The number of equationally additive clones on finite sets

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Basic concepts

Algebraic sets over clone $F \leq \mathcal{O}_A$
(= solution sets of systems of equations over F)

$\varrho \subseteq A^n$ algebraic $\iff$ $\varrho = \{\mathrm{x} \in A^n \mid \forall i \in I: \quad f_i(\mathrm{x}) = g_i(\mathrm{x})\}$

for some $f_i, g_i \in F^{(n)}$ ($i \in I$, I any set).

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$\varrho_{12} = \{(x_1, x_2, x_3, x_4) \in A^4 \mid x_1 = x_2\}$

algebraic over any clone; solution set of 1 equation:

$$e_1^{(4)}(x_1, x_2, x_3, x_4) = e_2^{(4)}(x_1, x_2, x_3, x_4).$$

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$$e_3^{(4)}(x_1, x_2, x_3, x_4) = e_4^{(4)}(x_1, x_2, x_3, x_4).$$

Equationally additive clones

$\mathrm{Alg}^{(n)} F := \{ \varrho \subseteq A^n \mid \varrho \text{ algebraic over } F \}$ $\quad \mathrm{Alg}\, F := \bigcup_{n \in \mathbb{N}_+} \mathrm{Alg}^{(n)} F$

Algebraic equivalence of clones $F, G \leq \mathcal{O}_A$

$F \equiv_{\mathrm{alg}} G$ algebraically equivalent $\iff \mathrm{Alg}\, F = \mathrm{Alg}\, G$
(same algebraic geometry)

Theorem: for finite A: Pinus, 2016

$|\{ F \leq \mathcal{O}_A \mid F \text{ 'equationally additive'} \} / \equiv_{\mathrm{alg}}| < \aleph_0$.

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Clone $F \leq \mathcal{O}_A$ equationally additive

$\iff \forall n \in \mathbb{N}_+ \, \forall \varrho, \sigma \in \mathrm{Alg}^{(n)} F: \quad \varrho \cup \sigma \in \mathrm{Alg}^{(n)} F$
(algebraic sets closed under finite unions)

Easy consequence

For a clone $F \leq \mathcal{O}_A$

F equationally additive

$$\Longrightarrow \Delta_A^{(4)} = \{(x_1, x_2, x_3, x_4) \in A^4 \mid x_1 = x_2 \text{ or } x_3 = x_4\} \in \mathrm{Alg}^{(4)} F$$

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We know

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- $\implies \Delta_A^{(4)} = \varrho_{12} \cup \varrho_{34} \in \mathrm{Alg}^{(4)} F$

since F is equationally additive

A characterisation of equational additivity

Theorem Danijarova, Mjasnikov, Remeslennikov, 2010

A clone $F \leq \mathcal{O}_A$ is equationally additive $\iff$ $\Delta_A^{(4)} \in \mathrm{Alg}\, F$

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A clone $F \leq \mathcal{O}_A$ is equationally additive $\iff \Delta_A^{(4)} \in \operatorname{Alg} F$

In a field

- $\varrho = \{a \in A^n \mid \forall i \in I: \quad f_i(a) = 0\} \in \operatorname{Alg} F$
- $\sigma = \{a \in A^n \mid \forall j \in J: \quad g_j(a) = 0\} \in \operatorname{Alg} F$
- $\implies \varrho \cup \sigma = \{a \in A^n \mid \forall i \in I\, \forall j \in J: \quad f_i(a) \cdot g_j(a) = 0\}$

A characterisation of equational additivity

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A clone $F \leq \mathcal{O}_A$ is equationally additive $\iff$ $\Delta_A^{(4)} \in \operatorname{Alg} F$

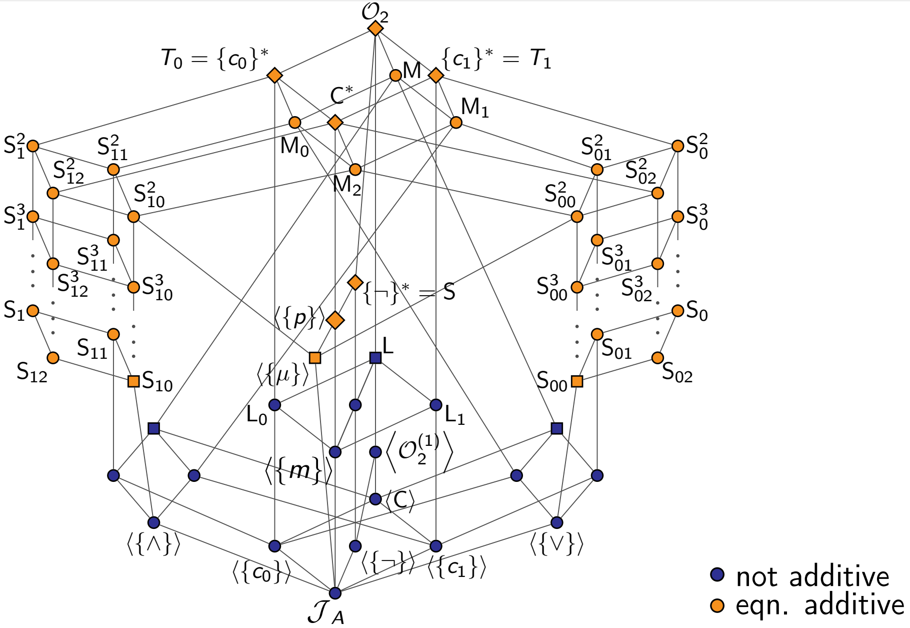
In general

- $\varrho = \{a \in A^n \mid \forall i \in I: \quad f_i(a) = f_i'(a)\} \in \operatorname{Alg} F$
- $\sigma = \left\{a \in A^n \,\middle|\, \forall j \in J: \quad g_j(a) = g_j'(a)\right\} \in \operatorname{Alg} F$
- $\Delta_A^{(4)} = \{a \in A^4 \mid \forall k \in K: \quad h_k(a) = h_k'(a)\} \in \operatorname{Alg} F$
- $\implies \varrho \cup \sigma = \Big\{a \in A^n \,\Big|\, \forall k \in K\, \forall i \in I\, \forall j \in J:$
  $h_k(f_i(a), f_i'(a), g_j(a), g_j'(a)) = h_k'(f_i(a), f_i'(a), g_j(a), g_j'(a))\Big\}$

Goal

How many equationally additive clones are there on a finite set?

Boolean equationally additive clones (AAA 102)



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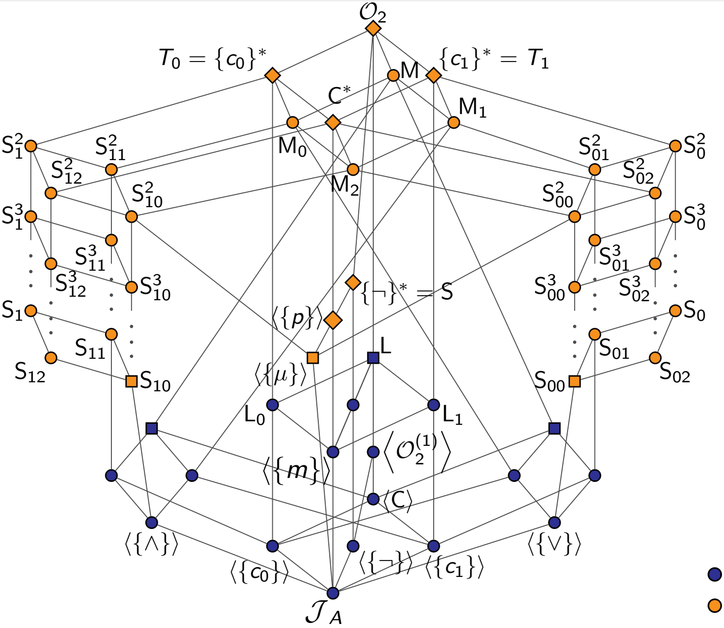

number eqn. add. clones $= \aleph_0$

● not additive
● eqn. additive

Characterisation for Boolean clones

Theorem (Aichinger, Rossi, MB)

For all clones $F \leq \mathcal{O}_2$:

$$
\begin{aligned}
F \text{ eqn. additive} &\iff \mathsf{S}_{00} \subseteq F \text{ or } \mathsf{S}_{10} \subseteq F \text{ or } \langle\{\mu\}\rangle_{\mathcal{O}_2} \subseteq F \\
&\iff ((x,y,z) \mapsto x \vee (y \wedge z)) \in F \text{ or} \\
&\qquad ((x,y,z) \mapsto x \wedge (y \vee z)) \in F \text{ or} \\
&\qquad \text{majority } \mu \in F \\
&\iff \exists f \in F^{(3)}\colon f(x,x,y) \approx x \approx f(x,y,x) \wedge \\
&\qquad\qquad f(y,x,x) \approx f(x,y,f(y,x,x)) \\
&\iff F \not\subseteq \langle\{\wedge, c_0, c_1\}\rangle_{\mathcal{O}_2} \text{ and} \\
&\qquad F \not\subseteq \langle\{\vee, c_0, c_1\}\rangle_{\mathcal{O}_2} \text{ and} \\
&\qquad F \not\subseteq \mathsf{L}
\end{aligned}
$$

Demonstrating equational additivity explicitly

$S_{00} = \langle\{f\}\rangle_{\mathcal{O}_2}, \qquad f(x,y,z) = x \vee (y \wedge z)$

$$\Delta_2^{(4)} = \left\{ (x_1,x_2,x_3,x_4) \in \{0,1\}^4 \;\middle|\; \begin{array}{l} f(x_3,x_4,x_1) = f(x_3,x_4,x_2) \\ f(x_4,x_3,x_1) = f(x_4,x_3,x_2) \end{array} \right\}$$

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$\langle\{\mu\}\rangle_{\mathcal{O}_2}, \qquad \mu$ Boolean majority

$$\Delta_2^{(4)} = \left\{ (x_1,x_2,x_3,x_4) \in \{0,1\}^4 \;\middle|\; \mu(x_3,x_4,x_1) = \mu(x_3,x_4,x_2) \right\}$$

The number of equationally additive clones on finite sets

$\lvert A\rvert$	$\lvert\mathcal{L}_A\rvert$	$\lvert{\uparrow_{\mathcal{L}_A}}\{\mathsf{C}\}\rvert$	eqn. additive	eqn. additive $\supseteq \mathsf{C}$
2	$\aleph_0$	7	$\aleph_0$	2

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A new construction of $2^{\aleph_0}$ clones on $\{0,1,2\}$

Relational approach: $F = \mathrm{Pol}_3\, Q$, where $Q \subseteq \{\varrho_3, \varrho_4, \varrho_5, \ldots\}$

Wish list

- $\mathrm{C} \subseteq F \implies$ all relations ϱ_k reflexive
- F eqn. add. $\implies$ need $G \subseteq F$ defining $\Delta_3^{(4)}$, $\Rightarrow Q \subseteq \mathrm{Inv}_3\, G$
  - $G = \{f\}$, $f \in \mathcal{O}_3^{(3)}$
  - use equations from the Boolean case:

$$\begin{aligned} f(x_1,x_2,x_3) &= f(x_1,x_2,x_4)\\ f(x_2,x_1,x_3) &= f(x_2,x_1,x_4)\\ f(x_3,x_4,x_1) &= f(x_3,x_4,x_2)\\ f(x_4,x_3,x_1) &= f(x_4,x_3,x_2) \end{aligned}$$

  - $\implies f \in \mathrm{Pol}_3^{(3)}\{\varrho_k\}$
- $|\{\mathrm{Pol}_3\, Q \mid Q \subseteq \{\varrho_3, \varrho_4, \ldots\}\}| = 2^{\aleph_0} \impliedby Q \mapsto \mathrm{Pol}_3\, Q$ inj.

Step 1: ensuring equational additivity

$$f\colon \{0,1,2\}^3 \to \{0,1,2\}$$
$$f(0,2,0) := 2$$
$$f(0,1,1) := 2$$
$$f(1,2,2) := 2$$
$$f(x,y,z) := x \text{ else}$$

$$\implies \Delta_3^{(4)} = \left\{ (x_1,x_2,x_3,x_4) \in 3^4 \;\middle|\; \begin{array}{l} f(x_1,x_2,x_3) = f(x_1,x_2,x_4) \\ f(x_2,x_1,x_3) = f(x_2,x_1,x_4) \\ f(x_3,x_4,x_1) = f(x_3,x_4,x_2) \\ f(x_4,x_3,x_1) = f(x_4,x_3,x_2) \end{array} \right\}$$

$$f \in F \implies F \text{ equationally additive}$$

The relations ϱ_k for $k \geq 2$

Forbidden set

$$B_k := \left\{ r \in \{0,1\}^k \;\middle|\; 3 \leq \mathrm{w}(r) \leq k-1 \right\}$$

0-1-tuples of Hamming weight ≥ 3, but not $\underline{1} = (1, \ldots, 1)$

k-ary relation ϱ_k

$$\varrho_k := \{0,1,2\}^k \setminus \left(B_k \cup \left\{ e_1^k \right\}\right)$$

only proper 0-1-tuples are removed!
every tuple with a 2 is in ϱ_k

$$e_i^k := \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

Step 2: $k \geq 2$ $\quad C \cup \{f\} \subseteq \mathrm{Pol}_3\{\varrho_k\}$

- ϱ_k is reflexive (only proper 0-1-tuples get removed)
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- Consider
$$u_1 := f(x_1, y_1, z_1)$$
$$\vdots$$
$$u_k := f(x_k, y_k, z_k)$$

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- at least one row is special $\implies (u_1, \ldots, u_k) \in \varrho_k$ has a 2
- no row is special $\implies (u_1, \ldots, u_k) = (x_1, \ldots, x_k) \in \varrho_k$
- $\implies f \in \mathrm{Pol}_3\{\varrho_k\} \implies \forall Q \subseteq \{\varrho_2, \varrho_3, \ldots\}$: $\mathrm{Pol}_3\, Q$ eqn. add.

Step 3

$\mathcal{L}_f := \{\, F \leq \mathcal{O}_3 \;\mid\; f \in F \supseteq \mathrm{C} \}$ (for ternary f guaranteeing eqn. additivity)

Want: Injectivity of

$$\begin{aligned} \Phi\colon \quad \mathfrak{P}(\mathbb{N}_{\geq 3}) &\longrightarrow \mathcal{L}_f \\ I &\longmapsto \mathrm{Pol}_3\{\, \varrho_i \mid i \in I \} \end{aligned}$$

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Will show: dual order embedding

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- $I \subseteq J \Rightarrow \Phi(I) = \mathrm{Pol}_A\{ \varrho_i \mid i \in I\} \supseteq \mathrm{Pol}_A\{ \varrho_j \mid j \in J\} = \Phi(J)$ trivial

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- Assume $\Phi(I) \supseteq \Phi(J)$ and $\exists i \in I\colon i \notin J, \quad \Longrightarrow \quad J \subseteq \mathbb{N}_{\geq 3} \setminus \{i\}$

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- $I \subseteq J \Rightarrow \Phi(I) = \mathrm{Pol}_A\{ \varrho_i \mid i \in I \} \supseteq \mathrm{Pol}_A\{ \varrho_j \mid j \in J \} = \Phi(J)$ trivial
- Assume $\Phi(I) \supseteq \Phi(J)$ and $\exists i \in I \colon i \notin J$, $\implies J \subseteq \mathbb{N}_{\geq 3} \setminus \{i\}$
- Then

$$\Phi(\{i\}) \supseteq \Phi(I) \supseteq \Phi(J) \supseteq \Phi(\mathbb{N}_{\geq 3} \setminus \{i\})$$

Step 3

$\mathcal{L}_f := \{\, F \leq \mathcal{O}_3 \mid f \in F \supseteq \mathrm{C} \}$ (for ternary f guaranteeing eqn. additivity)

Will show: dual order embedding

$$\begin{array}{rccc} \Phi\colon & \mathfrak{P}(\mathbb{N}_{\geq 3}) & \longrightarrow & \mathcal{L}_f \\ & I & \longmapsto & \mathrm{Pol}_3\{\, \varrho_i \mid i \in I\} \end{array}$$

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$$\Phi(\{i\}) \supseteq \Phi(I) \supseteq \Phi(J) \supseteq \Phi(\mathbb{N}_{\geq 3} \setminus \{i\}) \overset{\forall}{\ni} g_{i-1}$$

Step 3

$\mathcal{L}_f := \{ F \leq \mathcal{O}_3 \mid f \in F \supseteq C \}$ (for ternary f guaranteeing eqn. additivity)

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$$\Phi(\{i\}) \supseteq \Phi(I) \supseteq \Phi(J) \supseteq \Phi(\mathbb{N}_{\geq 3} \setminus \{i\}) \overset{\forall}{\ni} g_{i-1}$$

- But we will construct

$$g_{i-1} \in \mathrm{Pol}_3\{ \varrho_k \mid k \neq i \} \setminus \mathrm{Pol}_3\{\varrho_i\}$$

The functions g_n for $n \geq 2$

$$g_n \colon \{0,1,2\}^n \to \{0,1,2\}$$

$$\begin{aligned} g_n(1,1,\dots,1) &:= 1 \\ g_n(1,0,\dots,0) &:= 0 \\ g_n(0,1,\dots,0) &:= 0 \\ \vdots \quad & \quad \vdots \\ g_n(0,\dots,0,1) &:= 0 \\ g_n(x_1,\dots,x_n) &:= 2 \text{ else} \end{aligned}$$

- 1 on $(1,\dots,1)$
- 0 on unit tuples e_i^n
- 2 else

Step 4: g_n does not preserve ϱ_{n+1}

$\forall n \geq 2: \quad g_n \notin \mathrm{Pol}_3\{\varrho_{n+1}\}$

$$
\begin{array}{lccccll}
g_n(& 1 & 1 & \cdots & 1 &) = & 1 \\
g_n(& 1 & 0 & \cdots & 0 &) = & 0 \\
g_n(& 0 & 1 & \cdots & 0 &) = & 0 \\
 & \vdots & \vdots & \ddots & \vdots & & \\
g_n(& 0 & 0 & \cdots & 1 &) = & 0 \\
 & \in & \in & \cdots & \in & & \notin \\
 & \varrho_{n+1} & \varrho_{n+1} & \cdots & \varrho_{n+1} & & \varrho_{n+1}
\end{array}
$$

Definition of g_n:

- $g_n(\underline{1}) = 1$
- $g_n(e_i^n) = 0$
- $g_n(x) = 2$ else

Forbidden in $\varrho_{n+1} \subseteq 3^{n+1}$:

- e_1^{n+1}
- 0-1-tuples $\neq \underline{1}$ with ≥ 3 entries 1

Step 5: g_n preserves everything else

$\forall n \geq 2 \, \forall k \in \mathbb{N}_+ \setminus \{n+1\}\colon \quad g_n \in \operatorname{Pol}_3 \varrho_k$

We show: $\forall n \geq 2 \, \forall k \neq n+1 \, \forall r_1, \ldots, r_n \in \{0,1,2\}^k$:

$$g_n \circ (r_1, \ldots, r_n) \notin \varrho_k \implies \exists 1 \leq i \leq n\colon \quad r_i \notin \varrho_k$$

Step 5: g_n preserves everything else

$\forall n \geq 2 \, \forall k \in \mathbb{N}_+ \setminus \{n+1\}: \quad g_n \in \mathrm{Pol}_3\, \varrho_k$

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$$g_n \circ (r_1, \ldots, r_n) \notin \varrho_k \implies \exists 1 \leq i \leq n: \quad r_i \notin \varrho_k$$

i.e.,

$$g_n \circ (r_1, \ldots, r_n) \in B_k \cup \left\{e_1^k\right\} \implies \exists 1 \leq i \leq n: \quad r_i \in B_k \cup \left\{e_1^k\right\}$$

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$\forall n \geq 2 \, \forall k \in \mathbb{N}_+ \setminus \{n+1\}\colon \quad g_n \in \operatorname{Pol}_3 \varrho_k$

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Strategy

- Fix $n \geq 2$ and g_n.
- Consider all ϱ_k where

  $1 \leq k \leq n$ Show: g_n preserves ϱ_k

  $k \geq n+2$ Show: g_n preserves ϱ_k

Step 5.1: $n \geq 2$, $1 \leq k \leq n$: $g_n \in \mathrm{Pol}_3\{\varrho_k\}$

5.1.1: $g_n \circ (r_1, \ldots, r_n) = e_1^k \notin \varrho_k$

$$g_n(\quad) = 1$$
$$g_n(\qquad) = 0$$
$$\vdots$$
$$g_n(\qquad\qquad) = 0$$

Definition of g_n:

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Forbidden in $\varrho_k \subseteq 3^k$:

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- 0-1-tuples $\neq \underline{1}$ with ≥ 3 entries 1

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$I := \{i_1, \ldots, i_{k-1}\} \subseteq \{1, \ldots, n\}; \qquad |I| \leq k-1 < n$

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$I := \{i_1, \ldots, i_{k-1}\} \subseteq \{1, \ldots, n\}$; $\quad |I| \leq k-1 < n$

$\implies \exists i \in \{1, \ldots, n\} \setminus I: \quad r_i = e_1^k \notin \varrho_k$

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- 0-1-tuples $\neq \underline{1}$ with ≥ 3 entries 1

Step 5.1: $n \geq 2$, $1 \leq k \leq n$: $\quad g_n \in \mathrm{Pol}_3\{\varrho_k\}$

5.1.2: $g_n \circ (r_1, \ldots, r_n) \in B_k$, i.e., $3 \leq w := \mathrm{w}(g_n \circ (r_1, \ldots, r_n)) < k$

$$\left.\begin{array}{r} g_n(\quad) = 1 \\ \cdots \\ g_n(\quad) = 1 \end{array}\right\} w \geq 3$$

$$g_n(\quad) = 0$$

$$\cdots$$

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$$\cdots$$

$$g_n(e^n_{i_{k-w}}) = 0$$

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$$\left.\begin{array}{r} g_n(e^n_{i_1}) = 0 \\ \cdots \\ g_n(e^n_{i_{k-w}}) = 0 \end{array}\right\} k - w \leq k - 3 < n$$

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$I := \{i_1, \ldots, i_{k-w}\} \subseteq \{1, \ldots, n\}$; $\qquad |I| \leq k - w \leq k - 3 < n$

$\Longrightarrow \ \exists i \in \{1, \ldots, n\} \setminus I: \quad \mathrm{w}(r_i) = w$

Definition of g_n:

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Forbidden in $\varrho_k \subseteq 3^k$:

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- 0-1-tuples $\neq \underline{1}$ with w entries 1 $(3 \leq w < k)$

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$\implies \exists i \in \{1, \ldots, n\} \setminus I: \quad \mathrm{w}(r_i) = w \implies r_i \in B_k \implies r_i \notin \varrho_k$

Definition of g_n:

- $g_n(\underline{1}) = 1$
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Forbidden in $\varrho_k \subseteq 3^k$:

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Step 5.2: $n \geq 2$, $k \geq n+2$: $g_n \in \mathrm{Pol}_3\{\varrho_k\}$

5.2.1: $g_n \circ (r_1, \ldots, r_n) = e_1^k \notin \varrho_k$

$$g_n(\quad) = 1 \;\}1$$

$$g_n(\quad) = 0$$

$$\vdots$$

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$$\begin{aligned} g_n(\underline{1}) &= 1 \quad \}1 \\ \left.\begin{aligned} g_n(e_{i_1}^n) &= 0 \\ &\vdots \\ g_n(e_{i_{k-1}}^n) &= 0 \end{aligned}\right\} & k-1 \geq n+1 > n \end{aligned}$$

$i_1, \ldots, i_{k-1} \in \{1, \ldots, n\}$ cannot all be distinct;

Definition of g_n:

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$\implies \exists 1 \leq j \leq n\colon \quad \mathrm{w}(r_j) \geq 3$;

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$i_1, \ldots, i_{k-1} \in \{1, \ldots, n\}$ cannot all be distinct; $\exists \nu < \mu : j := i_\nu = i_\mu$
$\Longrightarrow \exists 1 \leq j \leq n$: $\mathrm{w}(r_j) \geq 3$; if $r_j \neq \underline{1}$, then $r_j \in B_k$, $r_j \notin \varrho_k$

Definition of g_n:

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5.2.1: $g_n \circ (r_1, \ldots, r_n) = e_1^k \notin \varrho_k$

$$\left.\begin{aligned} g_n(\underline{1}) &= 1 \end{aligned}\right\}1$$

$$\left.\begin{aligned} g_n(e^n_{i_1}) &= 0 \\ &\vdots \\ g_n(e^n_{i_{k-1}}) &= 0 \end{aligned}\right\} k-1 \geq n+1 > n$$

$i_1, \ldots, i_{k-1} \in \{1, \ldots, n\}$ cannot all be distinct; $\exists \nu < \mu \colon j := i_\nu = i_\mu$

$\Longrightarrow \exists 1 \leq j \leq n\colon \quad \mathrm{w}(r_j) \geq 3; \qquad$ if $r_j \neq \underline{1}$, then $r_j \in B_k$, $r_j \notin \varrho_k$

if $r_j = \underline{1}$, then $i_1 = \ldots = i_{k-1} = j$, $\overset{n\geq 2}{\Longrightarrow}$ $\exists l \neq j\colon r_l = e_1^k \notin \varrho_k$

Definition of g_n:

- $g_n(\underline{1}) = 1$
- $g_n(e_i^n) = 0$
- $g_n(x) = 2$ else

Forbidden in $\varrho_k \subseteq 3^k$:

- e_1^k
- 0-1-tuples $\neq \underline{1}$ with ≥ 3 entries 1

Step 5.2: $n \geq 2$, $k \geq n+2$: $g_n \in \mathrm{Pol}_3\{\varrho_k\}$

5.2.2: $g_n \circ (r_1, \ldots, r_n) \in B_k$, i.e., $3 \leq w := \mathrm{w}(g_n \circ (r_1, \ldots, r_n)) < k$

$$\left.\begin{array}{r} g_n(\quad) = 1 \\ \cdots \\ g_n(\quad) = 1 \end{array}\right\} w \geq 3$$

$$g_n(\quad) = 0$$

$$\cdots$$

$$g_n(\quad) = 0$$

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Forbidden in $\varrho_k \subseteq 3^k$:

- e_1^k
- 0-1-tuples $\neq \underline{1}$ with w entries 1 ($3 \leq w < k$)

Step 5.2: $n \geq 2$, $k \geq n+2$: $\quad g_n \in \mathrm{Pol}_3\{\varrho_k\}$

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$$g_n(e^n_{i_1}) = 0$$
$$\cdots$$
$$g_n(e^n_{i_{k-w}}) = 0$$

Definition of g_n:

- $g_n(\underline{1}) = 1$
- $g_n(e^n_i) = 0$
- $g_n(x) = 2$ else

Forbidden in $\varrho_k \subseteq 3^k$:

- e^k_1
- 0-1-tuples $\neq \underline{1}$ with w entries 1 ($3 \leq w < k$)

Step 5.2: $n \geq 2$, $k \geq n+2$: $g_n \in \mathrm{Pol}_3\{\varrho_k\}$

5.2.2: $g_n \circ (r_1, \ldots, r_n) \in B_k$, i.e., $3 \leq w := \mathrm{w}(g_n \circ (r_1, \ldots, r_n)) < k$

$$\left.\begin{array}{r} g_n(\underline{1}) = 1 \\ \cdots \\ g_n(\underline{1}) = 1 \end{array}\right\} w \geq 3$$

$$\left.\begin{array}{r} g_n(e^n_{i_1}) = 0 \\ \cdots \\ g_n(e^n_{i_{k-w}}) = 0 \end{array}\right\} k - w \geq 1$$

Definition of g_n:

- $g_n(\underline{1}) = 1$
- $g_n(e^n_i) = 0$
- $g_n(x) = 2$ else

Forbidden in $\varrho_k \subseteq 3^k$:

- e^k_1
- 0-1-tuples $\neq \underline{1}$ with w entries 1 $(3 \leq w < k)$

Step 5.2: $n \geq 2$, $k \geq n+2$: $g_n \in \mathrm{Pol}_3\{\varrho_k\}$

5.2.2: $g_n \circ (r_1, \ldots, r_n) \in B_k$, i.e., $3 \leq w := \mathrm{w}(g_n \circ (r_1, \ldots, r_n)) < k$

$$\left.\begin{array}{c} g_n(\underline{1}) = 1 \\ \cdots \\ g_n(\underline{1}) = 1 \end{array}\right\} w \geq 3$$

$$\left.\begin{array}{c} g_n(e^n_{i_1}) = 0 \\ \cdots \\ g_n(e^n_{i_{k-w}}) = 0 \end{array}\right\} k - w \geq 1$$

$$n \geq 2 \implies \exists i \in \{1, \ldots, n\} \setminus \{i_1\}: \quad 3 \leq w \leq \mathrm{w}(r_i) < k$$

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5.2.2: $g_n \circ (r_1, \ldots, r_n) \in B_k$, i.e., $3 \leq w := \mathrm{w}(g_n \circ (r_1, \ldots, r_n)) < k$

$$\left.\begin{array}{r} g_n(\underline{1}) = 1 \\ \cdots \\ g_n(\underline{1}) = 1 \end{array}\right\} w \geq 3$$

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$n \geq 2 \implies \exists i \in \{1, \ldots, n\} \setminus \{i_1\}$: $3 \leq w \leq \mathrm{w}(r_i) < k$

$\implies r_i \in B_k \implies r_i \notin \varrho_k$

Definition of g_n:

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- $g_n(x) = 2$ else

Forbidden in $\varrho_k \subseteq 3^k$:

- e^k_1
- 0-1-tuples $\neq \underline{1}$ with w entries 1 $(3 \leq w < k)$

Finishing

We have shown:

- $\forall n \geq 2\ \forall k\ \forall r_1, \ldots, r_n \in \{0,1,2\}^k$:

	$k \leq n$	$k \geq n+2$
	$g_n \circ (r_1, \ldots, r_n) = e_1^k$	$\exists 1 \leq i \leq n$:
	$g_n \circ (r_1, \ldots, r_n) \in B_k$	$r_i \notin \varrho_k$

- $\forall n \geq 2\ \forall k \neq n+1\ \forall r_1, \ldots, r_n \in \varrho_k$: $\quad g_n \circ (r_1, \ldots, r_n) \in \varrho_k$
- $\forall n \geq 2\ \forall k \neq n+1$: $\quad g_n \in \mathrm{Pol}_3\{\varrho_k\}$
- $\forall n \geq 2$: $\quad g_n \in \mathrm{Pol}_3\{\, \varrho_k \mid k \neq n+1 \}$
- $\forall n \geq 2$: $\quad g_n \notin \mathrm{Pol}_3\{\varrho_{n+1}\}$
- $\{\varrho_3, \varrho_4, \varrho_5, \ldots\} \supseteq Q \mapsto \mathrm{Pol}_3\, Q \ni f$ is injective

The end

Thank you…

…~~very~~uncountably much for listening

Questions, comments and remarks…

…are most welcome