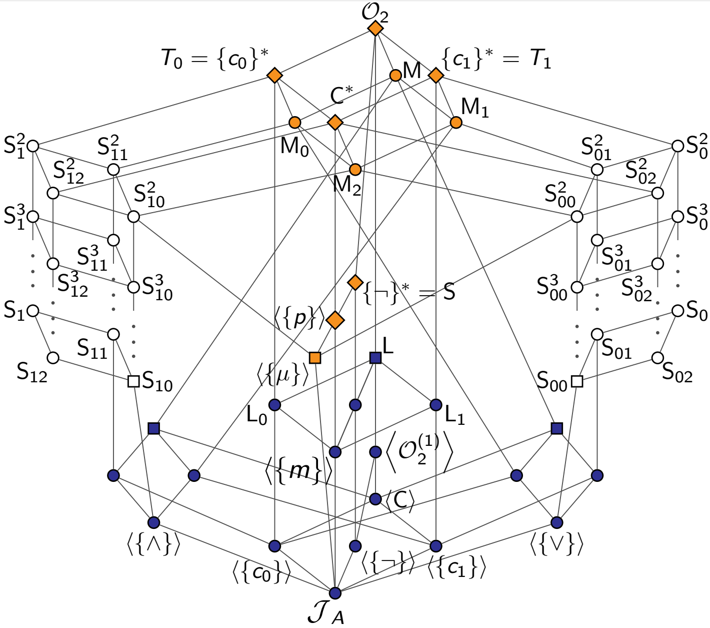

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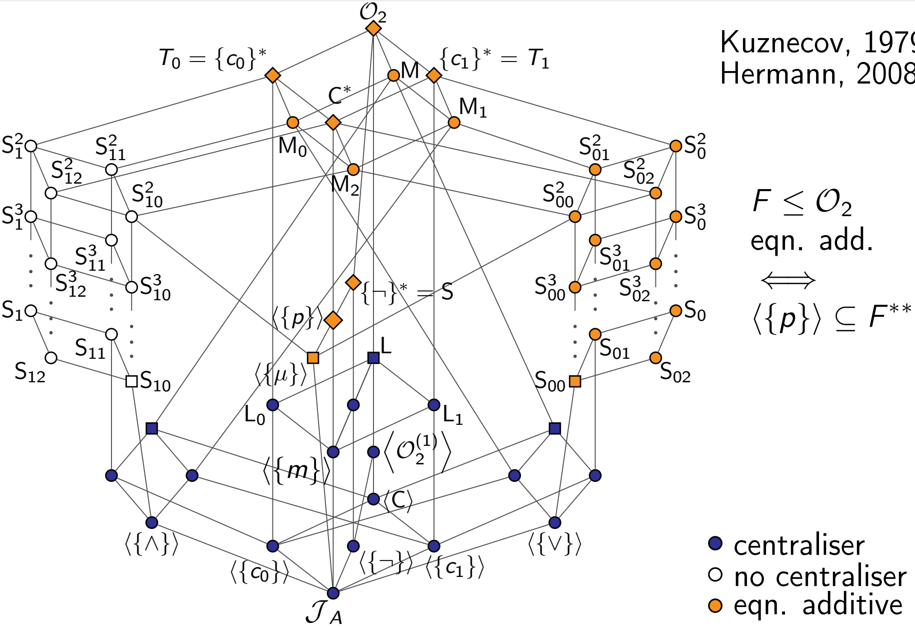

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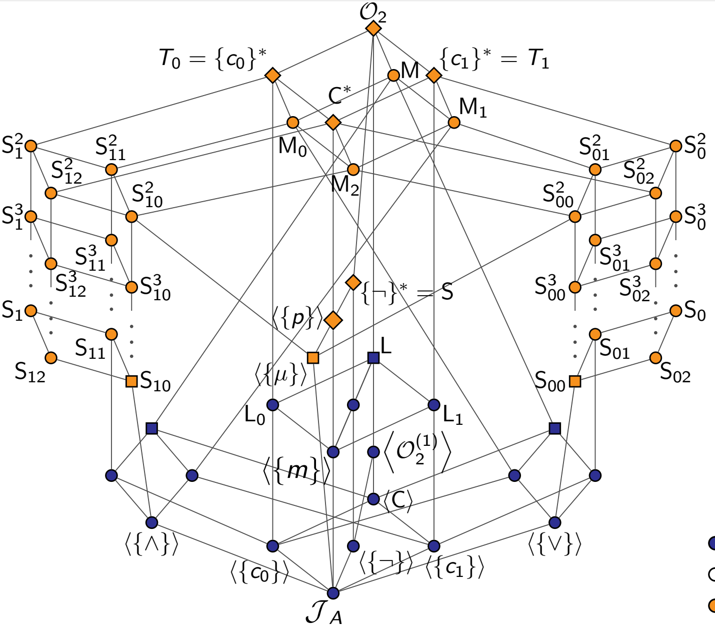

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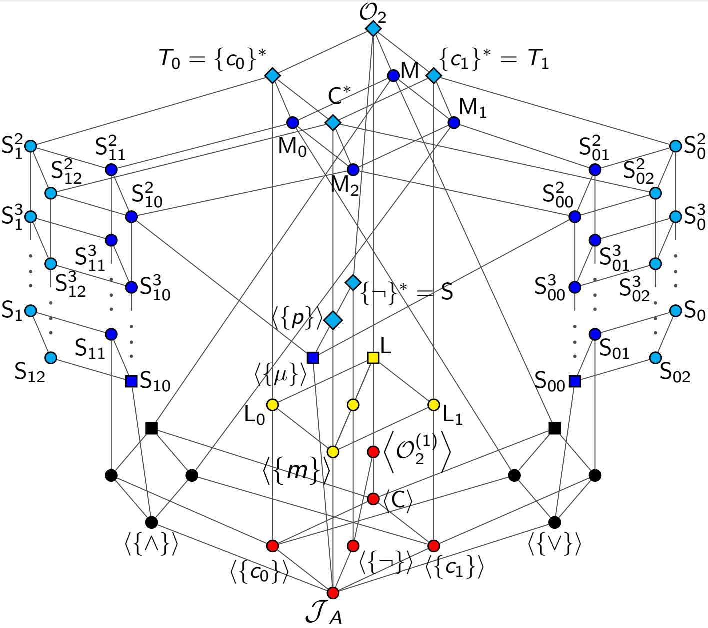

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- 1: unary
- 2: vector space
- 3: Boolean
- 4: lattice
- 5: semilattice

Characterisation for Boolean clones part 2

For $F \leq \mathcal{O}_2$ — Aichinger, Rossi, MB

$$
\begin{aligned}
F \text{ eqn. additive} \quad &\Longleftrightarrow \quad \mathsf{S}_{00} \in F \text{ or } \mathsf{S}_{10} \in F \text{ or } \langle\{\mu\}\rangle_{\mathcal{O}_2} \in F \\
&\Longleftrightarrow \quad ((x,y,z) \mapsto x \vee (y \wedge z)) \in F \text{ or} \\
&\qquad\quad ((x,y,z) \mapsto x \wedge (y \vee z)) \in F \text{ or} \\
&\qquad\quad \text{majority } \mu \in F \\
&\Longleftrightarrow \quad \exists f \in F^{(3)} \setminus \{e_1^{(3)}\} \colon f(x,x,y) \approx x \approx f(x,y,x) \\
&\Longleftrightarrow \quad F \not\subseteq \langle\{\wedge, c_0, c_1\}\rangle_{\mathcal{O}_2} \text{ and} \\
&\qquad\quad F \not\subseteq \langle\{\vee, c_0, c_1\}\rangle_{\mathcal{O}_2} \text{ and} \\
&\qquad\quad F \not\subseteq \mathsf{L} \\
&\Longleftrightarrow \quad \langle\{0,1\}; F\rangle \text{ has TCT-type} \\
&\qquad\quad 3 \text{ (Boolean algebra) or} \\
&\qquad\quad 4 \text{ (lattice)}
\end{aligned}
$$

Demonstrating equational additivity explicitly

$S_{00} = \langle \{f\} \rangle_{\mathcal{O}_2}, \qquad f(x, y, z) = x \vee (y \wedge z)$

$$\Delta_2^{(4)} = \left\{ (x_1, x_2, x_3, x_4) \in \{0, 1\}^4 \;\middle|\; \begin{array}{l} f(x_3, x_4, x_1) = f(x_3, x_4, x_2) \\ f(x_4, x_3, x_1) = f(x_4, x_3, x_2) \end{array} \right\}$$

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$\langle \{\mu\} \rangle_{\mathcal{O}_2}, \qquad \mu$ Boolean majority

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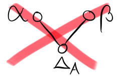
Putting results about TCT-types into perspective

Proposition — Aichinger, Rossi, MB

$F \leq \mathcal{O}_A$ equationally additive
$\Longrightarrow \forall \alpha, \beta \in \mathrm{Con}\langle A; F\rangle \setminus \{\Delta_A\}: \qquad \alpha \cap \beta \supsetneq \Delta_A.$



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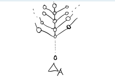

Consequence:

$\mathbf{A}$ equationally additive, $2 \leq |A| < \aleph_0 \implies \mathbf{A}$ subdirectly irreducible

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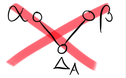

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For finite algebras A with Taylor term operation

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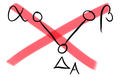

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Boolean TCT-type results: $\qquad F \leq \mathcal{O}_2$ eqn. add. $\iff$ F type 3 or 4
$=$ results about monoliths of two-element s.i. algebras

More TCT-related facts we can prove

$|A| \geq 2, \qquad F = \langle F^{(1)} \rangle_{\mathcal{O}_A} \leq \mathcal{O}_A$ (ess. at most unary)

$\Longrightarrow$ F not equationally additive

A finite minimal algebra

$\mathrm{Clo}(\mathbf{A})$ equationally additive $\iff$ $\mathrm{typ}(\mathbf{A}) \in \{3, 4\}$

A finite E-minimal algebra

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The number of equationally additive clones on finite sets

$\lvert A\rvert$	$\lvert\mathcal{L}_A\rvert$	$\lvert\uparrow_{\mathcal{L}_A}\{\mathrm{C}\}\rvert$	eqn. additive	eqn. additive $\supseteq$ C
2	$\aleph_0$	7	$\aleph_0$	2

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3	$2^{\aleph_0}$	$2^{\aleph_0}$	$2^{\aleph_0}$	?
≥ 4	$2^{\aleph_0}$	$2^{\aleph_0}$	$2^{\aleph_0}$	$2^{\aleph_0}$

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The end

Thank you. . .

. . . very much for listening

Questions, comments and remarks. . .

. . . most welcome