Singular trajectories in laminar-turbulent transition

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Introduction

❖ **Boundary layer theory (high-Re asymptotics)**

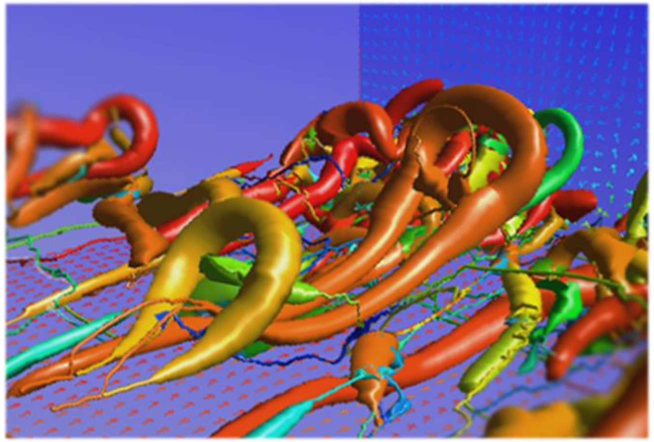
- simplification of Navier-Stokes equations for locally separated flows [Braun, 2004]

⇒ **Fisher-(KPP) equation** [Fisher, 1937]

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial z^2} = u - u^2$$

z … spanwise direction
t … time
u … vorticity / wall shear stress

- inherits the dynamics of hairpin vortex generation (induced transition to turbulence)



Vortices near the point of laminar turbulent transition [Rist, 2022]

❖ **Propagation of singularities $z_p(t)$**

- study their motion after finite time blow-up of u
- marks the creation and the trajectory of the first vortex kernel (at the wall)
- use method of matched asymptotic expansions

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Blow-up and pair generation

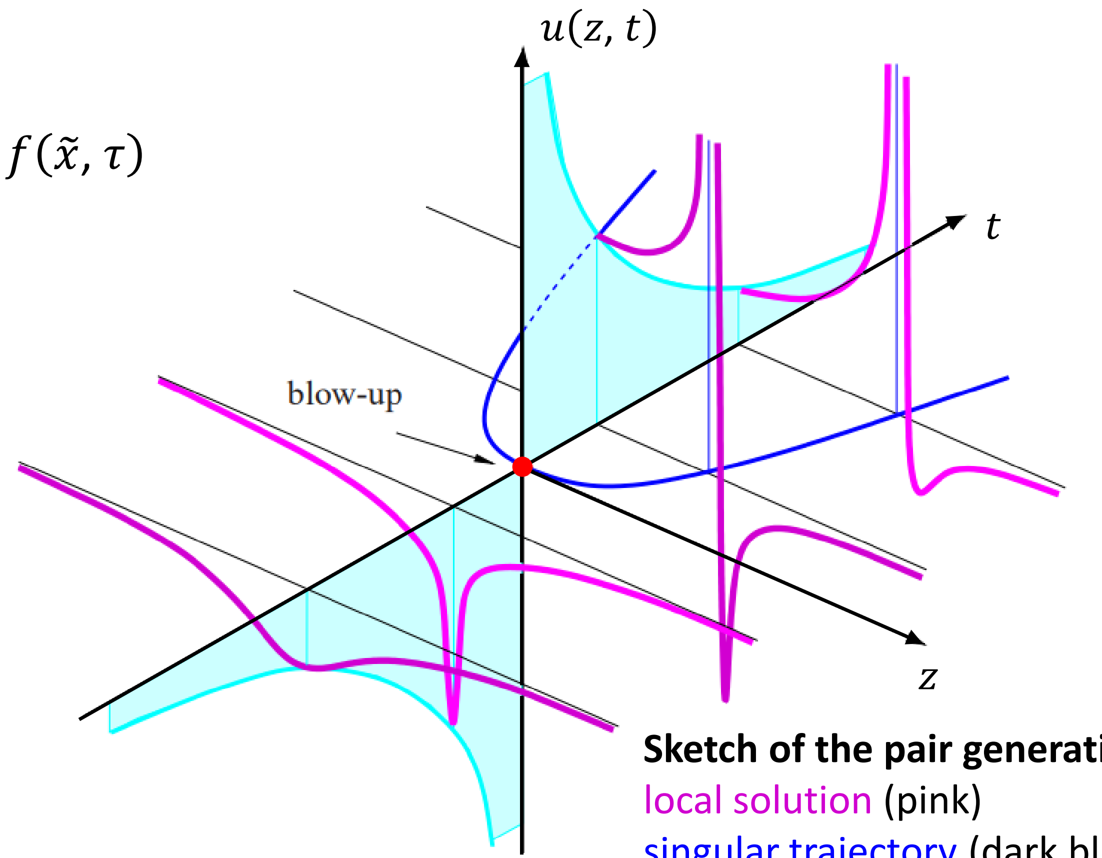
❖ **Finite time blow-up** at $[z_s, t_s]$ satisfies $u \sim \dfrac{1}{t - t_s} f(\tilde{x}, \tau)$

- blow-up variable $\tilde{x} = \frac{z - z_s}{\sqrt{\tau |t - t_s|}}$,
- regulation on an exponentially short time scale [Hocking, 1972]

$$\tau = -\ln|t - t_s| \sim O(1)$$

- extension after t_s through symmetry

$$u(z, t_s - t) \sim -u(iz, t - t_s)$$



Sketch of the pair generation:
local solution (pink)
singular trajectory (dark blue)

❖ **Asymptotic solution**, expansion in τ [Braun, 2004]

$$f(\tilde{x}, \tau) \sim \frac{8}{8 \pm \tilde{x}^2} \mp \frac{10\tilde{x}^2}{(8 \pm \tilde{x}^2)^2} \frac{\ln(\tau)}{\tau} + \frac{16 \mp C_1 \tilde{x}^2 \pm 8\tilde{x}^2 \ln|8 - \tilde{x}^2|}{(8 \pm \tilde{x}^2)^2} \frac{1}{\tau} + o\left(\frac{1}{\tau}\right), \qquad t - t_s \to 0^{\mp}$$

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Singular trajectory expansion

❖ **Expansion after t_s**

$$f(\tilde{x},\tau) \sim \boxed{\frac{8}{8-\tilde{x}^2}} + \boxed{\frac{10\tilde{x}^2}{(8-\tilde{x}^2)^2}\frac{\ln(\tau)}{\tau}} + \frac{16 + C_1\tilde{x}^2 - 8\tilde{x}^2\ln|8-\tilde{x}^2|}{(8-\tilde{x}^2)^2}\frac{1}{\tau} + o\left(\frac{1}{\tau}\right), \qquad \tau \to \infty$$

breaks down near the singularity $\tilde{x} = \pm\sqrt{8}$

❖ **Method of matched asymptotic expansion**

- all orders are of equal magnitude $\Rightarrow$ first inner scaling $\left(\tilde{x} \mp \sqrt{8}\right) = \gamma\,\frac{\ln(\tau)}{\tau}$
- zoom in towards the breakdown area $\gamma \sim O(1)$ $\Rightarrow$ first asymptotic sublayer
- match expansions in the overlap area

❖ **Stepwise correction of singularity motion**

$$\tilde{x}_p \sim \pm\sqrt{8}\left[1 - \frac{1}{8}\frac{\ln(\tau)}{\tau} + \frac{\zeta_s}{4}\frac{1}{\tau} + o\left(\frac{1}{\tau}\right)\right]$$

- Second sublayer $g(\zeta,\tau)$ is non-analytic
- Connection between ζ_s and C_1 requires numerical treatment

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Second sublayer

❖ **Study leading order eq.** $$\frac{d^2 g}{d\zeta^2} + \frac{dg}{d\zeta} = g^2$$

$$g \sim \begin{cases} \boxed{-\frac{1}{\zeta} + \frac{\ln(\zeta^2)}{\zeta^2}} + \frac{F_1}{\zeta^2}, & \zeta \to \pm\infty \\ \boxed{\frac{6}{(\zeta-\zeta_s)^2} - \frac{6/5}{\zeta-\zeta_s}} - \frac{1}{50}, & \zeta - \zeta_s \to 0 \end{cases}$$

- matching $\Rightarrow$ $F_1(C_1)$ same at both far field areas
- translation invariance of g $\Rightarrow$ $F_1 + \zeta_s = A$

❖ **Separate hindering terms** (w.l.o.g. set $\zeta_s = 0$):

$$g = f + \frac{6}{\zeta^2} - \frac{6/5}{\zeta} + \frac{\zeta/5}{\zeta^2+1} + \frac{\ln[\zeta^2+1]}{\zeta^2+1}$$

❖ **Computation of f with Chebyshev collocation**

- Spatial mapping $(-\infty,\infty) \to [-1,1]$

$$\zeta = B\tan\left(\frac{\pi}{2}s\right)$$

- Gauss-Lobatto points:

$$s_j = \cos\left(\frac{j\pi}{n}\right), \qquad n \in [0,\dots,N]$$

- Access to F_1 through derivative

$$A = \frac{2B^2}{\pi^2}\frac{d^2 f}{ds^2}(\pm 1)$$

- Boundary/Interior conditions

$$\left.\frac{d^2 f}{ds^2}\right|_{s=-1}^{1} = 0, \qquad f(0) = -\frac{1}{50}$$

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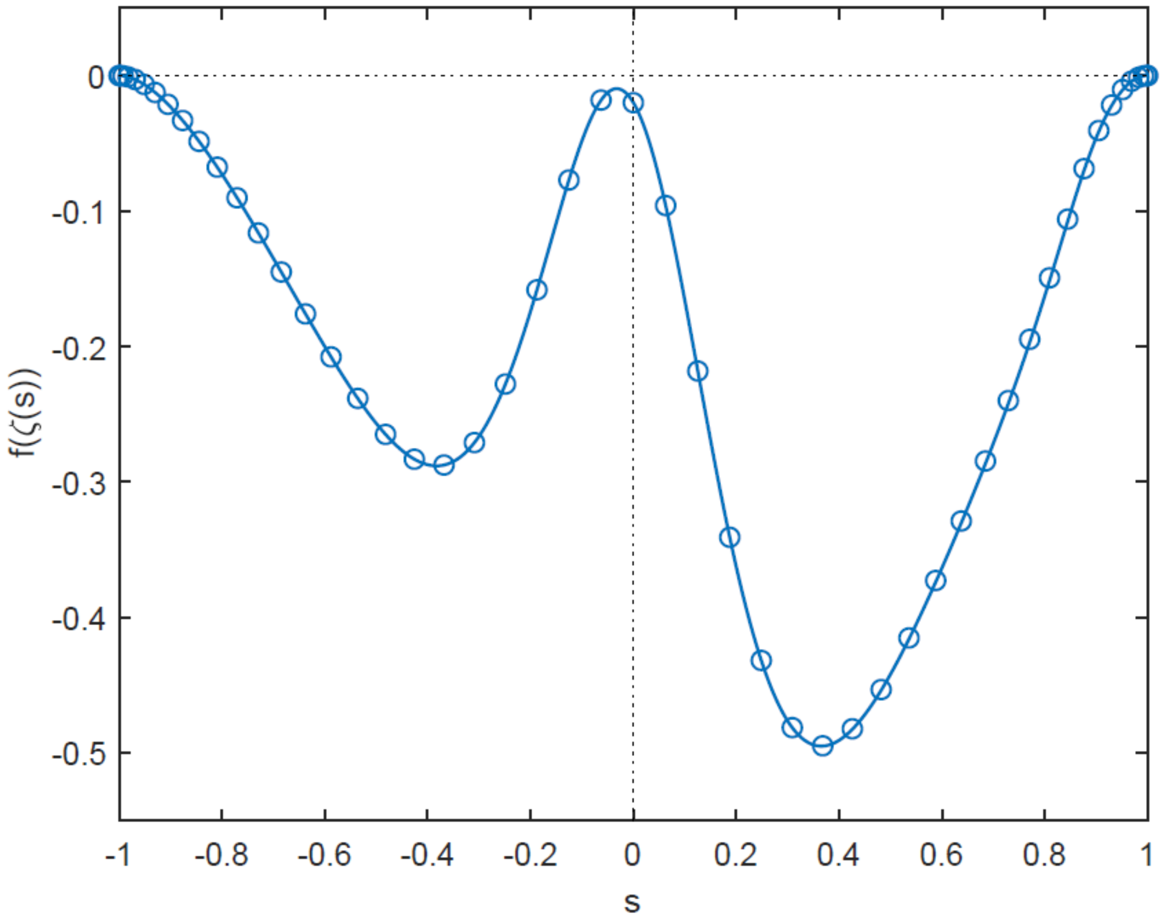
Second sublayer



$f(\zeta(s))$ for $N = 50$ Gauss-Lobatto points (circles), $B = 2$.

❖ **Computation of f with Chebyshev collocation**

- Spatial mapping $(-\infty, \infty) \rightarrow [-1,1]$

$$\zeta = B \tan\left(\frac{\pi}{2} s\right)$$

- Gauss-Lobatto points:

$$s_j = \cos\left(\frac{j\pi}{n}\right), \qquad n \in [0, \dots, N]$$

- Access to F_1 through derivative

$$A = \frac{2B^2}{\pi^2} \frac{d^2 f}{ds^2}(\pm 1) = F_1 + \zeta_s \approx -0.056$$

- Boundary/Interior conditions

$$\left.\frac{d^2 f}{ds^2}\right|_{s=-1}^{1} = 0, \qquad f(0) = -\frac{1}{50}$$

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Singular trajectory expansion

❖ **Final result:**

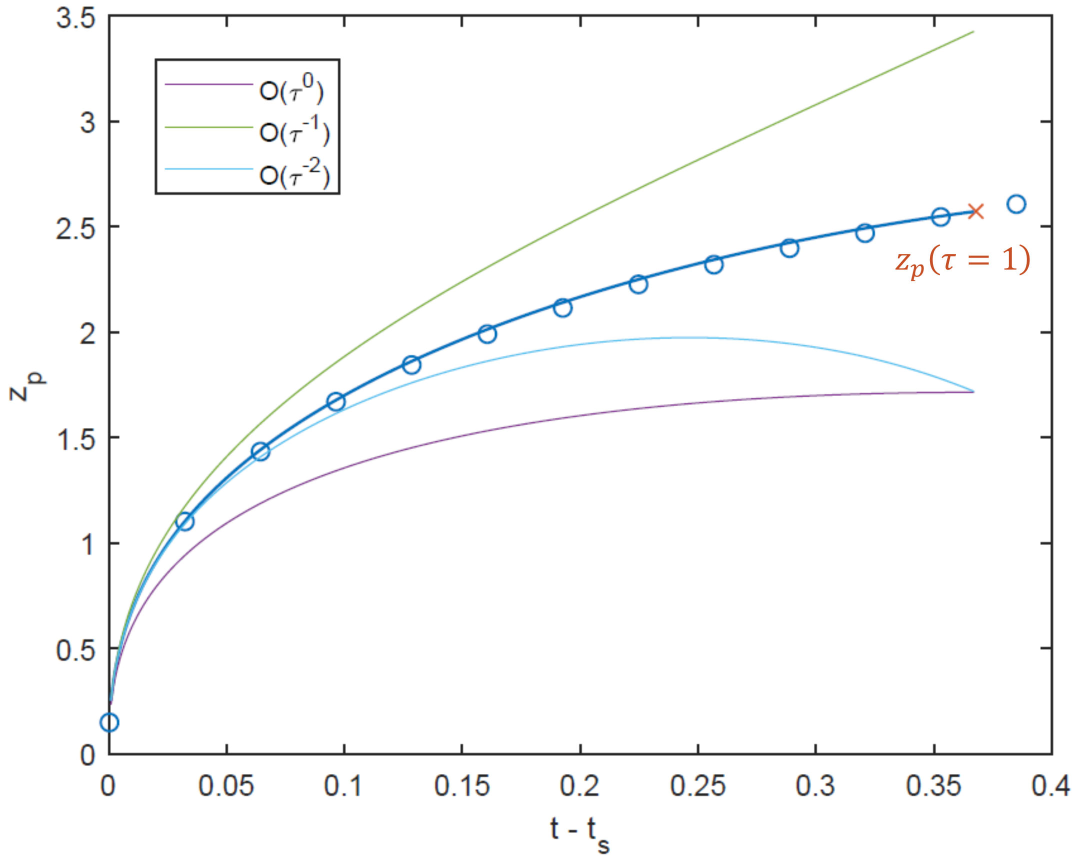
$$\underbrace{\frac{z_p}{\pm\sqrt{8\tau(t-t_s)}}}_{=\tilde{x}_p/\sqrt{8}} = 1 - \frac{1}{8}\frac{\ln(\tau)}{\tau} + \frac{\zeta_s}{4}\frac{1}{\tau} + o\left(\frac{1}{\tau}\right), \qquad \tau \to \infty$$

with $\zeta_s \approx \frac{C_1}{2} - \frac{20}{3}$

❖ **Compare results to known data**

- still constants C_i are unknown
- estimate series limit using guesses
  - temporal limit $\tau_{fin} = 1$
  - geometric series of main contributions

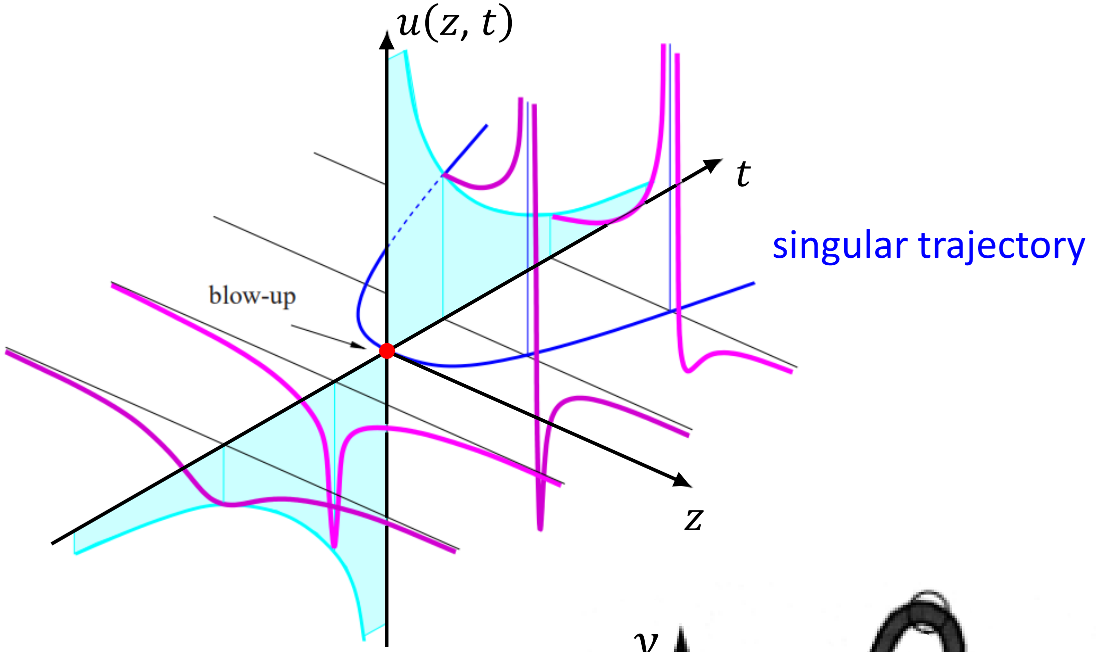
$\Rightarrow$ temporal limiting value $z_p(\tau = 1) = 3\sqrt{\frac{2}{e}}$



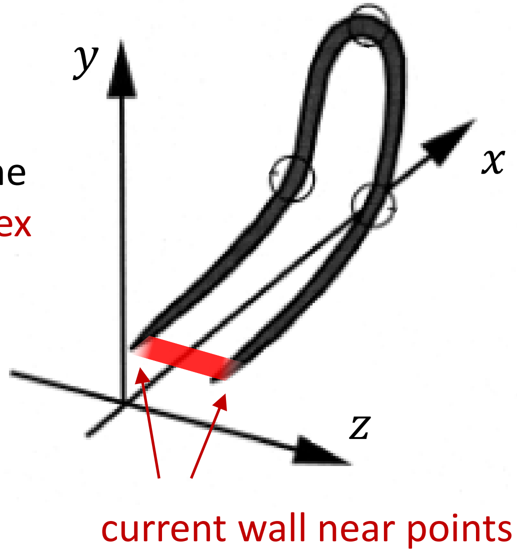
Comparison of our analytic estimate with numerical values (circles) from [Weideman, 2003]

Summary

- Fisher equation inherits essential nonlinear dynamics of hairpin vortex creation
- Finite time blow-up requires logarithmic time scale, continuation after blow-up through symmetry
- Resulting in a pair of singularities that propagate away from creation point $\Rightarrow$ hairpin vortex
- Estimate initial singular trajectory through successive inner expansions (asymptotic sublayers)
- With temporal range of validity: $t - t_s \in [0, e^{-1}]$
  $\rightarrow$ What happens afterwards?



Comparison:
Singular trajectory describes the temporal evolution of the vortex kernel "at the surface" .



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Thank you!