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ORIGINAL ARTICLE

Analytical approach of merging a different number of storage aisle under a fully sequenced order

Michael Eder[1]

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Abstract
This paper presents a method to determine the throughput of a multiple-aisle storage system. The main key feature of this approach is that the sequence of the ordered order is to be strictly adhered and the buffer-slots in front of every storage aisle are considered. Therefore, a merging process with several input streams with limited capacity is modelled. The invented approach is based on a superposition of different queueing systems with limited capacity, such as a M/M/1/K-queueing model. The accuracy of the invented approach is given by a comparison to a discrete event simulation. The approximation quality is very high, through an estimation error of less than 10% for some configurations and less than 2% for over 90% of all examined configurations. An example is presented to show the influence of the number of aisles and their capacity. The result is that the capacity of the aisle has a prime influence on the throughput, especially at a higher number of aisles. This approach serves a decision tool to determine the throughput of a multi-aisle storage system in an accurate and a simple manner.

Keywords Automated warehouses · Queue modelling · Performance analysis · Storage aisle performance

1 Introduction

Warehouses are a very important part of meeting the supply chain's target. To meet this target, different aisle based storage and retrieval systems, such as automated storage and retrieval systems (AS/RS) for example, shuttle based storage and retrieval systems (SBS/RS) are developed. These systems replaced the traditional picker to port warehousing system to save labor and space [1–3]. Normally, storage systems are composed of different aisles which work independently from each other. This can be seen in Fig. 1, where a SBS/RS with three aisles is shown. The performance of such a system is mostly defined as the summation of performing of the separate aisle [4, 5]. This assumption is right if the storage assortment is not too various. If an order can be retrieved out of one aisle, this assumption is right, but if the assortment is very different and, for example, all storage aisles are needed to retrieve the ordered order in the right order, some more considerations are necessary. The fully sequenced retrieval leads to a dependency of the storage aisles among themselves. Further, this causes waiting times in the storage aisles. The aim of this paper is to give a decision tool to evaluate the throughput of a storage system with a different number of physical independent storage aisles with the assumption that the retrieval process has to hold exact the ordered order. A typical usage of such a system can be seen in Fig. 1. Here, a storage system with three aisles is shown with its pre-storage area. The passing from the storage aisles to the pre-storage area is the point that is examined in more detail in this paper.

The basis of this decision tool is a merging process with different incoming streams. These streams are again based on Markov chains with limited capacity. The idea of this is to make a superposition of these Markov chains.

The storage process is not considered in this study because the totes are stored randomly. This leads to an independency over all storage aisles. Because of this, every aisle can be treated without the pre-storage area. This can be done by a simple storage queue with limited capacity, e.g., a M/M/1/K-queueing model.

The paper is organized as follows. Section 2 summarizes the three major topics in the literature that regard parts

✉ Michael Eder
michael.eder@tuwien.ac.at

1 Institute of Engineering Design and Product Development, TU Wien, Lehargasse 6 Obj. 7, Wien, 1060, Austria

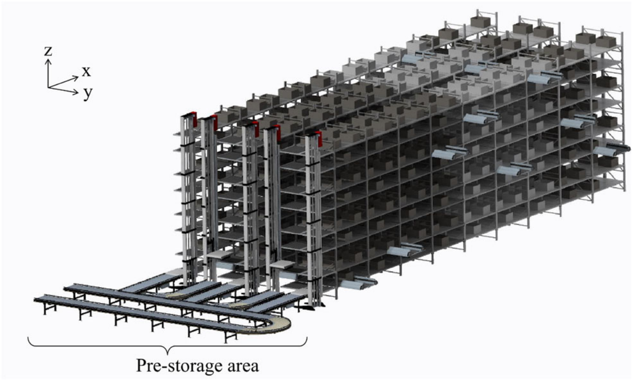

Fig. 1 Pre-storage area of a Shuttle-based storage and retrieval system [6]

of the invented analytical approach. Following Section 3 depicts the description of the discussed system in real. In Section 4, the invented analytical approach is depicted and the main results of the analysis of the depicted queueing system are shown. Section 5 deals with different examples which describe the manner of the discussed system and here also the accuracy of the invented approach is discussed. The conclusion section (Section 6) summarizes the outcomes of this paper and gives a forecast to the future research questing concerning this greater topic. In the Appendix, additional data that was used in the invention phase of the depicted approach is shown.

2 Literature review

There are three different topics in the literature that occur which are interesting to mention in this paper.

The first topic is the merging of different flow streams. The oldest paper that has to be mentioned is from [7]. He gives the basis for the calculation of superposition or merging of different streams. Here Helber [8, 9] deal with push merge operations on production systems with limited buffer capacity. In this publication, a priority ranking between the incoming streams is used. In further publication [10], develop an exact algorithm to compute the throughput of systems with three incoming streams and a shared buffer. There are still more papers in this field of research, e.g., [11–14]. But there is no paper which deals with a merging system that has an underlying order sequence that has to be kept.

The second topic that has to mention is Markov chains with limited capacity. In this field, the oldest paper that has to be mentioned by Rath et al. [15], who discuss a queueing system with a finite waiting room. This M/M/C/K model has several service stations C and a capacity K that has to be higher than the number of service stations. One of the first publications that deal with the analytical solution of the throughput of an open queueing model with limited capacity is from Nagarajan and Towsley [16]. This paper presents an equation that solves the queueing problem simply.

The third topic which is interesting is the usage of queueing model with limited capacity of storage systems and the usage of queueing networks on storage systems to evaluate the performance of a different number of aisles. Queueing models with limited capacity are discussed in the publication by Eder [17–20] to evaluate the throughput of different shuttle based storage and retrieval systems (SBS/RS). He describes on the one hand in [17] that the throughput of a certain number of tiers is not only the summation of the throughput of the single tiers, but there is not more discussion about this topic. He assumes in [18] that the throughput out of a different number of aisles is the summation of the throughput of the single aisles. This is definitely true if an order can be served from an aisle, but if all aisles are involved. This does not have to be true. The other part of the usage of queueing networks takes place in the discussion of storage systems with open queueing networks. Approaches like this can be found, e.g., in [21–24]. These papers deal with different shaping of the queueing networks, but they all try to evaluate the performance of a different number of aisles. These approaches have the disadvantage that they are discrete in time and so a numerical solution has to be found.

This literature overview shows that the evaluation of the throughput out of a different number of aisles with the restriction that the totes have to be in the ordered order sequence has not been discussed before. The idea of this paper is to describe a storage system with a different number of aisles through a merging system with a different number of streams. These streams have one buffer with a limited capacity. In order to meet all these tasks, a certain number of open queueing models with limited capacity should be merged through a service station.

3 System description

The system depicted in this article describes merging the different streams of totes out of storage aisles. This procedure is in the pre-storage area of an automated storage system, as shown in Fig. 1. The ordered tote arrives in a different sequence from the aisles (called stream in this paper) and the following conveyor has to transport the ordered tote in the ordered order to the following stations like a picking station. It differs from the incoming sequence because of the spread of the aisles. Therefore, the throughput of the merging conveyor is not only depending on the performance itself or the performance of the upstream aisles. Here also the sequence of the incoming tote and the buffer space in front of the conveyor have a high influence on the whole throughput. The service sequence of the individual aisles follows the ordering sequence, and this means that all aisles are subject to an exponential operating sequence.

The merging process itself can be described with a combination of a different number of waiting systems with limited capacity, with an intersection at the end of each waiting system. In Fig. 2, the merging process of the different waiting systems is depicted. The capacity of an aisle is 1 for the merging point plus the number of waiting spaces between the merging point and the upstream conveyor.

The assumptions are as follows:

- The service time of the merging point is under a common distribution.
- The orders are distributed equally over all aisles.
- The order has to be retrieved in the ordered order, also known as full sequence of the order.
- The incoming streams arrive under an exponential time distribution.
- All aisles have the same number of buffer in front of the merging point.
- All incoming streams have the same inter-arrival time.
- All influences of the pre-storage area except the merging point are not considered.
- All influences of the aisle except the retrieval stream are not considered.

4 Analytical approach

The analytical approach that is depicted in this section is based on the calculation of the throughput of a waiting system with limited capacity. The underlying assumptions of this queueing system can be seen in [25]. The notations for the analytical approach are given in Table 1

The first part that has to be calculated is the inter-arrival time to the merging point ($t_{interarrival}$).

$$t_{interarrival} = \frac{t_{service\_time\_per\_aisle}}{n_{aisles}} \tag{1}$$

This equation contains the service time of the aisle that is situated before the merging point ($t_{service\_time\_per\_aisle}$) times the number of aisle (n_{aisles}).

The second part of this calculation is the utilization rate of the merging point (ρ).

$$\rho = \frac{t_{interarrival}}{t_{service\_time\_merge}} \tag{2}$$

This part contains the relation of the inter-arrival time ($t_{interarrival}$) to the service time of the merging point ($t_{service\_time\_merge}$).

The third part of this calculation is the capacity of a single aisle of the merging point. It contains the numbers of buffers in the aisle (n_{buffer}).

$$K = 1 + n_{buffer} \tag{3}$$

The throughput of an aisle is according to the throughput equation [26]:

$$\vartheta_{aisle} = \vartheta_{MM1K} = \frac{1}{t_{interarrival}} \cdot \frac{1-\rho^K}{1-\rho^{K+1}} \tag{4}$$

The waiting process of such a waiting system with limited capacity is shown in Fig. 3.

The throughput of the merging point is basically the throughput of a single aisle ϑ_{aisle} times the number of aisles n_{aisles} raised to the exponent of X. The coefficient X represents the performance reduction because of the influence of the fully sequenced order. This performance reduction can be seen in Fig. 4. Here the throughput

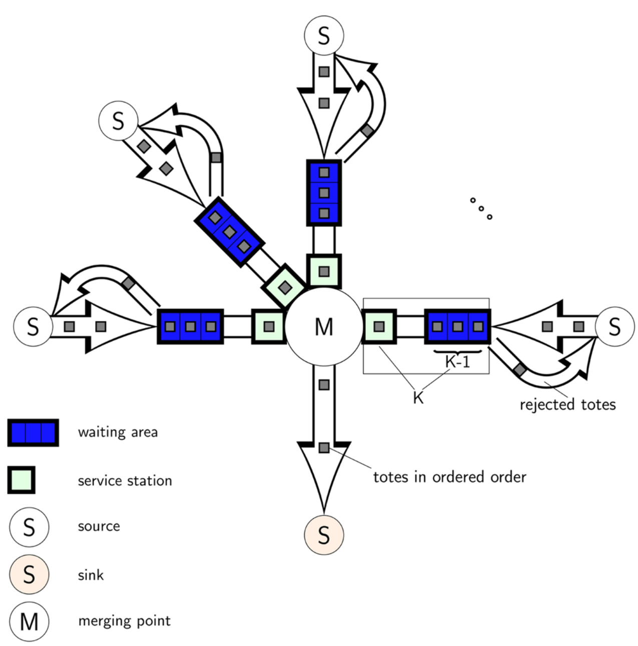

Fig. 2 Merging system with a combination of open queueing systems with limited capacity

Table 1 Notation of the multi-aisle merging system

ρ	Utilization rate of the merging point
ϑ_{aisle}	Throughput of a single aisle
ϑ_{MM1K}	Throughput of a single aisle
ϑ_{system}	Throughput of the merging point
K	Capacity of an aisle
n_{aisles}	Number of aisles in the merging system
n_{buffer}	Number of buffer-slots in each aisle
$t_{interarrival}$	Inter-arrival time to the merging point
$t_{service\_time\_merge}$	Service-time of the merging point
$t_{service\_time\_per\_aisle}$	Service-time of every aisle
X	Coefficient to consider the influence of the full sequence

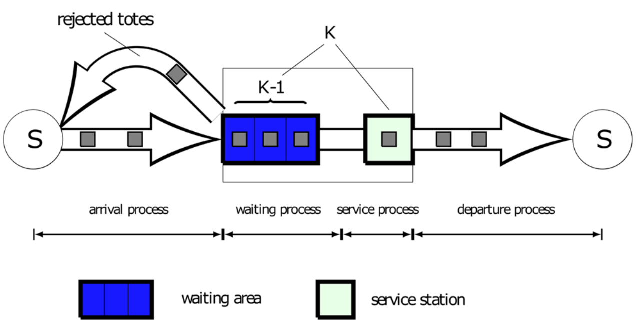

Fig. 3 Open queueing model with limited capacity [17]

of a different number of aisles is depicted to show the influence of the utilization rate of the merging point and a capacity of 2. The underlying data is depicted in Table 2. The last column in this table depicts the result from a merging process with no underlying sequence. This is only a theoretical result with no influence of the different arrival streams among themselves and an infinitely high utilization rate of the merging point. As seen, lower utilization rates deliver a higher influence on the throughput of several aisles against to a higher utilization rate.

The main equation is based on a multiplication of the throughput of an aisle with the number of aisles. To take the full sequence into account the number of aisles has an exponent X. This factor describes the influence of the fully sequenced ordered order.

$$\vartheta_{system} = \vartheta_{aisle} \cdot n_{aisles}^{X} \tag{5}$$

The merging process that is described above is shown in Fig. 2. The number of open queueing systems is not limited to a certain number.

$$X = 1 - \left(1 + \frac{1}{n_{aisles}^{20}}\right) \cdot \frac{1}{K^{\frac{(1-\rho^5)\cdot 25}{(1-\rho^4)\cdot 29}}} \tag{6}$$

This factor X is gained through a comparative analysis with a discrete event simulation. Over 3000 scenarios with in sum over 30,000 individual results were used in this analysis. The basis of this simulation is described in Section 5. The following equation represents the throughput of the merging system based on the number of aisles, the capacity and the utilization rate of the upstream queueing system.

$$\vartheta_{system} = \vartheta_{aisle} \cdot n_{aisles}^{1-\left(1+\frac{1}{n_{aisles}^{20}}\right)\cdot\frac{1}{K^{\frac{(1-\rho^5)\cdot 25}{(1-\rho^4)\cdot 29}}}} \tag{7}$$

5 Numerical study and discussion

The performance of a multiple-aisle storage system is of key importance during the design process of such systems. Understanding the impact of the different number of aisle and the limitation that all totes have to be retrieved in the ordered order (fully sequenced) helps to design an economically and ecologically ideal storage system. Thus, a storage system with a different number of storage aisle has been modelled in a discrete event simulation (DES). Therefore, the inter-arrival time was set to $t_a = 10$ min. The service time of the merging point was set according to the number of aisles (n_a) and the utilization rate (ρ). For example, if there are 5 aisles ($n_a = 5$) and the utilization rate is 2 ($\rho = 2$) the service time of the merging point was set to $t_s = 4$ min. One the first sight this looks false, but the real inter-arrival time of all 5 aisles $\sum t_a = 2$ min. The simulation time was set to $t_{sim} = 200$ h, what means 6000 arrivals of totes. To eliminate the transient phase, the first 1000 arrivals were cut off from the analysis. All

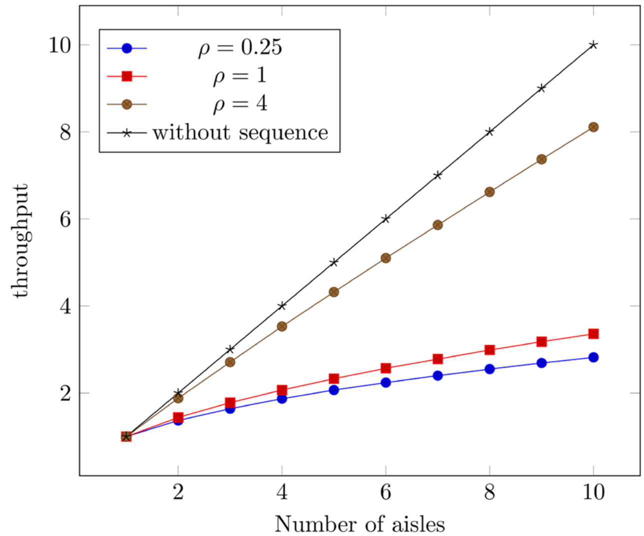

Fig. 4 Throughput of a merging system depending on the capacity of the queueing systems

Table 2 Throughput of a merging system depending on the capacity of the queueing systems

		$\rho = 0,25$	$\rho = 1$	$\rho = 4$	Reference without sequence
Number of aisles (n_{aisle})	1	1,00	1,00	1,00	1,00
	2	1,37	1,44	1,88	2,00
	3	1,64	1,78	2,71	3,00
	4	1,87	2,07	3,53	4,00
	5	2,07	2,33	4,32	5,00
	6	2,24	2,57	5,10	6,00
	7	2,40	2,78	5,86	7,00
	8	2,55	2,99	6,62	8,00
	9	2,69	3,18	7,37	9,00
	10	2,82	3,36	8,11	10,00

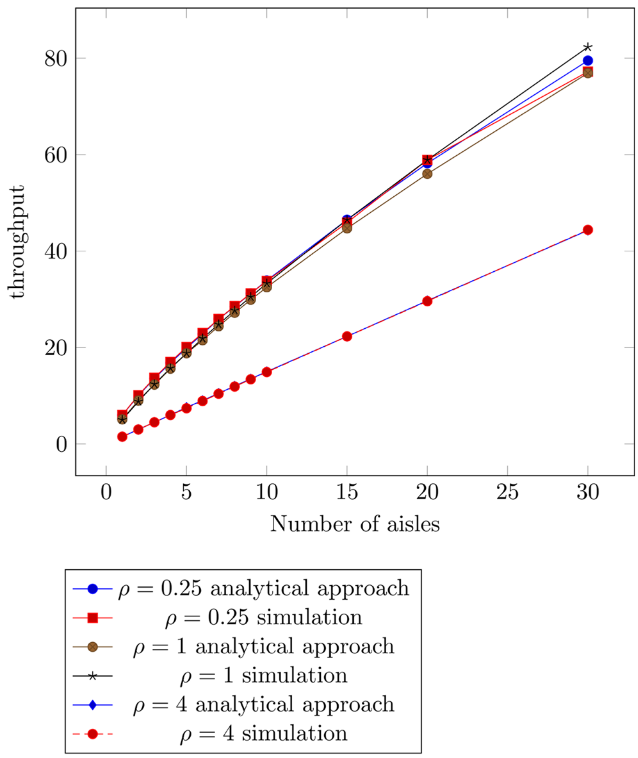

Fig. 5 Throughput according to the presented approach versus discrete event simulation of a merging system with a fixed capacity of $K = 5$

scenarios were made 10 times to get an accurate result. The inter-arrival times and the service times were taken under an exponential distribution. The ordered order was set to a uniform distribution of all aisles. This lead to an approximation of the merging queueing systems with limited capacity as it is shown in Fig. 2.

5.1 Throughput of the storage system among the different number of aisles

The aim of this part is to show the accuracy of the invented approach compared to the DES. The results of three different utilization rates of the merging system over a different number of aisles are visualized in Fig. 5. The data of this figure is depicted in Table 3.

As seen in Fig. 4, the two curves of the results from the invented approach and the results from the discrete event simulation are almost congruent. The estimation error is in the low single digits. The values of the estimation error are depicted in Table 3.

Another thing that can be seen is that the influence of the number of ordered order sequence increases with the number of aisles. This is an expected effect of the fully sequence, because of the higher number of acting sources. Also, higher influence of the sequence can be seen at a lower utilization rate of the merging system. This can be explained because at low utilization rates, the desired aisle can be empty and it takes several time for a new arrival. If the utilization rate is higher, the queueing system is fuller, and an emptiness is less, probably. So the throughput is less depending on the fully sequence.

5.2 Throughput of the storage system among the different number of buffer-slots/capacity of the incoming queueing system

This part addresses the influence of the different capacities of the incoming queueing systems on the throughput of the merging system. The results of three different utilization

Table 3 Throughput according to the presented approach versus discrete event simulation of a merging system with a fixed capacity of $K = 5$

		Analytical approach			Discrete event simulation			Estimation error		
		$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$
Number of aisles	1	$6,0\frac{1}{h}$	$5,1\frac{1}{h}$	$1,5\frac{1}{h}$	$6,0\frac{1}{h}$	$5,0\frac{1}{h}$	$1,5\frac{1}{h}$	−0,6%	−1,5%	−0,2%
(n_{aisle})	2	$10,0\frac{1}{h}$	$9,0\frac{1}{h}$	$3,0\frac{1}{h}$	$10,1\frac{1}{h}$	$8,8\frac{1}{h}$	$3,0\frac{1}{h}$	1,3%	−1,2%	−0,2%
	3	$13,6\frac{1}{h}$	$12,3\frac{1}{h}$	$4,5\frac{1}{h}$	$13,7\frac{1}{h}$	$12,4\frac{1}{h}$	$4,5\frac{1}{h}$	1,0%	0,1%	0,2%
	4	$16,8\frac{1}{h}$	$15,6\frac{1}{h}$	$6,0\frac{1}{h}$	$17,0\frac{1}{h}$	$15,7\frac{1}{h}$	$6,0\frac{1}{h}$	0,9%	0,2%	−0,5%
	5	$19,9\frac{1}{h}$	$18,8\frac{1}{h}$	$7,6\frac{1}{h}$	$20,1\frac{1}{h}$	$18,8\frac{1}{h}$	$7,4\frac{1}{h}$	1,0%	−0,2%	−1,7%
	6	$22,8\frac{1}{h}$	$21,5\frac{1}{h}$	$9,0\frac{1}{h}$	$23,0\frac{1}{h}$	$21,9\frac{1}{h}$	$8,9\frac{1}{h}$	1,1%	1,8%	−1,0%
	7	$25,9\frac{1}{h}$	$24,4\frac{1}{h}$	$10,5\frac{1}{h}$	$25,9\frac{1}{h}$	$24,8\frac{1}{h}$	$10,4\frac{1}{h}$	−0,3%	1,6%	−1,1%
	8	$28,5\frac{1}{h}$	$27,2\frac{1}{h}$	$12,0\frac{1}{h}$	$28,6\frac{1}{h}$	$27,7\frac{1}{h}$	$11,9\frac{1}{h}$	0,2%	1,7%	−0,8%
	9	$31,2\frac{1}{h}$	$29,9\frac{1}{h}$	$13,5\frac{1}{h}$	$31,2\frac{1}{h}$	$30,5\frac{1}{h}$	$13,4\frac{1}{h}$	0,1%	2,2%	−1,1%
	10	$33,9\frac{1}{h}$	$32,5\frac{1}{h}$	$15,0\frac{1}{h}$	$33,8\frac{1}{h}$	$33,3\frac{1}{h}$	$14,9\frac{1}{h}$	−0,3%	2,5%	−0,7%
	15	$46,5\frac{1}{h}$	$44,7\frac{1}{h}$	$22,3\frac{1}{h}$	$45,9\frac{1}{h}$	$46,5\frac{1}{h}$	$22,3\frac{1}{h}$	−1,4%	4,0%	−0,1%
	20	$58,2\frac{1}{h}$	$56,0\frac{1}{h}$	$29,7\frac{1}{h}$	$56,9\frac{1}{h}$	$58,9\frac{1}{h}$	$29,6\frac{1}{h}$	−2,2%	4,9%	−0,2%
	30	$79,5\frac{1}{h}$	$76,9\frac{1}{h}$	$44,3\frac{1}{h}$	$77,2\frac{1}{h}$	$82,3\frac{1}{h}$	$44,4\frac{1}{h}$	−3,0%	6,6%	0,2%

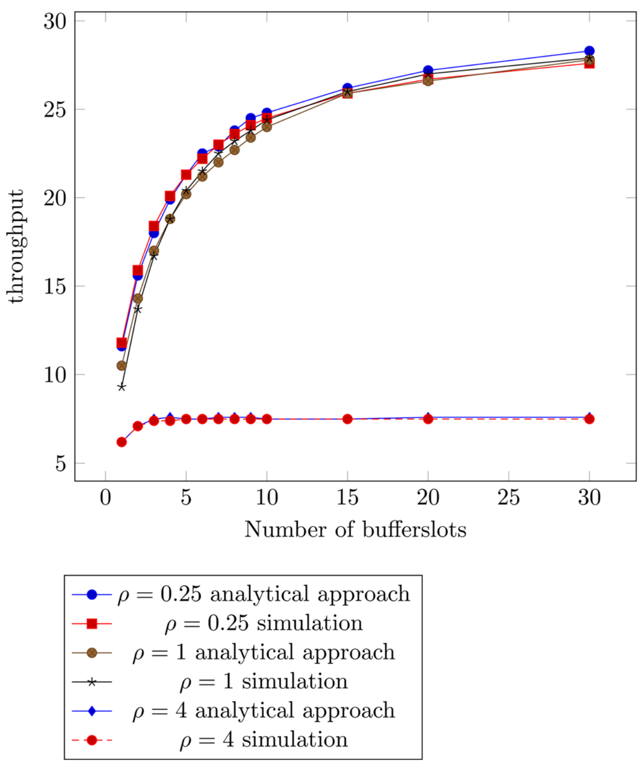

Fig. 6 Throughput according to the presented approach versus discrete event simulation of a merging system with 5 aisles and varied capacity

rates of the merging system over a different number of buffer-slots of the queueing systems are visualized in Fig. 6. The data of this figure is depicted in Table 4.

As seen in Fig. 6 the curves of the invented approach and the curves from the DES are almost congruent, as in Fig. 5. The estimation error is mostly under 10%, only at a utilization rate of $\rho = 1$ and a capacity of the incoming queueing system of $K = 2$ the estimation error is around 12%. The influence of the capacity of the queueing systems is highest on merging systems with a low utilization rate $\rho < 1$. Here, the influence is very up to 10 buffer-slot at each incoming stream. At higher utilization rate, e.g., $\rho = 4$, the influence of the capacity decreases and over a certain number of 3 buffer-slots there is no more effect. Because of this observation, it is very important to know the influence of the number of buffer-slots in the incoming queueing systems to achieve an economic storage system.

5.3 Throughput of the storage system among different number of aisles and different number of buffer-slots

This part of the paper depicts the impact of the number of aisles in combination with the number of buffer-slots in the aisles. In Figs. 7, 8 and 9) the combination of these two parameters is shown.

As seen in these illustrations common form of the shape of the results from the invented approach has a strong similarity. The course of the shape is smothered at utilization rate $\rho = 0.5$ (Fig. 7) and $\rho = 1$ (Fig. 8) as at the higher utilization rate $\rho = 2$ (Fig. 9). This means that the influence of the buffer-slots is higher at low utilization rates, see also Section 5.2. What also can be seen is that with a few aisles, the impact of the buffer-slots is even less than with a high number of aisles. Because of these observations, of the influence of the buffer-slots on the throughput dependent on the number of aisles, the decision in the design process of a storage system of how many buffer-slots should be made, can be grounded on accurate data to avoid too less

Table 4 Throughput according to the presented approach versus discrete event simulation of a merging system with 5 aisles and varied capacity

		Analytical approach			Discrete event simulation			Estimation error		
		$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$
Number of buffer-slots	1	$11,6\frac{1}{h}$	$10,5\frac{1}{h}$	$6,2\frac{1}{h}$	$11,8\frac{1}{h}$	$9,3\frac{1}{h}$	$6,2\frac{1}{h}$	1,5%	– 12,8%	0,0%
(n_{buffer})	2	$15,6\frac{1}{h}$	$14,3\frac{1}{h}$	$7,1\frac{1}{h}$	$15,9\frac{1}{h}$	$13,7\frac{1}{h}$	$7,1\frac{1}{h}$	2,2%	– 4,3%	0,9%
	3	$18,0\frac{1}{h}$	$17,0\frac{1}{h}$	$7,5\frac{1}{h}$	$18,4\frac{1}{h}$	$16,7\frac{1}{h}$	$7,4\frac{1}{h}$	2,2%	– 1,8%	– 1,6%
	4	$19,9\frac{1}{h}$	$18,8\frac{1}{h}$	$7,6\frac{1}{h}$	$20,1\frac{1}{h}$	$18,8\frac{1}{h}$	$7,4\frac{1}{h}$	1,0%	– 0,2%	– 1,7%
	5	$21,3\frac{1}{h}$	$20,2\frac{1}{h}$	$7,5\frac{1}{h}$	$21,3\frac{1}{h}$	$20,4\frac{1}{h}$	$7,5\frac{1}{h}$	0,1%	0,8%	– 0,6%
	6	$22,5\frac{1}{h}$	$21,2\frac{1}{h}$	$7,5\frac{1}{h}$	$22,2\frac{1}{h}$	$21,5\frac{1}{h}$	$7,5\frac{1}{h}$	– 1,0%	1,6%	0,3%
	7	$22,9\frac{1}{h}$	$22,0\frac{1}{h}$	$7,6\frac{1}{h}$	$23,0\frac{1}{h}$	$22,5\frac{1}{h}$	$7,5\frac{1}{h}$	0,2%	2,2%	– 1,4%
	8	$23,8\frac{1}{h}$	$22,7\frac{1}{h}$	$7,6\frac{1}{h}$	$23,6\frac{1}{h}$	$23,2\frac{1}{h}$	$7,5\frac{1}{h}$	– 0,8%	2,3%	– 1,1%
	9	$24,5\frac{1}{h}$	$23,4\frac{1}{h}$	$7,6\frac{1}{h}$	$24,1\frac{1}{h}$	$23,8\frac{1}{h}$	$7,5\frac{1}{h}$	– 1,8%	1,7%	– 1,3%
	10	$24,8\frac{1}{h}$	$24,0\frac{1}{h}$	$7,5\frac{1}{h}$	$24,5\frac{1}{h}$	$24,4\frac{1}{h}$	$7,5\frac{1}{h}$	– 1,3%	1,6%	0,5%
	15	$26,2\frac{1}{h}$	$25,9\frac{1}{h}$	$7,5\frac{1}{h}$	$25,9\frac{1}{h}$	$26,0\frac{1}{h}$	$7,5\frac{1}{h}$	– 1,3%	0,7%	– 0,4%
	20	$27,2\frac{1}{h}$	$26,6\frac{1}{h}$	$7,6\frac{1}{h}$	$26,7\frac{1}{h}$	$27,0\frac{1}{h}$	$7,5\frac{1}{h}$	– 1,6%	1,1%	– 1,0%
	30	$28,3\frac{1}{h}$	$27,8\frac{1}{h}$	$7,6\frac{1}{h}$	$27,6\frac{1}{h}$	$27,9\frac{1}{h}$	$7,5\frac{1}{h}$	– 2,4%	0,4%	– 1,4%

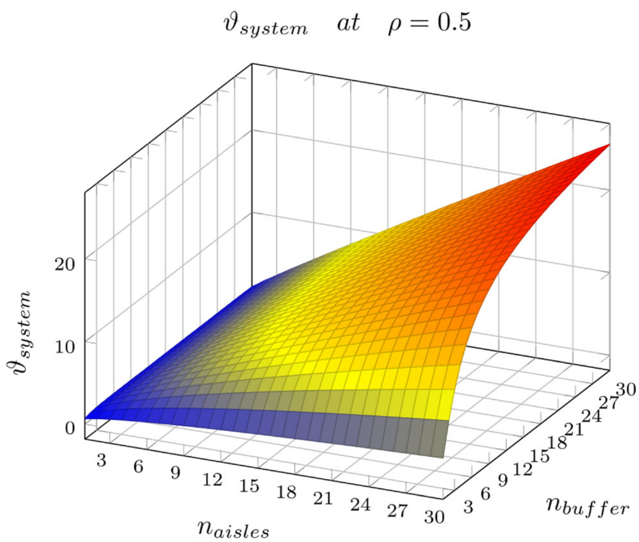

Fig. 7 Throughput according to the presented approach over a different number of aisles and varied capacity at a utilization rate $\rho = 0.5$

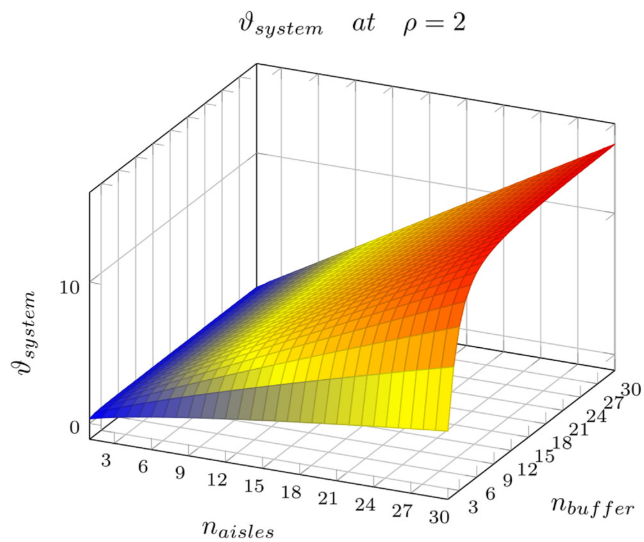

Fig. 9 Throughput according to the presented approach over a different number of aisles and varied capacity at a utilization rate $\rho = 2$

or too much slots. The higher benefit of increasing the number of buffer-slots with a few compared to the benefit if a certain higher number is already available is the result here. The special question in this case is the marginal costs of the additional buffer locations compared to the additional throughput capacity for these locations. This leads to a more economical storage system.

6 Conclusion

The high demands on storage systems in terms of performance and the required storage capacity present the manufacturers of such systems with many challenges. What is special here is the ever-changing selection of articles and thus the ever-growing assortment that is to be stored and then retrieved as required. This means that an order no longer has to be compiled from one storage aisle alone, but also distributed across the entire warehouse. As a result, the performance of the entire storage system is decisive for the retrieval performance. Therefore, it is necessary to know the throughput of the system at a fully ordered order sequence.

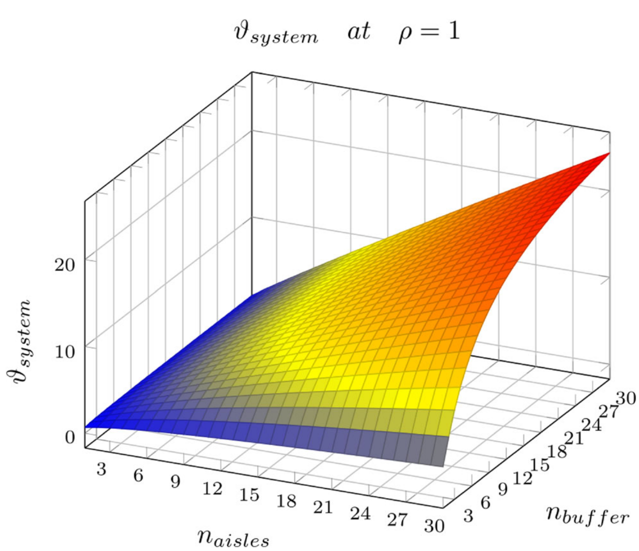

Fig. 8 Throughput according to the presented approach over a different number of aisles and varied capacity at a utilization rate $\rho = 1$

This article presents a method to determine the retrieval throughput of a storage system with a certain number of aisle. The basis of the invented approach is the superposition of queueing systems with limited capacity. This approach takes the dependencies of the aisles among themselves. Therefore, an exponential approach was chosen. The main input parameters are the number of aisles, the throughput parameters of the aisles and the number of buffer-slots in each aisle, respectively, the capacity of the incoming retrieval stream. In the first step, the time distributions are set to an exponential distribution. The approach developed here allows the marginal costs for an additional buffer space to be compared with the gain in additional performance or the gain in independence of the different storage aisles. It is thus possible to design a storage system that achieves the desired performance with the least effort. The main outcome of the investigations shows that the gain in performance with additional buffer locations decreases with the number of them. However, the influence of the buffer areas is greater, especially with a lower utilization rate of the following conveyors than with a higher utilization rate or with planned congestion in front of the pre-storage area.

Further work will be dedicated to storage systems with a common time distribution of the incoming streams from the storage aisles. As an outlook, the invented approach could be advanced to a system where the fully ordered sequence is unnecessary in its full form.

Appendix: Additional numerical data

The additional data listed here should serve as a further explanation of the approximation accuracy. Tables 5, 6, 7 and 8 show the results of the DES versus the analytical calculation. Here are the data for a capacity of 2 (Table 5) and 10 (Table 6) for the different number of axes. However, the data for 2 (Table 7) and 10 (Table 8) axes are given for the different capacities. All these results show that the approximation accuracy is very high because of the estimation error of less than 10% in over 90% of the discussed scenarios. The estimation error is only higher on configurations which have a very low capacity (see Table 5) and have a utilization rate which is around $\rho = 1$. Higher capacities and utilization rates are unequal to $\rho = 1$, lead to an accurate approximation quality.

Table 5 Throughput according to the presented approach versus discrete event simulation of a merging system with a fixed capacity of $K = 2$

		Analytical approach			Discrete event simulation			Estimation error		
		$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$
Number of aisles	1	$5,7\frac{1}{h}$	$4,0\frac{1}{h}$	$1,4\frac{1}{h}$	$5,7\frac{1}{h}$	$4,0\frac{1}{h}$	$1,4\frac{1}{h}$	– 0,4%	– 0,3%	0,2%
(n_{aisle})	2	$7,8\frac{1}{h}$	$6,3\frac{1}{h}$	$2,8\frac{1}{h}$	$7,8\frac{1}{h}$	$5,8\frac{1}{h}$	$2,7\frac{1}{h}$	0,6%	– 9,1%	– 2,5%
	3	$9,3\frac{1}{h}$	$8,0\frac{1}{h}$	$3,9\frac{1}{h}$	$9,4\frac{1}{h}$	$7,1\frac{1}{h}$	$3,9\frac{1}{h}$	0,8%	– 12,0%	– 1,8%
	4	$10,5\frac{1}{h}$	$9,4\frac{1}{h}$	$5,1\frac{1}{h}$	$10,7\frac{1}{h}$	$8,3\frac{1}{h}$	$5,0\frac{1}{h}$	1,5%	– 13,5%	– 1,3%
	5	$11,6\frac{1}{h}$	$10,5\frac{1}{h}$	$6,2\frac{1}{h}$	$11,8\frac{1}{h}$	$9,3\frac{1}{h}$	$6,2\frac{1}{h}$	1,5%	– 12,8%	0,0%
	6	$12,8\frac{1}{h}$	$11,6\frac{1}{h}$	$7,1\frac{1}{h}$	$12,8\frac{1}{h}$	$10,3\frac{1}{h}$	$7,3\frac{1}{h}$	0,1%	– 13,4%	3,2%
	7	$13,5\frac{1}{h}$	$12,6\frac{1}{h}$	$8,0\frac{1}{h}$	$13,7\frac{1}{h}$	$11,1\frac{1}{h}$	$8,4\frac{1}{h}$	1,5%	– 12,9%	4,0%
	8	$14,6\frac{1}{h}$	$13,3\frac{1}{h}$	$8,8\frac{1}{h}$	$14,6\frac{1}{h}$	$11,9\frac{1}{h}$	$9,5\frac{1}{h}$	– 0,1%	– 11,6%	6,5%
	9	$15,3\frac{1}{h}$	$14,3\frac{1}{h}$	$9,7\frac{1}{h}$	$15,4\frac{1}{h}$	$12,7\frac{1}{h}$	$10,5\frac{1}{h}$	0,4%	– 12,5%	7,9%
	10	$16,2\frac{1}{h}$	$15,1\frac{1}{h}$	$10,4\frac{1}{h}$	$16,1\frac{1}{h}$	$13,4\frac{1}{h}$	$11,6\frac{1}{h}$	– 0,6%	– 12,0%	10,4%
	15	$19,8\frac{1}{h}$	$18,5\frac{1}{h}$	$14,1\frac{1}{h}$	$19,4\frac{1}{h}$	$16,6\frac{1}{h}$	$16,7\frac{1}{h}$	– 2,1%	– 11,4%	15,7%
	20	$22,1\frac{1}{h}$	$21,3\frac{1}{h}$	$17,2\frac{1}{h}$	$22,1\frac{1}{h}$	$19,3\frac{1}{h}$	$21,8\frac{1}{h}$	– 0,4%	– 9,8%	21,1%
	30	$27,1\frac{1}{h}$	$26,5\frac{1}{h}$	$22,0\frac{1}{h}$	$26,5\frac{1}{h}$	$23,9\frac{1}{h}$	$31,5\frac{1}{h}$	– 2,5%	– 10,7%	30,0%

Table 6 Throughput according to the presented approach versus discrete event simulation of a merging system with a fixed capacity of $K = 10$

		Analytical approach			Discrete event simulation			Estimation error		
		$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$
Number of aisles	1	$6,0\frac{1}{h}$	$5,4\frac{1}{h}$	$1,5\frac{1}{h}$	$6,0\frac{1}{h}$	$5,5\frac{1}{h}$	$1,5\frac{1}{h}$	0,3%	0,3%	– 1,0%
(n_{aisle})	2	$10,9\frac{1}{h}$	$10,2\frac{1}{h}$	$3,0\frac{1}{h}$	$10,9\frac{1}{h}$	$10,3\frac{1}{h}$	$3,0\frac{1}{h}$	0,4%	0,5%	– 0,2%
	3	$15,6\frac{1}{h}$	$14,9\frac{1}{h}$	$4,6\frac{1}{h}$	$15,5\frac{1}{h}$	$14,9\frac{1}{h}$	$4,5\frac{1}{h}$	– 1,0%	0,2%	– 1,7%
	4	$20,1\frac{1}{h}$	$19,1\frac{1}{h}$	$6,1\frac{1}{h}$	$19,9\frac{1}{h}$	$19,4\frac{1}{h}$	$6,0\frac{1}{h}$	– 1,1%	1,6%	– 0,9%
	5	$24,5\frac{1}{h}$	$23,4\frac{1}{h}$	$7,6\frac{1}{h}$	$24,1\frac{1}{h}$	$23,8\frac{1}{h}$	$7,5\frac{1}{h}$	– 1,8%	1,7%	– 1,3%
	6	$28,5\frac{1}{h}$	$27,9\frac{1}{h}$	$9,1\frac{1}{h}$	$28,2\frac{1}{h}$	$28,2\frac{1}{h}$	$9,0\frac{1}{h}$	– 1,2%	1,1%	– 1,2%
	7	$32,5\frac{1}{h}$	$31,7\frac{1}{h}$	$10,7\frac{1}{h}$	$32,2\frac{1}{h}$	$32,4\frac{1}{h}$	$10,5\frac{1}{h}$	– 0,9%	2,4%	– 1,5%
	8	$36,9\frac{1}{h}$	$35,3\frac{1}{h}$	$12,0\frac{1}{h}$	$36,1\frac{1}{h}$	$36,7\frac{1}{h}$	$12,0\frac{1}{h}$	– 2,0%	3,9%	– 0,2%
	9	$40,9\frac{1}{h}$	$39,9\frac{1}{h}$	$13,7\frac{1}{h}$	$40,0\frac{1}{h}$	$40,8\frac{1}{h}$	$13,5\frac{1}{h}$	– 2,3%	2,3%	– 1,3%
	10	$44,8\frac{1}{h}$	$43,6\frac{1}{h}$	$15,1\frac{1}{h}$	$43,8\frac{1}{h}$	$45,0\frac{1}{h}$	$15,0\frac{1}{h}$	– 2,2%	3,0%	– 0,8%
	15	$63,9\frac{1}{h}$	$62,1\frac{1}{h}$	$22,4\frac{1}{h}$	$62,2\frac{1}{h}$	$65,2\frac{1}{h}$	$22,5\frac{1}{h}$	– 2,8%	4,8%	0,3%
	20	$82,7\frac{1}{h}$	$80,2\frac{1}{h}$	$30,3\frac{1}{h}$	$79,7\frac{1}{h}$	$84,9\frac{1}{h}$	$30,0\frac{1}{h}$	– 3,8%	5,5%	– 1,1%
	30	$117,9\frac{1}{h}$	$114,7\frac{1}{h}$	$45,4\frac{1}{h}$	$113,1\frac{1}{h}$	$123,1\frac{1}{h}$	$44,9\frac{1}{h}$	– 4,2%	6,9%	– 1,0%

Table 7 Throughput according to the presented approach versus discrete event simulation of a merging system with 2 aisles and varied capacity

		Analytical approach			Discrete event simulation			Estimation error		
		$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$
Number of buffer-slots	1	$7,8\frac{1}{h}$	$6,3\frac{1}{h}$	$2,8\frac{1}{h}$	$7,8\frac{1}{h}$	$5,8\frac{1}{h}$	$2,7\frac{1}{h}$	0,6%	– 9,1%	– 2,5%
(n_{buffer})	2	$8,9\frac{1}{h}$	$7,6\frac{1}{h}$	$2,9\frac{1}{h}$	$9,1\frac{1}{h}$	$7,3\frac{1}{h}$	$2,9\frac{1}{h}$	1,3%	– 4,7%	– 0,4%
	3	$9,5\frac{1}{h}$	$8,4\frac{1}{h}$	$3,1\frac{1}{h}$	$9,7\frac{1}{h}$	$8,2\frac{1}{h}$	$3,0\frac{1}{h}$	1,7%	– 2,0%	– 2,6%
	4	$10,0\frac{1}{h}$	$9,0\frac{1}{h}$	$3,0\frac{1}{h}$	$10,1\frac{1}{h}$	$8,8\frac{1}{h}$	$3,0\frac{1}{h}$	1,3%	– 1,2%	– 0,2%
	5	$10,2\frac{1}{h}$	$9,4\frac{1}{h}$	$3,0\frac{1}{h}$	$10,4\frac{1}{h}$	$9,3\frac{1}{h}$	$3,0\frac{1}{h}$	1,1%	– 1,0%	– 1,1%
	6	$10,5\frac{1}{h}$	$9,8\frac{1}{h}$	$3,0\frac{1}{h}$	$10,5\frac{1}{h}$	$9,6\frac{1}{h}$	$3,0\frac{1}{h}$	0,5%	– 1,5%	– 0,2%
	7	$10,7\frac{1}{h}$	$9,9\frac{1}{h}$	$3,1\frac{1}{h}$	$10,7\frac{1}{h}$	$9,9\frac{1}{h}$	$3,0\frac{1}{h}$	0,2%	0,3%	– 2,2%
	8	$10,8\frac{1}{h}$	$10,1\frac{1}{h}$	$3,0\frac{1}{h}$	$10,8\frac{1}{h}$	$10,1\frac{1}{h}$	$3,0\frac{1}{h}$	– 0,1%	– 0,1%	0,2%
	9	$10,9\frac{1}{h}$	$10,2\frac{1}{h}$	$3,0\frac{1}{h}$	$10,9\frac{1}{h}$	$10,3\frac{1}{h}$	$3,0\frac{1}{h}$	0,4%	0,5%	– 0,2%
	10	$10,9\frac{1}{h}$	$10,5\frac{1}{h}$	$3,1\frac{1}{h}$	$11,0\frac{1}{h}$	$10,4\frac{1}{h}$	$3,0\frac{1}{h}$	0,8%	– 0,3%	– 2,3%
	15	$11,1\frac{1}{h}$	$11,0\frac{1}{h}$	$3,0\frac{1}{h}$	$11,3\frac{1}{h}$	$10,9\frac{1}{h}$	$3,0\frac{1}{h}$	1,2%	– 0,7%	– 0,7%
	20	$11,5\frac{1}{h}$	$11,1\frac{1}{h}$	$3,0\frac{1}{h}$	$11,4\frac{1}{h}$	$11,2\frac{1}{h}$	$3,0\frac{1}{h}$	– 0,7%	0,1%	– 0,9%
	30	$11,7\frac{1}{h}$	$11,5\frac{1}{h}$	$3,0\frac{1}{h}$	$11,6\frac{1}{h}$	$11,4\frac{1}{h}$	$3,0\frac{1}{h}$	– 0,8%	– 0,7%	– 0,6%

Table 8 Throughput according to the presented approach versus discrete event simulation of a merging system with 10 aisles and varied capacity

		Analytical approach			Discrete event simulation			Estimation error		
		$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$	$\rho = 0,25$	$\rho = 1$	$\rho = 4$
Number of buffer-slots	1	$16,2\frac{1}{h}$	$15,1\frac{1}{h}$	$10,4\frac{1}{h}$	$16,1\frac{1}{h}$	$13,4\frac{1}{h}$	$11,6\frac{1}{h}$	– 0,6%	– 12,0%	10,4%
(n_{buffer})	2	$23,8\frac{1}{h}$	$22,7\frac{1}{h}$	$13,6\frac{1}{h}$	$24,3\frac{1}{h}$	$22,2\frac{1}{h}$	$14,1\frac{1}{h}$	2,1%	– 2,0%	3,2%
	3	$29,5\frac{1}{h}$	$28,6\frac{1}{h}$	$14,7\frac{1}{h}$	$29,9\frac{1}{h}$	$28,6\frac{1}{h}$	$14,7\frac{1}{h}$	1,3%	0,2%	0,1%
	4	$33,9\frac{1}{h}$	$32,5\frac{1}{h}$	$15,0\frac{1}{h}$	$33,8\frac{1}{h}$	$33,3\frac{1}{h}$	$14,9\frac{1}{h}$	– 0,3%	2,5%	– 0,7%
	5	$37,3\frac{1}{h}$	$35,7\frac{1}{h}$	$15,2\frac{1}{h}$	$36,8\frac{1}{h}$	$36,8\frac{1}{h}$	$14,9\frac{1}{h}$	– 1,4%	3,2%	– 1,7%
	6	$39,4\frac{1}{h}$	$38,3\frac{1}{h}$	$15,1\frac{1}{h}$	$39,1\frac{1}{h}$	$39,6\frac{1}{h}$	$15,0\frac{1}{h}$	– 0,8%	3,2%	– 0,6%
	7	$41,4\frac{1}{h}$	$40,3\frac{1}{h}$	$15,1\frac{1}{h}$	$41,0\frac{1}{h}$	$41,7\frac{1}{h}$	$15,0\frac{1}{h}$	– 1,1%	3,5%	– 0,5%
	8	$43,5\frac{1}{h}$	$42,0\frac{1}{h}$	$15,2\frac{1}{h}$	$42,5\frac{1}{h}$	$43,5\frac{1}{h}$	$15,0\frac{1}{h}$	– 2,3%	3,6%	– 1,1%
	9	$44,8\frac{1}{h}$	$43,6\frac{1}{h}$	$15,1\frac{1}{h}$	$43,8\frac{1}{h}$	$45,0\frac{1}{h}$	$15,0\frac{1}{h}$	– 2,2%	3,0%	– 0,8%
	10	$46,1\frac{1}{h}$	$44,4\frac{1}{h}$	$15,1\frac{1}{h}$	$44,9\frac{1}{h}$	$46,2\frac{1}{h}$	$15,0\frac{1}{h}$	– 2,6%	3,9%	– 0,7%
	15	$50,0\frac{1}{h}$	$49,4\frac{1}{h}$	$15,2\frac{1}{h}$	$48,7\frac{1}{h}$	$50,3\frac{1}{h}$	$15,0\frac{1}{h}$	– 2,8%	1,8%	– 1,4%
	20	$52,1\frac{1}{h}$	$51,4\frac{1}{h}$	$15,1\frac{1}{h}$	$50,8\frac{1}{h}$	$52,5\frac{1}{h}$	$15,0\frac{1}{h}$	– 2,5%	2,1%	– 0,6%
	30	$54,4\frac{1}{h}$	$54,3\frac{1}{h}$	$15,1\frac{1}{h}$	$53,3\frac{1}{h}$	$54,9\frac{1}{h}$	$15,0\frac{1}{h}$	– 2,1%	1,1%	– 0,5%

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Declarations

Conflict of Interests The authors declare no competing interests.

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