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On the solution stability of parabolic optimal control problems*

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Abstract

The paper investigates stability properties of solutions of optimal control problems for semilinear parabolic partial differential equations. Hölder or Lipschitz dependence of the optimal solution on perturbations are obtained for problems in which the equation and the objective functional are affine with respect to the control. The perturbations may appear in both the equation and in the objective functional and may nonlinearly depend on the state and control variables. The main results are based on an extension of recently introduced assumptions on the joint growth of the first and second variation of the objective functional. The stability of the optimal solution is obtained as a consequence of a more general result obtained in the paper – the proved metric subregularity of the mapping associated with the system of first order necessary optimality conditions. This property also enables error estimates for approximation methods. Lipschitz estimate for the dependence of the optimal control on the Tikhonov regularization parameter is obtained as a by-product.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $1 \le n \le 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. For a finite $T > 0$, denote by $Q := \Omega \times (0,T)$ the space-time cylinder and by $\Sigma := \partial\Omega \times (0,T)$ its lateral boundary. In the present paper, we investigate the following optimal control problem:

$$\text{(P)} \quad \min_{u\in\mathcal{U}} \left\{ J(u) := \int_Q L(x,t,y(x,t),u(x,t))\,\mathrm{d}x\,\mathrm{d}t \right\}, \tag{1.1}$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + f(\cdot,y) &= u \quad \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) &= y_0 \quad \text{on } \Omega. \end{cases} \tag{1.2}$$

Denote by y_u the unique solution to the semilinear parabolic equation (1.2) that corresponds to control $u \in L^r(Q)$, where r is a fixed number satisfying the inequality $r > 1 + \frac{n}{2}$. For functions $u_a, u_b \in L^\infty(Q)$ such that $u_a < u_b$ a.e in Q, the set of feasible controls is given by

$$\mathcal{U} := \{u \in L^\infty(Q)\,|\; u_a \le u \le u_b \;\text{ for a.a. } (x,t) \in Q\}. \tag{1.3}$$

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The objective integrand in (1.1) is defined as

$$L(x,t,y,u) := L_0(x,t,y) + (my+g)u, \tag{1.4}$$

where m is a number, g is a function in $L^\infty(Q)$ and L_0 satisfies appropriate smoothness condition (see Assumption 2 in Subsection 1.1).

The goal of the present paper is to obtain stability results for the optimal solution of problem (1.1)–(1.3). The meaning of "stability" we focus on, is as follows. Given a reference optimal control $\bar u$ and the corresponding solution $y_{\bar u}$, the goal is to estimate the distance (call it Δ) from the optimal solutions (u, y_u) of a disturbed version of problem (1.1)–(1.3) to the pair $(\bar u, y_{\bar u})$, in terms of the size of the perturbations (call it δ). The perturbations may enter either in the objective integrand or in the state equation, and the meaning of "distance" and "size" in the previous sentence will be clarified in the sequel in terms of appropriate norms. If an estimation $\Delta \le \text{const}.\delta^\theta$ holds with $\theta \in (0,1)$, we talk about *Hölder stability*, while in the case $\theta = 1$ we have *Lipschitz stability.*

A powerful technique for establishing stability properties of the solutions of optimization problems is based on regularity properties of the system of first order necessary optimality conditions (see e.g. [18]). In the case of problem (1.1)–(1.3), these are represented by a *differential variational inequality* (see e.g. [16, 24]), consisting of two parabolic equations (the primal equation (1.1) and the corresponding adjoint equation) and one variational inequality representing the condition for minimization of the Hamiltonian associated with the problem. The Lipschitz or Hölder stability of the solution of problem (1.1)–(1.3) is then a consequence of the property of *metric subregularity* (see [15, 18]) of the mapping defining this differential variational inequality. An advantage of this approach is that it unifies in a compact way the study of stability of optimal solutions under a variety of perturbations (linear or nonlinear). Therefore, the main result in the present paper focuses on conditions for metric subregularity of the mapping associated with the first order optimality conditions for problem (1.1)–(1.3). These conditions are related to appropriate second order sufficient optimality conditions, which are revisited and extended in the paper. Several results for stability of the solutions are obtained as a consequence.

The commonly used second order sufficient optimality conditions for ODE or PDE optimal control problems involve a *coercivity condition*, requiring strong positive definiteness of the objective functional as a function of the control in a Hilbert space. We stress that problem (1.1)–(1.3) is affine with respect to the control variable and such a coercivity condition is not fulfilled. The theory of sufficient optimality theory and the regularity theory for affine optimal control of ODE systems have been developed in the past decade, see [23] and the bibliography therein. Sufficient conditions for weak or strong local optimality for optimal control problems with constraints given by elliptic or parabolic equations are developed in [2, 3, 4, 8, 10, 12, 17]. A detailed discussion thereof is provided in Section 2.1. In contrast with the elliptic setting, there are only a few stability results for semilinear parabolic optimal control problems. Progress in this regard for a tracking type objective functional was made for instance in [9, 10] where stability with respect to perturbations in the objective functional was studied, and in [11], where stability with respect to perturbations in the initial data was investigated. We mention that for a linear state equation and a tracking type objective functional, Lipschitz estimates were obtained in [29] under an additional assumption on the structure of the optimal control. More comprehensive discussion about the sufficiency theory and stability can be found in Section 2.

The main novelty in the present paper is the study of the subregularity property of the optimality mapping associated with problem (1.1)–(1.3). In contrast with the case of coercive problems, our assumptions in the affine case jointly involve the first and the second order variations of the objective functional with respect to the control. These assumptions are weaker than the ones in the existing literature in the context of sufficient optimality conditions, however, they are strong enough to imply metric subregularity of the optimality mapping. The subregularity result is used to obtain new Hölder- and Lipschitz estimates for the solution of the considered optimal control problem. An error estimate for the Tikhonov regularization is obtained as a consequence.

The obtained subregularity result provides a base for convergence and error analysis for discretization methods applied to problem (1.1)–(1.3). The point is, that numerical solutions of the discretized versions of the problem typically satisfy approximately first order optimality conditions for the discretized problem and after appropriate embedding in the continuous setting (1.1)–(1.3), satisfy the optimality conditions for the latter problem with a residual depending on the approximation and the discretization error. Then the subregularity property of the optimality mapping associated with (1.1)–(1.3) provides an error estimate. Notice that the (Lipschitz) stability of the solution alone is not enough for such a conclusion, and this is an important motivation for studying subregularity of the optimality mapping rather than only stability of the solutions. However, we do not go into this subject, postponing it to a later paper based on the present one.

The paper is organized as follows. The analysis of the optimal control problem (1.1)–(1.3) begins in Section 2. We recall the state of the art regarding second order sufficient conditions for weak and strong (local) optimality, as well as known sufficient conditions for stability of optimal controls and states under perturbations. In Section 3 we formulate and discuss the assumptions on which our further analysis on sufficiency and stability is based. The strong subregularity of the optimality mapping is proved in Section 4. In Section 5, we obtain stability results for the optimal control problem under non-linear perturbations, postponing some technicalities to Assumption A. Finally, we support the theoretical results with some examples.

1.1 Preliminaries

We begin with some basic notations and definitions. Given a non-empty, bounded and Lebesgue measurable set $X \subset \mathbb{R}^n$, we denote by $L^p(X)$, $1 \le p \le \infty$, the Banach spaces of all measurable functions $f : X \to \mathbb{R}$ for which the usual norm $\|f\|_{L^p(X)}$ is finite. For a bounded Lipschitz domain $X \subset \mathbb{R}^n$ (that is, a set with Lipschitz boundary), the Sobolev space $H_0^1(X)$ consists of functions that vanish on the boundary (in the trace sense) and that have weak first order derivatives in $L^2(X)$. The space $H_0^1(X)$ is equipped with its usual norm denoted by $\|\cdot\|_{H_0^1(X)}$. By $H^{-1}(X)$ we denote the topological dual of $H_0^1(X)$, equipped with the standard norm $\|\cdot\|_{H^{-1}(X)}$. Given a real Banach space Z, the space $L^p(0,T;\, Z)$ consist of all strongly measurable functions $y : [0,T] \to Z$ that satisfy

$$\|y\|_{L^p(0,T;\, Z)} := \Big(\int_0^T \|y(t)\|_Z^p \,\mathrm{d}t \Big)^{\frac{1}{p}} < \infty \qquad \text{if } 1 \le p < \infty,$$

or, for $p = \infty$,

$$\|y\|_{L^\infty(0,T;\, Z)} := \inf\{M \in \mathbb{R} \mid \|y(t)\|_Z \le M \text{ a.e } t \in (0,T)\} < \infty.$$

The Hilbert space $W(0,T)$ consists of all of functions in $L^2(0,T;\, H_0^1(\Omega))$ that have a distributional derivative in $L^2(0,T;\, H^{-1}(\Omega))$, i.e.

$$W(0,T) := \left\{ y \in L^2(0,T; H_0^1(\Omega)) \Big| \ \frac{\partial y}{\partial t} \in L^2(0,T; H^{-1}(\Omega)) \right\},$$

which is endowed with the norm

$$\|y\|_{W(0,T)} := \|y\|_{L^2(0,T;H_0^1(\Omega))} + \|\partial y/\partial t\|_{L^2(0,T;H^{-1}(\Omega))}.$$

The Banach space $C([0,T];\, L^2(\Omega))$ consists of all continuous functions $y : [0,T] \to L^2(\Omega)$ and is equipped with the norm $\max_{t\in[0,T]} \|y(t)\|_{L^2(\Omega)}$. It is well known that $W(0,T)$ is continuously embedded in $C([0,T];\, L^2(\Omega))$ and compactly embedded in $L^2(Q)$. For proofs and further details regarding spaces involving time, see [14, 20, 27, 30, 31].

The following assumptions, close to those in [2, 5, 6, 8, 10, 11, 12, 13], are standing in all the paper, together with the inequality

$$r > \max\left\{2, 1 + \frac{n}{2}\right\} \tag{1.5}$$

for the real number r that appears in some assumptions and many statements below (we also remind that $n \in \{1, 2, 3\}$).

Assumption 1. *The operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$, is given by*

$$\mathcal{A} = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{i,j}(x)\partial_{x_i}),$$

where $a_{i,j} \in L^\infty(\Omega)$ satisfy the uniform ellipticity condition

$$\exists \lambda_{\mathcal{A}} > 0 : \; \lambda_{\mathcal{A}}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \;\; \forall \xi \in \mathbb{R}^n \;\text{ and a.a. } x \in \Omega.$$

The matrix with components $a_{i,j}$ is denoted by A.

The functions f, $L_0 : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ of the variables (x, t, y), and the "initial" function y_0 have the following properties.

Assumption 2. *For every $y \in \mathbb{R}$, the functions $f(\cdot, \cdot, y) \in L^r(Q)$, $L_0(\cdot, \cdot, y) \in L^1(Q)$, and $y_0 \in L^\infty(\Omega)$. For a.e. $(x, t) \in Q$ the first and the second derivatives of f and L_0 with respect to y exist and are locally bounded and locally Lipschitz continuous, uniformly with respect to $(x, t) \in Q$. Moreover, $\frac{\partial f}{\partial y}(x, t, y) \geq 0$ for a.e. $(x, t) \in Q$ and for all $y \in \mathbb{R}$.*

Remark 1. *The last condition in Assumption 2 can be relaxed in the following way:*

$$\exists C_f \in \mathbb{R} : \; \frac{\partial f}{\partial y}(x, t, y) \geq C_f \quad \text{a.a. } (x, t) \in Q \text{ and } \forall y \in \mathbb{R}, \tag{1.6}$$

see [3, 8]. However, this leads to complications in the proofs.

1.2 Facts regarding the linear and the semilinear equation

Let $0 \leq \alpha \in L^\infty(Q)$ and $u \in L^2(Q)$. We first consider solutions of the following linear variational equality for $y \in W(0, T)$ with $y(\cdot, 0) = 0$:

$$\int_0^T \langle \frac{\partial y}{\partial t}, \psi \rangle \, dt + \int_0^T \langle \mathcal{A}y, \psi \rangle \, dt = \int_0^T \langle u, \psi \rangle \, dt - \int_0^T \langle \alpha y, \psi \rangle \, dt \tag{1.7}$$

for all $\psi \in L^2(0, T, H_0^1(\Omega))$, that is, for weak solutions of the equation (1.2) with $f(x, t, y) := \alpha(x, t)y$ and $y_0 = 0$.

Theorem 1. *Let $0 \leq \alpha \in L^\infty(Q)$ be given.*

1. *For each $u \in L^2(Q)$ the linear parabolic equation (1.7) has a unique weak solution $y_u \in W(0, T)$. Moreover, there exists a constant $\hat{C} > 0$ independent of u and α such that*

$$\|y_u\|_{L^2(0,T,H_0^1(\Omega))} \leq \hat{C}\|u\|_{L^2(Q)}. \tag{1.8}$$

2. *If, additionally, $u \in L^r(Q)$ (we remind (1.5)) then the weak solution y_u of (1.7) belongs to $W(0, T) \cap C(\bar{Q})$. Moreover, there exists a constant $C_r > 0$ independent of u and α such that*

$$\|y_u\|_{L^2(0,T,H_0^1(\Omega))} + \|y_u\|_{C(\bar{Q})} \leq C_r\|u\|_{L^r(Q)}. \tag{1.9}$$

Besides the independence of the constants $\hat{C}$, and C_r on α all claims of the theorem are well known, see [28, Theorem 3.13, Theorem 5.5]. A proof of a similar independence statement can be found in [2] for a linear elliptic PDE of non-monotone type.

Proof. For convenience of the reader, we prove that the estimates are independent of α. This is done along the lines of the proof of [2, Lemma 2.2]. By $y_{0,u}$ we denote a solution of (1.7) for $\alpha \equiv 0$. It is well known that in this case there exist constants $C_r, \hat{C} > 0$ such that

$$\|y_{0,u}\|_{C(\bar{Q})} \leq C_r \|u\|_{L^r(Q)}, \quad \|y_{0,u}\|_{L^2(Q)} \leq \hat{C}\|u\|_{L^2(Q)}.$$

To apply this, we decompose u in positive and negative parts, $u = u^+ - u^-$, $u^+, u^- \geq 0$. By the weak maximum principle [14, Theorem 11.9], it follows that $y_{\alpha,u^+}, y_{\alpha,u^-} \geq 0$. Again by the weak maximum principle, the equation

$$\frac{\partial}{\partial t}(y_{\alpha,u^+} - y_{0,u^+}) + \mathcal{A}(y_{\alpha,u^+} - y_{0,u^+}) + \alpha(y_{\alpha,u^+} - y_{0,u^+}) = -\alpha y_{0,u^+}$$

implies $0 \leq y_{\alpha,u^+} \leq y_{0,u^+}$, thus $\|y_{\alpha,u^+}\|_{C(\bar{Q})} \leq \|y_{0,u^+}\|_{C(\bar{Q})}$. By the same reasoning, it follows that $0 \leq y_{\alpha,u^-} \leq y_{0,u^-}$ and $\|y_{\alpha,u^-}\|_{C(\bar{Q})} \leq \|y_{0,u^-}\|_{C(\bar{Q})}$. Hence,

$$\begin{aligned}\|y_{\alpha,u}\|_{C(\bar{Q})} &\leq \|y_{\alpha,u^+}\|_{C(\bar{Q})} + \|y_{\alpha,u^-}\|_{C(\bar{Q})} \leq \|y_{0,u^+}\|_{C(\bar{Q})} + \|y_{0,u^-}\|_{C(\bar{Q})} \\ &\leq C_r(\|u^+\|_{L^r(Q)} + \|u^-\|_{L^r(Q)}) \leq 2C_r\|u\|_{L^r(Q)}.\end{aligned}$$

The estimate for $L^2(0,T,H_0^1(\Omega))$ can be obtained by similar arguments as in [2]. □

The next lemma is motivated by an analogous result for linear elliptic equations [2, Lemma 2.3], although, according to the nature of the parabolic setting, the interval of feasible numbers s, is smaller.

Lemma 2. *Let $u \in L^r(Q)$ and $0 \leq \alpha \in L^\infty(Q)$. Let y_u be the unique solution of (1.7) and let p_u be a solution of the problem*

$$\begin{cases} -\frac{\partial p}{\partial t} + \mathcal{A}^* p + \alpha p = u \quad in \quad Q, \\ p = 0 \ on \ \Sigma, \ p(\cdot,T) = 0 \quad on \ \Omega. \end{cases} \tag{1.10}$$

Then, for any $s_n \in [1, \frac{n+2}{n})$ there exists a constant $C_{s_n'} > 0$ independent of u and α such that

$$\max\{\|y_u\|_{L^{s_n}(Q)}, \|p_u\|_{L^{s_n}(Q)}\} \leq C_{s_n'}\|u\|_{L^1(Q)}. \tag{1.11}$$

Here s_n' denotes the Hölder conjugate of s_n.

Proof. First we observe that by Theorem 1, $y_u \in C(\bar{Q}) \cap W(0,T)$ and as a consequence, $|y_u|^{s_n-1}\mathrm{sign}(y_u) \in L^{s_n'}(Q)$. Moreover, $s_n < \frac{n+2}{n}$ implies that $s_n' > 1 + \frac{n}{2}$. By change of variables, see for instance [28, Lemma 3.17], a solution of equation (1.10) transforms into a solutions of (1.7). Thus according to Theorem 1, the solution q of

$$\begin{cases} -\frac{\partial q}{\partial t} + \mathcal{A}^* q + \alpha q = |y_u|^{s_n-1}\mathrm{sign}(y_u) \ \text{ in } \ Q, \\ q = 0 \text{ on } \Sigma, \ q(\cdot,T) = 0 \ \text{ on } \Omega. \end{cases}$$

belongs to $W(0,T) \cap C(\bar{Q})$ and satisfies

$$\|q\|_{C(\bar{Q})} \leq C_{s_n'}\||y_u|^{s_n-1}\mathrm{sign}(y_u)\|_{L^{s_n'}(Q)} = C_{s_n'}\|y_u\|_{L^{s_n}(Q)}^{s_n-1},$$

where $C_{s_n'}$ is independent of a and v. Using these facts we derive the equalities

$$\begin{aligned}\|y_u\|_{L^{s_n}(Q)}^{s_n} &= \int_Q |y_u|^{s_n}\,\mathrm{d}x = \left\langle -\frac{\partial q}{\partial t} + \mathcal{A}^* q + \alpha q, y_u\right\rangle = \left\langle \frac{\partial y_u}{\partial t} + \mathcal{A} y_u + \alpha y_u, q\right\rangle \\ &= \int_Q uq\,\mathrm{d}x \leq \|u\|_{L^1(Q)}\|q\|_{C(\bar{Q})} \leq C_{s_n'}\|u\|_{L^1(Q)}\|y_u\|_{L^{s_n}(Q)}^{s_n-1}.\end{aligned}$$

This proves (1.11) for y_u. To obtain (1.11) for p_u, one tests (1.10) with a weak solution of

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y + \alpha y = |q_u|^{s_n - 1}\text{sign}(q_u) \text{ in } Q, \\ y = 0 \text{ on } \Sigma,\ y(\cdot, 0) = 0 \text{ on } \Omega, \end{cases}$$

and argues in an analogous way. □

Below we remind several results for the semilinear equation (1.2), which will be used further. A proof of the next theorem can be found in [5, Theorem 2.1] or [28, Theorem 2.1].

Theorem 3. *For any $u \in L^2(Q)$ the semilinear parabolic initial-boundary value problem* (1.2) *has a unique weak solution $y_u \in W(0,T)$. If $u \in L^r(Q)$ (see* (1.5)*) then $y_u \in W(0,T) \cap L^\infty(Q)$. If additionally $y_0 \in C(\bar{\Omega})$, then $y_u \in C(\bar{Q})$. Moreover, there exists a constant $D_r > 0$, independent of u, f, y_0 such that*

$$\|y_u\|_{W(0,T)} + \|y_u\|_{L^\infty(Q)} \le D_r\big(\|u\|_{L^r(Q)} + \|f(\cdot,\cdot,0)\|_{L^r(Q)} + \|y_0\|_{L^\infty(\Omega)}\big). \tag{1.12}$$

Finally, if $u_k \rightharpoonup u$ weakly in $L^r(Q)$, then

$$\|y_{u_k} - y_u\|_{L^\infty(Q)} + \|y_{u_k} - y_u\|_{L^2(0,T;H_0^1(\Omega))} \to 0. \tag{1.13}$$

The differentiability of the control-to-state operator under the assumptions 1 and 2 is well known, see among others [8, Theorem 2.4].

Theorem 4. *The control-to-state operator $\mathcal{G} : L^r(Q) \to W(0,T) \cap L^\infty(Q)$, defined as $\mathcal{G}(v) := y_v$, is of class C^2 and for every $u, v, w \in L^r(Q)$, it holds that $z_{u,v} := \mathcal{G}'(u)v$ is the solution of*

$$\begin{cases} \frac{dz}{dt} + \mathcal{A}z + f_y(x,t,y_u)z = v \text{ in } Q, \\ z = 0 \text{ on } \Sigma,\ z(\cdot,0) = 0 \text{ on } \Omega \end{cases} \tag{1.14}$$

and $\omega_{u,(v,w)} := \mathcal{G}''(u)(v,w)$ is the solution of

$$\begin{cases} \frac{dz}{dt} + \mathcal{A}z + f_y(x,t,y_u)z = -f_{yy}(x,t,y_u)z_{u,v}z_{u,w} \text{ in } Q, \\ z = 0 \text{ on } \Sigma,\ z(\cdot,0) = 0 \text{ on } \Omega. \end{cases} \tag{1.15}$$

In the case $v = w$, we will just write $\omega_{u,v}$ instead of $\omega_{u,(v,v)}$.

Remark 2. *By the boundedness of $\mathcal{U}$ in $L^\infty(Q)$ and by Theorem 3, there exists a constant $M_{\mathcal{U}} > 0$ such that*

$$\max\{\|u\|_{L^\infty(Q)}, \|y_u\|_{C(\bar{Q})}\} \le M_{\mathcal{U}} \quad \forall u \in \mathcal{U}. \tag{1.16}$$

1.3 Estimates associated with differentiability

We employ results of the last subsection to derive estimates for the state equation (1.2) and its linearisation (1.14). These estimates constitute a key ingredient to derive stability results in the later sections. The next lemma extends [2, Lemma 2.7] from elliptic equations to parabolic ones.

Lemma 5. *The following statements are fulfilled.*

(i) There exists a positive constant M_2 such that for every $u, \bar{u} \in \mathcal{U}$ and $v \in L^r(Q)$

$$\|z_{u,v} - z_{\bar{u},v}\|_{L^2(Q)} \le M_2 \|y_u - y_{\bar{u}}\|_{C(\bar{Q})} \|z_{\bar{u},v}\|_{L^2(Q)}. \tag{1.17}$$

(ii) Let $X = C(\bar{Q})$ or $X = L^2(Q)$. Then there exists $\varepsilon > 0$ such that for every $u, \bar{u} \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{C(\bar{Q})} < \varepsilon$ the following inequalities are satisfied

$$\|y_u - y_{\bar{u}}\|_X \le 2\|z_{\bar{u},u-\bar{u}}\|_X \le 3\|y_u - y_{\bar{u}}\|_X, \tag{1.18}$$

$$\|z_{\bar{u},v}\|_X \le 2\|z_{u,v}\|_X \le 3\|z_{\bar{u},v}\|_X. \tag{1.19}$$

The proof, that is a consequence of Lemma 28, is given in Appendix A.

2 The control problem

The optimal control problem (1.1)-(1.3) is well posed under assumptions 1 and 2. Using the direct method of calculus of variations one can easily prove that there exists at least one global minimizer, see [28, Theorem 5.7]. On the other hand, the semilinear state equation makes the optimal control problem nonconvex, therefore we allow global minimizers as well as local ones. In the literature, weak and strong local minimizers are considered.

Definition 1. *We say that $\bar u \in \mathcal{U}$ is an $L^r(Q)$-weak local minimum of problem* (1.1)-(1.3)*, if there exists some $\varepsilon > 0$ such that*

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U} \text{ with } \|u - \bar u\|_{L^r(Q)} \le \varepsilon.$$

We say that $\bar u \in \mathcal{U}$ a strong local minimum of (P) *if there exists $\varepsilon > 0$ such that*

$$J(\bar u) \le J(u) \quad \forall u \in \mathcal{U} \text{ with } \|y_u - y_{\bar u}\|_{L^\infty(Q)} \le \varepsilon.$$

We say that $\bar u \in \mathcal{U}$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \ne \bar u$.

Relations between these types of optimality are obtained in [3, Lemma 2.8].

As a consequence of Theorem 4 and the chain rule, we obtain the differentiability of the objective functional with respect to the control.

Theorem 6. *The functional $J : L^r(Q) \longrightarrow \mathbb{R}$ is of class C^2. Moreover, given $u, v, v_1, v_2 \in L^r(Q)$ we have*

$$J'(u)v = \int_Q \Big(\frac{dL_0}{dy}(x,t,y_u) + mu\Big) z_{u,v} + (my_u + g)v \,\mathrm{d}x\,\mathrm{d}t \tag{2.1}$$

$$= \int_Q (p_u + my_u + g)v \,\mathrm{d}x\,\mathrm{d}t, \tag{2.2}$$

$$J''(u)(v_1,v_2) = \int_Q \Big[\frac{\partial^2 L}{\partial y^2}(x,t,y_u,u) - p_u \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\Big] z_{u,v_1} z_{u,v_2} \,\mathrm{d}x\,\mathrm{d}t \tag{2.3}$$

$$+ \int_Q m(z_{u,v_1} v_2 + z_{u,v_2} v_1) \,\mathrm{d}x\,\mathrm{d}t, \tag{2.4}$$

Here, $p_u \in W(0,T) \cap C(\bar Q)$ is the unique solution of the adjoint equation

$$\begin{cases} -\dfrac{dp}{dt} + \mathcal{A}^* p + \dfrac{\partial f}{\partial y}(x,t,y_u)p = \dfrac{\partial L}{\partial y}(x,t,y_u,u) \text{ in } Q, \\ p = 0 \text{ on } \Sigma, \ p(\cdot,T) = 0 \text{ on } \Omega. \end{cases} \tag{2.5}$$

We introduce the Hamiltonian $Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (x,t,y,p,u) \mapsto H(x,t,y,p,u) \in \mathbb{R}$ in the usual way:

$$H(x,t,y,p,u) := L(x,t,y,u) + p(u - f(x,t,y)).$$

The local form of the Pontryagin type necessary optimality conditions for problem (1.1)-(1.3) in the next theorem is well known (see e.g. [3, 8, 28]).

Theorem 7. *If $\bar u$ is a weak local minimizer for problem* (1.1)-(1.3)*, then there exist unique elements $\bar y, \bar p \in W(0,T) \cap C(\bar Q)$ such that*

$$\begin{cases} \frac{d\bar y}{dt} + \mathcal{A}\bar y + f(x,t,\bar y) = \bar u \text{ in } Q, \\ \bar y = 0 \text{ on } \Sigma, \ \bar y(\cdot,0) = y_0 \text{ on } \Omega. \end{cases} \tag{2.6}$$

$$\begin{cases} \dfrac{d\bar p}{dt} + \mathcal{A}^* \bar p = \dfrac{\partial H}{\partial y}(x,t,\bar y,\bar p,\bar u) \text{ in } Q, \\ \bar p = 0 \text{ on } \Sigma, \ \bar p(\cdot,T) = 0 \text{ on } \Omega. \end{cases} \tag{2.7}$$

$$\int_Q \frac{\partial H}{\partial u}(x,t,\bar y,\bar p,\bar u)(u - \bar u) \,\mathrm{d}x\,\mathrm{d}t \ge 0 \quad \forall u \in \mathcal{U}. \tag{2.8}$$

2.1 Sufficient conditions for optimality and stability

In this subsection we discuss the state of the art in the theory of sufficient second order optimality conditions in PDE optimal control, as well as related stability results for the optimal solution. For this purpose, we recall the definitions of several cones that are useful in the study of sufficient conditions. Given a triplet $(\bar{y}, \bar{p}, \bar{u})$ satisfying the optimality system in Theorem 7, and abbreviating $\frac{\partial \bar{H}}{\partial u}(x,t) := \frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})$, we have from (2.8) that almost everywhere in Q

$$\bar{u} = u_a \ \text{ if } \ \frac{\partial \bar{H}}{\partial u} > 0 \quad \text{and} \quad \bar{u} = u_b \ \text{ if } \ \frac{\partial \bar{H}}{\partial u} < 0.$$

This motivates to consider the following set

$$\Big\{ v \in L^2(Q) \Big| v \geq 0 \text{ a.e. on } [\bar{u} = u_a] \text{ and } v \leq 0 \text{ a.e. on } [\bar{u} = u_b] \Big\}. \tag{2.9}$$

Sufficient second order conditions for (local) optimality based on (2.9) are given in [8, 3, 10]. Following the usual approach in mathematical programming, one can define the critical cone at $\bar{u}$ as follows:

$$C_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.9) and } v(x,t) = 0 \text{ if } \Big| \frac{\partial \bar{H}}{\partial u}(x,t) \Big| > 0 \Big\}.$$

Obviously, this cone is trivial if $\frac{\partial \bar{H}}{\partial u}(x,t) \neq 0$ for a.e. (x,t) (which implies bang-bang structure of $\bar{u}$) thus no additional information can be gained based on $C_{\bar{u}}$. To address this issue, it was proposed in [19, 21] to consider larger cones on which second order conditions can be posed. Namely, for $\tau > 0$ one defines

$$D^\tau_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.9) and } v(x,t) = 0 \text{ if } \Big| \frac{\partial \bar{H}}{\partial u}(x,t) \Big| > \tau \Big\}, \tag{2.10}$$

$$G^\tau_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.9) and } J'(\bar{u})(v) \leq \tau \|z_{\bar{u},v}\|_{L^1(Q)} \Big\}, \tag{2.11}$$

$$E^\tau_{\bar{u}} := \Big\{ v \in L^2(Q) \Big| v \text{ satisfies (2.9) and } J'(\bar{u})(v) \leq \tau \|z_{\bar{u},v}\|_{L^2(Q)} \Big\}, \tag{2.12}$$

$$C^\tau_{\bar{u}} := D^\tau_{\bar{u}} \cap G^\tau_{\bar{u}}. \tag{2.13}$$

The cones $D^\tau_{\bar{u}}$, $E^\tau_{\bar{u}}$ and $G^\tau_{\bar{u}}$ were introduced in [4, 10] as extensions of the usual critical cone. It was proven in [4, 9, 10] that the condition:

$$\exists \delta > 0, \tau > 0 \quad \text{such that} \quad J''(\bar{u})v^2 \geq \delta \|z_{\bar{u},v}\|^2_{L^2(Q)} \quad \forall v \in G \tag{2.14}$$

is sufficient for weak (in the case $G = D^\tau_{\bar{u}}$) or strong (in the case $G = E^\tau_{\bar{u}}$) local optimality in the elliptic and parabolic setting. Most recently, the cone $C^\tau_{\bar{u}}$ was defined in [3] and also used in [6]. It was proved in [3], that (2.14) with $C = C^\tau_{\bar{u}}$ is sufficient for strong local optimality.
Under (2.14) it is possible to obtain some stability results. In [9] and [10] the authors obtain Lipschitz stability in the $(L^2 - L^\infty)$-sense for the states[1], under perturbations appearing in a tracking type objective functional and under the assumption that the perturbations are Lipschitz. Further they obtain Hölder stability for the states under a Tikhonov type perturbation. Hölder stability under (2.14) with exponent 1/2 was proved in [11] with respect to perturbations in the initial condition.

To improve the stability results an additional assumption is needed. This role is usually played by the structural assumption on the adjoint state or generally on the derivative of the Hamiltonian with respect to the control. In the case of an elliptic state equation, [25] uses the structural assumption

$$\exists \kappa > 0 \text{ such that} \quad \Big| \Big\{ x \in \Omega : \Big| \frac{\partial \bar{H}}{\partial u} \Big| \leq \varepsilon \Big\} \Big| \leq \kappa\varepsilon \quad \forall \varepsilon > 0. \tag{2.15}$$

[1] For $p, r \in [1, \infty]$, we speak of stability in the $L^p - L^r$-sense for the optimal states $\bar{y}$ with respect to perturbations (may appear in the equation or the objective) ξ, if there exists a constant $\kappa > 0$ such that $\|y^\xi - \bar{y}\|_{L^p(Q)} \leq \kappa \|\xi\|_{L^r(Q)}$, for all ξ that are sufficiently small. Here, y^ξ denotes the state corresponding to the perturbation ξ. We use this expression analogously for the optimal controls.

In the parabolic case this assumption (with Ω replaced with Q) is used in [11]. We recall that the assumption (2.15) implies that $\bar{u}$ is of bang-bang type. Further, (2.15) implies the existence of a constant $\tilde{\kappa} > 0$ such that the following growth property holds:

$$J'(\bar{u})(u-\bar{u}) \geq \tilde{\kappa}\|u-\bar{u}\|^2_{L^1(X)} \ \forall u \in \mathcal{U}. \tag{2.16}$$

For a proof see [1], [22] or [26]. If the control constraints satisfy $u_a < u_b$ almost everywhere on Q, both conditions, (2.15) and (2.16) are equivalent, see [17, Proposition 6.4]. In [25], using (2.15) and (2.14) with $G = D^\tau_{\bar{u}}$, the authors proof L^1-Lipschitz stability of the controls for an elliptic semilinear optimal control problem under perturbations appearing simultaneously in the objective functional and the state equation. Assuming (2.15), (2.14) may also be weakened to the case of negative curvature,

$$\exists \delta < \tilde{\kappa},\ \exists \tau > 0 \ \text{ such that } \ J''(\bar{u})v^2 \geq -\delta\|v\|^2_{L^1(\Omega)} \ \ \forall v \in C^\tau_{\bar{u}}. \tag{2.17}$$

In [12], [13] it was proved that (2.15) together with (2.17) implies, for the semililnear elliptic case, weak local optimality in $L^1(\Omega)$. Lipschitz stability results were also obtained in [17] in the elliptic case. Finally, for a semilinear parabolic equation with perturbed initial data, [11, Theorem 4.6] obtains, under (2.14) and (2.15), $L^2 - L^2$ and $L^1 - L^2$-Hölder stability (see Footnote 1), with exponent $\frac{2}{3}$, for the optimal states and controls respectively. Additionally, Lipschitz dependence is obtained on perturbations in $L^\infty(Q)$.

3 A unified sufficiency condition

In this section, we introduce an assumption that unifies the first and second order conditions presented in the previous section.

Assumption 3. *For a number $k \in \{0,1,2\}$, at least one of the following conditions is fulfilled:*

(A_k): There exist constants $\alpha_k, \gamma_k > 0$ such that

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)}\|u-\bar{u}\|^{2-k}_{L^1(Q)} \tag{3.1}$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \alpha_k$.

(B_k): There exist constants $\tilde{\alpha}_k, \tilde{\gamma}_k > 0$ such that (3.1) *holds for all $u \in \mathcal{U}$ such that $\|u-\bar{u}\|_{L^1(Q)} < \tilde{\alpha}_k$.*

In the context of optimal control of PDE's the assumptions (A_0) and (B_0) were first introduced in [17] and for $k = 1, 2$ in [2]. Assumption 3(B_0) originates from optimal control theory of ODE's where it was first introduced in [23] to deal with nonlinear affine optimal control problems. The cases $k = 1, 2$ are extensions, adapted to the nature of the PDE setting, while the case $k = 0$ can be hard to verify if a structural assumption like (2.15) is not imposed. The assumptions corresponding to $k = 1, 2$ are applicable for the case of optimal controls that need not be bang-bang, especially the case $k = 2$ seems natural for obtaining state stability. Assumption (A_k) implies strong (local) optimality, while Assumption (B_k) leads to weak (local) optimality. As seen below, in some cases the two assumptions are equivalent.

For an optimal control problem subject to an semilinear elliptic equation the claim of the next proposition with $k = 0$ was proven in [2, Proposition 5.2].

Proposition 8. *For any $k \in \{0,1,2\}$, Assumption (A_k) implies (B_k). If $\bar{u}$ is bang-bang (that is, $\bar{u}(x,t) \in \{u_a(x,t), u_b(x,t)\}$ for a.e. $(x,t) \in Q$) then assumptions (A_k) and (B_k) are equivalent.*

The proof is given in Appendix A.

Remark 3. *We compare the items in Assumption 3 to the ones using* (2.15) *and* (2.17) *or* (2.14).

1. *Assumption 3(A_0) is implied by the structural assumption* (2.15) *and also allows for negative curvature, similar to* (2.17)*. For details see [17, Theorem 6.3].*

2. *Assumption 3(A_1) is implied by the structural assumption* (2.15) *together with* (2.14)*. This is clear by* (2.16) *and by using v and w as defined in Lemma 13 and arguing as in Corollary 14, both presented below in this section.*

3. *Assumption 3(A_2) is implied by* (2.14) *together with the first order necessary condition.*

3.1 Sufficiency for optimality of the unified condition

In this subsection we show that assumptions 3(A_k) and (B_k) are sufficient either for strict weak or strict strong local optimality, correspondingly.

Theorem 9. *The following holds.*

1. *Let $m = 0$ in* (1.4)*. Let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (2.6)–(2.8) *and Assumption 3(A_k) with some $k \in \{0, 1, 2\}$. Then, there exist $\varepsilon_k, \kappa_k > 0$ such that:*

$$J(\bar{u}) + \frac{\kappa_k}{2} \|y_u - \bar{y}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)} \leq J(u) \tag{3.2}$$

for all $u \in \mathcal{U}$ such that $\|y_u - \bar{y}\|_{C(\bar{Q})} < \varepsilon_k$.

2. *Let $\bar{u} \in \mathcal{U}$ satisfy the optimality conditions* (2.6)–(2.8) *and Assumption 3(B_k) with some $k \in \{0, 1, 2\}$. Then, there exist $\varepsilon_k, \kappa_k > 0$ such that* (3.2) *holds for all $u \in \mathcal{U}$ such that $\|u - \bar{u}\|_{L^1(Q)} < \varepsilon_k$.*

Before presenting a proof of Theorem 9, we establish some technical results. The following lemma was proved for various types of objective functionals, see e.g. [10, Lemma 6],[9, Lemma 3.11]. Nevertheless, our objective functional is more general, therefore we present in Appendix A an adapted proof.

Lemma 10. *Let $\bar{u} \in \mathcal{U}$. The following holds.*

1. *Let $m = 0$ hold. For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \leq \rho \|z_{\bar{u}, u-\bar{u}}\|^2_{L^2(Q)} \tag{3.3}$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \varepsilon$ and $\theta \in [0, 1]$.

2. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that* (3.3) *holds for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \varepsilon$ and $\theta \in [0, 1]$.*

For the assumptions with $k \in \{0, 1\}$, we need the subsequent corollary, which is also given in Appendix A.

Corollary 11. *Let $\bar{u} \in \mathcal{U}$. The following holds for $m = 0$:*

1. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \leq \rho \|z_{\bar{u}, u-\bar{u}}\|_{L^2(Q)} \|u - \bar{u}\|_{L^1(Q)} \tag{3.4}$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \varepsilon$ and for all $\theta \in [0, 1]$.

2. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2| \leq \rho \|u - \bar{u}\|^2_{L^1(Q)} \tag{3.5}$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \varepsilon$ and for all $\theta \in [0, 1]$.

The same assertions hold for $m \neq 0$ if one requires $\|u - \bar{u}\|_{L^1(Q)}$ to be small instead of $\|y_u - \bar{y}\|_{C(\bar{Q})}$.

The next lemma clams that Assumption 3 implies a growth similar to (3.2) of the first derivative of the objective functional in a neighborhood of $\bar{u}$.

Lemma 12. *The following claims are fulfilled.*

1. *Let $m = 0$ and $\bar{u}$ satisfy assumption (A_k), for some $k \in \{0, 1, 2\}$. Then, there exist $\bar{\alpha}_k, \bar{\gamma}_k > 0$ such that*
$$J'(u)(u - \bar{u}) \geq \bar{\gamma}_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)} \tag{3.6}$$
for every $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \bar{\alpha}_k$.
2. *Let $\bar{u}$ satisfy assumption (B_k) for some $k \in \{0, 1, 2\}$. Then, there exist $\bar{\alpha}_k, \bar{\gamma}_k > 0$ such that* (3.6) *holds for every $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \bar{\alpha}_k$.*

Proof. Since J is of class C^2 we can use the mean value theorem to infer the existence of a function $\theta : Q \to [0, 1]$ such that
$$J'(u)(u - \bar{u}) - J'(\bar{u})(u - \bar{u}) = J''(\bar{u} + \theta(u - \bar{u}))(u - \bar{u})^2$$
and under (A_k) in Assumption 3, we infer the existence of positive constants γ_k and α_k such that
$$\begin{aligned} J'(u)(u - \bar{u}) &= J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 + [J'(u)(u - \bar{u}) - J'(\bar{u})(u - \bar{u}) - J''(\bar{u})(u - \bar{u})^2] \\ &\geq \gamma_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)} - |[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})](u - \bar{u})^2|, \end{aligned}$$
for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \alpha_k$. Using Lemma 10, we obtain that
$$J'(u)(u - \bar{u}) \geq (\gamma_k - \rho_k) \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)}$$
for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \bar{\alpha}_k$ and $\bar{\alpha_k} := \min\{\alpha_k, \varepsilon_k\}$, where $\varepsilon_k > 0$ is chosen such that $\bar{\gamma}_k := \gamma_k - \rho_k > 0$ holds. Using Corollary 11 and the estimate $\|y_u - \bar{y}\|_{L^\infty(Q)} \leq C_r (2M_{\mathcal{U}})^{\frac{r-1}{r}} \|u - \bar{u}\|^{\frac{1}{r}}_{L^1(Q)}$, proves the case for (3.6). ∎

Finally, we conclude this subsection with the proof of Theorem 9.

Proof of Theorem 9. Using the Taylor expansion and the optimality condition $J'(\bar{u})(u - \bar{u}) \geq 0$ we have
$$J(u) = J(\bar{u}) + J'(\bar{u})(u - \bar{u}) + \frac{1}{2} J''(u_\theta)(u - \bar{u})^2 \geq J(\bar{u}) + \frac{1}{2} J'(\bar{u})(u - \bar{u}) + \frac{1}{2} J''(u_\theta)(u - \bar{u})^2$$
where $u_\theta := \bar{u} + \theta(u - \bar{u})$, with $\theta : Q \to [0, 1]$. We continue this inequality, using that by Assumption 3 there exist $\alpha_k > 0$ and $\gamma_k > 0$ such that (3.2) holds:
$$\begin{aligned} J(u) &\geq J(\bar{u}) + \frac{1}{2}[J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2] + \frac{1}{2}[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2] \\ &\geq J(\bar{u}) + \frac{\gamma_k}{2} \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)} - \frac{1}{2} \big| [J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2 \big| \end{aligned}$$
for all $u \in \mathcal{U}$ with either $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_k$ or $\|u - \bar{u}\|_{L^1(Q)} < \alpha_k$, depending on the chosen assumption (A_k) or (B_k). Now, either by Lemma 10 or Corollary 11 (depending on the assumption) there exist $\varepsilon > 0$ and $\bar{\gamma}_k < \gamma_k$ such that
$$|[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2| \leq \bar{\gamma}_k \|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)} \|u - \bar{u}\|^{2-k}_{L^1(Q)}$$

for every $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{C(\bar{Q})} < \varepsilon$. We may choose $\bar{\alpha}_k > 0$ and $\bar{\gamma}_k > 0$ according to Lemma 12 and depending on the chosen assumption therein. Inserting this estimate in the above expression and applying (1.18) gives

$$J(u) \geq J(\bar{u}) + \frac{1}{2}(\gamma_k - \bar{\gamma}_k)\|z_{\bar{u},u-\bar{u}}\|^k_{L^2(Q)}\|u - \bar{u}\|^{2-k}_{L^1(Q)} \geq J(\bar{u}) + \frac{3(\gamma_k - \bar{\gamma}_k)}{4}\|y_u - \bar{y}\|^k_{L^2(Q)}\|u - \bar{u}\|^{2-k}_{L^1(Q)},$$

for all $u \in \mathcal{U}$ with either $\|y_u - \bar{u}\|_{L^\infty(Q)} < \min\{\varepsilon, \bar{\alpha}_k\}$ or $\|u - \bar{u}\|_{L^1(Q)} < \min\{\frac{\varepsilon^r}{C_r^r(2M_{\mathcal{U}})^{(r-1)}}, \bar{\alpha}_k\}$ depending on the selected $k \in \{0, 1, 2\}$. To complete the proof of the second claim of the theorem we use that

$$\|y_u - \bar{y}\|_{L^\infty(Q)} \leq C_r(2M_{\mathcal{U}})^{\frac{r-1}{r}}\|u - \bar{u}\|^{\frac{1}{r}}_{L^1(Q)}$$

to apply Lemma 10 or Corollary 11 depending on $k \in \{0, 1, 2\}$.

□

3.2 Some equivalence results for the assumptions on cones

In this subsection we show that some of the items in Assumption 3 can be formulated equivalently on the cones $D^\tau_{\bar{u}}$ or $C^\tau_{\bar{u}}$ respectively. This applies to (B_k) or to (A_k) depending on whether the objective functional explicitly depends on the control or not. We need the next lemma, the proof of which uses a result from [7].

Lemma 13. *Let $\bar{u} \in \mathcal{U}$ satisfy the first order optimality condition* (2.6)-(2.8) *and let $u \in \mathcal{U}$ be given. For $\tau > 0$, we define*

$$v := \begin{cases} 0 & \text{on} \quad [\ |\frac{\partial \bar{H}}{\partial u}| > \tau\ ], \\ u - \bar{u} & \text{else}, \end{cases}$$

and $w := u - \bar{u} - v$. Let $\varepsilon > 0$ be given. Then there exists a constant $C > 0$ such that

$$\max\{\|z_{\bar{u},w}\|_{L^\infty(Q)},\ \|z_{\bar{u},v}\|_{L^\infty(Q)}\} < C\max\{\varepsilon, \varepsilon^{\frac{1}{r}}\} \tag{3.7}$$

for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \varepsilon$. Let $\varepsilon_0 > 0$ be such that (1.18) *holds. If the control does not appear explicitly in* (1.1) *(that is, $m = g = 0$ in* (1.4)*), then* (3.7) *holds for all $u \in \mathcal{U}$ such that $u - \bar{u} \in G^\tau_{\bar{u}}$ and $\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)} < \varepsilon_0$.*

Proof. We define $\tilde{u}, \hat{u} \in \mathcal{U}$ by

$$\tilde{u} := \begin{cases} \bar{u} & \text{on} \quad [\ |\frac{\partial \bar{H}}{\partial u}| > \tau\ ], \\ u & \text{else}. \end{cases} \qquad \hat{u} := \begin{cases} u & \text{on} \quad [\ |\frac{\partial \bar{H}}{\partial u}| > \tau\ ], \\ \bar{u} & \text{else}. \end{cases}$$

Observe that $v = \tilde{u} - \bar{u}$, $w = \hat{u} - \bar{u}$ and $u - \bar{u} = v + w$. It is trivial by construction that $\|v\|_{L^1(Q)},\ \|w\|_{L^1(Q)} \leq \|u - \bar{u}\|_{L^1(Q)}$. On the other hand, by (1.18), $\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)} < \varepsilon$ implies $\|y_u - y_{\bar{u}}\|_{L^\infty(Q)} < 2\varepsilon$. If $m, g = 0$, we can argue as in [7] using $u - \bar{u} \in G^\tau_{\bar{u}}$ and the definition of w, to estimate

$$\tau\|w\|_{L^1(Q)} \leq J'(\bar{u})(u - \bar{u}) \leq \tau\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)}.$$

Thus by Theorem 1 and (1.16)

$$\|z_{\bar{u},w}\|_{L^\infty(Q)} \leq \begin{cases} C_0\|z_{\bar{u},u-\bar{u}}\|^{1/r}_{L^\infty(Q)} & \text{if} \quad m, g = 0,\ u - \bar{u} \in G^\tau_{\bar{u}}, \\ C_0\|u - \bar{u}\|^{1/r}_{L^1(Q)} & \text{else}, \end{cases}$$

with $C_0 := C_r(2M_{\mathcal{U}})^{\frac{r-1}{r}}$. For $z_{\bar{u},v}$, we estimate with $C := 2(C_0 + 1)$

$$\|z_{\bar{u},v}\|_{L^\infty(Q)} \leq \|z_{\bar{u},v+w}\|_{L^\infty(Q)} + \|-z_{\bar{u},w}\|_{L^\infty(Q)} \leq C\max\{\varepsilon, \varepsilon^{\frac{1}{r}}\}.$$

In the second case the estimate holds trivially. □

Now we continue with the equivalence properties.

Corollary 14. *For $k \in \{0,2\}$, Assumption 3(B_k) is equivalent to the following condition ($\bar{B}_k$): there exist constants $\alpha_k, \gamma_k, \tau > 0$ such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_k \|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}^k \|u-\bar{u}\|_{L^1(Q)}^{2-k}, \tag{3.8}$$

for all $u \in \mathcal{U}$ for which $(u-\bar{u}) \in D_{\bar{u}}^{\tau}$ and $\|u-\bar{u}\|_{L^1(Q)} < \alpha_k$.

Proof. Let $k \in \{0,2\}$. If (B_k) holds then ($\bar{B}_k$) is obviously also fulfilled. Now let ($\bar{B}_k$) hold. The numbers $\tilde{\alpha}_k$ and $\tilde{\gamma}_k > 0$ will be chosen later so that assumption (B_k) will hold with these numbers. For now we only require that $0 < \tilde{\alpha}_k < \alpha_k$. Choose an arbitrary $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(Q)} < \tilde{\alpha}_k$. We only need to prove (3.1) in the case $u - \bar{u} \notin D_{\bar{u}}^{\tau}$. Take v and w as defined in Lemma 13. Clearly by definition $v \in D_{\bar{u}}^{\tau}$. As a direct consequence of (2.3)-(2.4) and Assumption 1 and 2 there exists a constant $C_0 > 0$ such that

$$|J''(\bar{u})(w)^2| \leq C_0 \|z_{\bar{u},w}\|_{L^\infty(Q)} \|w\|_{L^1(Q)}, \tag{3.9}$$

$$|J''(\bar{u})(w,v)| \leq C_0 \|z_{\bar{u},v}\|_{L^\infty(Q)} \|w\|_{L^1(Q)}. \tag{3.10}$$

We estimate

$$\Big| J''(\bar{u})(w)^2 + 2J''(\bar{u})(w,v) \Big| \leq 3C_0 (\|z_{\bar{u},w}\|_{L^\infty(Q)} + \|z_{\bar{u},v}\|_{L^\infty(Q)}) \|w\|_{L^1(Q)} \tag{3.11}$$

Since $\tilde{\alpha}_k < \alpha_k$ and $v \in D_{\bar{u}}^{\tau}$ we may apply (3.8) with v instead of $u - \bar{u}$. Using also (3.11), we estimate

$$\begin{aligned}
&J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 = J'(\bar{u})(v+w) + J''(\bar{u})(v+w)^2 \\
&\geq J'(\bar{u})(v) + J'(\bar{u})(w) + J''(\bar{u})(v)^2 + J''(\bar{u})(w)^2 + 2J''(\bar{u})(w,v) \geq \gamma_k \|z_{\bar{u},v}\|_{L^2(Q)}^k \|v\|_{L^1(Q)}^{2-k} + \tau \|w\|_{L^1(Q)} \\
&- 3C_0 (\|z_{\bar{u},w}\|_{L^\infty(Q)} + \|z_{\bar{u},v}\|_{L^\infty(Q)}) \|w\|_{L^1(Q)} \geq \gamma_k \|z_{\bar{u},v}\|_{L^2(Q)}^k \|v\|_{L^1(Q)}^{2-k} + \frac{\tau}{2} \|w\|_{L^1(Q)}.
\end{aligned}$$

In the last inequality we use that by choosing $\tilde{\alpha}_k > 0$ sufficiently small we may ensure that

$$\tau - 3C_0 (\|z_{\bar{u},w}\|_{L^\infty(Q)} + \|z_{\bar{u},v}\|_{L^\infty(Q)}) \geq \tau - 3C_0 C \max\{\tilde{\alpha}, \tilde{\alpha}^{\frac{1}{r}}\} \geq \frac{\tau}{2}.$$

This is implied by the inequalities $\|z_{\bar{u},w}\|_{L^\infty}$, $\|z_{\bar{u},v}\|_{L^\infty(Q)} \leq C_r \tilde{\alpha}_k^{\frac{1}{r}}$ resulting from Lemma 13. Further, we find

$$\|w\|_{L^1(Q)} \geq \begin{cases} \frac{1}{2M_{\mathcal{U}}} \|w\|_{L^1(Q)}^2 \\ \frac{1}{C_r (2M_{\mathcal{U}})^{(1/r)}} \|z_{\bar{u},w}\|_{L^2(Q)}^2, \end{cases}$$

where we used that $\|u-\bar{u}\|_{L^1(Q)} < 2M_{\mathcal{U}}$ for all $u \in \mathcal{U}$ and

$$\|z_{\bar{u},w}\|_{L^2(Q)}^2 \leq \|z_{\bar{u},w}\|_{L^1(Q)} \|z_{\bar{u},w}\|_{L^\infty(Q)} \leq \|w\|_{L^1(Q)} C_r (2M_{\mathcal{U}})^{1/r}.$$

For $k = 0$:

$$\begin{aligned}
J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_0 \|v\|_{L^1(Q)}^2 + \frac{\tau}{2M_{\mathcal{U}}} \|w\|_{L^1(Q)}^2 &\geq \min\Big\{\gamma_0, \frac{\tau}{2M_{\mathcal{U}}}\Big\} (\|v\|_{L^1(Q)}^2 + \|w\|_{L^1(Q)}^2) \\
&\geq \frac{2}{3} \min\Big\{\gamma_0, \frac{\tau}{2M_{\mathcal{U}}}\Big\} (\|u-\bar{u}\|_{L^1(Q)}^2.
\end{aligned}$$

For $k = 2$:

$$\begin{aligned}
&J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_2 \|z_{\bar{u},v}\|_{L^2(Q)}^2 + \frac{\tau}{2} \|w\|_{L^1(Q)} \\
&\geq \min\Big\{\gamma_2, \frac{\tau}{2C_r (2M_{\mathcal{U}})^{(1/r)}}\Big\} (\|z_{\bar{u},v}\|_{L^2(Q)}^2 + \|z_{\bar{u},w}\|_{L^2(Q)}^2) \geq \frac{2}{3} \min\Big\{\gamma_2, \frac{\tau}{2C_r (2M_{\mathcal{U}})^{(1/r)}}\Big\} \|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)}^2.
\end{aligned}$$

This proves that (3.1) is satisfied with an appropriate number $\tilde{\gamma}_k$. □

If the control does not appear explicitly in the objective functional, we obtain a stronger result.

Corollary 15. *Let $m, g = 0$. Then Assumption 3(A_2) is equivalent to the following condition ($\bar{A}_2$): there exist constants $\alpha_2, \gamma_2, \tau > 0$ such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq \gamma_2 \|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)} \tag{3.12}$$

for all $u \in \mathcal{U}$ for which $(u-\bar{u}) \in C^\tau_{\bar{u}}$ and $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_2$.

Proof. It is obvious that (A_2) implies $(\bar{A}_2)$. For the reverse, if $u - \bar{u} \in C^\tau_{\bar{u}}$ the estimate holds trivially. We need to consider the cases $u - \bar{u} \notin G^\tau_{\bar{u}}$ and $u - \bar{u} \notin D^\tau_{\bar{u}}$ with $u - \bar{u} \in G^\tau_{\bar{u}}$. For the first, we argue as follows. Since $u - \bar{u} \notin G^\tau_{\bar{u}}$ it holds

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u}) > \tau \|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)} \geq \frac{\tau}{2C_r|Q|^{\frac{1}{r}} M_{\mathcal{U}}} \|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}.$$

For the second case $u - \bar{u} \in G^\tau_{\bar{u}}$ and $u - \bar{u} \notin D^\tau_{\bar{u}}$, let $\tilde{\alpha} > 0$ be smaller than α_2, so that (3.12) and the prerequisite of Lemma 13 is satisfied. We define w, v as in Lemma 13. By the choice of α_2, Lemma 13 gives the existence of a constant $C > 0$ such that $\|z_{\bar{u},u-\bar{u}}\|_{L^\infty} < \alpha_2$ implies

$$\max\{\|z_{\bar{u},w}\|_{L^\infty(Q)},\ \|z_{\bar{u},v}\|_{L^\infty(Q)}\} < C \max\{\alpha_2, \alpha_2^{\frac{1}{r}}\}.$$

Now we can proceed by the same arguments as in Corollary 14

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 = J'(\bar{u})(v+w) + J''(\bar{u})(v+w)^2 \geq \gamma_2 \|z_{\bar{u},v}\|^2_{L^2(Q)} + \frac{\tau}{2}\|w\|_{L^1(Q)}.$$

Finally, we use the estimate

$$\|z_{\bar{u},w}\|^2_{L^2(Q)} \leq \|z_{\bar{u},w}\|_{L^1(Q)} \|z_{\bar{u},w}\|_{L^\infty(Q)} \leq \|w\|_{L^1(Q)} C_r (2M_{\mathcal{U}})^{(1/r)}$$

to find

$$\begin{aligned} J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &\geq \gamma_2 \|z_{\bar{u},v}\|^2_{L^2(Q)} + \frac{\tau}{2C_r(2M_{\mathcal{U}})^{(1/r)}} \|w\|_{L^1(Q)} \\ &\geq \min\Big\{\gamma_2, \frac{\tau}{2C_r(2M_{\mathcal{U}})^{(1/r)}}\Big\}(\|z_{\bar{u},v}\|^2_{L^2(Q)} + \|z_{\bar{u},w}\|^2_{L^2(Q)}) \\ &\geq \min\Big\{\gamma_2, \frac{\tau}{2C_r(2M_{\mathcal{U}})^{(1/r)}}\Big\}(\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}, \end{aligned}$$

for all $(u-\bar{u}) \in C^\tau_{\bar{u}}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha_2$. ∎

4 Strong metric Hölder subregularity and auxiliary results

We study the strong metric Hölder subregularity property (SMHSr) of the optimality map. This is an extension of the strong metric subregularity property (see, [18, Section 3I] or [15, Section 4]) dealing with Lipschitz stability of set-valued mappings. The SMHSr property is especially relevant to the parabolic setting where Lipschitz stability may fail.

4.1 The optimality mapping

We begin by defining some operators used to represent the optimality map in a more convenient way. This is done analogously to [17, Section 2.1]. Given the initial data y_0 in (1.2), we define the set

$$D(\mathcal{L}) := \Big\{ y \in W(0,T) \cap L^\infty(Q) \Big| \ \Big(\frac{d}{dt} + \mathcal{A}\Big) y \in L^r(Q), y(\cdot, 0) = y_0 \Big\}.$$

To shorten notation, we define $\mathcal{L} : D(\mathcal{L}) \to L^r(Q)$ by $\mathcal{L} := \frac{d}{dt} + \mathcal{A}$. Additionally, we define the operator $\mathcal{L}^* : D(\mathcal{L}^*) \to L^r(Q)$ by $\mathcal{L}^* := (-\frac{d}{dt} + \mathcal{A}^*)$, where

$$D(\mathcal{L}^*) := \Big\{p \in W(0,T) \cap L^\infty(Q) \Big| \Big(-\frac{d}{dt} + \mathcal{A}^*\Big)p \in L^r(Q), p(\cdot, T) = 0\Big\}.$$

With the operators $\mathcal{L}$ and $\mathcal{L}^*$, we recast the semilinear state equation (1.2) and the linear adjoint equation (2.7) in a short way:

$$\mathcal{L}y = u - f(\cdot, y)$$
$$\mathcal{L}^*p = L_y(\cdot, y_u, u) - pf_y(\cdot, y_u) = \frac{\partial H}{\partial y}(\cdot, y_u, p, u).$$

The normal cone to the set $\mathcal{U}$ at $u \in L^1(Q)$ is defined in the usual way:

$$N_{\mathcal{U}}(u) := \begin{cases} \{\nu \in L^\infty(Q) \big| \int_Q \nu(v-u)\,\mathrm{d}x\,\mathrm{d}t \leq 0 \quad \forall v \in \mathcal{U}\} & \text{if } u \in \mathcal{U}, \\ \emptyset & \text{if } u \notin \mathcal{U}. \end{cases}$$

The first order necessary optimality condition for problem (1.1)-(1.3) in Theorem 7 can be recast as

$$\begin{cases} 0 &= \mathcal{L}y + f(\cdot, y) - u, \\ 0 &= \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot, y, p, u), \\ 0 &\in H_u(\cdot, y, p) + N_{\mathcal{U}}(u). \end{cases} \tag{4.1}$$

For (4.1) to make sense, a solution (y, p, u) must satisfy $y \in D(\mathcal{L})$, $p \in D(\mathcal{L}^*)$ and $u \in \mathcal{U}$. For a local solution $\bar{u} \in \mathcal{U}$ of problem (1.1)-(1.3), by Theorem 7, the triple $(y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ is a solution of (4.1). We define the sets

$$\mathcal{Y} := D(\mathcal{L}) \times D(\mathcal{L}^*) \times \mathcal{U} \quad \text{and} \quad \mathcal{Z} := L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega), \tag{4.2}$$

and consider the set-valued mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ given by

$$\Phi\begin{pmatrix} y \\ p \\ u \end{pmatrix} := \begin{pmatrix} \mathcal{L}y + f(\cdot, y) - u \\ \mathcal{L}^*p - \frac{\partial H}{\partial y}(\cdot, y, p, u) \\ \frac{\partial H}{\partial u}(\cdot, y, p, u) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{4.3}$$

With the abbreviation $\psi := (y, p, u)$, the system (4.1) can be rewritten as the inclusion $0 \in \Phi(\psi)$. Our goal is to study the stability of system (4.1), or equivalently, the stability of the solutions of the inclusion $0 \in \Phi(\psi)$ under perturbations. For elements $\xi, \eta \in L^r(\Omega)$ and $\rho \in L^\infty(\Omega)$ we consider the perturbed system

$$\begin{cases} \xi &= -\mathcal{L}y + f(\cdot, y) - u, \\ \eta &= -\mathcal{L}p + \frac{\partial H}{\partial y}(\cdot, y, p, u), \\ \rho &\in \frac{\partial H}{\partial u}(\cdot, y, p) + N_{\mathcal{U}}(u), \end{cases} \tag{4.4}$$

which is equivalent to the inclusion $\zeta := (\xi, \eta, \rho) \in \Phi(\psi)$.

Definition 2. *The mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is called the* optimality mapping *of the optimal control problem (1.1)-(1.3).*

Theorem 16. *For any perturbation $\zeta := (\xi, \eta, \rho) \in L^r(Q) \times L^r(Q) \times L^\infty(Q)$ there exists a triple $\psi := (y, p, u) \in \mathcal{Y}$ such that $\zeta \in \Phi(\psi)$.*

Proof. We consider the optimal control problem

$$\min_{u \in \mathcal{U}} \Big\{\mathcal{J}(u) + \int_Q \eta y \;\mathrm{dxdt} - \int_Q \rho u \;\mathrm{dxdt}\Big\},$$

subject to

$$\begin{cases} \mathcal{L}y + f(x,t,y) = u + \xi \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \ y(\cdot,0) = y_0 \text{ in } \Omega. \end{cases}$$

Under assumptions 1 and 2, we have by standard arguments the existence of a global solution $\tilde{u}$. Then $\tilde{u}$ and the corresponding state $y_{\tilde{u}}$ and adjoint state $p_{\tilde{u}}$ satisfy (4.4). □

The following extension of the previous theorem can be proved along the lines of [17, Theorem 4.12].

Theorem 17. *Let Assumption 3(A_0) hold. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\zeta \in B_Z(0;\delta)$ there exists $\psi \in B_Y(\bar{\psi};\varepsilon)$ satisfying the inclusion $\zeta \in \Phi(\psi)$.*

4.2 Strong metric Hölder subregularity: main result

This subsection contains one of the main results in this paper: estimates of the difference between the solutions of the perturbed system (4.4) and a reference solution of the unperturbed one, (4.1), by the size of the perturbations. This will be done using the notion of *strong metric Hölder subregularity* introduced in the next paragraphs.

Given a metric space $(\mathcal{X}, d_{\mathcal{X}})$, we denote by $B_{\mathcal{X}}(c,\alpha)$ the closed ball of center $c \in \mathcal{X}$ and radius $\alpha > 0$. The spaces $\mathcal{Y}$ and $\mathcal{Z}$, introduced in (4.2), are endowed with the metrics

$$\begin{aligned} d_{\mathcal{Y}}(\psi_1,\psi_2) &:= \|y_1 - y_2\|_{L^2(Q)} + \|p_1 - p_2\|_{L^2(Q)} + \|u_1 - u_2\|_{L^1(Q)}, \\ d_{\mathcal{Z}}(\zeta_1,\zeta_2) &:= \|\xi_1 - \xi_2\|_{L^2(Q)} + \|\eta_1 - \eta_2\|_{L^2(Q)} + \|\rho_1 - \rho_2\|_{L^\infty(Q)}, \end{aligned} \tag{4.5}$$

where $\psi_i = (y_i, p_i, u_i)$ and $\zeta_i = (\xi_i, \eta_i, \rho_i)$, $i \in \{1,2\}$. From now on, we denote $\bar{\psi} := (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ to simplify notation.

Definition 3. *Let $\bar{\psi}$ satisfy $0 \in \Phi(\bar{\psi})$. We say that the optimality mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is* strongly metrically Hölder subregularity *(SMHSr) at $(\bar{\psi}, 0)$ with exponent $\theta > 0$ if there exist positive numbers α_1, α_2 and κ such that*

$$d_{\mathcal{Y}}(\psi, \bar{\psi}) \le \kappa d_{\mathcal{Z}}(\zeta, 0)^\theta$$

for all $\psi \in B_{\mathcal{Y}}(\bar{\psi};\, \alpha_1)$ and $\zeta \in B_{\mathcal{Z}}(0;\, \alpha_2)$ satisfying $\zeta \in \Phi(\psi)$.

Notice that applying the definition with $\zeta = 0$ we obtain that $\bar{\psi}$ is the unique solution of the inclusion $0 \in \Phi(\psi)$ in $B_{\mathcal{Y}}(\bar{\psi};\, \alpha_1)$. In particular, $\bar{u}$ is a strict local minimizer for problem (1.1)-(1.3).

In the next assumption we introduce a restriction on the set of admissible perturbations, call it Γ, which is valid for the remaining part of this section.

Assumption 4. *For a fixed positive constant C_{pe}, the admissible perturbation $\zeta = (\xi, \eta, \rho) \in \Gamma \subset \mathcal{Z}$ satisfy the restriction*

$$\|\xi\|_{L^r(Q)} \le C_{pe}. \tag{4.6}$$

For any $u \in \mathcal{U}$ and $\zeta \in \Gamma$ we denote by $(y_u^\zeta, p_u^\zeta, u)$ a solution of the first two equations in (4.4). Using (1.12) in Theorem 3 we obtain the existence of a constant K_y such that

$$\|y_u^\zeta\|_{L^\infty(\bar{Q})} \le K_y \quad \forall u \in \mathcal{U} \ \ \forall \zeta \in \Gamma. \tag{4.7}$$

Then for every $u \in \mathcal{U}$, every admissible disturbance ζ, and the corresponding solution y of the first equation in (4.4) it holds that $(y_u^\zeta(x,t), u(x,t)) \in R := [-K_y, K_y] \times [u_a, u_b]$.

Remark 4. *We apply the local properties in Assumption 2 to the interval $[-K_y, K_y]$, and denote further by $\bar{C}$ a constant that majorates the bounds and the Lipschitz constants of f and L_0 and their first and second derivatives with respect to $y \in [-K_y, K_y]$.*

By increasing the constant K_y, if necessary, we may also estimate the adjoint state:

$$\|p_u^\zeta\|_{L^\infty(\bar{Q})} \leq K_y(1 + \|\eta\|_{L^r}(Q)) \quad \forall u \in \mathcal{U} \;\; \forall \zeta \in \Gamma.$$

This follows from Theorem 1 with $\alpha = -\frac{\partial f}{\partial y}(x, t, y_u^\zeta)$ and with $\frac{\partial L}{\partial y}(x, t, y_u^\zeta, u)$ at the place of u.

We need some technical lemmas before stating our main result.

Lemma 18. *Let $u \in \mathcal{U}$ be given and $v, \eta \in L^r(Q)$, $\xi \in L^\infty(Q)$. Consider solutions y_u, y_u^ξ, p_u and p_u^η of the equations*

$$\begin{cases} \mathcal{L}y + f(\cdot, y) &= u + \xi, \\ \mathcal{L}y + f(\cdot, y) &= u, \end{cases} \quad \begin{cases} \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot, y_u^\xi, p, u) &= \eta, \\ \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot, y_u, p, u) &= 0, \end{cases} \tag{4.8}$$

and solutions $z_{\bar{u},v}^\xi$, $z_{\bar{u},v}$ of

$$\begin{cases} \mathcal{L}z + f_y(\cdot, y_u^\xi)z &= v, \\ \mathcal{L}z + f_y(\cdot, y_u)z &= v. \end{cases} \tag{4.9}$$

There exists constants $\beta_i > 0$, $i \in \{1, 2\}$, independent of $\zeta \in \Gamma$, such that the following inequalities hold

$$\|y_u^\xi - y_u\|_{L^2(Q)} \leq \hat{C}\|\xi\|_{L^2(Q)}, \tag{4.10}$$

$$\|z_{u,v}^\xi - z_{u,v}\|_{L^2(Q)} \leq \beta_1 \|\xi\|_{L^r(Q)} \|z_{u,v}\|_{L^2(Q)}, \tag{4.11}$$

$$\|z_{u,v}^\xi - z_{u,v}\|_{L^s(Q)} \leq \beta_1 \|\xi\|_{L^2(Q)} \|z_{u,v}\|_{L^2(Q)}, \tag{4.12}$$

$$\|p_u^\eta - p_u\|_2 \leq \beta_2(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}), \tag{4.13}$$

where $\hat{C}$ is the constant given in (1.8) *and $s \in [1, \frac{n+2}{n})$.*

Proof. Subtracting the state equations in (4.8) and using the mean value theorem we obtain

$$\frac{d}{dt}(y_u^\xi - y_u) + \mathcal{A}(y_u^\xi - y_u) + \frac{\partial f}{\partial y}(x, t, y_\theta)(y_u^\xi - y_u) = \xi.$$

Then, (1.8) implies (4.10). To prove (4.11) we subtract the equations (4.9) satisfied by $z_{u,v}^\xi$ and $z_{u,v}$ to obtain

$$\frac{d}{dt}(z_{u,v}^\xi - z_{u,v}) + \mathcal{A}(z_{u,v}^\xi - z_{u,v}) + \frac{\partial f}{\partial y}(x, t, y_u^\xi)(z_{u,v}^\xi - z_{u,v}) = \Big[\frac{\partial f}{\partial y}(x, t, y_u) - \frac{\partial f}{\partial y}(x, t, y_u^\xi)\Big] z_{u,v}.$$

Now, using (1.8), the mean value theorem, and (4.6) we obtain

$$\begin{aligned} \|z_{u,v}^\xi - z_{u,v}\|_{L^2(Q)} &\leq \hat{C}\Big\|\Big[\frac{\partial f}{\partial y}(x, t, y_u) - \frac{\partial f}{\partial y}(x, t, y_u^\xi)\Big] z_{u,v}\Big\|_{L^2(Q)} \leq \hat{C}\bar{C}\|(y_u^\xi - y_u) z_{u,v}\|_{L^2(Q)} \\ &\leq \hat{C}\bar{C}\|y_u^\xi - y_u\|_{L^\infty(Q)} \|z_{u,v}\|_{L^2(Q)} \leq C_r \hat{C}\bar{C}\|\xi\|_{L^r(Q)} \|z_{u,v}\|_{L^2(Q)}. \end{aligned}$$

The proof for estimate (4.12) follows by the same argumentation but using (1.11). We denote by $\beta_1 >$ the maximum of the constants appearing in the estimate above and its analog for (4.12). Finally, we subtract the adjoint states and employ the mean value theorem to find

$$\begin{aligned} &-\frac{d}{dt}(p_u^\eta - p_u) + \mathcal{A}^*(p_u^\eta - p_u) + \frac{\partial f}{\partial y}(x, t, y_u^\xi)(p_u^\eta - p_u) \\ &= \frac{\partial^2 L}{\partial y^2}(x, t, y_\theta)(y_u^\xi - y_u) + \frac{\partial^2 f}{\partial y^2}(x, t, y_\theta)(y_u^\xi - y_u) p_u + \eta. \end{aligned}$$

The claim follows using (1.8), (1.16), and (4.7) to estimate

$$\|p_u^\eta - p_u\|_{L^2(Q)} \le (\hat{C}^2\bar{C} + M_{\mathcal{U}}\hat{C}^2\bar{C} + \hat{C})(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}).$$

□

Lemma 19. *Let $s \in [1, \frac{n+2}{n}) \cap [1,2]$. Let $u \in \mathcal{U}$ and let y_u, p_u be the corresponding state and adjoint state. Further, let y_u^ζ and p_u^ζ be solutions to the perturbed state and adjoint equation in (4.4) for the control u. There exist constants $C, \tilde{C} > 0$, independent of $\zeta \in \Gamma$, such that for $v \in \mathcal{U}$, the following estimates hold.*

1. *For $m = 0$ in (1.4):*

$$\Big|\int_Q \Big(\frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta)\Big)(v-u)\,\mathrm{d}x\,\mathrm{d}t\Big|$$
$$\le C(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)})\|z_{u,u-v}\|_{L^2(Q)} \tag{4.14}$$
$$\le \tilde{C}(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)})\|v-u\|_{L^1(Q)}^{\frac{3s-2}{2s}}. \tag{4.15}$$

2. *For a general $m \in \mathbb{R}$:*

$$\Big|\int_Q \Big(\frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta)\Big)(v-u)\,\mathrm{d}x\,\mathrm{d}t\Big| \le \tilde{C}(\|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)})\|v-u\|_{L^1(Q)}. \tag{4.16}$$

Proof. We consider the first case, $m = 0$. We begin with integrating by parts

$$\Big|\int_Q \Big(\frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\zeta)\Big)(v-u)\,\mathrm{d}x\,\mathrm{d}t\Big|$$
$$\le \Big|\int_Q \Big[\frac{\partial L_0}{\partial y}(x,t,y_u)z_{u,u-v} - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta)z_{u,u-v}^\zeta\Big]\,\mathrm{d}x\,\mathrm{d}t\Big| + \Big|\int_Q z_{u,u-v}^\zeta \eta\,\mathrm{d}x\,\mathrm{d}t\Big|$$
$$\le \int_Q \Big|\frac{\partial L_0}{\partial y}(x,t,y_u) - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta)\Big|\Big|z_{u,u-v}\Big|\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Big|\frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) + \eta\Big|\Big|z_{u,u-v} - z_{u,u-v}^\zeta\Big|\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \Big|\int_Q \eta z_{u,u-v}\,\mathrm{d}x\,\mathrm{d}t\Big| = I_1 + I_2 + I_3.$$

For the first term we use the Hölder inequality, the mean value theorem, (1.11), (1.16), and (4.10) to estimate

$$I_1 \le \int_Q \Big|\frac{\partial L_0}{\partial y}(x,t,y_u) - \frac{\partial L_0}{\partial y}(x,t,y_u^\zeta)\Big||z_{u,u-v}|\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \bar{C}\|y_u^\zeta - y_u\|_{L^2(Q)}\|z_{u,u-v}\|_{L^2(Q)} \le \bar{C}\hat{C}\|\xi\|_{L^2(Q)}\|z_{u,u-v}\|_{L^2(Q)}$$
$$\le \bar{C}\hat{C}C_{s'}^{1+\frac{2-s}{2}}(2M_{\mathcal{U}})^{\frac{(s'-1)(2-s)}{2s'}}\|\xi\|_{L^2(Q)}\|u-v\|_{L^1(Q)}^{1+\frac{s-2}{2s}}.$$

Here we used that by Theorem 1 and Lemma 1.11 it holds

$$\|z_{u,u-v}\|_{L^2(Q)} \le \|z_{u,u-v}\|_{L^\infty(Q)}^{\frac{2-s}{2}}\|z_{u,u-v}\|_{L^s(Q)}^{\frac{s}{2}} \le C_{s'}^{1+\frac{2-s}{2}}(2M_{\mathcal{U}})^{\frac{(s'-1)(2-s)}{2s'}}\|u-v\|_{L^1(Q)}^{\frac{2-s}{2s'}+\frac{s}{2}},$$

and noticing that $\frac{2-s}{2s'} + \frac{s}{2} = 1 - \frac{2-s}{2s}$. The second term is estimated by using (1.16), Hölder's inequality, and (4.11):

$$I_2 \le \int_Q \Big|\frac{\partial L_0}{\partial y}(x,t,y_u^\zeta) + \eta\Big|\Big|z_{u,u-v}^\zeta - z_{u,u-v}\Big|\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \beta_1 \max\{d_1, |Q|^{\frac{1}{r}}C_{pe}\}(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)})\|z_{u,u-v}\|_{L^2(Q)}$$
$$\le d_2(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)})\|u-v\|_{L^1(Q)}^{1+\frac{s-2}{2s}},$$

where $d_1 := \|\psi_{M_\mathcal{U}}\|_{L^{s'}(Q)}$ and $d_2 := \beta_1 \max\{d_1, |Q|^{\frac{1}{r}} C_{pe}\} C_{s'}^{1+\frac{2-s}{2}} (2M_\mathcal{U})^{\frac{(s'-1)(2-s)}{2s'}}$. For last term we estimate

$$I_3 \leq \Big| \int_Q \eta z_{u,u-v} \,\mathrm{d}x \,\mathrm{d}t \Big| \leq \|z_{u,u-v}\|_{L^2(Q)} \|\eta\|_{L^2(Q)}.$$

We prove the second case (4.16). By applying (1.9) and arguing as in the proof of (4.10) and (4.13) but for r, we infer the existence of a constant, again denoted by $\tilde{C} > 0$, such that:

$$\begin{aligned}
&\Big| \int_Q \Big(\frac{\partial H}{\partial u}(x,t,y_u,p_u) - \frac{\partial H}{\partial u}(x,t,y_u^\zeta,p_u^\eta) \Big)(v-u) \,\mathrm{d}x \,\mathrm{d}t \Big| \\
&= \Big| \int_Q \Big[p_u - p_u^\eta + m(y_u - y_u^\zeta) \Big](v-u) \,\mathrm{d}x \,\mathrm{d}t \Big| \\
&\leq \|p_u - p_u^\eta + m(y_u - y_u^\zeta)\|_{L^\infty(Q)} \|u - \bar{u}\|_{L^1(Q)} \\
&\leq \tilde{C}(\|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)}) \|v - u\|_{L^1(Q)}.
\end{aligned}$$

□

The main result in the paper follows.

Theorem 20. *Let assumption 3(A_0) be fulfilled for the reference solution $\bar{\psi} = (\bar{y}, \bar{p}, \bar{u})$ of $0 \in \Phi(\psi)$. Then the mapping Φ is strongly metrically Hölder subregular at $(\bar{\psi}, 0)$. More precisely, for every $\varepsilon \in (0, 1/2]$ there exist $\alpha_n > 0$ and κ_n (with α_1 and κ_1 independent of ε) such that for all $\psi \in \mathcal{Y}$ with $\|u - \bar{u}\|_{L^1(Q)} \leq \alpha_n$ and $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$, the following inequalities are satisfied.*

1. *In the case $m = 0$ in (1.4):*

$$\|\bar{u} - u\|_{L^1(Q)} \leq \kappa_n \Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)^{\theta_0}, \tag{4.17}$$

$$\|y_{\bar{u}} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar{u}} - p_u^\zeta\|_{L^2(Q)} \leq \kappa_n \Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)^{\theta}, \tag{4.18}$$

where

$$\theta_0 = \theta = 1 \qquad \text{if } n = 1, \tag{4.19}$$

$$\theta_0 = \theta = 1 - \varepsilon \qquad \text{if } n = 2, \tag{4.20}$$

$$\theta_0 = \frac{10}{11} - \varepsilon, \;\; \theta = \frac{9}{11} - \varepsilon \qquad \text{if } n = 3. \tag{4.21}$$

2. *In the general case $m \in \mathbb{R}$:*

$$\|\bar{u} - u\|_{L^1(Q)} \leq \kappa_n \Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)} \Big), \tag{4.22}$$

$$\|y_{\bar{u}} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar{u}} - p_u^\zeta\|_{L^2(Q)} \leq \kappa_n \Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)} \Big)^{\theta_0}. \tag{4.23}$$

Proof. We begin with the proof for $m = 0$. We select $\alpha_1 < \tilde{\alpha}_0$ according to Lemma 12. Let $\zeta = (\xi, \eta, \rho) \in \mathcal{Z}$ and $\psi = (y_u^\zeta, p_u^\zeta, u)$ with $\|u - \bar{u}\|_{L^1(Q)} \leq \alpha_1$ such that $\zeta \in \Phi(\psi)$, i.e.

$$\left\{ \begin{array}{rcl}
\xi &=& \mathcal{L} y_u^\zeta + f(\cdot,\cdot,y_u^\zeta) - u, \\
\eta &=& \mathcal{L}^* p_u^\zeta - \frac{\partial H}{\partial y}(\cdot, y_u^\zeta, p_u^\zeta, u), \\
\rho &\in& \frac{\partial H}{\partial u}(\cdot, y_u^\zeta, p_u^\zeta) + N_\mathcal{U}(u).
\end{array} \right.$$

Let y_u and p_u denote the solutions to the unperturbed problem with respect to u, i.e.

$$u = \mathcal{L}y_u + f(\cdot,\cdot,y_u) \text{ and } 0 = \mathcal{L}^* p_u - \frac{\partial H}{\partial y}(\cdot, y_u, p_u, u).$$

By Lemma 18, there exists $\hat{C}, \beta_2 > 0$ independent of ψ and ζ such that

$$\|y_u^\zeta - y_u\|_{L^2(Q)} + \|p_u^\zeta - p_u\|_{L^2(Q)} \le (\hat{C} + \beta_2)\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big). \tag{4.24}$$

By the definition of the normal cone, $\rho \in \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta) + N_{\mathcal{U}}(u)$ is equivalent to

$$0 \ge \int_Q (\rho - \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta))(w-u) \quad \forall w \in \mathcal{U}.$$

We conclude for $w = \bar{u}$,

$$\begin{aligned} 0 &\ge \int_Q \frac{\partial H}{\partial u}(\cdot,\cdot,y_u,p_u)(u-\bar{u}) + \int_Q (\rho + \frac{\partial H}{\partial u}(\cdot,\cdot,y_u,p_u) - \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta))(\bar{u}-u) \\ &\ge J'(u)(u-\bar{u}) - \|\rho\|_{L^\infty(Q)}\|\bar{u}-u\|_{L^1(Q)} - \Big| \int_Q (\frac{\partial H}{\partial u}(\cdot,\cdot,y_u,p_u) - \frac{\partial H}{\partial u}(\cdot,\cdot,y_u^\zeta,p_u^\zeta))(\bar{u}-u)\,\mathrm{d}x\,\mathrm{d}t\Big|. \end{aligned} \tag{4.25}$$

By Lemma 19, we have an estimate on the third term. Since $\|u-\bar{u}\|_{L^1(Q)} < \tilde{\alpha}_0$, we estimate by Lemma 12 and Lemma 19

$$\|u-\bar{u}\|_{L^1(Q)}^2\tilde{\gamma} \le J'(u)(u-\bar{u}) \le \tilde{C}\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)\|u-\bar{u}\|_{L^1(Q)}^{1+(s-2)/(2s)} + \|\rho\|_{L^\infty(Q)}\|\bar{u}-u\|_{L^1(Q)}$$

and consequently for an adapted constant, denoted in the same way

$$\|\bar{u}-u\|_{L^1(Q)} \le \tilde{C}\Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)^{\frac{2s}{s+2}}.$$

To estimate the states, we use the estimate for the controls. We notice that $(2-s)/(2s') + s/2 = 1 + (s-2)(2s)$ and obtain

$$\|y_{\bar{u}} - y_u\|_{L^2(Q)} \le \|y_{\bar{u}} - y_u\|_{L^\infty(Q)}^{\frac{2-s}{2}}\|y_{\bar{u}} - y_u\|_{L^s(Q)}^{\frac{s}{2}} \le C_r^{\frac{2-s}{2}}\|\bar{u}-u\|_{L^1(Q)}^{1+\frac{s-2}{2s}}. \tag{4.26}$$

Thus, for a constant again denoted by $\tilde{C}$ and with $(1+\frac{s-2}{2s})\frac{2s}{s+2} = \frac{3s-2}{2+s}$,

$$\|y_{\bar{u}} - y_u\|_{L^2(Q)} \le \tilde{C}\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} + \|\rho\|_{L^\infty(Q)}\Big)^{\frac{3s-2}{2+s}}.$$

Next, we realize that by Lemma 18 and (4.2)

$$\|y_{\bar{u}} - y_u^\zeta\|_{L^2(Q)} \le \|y_{\bar{u}} - y_u\|_{L^2(Q)} + \|y_u - y_u^\zeta\|_{L^2(Q)} \le \max\{\tilde{C},\hat{C}\}\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} + \|\rho\|_{L^\infty(Q)}\Big)^{\frac{3s-2}{2+s}}.$$

Using $\|p_{\bar{u}} - p_u\|_{L^2(Q)} \le \hat{C}\|y_{\bar{u}} - y_u\|_{L^2(Q)}$ and (4.13), the same estimate holds for the adjoint state

$$\|p_{\bar{u}} - p_u^\zeta\|_{L^2(Q)} \le \|p_{\bar{u}} - p_u\|_{L^2(Q)} + \|p_u - p_u^\zeta\|_{L^2(Q)} \le (\hat{C}\tilde{C} + \beta_2)\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} + \|\rho\|_{L^\infty(Q)}\Big)^{\frac{3s-2}{2+s}},$$

subsequently we define $\kappa := \max\{\tilde{C},\hat{C}\}$. Finally, we consider the case $m \neq 0$. Using estimate 4.16 in (4.25) and arguing from that as for the case $m = 0$, we infer the existence of a constant $\tilde{C} > 0$ such that

$$\|u-\bar{u}\|_{L^1(Q)} \le \tilde{C}\Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^r(Q)} + \|\eta\|_{L^r(Q)}\Big).$$

This implies under (4.26) the estimate for the states and adjoint-states

$$\|y_{\bar u} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar u} - p_u^\zeta\|_{L^2(Q)} \le \max\{\tilde C, \hat C \tilde C + \beta_2\}\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} + \|\rho\|_{L^\infty(Q)}\Big)^{1+(s-2)/(2s)}.$$

To determine θ and θ_0 we notice that the functions $s \to \frac{s-2}{2s}$ and $s \to \frac{3s-2}{2+s}$ are monotone. Inserting the value for $\frac{n+2}{2}$ for each case $n \in \{1, 2, 3\}$ completes the proof. □

To obtain results under Assumption 3 for $k \in \{1, 2\}$, we need additional restrictions. We either don't allow perturbations ρ (appearing in the inclusion in (4.4)) or they need to satisfy

$$\rho \in D(\mathcal{L}^*). \tag{4.27}$$

Theorem 21. *Let $m = 0$ and let some of the assumptions $(A_1), (B_1)$ and $(A_2), (B_2)$ be fulfilled for the reference solution $\bar\psi = (\bar y, \bar p, \bar u)$ of $0 \in \Phi(\psi)$. Let, in addition, the set Γ of feasible perturbations be restricted to such $\zeta \in \Gamma$ for which the component ρ is either zero or satisfies (4.27). The numbers α_n, κ_n and ε are as in Theorem 20. Then the following statements hold for $n \in \{1, 2, 3\}$:*

1. Under Assumption 3, cases (A_1) and (B_1), the estimations

$$\|\bar u - u\|_{L^1(Q)} \le \kappa_n\Big(\|\mathcal{L}^*\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big),$$
$$\|y_{\bar u} - y_u^\zeta\|_{L^2(Q)} + \|p_{\bar u} - p_u^\zeta\|_{L^2(Q)} \le \kappa_n\Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)^{\theta_0},$$

with θ_0 as in Theorem 20, hold for all $u \in \mathcal{U}$ with $\|y_u - \bar y\|_{L^\infty(Q)} < \alpha_n$, in the case of (A_1), or $\|u - \bar u\|_{L^1(Q)} < \alpha_n$ in the case (B_1), and for all $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.

2. Under Assumption 3, cases (A_2) and (B_2), the estimation

$$\|\bar y - y_u^\zeta\|_{L^2(Q)} + \|\bar p - p_u^\zeta\|_{L^2(Q)} \le \kappa_n\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} + \|\mathcal{L}^*\rho\|_{L^2(Q)}\Big)$$

hold for all $u \in \mathcal{U}$ with $\|y_u - \bar y\|_{L^\infty(Q)} < \alpha_n$, in the case of (A_2), or $\|u - \bar u\|_{L^1(Q)} < \alpha_n$ in the cases (B_2), and for all $\zeta \in \Gamma$ satisfying $\zeta \in \Phi(\psi)$.

Proof. We first notice that if the perturbation ρ satisfies (4.27), it holds

$$\int_Q \rho(u - \bar u)\,\mathrm{d}x\,\mathrm{d}t = \int_Q ((\frac{d}{dt} + \mathcal{A})z_{\bar u, u-\bar u} + f_y(x,t,y_{\bar u})z_{\bar u,u-\bar u})\rho\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q ((-\frac{d}{dt} + \mathcal{A}^*)\rho + f_y(x,t,y_{\bar u})\rho)z_{\bar u,u-\bar u}\,\mathrm{d}x\,\mathrm{d}t.$$

Thus

$$\Big|\int_Q \rho(u - \bar u)\,\mathrm{d}x\,\mathrm{d}t\Big| \le \|z_{\bar u,u-\bar u}\|_{L^2(Q)}(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|f_y(x,t,y_{\bar u})\|_{L^\infty(Q)}\|\rho\|_{L^2(Q)}).$$

Under Assumption (A_1), we can proceed as in the proof of Theorem 20 using Lemma 12 and (4.15) in Lemma 19, to infer the existence of constants $\alpha, \kappa_1 > 0$ such that

$$\|\bar u - u\|_{L^1(Q)} \le \kappa_1\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big),$$

and by standard estimates the existence of a constant $\hat C > 0$ and using (1.18)

$$\|y_{\bar u} - y_u\|_{L^2(Q)} + \|p_{\bar u} - p_u\|_{L^2(Q)} \le \hat C\|y_{\bar u} - y_u\|_{L^2(Q)} \le 2\hat C\|z_{u,u-\bar u}\|_{L^2(Q)}$$
$$\le 2\hat C\kappa_1^{\frac{2s}{s+2}}\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big)^{\frac{2s}{s+2}},$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha$ or $\|u - \bar{u}\|_{L^1(Q)} < \alpha$ depending on the assumption. From here on, one can proceed as in the proof of Theorem 20 and define the final constant $\kappa > 0$ and the exponent θ_0 accordingly. Finally, by similar reasoning, under Assumption (A_2) with Lemma 12 and Lemma 19, one obtains the existence of a constant $\kappa > 0$ such that

$$\|y_{\bar{u}} - y_u\|_{L^2(Q)} + \|p_{\bar{u}} - p_u\|_{L^2(Q)} \leq \kappa\Big(\|\mathcal{L}^*\rho\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)}\Big),$$

for all $u \in \mathcal{U}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \alpha$ or $\|u - \bar{u}\|_{L^1(Q)} < \alpha$. Again, proceeding as in Theorem 20 and increasing the constant κ if needed, proves the claim. □

Remark 5. *Theorems 20 and 21 concern perturbations which are functions of x and t only. On the other hand, [15, Theorem] suggests that SMHSr implies a similar stability property under classes of perturbations that depend (in a non-linear way) on the state and control. This fact will be used and demonstrated in the next section.*

5 Stability of the optimal solution

In this section we obtain stability results for the optimal solution under non-linear perturbations in the objective functional. Namely, we consider a disturbed problem

$$(\mathrm{P}_\zeta)\ \min_{u\in\mathcal{U}} J_\zeta(u) := \int_Q [L(x,t,y(x,t),u(x,t)) + \eta(x,t,y_u(x,t),u(x,t))]\,\mathrm{d}x\,\mathrm{d}t, \tag{5.1}$$

subject to

$$\begin{cases} \frac{dy}{dt} + \mathcal{A}y + f(x,t,y) = u + \xi \ \text{ in } \ Q, \\ y = 0 \ \text{ on } \ \Sigma,\ y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{5.2}$$

where $\zeta := (\xi, \eta)$ is a perturbation. The corresponding solution will be denoted by y_u^ζ. In contrast with the previous section, the perturbation η may be state and control dependent. For this reason, here we change the notation of the set of admissible perturbations to $\hat{\Gamma}$. However, Assumption 4 will still be valid for the set $\hat{\Gamma}$. We also use the notations C_{pe}, K_y and R with the same meaning as in Subsection 4.2.

In addition to Assumption 4 we require the following that holds through the reminder of the section.

Assumption 5. *The perturbation $\eta \in L^1(Q \times R)$ for every $(\xi, \eta) \in \hat{\Gamma}$. For a.e. $(x,t) \in Q$ the function $\eta(x,t,\cdot,\cdot)$ is of class C^2 and is convex with respect to the last argument, u. Moreover, the functions $\frac{\partial\eta}{\partial y}$ and $\frac{\partial^2\eta}{\partial y^2}$ are bounded on $Q \times R$, and the second one is continuous in $(y,u) \in R$, uniformly with respect to $(t,x) \in Q$.*

Due to the linearity of (5.2) and the convexity of the objective functional (5.1) with respect to u, the proof of the next theorem is standard.

Theorem 22. *For perturbations $\zeta \in \hat{\Gamma}$ satisfying Assumption 5, the perturbed problem (P_ζ) has a global solution.*

In the next two theorems, we consider sequences of problems $\{(\mathrm{P}_{\zeta_k})\}$ with $\zeta_k \in \hat{\Gamma}$. The proofs repeat the arguments in [2, Theorem 4.2, Theorem 4.3].

Theorem 23. *Let a sequence $\{\zeta_k \in \hat{\Gamma}\}_k$ converge to zero in $L^2(Q) \times L^2(Q \times R)$ and let u_k be a local solution of problem (P_{ζ_k}), $k = 1, 2, \ldots$. Then any control $\bar{u}$ that is a weak* limit in $L^\infty(Q)$ of this sequence is a week local minimizer in problem (P), and for the corresponding solutions it holds that $y_{u_k} \to \bar{y}$ in $L^2(0,T;\, H_0^1(\Omega)) \cap L^\infty(Q)$.*

Theorem 24. *Let $\{\zeta_k\}_k$ be as in Theorem 23. Let $\bar{u}$ be a strict strong local minimizer of* (P)*. Then there exists a sequence of strong local minimizers $\{u_k\}$ of problems (P_{ζ_k}) such that $u_k \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(Q)$ and y_{u_k} converges strongly in $L^2(0,T;\, H_0^1(\Omega)) \cap L^\infty(Q)$.*

The next theorem is central in this section.

Theorem 25. *Let assumption 3(A_0) be fulfilled for the reference weakly optimal control $\bar{u}$ in problem (P) and the corresponding $\bar{y}$ and $\bar{p}$. Then there exist positive numbers α and C for which the following is fulfilled. For every perturbation $\zeta \in \hat{\Gamma}$ and for every weak local solution u_ζ of problem (P_ζ) with $\|u_\zeta - \bar{u}\|_{L^1(Q)} \leq \alpha$, the following estimates hold:*

1. *If $m = 0$ in (1.4):*

$$\|\bar{u} - u_\zeta\|_{L^1(Q)} \leq C\Big[\|\xi\|_{L^2(Q)} + \|\|\frac{d}{dy}\eta\|_{L^\infty(R)}\|_{L^2(Q)} + \|\frac{d}{du}\eta\|_{L^\infty(Q\times R)}\Big]^{\theta_0},$$

$$\|\bar{y} - y_{u_\zeta}\|_{L^2(Q)} \leq C\Big[\|\xi\|_{L^2(Q)} + \|\|\frac{d}{dy}\eta\|_{L^\infty(R)}\|_{L^2(Q)} + \|\frac{d}{du}\eta\|_{L^\infty(Q\times R)}\Big]^{\theta}.$$

2. *For $m \in \mathbb{R}$:*

$$\|\bar{u} - u_\zeta\|_{L^1(Q)} \leq C\Big[\|\xi\|_{L^r(Q)} + \|\|\frac{d}{dy}\eta\|_{L^\infty(R)}\|_{L^r(Q)} + \|\frac{d}{du}\eta\|_{L^\infty(Q\times R)}\Big],$$

$$\|\bar{y} - y_{u_\zeta}\|_{L^2(Q)} \leq C\Big[\|\xi\|_{L^r(Q)} + \|\|\frac{d}{dy}\eta\|_{L^\infty(R)}\|_{L^r(Q)} + \|\frac{d}{du}\eta\|_{L^\infty(Q\times R)}\Big]^{\theta_0}.$$

Here θ_0 and θ are defined as in Theorem 20.

Proof. The local solution $(\bar{y}, \bar{u})$ satisfies, together with the corresponding adjoint variable, the relations (4.1). Similarly, (y_{u_ζ}, u_ζ) satisfies, together with the corresponding p_{u_ζ} the perturbed optimality system (4.4) with the left-hand side given by the triple

$$\begin{pmatrix} \xi(\cdot) \\ \frac{d}{dy}(\eta(\cdot, y_{u_\zeta}(\cdot), u_\zeta(\cdot)) \\ \frac{d}{du}(\eta(\cdot, y_{u_\zeta}(\cdot), u_\zeta(\cdot)). \end{pmatrix} \tag{5.3}$$

Since it is assumed that $\|u_\zeta - \bar{u}\|_{L^1(Q)} \leq \alpha$ we may apply Theorem 20 (here we choose the same α as in this theorem) to prove the inequalities in the theorem. □

The proof of theorems 26 and 27 follows in the same spirit but using Theorem 21 instead of Theorem 20. We make an additional assumption for the perturbation η in the objective functional, namely, that $\rho := \frac{d}{du}(\eta(\cdot, y_{u_\zeta}(\cdot), u_\zeta(\cdot))$ satisfies (4.27), i.e.

$$\frac{d}{du}(\eta(\cdot, y_{u_\zeta}(\cdot), u_\zeta(\cdot)) \in D(\mathcal{L}^*). \tag{5.4}$$

For an explanation of the condition (5.4), we refer to the proof of Theorem 21.

Theorem 26. *Let $m = 0$ and Assumption 3(A_1) be fulfilled for the reference strongly optimal control $\bar{u}$ in problem (P). Then there exist positive numbers α and C for which the following is fulfilled. For every perturbation $\zeta \in \hat{\Gamma}$ and for every local solution u_ζ of problem (P_ζ) with $\|y_{u_\zeta} - \bar{y}\|_{L^\infty(Q)} \leq \alpha$, the following estimates hold.*

$$\|\bar{u} - u_\zeta\|_{L^1(Q)} \leq C\Big(\|\mathcal{L}^*\frac{d}{du}(\eta(\cdot, y_{u_\zeta}(\cdot), u_\zeta(\cdot))\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\|\frac{d}{dy}\eta\|_{L^\infty(R)}\|_{L^2(Q)}\Big)$$

and all together

$$\|\bar{y} - y_{u_\zeta}\|_{L^2(Q)} \leq C\Big(\|\mathcal{L}^*\frac{d}{du}(\eta(\cdot, y_{u_\zeta}(\cdot), u_\zeta(\cdot))\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\|\frac{d}{dy}\eta\|_{L^\infty(R)}\|_{L^2(Q)}\Big)^{\theta_0},$$

where θ_0 is defined in Theorem 20.

Theorem 27. *Let $m = 0$ and let Assumption 3(A_2) be fulfilled for the reference strongly optimal control $\bar{u}$ in problem (P). Then there exist positive numbers α and C for which the following is fulfilled. For every perturbation $\zeta \in \hat{\Gamma}$ and for every local solution u_ζ of problem (P_ζ) with $\|y_{u_\zeta} - \bar{u}\|_{L^\infty(Q)} \le \alpha$, the following estimates hold:*

$$\|\bar{y} - y_{u_\zeta}\|_{L^2(Q)} \le C\Big(\|\mathcal{L}^* \frac{d}{du}(\eta(\cdot, y_{u_\zeta}(\cdot), u_\zeta(\cdot))\|_{L^2(Q)} + \|\xi\|_{L^2(Q)} + \|\|\frac{d}{dy}\eta\|_{L^\infty(R)}\|_{L^2(Q)}\Big).$$

Remark 6. *The constraint that u_ζ needs to be close to the reference solution $\bar{u}$ in the theorems above is not a big restriction. This is clear, since Assumption 3 implies that $\bar{u}$ satisfies* (3.2)*. Hence, $\bar{u}$ is a strict strong local minimizer of* (P) *and, consequently, Theorem 24 ensures the existence of a family $\{u_{\zeta_k}\}$, $\zeta_k \in \hat{\Gamma}$, of strong local minimizers of problems* (P_ζ) *satisfying the conditions of Theorem 20 or 21.*

Example 1 (Tikhonov regularization)**.** *We consider the optimal control problem*

$$(\mathrm{P}_\lambda)\ \min_{u \in \mathcal{U}} J_\lambda(u) := \int_Q L(x,t,y(x,t),u(x,t)) + \frac{\lambda}{2}\int_Q u(x,t)^2 \,\mathrm{d}x\,\mathrm{d}t,$$

subject to (1.2) *and* (1.3)*. As before, $\bar{u}$ denotes a strict strong solution of problem (P)$\equiv$ (P_0). We assume that $\bar{u}$ satisfies Assumption 3(A_0). From Theorem 24 we know that for every sequence $\lambda_k > 0$ converging to zero there exists a sequence of strong local minimizer $\{u_{\lambda_k}\}_{k=1}^\infty$ such that $u_k \to \bar{u}$ in $L^1(Q)$ for $k \to \infty$, thus for a sufficiently large k_0 we have that for all $k > k_0$*

$$\|y_{\bar{u}} - y_{u_k}\|_{L^2(Q)} + \|p_{\bar{u}} - p_{u_k}\|_{L^2(Q)} \le C\Big(\lambda_k\Big)^\theta,$$

$$\|\bar{u} - u_k\|_{L^1(Q)} \le C\lambda_k,$$

where θ is defined in Theorem 20.

6 Examples

Here we present two examples that show particular applications in which different assumptions are involved.

Example 2 (Negative curvature)**.** *We begin with an optimal control problem, that has negative curvature. The parabolic equation has the form*

$$\begin{cases} \frac{dy}{dt} + \mathcal{A}y + \exp(y) & = u \quad \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) & = y_0 \quad \text{on } \Omega. \end{cases} \tag{6.1}$$

Let $0 \le g \in L^2(Q)$ be a function satisfying the structural assumption (2.15)*. We consider the optimal control problem*

$$\min_{u \in \mathcal{U}} \Big\{ J(u) := \int_Q (y_u + gu)\,\mathrm{d}x\,\mathrm{d}t \Big\}$$

subject to (6.1) *and with control constraints*

$$\mathcal{U} := \{u \in L^\infty(Q) |\ 0 \le u_a \le u \le u_b \ \text{ for a.a. } (x,t) \in Q\}. \tag{6.2}$$

By the weak maximum principle $y_{u_a} - y_u \le 0$ for all $u \in \mathcal{U}$ and $\bar{u} := u_a$ constitutes an optimal solution. Further, by the weak maximum principle, the adjoint-state $\bar{p}$ and the linearized states $z_{\bar{u},u-\bar{u}}$ for all $u \in \mathcal{U}$, are non-negative. Moreover, we have

$$J'(\bar{u})(u - \bar{u}) = \int_Q (\bar{p} + g)(u - \bar{u})\,\mathrm{d}x\,\mathrm{d}t \ge 0,$$

$$J''(\bar{u})(u-\bar{u})^2 = \int_Q w_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t = \int_Q -\bar{p}\exp(\bar{y})z^2_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t < 0,$$

for all $u \in \mathcal{U}$. Since g satisfies the structural assumption, there exists a constant $C > 0$ such that

$$\int_Q g(u-\bar{u})\,\mathrm{d}x\,\mathrm{d}t \geq C\|u-\bar{u}\|^2_{L^1(Q)} \quad \forall u \in \mathcal{U}.$$

On the other hand, integrating by parts we obtain

$$\int_Q \bar{p}(u-\bar{u})\,\mathrm{d}x\,\mathrm{d}t = \int_Q z_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t. \tag{6.3}$$

If for $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(Q)}$ or $\|y_u - \bar{y}\|_{L^\infty(Q)}$ sufficiently small such that

$$\frac{1}{2\|\bar{p}\exp(\bar{y})\|_{L^\infty(Q)}} > \|z_{\bar{u},u-\bar{u}}\|_{L^\infty(Q)},$$

we can absorb the term $J''(\bar{u})(u-\bar{u})^2$ by estimating

$$\begin{aligned}\int_Q \bar{p}(u-\bar{u})\,\mathrm{d}x\,\mathrm{d}t + J''(\bar{u})(u-\bar{u}) &= \int_Q z_{\bar{u},u-\bar{u}}(1-\bar{p}\exp(\bar{y})z_{\bar{u},u-\bar{u}})\,\mathrm{d}x\,\mathrm{d}t & (6.4)\\ &\geq \frac{1}{2}\int_Q z_{\bar{u},u-\bar{u}}\,\mathrm{d}x\,\mathrm{d}t \geq \frac{K}{2}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}, & (6.5)\end{aligned}$$

where the last inequality is a consequence of the boundedness of $\mathcal{U} \subset L^\infty(Q)$ that implies the existence of a constant $K > 0$ such that $\|z_{\bar{u},u-\bar{u}}\|_{L^1(Q)} \geq K\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}$ for all $u \in \mathcal{U}$. Altogether, we find

$$\begin{aligned}J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 &\geq C\|u-\bar{u}\|^2_{L^1(Q)} + \frac{K}{2}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(Q)}\\ &\geq \sqrt{\frac{CK}{2}}\|u-\bar{u}\|_{L^1(Q)}\|z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \quad \forall u \in \mathcal{U}.\end{aligned}$$

Thus, Assumption 3(A_1) is fulfilled and we can apply Theorem 21 to obtain a stability result.

Example 3 (State stability)**.** *We consider a tracking type objective functional where the control does not appear explicitly and for which we will verify (A_2). As perturbations we consider functions $\zeta = (\xi, \eta, \rho) \in D(\mathcal{L}^*) \times L^r(Q) \times L^r(Q) \times D(\mathcal{L}^*)$. Denote by y_d the solution of this equation with $u = u_a$ and consider the problem*

$$\min_{u\in\mathcal{U}} \Big\{ J(u) := \frac{1}{2}\int_Q (y(x,t) - y_d(x,t))^2\,\mathrm{d}x\,\mathrm{d}t + \int_Q \eta y\,\mathrm{d}x\,\mathrm{d}t + \int_Q \rho u\,\mathrm{d}x\,\mathrm{d}t \Big\},$$

subject to the same constraints as inn Example 2. For a local minimizer $\bar{u}$ of the unperturbed problem ($\zeta = 0$), it holds

$$\begin{aligned}J'(\bar{u})(u-\bar{u}) &= \int_Q (\bar{y}(x,t) - y_d(x,t))z_{\bar{u},u-\bar{u}}\ dxdt \geq 0 \quad \forall u \in \mathcal{U},\\ J''(\bar{u})(u-\bar{u}) &= \int_Q (\bar{y}(x,t) - y_d(x,t))w_{\bar{u},u-\bar{u}} + z^2_{\bar{u},u-\bar{u}}\ dxdt\\ &= \int_Q (1-\bar{p}\,exp(\bar{y}))z^2_{\bar{u},u-\bar{u}}\ dxdt \quad \forall u \in \mathcal{U},\end{aligned}$$

where p solves

$$\begin{cases} -\frac{d\bar{p}}{dt} + \mathcal{A}^*\bar{p} + \exp(\bar{y})\bar{p} = \bar{y} - y_d & in\ Q, \\ \bar{p} = 0\ \ on\ \Sigma,\ \ p(\cdot, T) = 0 & on\ \Omega. \end{cases}$$

If the optimal state tracks y_d such that $\|\bar{y} - y_d\|_{L^r(Q)} \le \frac{1}{2C_r\|\exp(\bar{y})\|_{L^\infty(Q)}}$ we find that (A_2) holds. From Theorem 26 we obtain the existence of a constant $\kappa > 0$ such that

$$\|y_{\bar{u}} - y_\zeta\|_{L^2(Q)} + \|p_{\bar{u}} - p_\zeta\|_{L^2(Q)} \le \kappa\Big(\|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} + \|\mathcal{L}^*\rho\|_{L^2(Q)}\Big),$$

for every perturbation $\zeta \in \hat{\Gamma}$ and for every local solution u_ζ of problem (P) with $\|y_{u_\zeta} - \bar{u}\|_{L^\infty(Q)} \le \alpha$.

A Appendix

Lemma 28. *Suppose $r > 1 + \frac{n}{2}$ and $s \in [1, \frac{n+2}{n}) \cap [1, 2]$. The following statement is fulfilled for all $u, \bar{u} \in \mathcal{U}$. There exist positive constants K_r, M_s and $N_{r,s}$ depending on s and r such that*

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{C(\bar{Q})} \le K_r\|y_u - y_{\bar{u}}\|^2_{L^{2r}(Q)}, \tag{A.1}$$

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^s(Q)} \le M_s\|y_u - y_{\bar{u}}\|^{2-s}_{C(\bar{Q})}\|y_u - y_{\bar{u}}\|^s_{L^s(Q)}, \tag{A.2}$$

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^2(Q)} \le N_{r,s}\|y_u - y_{\bar{u}}\|^{2-\frac{s^2}{2}}_{C(\bar{Q})}\|y_u - y_{\bar{u}}\|^{\frac{s}{2}}_{L^s(Q)}. \tag{A.3}$$

Proof. Let us denote $\phi := y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}} \in W(0,T) \cap C(\bar{Q})$. From the equations satisfied by the three functions and by the mean value theorem ϕ satisfies

$$\frac{d\phi}{dt} + \mathcal{A}\phi + \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\phi = \Big[\frac{\partial f}{\partial y}(x,t,y_{\bar{u}}) - \frac{\partial f}{\partial y}(x,t,y_\theta)\Big](y_u - y_{\bar{u}}),$$

where $y_\theta(x,t) = y_{\bar{u}}(x,t) + \theta(x,t)(y_u(x,t) - y_{\bar{u}}(x,t))$ with $\theta : Q \longrightarrow [0,1]$ measurable. Applying again the mean value theorem we obtain

$$\frac{d\phi}{dt} + \mathcal{A}\phi + \frac{\partial f}{\partial y}(x,t,y_{\bar{u}})\phi = \frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)(y_u - y_{\bar{u}})^2$$

with $y_\vartheta(x,t) = y_{\bar{u}}(x,t) + \vartheta(x,t)(y_\theta(x,t) - y_{\bar{u}}(x,t))$ and $\vartheta : Q \longrightarrow [0,1]$ measurable. By Theorem 1 and Remark 4 we infer the existence of constants $C_r, \bar{C}$ independent of $u, \bar{u} \in \mathcal{U}$ and $\frac{\partial f}{\partial y}(x,t,y_{\bar{u}})$ such that

$$\|\phi\|_{C(\bar{Q})} \le C_r\bar{C}\|(y_u - y_{\bar{u}})^2\|_{L^r(Q)} = C_r\bar{C}\|y_u - y_{\bar{u}}\|^2_{L^{2r}(Q)},$$

which proves (A.1) with $K_r := C_r\bar{C}$. To prove (A.2), we use Lemma 2, Remark 4 and (1.16) to obtain

$$\|\phi\|_{L^s(Q)} \le C_{s'}\bar{C}\|(y_u - y_{\bar{u}})^2\|_{L^1(Q)} = C_{s'}\bar{C}\|y_u - y_{\bar{u}}\|^2_{L^2(Q)} \le C_{s'}\bar{C}\|y_u - y_{\bar{u}}\|^{2-s}_{C(\bar{Q})}\|y_u - y_{\bar{u}}\|^s_{L^s(Q)}. \tag{A.4}$$

Taking $M_s := C_{s'}\bar{C}$, (A.2) follows. The inequality, (A.3), follows from (A.2) and (A.1) of Lemma 28 by estimating

$$\begin{aligned} \|\phi\|_{L^2(Q)} \le \|\phi\|^{\frac{2-s}{2}}_{C(\bar{Q})}\|\phi\|^{\frac{s}{2}}_{L^s(Q)} &\le K_r^{\frac{2-s}{2}}\|y_u - y_{\bar{u}}\|^{\frac{2(2-s)}{2}}_{L^{2r}(Q)}\Big[M_s^{\frac{s}{2}}\|y_u - y_{\bar{u}}\|^{\frac{(2-s)s}{2}}_{C(\bar{Q})}\|y_u - y_{\bar{u}}\|^{\frac{s^2}{2}}_{L^s(Q)}\Big] \\ &\le K_r^{\frac{(2-s)}{2}}M_s^{\frac{s}{2}}|Q|^{\frac{2-s}{2r}}\|y_u - y_{\bar{u}}\|^{2-s+\frac{(2-s)s}{2}}_{C(\bar{Q})}\|y_u - y_{\bar{u}}\|^{\frac{s^2}{2}}_{L^s(Q)}. \end{aligned}$$

Defining $N_{r,s} := K_r^{\frac{(2-s)}{2}}M_s^{\frac{s}{2}}|Q|^{\frac{2-s}{2r}}$ and noticing that $2 - s + \frac{(2-s)s}{2} = 2 - \frac{s^2}{2}$ proves the claim. □

Proof. of Proposition 5. We prove (1.17) by applying Theorem 1 to $\psi := z_{\bar u,v} - z_{u_\theta,v}$, that solves

$$\frac{d\psi}{dt} + \mathcal{A}\psi + \frac{\partial f}{\partial y}(x,t,y_{\bar u})\psi = \Big[\frac{\partial f}{\partial y}(x,t,y_{u_\theta}) - \frac{\partial f}{\partial y}(x,t,y_{\bar u})\Big] z_{u_\theta,v} = \frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)(y_{\bar u} - y_{u_\theta}) z_{u_\theta,v}. \tag{A.5}$$

To prove (1.18), we use (A.3) with $s = \sqrt{2}$ to estimate

$$\|y_u - y_{\bar u}\|_{L^2(Q)} \le \|\phi\|_{L^2(Q)} + \|z_{\bar u,u-\bar u}\|_{L^2(Q)} \le N_{r,\sqrt{2}} \|y_u - y_{\bar u}\|_{C(\bar Q)} \|y_u - y_{\bar u}\|_{L^{\sqrt{2}}(Q)} + \|z_{\bar u,u-\bar u}\|_{L^2(Q)}.$$

Using fact that by the Hölder inequality $\|y_u - y_{\bar u}\|_{L^{\sqrt{2}}(Q)} \le |Q|^{\frac{1}{\sqrt{2}} - \frac{1}{2}} \|y_u - y_{\bar u}\|_{L^2(Q)}$, the claim follows. For the other direction, we select again $s = \sqrt{2}$ in (A.3) and find

$$\begin{aligned}
\|z_{\bar u,u-\bar u}\|_{L^2(Q)} &\le \|\phi\|_{L^2(Q)} + \|y_u - y_{\bar u}\|_{L^2(Q)} \\
&\le N_{r,\sqrt{2}} \|y_u - y_{\bar u}\|_{C(\bar Q)} \|y_u - y_{\bar u}\|_{L^{\sqrt{2}}(Q)} + \|y_u - y_{\bar u}\|_{L^2(Q)} \\
&\le \left(N_{r,\sqrt{2}} |Q|^{\frac{1}{\sqrt{2}} - \frac{1}{2}} \|y_u - y_{\bar u}\|_{C(\bar Q)} + 1 \right) \|y_u - y_{\bar u}\|_{L^2(Q)}.
\end{aligned}$$

Finally, for (1.19) we use (1.17) and estimate

$$\|z_{\bar u,v}\|_{L^2(Q)} \le \|z_{\bar u,v} - z_{u,v}\|_{L^2(Q)} + \|z_{u,v}\|_{L^2(Q)} \le K_2 \sqrt[2]{|Q|} \|y_u - y_{\bar u}\|_{C(\bar Q)} \|z_{\bar u,v}\|_{L^2(Q)} + \|z_{u,v}\|_{L^2(Q)}.$$

Choosing $\varepsilon = [2K_2 \sqrt[2]{|Q|}]^{-1}$ proves the first part. The second inequality follows in a similar way. The estimates with respect to the $\|\cdot\|_{L^\infty(Q)}$ follow by similar reasoning, using (A.1). □

Proof. of Proposition 8. Let us prove first the implication $(A_k) \Rightarrow (B_k)$ for any $k \in \{0,1,2\}$. Given $u \in \mathcal{U}$, by the mean value theorem

$$\frac{d(y_u - \bar y)}{dt} + \mathcal{A}(y_u - \bar y) + \frac{\partial f}{\partial y}(x, \bar y + \theta(y_u - \bar y))(y_u - \bar y) = u - \bar u.$$

Using (1.9) in Theorem 1 we obtain that

$$\|y_u - \bar y\|_{C(\bar Q)} \le C_r \|u - \bar u\|_{L^r(Q)} \le C_r (2M_{\mathcal{U}})^{\frac{r-1}{r}} \|u - \bar u\|^{\frac{1}{r}}_{L^1(Q)}.$$

Then, by $\tilde\alpha_k := \frac{\alpha_k^r}{C_r^r (2M_{\mathcal{U}})^{r-1}}$, we obtain that (A_k) implies (B_k) with $\gamma_k = \tilde\gamma_k$.

To prove the converse implication, $(B_k) \Rightarrow (A_k)$, we assume that (B_k) holds, but (A_k) fails. Then for every integer $l \ge 1$ there exists an element $u_l \in \mathcal{U}$ such that

$$J'(\bar u)(u_l - \bar u) + J''(\bar u)(u_l - \bar u)^2 < \frac{1}{l} \|u_l - \bar u\|^{2-k}_{L^1(Q)} \|z_{\bar u,u_l - \bar u}\|^k_{L^2(Q)} \quad \text{and} \quad \|y_{u_l} - \bar y\|_{C(\bar Q)} < \frac{1}{l}. \tag{A.6}$$

Since $\{u_l\}_{l=1}^\infty \subset \mathcal{U}$ is bounded in $L^\infty(Q)$, we can extract a subsequence, denoted in the same way, such that $u_l \stackrel{*}{\rightharpoonup} u$ in $L^\infty(Q)$. On one side, (A.6) implies that $y_{u_l} \to \bar y$ in $L^\infty(Q)$. On the other side, $u_l \stackrel{*}{\rightharpoonup} u$ in $L^\infty(Q)$ implies weak convergence in $L^r(Q)$. From (1.13), the convergence $y_{u_l} \to y_u$ in $L^\infty(Q)$ follows. Then, $y_u = \bar y$ and, consequently, $u = \bar u$ holds. But Assumption(B_0) implies that $\bar u$ is bang-bang, and hence the weak convergence $u_l \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(Q)$ yields the strong convergence $u_l \to \bar u$ in $L^1(Q)$; see [17, Proposition 4.1 and Lemma 4.2]. Then, for $k = 0$, (A.6) contradicts (B_0). The same argument holds for (B_1) and (B_2) under the additional condition that $\bar u$ is bang-bang and noticing that $\|z_{\bar u,u_l - \bar u}\|_{C(\bar Q)} \le 3/2 \|y_{u_l} - \bar y\|_{C(\bar Q)}$ by Lemma 5.

□

A proof of the following Lemma can be found in [2, Lemma 3.5] or [8, Lemma 3.5].

Lemma 29. *Given $\bar u \in \mathcal{U}$ with associated state $\bar y$. Then, the following estimate holds*

$$\|y_{\bar u+\theta(u-\bar u)} - \bar y\|_{C(\bar Q)} \le B\|y_u - \bar y\|_{C(\bar Q)} \quad \forall \theta \in [0,1] \quad \text{and} \quad \forall u \in \mathcal{U}, \tag{A.7}$$

where $B := (2C_r\bar C\sqrt[r]{|Q|}M_{\mathcal{U}} + 1)$, C_r is the constant of Lemma 2 and $\bar C$ is the one from Remark 4.

We proof the analogous statement for the adjoint-state. For an elliptic state equation, it was also done in [2, Lemma 3.7].

Lemma 30. *Given $\bar u \in \mathcal{U}$ with associated state $\bar y$ and adjoint-state $\bar p$, there exists a constant $\tilde B > 0$ such that*

$$\|p_{\bar u+\theta(u-\bar u)} - \bar p\|_{C(\bar Q)} \le \tilde B(\|y_u - \bar y\|_{C(\bar Q)} + |m|\|u-\bar u\|_{L^1(Q)}^{\frac{1}{r}}), \tag{A.8}$$

for all $\theta \in [0,1]$ and $u \in \mathcal{U}$.

Proof. Let us prove (A.8). Given $u \in \mathcal{U}$ and $\theta \in [0,1]$, let us denote $u_\theta = \bar u + \theta(u-\bar u)$, $y_\theta = y_{u_\theta}$, and $p_\theta = p_{u_\theta}$. Subtracting the equations satisfied by p_θ and $\bar p$ we get with the mean value theorem

$$\begin{aligned}
&-\frac{d}{dt}(p_\theta - \bar p) + \mathcal{A}^*(p_\theta - \bar p) + \frac{\partial f}{\partial y}(x,t,\bar y)(p_\theta - \bar p) = \frac{\partial L}{\partial y}(x,t,y_\theta,u_\theta) - \frac{\partial L}{\partial y}(x,t,\bar y,\bar u)\\
&+\Big[\frac{\partial f}{\partial y}(x,t,\bar y) - \frac{\partial f}{\partial y}(x,t,y_\theta)\Big]p_\theta\\
&=\Big[\frac{\partial^2 L}{\partial y^2}(x,t,y_\vartheta) - p_\theta\frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)\Big](y_\theta - \bar y) + m(u_\theta - \bar u),
\end{aligned}$$

where $y_\vartheta = \bar y + \vartheta(y_\theta - \bar y)$ for some measurable function $\vartheta : Q \longrightarrow [0,1]$. Now, we can apply again Theorem 1 and Remark 4 to conclude from the above equation

$$\begin{aligned}
\|p_\theta - \bar p\|_{C(\bar Q)} &\le C_r(\bar C + M_{\mathcal{U}}\bar C)\sqrt[r]{|Q|}\|y_\theta - \bar y\|_{C(\bar Q)} + |m|\theta C_r\|u-\bar u\|_{L^r(Q)}\\
&\le \tilde B\|y_u - \bar y\|_{C(\bar Q)} + |m|\|u-\bar u\|_{L^1(Q)},
\end{aligned}$$

where $\tilde B := C_r((\bar C + M_{\mathcal{U}}\bar C)\sqrt[r]{|Q|}B + (2M_{\mathcal{U}})^{\frac{r-1}{r}})$, with B being the constant from Lemma 29. Then, (A.8) follows by applying Lemma 29. $\square$

Proof. of Lemma 10. The second variation of the objective functional is given by Theorem 6. Let us denote u_θ, y_θ, and φ_θ as in the proof of Lemma 30. From (2.4) we obtain

$$\begin{aligned}
&|[J''(\bar u + \theta(u-\bar u)) - J''(\bar u)](u-\bar u)^2|\\
&\le \int_Q \Big|\Big[\frac{\partial^2 L_0}{\partial y^2}(x,t,y_\theta) - \frac{\partial^2 L_0}{\partial y^2}(x,t,\bar y)\Big]z^2_{u_\theta,u-\bar u}\Big|\,\mathrm{d}x\,\mathrm{d}t + \int_Q\Big|(\bar\varphi - \varphi_\theta)\frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)z^2_{u_\theta,u-\bar u}\Big|\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q\Big|\bar\varphi\Big[\frac{\partial^2 f}{\partial y^2}(x,t,\bar y) - \frac{\partial^2 f}{\partial y^2}(x,t,y_\theta)\Big]z^2_{u_\theta,u-\bar u}\Big|\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_Q\Big|\Big[\frac{\partial^2 L_0}{\partial y^2}(x,t,\bar y) - \bar\varphi\frac{\partial^2 f}{\partial y^2}(x,t,\bar y)\Big](z^2_{u_\theta,u-\bar u} - z^2_{\bar u,u-\bar u})\Big|\,\mathrm{d}x\,\mathrm{d}t + 2\Big|\int_Q(u-\bar u)m\Big[z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\Big]\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

We consider the case $m = 0$ first. Let us consider the terms I_i, $i \in \{1,..,4\}$. For I_1, we deduce from Remark 4, (A.7), and (1.19) that for every $\rho_1 > 0$ there exists $\varepsilon_1 > 0$ such that

$$I_1 \le \rho_1\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar y\|_{C(\bar Q)} < \varepsilon_1.$$

To deal with I_2, we use Remark 4, (1.19), and (A.8) to obtain for every $\rho_2 > 0$ the existence of a $\varepsilon_2 > 0$ such that
$$I_2 \le \rho_2 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar y\|_{C(\bar Q)} < \varepsilon_2.$$
The estimate for I_3 follows from (1.19) and Remark 4. Thus for every $\rho_3 > 0$, there exists $\varepsilon_3 > 0$ with
$$I_3 \le \rho_3 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar y\|_{C(\bar Q)} < \varepsilon_3.$$
For I_4 we infer by Remark 4, (A.3), (1.19) and (A.7) that for every $\rho_4 > 0$ there exists $\varepsilon_4 > 0$ such that
$$\begin{aligned} I_4 &\le (\bar C + M_{\mathcal U}\bar C)\|z_{u_\theta,u-\bar u} + z_{\bar u,u-\bar u}\|_{L^2(Q)}\|z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\|_{L^2(Q)} \\ &\le \frac{C_2 5}{2}(\bar C + M_{\mathcal U}\bar C)\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|y_\theta - \bar y\|_{C(\bar Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)} \\ &\le \rho_4 \|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|y_u - \bar y\|_{C(\bar Q)} < \varepsilon_4. \end{aligned}$$
Taking ρ_i small enough such that $I_i < \frac{\varrho}{4}$ for every $i \in \{1,..,4\}$ and setting $\varepsilon = \min_{1\le i\le 4}\varepsilon_i$, the first claim follows.
For the case $m \neq 0$, we need to additionally estimate I_5 and reconsider the term I_2. We recall that for the case $m \neq 0$, we assume that $\|u-\bar u\|_{L^1(Q)}$ is sufficiently small. To estimate I_5 we use that $z_{\bar u,v}$ satisfies equation (1.14) and that $\psi := z_{\bar u,u-\bar u} - z_{u_\theta,u-\bar u}$ solves
$$\frac{d\psi}{dt} + \mathcal A\psi + \frac{\partial f}{\partial y}(x,t,y_{\bar u})\psi = \Big[\frac{\partial f}{\partial y}(x,t,y_{u_\theta}) - \frac{\partial f}{\partial y}(x,t,y_{\bar u})\Big] z_{u_\theta,u-\bar u} = \frac{\partial^2 f}{\partial y^2}(x,t,y_\vartheta)(y_{\bar u} - y_{u_\theta}) z_{u_\theta,u-\bar u}, \tag{A.9}$$
where we used the mean value theorem to infer the existence of a function ϑ such that (A.9) holds. We use Remark 4, (1.19), Lemma 2 and (A.7) to estimate
$$\begin{aligned} &2\Big|\int_Q (u-\bar u)m\Big[z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\Big]\,\mathrm{d}x\,\mathrm{d}t\Big| \le 2|m|\|u-\bar u\|_{L^{s'}(Q)}\|z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\|_{L^s(Q)} \\ &\le 2|m|(2M_{\mathcal U})^{\frac{s'-1}{s'}}\|u-\bar u\|^{\frac{1}{s'}}_{L^1(Q)}\|z_{u_\theta,u-\bar u} - z_{\bar u,u-\bar u}\|_{L^s(Q)} \\ &\le |m|\bar C C_{s'} B(2M_{\mathcal U})^{\frac{s'-1}{s'}}\|u-\bar u\|^{\frac{1}{s'}}_{L^1(Q)}\|y_{u_\theta} - \bar y\|_{L^2(Q)}\|z_{\bar u_\theta,u-\bar u}\|_{L^2(Q)} \\ &\le \rho_5\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|u-\bar u\|_{L^1(Q)} < \varepsilon_5. \end{aligned}$$
We remark, that to make the last step, we used that (A.7) holds also if the $\|\cdot\|_{L^\infty(Q)}$-norm is exchanged with the $\|\cdot\|_{L^2(Q)}$-norm. This can be seen in the proof of [2, Lemma 3.5]. The validity of the estimates for I_i for $i \in \{1,3,4\}$ holds, noticing that by (1.9), $\|u-\bar u\|_{L^1(Q)} < \frac{\varepsilon^r}{C_r^r(2M_{\mathcal U})^{\frac{r-1}{2r}}}$, implies $\|y_u - \bar y\|_{C(\bar Q)} < \varepsilon$. For the term I_2 we use Remark 4, (1.19), and (A.8), to find for any $\rho_2 > 0$ a $\varepsilon_2 > 0$ such that
$$I_2 \le \frac{9}{4}\bar C\tilde B(C_r(2M_{\mathcal U})^{\frac{r-1}{r}} + |m|)\|u-\bar u\|^{\frac{1}{r}}_{L^1(Q)}\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \le \rho_2\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} \quad \text{if} \quad \|u-\bar u\|_{L^1(Q)} < \varepsilon_2. \tag{A.10}$$
Taking $\varepsilon := \min_{1\le i\le 5}\varepsilon_i$, completes the proof. ☐

Proof. of Corollary 11. Let $s \in [1, \frac{n+2}{n}) \cap [1,2]$. We first consider the case $m = 0$. Using that L_0 and f satisfy the assumption in Remark 4 and arguing as in the proof of Lemma 10, there exists $\varepsilon > 0$ and a constant $P > 0$ such that
$$|[J''(\bar u + \theta(u-\bar u)) - J''(\bar u)](u-\bar u)^2| < P\|y_u - y_{\bar u}\|_{L^\infty(Q)}\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)}$$
for all $u \in \mathcal U$ with $\|y_u - y_{\bar u}\|_{L^\infty(Q)} < \varepsilon$. To prove (3.4), we select $l_1, l_2 \ge 0$ with $l_1 + l_2 = 1$ and use the estimate
$$\|z_{\bar u,u-\bar u}\|_{L^2(Q)} \le \|z_{\bar u,u-\bar u}\|^{\frac{2-s}{2}}_{C(\bar Q)}\|u-\bar u\|^{\frac{s}{2}}_{L^1(Q)}. \tag{A.11}$$

By (A.11), (1.9), (1.11) and (A.3), we find

$$\begin{aligned}\|y_u-\bar y\|_{C(\bar Q)}\|z_{\bar u,u-\bar u}\|^2_{L^2(Q)} &\le \|y_u-\bar y\|_{C(\bar Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|z_{\bar u,u-\bar u}\|^{(2-s)/2}_{C(\bar Q)}\|u-\bar u\|^{s/2}_{L^1(Q)}\\ &\le C_{s'}\sup_{\mathcal U}\|u-\bar u\|^{(s-1)/(s')}_{L^\infty(Q)}\|y_u-\bar y\|^{l_1+l_2}_{C(\bar Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|u-\bar u\|^{(2-s)/(2s')+s/2}_{L^1(Q)}\\ &\le C^2_{s'}\tilde M_{\mathcal U}\|y_u-\bar y\|^{l_1}_{C(\bar Q)}\|z_{\bar u,u-\bar u}\|_{L^2(Q)}\|u-\bar u\|^{l_2/s'}_{L^1(Q)}\|u-\bar u\|^{(2-s)/(2s')}_{L^1(Q)}\|u-\bar u\|^{s/2}_{L^1(Q)},\end{aligned} \tag{A.12}$$

with $\tilde M := M_{\mathcal U}^{\frac{s-1}{s'}(l_2+\frac{2-s}{2})}$. We select l_2 such that

$$\frac{l_2}{s'}+\frac{2-s}{2s'}+\frac{s}{2}=1.$$

Using $1/s'=1-1/s$, this is equivalent to $(1+l_2)(1-1/s)+s/2(1-1+1/s)=1$, thus we find

$$l_2=s'/2-1.$$

Defining $\varepsilon := \frac{1}{C^2_{s'}\tilde M}\rho^{\frac{1}{l_1}}$ proves the first claim. For the proof of (3.5) we use (1.9), (1.11) and (A.3) to infer

$$\begin{aligned}&\|y_u-\bar y\|_{C(\bar Q)}\|z_{\bar u,v}\|^2_{L^2(Q)} \le C_{s'}\|y_u-\bar y\|_{C(\bar Q)}\|z_{\bar u,v}\|^{(2-s)}_{C(\bar Q)}\|u-\bar u\|^s_{L^1(Q)}\\ &\le C^2_{s'}M_{\mathcal U}^{\frac{s'-1}{s'}}\|y_u-\bar y\|^{l_1+l_2}_{C(\bar Q)}\|u-\bar u\|^{(2-s)/s'}_{L^1(Q)}\|u-\bar u\|^s_{L^1(Q)}\\ &\le C^3_{s'}\tilde M\|y_u-\bar y\|^{l_1}_{C(\bar Q)}\|u-\bar u\|^{l_2/s'}_{L^1(Q)}\|u-\bar u\|^{(2-s)/s'}_{L^1(Q)}\|u-\bar u\|^s_{L^1(Q)},\end{aligned} \tag{A.13}$$

with $\tilde M := M_{\mathcal U}^{\frac{s-1}{s'}(l_2+2-s)}$. Select l_2 such that

$$\frac{l_2}{s'}+\frac{2-s}{s'}+s=2.$$

By $\frac{1}{s'}=1-\frac{1}{s}$, this is equivalent to $l_2=\frac{2-s}{s-1}$. Defining $\varepsilon := \frac{1}{C^3_{s'}\tilde M}\rho^{\frac{1}{l_1}}$ proves the case for $m=0$. For $m\neq 0$, we recall, that the $L^1(Q)$-distance of the controls is assumed to be sufficiently small. But by the estimate (1.9), this implies that the states are close and we proceed as displayed. □

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