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Numerical method for Weak Gravitational Formulation

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Kurzfassung

Diese Arbeit beschäftigt sich mit der Anwendung der Finite Elemente Methoden im Zusammenhang mit der Numerischen Berechnung von Gravitationswellen.

Die eingehenden Probleme von Simulationen in der Allgemeinen Relativitätstheorie (GR) ergeben sich aus der Tatsache, dass die Einstein-Feldgleichungen (EFE) stark nichtlinear sind. Ein möglicher Ansatz zur Lösung des EFE-Problems besteht daher darin, eine stabile lineare Methode und einen nichtlinearen Löser, der auf dieser Methode basiert zu entwickeln und zu zeigen, dass die kontinuierlichen Lösungen von den diskreten approximiert können. Eine Möglichkeit, solche linearen Methoden zu erstellen, besteht darin, nach Ähnlichkeiten mit den bereits bekannten Methoden zu suchen. Aus diesem Grund beschäftigt sich diese Arbeit mit dem Vergleich zwischen der Finite-Elemente-Methode für die linearisierten Maxwell-Gleichungen (LME) und Gravitationswellen (GW). Nach einer kurzen Einleitung der Theorien, werden die grundlegenden Objekte für die Variationsumgebung vorgestellt: die Sobolev-Räume und das Finite Element, welche die benötigter Differentialoperatoren auf natürliche Weise erweitern. Besonderes Augenmerk wird auf den Neuesten der vier FEs, den Regge-Elementraum, gelegt: Hier wird die formale Ableitungstheorie fast vollständig bis zur Definition einer anwendbaren Basis für reale Implementierungszwecke abgedeckt. Weiters, wird der Fokus auf die schwache Definition der Operatoren verlagert, nämlich die Definition von Triangulationen. Diese führen zu einem symplektischen System, welches implementiert und gelöst werden muss. Die Dualität sieht gebrochen aus, da die LMEs Permittivität als möglichen Passage-Operator zwischen den Räumen H(curl) und H(div) verwenden. Unter Verwendung einiger Ergebnisse wird die Dualität zwischen den Räumen H(curlcurl) und H(divdiv) vom S-Operator wieder aufgebaut. Das Galerkin-Verfahren liefert ein symplektisches System gewöhnlicher Differentialgleichungen erster Ordnung. Abschließend wird im letzten Teil dieser Arbeit über mögliche Ergbnisse der vorgestellten Simulationen diskutiert. Die Implementierung und Durchführung aller numerischen Berechnungen wurde mit Hilfe der finiten Elementen Bibliothek Netgen/NGsolve durchgeführt.

Abstract

This work is aimed at addressing the issues arising from the application of the Finite Element Method in the context of Gravitational Wave (GW) simulation.

The inherent problems of simulations in general relativity (GR) stem from the fact that Einstein Field Equations (EFE) are strongly nonlinear, therefore one possible approach to solve the EFE is to develop a stable linear method, a nonlinear solver based upon that method and to show that the continuous solutions are well approximated by the discrete ones. One possible way to create such linear methods is to look for similarities with the already known methods. For this reason, this paper addresses the comparison between the finite element method for the linearized Maxwell equations (LME) and gravitational waves. After a short presentation of the theories, the basic objects for the variational setting will be introduced: the Sobolev spaces and the Finite Elements, which extend the differential operators involved in the equations in a natural way. Particular attention is paied to the newest among the four discrete spaces, the Regge element; here, the formal derivation is covered almost completely up to the definition of an purposes, it follows an implementable definition ford Dofs and Shape Functions. Further, the focus is shifted on the weak definition of the operators, namely the definition on triangulations, which leads to a symplectic system. The duality looks broken since the LMEs use permittivity as a passage operator between the spaces H(curl) and H(div); later the duality with the spaces H(curlcurl) and H(divdiv) is re-established by the operator S. The Galerkin procedure leads to a first order symplectic system of ordinary differential equations .Consequently, the last part of the theory, covers the creation of possible symplectic methods, in particular, the attempt to solve the problem on a numeric solver will be expounded. The final part of this paper examines the outcomes of a basic simulation. After presenting some basic facts on how to implement a simulation in Netgen/NGsolve via the package NGs-py, plots about the conservation of energy are discussed.

Acknowledgement

This endeavor would not have been possible without the teaching of my first mentor, P.Oak, who first taught me the art of patience: "*C'è un luogo e un momento per ogni cosa*".

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

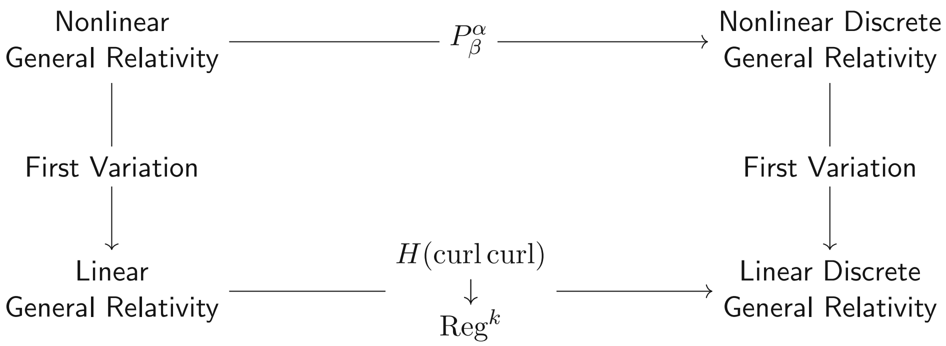
Wien, am 25/10/2022

Edoardo Bonetti

Contents

1. Introduction

The theory of GR is the theory of gravity, was developed by Einstein at the end of 1915 and published as final draft in 1916[Ein23], his work aims to extend the previous works, published in 1906 on special relativity, in the considering now a possible variation of the gravitational field. Immediately the after his publication all the scientific community started working on the topic, finding theoretical results that have been proved to be correct results of a well posed theory. The latest examples are the detection of the gravitational waves in the LIGO experiment[Abb+16] and the first picture of the black hole M87 at the center of a galaxy[Col+19]. These results are obtained by a deeper and deeper understanding of GR, in the particular case of the wave detection there is the problem concerning the frequency that one needs to look for when is seeking for gravitational waves, to do so one needs to simulate binary black holes/neutron stars spiraling into each other, this is topic of research of numerical relativity[Sch22]. High quality finite difference methods have been developed since the '60, but so far no satisfactory results are presented in the framework of FE, but an important step towards a possible FEM simulation was done in the year 2018 when L. Lizao proposed in his Ph.D. thesis a new finite element for the discretization of the space $H(\text{inc})$. In his thesis Lizao introduces a generalization of the finite element extrapolated in a paper [Chr11] by S. Christiansen. This paper is primarily centered around the formalization of a discrete space, introduced informally in the works of T. Regge[Reg61], in which the inc operator is well defined in a discrete, this space is known as aka Reg^0. A more naive motivation to study the discretization of the $H(\text{curl}\,\text{curl})$ is given by the fact that the de Rham sequence2 it is well defined and there is a way to pass between the spaces in the sequence, but the complex involving the $H(\text{div}\,\text{div})$, $H(\text{curl}\,\text{div})$ and $H(\text{curl}\,\text{curl})$ is still under research. The idea behind developing a good numerical scheme for GR is quite easy, after performing a discretization on the nonlinear equations, furthermore the procedure can be the definition of a good linear method to approximate the nonlinear one. In general the previous is not the only way: the discretization can be done before the linearization and not after. Usually there is hope that the relation between the two approaches lead to the same result.



The thesis is divided into four parts, the first one concerns the table of notation in this introduction the description of the physical theories in a smooth setting. The derivation of the ME is quite simple, therefore the main focus is on the GR. It is shown how the naive linearization leads to a resulting system, but one of the two components of the system loses its physical meaning. The two theories have in common a symilar first integral that is used to in the final part to check the correctness of the implementation.
The second part is devoted to the exploration of the FEs used to discretize the system, in particular we give a review of the theory in a schematic way, then the implementable elements are given. The finite element Reg^k is the more complicated among all, therefore a specific discussion is reserved.
The third part opens with an important question on how to formally derive a theory for differential equations involving different hilbert spaces. The question is partially answered. Moreover

The finite element method [GLS20]

This thesis fit into the research field of numerical relativity, are interested in the numerical analysis and implementation of the Einstein Field Equation more then their derivation, nevertheless it important to have at least a basic idea of where the Einstein equations come from to better understand the results of the simulations. The derivtion of the fully non-linear equation can be found in many text books, for example [Cal00] or [Car19].

Notation: In this chapter it is listed the most common notation used throughout the thesis, it is a mixture of notation coming from Finite element Method and Analysis and Differential geometry. We indicate with Greek indices the spacetime components and with the Latin ones only the space components, the Einstein index summation notation is taken without the common rule of summation on different level indices:

$$\sum_i a_{ki} b_{li} =: a_{ki} b_{li} \tag{1.1}$$

We define the anti-symmetric and symmetric relation on the indices as:

$$T_{[a..b]} := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{sign \sigma} T_{\sigma a \ldots \sigma b} \tag{1.2}$$

$$T_{(a..b)} := \frac{1}{n!} \sum_{\sigma \in S_n} T_{\sigma a \ldots \sigma b} \tag{1.3}$$

Where n is the number of indies and S_n is the set of permutations. The space of vector valued functions $C(\Omega;\mathbb{R}^d)$ can be written as $[C(\Omega)]^d$ or also as $C(\Omega) \otimes \mathbb{V}^d$ without distinctions, the same holds true for thespace of matrix valued functions $C(\Omega;\mathbb{R}^{d\times d}) = [C(\Omega)]^d = C(\Omega) \otimes \mathbb{S}^d$. When the dimension of the vector/matrix function known we can omit it. Usually the test functions are denoted with Greek letters, usually ϕ, ψ; The same notation is used for the smooth functions. in general in the smooth settings the when one has more than one partial derivative they are grouped together $\partial_\alpha \partial_\beta \to \partial_{\alpha\beta}$, the same holds true for mixed indices $\partial_\alpha \partial^\beta \to \partial_\alpha^\beta$.

In the above picture we give an idea of how to simulate general relativity, the blue parts are possible future work, the orange one are topics we aim to present in this thesis.

Spaces:	
$\mathbb{P}^k(\Omega)$	space of polynomials of degree k
$\mathbb{V}^d(\Omega)$	vector space
$\mathbb{M}^{d\times d}(\Omega)$	scalar valued matrix space
$\mathbb{S}^d(\Omega)$	scalar valued symmetric matrix space
Electromagnetism:	
μ	magnetic inductivity
ε	electric permettivity
E	electric field
B	magnetic field
D	displacement field
H	magnetic Induction Field
$\mathcal{H}$	Hamiltonian
General Relativity:	
$g_{\alpha\beta}$	spacetime metric tensor
$\eta_{\alpha\beta}$	Minkowskian metric tensor
γ_{ij}	space metric induced on the foliation
κ_{ij} or K_{ij}	intrinsic curvature
P_α^β	projection related to a foliation
Algebraic Operations and Differential Geometry:	
$\odot$	symmetric product of tensors
$\oslash$	dyadic product of tensors
$\otimes$	tensor product
$\pounds$	Lie derivative
Finite Element:	
V, E, F, T	vertex, Edge, Face, Tetrahedron (triangle in the 2d case)
$\mathcal{V}, \mathcal{E}, \mathcal{F}, \mathcal{T}$	Collection of all vertex, Edge, Face, Tetrahedron (triangle in the 2d case)
λ_i	baricentric coordinate related to the vertex V_i

Table 1.1.: names associated with mathematical symbols

Part I.

Introduction to the Theory of Electrodynamics and Gravitation

2. Electrodynamics

Scope of the chapter:
In this section the theory of electromagnetism is quickly introduced, the aim here is to obtain a linearization of the equations and the extrapolation of a wave equation for the field E, further an Hamiltonian is given for the wave equation.

The modern theory of electrodynamics find its roots in the 19 century as a result of many scientists[Byr+15], in our present days we refer to laws governing the the Electromagnetism as Maxwell equations, in honor to the huge effort of J.K. Maxwell, that was able to first describe almost fully the mathematics of this physical behaviour. Despite the fact that now, two centuries later,we have a mathematically more accurate description of the electromagnetic phenomenons, the equations describing the world remain the same

Definition 1 (Maxwell Equations (ME))**.**

$$\operatorname{curl} E = -\frac{\partial B}{\partial t} \tag{2.1a}$$

$$\operatorname{div} D = \rho \tag{2.1b}$$

$$\operatorname{curl} H = \frac{\partial D}{\partial t} j \tag{2.1c}$$

$$\operatorname{div} B = 0 \tag{2.1d}$$

The ME are constructed geometrically starting with the construction of infinitesimal quantities[Gri05], and some important aspect about the boundary behaviour are extrapolated, in particular using the "Gaussian pillar box" and the "rectangular path" approaches[Sch09a] one can demonstrate the tangential continuity for the electric field E and the normal continuity for the displacement field D, or in symbols:

$$E_- \times n_- = E_+ \times n_+ \tag{2.2a}$$

$$D_- \cdot n_- = D_+ \cdot n_+ \tag{2.2b}$$

A similar result is obtained for the Magnetic part of the equations; moreover the ME system is not complete in the sense that there is not enough information to solve the system, in fact in the equations 2.1a and 2.1c can be grouped in a separate set of the equations independent from 2.1b and 2.1d, but from experience we expect that they are be somewhat linked. To overcome the issue one requires measurements! The real world dictates the constitutive laws for coupling the different fields:

Definition 2 (constitutive laws of Electromagnetism)**.**

$$B = B(H) \tag{2.3a}$$
$$D = D(E) \tag{2.3b}$$
$$j = j(E) \tag{2.3c}$$

from one hand these laws give enough information to solve the system, on the other they are not expressed in an explicit way; to have some usefull laws we need more assumptions.

The process of linearization:
We aim to create a linear wave equation (in empty space) and, since the ME are in general linear on their own, we can require to have linear constitutive laws too, in particular assume:

- In absence of strong electric/magnetic fields in substance that constitute our domain has a weak residual magnetization/polarization.
- The substance is isotropic and the constitutive laws are linear.
- There is no real current in the substance.
- The properties of the substance remain constant in time.

The above assumptions can be expressed in symbols such as:

Definition 3 (Linear constitutive equations for electromagnetism in Vacuum)**.**

$$B = \mu H \tag{2.4a}$$
$$D = \varepsilon E \tag{2.4b}$$
$$j = 0 \tag{2.4c}$$

Remark 2.0.1. *In this above the quantities μ, ε, that are usually matrices in order to describe anisotropic materials*[1]*,in this setting they are scalars. This fact comes directly from the isotropic hypothesis*[2]*.*

We can modify the Maxwell's equations 1 using the material laws 3 to eliminate B, D we get:

$$\mu \frac{\partial H}{\partial t} = -\operatorname{curl} E \tag{2.5}$$
$$\varepsilon \frac{\partial E}{\partial t} = \operatorname{curl} H \tag{2.6}$$

On its own this can induce a possible symplectic scheme[3], but in this form it is involving elements that are well defined for the curl. We would like to obtain a different scheme,

[1] In [Bos10] the construction of the spaces is linked to the theory of linear forms and μ, ε are seen as the Hodge star operator.

[2] Instead of scalars one can interpret them as scalar matrices, namely matrices of the form εI, where ε is now a scalar and I is a matrix.

[3] This kind of system can lead to very efficient results, both in time and accuracy [Cod+18].

involving the curl curl operator and we aim to cancel out the H part. To do so one one get ideas from the classic second order (in time) separation: applying the $\operatorname{curl}\mu^{-1}$ to the first equation in 1, and applying $\operatorname{curl}\varepsilon^{-1}$ to the third one we obtain 2 independent second order in time differential equations. We only consider the electric one

Definition 4 (Maxwell Electric Waves)**.**

$$\varepsilon\ddot{E} = -\operatorname{curl}\mu^{-1}\operatorname{curl} E \tag{2.7a}$$

$$\operatorname{div}\varepsilon E = 0 \tag{2.7b}$$

using vector calculus identities and the constancy in time of the scalar coefficients, the right hand side can be rewritten as $\mu^{-1}\left(\Delta^2 - \nabla(\nabla\cdot E)\right)$. since it is a second order in time equation and it has a similar form to the wave equation, we refer to it as Electric Wave Equation, or electric wave (EW). This second order differential equation can be solved using a second derivative scheme, but one scheme the one we are interested in is the decomposition into a system of first order in time differential equations. Starting from the above 4 and using the constitutive law 3 for the electric field:

$$\frac{\partial\dot{D}}{\partial t} = -\operatorname{curl}\mu^{-1}\operatorname{curl} E \tag{2.8a}$$

$$\dot{E} = \varepsilon^{-1}\frac{\partial D}{\partial t} \tag{2.8b}$$

$$\operatorname{div}\varepsilon E = 0 \tag{2.8c}$$

The constraint $\operatorname{div} D = 0$ is preserved in time, therefore the following holds true:

Remark 2.0.2. *If (E, D) is a solution of the above system 2.8 then the $\operatorname{div}\partial_t D = 0$ for all t; and vice versa if the time derivative of the solution is zero in time, then if we start with the initial condition $(\operatorname{div} D)_{|t=0} = 0$ then $\operatorname{div} D = 0$ for all t.*

Using the remark above, the notation $\frac{\partial D}{\partial t} = d$ and for conformity $E = e$, we have :

$$\dot{d} = -\operatorname{curl}\mu^{-1}\operatorname{curl} e \tag{2.9a}$$

$$\dot{e} = \varepsilon^{-1} d \tag{2.9b}$$

$$\operatorname{div} d = 0 \tag{2.9c}$$

Remark 2.0.3. *Since $d = \frac{\partial D}{\partial t}$, and since there are no actual current $j = 0$ one gives the following interpretation to the above system of equations: The electric field tells the current how to flow; the current tells the electric field how to change.*

One fact that has not been discussed yet is the boundary conditions: the main goal of this thesis is to show similarities between the wave equations in the space $\mathbb{R}^3 \times [0, T]$, in particular the initial conditions are bump-functions, therefore there is a time certain time T^* in which the boundary conditions have negligible impact on the solution. If time T^*

is exceeded then the the wave loses its physical meaning but one can still obtain a good behaviour of the method. Therefore zero boundary conditions may be applied[4]

We can now state the classic form of the problem:

Problem 2.0.4 (Classic Maxwell Wave Problem in Vacuum). *Let Ω be a compact simply connected domain, let $e_0 \in C^2(\Omega;\mathbb{R}^3)$ and $d_0 \in C^1(\Omega;\mathbb{R}^3)$ be initial conditions, find $e \in C^1((0,T);C^2(\Omega;\mathbb{R}^3))$ and $d_0 \in C^1((0,T);C^1(\Omega;\mathbb{R}^3))$ such that:*

$$\begin{aligned} \dot{d} &= -\operatorname{curl}\mu^{-1}\operatorname{curl} e && \text{in } \Omega\times[0,T] \\ \dot{e} &= \varepsilon^{-1} d && \text{in } \Omega\times[0,T] \\ \operatorname{div} d &= 0 && \text{in } \Omega\times[0,T] \\ e &= e_0 && \text{in } \Omega, t=0 \\ d &= d_0 && \text{in } \Omega, t=0 \end{aligned} \tag{2.10}$$

let (e,d) be solution of the above problem in finite time T^* (with zero boundary conditions), furthermore the convention $\varepsilon = \mu = 1$ is introduced. Then one can use the usual $\mathbb{L}_2(\Omega)\otimes\mathbb{V}^d$ inner product to derive a first integral[5]

Proposition 1. *The following quantity is referred to as Hamiltonian*

$$\mathcal{H}^{LME}(e,d) := \frac{1}{2}\|d\|^2 + \frac{1}{2}\|\operatorname{curl} e\|^2 \tag{2.11}$$

The Hamiltonian $\mathcal{H}$ is a first integral for the problem 2.0.4

Proof. Using the $\mathbb{L}_2$ inner product

$$\begin{aligned} \ddot{e} + \operatorname{curl}\operatorname{curl} e &= 0 \\ (\ddot{e},\dot{e}) + (\operatorname{curl}\operatorname{curl} e,\dot{e}) &= 0 \\ (\ddot{e},\dot{e}) + (\operatorname{curl} e,\operatorname{curl}\dot{e}) &= 0 \\ \frac{d}{dt}\left(\frac{1}{2}\|d\|^2 + \frac{1}{2}\|\operatorname{curl} e\|^2\right) &= 0 \\ \frac{d}{dt}\mathcal{H}^{LEFE}(e,d) &= 0 \end{aligned}$$

□

The third passage holds true thanks to the homogeneous boundary conditions of e.

To end the chapter there is one detail that need to be pointed out: the problem 2.0.4 needs to be well defined. In the smooth setting there is no problem, in fact the curl curl-operator leads to a function that is differentiable in time. Moreover the divergence acts on the field d that is well defined thanks to the completeness of the de Rahm sequence

$$C^\infty(\Omega) \xrightarrow{\nabla} C^\infty(\Omega)\otimes\mathbb{V}^3 \xrightarrow{\operatorname{curl}} C^\infty(\Omega)\otimes\mathbb{V}^3 \xrightarrow{\operatorname{div}} C^\infty(\Omega) \tag{2.12}$$

Therefore this fact needs to be ensured in a more generic setting.

[4] A possible future research could be implementing absorbing boundary conditions (ABC) or even Perfectly matched layers (PML)

[5] the first integral I is a quantity that remains constant in time along solutions of the differential equation.

3. Introduction to General Relativity

Disclaimer:
The theory of general relativity mathematically intricate, the derivation of the Einstein Field Equations is mathematically challenging, but mostly it is not relevant for the goal of this thesis. Therefore here are I faithfully report the main conceptual road to derive the EFE and if the reader is interested I will point out some literature about the specific topic.

Scope of the chapter:
The in this chapter the road to obtain General Relativity is presented as a conceptual map, after justifying the EFE there is a naive approach to a possible linearization, to do so I follow the idea given in Li Lizao Ph.D. thesis [Li18]. After the introduction of the linearized EFE (LEFE) the resulting problem is quite similar to the LME problem, therefore a possible first integral is given for this problem too. The naive approach leads to the losing of physical interpretation regarding the field κ, the last part of the chapter is used to explain derive a proper linearization of the EFE using the ADM formalism[Alc08].

The reader needs to be informed here that from now on the thesis will include heavy notation inherited from differential geometry, the most important notation are fully covered in the appendix **??** and the reader is invited to read quickly through this chapter in the appendix to get used to the notation and to see how to handle the index notation.

3.1. The Derivation of the Einstein Field Equation

In this section we give a quick overview of the approach used in [Sch22]: the starting principles to introduce to the theory of gravity are:

1. **Principle of general covariance**, which says that the laws of physics must be the same for all the observers.
2. **Principle of equivalence** that says that all observers fall with the same acceleration in a gravitational field regardless of their mass.
3. **Mach's principle** which states that the local inertial properties of physical objects must be determined by the total distribution of matter in the Universe.

It is a long road to show that each of the above leads to an important result, therefore we are not going to derive them but just to present the implications:

1. the first principle implies tensor form of the theory.
2. the principle of equivalence implies the time and space to be parts of a bigger geometrical thing that is the space-time.

3. the last principle links the distribution of mass and energy with the geometry of the space-time.

Using the same approach presented in the ending page of last chapter we substitute all constants to 1. Then one has generally two possibilities to derive the EFE, the first one is to use a minimization approach of the action

$$\mathcal{S}(g) = \frac{1}{16\pi} \int_{\mathbb{R}^4} R(g_{\mu\nu})\sqrt{\det(g_{\mu\nu})}$$

where R is the Ricci scalar or, alternatively, can use a more geometrical approach based upon the comparison between the known potential gravitational equation, introduced by Newton, and a generic differential operator Op that acts on g metric of the spacetime. Alternatively one can derive the EFE 5 following the book from Schultz [Sch22], one needs to start from the Classic gravitational field express in terms of scalar potential ϕ and the density of mass ρ

$$\nabla^2\phi = 4\pi G\rho \tag{3.1}$$

And somehow manage to derive a differential equation of the form

$$Op(g) = \tilde{k}T \tag{3.2}$$

Where Op is a generic differential operator and $\tilde{k}$ a scalar, both yet to be found, such that in a weak limit, i.e. substituting in both the differential equations the conditions used by Newton to derive 3.1.

Definition 5 (The Einstein Field Equation (EFE))**.**

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3.3a}$$

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{3.3b}$$

$$R := g^{\mu\nu} R_{\mu\nu} \tag{3.3c}$$

Idea of derivation for the EFE:. Since Op must be a two index tensor one can think to use the Ricci curvature tensorA.20b as possible guess, but the right hand side of the equation dictates the tensor $G_{\mu\nu}$ must be divergence free, to solve the problem of finding a divergence free 2-index tensor field one uses the Bianchi's identity39. In particular this identity can be rewritten as:

$$\nabla_\mu R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta}$$

to ensures to find a 2-index differential operator that is divergence free, the operator $G_{\mu\nu}$. In particular one can prove that the the operator $G_{\mu\nu}$ is unique. □

3.2. The Linearization of the Einstein Field Equation

In this section it is explained into details how to derive the linearized version of the Einsetin field equations using a weak field theory; let the metric of spacetime be perturbed:

$$g_{\mu\nu} = \eta_{\mu\nu} + sh_{\mu\nu} \quad \text{For } s > 0 \text{ small} \tag{3.4}$$

in [Car19] there are two distinct weak field theories, the first one is also known as **Newtonian limit**, it requires an inertial frame of reference such that the field can be considered as a perturbation of the Minkowski space, the test particle has to move slow w.r.t. the speed of light and the field $h_{\alpha\beta}$ is static. The latter hypothesis is dropped in the second hypothesis leaving us with the (**linearized gravity theory**). The second hypothesis is a restriction on the motion of the test particle that now is allowed to move relatively fast, it is required to observe a motion in otherwise static test particle (gravitational wave).

the physical framework is the linearized gravity one,it is introduced the Einstein operator (not to be confused with the einstein tensor5):

$$(\mathrm{ein}\, h)_{\mu\nu} := \left.\frac{d}{ds}\right|_{s=0} G_{\mu\nu}$$

it is the first variation of the Einstein tensor given by η in the direction $h_{\alpha\beta}$

Proposition 2. *Let $g_{\mu\nu} = \eta_{\mu\nu} + sh_{\mu\nu}$ for some constant symmetric 2-tensor field $h_{\mu\nu}$ with $s \in \mathbb{R}$ and $G_{\mu\nu}$ is the Einstein tensor. Then the following holds true:*

$$(2\,\mathrm{ein}\, h)_{\mu\nu} = -\partial^{\lambda}_{\lambda} h_{\mu\nu} + \partial^{\lambda}_{\mu} h_{\lambda\nu} + \partial^{\lambda}_{\nu} h_{\lambda\mu} - \partial_{\mu\nu} h^{\alpha}_{\alpha} - \partial^{\lambda}_{\lambda} h_{\mu\nu} - \eta_{\mu\nu}\partial^{\alpha\beta} h_{\alpha\beta} + \eta_{\mu\nu}\partial^{\lambda}_{\lambda} h^{\alpha}_{\alpha} \tag{3.5}$$

The proof is given in [Li18] page 123,as an immediate result of proposition 5.4 , 5.5 and lemma 5.2 in [Li18]; here it is shown only an idea of proof.

idea of proof. The first thing to notice is that

$$g_{\mu\nu} = \eta_{\mu\nu} + sh_{\mu\nu} \implies g^{\mu\nu} = \eta_{\mu\nu} - sh_{\mu\nu} + \mathcal{O}(s^2) \tag{3.6}$$

Then it is possible to divide G into the first order and second order in $\Gamma^{\gamma}_{\alpha\beta}$ and get rid of all the terms with s except the one that show the first power. using that

$$\partial_\alpha \delta_{\mu\nu} \implies g_{\mu\beta}\partial_\alpha g^{\beta\nu} = g^{\beta\nu}\partial_\alpha g_{\mu\beta} \tag{3.7}$$

It is immediate the result ☐

Since the energy tensor is zero (we are in the vacuum)

$$(\mathrm{ein}\, h)_{\mu\nu} = 0 \tag{3.8}$$

and using the assumption of η constant Minkowskian background metric and some notions in appendix A.1

$$(2\,\mathrm{ein}\, h)_{\mu\nu} = -\underbrace{\partial^{\lambda}\partial_{\lambda} h_{\mu\nu}}_{\Delta_\eta h} + \underbrace{\partial_{\mu}\partial^{\lambda} h_{\lambda\nu} + \partial_{\nu}\partial^{\lambda} h_{\lambda\mu}}_{2\varepsilon\, \mathrm{div}_\eta h} - \underbrace{\partial_{\mu}\partial_{\nu} h^{\alpha}_{\alpha}}_{\nabla\nabla\, \mathrm{tr}\, h} - \underbrace{\eta_{\mu\nu}\partial^{\alpha}\partial^{\beta} h_{\alpha\beta}}_{\eta\, \mathrm{div}_\eta \mathrm{div}_\eta h} + \underbrace{\eta_{\mu\nu}\partial^{\lambda}\partial_{\lambda} h^{\alpha}_{\alpha}}_{\eta\Delta_\eta \mathrm{tr}_\eta h} \tag{3.9}$$

in compact notation:

$$2\,\mathrm{ein}\,h = -\Delta h + 2\varepsilon\,\mathrm{div}\,h - \nabla\nabla\,\mathrm{tr}\,h - \eta\,\mathrm{div}\,\mathrm{div}\,h + \eta\Delta\,\mathrm{tr}\,h \tag{3.10}$$

The naive 1+3 decomposition

The idea of a decomposition in the time and, separately, in space seems to be a natural choice in which we should head to, GR is in fact intended as an extension of the newtonian's dynamic. The division of space and time as "comunicating", but separate identities takes the name of **1+3 decomposition** or **1+3 formalism**. The best choice to start such a decomposition is to find a coordinate system such that the metric is decomposed as follows:

- h_{00} is the time dependant part,
- h_{0i}, h_{i0} are the mixed entries
- h_{ij} are the time dependant entries.

then the direction of the perturbation of the metric g can be split into the

$$h = \begin{pmatrix} \alpha & \beta^T \\ \beta & \gamma \end{pmatrix} \tag{3.11}$$

When the spacial dimensions are just 3, instead of a generic m-dimension, the linearized einstein equation above assumes a particularly interesting form.

Proposition 3. *Under the hypothesis of proposition 2, let the dimension of the manifold be $m = 3$ and let $h_{\mu\nu}$ be decomposed as in 3.11, then*

$$2\,\mathrm{ein} = 0 \tag{3.12}$$

Holds true if and only if the following holds true

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,S\gamma = 0 \\ \mathrm{div}\,S\,(\gamma' - 2\varepsilon\beta) = 0 \\ S\gamma'' = -\,\mathrm{inc}\,\gamma - S\nabla\nabla\alpha + 2S\varepsilon\beta' \end{cases} \tag{3.13}$$

[1]

Remark 3.2.1. *The equation 3.13 is to be interpreted as a constrained evolution equation:*

$$\begin{cases} \left.\begin{aligned} \mathrm{div}\,\mathrm{div}\,S\gamma &= 0 \\ \mathrm{div}\,S\,(\gamma' - 2\varepsilon\beta) &= 0 \end{aligned}\right\} & \textit{constraints} \\ S\gamma'' = -\,\mathrm{inc}\,\gamma - S\nabla\nabla\alpha + 2S\varepsilon\beta' & \textit{evolution equation} \end{cases} \tag{3.14}$$

Moreover, if the constraints are satisfied at time $t = 0$, then they are propagated for time $t > 0$ by the evolution equation

[1] The inc operator is known in literature as Incompatibility Operator or e Saint-Venant Operator [AV19], it is an operator that acts on symmetric square matrices in the following way $\mathrm{inc}(\gamma) = \mathrm{curl}(\mathrm{curl}\,\gamma)^T$ with curl that acts row-by-row on a square matrix, but more on that in the following chapter.

Proof. the following notation is used in the proof:

$$v_{(4)} = \begin{pmatrix} \mathrm{v} \\ v \end{pmatrix}$$

for the decomposition in the time (1-dimensional) and space (3-dimensional) parts, the zero-th partial derivative is denoted by $\cdot'$.

Using the lemma in appendix we can show that the 2 ein operator has a more compact form A.27:

$$\mathrm{ein} = \underbrace{-\frac{1}{2} J \Delta}_{\mathrm{i}} + \underbrace{J \varepsilon \operatorname{div} J}_{\mathrm{ii}} \tag{3.15}$$

each member of the equation can be split into the time-time, time-space and space-space:
part i:
Using:

$$\Delta_{(4)} \begin{pmatrix} \alpha & \beta^T \\ \beta & \alpha \end{pmatrix} = - \begin{pmatrix} \alpha'' & \beta^{T''} \\ \beta'' & \gamma'' \end{pmatrix} + \begin{pmatrix} \Delta\alpha & \Delta\beta^T \\ \Delta\beta & \Delta\alpha \end{pmatrix}$$

It is immediate to obtain

$$-\frac{1}{2} J \Delta_{(4)} \begin{pmatrix} \alpha & \beta^T \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\left(\alpha'' - \Delta\alpha + \operatorname{tr}\gamma'' - \Delta \operatorname{tr}\gamma\right) & \frac{1}{2}\beta^{T''} - \Delta\beta^T \\ \frac{1}{2}\beta'' - \Delta\beta & -\frac{1}{2}\left(J\gamma'' - \Delta J\gamma\right) + \frac{1}{4}\mathrm{I}\left(\alpha'' - \Delta\alpha\right) \end{pmatrix}$$

part ii:
Using

$$\operatorname{div}_{(4)} \begin{pmatrix} \alpha & \beta^T \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} -\alpha' + \operatorname{div}\beta \\ -\beta' + \operatorname{div}\gamma \end{pmatrix}$$

and

$$\varepsilon_{(4)} \begin{pmatrix} v \\ \mathrm{v} \end{pmatrix} = \begin{pmatrix} -v' & \frac{1}{2}\left(\nabla v + \mathrm{v}'\right) \\ \frac{1}{2}\left(\nabla v + \mathrm{v}'\right) & \varepsilon \mathrm{v} \end{pmatrix}$$

It is possible to obtain:

$$\begin{aligned}
&J\varepsilon \operatorname{div} J \begin{pmatrix} \alpha & \beta^T \\ \beta & \gamma \end{pmatrix} \\
&= J\varepsilon_{(4)} \operatorname{div}_{(4)} \begin{pmatrix} \frac{1}{2}\left(\alpha + \operatorname{tr}\gamma\right) & \beta^T \\ \beta & J\gamma + \frac{1}{2}\mathrm{I}\alpha \end{pmatrix} \\
&= J\varepsilon_{(4)} \begin{pmatrix} -\frac{1}{2}\left(\alpha' + \operatorname{tr}\gamma'\right) + \operatorname{div}\beta \\ -\beta' + \operatorname{div} J\gamma + \frac{1}{2}\nabla\nabla\alpha \end{pmatrix} \\
&= J_{(4)} \begin{pmatrix} -\frac{1}{2}\left(\alpha'' + \operatorname{tr}\gamma''\right) + \operatorname{div}\beta' & \frac{1}{2}\left(-\beta'' + -\nabla\operatorname{tr}\gamma' + \nabla\operatorname{div}\beta\right)^T \\ \frac{1}{2}\left(-\beta'' + -\nabla\operatorname{tr}\gamma' + \nabla\operatorname{div}\beta\right) & -\varepsilon\beta' + \varepsilon\operatorname{div} J\gamma + \frac{1}{2}\nabla\nabla\alpha \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4}\left(-\alpha'' - \operatorname{tr}\gamma'' + \Delta\alpha\right) + \frac{1}{2}\operatorname{div}\operatorname{div} J\gamma' & \frac{1}{2}\left(-\beta'' + \operatorname{div}\gamma' - \nabla\operatorname{tr}\gamma' + \nabla\operatorname{div}\beta\right)^T \\ \frac{1}{2}\left(-\beta'' + \operatorname{div}\gamma' - \nabla\operatorname{tr}\gamma' + \nabla\operatorname{div}\beta\right) & J\varepsilon\operatorname{div} J\gamma + \frac{1}{2}J\nabla\nabla\alpha - \frac{1}{4}\mathrm{I}(\alpha'' + \operatorname{tr}\gamma'') - S\varepsilon\beta' \end{pmatrix}
\end{aligned}$$

Summing i and ii we obtain:
The time-time component:

$$(\operatorname{ein} h)_{00} = \frac{1}{4}\left(\alpha'' - \Delta\alpha + \operatorname{tr}\gamma'' - \Delta \operatorname{tr}\gamma\right) + \frac{1}{4}\left(-\alpha'' - \operatorname{tr}\gamma'' + \Delta\alpha\right) + \frac{1}{2}\operatorname{div}\operatorname{div} J\gamma'$$
$$= \frac{1}{2}\operatorname{div}\operatorname{div} J\gamma'$$

the mixed component:

$$\begin{aligned}
(\operatorname{ein} h)_{\cdot 0} &= \frac{1}{2}\left(\beta'' - \Delta\beta\right) + \frac{1}{2}\left(-\beta'' + \operatorname{div}\gamma' - \nabla\operatorname{tr}\gamma' + \nabla\operatorname{div}\beta\right)\\
&= \frac{1}{2}\Bigg(\underbrace{-\Delta\beta + \nabla\operatorname{div}\beta}_{\operatorname{curl}\operatorname{curl}\beta} + \operatorname{div}\gamma' - \underbrace{\nabla\operatorname{tr}\gamma'}_{\operatorname{div}(\operatorname{tr}\gamma)\mathrm{I}}\Bigg)\\
&= \frac{1}{2}\Bigg(\underbrace{\operatorname{curl}\operatorname{curl}\beta}_{-2\operatorname{div} S\varepsilon\beta'} + \underbrace{\operatorname{div}\gamma' + \operatorname{div}(\operatorname{tr}\gamma)\mathrm{I}}_{\operatorname{div} S\gamma'}\Bigg)\\
&= \operatorname{div} S\left(\gamma' - 2\varepsilon\beta\right)
\end{aligned}$$

For the space-space entries

$$\begin{aligned}
(\operatorname{ein} h)_{\cdot\cdot} =& \frac{1}{2}\left(J\gamma'' - \Delta J\gamma + \frac{1}{4}\mathrm{I}\left(\alpha'' - \Delta\alpha\right)\right) + \left(J\varepsilon\operatorname{div} J\gamma + \frac{1}{2}J\nabla\nabla\alpha - \frac{1}{4}\mathrm{I}(\alpha'' + \operatorname{tr}\gamma'') - S\varepsilon\beta'\right)\\
=& \frac{1}{2}\Bigg(\underbrace{J\gamma'' - \frac{1}{2}\mathrm{I}\operatorname{tr}\gamma''}_{S\gamma''} + \underbrace{J\nabla\nabla\alpha - \frac{1}{2}\mathrm{I}\alpha}_{S\nabla\nabla\alpha} + \underbrace{2J\varepsilon\operatorname{div} J\gamma}_{2\operatorname{ein}\gamma + J\Delta} - S\varepsilon\beta'\Bigg)
\end{aligned}$$

Using $2\operatorname{ein} = \operatorname{inc}$ we conclude that

$$2\operatorname{ein}\begin{pmatrix}\alpha & \beta^T\\ \beta & \gamma\end{pmatrix} = \begin{pmatrix}\operatorname{div}\operatorname{div} S\gamma & \left(\operatorname{div} S\left(\gamma' - 2\varepsilon\beta\right)\right)^T\\ \operatorname{div} S\left(\gamma' - 2\varepsilon\beta\right) & S\gamma'' + \operatorname{inc}\gamma + S\nabla\nabla\alpha - 2S\varepsilon\beta'\end{pmatrix}$$

□

There is an important remark regarding the equations 3.2.1, the degrees of freedom, to be intended here as possible coordinate system, loosely speaking allow to fix 7 entries in the 4×4 matrix, this fact can be seen from the following remark

Remark 3.2.2. *Consider a second (euclidean) coordinate system and a linear re-parameterization only of the time component:*

$$\begin{cases}\hat{t} = & Ht + F^T x\\ \hat{x} = & x\end{cases}$$

Then the pulled-back of the metric does not influence the space-space component:

$$\begin{pmatrix} H & F^T \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} H & 0 \\ F & I \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\beta} & \gamma \end{pmatrix} \tag{3.16}$$

If we find a solution of the Cauchy problem with given α and β ,then

$$\hat{\gamma}(t) = \gamma \int_0^t \int_0^s \nabla\nabla\alpha(v)dvds - 2\int_0^t \varepsilon\beta(s)ds \tag{3.17}$$

Solves the same problem with vanishing α and β.

Thanks to the above remark one can reformulate the family of problems given by 3.2.1, depending on α and β in a single problem

Problem 3.2.3 (Homogeneous LEFE Problem)**.** *Given a symmetric field $\gamma_0 \in C^1$ and $\gamma_1 \in C^1$ satisfying the equations:*

$$\begin{cases} \operatorname{div}\operatorname{div} S\gamma_0 = 0 \\ \operatorname{div} S\gamma_1 = 0 \end{cases} \tag{3.18}$$

Find a symmetric matrix field $\gamma_(t) \in C^1(C^2(\Omega;\mathbb{R}^3);C^1(\Omega;\mathbb{R}^3))$ $\gamma(0) = \gamma_0$ and $\gamma'(0) = \gamma_1$ such that

$$S\gamma'' = -\operatorname{inc}\gamma \tag{3.19}$$

Remark 3.2.4 (It has the "form" of a wave)**.** *The equations are of the second order and un-damped, moreover using equation 3.10 therefore one can interpret them in a waves ,moreover the in the same way as we do for the Maxwell wave equations.*

Using the invertibility of the S operator, eq.A.25b, and using the remark 3.2.1, we can define a new symplectic problem:

Problem 3.2.5. *Given symmetric fields: $\gamma_0 \in C(\Omega)$ and $\gamma_1 \in C(\Omega)$ satisfying the equations:*

$$\begin{cases} \operatorname{div}\operatorname{div} S\gamma_0 = 0 \\ \operatorname{div} S\gamma_1 = 0 \end{cases} \tag{3.20}$$

Find symmetric matrix fields $\gamma_(t) \in C^1(C^2(\Omega), C^1(\Omega)), \kappa(t) \in C^1(C^2(\Omega), C^1(\Omega)) : \gamma(0) = \gamma_0$ and $\kappa'(0) = J\gamma_1$

$$\dot{\kappa} = -\operatorname{inc}\gamma \tag{3.21a}$$

$$\dot{\gamma} = J\kappa \tag{3.21b}$$

$$\operatorname{div}\kappa = 0 \tag{3.21c}$$

It is now possible to conclude the section with an observation about the energy conservation in the smooth setting.

Remark 3.2.6. *Let* $\langle A, B\rangle_F := \sum_{ij} A_{ij}B_{ij}$ *be the Frobenious inner product between matrices. Then the quantity:*

$$\mathcal{H}(\gamma,\kappa) = \frac{1}{2}\|\kappa\|_F^2 - \frac{1}{4}\|\operatorname{tr}\kappa\|^2 + \frac{1}{2}\left\langle \operatorname{curl}(\gamma)^T, \operatorname{curl}(\gamma)\right\rangle_F$$

Is constant in time.

Proof.

$$\begin{aligned} S(\ddot{\gamma}) + \operatorname{inc}\gamma =& 0 \\ \langle S(\ddot{\gamma}), \dot{\gamma}\rangle + \langle \operatorname{inc}\gamma, \dot{\gamma}\rangle =& 0 \\ \frac{d}{dt}\frac{1}{2}\langle\kappa), J\kappa\rangle + \frac{d}{dt}\frac{1}{2}\left\langle \operatorname{curl}\gamma^T, \operatorname{curl}\gamma\right\rangle =& 0 \end{aligned}$$

The last passage is done using the fact that both inc and J are self adjoint. Using the definition of J

$$\langle J(A), B\rangle_F = \langle A - \operatorname{tr}(A)/2I, B\rangle_F = \langle A, B\rangle - \left\langle \frac{1}{2}(\operatorname{tr} A), \operatorname{tr} B\right\rangle_F$$

□

so far we no physical meaning is given to κ.BUt the next section will finally solve this problem.

3.3. The formal 1+3 derivation: The AMD Formalism

For a proper splitting of the metric one introduces a formalism, i.e. the splitting by mean of formal/mathematically consistent theories. Some of the most common are.

1. **1 + 3 formalism** One divides the spacetime into space and time.
2. **conformal formalism** makes use of angles preserving maps to treat the unbounded M^4, minkowskian spacetime, as if it were a bounded domain.
3. **characteristic formalism** split the spacetime into lightcones

Only the firs formalism is faced here, in particular the main idea is extracted from the book[Alc08]. The first problem one is facing in GR: space and time are dealt with with the same footing, the solution of the Cauchy problem needs to take care of something that has initial conditions (and maybe boundaries conditions). This process of splitting is known as $3+1$-formalism.

We consider the spacetime metric $g_{\alpha\beta}$, since the metric induces a metric on every sub-manifold we create a parametrization of the spacetime such that on each 3D sub-manifold, parametrized using the worldline of a test particle (for example), one has that the induced inner product is positively defined (AKA spacelike). if the spacetime can be foliated in this way one refers to is as globally hyperbolic, in particular a globally hyperbolic manifold has a preferential direction of travel ... foreward in time.

The time function t is the parameter function used to foliate the spacetime. It doesn't have to be a proper time in general, or any kind of preferred time of observation. For exaple can be the time mesured by a free falling particle (geodesic time) or can a generic one.

Now that we have a possible foliation let's consider what is going to happen between infinitesimally close hypersurfaces Σ_t and Σ_{t+dt}.
The splitting of the metric follows the following:

1. γ_{ij} induces a proper measurement for vectors (and covectors as a consequence) lying on the same hypersurface :

$$dl^2 = \gamma_{ij} dx^i dx^j \tag{3.22}$$

2. The lapse of proper time is the change in time measured by observers that at rest in space:

$$d\tau = \alpha(t, x^i) dt \tag{3.23}$$

α is known as the “lapse function”.

3. β^i represents the relative velocity between the observers at rest in space and the line normal to the hypesurface:

$$x^i_{t+dt} = x_i - \beta^i(t, x^j) dt \tag{3.24}$$

The 3-vector β^i is known as the “shift vector”

The metric can be expressed in terms of functions:

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix} \tag{3.25}$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta_i}{\alpha^2} \\ \frac{\beta_j}{\alpha^2} & \gamma_{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{3.26}$$

similarly the vector normal to the hypersurface becomes:

$$n^\nu = (\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}) \tag{3.27}$$

$$n_\nu = (-\alpha, \vec{0}) \tag{3.28}$$

$$n^\nu n_\nu = -1 \tag{3.29}$$

Extrinsic and Intrinsic curvature

There are in general two kinds of curvatures, the one that depends on the only on the sub-manifold Σ_t and the one that uses the embedding space as a super-space, they are referred to as intrinsic and extrinsic curvature.

Usually the intrisic curvature concerns only the Riemannian tensor built around γ_{ij}, contrary the extrinsic curvature, or extrinsic curvature tensor, is defined starting from the normal vector.

In order to define K itself one needs to define a projection onto the hypersurface

$$P^\alpha_\beta := \delta^\alpha_\beta + n^\alpha n_\beta \tag{3.30}$$

To show that the projection just defined is a projection onto the hypersurface one considers a projection of a generic vector via P^β_α and considers the normal component using the fact that n^μ is time-like and has norm one

$$\left(P^\alpha_\beta v^\beta\right) n_\alpha = \delta^\alpha_\beta + n^\alpha v^\beta n_\beta n_\alpha = 0 \tag{3.31}$$

To project a tensor one needs to project each component

$$PT_{\alpha\beta} = P^\mu_\alpha P^\nu_\beta T_{\mu\nu} \tag{3.32}$$

Definition 6 (Extrinsic curvature)**.** *The extrinsic curvature is defined as the projection applied to the covariant derivative of the normal to the hypersurface:*

$$K_{\alpha\beta} := -P\nabla_\alpha n_\beta = -(\nabla_\alpha n_\beta + n_\alpha n^\nu \nabla_\nu n_\beta) \tag{3.33}$$

As an obvious consequence from the definition we have

Remark 3.3.1.

$$n^\alpha K_{\alpha\beta} = 0 \tag{3.34a}$$

$$K^{00} = K^{0i} = 0 \tag{3.34b}$$

$$K_{\alpha\beta} = K_{\beta\alpha} \tag{3.34c}$$

The last one is not immediate, to prove it one considers the expansion of the covariant derivative of the normals using the Christoffel symbols.

Using the notation in the previous chapter and using the definition of the normal vector defined before and using the metrics we reduce the definition to :

$$\kappa_{ij} = \frac{1}{2\alpha}\left(-\partial_t \gamma_{ij} + D_i\beta_j + D_j\beta_i\right) \tag{3.35}$$

Where: D_i is the induced covariant derivative in 3d.

$$\partial_t \gamma_{ij} = -2\alpha\kappa_{ij} + D_i\beta_j + D_j\beta_i \tag{3.36}$$

It [Alc08] it is proven the following result:

$$(\partial_t - \pounds_{\vec{\beta}})\gamma_{ij} = -2\alpha\kappa_{ij} \tag{3.37}$$

or in other words:

$$(\pounds_{\alpha\vec{n}})\gamma_{ij} = -2\alpha\kappa_{ij} \tag{3.38}$$

Therefore the time variation of the metric observed when at rest in space is exactly the extrinsic curvature.

In particular this equation here looks particularly similar to one equation of our previous derivation 3.21 where we set the

The Arnowitt-Deser-Misner formalism

The project the EFE in empty space into the various components of the spacetime is done using the projection P_α^β and the n^α:

$$n^\alpha n^\beta G_{\alpha\beta} = 0 \tag{3.39}$$

$$P\left(n^\alpha G_{\alpha\beta}\right) = 0 \tag{3.40}$$

$$P\left(G_{\alpha\beta}\right) = 0 \tag{3.41}$$

Called respectively the normal, mixed and hyper-surface projections. The derivation is quite long and involved, therefore only the final result will be given:

Definition 7 (Arnowitt-Deser-Misner (ADM) equations)**.** *The following set of equations is called Arnowitt-Deser-Misner equations*

$$\begin{cases} (\partial_t - \pounds_{\vec{\beta}})\gamma_{ij} = -2\alpha K_{ij} \\ \left(\partial_t - \pounds_{\vec{\beta}}\right) K_{ij} = -D_i D_j \alpha + \alpha\left(R_{ij}^{(3)} - 2K_{ia}K_j^a + K_{ij} trK\right) + 4\pi\alpha\left(\gamma_{ij}(trS - \rho) - 2S_{ij}\right) \\ R^{(3)} + (\mathrm{tr}\,\kappa)^2 - \kappa_{ij}\kappa^{ij} = 0 \\ D_j\left(\kappa^{ij} - \gamma^{ij} tr\kappa\right) = 0 \end{cases} \tag{3.42}$$

The first two equations are intended as evolution equation, meanwhile the last two are referred to as energy (or Hamiltonian) constraint and momentum constraint.

So far only the projection onto the different parts of the matrix has being made, we need to linearize it and to check if it is indeed equivalent to the first derivation. The same hypothesis circa the framework are given: small perturbation, minkowskian background metric, the lapse function is $= 1$, there is no shift functions $\vec{\beta} = 0...$. The linearized ADM evolution equations are:

Definition 8 (Linearized ADM)**.**

$$\partial_t \gamma_{ij} = -2\kappa_{ij} \tag{3.43a}$$

$$\partial_t \kappa_{ij} = \frac{1}{2}\left(\nabla^2_{flat}\gamma_{ij} - \partial_i\partial_k\gamma_{jk} - \partial_j\partial_k\gamma_{ik} + \partial_i\partial_j\gamma_{rr}\right) \tag{3.43b}$$

Remark 3.3.2. *Here there is a subtle abuse of notation, usually the linearized ADM are derived creating the variation of the 3-metric $\gamma_{ij} = \delta_{ij} + \lambda_{ij}$ but to have a comparable result with the Li derivation I decided to keep γ.*

The above can be written in a more compact form :

$$\begin{cases} \dot{\gamma} &= -2\kappa \\ \dot{\kappa} &= \frac{1}{2}\left(-\Delta\gamma + 2\varepsilon\,\mathrm{div}(\gamma) - \nabla\nabla\,\mathrm{tr}\,\gamma\right) \end{cases} \tag{3.44}$$

Proposition 4 (Equivalence of the results)**.** *The evolution equation for linearized ADM 8 and for Lizao 3.2 are equivalent.*

Proof. Transforming back the Lizao equation to the preceding equation, and inverting the operator S

$$\ddot{\gamma} = -J\,\mathrm{inc}\,\gamma$$

and deriving in time the first component of 8

$$\ddot{\gamma} = -\cancel{2}\frac{1}{\cancel{2}}\left(-\Delta\gamma + 2\varepsilon\,\mathrm{div}(\gamma) - \nabla\nabla\,\mathrm{tr}\,\gamma\right) = \Delta\gamma - 2\varepsilon\,\mathrm{div}(\gamma) + \nabla\nabla\,\mathrm{tr}\,\gamma$$

Comparing the right hand sides there should result an equality.

$$-2J\,\mathrm{ein}\,h = \Delta h - 2\varepsilon\,\mathrm{div}(h) + \nabla\nabla trh \tag{3.45}$$

Fortunately we know already the form of ein and since we can invert J we have to prove:

$$\begin{aligned}
-2\,\mathrm{ein}\,h &= S\Delta h - S2\varepsilon\,\mathrm{div}(h) + S\nabla\nabla\,\mathrm{tr}\,h \\
\left(\cancelto{b}{\Delta\,\mathrm{tr}\,h} - \mathrm{div}\,\mathrm{div}\,h\right) Id + 2\varepsilon\,\mathrm{div}\,h - \nabla\nabla\,\mathrm{tr}\,h - \cancelto{a}{\Delta h} &= -\cancelto{a}{\Delta h} + \cancelto{b}{\Delta\,\mathrm{tr}\,h I} + S2\varepsilon\,\mathrm{div}(h) - S\nabla\nabla\,\mathrm{tr}\,h \\
-\,\mathrm{div}\,\mathrm{div}\,hId + \cancelto{c}{2\varepsilon\,\mathrm{div}(h)} - \nabla\nabla\,\mathrm{tr}\,h &= \cancelto{c}{2\varepsilon\,\mathrm{div}(h)} - \mathrm{tr}\,2\varepsilon\,\mathrm{div}(h) - S\nabla\nabla\,\mathrm{tr}\,h \\
-\,\mathrm{div}\,\mathrm{div}\,hId - \cancelto{d}{\nabla\nabla\,\mathrm{tr}\,h} &= -\,\mathrm{tr}\,2\varepsilon\,\mathrm{div}(h) - \cancelto{d}{\nabla\nabla\,\mathrm{tr}\,h} + (\mathrm{tr}\,\nabla\nabla\,\mathrm{tr}\,h)Id \\
-\,\mathrm{div}\,\mathrm{div}\,hId &= -\,\mathrm{tr}\,2\varepsilon\,\mathrm{div}(h) + (\mathrm{tr}\,\nabla\nabla\,\mathrm{tr}\,h)Id
\end{aligned}$$

The last part cancels with the usual lemma of algebraic identities A.1.1 □

Part II.

Finite Element Spaces and their Discretization

4. Some Finite Elements

Scope of the Chapter This chapter is auxiliary to the next one, in here the most basic concepts of Finite Element Methods are introduced: the dual space of the test functions, the Sobolev spaces, the definition of finite element, and so on[1]. More in details the spaces that are treated are : H(curl), H(div), H(curl curl), H(div div). Since they are the one in which which at least one the operators of the previous equations are defined. The discretization is introduced for the lowest order shape functions for all FEs[2]. In particular there is a more sophisticated introduction of the new element $H(\text{inc})$, since it is the newest it had a big impact on the theory.

Let $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ be the space of test functions, let its dual be $\mathcal{D}'$, called the space of distributions[3] the space is identified as the space of bounded linear functionals that act on elements of $\mathcal{D}$. In particular in this big space -to give some comparison it is bigger than the space of measures- it is always possible to define a differentiation of arbitrary order

Definition 9 (Generalized and Weak Derivative)**.** *Let $\vec{\alpha}$ be a multi-index, let $u \in \mathcal{D}'$ then one defines the function $g \in \mathcal{D}'$ and call it generalized derivative of u in the following way:*

$$\langle g, \phi \rangle_{\mathcal{D}' \times \mathcal{D}} = (-1)^{|\vec{\alpha}|} \langle u, D^\alpha \phi \rangle_{\mathcal{D}' \times \mathcal{D}} \quad \forall \phi \in \mathcal{D} \tag{4.1}$$

, if $u \in L_1^{loc}(\Omega)$ and if $g \in L_1^{loc}(\Omega)$ then we refer to g as the weak derivative of u if:

$$\int_\Omega g\phi = (-1)^{|\vec{\alpha}|} \int_\Omega u D^\alpha \phi \quad \forall \phi \in \mathcal{D} \tag{4.2}$$

,

Starting from this definition one builds the definitions of the special Sobolev spaces[ZT17]

$$W_p^k(\Omega) := \left\{ u \in L_1^{loc} : \|u\|_{W_p^k(\Omega)} < \infty \right\} \tag{4.3}$$

in particular they are Banach spaces, with except to $p = 2$ that is also a Hilbert space and it is indicated by H^k.By means of the trace it is possible to form a sequence of inclusions

$$\mathcal{D} \cdots \subset H^2 \subset H^1 \subset H^0 = \mathbb{L}_2 \subset H^{-1} \subset H^{-2} \subset \cdots \mathcal{D}'$$

where $H^{-k} = (H_0^k)^*$ using the duality pairing induced by the inner product of the H^k spaces[BB11].

[1] The stability, the Existence and uniqueness are not discussed but cited.
[2] citations are given for where to find the higher order shape functions.
[3] The creation of this particular dual space is not as immediate as it seems, the theory suggests to build a Frechét space.[ZT17], Chapter 10.

The next section is centered around giving a recap of the certain Hilbert-Sobolev spaces and their discretization. Here a quick scheme of the spaces involved:

$$H(\mathrm{div}, \Omega) \longrightarrow RT^k(\mathcal{T}) \quad Ravier - Thomas \tag{4.4a}$$
$$H(\mathrm{curl}, \Omega) \longrightarrow N_I^k(\mathcal{T}) \quad Nedelecofthefirstkind \tag{4.4b}$$
$$H(\mathrm{div}\,\mathrm{div}, \Omega) \longrightarrow NN^k(\mathcal{T}) \quad Pechstein - Schoeberl \tag{4.4c}$$
$$H(\mathrm{curl}\,\mathrm{curl}, \Omega) \longrightarrow \mathrm{Reg}^k(\mathcal{T}) \quad Regge - Lizao \tag{4.4d}$$

where k is the order of the FEs involves, Ω is the manifold of definition of the of the problem (domain), $\mathcal{T}$ is the triangulation of the space Ω. All the above are conforming with respect to their natural definition.[4]

Definition 10 (Ciarlet's Finite Element). *A finite element is a triple (T, V_T, Ψ_T) where T is a bounded set, V_T is a function space on T of finite dimension and $\Psi_T = (\psi_T^i)$ is a collection of linearly indeperndant functionals on V_T*

We want to find a basis (ϕ_T^i) for V_T such that with (ψ_T^i) it becomes a bi-orthogonal to the functionals, moreover set T to be a tetrahedron since in a further definition we will need the set T to be a simplex and we define a set polynomial spaces to approximate the continuous spaces. To give an explicit basis for V_T usually auxiliary families of polynomials are given

4.0.1. Hierarchical polynomials

Definition 11 (Legendre Polynomials, Integrated Legendre Polynomials). *The Legendre polynomials are defined in the following recursive way*

$$P_0(x) = 1 \tag{4.5}$$
$$P_1(x) = x \tag{4.6}$$
$$P_i(x) = \frac{2i-1}{i} P_{i-1}(x) - \frac{i-1}{i} P_{i-2}(x) \tag{4.7}$$

The integrated Legendre polynomials are defined as

$$L_0(x) = -1 \tag{4.8}$$
$$L_1(x) = x \tag{4.9}$$
$$L_i(x) = \int_{-1}^{x} P_{i-1} \tag{4.10}$$

The main difference between constructing a basis for a square and a triangle is that on the in a square one can take as basis the simple product between the basis of the edges, the same thing is not possible for the triangle, since in the reference triangle the point $(1, 1)$ is out of the set! Therefore one must scale the polynomials when we detach from one side.

[4]In a FEs there is a natural condition of continuity from one cell to the adjacent one, in general this condition can be broken, but here it is not.

Definition 12 (Scaled Legendre and Scaled Integrated Legendre Polynomial)**.** *We define the Scaled Legendre and Scaled Integrated Legendre Polynomial of order i as follows:*

$$P_i^S(x,t) = P_i(\frac{x}{t})t^i \tag{4.11}$$

$$L_i^S(x,t) = L_i(\frac{x}{t})t^i \tag{4.12}$$

Definition 13 (Jacobi polynomials and scaled Jacobi polynomials)**.** *Let $\alpha, \beta > -1$, let $\omega = (1-x)^\alpha(1-x)^\beta$ then we define the Jacobi polynomials and scaled Jacobi polynomials as:*

$$P_n^{00}(x) = L_n(x) \tag{4.13a}$$

$$P_n^{\alpha\beta}(x) = \frac{(-1)^n}{\omega(x)2^n n!}\frac{d^n}{dx^n}(\omega(x)(1-x^2)^n) \tag{4.13b}$$

$$P_n^{\alpha\beta,S}(x,t) = t^n P_n^{\alpha\beta}(x/t) \tag{4.13c}$$

Definition 14 ((Transformed) Daubinier polynomials of order k and)**.** *let $k > -1$ called order, let $i, j > -1$ such that $i + j \leq k$*

$$\phi_{ij}^D(x) = L_i^S(x,y)P_j^{(2i+1)0}(2y-1)$$

Let $i, j > -1$ such that $i + j + n \leq k$

$$\phi_{ijn}^{D,3}(\lambda_1,\lambda_2,\lambda_3) := L_i^S(\lambda_2-\lambda_1,\lambda_1+\lambda_2)P_j^{(2i+1)S}(\lambda_3-\lambda_1-\lambda_2, 1-\lambda_4)P_n^{2i+2j+2}(1-2(1-\lambda_4))$$

Where $\lambda_1, \lambda_2, \lambda_3$ are respectively the baricentric coordinate on the ref. tetrahedron in 3D of the vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.

Moreover $\mathbb{P}^k*$ is defined as the set of homogeneous polynomials of degree k.

4.1. A recap on the spaces

Using the auxiliary families of polynomials one can buils the FES by means of the bi-orthogonal basis. In particular one needs to prove two things in order to create a "good space":

- The number of dofs and the number of elements spanning V_T are the same: usually this is done counting!
- The dofs need to be linearly independent: for this proof usually one starts assuming that for a generic function in that space the action of all dofs make it vanish and extract the fact that it has to be the zero-polynomial.
- The preservation of the continuity under linear affine transformations: If one maps the reference element onto another element they must be affine equivalent, therefore there is the necessity of a map from the reference element to the another element that preserves the dofs. This is done using the piola transormation[Cia88] and the covariant transformation[SR16]

$H(\text{div})$ and $\mathcal{RT}^k$

$$H(\text{div};\Omega) = \left\{ d \in \left[\mathbf{L}^2(\Omega)\right]^3 e : \text{div}\, d \in \left[\mathbf{L}^2(\Omega)\right]^d \right\} \tag{4.14a}$$

$$\mathcal{RT}^k(T) = \left\{ d = a + b\vec{x} \,|\, a \in [\mathbb{P}^k(\mathcal{T})]^3 \,,\, b \in \mathbb{P}^{k,*}(\mathcal{T}) \,,\, \text{normal continuous on} F, \forall F \in \mathcal{F} \right\} \tag{4.14b}$$

The transformation needed to pass from the reference tertahedron to a generic one is $\phi : x \mapsto Ax + b$, where $A \in \mathbb{M}^{3\times 3}$ is a constant matrix and $b \in \mathbb{V}^3$ is a constant vector. Here intention is to see A as the differential of the map ϕ, therefore the transformation that acts between the triplets is well is defined as:

$$H(\text{div}, \hat{T}) \ni \hat{d} \mapsto d = \det(A)^{-1}(A\hat{d}) \circ \phi^{-1} \tag{4.15}$$

- Lowest order dofs

$$\psi_{\alpha\beta\gamma} : v \mapsto \int_{F_{\alpha\beta\gamma}} v \cdot n \tag{4.16}$$

Where $F_{\alpha\beta\gamma}$ is a face identified via V_α, V_β and V_γ

- Lowest order shape functions

An implementable higher order dofs and shape functions, for tetraherons can be found in [BBF+13].

The distributive div assuming tangential continuity

$$\langle \text{div}\, u, \phi \rangle = \sum_T \int_T \text{div}\, u, \phi \quad \forall \phi \in H(\text{div}, \mathcal{T}) \tag{4.17}$$

$H(\text{curl})$ and $\mathcal{N}_I^k$

$$H(\text{curl};\Omega) = \left\{ e \in \left[\mathbf{L}^2(\Omega)\right]^3 : \text{curl}\, e \in \left[\mathbf{L}^2(\Omega)\right]^3 \right\} \tag{4.18}$$

$$\mathcal{N}_I^k(T) = \left[\mathbb{P}^k(T)\right]^3 \otimes \{e \in [\mathbb{P}^{k+1,*}] \,:\, e \cdot x = 0\} \tag{4.19}$$

$$\mathcal{N}_I^k(\mathcal{T}) = \left\{ e \in \prod_{T \in \mathcal{T}} \mathcal{N}_I^k(T) \,|\, e_t \text{ is continuous} \right\} \tag{4.20}$$

The transformation here is called Covariant transformation :

$$H(\text{curl}, \hat{T}) \ni \hat{e} \mapsto e = A^{-T}\hat{e} \circ \phi^{-1} \tag{4.21}$$

- Lowest order dofs

$$\psi_{E_{\alpha\beta}} : v \to \int_{E_{\alpha\beta}} v \cdot \tau ds \tag{4.22}$$

Where $E_{\alpha\beta}$ is the oriented segment with vertices V_α , V_β

- Lowest order shape functions

$$\phi_{\alpha\beta,0} := \lambda_\alpha \nabla \lambda_\beta - \lambda_\beta \nabla \lambda_\alpha \tag{4.23}$$

For higeh order dofs and shape functions refer to [SZ05].

The distributive form of the curl assuming tangential continuity across the faces.

$$\langle \operatorname{curl} u, \phi \rangle = \sum_T \int_T \operatorname{curl} u, \phi \quad \forall \phi \in H(\operatorname{curl}, \mathcal{T}) \tag{4.24}$$

$H(\operatorname{div}\operatorname{div})$ **and** NN^k

$$H(\operatorname{div}\operatorname{div};\Omega) = \left\{ \kappa \in \left[\mathbf{L}^2(\Omega)\right]^{3\times 3} \mid \operatorname{div}\operatorname{div}\kappa \in H^{-1}(\Omega) \right\} \tag{4.25}$$

$$NN^k(T) = \left\{ [\mathbb{P}^k(T)]^{3\times 3}_{sym} \mid [\![\kappa_{nn}]\!]_F = 0 \,\forall F \in \mathcal{F} \right\} \tag{4.26}$$

The transformation is a double Piola one:

$$H(\operatorname{div}\operatorname{div}, \hat{T}) \ni \hat{\kappa} \mapsto \kappa = \det(A)^{-2}(A\hat{d}A^T) \circ \phi^{-1} \tag{4.27}$$

A possible conforming implementation can be found in [CH20].

The distributive form for the divergence is:

$$\langle \operatorname{div} \kappa, \phi \rangle = \sum_T \langle \operatorname{div} \kappa, \phi \rangle - \sum_{\partial T} \langle \kappa_{nt}, \phi_t \rangle \tag{4.28a}$$

$$= \sum_T \langle \operatorname{div} \kappa, \phi \rangle - \sum_{\partial T} \langle \kappa_{nt}, \phi_t \rangle \tag{4.28b}$$

As mentined in the introduction and in the abstract it is now time to derive formally the regge elements.

4.2. The Regge elements

a brief history of the new element:

The roots for this element can be traced back to a paper [Reg61] by Tullio Regge in which he was considering a way to approximate a curved space on a polyhedrons, further developement of the theory, such as seeing the space-time as a complex of simplices, are referred as Regge Calculus[**empty citation**]; In 2011 Snorre H. Christiansen [Chr11] developed a discrete setting on which one is able to discretise the curl T curl operator in such a way that the it converges more and more to the continuous operator, the paper is the first formal description of the space Reg^0. The first description of the generalized regge finite element is given in [Li18], here Lizao sets his work in the differential geometry framework,he starts from the definition of simplicial finite element and from a particular choice of the degrees of freedom he is able to obtain different properties for this Finite Element:

- assemblability in a simplicial mesh.
- The set of degrees of freedom is unisolvent.
- The tangential part has to be single valued on the faces of positive dimension, AKA is tangential-tangential continuous.

They are proven using the technique of geometric decomposition of the space of polynomials and using the pullback on the faces, in this way one obtains the previous properties as direct corollaries. For more in depth explanation please cfr page 6-40 of [Li18]. in particular at we need some implementable basis for the Regge space.

Li Derivation of Regge FES

This part deals with the coordinate-free derivation of the Regge Finite Element Space of degree k Reg^k.

Let T be a simplex of generic space dimension m, a subsimplex of T is noted ad f, F is referred to $(m-1)$-simplex and E for 1- simplex. Let $\mathbb{P}^k(f)$ be the space of polynomials on f of degree k or less, in r-variables where r is the dimension of f.

Definition 15 (Simplicial Finite Element)**.** *It is a triple* (T, V, Σ) *with:*

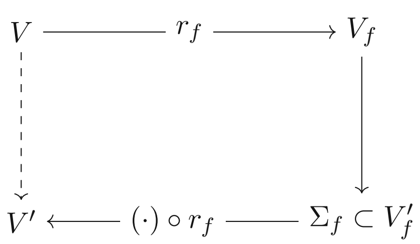
1. *T is a simplex.*
2. *V is a finite dimensional function space on T.*
3. *$\Sigma = ((r_f, \Sigma_f))_{f \subset T}$ is a collection indexed on the subsimplex of T with:*

$$r_f : V \to V_f$$

With V_f some function space and $\Sigma_f \subset V_f'$ such that Σ solves the unisolvency condition:

$$V' = \bigoplus_{f \subset T} (u \mapsto l(r_f(u)) | l \in \Sigma_f) \tag{4.29}$$

Remark 4.2.1. *Most of the time the finite element, in Ciarlet sense4, there is the necessity to define the dofs a priori and only after the possible shape funcitons to span the domain. Moreover the dofs can be substantially different one another, but they keep in common one simple fact: they use the facets of the element! Therefore the idea of defining the dofs on the in the above sense4.2 seems a good starting point. Instead of giving directly a dual space of V one pass through this construction: $a \in V'$ can be indexed on the face it belongs to $a_f^i = l_f^i \circ r_f$ for a certain index i such that $l_f^i \in \Sigma_f$.*



One of the fundamental ideas is that each facet of a simplex is a simplex itself, there we can define an inclusion map such that given a d-simplex T and given a k -face f we define an inclusion map from the face to the simplex $\iota_{f\to T}$.

Since the inclusion is a smooth map one can define its pullback

Definition 16. *Let $g \in \mathbb{S}\otimes C^{\infty}(T)$, let f be a subsimplex of T ad $\iota_{f\to T}$ the inclusion, then the pullback $\iota_f^* g := \iota_{f\to T}^* g \in \mathbb{S}\otimes C^{\infty}(f)$ is called tangential-tangential part of g at f*

it assigns at each point $p \in f$ a symmetric bilinear form on vectors tangent to f .

Remark 4.2.2. *Let M manifold and N sub-manifold, then the inclusion map $\iota_{M\to N}$: $M \ni x \overset{x}{\hookrightarrow} \in N$, then for all $p \in M$, $T_pM \subset T_pN$ in the following sense: Let $v \in T_pM$ be a derivation, then there exists an open neighbour U_p of $p \in M$ such that it is mapped to U_p in N and a smooth function $f \in C(U_p)$:*

$$T_pN \ni \left(d\iota_{M\to N}\right)_p v\,[f] := v\left(f\circ\iota_{M\to N}\right)_p = v\,(f)_p \in T_pM \tag{4.30}$$

As a direct consequence, for every k-inear form g defined on T_pN and for every $(v_j)_{j=1}^k$ derivations in T_pM we have that g is a k-form on T_pN too:

$$\iota^* g\left((v_j)_{j=1}^k\right) = g\left((v_j)_{j=1}^k\right)$$

The definition 4.2 "suggests" to use somehow the just defined pullback to define a Finite Element, as folklore we start with the family of polynomials of degree r on the k-simplex T : $\mathbb{P}^r(T)$, this space has dimension

$$\dim \mathbb{P}^r(T) = \binom{d+r}{d}.$$

The space of polynomial symmetric 2-tensor fields of degree r can be identified given via tensor product of the space of symmetric covariant 2-tensor fields in $\mathbb{R}^d$,AKA $\mathbb{S}^d$, and the previous space of polynomials. The dimension of $\mathbb{S}^d$ is obviously $\frac{(d+1)d}{2}$ that can be rewritten in a more elegant way as $\binom{d+1}{2}$, therefore the space that we need to introduce

$$\mathbb{P}^r\mathbb{S}(T) := \mathbb{P}^r(T)\otimes\mathbb{S}^d$$

with

$$\dim \mathbb{P}^r\mathbb{S}(T) = \binom{d+1}{2}\binom{d+r}{d} \tag{4.31}$$

Let $\mathcal{T}$ be a mesh of dimension d, we define a symmetric covariant 2-tensor fields of degree r or less a function such that assigns at each cell $T \in \mathcal{T}$ an element of $\mathbb{P}^r\mathbb{S}(T)$. In particular

$$\mathbb{P}^r\mathbb{S}(\mathcal{T}) := \left\{\gamma : \gamma_{|_T} \in \mathbb{P}^r\mathbb{S}(T)\right\}$$

Definition 17 (Finite Element Family)**.** *The finite element family $\mathscr{F}$ is a function defined on a collection of simplices $D(\mathscr{F})$ such that, given a simplex $T \in D(\mathscr{F})$ it returns the finite element associated to the simplex $\mathscr{F}(T)$.*

Definition 18. *A finite element is called assemblable on $\mathcal{T}$ if, for all $T_1, T_2 \in \mathcal{T}$, $T_1, T_2 \in D(\mathscr{F})$ and if $f = T_1 \cap T_2$ then $\mathscr{F}(T_1)$ and $\mathscr{F}(T_2)$ give the same $r_f(V)$ and Σ_f on f.*

If the space is assemblable than each function g in the final space $\mathscr{F}(\mathcal{T})$ must respect the following:

$$l \circ r_f(g_{|T_1}) = l \circ r_f(g_{|T_2}) \quad \forall l \in \Sigma_f \tag{4.32}$$

We have all the ingredients to define the generalized Regge finite element:

Definition 19 (Generalized Regge Finite Element)**.** *The Generalized Regge Finite Element of degree r is a simplicial finite element*

$$(T, V, \Sigma) \tag{4.33}$$

Where:

1. *T be a d-simplex in $\mathbb{R}^d$*
2. *$V := \mathbb{P}^r\mathbb{S}(T)$ shape functions*
3. *the degrees of freedom assigned on each k-face f of T with $k \geq 1$ are:*

$$r_f := \iota^*_{f \to T} : \mathbb{P}^r\mathbb{S}(T) \to \mathbb{P}^r\mathbb{S}(f) \tag{4.34}$$

$$\Sigma_f := \left(g \mapsto \int_f g : q \,|\, q \in \mathbb{P}^{r-k+1}\mathbb{S}(f) \right)_{f \subset T} \tag{4.35}$$

When working with finite elements we need unisolvency, to do so one must first check that the dimension of the shape functions is the same as the dimension of the Dofs.

Lemma 4.2.3. *The following holds true:*

$$\sum_{k=1}^{d} \binom{d-1}{d-k} \binom{r+1}{k} = \binom{r+d}{d} \tag{4.36}$$

Proof. To prove it we need the following identity

$$(1+x)^{d-1}(1+x)^{r+1} = (1+x)^{d+r} \tag{4.37}$$

$$\tag{4.38}$$

and the Newton's binomial expansion:

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \tag{4.39}$$

Then using 4.39 in 4.37 we obtain:

$$\left(\sum_{j=0}^{d-1}\binom{d-1}{j}x^j\right)\left(\sum_{l=0}^{r+1}\binom{r+1}{l}x^l\right)=\left(\sum_{m=0}^{d+r}\binom{d+r}{m}x^m\right) \tag{4.40}$$

the above ere equal if and only if the element with same exponent in the x variable are the same, this means that collecting the x^m on both sides yields to:

$$\sum_{l+j=m}\left(\binom{d-1}{j}\binom{r+1}{l}x^m\right)=\left(\sum_{m=0}^{d+r}\binom{d+r}{m}x^m\right) \tag{4.41}$$

Replacing $j = d - k$ in the first coefficient the proof is complete. □

Proposition 5. *The dimension of shape functions is equal to the number of dofs*

Proof. The dimension of $\mathbb{P}^r\mathbb{S}$ has been proven (4.2) to be:

$$\dim \mathbb{P}^r\mathbb{S}(T)=\binom{d+1}{2}\binom{d+r}{d}$$

To count the number of k-faces in a d-simplex one uses the following "trick" observing that, to create a k-simplex from a d-simplex, one may pick $k+1$ vertices among the $d+1$ that form the d-simplex, then the number of k faces equals the number of combinations are $\binom{n+1}{k+1}$. In particular the total number of Dofs are:

$$\sum_{k=1}^{d}\binom{n+1}{k+1}\dim \mathbb{P}^{r-k+1}\mathbb{S}(f)=\sum_{k=1}^{d}\binom{n+1}{k+1}\binom{r+1}{k}\binom{k+1}{2} \tag{4.42}$$

Modifying the first and last members of the product on the right hand side:

$$\begin{aligned}\binom{d+1}{k+1}\binom{k+1}{2}&=\frac{(d+1)!}{(k+1)!(d-k)!}\,\frac{(k+1)!}{(k-1)!2!}\\&=\frac{(d+1)!}{(d-1)!2!}\,\frac{(d-1)!}{(k-1)!(d-k)!}\\&=\binom{d+1}{2}\binom{d-1}{d-k}\end{aligned}$$

Inserting back into the equation 4.42 and using the above technical lemma 4.2.3 we end the proof. □

Since the third member of the triplet (T, V, Σ) depends only on the facets $f \subset \Sigma$ then the space is assemblable:

$$\iota^*_{f\to T_1}(\mathbb{P}^r\mathbb{S}(T_1))=\mathbb{P}^r\mathbb{S}(f)=\iota^*_{f\to T_2}(\mathbb{P}^r\mathbb{S}(T_2)) \quad \text{, for all common } k\text{-face } f\subset T_1\cap T_2 \tag{4.43}$$

The above identification is not so immediate to prove, therefore we redirect the reader to the original paper [Li18].

$H(\text{inc})$ and Reg^k : Neunteufel Implementation

The following is the implementabe finite element space given by M. Neunteufel in [Neu21]

$$H(\text{inc}\,.\Omega) = \left\{\gamma \in \left[\mathbf{L}^2(\Omega)\right]^{3\times 3}_{ysm} \mid \text{curl}(\text{curl}\,\gamma)^T \in [H^{-1}(\Omega)]^{\star d\times\star d}\right\} \tag{4.44}$$

$$NN^k(T) = \left\{[\mathbb{P}^k(T)]^{3\times 3}_{sym} \mid [\![\kappa_{nn}]\!]_F = 0\,\forall F \in \mathcal{F}\right\} \tag{4.45}$$

The transformation that one needs to take into account is a double covariant transformation:

$$H(\text{inc}, \hat{T}) \ni\mapsto \gamma = (A^{-T}\hat{\gamma}A^1) \tag{4.46}$$

The dofs are defined for $n = 0...k$:

$$\psi_{\bar{E}_i,n} : g \mapsto \int_{\hat{E}_i} g : q_{\hat{E},n} t_{\hat{E}} \otimes t_{\hat{E}} \quad \psi_{\bar{F}_i,n} : g \mapsto \int_{\hat{F}_i} g : q_{\hat{F},n} \quad \psi_{\bar{T},n} : g \mapsto \int_{\hat{T}} g : q_{\hat{T},n} \tag{4.47}$$

Where $q_{\hat{E}_i,n}$, $q_{\hat{F}_i,n}$, $q_{\hat{T},n}$ form a, possibly hierarchical, basis for

- $\mathbb{P}^k(E_i)$
- $(\mathbb{P}^{k-1}(F_i) \otimes \mathbb{S}^3_{sym}) \bigcap \hat{n}_F^{\perp}$
- $(\mathbb{P}^{k-1}(T) \otimes \mathbb{S}^3_{sym})$

The shape functions spanning $\mathbb{P}^k \otimes \mathbb{S}^3$ are given by

- **The edges shape functions:** of the edge $E_i = [V_1, V_2]$ for $r = 1...k$

$$\phi_{E_i,0} = -\nabla\lambda_{V_1} \odot \nabla\lambda_{V_2} \tag{4.48}$$

$$\phi_{E_i,r} = -L^S_r(\lambda_{V_2} - \lambda_{V_1}, \lambda_{V_2} + \lambda_{V_1})\nabla\lambda_{V_1} \odot \nabla\lambda_{V_2} \tag{4.49}$$

- **The edges facet functions:** for the face $F_i = [V_1, V_2, V_3]$ for $r + s \leq k - 1$

$$\phi_{F_i^1,r,s} = -\phi^D_{r,s}(\lambda_{V_2} - \lambda_{V_1}, \lambda_{V_2} + \lambda_{V_1})\lambda_{V_1}\nabla\lambda_{V_2} \odot \nabla\lambda_{V_2} \tag{4.50}$$

$$\phi_{F_i^2,r,s} = -\phi^D_{r,s}(\lambda_{V_2} - \lambda_{V_1}, \lambda_{V_2} + \lambda_{V_1})\lambda_{V_2}\nabla\lambda_{V_3} \odot \nabla\lambda_{V_2} \tag{4.51}$$

$$\phi_{F_i^3,r,s} = -\phi^D_{r,s}(\lambda_{V_2} - \lambda_{V_1}, \lambda_{V_2} + \lambda_{V_1})\lambda_{V_3}\nabla\lambda_{V_1} \odot \nabla\lambda_{V_2} \tag{4.52}$$

- **The inner functions:** for the face for $r + s + t \leq k - 2$

$$\phi_{T^1,r,s,t} = -\phi^{D,3}_{r,r,t}(\lambda_1, \lambda_2, \lambda_3)\lambda_1\lambda_2(\lambda_3 \odot \lambda_4) \tag{4.53}$$

$$\phi_{T^2,r,s,t} = -\phi^{D,3}_{r,r,t}(\lambda_1, \lambda_2, \lambda_3)\lambda_2\lambda_3(\lambda_4 \odot \lambda_1) \tag{4.54}$$

$$\phi_{T^3,r,s,t} = -\phi^{D,3}_{r,r,t}(\lambda_1, \lambda_2, \lambda_3)\lambda_3\lambda_4(\lambda_1 \odot \lambda_2) \tag{4.55}$$

$$\phi_{T^4,r,s,t} = -\phi^{D,3}_{r,r,t}(\lambda_1, \lambda_2, \lambda_3)\lambda_4\lambda_1(\lambda_2 \odot \lambda_3) \tag{4.56}$$

$$\phi_{T^5,r,s,t} = -\phi^{D,3}_{r,r,t}(\lambda_1, \lambda_2, \lambda_3)\lambda_1\lambda_3(\lambda_2 \odot \lambda_4) \tag{4.57}$$

$$\phi_{T^6,r,s,t} = -\phi^{D,3}_{r,r,t}(\lambda_1, \lambda_2, \lambda_3)\lambda_2\lambda_4(\lambda_1 \odot \lambda_3) \tag{4.58}$$

Part III.

The Variational Formulation

5. Variational Formulation:

The problem in weakening a system of differential equations is that sometimes the LHS of the system lies in a different space than the RHS. To solve the problem one can find a proper minimization problem for the continuous setting, embed it in the weakened setting and derive a "new" system of differential equations that will lead to the wanted result. Here we will use a different approach: The problems presented so far can be formulated in terms of this generic continuous setting:

Problem 5.0.1 (Abstract Variational Problem)**.** *Let U, V, W hilbert space such that $U, V \subset W$, let F and $\bar{G}$ be continuous functions*

$$\dot{u} = v \tag{5.1}$$
$$\dot{v} = v \tag{5.2}$$
$$\tag{5.3}$$

5.1. Abstract framework

The abstract framework for stationary differential equations is well understood, in particular one aims to extend the theory for non-stationary systems. For a Let $u \in V$ generic element of a Hilbert space, let $f : V \times V \to \mathbb{R}$.

$$(\ddot{u}, \phi) = \langle Fu, \phi \rangle \tag{5.4}$$

Where $F : V \to V^*$ is the map induced by f. using the Riesz representation theorem one can realize that the above isomorphism as follows.

$$\ddot{u} = \mathscr{I}_V F u$$

Remark 5.1.1. *Given a basis $(e_i)_i$ and a basis for the dual $(e^j)_j$ (notice that we don't require this one to be the dual basis)*

$$\forall l \in V^* \mathscr{I} l := \sum_i l(e_i) e_i \in V \tag{5.5}$$

in case one has a finite dimensional space one obtains the relation:

$$\forall \vec{l} \in V^* : \mathscr{I} \vec{l} := M_{V^*,V} \vec{l} \in V \tag{5.6}$$

With $(M_{V^*,V})_{ij} = l_j e^j(e_i)$, Therefore one obtains $\mathscr{I} = M^{-1}_{V^*,V}$

To be more formal the previous should be expressed as a row vector, but we will not make this distinction.

Therefore when discretised one can use the following expression

$$u^{n+1} - 2u^n + u^{n-1} = \tau F_h u^n \tag{5.7}$$
$$+\text{initial conditions} \tag{5.8}$$

Where F_h is the discrete operator $F_h : V_h \to V_h^*$.

If we consider a system in $(u, v) \in U \times V$[1]

$$u' = F(v) \tag{5.9}$$
$$v' = G(u) \tag{5.10}$$

Then we have to assueme that $(\phi, \psi) \in U \times V$, moreover let $F : V \to V^*$,$G : U \to U^*$ and they are given by $f : V \times V \to \mathbb{R}$,$G : U \times U \to \mathbb{R}$

$$(u', \phi) = \langle F(v), \phi\rangle \tag{5.11}$$
$$(v', \psi) = \langle G(u), \psi\rangle \tag{5.12}$$

Obviously the duality must hold true in the hypothesis.

The functions $F(v)$ and $G(u)$ are maps from one pre-dual to the opposite dual space then one needs to treat them as a different equation in our system:

$$(u', \phi) = \langle F_v, \phi\rangle \tag{5.13}$$
$$(F_v, \psi) = f(v, \psi) \tag{5.14}$$
$$(v', \psi) = \langle G_u, \psi\rangle \tag{5.15}$$
$$(G_u, \psi) = g(u, \psi) \tag{5.16}$$

In abstract notation can be written:

$$\mathscr{I}_U^{-1} u' = M_{V\to U} F_v \tag{5.17}$$
$$\mathscr{I}_V^{-1} F_v = \bar{f} v \tag{5.18}$$
$$\mathscr{I}_V^{-1} v' = M_{U\to V} G_u \tag{5.19}$$
$$\mathscr{I}_U^{-1} G_u = \bar{g} u \tag{5.20}$$
$$\tag{5.21}$$

In the end

$$u' = \mathscr{I}_U M_{V\to U} \mathscr{I}_V \bar{f} v \tag{5.22}$$
$$v' = \mathscr{I}_V M_{U\to V} \mathscr{I}_U \bar{g} u \tag{5.23}$$

[1] In this chapter we use the convention that $(u, v) \in U \times V$ and U, V (and their dual) are in fact a shorthand for $AC([0, T]) \times V$.

The discretization of the equation above acquires a trivial form thanks to the remark and interpreting the passage matrix $M_{U\to V}$ as the matrix obtained by the inner product of the one basis in the pre-dual and the other in the dual :

$$(M_{U\to V})_{ij} = \psi_i(\phi^j) \tag{5.24}$$

Obviously one obtains that

$$M_{U\to V} = M_{V\to U}^T \tag{5.25}$$

Proof. The first is given by the bilinear form

$$m^{(1)} : U^* \times V \to \mathbb{R} \tag{5.26}$$
$$M^{(1)} : U^* \to V^* \tag{5.27}$$
$$\tag{5.28}$$

The second one is given by the bilinear form

$$m^{(2)} : U \times V^* \to \mathbb{R} \tag{5.29}$$
$$M^{(2)} : V^* \to U^* \tag{5.30}$$

□

In case one is studying a constrain evolution equation of the from:

$$\bar{G}\ddot{u} = F(u) \tag{5.31}$$
$$C(u) = 0 \tag{5.32}$$

Where $\bar{G}$ is invertible, with inverse G, and the functional $C : U \to Z^* \subset W$ is induced by a bounded bilinear form $c(u,z) \to \mathbb{R}$, then the above system would result in the continuous system of equations:

$$u' = F(v) \tag{5.33}$$
$$v' = G(u) \tag{5.34}$$
$$C(u) = 0 \tag{5.35}$$

then this theory is underdeveloped so far[2], it will be expanded in future research, so we are forced to use the more known approach that is the minimization of functionals:

$$\min_{C(\dot{u})} \|u' - \langle F_v, \phi\rangle\|^2 \tag{5.36}$$
$$(F_v, \psi) = f(v, \psi) \tag{5.37}$$
$$(v', \psi) = \langle G_u, \psi\rangle \tag{5.38}$$
$$\min_{C(u)} \|G_u - G(u\|^2 \tag{5.39}$$

[2] I couldn't find any books covering this topic, if you find any you are more than welcome to point me towards it via my email.

Using the same discretization as before one creates the same system as in 5.22 but now the matrix operator $\mathscr{I}_U$ is to be intended as the sub-matrix $\tilde{M}_{UU}$ of the composit matrix:

$$\tilde{M} = \begin{pmatrix} \tilde{M}_{UU} & \tilde{M}_{ZU} \\ \tilde{M}_{UZ} & \tilde{M}_{ZZ} \end{pmatrix} = \begin{pmatrix} M_{UU} & M_C^T \\ M_C & 0 \end{pmatrix}^{-1} \tag{5.40}$$

Where the last matrix is to be intended as the discrete projection into the constrained manifold.

Since we skipped many calculations -I invite the reader to try on his/her own- it is mandatory to point out that the invertibility of the matrix $\tilde{M}$ or the solvability of the system di per se is given by one or more of the following existence and uniqueness theorems:

- Riesz' representation theorem
- Lax-Milgram
- Babuska-Aziz
- Brezzi
- Fredholm

The statement can be found in the notes of professor Schoeberl[Sch09a][Sch09b]

We are now ready for discussing the variational setting for each individual set of equation:

5.2. Variational Maxwell Wave Equation

Problem 5.2.1. *Find* $d \in H(\mathrm{div}), e \in H(\mathrm{curl}), v \in \mathbb{L}_2/\mathbb{R}$ *such that:*

$$\min_{d \in H(\mathrm{div}), \mathrm{div}\, d = 0} \frac{1}{2} \|\dot{d} + \mathrm{curl}\, \mu^{-1} \,\mathrm{curl}\, e\|_{\mathbb{L}_2}^2 \tag{5.41a}$$

$$\min_{e \in H(\mathrm{curl})} \frac{1}{2} \|\dot{e} - \varepsilon^{-1} d\|^2 \tag{5.41b}$$

Since we are now in the setting of continuous finite element we require the above to be continuous functions in the respective spaces $H(\mathrm{curl})$ and $H(\mathrm{div})$, therefore the above make no sense unless one talks about the minimization problem:

The above contains one abuse of notation, the time dependance has not being introduced, the class function $H(\mathrm{curl})$ needs to be tensor-multiplied by $C^1([0,T])$ where T is a final time.

The problem 5.2, with the convention that all functions denominated with letters e live in $H(\mathrm{curl})$ and the same for d's, can be split into different components to have a more organized system:

$$\min_{\hat{e}\in H(\text{curl})} \frac{1}{2}\|\hat{e} + \operatorname{curl}\mu^{-1}\operatorname{curl} e\|_{L_2}^2 \tag{5.42a}$$

$$\min_{\hat{d}\in H(\text{div})} \frac{1}{2}\|\hat{d} - \varepsilon^{-1} d\|_{L_2}^2 \tag{5.42b}$$

$$\min_{\tilde{d}\in H(\text{div})} \frac{1}{2}\|\tilde{d} - \hat{e}\|_{L_2}^2 \tag{5.42c}$$

$$\min_{\tilde{e}\in H(\text{curl})} \frac{1}{2}\|\tilde{e} - \hat{d}\|_{L_2}^2 \tag{5.42d}$$

$$\min_{\dot{e}\in H(\text{curl})} \frac{1}{2}\|\dot{e} - \tilde{e}\|_{L_2}^2 \tag{5.42e}$$

$$\min_{\operatorname{div} d=0} \frac{1}{2}\|\dot{d} - \tilde{d}\|_{L_2}^2 \tag{5.42f}$$

The method of Lagrangian multipliers give birth to the following continuous method:

$$(\hat{e},\psi) = -\left\langle \operatorname{curl}\mu^{-1}\operatorname{curl} e, \psi\right\rangle \quad \forall\psi\in H(\text{curl}) \tag{5.43a}$$

$$(\hat{d},\phi) = (\varepsilon^{-1} d, \phi) \quad \forall\phi\in H(\text{div}) \tag{5.43b}$$

$$(\tilde{d},\phi) = (\hat{e},\phi) \quad \forall\phi\in H(\text{div}) \tag{5.43c}$$

$$(\tilde{e},\psi) = (\hat{d},\psi) \quad \forall\psi\in H(\text{curl}) \tag{5.43d}$$

$$(\dot{e},\psi) = (\tilde{e},\psi) \quad \forall\psi\in H(\text{curl}) \tag{5.43e}$$

$$\begin{pmatrix} (\dot{d},\phi) & \langle\phi,u\rangle \\ \langle d,v\rangle & \end{pmatrix} = \left((\tilde{e},\phi)\right) \quad \begin{matrix}\forall\phi\in H(\text{div}) \\ \forall u\in\mathbb{R}\end{matrix} \tag{5.43f}$$

In discrete form it has the following form:

$$\vec{d}^{n+1} = \vec{d}^{n} - \bar{M}_d^{-1} M_{dc} M_c^{-1} M_{\mu,K} \tag{5.44}$$

$$\vec{e}^{n+1} = \vec{e}^{n} - M_c^{-1} M_{cd} \bar{M}_d^{-1} M_{\varepsilon} \tag{5.45}$$

In particular $\bar{M}_d^{-1}$ is the matrix defined as the restriction of the projection onto the divergence free part :

$$\bar{M}_d^{-1} = \begin{pmatrix} I & 0 \end{pmatrix} M \begin{pmatrix} I \\ 0 \end{pmatrix} \tag{5.46}$$

5.3. Variational Graviatational Wave Equation

In previous chapter one fundamental fact is the well defined interplay among all the function spaces that are used. While the spaces $\mathbb{L}_2, H(\text{div})$ and $H(\text{curl})$ are linked using the well known de Rham sequence 2, here this fact does not hold true, but fortunately the algebraic operation S comes into play. The opertation S can be understood as a rotation operator, it is shown in the discrete case to map Regge elements to Schoeberl Pechstein elements $S : \text{Reg}^k \to NN^k$[Li18], and obviously it works quite well in the smooth case, here we show a possible proof to extend it to the continuous case.

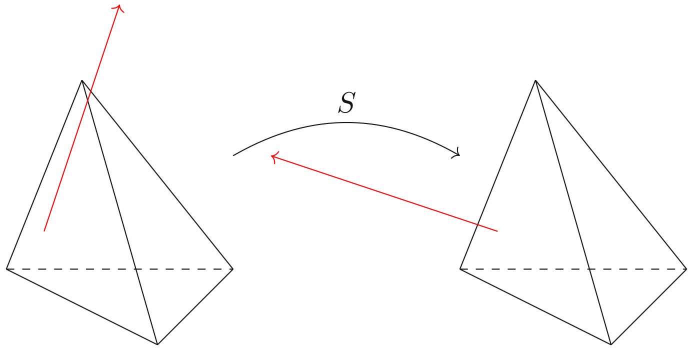


Figure 5.1.: Map from $\mathrm{Reg}^0 \to NN^0$ transforms tt-continuous dofs into nn-continuous dofs

Lemma 5.3.1 (Unpublished result). [3] *Let $g \in H(\mathrm{inc})$ then there exists a decomposition*

$$g = g^{(1)} + \varepsilon(u) + \nabla\nabla(w) \tag{5.47}$$

Where $g^{(1)} \in H^1 \otimes \mathbb{S}$,$u \in H^1 \otimes \mathbb{V}$ and $w \in H^1$

The above lemma and lemma A.1.1 have as a consequence a the following theorem

Theorem 5.3.2. *Let $S : H(\mathrm{inc}) \to H(\mathrm{div\,div})$ is continuous.*

Proof. Let $g \in H(\mathrm{inc})$ be decomposed via lemma 5.3.1: $g = g^{(1)} + \varepsilon(u) + \nabla\nabla(w)$ To prove that S is $H(\mathrm{div\,div})$ valued we need to show that Sg is bounded in $\mathbb{L}_2$, but this is trivial from the fact that the norm act on the components element-wise of the matrix g. To prove now that $\mathrm{inc}\, Sg \in H^{-1}$ we use the decomposition:

- $\mathrm{div\,div}\, Sg^{(1)} = -2\,\mathrm{tr\,inc}\, g^{(1)} \in H^{-1}$
- $\mathrm{div\,div}\, S\varepsilon u \overset{(A.25f)}{=} 0 \in H^{-1}$
- $\mathrm{div\,div}\, S\nabla\nabla w \overset{(A.25e)}{=} 0 \in H^{-1}$

To prove the continuity we use the definition of the operator norm and check when it is bounded:

$$\|S\|_{H(cc)\to H(dd)} := \sup_{g\in H(cc)} \frac{\|Sg\|_{Hdd}}{\|g\|_{Hcc}} < \infty$$

The norms here are the norms induced by the operator given in the definition. Let

[3] This result has not been published yet, the authors of the paper that will publish it are: Jay Gopalakrishnan, Philip L. Lederer, Astrid Pechstein, Joachim Schoeberl and Michael Neunteufel

$g \in H(\mathrm{inc})$

$$
\begin{aligned}
\|Sg\|^2_{Hdd} &= \|Sg\|^2_{L_2} + \|\operatorname{div}\operatorname{div} Sg\|^2_{H^{-1}} \\
&\preceq \|g\|^2_{L_2} + \|\operatorname{div}\operatorname{div} Sg^{(1)}\|^2_{H^{-1}} \\
&\preceq \|g\|^2_{L_2} + \|\operatorname{tr}\operatorname{inc} g^{(1)}\|^2_{H^{-1}} \\
&\preceq \|g\|^2_{L_2} + \|\operatorname{inc} g^{(1)}\|^2_{H^{-1}} \\
&= \|g\|^2_{L_2} + \|\operatorname{inc} g^{(1)} + \operatorname{inc}\varepsilon u\|^2_{H^{-1}} \\
&= \|g\|^2_{L_2} + \|\operatorname{inc} g^{(1)} + \operatorname{inc}\varepsilon u + \operatorname{inc}\nabla\nabla w\|^2_{H^{-1}} \\
&= \|g\|^2_{L_2} + \|\operatorname{inc} g\|^2_{H^{-1}} \\
&= \|g\|^2_{Hcc}
\end{aligned}
$$

From here follows immediately the continuity All the above referenced equations are proved in *A.1.1*. □

We start with equations given by the problem 3.2, expressing them in terms of minimization problem the following are obtained:

$$\dot{\kappa} = -\operatorname{inc}\gamma \tag{5.48}$$

$$\dot{\gamma} = J\kappa \tag{5.49}$$

$$\operatorname{div}\kappa = 0 \tag{5.50}$$

Remark 5.3.3. *To introduce the final time-stepping method one needs to define the triangulation-wise distributive* inc *operator and the divergence free constrain (also in the composite distributive sense).*

Using the convention that all the κ^n's are in the $H(\operatorname{div}\operatorname{div})$ space and all the γ's are in the $H(\operatorname{curl}\operatorname{curl})$, and performing the same steps that we performed for the maxewll's equations 5.2 the resulting problem becomes:

Problem 5.3.4. *Given adequate initial conditions, find* $(\phi, \psi, q) \in H(\operatorname{curl}\operatorname{curl}) \times H(\operatorname{div}\operatorname{div}) \times H(\operatorname{curl})$ *such that*

$$
\begin{cases}
\langle \tilde{\gamma}^n, \phi \rangle &= \langle \operatorname{inc}\gamma^n, \phi \rangle \\
\langle \tilde{\kappa}^n, \psi \rangle &= \langle J\kappa^n, \psi \rangle
\end{cases} \tag{5.51}
$$

$$
\begin{cases}
\langle \hat{\kappa}^n, \psi \rangle &= \langle \tilde{\gamma}^n, \psi \rangle \\
\langle \hat{\gamma}^n, \phi \rangle &= \langle \tilde{\kappa}^n, \phi \rangle
\end{cases} \tag{5.52}
$$

$$
\begin{cases}
\left\langle \kappa^{n+1}, \psi \right\rangle + \langle \operatorname{div}\psi, p \rangle &= \langle \kappa^n - \tau\hat{\kappa}^n, \psi \rangle \\
\left\langle \operatorname{div}\kappa^{n+1}, q \right\rangle &= \langle 0, q \rangle \\
\left\langle \gamma^{n+1}, \phi \right\rangle &= \langle \gamma^n + \tau\hat{\gamma}^n, \phi \rangle
\end{cases} \tag{5.53}
$$

Or in a matricial form:

$$\begin{cases} M_{\gamma\gamma}\tilde{\gamma}^n &= M^{\text{inc}}\gamma^n \\ M_{\kappa\kappa}\tilde{\kappa}^n &= M^J\kappa^n \end{cases} \tag{5.54}$$

$$\begin{cases} M_{\kappa\kappa}\hat{\kappa}^n &= M_{\kappa\gamma}\tilde{\gamma}^n \\ M_{\gamma\gamma}\hat{\gamma}^n &= M_{\gamma\kappa}\tilde{\kappa}^n \end{cases} \tag{5.55}$$

$$\begin{cases} \begin{pmatrix} M_{\kappa\kappa} & (M^{\text{div}}_{\kappa c})^T \\ M^{\text{div}}_{\kappa c} & -C \end{pmatrix} \begin{pmatrix} \kappa^{n+1} \\ p \end{pmatrix} = \begin{pmatrix} M_{\kappa\kappa} & 0 \\ 0 & M_{cc} \end{pmatrix} \begin{pmatrix} \kappa^n - \tau\hat{\kappa}^n \\ 0 \end{pmatrix} \\ \gamma^{n+1} = \gamma^n + \tau\hat{\gamma}^n \end{cases} \tag{5.56}$$

With $M^{\text{div}}_{\kappa c}$ the Mass matrix given by $\langle \operatorname{div}\psi, q\rangle$, meanwhile the sub-matrix C is the one used to derive a more stable method, the higher stability comes from the fact one theorem known as extended Brezzi theorem

$$\begin{cases} \begin{pmatrix} \psi^{n+1} \\ p \end{pmatrix} = \begin{pmatrix} M_{\kappa\kappa} & (M^{\text{div}}_{\kappa c})^T \\ M^{\text{div}}_{\kappa c} & -C \end{pmatrix}^{-1} \begin{pmatrix} M_{\kappa\kappa} & 0 \\ 0 & M_{cc} \end{pmatrix} \begin{pmatrix} \kappa^n - \tau(M_{\kappa\kappa})^{-1}M_{\kappa\gamma}(M_{\gamma\gamma})^{-1}M^{\text{inc}}\gamma^n \\ 0 \end{pmatrix} \\ \gamma^{n+1} = \gamma^n + \tau(M_{\gamma\gamma})^{-1}M_{\gamma\kappa}(M_{\kappa\kappa})^{-1}M^J\kappa^n \end{cases} \tag{5.57}$$

The equation is the one implemented in the code.

Constraint: To derive the constraint one needs to work in the correct spaces, the literature suggests to pick $H(\operatorname{div}\operatorname{div})$ and $H(\operatorname{curl})$, the idea comes from the following theorem:

Theorem 5.3.5. *Let $u \in H(\operatorname{div}\operatorname{div})$, the duality:*

$$\langle \operatorname{div} u, \phi\rangle_{H(\text{curl})^* \times H(\text{curl})} \tag{5.58}$$

is well defined

The theorem can be found in [PS11] and with it it follows that for a triangulation $\mathcal{T}$ of a simply connected domain Ω, if the function is piece-wise smooth and tn-continuous across interfaces, then the divergence is well defined as:[4]

$$\langle \operatorname{div} u, \phi\rangle = \sum_T \int \operatorname{div} u \cdot \phi - \int_{\partial T} u_{nt}\phi_t \tag{5.59}$$

The derivation of the a triangulation-wise inc operator is quite involved, the idea to derive it is very similar for to the derivation of the div div for functions in $H(\operatorname{div}\operatorname{div})$, the only problem lies in the matrix correct decomposition of the matrix-valued functions and the action of the curl on them.

[4]It is quite simple to define a hybridised method for the div-free constraint, the implementation for the test case does not show a big impact, therefore I will leave this topic out of the scope of this thesis.

5.3.1. Definition of the inc operator

Remark 5.3.6. *When one tries to integrate by parts a function on a tetrahedron realizes immediately that the functions get smeared on the faces, if one integrates by part one more time the elements involved are the edges, in the next part of this section I invite the reader to pay particular attention to these details.*

the two dimensional case is quite simple, it involves only the edges and the the vertices. The the decomposition of the inc operator is proved in the previous chapter to introduce the 3 + 1 decomposition, here we want to extend it to a weak operator; to do so one observation is particularly relevant: the dimension of the co-domain of the operator differ if the domain of the domain is different. In this subsection there is a quick introduction on the the definition of $\operatorname{inc} : C^\infty(\Omega) \otimes \mathbf{S}^{d\times d} \ni \phi \to \operatorname{inc}\phi \in C^\infty(\Omega) \otimes \mathbf{S}^{\star d\times \star d}$ with $\Omega \subset \mathbb{R}^d$ and the star operator $\star d = \frac{(d-1)d}{2}$. Later there is a discussion on how to derive the distributional form.

Definition 20 (vector curl operator)**.** *Let* $\phi \in C^\infty(\Omega) \otimes \mathbb{V}^d$ *, then the operator curl is defined as follows*

$$\begin{cases} \operatorname{curl}\phi = \varepsilon_{jk}\partial_j\phi_k & d = 2 \\ (\operatorname{curl}\phi)_i = \varepsilon_{ijk}\partial_j\phi_k & d = 3 \end{cases} \tag{5.60}$$

Notice that the curl for a 2-dimensional space is a scalar, meanwhile the curl for a 3-d space is a 3-d vector. There exists another notion that is often confused with curl, that is the rot. Formally can be defined as the adjoint of the curl, but for now only the definition as a differential operator is given.

Definition 21 (vector rot operator)**.** *Let* $\phi \in C^\infty(\Omega)$ *, then the we define the curl operator as follows*

$$(\operatorname{rot}\phi)_i = \varepsilon_{ij}\partial_j\phi \tag{5.61}$$

The natural extension of the divergence for matrix valued functions is the action of the divergence on each row, here the same concept is used to create the extension of curl.

Definition 22 (matrix curl operator)**.** *Let* $\phi \in C^\infty(\Omega) \otimes \mathbf{S}^{d\times d}$ *, then the we define the curl operator as follows*

$$\begin{cases} (\operatorname{curl}\phi)_i = \varepsilon_{jk}\partial_j\phi_{ik} & d = 2 \\ (\operatorname{curl}\phi)_{ij} = \varepsilon_{jmn}\partial_m\phi_{in} & d = 3 \end{cases} \tag{5.62}$$

The adjoint in case $d = 3$ remains the same, we can therefore define a rot operator in case of a 2×2 dimensional matrix:

Definition 23 (matrix rot operator)**.** *Let* $\phi \in C^\infty(\Omega) \times \mathbb{V}^2$ *, then the we define the curl operator as follows*

$$(\operatorname{rot}\phi)_{ij} = \varepsilon_{jk}\partial_k\phi_i \tag{5.63}$$

Remark 5.3.7. *The all the above operators, where $d = 2$, can be written in a more friendly way using lower and upper case:*

$$curl: C^\infty(\Omega)^2 \ni \begin{pmatrix} \phi_1 \\ u_2 \end{pmatrix} \longmapsto \partial_1\phi_2 - \partial_2\phi_1 \in C^\infty(\Omega)$$

$$Curl: C^\infty(\Omega)^{2\times 2} \ni \begin{pmatrix} \phi_{11} & u_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \partial_1\phi_{12} - \partial_2\phi_{11} \\ \partial_1\phi_{22} - \partial_2\phi_{21} \end{pmatrix} \in C^\infty(\Omega)^2$$

$$rot: C^\infty(\Omega)^1 \ni \phi \longmapsto \begin{pmatrix} \partial_2\phi \\ -\partial_1\phi \end{pmatrix} \in C^\infty(\Omega)^2$$

$$Rot: C^\infty(\Omega)^2 \ni \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longmapsto \begin{pmatrix} \partial_2\phi_1 & -\partial_1\phi_1 \\ \partial_2\phi_2 & -\partial_1\phi_2 \end{pmatrix} \in C^\infty(\Omega)^{2\times 2}$$

The curl *lower the dimension of the co-domain, the* rot *enlarges it.*

As mentioned above the inc operator in $\mathbb{R}^d$ with $d = 2, 3$ is defined as composition of curl's:

Definition 24 (inc operator)**.** *Let $\phi \in C^\infty(\Omega) \otimes \mathbf{S}^{d\times d}$, then the we define the curl operator as follows*

$$\operatorname{inc}\phi := \operatorname{curl}(\operatorname{curl}(\phi^T))^T \tag{5.64}$$

As mentioned before the curl and rot are dual operators in $d = 2$; morover the curl is self-dual 3d

Theorem 5.3.8. *let one or (exclusive) the other:*

- $\phi \in C^\infty(\Omega) \times \mathbb{V}^d$ *and* $\psi \in C^\infty(\Omega) \times \mathbb{V}^{\star d}$
- $\phi \in C^\infty(\Omega) \times \mathbb{S}^d$ *and* $\psi \in C^\infty(\Omega) \times \mathbb{S}^{\star d\times d}$

Let one function to vanish at the boundary $\partial\Omega$, then the following holds true:

$$\langle \operatorname{curl}\phi, \psi\rangle_{\mathbb{L}^d(\Omega)} = \langle \phi, \operatorname{curl}^* \psi\rangle_{\mathbb{L}^d(\Omega)} \tag{5.65}$$

Where $\operatorname{curl}^* = \operatorname{curl}$ *for* $d = 3$ *and* $\operatorname{curl}^* = \operatorname{rot}$ *for* $d = 2$*.*

Proof. A direct consequence of next lemmas setting choosing zero boundary conditions. □

Some integral identities

Proposition 6. *Let $\phi, \psi \in C^(\Omega) \otimes \mathbb{V}^3$, then the integration by parts of the curl is:*

$$\int_\Omega \operatorname{curl}(\phi)\cdot\psi = \int_\Omega \operatorname{curl}(\psi)\cdot\phi + \int_{\partial\Omega} (\phi\times\psi)\cdot n \tag{5.66}$$

Proof.

$$\begin{aligned}\int_\Omega \operatorname{curl}(\phi)\cdot\psi &= \int_\Omega \varepsilon_{ijk}\partial_j\phi_k\psi_i \\ &= -\int_\Omega \phi_k\varepsilon_{ijk}\partial_j\psi_i + \int_\Omega \partial_j(\phi_k\varepsilon_{ijk}\psi_i) \\ &= \int_\Omega \phi\operatorname{curl}(\psi) + \int_{\partial\Omega} n_j(\varepsilon_{jki}\phi_k\psi_i)\end{aligned}$$

□

The 2d case becomes

Proposition 7. *Let $u \in C^\infty(\Omega)^{2\times}$, $u \in C^\infty(\Omega)^2$ and $f \in C^\infty(\Omega)$ then the following hold true:*

$$\int_\Omega \operatorname{curl}(u)f = \int_{\partial\Omega}(u\cdot t)f + \int_\Omega u\cdot\operatorname{rot}(f) \tag{5.67}$$

$$\int_\Omega \operatorname{Curl}(\sigma)\cdot u = \int_\Omega \sigma\cdot\operatorname{Rot}(u) + \int_{\partial\Omega}(\sigma t)\cdot u \tag{5.68}$$

Proof.

$$\begin{aligned}\int_\Omega \operatorname{curl}(u)f &= \int_\Omega(\partial_1u_2 - \partial_2u_1)f \\ &= \int_\Omega(\partial_1(fu_2) + \partial_2(-fu_1) - \int_\Omega u_2(\partial_1 f) + u_1(-\partial_2 f) \\ &= \int_\Omega \operatorname{div}\begin{pmatrix} fu_2 \\ -fu_1\end{pmatrix} + \int_\Omega \begin{pmatrix} u_1 \\ u_2\end{pmatrix}\cdot\begin{pmatrix}\partial_2 f \\ -\partial_1 f\end{pmatrix} \\ &= \int_{\partial\Omega} f\begin{pmatrix} u_2 \\ -u_1\end{pmatrix}\cdot n + \int_\Omega u\cdot\operatorname{rot}(f) + \\ &= \int_{\partial\Omega}(u\cdot t)f + \int_\Omega u\cdot\operatorname{rot}(f)\end{aligned}$$

The second equation is immediate using $(t^1, t^2) = (n^2, -n^2)$. □

Using the definition of generalize derivative

Remark 5.3.9. *And also the definition of the inc operator for 2×2 metrics is derived from the composition of the curl operators and is:*

$$\int_\Omega \operatorname{inc}(\sigma)\phi = \int_\Omega \operatorname{curl}\operatorname{Curl}(\sigma)\phi = \int_\Omega \sigma\operatorname{Rot}\operatorname{rot}(\phi) \quad \forall\phi\in C_0^\infty(\Omega)$$

The above can be used to derive the distributional triangulation-wise inc operator for 2-metrics

Theorem 5.3.10. *Let σ be a tt-continuous 2-metric, let the boundary values be well defined and suppose that the* inc *operator is well defined in the interior of the triangulation. Then the distributional inc operator has the following form*

$$\langle \operatorname{inc}\sigma, \phi\rangle_\Omega = \sum_T \int_T \operatorname{inc}(\sigma)\phi - \int_{\partial T} \operatorname{Curl}(\sigma)\cdot t\phi - \int_{\partial T} t\cdot\nabla(\sigma_{nt})\phi + (\sigma_{nt}\phi)\Big|_{V_1}^{V_2} \tag{5.69}$$

Proof. Splitting onto the triangulation becomes:

$$\sum_T \int_T \sigma \operatorname{Rot}\operatorname{rot}(\phi) \sum_T \int_T \operatorname{Curl}(\sigma)\cdot\operatorname{rot}(\phi) - \int_{\partial T} \sigma_t \cdot \operatorname{rot}(\phi)$$

We now split in tangential and normal componen the last integral, sinco the derivative is allowed only on the faces of the triangle and not accross:

$$\sum_T \int_T \operatorname{Curl}(\sigma)\cdot\operatorname{rot}(\phi) - \int_{\partial T} \sigma_{tt}\operatorname{rot}(\phi)\cdot t + \sigma_{nt}\operatorname{rot}(\phi)\cdot n$$

Since ϕ is smooth , σ_{tt} is continuous accross the interfaces and $\operatorname{rot}(\phi)\times t = n\cdot\nabla\phi$ that cancels because the normal changes sign... then...

$$\sum_T \int_T \operatorname{Curl}(\sigma)\cdot\operatorname{rot}(\phi) - \int_{\partial T} \sigma_{nt}\operatorname{rot}(\phi)\cdot n$$

Similarly $\operatorname{rot}(\phi)\cdot n = -t\cdot\nabla\phi$

$$\sum_T \int_T \operatorname{Curl}(\sigma)\cdot\operatorname{rot}(\phi) + \int_{\partial T} \sigma_{nt} t\cdot\nabla\phi$$

$$\sum_T \int_T \operatorname{curl}\,\operatorname{Curl}(\sigma)\phi - \int_{\partial T} \operatorname{Curl}(\sigma)\cdot t\phi + \int_{\partial T} \sigma_{nt} t\cdot\nabla\phi$$

$$\sum_T \int_T \operatorname{inc}(\sigma)\phi - \int_{\partial T} \operatorname{Curl}(\sigma)\cdot t\phi - \int_{\partial T} t\cdot\nabla(\sigma_{nt})\phi + \int_{\partial T} t\cdot\nabla(\sigma_{nt}\phi)$$

that is:

$$\sum_T \int_T \operatorname{inc}(\sigma)\phi - \int_{\partial T} \operatorname{Curl}(\sigma)\cdot t\phi - \int_{\partial T} t\cdot\nabla(\sigma_{nt})\phi + (\sigma_{nt}\phi)\Big|_{V_1}^{V_2}$$

Where $|_{V_1}^{V_2}$ Is the two vertices evaluation. The previous suggest something important about the distributive subspace where we need to look for the solution and the discrete solution, that is a space that allows point evaluation in the 2d case and edge evaluation in the 3d case. In particular one need to relate on the Sobolev embedding theorem to understand a minimum necessary requirement for this particular operator to be well defined. □

suppose to have two smooth matrix field γ, ϕ (here we don't require symmetry) then the integration by parts on a generic domain Ω becomes:

Proposition 8. *Let γ, ϕ be smooth matrix valued functions in $M^{d\times d}$, where $d = 3$ and let $\Omega \subset \mathrm{R}^d$ open, then it follows that:*

$$\int_\Omega \gamma : \operatorname{curl}\phi = \int_\Omega \operatorname{curl}\gamma : \phi - \int_{\partial\Omega} \gamma C_n : \phi \tag{5.70}$$

Proof.

$$\begin{aligned} \int \gamma : \operatorname{curl}\phi &= \int \gamma_{ij}\varepsilon_{jmn}\partial_m\phi_{in} && \text{definition o} \\ &= \int \partial_m \left(\varepsilon_{jmn}\gamma_{ij}\phi_{in}\right) + \int \left(\varepsilon_{nmj}\partial_m\gamma_{ij}\right)\phi_{in} && \text{integration by parts and levi} \\ &= \int_\partial \left(-\varepsilon_{jnm}n_m\gamma_{ij}\phi_{in}\right) + \int \left(\varepsilon_{nmj}\partial_m\gamma_{ij}\right)\phi_{in} && \text{Integration of partial derivative and levi civita s} \\ &= \int_\partial -\left(C_n\right)_{jn}\gamma_{ij}\phi_{in} + \int \left(\operatorname{curl}\gamma\right)_{in}\phi_{in} && \text{de} \\ &= -\int_\partial \gamma C_n : \phi + \int \operatorname{curl}\gamma : \phi && \text{compac} \end{aligned}$$

□

We need to discuss the case in which one needs to apply the above result to a surface, in this case one gets some help from the famous area's formula: Let a function $f : \mathrm{R}^m \to \mathrm{R}$ be a scalar, let $\Phi : \Sigma' to \Sigma$ a smooth function, then it holds true that

$$\int_{\Phi(\Sigma')} f \circ \Phi \, J_\Phi = \int_{\Phi(\Sigma')=\Sigma} f \tag{5.71}$$

Where J_Φ is called transformation factor and it is defined by means of $(D\Phi)_{ij} = \partial_j \Phi_i$ in the following way

$$J_\Phi : \Sigma \ni x \longmapsto \left(\det D\Phi(x)^T \cdot D\Phi(x)\right) \mathbb{R} \tag{5.72}$$

Therefore it is important to define the curl not in the 3 possible directions but on a subspace (using certain projections).

In appendix **??** , in particular theorem B.0.5, gives us a way to treat the surface integration by parts:

$$\langle \gamma, \operatorname{curl}(\phi Q)P \rangle_{\mathbb{L}^2(\Sigma)} = \langle \operatorname{curl}(\gamma P)Q, \phi \rangle_{\mathbb{L}^2(\Sigma)} + \langle \gamma n, \phi Q t_{\partial\Sigma} \rangle_{\mathbb{L}^2(\partial\Sigma)} \tag{5.73}$$

Let now $\mathcal{J} = \operatorname{inc} u$ be the inc operator in distributional form defined on Ω, suppose that it is well defined on a triangulation on Ω and suppose that u is tt-coninuous; then supposing v smooth with compact support, a first approximation of the distributional inc would be:

Theorem 5.3.11. *Let $\mathcal{T}$ be an affine triangulation of Ω, let γ_i be a function defined on all the simplices T of the triangualtion such that the* inc *operator is well defined in the interior of the triangulation. Suppose also that the adjacent trianles have continuous tangrential-tangential functions γ_i, i.e.*

$$P_i\gamma_i P_i = P_j\gamma_j P_j\,, \quad \textit{for ajacent trinagles } T_i, T_j \tag{5.74}$$

Then the inc operator is well defined on the whole domain Ω in the distributional sense, in particular it holds that:

$$\langle \operatorname{inc}\gamma, \phi\rangle = \sum_T \left\{ \int_T \operatorname{inc}\gamma : \phi - \sum_{F\subset T}\left[\int_F [\operatorname{curl}\gamma]_{QQ} : [C_n\phi]_{QQ} - \operatorname{rot}_F([C_n\gamma]_{PQ}) : [\phi]_{QQ} - \sum_{E\subset F}\int_E [C_n\gamma]_{PQ}\, n\cdot[\phi]_{QQ}\, t_E\right]\right\} \tag{5.75}$$

$\forall \phi \in C_0^\infty M^{d\times d}_{Sym}$

Proof.

$$\begin{aligned}
\langle \operatorname{inc}\gamma, \phi\rangle_{\mathbb{L}^2(\Omega)} &= \int_\Omega \gamma : \operatorname{inc}\phi && \text{definition of distributive inc}\\
&= \sum_T \int_T \gamma : \operatorname{curl}(\operatorname{curl}\phi)^T && \text{expansion of inc}\\
&= \sum_T \int_T \operatorname{curl}\gamma : (\operatorname{curl}\phi)^T - \int_{\partial T}\gamma C_n : (\operatorname{curl}\phi)^T && \text{theorem 8}\\
&= \sum_T \int_T \operatorname{inc}\gamma : \phi - \int_{\partial T}(\operatorname{curl}\gamma)^T C_n : \phi + \int_{\partial T} C_n\gamma : \operatorname{curl}\phi && \text{theorem 8 and transpose}\\
&= \sum_T \int_T \operatorname{inc}\gamma : \phi + \underbrace{\int_{\partial T} C_n \operatorname{curl}\gamma : \phi}_{(1)} + \underbrace{\int_{\partial T} C_n\gamma : \operatorname{curl}\phi}_{(2)} && \text{theorem 8 and transpose}
\end{aligned}$$

The treatment of (1) and (2) is different in nature, the first one needs to understand what is the useful decomposition of the curl of γ, the second one is more subtle and requires the understanding of when it is possible to split an integral by parts on a surface.

(1): We indirectly the decompositions B.4a , B.4b and B.4c

$$\begin{aligned}
\sum_T \int_{\Sigma_T := \partial T} C_n \operatorname{curl}\gamma : \phi &= -\sum_T \int_{\Sigma_T} \operatorname{curl}\gamma : C_n\phi \quad \text{adjoint of } C_n\\
&= -\sum_T \int_{\Sigma_T} Q\,[\operatorname{curl}\gamma] : Q\,[C_n\phi] \quad C_n \text{ has same image of } Q\\
&= -\sum_T \int_{\Sigma_T} Q\,[\operatorname{curl}\gamma]\,Q : Q\,[C_n\phi]\,Q
\end{aligned}$$

In the last step is used that remark B.0.2 implies continuity of $Q\,[\operatorname{curl}\gamma]\,P$

(2): We make use of 5.73

$$\sum_T \int_{\Sigma_T := \partial T} C_n\gamma : \operatorname{curl}\phi = \sum_T \int_{\Sigma_T} Q\,[C_n\gamma] : Q\,[\operatorname{curl}\phi] \qquad C_n \text{ has same image of } Q$$

$$= \sum_T \int_{\Sigma_T} Q\,[C_n\gamma]\,P : Q\,[\operatorname{curl}\phi]\,P \qquad \gamma \text{ is tt-continuous}$$

$$= \sum_T \int_{\Sigma_T} Q\,[C_n\gamma]\,P : [\operatorname{curl} Q\phi]\,P \qquad \Sigma_T \text{ is affine linear + remark B.0.3}$$

Now using theorem 5.73

$$(2) = \sum_T \int_{\Sigma_T} Q\,[\operatorname{curl}(Q\,[C_n\gamma]\,P)]\,Q : Q\,[\phi]\,Q + \sum_T \int_{\partial\Sigma_T} Q\,[C_n\gamma]\,Pn \cdot Q\,[\phi]\,Qt_{\partial\Sigma_T}$$

The square bracket here have no meaning, they are used only to indicate some sort of projection that acts on the sides.

To deal with a more readable formula we use the fact that the surface of each tetrahedron T can be split into a sum of facets triangles $\partial T = \sum F$, also each perimeter of each facet can be split up into several edges $\partial F = \sum E$. Moreover we use the more compact notation

$$A\gamma B = [\gamma]_{BA}$$

To indicate a sort of action of 2 matrices A, B onto a matrix.

$$\begin{cases} (1) & = -\sum_T \sum_{F\subset T} \int_F [\operatorname{curl}\gamma]_{QQ} : [C_n\phi]_{QQ} \\ (2) & = \sum_T \sum_{F\subset F} \int_F [\operatorname{curl}(Q\,[C_n\gamma]\,P)]_{QQ} : [\phi]_{QQ} + \sum_T \sum_{F\subset T} \sum_{E\subset F} \int_E [C_n\gamma]_{PQ}\, n \cdot [\phi]_{QQ}\, t_E \end{cases} \tag{5.76}$$

using the notation B.8a

$$[\operatorname{curl}(Q\,[C_n\gamma]\,P)]_{QQ} = \operatorname{curl}([C_n\gamma]_{PQ})$$

We can rewrite the second part of (2)

$$\sum_T \sum_{F\subset F} \int_F \operatorname{rot}_F([C_n\gamma]_{PQ}) : [\phi]_{QQ} + \sum_T \sum_{F\subset T} \sum_{E\subset F} \int_E [C_n\gamma]_{PQ}\, n \cdot [\phi]_{QQ}\, t_E \tag{5.77}$$

Lastly , since we could have some elements in one sub-simplex, then we indicate the γ

□

Remark 5.3.12. *So far this result can be found in different papers under the name of Green identity [GK09] or [AAB22]. Some other important results can be found in [AAW08] or in [CH22], but only in the last one the idea of using the smearing the operator on the edges is used, in particular the assumption is that the space in which we are setting the test functions is H^2 and therefore all the derivatives at the second order are allowed. We will see that this requirement is too much and the space where we can search for test functions can be reduced to $H^{1+\varepsilon}$.*

Conservation of energy

To test if the method gives correct results one can try to figure out some invariants that hold true in all cases: smooth, continuous and discrete setting. In the smooth setting, the Hamiltonian, turns out to be found a first integral. In the following it is explained why the it holds true for the other variational case.

Starting with the equation

$$S\ddot{\gamma} = -\operatorname{inc}\gamma$$

We want to derive a quantity that is preserved in time, multiplying the previous with : $\dot{\gamma}$

$$\begin{aligned}\langle S\ddot{\gamma} + \operatorname{inc}\gamma, \dot{\gamma}\rangle_F &= 0\\ \langle S\ddot{\gamma}, \dot{\gamma}\rangle_F + \langle \operatorname{inc}\gamma, \dot{\gamma}\rangle_F &= 0\\ \langle \dot{\kappa}, J\kappa\rangle_F + \langle \operatorname{inc}\gamma, \dot{\gamma}\rangle_F &= 0\end{aligned}$$

Now since the operators J and inc are self adjoint, namely :

$$\begin{aligned}\langle S(\gamma), \eta\rangle &= \langle \gamma, S(\eta)\rangle\\ \langle \operatorname{inc}(\gamma), \eta\rangle &= \langle \gamma, \operatorname{inc}(\eta)\rangle\end{aligned}$$

Therefore it holds true that:

$$\begin{aligned}\frac{d}{dt}\langle S(\gamma), \eta\rangle &= \left\langle \frac{d}{dt}S(\gamma), \eta\right\rangle + \left\langle S(\gamma), \frac{d}{dt}\eta\right\rangle\\ &= \left\langle S(\frac{d}{dt}\gamma), \eta\right\rangle + \left\langle S(\gamma), \frac{d}{dt}\eta\right\rangle\\ &= \left\langle \frac{d}{dt}\gamma, S(\eta)\right\rangle + \left\langle S(\gamma), \frac{d}{dt}\eta\right\rangle\end{aligned}$$

And similarly for inc.

Therefore the previous becomes:

$$\begin{aligned}\frac{d}{dt}\frac{\langle \kappa, J\kappa\rangle_F}{2} + \frac{d}{dt}\frac{\langle \operatorname{inc}\gamma, \gamma\rangle_F}{2} &= 0\\ \frac{d}{dt}\frac{1}{2}\left(\langle \kappa, \kappa\rangle_F - \frac{\langle \operatorname{tr}\kappa, \operatorname{tr}\kappa\rangle_F}{2} + \langle \operatorname{inc}\gamma, \gamma\rangle_F\right) &= 0\end{aligned}$$

If one considers the integration in space

$$\begin{aligned}\left\langle \operatorname{curl}(\operatorname{curl}\gamma)^T, \gamma\right\rangle_\Omega &= \left\langle (\operatorname{curl}\gamma)^T, \operatorname{curl}(\gamma)\right\rangle_\Omega - \left\langle \operatorname{curl}(\gamma)^T C_n, \gamma\right\rangle\\ &= \left\langle (\operatorname{curl}\gamma)^T, \operatorname{curl}(\gamma)\right\rangle_\Omega + \left\langle \operatorname{curl}(\gamma)^T C_n^T, \gamma\right\rangle\\ &= \left\langle (\operatorname{curl}\gamma)^T, \operatorname{curl}(\gamma)\right\rangle_\Omega + \left\langle (C_n \operatorname{curl}(\gamma))^T, \gamma\right\rangle\\ &= \left\langle (\operatorname{curl}\gamma)^T, \operatorname{curl}(\gamma)\right\rangle_\Omega + \langle (C_n \operatorname{curl}(\gamma)), \gamma\rangle\\ &= \left\langle (\operatorname{curl}\gamma)^T, \operatorname{curl}\gamma\right\rangle_\Omega\end{aligned}$$

The last part is given by the fact that at the boundary one considers homogenous dirichlet BND for $\gamma = 0$ and/or homogeneous Neumann BND tt-part of γ is zero $\implies C_n\gamma = 0 \implies \mathrm{curl}(C_n\gamma) = 0$

Therefore we can define a functional $\mathcal{H}(\gamma,\kappa) := \frac{1}{2}\left(\|\kappa\|_F^2 - \frac{(\mathrm{tr}\,\kappa)^2}{2} + \left\langle \mathrm{curl}^T\gamma, \mathrm{curl}\,\gamma\right\rangle_F\right)$ and the system that we need to solve is the symplectic one :

$$\frac{d}{dt} = \frac{\partial}{\partial\gamma}\mathcal{H}(\gamma,\kappa) \tag{5.78a}$$

$$\frac{d}{dt} = -\frac{\partial}{\partial\kappa}\mathcal{H}(\gamma,\kappa) \tag{5.78b}$$

It is easy to prove that the above is equivalent to

$$\dot{\kappa} = -\,\mathrm{inc}\,\gamma$$
$$\dot{\gamma} = J\kappa$$

with the notation $\gamma \to g$ and $\kappa \to k$ for the discrete approximation, the following implementable system can be found :

$$\begin{cases} \dot{k} &= -\,\mathrm{inc}\,g \\ \dot{g} &= Jk \end{cases} \implies \begin{cases} \dot{k} &= -\bar{M}_{dd}^{-1}M_{dc}M_{cc}^{-1}M_{\mathrm{inc}}g \\ \dot{g} &= M_{cc}^{-1}M_{cd}\bar{M}_{dd}^{-1}M_{\mathrm{J}}k \end{cases} \tag{5.79}$$

In particular we insert here the projection matrix onto the divergence free constraints: $\bar{M}_d d = \begin{pmatrix} Id & 0\end{pmatrix}\mathrm{Proj}\begin{pmatrix} Id \\ 0\end{pmatrix}$ with *Proj* the projection matrix.

Proposition 9. *The discrete energy*

$$\mathcal{E}_h(g,k) = \frac{1}{2}\left(\langle k, Jk\rangle_F + \langle \mathrm{inc}_h\, g, g\rangle_F\right) \tag{5.80}$$

Is preserved

Proof.

$$\frac{d}{dt}\mathcal{E}_h(g,k) = \left\langle \dot{k}, k\right\rangle_{J_h} + \langle \dot{g}, g\rangle_{\mathrm{inc}_h} \tag{5.81}$$

$$= \left\langle \dot{k}, k\right\rangle_{J_h} + \langle \dot{g}, g\rangle_{\mathrm{inc}_h} \tag{5.82}$$

$$= \left\langle -\bar{M}_{dd}^{-1}M_{dc}M_{cc}^{-1}M_{\mathrm{inc}}g, k\right\rangle_{J_h} + \left\langle M_{cc}^{-1}M_{cd}\bar{M}_{dd}^{-1}M_{\mathrm{J}}, g\right\rangle_{\mathrm{inc}_h} \tag{5.83}$$

$$= -\left(\bar{M}_{dd}^{-1}M_{dc}M_{cc}^{-1}M_{\mathrm{inc}}g\right)^T M_J k + \left(M_{cc}^{-1}M_{cd}\bar{M}_{dd}^{-1}M_{\mathrm{J}}\right)^T M_{Inc}g \tag{5.84}$$

$$= -\left(M_{\mathrm{inc}}g\right)^T\left(M_{cc}^{-1}M_{cd}\bar{M}_{dd}^{-1}M_J k\right) + \left(M_{cc}^{-1}M_{cd}\bar{M}_{dd}^{-1}M_{\mathrm{J}}\right)^T M_{Inc}g \tag{5.85}$$

$$= 0 \tag{5.86}$$

□

Therefore the continuous and the discrete case follow the same rule for the conservation of energy. In particular, since we lack of possible test for the implementation[5] the conservation of the energy can be a valid test.

[5] I couldn't find an analytical solution for a wave that starts with a bump function initial condition.

Part IV.

Implementation and Results

6. Symplectic Integrators

In this chapter I developed exercise-like some symplectic integrators, I don't give any theorem regarding the stability, only hands on computations for obtaining higher order methods instead of the ususal symplectic euler used for solving the numerical problems obtained.[1]

6.1. Runge Kutta Methods:

The FEM procedure to discretize a PDE brings inevitably to a system of ODEs of the form:

$$\begin{cases} y'(t) & = f(t, y(t)) \\ y(t_0) & = y_0 \end{cases} \tag{6.1}$$

One of the most widely used class of numerical one-step methods is the Runge Kutta methods, in this appendix we give some results that are important to create a quick solver given the butcher tableau of a method.

Definition 25 (m-stage Runge Kutta Method)**.** *The m-stages Runge Kutta method is a one-step method*

$$\Phi := \sum_j k_j b_j \tag{6.2}$$

$$y_{n+1} := y_n + h_n \Phi(t_n, y_n, h_n) \tag{6.3}$$

$$\tag{6.4}$$

Where Φ is called incremental function ,b_j are the called weights of the stage k_j, i.e. the stages are defined in the following way:

$$k_i = f(t + c_i h, y + h \sum_j A_{ij} k_j) \tag{6.5}$$

where c_i are called nodes.

Since the method is defined using only A_{ij}, b_j and c_j then it is usually written as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

[1]This derivations will be revisited in future work to solve possible nonlinear equations.

Obviously at each time step one has to find the root of the equation 6.5, but one can in general notice that for fixed-triangulation $\mathcal{T}$ and for time independent quantities we have that the pde is linear of the form :

$$k_i = M\left(y + h\sum_j A_{ij}k_j\right) \tag{6.6}$$

for some Msquared matrix This problem is indeed a hidden matrix problem, but before showing that we need to introduce some mathematical object such as the kronecker product

$$\otimes : \mathbb{R}^{a\times b} \times \mathbb{R}^{c\times d} \ni \left(\begin{pmatrix} E_{11} & \cdots & E_{1b} \\ \vdots & \ddots & \vdots \\ E_{a1} & \cdots & E_{ab}\end{pmatrix}, \begin{pmatrix} F_{11} & \cdots & F_{1d} \\ \vdots & \ddots & \vdots \\ F_{c1} & \cdots & F_{cd}\end{pmatrix}\right) \longmapsto \begin{pmatrix} E_{11}F & \cdots & E_{1b}F \\ \vdots & \ddots & \vdots \\ E_{a1}F & \cdots & E_{ab}F\end{pmatrix}, \mathbb{R}^{ac\times bd} \tag{6.7}$$

And the vectorization operator is the operator that returns all the columns of a matrix merged in a long vector :

$$\text{vec} : \mathbb{R}^{d\times m} \ni \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{d1} & \cdots & A_{dm}\end{pmatrix} \longmapsto (A_{11}, \cdots, A_{1d}, \cdots\cdots A_{1m}, \cdots, A_{dm})^T \in \mathbb{R}^{dm} \tag{6.8}$$

Obviously it is possible to define the inverse operator if one specifies the right dimension size.

Remark 6.1.1. *It is possible to prove that :*

$$vec\,(ABC) = (C^T \otimes A)\,vec\,(B) \tag{6.9}$$

Lemma 6.1.2 (Equivalence of Solutions)**.** $(k_j)_j^m \subset \mathbb{R}^d$ *is a solution of*

$$k_i = M(y + h\sum_j A_{ij}k_j)\forall i \in 1, ..., s \tag{6.10}$$

if and only if $K = [k_1, ..., k_m] \in \mathbb{R}^{d\times m}$ *is a matrix solution of the following equation:*

$$(I_m \otimes I_d)\,K = M(y + \sum_j A_{ij}k_j) \quad \forall i \in 1, ..., s \tag{6.11}$$

Proof.

$$
\begin{aligned}
k_j &= M\left(y + h\sum_j A_{jr}k_r\right) && \text{vector equatio} \\
k_j &= My + h\sum_j A_{jr}\,M\,k_r && \text{a} \\
(k_j)_i &= (My)_i + h\sum_j A_{jr}\,(M\,k_r)_i && \text{selecting only the i-th row of the} \\
k_{ij} &= (My)_i + h\sum_j A_{jr}M_{is}(k_r)_s && \text{conventio} \\
k_{ij} - h\sum_j A_{jr}M_{is}k_{sr} &= (My)_i \mathbb{1}_j && \text{use of row-ve} \\
k - hMkA^T &= (My)\otimes \mathbb{1}(m) && \text{definition} \\
I_d \otimes I_m \text{vec} k - h(M\otimes A)\,\text{vec} k &= \text{vec}\left((My)\otimes \mathbb{1}(m)\right) && \text{v} \\
\text{vec}(k) &= \left(I_{dm} - hM\otimes A\right)^{-1}\text{vec}\left((My)\otimes \mathbb{1}(m)\right) && \text{Big vect}
\end{aligned}
$$

Where I is the identity matrix of dimension $d\cdot m \times d\cdot m$ and $M\otimes A$ is the tensor product of M and A; Also $\mathbb{1}_m = (1, ..., 1)$ for m- times. □

Therefore the a generic time stepping method can be written as follows:

$$
\begin{aligned}
\text{vec }(k) &= (I - hM\otimes A)^{-1}\ \text{vec }(My\otimes \mathbb{1}_m) && \text{Matrix-form equation} \quad (6.12)\\
y^{n+1} &= y^n + hkb && \text{vector equation for stages} \quad (6.13)\\
& && (6.14)
\end{aligned}
$$

Remark 6.1.3. *The method can work with any m-stage Runge Kutta, implicit or explicit, on a linear differential equation.*

In particular it is now possible to use composite Runge Kutta that result in symplectic integrators, such as the symplectic Euler and the Gaussian integrator. Later, in the implementation section, will be specified a link for all the implementations.

We now define two symplectic integrators for a separable Hamiltonian that are used in the implementation, it will not follow the proof for the order.

Definition 26. *The* 2^{nd} *order Leap Frog and* 4^{th} *order Yoshida Integrators are coded in the following way:*

1. ***Leap Frog:***

```
def LeapFrogStep(tau,u,Mu,v,Mv, dual = False):
    # SE1
    u.vec.data += tau/2*Mv*v.vec
    v.vec.data += tau/2*Mu*u.vec
    # SE2
    v.vec.data += tau/2*Mu*u.vec
    u.vec.data += tau/2*Mv*v.vec
```

2. ***Yoshida:***

```
def YoshidaIntegratorStep(tau,u,Mu,v,Mv):
    w0 = - 2**-(1/3)/(2-2**-(1/3))
    w1 = 1/(2-2**-(1/3))

    c= [w1/2 , (w0+w1)/2  , (w0+w1)/2 , w1/2 ]
    d= [w1,w0,w1]

    u.vec.data += c[0]*tau*Mv*v.vec
    v.vec.data += d[0]*tau*Mu*u.vec

    u.vec.data += c[1]*tau*Mv*v.vec
    v.vec.data += d[1]*tau*Mu*u.vec

    u.vec.data += c[2]*tau*Mv*v.vec
    v.vec.data += d[2]*tau*Mu*u.vec

    u.vec.data += c[3]*tau*Mv*v.vec
    return u,v
```

7. Implementation and Results

Scope of the Chapter in the following chapter there is a brief discussion on the coding fallacy that may occur when implementing in a naive way. Some ideas for the stabilization of the numerical method are given and in the end the chapter is concluded presenting some numerical results.

The implementation is done using NGsolve[Sch14], all the codes I wrote are available in my Github in the repository Master_Thesis[1]. The discussion about the code is

Implementation aspects

The generic structure of a finite element implementation has to respect the following points:

1. **define a proper mesh over a geometry**: usually a square, cube or sphere; in any case something simple and if one is working with differential operator needs to keep in mind that domains that are not simply connected may result in complications[GK09]
2. **define the spaces of interest:** Particular attention has to be payed on the order of the various spaces, the goal is to define spaces that work well together (namely respect the Brezzi theorem or Babuska-Aziz for the discrete case).
3. **create the matrix operator via the assembly process:** in this step is always better to use a functions to obtain cleaner codes.
4. **define enough test functions** obviously there is the need of at least one grid-function to store the result.

At this point there are only two thing left to discuss, one is the possible complications arising in the implementation, the second one is the methods used for the time-stepping and visualize some results. The first pointed out in the next remark.

Remark 7.0.1. *During the implementation there is the possibility to encounter problems linked to the orientation of the elements of the triangles/tetrahedrons if there is no consideration for internal ordering.*

Since it is difficult to present with clarity the problem in three dimensions I will present it in 2d. The following snippet of code shows the creation of the inc operator acting on a 2-metric:

```
u,v = fesH1.TnT()
```

[1]the collection of codes is quite big. My intention was, and still is, to create a collection of examples to enlarge the documentation.

```
# Mass matrix
a = BilinearForm(fesH1)
a += InnerProduct(u,v)*dx

# linear form induced by the metric gfG
f = LinearForm(fesH1)

# triangle inc part
f += (gfG.Operator("inc"))*v*dx

# edge curl part
t = specialcf.tangential(2)
f += -v*InnerProduct(curl(gfG),t)*dx(element_vb=BND)

# edge tangential gradient part
n = specialcf.normal(2)
tdergfG = grad(gfG)[t,:]
print(tdergfG.dims)
f += -v*InnerProduct( t,tdergfG.Reshape((2,2))*n)*dx(element_vb=BND)

# vertex elements
Vt = specialcf.VertexTangentialVectors(2)
    # I need the normals to be out from the triangle:
Vn0 =  (CF((0,-1,1,0), dims= (2,2))*Vt[0,:])
Vn1 =  (CF((0,1,-1,0), dims= (2,2))*Vt[1,:])
f += v*InnerProduct( Vt[0,:],gfG*Vn0)*dx(element_vb=BBND)
f += v*InnerProduct( Vt[1,:],gfG*Vn1)*dx(element_vb=BBND)
```

First of all the focus needs to be directed to the space of definition of the linear form f, lemma 5.3.10 suggests that the space for the linear form is not the $H(\mathrm{inc})$ space but H^1. Second observation is directed to the definition of the normals, here the first one is rotated counterclock-wise and the other clock-wise, the idea is that when we pick a triangle if one rotates always on one side a tangent to obtain a normal then the triangle will have an inward and an outward pointing vectors that may not coincide on a triangulation!

In 5.3 it is introduced a component in the *Proj* matrix, in the code it is implemented using the bilinear form:

```
pT, dpT = HCurl.tnt

# some other code

PROJ += -1*(pT*dpT )*dx
PROJ += -1*curl(pT)*curl(dpT)*dx
```

heuristically speaking this helps the small eigenvalues of the matrix to shift further from the zero on one hand. On the other hand helps during the factorization process. The flag that is used in the code is

```
inverse = "sparsecholesky"
```

This factorization does not check for pivotal zeros in the matrix, therefore the code can run badly for implementation errors.

Another important element that was added in the implementation is the use of a damping matrix, it is used to regularize the result in the same style as a the heat equation damping matrix; in particular this matrix comes out from the formulation when one tries to insert in an explicit way the constraint in the evolution equation.

Definition 27 (Damped System)**.** *The damped system relative to the system 3.21 is defined as:*

$$\dot{\gamma} = J\kappa - \lambda\mathscr{D}\gamma \tag{7.1a}$$

$$\dot{\kappa} = -\operatorname{inc}\gamma \tag{7.1b}$$

$$\operatorname{div}\kappa = 0 \tag{7.1c}$$

Where the operator $\mathscr{D}$ is defined as follows

$$\mathscr{D} : H(\text{inc}) \ni \gamma \longrightarrow \left\{ \langle \mathscr{D}\gamma, \phi \rangle_{H^{-1}(\text{div}),H(\text{curl})} = (\operatorname{div} S\gamma, \operatorname{div} S\phi) \quad \forall\, \phi \in H(\text{curl}) \right\} \in (H(\text{inc}))^*$$

The above definition takes in consideration that div : $H(dd) \to H^{-1}(\text{div}) = H(curl)^*$ therefore a good way to define this operator is using the pairing in the dual of $H(\text{curl})$, moreover λ is a tuning parameter. Following the line dictated by 5.2 one obtains the following result.

Proposition 10. *Let $\mathscr{D}$ be the damping operator defined above in 7, then the discretized form of $\mathscr{D}$, indicated by $\mathbb{D}$, is of the following form*

$$\mathbb{D} = \mathbb{B}^T M_c^{-1} \mathbb{B} \tag{7.2}$$

Where $\mathbb{B} = (Sg, \nabla e)$ for $(g, e) \in H(\text{inc}) \times H(\text{curl})$ is the passage matrix and $M_c = (e, \hat{e})$ for $d, \hat{d} \in H(\text{curl})$ is the mass matrix.

Intuitive proof: Let $\hat{\gamma} = \mathscr{D}$ then using the inner product in $H(\text{inc})$:

$$(\hat{\gamma}, \Psi) = \langle \mathscr{D}\gamma, \Psi \rangle = (\operatorname{div} S\gamma, \operatorname{div} S\Psi)_{H^{-1}(\text{div})}$$

Since $H(\text{curl})^* \simeq H^{-1}(\text{div})$ then we can transfer the above inner product

$$\langle \mathscr{D}\gamma, \Psi \rangle = (\operatorname{div} S\gamma, \operatorname{div} S\Psi)_{H(\text{curl})^*}$$

Using the natural definition of inner product in the the dual space the

$$\langle \mathscr{D}\gamma, \phi \rangle = \left(\mathscr{J}_{H(c)} \operatorname{div} S\gamma, \mathscr{J}_{H(c)} \operatorname{div} S\Psi \right)_{H(\text{curl})}$$

Now each of the 2 entries on the right hand side of the above can be described in terms of an element in $H(\text{curl})$, namely $u_\gamma = \mathscr{I}_{H(c)} \operatorname{div} S\gamma$, in particular

$$(u_\gamma, \phi)_{H(c)} = \langle \operatorname{div} S\gamma, \phi \rangle_{H(c)} = (S\gamma, \nabla\phi) - \int_{\partial T} (S\gamma)_{nn} \phi_n$$

The discretization becomes

$$\begin{aligned} M_c u_\gamma &= \mathbb{B}\gamma \\ M_c u_\Psi &= \mathbb{B}\Psi \\ (\mathscr{I}_{H(cc)}\gamma)^T M_{cc}\Psi &= u_\gamma^T M_c u_\Psi \end{aligned}$$

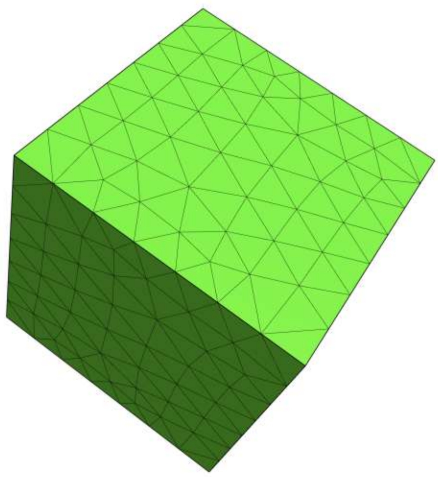
Imply immediately the theorem. □

Remark 7.0.2. *The above theorem allow us to define a general rule. When working in the discrete setting of iterable finite elements methods the definition that one except is the following $F_h : V_h \to V_h^*$ and not $F_h : V_h \to V_h$ since the test functions are always going to be paired in via a duality paring and only then the definition of the paring is given. Therefore even if the solution is given in a pre-dual setting one works mainly the dual one.*

Results:

As mentioned above the results are just simple waves. We run two tests. one for performance check and the second one for quality comparison.

- **Domain and Mesh**: As mentioned at the beginning of the chapter the domain is a simply connected cube of edge length 2.



- the initial function coefficient function is built using a function that vanishes at the boundary $f(\vec{x})\alpha \exp(-\beta(\|\vec{x}\|^2))$ where $\beta = 20$, $\alpha = 1/10$, this is insert into a 3×3 zero matrix as coefficient in position $(0, 0)$, After doing so the symmetric curl is applyed and the resulting metric is used as initial condition γ_0, meanwhile κ_0 is set to zero. After deriving the Frobenious norm the it is possible to derive the maximum and minimum of the norm. In particular it can be found that the maximum norm of the initial condition, in particular along the $[z = 0]$ plane the maximum is equally distributed on a cirle of radious $r = (2\beta)^{-1/2}$.

- The spaces are taken with order $k = 2$, the time step size is 0.02 and the final time is 1.2 seconds.
- The method implemented for the time stepping is the leap frog.
- The damping parameter is set to $\frac{dt}{10} \frac{maxh}{(order+1)^2}$. and the damping process is applied 10 times.
- The program is run on a cpu i9-9980HK 2.40GHz, parallelized on 8 cores with 16 threads in tota. The runtime for the is 1 minute and 42 seconds.

The final result is compared to a reference test: the order is uppered by one, the time stepping is divided by two and the fourth Yosida integrator substitutes the second order one, then the result is the following.

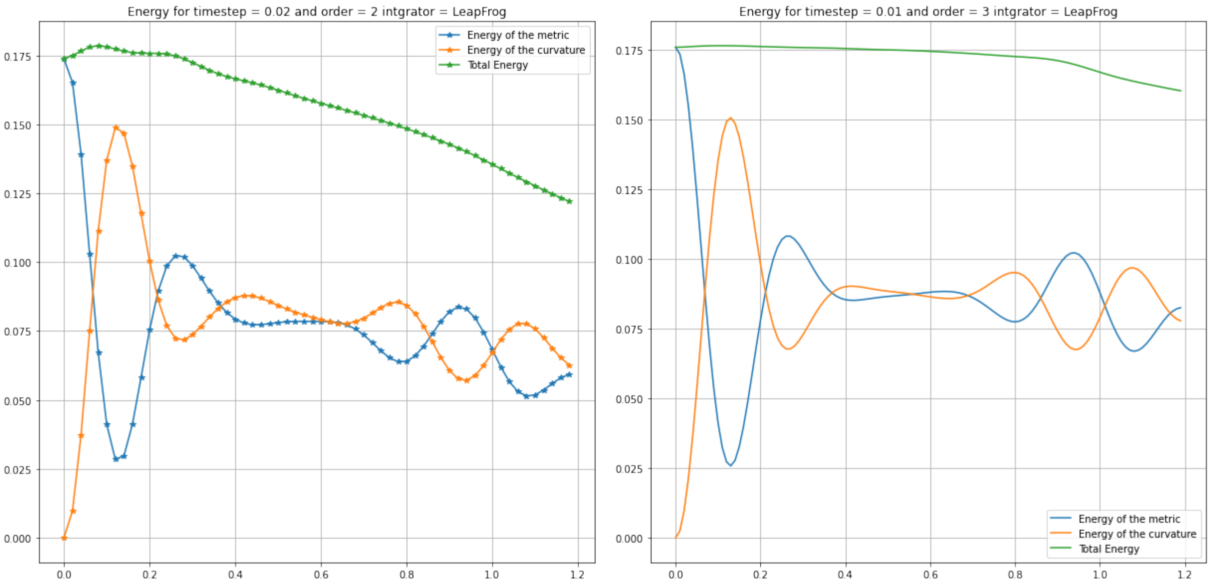


Figure 7.1.: The blue line is given by the first component of the Hamiltonian 9, the orange line by the second one. The green line is the sum, as one notice is almost constant with a slight curvature downward, in particular this bending is given by the damping condition.

Picture **??** shows that there is a problem at the starting point, in fact it looks like the energies do not start at the same value. But this is a problem of interpolation of the initial coefficient function.

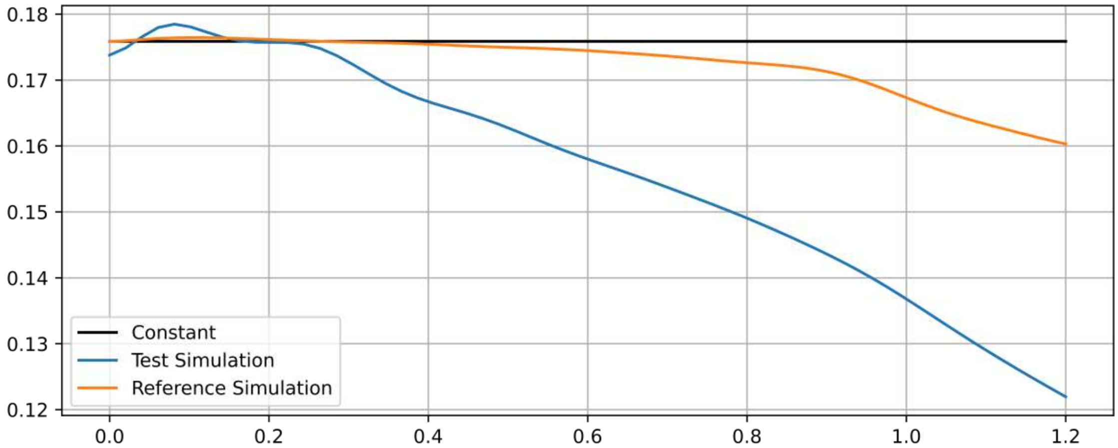


Figure 7.2.: Here an overlapping comparison of the tests.

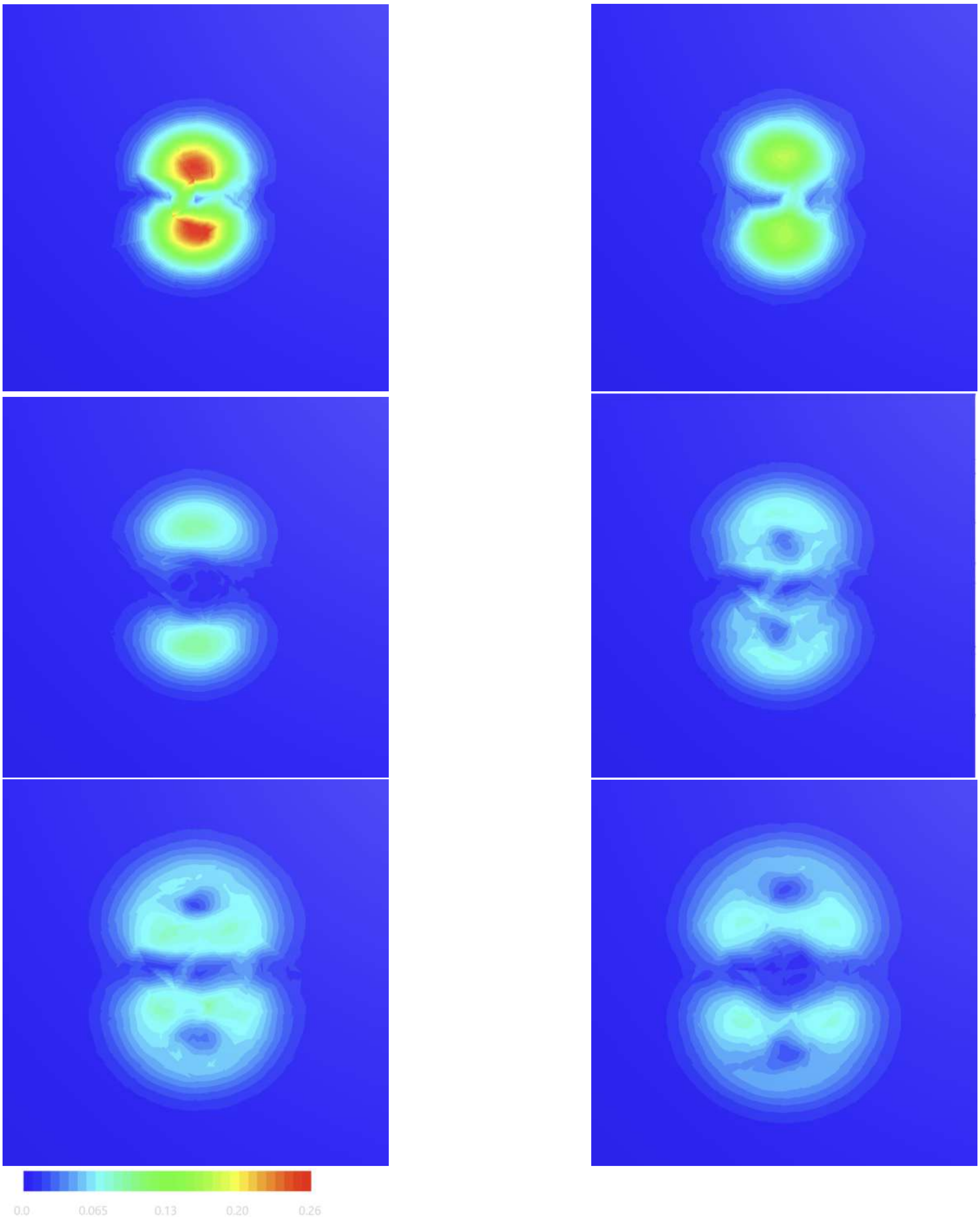


Figure 7.3.: View of the wave perpendicular to the direction of propagation, above all the magnitude of the metric. Seeing the animated motion one notice a sort of bubbling outward of the "low pressure" zones inside the metric.

8. Conclusions

Throughout the thesis we have pointed out a sort of symmetry between the chosen problems. The smooth setting allow us to see that both the equations obtain the known "wave" form, in particular, for both problems, it is possible to figure out a possible first integral (energy/Hamiltonian) that is preserved in time. One difference that is seen in the linearization is given by the fact that the ME are in linear in time, and the only possible nonlinear part is given by the matter interaction that, only at this point, is set linear by choice. Meanwhile the EFE are nonlinear by nature, the linearization process does not have a natural preference, therefor one can chose one ot the other of the following:

- Fist linearize the equations by mean of the first variation, and only then pick a reference of frame such that the metric is split in time, space and mixed components.
- Or first set up the splitting of the metric by means of the project by mean of a mixed tensor object and only then linearize the problem.

In the second part of the chapter there is an introduction of the Sobolev spaces used and their discretization FE. Here the complexity relies on two factors, the first one is the huge degrees of freedom involved in the latest FEs, the second involves the continuity of the latter spaces: namely they are functions that are continuous in a functional way.

In the third part it is observed the complexity of the modelling, in particular there is no simple approach to create a model that fits well for all cases. In fact the the methods introduced, particularly the first ones, are based upon the existence of one space that acts as a super-space of the two considered in the equation. Moreover it is shown that the definition of one distributive pairing can be quite long to derive when considering higher order differential operator.

In the last chapter some results about the implementation are given, the only thing remaining to do is to define PML boundary conditions and compare the method with an analytical solution.

A. Differential Geometry

This chapter is useful for those who are rusty on the subject in terms of definitions or notation.

Definition 28 (Directional Derivative). *Let $U \subset \mathbf{R}^d$ open, $a = (a_i)_{i=1}^d$ point in U, $v = (v_i)_{i=1}^d$ in $\mathbf{R}^d$ called direction and a function $F : U \to \mathbf{R}$. If the composit function $F(a+tv)$ is well defined in a neighborhood of $t = 0$ and there exist the limit*

$$v(F)_a := \lim_{t \to 0} \frac{F(s+tv) - F(a)}{t} \tag{A.1}$$

Then the limit is called directional derivative of F in a in direction v.

The coordinate functions in $\mathbf{R}^d$ are noted by $x_i : \mathbf{R}^d \to \mathbf{R}$, they return the i-th component of the element $a = (a_j)_{j=0}^d$, therefore if one canonical direction is chosen then $e_i(F)_a$ con be written in the following form:

$$\frac{\partial F}{\partial x_i}(a) := e_i(F)_a$$

And it is referred to as j-th partial derivative.

Definition 29 (Local chart). *Let M be a topological space. A pair (U, Φ_U), with U open in M and Φ_U omeomorphic form U to an open subset op $\mathbf{R}^d$ is called d-local chart, or d-system of local coordinates on M. Two local charts are said to be C^k compatible if the intersection of the sets are empty or the application*

$$\Phi_V \circ \Phi^{-1} : \Phi_U(U \cap V) \to \Phi_V(U \cap V)$$

Is a c^k-diffeomorhism.
With abuse of notation only the function Φ_U is referred to local chart and it is written as

$$\Phi_U(p) = (x_1(p), \cdots, x_d(p))$$

Where $(x_i : U \to \mathbf{R})_{i=1}^d$ are the components of the application Φ_U and are referred to as local coordinates in U.

Definition 30 (n-simplex in). *the convex hull of $n+1$ points in generic position in $\mathbf{R}^d$ is called n-simplex*

Definition 31 (mesh). *A mesh $\mathcal{T}$ is a collection of simplices in $\mathcal{R}^d$ satisfying:*

- *Any face of 2 simplex in $\mathcal{T}$ is a simplex in $\mathcal{T}$*

- *the intersection of any 2 simplices of $\mathcal{T}$ is a common face of both simplices $\mathcal{T}$.*
- *The union of all simplices of $\mathcal{T}$ is a submanifold of dimension n in $\mathbb{R}^d$.*

The integer n is called topological dimension of $\mathcal{T}$, d is the geometric dimension of $\mathcal{T}$.

In the thesis the previous dimensions are equal unsless specified otherwise.

Definition 32 (Triangulation)**.** *the manifold specified by $\mathcal{T}$ is called carrier of $\mathcal{T}$ and it is denoted using $|\mathcal{T}|$. Let M be a manifold diffeomorphic to $|\mathcal{T}|$, for a certain mesh $\mathcal{T}$ then we refer to $\mathcal{T}$ as triangulation of M*

Definition 33 (k-tensor)**.** *Let M be a smooth manifold. For any point $p \in M$ one can define a covariant tensor qt p as a real k-linear form on the tangent space T_pM. A covariant tensor field on M is a is function on M assigning to each point p a covariant K-tensor at p.*

Similarly to the identification of the space of smooth vector field with $\mathbb{V} \otimes C^\infty$ The space of symmetric covariant 2-tensor fields can be identified with denoted using $\mathbf{S}^n \otimes C^\infty$, where $\mathbb{S}^n$ is here used to denote the space of symmetric square matrices.

Definition 34 (differential)**.** *Let $\phi : M \to N$ be a smooth function between smooth manifolds. At every point $p \in M$, the differential is a linear map $(d\phi)_p : T_pMT_{\phi(p)}N$ defined in the following way: Given any $f : U \to \mathbb{R}$ with U open neighbour of $\phi(p)$ and given any derivation $v \in T_pM$:*

$$(d\phi)_p v[f := v(f \circ \phi)]$$

The above induces map (ϕ^*) from the space of k-tensors at $\phi(p) \in N$ to covariant k-tensors at $p \in M$ in the following way

Definition 35 (pullback)**.** *For any k-tensor g at $\phi(p) \in N$ and any $(u_1, \cdots, u_k) \in \bigtimes_{i=1}^k T_pM$, we define the pullback of g at p:*

$$(\phi^*)_p g(u_1, \cdots, u_k) := g(d\phi_p(u_1), \cdots, d\phi_p(u_k))$$

We define the pullback of g under ϕ the function that point-wise returns $(\phi^*)_p$, it is noted using $\phi^* g$

A.1. Tensor calculus Formalism

Let an n-dimensional vector space V, spanned by a particular base (e_i) and its dual V^* spanned by (e^i) such that $e^i(e_j) = \delta^i_j$ (called bi-orthogonal basis). We know that an element $v \in V$ can be written in a unique way as:

$$v = v^i e_i \tag{A.2}$$

Where v^j can be seen as the measurement of the vector v w.r.t. the particular choice of the basis, for all $j = 1...n$ in fact if we "measure" the $j-$th component of the vector:

$$e^j(v) = e^j(v^i e_i) = v^i e^j(e_i) = v^i \delta^j_i = v^j$$

Fixing a basis for V then a bilinear form $g : V \times V \to \mathbf{R}$ can be represend in terms of its evaluation on the basis:

$$g_{ij} = g(e_i, e_j) \tag{A.3}$$

If the requirements are strong enough (for example the matrix is non-degenerate and symmetric) then the matrix is invertible and the inverse is written as g^{ij}.
Without diving too much into details one sees that for a non-degenerate symmetric bilinear form one can build a Riesz-like isomorphism

$$w := g(\cdot, v)\text{define a co-vector from a vector} \tag{A.4}$$
$$w_i e^i(e_k) = g(e_k, v^j e_j)\text{express it in coordinates} \tag{A.5}$$
$$w_i \delta^i_k = v^j g(e_k, e_j)\text{use bi-orthogonality} \tag{A.6}$$
$$w_k = v^j g_{kj} \tag{A.7}$$
$$\tag{A.8}$$

This is the k-th component of the induced co-vector, using the invertibility hypothesis we obtain:

$$v_i = g_{ij} v^j \tag{A.9}$$
$$v^i = g^{ij} v_j \tag{A.10}$$
$$\tag{A.11}$$

the partial derivatives obtain a similar notation:

$$\partial_\alpha := \frac{\partial}{\partial x_\alpha} \tag{A.12}$$
$$\partial^\alpha := g^{\alpha\beta} \partial_\beta \tag{A.13}$$

It is important to notice that the above is used just as a notation! Here we notice that at each point $p \in \mathcal{M}$ of the manifold one has that $g_p : T_p\mathcal{M} \times T_p\mathbf{M} \to \mathbf{R}$, therefore the uppering and lowering of the index as explained above make sense.
When one performs a derivation of vector, or co-vector, in a proper way needs to take into account that there are two things that may change: the field itself and the frame of reference, therefore the proper derivation takes into consideration both:

Definition 36 (Christoffel Symbols)**.**

$$\Gamma^\alpha_{\beta\gamma} := \frac{g^{\alpha\mu}}{2} \left(\frac{g_{\beta\mu}}{\partial x^\gamma} + \frac{g_{\gamma\mu}}{\partial x^\beta} - \frac{g_{\beta\gamma}}{\partial x^\mu} \right) \tag{A.14}$$

The Christoffel symbols are essential to obtain a proper derivation of the field, they are helpful in the derivation of (m, n)-tensors in general, since the indices become incomprehensible I will report only the (covariant)-deirivation for the vector, covector and (m, n)-tensors with $m + n = 2$.

Definition 37 (covariant derivative of vector, covector and tensors:).

$$v^{\alpha}_{;\beta} = \nabla_{\beta} := \frac{\partial v^{\alpha}}{\partial x^{\beta}} + v^{\mu}\Gamma^{\alpha}_{\beta\mu} \tag{A.15}$$

$$v\alpha;\beta := \frac{\partial v_{\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}v^{\mu} \tag{A.16}$$

$$T^{\mu\nu}_{;\alpha} := \partial_{\alpha}T^{\mu\nu} + \Gamma^{\mu}_{\alpha\beta}T^{\beta\nu} + \Gamma^{\nu}_{\alpha\beta}T^{\mu\beta} \tag{A.17}$$

$$T_{\mu\nu;\alpha} := \partial_{\alpha}T_{\mu\nu} - \Gamma^{\alpha}_{\mu\beta}T_{\beta\nu} - \Gamma^{\nu}_{\alpha\beta}T^{\mu\beta} \tag{A.18}$$

$$T^{\mu}_{\nu;\alpha} := \partial_{\alpha}T^{\mu}_{\nu} + \Gamma^{\mu}_{\alpha\beta}T^{\beta}_{\nu} - \Gamma^{\beta}_{\alpha\nu}T^{\mu}\beta \tag{A.19}$$

Some other important object for the derivation of the EFE are

Definition 38 (Riemann Tensor, Ricci Tensor).

$$R^{\sigma}_{\mu\nu\rho} := \partial_{[\nu}\Gamma^{\sigma}_{\mu]\rho} + \Gamma^{\alpha}_{\rho[\mu}\Gamma^{\sigma}_{\nu]\alpha} \tag{A.20a}$$

$$R_{\mu\nu} := R^{\lambda}_{\mu\lambda\nu} = \partial_{[\lambda}\Gamma^{\lambda}_{\mu]\nu} + \Gamma^{\lambda}_{\nu[\mu}\Gamma^{\sigma}_{\lambda]\alpha} \tag{A.20b}$$

Definition 39 (Bianchi's identity).

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \tag{A.21}$$

During the derivation in the setting of the 3+1 formalism we need to make useage of the lie derivative, here I will dive the definition applied to vectors, covectors and tensors in the same way that I did in definition A.1

Definition 40.

$$\pounds_{\vec{u}}\,\phi = u^{\mu}\partial_{\mu}\phi \tag{A.22a}$$

$$\pounds_{\vec{u}}\,v^{\alpha} = u^{\mu}\partial_{\mu}v^{\alpha} - v^{\mu}\partial_{\nu}u^{\alpha} \tag{A.22b}$$

$$\pounds_{\vec{u}}\,v_{\alpha} = u^{\mu}\partial_{\mu}v_{\alpha} + v_{\mu}\partial_{\alpha}u^{\mu} \tag{A.22c}$$

$$\pounds_{u}\,T_{\alpha\beta} = u^{\mu}\partial_{\mu}T_{\alpha\beta} + T_{\alpha\mu}\partial_{\beta}u^{\mu} + T_{\mu\beta}\partial_{\alpha}u^{\mu} \tag{A.22d}$$

It is now time to introduce some calculus differential operator; as mentioned before the background metric is the Minkowskian one, suppose also that that the space is endowed with a metric g. Most of the time in general relativity the huge number of indices creates confusion, therefore it is a good practice to introduce some names for usual differential operators on a manifold equipped with a metric

Definition 41 (Differential Operators in pseudu-Riemannian Geometry). *Let a manifold $\mathcal{M}$ be equipped with a non-degenerate metric g,and let u, v, w be respectively a scalar, a vector and a matrix fields defined on $(\mathcal{M}, g)$; then we can define the following differential* *operators*[1]*:*

$$
\begin{aligned}
(\nabla u)_\alpha :=& \partial_\alpha u && \textit{gradient} && \text{(A.23a)}\\
(\nabla\nabla u)_{\alpha\beta} :=& \partial_\alpha\partial_\beta u && \textit{Hessian} && \text{(A.23b)}\\
(\nabla^{\mathbf{S}} v)_{\alpha\beta} = (\varepsilon v)_{\alpha\beta} :=& \frac{1}{2}\left(\partial_\alpha v_\beta + \partial_\beta v_\alpha\right) && \textit{symmetric gradient} && \text{(A.23c)}\\
\operatorname{div}_g v :=& g^{\alpha\beta}\partial_\alpha u_\beta = \partial^\beta u_\beta && \textit{vector divergence} && \text{(A.23d)}\\
(\operatorname{div}_g w)_\beta :=& g^{\alpha\lambda}\partial_\lambda w_{\alpha\beta} && \textit{matrix divergence} && \text{(A.23e)}\\
\Delta u :=& g^{\alpha\beta}\partial_\alpha\partial_\beta u && \textit{Laplacian} && \text{(A.23f)}\\
(\Delta v)_\alpha :=& g^{\lambda\beta}\partial_\lambda\partial_\beta v_\alpha && \textit{Vector Laplacian} && \text{(A.23g)}\\
(\Delta v)_{\alpha\beta} :=& g^{\lambda\mu}\partial_\lambda\partial_\mu v_{\alpha\beta} && \textit{Matrix Laplacian} && \text{(A.23h)}\\
(\operatorname{tr}_g w) :=& g^{\alpha\beta} w_{\alpha\beta} && \textit{Trace} && \text{(A.23i)}
\end{aligned}
$$

And the following algebraic operators:

$$(S_g w) := w - (\operatorname{tr}_g u)\, g \tag{A.23j}$$

$$(J_g w) := w - \frac{1}{2}(\operatorname{tr}_g u)\, g \tag{A.23k}$$

Proposition 11.

$$\frac{-2}{d-2}\operatorname{tr}\operatorname{ein}_g \gamma = \operatorname{div}\operatorname{div} S\gamma \tag{A.24}$$

Proof.

$$
\begin{aligned}
\operatorname{tr}_g 2\operatorname{ein} w &= -\partial^\lambda\partial_\lambda w^\nu_\nu + \partial^\nu\partial^\lambda w_{\lambda\nu} + \partial^\mu\partial^\lambda w_{\lambda\mu} - \partial^\nu\partial_\nu w^\alpha_\alpha - \delta^\mu_\mu\partial^\alpha\partial^\beta w_{\alpha\beta} + \delta^\mu_\mu\partial^\lambda\partial_\lambda w^\alpha_\alpha\\
&= (d-2)\partial^\lambda\partial_\lambda w^\nu_\nu - (d-2)\partial^\nu\partial^\lambda w_{\lambda\nu}\\
&= -(d-2)\left(\operatorname{div}_g\operatorname{div}_g w - \operatorname{div}_g\operatorname{div}_g(\operatorname{tr}_g g)\right)\\
&= -(d-2)\operatorname{div}_g\operatorname{div}_g Sw
\end{aligned}
$$

□

It can be seen as as the definitions above are the same if we use the usual Euclidean metric, moreover we can achieve new differential identities.

Lemma A.1.1 (algebraic identities)**.** *In m dimension and under constant metric g the*

[1] An alternative way to define these operators is using the exterior calculus.

following identities hold true:

$$J^{-1}w = w - \frac{1}{m-2}g(\operatorname{tr} u) \tag{A.25a}$$

$$S^{-1}w = w - \frac{1}{m-1}g(\operatorname{tr} u) \tag{A.25b}$$

$$\operatorname{div}\varepsilon = \frac{1}{2}\Delta + \frac{1}{2}\nabla\operatorname{div} \tag{A.25c}$$

$$\operatorname{div} J\varepsilon = \frac{1}{2}\Delta \tag{A.25d}$$

$$\operatorname{div} S\nabla\nabla = 0 \tag{A.25e}$$

$$\operatorname{div}\operatorname{div} S\varepsilon = 0 \tag{A.25f}$$

$$\operatorname{div}\operatorname{ein} = 0 \tag{A.25g}$$

$$\operatorname{ein}\varepsilon = 0 \tag{A.25h}$$

$$\Delta_g \operatorname{tr}_g = \operatorname{tr}_g \Delta_g \tag{A.25i}$$

$$\operatorname{tr}_g \nabla^{\mathbf{S}} v = \operatorname{div}_g v \tag{A.25j}$$

$$\nabla^{\mathbf{S}} \operatorname{div}_g \operatorname{tr}_g = \nabla\nabla \operatorname{tr}_g \tag{A.25k}$$

$$\operatorname{ein}_g \nabla\nabla = 0 \tag{A.25l}$$

Proof.

$$\begin{aligned}
(\operatorname{div}_g \varepsilon v)_{\alpha\beta} &= g^{\alpha\lambda}\partial_\lambda (\varepsilon v)_{\alpha\beta} \\
&= g^{\alpha\lambda}\partial_\lambda \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) \\
&= \frac{1}{2}g^{\alpha\lambda}\partial_\lambda\partial_\alpha v_\beta + \frac{1}{2}g^{\alpha\lambda}\partial_\lambda\partial_\beta v_\alpha\beta \\
&= \frac{1}{2}(\Delta_g v)_\beta + \frac{1}{2}\partial_\beta g^{\alpha\lambda}\partial_\lambda v_\alpha\beta
\end{aligned}$$

$$\begin{aligned}
(\operatorname{div}_g J\varepsilon v)_{\alpha\beta} &= g^{\alpha\lambda}\partial_\lambda (J\varepsilon v)_{\alpha\beta} \\
&= g^{\alpha\lambda}\partial_\lambda (v)_{\alpha\beta} - \frac{1}{2}g^{\alpha\lambda}\partial_\lambda (g \operatorname{tr}_g \varepsilon v)_{\alpha\beta} \\
&= \frac{1}{2}(\Delta_g v)_\beta + \frac{1}{2}\partial_\beta g^{\alpha\lambda}\partial_\lambda v_\alpha\beta - \frac{1}{2}g^{\alpha\lambda}\partial_\lambda (g \operatorname{tr}_g \varepsilon v)_{\alpha\beta} \\
&= \frac{1}{2}(\Delta_g v)_\beta + \frac{1}{2}\partial_\beta g^{\alpha\lambda}\partial_\lambda v_\alpha\beta - \frac{1}{2}g^{\alpha\lambda}\partial_\lambda g_{\alpha\beta}g^{\alpha\beta}\partial_\beta v_\alpha \\
&= \frac{1}{2}(\Delta_g v)_\beta + \frac{1}{2}\partial_\beta g^{\alpha\lambda}\partial_\lambda v_\alpha\beta - \frac{1}{2}\partial_\beta g^{\alpha\lambda}\partial_\lambda g_{\alpha\beta}g^{\alpha\beta}v_\alpha \\
&= \frac{1}{2}(\Delta_g v)_\beta
\end{aligned}$$

$$\begin{aligned}
\operatorname{div} S\nabla\nabla u &= g^{\alpha\lambda}\partial_\lambda \left(S\nabla\nabla u\right)_{\alpha\beta} \\
&= g^{\alpha\lambda}\partial_\lambda \left(\nabla\nabla u\right)_{\alpha\beta} - g^{\alpha\lambda}\partial_\lambda \left(g \operatorname{tr}_g(\nabla\nabla u)\right)_{\alpha\beta} \\
&= g^{\alpha\lambda}\partial_\lambda\partial_\alpha\partial_\beta u - g^{\alpha\lambda}\partial_\lambda g_{\alpha\beta} g^{\mu\nu}\partial_\mu\partial_\nu u \\
&= g^{\alpha\lambda}\partial_\lambda\partial_\alpha\partial_\beta u - \underbrace{g^{\alpha\lambda} g_{\alpha\beta}}_{\delta^\lambda_\beta} g^{\mu\nu}\partial_\lambda\partial_\mu\partial_\nu u \\
&= g^{\alpha\lambda}\partial_\lambda\partial_\alpha\partial_\beta u - g^{\mu\nu}\partial_\beta\partial_\mu\partial_\nu u \\
&= \partial_\beta \left(g^{\alpha\lambda}\partial_\lambda\partial_\alpha - g^{\mu\nu}\partial_\mu\partial_\nu u\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}\operatorname{div} S\varepsilon v &= g^{\alpha\beta}\partial_\alpha \left(\operatorname{div}_g S\varepsilon v\right)_\beta \\
&= g^{\alpha\beta}\partial_\alpha g^{\mu\nu}\partial_\nu \left(S\varepsilon v\right)_{\mu\beta} \\
&= g^{\alpha\beta}\partial_\alpha g^{\mu\nu}\partial_\nu \left(\varepsilon v\right)_{\mu\beta} - g^{\alpha\beta}\partial_\alpha g^{\mu\nu}\partial_\nu \left(g \operatorname{tr}_g(\varepsilon v)\right)_{\mu\beta} \\
&= \frac{1}{2} g^{\alpha\beta}\partial_\alpha g^{\mu\nu}\partial_\nu \left(\partial_\mu v_\beta + \partial_\beta v_\mu\right) - g^{\alpha\beta}\partial_\alpha g^{\mu\nu}\partial_\nu \left(g_{\mu\beta} g^{rs}(\varepsilon v)_{rs}\right) \\
&= \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}\partial_\alpha\partial_\nu \left(\partial_\mu v_\beta + \partial_\beta v_\mu\right) - g^{\alpha\beta} \underbrace{g_{\mu\beta} g^{\mu\nu}}_{\delta^\nu_\beta} \partial_\alpha\partial_\nu \left(g^{rs}(\varepsilon v)_{rs}\right) \\
&= \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}\partial_\alpha\partial_\nu \left(\partial_\mu v_\beta + \partial_\beta v_\mu\right) - \frac{1}{2} g^{\alpha\beta} g^{rs}\partial_\alpha\partial_\beta \left(\partial_r v_s + \partial_s v_r\right) \\
&= \frac{1}{2}\partial^\beta\partial_\nu \left(\partial^\nu v_\beta + \partial_\beta v^\nu\right) - \frac{1}{2}\partial^\beta\partial_\beta \left(\partial^s v_s + \partial_s v^s\right) \\
&= \frac{1}{2}\partial^\beta\partial_\nu \left(\partial^\nu v_\beta\right) - \frac{1}{2}\partial^\beta\partial_\beta \left(\partial^s v_s\right) + \frac{1}{2}\partial^\beta\partial_\nu \left(\partial_\beta v^\nu\right) - \frac{1}{2}\partial^\beta\partial_\beta \left(\partial_s v^s\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Delta_g \operatorname{tr}_g \gamma &= \operatorname{tr}_g \Delta_g \gamma \\
g^{\alpha\beta}\partial_\alpha\partial_\beta \operatorname{tr}_g \gamma &= g^{\alpha\beta}(\Delta_g)_{\alpha\beta}\gamma \\
g^{\alpha\beta}\partial_\alpha\partial_\beta g^{\mu\nu}\gamma_{\mu\nu} &= g^{\alpha\beta}(g^{\mu\nu}\partial_\mu\partial_\nu\gamma_{\alpha\beta})
\end{aligned}$$

we conclude the above renaming the indices.

$$\begin{aligned}
\operatorname{tr}_g \nabla^{\mathbf{S}} v &= \operatorname{div}_g v \\
\frac{1}{2} g^{\alpha\beta}\left(\partial_\alpha v_\beta + \partial_\beta v_\alpha\right) &= \partial^\beta v_\beta \\
\frac{1}{2}\left(\partial^\beta v_\beta + \partial^\alpha v_\alpha\right) &= \partial^\beta v_\beta
\end{aligned}$$

$$\begin{aligned}\left(\nabla^{\mathbf{S}}\operatorname{div}_g g \operatorname{tr}_g \gamma\right)_{\mu\nu} =& \frac{1}{2}\left(\partial_\mu(\operatorname{div}_g \operatorname{tr}_g \gamma)_\nu + \partial_\nu(\operatorname{div}_g \operatorname{tr}_g \gamma)_\mu\right)\\ =& \frac{1}{2}\left(\partial_\mu \partial^\lambda g_{\lambda\nu} g^{\alpha\beta}\gamma_{\alpha\beta} + \text{same with } \mu \text{ and } \nu \text{ switched:}\right)\\ =& \frac{1}{2}\left(\partial_\mu \partial_\nu g^{\alpha\beta}\gamma_{\alpha\beta} + \text{same with } \mu \text{ and } \nu \text{ switched:}\right)\\ =& \partial_\mu\partial_\nu g^{\alpha\beta}\gamma_{\alpha\beta}\\ =& \nabla\nabla \operatorname{tr}_g \gamma\end{aligned}$$

For the next one we observe that $(\nabla\nabla w)_{\alpha\beta} = \partial_{\alpha\beta}$ in index form, and $\operatorname{tr}_g(\nabla\nabla w) = (\nabla\nabla w)^\alpha_\alpha = \partial^\alpha_\alpha w$, therefore the next equation can be written without taking into account the w part

$$(\operatorname{ein}_g \nabla\nabla)_{\mu\nu} = -\partial^\lambda_{\lambda\mu\nu} + \partial^\lambda_{\mu\lambda\nu} + \partial^\lambda_{\nu\lambda\mu} - \partial^\alpha_{\mu\nu\alpha} - g_{\mu\nu}\partial^{\alpha\beta}_{\alpha\beta} + g_{\mu\nu}\partial^{\lambda\alpha}_{\lambda\alpha} = 0 \tag{A.26}$$

For the algebraic oprators it is sufficient to show that th composition results in the identity operator, since it is always the same thing we just need to prove it for just one composition

$$\begin{aligned}J_g J_g^{-1}\gamma =& J_g\left(\gamma - \frac{1}{m-2} g(\operatorname{tr}\gamma)\right)\\ =& \left(\gamma - \frac{\operatorname{tr}\,\gamma}{2} g\right) - \left(\frac{1}{m-2} g(\operatorname{tr}\gamma) - \frac{1}{2}\frac{1}{m-2}(g\operatorname{tr}(g\operatorname{tr}\gamma))\right)\\ =& \gamma - \left(\frac{\operatorname{tr}\,\gamma}{2} g + \frac{1}{m-2} g(\operatorname{tr}\gamma)\right) + \frac{1}{2}\frac{1}{m-2}(g\operatorname{tr}(g\operatorname{tr}\gamma))\\ =& \gamma - \frac{m}{2(m-2)} g\operatorname{tr}\,\gamma + \frac{1}{2}\frac{1}{m-2}(g\operatorname{tr}(g\operatorname{tr}\gamma))\end{aligned}$$

Since

$$\operatorname{tr}_g(g\operatorname{tr}_g\gamma) = g^{\alpha\beta}(g\operatorname{tr}_g\gamma)_{\alpha\beta} = g^{\alpha\beta}g_{\alpha\beta}g^{\mu\nu}\gamma_{\mu\nu} = m\cdot\operatorname{tr}_g\gamma$$

□

Lemma A.1.2. *the* ein_g *operator on a constant background metric* g *can be written in the following compact forms:*

$$2\operatorname{ein}_g = (\Delta \operatorname{tr} - \operatorname{div}\operatorname{div})\, g + 2\varepsilon \operatorname{div}\phi - \nabla\nabla \operatorname{tr}\phi - \Delta\phi \tag{A.27}$$

$$2\operatorname{ein}_g = -J\Delta + 2J\varepsilon \operatorname{div} J \tag{A.28}$$

(A.29)

In particular the 3dimensional case with Euclidean background metric will result in:

$$\operatorname{inc} = (\Delta \operatorname{tr} - \operatorname{div}\operatorname{div})\, I + 2\varepsilon \operatorname{div}\phi - \nabla\nabla \operatorname{tr}\phi - \Delta\phi \tag{A.30}$$

Where $\operatorname{inc} = 2\operatorname{ein}$ *in the 3-dimensional case.*

The equation in A.30 is a consequence of the first equation in A.27. But we will still prove it in a different way ton show how to use properly the δ-of-kronecker.

i. The first equation can be proven easily with:

$$\begin{aligned}(2\,\mathrm{ein}\,\gamma)_{\mu\nu} &= -\partial^\lambda\partial_\lambda\gamma_{\mu\nu} + \partial_\mu\partial^\lambda\gamma_{\lambda\nu} + \partial_\nu\partial^\lambda\gamma_{\lambda\mu} - \partial_\mu\partial_\nu\gamma^\alpha_\alpha - g_{\mu\nu}\partial^\alpha\partial^\beta\gamma_{\alpha\beta} + g_{\mu\nu}\partial^\lambda\partial_\lambda\gamma^\alpha_\alpha \\ &= -\underbrace{\partial_\eta g^{\lambda\eta}\partial_\lambda\gamma_{\mu\nu}}_{\Delta_g\gamma} + \underbrace{\partial_\mu\partial_\eta g^{\lambda\eta}\gamma_{\lambda\nu} + \partial_\nu\partial_\eta g^{\lambda\eta}\gamma_{\lambda\mu}}_{2\nabla^{\mathbf{S}}\operatorname{div}_g\gamma} - \underbrace{\partial_\mu\partial_\nu h_{\alpha\beta}g^{\alpha\beta}}_{\nabla\nabla\operatorname{tr}_g\gamma} \\ &\quad - \underbrace{g_{\mu\nu}\partial_\eta g^{\alpha\eta}\partial_\rho g^{\rho\beta}\gamma_{\alpha\beta}}_{g\operatorname{div}_g\operatorname{div}_g\gamma} + \underbrace{g_{\mu\nu}\partial_\eta g^{\lambda\eta}\partial_\lambda\gamma_{\alpha\beta}g^{\alpha\beta}}_{g\Delta_g\operatorname{tr}_g\gamma}\end{aligned}$$

The second identity is proven using $\Delta_g \operatorname{tr}_g = \operatorname{tr}_g \Delta_g$

$$J\Delta\gamma = \Delta\gamma - g\operatorname{tr}_g\Delta_g\gamma \tag{A.31}$$

$$= \Delta\gamma - \frac{1}{2}g\Delta_g\operatorname{tr}_g\gamma \tag{A.32}$$

$$\tag{A.33}$$

therefore substituting in the identity it reduces to :

$$2\,\mathrm{ein} = -\underbrace{\Delta_g\gamma + \frac{1}{2}g\Delta_g\operatorname{tr}_g\gamma}_{J\Delta_g\gamma} + 2\nabla^{\mathbf{S}}\operatorname{div}_g\gamma - \nabla\nabla\operatorname{tr}_g\gamma - g\operatorname{div}_g\operatorname{div}_g\gamma + \frac{1}{2}g\Delta_g\operatorname{tr}_g\gamma$$

And we need to prove that

$$2J\varepsilon\operatorname{div}J = +2\nabla^{\mathbf{S}}\operatorname{div}_g\gamma - \nabla\nabla\operatorname{tr}_g\gamma - g\operatorname{div}_g\operatorname{div}_g\gamma + \frac{1}{2}g\Delta_g\operatorname{tr}_g\gamma$$

The usual expansion is used to prove it

$$\begin{aligned}(2J\varepsilon\operatorname{div}J\gamma)_{\mu\nu} &= (2\varepsilon\operatorname{div}J\gamma)_{\mu\nu} - g_{\mu\nu}(\operatorname{tr}\varepsilon\operatorname{div}J\gamma) \\ &= (2\varepsilon\operatorname{div}\gamma)_{\mu\nu} - \underbrace{(\varepsilon\operatorname{div}g\operatorname{tr}\gamma)_{\mu\nu}}_{\nabla\nabla\operatorname{tr}\gamma} - g_{\mu\nu}\left(\underbrace{\operatorname{tr}\varepsilon}_{\operatorname{div}}\operatorname{div}\gamma\right) + \frac{1}{2}g_{\mu\nu}(\operatorname{tr}\varepsilon\operatorname{div}g\operatorname{tr}\gamma) \\ &= (2\varepsilon\operatorname{div}\gamma)_{\mu\nu} - \underbrace{(\varepsilon\operatorname{div}g\operatorname{tr}\gamma)_{\mu\nu}}_{\nabla\nabla\operatorname{tr}\gamma} - g_{\mu\nu}\left(\underbrace{\operatorname{tr}\varepsilon}_{\operatorname{div}}\operatorname{div}\gamma\right) + \frac{1}{2}g_{\mu\nu}(\operatorname{tr}\varepsilon\operatorname{div}g\operatorname{tr}\gamma)\end{aligned}$$

□

ii.

$$(\operatorname{inc}\phi)_{ij} = \varepsilon_{imn}\varepsilon_{jkl}\partial_{mk}\phi_{nl} \tag{A.34}$$

$$= \begin{vmatrix} \delta_{ij} & \delta_{ik} & \delta_{il} \\ \delta_{mj} & \delta_{km} & \delta_{lm} \\ \delta_{nj} & \delta_{kn} & \delta_{ln} \end{vmatrix} \partial_{mk}\phi_{nl} \tag{A.35}$$

$$= \{\delta_{ij}(\delta_{km}\delta_{ln} - \delta_{lm}\delta_{kn}) - \delta_{ik}(\delta_{mj}\delta_{ln} - \delta_{lm}\delta_{nj}) + \delta_{il}(\delta_{mj}\delta_{kn} - \delta_{km}\delta_{nj})\}\,\partial_{mk}\phi_{nl} \tag{A.36}$$

$$= \delta_{ij}(\partial_{mm}\phi_{nn} - \partial_{mn}\phi_{mn}) - \delta_{ik}(\partial_{jk}\phi_{nn} - \partial_{mk}\phi_{jm}) + \delta_{il}(\partial_{jk}\phi_{kl} - \partial_{kk}\phi_{jl}) \tag{A.37}$$

$$= \underbrace{\delta_{ij}(\partial_{mm}\phi_{nn} - \partial_{mn}\phi_{mn})}_{\mathrm{I}(\Delta \operatorname{tr}\gamma - \operatorname{div}\operatorname{div}\gamma)} + \underbrace{\partial_{mi}\phi_{jm} + \partial_{jk}\phi_{ki}}_{2\varepsilon \operatorname{div}\gamma} - \underbrace{\partial_{ji}\phi_{nn}}_{\nabla\nabla \operatorname{tr}\gamma} - \underbrace{\partial_{kk}\phi_{ji}}_{\Delta\gamma} \tag{A.38}$$

(A.39)

□

B. Matrix Decomposition and Integral Decomposition

We introduce some notation for later purpose: Let $\mathcal{T}$ be a triangulation and let n be the normal to a face of a triangle $T \in \mathcal{T}$, then we define the normal projection as

$$P_n = n \otimes n$$

and the projection on the tangent space as

$$P_t = Id - P_n.$$

Using the levi-civita tensor

$$\varepsilon_{ijk}$$

and the Einstein summation notation we define the matrix C_u as the matrix rappresentatin of the operator $\cdot \times u$ that acts on v. Then it can be observed that

$$C_u v = v \times u \implies (C_u)_{ij} v_j = [C_u v]_i = [v \times u]_i = \varepsilon_{ijk} v_j u_k$$

since they have to be equal for every v_j then :

$$(C_u)_{ij} = \varepsilon_{ijk} u_k$$

Sometimes this operator is denoted in literature with $skew(u)$. In particular can be seen as

$$C_n^2 = -P_n$$

Proof.

$$\varepsilon_{ikl} n_l \varepsilon_{kjr} n_r = \varepsilon_{ikl} \varepsilon_{kjr} n_r n_l = -\varepsilon_{kil} \varepsilon_{kjr} n_r n_l$$

using

$$\varepsilon_{kil} \varepsilon_{kjr} n_l n_r = \begin{vmatrix} \delta_{ij} & \delta_{ir} \\ \delta_{lj} & \delta_{lr} \end{vmatrix} n_l n_r = \delta_{ij}\delta_{lr} - \delta_{ir}\delta_{lj} = \begin{cases} i = j, l = r, i \neq r \implies & +1 - n_i n_i \\ i = r, j = l, i \neq j \implies & -n_i n_j \end{cases}$$

$$-\varepsilon_{kil} \varepsilon_{kjr} n_r n_l = -(\delta_{ij} - (n \otimes n)_{ij}) = -P_n$$

□

Since the operation $skew(n)$ acts on a vector v as if it were the vector product with n then its action projects v onto the space normal to n. Therefore we expect that

$$C_n P_= 0 = P_t C_n = 0 \text{ and } C_n P_n = P_n C_n = C_n$$

Proof. The first one follows directly form:

$$C_n P_n v = P_n v \times n = (n \otimes n) v \times n$$

the ith entry has:

$$\varepsilon_{ijk}((n \otimes n)v)_j n_k = \varepsilon_{ijk} n_j n_l v_l n_k = (\varepsilon_{ijk} n_j n_k)(n_l v_l) = n \times n(n \cdot v) = 0$$

Similarly:

$$P_n C_n v = P_n v \times n = (n \otimes n)(v \times n) =$$

and each entry:

$$n_i n_j \varepsilon_{jmn} v_m n_n = n_i v_m \varepsilon_{mnj} n_j n_n = n_i(v \cdot (n \times n)) = 0$$

At last : $C_n = IdC_n = P_n C_n + P_t C_n = P_t C_n$ □

We define curl and curl* of a matrix field function as the curl row-wise and resp. column-wise:

$$\begin{cases} (\operatorname{curl} S)_{ij} := \varepsilon_{jmn} \partial_m A_{in} \\ (\operatorname{curl}^* S)_{ij} := \varepsilon_{imn} \partial_m A_{nj} \end{cases}$$

And the inc operator as the composition of the two previous operators:

$$\operatorname{inc} u := \operatorname{curl}^* \operatorname{curl} u = \operatorname{curl} \operatorname{curl}^* u$$

It is easily provable the second equality in the previous line.
For later purpose that
$\langle AB, C\rangle = AB : C = A_{ik} B_{kj} C_{ij} = B_{kj} A_{ik} C_{ij} = B : A'C = \langle B, A'C\rangle$.
Suppose $\Omega = \mathbb{R} \times \mathbb{R}^-$, then the normal would be $n = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, then a matrix field u can be decomposed into the the normal and tangential components in the following way:
A left projection decomposition

$$u = \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{pmatrix} = P_n u + Q_n u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u_{20} & u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ 0 & 0 & 0 \end{pmatrix} \tag{B.1}$$

Similarly a left projection would be equal to:

$$u = \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{pmatrix} = uP_n + uQ_n = \begin{pmatrix} 0 & 0 & u_{02} \\ 0 & 0 & u_{12} \\ 0 & 0 & u_{22} \end{pmatrix} + \begin{pmatrix} u_{00} & u_{01} & 0 \\ u_{10} & u_{11} & 0 \\ u_{20} & u_{21} & 0 \end{pmatrix} \tag{B.2}$$

The curl of the same matrix field can be decomposed into 3 parts:

$$\operatorname{curl} u = \begin{pmatrix} \partial_1 u_{02} - \partial_2 u_{01} & \partial_2 u_{00} - \partial_0 u_{02} & \partial_0 u_{01} - \partial_1 u_{00} \\ \partial_1 u_{12} - \partial_2 u_{11} & \partial_2 u_{10} - \partial_0 u_{12} & \partial_0 u_{11} - \partial_1 u_{10} \\ \partial_1 u_{22} - \partial_2 u_{21} & \partial_2 u_{20} - \partial_0 u_{22} & \partial_0 u_{21} - \partial_1 u_{20} \end{pmatrix} \tag{B.3}$$

Remark B.0.1. *In sight of the next chapter in the appendix* **??** *we point out that in this particular case the only possible differentiation by parts are possible in the tangent plane, therefore in those entries that have only ∂_0 or ∂_1; In particular if the considered function can be decomposed onto the normal and the tangential parts then it is always a good starting point to do it. (??)*

In case we consider the curl of the single part of the decomposition B.2 we would obtain:

$$\operatorname{curl}(uP)\,Q = \begin{pmatrix} \partial_1 u_{02} & -\partial_0 u_{02} & 0 \\ \partial_1 u_{12} & -\partial_0 u_{12} & 0 \\ \partial_1 u_{22} & -\partial_0 u_{22} & 0 \end{pmatrix} \tag{B.4a}$$

$$\operatorname{curl}(uQ)\,P = \begin{pmatrix} 0 & 0 & \partial_0 u_{01} - \partial_1 u_{00} \\ 0 & 0 & \partial_0 u_{11} - \partial_1 u_{10} \\ 0 & 0 & \partial_0 u_{21} - \partial_1 u_{20} \end{pmatrix} \tag{B.4b}$$

$$\operatorname{curl}(uQ)\,Q = \begin{pmatrix} -\partial_2 u_{01} & \partial_2 u_{00} & 0 \\ -\partial_2 u_{11} & \partial_2 u_{10} & 0 \\ -\partial_2 u_{21} & \partial_2 u_{20} & 0 \end{pmatrix} \tag{B.4c}$$

Remark B.0.2. *From equation B.4b it is easy to see that any piece-wise function defined on a triangualtion that is t-t continuous has $Q\operatorname{curl}(\gamma)P$ part that is continuous.*

Remark B.0.3. *It is important to remember that we are working on sub-simplices, therefore one can assume the operator $A \in \{P_n, Q_n, C_n\}$ to be constant, but in general for constant matrices the following hold true:*

$$A\operatorname{curl}(\gamma) = \operatorname{curl}(A\gamma) \tag{B.5}$$

But it does not hold true for right applied operators!

With the above remark it is easy to find all the other possible subdivision of the curl applying a left projection, this is helpful in the discussion of the distributional inc Big part of modern finite element methods deal with finding the appropriate spaces where one can build the bilinear form needed in the theory. To do so one needs to inevitably step into the world of integration by part and integral identities of differential operators, most of the time one supposes the functions to be smooth enough, and only when the operator is used in a different setting then some hypothesis about the actual differentiability are stated; therefore in this chapter we always require the treated functions to be "smooth enough".

Theorem B.0.4. *let $\gamma, \phi \in C^\infty(\Omega; M_{d\times d})$ matrix fields defined on the simplex $\Omega \subset \mathbb{R}$, then the following holds true:*

$$\langle \gamma, \operatorname{curl}\phi\rangle_{\mathbb{L}^2(\Omega)} = \langle \operatorname{curl}\gamma, \phi\rangle_{\mathbb{L}^2(\Omega)} + \langle \gamma, \phi C_n\rangle_{\mathbb{L}^2(\partial\Omega)} \tag{B.6}$$

Proof.

$$
\begin{aligned}
\langle \gamma, \operatorname{curl} \phi \rangle_{\mathbb{L}^2(\Omega)} &= \int_\Omega \gamma_{ij} \varepsilon_{jmn} \partial_m \phi_{in} && \text{definition of matrix-curl} \\
&= \int_\Omega -\phi_{in} \varepsilon_{jmn} \partial_m \gamma_{ij} + \partial_m \varepsilon_{jmn} \gamma_{ij} \phi_{in} && \text{by part} \\
&= \int_\Omega +\phi_{in} \varepsilon_{nmj} \partial_m \gamma_{ij} + \int_{\partial\Omega} n_m \varepsilon_{jmn} \gamma_{ij} \phi_{in} && \text{Lemma of Gauss for divergence} \\
&= \langle \phi, \operatorname{curl} \gamma \rangle_{\mathbb{L}^2(\Omega)} + \int_{\partial\Omega} \gamma_{ij} \phi_{in} \varepsilon_{njm} n_m && \text{Rearrange the indices}
\end{aligned}
$$

□

Almost the same theorem holds for the surface setting

Theorem B.0.5. *let $\gamma, \phi \in C^\infty(\Sigma; M_{d\times d})$ matrix fields defined some simplex of dimension 2 embedded in 3 dimensions, $\Sigma \subset \mathbb{R}^3$, then the following holds true:*

$$
\langle \operatorname{curl}(\gamma P)Q, \phi \rangle_{\mathbb{L}^2(\Sigma)} = \langle \gamma, \operatorname{curl}(\phi Q)P \rangle_{\mathbb{L}^2(\Sigma)} - \langle \gamma n, Q\phi t_{\partial\Sigma} \rangle_{\mathbb{L}^2(\partial\Sigma)} \tag{B.7}
$$

Proof.

$$
\begin{aligned}
&\langle \operatorname{curl}(\gamma P), \phi Q \rangle_{\mathbb{L}^2(\Sigma)} \\
&= \int_\Sigma \varepsilon_{jmn} \partial_m (\gamma P)_{in} (\phi Q)_{ij} && \text{definition of matrix-curl} \\
&= \int_\Sigma \varepsilon_{jmn} \partial_m ((\gamma P)_{in} (\phi Q)_{ij}) - \int_\Sigma (\gamma P)_{in} \varepsilon_{jmn} \partial_m (\phi Q)_{ij} && \text{by parts} \\
&= \int_\Sigma \varepsilon_{jmn} \partial_m (\gamma_{ir} n_r n_n (\phi Q)_{ij}) + \int_\Sigma \gamma P : \operatorname{curl}(\phi Q) && \text{def. of curl and P} \\
&= - \int_\Sigma n_n \varepsilon_{nmj} \partial_m (\gamma_{ir} n_r (\phi Q)_{ij}) + \langle \gamma, \operatorname{curl}(\phi Q) P \rangle_{\mathbb{L}^2(\Sigma)} && \text{independence of } n_n \\
&= - \int_\Sigma n \cdot \operatorname{curl}((\phi Q)^T \gamma n) + \langle \gamma, \operatorname{curl}(\phi Q) P \rangle_{\mathbb{L}^2(\Sigma)} && \text{matrix form} \\
&= - \int_{\partial\Sigma} t_{\partial\Sigma}^T (\phi Q)^T \gamma n) + \langle \gamma, \operatorname{curl}(\phi Q) P \rangle_{\mathbb{L}^2(\Sigma)} && \text{curl theorem}
\end{aligned}
$$

We were able to integrate by parts since $\operatorname{curl}(\gamma P)Q$ and $\operatorname{curl}(\gamma Q)P$ are defined in the tangent space of Σ, i.e. are well defined . □

In particular since the operators $curl(\gamma P)Q$ and its twin $curl(\gamma Q)P$ will come out very often then it is useful to give them a name, in particular

$$
\operatorname{curl}_\Sigma \gamma = \operatorname{curl}(\gamma P)Q \tag{B.8a}
$$

$$
\operatorname{rot}_\Sigma \gamma = \operatorname{curl}(\gamma Q)P \tag{B.8b}
$$

The name resembles the notion in section **??** where in the 2-dimensional case the curl reduces the dimensions from 2 to 1 (here we have only the last column with possible non-zero entries), and rot increases the dimension from 1 to 2 (in this case the possible non-empty entries are the first 2).

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