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Modellierung vager natürlichsprachlicher Quantoren über Dialogspiele und Fuzzy Logik

DISSERTATION

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Matthias F. J. Hofer, MSc
Matrikelnummer 01476075

an der Fakultät für Informatik

der Technischen Universität Wien

Betreuung: AO. Prof. Dr. Techn. Chris Fermüller
Zweitbetreuung: Dr. Techn. Magdalena Ortiz

Diese Dissertation haben begutachtet:

Thomas Vetterlein

Carles Noguera

Chris Fermüller

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Matthias F. J. Hofer

Modeling vague Natural Language Quantifiers via Dialogue Games and Fuzzy Logic

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Matthias F. J. Hofer, MSc
Registration Number 01476075

to the Faculty of Informatics

at the TU Wien

Advisor: AO. Prof. Dr. Techn. Chris Fermüller
Second advisor: Dr. Techn. Magdalena Ortiz

The dissertation has been reviewed by:

Thomas Vetterlein

Carles Noguera

Chris Fermüller

Vienna, 15th February, 2019

Matthias F. J. Hofer

Erklärung zur Verfassung der Arbeit

Matthias F. J. Hofer, MSc
Wohlmutstrasse 19-21, Top 35
1020 Wien
Österreich

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Matthias F. J. Hofer

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Kurzfassung

In der natürlichen Sprache (NL) werden Quantoren oft benutzt um Sachverhalte, wie zum Beispiel “Viele Leute mögen Fussball” und “Ungefähr die Hälfte der Menschheit ist weiblich”, darzustellen. Die Ausdrücke *viele* und *wenige* bedeuten, dass eine gewisse Menge relativ gesehen groß, beziehungsweise klein ist. Die Semantik dieser zwei Quantoren ist nicht endgültig festgelegt, sondern hängt insbesondere von Kontextinformationen ab. In ähnlicher Weise verhalten sich Quantoren wie *ungefähr die Hälfte* und *fast alle*, insofern als dass die Toleranzen die, bezüglich entsprechender Statements, akzeptabel sind, von Situation zu Situation verschieden sein können.
Fuzzy Logik wird oft herangezogen um solche NL Ausdrücke zu modellieren, insbesondere zeitgenössische t-norm basierte Mathematische Fuzzy Logik (MFL). Hintikka hat die klassische Logik (CL), und Giles hat die Łukasiewicz Logik, eine MFL, spiel semantisch ausgedrückt. Die Gemeinsamkeit ist dabei ein Zweispieler-Nullsummenspiel mit perfekter Information, wobei die zwei Spieler strategisch vorgehen. Fermüller und Roschger haben Giles’s Spiel um einen dritten nicht-strategischen Spieler erweitert und damit eingeführt was wir das Zufallszeugenauswahlprinzip nennen.
Dieses eben genannte Prinzip erlaubt es uns zwei weiter MFLen, Gödel und Produkt Logik, spiel semantisch auszudrücken. Das erreichen wir mittels propositionaler Quantoren, welche es uns ermöglichen den Delta-Operator zu modellieren, welcher im Wesentlichen ein Projektionsoperator ist, der unstetige Wahrheitsfunktionen hervorruft. Diesen benötigen wir um die Gödel Implikation in Giles’s System auszudrücken. Darüber hinaus modellieren wir die Multiplikation und Division von Wahrheitsfunktionen mittels dem propositionalen Quantor der auf dem Zufallszeugenauswahlprinzip basiert, dem Delta-Operator und dem propositionalen Existenzquantor. Das ermöglicht es uns die Konnektive der Produkt Logik zu definieren. Auf diesen Resultaten aufbauend, zeigen wir wie man alle MFLen die auf endlichen Darstellungen basieren in unserem System definieren kann.
Aufbauend auf der erweiterten Ausdrucksstärke, modellieren wir zusätzlich eine Vielzahl von NL Quantoren in unserem System. Dieses Unterfangen betreiben wir schrittweise, so dass die ordentliche Interpretierbarkeit von entsprechenden Aussagen garantiert bleibt. Zunächst modellieren wir semi-fuzzy Quantoren, das sind diejenigen die nur klassische Argumente annehmen, das heißt Prädikate die entweder (vollständig) wahr oder falsch sind. Diese werden dann in systematischer Art und Weise mittels Quantifier Fuzzification Mechanisms (QFMs) zu fully-fuzzy Quantoren erweitert.
Als abschließenden Beitrag dieser Arbeit definieren und testen (mittels Implementierung)

wir eine Abfragesprache die Quantoren basierend auf dem Zufallszeugenauswahlprinzip beinhaltet. Die Resultate zeigen, dass probabilistische Auswertungen nicht nur geeignet sind für Modelle vager NL Quantoren, sondern auch die Auswertungsdauer von großen Datenmengen verringern können.

Abstract

In natural language (NL), quantifiers are often used to make statements about states of affairs, like “Many people like football”, and “About half the people are female”. In particular, *many* and *few* express that some set of objects is relatively big, or small respectively. The semantics of those two quantifiers is not fixed once and for all, but rather depends on contextual information. Likewise, quantifiers like *about half* and *almost all* show a comparable behavior, as the tolerance margins that make corresponding statements acceptable can change from one situation to another.

Fuzzy logic is often used to model such NL constructs, in particular contemporary t-norm based mathematical fuzzy logics (MFLs). Hintikka expressed Classical Logic (CL) game semantically, and Giles expressed Łukasiewicz logic (Ł), a MFL, game semantically. The shared underpinning is a two player zero sum game of perfect information, where the two players act strategically. Fermüller and Roschger have augmented Giles's game by a third non-strategic player, thereby introducing what we call the *random witness selection principle* into the framework.

The latter principle allows us to also express two other MFLs, Gödel logic and Product logic, game semantically. We achieve this by allowing for propositional quantification, which enables us to model the Delta operator, which is basically a projection operator, evoking discontinuous truth functions. This is needed to express Gödel implication in Giles's framework. Moreover, the propositional quantifier based on the random witness selection principle, together with the Delta operator and the existential propositional quantifier, allows us to model multiplication and division of truth functions, which we need to define the connectives of Product logic. Building on this result, we show how to define all MFLs that are finitely representable in our framework.

Furthermore, the gained expressibility is used to model a variety of NL quantifiers within the framework. This pursuit is conducted in a step-by-step manner, that guarantees neat interpretability of statements. First, we model semi-fuzzy quantifiers, i.e. quantifiers that can only take classical arguments, i.e. predicates that evaluate to either (definitely) true or false. Then we lift those to fully-fuzzy quantifiers in a systematic and principle guided way, by means of quantifier fuzzification mechanisms (QFMs).

As a final contribution of this thesis, we define and test, by means of an implementation, a full-fledged query language, featuring quantifiers based on the random witness selection principle. The results show that probabilistic evaluations not only are suitable to model vagueness in NL, but also increase efficiency in presence of large amounts of data.

"Bildung ist die Fähigkeit, die verborgenen Zusammenhänge zwischen den Phänomenen wahrzunehmen."

Václav Havel

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CHAPTER 1

Introduction

Games for logics can help to motivate the semantics of connectives and quantifiers. Hintikka characterized Classical Logic (CL) as a two player zero-sum game with perfect information [Hin73], which is perhaps the best known example, of a game based interpretation of logics. He achieved this by only adhering to two strategic players and a notion of role switch. However, it is well known, that in CL one cannot express relative (or proportional) quantifiers, like e.g. “at least half”, which nevertheless appear frequently in Natural Language (NL). Another feature not present in CL is intermediate truth values, which are standard in contemporary t-norm based Mathematical Fuzzy Logic (MFL) [CHN11]. Giles’s game for Łukasiewicz logic [Gil74, Gil77], which also is a two player zero-sum game with perfect information, gives a game semantic representation of the only MFL that features continuous truth functions for all connectives and quantifiers, and can hence be considered singular. While Giles’s game does not feature role switches anymore, it can still be understood as strictly more expressive than the Hintikka game. However, although Łukasiewicz logic can handle intermediate truth values, and comparisons thereof by virtue of the implication rule, it does not encompass relative quantifiers, neither does it allow for expressing intensional, i.e. context dependent, quantifiers, like “many” and “few”, which are also used a lot in NL. Expressing such quantifiers formally, as well as finding adequate semantics, remains an open challenge, to which we contribute by building on a randomized version of Giles’s game, introduced by Fermüller and Roschger [FR12, FR14]. Such a game adds a third non-strategic player to the setting, referred to as *Nature*. We will call this feature *random witness selection principle*. The framework resulting from this principle will be used to derive the main contributions of this thesis, namely:

(C1) definability[1] of (continuous) t-norm based MFLs that are finitely representable,

[1] One may also speak of term-definability, see Remark 8.

(C2) modeling of a great number of NL quantifiers, and

(C3) a probabilistic query language for practical application.

1.1 Logic, NL quantifiers, and vagueness

The aim of this thesis is to model certain NL expressions, using formal languages, within a coherent logical framework. While establishing the setting amounts to a technical challenge, finding adequate semantics for such expressions is of more conceptual nature, especially when the expressions are inherently vague.

We are predominately concerned with quantification in NL, rather than with other NL expressions, like adjectives, adverbs and nouns, expressing properties[2]. Nevertheless, we will point out connections and similarities where suitable, e.g. in Section 1.3 below. The binding aspect of our investigations is vagueness, or uncertainty, i.e. quantifiers (or predicates) that do not have a clear and indisputable semantics. To clarify that, let us look at some explanatory NL sentences. Consider first the utterance "All men are mortal". We assume there is no issue in accepting it as a true statement. Dismantled, we have a quantifier "for all" ($\forall$), and two predicates, "being human" and "being mortal". Also, here we assume there is a (finite) domain of discourse (D), which at least contains all humans. Moreover, we assume the meaning of "for all" needs no explanation, while it seems important to elaborate on the fact, that we furthermore assume that for any object in D[3], there is a clear, or crisp, answer as to the question whether it is a human or not, meaning that there are no borderline cases. The same should hold for the property "being mortal", one either (fully) is or (fully) is not mortal. We will sometimes call this characteristic bivalency. More formally one speaks about *classical* predicates, those that, upon evaluation with respect to some object from the domain, have exactly one of two different truth values, which are 0 for *false* and 1 for *true*. The same applies to the level of quantification, as long as we consider quantifiers that are as simple to conceptualize as "for all" (applied to classical predicates). But, there are other quantifiers in NL, like e.g. "almost all", for which it can make sense to consider truth functions that are not bivalent. One aspect of this is that, one finds that not everybody agrees upon the meaning of a statement like "Almost all US citizens speak English". Also, even the same person that, on one side, agrees with the utterance, can still claim that, on the other side, there are "Many US citizens who do not speak English". If one, like Peterson [Pet00], proclaims a semantics of "many" that is equivalent with "majority", this would not even be possible, as long as one agrees that "almost all" implies "majority". This entails that there are different readings of quantifying expression, each of which corresponds to a certain truth function. The overlapping of more than one (crisp) truth function can be modeled by truth functions that take on values within the real unit interval. In Chapter

[2]The simple reason for this is that, the project that funded the research of this thesis – MOVAQ, Modeling Vague Quantifiers in Mathematical Fuzzy Logic – did not encompass predicates.

[3]Throughout this thesis, we will always identify constants from a domain D, and the respective objects that the constants refer to. The set of objects is also called D.

4, we introduce quantifier models [M3], which are based on the this idea, which is inspired by Lawry's voting semantics [Law98], supervaluationistic models [Fin75, FK06, Smi08], and Vetterlein's work on granularity levels [Vet11].

In our NL oriented account of fuzziness, intermediate truth values are not meant to reflect any sort of inherent fuzziness in the world, but rather serve to increase the robustness of evaluations of statements. One particularly persuasive example is the following. Think of a library with exactly 1.000.103 books in stock. Let us further assume that exactly 500.000 of them are written in English, while the others are all written in Czech. Now, on a guided tour through the library someone asks "How many of the books you have here are actually written in Czech?", to which the guide replies "Half of them". Is he wrong? Yes, logically speaking 500.103 is the half of 1.000.103 as much as +5 is the square root −1, namely simply not at all. But do we want his statement to be rendered wrong? Not really, as it is very close to being right. Which leaves us with two options. We could allow the guide to amend his answer to "About half of them", assuming the deviation of less than 0.1% is acceptable for the people who take the tour, which would make the guide's statement fully true, or, we assume there is a truth function for the quantifier "Half", that continuously goes down from 1 to 0, as the proportion of witnesses recedes from the value $\frac{1}{2}$, which would, depending on the particular model taken, most likely still lead to a truth value very close to 1. Similarly, having a set of objects in the background, all but one of which have a certain crisp property A, we would like a statement of the form "Almost all objects have the property A" to be true, while the truth value should approach 0 monotonically and continuously, as the number of objects fulfilling property A decreases. The idea to relate vagueness and closeness has been discussed at length by Smith [Smi08].

1.2 Witness selection principles

Let us take a look at the game semantic selection principles. The existential quantifier "there is" (∃) is modeled as counting quantifier, i.e. existentially quantified formulas are true in a given model if there is at least one witness in the model that makes the formula true. As an example, we consider the statement "There is a country that has more than one capital". The meaning of "country" and "having more than one capital" is supposed to be crisp again, as the one of the quantifier "there is". To verify the statement, it suffices if the proponent finds a single witnessing country that has more than one capital, and if she can name one, e.g. Chile, she succeeded. Similarly, the universal quantifier "for all" can be explained as based on a game involving witness selection as well. Assume someone, let us call him Bob, asserts "All countries have more than one capital". Then, an opponent, let us call her Mary, can disprove the claim by simply pointing to any country that does not have this property, e.g. France. This persuasive connection of proving statements and (strategic) witness selection can be applied to model many different logics, in particular Classical Logic [Hin73], but also the more expressive Łukasiewicz logic [Gil74]. While the latter is a fuzzy logic, hence statements can have intermediate truth values within the real unit interval, we can not directly refer to the proportion of witnesses for a certain property. This remaining lack of expressibility can be countered by introducing what we call *random witness selection principle*. This will come along as a new, non-strategic, actor, or player, in the game of proving statements. Non-strategic, here, means without reason (or without vested interest), hence any witness is selected with equal probability, which in turn means we introduce a uniform probability distribution into our logical framework. Consider "You can come by at any time", uttered by someone, let us say Bob, to someone else, let us say Mary. This can be read as "Pick a random time to come by, and I'll be there". In the model we describe, this statement is only fully true if all time instances to visit Bob can completely (uniformly) randomly be chosen and all of them will work, i.e. Bob will be there. One the other hand, if Bob actually only is available half of the time, there is still a certain probability, namely $\frac{1}{2}$, that the randomly chosen time instance works, which we will model in a way that allows us to say, that the overall truth value of Bob's statement has decreased to $\frac{1}{2}$. Consequently, Bob's statement will not be entirely wrong as long as there is at least one time instance that would work for Mary to come visit him, although almost, as the probability for Mary to catch it is then conceivably small. Hence, at least in a certain way, the new quantifier (Π), based on the random witness selection principle, represents the natural language expression "any".

1.3 Vagueness and properties

The constituting feature of fuzzy logics is that statements can have intermediate truth values, different from 0 and 1. This is often considered to be controversial in itself. However, there are convincing examples that may serve to justify this characteristic. Let us think for example of a pair of socks, which is gray. We assume two predicate symbols to be present in our language, "black" and "white". If they are crisp, or classical, i.e. each sock must be either fully black or fully white to fulfill the respective property, the pair leads to a negative evaluation regarding both of these predicates. Alternatively, we can interpret the predicates as fuzzy, i.e. they can take on intermediate truth values. In that case, dependent on the actual shade of gray the socks have, both predicates can be partly true or even have the same truth value. The latter is the case when both truth values are $\frac{1}{2}$, which means the socks are neither fully white nor black, but they are white as much as they are black.

Another way to think about vague properties, such as "tall" or "rich", is the following. People have reasons for stating vague propositions, although not everybody is always aware of their own. That is to say that there are always crisp background evaluation criteria attached to vague propositions, such as "Donald is rich", or "John is tall". It can be as easy as just one other crisp predicate that justifies a statement, like the former, involving the property rich. For example, people may accept Donald as a rich person, because he has more than one million dollars, or because he is richer than everybody they know. Also, for judging someone to be tall, the reason may be that he or she who utters the statement predominantly socializes with shorter people than John, or lives in a society with mostly shorter people. While these are very simple accounts, crisp background criteria can also be arranged more complicatedly. A statement like "John is a football fan" could be acceptable for someone because John fulfills a number of criteria from a fixed list of such, not even necessarily a determined number. It can also be a vague statement again, like "At least about half of the criteria from the fixed list of (crisp) criteria are (fully) fulfilled". In particular, the truth value of such a statement can be modeled in a way that allows for intermediate truth values, which then is in full accordance with the idea of vague predicates. Also, different truth functions, that correspond to a certain predicate, can be overlapped again, in the sense that the global truth function is obtained by overlapping several local ones, each of which corresponds to a certain reading or interpretation by some agent. This perspective will be fully expressible in what we will describe and develop in this thesis, it simply needs to be formalized and we are done.

1.4 Thesis structure

This thesis is structured in the following way. In Chapter 2, we will introduce all the necessary terminology and background material, like the Hintikka game for CL [Hin73], Giles's game for Łukasiewicz logic [Gil74], and extensions thereof that already feature the random witness selection principle [FR14, Fer14]. In Chapter 3, we prove the main technical result of the thesis, namely the definability of Gödel logic and Product logic [CHN11] within an extension of Giles's game with the random witness selection principle. We will call the corresponding logic, which particularly features also propositional quantifiers, $Ł_\alpha(\Pi)$. We show that, as a consequence of the definability results, also all fuzzy logics, that are based on a finite ordinal sum of the three basic t-norm based fuzzy logics (Gödel, Product and Łukasiewicz), are definable within $Ł_\alpha(\Pi)$. Chapter 4 develops a large number of quantifier models in a principled manner, starting from the easiest cases of counting quantifiers, and ending with a full account of the intensional quantifiers "many" and "few", while predicates, i.e. arguments to quantifiers, are always assumed to be crisp. Such quantifiers are called semi-fuzzy. Chapter 5 deals with the lifting of semi-fuzzy quantifiers to fully-fuzzy ones, i.e. those that can take also fuzzy arguments. This is accomplished by means of so called quantifier fuzzification mechanisms (QFMs), which comply with certain criteria, that we introduce, discuss and evaluate regarding their adequateness. Eventually, in Chapter 6, we introduce a full blown query language that can deal with probabilistic quantifier models based on the random witness selection principle, and test it on real life data. Chapter 7 provides the reader with a summary of the presented material and an outlook to future work.

1.4.1 Publications related to this thesis

Much of the original material of this thesis builds upon the author's conference and journal contributions, and roughly relates to the chapter structure as follows:

- Chapter 3: [Hof18]
- Chapter 4: [Hof15, Hof16b, Hof16a, FH17]

- Chapter 5: [BFH18]
- Chapter 6: [FHO17]

However, some of the chapters, especially Chapter 4, contain also material that has not been published beforehand.

CHAPTER 2

From Hintikka's game to the $\mathcal{RG}$-game

In this and the following two chapters we are going to introduce a variety of logics. Some are well established already, e.g. Classical Logic, of which we assume the reader to be familiar with. Other logics will most likely be completely new to the reader, as for example the logic which will be denoted by $Ł_\alpha(\Pi)$. However, all of those follow the same pattern. Firstly, all have an underpinning language. By introducing the language, we will also give the syntactic formation rules for each logic. Then, a truth truth functional semantics will be specified for each logic, based on which we will define what it means for a formula to be valid within the formalism. The results of this thesis are concerned with characterizing our logics with a different semantics than the truth functional one, namely via game semantics, and hence we spare the notion of derivability.

We start by revisiting Hintikka's game for classical logic and continue to extend it step by step to the $\mathcal{RG}$-game, which, as an extension of Giles's game, corresponds to a randomized extension of Łukasiewicz logic. The complementary presentation of game semantics and truth functional semantics is supposed to help in understanding and motivating the nature of the logics itself. Game rules justify truth functions of respective connectives and quantifiers in a tangible manner, instead of just stating them in an ad hoc fashion, where the correspondence of both is the key feature of the presented perspective.

2.1 Hintikka game

Maybe the most fundamental, but certainly the best known game is Hintikka's characterization of truth in a model for classical first order sentences[1] [Hin73]. The setting is

[1] A sentence is a formula without free variables.

a zero sum game of perfect information with two strategic players, *I* and *You*[2], one in the role of the proponent **P** and the respective other in the role of the opponent **O**, who stepwise reduce a given formula to one of its immediate subformulas, until an atomic formula is reached. If the outermost connective (or quantifier) is $\wedge$ (or $\forall$), **O** may decide which conjunct (or instance of the quantified formula) the game continues with. Likewise, if the outermost connective (or quantifier) is $\vee$ (or $\exists$), **P** may decide which disjunct (or instance of the quantified formula) the game continues with.

If a player has asserted a negated formula $\neg F$, the game continues with her giving up responsibility for the claim, while the other player has to assert the formula F in return. One can think of it as a role switch of players, while removing the negation in front of a formula. When an atomic formula is reached, the player in the role of **P** wins, if it is true in the given model, otherwise she loses. This procedure we call $\mathcal{H}$-game. We will now give a more formal characterization of this game.

The two players, called *I (Myself)* and *You*, can both act either as *proponent* **P** or as *opponent* **O** with respect to a formula F built up from atoms, using the binary connectives $\wedge$, $\vee$, as well as the unary connective $\neg$ and the quantifiers $\forall$, $\exists$. Initially, *I* act as **P** and *You* act as **O**. *My* initial aim — or, more generally, **P**'s aim at any state of the game — is to show that the present formula is true in a given (classical) interpretation $\mathcal{M}$, which consists of a *finite domain* D, a *variable assignment* $\xi_{\mathcal{M}}$ that assigns elements of the domain to all free object variables, and a *signature interpretation* Φ that assigns relations $\tilde{R} : D^n \to \{0, 1\}$ to n-ary predicate symbols R[3]. The values represent payoffs, where payoff 0 is associated to falsehood and payoff 1 to truth. Also, Φ assigns domain elements to constant symbols. For simplicity, we will assume that there is a unique constant[4] for every element of the domain D of $\mathcal{M}$. The following rules refer to the outermost connective or quantifier of the *current formula*.

$\mathcal{R}^{\mathcal{H}}_{\wedge}$: If the current formula is $F \wedge G$, then **O** chooses whether the game continues with either F or G.

$\mathcal{R}^{\mathcal{H}}_{\vee}$ If the current formula is $F \vee G$, then **P** chooses whether the game continues with either F or G.

$\mathcal{R}^{\mathcal{H}}_{\neg}$: If the current formula is $\neg F$, the game continues with F, and the roles of the players are switched: the player who is currently acting as **P**, acts as **O** at the next state, and vice versa for the current **O**.

$\mathcal{R}^{\mathcal{H}}_{\forall}$: If the current formula is $\forall x F(x)$, then **O** chooses a constant $c \in D$ and the game continues with $F(c)$.

[2]Hintikka uses *Nature* and *Myself* as names for the players and *Verifier* and *Falsifier* for the two roles, while our terminology follows the handbook article in [CFN15].

[3]Note that we often call such predicates crisp.

[4]The set of constants, representing objects from the domain D, will be called D again.

$\mathcal{R}^{\mathcal{H}}_{\exists}$: If the current formula is $\exists x F(x)$, then **P** chooses a constant $c \in D$ and the game continues with $F(c)$.

Except[5] for $\mathcal{R}^{\mathcal{H}}_{\neg}$, the players' roles remain unchanged. The game ends when the current formula is atomic. The player who is acting as **P** at the final state *wins* and the other player (acting as **O**) *loses*, if this atomic formula is true in $\mathcal{M}$. We associate payoff 1 with winning and payoff 0 with losing. We also include the falsity constant $\perp$ among the atomic formulas, signifying definite payoff 0. The game starting with formula F is called the *$\mathcal{H}$-game for F under $\mathcal{M}$*.

The truth functional semantics of classical logic extends any given assignment of truth values in $\{0, 1\}$ to atomic formulas as follows:

- $v_{\mathcal{M}}(F \wedge G) = \min(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G))$
- $v_{\mathcal{M}}(F \vee G) = \max(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G))$
- $v_{\mathcal{M}}(\neg F) = 1 - v_{\mathcal{M}}(F)$
- $v_{\mathcal{M}}(\perp) = 0$

At the first order level, we may define the semantics of the universal and the existential quantifier as follows, where we identify the elements of the domain D with constants:

- $v_{\mathcal{M}}(\forall x F(x)) = \min_{c \in D}(v_{\mathcal{M}}(F(c)))$
- $v_{\mathcal{M}}(\exists x F(x)) = \max_{c \in D}(v_{\mathcal{M}}(F(c)))$

The following definition presents the notion of logical consequence and in particular the set of tautologies in classical logic, i.e. its valid formulas.

Definition 1 *In classical logic, a formula F is called a logical consequence of a set of formulas Γ, written $\Gamma \models_{cl} F$, if for every evaluation $v_{\mathcal{M}}$ we have:*

If $v_{\mathcal{M}}(G) = 1$ for all $G \in \Gamma$, then also $v_{\mathcal{M}}(F) = 1$.

In particular, a formula F is called valid if for all evaluations $v_{\mathcal{M}}$ we have $v_{\mathcal{M}}(F) = 1$. We denote that circumstance by $\models_{cl} F$.

The following theorem explains the fundamental relation between Hintikka's game semantics and the truth functional semantics of classical logic.

[5] Note the decoration of the upper index of the $\mathcal{R}$ preceding the rules, which indicates the setting they relate to.

Theorem 1 *[Hin73] A formula F is true in a (classical) interpretation $\mathcal{M}$ (also written as $v_{\mathcal{M}}(F) = 1$) iff* I *have a winning strategy in the $\mathcal{H}$-game for F under $\mathcal{M}$.*

Note that there is no explicit implication rule. However, one can define the implication $F \to G$ of two formulas F and G as $\neg F \vee G$, which corresponds to the implication of classical logic.

2.2 Game for Kleene-Zadeh logic

As explained in [FR14], we can go from classical to fuzzy interpretations, or Kleene-Zadeh interpretations, by letting the *signature interpretation* assign fuzzy relations $\tilde{R} : D^n \to [0, 1]$ to n-ary predicate symbols R, generalizing payoffs from $\{0, 1\}$ to $[0, 1]$. At the propositional level, Hintikka's result then directly generalizes to Kleene-Zadeh logic, which extends any given assignment of values in $[0, 1]$ to atomic formulas, as before, as:

- $v_{\mathcal{M}}(F \wedge G) = \min(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G))$
- $v_{\mathcal{M}}(F \vee G) = \max(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G))$
- $v_{\mathcal{M}}(\neg F) = 1 - v_{\mathcal{M}}(F)$
- $v_{\mathcal{M}}(\bot) = 0$

At the first order level, we may define the semantics of the universal and the existential quantifier as follows, where we again identify the elements of the domain D with constants:

- $v_{\mathcal{M}}(\forall x F(x)) = \inf_{c \in D}(v_{\mathcal{M}}(F(c)))$
- $v_{\mathcal{M}}(\exists x F(x)) = \sup_{c \in D}(v_{\mathcal{M}}(F(c)))$

The game rules remain unchanged. As we restrict our attention to finite domains we may assume that there are witnessing constants for all suprema and infima.

Definition 2 *w is called the* value *of the game for player $\boldsymbol{X}$, if $\boldsymbol{X}$ has a strategy that guarantees a payoff of at least w for $\boldsymbol{X}$, while the opponent player has a strategy that ensures that $\boldsymbol{X}$'s payoff is at most w.*

The upcoming definition determines the notion of logical consequence and in particular the set of tautologies in Kleene-Zadeh logic, i.e. its valid formulas.

Definition 3 *In Kleene-Zadeh logic, a formula F is called a logical consequence of a set of formulas Γ, written $\Gamma \models_{KlZ} F$, if for every evaluation $v_{\mathcal{M}}$ we have:*

If $v_{\mathcal{M}}(G) = 1$ for all $G \in \Gamma$, then also $v_{\mathcal{M}}(F) = 1$.

In particular, a formula F is called valid if for all evaluations $v_{\mathcal{M}}$ we have $v_{\mathcal{M}}(F) = 1$. We denote that circumstance by $\models_{KlZ} F$.

Interestingly, the notion of validity for Kleene-Zadeh logic only leads to a (non-empty) set of tautologies if we, as we do, allow for having $\bot$ explicitly, for otherwise there are no valid formulas at all [CFN15].

Theorem 2 *[FR14] A formula F evaluates to $v_{\mathcal{M}}(F) = w$ in a* Kleene-Zadeh*-interpretation $\mathcal{M}$ iff the $\mathcal{H}$-game for F with payoffs matching $\mathcal{M}$ has value w for* Myself.

Remark 1 *As it will often make sense to distinguish bivalent formulas from fuzzy formulas, i.e. such that can take only one of two definite truth values, 0 or 1, and those that can take arbitrary truth values within the real unit interval, we will use the notation $\hat{F}$ for the first, and F for the latter. The hat on a formula hence tells us that the formula is supposed to be strictly crisp or classical.*

Remark 2 *In the introduction, we already used a terminology that referred to the game semantic selection principles as witness selection. In the remainder of this thesis, we will sometimes use the terms* positive witness *and* negative witness*, with respect to a quantified crisp formula $\hat{F}$. Then, the term witness means a constant referring to an object from the domain that is substituted into $\hat{F}$ for the variable in the scope of the quantifier. If it makes $\hat{F}$ true, we call it a positive witness and if it makes $\hat{F}$ false, we call it a negative witness.*

2.3 Giles game

Giles's game for Łukasiewicz logic [FR14, Gil74], also is, like Hintikka's game, a zero sum game of perfect information, which describes the stepwise reduction of complex logical assertions into atomic ones in a rule guided dialogue between two players. The players' payoff at a final state of the game is specified in terms of the total expected loss of money if each player bets on the success of dispersive experiments corresponding to atomic formulas [Gil82]. 'Dispersive' means that the (yes/no) results may differ upon repetition; but a fixed failure probability is associated with each experiment, which is conceptualized as the risk of the atomic assertion. In this manner Giles succeeded in *deriving* the truth functions of the logical connectives of Łukasiewicz logic, or simply Ł, from first principles about reasoning, rather than just imposing them in an *ad hoc* fashion [Gil74]. The formal structure of the game is the following.

Giles's game ($\mathcal{G}$-game) shares some central ideas with the $\mathcal{H}$-game, but features not only a more complex evaluation of final game states, but also a considerably more complex notion of game states in principal. Whereas a state of an $\mathcal{H}$-game is fully determined by a single formula and the current distribution of roles (**O**/**P**) between the two players (*You*, *I*), a state of the $\mathcal{G}$-game is always of the following form:

$$[F_1, ..., F_n \mid G_1, ..., G_m] \tag{2.1}$$

where the $F's$ are the *asserted formulas* of the player *You* and the $G's$ are the *asserted formulas* of the player *I*. Independently, $[F_1, ..., F_n]$ is the tenet of the player *You*, and $[G_1, ..., G_m]$ is the tenet of the player *I*. Both of the tenets contain multisets of formulas, i.e. formulas can occur more than once in each tenet. Role switches never take place, i.e. *I* always is the proponent of the formulas of her tenet and the opponent of those in the tenet of *You*, and vice versa for *You*. By applying the rules of the game a state gets decomposed into a final state where only atomic formulas remain. This means that any of the compound formulas, upon decomposition, is removed from its tenet and the rules determine what they are replaced with. Final states of the game are of the following form:

$$[A_1, ..., A_{n'} \mid B_1, ..., B_{m'}] \tag{2.2}$$

where the $A's$ are atoms that are asserted by the player *You*, and the $B's$ are those atoms that are asserted by the player *I*.

Formally, one needs a notion of a *regulation* [FM09] to make the procedure a game. Following [FM09], such regulations are functions that map any non-final game state into one of the two labels **Y** and **I**. The first one signifies that the player *You* is the next one to initiate a state transition and the latter signifies that the player *I* is up to do so. A regulation is called *consistent* if any state labeled with a **Y** features a compound formula that the player *You* can choose as a next one to be decomposed, i.e. there is a compound formula in the tenet of the player *I*. Vice versa, any state labeled with an **I** features a compound formula that the player *I* can choose as a next one to be decomposed, i.e. there is a compound formula in the tenet of the player *You*. In that sense, one may speak of an intermediary state which connects two states. When transitioning from a state S into a state S', the intermediary state is the state S with one particular non-atomic formula being singled out as the next one to be decomposed. Then, a *game form* $G([\Gamma|\Omega], \rho)$, where Γ and Ω are multisets of formulas and ρ is a consistent regulation, is a tree with $[\Gamma|\Omega]$ being the root and all possible final states constitute the set of leafs. An actual game is a game form together with a *risk value assignment* for atomic formulas, which will be introduced now.

To each atomic formula A of the signature of the language, we associate a dispersive binary experiment E_A. Here, binary means that the result always is either 0, i.e. the experiment failed, or 1, i.e. the experiment succeeded, indicating whether A is false or true in this run of the game. The players have to pay 1€ to the respective other player for each atom in their tenet for which the experiment fails.

To each experiment we associate a failure probability $\pi(E_A)$. For example, for the falsum symbol[6] $\bot$ we assume there is an experiment that always fails, i.e. $\pi(E_\bot) = 1$. These failure probabilities are what we also call risks of atomic assertions. In symbols, the risk of an atom A is $\langle A \rangle$, hence $\langle A \rangle = \pi(E_A)$. In this sense we talk about expected losses (or risks) and expected gains (or payoffs) of assertions, and in particular of a *game with randomized payoff*. The relation between risk an payoff is that the risk of player I is the payoff of player *You*, and the payoff of player I is the risk of player *You*. We thereby guarantee that the game actually is a zero-sum game. A risk value assignment $\langle \cdot \rangle$ assigns values in $[0, 1]$ to all atoms of the signature of the language. We say a risk value assignment matches an interpretation $\mathcal{M}$ is for all atoms A we have that $v_{\mathcal{M}}(A) = 1 - \langle A \rangle$.

When the final state of a $\mathcal{G}$-game has been reached, the expected risk, from the perspective of the player I, of a game is computed as:

$$\langle A_1, ..., A_{n'} \mid B_1, ..., B_{m'} \rangle = \sum_{1 \leq i \leq m'} \langle B_i \rangle - \sum_{1 \leq i \leq n'} \langle A_i \rangle \tag{2.3}$$

A branch of the game tree is sometimes referred to as individual run of the game. To clarify the structure of the game, let us look at the following example.

Example 1 *Let A, B be two atoms. We consider the state $[A, A \,|\, B]$. This game state already is final, where the player* You *has asserted two copies of the atom A and the player* I *has asserted the atom B once. Let us assume (1) that $\pi(E_A) = 0.1$ and $\pi(E_B) = 0.4$, i.e. the failure probability of the experiments associated to the atoms A, B is* 0.1 *and* 0.4 *respectively. This then entails that the expected loss of money from the perspective of the player* I *is* 0.2€*, because $\langle A, A \mid B \rangle = 0.2$. If on the other hand (2) the risks of both atoms are equally* 0.4*, i.e. $\langle A \rangle = \langle B \rangle = 0.4$, we get $\langle A, A \mid B \rangle = -0.4$, which means the player* I *may except an average gain of* 0.4€*, which is associated to a payoff of* 0.4.

Definition 4 *A game with randomized payoff is r-valued for player $\boldsymbol{X}$, if for every $\epsilon > 0$, $\boldsymbol{X}$ has a strategy that guarantees that her expected loss is at most $(r + \epsilon)$€, while her opponent has a strategy that ensures that her expected loss is at least $(r - \epsilon)$€. We call r the risk for $\boldsymbol{X}$ in that game.*

The reference to $\epsilon > 0$ can be spared (that means we can safely read the same definition while just deleting "for every $\epsilon > 0$,", as well as the two subsequent appearances of ϵ), unless one considers infinite domains. For infinite domains the problem arises, that, in case of $\forall$ and $\exists$, there might be no witnesses for the suprema and infima, i.e. **O** potentially gets into the situation where, whatever she chooses for $c \in D$, there always is a better $c' \in D$ with $\langle F(c') \rangle > \langle F(c) \rangle$ (or for **P** with "$<$"). Later, from Section 3.2 on, when we

[6] The falsum symbol $\bot$ is now part of the language.

introduce the $\mathcal{NRG}$-game, we will employ Definition 2 as it stands, for reasons to be explained then.

The $\mathcal{G}$-game rules for (weak) conjunction ($\wedge$), disjunction ($\vee$), and for the universal and existential quantifiers ($\forall$ and $\exists$) directly correspond to the rules $\mathcal{R}^{\mathcal{H}}_{\wedge}$, $\mathcal{R}^{\mathcal{H}}_{\vee}$, $\mathcal{R}^{\mathcal{H}}_{\forall}$, and $\mathcal{R}^{\mathcal{H}}_{\exists}$. However, since the terminology slightly changed regarding the $\mathcal{H}$-game, we restate them here to avoid ambiguities:

$\mathcal{R}^{\mathcal{G}}_{\wedge}$: If **P** asserts $F \wedge G$, then, if **O** attacks, **O** chooses whether **P** has to assert either F or G.

$\mathcal{R}^{\mathcal{G}}_{\vee}$: If **P** asserts $F \vee G$, then, if **O** attacks, **P** chooses whether **P** has to assert either F or G.

$\mathcal{R}^{\mathcal{G}}_{\forall}$: If **P** asserts $\forall x F(x)$, then, if **O** attacks, **O** chooses a constant $c \in D$ and **P** has to assert $F(c)$.

$\mathcal{R}^{\mathcal{G}}_{\exists}$: If **P** asserts $\exists x F(x)$, then, if **O** attacks, **P** chooses a constant $c \in D$ and **P** has to assert $F(c)$.

The new game rules of the $\mathcal{G}$-game are the following [Fer09, FM09]:

$\mathcal{R}^{\mathcal{G}}_{\rightarrow}$: If **P** asserts $F \rightarrow G$, then, if **O** attacks, **O** has to assert F, and **P** has to assert G.

$\mathcal{R}^{\mathcal{G}}_{\&}$: If **P** asserts $F \& G$, then, if **O** attacks, **P** must assert F and G, or $\bot$ instead.

By *attacking* a complex formula, **O** triggers a defense by **P**, according to the rules. By this mechanism, complex formulas are decomposed into subformulas. The hedge "if **O** attacks" refers to the so called *principle of limited liability*:

- On one side, this principle means it is not necessary that **O** attack every formula asserted by **P**. Instead of attacking a formula, **O** can just grant it, in case it does not carry any risk for **P**. This then entails that the formula just is removed from **P**'s tenet. From the implication rule, we can see how this principle plays out. Assume we have the state $[\mid A \rightarrow B]$, with A, B being atoms with risks $\langle A \rangle = 0.8$ and $\langle B \rangle = 0.5$. If **O** attacked the formula, the subsequent state would be $[A \mid B]$, which is a final state with risk $\langle B \rangle - \langle A \rangle = -0.3$, i.e. **P** could expect a net payment by **O**. To avoid this, **O** can just grant **P** this formula, to hedge her own loss. In this case the formula simply is removed from **P**'s tenet, i.e. the subsequent state is $[\mid]$, with risk 0 for **P**, i.e. payoff 1, which in turn means the formula is true.

- The other side of this principle states that **P** can hedge her loss with regard to subsequently making assertions that have a risk greater than 1. Here, with the strong conjunction rule, it is stated explicitly that **P** can assert $\perp$ instead of both F and G, in case their risks sum up to a number greater than 1. This principle always remains in place throughout this thesis, although sometimes only implicitly [CFN15].

Remark 3 *The negation $\neg F$ of a formula F is defined as $F \to \perp$.*

Remark 4 *As Definition 4 insinuates, playing rationally means the following. Players always try to minimize their risk that results from further assertions related to the decomposition of formulas, and complementarily, they try to maximize their payoffs throughout the game. Hence, a strategy for the player* I *of a game form $G([\Gamma|\Omega], \rho)$ is a part of the corresponding tree of the game form, where for any state labeled with **I** there is only one immediate successor node. A strategy is called* winning *for the player* I *for a risk value assignment $\langle\cdot\rangle$ if we have $\langle A_1, \ldots A_n\rangle \geq \langle B_1, \ldots, B_m\rangle$ for any of the leaf nodes $[A_1, \ldots A_n|B_1, \ldots, B_m]$. Strategies (winning strategies) for the player* You *are characterized by analogy.*

The truth functional semantics of Łukasiewicz logic is given as follows:

$v_{\mathcal{M}}(\perp) = 0$

$v_{\mathcal{M}}(F \wedge G) = \min(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G))$

$v_{\mathcal{M}}(F \vee G) = \max(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G))$

$v_{\mathcal{M}}(\forall x F(x)) = \inf_{c \in D} v_{\mathcal{M}}(F(c))$

$v_{\mathcal{M}}(\exists x F(x)) = \sup_{c \in D} v_{\mathcal{M}}(F(c))$

$v_{\mathcal{M}}(F \to G) = \min(1, 1 - v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G))$

$v_{\mathcal{M}}(F \& G) = \max(0, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G) - 1)$

The upcoming definition determines the notion of logical consequence and in particular the set of tautologies in Łukasiewicz logic, i.e. its valid formulas.

Definition 5 *In Łukasiewicz logic, a formula F is called a logical consequence of a set of formulas Γ, written $\Gamma \models_{Ł} F$, if for every evaluation $v_{\mathcal{M}}$ we have:*

If $v_{\mathcal{M}}(G) = 1$ for all $G \in \Gamma$, then also $v_{\mathcal{M}}(F) = 1$.

In particular, a formula F is called valid if for all evaluations $v_{\mathcal{M}}$ we have $v_{\mathcal{M}}(F) = 1$. We denote that circumstance by $\models_{Ł} F$.

For this described $\mathcal{G}$-game, one can give the following characterization of Łukasiewicz logic:

Theorem 3 *[Gil74],[FM09] A Ł formula F is evaluated to $v_{\mathcal{M}}(F) = w$ in a Ł-interpretation $\mathcal{M}$ iff every $\mathcal{G}$-game for F, i.e. a game starting in the state $[|F]$, is $(1-w)$-valued for the player* I *under risk value assignment $\langle\cdot\rangle$ matching $\mathcal{M}$.*

Although the proof has been presented already [FM09], we show it here again to demonstrate the mechanisms. In particular, we repeat the proof to prepare for related proofs later in this thesis, which build upon this one. Hence, to remain self-contained, it seems necessary to have the important terminology introduced.

Proof:
According to Definition 4, to determine the value of a game for the proponent player I of a formula F, we have to compute the minimal final risk she can enforce, while her opponent player *You* tries to maximize it. This corresponds to a game that is starting in state $[|F]$.

For every final state, i.e. states $[A_1, \ldots, A_m | B_1, \ldots, B_n]$ where all A's and B's are atoms, the player I wins if her expected loss (associated risk) $\langle A_1, \ldots, A_m | B_1, \ldots, B_n \rangle = \sum_{i=1}^{n} \langle B_i \rangle - \sum_{i=1}^{m} \langle A_i \rangle$ is smaller or equal zero, i.e. non-positive. The minimal final risk that the player I can enforce in any state S by playing optimally can be calculated by taking into consideration (1) the maximum over all risks associated with the successor states of S that the player *You* can enforce by a move at S, and (2) the fact that I can enforce the minimum over the risks of successor states that correspond to her possible moves. Therefore, we will show that the notion of minimal risks for player I can be extended from final states to arbitrary states $[\Omega|\Omega']$ (Ω, Ω' multisets of formulas), such that the following conditions hold (Γ, Γ' multisets of formulas). The left-hand side of the equality symbols always shows the current state of the game, while the respective formula not in Γ and Γ' is the one currently being singled out according to the consistent regulation.

1a: $\langle \Gamma \mid \Gamma', G \wedge H \rangle = \max(\langle \Gamma \mid \Gamma', G \rangle, \langle \Gamma \mid \Gamma', H \rangle)$

2a: $\langle \Gamma \mid \Gamma', G \vee H \rangle = \min(\langle \Gamma \mid \Gamma', G \rangle, \langle \Gamma \mid \Gamma', H \rangle)$

3a: $\langle \Gamma \mid \Gamma', G \rightarrow H \rangle = \max(\langle \Gamma \mid \Gamma' \rangle, \langle \Gamma, G \mid \Gamma', H \rangle)$

4a: $\langle \Gamma \mid \Gamma', G \& H \rangle = \min(\langle \Gamma \mid \Gamma', \bot \rangle, \langle \Gamma \mid \Gamma', G, H \rangle)$

5a: $\langle \Gamma \mid \Gamma', \forall x G(x) \rangle = \sup_{c \in D} \langle \Gamma \mid \Gamma', G(c) \rangle$

6a: $\langle \Gamma \mid \Gamma', \exists x G(x) \rangle = \inf_{c \in D} \langle \Gamma \mid \Gamma', G(c) \rangle$

1b: $\langle \Gamma, G \wedge H \mid \Gamma' \rangle = \min(\langle \Gamma, G \mid \Gamma' \rangle, \langle \Gamma, H \mid \Gamma' \rangle)$

2b: $\langle \Gamma, G \vee H \mid \Gamma' \rangle = \max(\langle \Gamma, G \mid \Gamma' \rangle, \langle \Gamma, H \mid \Gamma' \rangle)$

3b: $\langle \Gamma, G \to H \mid \Gamma' \rangle = \min(\langle \Gamma \mid \Gamma' \rangle, \langle \Gamma, H \mid \Gamma', G \rangle)$

4b: $\langle \Gamma, G \& H \mid \Gamma' \rangle = \max(\langle \Gamma, \bot \mid \Gamma' \rangle, \langle \Gamma, G, H \mid \Gamma' \rangle)$

5b: $\langle \Gamma, \forall x G(x) \mid \Gamma' \rangle = \inf_{c \in D} \langle \Gamma, G(c) \mid \Gamma' \rangle$

6b: $\langle \Gamma, \exists x G(x) \mid \Gamma' \rangle = \sup_{c \in D} \langle \Gamma, G(c) \mid \Gamma' \rangle$

It is now to be checked that $\langle | \rangle$ is well-defined and unique. Moreover, risk values and risk assignments have to be connected with truth values and valuations.

Let us extend the semantics of Ł formulas to multisets Ω of formulas as follows:

$v_{\mathcal{M}}(\Omega) = \sum_{G \in \Omega} v_{\mathcal{M}}(G)$

Risk value assignments are placed in one to one correspondence with truth value assignments via the mapping $\langle A \rangle = 1 - v_{\mathcal{M}}(A)$, which then extends to:

$\langle A_1, \dots, A_m | B_1, \dots, B_n \rangle = n - m + v_{\mathcal{M}}([A_1, \dots, A_m]) - v_{\mathcal{M}}([B_1, \dots, B_n])$.

Accordingly, we define the following function for arbitrary states:

$\langle \Omega | \Omega' \rangle = |\Omega'| - |\Omega| + v_{\mathcal{M}}(\Omega) - v_{\mathcal{M}}(\Omega')$.

Note that $v_{\mathcal{M}}(F) = v_{\mathcal{M}}([F]) = 1$ iff $\langle | F \rangle \leq 0$.

We can now check all the conditions $1a, 1b, \dots, 6a, 6b$, i.e. perform the induction step regarding the induction over the complexity of states, i.e. the number of appearing connectives and quantifiers, given the players play rationally:

Cases $1a, 1b$ (cases $2a, 2b$ work by analogy):

$\langle \Gamma \mid \Gamma', G \wedge H \rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(G \wedge H) =$

$= \langle \Gamma | \Gamma' \rangle + 1 - v_{\mathcal{M}}(G \wedge H) =$

$= \langle \Gamma | \Gamma' \rangle + 1 - \min(v_{\mathcal{M}}(G), v_{\mathcal{M}}(H)) =$

$= \langle \Gamma | \Gamma' \rangle + \max(1 - v_{\mathcal{M}}(G), 1 - v_{\mathcal{M}}(H)) =$

$= \langle \Gamma | \Gamma' \rangle + \max(\langle G \rangle, \langle H \rangle) =$

$= \max(\langle \Gamma \mid \Gamma', G \rangle, \langle \Gamma \mid \Gamma', H \rangle)$

$\langle \Gamma, G \wedge H \mid \Gamma' \rangle = |\Gamma'| - |\Gamma| - 1 + v_{\mathcal{M}}(\Gamma) + v_{\mathcal{M}}(G \wedge H) - v_{\mathcal{M}}(\Gamma') =$

$= \langle \Gamma | \Gamma' \rangle - 1 + v_{\mathcal{M}}(G \wedge H) =$

$= \langle \Gamma | \Gamma' \rangle - (1 - \max(v_{\mathcal{M}}(G), v_{\mathcal{M}}(H))) =$

$= \langle \Gamma | \Gamma' \rangle - \min(1 - v_{\mathcal{M}}(G), 1 - v_{\mathcal{M}}(H)) =$
$= \langle \Gamma | \Gamma' \rangle - \min(\langle G \rangle, \langle H \rangle) =$
$= \min(\langle \Gamma, G \mid \Gamma' \rangle, \langle \Gamma, H \mid \Gamma' \rangle)$

Cases $3a, 3b$:

$\langle \Gamma \mid \Gamma', G \to H \rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(G \to H) =$
$= \langle \Gamma | \Gamma' \rangle + 1 - v_{\mathcal{M}}(G \to H) =$
$= \langle \Gamma | \Gamma' \rangle + 1 - \min(1, 1 - v_{\mathcal{M}}(G) + v_{\mathcal{M}}(H)) =$
$= \langle \Gamma | \Gamma' \rangle + \max(0, v_{\mathcal{M}}(G) - v_{\mathcal{M}}(H)) =$
$= \langle \Gamma | \Gamma' \rangle + \max(0, \langle G | H \rangle) =$
$= \max(\langle \Gamma \mid \Gamma' \rangle, \langle \Gamma, G \mid \Gamma', H \rangle)$

$\langle \Gamma, G \to H \mid \Gamma' \rangle = |\Gamma'| - |\Gamma| - 1 + v_{\mathcal{M}}(\Gamma) + v_{\mathcal{M}}(G \to H) - v_{\mathcal{M}}(\Gamma') =$
$= \langle \Gamma | \Gamma' \rangle - 1 + v_{\mathcal{M}}(G \to H) =$
$= \langle \Gamma | \Gamma' \rangle - 1 + \min(1, 1 - v_{\mathcal{M}}(G) + v_{\mathcal{M}}(H)) =$
$= \langle \Gamma | \Gamma' \rangle - 1 + \min(1, 1 + \langle H | G \rangle) =$
$= \langle \Gamma | \Gamma' \rangle + \min(0, \langle H | G \rangle) =$
$= \min(\langle \Gamma \mid \Gamma' \rangle, \langle \Gamma, H \mid \Gamma', G \rangle)$

Cases $4a, 4b$:

$\langle \Gamma \mid \Gamma', G \& H \rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(G \& H) =$
$= \langle \Gamma | \Gamma' \rangle + 1 - v_{\mathcal{M}}(G \& H) =$
$= \langle \Gamma | \Gamma' \rangle + 1 - \max(0, v_{\mathcal{M}}(G) + v_{\mathcal{M}}(H) - 1) =$
$= \langle \Gamma | \Gamma' \rangle + \min(1, (1 - v_{\mathcal{M}}(G)) + (1 - v_{\mathcal{M}}(H))) =$
$= \langle \Gamma | \Gamma' \rangle + \min(1, \langle | G, H \rangle) =$
$= \min(\langle \Gamma \mid \Gamma', \bot \rangle, \langle \Gamma \mid \Gamma', G, H \rangle)$

$\langle \Gamma, G \& H \mid \Gamma' \rangle = |\Gamma'| - |\Gamma| - 1 + v_{\mathcal{M}}(\Gamma) + v_{\mathcal{M}}(G \& H) - v_{\mathcal{M}}(\Gamma') =$
$= \langle \Gamma | \Gamma' \rangle - 1 + \max(0, v_{\mathcal{M}}(G) + v_{\mathcal{M}}(H) - 1) =$
$= \max(\langle \Gamma, \bot \mid \Gamma' \rangle, \langle \Gamma, G, H \mid \Gamma' \rangle)$

Cases $5a, 5b$ (cases $6a, 6b$ work by analogy):

$\langle\Gamma \mid \Gamma', \forall x G(x)\rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(\forall x G(x)) =$
$= \langle\Gamma|\Gamma'\rangle + 1 - \inf_{c\in D} v_{\mathcal{M}}(G(c)) =$
$= \langle\Gamma|\Gamma'\rangle + \sup_{c\in D}(1 - v_{\mathcal{M}}(G(c))) =$
$= \langle\Gamma|\Gamma'\rangle + \sup_{c\in D}\langle G(c)\rangle =$
$= \sup_{c\in D}\langle\Gamma \mid \Gamma', G(c)\rangle$

$\langle\Gamma, \forall x G(x) \mid \Gamma'\rangle = |\Gamma'| - |\Gamma| - 1 + v_{\mathcal{M}}(\Gamma) + v_{\mathcal{M}}(\forall x G(x)) - v_{\mathcal{M}}(\Gamma') =$
$= \langle\Gamma|\Gamma'\rangle - 1 + \inf_{c\in D} v_{\mathcal{M}}(G(c)) =$
$= \langle\Gamma|\Gamma'\rangle - (1 - \inf_{c\in D}(1 - \langle G(c)\rangle)) =$
$= \langle\Gamma|\Gamma'\rangle - \sup_{c\in D}\langle G(c)\rangle =$
$= \inf_{c\in D}\langle\Gamma, G(c) \mid \Gamma'\rangle$

These conditions hold independently of the order in which compound formulas are decomposed. Therefore, if $v_{\mathcal{M}}(F) = 1$, there is a winning strategy for the player I with risk assignment $\langle\cdot\rangle$, for any regulation. Also, recall that F is valid if and only if $\langle|F\rangle \leq 0$ for every valuation $v_{\mathcal{M}}$. Since this covers all possible risk value assignments $\langle\cdot\rangle$, the theorem follows. □

Note that we can define a strong disjunction connective $F \oplus G = \neg(\neg F \& \neg G)$, which then has the truth function $v_{\mathcal{M}}(F \oplus G) = \min(1, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G))$. One can also define the strong disjunction and the strong conjunction connectives directly via the implication connective, as we will do later, in Section 3.2, when we introduce the logic $Ł_\alpha(\Pi)$.

Remark 5 *Starting from a given logic* L*, we use the notation* L(Q) *for the logic that results from augmenting the language of* L *with the symbol Q, standing for a (unary) quantifier. The semantics of Q will be defined truth functionally as well as characterized by suitable rules of a corresponding semantic game. This notation will help achieving a concise notation.*

Remark 6 *For two formulas F, G and a fixed interpretation $\mathcal{M}$, we will sometimes write $F \equiv G$ to abbreviate that $v_{\mathcal{M}}((F \to G) \wedge (G \to F)) = 1$.*

2.4 Games with random choices - $\mathcal{RG}$-game

In [FR14], the authors proposed a further randomization of the $\mathcal{G}$-game, going beyond randomized payoffs for atomic formulas, by introducing a new non-strategic player **N**, called *Nature*. The player and its role are identical. It comes about as a new selection principle. Unlike the rational players, *Nature* never asserts any formula, and, upon call, decides without reason (or without vested interest) how the game continues. The overall structure of this augmented version of the $\mathcal{G}$-game remains unchanged, we only add two new rules to the framework, one for the object quantifier (Π) and one for the propositional connective (π) [Fer14], associated to the new principle, which we call *random witness selection principle*:

$\mathcal{R}_{\Pi}^{\mathcal{RG}}$**:** If **P** asserts $\Pi x F(x)$ then, if **O** attacks, **N** (uniformly) randomly chooses a constant $c \in D$, and **P** has to assert $F(c)$.

$\mathcal{R}_{\pi}^{\mathcal{RG}}$**:** If **P** asserts $F\pi G$, then, if **O** attacks, **N** chooses (uniformly) randomly whether **P** has to assert F or G.

Syntactically, we are considering the language of first-order Łukasiewicz logic enriched with the quantifier symbol Π and a symbol for the binary connective π. Then, we formally define Ł(Π, π) by specifying the syntactic rules and the semantics of Π and π:

Definition 6 *For a finite domain D we define:*

$$v_{\mathcal{M}}(\Pi x F(x)) = \frac{\sum_{c \in D} v_{\mathcal{M}}(F(c))}{|\,D\,|} \tag{2.4}$$

$$v_{\mathcal{M}}(F\pi G) = \frac{v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G)}{2} \tag{2.5}$$

Furthermore, for future reference, we also define[7]*:* $Prop_{\mathcal{M}}(F) = \frac{\sum_{c \in D} v_{\mathcal{M}}(F(c))}{|D|}$.

The upcoming definition determines the notion of logical consequence and in particular the set of tautologies of Ł(Π, π), i.e. its valid formulas.

Definition 7 *In Ł*(Π, π)*, a formula F is called a logical consequence of a set of formulas* Γ*, written* $\Gamma \models_{Ł(\Pi,\pi)} F$*, if for every evaluation* $v_{\mathcal{M}}$ *we have:*

If $v_{\mathcal{M}}(G) = 1$ *for all* $G \in \Gamma$*, then also* $v_{\mathcal{M}}(F) = 1$*.*

In particular, a formula F is called valid if for all evaluations $v_{\mathcal{M}}$ *we have* $v_{\mathcal{M}}(F) = 1$*. We denote that circumstance by* $\models_{Ł(\Pi,\pi)} F$*.*

[7] Note that, for crisp formulas $\hat{F}$, $Prop_{\mathcal{M}}(\hat{F})$ represents the proportion of elements from the domain that fulfill formula $\hat{F}$.

The focus on finite domains has two important reasons. One is the intention to model natural language quantifiers. In linguistics and natural language modeling, one usually restricts attention to finite domains [KH98], as explained in the introduction. The other, more technical, reason is the principle of *quantifier logicality*[8]. It demands that quantifier evaluations be independent of the identity of the constants [PW06], i.e. the evaluation only depends on the quantity of positive witnesses of a scope predicate, but not on which constants have the respective property. As an example consider the quantifier "about half" and a set of constants referring to all humans on the planet. Evaluating the statement "About half of all humans are female" should only depend on quantitative considerations but never on the assignment of constants to individual humans. This principle reduces the possible probability distributions, that **N** can represent, to the uniform one. This in turn forbids to consider arbitrary domains, as we will discuss later in more detail in Section 3.5.1.

To fully grasp the meaning of the following theorem, it is important to note that the quantifier Π and the connective π neatly fit with the realm of games with randomized payoff. Every single run of a game for a formula F involving these expressions can lead to different results, even if all atomic formulas in F are classical (i.e. assume only 0 or 1 as truth values). Nevertheless, we can talk about an expected payoff associated to such formulas, just as before.

Theorem 4 *[FR14] A Ł(Π, π) formula F is evaluated to $v_{\mathcal{M}}(F) = w$ in a fuzzy interpretation $\mathcal{M}$ iff every $\mathcal{G}$-game, augmented by the two rules $\mathcal{R}_{\Pi}^{\mathcal{RG}}$ and $\mathcal{R}_{\pi}^{\mathcal{RG}}$, for F, i.e. a game starting in the state $[|F]$, is $(1-w)$-valued for the player* I *under risk value assignment $\langle\cdot\rangle$ matching $\mathcal{M}$.*

Similarly to Theorem 3, the arguments as to the adequacy of the two rules have already been presented in their respective papers. However, to remain self-contained, we repeat them here.

Proof:
In addition to the proof of Theorem 3, we need to show the following four properties:

1a: $\langle \Gamma \mid \Gamma', \Pi x G(x)\rangle = \frac{1}{|D|}\sum_{c\in D}\langle \Gamma \mid \Gamma', G(c)\rangle$

2a: $\langle \Gamma \mid \Gamma', G\pi H\rangle = \frac{1}{2}\cdot(\langle \Gamma \mid \Gamma', G\rangle + \langle \Gamma \mid \Gamma', H\rangle)$

1b: $\langle \Gamma, \Pi x G(x) \mid \Gamma'\rangle = \frac{1}{|D|}\sum_{c\in D}\langle \Gamma, G(c) \mid \Gamma'\rangle$

2b: $\langle \Gamma, G\pi H \mid \Gamma'\rangle = \frac{1}{2}\cdot(\langle \Gamma, G \mid \Gamma'\rangle + \langle \Gamma, H \mid \Gamma'\rangle)$

Recall that these conditions result from the interpretation of the game rule. For case $1a$, since **N** chooses a constant $c \in D$ uniformly randomly, the risk on the left-hand side of

[8] Later, in Definition 41, this property will be introduced formally for arbitrary quantifiers

the equality, for the proponent player I, must be the average over the individual risks. Similarly for case $1b$. Analogous arguments apply to cases $2a, 2b$.

Hence, for case $1a$, we compute (case $1b$ works by analogy):

$$\langle \Gamma \mid \Gamma', \Pi x G(x)\rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(\Pi x G(x)) =$$
$$= \langle \Gamma|\Gamma'\rangle + 1 - v_{\mathcal{M}}(\Pi x G(x)) =$$
$$= \langle \Gamma|\Gamma'\rangle + 1 - \frac{\sum_{c\in D} v_{\mathcal{M}}(G(c))}{|D|} =$$
$$= \langle \Gamma|\Gamma'\rangle + 1 - \frac{\sum_{c\in D}(1-\langle G(c)\rangle)}{|D|} =$$
$$= \langle \Gamma|\Gamma'\rangle + \frac{\sum_{c\in D}\langle G(c)\rangle}{|D|} =$$
$$= \frac{1}{|D|}\sum_{c\in D}\langle \Gamma \mid \Gamma', G(c)\rangle$$

Case $2a$ (case $2b$ works by analogy):

$$\langle \Gamma \mid \Gamma', G\pi H\rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(G\pi H) =$$
$$= \langle \Gamma|\Gamma'\rangle + 1 - \tfrac{1}{2}(v_{\mathcal{M}}(G) + v_{\mathcal{M}}(H)) =$$
$$= \langle \Gamma|\Gamma'\rangle + 1 - \tfrac{1}{2}((1 - \langle G\rangle) + (1 - \langle H\rangle)) =$$
$$= \tfrac{1}{2}\cdot(\langle \Gamma \mid \Gamma', G\rangle + \langle \Gamma \mid \Gamma', H\rangle) \quad \square$$

In [FR14], the authors predominately focus on quantifiers applied to crisp scope formulas, a practice supported by Glöckner [Glö06] and others (e.g. [DHBB03, DRSV14]), in order to avoid unclarity regarding the interpretation of statements. Recall that quantifiers that, upon evaluation, can take intermediate truth values, even if restricted to crisp (or classical) arguments, are called *semi-fuzzy quantifiers*.

In [Bal16], Baldi introduced a hypersequent calculus for Kleene-Zadeh logic enriched with π. In the following, we will refer to this randomized version of Giles's game as $\mathcal{RG}$-game. One can see that propositions representing arbitrary truth values within the real unit interval can only be approximated. That means that for any $\epsilon > 0$ we can give a formula based π with a truth value that has a difference of the value, which we intend to approximate, of at most ϵ. As an example, let us consider the rational value $\frac{1}{3}$. We cannot express it using only the binary π as well as $\bot, \top$ (for $\top$ we use $\bot \to \bot$, i.e. $v_{\mathcal{M}}(\top) = 1$). However, we can define formulas that evaluate to $\frac{i}{2^j}$ with $i \in \{0, \ldots, 2^j\}$ where j is any positive integer, i.e. all dyadic rationals. It is well known that those are dense in the set of real numbers.

In [FM15], the authors show how rational truth values can propositionally be obtained as equilibrium values in a setting that augments both Łukasiewicz logic and IF logic [MSS11].

CHAPTER 3

The $\mathcal{NRG}$-game and t-norm based fuzzy logics

3.1 Fuzzy logics based on t-norms

In the Handbook of Mathematical Fuzzy Logic [CHN11], *t-norm* based fuzzy logics play a prominent role. T-norms are functions that are used as truth functions of conjunction connectives. To that end, one wants such functions to be commutative, as the truth of "A and B" should be the same as the one of "B and A", for two properties A, B. Also, "A and (B and C)" should always be as true as "(A and B) and C", for all properties A, B, C, hence one wants associativity. Furthermore, the conjunction with a true statement should not change the original truth value, and if A has a higher truth value than B, then "A and C" should have a higher truth value than "B and C", for all properties C. According to these principles, the definition of a t-norm is the following:

Definition 8 *A binary function* $* : [0,1]^2 \to [0,1]$ *is a t-norm if it is commutative, associative, monotone, and* 1 *is its unit element. More formally, in the respective order, we require:*

- $a * b = b * a$
- $a * (b * c) = (a * b) * c$
- $a * b \leq a' * b'$ *if* $a \leq a'$ *and* $b \leq b'$
- $a * 1 = a$

If $*$ is furthermore continuous on $[0,1]^2$ we call it a *continuous t-norm*. The following three are the Gödel, Product and Łukasiewicz t-norms respectively[1], which are all continuous:

- $x *_G y = \min(x, y)$
- $x *_P y = x \cdot y$
- $x *_Ł y = \max(0, x + y - 1)$

Theorem 5 *[CHN11] For any continuous t-norm $*$ there is a unique function $\Rightarrow$: $[0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$,*

$$z * x \leq y \quad \text{iff} \quad z \leq x \Rightarrow y. \tag{3.1}$$

The function $\Rightarrow$ is called the *residuum* of $*$. As t-norms are used to model the *conjunction* connective, $\Rightarrow$ can arguably be understood as respective *implication* connective [CHN11]. From this we can straightforwardly derive the residua of Gödel t-norm, Product t-norm and Łukasiewicz t-norm respectively[2]. For $x \leq y$ they all evaluate to 1, while for $x > y$ we have:

- $x \Rightarrow_G y = y$
- $x \Rightarrow_P y = y/x$
- $x \Rightarrow_Ł y = 1 - x + y$

Note that the residuum of the Łukasiewicz t-norm is the only of theses three that is continuous. However, it is enough to have a continuous t-norm $*$ to define *min* and *max* in terms of $*$ and and its residuum $\Rightarrow$ [CHN11]. Therefor, the three fuzzy logics can be understood as based on those two functions, t-norm and residuum, for (strong) conjunction and implication, while entailing also connectives for weak conjunction (modeled by *min*) and weak disjunction (modeled by *max*). Furthermore, the following relation between a t-norm and its residuum can be derived:

$$x \Rightarrow y = \max\{z : z * x \leq y\}. \tag{3.2}$$

Fuzzy logics that are based on continuous t-norms and their residuum are among the so called Mathematical Fuzzy Logic[3] (MFL). We will use the following notation for the rest

[1]The subscripts of the t-norms have the following meaning: G stands for Gödel, P for Product and Ł for Łukasiewicz

[2]The subscripts of the residua have the following meaning: G stands for Gödel, P for Product and Ł for Łukasiewicz

[3]The family of all MFLs is captured in a somewhat looser way, where one merely demands that the real unit interval forms the set of admissible truth values, and truth values of formulas can be determined by truth values of their subformulas via application of truth functions. See [Háj98, CHN11]. However, the most important conditions on truth functions are the ones described here, namely those that constitute t-norms.

of the thesis[4] to express the connectives with truth functions min, max, $\Rightarrow_\circ$ and $*_\circ$ (if different from min), where $\circ \in \{G, P, Ł\}$:

- for Gödel logic: $\wedge, \vee, \rightarrow^G$
- for Product logic: $\wedge, \vee, \rightarrow^P, \&^P$
- for Łukasiewicz logic: $\wedge, \vee, \rightarrow^Ł, \&^Ł$

The negation of a formula F is always defined as $F \rightarrow^\circ \bot$[5], while the universal and existential quantifier, $\forall$ and $\exists$, are uniformly expressed as before, in Section 2.2, for the case of the Kleene-Zadeh logic, i.e. their truth functions correspond to inf and *sup* respectively.

In the general case, this setting admits infinite domains, which are difficult when Π is to be given a meaningful semantics, as it is not always clear what a uniform distribution on an arbitrary set should be. Actually, on $\mathbb{N}$ and $\mathbb{R}$, it is not possible to define a uniform distribution in a meaningful way. As the remainder of this section is devoted to characterizing a wide class of fuzzy logics game semantically, where a key feature of the game is this random operator, we have to keep the restriction to finite domains. Hence, when we, in the following, talk about Gödel, Product and Łukasiewicz logic, we always mean these logics restricted to finite domains.

Employing Mostert-Shields' Theorem [CHN11, EGM04], there exists a representation for an arbitrary continuous t-norm as an ordinal sum of the three basic t-norms just introduced. We will show how we can define connectives with truth functions that evaluate in the same way as those given by a continuous t-norm which is representable as a finite ordinal sums, as well as their respective residua, in our setting which will formally be introduced in the following section. Furthermore, we will discuss means to extend our result to the case that admits even infinite ordinal sums.

3.2 $\mathcal{NRG}$-game, or: the logic $Ł_\alpha(\Pi)$

While $Ł(\Pi)$ features the connectives of Łukasiewicz logic, the ones for the other two basic t-norm based fuzzy logics are not all immediately at hand. From the game perspective, one can obtain an expressibility that is great enough to define also the truth functions of both Gödel and Product logic, by augmenting the set of game rules. First, we add an object quantifier generalizing strong Łukasiewicz conjunction, and then, for every object quantifier, we introduce a propositional version of it, i.e. the quantified formula may depend on a propositional variable [BCPV01]. We formally specify the syntax, of what

[4] If $\rightarrow$ or & appear without decoration, they are meant to be the Łukasiewicz ones.

[5] In [CHN11] this negation is called residual negation, as opposed to involutive negation.

we call[6] $Ł_\alpha(\Pi)$, and give a truth functional semantics for all connectives and quantifiers. Then, we characterize that logic game semantically with what we call the $\mathcal{NRG}$-game [Hof18].

Formulas are built by composition from the following expressions:

$$\gamma ::= \Lambda \mid w \mid P(\vec{t}) \mid \gamma \to \gamma \mid \gamma \wedge \gamma \mid \gamma \vee \gamma \mid \gamma\pi\gamma \mid \forall v\gamma \mid \exists v\gamma \mid \Pi v\gamma \mid \&v\gamma \mid$$
$$\forall w\gamma \mid \exists w\gamma \mid \mathbf{\Pi} w\gamma \mid \mathbf{\&} w\gamma$$

We have a meta variable Λ for constant symbols that stand for truth constants p_α with truth values[7] $\alpha \in [0,1]$, and P is our meta variable for predicate symbols. Then, $\vec{t}$ is a sequence of terms (either constant symbols or object variables), matching the arity of the preceding predicate symbol. Also, v is our meta variable for object variables, which we usually name $x, y, \ldots$, and w is our meta variable for propositional variables, which we usually name $p, q, \ldots$. We only assume countably many variables, while the language is uncountable due to the uncountably many constants that are represented by Λ. In the same way that object variables are placeholders for constants that refer to elements from the domain, propositional variables are placeholders for truth constants that refer to truth values from the real unit interval. We refer to the corresponding logic, for which we are subsequently going to introduce its truth functional semantics and its game semantics (as well as its notion of validity), of this language as $Ł_\alpha(\Pi)$. For any fixed interpretation $\mathcal{M}$, we always demand that its domain D be finite. The variable assignment $\xi_\mathcal{M}$ has to be extended in a way that it now assigns truth constants to free propositional variables, too. Truth functionally, atomic formulas are treated as in Łukasiewicz logic, i.e. for any atom A we have $v_\mathcal{M}(A) \in [0,1]$. Also, for all $p_\alpha \in \Lambda$ we have $v_\mathcal{M}(p_\alpha) = \alpha \in [0,1]$, and the rest of the truth functional semantics is defined as follows:

$$v_\mathcal{M}(F \to G) = \min(1, 1 - v_\mathcal{M}(F) + v_\mathcal{M}(G))$$

$$v_\mathcal{M}(F \wedge G) = \min(v_\mathcal{M}(F), v_\mathcal{M}(G))$$

$$v_\mathcal{M}(F \vee G) = \max(v_\mathcal{M}(F), v_\mathcal{M}(G))$$

$$v_\mathcal{M}(F\pi G) = \frac{v_\mathcal{M}(F)+v_\mathcal{M}(G)}{2}$$

$$v_\mathcal{M}(\forall x F(x)) = \inf_{c\in D} v_\mathcal{M}(F(c))$$

$$v_\mathcal{M}(\exists x F(x)) = \sup_{c\in D} v_\mathcal{M}(F(c))$$

[6] Note that the chosen notation constitutes a technical term, i.e. it represents exactly one logic and not a family. The α in the subscript indicates the dependency of some quantifiers on propositional variables rather than on object variables, which evaluate to numbers within the real unit interval, which we regularly denote by α.

[7] We use the common notation $\bot$ and $\top$ to express those truth value constants p_0 and p_1, with truth values 0 and 1.

$v_\mathcal{M}(\Pi x F(x)) = \frac{1}{|D|}\sum_{c\in D} v_\mathcal{M}(F(c)) = Prop_\mathcal{M}(F)$

$v_\mathcal{M}(\mathcal{\&} x F(x)) = \max(0, 1 - \sum_{c\in D}(1 - v_\mathcal{M}(F(c))))$

$v_\mathcal{M}(\forall p F(p)) = \inf_{\alpha\in[0,1]} v_\mathcal{M}(F(p_\alpha))$

$v_\mathcal{M}(\exists p F(p)) = \sup_{\alpha\in[0,1]} v_\mathcal{M}(F(p_\alpha))$

$v_\mathcal{M}(\mathbf{\Pi} p F(p)) = \int_0^1 v_\mathcal{M}(F(p_\alpha))d\alpha$

$v_\mathcal{M}(\mathbf{\&} p F(p)) = \max(0, 1 - \sup\{\sum_{\alpha\in J}(1 - v_\mathcal{M}(F(p_\alpha))) : J \subset [0,1], J \text{ finite}\})$

The last four quantifiers we refer to as propositional quantifiers. They are defined in the style of the other four quantifiers, to which we refer as object quantifiers, as they range over the set of constants that in turn refer to objects from the domain. Complementarily, the propositional quantifiers range over the set of truth constants that represents the truth values from the real unit interval. While $\mathcal{\&}$ generalizes strong Łukasiewicz conjunction to the object quantifier level, **&** does generalize $\mathcal{\&}$ to the level of propositional quantifiers. Similarly, $\forall, \exists$ and $\mathbf{\Pi}$ generalize their respective object quantifiers to the level of propositional quantifiers. This may still seem arbitrary, but the motivation can be seen in two different and complementary ways. For one, it is the goal of the present chapter to provide with a logic that increases the expressive power of the previously introduced logics, which will be achieved by these propositional quantifiers, as we will see shortly. On the other side, the particular choice of the semantics is based on the selection principles that correspond to the players of the $\mathcal{RG}$-game. Once we introduced the alternative (game) semantics, this will become rather obvious.

For the truth function of $\mathbf{\Pi}$ we chose the Lebesgue integral. We thereby guarantee existence, as $v_\mathcal{M}(F(p_\alpha)) \leq 1$ for all formulas F, which do not yet involve the propositional quantifier $\mathbf{\Pi}$, and $\alpha \in [0,1]$. Hence $\int_0^1 v_\mathcal{M}(F(p_\alpha))d\alpha \leq \int_0^1 d\alpha = 1 < \infty$. Again, this truth function generalizes the truth function of Π. The motivation of both is best understood from their game semantic characterization, which will be given further down.

For the truth function of **&**, note that, although the *supremum* may be infinite, the *maximum* still always takes values within the real unit interval.

Equivalence $F \leftrightarrow G$, of two formulas F and G, is defined as $(F \to G) \wedge (G \to F)$, i.e. $v_\mathcal{M}(F \leftrightarrow G) = 1 - \mid \min(v_\mathcal{M}(F), v_\mathcal{M}(G)) - \max(v_\mathcal{M}(F), v_\mathcal{M}(G)) \mid$, and the negation $\neg F$ of a formula F as $F \to \bot$, i.e. $v_\mathcal{M}(\neg F) = 1 - v_\mathcal{M}(F)$. Then we may define:

- $F\&G = \neg(F \to \neg G)$, leading to: $v_\mathcal{M}(F\&G) = \max(0, v_\mathcal{M}(F) + v_\mathcal{M}(G) - 1)$
- $F \oplus G = \neg F \to G$, leading to: $v_\mathcal{M}(F \oplus G) = \min(1, v_\mathcal{M}(F) + v_\mathcal{M}(G))$

- $F \ominus G = \neg(F \to G)$, leading to: $v_{\mathcal{M}}(F \ominus G) = \max(0, v_{\mathcal{M}}(F) - v_{\mathcal{M}}(G))$

Also, we can define k-ary versions of the connectives, associated to strategic players, as the simple iterations of the binary versions. For strong Łukasiewicz conjunction, this works as we have associativity, and hence the way brackets are placed does not affect the final truth value [Háj98].

- $\&_{i=1}^{k} F_i$, leading to: $v_{\mathcal{M}}(\&_{i=1}^{k} F_i) = \max(0, 1 - \sum_{i=1}^{k}(1 - v_{\mathcal{M}}(F_i)))$
- $[F_i]_{\oplus}^{1 \leq i \leq k}$, leading to: $v_{\mathcal{M}}([F_i]_{\oplus}^{1 \leq i \leq k}) = \min(1, \sum_{i=1}^{k} v_{\mathcal{M}}(F_i))$
- $\wedge_{i=1}^{k} F_i$, leading to: $v_{\mathcal{M}}(\wedge_{i=1}^{k} F_i) = \min\{v_{\mathcal{M}}(F_i) : 1 \leq i \leq k\}$
- $\vee_{i=1}^{k} F_i$, leading to: $v_{\mathcal{M}}(\vee_{i=1}^{k} F_i) = \max\{v_{\mathcal{M}}(F_i) : 1 \leq i \leq k\}$

Defining a k-ary π with a truth function that takes the average of k formulas cannot be done straightforwardly, as e.g., for three formulas F, G, H, we have $(F \pi G) \pi H \neq \frac{v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G) + v_{\mathcal{M}}(H)}{3}$. One would need to define such a k-ary ($k \geq 2$) π from the start. However, formulas with truth values that are arbitrarily close to any number in $[0, 1]$ can be defined via the binary π already, as we have seen at the end of Chapter 2. Syntactically, we just added uncountably many truth constants to the language of $Ł_{\alpha}(\Pi)$, while the expressive power of $Ł_{\alpha}(\Pi)$ already gives us all of them up to an infinitesimal error at most, i.e. an error that can be made arbitrarily small. A similar situation arises from Giles's game, where the risk of asserting an atomic formula is determined by an associated dispersive experiment [Gil82], which, by the very idea of tangible meaning, can only be executed finitely many times. Hence, rational and also irrational truth values appear only as an idealization.

It should further be noted that (1), we can not replace the implication connective $\to$ with one for strong conjunction & and define the implication via it, as that would need another rule for negation as well, and (2), for & is not idempotent[8], also *&* and **&** behave unlike usual quantifiers, as, e.g., if the quantified formula F is independent of the quantifier variable, the truth value of F might still be changed from the quantifier application. Object quantifiers based on t-norms are also discussed in [Got13].

We now characterize $Ł_{\alpha}(\Pi)$ game semantically, by giving game rules for the basic connectives and quantifiers, with payoffs for **P** matching the truth functional semantics. In that way, we show that a justification, regarding the choice of the truth functions, that is based on the selection principles the players adhere to, exists. As these selection principles are formulated with an appeal more intuitive than technical, one may even call them philosophically motivated. Furthermore we show that these three selection

[8] A binary operation $\star$ on a set A is called *idempotent* if $a \star a = a$ for all $a \in A$ [CHN11].

principles together are strong enough to express the discontinuous Gödel implication as well as Product implication and conjunction. The underlying language provides us with propositions p_α for all $\alpha \in [0,1]$ such that $v_\mathcal{M}(p_\alpha) = \alpha$. As usual, the proponent player optimizes her payoff, while the opponent tries to minimize it. In a way dual to that, the third player acts neutrally and selects uniformly randomly. We state the twelve rules of the $\mathcal{NRG}$-game, while the evaluation of states and atoms is as in the $\mathcal{RG}$-game (like the first seven rules). The $\mathcal{N}$ in $\mathcal{NRG}$ stands for *number*. For a fixed interpretation $\mathcal{M}$ with finite domain D, we then define:

$\mathcal{R}_\to^{\mathcal{NRG}}$**:** If **P** asserts $F \to G$, then **O** attacks by asserting F, obliging **P** to assert G.

$\mathcal{R}_\wedge^{\mathcal{NRG}}$**:** If **P** asserts $F \wedge G$, **O** chooses whether **P** must assert F or G.

$\mathcal{R}_\vee^{\mathcal{NRG}}$**:** If **P** asserts $F \vee G$, **P** chooses whether **P** must assert F or G.

$\mathcal{R}_\pi^{\mathcal{NRG}}$**:** If **P** asserts $F \pi G$, **N** chooses whether **P** must assert F or G.

$\mathcal{R}_\forall^{\mathcal{NRG}}$**:** If **P** asserts $\forall x F(x)$, **O** chooses $c \in D$, and **P** must assert $F(c)$.

$\mathcal{R}_\exists^{\mathcal{NRG}}$**:** If **P** asserts $\exists x F(x)$, **P** chooses $c \in D$, and **P** must assert $F(c)$.

$\mathcal{R}_\Pi^{\mathcal{NRG}}$**:** If **P** asserts $\Pi x F(x)$, **N** chooses $c \in D$, and **P** must assert $F(c)$.

$\mathcal{R}_{\&}^{\mathcal{NRG}}$**:** If **P** asserts $\& x F(x)$, then, if **O** attacks, **P** must assert all $F(c)$, $c \in D$, or $\bot$.

$\mathcal{R}_{\boldsymbol{\forall}}^{\mathcal{NRG}}$**:** If **P** asserts $\boldsymbol{\forall} p F(p)$, **O** chooses $p_\alpha \in \Lambda$, and **P** must assert $F(p_\alpha)$.

$\mathcal{R}_{\boldsymbol{\exists}}^{\mathcal{NRG}}$**:** If **P** asserts $\boldsymbol{\exists} p F(p)$, **P** chooses $p_\alpha \in \Lambda$, and **P** must assert $F(p_\alpha)$.

$\mathcal{R}_{\boldsymbol{\Pi}}^{\mathcal{NRG}}$**:** If **P** asserts $\boldsymbol{\Pi} p F(p)$, **N** chooses $p_\alpha \in \Lambda$, and **P** must assert $F(p_\alpha)$.

$\mathcal{R}_{\boldsymbol{\&}}^{\mathcal{NRG}}$**:** If **P** asserts $\boldsymbol{\&} p F(p)$, then **O** attacks by choosing a finite $J \subset \Lambda$ and **P** must assert $F(p_\alpha)$ for all $p_\alpha \in J$, or $\bot$.

Remark 7 *The motivation for defining such games is the tangibility of these rules. They are more tangible than truth functions, and often people understand the intended semantics better when it is explained in the realm of dialogue games between players. The implication rule is a particularly good example for that, but the instances of the random witness selection principle ($\pi, \Pi, \boldsymbol{\Pi}$) are also much easier to justify when explained as augmenting games with only strategic reasoning, as for example the $\mathcal{G}$-game. This is because in that way we can argue that the random witness selection complements the strategic witness* *selection, which should make intuitive sense. On the other hand, introducing truth functions that compute the average of values, hence leading to intermediate truth values in the real unit interval, may seem* ad hoc*, if no further background principle, that justifies this choice, is provided.*

The upcoming definition determines the notion of logical consequence and in particular the set of tautologies of $Ł_\alpha(\Pi)$, i.e. its valid formulas.

Definition 9 *In $Ł_\alpha(\Pi)$, a formula F is called a logical consequence of a set of formulas Γ, written $\Gamma \models_{Ł_\alpha(\Pi)} F$, if for every evaluation $v_\mathcal{M}$ we have:*

If $v_\mathcal{M}(G) = 1$ for all $G \in \Gamma$, then also $v_\mathcal{M}(F) = 1$.

In particular, a formula F is called valid if for all evaluations $v_\mathcal{M}$ we have $v_\mathcal{M}(F) = 1$. We denote that circumstance by $\models_{Ł_\alpha(\Pi)} F$.

The following theorem explains the correspondence of the truth functional semantics and the evaluation of these game rules, while the notion of risk is the one of Definition 4 as it stands.

Theorem 6 *[Hof18] A $Ł_\alpha(\Pi)$ formula F is evaluated to $v_\mathcal{M}(F) = w$ in a fuzzy interpretation $\mathcal{M}$ iff every $\mathcal{NRG}$-game for F, i.e. a game starting in the state $[|F]$, is $(1-w)$-valued for the player* I *under risk value assignment $\langle\cdot\rangle$ matching $\mathcal{M}$.*

Proof:
As we have Theorem 3 and Theorem 4, we only have to consider the cases where the $\mathcal{NRG}$-game, and hence the language of $Ł_\alpha(\Pi)$, differs from the $\mathcal{RG}$-game, and the language of $Ł(\Pi, \pi)$ respectively. This is the case for $\mathcal{E}, \forall, \exists, \mathbf{\Pi}$, and **&**. Truth value constants need no special treatment, as they behave like atoms. If the game for F starts, it is in the state $[|\ F]$. Leaving out the shared basics with the $\mathcal{RG}$-game (see Theorem 3 and Theorem 4), we have to treat the following ten cases (G, H are a $Ł_\alpha(\Pi)$ formulas, Γ, Γ' multisets of $Ł_\alpha(\Pi)$ formulas, and $\alpha = v_\mathcal{M}(p_\alpha)$).

1a: $\langle \Gamma \mid \Gamma', \mathcal{E}xG\rangle = \min(\langle \Gamma \mid \Gamma', \bot\rangle, \sum_{c \in D}\langle \Gamma \mid \Gamma', G(c)\rangle)$

2a: $\langle \Gamma \mid \Gamma', \forall pG\rangle = \sup_{\alpha \in [0,1]}\langle \Gamma \mid \Gamma', G(p_\alpha)\rangle$

3a: $\langle \Gamma \mid \Gamma', \exists pG\rangle = \inf_{\alpha \in [0,1]}\langle \Gamma \mid \Gamma', G(p_\alpha)\rangle$

4a: $\langle \Gamma \mid \Gamma', \mathbf{\Pi} pG\rangle = \int_0^1 \langle \Gamma \mid \Gamma', G(p_\alpha)\rangle d\alpha$

5a: $\langle \Gamma \mid \Gamma', \mathbf{\&} pG\rangle = \min(\langle \Gamma \mid \Gamma', \bot\rangle, \sup_{\{J \subset [0,1], J \text{ finite}\}} \sum_{\alpha \in J}\langle \Gamma \mid \Gamma', G(p_\alpha)\rangle)$

1b: $\langle\Gamma, \&xG \mid \Gamma'\rangle = \max(\langle\Gamma, \bot \mid \Gamma'\rangle, \sum_{c\in D}\langle\Gamma, G(c) \mid \Gamma'\rangle)$

2b: $\langle\Gamma, \forall pG \mid \Gamma'\rangle = \inf_{\alpha\in[0,1]}\langle\Gamma, G(p_\alpha) \mid \Gamma'\rangle$

3b: $\langle\Gamma, \exists pG \mid \Gamma'\rangle = \sup_{\alpha\in[0,1]}\langle\Gamma, G(p_\alpha) \mid \Gamma'\rangle$

4b: $\langle\Gamma, \mathbf{\Pi}pG \mid \Gamma'\rangle = \int_0^1\langle\Gamma, G(p_\alpha) \mid \Gamma'\rangle d\alpha$

5b: $\langle\Gamma, \mathbf{\&}pG \mid \Gamma'\rangle = \max(\langle\Gamma, \bot \mid \Gamma'\rangle, \sup_{\{J\subset[0,1], J \text{ finite}\}}\sum_{\alpha\in J}\langle\Gamma, G(p_\alpha) \mid \Gamma'\rangle)$

Recall that the conditions result from the interpretation of the respective game rules. For case 1a, the game rule prescribes to either assert $\bot$ or all instances $G(c)$ $(c \in D)$ if $\&xG(x)$ has been asserted previously, and as **P** can choose herself, she is able to enforce the minimum over these risks. Regarding case 2a, in the game, the opponent player *You* can enforce the maximum over all risks $\langle\Gamma|\Gamma', G(p_\alpha)\rangle$ $(\alpha \in [0,1])$. Similarly for case 3a, the proponent player I can enforce the minimum over the same risks as in case 2a. For case 4a, we have to consider that **N** samples uniformly randomly from the real unit interval, hence the average risk is computed as the integral over all results. Eventually for case 5a, the opponent player *You* chooses a set J corresponding to finitely many values from the real unit interval, and the proponent player I can choose whether to assert $\bot$ or $G(p_\alpha)$ for all $\alpha \in J$. The conditions for cases $1b - 5b$ are obtained by analogous arguments.

Case 1a (case 1b works by analogy):

$$\langle\Gamma \mid \Gamma', \&xG\rangle = |\Gamma'| + 1 - |\Gamma| + v_\mathcal{M}(\Gamma) - v_\mathcal{M}(\Gamma') - v_\mathcal{M}(\&xG) =$$
$$= \langle\Gamma|\Gamma'\rangle + 1 - \max(0, 1 - \textstyle\sum_{c\in D}(1 - v_\mathcal{M}(G(c)))) =$$
$$= \langle\Gamma|\Gamma'\rangle + \min(1, \textstyle\sum_{c\in D}\langle G(c)\rangle) =$$
$$= \min(\langle\Gamma \mid \Gamma', \bot\rangle, \textstyle\sum_{c\in D}\langle\Gamma \mid \Gamma', G(c)\rangle)$$

Case 2a (case 2b works by analogy):

$$\langle\Gamma \mid \Gamma', \forall pG\rangle = |\Gamma'| + 1 - |\Gamma| + v_\mathcal{M}(\Gamma) - v_\mathcal{M}(\Gamma') - v_\mathcal{M}(\forall pG) =$$
$$= \langle\Gamma|\Gamma'\rangle + 1 - \inf_{\alpha\in[0,1]}v_\mathcal{M}(G(p_\alpha)) =$$
$$= \langle\Gamma|\Gamma'\rangle + \sup_{\alpha\in[0,1]}\langle G(p_\alpha)\rangle =$$
$$= \sup_{\alpha\in[0,1]}\langle\Gamma \mid \Gamma', G(p_\alpha)\rangle$$

Case 3a (case 3b works by analogy):

$$\langle\Gamma \mid \Gamma', \exists pG\rangle = |\Gamma'| + 1 - |\Gamma| + v_\mathcal{M}(\Gamma) - v_\mathcal{M}(\Gamma') - v_\mathcal{M}(\exists pG) =$$
$$= \langle\Gamma|\Gamma'\rangle + 1 - \sup_{\alpha\in[0,1]}v_\mathcal{M}(G(p_\alpha)) =$$

$= \langle \Gamma | \Gamma' \rangle + \inf_{\alpha \in [0,1]} \langle G(p_\alpha) \rangle =$

$= \inf_{\alpha \in [0,1]} \langle \Gamma \mid \Gamma', G(p_\alpha) \rangle$

Case $4a$ (case $4b$ works by analogy):

$\langle \Gamma \mid \Gamma', \mathbf{\Pi} pG \rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(\mathbf{\Pi} pG) =$

$= \langle \Gamma | \Gamma' \rangle + 1 - \int_0^1 v_{\mathcal{M}}(G(p_\alpha)) d\alpha =$

$= \langle \Gamma | \Gamma' \rangle + \int_0^1 \langle G(p_\alpha) \rangle d\alpha =$

$= \int_0^1 \langle \Gamma \mid \Gamma', G(p_\alpha) \rangle d\alpha$

Case $5a$ (case $5b$ works by analogy):

$\langle \Gamma \mid \Gamma', \& pG \rangle = |\Gamma'| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Gamma') - v_{\mathcal{M}}(\& pG) =$

$= \langle \Gamma | \Gamma' \rangle + 1 - \max(0, 1 - \sup_{\{J \subset [0,1], J \text{ finite}\}} \sum_{\alpha \in J} (1 - v_{\mathcal{M}}(G(p_\alpha)))) =$

$= \langle \Gamma | \Gamma' \rangle + 1 - \max(0, 1 - \sup_{\{J \subset [0,1], J \text{ finite}\}} \sum_{\alpha \in J} \langle G(p_\alpha) \rangle) =$

$= \langle \Gamma | \Gamma' \rangle + \min(1, \sup_{\{J \subset [0,1], J \text{ finite}\}} \sum_{\alpha \in J} \langle G(p_\alpha) \rangle) =$

$= \min(\langle \Gamma \mid \Gamma', \bot \rangle, \sup_{\{J \subset [0,1], J \text{ finite}\}} \sum_{\alpha \in J} \langle \Gamma \mid \Gamma', G(p_\alpha) \rangle)$

Note that the notion of Definition 4 with $\epsilon > 0$ is needed here, for the cases $5a, 5b$. It is buried inside the *sup*, as here an approximation with infinitesimal error takes place. □

Remark 8 *In the remainder of this chapter, we will show how one can define the truth functions of certain MFLs by the truth functions of* Ł$_\alpha$(Π)*. In that sense, we may also speak about term-definability of a logic in another.*

3.3 Definability of Gödel logic in Ł$_\alpha$(Π)

Gödel implication can be defined in Łukasiewicz logic enriched with the Delta operator [CFN15, CHN11]. This operator allows for expressing discontinuities, and can be defined as follows:

Definition 10 *For* Ł$_\alpha$(Π) *formulas F we define:*

$$v_{\mathcal{M}}(\Delta F) = \begin{cases} 1 & \textit{if } v_{\mathcal{M}}(F) = 1 \\ 0 & \textit{otherwise} \end{cases} \tag{3.3}$$

Theorem 7 *[Hof18] For $Ł_\alpha(\Pi)$ formulas F, in which the propositional variable p does not occur, we have:*

$$v_{\mathcal{M}}(\Delta F) = v_{\mathcal{M}}(\&pF). \tag{3.4}$$

To prove this we need the following lemma[9]:

Lemma 1 *[Fol99] Let I be an uncountable index-set, and $f : I \to [0,1]$. If we have:*

$$\sum_{i\in I} f(i) < \infty, \text{ then for all but countably many } i \in I \text{ we also have } f(i) = 0. \tag{3.5}$$

We now prove Theorem 7:

Proof:
We prove this by case distinction:

Case (1): If $v_{\mathcal{M}}(F) = 1$, the sum in the truth function of $\&pF$ disappears, and we get $v_{\mathcal{M}}(\&pF) = 1$.

Case (2): If $v_{\mathcal{M}}(F) < 1$, then, in the limit, the sum of the truth function of $\&pF$ contains uncountably many positive terms, which by Lemma 1 means it is infinite, and hence $v_{\mathcal{M}}(\&pF) = 0$. □

Remark 9 *If we consider a game for $\&pF$ an interesting situation arises. If $v_{\mathcal{M}}(F) = 1$, the opponent player triggers her part of the principle of limited liability and will hence not attack. On the other hand, if $v_{\mathcal{M}}(F) < 1$, the proponent player triggers her part of the principle of limited liability and assert $\perp$ instead of a lot of F's with each having a positive risk. It is clear that, if F carries a positive risk, the opponent player can find a natural number m such that m times the risk of F is at least 1.*

Gödel conjunction is of course already part of $Ł(\Pi)$, and Gödel implication can be defined as follows:

Definition 11 *For two $Ł_\alpha(\Pi)$ formulas F, G, in which the propositional variable p does not occur, we define:*

$$F \twoheadrightarrow G = \&p(F \to G) \vee G. \tag{3.6}$$

Remark 10 *As the definition shows, we technically do not need the restriction to finite domains at all to define $\twoheadrightarrow$. The overall restriction to finite domains results from the presence of the object quantifier Π, as explained in the second half of Section 2.4, and in Section 3.5.1.*

[9]Note that the maybe unfamiliar sum over an uncountable index-set is defined as follows (using the terminology of the subsequent definition): $\sum_{i\in I} f(i) = \sup\{\sum_{i\in J} f(i) : J \in I \text{ with } J \subset I \text{ finite}\}$.

Remark 11 *Abusing notation, we will sometimes also write ΔF instead of $\&pF$, when it is clear from the context that we want to stay in $Ł_\alpha(\Pi)$, to ease the readability of formulas. In that case, we agree that inside the formula, inside the scope of that Δ (or better $\&p$), the propositional variable p does not occur.*

Theorem 8 *[Hof18] For two $Ł_\alpha(\Pi)$ formulas F, G, in which the propositional variable p does not occur, we have:*

$$v_{\mathcal{M}}(F \twoheadrightarrow G) = v_{\mathcal{M}}(F) \Rightarrow_G v_{\mathcal{M}}(G). \tag{3.7}$$

Proof:
To prove the theorem, we make the obvious case distinction:

(i): $v_{\mathcal{M}}(G) \geq v_{\mathcal{M}}(F)$

In that case, $v_{\mathcal{M}}(\&p(F \to G)) = 1$.

(ii): $v_{\mathcal{M}}(G) < v_{\mathcal{M}}(F)$

In that case, $v_{\mathcal{M}}(\&p(F \to G)) = 0$, hence the claim follows. □

Theorem 9 *[Hof18] Considering only interpretations with finite domains, we can define the truth functions of Gödel logic via the truth functions of $Ł_\alpha(\Pi)$.*

Proof:
As we have Gödel conjunction in $Ł_\alpha(\Pi)$, and since we can define a connective which evaluates like Gödel implication, the claim follows for Gödel logic restricted to finite domains through [EGM04], which gives us a finite axiomatization for any continuous t-norm and its residuum. □

3.3.1 Definability of Product logic in $Ł_\alpha(\Pi)$

As for Product logic, we have to be able to express multiplication and division, of real numbers in $[0, 1]$, on the truth functional level. For this, we can use the propositional quantifiers, especially **Π**. We can directly define:

Definition 12 *For two $Ł_\alpha(\Pi)$ formulas F and G, in which the propositional variables p and q do not occur*[10]*, we define:*

$$F \cdot G = \mathbf{\Pi} p \mathbf{\Pi} q(\Delta(p \to F) \wedge \Delta(q \to G)), \text{ and} \tag{3.8}$$

$$F \rightarrowtail G = \exists p(p \wedge \Delta(p \cdot F \to G)). \tag{3.9}$$

[10]Recall that we, abusing notation, write Δ instead of $\&p$. This convention implicitly demands that the formula in the scope of Δ (or $\&p$) be independent of the variable bound by $\&$.

From this construction we can see, that it were not enough to restrict Λ to countably many constants representing $[0,1] \cap \mathbb{Q}$, as the Lebesgue measure of that set is zero. Also, recall Remark 10.

Theorem 10 *[Hof18] For two $Ł_\alpha(\Pi)$ formulas F and G, as in Definition 12, we have:*

$$v_{\mathcal{M}}(F \cdot G) = v_{\mathcal{M}}(F) *_P v_{\mathcal{M}}(G), \tag{3.10}$$

$$v_{\mathcal{M}}(F \rightarrowtail G) = v_{\mathcal{M}}(F) \Rightarrow_P v_{\mathcal{M}}(G). \tag{3.11}$$

Proof:
For the conjunction connective $\cdot$, we assume, without loss of generality, that $v_{\mathcal{M}}(F) \leq v_{\mathcal{M}}(G)$. Also, $v_{\mathcal{M}}(p_\alpha) = \alpha, v_{\mathcal{M}}(p_{\alpha'}) = \alpha'$, with $\alpha, \alpha' \in [0,1]$. Then we have:

$v_{\mathcal{M}}(F \cdot G) = \int_0^1 \int_0^1 \min(v_{\mathcal{M}}(\Delta(p_\alpha \to F)), v_{\mathcal{M}}(\Delta(p_{\alpha'} \to G)))d\alpha d\alpha' =$
$= \int_0^1 (\int_0^{v_{\mathcal{M}}(F)} \min(1, v_{\mathcal{M}}(\Delta(p_{\alpha'} \to G)))d\alpha)d\alpha' =$
$= \int_0^1 (\int_0^{v_{\mathcal{M}}(F)} v_{\mathcal{M}}(\Delta(p_{\alpha'} \to G))d\alpha)d\alpha' = \int_0^1 v_{\mathcal{M}}(\Delta(p_{\alpha'} \to G))d\alpha' \cdot \int_0^{v_{\mathcal{M}}(F)} 1 d\alpha =$
$= \int_0^{v_{\mathcal{M}}(G)} 1 d\alpha' \cdot \int_0^{v_{\mathcal{M}}(F)} 1 d\alpha = v_{\mathcal{M}}(G) \cdot v_{\mathcal{M}}(F) = v_{\mathcal{M}}(F) *_P v_{\mathcal{M}}(G).$

For the implication connective $\rightarrowtail$ it is clear that $v_{\mathcal{M}}(F \rightarrowtail G) = 1$ if $v_{\mathcal{M}}(F) \leq v_{\mathcal{M}}(G)$. We can hence pay attention to the case where $v_{\mathcal{M}}(F) > v_{\mathcal{M}}(G)$:

$v_{\mathcal{M}}(F \rightarrowtail G) = \sup_{\alpha \in [0,1]} \min(v_{\mathcal{M}}(p_\alpha), v_{\mathcal{M}}(\Delta(p_\alpha \cdot F \to G))).$

From that line we see that the supremum is attained if $v_{\mathcal{M}}(p_\alpha \cdot F) = v_{\mathcal{M}}(G)$, which is exactly when $v_{\mathcal{M}}(p_\alpha) \cdot v_{\mathcal{M}}(F) = v_{\mathcal{M}}(G)$. Hence together,

$v_{\mathcal{M}}(F \rightarrowtail G) = v_{\mathcal{M}}(F) \Rightarrow_P v_{\mathcal{M}}(G).$ □

Theorem 11 *[Hof18] Considering only interpretations with finite domains, we can define the truth functions of Product logic via the truth functions of $Ł_\alpha(\Pi)$.*

Proof:
As we have defined connectives in $Ł_\alpha(\Pi)$ that model the implication and conjunction connective of Product logic, the claim follows for Product logic restricted to finite domains through [EGM04], which gives us a finite axiomatization for any continuous t-norm and its residuum. □

3.4 Definability of MFLs in Ł$_\alpha(\Pi)$

Continuous t-norms can be expressed as ordinal sums of the three basic t-norms, namely Gödel, Product and Łukasiewicz. This is what Mostert-Shields' Theorem gives us [CHN11]. We are going to represent t-norms that correspond to finite sums, as well as their residua, in Ł$_\alpha(\Pi)$. Then, we use [EGM04], to infer that, since we have a continuous t-norm and its residuum, a finite axiomatization for the corresponding fuzzy logic is effectively obtainable. Again, we only treat cases with finite domains.

Theorem 12 *[CHN11] Every continuous t-norm is isomorphic to an ordinal sum of the three basic t-norms (Gödel, Product and Łukasiewicz).*

Theorem 13 *[Hof18] Considering only interpretations with finite domains, we can define the truth functions of all fuzzy logics that are based on a continuous t-norm that is representable as a finite ordinal sum of the three basic ones, i.e. those corresponding to Gödel logic, Product logic and Łukasiewicz logic, via the truth functions of* Ł$_\alpha(\Pi)$.

Before we prove the theorem, let us recall Remark 10. The restriction to finite domains comes from the fact that we are dealing with Ł$_\alpha(\Pi)$, which has the object quantifier Π as an essential feature. Technically, one can also remove this object quantifier from the language, and only use the propositional version of it. Although in that way we can generalize the result to arbitrary domains, we thereby would lose an important component of our framework, which is particularly necessary when we want to model NL quantifiers.

Proof:
From Theorem 12 (Mostert-Shields' Theorem) [CHN11], we get, for any fixed continuous t-norm T, an ordinal sum decomposition $(*_{\iota_i}([a_i, b_i]))_{i\geq 1}$, $\iota_i \in \{\text{Ł}, P\}, i \geq 1$, such that we have:

$$x *_T y = \begin{cases} a_i + (b_i - a_i)(\frac{x-a_i}{b_i-a_i} *_{\text{Ł}} \frac{y-a_i}{b_i-a_i}) & \text{if } (x,y) \in [a_i, b_i]^2 \text{ and } \iota_i = \text{Ł} \\ a_i + (b_i - a_i)(\frac{x-a_i}{b_i-a_i} *_P \frac{y-a_i}{b_i-a_i}) & \text{if } (x,y) \in [a_i, b_i]^2 \text{ and } \iota_i = P \\ x *_G y & \text{otherwise} \end{cases} \tag{3.12}$$

We have to provide with a definition representing those continuous t-norms $*_T$ as Ł$_\alpha(\Pi)$ formulas for which there is a positive integer τ such that the ordinal sum consists of τ parts. The case $\tau = 1$ is trivial, so from now on we assume $\tau = k \geq 2$ fixed. For the appropriate truth constants, representing the interval bounds a and b, called[11] $\overline{a}$ and $\overline{b}$, we define for two Ł$_\alpha(\Pi)$ formulas F and G:

$F \&^{\iota}_{a,b} G =$

[11] Note that here, to ease readability, we employ a slightly different notation to refer to truth constants than usually.

$= \overline{a} \oplus ((\overline{b} \ominus \overline{a}) \cdot (((\overline{b} \ominus \overline{a}) \rightarrowtail (F \ominus \overline{a}))\&^{\iota}((\overline{b} \ominus \overline{a}) \rightarrowtail (G \ominus \overline{a})))).$

The connective $\&^T$ representing the full t-norm $*_T$ can now be defined as:

$F\&^T G =$

$= \vee_{i=1}^{k}(\Delta((\overline{a_i} \rightarrow F) \wedge (\overline{a_i} \rightarrow G) \wedge (F \rightarrow \overline{b_i}) \wedge (G \rightarrow \overline{b_i})) \wedge (F\&_{a_i,b_i}^{\iota_i} G)) \vee$

$\vee (\wedge_{i=1}^{k}(\neg\Delta(F \rightarrow \overline{b_i}) \vee \neg\Delta(G \rightarrow \overline{b_i}) \vee \neg\Delta(\overline{a_i} \rightarrow F) \vee \neg\Delta(\overline{a_i} \rightarrow G)) \wedge (F\&^G G)).$

It remains to verify that the $F\&^T G$ really possesses the truth function matching the definition of $*_T$. So, (1) assume there is an $1 \leq i \leq k$ such that $v_{\mathcal{M}}(F), v_{\mathcal{M}}(G) \in [a_i, b_i]$, which must be unique. That means:

$v_{\mathcal{M}}(\vee_{i=1}^{k}\Delta((\overline{a_i} \rightarrow F) \wedge (\overline{a_i} \rightarrow G) \wedge (F \rightarrow \overline{b_i}) \wedge (G \rightarrow \overline{b_i}))) = 1$, and

$v_{\mathcal{M}}(\wedge_{i=1}^{k}(\neg\Delta(F \rightarrow \overline{b_i}) \vee \neg\Delta(G \rightarrow \overline{b_i}) \vee \neg\Delta(\overline{a_i} \rightarrow F) \vee \neg\Delta(\overline{a_i} \rightarrow G)) = 0.$

Hence, $v_{\mathcal{M}}(F\&^T G) = v_{\mathcal{M}}(F\&_{a_i,b_i}^{\iota_i} G) = v_{\mathcal{M}}(\overline{a_i}) +$

$+ v_{\mathcal{M}}(\overline{b_i} \ominus \overline{a_i}) \cdot v_{\mathcal{M}}(((\overline{b_i} \ominus \overline{a_i}) \rightarrowtail (F \ominus \overline{a_i}))\&_{a_i,b_i}^{\iota_i}((\overline{b_i} \ominus \overline{a_i}) \rightarrowtail (G \ominus \overline{a_i}))).$

The truth values of the two conjuncts of $\&_{a_i,b_i}^{\iota_i}$ are:

$v_{\mathcal{M}}(((\overline{b_i} \ominus \overline{a_i}) \rightarrowtail (F \ominus \overline{a_i}))) = \frac{v_{\mathcal{M}}(F \ominus \overline{a_i})}{v_{\mathcal{M}}(\overline{b_i} \ominus \overline{a_i})} = \frac{v_{\mathcal{M}}(F) - v_{\mathcal{M}}(\overline{a_i})}{v_{\mathcal{M}}(\overline{b_i}) - v_{\mathcal{M}}(\overline{a_i})}$, and

$v_{\mathcal{M}}(((\overline{b_i} \ominus \overline{a_i}) \rightarrowtail (G \ominus \overline{a_i}))) = \frac{v_{\mathcal{M}}(G \ominus \overline{a_i})}{v_{\mathcal{M}}(\overline{b_i} \ominus \overline{a_i})} = \frac{v_{\mathcal{M}}(G) - v_{\mathcal{M}}(\overline{a_i})}{v_{\mathcal{M}}(\overline{b_i}) - v_{\mathcal{M}}(\overline{a_i})}.$

That gives us, that:

$v_{\mathcal{M}}(F\&^T G) = a_i + (b_i - a_i) \cdot (\frac{v_{\mathcal{M}}(F) - a_i}{b_i - a_i} *_{\iota_i} \frac{v_{\mathcal{M}}(G) - a_i}{b_i - a_i}).$

On the other hand (2), if there is no $1 \leq i \leq k$ such that $v_{\mathcal{M}}(F), v_{\mathcal{M}}(G) \in [a_i, b_i]$, then:

$v_{\mathcal{M}}(\vee_{i=1}^{k}\Delta((\overline{a_i} \rightarrow F) \wedge (\overline{a_i} \rightarrow G) \wedge (F \rightarrow \overline{b_i}) \wedge (G \rightarrow \overline{b_i}))) = 0$, and

$v_{\mathcal{M}}(\wedge_{i=1}^{k}(\neg\Delta(F \rightarrow \overline{b_i}) \vee \neg\Delta(G \rightarrow \overline{b_i}) \vee \neg\Delta(\overline{a_i} \rightarrow F) \vee \neg\Delta(\overline{a_i} \rightarrow G))) = 1.$

This in turn means, that:

$v_{\mathcal{M}}(F\&^T G) = v_{\mathcal{M}}(F\&^G G).$

This shows that we can express $\&^T$ in $Ł_\alpha(\Pi)$. It now remains to show that also the residuum of $\&^T$ is expressible in $Ł_\alpha(\Pi)$, which works as follows. For two formulas F and G we define:

$F \to^T G = \exists p(\Delta((p\&^T F) \to G) \wedge p)$.

Using $v_{\mathcal{M}}(p\&^T F) = v_{\mathcal{M}}(p) *_T v_{\mathcal{M}}(F)$ the truth function of $\to^T$ becomes ($\alpha = v_{\mathcal{M}}(p_\alpha)$):

$v_{\mathcal{M}}(F \to^T G) = \sup_{\alpha \in [0,1]}\{\alpha : v_{\mathcal{M}}(p_\alpha) *_T v_{\mathcal{M}}(F) \leq v_{\mathcal{M}}(G)\}$.

Therefore, by Equation 3.2, we have $v_{\mathcal{M}}(F \to^T G) = v_{\mathcal{M}}(F) \Rightarrow_T v_{\mathcal{M}}(G)$.

Furthermore, as is demonstrated in [EGM04], we can directly infer the axiomatization of a fuzzy logic, which is based on a continuous t-norm, from its t-norm and its residuum. □

3.5 Infinity and selecting subsequent states

In this last part of the section, we investigate some alternative approaches that may lead to similar results by other means, or just show how one can can use the selection principles associated to the three players to define certain constructs. Also, we argue why infinite domains are to be treated cautiously in the presence of Π, but most importantly, we look at a way of making the result about continuous t-norms more general, i.e. a way to express infinite ordinal sums. Note that the game rules $\mathcal{R}$ presented in this section do not carry any upper index. This is due to them not being part of any setting discussed so far. They only serve the purpose of illustrating alternative approaches.

3.5.1 Infinite domains

The object quantifier Π makes it difficult to consider infinite domains in the general case. Although there are many examples where one can define probability distributions over infinite sets, the uniform distribution on, e.g., $\mathbb{N}$ and $\mathbb{R}$ does not exist. Hence, we can not simply take the integral over an arbitrary domain D, and assume it were automatically defined. Actually, one needs to bring in the notion of measurability and hence σ-algebras, which goes beyond the scope of the present thesis, where we lay particular interest in a certain detachment from too much mathematics.

3.5.2 Infinitary game rules

Game rules that allow the players to choose from infinitely many options are nothing particularly special in principle. Having infinite domains, every choice of constants performed by **O** or **P**, i.e. those related to universal or existential quantifiers, corresponds to a selection from infinitely many possibilities. Therefore the following game rules appear natural.

$\mathcal{R}_{\vee^\infty}$**:** If **P** asserts $\vee_{i=1}^{\infty} F_i$, **P** chooses $i \geq 1$ and **P** asserts F_i.

$\mathcal{R}_{\wedge^\infty}$: If **P** asserts $\wedge_{i=1}^{\infty} F_i$, **O** chooses $i \geq 1$ and **P** asserts F_i.

The truth functions matching those game rules are the expected ones, except that, with respect to the k-ary versions, *max* and *min* become *sup* and *inf*, as no witnesses for the maximal or minimal truth value may exist. This is similar to the case where one has infinite domains. For that kind of scenario the notion of risk as introduced in Definition 4 as it stands, is again the appropriate one. Still, syntactically such infinitary constructs are to be justified, as for example via a schematic condition that all F_i are the same up to shuffling constants or changing the arity of the formula. In [Got13], the author considers infinitely long expressions as an alternative to generalizations of the kind here.

Infinite ordinal sums

In particular, although not executed entirely here, one can use infinitary rules for conjunction and disjunction to extend the result of Theorem 13 in a way that all MFLs based on continuous t-norms are covered. To that end, one merely has to replace $\wedge_{i=1}^{k}, \vee_{i=1}^{k}$, in the proof of Theorem 13, with $\wedge_{i=1}^{\infty}, \vee_{i=1}^{\infty}$, in case the ordinal sum corresponds to infinitely many different intervals. However, although the formulas in its scope follow a schematic pattern, a full characterization and an adequacy proof (i.e. a proof that the rules payoff-wise, for **P**, match their defined truth functions based on *inf* and *sup*) for $\wedge_{i=1}^{\infty}, \vee_{i=1}^{\infty}$ are not part of the present thesis.

A game rule for the Delta

Another application of $\wedge_{i=1}^{\infty}$ is a different characterization of the Delta operator. The following game rule would do the job:

$\mathcal{R}_{\Delta}$: If **P** asserts ΔF, **O** chooses $k \geq 1$ and **P** must assert k instances of F, or $\bot$.

Syntactically, this corresponds to an expression of the form $\wedge_{k=1}^{\infty}(\&_{j=1}^{k} F)$. Again, since we always assume the same formula F as base and then only adjust the number of times it appears within the scope of strong Ł conjunction, a schematic pattern is given.

Informally, if the risk of asserting F is exactly 0, no matter which $k \geq 1$ **O** chooses, also k instances of F do not entail any risk. On the other hand, if the risk of F is greater than zero, **O** only needs to go high enough and will always find a natural number which makes asserting that many instances of F, for **P**, at least as expensive as asserting $\bot$.

3.5.3 Truth constants and the k-ary π

For $k \geq 2$ and formulas F_i, $1 \leq i \leq k$, one can consider the following game rule:

$\mathcal{R}_{\pi^k}$: If **P** asserts $\pi_{i=1}^{k} F_i$, **N** chooses $1 \leq i \leq k$ and **P** must assert F_i.

The payoff for **P** matches the following truth function:

$$v_{\mathcal{M}}(\pi_{i=1}^{k} F_i) = \frac{\sum_{i=1}^{k} v_{\mathcal{M}}(F_i)}{k}. \tag{3.13}$$

This construct can be used, e.g., to define arbitrary rational truth value constants, as one can set any combination of $\perp$'s and $\top$'s for the F_i.

Note that here, as for the object quantifier Π, infinitely many options to choose from become really problematic, as the denominator of the defined truth function would become infinite.

CHAPTER 4

Quantifier Models

Models for vague natural language quantifiers can hardly be unique. One uses expressions like “almost all”, “about half”, “at least about a third” and “many” frequently and naturally in spoken and written language. Formal models for those are a tool of meta-level precisifications, that can be used to explain the discrepancy in speakers using them with different background meanings. For example, let us consider the two statements “Almost all free tigers are endangered”, and “Almost all children like chocolate”. The tolerance margin, i.e. the number of negative witnesses (or their percentage) that is still tolerable for a speaker to accept the respective statements to be true, neither is an absolute number, nor is it a fixed percentage of the underlying base set, here, the set of free-living tigers and the set of children respectively. We are going to conceptualize this freedom of defining quantifier semantics by allowing for different *readings* for each natural language quantifier. To do so in a systematic manner, we recall [FR14] the hierarchy of quantification, was introduced by Liu and Kerre [LK98] fo fuzzy quantifiers. Since the models of vague quantifier expressions might - in principle - not be based on fuzzy logic, we prefer to use “vague”, where Liu and Kerre have “fuzzy”. However we retain “crisp” for “precise”.

Type I: The quantifier and its scope are crisp.

Type II: The quantifier is crisp, but its scope may be vague.

Type III: The quantifier is vague, but its scope is crisp.

Type IV: The quantifier and its scope may be vague.

Although vagueness (and crispness) on the level of natural language and intermediate truth values of formulas are not necessarily related, we use the mentioned classification to concisely refer to the structure of the following definition. Subsequently, we are going to explain in greater detail how we intend to relate *vagueness* and *intermediate truth values*, namely by embedding the former classification into the latter, by means of defining appropriate versions of respective natural language quantifier expressions formally, respecting the potential differences that speakers may attribute to them.

Definition 13 *Let a fuzzy interpretation $\mathcal{M}$ and some language comprising symbols $\hat{F}, F, \mathsf{Q}, \ldots$ be given. Then we say:*

- *A quantifier Q is of Type I if it is defined only for classical formulas $\hat{F}$, and for all these we have $v_{\mathcal{M}}(\mathsf{Q}x\hat{F}(x)) \in \{0,1\}$.*
- *A quantifier Q is of Type II if it is defined for all formulas F, and for all these we have $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \in \{0,1\}$.*
- *A quantifier Q is of Type III if it is defined only for classical formulas $\hat{F}$, and for all these we have $v_{\mathcal{M}}(\mathsf{Q}x\hat{F}(x)) \in [0,1]$.*
- *A quantifier Q is of Type IV if it is defined for all formulas F, and for all these we have $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \in [0,1]$.*

Quantifiers of Type III are also called semi-fuzzy *and quantifiers of Type IV are also called* fully-fuzzy*.*

Note that the foregoing definition does not assume any particular logic yet. It is more thought to be a generic classification that may be used for different frameworks.

Remark 12 *Throughout this chapter, we will always have an interpretation $\mathcal{M}$ fixed in the background. This in particular means that D always denotes the corresponding (finite) domain, or set of constants referring to objects from the domain (which is also called D), respectively. Moreover, for any definition, theorem and corollary, we will make explicit which logical framework we refer to. In case this is not made explicit, we refer to a generic situation, i.e. definitions or statements refer to any framework, as for example just above in Definition 13.*

Another line alongside which we can distinguish quantifier expressions is *extensionality* and *intensionality* [FK96]. By the former we mean *absolute quantifiers* (or counting quantifiers) and *relative quantifiers* (or proportional quantifiers), i.e. those that relate only to absolute numbers, or proportions respectively, of witnessing constants for the quantifier's scope within the domain of some interpretation $\mathcal{M}$. By the latter, the intensional quantifiers, we mean such quantifiers that are not of the extensional sort, e.g. "many" and "few". For these, as we will discuss shortly in greater detail, we will require a more refined analysis of the distribution of witnesses for a quantifier's scope with respect to $\mathcal{M}$. However, extensional quantifiers can be seen as a special case of intensional quantifiers, in the following sense. We model intensional quantifiers in a way that allows us to incorporate contextual information. Models of quantifiers where this information is empty, i.e. it does not contribute to the evaluation, can be considered to be extensional. On one side that means, absolute and relative quantifiers can be seen as trivial cases of intensional ones, and intensional quantifiers without extra information from context can be seen as extensional models of intensional quantifiers. Consequently, the notion of absolute and relative quantifiers also applies to intensional quantifiers, while, instead of singling out particular names for them, we rather only speak of absolute and relative models of intensional quantifiers.

One more important feature is the arity of a quantifier Q. Arity refers to the number of arguments[1] a quantifier can take. In [PW06], in its most general form, a quantified sentence $\mathsf{Q}x_1, \dots x_n(F_1, \dots, F_m)$ features a quantifier Q, which is binding variables $x_1, \dots, x_n$, that may occur in several argument formulas $F_1, \dots, F_m$. Such quantifiers are then said to be of Type $\langle a_1, \dots, a_m \rangle$, where a_i is the arity of the formula F_i $(1 \leq i \leq m)$. In natural language, binary quantification[2] (Type $\langle 1, 1 \rangle$ quantification), where only one variable is bound, is the most common form, as e.g. in "All humans are mortal". In the example, the predicate "human" is the range (restricting) predicate (let us called it H), whereas "mortal" takes the place of what is usually called the scope predicate (let us called it M). Then, a possible way to write the statement as a formula is $\forall x(H(x), M(x))$. For unary quantification, i.e. Type $\langle 1 \rangle$ quantification, there only is a scope predicate, and the quantification ranges over the whole domain, as e.g. in the statement "All [objects from the domain] are blue".

We will mostly focus on unary and binary quantifiers, while the general case of n-ary quantifiers, with $n \geq 3$, only receives slight attention, as NL examples in that direction hardly exist in frequently used language. However, we will show how certain expressions of that kind, that are not expressible in Classical Logic, can neatly be expressed within our framework $Ł_\alpha(\Pi)$. Summarizing the different angles to quantifier classification, we give the following list:

- Quantifiers of Type $I - IV$ (w.r.t. Definition 13).
- Extensional and intensional quantifiers.
- The arity of a quantifier.

For extensional quantifiers, i.e. the relative and absolute ones, we will first treat the unary cases and only then work out how to deal with binary quantification, while for the

[1] Note that we sometimes refer to a quantifier's scope as the quantifier's argument(s). The reason for that is that in [?] quantifiers are regarded as functions, as no distinction between syntax and semantics is performed there.

[2] In [Zad85] Zadeh speaks of a classification of quantifiers into the first kind, second kind, third kind, etc., rather than of unary, binary, ternary, etc., quantification. However, this is in conflict with Zadeh's own earlier terminology in [Zad83].

intensional quantifiers “many” and “few”, we will start with the binary case, which then subsumes the unary case as well. This choice is conceptually motivated by the fact that for extensional quantifiers an additional range to the scope reduces to simply restricting the domain of discourse, which is fairly straightforward, while for our intensional quantifiers the situation is more complex, as we will see when we introduce respective readings. Quantifiers with an arity higher than 2 are considered in Section 4.5.3.

We consider the following basic natural language quantifiers ($k \geq 0, q \in [0, 1]$), where the abbreviations *Abs,Rel* and *Int* indicate that they encompass *absolute, relative* and *intensional* quantifier expressions respectively:

Abs1: “exactly k”, “at least/ at most k”, “more than/ less than k”.

Abs2: “about k”, “at least/ at most about k”, “more than/ less than about k”.

Rel1: “exactly $(q \cdot 100)\%$”, “at least/ at most $(q \cdot 100)\%$”, “more than/ less than $(q \cdot 100)\%$”.

Rel2: “about $(q \cdot 100)\%$”, “at least/ at most about $(q \cdot 100)\%$”, “more than/ less than about $(q \cdot 100)\%$”.

Int1: “many”, “few”.

Additionally to those, we can now fix some more quantifier expressions that are common in natural language ($k_1, k_2 \geq 0$ with $k_1 < k_2$, and $q_1, q_2 \in [0, 1]$ with $q_1 < q_2$):

Abs3: “more than k_1 and less than k_2”, “more than k_1 and at most k_2”, “at least k_1 and less than k_2”, “at least k_1 and at most k_2”.

Rel3: “more than $(q_1 \cdot 100)\%$ and less than $(q_2 \cdot 100)\%$”, “more than $(q_1 \cdot 100)\%$ and at most $(q_2 \cdot 100)\%$”, “at least $(q_1 \cdot 100)\%$ and less than $(q_2 \cdot 100)\%$”, “at least $(q_1 \cdot 100)\%$ and at most $(q_2 \cdot 100)\%$”.

Int2: “several”, “various”, “multiple”, “heaps of”, “loads of”.

We are now taking up the discussion again on the relation of vagueness in natural language and intermediate truth values in fuzzy logic. Type I-IV quantification has been used twice. The first and more informal hierarchy relates more to vagueness in natural language, while the second and more formal one rather relates to formulas with either classical or intermediate truth values, which is closer to the originally intended distinctions of Liu and Kerre, i.e. the one of Definition 13. In the informal hierarchy of quantification, the term *vague* is referring to an inherent vagueness of NL statements, which often is expressed with the hedge “about” and also implicit when we use quantifiers like “many” and ”few”. *Crispness*, on the other hand, refers to formulas that only assume one of two definite truth values, namely 0 for false, and 1 for true. On the quantifier level, “for all” and “exactly 10” are good examples, while on the propositional level it seems safe to say that “being mortal” or “being more than 20 years old” are crisp[3] in the sense that these properties are either completely fulfilled or unfulfilled by any conceivable witness. Definition 13 does not directly reflect these concepts. There, “vague” is changed to “can attain intermediate truth values” while crisp still expresses “bivalency”. Since the expressions $[Abs1], [Abs2], [Abs3], [Rel1], [Rel2], [Rel3], [Int1]$ and $[Int2]$ are NL quantifiers ($[Abs2]$ and $[Rel2]$ relate to vague ones, the others to crisp ones) which we intend to model in terms of fuzzy logic, we have to consider the following. There are four different cases for the quantifier level, while we always assume quantifier arguments to be crisp:

- vague NL quantifiers can be of Type I (w.r.t. Definition 13), i.e. bivalent:

  This is the way linguists usually treat vagueness, namely by not allowing for intermediate truth values, even if the quantifier under consideration is clearly vague, like “many”. Evaluations are based on (crisp) truth conditions., i.e. vagueness is taken care of by considering context as an independent parameter of a bivalent model.

- vague NL quantifiers can be of Type III (w.r.t. Definition 13), i.e. semi-fuzzy:

  This is how fuzzy logicians often proceed, namely by giving truth functions for vague quantifiers in the following way. The truth function of the relative quantifier “about half” is 1 within a smallish area (for the acceptable tolerance) around the value $\frac{1}{2}$, and then continuously approaches 0 on both sides. See e.g. Figure 4.4.

- crisp NL quantifiers can be of Type I (w.r.t. Definition 13), i.e. bivalent:

  This is fairly intuitive and done by linguists and fuzzy logicians. The reasons seem clear.

- crisp NL quantifiers can be of Type III (w.r.t. Definition 13), i.e. semi-fuzzy:

  This, again, is something that linguists do not commonly do, while fuzzy logicians find this very normal. Perhaps the best reason is to achieve a certain level of robustness of evaluations. Recall the introduction’s example involving a library and 1.000.103 books.

A comprehensive study of the literature, see e.g. [vG08, Ali81, dC05], all lexica in the world, e.g. [Ada79], and extensive talk with several specimens, reveals that people’s use of quantifier expressions is not limited to the ones introduced above. However, it often

[3]Note that we fequently use the same term to refer to classical, i.e. bivalent, formulas, too.

happens that certain expressions have assigned certain meanings that are already covered by our chosen quantifiers, like for example the quantifier expression “most”, which has at least two different interpretations or readings. For one, it is often understood as simple majority [Pet00], i.e. for a set of 100 balls, we can say “Most of the balls are green” as soon as “at least 51”, or “more than half” of the balls have the property of being green. Secondly, “most” often is read in the sense of “many”. For example, consider the statement “Most people that have voted for Hillary Clinton in 2016 are unhappy that Donald Trump won the presidential election.” Here, “most” expresses not the fact that a simple majority of Clinton voters are unhappy with the election’s outcome, rather it insinuates that a vast majority or even “almost all” voters are unhappy. Taking this interpretation as one of the possible readings of “many” entails a respective semantics that depends on a certain threshold value [FK96], which in turn can be seen as [$Rel1$] or [$Rel2$] expressions. On the other hand, “many” can be interpreted intensionally, based on a notion of comparison, as we will handle it further down in this chapter’s Section 4.6. Unless stated otherwise, for us, “most” will always refer to the “(simple) majority” reading.

4.1 Extensional absolute crisp quantifiers

For absolute (unary) Type I quantifiers, we consider the following base quantifiers, called [$Abs1$] above:

- “exactly k”, “at most k”, “less than k”, “at least k”, “more than k”.

Using those, we can also define various other quantifiers, by means of conjunction, called [$Abs3$] above:

- “more than k_1 and less than k_2”, “more than k_1 and at most k_2”, “at least k_1 and less than k_2”, “at least k_1 and at most k_2”.

Based on the game rule for $E2$ in [FH17], we can give a quantifier game rule $\mathcal{R}^{\mathcal{NRG}}_{\exists^{\geq k}}$, for absolute (unary) Type I quantifiers, as follows[4] (recall that the superscript of an $\mathcal{R}$ preceding a rule indicates the associated game, here the $\mathcal{NRG}$-game):

$\mathcal{R}^{\mathcal{NRG}}_{\exists^{\geq k}}$: If **P** asserts $\exists^{\geq k} x \hat{F}(x)$, then, if **O** attacks, **P** chooses k different constants $c_1, \ldots, c_k$, and then **O** chooses one of these constants, say c_i, and **P** has to assert $\hat{F}(c_i)$.

[4] Note that we do not give a theorem on the correspondence of this game rule with its associated truth function. This is because we do not wish to add this rule formally to $Ł_\alpha(\Pi)$, as the formula leading to the intended truth function is already expressible in $Ł_\alpha(\Pi)$. However, it is quite immediate that the payoff for the proponent player matches the truth function of the defined expression introduced just below.

We introduce the following terminology for notational convenience.

Definition 14 *For a crisp formula $\hat{F}$, we define*[5]*:* $\|\hat{F}\| = \sum_{c \in D} v_{\mathcal{M}}(\hat{F}(c))$.

Also, for a condition A, we use $\mathbb{I}_{(A)}$ as the indicator function for A:

$$\mathbb{I}_{(A)} = \begin{cases} 1 & \text{if } A \text{ holds} \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Note that this latter function merely serves as a convenient notation, while a condition A can be expressed in any way that admits assessing whether it holds or not.

We now define the following well known scheme [Glö06], assuming equality $=$ in the language:

Definition 15 *For a crisp $Ł_\alpha(\Pi)$ formula $\hat{F}$, and $1 \leq k \leq |D|$, we define:*

$$\exists^{\geq k} x \hat{F}(x) = \exists x_1 \ldots \exists x_k (\bigwedge_{i=1}^{k} \hat{F}(x_i) \wedge \bigwedge_{i \neq j} (x_i \neq x_j)). \tag{4.2}$$

Also, we define $\exists^{\geq 0} = \top$.

It is relatively easy to see that the following theorem holds:

Theorem 14 *[Glö06] For a crisp formula $\hat{F}$, and $0 \leq k \leq |D|$, we have:*

$$v_{\mathcal{M}}(\exists^{\geq k} x \hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| \geq k)}. \tag{4.3}$$

Game semantically, for a crisp formula $\hat{F}$, it is clear that **P** has a winning strategy for the game for $\exists^{\geq k} x \hat{F}(x)$, if and only if there are at least k positive witnesses for $\hat{F}$ in D. Hence, the payoff for **P** matches the truth value of the truth function just above.

Then, we define the following:

Definition 16 *For a crisp $Ł_\alpha(\Pi)$ formula $\hat{F}$, and $0 \leq k \leq |D| - 1$, we define:*

- $\exists^{\leq k} x \hat{F}(x) = \neg \exists^{\geq k+1} x \hat{F}(x)$
- $\exists^{> k} x \hat{F}(x) = \exists^{\geq k+1} x \hat{F}(x)$

[5] Note that this represents the cardinality of the extension of the $\hat{F}$.

- $\exists^{<k+1} x\hat{F}(x) = \exists^{\leq k} x\hat{F}(x)$
- $\exists^{k} x\hat{F}(x) = \exists^{\geq k} x\hat{F}(x) \wedge \exists^{\leq k} x\hat{F}(x)$

The remaining special cases, $\exists^{\leq |D|}, \exists^{>-1}, \exists^{<|D|+1}$ *are constantly true, while* $\exists^{>|D|}, \exists^{<0}$ *are constantly false. The quantifier* $\exists^{|D|}$ *is the same as* $\forall$*. Then, for* $0 \leq k_1, k_2 \leq\mid D\mid$ *with* $k_1 < k_2$*, we define:*

- $\exists^{[k_1,k_2]} x\hat{F}(x) = \exists^{\geq k_1} x\hat{F}(x) \wedge \exists^{\leq k_2} x\hat{F}(x)$
- $\exists^{(k_1,k_2]} x\hat{F}(x) = \exists^{>k_1} x\hat{F}(x) \wedge \exists^{\leq k_2} x\hat{F}(x)$
- $\exists^{[k_1,k_2)} x\hat{F}(x) = \exists^{\geq k_1} x\hat{F}(x) \wedge \exists^{<k_2} x\hat{F}(x)$
- $\exists^{(k_1,k_2)} x\hat{F}(x) = \exists^{>k_1} x\hat{F}(x) \wedge \exists^{<k_2} x\hat{F}(x)$

Theorem 15 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$*, and* $0 \leq k, k_1, k_2 \leq\mid D\mid$ *with* $k_1 < k_2$*, we have:*

- $v_{\mathcal{M}}(\exists^{\leq k} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| \leq k)}$
- $v_{\mathcal{M}}(\exists^{>k} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| > k)}$
- $v_{\mathcal{M}}(\exists^{<k} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| < k)}$
- $v_{\mathcal{M}}(\exists^{k} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| = k)}$
- $v_{\mathcal{M}}(\exists^{[k_1,k_2]} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| \in [k_1,k_2])}$
- $v_{\mathcal{M}}(\exists^{(k_1,k_2]} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| \in (k_1,k_2])}$
- $v_{\mathcal{M}}(\exists^{[k_1,k_2)} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| \in [k_1,k_2))}$
- $v_{\mathcal{M}}(\exists^{(k_1,k_2)} x\hat{F}(x)) = \mathbb{I}_{(\|\hat{F}\| \in (k_1,k_2))}$

Proof:
This is an immediate consequence of Theorem 14. □

Remark 13 *The quantifier expressions that were introduced as* $[Abs2]$ *above, can also be understood as absolute (unary) Type I quantifiers, while the choices one can take, e.g. for "about 10", are rather limited. The most obvious way consists in defining fixed tolerance margins and simply express them by means of* $[Abs3]$ *expressions.*

Theorem 16 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$*, and* $0 \leq k, k' \leq\mid D\mid$ *with* $k + k' =\mid D\mid$*, we have*[6]*:*

- $\exists^{\leq k} x\hat{F}(x) \equiv \neg\exists^{<k'} x\neg\hat{F}(x)$
- $\exists^{\geq k} x\hat{F}(x) \equiv \neg\exists^{>k'} x\neg\hat{F}(x)$

Proof:
For k, k' and $\hat{F}$ given according to the setting of the theorem, we start by noting:

$\neg\exists^{>k} x\hat{F}(x) \equiv \exists^{\leq k} x\hat{F}(x)$, and $\exists^{>k} x\hat{F}(x) \equiv \exists^{<k'} x\neg\hat{F}(x)$.

Together they prove the first item of the theorem. Then we see that:

$\exists^{>k} x\hat{F}(x) \equiv \neg\exists^{\geq k'} x\neg\hat{F}(x)$.

Applying outer-negation and replacing $\hat{F}$ with the formula $\neg\hat{F}$, yields the desired remaining second item. $\square$

4.2 Extensional relative crisp quantifiers

For *relative quantifiers*, we consider the following base expressions, introduced above as [$Rel1$]:

- "exactly $(q\cdot 100)\%$", "at most $(q\cdot 100)\%$", "less than $(q\cdot 100)\%$", "at least $(q\cdot 100)\%$", "more than $(q \cdot 100)\%$".

Again, from those we can define various other quantifiers, introduced above as [$Rel3$], by means of conjunction:

- "more than $(q_1 \cdot 100)\%$ and less than $(q_2 \cdot 100)\%$", "more than $(q_1 \cdot 100)\%$ and at most $(q_2 \cdot 100)\%$", "at least $(q_1 \cdot 100)\%$ and less than $(q_2 \cdot 100)\%$", "at least $(q_1 \cdot 100)\%$ and at most $(q_2 \cdot 100)\%$".

For relative (unary) Type I quantifiers, we define the following quantifier models, based on the Π quantifier:

Definition 17 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$ *and* $\overline{q} \in \Lambda \setminus \{\top, \bot\}$*, with* $v_{\mathcal{M}}(\overline{q}) = q$*, we define:*

[6] Recall that $\equiv$ means the equivalence of two formulas, i.e. their truth values coincide.

- $\exists^{\geq q} x \hat{F}(x) = \Delta(\overline{q} \to \Pi x \hat{F}(x))$
- $\exists^{\leq q} x \hat{F}(x) = \Delta(\Pi x \hat{F}(x) \to \overline{q})$
- $\exists^{> q} x \hat{F}(x) = \neg\Delta(\Pi x \hat{F}(x) \to \overline{q})$
- $\exists^{< q} x \hat{F}(x) = \neg\Delta(\overline{q} \to \Pi x \hat{F}(x))$
- $\exists^{q} x \hat{F}(x) = \Delta(\Pi x \hat{F}(x) \leftrightarrow \overline{q})$

For the remaining boundary cases we have: $\exists^{\leq 1}, \exists^{\geq 0}$ *are constantly true.* $\exists^{<0}, \exists^{>1}$ *are constantly false.* $\exists^{\geq 1}, \exists^{1}$ *can be associated to* $\forall$*.* $\exists^{\leq 0}, \exists^{0}$ *can be associated to* $\neg\exists$*.* $\exists^{>0}$ *can be associated to* $\exists$*, and* $\exists^{<1}$ *can be associated to* $\neg\forall$*. Then, for* $\overline{q_1}, \overline{q_2} \in \Lambda$ *with* $v_{\mathcal{M}}(\overline{q_1}) = q_1, v_{\mathcal{M}}(\overline{q_2}) = q_2$*, and* $q_1 < q_2$*, we define:*

- $\exists^{[q_1,q_2]} x \hat{F}(x) = \exists^{\geq q_1} x \hat{F}(x) \wedge \exists^{\leq q_2} x \hat{F}(x)$
- $\exists^{(q_1,q_2]} x \hat{F}(x) = \exists^{> q_1} x \hat{F}(x) \wedge \exists^{\leq q_2} x \hat{F}(x)$
- $\exists^{[q_1,q_2)} x \hat{F}(x) = \exists^{\geq q_1} x \hat{F}(x) \wedge \exists^{< q_2} x \hat{F}(x)$
- $\exists^{(q_1,q_2)} x \hat{F}(x) = \exists^{> q_1} x \hat{F}(x) \wedge \exists^{< q_2} x \hat{F}(x)$

Theorem 17 *For a crisp* $\text{Ł}_\alpha(\Pi)$ *formula* $\hat{F}$ *and* $\overline{q}, \overline{q_1}, \overline{q_2} \in \Lambda$*, with* $v_{\mathcal{M}}(\overline{q}) = q, v_{\mathcal{M}}(\overline{q_1}) = q_1, v_{\mathcal{M}}(\overline{q_2}) = q_2$*, and* $q_1 < q_2$*, we have:*

- $v_{\mathcal{M}}(\exists^{\geq q} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in [q,1])}$
- $v_{\mathcal{M}}(\exists^{\leq q} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in [0,q])}$
- $v_{\mathcal{M}}(\exists^{> q} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in (q,1])}$
- $v_{\mathcal{M}}(\exists^{< q} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in [0,q))}$
- $v_{\mathcal{M}}(\exists^{q} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) = q)}$
- $v_{\mathcal{M}}(\exists^{[q_1,q_2]} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in [q_1,q_2])}$
- $v_{\mathcal{M}}(\exists^{(q_1,q_2]} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in (q_1,q_2])}$
- $v_{\mathcal{M}}(\exists^{[q_1,q_2)} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in [q_1,q_2))}$
- $v_{\mathcal{M}}(\exists^{(q_1,q_2)} x \hat{F}(x)) = \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \in (q_1,q_2))}$

Proof:
The results are obtained by basic computations, however, to demonstrate the procedure, we prove the fourth statement:

First, we have to note that we have to consider a crisp, i.e. bivalent, statement. Consequently, the truth value must be either 0 or 1. It is 1 if and only if $v_{\mathcal{M}}(\Delta(\overline{q} \to \Pi x \hat{F}(x))) = 0$. This is the case if $Prop_{\mathcal{M}}(\hat{F}) < q$. $\square$

Remark 14 *Quantifiers from* $[Rel2]$ *can also be conceived as relative (unary) Type I quantifiers. The relative (unary) Type III quantifier models (*$[Models\ 3]$*), based on granular hierarchies, to be introduced below in Section 4.4.3, can serve as convenient models for such expressions, when the granular hierarchy only contains one element.*

Theorem 18 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$*, and* $q, q' \in [0,1]$ *with* $q + q' = 1$*, we have:*

- $\exists^{\leq q} x \hat{F}(x) \equiv \neg\exists^{<q'} x \neg\hat{F}(x)$
- $\exists^{\geq q} x \hat{F}(x) \equiv \neg\exists^{>q'} x \neg\hat{F}(x)$

Proof:
For q, q' and $\hat{F}$ given according to the setting of the theorem, we start by noting:

$\neg\exists^{>q} x \hat{F}(x) \equiv \exists^{\leq q} x \hat{F}(x)$, and $\exists^{>q} x \hat{F}(x) \equiv \exists^{<q'} x \neg\hat{F}(x)$.

Together they prove the first item of the theorem. Then we see that:

$\exists^{>q} x \hat{F}(x) \equiv \neg\exists^{\geq q'} x \neg\hat{F}(x)$.

Applying outer-negation and replacing $\hat{F}$ with the formula $\neg\hat{F}$, yields the desired remaining second item. $\square$

4.3 Extensional absolute semi-fuzzy quantifiers

The base quantifier expressions for this section are the following, introduced above as $[Abs2]$:

- "about k", "at least/ at most about k", "more than/ less than about k".

Of course, these can also be understood as absolute (unary) Type I quantifiers, as, e.g. "about 5" could clearly be defined as "at least 4 and at most 6." (tolerance 1), while our focus in this section goes beyond this interpretation of solely adding fixed tolerances. The main idea of Type III quantification is the one of having intermediate truth values for statements like "About 5 students passed the exam", as for example 0.9, if actually 3 students passed the exam, where only a tolerance of 1 is fully acceptable. Formally, we can model these quantifiers employing a similar game rule as before for absolute (unary) Type I quantifiers[7]:

$\mathcal{R}^{\mathcal{NRG}}_{\tilde{\exists}^{\geq k}}$**:** If **P** asserts $\tilde{\exists}^{\geq k} x \hat{F}(x)$, then, if **O** attacks, **P** chooses k different constants $c_1, \ldots, c_k$, and then **N** chooses one of these constants, say c_i, and **P** has to assert $\hat{F}(c_i)$.

This rule is based on the rule of Section 4.1. While there the opponent player **O** chooses among the constants selected by **P**, it is now nature **N** who does so. This is meant to relax the evaluation criteria and hence to increase robustness. The same principle applies to the definitions of quantifiers further down in this section, where the junction of two statements is modeled by π rather than $\wedge$.

As before, we can express the quantifiers captured by this rule schema syntactically, assuming again $=$ in the language:

Definition 18 *For a crisp* $\text{Ł}_\alpha(\Pi)$ *formula* $\hat{F}$*, and* $1 \leq k \leq \mid D \mid$*, we define:*

$$\tilde{\exists}^{\geq k} x \hat{F}(x) = \exists x_1 \ldots \exists x_k (\pi^k_{i=1} \hat{F}(x_i) \wedge \bigwedge_{i \neq j} (x_i \neq x_j)). \tag{4.4}$$

Also, we define $\tilde{\exists}^{\geq 0} = \top$*.*

It is then relatively easy to see that the following theorem holds:

Theorem 19 *For a crisp* $\text{Ł}_\alpha(\Pi)$ *formula* $\hat{F}$*, and* $1 \leq k \leq \mid D \mid$*, we have:*

$$v_{\mathcal{M}}(\tilde{\exists}^{\geq k} x \hat{F}(x)) = \frac{1}{k} \cdot \sup\{ \sum_{c \in D_k} v_{\mathcal{M}}(\hat{F}(c)) : D_k \subseteq D \textit{ such that } \mid D_k \mid = k\}. \tag{4.5}$$

Also, it holds $v_{\mathcal{M}}(\tilde{\exists}^{\geq 0} x \hat{F}(x)) = 1$*.*

[7] Note that we do not give a theorem on the correspondence of this game rule with its associated truth function. This is because we do not wish to add this rule formally to the setting, as the formula leading to the intended truth function is already expressible in $\text{Ł}_\alpha(\Pi)$. However, it is quite immediate that the payoff for the proponent player matches the truth function of the defined expression introduced just below.

Proof:
Since $1 \leq k \leq\mid D\mid$, it is of course possible to fulfill the right conjunct, $\bigwedge_{i\neq j}(x_i \neq x_j)$, of the formula. Also, it has to be fulfilled in order not to render the whole statement wrong entirely. This means there are sets $D_k \subseteq D$ of cardinality k of constants that make for k instances of $\hat{F}$, and due to the existence quantifiers, a set that maximizes the truth value of $\pi^k_{i=1}\hat{F}(x_i)$, which is $\frac{1}{k} \cdot \sum_{c\in D_k} v_{\mathcal{M}}(\hat{F}(c))$, determines the eventual truth value. □

By closely looking at both constructs, the game rule and the syntactic representation, one can see that the (expected) payoff for **P** matches the defined truth function. Then, as before, we can define the corresponding quantifiers as follows:

Definition 19 *For a crisp* Ł$_\alpha$(Π) *formula* $\hat{F}$, *and* $0 \leq k \leq\mid D\mid -1$ *we define:*

- $\tilde{\exists}^{\leq k} x\hat{F}(x) = \tilde{\exists}^{\geq |D|-k} x\neg\hat{F}(x)$
- $\tilde{\exists}^{> k} x\hat{F}(x) = \tilde{\exists}^{\geq k+1} x\hat{F}(x)$
- $\tilde{\exists}^{< k+1} x\hat{F}(x) = \tilde{\exists}^{\geq |D|-k} x\neg\hat{F}(x)$
- $\tilde{\exists}^{k} x\hat{F}(x) = \tilde{\exists}^{\geq k} x\hat{F}(x) \ \pi \ \tilde{\exists}^{\leq k} x\hat{F}(x)$

The remaining special cases $\tilde{\exists}^{>|D|}, \tilde{\exists}^{<0}$ *are constantly false.* $\tilde{\exists}^{\leq |D|}$ *are constantly true, and* $\tilde{\exists}^{|D|}$ *can be associated to* Π. *Then, for* $0 \leq k_1, k_2 \leq\mid D\mid$, *with* $k_1 < k_2$, *we define:*

- $\tilde{\exists}^{[k_1,k_2]} x\hat{F}(x) = \tilde{\exists}^{\geq k_1} x\hat{F}(x) \ \pi \ \tilde{\exists}^{\leq k_2} x\hat{F}(x)$
- $\tilde{\exists}^{(k_1,k_2]} x\hat{F}(x) = \tilde{\exists}^{> k_1} x\hat{F}(x) \ \pi \ \tilde{\exists}^{\leq k_2} x\hat{F}(x)$
- $\tilde{\exists}^{[k_1,k_2)} x\hat{F}(x) = \tilde{\exists}^{\geq k_1} x\hat{F}(x) \ \pi \ \tilde{\exists}^{< k_2} x\hat{F}(x)$
- $\tilde{\exists}^{(k_1,k_2)} x\hat{F}(x) = \tilde{\exists}^{> k_1} x\hat{F}(x) \ \pi \ \tilde{\exists}^{< k_2} x\hat{F}(x)$

Theorem 20 *For a crisp* Ł$_\alpha$(Π) *formula* $\hat{F}$, *and* $0 \leq k, k_1, k_2 \leq\mid D\mid -1$, *with* $k_1 < k_2$, *we have:*

- $v_{\mathcal{M}}(\tilde{\exists}^{\leq k} x\hat{F}(x)) =$
  $= \frac{1}{|D|-k} \cdot \sup\{\sum_{c\in D_{|D|-k}} v_{\mathcal{M}}(\neg\hat{F}(c)) : D_{|D|-k} \subseteq D \ \textit{such that} \mid D_{|D|-k}\mid=\mid D\mid -k\}$

- $v_{\mathcal{M}}(\tilde{\exists}^{> k} x\hat{F}(x)) =$
  $= \frac{1}{k+1} \cdot \sup\{\sum_{c\in D_{k+1}} v_{\mathcal{M}}(\hat{F}(c)) : D_{k+1} \subseteq D \ \textit{such that} \mid D_{k+1}\mid= k+1\}$

- $v_{\mathcal{M}}(\tilde{\exists}^{<k+1} x\hat{F}(x)) =$
  $= \frac{1}{|D|-k} \cdot \sup\{\sum_{c \in D_{|D|-k}} v_{\mathcal{M}}(\neg\hat{F}(c)) : D_{|D|-k} \subseteq D \textit{ such that } | D_{|D|-k} |=| D | -k\}$

- $v_{\mathcal{M}}(\tilde{\exists}^{k} x\hat{F}(x)) = \frac{v_{\mathcal{M}}(\tilde{\exists}^{\geq k} x\hat{F}(x)) + v_{\mathcal{M}}(\tilde{\exists}^{\leq k} x\hat{F}(x))}{2}$

The truth functions of $\tilde{\exists}^{>|D|}, \tilde{\exists}^{<0}$ are always 0. The one of $\tilde{\exists}^{\leq|D|}$ is always 1, and the one of $\tilde{\exists}^{|D|}$ is $Prop_{\mathcal{M}}$. Then, for $0 \leq k_1, k_2 \leq| D |$, with $k_1 < k_2$, we have:

- $v_{\mathcal{M}}(\tilde{\exists}^{[k_1,k_2]} x\hat{F}(x)) = \frac{v_{\mathcal{M}}(\tilde{\exists}^{\geq k_1} x\hat{F}(x)) + v_{\mathcal{M}}(\tilde{\exists}^{\leq k_2} x\hat{F}(x))}{2}$

- $v_{\mathcal{M}}(\tilde{\exists}^{(k_1,k_2]} x\hat{F}(x)) = \frac{v_{\mathcal{M}}(\tilde{\exists}^{> k_1} x\hat{F}(x)) + v_{\mathcal{M}}(\tilde{\exists}^{\leq k_2} x\hat{F}(x))}{2}$

- $v_{\mathcal{M}}(\tilde{\exists}^{[k_1,k_2)} x\hat{F}(x)) = \frac{v_{\mathcal{M}}(\tilde{\exists}^{\geq k_1} x\hat{F}(x)) + v_{\mathcal{M}}(\tilde{\exists}^{< k_2} x\hat{F}(x))}{2}$

- $v_{\mathcal{M}}(\tilde{\exists}^{(k_1,k_2)} x\hat{F}(x)) = \frac{v_{\mathcal{M}}(\tilde{\exists}^{> k_1} x\hat{F}(x)) + v_{\mathcal{M}}(\tilde{\exists}^{< k_2} x\hat{F}(x))}{2}$

Proof:
This is an immediate consequence of Theorem 19. □

Corollary 1 *For a crisp $Ł_\alpha(\Pi)$ formula $\hat{F}$ and $0 \leq k, k_1, k_2 \leq| D |$, with $k_1 < k_2$, we have:*

- $v_{\mathcal{M}}(\tilde{\exists}^{\geq k} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| \geq k$
- $v_{\mathcal{M}}(\tilde{\exists}^{\leq k} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| \leq k$
- $v_{\mathcal{M}}(\tilde{\exists}^{> k} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| > k$
- $v_{\mathcal{M}}(\tilde{\exists}^{< k} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| < k$
- $v_{\mathcal{M}}(\tilde{\exists}^{k} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| = k$
- $v_{\mathcal{M}}(\tilde{\exists}^{[k_1,k_2]} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| \in [k_1, k_2]$
- $v_{\mathcal{M}}(\tilde{\exists}^{(k_1,k_2]} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| \in (k_1, k_2]$
- $v_{\mathcal{M}}(\tilde{\exists}^{[k_1,k_2)} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| \in [k_1, k_2)$
- $v_{\mathcal{M}}(\tilde{\exists}^{(k_1,k_2)} x\hat{F}(x)) = 1$ iff $\|\hat{F}\| \in (k_1, k_2)$

Proof:
The results are obtained by basic computations, however, to demonstrate the procedure, we prove the first statement:

We prove by case distinction. Assume there are at least k positive witnesses of $\hat{F}$. That means there is a set $D_k \subseteq D$ such that $\mid D_k \mid= k$ with $\sum_{c \in D_k} v_{\mathcal{M}}(\hat{F}(c)) = k$. Since $\sum_{c \in D'} v_{\mathcal{M}}(\hat{F}(c)) \leq k$ for all $D' \subseteq D$ with $\mid D' \mid= k$, we have $v_{\mathcal{M}}(\tilde{\exists}^{\geq k} x \hat{F}(x)) = \frac{1}{k} \cdot k = 1$. On the other hand, if there are less than k positive witnesses of $\hat{F}$, we have $\sum_{c \in D_k} v_{\mathcal{M}}(\hat{F}(c)) = x < k$. Hence, $v_{\mathcal{M}}(\tilde{\exists}^{\geq k} x \hat{F}(x)) = \frac{1}{k} \cdot x < 1$. □

Remark 15 *Additional quantifiers, like e.g. "more than about k_1 and at most about k_1", can freely be define by means of conjunction.*

4.4 Extensional relative semi-fuzzy quantifiers

This section's prototypical quantifiers are the following relative (unary) Type III quantifiers, introduced as [$Rel2$] above:

- "about $(q \cdot 100)\%$", "at least/ at most about $(q \cdot 100)\%$", "more than/ less than about $(q \cdot 100)\%$".

Those can be modeled in various ways and then freely be combined to arrive at models for, e.g. "at least about a third and at most about half". Perhaps the simplest way of getting adequate models is to take the models for the relative (unary) Type I quantifiers and just remove all the Deltas from the definitions (note that in this case we will not anymore distinguish between "more than/ less than about" and "at least/ at most about"). Other methods comprise the models of *blind choice* and *deliberate choice* quantifiers [FR12, FR14], a method based on the idea of granular hierarchies [Yao01, Kee09, XYJ13], and methods based on an augmented version of the Π quantifier, which corresponds to random samples without replacement, called $\Pi^{j,k}$. We summarize our classes of relative (unary) Type III quantifiers as follows:

M 1: Quantifiers based extensional relative crisp quantifiers and dropping the Deltas.

M 2: Blind Choice and Deliberate Choice quantifiers.

M 3: Quantifiers based on granularity levels.

M 4: Quantifiers based on random sampling without replacement and strong conjunction.

M 5: Quantifiers based on random sampling without replacement and strong disjunction.

M 6: Quantifiers for querying databases.

4.4.1 Quantifiers based extensional relative crisp quantifiers and dropping the Deltas

This section's models are based on those of Section 4.2. The main difference is that we removed the Delta operator that made the expressions crisp. As a consequence, the distinction between "more than/ less than about" and "at least/ at most about" becomes rather insignificant and is hence omitted.

Definition 20 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$ *and* $\overline{q}, \overline{q_1}, \overline{q_2} \in \Lambda$, *with* $v_{\mathcal{M}}(\overline{q}) = q, v_{\mathcal{M}}(\overline{q_1}) = q_1, v_{\mathcal{M}}(\overline{q_2}) = q_2$, *and* $q_1 < q_2$, *we define:*

- $\tilde{\exists}^{\geq q} x\hat{F}(x) = \overline{q} \to \Pi x\hat{F}(x)$
- $\tilde{\exists}^{\leq q} x\hat{F}(x) = \Pi x\hat{F}(x) \to \overline{q}$
- $\tilde{\exists}^{q} x\hat{F}(x) = \Pi x\hat{F}(x) \leftrightarrow \overline{q}$
- $\tilde{\exists}^{[q_1,q_2]} x\hat{F}(x) = \tilde{\exists}^{\geq q_1} x\hat{F}(x) \wedge \tilde{\exists}^{\leq q_2} x\hat{F}(x)$

Theorem 21 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$ *and* $\overline{q}, \overline{q_1}, \overline{q_2} \in \Lambda$, *with* $v_{\mathcal{M}}(\overline{q}) = q, v_{\mathcal{M}}(\overline{q_1}) = q_1, v_{\mathcal{M}}(\overline{q_2}) = q_2$, *and* $q_1 < q_2$, *we have:*

- $v_{\mathcal{M}}(\tilde{\exists}^{\geq q} x\hat{F}(x)) = \min(1, 1 - q + Prop_{\mathcal{M}}(\hat{F}))$
- $v_{\mathcal{M}}(\tilde{\exists}^{\leq q} x\hat{F}(x)) = \min(1, 1 + q - Prop_{\mathcal{M}}(\hat{F}))$
- $v_{\mathcal{M}}(\tilde{\exists}^{q} x\hat{F}(x)) = 1 - \max(Prop_{\mathcal{M}}(\hat{F}), q) + \min(Prop_{\mathcal{M}}(\hat{F}), q)$
- $v_{\mathcal{M}}(\tilde{\exists}^{[q_1,q_2]} x\hat{F}(x)) = \min(v_{\mathcal{M}}(\tilde{\exists}^{\geq q_1} x\hat{F}(x)), v_{\mathcal{M}}(\tilde{\exists}^{\leq q_2} x\hat{F}(x)))$

Proof:
The results are obtained by basic computations. □

Corollary 2 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$ *and* $\overline{q}, \overline{q_1}, \overline{q_2} \in \Lambda$, *with* $v_{\mathcal{M}}(\overline{q}) = q, v_{\mathcal{M}}(\overline{q_1}) = q_1, v_{\mathcal{M}}(\overline{q_2}) = q_2$, *and* $q_1 < q_2$, *we have:*

- $v_{\mathcal{M}}(\tilde{\exists}^{\geq q} x\hat{F}(x)) = 1$ iff $Prop_{\mathcal{M}}(\hat{F}) \in [q, 1]$
- $v_{\mathcal{M}}(\tilde{\exists}^{\leq q} x\hat{F}(x)) = 1$ iff $Prop_{\mathcal{M}}(\hat{F}) \in [0, q]$
- $v_{\mathcal{M}}(\tilde{\exists}^{q} x\hat{F}(x)) = 1$ iff $Prop_{\mathcal{M}}(\hat{F}) = q$
- $v_{\mathcal{M}}(\tilde{\exists}^{[q_1,q_2]} x\hat{F}(x)) = 1$ iff $Prop_{\mathcal{M}}(\hat{F}) \in [q_1, q_2]$

Proof:
The results are obtained by basic computations. □

Remark 16 *As the Corollary shows, these models need a tolerance margin to be added manually if one wants to have one. This can be achieved straightforwardly. Also note that, since we assume an about-hedge for the basic NL quantifiers, we spare strict comaprisons, as they seem not to contribute anything significant to the intended meaning.*

4.4.2 Blind Choice and Deliberate Choice quantifiers

This section's models are based on [FR14]. While blind choice quantifiers are merely restated for the sake of a self-contained and sufficiently complete presentation, we consider deliberate choice quantifiers in two different settings. The first account models them in what we will call the randomized Hintikka game, i.e. Hintikka's game with the Π quantifier over fuzzy interpretations. The second account will model deliberate choice quantifiers directly in $Ł_\alpha(\Pi)$.

Blind Choice Quantifiers

One class of semi-fuzzy quantifiers is blind choice quantifiers. In [FR14] the authors introduce two blind choice quantifier rule schemata and show how all other possible ones can be reduced to them. Those two are G^k_m and L^k_m, for $k, m \in \mathbb{N}$[8]:

$\mathcal{R}^{\mathcal{RG}}_{G^k_m}$**:** If **P** asserts $G^k_m x\hat{F}(x)$, then, if **O** attacks, **N** chooses a list of $m+k$ occurrences of (not necessarily different) constants $c_1, \ldots, c_{m+k}$. Then, **O** must assert $\neg\hat{F}(c_i)$ for all $i \in \{1, \ldots, m\}$, and **P** must assert $\hat{F}(c_i)$ for all $i \in \{m+1, \ldots, m+k\}$.

$\mathcal{R}^{\mathcal{RG}}_{L^k_m}$**:** If **P** asserts $L^k_m x\hat{F}(x)$, then, if **O** attacks, **N** chooses a list of $k+m$ occurrences of (not necessarily different) constants $c_1, \ldots, c_{k+m}$. Then, **O** must assert $\hat{F}(c_i)$ for all $i \in \{1, \ldots, k\}$, and **P** must assert $\neg\hat{F}(c_i)$ for all $i \in \{k+1, \ldots, k+m\}$.

The idea behind blind choice quantifiers is that the players place their bets unaware of the identity of the sampled constants. Hence, the defined truth functions, that, for **P**, payoff-wise match these game rules, are:

- $v_{\mathcal{M}}(G^k_m x\hat{F}(x)) = \min(1, \max(0, 1-k+(m+k)\cdot \mathit{Prop}_{\mathcal{M}}(\hat{F})))$
- $v_{\mathcal{M}}(L^k_m x\hat{F}(x)) = \min(1, \max(0, 1+k-(m+k)\cdot \mathit{Prop}_{\mathcal{M}}(\hat{F})))$

[8]Note that the following rules adhere to the principle of limited liability, i.e. **P** can always hedge her loss by asserting ⊥.

Their aim then is to derive a representation theorem, in terms of the connectives and quantifiers from the randomized Giles game from [FR14], for each of them. While the ones in [FR14] are incorrect, corrected versions in Fermüller's handbook article in [CFN15] make use of truth constants.

Theorem 22 *[CFN15] The blind choice quantifiers L_m^k and G_m^k for all $m, k \geq 1$ can be expressed in Ł(Π) enriched by certain truth constants via the following reductions ($\overline{a}$ denotes the truth constant for $a \in [0,1]$):*

- $v_{\mathcal{M}}(G_m^k x \hat{F}(x)) = v_{\mathcal{M}}([\neg(\Pi x \hat{F}(x) \to \overline{(k-1)/(m+k)})]_{\oplus}^{m+k})$
- $v_{\mathcal{M}}(L_m^k x \hat{F}(x)) = v_{\mathcal{M}}([\neg(\overline{(1+k)/(m+k)} \to \Pi x \hat{F}(x))]_{\oplus}^{m+k})$

Remark 17 *In the theorem, it is clearly enough to restrict to truth constants referring to the rationals within the real unit interval. Also, for G_m^k if $k = 0$ and for L_m^k if $m = 0$, the truth functions become constantly* 1*. On the other hand, (1) for G_m^k if $m = 0$ the truth function becomes* $\min(1, \max(0, 1 - k + k \cdot \mathit{Prop}_{\mathcal{M}}(\hat{F})))$*, and (2) for L_m^k if $k = 0$ the truth function becomes* $\min(1, \max(0, 1 - m \cdot \mathit{Prop}_{\mathcal{M}}(\hat{F})))$*,*

Deliberate Choice Quantifiers: Representation in $KlZ(\Pi)$

Deliberate choice quantifiers are another important class of semi-fuzzy quantifiers that are based on the random selection principle [FR14, CFN15]. They have been introduced extending Giles's game enriched with the Π quantifier, which we call $\mathcal{RG}$-game[9]. Players place rational bets on numbers of random instances, i.e. unlike with blind choice quantifiers, players now know about the identity of constants a prior to betting on them. Thereby they provide payoff schemes[10] that are supposed to model natural language quantifiers, like "about half" or "about a third".

$\mathcal{R}_{\Pi_m^k}^{\mathcal{RG}}$**:** If **P** asserts $\Pi_m^k x \hat{F}(x)$, then, if **O** attacks, **N** (uniformly) randomly chooses $k + m$ (not necessarily different) constants. Then, **P** picks k of those constants, say $c_1, \ldots, c_k$ and asserts $\hat{F}(c_1), \ldots, \hat{F}(c_k)$, while also asserting $\neg\hat{F}(c_1'), \ldots, \neg\hat{F}(c_m')$, where $c_1', \ldots, c_m'$ are the remaining constants.

In [FR14], the authors show that the associated payoff for **P** matches the following truth function:

$$v_{\mathcal{M}}(\Pi_m^k x \hat{F}(x)) = \binom{k+m}{k} \cdot (\mathit{Prop}_{\mathcal{M}}(\hat{F}))^k \cdot (1 - \mathit{Prop}_{\mathcal{M}}(\hat{F}))^m. \qquad (4.6)$$

[9]Note that there the underlying language is the one of Ł(Π).

[10]Note that the following rule adheres to the principle of limited liability, i.e. **P** can always hedge her loss by asserting ⊥.

Taking Π as semi-fuzzy quantifier, i.e. the scope of Π must be crisp, there is no representation of deliberate choice quantifiers in Ł(Π), while when taking Π as fully-fuzzy quantifier, i.e. fuzzy scopes are permitted, we can express them in Ł(Π). However, as we have shown in [FH17], to that end we do not make use of strong Łukasiewicz conjunction or Łukasiewicz implication, which means we can represent deliberate choice quantifiers in what we call the $\mathcal{RH}$-game. This game is Hintikka's game, enriched with a rule for the quantifier Π, over fuzzy interpretations. Therefore, the corresponding logic is Kleene-Zadeh logic enriched with the Π quantifier, with the defined truth function $Prop_{\mathcal{M}}$. We call this logic $KlZ(\Pi)$, while logical consequence and validity are captured by strict analogy with the logic KlZ.

Let us consider the following two rule schemata[11]:

$\mathcal{R}^{\mathcal{RH}}_{\Pi^{\wedge}_k}$: If the current formula is $\Pi^{\wedge}_k xF(x)$, then **N** (uniformly) randomly chooses k (not necessarily different) constants, and then the player acting as **O** can decide for which constant the game continues with $F(c)$.

$\mathcal{R}^{\mathcal{RH}}_{\Pi^{\vee}_k}$: If the current formula is $\Pi^{\vee}_k xF(x)$, then **N** (uniformly) randomly chooses k (not necessarily different) constants, and then the player acting as **P** can decide for which constant the game continues with $F(c)$.

One can take these two rule schemata as augmenting the $\mathcal{RH}$-game, given that we also augment the underlying language with the respective symbols. However, we will see shortly, in the proof of Theorem 23, how these quantifiers can syntactically be expressed as $KlZ(\Pi)$ formulas, without augmenting the language. Theorem 24 states the explicit correspondence between rules adhering to the general Rule Scheme, defined below just before Theorem 24, and its syntactic representations.

Then, the following theorem holds:

Theorem 23 *For crisp $KlZ(\Pi)$ formulas $\hat{F}$, and $k \geq 0$, we have:*

- $v_{\mathcal{M}}(\Pi^{\wedge}_k x\hat{F}(x)) = Prop_{\mathcal{M}}(\hat{F})^k$
- $v_{\mathcal{M}}(\Pi^{\vee}_k x\hat{F}(x)) = 1 - (1 - Prop_{\mathcal{M}}(\hat{F}))^k$

Proof:
It is relatively straightforward to see that the payoff for the proponent, associated to the rules $\mathcal{R}^{\mathcal{RH}}_{\Pi^{\wedge}_k}, \mathcal{R}^{\mathcal{RH}}_{\Pi^{\vee}_k}$, matches the truth functions of the following nested formulas:

[11] In the upcoming two rule schemata, note that **N** chooses the constants with replacement, i.e. the chosen constants are not necessarily different. This corresponds to independent choices.

- $\Pi x_1 \dots \Pi x_k(\hat{F}(x_1) \wedge \dots \wedge \hat{F}(x_k))$
- $\Pi x_1 \dots \Pi x_k(\hat{F}(x_1) \vee \dots \vee \hat{F}(x_k))$

This is due to the fact that, in the rules, **N** first chooses k constants independently, and then **O** (or **P** respectively) can minimize (or maximize respectively) **P**'s payoff.

Hence:

$v_{\mathcal{M}}(\Pi_k^{\wedge} x \hat{F}(x)) = \frac{1}{|D|^k} \cdot \sum_{c^1 \in D} \cdots \sum_{c^k \in D} \min(v_{\mathcal{M}}(\hat{F}(c_i^1)), \dots, v_{\mathcal{M}}(\hat{F}(c_j^k)))$, and

$v_{\mathcal{M}}(\Pi_k^{\vee} x \hat{F}(x)) = \frac{1}{|D|^k} \cdot \sum_{c^1 \in D} \cdots \sum_{c^k \in D} \max(v_{\mathcal{M}}(\hat{F}(c_i^1)), \dots, v_{\mathcal{M}}(\hat{F}(c_j^k)))$.

For the first equation, we observe that (1), since $\hat{F}$ is a crisp formula, i.e. true or false for each constant, the minimum is zero if one or more constants lead to a negative evaluation of $\hat{F}$. Let us fix the number of constants leading to positive evaluations as $z \in \{0, \dots, \mid D \mid\}$ (note that $Prop_{\mathcal{M}}(\hat{F}) = \frac{z}{|D|}$). For the second equation, we observe that (2) $v_{\mathcal{M}}(\Pi_1^{\vee} x \hat{F}(x)) = v_{\mathcal{M}}(\Pi x \hat{F}(x))$, and that the following reduction equation holds for all $k > 1$:

$v_{\mathcal{M}}(\Pi_k^{\vee} x \hat{F}(x)) = Prop_{\mathcal{M}}(\hat{F}) + (1 - Prop_{\mathcal{M}}(\hat{F})) \cdot v_{\mathcal{M}}(\Pi_{k-1}^{\vee} x \hat{F}(x))$.

We may hence conclude, that ($Prop_{\mathcal{M}}(\hat{F}) = p = 1 - q$):

$v_{\mathcal{M}}(\Pi_k^{\wedge} x \hat{F}(x)) =^{(1)} \frac{z^k}{|D|^k} = Prop_{\mathcal{M}}(\hat{F})^k$, and

$v_{\mathcal{M}}(\Pi_k^{\vee} x \hat{F}(x)) =^{(2)} (1-q) \cdot \sum_{i=0}^{k-1} q^k = (1-q) \cdot \frac{1-q^k}{1-q} = 1 - (1 - Prop_{\mathcal{M}}(\hat{F}))^k$. $\square$

Corollary 3 *For crisp $KlZ(\Pi)$ formulas $\hat{F}$, and $k \geq 0$, we have:*

$$v_{\mathcal{M}}(\Pi_k^{\vee} x \hat{F}(x)) = v_{\mathcal{M}}(\neg \Pi_k^{\wedge} x \neg \hat{F}(x)) \tag{4.7}$$

The corollary follows directly from Theorem 23. The winning conditions for the player I are, that, in case of $\Pi_k^{\wedge}$, all independently and (uniformly) randomly chosen constants lead to a positive evaluation of $\hat{F}$, or, for $\Pi_k^{\vee}$, that there is at least one such constant. In the following, we show how we can produce more involved game rules that allow for expressing binomially distributed success probabilities for the player I of winning the game. The general rule schema, provided in [FH17], captures all possible formulas built from variants[12] of a formula F by using only $\wedge, \vee, \neg, \forall, \exists$, as well as π and Π. It

[12] A variant of $F(x)$ results from uniformly renaming all free occurrences of the exhibited variable x in F. In particular $F(x)$ and $F(y)$ denote variants of each other.

should be noted, that the Rule Scheme admits rules corresponding to game rules for the propositional connectives $\wedge, \vee, \pi$, of arity k in [FH17], and hence here in this section. The general Rule Scheme can be expressed in the following form:

Rule Scheme for $QxF(x)$:

Each move of the rule refers to some $\mathbf{X} \in \{\mathbf{O}, \mathbf{P}, \mathbf{N}\}$, where the player (in role) $\mathbf{X}$ may do just one of two things:
(A) choose a constant, or else
(B) choose among a finite number of given options for continuation; i.e. $\mathbf{X}$ chooses either a subsequent move or a particular instance $F(c)$ (plus potential role switch between I and You) with which the rule ends, where c is from a specified number of previously chosen constants.

The following theorem is then quite natural:

Theorem 24 *[FH17] Let G be a formula built up from variants of a given formula $F(x)$ using the connectives $\wedge_{i=1}^{k}$, $\vee_{i=1}^{k}$, $\pi_{i=1}^{k}$, ($k \geq 2$), $\neg$ and the quantifiers $\forall$, $\exists$, and Π, such that all exhibited variable occurrences are bound. Then G translates into a game rule instantiating the scheme for the quantifier Q_G specified by $v_{\mathcal{M}}(Q_G xF(x)) = v_{\mathcal{M}}(G)$.*

Proof:
By analogy with the arguments in [FR14] and [FH17], we can interpret $\mathcal{RH}$-games as restricted versions of $\mathcal{NRG}$-games in the following way. We disregard the rules for implication, the object quantifier corresponding to strong Łukasiewicz conjunction, and all propositional quantifiers. Then, any state is either of the form $[|H]$, indicating a role distribution $I : \mathbf{P}/You : \mathbf{O}$, or of the form $[H|]$, indicating a role distribution $You : \mathbf{P}/I : \mathbf{O}$, for a formula H. To capture negation, we let states alternate from $[|\neg F]$ to $[F|]$ and from $[\neg F|]$ to $[|F]$, i.e. we perform a role switch. When we arrive at the atomic level, after having decomposed the initial formula G to an atomic formula A, if the state is $[|A]$, the player I wins if $\langle A\rangle = 1 - v_{\mathcal{M}}(A) = 0$, and loses if $\langle A\rangle = 1 - v_{\mathcal{M}}(A) = 1$. Accordingly, if the state is $[A|]$, the player *You* wins if $\langle A\rangle = 1 - v_{\mathcal{M}}(A) = 0$, and loses if $\langle A\rangle = 1 - v_{\mathcal{M}}(A) = 1$.

In the proof of Theorem 4, the argument for π extends straightforwardly to the k-ary case. Then we need to note that we can push negations in G to the atomic level, as we have the following equivalences.

- $\neg(\wedge_{i=1}^{k} F_i) \equiv \vee_{i=1}^{k} \neg F_i$
- $\neg(\vee_{i=1}^{k} F_i) \equiv \wedge_{i=1}^{k} \neg F_i$
- $\neg(\pi_{i=1}^{k} F_i) \equiv \pi_{i=1}^{k} \neg F_i$

- $\neg\forall x F(x) \equiv \exists x \neg F(x)$
- $\neg\exists x F(x) \equiv \forall x \neg F(x)$
- $\neg\Pi x F(x) \equiv \Pi x \neg F(x)$

Consider G' resulting from G by pushing all occurrences of $\neg$ to the inside as far as possible. Then, we arrive at G'' by replacing all consecutive occurrences of negation symbols, in front of atoms, that correspond to an uneven number, with one negation symbol. Also, we entirely remove all consecutive occurrences of negation symbols, in front of atoms, that are multiples of two. Then, we use the following correspondence to build a rule for G'', which is equivalent to G, adhering to the Rule Scheme. The quantifiers $\forall, \exists, \Pi$ correspond to moves of type (A) with a player in role **O**, **P** or **N** respectively. Analogously, the connectives $\wedge_{i=1}^{k}, \vee_{i=1}^{k}, \pi_{i=1}^{k}$ correspond to moves of type (B) with a player in role **O**, **P** or **N** respectively. When all connectives and quantifiers are eliminated, we check whether there is a role switch to be performed, and eventually, the game ends with the evaluation of the remaining atomic formula, as described above. $\square$

The rules $\mathcal{R}^{\mathcal{RH}}_{\Pi_k^\wedge}, \mathcal{R}^{\mathcal{RH}}_{\Pi_k^\vee}$ can now be identified with the instantiation of the Rule Scheme with respect to the formulas presented at the beginning of the proof of Theorem 23. Let us consider another example.

Example 2 *Let* $G = \Pi x(\neg F(x) \pi \exists y(F(y) \vee \forall z \neg F(z)))$. *The corresponding rule that can be extracted with respect to the Rule Scheme is this one:*

$\mathcal{R}^{\mathcal{RH}}_{Q_G}$**:** If the current formula is $\mathsf{Q}_G x F(x)$, then

1 [type (A) move for **N**]: **N** chooses a constant c.

2 [type (B) move for **N**]: **N** chooses between 3 and 4.

3 [end]: The game continues with $F(c)$ after role switch.

4 [type (A) move for **P**]: **P** chooses a constant d.

5 [type (B) move for **P**]: **P** chooses between 6 and 7.

6 [end]: The game continues with $F(d)$ (no role switch).

7 [type (A) move for **O**]: **O** chooses a constant e.

8 [end]: The game continues with $F(e)$ after role switch.

In [FH17], we showed how the following recursive definition of quantifiers leads to a truth function matching the one of the deliberate choice quantifiers (recall $\Pi_1^\wedge x F(x) = \Pi_1^\vee x F(x) = \Pi x F(x)$):

Definition 21 *For a crisp $KlZ(\Pi)$ formula $\hat{F}$, and $k, m \geq 0$, the quantifiers Π^k_m are given by:*

$$\Pi^k_0 x\hat{F}(x) = \Pi^{\wedge}_k x\hat{F}(x), \Pi^0_m x\hat{F}(x) = \Pi^{\wedge}_m x\neg\hat{F}(x), \text{ and}$$
$$\Pi^{k+1}_{m+1} x\hat{F}(x) = \Pi x((\hat{F}(x) \wedge \Pi^k_{m+1} y\neg\hat{F}(y)) \vee (\neg\hat{F}(x) \wedge \Pi^{k+1}_m y\hat{F}(y)).$$

Note that we again use the symbol Π^k_m in this setting of Hintikka's game enriched with the Π quantifier, as we did for the deliberate choice quantifiers, which are objects of Ł(Π). This seems uncontroversial as the distinction of settings is made explicit throughout the presentation.

Theorem 25 *[FH17] Applied to crisp $KlZ(\Pi)$ formulas $\hat{F}$, and $k, m \geq 0$, the truth functions for Π^k_m are as follows:*

$$v_{\mathcal{M}}(\Pi^k_m x\hat{F}(x)) = \binom{k+m}{k} \cdot Prop_{\mathcal{M}}(\hat{F})^k \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))^m. \tag{4.8}$$

Proof:
The two base cases, where either k or m are 0, follow from Theorem 23. For the case $k = 1$ we argue as follows. Writing v^k_m for $v_{\mathcal{M}}(\Pi^k_m x\hat{F}(x))$,$\overline{v}^k_m$ for $v_{\mathcal{M}}(\Pi^k_m x\neg\hat{F}(x))$ and p for $Prop_{\mathcal{M}}(\hat{F})$ we obtain:

$$v^1_{m+1} = p \cdot \overline{v}^0_{m+1} + (1 - p) \cdot v^1_m$$

and hence by Theorem 23 and induction on m, we have:

$v^1_{m+1} = p \cdot (1 - p)^{m+1} + (1 - p) \cdot \binom{m+1}{1} \cdot p \cdot (1 - p)^m =$
$= (m + 1 + 1) \cdot p \cdot (1 - p)^{m+1} = \binom{m+2}{1} \cdot p \cdot (1 - p)^{m+1}.$

The case for fixed $m = 1$ works by analogy. Generally, for $m > 1$,

$$v^{k+1}_m = p \cdot \overline{v}^k_m + (1 - p) \cdot v^{k+1}_{m-1}$$

and therefore, by induction and Footnote 14,

$v^{k+1}_m =$
$= p \cdot \binom{k+m}{k} \cdot p^k \cdot (1 - p)^m + (1 - p) \cdot \binom{k+1+m-1}{k+1} \cdot p^{k+1} \cdot (1 - p)^{m-1} =$
$= \binom{k+m+1}{k+1} \cdot p^{k+1} \cdot (1 - p)^m.$

The case for increasing k while fixing m works analogously. □

We now provide the explicit game rule schema, adhering to the Rule Scheme, with a payoff for the player I that matches the afore-mentioned truth functions of Π^k_m. The following rules are supposed to be understood as applied to a crisp scope formula $\hat{F}$. Let k, m be natural numbers greater than zero[13], $i \in \{1, \ldots, k+m\}$, and $j \in \{0, \ldots, m\}$, then we define the following game rule schema (starting with move α^0_1):

$\mathcal{R}^{\mathcal{RH}}_{\Pi^k_m}$: If the current formula is $\Pi^k_m x\hat{F}(x)$, then, for all $j \in \{0, \ldots, m-1\}$ with $i \in \{1+j, \ldots, k+j\}$:

α^j_i : **N** chooses a constant c_i.

β^j_i : **P** decides between γ^j_i and γ^{j+1}_i:

- γ^j_i: **O** decides whether to end the game with $F(c_i)$, or to continue the game with move α^j_{i+1}.
- γ^{j+1}_i: **O** decides whether to end the game with $F(c_i)$+role switch, or to continue the game with move α^{j+1}_{i+1}.

If $j = m$ (or $i - j = k + 1$):

α^j_i : **N** chooses $k + m + 1 - i$ (or $m - j$) constants.

β^j_i : **O** decides for which of these constants the game continues with $F(c)$ (or $F(c)$ + role switch).

We capture the behaviour of this game rule in the following theorem. Note that, since we are dealing with randomized payoff, we are employing Definition 4 rather than Definition 2. Although Definition 4 can be applied as it stands, we can here also spare the reference to $\epsilon > 0$ (recall that this just means we read the same definition while deleting the references to ϵ in it).

Theorem 26 *Let a crisp $KlZ(\Pi)$ formula $\hat{F}$ be given. Then, $v_{\mathcal{M}}(\Pi^k_m x\hat{F}(x)) = w$ in a fuzzy interpretation $\mathcal{M}$ iff the $\mathcal{RH}$-game for $\Pi^k_m x\hat{F}(x)$ is $(1-w)$-valued for the player* I *under risk value assignment $\langle\cdot\rangle$ matching $\mathcal{M}$..*

Proof:
To terminate the game for Π^k_m it is necessary that either the subscript or the superscript of Π^k_m becomes zero first, while Figure 4.2 shows how the decrementing of the scripts of Π^k_m corresponds to the incrementing of the scripts of α^j_i. Hence, we distinguish these two cases ($Prop_{\mathcal{M}}(\hat{F}) = p$):

1, The superscript of Π^k_m is zero first:

[13]Note that rules for either one of these numbers being zero, or even both, are trivial.

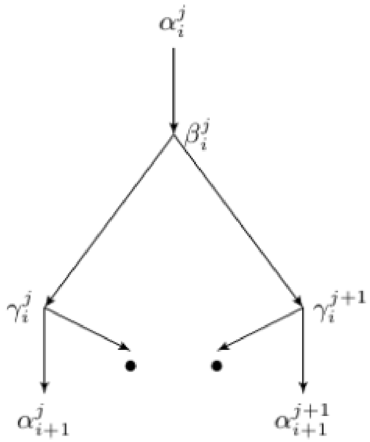

Figure 4.1: Flow of the game rule $\mathcal{R}^{\mathcal{RH}}_{\Pi^k_m}$, for fixed i.

Scripts get decremented by 1, hence to get a superscript equal to 0, i.e. being in one of the situations $\Pi^0_m, \dots, \Pi^0_1$, we have to come from one of the situations $\Pi^1_m, \dots, \Pi^1_1$ with probability p. To those, the player I comes with probabilities $\binom{k-1+0}{k-1} \cdot p^{k-1} \cdot (1-p)^0, \dots, \binom{k-1+(m-1)}{k-1} \cdot p^{k-1} \cdot (1-p)^{m-1}$. And from those, the player I departs with success probabilities $(1-p)^m, \dots, (1-p)^1$. Hence together, the success probabilities for the player I are:

$\sum_{i=0}^{m-1} \binom{k-1+i}{k-1} \cdot p^k \cdot (1-p)^m$.

2, The subscript of Π^k_m is zero first:

As scripts get decremented by 1, one gets to a subscript equal to 0, i.e. to one of the situations $\Pi^k_0, \dots, \Pi^1_0$, by coming from one of the situations $\Pi^k_1, \dots, \Pi^1_1$ with probability $1-p$. To those, the player I comes with probabilities $\binom{m-1+0}{0} \cdot p^0 \cdot (1-p)^{m-1}, \dots, \binom{m-1+(k-1)}{k-1} \cdot p^{k-1} \cdot (1-p)^{m-1}$. And from those, the player I departs with success probabilities $p^k, \dots, p^1$. Hence together, the success probabilities for the player I are:

$\sum_{i=0}^{k-1} \binom{m-1+i}{i} \cdot p^k \cdot (1-p)^m$.

It now remains to show, that:

$\binom{k+m}{k} = \sum_{i=0}^{m-1} \binom{k-1+i}{k-1} + \sum_{i=0}^{k-1} \binom{m-1+i}{i}$.

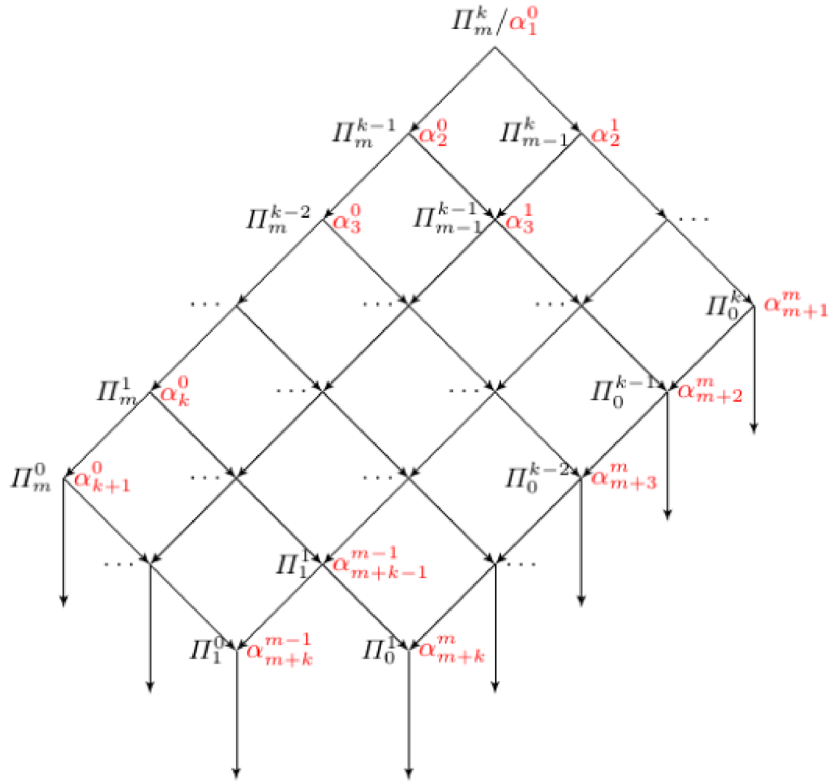

Figure 4.2: Flow of the game rule $\mathcal{R}^{\mathcal{RH}}_{\Pi^k_m}$ as i increases.

The cases where $m = 1$, or $k = 1$ are checked easily. For bigger numbers we prove the statement by induction, and (1) assume that $k > 1$ is fixed first. We now fix also m and assume (induction hypothesis IH) the equation holds. We now have to show that:

$\binom{k+m+1}{k} = \sum_{i=0}^{m} \binom{k-1+i}{k-1} + \sum_{i=0}^{k-1} \binom{m+i}{i}$.

The following computation holds[14]:

$\binom{k+m+1}{k} = \binom{k+m}{k-1} + \binom{k+m}{k} =^{(IH)} \binom{k+m}{k-1} + \sum_{i=0}^{m-1} \binom{k-1+i}{k-1} + \sum_{i=0}^{k-1} \binom{m-1+i}{i}$.

We compute:

$\sum_{i=0}^{k-1} \binom{m-1+i}{i} - \sum_{i=0}^{k-1} \binom{m+1}{i} = \sum_{i=0}^{k-1} (\binom{m-1+i}{i} - \binom{m+1}{i}) =$

[14] As we have $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$, for $n > 0$ and $0 \leq r < n$.

$= \sum_{i=1}^{k-1}(\binom{m-1+i}{i} - \binom{m+1}{i}) = -\sum_{i=1}^{k-1}\binom{m-1+i}{i-1}$.

The last equality follows directly from the equation of Footnote 14. Hence it remains to show:

$\binom{k+m}{k-1} = \binom{k-1+m}{k-1} + \sum_{i=1}^{k-1}\binom{m-1+i}{i-1}$.

After a simple rearrangement, it remains to show that:

$\sum_{i=0}^{k-2}\binom{m+i}{i} = \binom{k+m-1}{k-2}$,

which follows from a side-induction on $k > 1$:

For the side-induction start, we note that for $k = 2$ we have:

$\sum_{i=0}^{k-2}\binom{m+i}{i} = \binom{m}{0} = 1 = \binom{m+1}{0} = \binom{k+m-1}{k-2}$.

The side-induction hypothesis (sI.H.) is that, for an arbitrary but fixed $k \geq 2$, we have:

$\sum_{i=0}^{k-2}\binom{m+i}{i} = \binom{k+m-1}{k-2}$.

As to the side-induction step, we compute:

$\sum_{i=0}^{(k+1)-2}\binom{m+i}{i} = \sum_{i=0}^{k-2}\binom{m+i}{i} + \binom{m+k-1}{k-1} =^{sI.H.} \binom{k+m-1}{k-2} + \binom{m+k-1}{k-1} =$
$= \binom{k+m}{k-1} = \binom{(k+1)+m-1}{(k+1)-2}$.

The second last rewriting step is due to Footnote 14.

Part (2) of the induction, when $m > 1$ is fixed first, proceeds by analogy. □

Deliberate Choice Quantifiers: Representation in Ł$_\alpha(\Pi)$

The following theorem gives a representation theorem for deliberate choice quantifiers, in the Ł$_\alpha(\Pi)$ setting, that employs the defined $\cdot$ connective, rather than nesting of quantifiers.

Definition 22 *For a* Ł$_\alpha(\Pi)$ *formula F and a positive integer k we define:*

$$F^k = F \cdot ... \cdot F, \text{ with } k \text{ occurrences of } F. \tag{4.9}$$

It is clear that, $v_{\mathcal{M}}(F^k) = v_{\mathcal{M}}(F)^k$.

Theorem 27 *[Hof18]For crisp* $Ł_\alpha(\Pi)$ *formulas* $\hat{F}$*, we have:*

$$v_{\mathcal{M}}(\Pi^k_m x \hat{F}(x)) = v_{\mathcal{M}}([(\Pi x \hat{F}(x))^k \cdot (\Pi x \neg \hat{F}(x))^m]_{\oplus}^{1 \leq i \leq \binom{k+m}{m}}). \tag{4.10}$$

Proof:
$v_{\mathcal{M}}([(\Pi x \hat{F}(x))^k \cdot (\Pi x \neg \hat{F}(x))^m]_{\oplus}^{1 \leq i \leq \binom{k+m}{m}}) =$
$= \min(1, \binom{k+m}{m} \cdot v_{\mathcal{M}}((\Pi x \hat{F}(x))^k) \cdot v_{\mathcal{M}}((\Pi x \neg \hat{F}(x))^m)) =$
$= \min(1, \binom{k+m}{m} \cdot v_{\mathcal{M}}(\Pi x \hat{F}(x))^k \cdot v_{\mathcal{M}}(\Pi x \neg \hat{F}(x))^m) =$
$= \min(1, \binom{k+m}{m} \cdot Prop_{\mathcal{M}}(\hat{F})^k \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))^m) =$
$= \binom{k+m}{m} \cdot Prop_{\mathcal{M}}(\hat{F})^k \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))^m.$ □

4.4.3 Quantifiers based on granularity levels

An idea going back to Zadeh [Zad79], being carried out much in recent years, is granular computing [PSK08]. The idea is to attach a level of granularity to certain scenarios, hence making objects indistinguishable with respect to some (equivalence) relation, which corresponds to having tolerance margins, or intervals, around some crisp value. We extend this idea, which is applied to vague concepts in [Yao01], to model relative (unary) quantifiers of Type III [Hof16b], but also show how the same idea can be used to model relative (unary) Type I quantifiers [Hof16b].

Example 3 *The quantifier expression"about half" can be associated with several acceptance intervals, e.g.* $[37.5\%, 62.5\%]$, $[45\%, 55\%]$, $[49.5\%, 50.5\%]$*, or others. We can partition the unit interval in arbitrarily many different ways, where each partitioning then corresponds to some level of granularity. Having several such levels, we can talk about a granular hierarchy [Kee09, XYJ13, Yao01]. However, following everyday experience, we propose the following systematic refinement procedure:*

- *3-partitioning, e.g. "small numbers","about half", and "large numbers".*

  partitioning intervals: $[0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, 1]$

- *5-partitioning, e.g. "nearly none", "several", "about half", "most", "almost all".*

  partitioning intervals: $[0, \frac{1}{5}), \ldots, [\frac{4}{5}, 1]$

- *7-partitioning, e.g. "nearly none","small numbers", "several","about half","most", "large numbers", "almost all".*

  partitioning intervals: $[0, \frac{1}{7}), \ldots, [\frac{6}{7}, 1]$

- *tenner-partitioning: (About)* $0\%, 10\%, 20\%, \ldots, 90\%, 100\%$.

  partitioning intervals: $[0, \frac{1}{20}), [\frac{1}{20}, \frac{3}{20}), \ldots, [\frac{17}{20}, \frac{19}{20}), [\frac{19}{20}, 1]$

- *fiver-partitioning: (About)* $0\%, 5\%, 10\%, 15\%, \ldots, 90\%, 95\%, 100\%$.

  partitioning intervals: $[0, \frac{1}{40}), [\frac{1}{40}, \frac{3}{40}), \ldots, [\frac{37}{40}, \frac{39}{40}), [\frac{39}{40}, 1]$

- *oner-partitioning: (About)* $0\%, 1\%, 2\%, 3\%, \ldots, 98\%, 99\%, 100\%$.

  partitioning intervals: $[0, \frac{1}{200}), [\frac{1}{200}, \frac{3}{200}), \ldots, [\frac{197}{200}, \frac{199}{200}), [\frac{199}{40}, 1]$

- *decimal place-partitioning: (About)* $0\%, 0.1\%, 0.2\%, 0.3\%, \ldots, 99.8\%, 99.9\%, 100\%$.

  partitioning intervals: $[0, \frac{1}{2000}), [\frac{1}{2000}, \frac{3}{2000}), \ldots, [\frac{1997}{2000}, \frac{1999}{2000}), [\frac{1999}{2000}, 1]$

These classifications are, of course, somehow freely defined, and may hence be changed accordingly. However, taking for example the 3*-partitioning, an arbitrary set of objects and an arbitrary crisp property, e.g.* 100 *people and the property of being a student, one probably could agree that there might be someone who would consider a number of* 29 *students among the* 100 *people to be a relatively small number. Similarly for all other partitionings.*

To describe the semantics of some relative (unary) Type III quantifier Q*, we need to fix a finite number of such levels of granularity, say* $GL_1, \ldots, GL_{m_\mathsf{Q}}$*,* $m_\mathsf{Q} \geq 1$*, with respect to which we can evaluate respective statements. In the present case, for statements of the kind "about half (of the domain elements) fulfill property* $\hat{F}$*", we then have acceptance intervals*[15] *as follows:*

- $[\frac{1}{3}, \frac{2}{3})$ *(3-partitioning)*
- $[\frac{2}{5}, \frac{3}{5})$ *(5-partitioning)*
- $[\frac{3}{7}, \frac{4}{7})$ *(7-partitioning)*

[15] I.e. the statement is true if $Prop_\mathcal{M}(\hat{F})$ is an element of this acceptance interval.

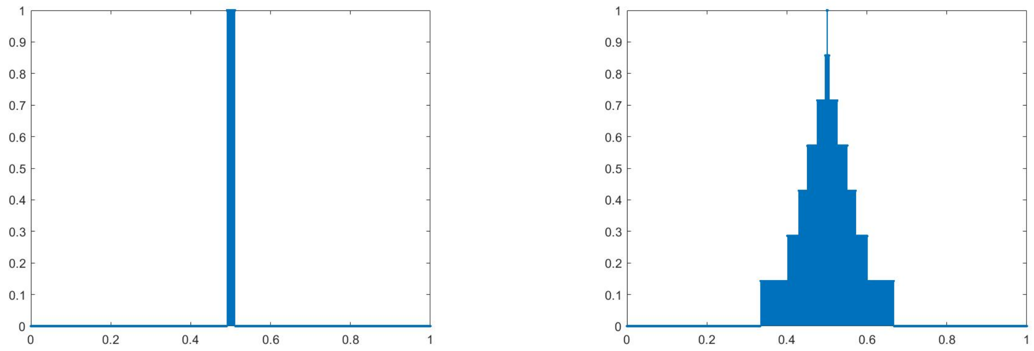

Figure 4.3: left: truth function of “about half” modeled by a granular hierarchy with only one element, i.e. only one granular level, namely $[0.49, 0.51]$; right: truth function of “about half” modeled by a granular hierarchy with seven distinct granular levels, namely the $3-, 5-, 7-$,tenner-,fiver-,oner- and deciaml place-partitioning of Example 3.

- $[45, 55)$ *(tenner-partitioning)*
- $[47.5, 52.5)$ *(fiver-partitioning)*
- $[49.5, 50.5)$ *(oner-partitioning)*
- $[49.95, 50.05)$ *(decimal place-partitioning)*

Remark 18 *It seems important to remark that different levels of granularity are not merely a tool to model that different agents may tolerate smaller or larger deviations regarding the same scenario. Different levels of granularity should also be seen as referring to entirely different situations, where once a smaller and another time a larger tolerance margin is what is needed to model vague quantifier expressions adequately. For the latter use, Vetterlein [Vet11] has contributed convincing arguments.*

Formally, we define a granularity level and granular hierarchies as follwos:

Definition 23 *A granularity level GL corresponds to a partitioning of the real unit interval $[0, 1]$ into finitely many disjoint intervals $A_1, \ldots, A_{m_{GL}}$, $m_{GL} \geq 1$, such that $\bigcup_{i=1}^{m_{GL}} A_i = [0, 1]$. A granular hierarchy g consists of a set of granularity levels.*

Definition 24 *Any relative (unary) Type III quantifier Q_g, based on a granular hierarchy g, comes with a set $\{GL_1, \ldots, GL_{m_\mathsf{Q}}\}$, $m_\mathsf{Q} \geq 1$, of granularity levels, corresponding to g, such that each such level has a unique acceptance interval I_z for Q_g. That means if Q_{g,GL_i} denotes the quantifier Q_g restricted to the granularity level GL_i, and $I_z =$*

$[z^-_{\mathsf{Q},g,GL_i}, z^+_{\mathsf{Q},g,GL_i}]$, with $\overline{z^-_{\mathsf{Q},g,GL_i}}, \overline{z^+_{\mathsf{Q},g,GL_i}} \in \Lambda$ such that $v_{\mathcal{M}}(\overline{z^-_{\mathsf{Q},g,GL_i}}) = z^-_{\mathsf{Q},g,GL_i}$ and $v_{\mathcal{M}}(\overline{z^+_{\mathsf{Q},g,GL_i}}) = z^+_{\mathsf{Q},g,GL_i}$, we define for a crisp formula from $Ł_\alpha(\Pi)$ enriched with a k-ary π for $k \geq 2$:

- $\mathsf{Q}_{g,GL_i} x\hat{F}(x) = \Delta((\Pi x\hat{F}(x) \to \overline{z^+_{\mathsf{Q},g,GL_i}}) \wedge (\overline{z^-_{\mathsf{Q},g,GL_i}} \to \Pi x\hat{F}(x)))$
- $\mathsf{Q}_g x\hat{F}(x) = \pi^{m_\mathsf{Q}}_{i=1} \mathsf{Q}_{g,GL_i} x\hat{F}$

Theorem 28 *For a crisp formula $\hat{F}$ from $Ł_\alpha(\Pi)$ enriched with a k-ary π for $k \geq 2$, and relative (unary) Type III quantifier, based on a granular hierarchy, we have:*

- $v_{\mathcal{M}}(\mathsf{Q}_{g,GL_i} x\hat{F}(x)) = \min(\mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \geq z^-_{\mathsf{Q},g,GL_i})}, \mathbb{I}_{(Prop_{\mathcal{M}}(\hat{F}) \leq z^+_{\mathsf{Q},g,GL_i})})$
- $v_{\mathcal{M}}(\mathsf{Q}_g x\hat{F}(x)) = \frac{\sum^{m_\mathsf{Q}}_{i=1} v_{\mathcal{M}}(\Delta((\Pi x\hat{F}(x) \to \overline{z^+_{\mathsf{Q},g,GL_i}}) \wedge (\overline{z^-_{\mathsf{Q},g,GL_i}} \to \Pi x\hat{F}(x))))}{m_\mathsf{Q}}$

Proof:
For the first claim, we note that, since there is a Delta in front of the conjunction, both conjuncts have to be fully true, in order to make the whole statement true. This is the case if both, $Prop_{\mathcal{M}}(\hat{F}) \geq z^-_{\mathsf{Q},GL_i}$ and $Prop_{\mathcal{M}}(\hat{F}) \leq z^+_{\mathsf{Q},GL_i}$, are true. This is exactly what the stated truth function expresses. The second claim follows directly from the first.
□

Remark 19 *When one restricts attention to granular hierarchies with only one element, one immediately gets models for relative (unary) Type I quantifiers.*

The models Q_g can of course be weakened or strengthened by replacing $\pi^{m_\mathsf{Q}}_{i=1}$ by $\vee^{m_\mathsf{Q}}_{i=1}$ or $\wedge^{m_\mathsf{Q}}_{i=1}$ respectively, according to intended usages. The merit of this approach to modeling vague natural language quantifiers is the flexibility that comes from allowing for different ways to make vagueness crisp. This reflects different opinions of distinct speakers, while the overlapping operation, modeled by a k-ary π connective, gives us the average of of individual crisp values as the final truth value of a statement.

4.4.4 Quantifiers based on random sampling without replacement and strong conjunction/ strong disjunction

Nature selects witnessing constants uniformly randomly. In a formula, each Π quantifier stands for choosing one constant from D. When more than one constant is chosen, due to the appearance of more than one Π quantifier, then, each time, *Nature* will choose again from the whole set of constants D, hence the picks are meant to be stochastically independent and performed with replacement (w.r.). New quantifier models arise when we allow for uniform random selection of constants, where *Nature* can choose more than one constant at once, hence without replacement (wo.r.), which then need to be different. Above, when we treated the case of relative (unary) Type I quantifiers, we have seen game rules, where **P** chooses several different constants. There, we already saw how it can make an important difference to demand that the witnesses not be identical, and the same applies when *Nature* acts, as the probabilities for certain events, that we are interested in, increase. For large domains it might seem neglectable, however for finite domains interesting consequences pop up. Unlike deliberate choice quantifiers, which in [FR14] undergo an artificial modification procedure to cope with the defect that the truth functions are always bound by certain numbers smaller than 1, the augmented (uniform) random selection principle allows for defining convincing models for relative semi-fuzzy quantifiers representing NL quantifiers like "about $(q \cdot 100)\%$", and "at least/ at most about $(q \cdot 100)\%$". Looking at it from the perspective of probability theory, the truth functions of the deliberate choice quantifiers are the probability distributions of binomially distributed random variables. The new approach of this chapter then corresponds to the so called hypergeometric distributions, which, for domains that grow toward an infinite cardinality, converge to the binomial distribution [Geo13]. This intuitively makes immediate sense, as for infinite domains the probability of choosing exactly one particular object is zero (when choosing uniformly randomly). We will investigate two classes of models. Both can be used for applications, while the first seems most appropriate when the domain of discourse is rather small but precision matters, hence the quantifier models of that kind have a focus on a clear semantics of vague quantifiers, while the latter uses a stochastic trick, that makes it usable even when the domain of discourse becomes very large, at the price that the semantics is much more bound to probabilistic expectations than to hard facts. The game rule, for what we call the $\Pi^{j,k}$ quantifier, is the following [Hof16a]:

$\mathcal{R}^{\mathcal{NRG}}_{\Pi^{j,k}}$**:** If **P** asserts $\Pi^{j,k} x \hat{F}(x)$, then, if **O** attacks, **N** chooses $1 \leq k \leq \mid D \mid$ different constants c, and then **P** has to assert $\hat{F}(c)$ for j different of the c's, or $\bot$ instead.

The upcoming theorem states, that the payoff for **P** associated to that game rule matches the following truth function:

$$v_{\mathcal{M}}(\Pi^{j,k} x \hat{F}(x)) = \frac{\sum_{i=j}^{k} \binom{|D| \cdot Prop_{\mathcal{M}}(\hat{F})}{i} \binom{|D| \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))}{k-i}}{\binom{|D|}{k}}. \tag{4.11}$$

Theorem 29 *Let $1 \leq j \leq k \leq \mid D \mid$ be given. A sentence $\Pi^{j,k} x \hat{F}(x)$, for a crisp $Ł_\alpha(\Pi)$ formula $\hat{F}$, is evaluated to $v_{\mathcal{M}}(\Pi^{j,k} x \hat{F}(x)) = w$ in a fuzzy interpretation $\mathcal{M}$ iff every $\mathcal{NRG}$-game augmented by rule ($\mathcal{R}^{\mathcal{RG}}_{\Pi^{j,k}}$) for $\Pi^{j,k} x \hat{F}(x)$ is $(1-w)$-valued for the player* I *under risk value assignment $\langle \cdot \rangle$ matching $\mathcal{M}$.*

Proof:
The proof builds upon the proof of Theorem 3, Theorem 4 and Theorem 6. As we have seen there, risks behave additively, which means that:

A: $\langle \Gamma \mid \Gamma', \Pi^{j,k} x \hat{F}(x) \rangle = \langle \Gamma \mid \Gamma' \rangle + \langle \mid \Pi^{j,k} x \hat{F}(x) \rangle$

B: $\langle \Gamma, \Pi^{j,k} x \hat{F}(x) \mid \Gamma' \rangle = \langle \Gamma \mid \Gamma' \rangle + \langle \Pi^{j,k} x \hat{F}(x) \mid \rangle$

Case A: When $\mathbf{N}$ (uniformly) randomly samples k different constants, we first of all have to distinguish two different cases. There are either at least j positive witnesses for the formula $\hat{F}$, or strictly less than j. In the latter case, the player I loses and since $\hat{F}$ is a crisp formula, we can assume that she triggers her part of the principle of limited liability, i.e. she asserts $\bot$. In total, there are $\binom{|D|}{k}$ ways for $\mathbf{N}$ to perform the sampling, i.e. there are as many branches of the game tree. For each $1 \leq i \leq k$, there are $\binom{|D| \cdot Prop_{\mathcal{M}}(\hat{F})}{i} \cdot \binom{|D| \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))}{k-i}$ branches with exactly i positive and $k - i$ negative witnesses of $\hat{F}$. Since we assume that $\langle \hat{F}(c) \rangle = 1 - v_{\mathcal{M}}(\hat{F}(c))$ for all $c \in D$ (induction hypothesis), the value for the player I is computed as follows:

$$\langle \mid \Pi^{j,k} x \hat{F}(x) \rangle = \sum_{i=0}^{j-1} \frac{\binom{|D| \cdot Prop_{\mathcal{M}}(\hat{F})}{i} \cdot \binom{|D| \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))}{k-i}}{\binom{|D|}{k}} = 1 - w.$$

We now compute w:

$$w = 1 - \langle \mid \Pi^{j,k} x \hat{F}(x) \rangle = 1 - \sum_{i=0}^{j-1} \frac{\binom{|D| \cdot Prop_{\mathcal{M}}(\hat{F})}{i} \cdot \binom{|D| \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))}{k-i}}{\binom{|D|}{k}} =$$

$$= \sum_{i=j}^{k} \frac{\binom{|D| \cdot Prop_{\mathcal{M}}(\hat{F})}{i} \cdot \binom{|D| \cdot (1 - Prop_{\mathcal{M}}(\hat{F}))}{k-i}}{\binom{|D|}{k}}.$$

The last equation is an instance of the distribution function of hypergeometric random variables. It follows from Vandermonde's identity, which states that for three natural numbers $m, r, k \in \mathbb{N}$ we have [CKM92]:

$$\binom{m+k}{r} = \sum_{i=0}^{r} \binom{m}{i} \cdot \binom{k}{r-i}. \tag{4.12}$$

Case B works by analogy. $\square$

Note that we have $\Pi x \hat{F}(x) = \Pi^{1,1} x \hat{F}(x)$. Also, it is important to note that the $\Pi^{j,k}$ quantifier can only be applied to crisp scopes, as the Π quantifier in its original intention [FR14]. This makes it a bit less general again, but our overall strategy, to treat semi-fuzzy quantifiers, and only later lift them to fully-fuzzy ones, by means of quantifier fuzzification mechanisms (QFMs), works well in accordance with that fact. We are only going to use the quantifier $\Pi^{j,k}$ in this chapter, but for reasons of symmetry and completeness, we also introduce the propositional connective that works along the same lines. For a finite set of m closed formulas $\hat{F}_i$, let us call it A_m, the game rule becomes:

$\mathcal{R}^{\mathcal{NRG}}_{\pi^{j,k}_{A_m}}$**:** If **P** asserts $\pi^{j,k}_{A_m}\hat{F}_i$, then, if **O** attacks, **N** uniformly randomly chooses $1 \leq k \leq m$ different $\hat{F}_i$'s from a set of m such, i.e. from A_m, and then **P** has to assert $\hat{F}_i$ for j different of those, or $\bot$ instead.

P's payoff associated to that game rule matches the following truth function, as the following theorem states ($q = \frac{1}{m} \cdot \sum_{i=1}^{m} v_{\mathcal{M}}(\hat{F}_i)$):

$$v_{\mathcal{M}}(\pi^{j,k}_{A_m}\hat{F}_i) = \frac{\sum_{i=j}^{k} \binom{m \cdot q}{i}\binom{m\cdot(1-q)}{k-i}}{\binom{m}{k}}. \tag{4.13}$$

Theorem 30 *Let $1 \leq j \leq k \leq m$ and A_m, as introduced just above, be given. A formula $\pi^{j,k}_{A_m}x\hat{F}_i$, for crisp and closed $Ł_\alpha(\Pi)$ formulas $\hat{F}_i$, $1 \leq i \leq m$, is evaluated to $v_{\mathcal{M}}(\pi^{j,k}_{A_m}\hat{F}_i) = w$ in a fuzzy interpretation $\mathcal{M}$ iff every $\mathcal{NRG}$-game augmented by rule ($\mathcal{R}^{\mathcal{RG}}_{\pi^{j,k}_{A_m}}$) for $\pi^{j,k}_{A_m}\hat{F}_i$ is $(1-w)$-valued for the player* I *under risk value assignment $\langle\cdot\rangle$ matching $\mathcal{M}$.*

Proof:
This works by analogy with the proof of Theorem 29. □

Remark 20 *The propositional connective $\pi^{j,k}_{A_m}$ is a good candidate to model vague NL properties, as we can use it to encode the necessity to fulfill a certain number of distinct crisp properties from a given list. In that way, we model borderline cases of predicates, i.e. such that are neither fully true nor fully false, but have intermediate truth values. A longer treatment of such borderline cases can e.g. be found in [KS97] or [Smi08].*

Quantifiers based on random sampling without replacement and strong conjunction

The quantifier expressions that we model in this section relate to those that we introduced as [*Rel*2] above, comprising[16] "almost all" (Q^{alal}_t), "nearly none" (Q^{neno}_t), "at least/ at most about $(q \cdot 100)$%" ($\mathsf{Q}^{\geq q}_t$/$\mathsf{Q}^{\leq q}_t$), and "about $(q \cdot 100)$%" (Q^{q}_t), all dependent on a tolerance value t [Hof16a].

Definition 25 *For a crisp $Ł_\alpha(\Pi)$ formula $\hat{F}$, $q \in [0,1]$, and $j, t \geq 1$ such that $j + t \leq \mid D \mid$, we define:*

[16] We interpret quantifiers "about 0%" as "nearly none" and "about 100%" as "almost all".

- $\mathsf{Q}_t^{alal} x\hat{F}(x) = \Pi^{j,j+t} x\hat{F}(x)$
- $\mathsf{Q}_t^{neno} x\hat{F}(x) = \Pi^{j,j+t} x\neg\hat{F}(x)$

If for $n = \mid D \mid$ we have $\lfloor n(1-q)\rfloor + j + t \leq n$, or $\lfloor nq\rfloor + j + t \leq n$ respectively, we also define:

- $\mathsf{Q}_t^{\geq q} x\hat{F}(x) = \Pi^{j,\lfloor n(1-q)\rfloor+j+t} x\hat{F}(x)$
- $\mathsf{Q}_t^{\leq q} x\hat{F}(x) = \Pi^{j,\lfloor nq\rfloor+j+t} x\neg\hat{F}(x)$

Eventually we set: $\mathsf{Q}_t^{q} x\hat{F}(x) = \mathsf{Q}_t^{\leq q} x\hat{F}(x) \ \wedge\ \mathsf{Q}_t^{\geq q} x\hat{F}(x)$.

The truth functions can be determined as follows:

Theorem 31 *For a crisp $Ł_\alpha(\Pi)$ formula $\hat{F}$, $q \in [0,1]$, $n = \mid D \mid$, and $j, t \geq 1$ such that $j + t \leq n$ (or $\lfloor n(1-q)\rfloor + j + t \leq n$, $\lfloor nq\rfloor + j + t \leq n$ respectively), we have:*

- $v_{\mathcal{M}}(\mathsf{Q}_t^{alal} x\hat{F}(x)) = \frac{\sum_{i=j}^{j+t}\binom{n\cdot p}{i}\binom{n\cdot(1-p)}{j+t-i}}{\binom{n}{j+t}}$
- $v_{\mathcal{M}}(\mathsf{Q}_t^{neno} x\hat{F}(x)) = \frac{\sum_{i=j}^{j+t}\binom{n\cdot(1-p)}{i}\binom{n\cdot p}{j+t-i}}{\binom{n}{j+t}}$
- $v_{\mathcal{M}}(\mathsf{Q}_t^{\geq q} x\hat{F}(x)) = \frac{\sum_{i=j}^{\lfloor n(1-q)\rfloor+j+t}\binom{n\cdot p}{i}\binom{n\cdot(1-p)}{\lfloor n(1-q)\rfloor+j+t-i}}{\binom{n}{\lfloor n(1-q)\rfloor+j+t}}$
- $v_{\mathcal{M}}(\mathsf{Q}_t^{\leq q} x\hat{F}(x)) = \frac{\sum_{i=j}^{\lfloor nq\rfloor+j+t}\binom{n\cdot(1-p)}{i}\binom{n\cdot p}{\lfloor nq\rfloor+j+t-i}}{\binom{n}{\lfloor nq\rfloor+j+t}}$
- $v_{\mathcal{M}}(\mathsf{Q}_t^{q} x\hat{F}(x)) =$
  $= \min(\frac{\sum_{i=j}^{\lfloor nq\rfloor+j+t}\binom{n\cdot(1-p)}{i}\binom{n\cdot p}{\lfloor nq\rfloor+j+t-i}}{\binom{n}{\lfloor nq\rfloor+j+t}}, \frac{\sum_{i=j}^{\lfloor n(1-q)\rfloor+j+t}\binom{n\cdot p}{i}\binom{n\cdot(1-p)}{\lfloor n(1-q)\rfloor+j+t-i}}{\binom{n}{\lfloor n(1-q)\rfloor+j+t}})$

Proof:
The results follow from basic computations. □

This semantic modeling of relative semi-fuzzy quantifiers is accurate in the sense that, the quantifiers evaluate to 1 if and only if the intended meaning is true. For example, for the quantifier “almost all” with tolerance $t = 1$ and a domain size of 100, the intended meaning is that a corresponding statement is (fully) true if and only if 99 or 100 objects from the domain fulfill the property from the quantified predicate. This means at most one negative witness is permitted. If that is true, this means that any sample of size greater or equal than two must have at least one positive witness. The tolerance value is absolute in the sense that this deviation leaves the truth value unchanged, while increasing j corresponds to stronger readings, i.e. it becomes harder (more unlikely) to win the game. A similar reasoning applies to the quantifiers “at least about $(q \cdot 100)\%$” and “at most about $(q \cdot 100)\%$”. This explains in particular the range dependence of these quantifier models. To check whether there are “at least (about) half” objects fulfilling a certain property, we have to look at more than half of the objects. To illustrate this, assume that $t = 0$ and $j = 1$, this means we consider at $\Pi^{1,50+1}$. If there are at least 50% positive instances of $\hat{F}$, for our example that means 50 or more, any sample of of size 51 has to contain at least one positive instance, since there are only 49 remaining ones. Similarly, if we sample 52 instances (without replacement), there would have to be at least 2 positive ones, and so on. Having a positive tolerance value t means sampling more instances, thus making it easier for **P** to win. If the statements are not completely true, the quantifiers behave monotonically, in the sense that truth values monotonically go down as the proportion more and more clears away from the acceptance intervals, as can be seen in Figure 4.4.

Corollary 4 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$*,* $q \in [0, 1]$*,* $n =\mid D \mid$*, and* $j, t \geq 1$ *such that* $j + t \leq n$ *(or* $\lfloor n(1 - q)\rfloor + j + t \leq n$*,* $\lfloor nq\rfloor + j + t \leq n$ *respectively), we have:*

- $v_{\mathcal{M}}(\mathsf{Q}^{alal}_t x\hat{F}(x)) = 1$ iff $v_{\mathcal{M}}(\exists^{\leq t} x\neg\hat{F}(x)) = 1$
- $v_{\mathcal{M}}(\mathsf{Q}^{neno}_t x\hat{F}(x)) = 1$ iff $v_{\mathcal{M}}(\exists^{\leq t} x\hat{F}(x)) = 1$
- $v_{\mathcal{M}}(\mathsf{Q}^{\geq q}_t x\hat{F}(x)) = 1$ iff $Prop_{\mathcal{M}}(\hat{F}) \geq \frac{\lceil nq\rceil - t}{n}$
- $v_{\mathcal{M}}(\mathsf{Q}^{\leq q}_t x\hat{F}(x)) = 1$ iff $Prop_{\mathcal{M}}(\hat{F}) \leq \frac{\lfloor nq\rfloor + t}{n}$
- $v_{\mathcal{M}}(\mathsf{Q}^{q}_t x\hat{F}(x)) = 1$ iff $Prop_{\mathcal{M}}(\hat{F}) \in [\frac{\lceil nq\rceil - t}{n}, \frac{\lfloor nq\rfloor + t}{n}]$

Proof:
Let $z = n \cdot p$, where $p = Prop_{\mathcal{M}}(\hat{F})$, and $n =\mid D \mid$. Recall that for $q \in [0, 1]$ we have $n = \lfloor n \cdot q\rfloor + \lceil n \cdot (1 - q)\rceil$. Also, recall that for three natural numbers $m, r, k \in \mathbb{N}$ we have Vandermonde's identity [CKM92]:

$$\binom{m+k}{r} = \sum_{i=0}^{r} \binom{m}{i} \cdot \binom{k}{r-i}. \tag{4.14}$$

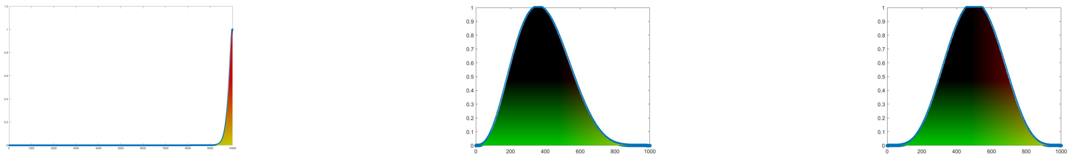

Figure 4.4: Left: truth function of “almost all” modeled by $\Pi^{100,101}$, $n = 10000$; Middle: truth function of “about a third” modeled by $\mathsf{Q}^{18,\{4,5,6\},\{6,7,8,9\}}$, $n = 1000$; Right: truth function of “about half” modeled by $\mathsf{Q}^{14,\{5,6,7\},\{7,8,9\}}$, $n = 1000$.

- For the first case, “almost all”, we note that $v_{\mathcal{M}}(\mathsf{Q}_t^{alal} x\hat{F}(x)) = 1$, for two numbers $j, t \geq 1$ with $j + t \leq n$, if and only if $\sum_{i=0}^{j-1} \binom{z}{i} \cdot \binom{n-z}{j+t-i} = 0$. As $\binom{z}{0} = 1$, this leads to the condition $z > n - j - t$, for otherwise the product of the binomial coefficients is not equal to zero, which it must be, because all summands of the sum are non-negative. This in turn gives $z \geq 1$, since $j + t \leq n$. Consequently, as $\binom{z}{1} \geq 1$, the next condition is $z > n - j - t + 1$, which gives $z \geq 2$. Continuing this argument leads to the condition $z > n - j - t + (j - 1) = n - t - 1$, which is equivalent to $t \geq n - z$, which precisely says that $Prop_{\mathcal{M}}(\neg\hat{F}) \cdot n \leq t$.

- This follows from the case “almost all”, by replacing $\hat{F}$ with $\neg\hat{F}$.

- Similarly to before, we note that we have the necessary and sufficient condition for $v_{\mathcal{M}}(\mathsf{Q}_t^{\geq q} x\hat{F}(x)) = 1$ equivalently as $\sum_{i=0}^{j-1} \binom{z}{i} \cdot \binom{n-z}{\lfloor n\cdot(1-q)\rfloor+j+t-i} = 0$. Again, since all summands of the sum are non-negative, and with the same reasoning as in the first case, we have the condition $n - z < \lfloor n \cdot (1-q)\rfloor + j + t - i$ for all $0 \leq i \leq j - 1$, hence in particular the strongest of these, namely $n - z < \lfloor n \cdot (1 - q)\rfloor + j + t - (j - 1)$, which is equivalent to $z \geq n - \lfloor n \cdot (1 - q)\rfloor - t = \lceil n \cdot q\rceil - t$.

- Analogously we get the condition $z < \lfloor n \cdot q\rfloor + j + t - (j - 1)$, which is equivalent to $z \leq \lfloor n \cdot q\rfloor + t$.

- This is obvious from the semantics of the ordinary conjunction connective $\wedge$.

□

Quantifiers based on random sampling without replacement and strong disjunction

This section’s models, for relative semi-fuzzy quantifiers, are based on the idea of evaluating quantified statements, like “About half of all humans are women”, on a small set of randomly chosen witnesses (in this case humans). In particular, for reasons to be explained shortly, we choose to read “*About* $\frac{k}{k+m}$”, for appropriate choices of k, m as "*almost* $\frac{k}{k+m}$ *(including the proportion* $\frac{k}{k+m}$*)* OR *slightly more than* $\frac{k}{k+m}$ *(also including the proportion* $\frac{k}{k+m}$*)*", where the OR is interpreted as strong Łukasiewicz disjunction. The reason for this design choice is the following. Assuming we are in a situation in which the example statement is intuitively true, e.g. if there were indeed exactly have of the humans male and the others female, choosing a small random sample of size s, will never tell us with certainty, i.e. with probability 1, that this is actually the case, as we only look at a fraction of all humans. One means to trick ourselves out of this dilemma is to add up more than one probability value with respect to overlapping events, which can be accomplished by employing strong Łukasiewicz disjunction. We now have to consider the following fact about the density function of the binomial distribution underpinning deliberate choice quantifiers:

Let $k, m \geq 1$ be given, and $f(x) = \binom{k+m}{m} \cdot x^k \cdot (1-x)^m$. Then its derivative is:

$$f'(x) = \binom{k+m}{m} \cdot k \cdot x^{k-1} \cdot (1-x)^m - \binom{k+m}{m} \cdot x^k \cdot m \cdot (1-x)^{m-1}. \tag{4.15}$$

To determine the maximum of $f(x)$ we set $f'(x) = 0$, which is equivalent to $x = \frac{k}{k+m}$.

Versions of deliberate choice quantifiers that are supposed to model "about half", always have $k = m$ ($\frac{k}{k+m} = \frac{1}{2}$), which means that their truth functions are bound by $\frac{1}{2}$, which in turn means that, even if we sum up to identical copies of such truth values, the result will still be bound by 1, i.e. can become 1 at most, and nothing higher. This is problematic, as the chosen approach is particularly supposed to model the vagueness component of the natural language quantifier "about half", which is completely ignored when the truth function does not model a tolerance area around one half. To put it in other words, only if the proportion of objects fulfilling the scope predicate is exactly $\frac{1}{2}$, the direct sum (modeled by strong Łukasiewicz disjunction) of two copies of deliberate choice quantifiers modeling "about half" would yield a truth value of 1 and no tolerance region were modeled by this method. The issue that can be identified is that deliberate choice quantifiers sample random witnesses with replacement. Choosing a different sampling mechanism, namely without replacement, directly eliminates this issue in the minimally invasive manner. The corresponding distribution is called hypergeometric distribution, which converges to the binomial distribution, when the size of the domain approaches infinity [Geo13]. The only difference lies in the underlying sampling mechanism. Imagine a bowl with 100 balls in it, 50 of which are green and the other 50 are beige. With the deliberate choice quantifier Π_1^1 we have a probability of $\frac{1}{2}$ to get one green and one beige ball, while when we sample two constants at once, the probability increases to $\frac{50}{99}$. This is because, when the two witnesses are selected, it can not happen that both are identical, as in the case before, when the two witnesses were selected with replacement. One can also see it as if the first draw doesn't matter, as only the second need be different from the first, but when one ball already misses, then one is left with 99, 50 of which are different from the first. Let us look at the following example:

Example 4 *Let us consider the quantifier "about a third", and a domain of size* 1000 *representing certain people. The statement we look at is "About a third of all people own a bike". The truth of this statement should be* 1 *if the proportion of all people that own a bike is exactly* $\frac{1}{3}$*. But it should also be* 1 *if the proportion deviates only a bit. One may also argue, that the tolerance toward deviations to smaller values of proportion is a bit smaller than the one to bigger such values. Concretely, on a scale* $1, 2, 3, 4, \ldots, 16, 17, 18$*, which means* $s = 18$*, the area for* almost a third (including exactly a third) *can be taken as* $M_1 = \{4, 5, 6\}$ *and the region corresponding to* slightly more than a third (including exactly a third) *as* $M_2 = \{6, 7, 8, 9\}$*. For a model for the quantifier* about half*, one can, for example, take the values* $s' = 14$ *and* $M_1' = \{5, 6, 7\}, M_2' = \{7, 8, 9\}$*. Both truth functions get depicted in Figure 4.4.*

Formally, we define the quantifier models as follows [Hof16a]:

Definition 26 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$*,* $s \leq n = |D|$ *and*[17] $M_1, M_2 \in \mathcal{P}(\{1, \ldots, s\}) \setminus \emptyset$*, we define:*

$\mathsf{Q}^{s,M_1,M_2} x\hat{F}(x) =$

$= \bigoplus_{i \in M_1} (\Pi^{i,s} x\hat{F}(x) \ominus \Pi^{i+1,s} x\hat{F}(x)) \oplus \bigoplus_{i \in M_2} (\Pi^{i,s} x\hat{F}(x) \ominus \Pi^{i+1,s} x\hat{F}(x)).$

Theorem 32 *For a crisp* $Ł_\alpha(\Pi)$ *formula* $\hat{F}$*,* $s \leq n = |D|$ *and* $M_1, M_2 \in \mathcal{P}(\{1, \ldots, s\}) \setminus \emptyset$*, we have:*

$$v_{\mathcal{M}}(\mathsf{Q}^{s,M_1,M_2} x\hat{F}(x)) = \min(1, \frac{\sum_{i \in M_1} \binom{n \cdot p}{i} \binom{n \cdot (1-p)}{s-i}}{\binom{n}{s}} + \frac{\sum_{i \in M_2} \binom{n \cdot p}{i} \binom{n \cdot (1-p)}{s-i}}{\binom{n}{s}}). \quad (4.16)$$

Proof:
The result follows from the observation, that, for $|D| = n$, we have:

$v_{\mathcal{M}}(\Pi^{i,s} x\hat{F}(x) \ominus \Pi^{i+1,s} x\hat{F}(x)) = \max(0, \frac{\sum_{l=i}^{s} \binom{n \cdot p}{l} \binom{n \cdot (1-p)}{s-l}}{\binom{n}{s}} - \frac{\sum_{l=i+1}^{s} \binom{n \cdot p}{l} \binom{n \cdot (1-p)}{s-l}}{\binom{n}{s}}) =$

$= \frac{\binom{np}{i} \binom{n(1-p)}{s-i}}{\binom{n}{s}}. \quad \square$

By instantiating Q^{s,M_1,M_2} in a clever way, we can get models for "about half" with truth functions that actually have a tolerance region around the value of proportion $\frac{1}{2}$. More formally, we define the conditions for "clever instantiation" as follows:

[17] $\mathcal{P}$ refers to the power set.

Definition 27 *Let $k, m \geq 1$ be given. For $s \leq n = \mid D \mid$ and $M_1, M_2 \in \mathcal{P}(\{1, \ldots, s\}) \setminus \emptyset$, Q^{s,M_1,M_2} is called* a clever instance for the quantifier "about $\frac{k}{k+m}$", *if, for $j = 1, 2$ and $1 \leq m_1 \leq m_2 \leq n$, we have:*

$$\frac{\sum_{i \in M_j} \binom{\lfloor n \cdot \frac{k}{k+m} \rfloor}{i} \binom{\lfloor n \cdot (1 - \frac{k}{k+m}) \rfloor}{s-i}}{\binom{n}{s}} > \frac{1}{2}, \text{ and} \tag{4.17}$$

$$M_1 = \{m_1, m_1 + 1, \ldots, \lfloor \frac{s \cdot k}{k+m} \rfloor\}, \text{ and } M_2 = \{\lfloor \frac{s \cdot k}{k+m} \rfloor, \ldots, m_2 - 1, m_2\}. \tag{4.18}$$

This condition can clearly be fulfilled as we can always adjust M_1 and M_2 rich enough, for any value $s \leq \mid D \mid$.

Regarding the case of "about half" of Example 4, this relates as follows. $\mid D \mid = 1000, s = 14, M_1 = \{5, 6, 7\}, M_2 = \{7, 8, 9\}, k = m$. Hence, the conditions are:

$\frac{\sum_{i=5}^{7} \binom{500}{i} \binom{500}{14-i}}{\binom{1000}{14}} > \frac{1}{2}$, and $\frac{\sum_{i=7}^{9} \binom{500}{i} \binom{500}{14-i}}{\binom{1000}{14}} > \frac{1}{2}$. Since,

$\frac{\sum_{i=5}^{7} \binom{500}{i} \binom{500}{14-i}}{\binom{1000}{14}} = \frac{\sum_{i=7}^{9} \binom{500}{i} \binom{500}{14-i}}{\binom{1000}{14}} = 0.517$,

$\mathsf{Q}^{14,\{5,6,7\},\{7,8,9\}}$ makes for a reasonable model for the vague natural language quantifier "about half", as it is a clever instance of the schema.

Similarly, regarding the case "about a third" from the previous example, we have the following setting. $\mid D \mid = 1000, s = 18, M_1 = \{4, 5, 6\}, M_2 = \{6, 7, 8, 9\}, 2 \cdot k = m$. Hence, the conditions are:

$\frac{\sum_{i=4}^{6} \binom{500}{i} \binom{500}{18-i}}{\binom{1000}{18}} > \frac{1}{2}$, and $\frac{\sum_{i=6}^{9} \binom{500}{i} \binom{500}{18-i}}{\binom{1000}{18}} > \frac{1}{2}$. Since,

$\frac{\sum_{i=4}^{6} \binom{500}{i} \binom{500}{18-i}}{\binom{1000}{18}} = 0.51$, and $\frac{\sum_{i=6}^{9} \binom{500}{i} \binom{500}{18-i}}{\binom{1000}{18}} = 0.55$,

$\mathsf{Q}^{18,\{4,5,6\},\{6,7,8,9\}}$ makes for a reasonable model for the vague natural language quantifier "about a third", as it is a clever instance of the schema.

Although these models work, they are still dependent on the size of the actual domain, which one usually wants to avoid, in favor of a semantics that uniformly works for any size of the domain. To that end, in particular taking into consideration practical applications, we give similar but domain size independent quantifier models as a next step.

4.4.5 Quantifiers for querying

We discuss the quantifier models of this section in Chapter 6 on queries, as we there introduce the corresponding query language that they are designed for [FHO17].

4.5 Beyond unary quantifiers: $\forall, \exists, \Pi$

In natural language, binary quantifiers are more common than unary ones. For example consider the statement "All humans are mortal". Uttering this assumes a domain of discourse that, besides all humans, contains also other objects. Otherwise one could spare the range restriction and simply say "Everything is mortal". For the *universal* and the *existential* quantifier we can, for two formulas $\hat{F}$ and G, where $\hat{F}$ is crisp, define[18] $\exists x(\hat{F}(x), G(x)) = \exists x(\hat{F}(x) \wedge G(x))$ and $\forall x(\hat{F}(x), G(x)) = \forall x(\hat{F}(x) \to G(x))$.

4.5.1 Binary Π

We show how one can express the binary semi-fuzzy quantifier ${}^{\hat{F}}\Pi$ from [FR14], a quantifier with a range not necessarily equal to the whole domain of discourse, in $\text{Ł}_\alpha(\Pi)$. Such quantifiers are very commonly motivated by examples from natural language, as e.g. "Almost all elephants are from Africa", or "About half of all humans are female".

The game rule for ${}^{\hat{F}}\Pi$ from [FR14] is the following:

$\mathcal{R}_{\hat{F}\Pi}$**:** If **P** asserts ${}^{\hat{F}}\Pi x G(x)$, then **P** has to assert $G(c)$ for a randomly picked element c from those that make $\hat{F}(c)$ true.

The defined truth function of ${}^{\hat{F}}\Pi$ is the following:

$$v_{\mathcal{M}}({}^{\hat{F}}\Pi x G(x)) = \frac{\sum_{c \in D} v_{\mathcal{M}}(\hat{F}(c) \wedge G(c))}{\sum_{c \in D} v_{\mathcal{M}}(\hat{F}(c))}. \tag{4.19}$$

This truth function matches **P**'s payoff regarding a game for ${}^{\hat{F}}\Pi x G(x)$.

Definition 28 *For two $\text{Ł}_\alpha(\Pi)$ formulas $\hat{F}, G$, where $\hat{F}$ is crisp, we define*[19]*:*

$${}^{2}\Pi x(\hat{F}(x), G(x)) = \Pi x \hat{F}(x) \rightarrowtail \Pi x(\hat{F}(x) \wedge G(x)). \tag{4.20}$$

The relation of $v_{\mathcal{M}}({}^{\hat{F}}\Pi x G(x))$ and $v_{\mathcal{M}}({}^{2}\Pi x(\hat{F}(x), G(x)))$ is the following:

Theorem 33 *[Hof18] For two $\text{Ł}_\alpha(\Pi)$ formulas $\hat{F}, G$, where $\hat{F}$ is crisp, we have:*

$$v_{\mathcal{M}}({}^{2}\Pi x(\hat{F}(x), G(x))) = v_{\mathcal{M}}({}^{\hat{F}}\Pi x G(x)). \tag{4.21}$$

[18] Note that this means that $v_{\mathcal{M}}(\exists x(\hat{F}(x), G(x))) = \sup_{c \in D}(\hat{F}(c) \wedge G(c))$ and $v_{\mathcal{M}}(\forall x(\hat{F}(x), G(x))) = \inf_{c \in D} \min(1, 1 - v_{\mathcal{M}}(\hat{F}(c)) + v_{\mathcal{M}}(G(c)))$.

[19] Recall that $\rightarrowtail$ is equivalent to $\to^{P}$

Proof:
Since $v_{\mathcal{M}}(\Pi x \hat{F}(x)) \geq v_{\mathcal{M}}(\Pi x(\hat{F}(x) \wedge G(x)))$:

$$v_{\mathcal{M}}(\Pi x \hat{F}(x) \rightarrowtail \Pi x(\hat{F}(x) \wedge G(x))) = \frac{v_{\mathcal{M}}(\Pi x(\hat{F}(x) \wedge G(x)))}{v_{\mathcal{M}}(\Pi x \hat{F}(x)))} =$$

$$= \frac{Prop_{\mathcal{M}}(\hat{F} \wedge G)}{Prop_{\mathcal{M}}(\hat{F})} = \frac{\frac{\sum_{c \in D} v_{\mathcal{M}}(\hat{F}(c) \wedge G(c))}{|D|}}{\frac{\sum_{c \in D} v_{\mathcal{M}}(\hat{F}(c))}{|D|}} = \frac{\sum_{c \in D} v_{\mathcal{M}}(\hat{F}(c) \wedge G(c))}{\sum_{c \in D} v_{\mathcal{M}}(\hat{F}(c))} . \quad \square$$

4.5.2 Remaining binary versions of quantifier models of this chapter

For the first case, absolute (unary) Type I quantifiers, and two formulas $\hat{F}, G$, where $\hat{F}$ is crisp, the binary version of the base quantifier can be defined as follows:

$$\exists^{\geq k} x(\hat{F}(x), G(x)) \equiv \exists x_1 \ldots \exists x_k (\bigwedge_{i=1}^{k} (\hat{F}(x_i) \wedge G(x_i)) \wedge \bigwedge_{i \neq j} (x_i \neq x_j)). \tag{4.22}$$

For the case of absolute (unary) Type III quantifiers, and two formulas $\hat{F}, G$, where $\hat{F}$ is crisp, the binary version of the base quantifier can be defined as follows:

$$\tilde{\exists}^{\geq k} x(\hat{F}(x), G(x)) \equiv \exists x_1 \ldots \exists x_k (\bigwedge_{i=1}^{k} \hat{F}(x_i) \wedge \pi_{i=1}^{k} G(x_i) \wedge \bigwedge_{i \neq j} (x_i \neq x_j)). \tag{4.23}$$

For relative (unary) Type I quantifiers and the relative (unary) Type III models $[Models\ 1] - [Models\ 3]$, we only need to replace occurrences $\Pi x G(x)$ with $\Pi x(\hat{F}(x), \hat{G}(x))$ to arrive at the binary versions, featuring $\hat{F}$ as scope. For the remaining cases of relative (unary) Type III models, $[Models\ 4]$ – $[Models\ 6]$, there are sadly no known representations in $Ł_{\alpha}(\Pi)$.

4.5.3 k-ary quantifiers, with $k \geq 3$

Although it is not the main topic of this thesis to investigate quantifiers of higher arities, we remark a few things on such expressions, particularly those that are discussed in Glöckner's monograph [Glö06] on fuzzy quantifiers. Further literature on higher arity quantifiers comprises for example Westerstahl [PW06]. It is not our aim to express all expressions that are sometimes considered in the literature within our framework, that would need much more work and might not even be possible. However, one example of a case that is called a *resumption quantifier* in [PW06] is the statement "Men are usually taller than women". One might want to argue about the actual meaning of this natural language sentence, but one possible reading certainly is this one: "On average men are taller than women", which can concisely be expressed as:

- $\Pi x(woman(x), tall(x)) \rightarrow \Pi x(man(x), tall(x))$.

The two main examples from [Glö06] are (1) "more A's than B's are C's", and (2) "most A's and B's are C's or D's". Reductions that only need binary quantifiers, which in turn can be reduced to unary ones in $Ł_\alpha(\Pi)$, are the following:

- $\Pi x(C(x), B(x)) \to \Pi x(C(x), A(x))$
- $\mathsf{Q}^{most}x(A(x), C(x) \vee D(x)) \wedge \mathsf{Q}^{most}x(B(x), C(x) \vee D(x))$

Yet another more involved example from [PW06] is this one: "At most three boys gave more dahlias than roses to Mary", which can be expressed as:

- $\exists^{\leq 3}x\Delta(\Pi y(rose(y) \wedge gaveToMary(x,y)) \to \Pi z(dahlia(z) \wedge gaveToMary(x,z)))$.

This is of course no comprehensive analysis of quantifiers with arity greater than two, but shows how expressions not expressible in standard frameworks, like Classical Logic, fit neatly in the very expressive setting of $Ł_\alpha(\Pi)$.

4.6 Intensional quantifiers: Many and Few

A quantifier pair, which is notoriously difficult to treat, is "many" and "few"([$Int1$]). Most people can agree that these two are vague quantifiers, but what exactly constitutes the vagueness and how it should be understood is disputed. Like before, we leave out the discussion of what vague means on the propositional level, and focus on the quantifiers themselves. As it is common practice in linguistics [KH98, KS86, Cru11], we only consider crisp arguments $\hat{F}$ to the quantifiers "many" and "few", which may yet evaluate to intermediate truth values, if conceptualized as Type III quantifiers, i.e. when they are semi-fuzzy. Restricting to this scenario does not reduce expressibility, it rather guarantees the neat interpretability of statements, and the lifting to fully-fuzzy quantifiers will be subject to general quantifier fuzzification mechanisms (QFMs). Also, as we have the modeling of natural language as our objective, and since it is needed to make sense of the comparison based semantics we will give shortly, we will have binary, or 2-place, quantifier expressions as our main type of statement. For example, consider "Many quantum physicists have visited The Eagle and Child"[20]. In our interpretation of the world, we don't want to restrict to one domain with only quantum physicists and to another one as soon as the subject changes. Rather we assume a (finite) domain with all objects under consideration, like all humans, animals, plants, and other sorts of items, and, whenever we make a quantified utterance, we tell the range restriction. There still is a lot of room to discuss what the semantics of "many" and "few" should then be. In [BC81], Barwise and Cooper gave a strong account of the theory of generalized

[20]For the interested reader, the Eagle and Child is an English Pub in Oxford were academics like to have their glass of ale.

quantifiers, but did not, whatsoever, contribute to the semantics of the vague quantifier pair “many” and “few”.

Keenan and Stavi, in [KS86], explain that, as the meaning of the quantifiers “many” and “few” is context dependent, i.e. their evaluation must be based on some notion of comparison, statements involving them cannot be interpreted and do not assume any determinate truth value at all. They come to this conclusion, as “in simple uses at least, the standard of comparison is usually not given”. Since we have a very expressive framework at our disposal and for people usually have no problem interpreting respective statements in real life [Glö06], we will not follow their dismissal but only keep in mind that a notion of comparison has to be established.

In [Pet00], Peterson, for example, considers the quantifiers “most”, “many” and “few”. His account is based on the relations of those quantifiers, which he depicts in something reminiscent of the Aristotelian square. Although he recognizes that “most” can be read as sheer “majority”, he predominately considers it equivalently with “almost all”, which then entails “many”. To be more concrete, “Most A’s are B’s” implies “Many A’s are B’s”. Furthermore, when the entity of interest is a mass noun rather than a count noun, he states that “much” and “little” replace “many” and “few”.

Westerstahl [PW06] explains “many” as symmetric to “few”, which, for him, means that, in case of an absolute account, there must be absolute thresholds, and for the relative account, that there must be relative thresholds. That means, for example, based on a domain of discourse with one hundred elements having a certain property A, that “Many A’s are B’s” were true if, say, 70 elements, or 70% of the elements have property B. And the same for “few”, just that there the threshold were to be chosen smaller, let us say as 30 elements, or 30% of elements, while the parametrization, of course, allows for any threshold values to be used.

Fernando and Kamp [FK96] model “many” differently. Although they also see Westerstahl’s simple interpretation, they go beyond it by introducing a notion of expectation, which is to give a means to determine threshold values instead of only claiming their existence and “that context spits out n”. These expectations come about as comparisons. For example, consider the statement “Many lawyers are doctors, as compared to criminals” [FK96], which is supposed to be true if and only if, within a fixed set of lawyers, there are more doctors than criminals.

Lappin’s [Lap00] intensional parametrization is broadly similar to the approach of Fernando and Kamp, as the intensionality comes about via contextual comparisons. Still, we chose to follow his account rather than any of the others, for, either the others are subsumed within his (or rather our refinement of it [Hof15]), or since it simply makes more sense here, as his the notion of comparisons fits in most neatly with $Ł_\alpha(\Pi)$, and can be developed to a mature state. In the remainder of this section, we will introduce Lappin’s method in detail and augment it to our own contribution to the matter.

To have all this a bit more down-to-earth, let us discuss some of the influentials that can go into the evaluation of a "many-quantified" statement. In the example above, the range is the set of quantum physicists (QP), and "has visited the Eagle and Child" ($hvEC$) is the scope predicate. Now, assuming (1) that from the set QP more than 50% have visited the Eagle and Child, the interpretation of "many", that assumes the semantics of it to be equivalent with simple majority, is fulfilled. This is of course usually too simple as a semantics for "many", but even Westerstahl's idea of always having a threshold value that just needs to be determined by context is unsatisfying, as no means at all to determine such a value is *a priori* at our disposal. It is just assumed to exist. Going beyond that, hence assuming that contextual information has to be taken into account formally, one arrives yet at a multitude of ways to model the semantics of "many". Fernando and Kamp argue that the expectation of someone who utters a statement must go into the evaluation. That means, for example, if Bob talks about that "Many quantum physicists have visited The Eagle and Child", he must have some implicit set of properties in mind to which we can compare, as for example "Many quantum physicists have visited The Eagle and Child, as compared to Mathematicians." Or "Many quantum physicists have visited The Eagle and Child, as compared to Psychologists or Sociologists". This is already quite similar the Lappin's approach, who extends that idea to not only comparison properties but also whole comparison situations. An example for that can be the following: "Many quantum physicists eat at Mc Donald's", while the statement is once evaluated over the situation "USA" and another time over the situation "France". Not knowing what the actual (absolute or relative) difference would be, it is clear that such situation dependent evaluations yield different results in general.

In fuzzy logic, generalized quantifiers are usually defined for fuzzy scopes, which we ignore, until we employ QFMs. Concerning versions of the quantifier "many" in fuzzy logic, most notably, Hajek's modal characterization of it [Háj98] has to be mentioned. Without giving all the technical details, it uses the average of evaluations of scope formulas over finitely many different worlds as a semantics for the quantified statement. We will capture the main intuition behind this approach by interpreting the different worlds as different situations, and the taking of an average will be, although in a much refined way, be expressed by means of the Π quantifier. This, as we will show shortly, will bring us especially much closer to linguistic interpretations of the quantifier "many", particularly the one of Lappin.

Also, we can express any non-context dependent truth function for "many" using Zadeh's S function [Glö06], see 4.6.7. That is, in particular, what Glöckner does, when defining a relative version of "many" [Glö06] (page 72). The absolute version of "many" defined on the same page also employs the S function, which is actually a mistake of Glöckner's, as S is not defined for values greater than 1 in its first argument position.

4.6.1 Lappin's parametrization of "many"

In [Lap00], Lappin defines a bivalent and binary intensional parametric interpretation of $\|\text{many}\|$, namely this one[21]:

$$\|B\|^{sa} \in \|\text{many}\|(\|A\|^{sa}) \text{ iff}$$
$$S \neq \emptyset, \text{ and for all } sn \in S, \quad |\ \|A\|^{sa} \cap \|B\|^{sa}\ | \geq |\ \|A\|^{sn} \cap \|B\|^{sn}\ | \quad (4.24)$$

Informally, this reads as follows. In an actual situation sa, there are many $A's$ that are $B's$ if and only if for all normative situations sn_i, of which there must be at least one, we have that the amount of $A's$ that are $B's$ in sn_i is smaller or equal than the amount of $A's$ that are $B's$ in sa (the actual situation). Hence, the (linguists) notation can, in our framework, be captured as follows:

- a situation s is a non-maximal possible world, hence we have $s \subseteq \mathcal{M}$.[22]
- sa is the actual situation (for one fixed statement).
- S is the set of all normative situations sn_i (for one fixed statement).
- for a crisp properties A and a situation s we define[23]: $\|A\|^s = \{a : s \vDash A(a)\}$.

In the remainder of this section, a set $S = \{sn_1, \ldots, sn_n\}$ of normative situations ($n \geq 1$), is always supposed to be such that for $1 \leq i, j \leq n$ with $i \neq j$ we have $D_{sn_i} \cap D_{sn_j} = \emptyset$. In particular, we require that $D_{sn_i} \cap D_{sa} = \emptyset$, for $1 \leq i \leq n$.

4.6.2 Extensional readings of "many": binary, Type I

In this setting we can distinguish two different types of statements. Those that refer to only one situation, namely sa, and those that take into consideration normative situations, sn. Although Lappin's parametrization is intensional as it stands, the restriction to only one situation of interest can be regarded as extensional special case within the intensional framework, where the set S contains only sa. The nine extensional readings of "many" in [Lap00] are the following[24]($\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$):

Definition 29 *For three crisp* $Ł_\alpha(\Pi)$ *formulas* $\hat{F}, \hat{G}, \hat{H}$, $q, q_1, q_2 \in [0,1], j \geq 1$ *and* $f : \mathbb{N}^+ \to \mathbb{N}^+$, *we define:*

[21] Here, the letters A and B stand for crisp properties, which we usually, and also in the remainder of this section, denote with $\hat{F}$ or $\hat{G}$.

[22] D_s will denote the domain D restricted to the situation s.

[23] Note that this is related but not the same as the notation of Definition 14. However, upon taking absolute values $|\cdot|$, for crisp formulas $\hat{F}$ and a situation s, we have: $|\ \|\hat{F}\|^s\ | = \sum_{c \in D_s} v_{\mathcal{M}}(\hat{F}(c))$.

[24] The first is due to Barwise and Cooper [BC81], while the second to fifth are taken and reformulated from Westerstahl (1985). The last four are Lappin's contribution.

m_{e1}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq (q \cdot 100)\% \ |\, \|\hat{F}\|^{sn} \,| \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq j\}$

m_{e2}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq i\cdot \,|\, \|\hat{F}\|^{sn} \,|, \text{with } i \in (0,1) \text{ fixed}\}$

m_{e3}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq \frac{|\|\hat{G}\|^{sn}|}{|D_{sn}|} \cdot \,|\, \|\hat{F}\|^{sn} \,|\}$

m_{e4}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq f(|\, D_{sn} \,|), f(|\, D_{sn} \,|) \in \mathbb{N}^{+}\}$

m_{e5}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq i\cdot \,|\, \|\hat{G}\|^{sn} \,|, \text{with } i \in (0,1) \text{ fixed}\}$

m_{e6}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq |\, \|\hat{F}\|^{sn} \cap \|\hat{H}\|^{sn} \,|\}$

m_{e7}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq |\, \|\hat{G}\|^{sn} \cap \|\hat{H}\|^{sn} \,|\}$

m_{e8}: $S = \{sn\colon sn{=}sa \ \& \ |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \geq |\, \|\hat{H}\|^{sn} \,|\}$

m_{e9}: $S = \{sn\colon sn{=}sa \ \& \ \frac{|\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}|}{|\|\hat{F}\|^{sn}|} \geq (q_1 \cdot 100)\% \ \& \frac{|\|\hat{G}\|^{sn} \cap \|\hat{H}\|^{sn}|}{|\|\hat{H}\|^{sn}|} = (q_2 \cdot 100)\% \ \& \ q_1 \geq q_2\}$

Note that, although reading m_{e4} relies on a function f with values in $\mathbb{N}^{+}$, we can safely assume that f takes on values in $\{1, \ldots, |\, D_{sa} \,|\}$. This is because we have $|\, \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} \,| \leq |\, D_{sa} \,|$. Hence, we can also assume that f acts simply as multiplication with a number $a \in \{\frac{1}{|D_{sa}|}, \ldots, \frac{|D_{sa}|}{|D_{sa}|}\}$ on $|\, D_{sa} \,|$.

Definition 30 *Let sa be an actual situation. A set of crisp $Ł_{\alpha}(\Pi)$ formulas $\hat{H}_i$, for $1 \leq i \leq k$ with $k \geq 1$, is defined to be a set of crisp* comparison sets*, or crisp comparison predicates, if the following holds: for all $1 \leq i, j \leq k$ with $i \neq j$ we have $|\ ||\hat{H}_i||^{sa} \cap ||\hat{H}_j||^{sa} \,| = \emptyset$.*

Then, for two crisp formulas $\hat{F}$ and $\hat{G}$, as well as $\hat{H}_i$, for $1 \leq i \leq k$ with $k \geq 1$, a set of comparison predicates with respect to an actual situation sa, we define:

- $I_{>} = \{i : i \in \{1, \ldots, k\}, |\, \|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa} \,| > |\, \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} \,|\}$, and $a = |I_{>}|$.
- $I_{=} = \{i : i \in \{1, \ldots, k\}, |\, \|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa} \,| = |\, \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} \,|\}$, and $b = |I_{=}|$.
- $I_{<} = \{i : i \in \{1, \ldots, k\}, |\, \|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa} \,| < |\, \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} \,|\}$, and $c = |I_{<}|$.
- $|H^{+}| = \max\{|\|\hat{H}_i\|^{sa} \cap \|\hat{F}\|^{sa}| : 1 \leq i \leq k\}$

The following readings are taken from [Hof15]:

Definition 31 *For $k \geq 1$, and crisp $Ł_\alpha(\Pi)$ formulas $\hat{F}, \hat{G}$ as well as $\hat{H}_i$, $1 \leq i \leq k$ comparison predicates with respect to an actual situation sa, we define:*

m_{e10}: $S = \{sn : sn = sa \ \& \ \ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| \geq |H^+|\}$

m_{e11}: $S = \{sn : sn = sa \ \& \ \ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| > |H^+|\}$

m_{e12}: $S = \{sn : sn = sa \ \& \ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| \geq |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|, \text{for most } 1 \leq i \leq k\}$

m_{e13}: $S = \{sn : sn = sa \ \& \ \sum_{i \in I_>} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,| < \sum_{i \in I_<} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|\}$

Example 5 *Let us consider a situation of some small children playing LEGO. A child could consider the number of red LEGO-bricks to be "many", if there are more of the red sort than of all other colors. That would be Lappin's interpretation of Equation 4.24, if he had introduced more than one comparison set (here it corresponds to m_{e10}; note that m_{e11} is simply a strengthening thereof). Still, it makes sense to talk about "many red LEGO-bricks", even if there is a color of which there are more LEGO-bricks. Reading m_{e12} weakens the condition by demanding that only a majority of all comparison sets features the respective property. Then, m_{e13} formalizes a deeper comparison of these numbers. Considering the actual situation of LEGO playing children, there would be many red LEGO bricks, if the heap of heaps of all LEGO bricks of which there are less than of the red sort, is bigger than the heap of heaps of all LEGO bricks of which there are more than of the red sort.*

4.6.3 Intensional readings of "many": binary, Type I

Definition 32 *For $S = \{sn_i : i \in \{1, \ldots, n\}\}$ a set of normative situations, $n \geq 1$, and two crisp $Ł_\alpha(\Pi)$ formulas $\hat{F}$ and $\hat{G}$, we define:*

- $I^{rel}_{\succ} = \{i : i \in \{1, \ldots, n\}, \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} > \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}\}$, and $\alpha = |I^{rel}_{\succ}|$.
- $I^{rel}_{\equiv} = \{i : i \in \{1, \ldots, n\}, \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} = \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}\}$, and $\beta = |I^{rel}_{\equiv}|$.
- $I^{rel}_{\prec} = \{i : i \in \{1, \ldots, n\}, \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} < \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}\}$, and $\gamma = |I^{rel}_{\prec}|$.
- $I^{abs}_{\succ} = \{i : i \in \{1, \ldots, n\}, |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,| > |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,|\}$, and $\alpha' = |I^{abs}_{\succ}|$.
- $I^{abs}_{\equiv} = \{i : i \in \{1, \ldots, n\}, |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,| = |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,|\}$, and $\beta' = |I^{abs}_{\equiv}|$.
- $I^{abs}_{\prec} = \{i : i \in \{1, \ldots, n\}, |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,| < |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,|\}$, and $\gamma' = |I^{abs}_{\prec}|$.

The intensional readings then are [Hof15]:[25]

Definition 33 *For two crisp* $Ł_\alpha(\Pi)$ *formulas* $\hat{F}$ *and* $\hat{G}$*, and* $S = \{sn_i : i \in \{1, \ldots, n\}\}$ *a set of normative situations,* $n \geq 1$*, we define:*

m_{i1}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and for } 1 \leq i \leq n, \text{ we have: } \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} \geq \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$$

m_{i2}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and for most } 1 \leq i \leq n : | \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} | \geq | \|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i} |$$

m_{i3}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and for most } 1 \leq i \leq n : \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} \geq \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$$

m_{i4}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and: } | \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} | > \max\{| \|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i} | : 1 \leq i \leq n\}$$

m_{i5}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and: } \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} > \max\{\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} : 1 \leq i \leq n\}$$

m_{i6}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and: } \sum_{i \in I^{abs}_{\prec}} | \|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i} | > \sum_{i \in I^{abs}_{\succ}} | \|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i} |$$

m_{i7}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and: } \sum_{i \in I^{rel}_{\prec}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} > \sum_{i \in I^{rel}_{\succ}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$$

To see that we are really going beyond Lappin's characterization, we prove the following theorem:

Theorem 34 *[Hof15] Let* $S = \{sn_1, \ldots, sn_n\}$ *be a set of normative situations. Furthermore, assume we want to employ the relative reading of "many" captured by the intensional reading* m_{i1}*. It is not possible to rewrite* S *into* $S' = \{sn'_1, \ldots, sn'_n\}$ *such that* S' *is a set of normative situations, and we can employ* m_{i1} *equivalently to Lappin's original characterization of "many", using this* S' *instead of* S*.*

[25] To not run into trouble, we employ the following common practice: For two sets M and N, we have: $\frac{|M \cap N|}{|N|} = 0$, if $|N| = 0$.

Proof:
Let $\hat{F}$ and $\hat{G}$ be fixed crisp formulas, we assume an actual situation sa and normative situations $sn \in S \neq \emptyset$ with:

- $\|\hat{F}\|^{sa} \neq \emptyset$, and $\|\hat{F}\|^{sn} \neq \emptyset$ for all $sn \in S$.
- for all $sn \in S : \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} \geq \frac{|\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}|}{|\|\hat{F}\|^{sn}|}$. (⊠)

We are looking for some S' such that:

- for all $sn' \in S' : |\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}| \geq |\|\hat{F}\|^{sn'} \cap \|\hat{G}\|^{sn'}|$.

Also, we can fix some sn from S for which we have equality in (⊠), and try to rewrite it into an sn', while sa is fixed from the very beginning. Thus, by assumption, $q = \frac{\|\hat{F}\|^{sa}}{\|\hat{F}\|^{sn}}$ is a non-negative rational number, depending particularly on the actual situation sa. Hence, it can be, that $q = \frac{1}{3}$, as a special case, as well as it may be the case, that $|\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}| = 26$. Since $|\|\hat{F}\|^{sn'} \cap \|\hat{G}\|^{sn'}| \in \mathbb{N}$, we can not have, that:

$|\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}| \frac{|\|\hat{F}\|^{sa}|}{|\|\hat{F}\|^{sn}|} = |\|\hat{F}\|^{sn'} \cap \|\hat{G}\|^{sn'}|$, since that would mean, that $\frac{26}{3} \in \mathbb{N}$, which is clearly wrong. This completes the argument. □

This shows, that there is more to achieve than Equation 4.24 suggests. However, some of our stated readings actually can be encoded in Lappin's original one, namely m_{i2} and m_{i4}. We decided not to hide them anyway, for the spectrum becomes more transparent when we have them listed altogether.

4.6.4 New readings of "many": binary, Type I

We now introduce one more extensional and two more intensional readings. The idea is that, for the extensional case, the proportion of $\hat{G}$'s (within the $\hat{F}$'s in sa) should be higher than the average of the proportions of the comparison sets $\hat{H}_i$'s (within the $\hat{F}$'s in sa, $1 \leq i \leq k$).

m_{e14}: $S = \{sn : sn = sa \ \& \ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| > \frac{1}{k} \cdot \sum_{i=1}^{k} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|\}$

Intensionally, we have the option to either look at absolute or relative numbers, but in both cases we compare the proportion of $\hat{G}$'s (within the $\hat{F}$'s):

m_{i8}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$S \neq \emptyset$, and: $|\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| > \frac{1}{n} \sum_{i=1}^{n} |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,|$

City	residents	German students	%	all students	rel. count
Vienna	$1.8 \cdot 10^6$	20000	$\sim \frac{11}{1000}$	60000	~ 0.33
NY	10^7	20010	$\sim \frac{2}{1000}$	150000	~ 0.13
London	10^7	12000	$\sim \frac{12}{10000}$	100000	~ 0.12
Paris	$2.2 \cdot 10^6$	10010	$\sim \frac{46}{10000}$	80000	~ 0.125

Table 4.1: 'German students' corresponds to $|\|\hat{G}\| \cap \|\hat{F}\||$, 'students' corresponds to $|\|\hat{F}\||$, and 'relative count' corresponds to $\frac{|\|\hat{G}\| \cap \|\hat{F}\||}{|\|\hat{F}\||}$.

m_{i9}: $\|\hat{G}\|^{sa} \in \|many\|(\|\hat{F}\|^{sa})$ iff

$$S \neq \emptyset, \text{ and: } \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} > \frac{1}{n} \sum_{i=1}^{n} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$$

Example 6 *Let us consider the natural language statement "Many students are German", and evaluate it with regard to an actual situation* $sa =$ Vienna *and the following set of normative situations* $S = \{sn_1, sn_2, sn_3\}$*, with* $sn_1 =$ New York*,* $sn_2 =$ London*, and* $sn_3 =$ Paris*. So, what we intend to evaluate is the following statement:*

"Many students are German (in Vienna, compared to NY, London and Paris)".

Let us consider the fictive numbers of Table 4.1. Using Equation 4.24, we are bound to evaluate the statement to false, unlike with, say m_{i1}*, since, even though there is a normative situation in which we have (absolutely) more German students than in the actual one, it is still true, that Vienna has (relatively) the most German students within all normative situations. This is still a plausible reading of "many". Since* m_{i1} *is just the relative version of Equation 4.24, it it obvious how the interpret the two (absolute and relative) weakenings* m_{i2} *and* m_{i3}*, as well as the two (absolute and relative) strengthenings* m_{i4} *and* m_{i5}*.* m_{i6} *and* m_{i7} *are the new intensional readings of "many" from [Hof15]. Note that, unlike before in the extensional case, here it makes a difference whether we introduce a relative version of* m_{i6}*, namely* m_{i7}*, for the normative situations are distinct. These respectively evaluate to:*

W.r.t. m_{i6}*:* "Many $\hat{F}$s are $\hat{G}$s" *is true , since* $10010 + 12000 > 20010$.

W.r.t. m_{i7}*:* "Many $\hat{F}$s are $\hat{G}$s" *is true , since* $0.12 + 0.125 + 0.13 > 0$.

The intensional readings which were not already introduced in [Hof15], i.e. m_{i8}, m_{i9}*, evaluate as follows:*

W.r.t. m_{i8}*:* "Many $\hat{F}$s are $\hat{G}$s" *is true , since* $20000 > \frac{10010+12000+20010}{3}$.

W.r.t. m_{i9}*:* "Many $\hat{F}$s are $\hat{G}$s" *is true , since* $20000 > \frac{0.13+0.12+0.125}{3}$.

The extensional reading that was not already introduced in [Hof15], i.e. m_{e14}*, needs comparison sets to be evaluated. Assuming those are the nationalities of other non Austrian students, we might expect that this would lead to* 1 *as a truth value of the statement as well.*

Example 7 *To explain these readings once more in detail, we look at two different statements, namely (st1) "Many people are French", and (st2) "Many people are Catholics". Of course, one way to understand those is as applying to the whole world, but we restrict our attention to Europe, to make things more meaningful. Now, let us say (1) the actual situation* sa *is fixed as the European Union (EU), while "people" and "French" are crisp properties. Then we might fix the comparison sets* C_j *as the crisp predicates expressing "from country j", where each* $j \in \{1, \dots, 27\}$ *represents one other EU country*[26]*. (st1) together with the implicit information now reads as "Many people in the EU are French, as compared to other nationalities from the EU.". Within the term "compared to", there still is room to choose from one of the extensional readings that employ several comparison sets, like* $m_{e10}, \dots, m_{e14}$*. The first two of those simply demand that there must be (not strictly or strictly) more French people in the EU, than of any other EU nationality, which is probably wrong, as there are supposedly more Germans. This again reflects Lappin's very strong condition.* m_{e12} *relaxes it to only requiring that 14, hence the majority of other EU nations have less citizens in the EU than France, which is probably true. The meaning of* m_{e13} *is somewhat more complicated. It compares those nations with less citizens with those that have more. If the sum of people from countries with fewer citizens is bigger than the sum of people from countries with higher population, then it renders (st1) true. The last reading,* m_{e14}*, takes the average of population sizes from all comparison countries and compares it to the French one. As the average population in the EU countries is about 18 million with a French population of about 67 million, this reading also evaluates (st1) to true.*

We now change the actual situation sa *from the EU to Spain. The comparison sets stay the same, but are now taken within* sa*, hence we look at the French population in Spain, compared to other EU nationalities within Spain, including the Spanish. According to Wikipedia*[27]*, Romania leads the list of EU immigrants in Spain, so* m_{e10} *and* m_{e11} *evaluate to false, but only six EU countries have more people in Spain than France, so* m_{e12} *evaluates to true. On the other side, those six have together about 1.6 million people living in Spain, while the remaining 21 altogether have only about a tenth of that, so* m_{e13} *evaluates to false.* m_{e14} *evaluates to true, as the average of EU citizens in Spain is clearly less than 100000, which is approximately the French population in Spain.*

We now keep the actual situation Spain, but look at the other 27 EU nations as normative situations sn_i*, and evaluate (st2). Again, according to Wikipedia*[28]*, there are about 32 million Catholics in Spain, and only France, Italy and Poland have more (about 40*

[26] In case you read this past 2019 and the UK has left the EU, then j ranges only until 26.

[27] https://en.wikipedia.org/wiki/Immigration_to_Spain

[28] https://en.wikipedia.org/wiki/Catholic_Church_by_country#Europe

million, 50 million and 33 million). The rest of the about 232 million Catholics in total in the EU distributes over the remaining 24 nations, and the average is about 8 million. Hence, Equation 4.24 evaluates to false, as does m_{i1}, as Italy has (relatively) clearly more Catholics than Spain. Similarly also m_{i4} and m_{i5} evaluate to false, since the only difference is the strictness in the inequalities. m_{i2} evaluates to true, as there are only three nation that have (absolutely) more Catholics than Spain. Also relatively, there are only seven nations that have more Catholics than Spain, therefore m_{i3} evaluates to true. Going through the numbers yields that both m_{i6} and m_{i7} evaluate to false, as the EU countries with (absolutely or relatively) more Catholics than Spain account for about 123 million (or an average of about 76%), while those with less Catholics only account for about 76 million (or an average of about 25%), while Spain's average is about 70%. The overall average though, is only about 8 million, or 45%, so both m_{i8} and m_{i9} evaluate to true.

4.6.5 Extensional readings of "few": binary, Type I

The respective definition for $\|\text{few}\|$, with respect to Equation 4.24 can be obtained by replacing $\geq$ with $<$[29]:

$$\|\hat{G}\|^{sa} \in \|\text{few}\|(\|\hat{F}\|^{sa}) \text{ iff}$$
$$Z \neq \emptyset, \text{ and for all } sn \in Z, \quad |\, \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} \,| < |\, \|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn} \,| \quad (4.25)$$

Definition 34 *For crisp* $\text{Ł}_\alpha(\Pi)$ *formulas* $\hat{F}, \hat{G}, \hat{H}, \hat{H}_i$, $1 \leq i \leq k$, *with* $k \geq 1$, *such that the* $\hat{H}_i$ *are comparison sets with respect to* sa, *and* $j \geq 1$ $q, q_1, q_2 \in [0,1]$, *as well as* $f : \mathbb{N}^+ \to \mathbb{N}^+$, *we define:*

f_{e1}: $Z = \{sn\colon sn = sa \;\&\; |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < (q \cdot 100)\% \,|\,\|\hat{F}\|^{sn}\,| \;\&\; |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < j\}$

f_{e2}: $Z = \{sn\colon sn = sa \;\&\; |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < l \cdot |\,\|\hat{F}\|^{sn}\,|, \text{with } l \in (0,1) \text{ fixed}\}$

f_{e3}: $Z = \{sn\colon sn = sa \;\&\; |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < \frac{|\|\hat{G}\|^{sn}|}{|D_{sn}|} \cdot |\,\|\hat{F}\|^{sn}\,|\}$

f_{e4}: $Z = \{sn\colon sn = sa \;\&\; |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < f(|\,D_{sn}\,|), f(|\,D_{sn}\,|) \in \mathbb{N}^+\}$

f_{e5}: $Z = \{sn\colon sn = sa \;\&\; |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < l \cdot |\,\|\hat{G}\|^{sn}\,|, \text{with } l \in (0,1) \text{ fixed}\}$

f_{e6}: $Z = \{sn\colon sn = sa \;\&\; |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < |\,\|\hat{F}\|^{sn} \cap \|\hat{H}\|^{sn}\,|\}$

[29] Before, regarding Equation 4.24, we kept Lappin's original notation for (crisp) predicates, namely A and B, while now we use our common notation for (crisp) formulas, namely $\hat{F}$ and $\hat{G}$. Also, we use Z, instead of S, in the definitions of the readings for "few", to make visible the difference from the definitions of the readings for "many".

f_{e7}: $Z = \{sn\colon sn{=}sa \ \&\ |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < |\,\|\hat{G}\|^{sn} \cap \|\hat{H}\|^{sn}\,|\}$

f_{e8}: $Z = \{sn\colon sn{=}sa \ \&\ |\,\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}\,| < |\,\|\hat{H}\|^{sn}\,|\}$

f_{e9}: $Z = \{sn\colon sn{=}sa \ \&\ \frac{|\|\hat{F}\|^{sn} \cap \|\hat{G}\|^{sn}|}{|\|\hat{F}\|^{sn}|} < (q_1 \cdot 100)\% \ \&\ \frac{|\|\hat{G}\|^{sn} \cap \|\hat{H}\|^{sn}|}{|\|\hat{H}\|^{sn}|} = (q_2 \cdot 100)\% \ \&\ q_1 < q_2\}$

f_{e10}: $Z = \{sn : sn = sa \ \&\ \ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| < |H^+|\}$

f_{e11}: $Z = \{sn : sn = sa \ \&\ \ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| \leq |H^+|\}$

f_{e12}: $Z = \{sn : sn{=}sa \ \&\ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| < |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|, \text{for most } 1 \leq i \leq k\}$

f_{e13}: $Z = \{sn : sn = sa \ \&\ \ \sum_{i\in I_>} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,| \geq \sum_{i\in I_<} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|\}$

f_{e14}: $Z = \{sn : sn = sa \ \&\ \ |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| \leq \frac{1}{k} \cdot \sum_{i=1}^{k} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|\}$

Note that, although reading f_{e4} relies on a function f with values in $\mathbb{N}^+$, we can safely assume that f takes on values in $\{1, \ldots, |\,D_{sa}\,|\}$. This is because we have $|\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| \leq |\,D_{sa}\,|$. Hence, we can also assume that f acts simply as multiplication with a number $a \in \{\frac{1}{|D_{sa}|}, \ldots, \frac{|D_{sa}|}{|D_{sa}|}\}$ on $|\,D_{sa}\,|$.

4.6.6 Intensional readings of "few": binary, Type I

Definition 35 *For two crisp $Ł_\alpha(\Pi)$ formulas $\hat{F}$ and $\hat{G}$, and $Z = \{sn_i : i \in \{1, \ldots, n\}\}$ as set of normative situations, $n \geq 1$, we define:*

f_{i1}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$Z \neq \emptyset$, and for $1 \leq i \leq n$, we have: $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} < \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$

f_{i2}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$Z \neq \emptyset$, and for most $1 \leq i \leq n : |\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| < |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,|$

f_{i3}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$Z \neq \emptyset$, and for most $1 \leq i \leq n$: $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} < \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$

f_{i4}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$Z \neq \emptyset$, and: $|\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| \leq \max\{|\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,| : 1 \leq i \leq n\}$

f_{i5}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$$Z \neq \emptyset, \text{ and: } \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} \leq \max\{\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} : 1 \leq i \leq n\}$$

f_{i6}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$$Z \neq \emptyset, \text{ and: } \sum_{i \in I_{\prec}^{abs}} |\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}| \leq \sum_{i \in I_{\succ}^{abs}} |\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|$$

f_{i7}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$$Z \neq \emptyset, \text{ and: } \sum_{i \in I_{\prec}^{rel}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} \leq \sum_{i \in I_{\succ}^{rel}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$$

f_{i8}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$$Z \neq \emptyset, \text{ and: } |\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}| \leq \frac{1}{n} \sum_{i=1}^{n} |\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|$$

f_{i9}: $\|\hat{G}\|^{sa} \in \|few\|(\|\hat{F}\|^{sa})$ iff

$$Z \neq \emptyset, \text{ and: } \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} \leq \frac{1}{n} \sum_{i=1}^{n} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$$

4.6.7 Representation of the readings of "many" and "few"

The above analysis of Lappin's account of "many" [Lap00], and the refinement of it [Hof15], show how one can define plausible readings for the two quantifiers "many" and "few". From a linguists perspective this might already be it, while our aim here is to translate the given characterizations into a formal logic framework, namely $Ł_\alpha(\Pi)$. As we will see, since some of the readings are relatively involvedly defined, we have to use $Ł_\alpha(\Pi)$ to cover the full spectrum of all 20 readings.

Type I, binary, "many" and "few"

Definition 36 *For crisp* $Ł_\alpha(\Pi)$ *formulas* $\hat{F}, \hat{G}, \hat{H}, \hat{H}_i$*, with* $i \in \{1, \ldots, k\}$*,* $k \geq 1$*, such that the* $\hat{H}_i$ *are comparison predicates with respect to* sa*, and a set* $S = \{sn_1, \ldots, sn_n\}$*,* $n \geq 1$*, of normative situations, we first define the following expressions:*

For any situation s *and two crisp formulas* $\hat{H}_a, \hat{H}_b$*, we define:*

- $v_{\mathcal{M}}(P_s(c)) = \mathbb{I}_{(c \in D_s)}$
- $q^s = \Pi x(P_s(x))$
- $q^s_{\hat{H}_a} = \Pi x(P_s(x) \wedge \hat{H}_a(x))$
- $q^s_{\hat{H}_a, \hat{H}_b} = \Pi x(P_s(x) \wedge \hat{H}_a(x) \wedge \hat{H}_b(x))$

For two values $a, b \in [0,1]$ and $d \in (0,1)$, we use $\overline{q_a}, \overline{q_b}$ and $\overline{q_d}$ for truth value constants that evaluate to a, b and d respectively. Similarly, for a value $c \in \{\frac{1}{|D_{sa}|}, \ldots, \frac{|D_{sa}|}{|D_{sa}|}\}$, we use $\overline{q_c}$ for the truth constant that evaluates to c. Also, we may assume that there is a situation $s_{|j|}$ such that $|D_{s_{|j|}}| = j$, for all $1 \leq j \leq |D|$. Then[30]*:*

Base (binary) Type I reading of "many":

- $\mathsf{Q}_0^m x(\hat{F}(x), \hat{G}(x)) = \wedge_{i=1}^n \Delta(q_{\hat{F},\hat{G}}^{sn_i} \to q_{\hat{F},\hat{G}}^{sa})$

Extensional (binary) Type I readings of "many":

- $\mathsf{Q}_{e1}^m x(\hat{F}(x), \hat{G}(x)) = \Delta((\overline{q_a} \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa})) \wedge (q^{s_{|j|}} \to q_{\hat{F},\hat{G}}^{sa}))$
- $\mathsf{Q}_{e2}^m x(\hat{F}(x), \hat{G}(x)) = \Delta(\overline{q_d} \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa}))$
- $\mathsf{Q}_{e3}^m x(\hat{F}(x), \hat{G}(x)) = \Delta((q^{sa} \rightarrowtail q_{\hat{G}}^{sa}) \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa}))$
- $\mathsf{Q}_{e4}^m x(\hat{F}(x), \hat{G}(x)) = \Delta((\overline{q_c} \cdot q^{sa}) \to (q_{\hat{F},\hat{G}}^{sa}))$
- $\mathsf{Q}_{e5}^m x(\hat{F}(x), \hat{G}(x)) = \Delta(\overline{q_d} \to (q_{\hat{G}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa}))$
- $\mathsf{Q}_{e6}^m x(\hat{F}(x), \hat{G}(x)) = \Delta(q_{\hat{F},\hat{H}}^{sa} \to q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e7}^m x(\hat{F}(x), \hat{G}(x)) = \Delta(q_{\hat{G},\hat{H}}^{sa} \to q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e8}^m x(\hat{F}(x), \hat{G}(x)) = \Delta(q_{\hat{H}}^{sa} \to q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e9}^m x(\hat{F}(x), \hat{G}(x)) = \Delta((\overline{q_a} \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa})) \wedge (\overline{q_b} \leftrightarrow (q_{\hat{H}}^{sa} \rightarrowtail q_{\hat{G},\hat{H}}^{sa})) \wedge (\overline{q_b} \to \overline{q_a}))$
- $\mathsf{Q}_{e10}^m x(\hat{F}(x), \hat{G}(x)) = \Delta(\vee_{i=1}^k q_{\hat{F},\hat{H}_i}^{sa} \to q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e11}^m x(\hat{F}(x), \hat{G}(x)) = \neg\Delta(q_{\hat{F},\hat{G}}^{sa} \to \vee_{i=1}^k q_{\hat{F},\hat{H}_i}^{sa})$
- $\mathsf{Q}_{e12}^m x(\hat{F}(x), \hat{G}(x)) = \neg\Delta(\pi_{i=1}^k \Delta(q_{\hat{F},\hat{H}_i}^{sa} \to q_{\hat{F},\hat{G}}^{sa}) \to (\top \pi \bot))$

[30] Recall that $\rightarrowtail$ is equivalent with $\to^P$.

- $\mathsf{Q}^m_{e13}x(\hat{F}(x),\hat{G}(x)) =$
$= \neg\Delta([q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{H}_i})]^{\oplus}_{1\leq i\leq k} \to [q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq k})$

- $\mathsf{Q}^m_{e14}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to \pi^k_{i=1}q^{sa}_{\hat{F},\hat{H}_i})$

Intensional (binary) Type I readings of "many":

- $\mathsf{Q}^m_{i1}x(\hat{F}(x),\hat{G}(x)) = \wedge^n_{i=1}\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))$

- $\mathsf{Q}^m_{i2}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\pi^n_{i=1}\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}}) \to (\top\pi\bot))$

- $\mathsf{Q}^m_{i3}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\pi^n_{i=1}\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})) \to (\top\pi\bot))$

- $\mathsf{Q}^m_{i4}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to \vee^n_{i=1}q^{sn_i}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^m_{i5}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \vee^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))$

- $\mathsf{Q}^m_{i6}x(\hat{F}(x),\hat{G}(x)) =$
$= \neg\Delta([q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sn_i}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n} \to [q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n})$

- $\mathsf{Q}^m_{i7}x(\hat{F}(x),\hat{G}(x)) =$
$= \neg\Delta([(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n} \to$
$\to [(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n})$

- $\mathsf{Q}^m_{i8}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to \pi^n_{i=1}q^{sn_i}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^m_{i9}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \pi^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))$

Base (binary) Type I reading of "few":

- $\mathsf{Q}^f_0x(\hat{F}(x),\hat{G}(x)) = \wedge^n_{i=1}\neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})$

Extensional (binary) Type I readings of "few":

- $\mathsf{Q}^f_{e1}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\overline{q_a} \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})) \wedge \neg\Delta(q^{s_{|j|}} \to q^{sa}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^f_{e2}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\overline{q_d} \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))$
- $\mathsf{Q}^f_{e3}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta((q^{sa} \rightarrowtail q^{sa}_{\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))$
- $\mathsf{Q}^f_{e4}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta((\overline{q_c} \cdot q^{sa}) \to (q^{sa}_{\hat{F},\hat{G}}))$
- $\mathsf{Q}^f_{e5}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta((\overline{q_d} \to (q^{sa}_{\hat{G}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})))$
- $\mathsf{Q}^f_{e6}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(q^{sa}_{\hat{F},\hat{H}} \to q^{sa}_{\hat{F},\hat{G}})$
- $\mathsf{Q}^f_{e7}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(q^{sa}_{\hat{G},\hat{H}} \to q^{sa}_{\hat{F},\hat{G}})$
- $\mathsf{Q}^f_{e8}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(q^{sa}_{\hat{H}} \to q^{sa}_{\hat{F},\hat{G}})$
- $\mathsf{Q}^f_{e9}x(\hat{F}(x),\hat{G}(x)) =$
  $= \neg\Delta(\overline{q_a} \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})) \wedge \Delta(\overline{q_b} \leftrightarrow (q^{sa}_{\hat{H}} \rightarrowtail q^{sa}_{\hat{G},\hat{H}})) \wedge \neg\Delta(\overline{q_b} \to \overline{q_a})$
- $\mathsf{Q}^f_{e10}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\vee^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})$
- $\mathsf{Q}^f_{e11}x(\hat{F}(x),\hat{G}(x)) = \Delta(q^{sa}_{\hat{F},\hat{G}} \to \vee^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i})$
- $\mathsf{Q}^f_{e12}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\pi^k_{i=1} \neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}}) \to (\top\pi\bot))$
- $\mathsf{Q}^f_{e13}x(\hat{F}(x),\hat{G}(x)) =$
  $= \Delta([q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{H}_i})]^{\oplus}_{1 \le i \le k} \to [q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1 \le i \le k})$
- $\mathsf{Q}^f_{e14}x(\hat{F}(x),\hat{G}(x)) = \Delta(q^{sa}_{\hat{F},\hat{G}} \to \pi^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i})$

Intensional (binary) Type I readings of "few":

- $\mathsf{Q}^f_{i1}x(\hat{F}(x),\hat{G}(x)) = \wedge^n_{i=1} \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))$
- $\mathsf{Q}^f_{i2}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\pi^n_{i=1} \neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}}) \to (\top\pi\bot))$
- $\mathsf{Q}^f_{i3}x(\hat{F}(x),\hat{G}(x)) = \neg\Delta(\pi^n_{i=1} \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})) \to (\top\pi\bot))$

- $\mathsf{Q}^f_{i4}x(\hat{F}(x), \hat{G}(x)) = \Delta(q^{sa}_{\hat{F},\hat{G}} \to \vee^n_{i=1} q^{sn_i}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^f_{i5}x(\hat{F}(x), \hat{G}(x)) = \Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \vee^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))$

- $\mathsf{Q}^f_{i6}x(\hat{F}(x), \hat{G}(x)) =$
  $= \Delta([q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sn_i}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n} \to [q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n})$

- $\mathsf{Q}^f_{i7}x(\hat{F}(x), \hat{G}(x)) =$
  $= \Delta([(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n} \to$
  $\to [(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n})$

- $\mathsf{Q}^f_{i8}x(\hat{F}(x), \hat{G}(x)) = \Delta(q^{sa}_{\hat{F},\hat{G}} \to \pi^n_{i=1} q^{sn_i}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^f_{i9}x(\hat{F}(x), \hat{G}(x)) = \Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \pi^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))$

Theorem 35 *The quantifiers Q^m_ζ and Q^f_ζ, for $\zeta \in \{0, e1, \ldots, e14, i1, \ldots, i9\}$, formalize the corresponding readings m_ζ, f_ζ for "many" and for "few".*

Proof:
We first note that:

$v_{\mathcal{M}}(q^s) = \frac{|D_s|}{|D|}$

$v_{\mathcal{M}}(q^s_{\hat{H}_a}) = \frac{\sum_{c\in D_s} v_{\mathcal{M}}(\hat{H}_a(c))}{|D|} = \frac{|\|\hat{H}_a\|^s|}{|D|}$

$v_{\mathcal{M}}(q^s_{\hat{H}_a,\hat{H}_b}) = \frac{\sum_{c\in D_s} v_{\mathcal{M}}(\hat{H}_a(c)\wedge\hat{H}_b(c))}{|D|} = \frac{|\|\hat{H}_a\|^s\cap\|\hat{H}_b\|^s|}{|D|}$

$v_{\mathcal{M}}(q^s_{\hat{H}_a} \rightarrowtail q^s_{\hat{H}_a,\hat{H}_b}) = \frac{|\|\hat{H}_a\|^s\cap\|\hat{H}_b\|^s|/|D|}{|\|\hat{H}_a\|^s|/|D|} = \frac{|\|\hat{H}_a\|^s\cap\|\hat{H}_b\|^s|}{|\|\hat{H}_a\|^s|}$

We now prove case by case:

- $v_{\mathcal{M}}(\mathsf{Q}^m_0 x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\wedge^n_{i=1}\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}}))$

  This evaluates to 1 if and only if for all $1 \leq i \leq n$ it is true that:

  $\frac{|\|\hat{F}\|^{sa}\cap\|\hat{G}\|^{sa}|}{|D|} \geq \frac{|\|\hat{F}\|^{sn_i}\cap\|\hat{G}\|^{sn_i}|}{|D|}$, which holds if and only if

  $|\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| \geq |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,|$, for all $1 \leq i \leq n$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e1}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\Delta((\overline{p_a} \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})) \wedge (q^{s_{|j|}} \to q^{sa}_{\hat{F},\hat{G}})))$

  This evaluates to 1 if and only if, for a fixed $a \in [0,1]$ and $1 \leq j \leq\mid D\mid$:

  $a \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{F}\|^{sa}|/|D|}$, and $\frac{|D_{s_{|j|}}|}{|D|} = \frac{j}{|D|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|}$.

  This holds if and only if:

  $a \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}$, and $\mid D_{s_{|j|}}\mid = j \leq\mid \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\mid$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e2}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\Delta((\overline{p_d} \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))))$

  This evaluates to 1 if and only if, for a fixed $d \in (0,1)$:

  $d \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{F}\|^{sa}|/|D|}$. This holds if and only if $d \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e3}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\Delta((q^{sa} \rightarrowtail q^{sa}_{\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})))$

  This evaluates to 1 if and only if:

  $\frac{|\|\hat{G}\|^{sa}|/|D|}{|D_{sa}|/|D|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{F}\|^{sa}|/|D|}$. This holds if and only if $\frac{|\|\hat{G}\|^{sa}|}{|D_{sa}|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e4}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\Delta((\overline{q_c} \cdot q^{sa}) \to (q^{sa}_{\hat{F},\hat{G}})))$

  This evaluates to 1 if and only if, for a value $c \in \{\frac{1}{|D_{sn}|},\ldots,\frac{|D_{sn}|}{|D_{sn}|}\}$:

  $\frac{c\cdot|D_{sa}|}{|D|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|}$. This holds if and only if $c\cdot \mid D_{sa}\mid = f(\mid D_{sa}\mid) \leq\mid \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\mid$

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e5}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\Delta((\overline{p_d} \to (q^{sa}_{\hat{G}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))))$

  This evaluates to 1 if and only if, for a fixed $d \in (0,1)$:

  $d \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{G}\|^{sa}|/|D|}$. This holds if and only if $d \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{G}\|^{sa}|}$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e6}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\Delta(q^{sa}_{\hat{F},\hat{H}} \to q^{sa}_{\hat{F},\hat{G}}))$

  This evaluates to 1 if and only if:

  $1 \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{F}\|^{sa} \cap \|\hat{H}\|^{sa}|/|D|}$. This holds if and only if:

  $\mid \|\hat{F}\|^{sa} \cap \|\hat{H}\|^{sa}\mid \leq\mid \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\mid$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e7}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\Delta(q^{sa}_{\hat{G},\hat{H}} \to q^{sa}_{\hat{F},\hat{G}}))$

  This evaluates to 1 if and only if:

  $1 \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{G}\|^{sa} \cap \|\hat{H}\|^{sa}|/|D|}$. This holds if and only if:

  $| \|\hat{G}\|^{sa} \cap \|\hat{H}\|^{sa} | \leq | \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} |$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e8}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\Delta(q^{sa}_{\hat{H}} \to q^{sa}_{\hat{F},\hat{G}}))$

  This evaluates to 1 if and only if:

  $1 \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{H}\|^{sa}|/|D|}$. This holds if and only if:

  $| \|\hat{H}\|^{sa} | \leq | \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} |$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e9}x(\hat{F}(x), \hat{G}(x))) =$
  $v_{\mathcal{M}}(\Delta((\overline{p_a} \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})) \wedge (\overline{p_b} \leftrightarrow (q^{sa}_{\hat{H}} \rightarrowtail q^{sa}_{\hat{G},\hat{H}})) \wedge (\overline{p_b} \to \overline{p_a})))$

  This evaluates to 1 if and only if, for fixed $a, b \in [0,1]$:

  $a \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|/|D|}{|\|\hat{F}\|^{sa}|/|D|}$, and $b = \frac{|\|\hat{G}\|^{sa} \cap \|\hat{H}\|^{sa}|/|D|}{|\|\hat{H}\|^{sa}|/|D|}$, and $b \leq a$.

  This holds if and only if:

  $a \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}$, and $b = \frac{|\|\hat{G}\|^{sa} \cap \|\hat{H}\|^{sa}|}{|\|\hat{H}\|^{sa}|}$, and $b \leq a$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e10}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\Delta(\vee^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}}))$

  This evaluates to 1 if and only if for all $1 \leq i \leq k$ we have:

  $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}|}{|D|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|}$. This holds if and only if

  $| \|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa} | \leq | \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} |$, for all $1 \leq i \leq k$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e11}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to \vee^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i}))$

  This evaluates to 1 if and only if for all $1 \leq i \leq k$ we have:

  $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}|}{|D|} < \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|}$. This holds if and only if

  $| \|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa} | < | \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} |$, for all $1 \leq i \leq k$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e12}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta(\pi^k_{i=1}\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}}) \to (\top\pi\bot)))$

  This evaluates to 1 if and only if:

  $\frac{1}{2} < \frac{1}{k} \cdot \sum_{i=1}^{k} v_{\mathcal{M}}(\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}}))$. This is true if and only if there are at least l comparison predicates with $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}|}{|D|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|}$, for l such that $\frac{l}{k} > \frac{1}{2}$. Hence, there must be a majority of comparison predicates, which is exactly the semantics of "most".

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e13}x(\hat{F}(x), \hat{G}(x))) =$
  $= v_{\mathcal{M}}(\neg\Delta([q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{H}_i})]^{\oplus}_{1\leq i\leq k} \to$
  $\to [q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq k})) = 1 - \mathbb{I}_{\{1=\min(1,1-v+w)\}}$, with v, w:

  $v = v_{\mathcal{M}}([q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{H}_i})]^{\oplus}_{1\leq i\leq k}) = \sum_{i\in I_<} \frac{|\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}|}{|D|}$, and

  $w = v_{\mathcal{M}}([q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq k}) = \sum_{i\in I_>} \frac{|\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}|}{|D|}$.

  Hence, $v_{\mathcal{M}}(\mathsf{Q}^m_{e13}x(\hat{F}(x), \hat{G}(x)))$ evaluates to 1 if and only if $v > w$, which is exactly when $\sum_{i\in I_<} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,| > \sum_{i\in I_>} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{e14}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to \pi^k_{i=1}q^{sa}_{\hat{F},\hat{H}_i}))$

  This evaluates to 1 if and only if:

  $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|} > \frac{1}{k}\sum_{i=1}^{k} \frac{|\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}|}{|D|}$. This is true if and only if,

  $|\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| > \frac{1}{k}\sum_{i=1}^{k} |\,\|\hat{F}\|^{sa} \cap \|\hat{H}_i\|^{sa}\,|$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i1}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\wedge^n_{i=1}\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})))$

  This evaluates to 1 if and only if for all $1 \leq i \leq n$:

  $\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}$

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i2}x(\hat{F}(x), \hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta(\pi^n_{i=1}\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}}) \to (\top\pi\bot)))$

  This evaluates to 1 if and only if:

  $\frac{1}{2} < \frac{1}{n} \cdot \sum_{i=1}^{n} v_{\mathcal{M}}(\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}}))$. This is true if and only if there are at least l normative situations, with $\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|D|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|}$, for l such that $\frac{l}{n} > \frac{1}{2}$. Hence, there must be a majority of normative situations, which is exactly the semantics of "most".

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i3}x(\hat{F}(x),\hat{G}(x))) =$
  $= v_{\mathcal{M}}(\neg\Delta(\pi^n_{i=1}\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})) \to (\top\pi\bot)))$

  This evaluates to 1 if and only if:

  $\frac{1}{2} < \frac{1}{n} \cdot \sum_{i=1}^n v_{\mathcal{M}}(\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})))$. This is true if and only if there are at least l normative situations, with $\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} \leq \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}$, for l such that $\frac{l}{n} > \frac{1}{2}$. Hence, there must be a majority of normative situations, which is exactly the semantics of "most".

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i4}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to \vee^n_{i=1} q^{sn_i}_{\hat{F},\hat{G}}))$

  This evaluates to 1 if and only if for all $1 \leq i \leq n$ we have:

  $\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|D|} < \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|}$. This holds if and only if:

  $|\,\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}\,| > \max_{1\leq i\leq n}\{|\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,|\}$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i5}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \vee^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})))$

  This evaluates to 1 if and only if for all $1 \leq i \leq n$ we have:

  $\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} < \frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|}$. This holds if and only if:

  $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} > \max_{1\leq i\leq n}\{\frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}\}$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i6}x(\hat{F}(x),\hat{G}(x))) =$
  $= v_{\mathcal{M}}(\neg\Delta([q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sn_i}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n} \to$
  $\to [q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n})) = 1 - \mathbb{I}_{\{1=\min(1,1-v+w)\}}$, with v, w:

  $v = v_{\mathcal{M}}([q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sn_i}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n}) = \sum_{i\in I^{abs}_{\prec}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|D|}$, and

  $w = v_{\mathcal{M}}([q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n}) = \sum_{i\in I^{abs}_{\succ}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|D|}$.

  Hence, $v_{\mathcal{M}}(\mathsf{Q}^m_{i6}x(\hat{F}(x),\hat{G}(x)))$ evaluates to 1 if and only if $v > w$, which is exactly when $\sum_{i\in I^{abs}_{\prec}} |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,| > \sum_{i\in I^{abs}_{\succ}} |\,\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}\,|$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i7}x(\hat{F}(x),\hat{G}(x))) =$
  $= v_{\mathcal{M}}(\neg\Delta([(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))]^{\oplus}_{1\le i\le n} \to$
  $\to [(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))]^{\oplus}_{1\le i\le n})) =$
  $= 1 - \mathbb{I}_{\{1=min(1,1-v+w)\}}$, with v, w:
  $v = v_{\mathcal{M}}([(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))]^{\oplus}_{1\le i\le n}) =$
  $= \sum_{i\in I^{rel}_{\prec}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|D|\cdot|\|\hat{F}\|^{sn_i}|} \le 1$, and
  $w = v_{\mathcal{M}}([(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))]^{\oplus}_{1\le i\le n}) =$
  $= \sum_{i\in I^{rel}_{\succ}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|D|\cdot|\|\hat{F}\|^{sn_i}|} \le 1.$

  Hence, $v_{\mathcal{M}}(\mathsf{Q}^m_{i7}x(\hat{F}(x),\hat{G}(x)))$ evaluates to 1 if and only if $v > w$, which is exactly when $\sum_{i\in I^{rel}_{\prec}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|} > \sum_{i\in I^{rel}_{\succ}} \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i8}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to \pi^n_{i=1} q^{sn_i}_{\hat{F},\hat{G}}))$

  This evaluates to 1 if and only if:

  $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|D|} > \frac{1}{n}\sum_{i=1}^n \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|D|}$. This is true if and only if,

  $|\, \|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa} \,| > \frac{1}{n}\sum_{i=1}^n |\, \|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i} \,|$.

- $v_{\mathcal{M}}(\mathsf{Q}^m_{i9}x(\hat{F}(x),\hat{G}(x))) = v_{\mathcal{M}}(\neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \pi^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})))$

  This evaluates to 1 if and only if $\frac{|\|\hat{F}\|^{sa} \cap \|\hat{G}\|^{sa}|}{|\|\hat{F}\|^{sa}|} > \frac{1}{n}\sum_{i=1}^n \frac{|\|\hat{F}\|^{sn_i} \cap \|\hat{G}\|^{sn_i}|}{|\|\hat{F}\|^{sn_i}|}$.

The cases for “few” work analogously. □

Type III, binary, “many” and “few”

Although not common in linguistic accounts, “many” and “few” can be understood as Type III quantifiers, too.

Zadeh [Zad65] initiated the studies in fuzzy logic, which have since been developed incredibly far until today’s state of a formal Mathematical Fuzzy Logic (MFL) [CHN11]. To that end, major contributions have been made by Hajek [Háj98], Cintula and Noguera [CHN11], especially when developing a systematic t-norm based account to fuzzy logic, which is today at the heart of any MFL. Albeit, although fuzzy logic was (and still is) intended to model natural reasoning more flexibly, no case closing contributions have been made in regard of the semantics of "many" and "few". This is not to say none have been made at all. In fact, several readings we discussed beforehand, have been formalized already, e.g. by means of Zadeh's S function[Glö06], which is basically a parametrized function from the real unit interval to itself, with function values starting in the origin and ending up in the point $(1,1)$. More formally, dependent on two parameters a, b, it is defined as follows:

$S : [0,1]^3 \rightarrow [0,1]$, with:

$$S(x,a,b) = \begin{cases} 0 & x \leq a \\ 2 \cdot (\frac{x-a}{b-a})^2 & a < x \leq \frac{a+b}{2} \\ 1 - 2 \cdot (\frac{x-b}{b-a})^2 & \frac{a+b}{2} < x \leq b \\ 1 & x > b \end{cases}$$

This is to be read as follows. Regarding a statement of the form "Many A's are B's", the value x represents the proportion of $A's$ that are $B's$, hence $\frac{||A \cap B||}{||A||}$, while the value $S(x,a,b)$ models the respective truth value of the statement for any x. Since S is clearly a monotonically increasing function, starting in the origin and ending up in $(1,1)$, it is possible to model all sorts of readings for "many" with it. Still, again, we are left with the determination of the parameters a and b, which is unsatisfying as there is no described way as to how to get to them.

Then, there is Hajek's way [Háj98], which basically consists in shifting the problem away, similarly to most of the other strategies discussed so far. He introduces a set of possible worlds, each of which evaluates respective "many"-quantified statements classically, i.e. the evaluation yields either 0 or 1. Eventually, one takes the average of these values, interpreting it as the probability that a respective statement is true in a randomly picked world. Although this makes a lot of sense in principle, we still don't know how the get to the individual truth values in each world. Looks like we go round in circles, as also Glöckner [Glö06] and, more recently, Delgado *et al.* [DRSV14] do not exceed these limits.

Following our own account, based on Lappin's, going from Type I to Type III can simply be done by making the evaluation criteria non-crisp, i.e. removing some Deltas from the Type I definitions. We can not just say all Deltas, since some of the encoded conditions have to remain crisp. As a consequence we list all Type III quantifier definitions for "many" and "few".

Definition 37 *Let crisp* $Ł_\alpha(\Pi)$ *formulas* $\hat{F}, \hat{G}, \hat{H}, \hat{H}_i$*, with* $i \in \{1, \ldots, k\}$*,* $k \geq 1$*, such that the* $\hat{H}_i$ *are comparison predicates with respect to* sa*, and a set* $S = \{sn_1, \ldots, sn_n\}$*,*

$n \geq 1$, of normative situations, be given. For two values $a, b \in [0,1]$ and $d \in (0,1)$, we use $\overline{q_a}, \overline{q_b}$ and $\overline{q_d}$ for truth constants that evaluate to a, b and d respectively. Similarly, for a value $c \in \{\frac{1}{|D_{sa}|}, \ldots, \frac{|D_{sa}|}{|D_{sa}|}\}$, we use $\overline{q_c}$ for the truth constant that evaluates to c. Also, we may assume that there is a situation $s_{|j|}$ such that $|D_{s_{|j|}}| = j$, for all $1 \leq j \leq |D|$. Then[31]:

Base (binary) Type III reading of "many":

- $\mathsf{Q}_0^M x(\hat{F}(x), \hat{G}(x)) = \wedge_{i=1}^n (q_{\hat{F},\hat{G}}^{sn_i} \to q_{\hat{F},\hat{G}}^{sa})$

Extensional (binary) Type III readings of "many":

- $\mathsf{Q}_{e1}^M x(\hat{F}(x), \hat{G}(x)) = (\overline{q_a} \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa})) \wedge (q^{s_{|j|}} \to q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e2}^M x(\hat{F}(x), \hat{G}(x)) = \overline{q_d} \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e3}^M x(\hat{F}(x), \hat{G}(x)) = (q^{sa} \rightarrowtail q_{\hat{G}}^{sa}) \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e4}^M x(\hat{F}(x), \hat{G}(x)) = (\overline{q_c} \cdot q^{sa}) \to (q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e5}^M x(\hat{F}(x), \hat{G}(x)) = \overline{q_d} \to (q_{\hat{G}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa})$
- $\mathsf{Q}_{e6}^M x(\hat{F}(x), \hat{G}(x)) = q_{\hat{F},\hat{H}}^{sa} \to q_{\hat{F},\hat{G}}^{sa}$
- $\mathsf{Q}_{e7}^M x(\hat{F}(x), \hat{G}(x)) = q_{\hat{G},\hat{H}}^{sa} \to q_{\hat{F},\hat{G}}^{sa}$
- $\mathsf{Q}_{e8}^M x(\hat{F}(x), \hat{G}(x)) = q_{\hat{H}}^{sa} \to q_{\hat{F},\hat{G}}^{sa}$
- $\mathsf{Q}_{e9}^M x(\hat{F}(x), \hat{G}(x)) = (\overline{q_a} \to (q_{\hat{F}}^{sa} \rightarrowtail q_{\hat{F},\hat{G}}^{sa})) \wedge (\overline{q_b} \leftrightarrow (q_{\hat{H}}^{sa} \rightarrowtail q_{\hat{G},\hat{H}}^{sa})) \wedge \Delta(\overline{q_b} \to \overline{q_a})$
- $\mathsf{Q}_{e10}^M x(\hat{F}(x), \hat{G}(x)) = \vee_{i=1}^k q_{\hat{F},\hat{H}_i}^{sa} \to q_{\hat{F},\hat{G}}^{sa}$
- $\mathsf{Q}_{e11}^M x(\hat{F}(x), \hat{G}(x)) = \vee_{i=1}^k q_{\hat{F},\hat{H}_i}^{sa} \to q_{\hat{F},\hat{G}}^{sa}$
- $\mathsf{Q}_{e12}^M x(\hat{F}(x), \hat{G}(x)) = (\top \pi \bot) \to \pi_{i=1}^k \Delta(q_{\hat{F},\hat{H}_i}^{sa} \to q_{\hat{F},\hat{G}}^{sa})$

[31]Recall that $\rightarrowtail$ is equivalent with $\to^P$.

- $\mathsf{Q}^M_{e13}x(\hat{F}(x),\hat{G}(x)) =$
  $= [q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq k} \to [q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{H}_i})]^{\oplus}_{1\leq i\leq k}$

- $\mathsf{Q}^M_{e14}x(\hat{F}(x),\hat{G}(x)) = \pi^k_{i=1}q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}}$

Intensional (binary) Type III readings of "many":

- $\mathsf{Q}^M_{i1}x(\hat{F}(x),\hat{G}(x)) = \wedge^n_{i=1}((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))$

- $\mathsf{Q}^M_{i2}x(\hat{F}(x),\hat{G}(x)) = (\top\pi\bot) \to \pi^n_{i=1}\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^M_{i3}x(\hat{F}(x),\hat{G}(x)) = (\top\pi\bot) \to \pi^n_{i=1}\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))$

- $\mathsf{Q}^M_{i4}x(\hat{F}(x),\hat{G}(x)) = \vee^n_{i=1}q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}}$

- $\mathsf{Q}^M_{i5}x(\hat{F}(x),\hat{G}(x)) = \vee^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^M_{i6}x(\hat{F}(x),\hat{G}(x)) =$
  $= [q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n} \to [q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sn_i}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n}$

- $\mathsf{Q}^M_{i7}x(\hat{F}(x),\hat{G}(x)) =$
  $= [(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n} \to$
  $\to [(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n}$

- $\mathsf{Q}^M_{i8}x(\hat{F}(x),\hat{G}(x)) = \pi^n_{i=1}q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}}$

- $\mathsf{Q}^M_{i9}x(\hat{F}(x),\hat{G}(x)) = \pi^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}})$

Base (binary) Type III reading of "few":

- $\mathsf{Q}^F_0x(\hat{F}(x),\hat{G}(x)) = \wedge^n_{i=1}(q^{sa}_{\hat{F},\hat{G}} \to q^{sn_i}_{\hat{F},\hat{G}})$

Extensional (binary) Type III readings of "few":

- $\mathsf{Q}^F_{e1}x(\hat{F}(x),\hat{G}(x)) = ((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \overline{p_a}) \wedge (q^{sa}_{\hat{F},\hat{G}} \to q^{s_{|j|}})$

- $\mathsf{Q}^F_{e2}x(\hat{F}(x),\hat{G}(x)) = (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \overline{q_d}$
- $\mathsf{Q}^F_{e3}x(\hat{F}(x),\hat{G}(x)) = (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sa} \rightarrowtail q^{sa}_{\hat{G}})$
- $\mathsf{Q}^F_{e4}x(\hat{F}(x),\hat{G}(x)) = (q^{sa}_{\hat{F},\hat{G}}) \to (\overline{q_c} \cdot q^{sa})$
- $\mathsf{Q}^F_{e5}x(\hat{F}(x),\hat{G}(x)) = (q^{sa}_{\hat{G}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \overline{q_d}$
- $\mathsf{Q}^F_{e6}x(\hat{F}(x),\hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{H}}$
- $\mathsf{Q}^F_{e7}x(\hat{F}(x),\hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{G},\hat{H}}$
- $\mathsf{Q}^F_{e8}x(\hat{F}(x),\hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{H}}$
- $\mathsf{Q}^F_{e9}x(\hat{F}(x),\hat{G}(x)) =$
  $= ((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \overline{q_a}) \wedge (\overline{q_b} \leftrightarrow (q^{sa}_{\hat{H}} \rightarrowtail q^{sa}_{\hat{G},\hat{H}})) \wedge \neg\Delta(\overline{q_b} \to \overline{q_a})$
- $\mathsf{Q}^F_{e10}x(\hat{F}(x),\hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to \vee^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i}$
- $\mathsf{Q}^F_{e11}x(\hat{F}(x),\hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to \vee^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i}$
- $\mathsf{Q}^F_{e12}x(\hat{F}(x),\hat{G}(x)) = (\top\pi\bot) \to \pi^k_{i=1}\neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})$
- $\mathsf{Q}^F_{e13}x(\hat{F}(x),\hat{G}(x)) =$
  $= [q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{H}_i})]^{\oplus}_{1\le i\le k} \to [q^{sa}_{\hat{F},\hat{H}_i} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{H}_i} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\le i\le k}$
- $\mathsf{Q}^F_{e14}x(\hat{F}(x),\hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to \pi^k_{i=1} q^{sa}_{\hat{F},\hat{H}_i}$

Intensional (binary) Type III readings of "few":

- $\mathsf{Q}^F_{i1}x(\hat{F}(x),\hat{G}(x)) = \wedge^n_{i=1}((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))$
- $\mathsf{Q}^F_{i2}x(\hat{F}(x),\hat{G}(x)) = (\top\pi\bot) \to \pi^n_{i=1}\neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})$
- $\mathsf{Q}^F_{i3}x(\hat{F}(x),\hat{G}(x)) = (\top\pi\bot) \to \pi^n_{i=1}\neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))$

- $\mathsf{Q}^F_{i4}x(\hat{F}(x), \hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to \vee^n_{i=1} q^{sn_i}_{\hat{F},\hat{G}}$

- $\mathsf{Q}^F_{i5}x(\hat{F}(x), \hat{G}(x)) = (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \vee^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})$

- $\mathsf{Q}^F_{i6}x(\hat{F}(x), \hat{G}(x)) =$
  $= [q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sa}_{\hat{F},\hat{G}} \to q^{sn_i}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n} \to [q^{sn_i}_{\hat{F},\hat{G}} \wedge \neg\Delta(q^{sn_i}_{\hat{F},\hat{G}} \to q^{sa}_{\hat{F},\hat{G}})]^{\oplus}_{1\leq i\leq n}$

- $\mathsf{Q}^F_{i7}x(\hat{F}(x), \hat{G}(x)) =$
  $= [(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n} \to$
  $\to [(q^{s_{|1|}} \cdot (q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})) \wedge \neg\Delta((q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}}) \to (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}))]^{\oplus}_{1\leq i\leq n}$

- $\mathsf{Q}^F_{i8}x(\hat{F}(x), \hat{G}(x)) = q^{sa}_{\hat{F},\hat{G}} \to \pi^n_{i=1} q^{sn_i}_{\hat{F},\hat{G}}$

- $\mathsf{Q}^F_{i9}x(\hat{F}(x), \hat{G}(x)) = (q^{sa}_{\hat{F}} \rightarrowtail q^{sa}_{\hat{F},\hat{G}}) \to \pi^n_{i=1}(q^{sn_i}_{\hat{F}} \rightarrowtail q^{sn_i}_{\hat{F},\hat{G}})$

(Non)-Monotonicity of the readings for "many" and "few"

Monotonicity of a quantifier is a very basic and core property that can be fulfilled or even should or must be fulfilled in certain cases where adequacy of the quantifier model is an objective. An non-decreasing quantifier has a truth function with non-negative slope, while a quantifier is non-increasing if it has a non-positive slope. In the next chapter, we will analyze properties of quantifiers in more detail. However, since here we are interested in particular (binary) quantifiers, namely "many" and "few", we introduce and use the notion of monotonicity (in proportion) of a quantifier in advance.[32].

Definition 38 *A binary semi-fuzzy quantifier* Q *is called non-decreasing in proportion, if for three crisp formulas* $\hat{F}, \hat{G}, \hat{H}$*, with* $Prop_{\mathcal{M}}(\hat{H} \wedge \hat{F}) \leq Prop_{\mathcal{M}}(\hat{H} \wedge G)$*, we have:*

$$v_{\mathcal{M}}(\mathsf{Q}x(\hat{H}(x), \hat{F}(x))) \leq v_{\mathcal{M}}(\mathsf{Q}x(\hat{H}(x), \hat{G}(x))). \tag{4.26}$$

Also, a semi-fuzzy quantifier Q *is called non-increasing in proportion, if for three formulas* $\hat{F}, \hat{G}, \hat{H}$*, with* $Prop_{\mathcal{M}}(\hat{H} \wedge \hat{F}) \leq Prop_{\mathcal{M}}(\hat{H} \wedge G)$*, we have:*

$$v_{\mathcal{M}}(\mathsf{Q}x(\hat{H}(x), \hat{F}(x))) \geq v_{\mathcal{M}}(\mathsf{Q}x(\hat{H}(x), \hat{G}(x))). \tag{4.27}$$

[32] In the next chapter we will define monotonicity in proportion for unary quantifiers. This will then technically be redundant, but for the presentation still be useful.

In [PW06], Westerstahl argues that "many" and "few" are particularly good examples for quantifiers that should have the monotonicity property, i.e. models for "many" should be non-decreasing and models for "few" should be non-increasing. This seems to make sense, as, for example, if one accepts that "Many people are destitute", then we would most likely also accept that "Many people are destitute or just poor". To rate the one or the other statement, we can perhaps safely use the same reading of "many", meaning the same comparison sets or normative situations. On the other side, when the quantified predicates relate to different contexts, this comparability most certainly will be lost. The following theorem captures the non-monotonic nature of this section's models, which is due to the very flexible way in which we can instantiate situations and comparison predicates.

Theorem 36 *None of the quantifier models of this section generally complies with their respective monotonicity properties.*

Proof:
For extensional readings of "many", it is relatively easy to see that the conditions for monotonicity (in proportion) can be violated by setting the actual situation accordingly. The issue arises from the fact that $Prop_{\mathcal{M}}$ always evaluates the whole domain D. This means that any statement of the form $Prop_{\mathcal{M}}(\hat{F} \wedge \hat{G}) \leq Prop_{\mathcal{M}}(\hat{F} \wedge G')$, for a semi-fuzzy quantifier Q and crisp $Ł_\alpha(\Pi)$ formulas $\hat{F}, \hat{G}, \hat{G}'$, can be rendered irrelevant with respect to evaluations of situation dependent quantifier models. This happens if we simultaneously have an actual situation sa such that $D_{sa} = \{c \in D : v_{\mathcal{M}}(\hat{F}(c) \wedge \hat{G}(c)) = 1\}$, and $\{c \in D : v_{\mathcal{M}}(\hat{F}(c) \wedge \hat{H}(c)) = 1\} \cap D_{sa} = \emptyset$. When normative situations enter the stage, the argument becomes even more obvious. The argument for readings of "few" works by analogy. □

The result of this theorem should not be taken as a defect of the quantifier models, but rather as a hint that most basic criteria, like monotonicity, should not be demanded from natural language expressions that are so very flexible by their very nature.

Remark 21 *In the beginning of this chapter, we outlined which quantifiers we are going to treat and analyze. It now remains to mention how those labeled as* $[Int2]$*, namely* several*,* various*,* multiple*,* heaps of*, and* loads of*, are supposed to be understood. Again, there are many different interpretations that can be witnessed in real life. Some are fairly straightforward, e.g.* several*,* various *and* multiple *can be read as* more than one*. The other two are somewhat similar to "many", and, as there are already plenty of readings to choose from, we claim that some of them may also apply to those. The use of* plenty *in the forgoing sentence also falls under this regime. Taking the present thesis as actual situation, just think of the comparison sets referring to the amount of readings we defined for other quantifiers, and verify that there are less.*

CHAPTER 5

Quantifier Fuzzification Mechanisms: QFMs

5.1 Desiderata for Quantifier Fuzzification Mechanisms

Lifting semi-fuzzy quantifiers to fully-fuzzy quantifiers should be performed in a systematic manner, following certain principles or axioms. This approach has been pursued most notably by Glöckner, who wrote a whole monograph on the topic [Glö06]. More recently, Delgado *et al.* [DRSV14] provided the community with a survey article, that summarizes the most important techniques in that regard concisely and in great detail. Also, Diaz-Hermida *et al.* [DHBCB04] dedicated themselves to this method. This chapter is based on [BFH18]. The framework we build upon throughout this chapter will be $Ł_\alpha(\Pi)$, unless clearly stated otherwise. Hence, we will keep that fact implicit, to shorten the presentation. The main notion, the Quantifier Fuzzification Mechanism (QFM), is defined as follows:

Definition 39 *A* quantifier fuzzification mechanism (QFM) $\mathcal{F}$ *assigns to each semi-fuzzy quantifier* Q *a corresponding fully-fuzzy quantifier* $\mathcal{F}(\mathsf{Q})$*.*

Focusing on the semantics that underpins the different QFMs that we will discuss and analyze in this chapter, we will only treat unary quantifiers, as done in [BFH18]. The lifting principles will be introduced as desiderata rather than axioms. Unlike Glöckner, we do not aim at a set of axioms that are minimal in the sense that none of them is derivable by the others. The reason is that we intend to have them listed transparently and systematically, one by one. Two of Glöckner's principles, or axioms in his terminology, called 'internal joins' and 'functional application', respectively, apply to quantifiers with arity greater than 1, and are hence omitted here. We further note that Glöckner does not distinguish properly between syntax and semantics, achieving a shorter presentation of matters, while we translate that terminology into one corresponding to our logical framework, following the practice of [BFH18]. According to Glöckner, "perhaps the most important axiom" is this one:

Correct Generalization: For all crisp formulas $\hat{F}$: $v_{\mathcal{M}}(\mathcal{F}(\mathsf{Q})x\hat{F}(x)) = v_{\mathcal{M}}(\mathsf{Q}x\hat{F}(x))$.

This desideratum expresses that the evaluation of the fuzzified quantifier $\mathcal{F}(\mathsf{Q})$ must coincide with the evaluation of the semi-fuzzy quantifier Q on crisp arguments.

For the following definition, remember that we identify domain elements with constants.

Definition 40 *For all crisp formulas $\hat{F}$ and all $c \in D$, we define the (Type I)* projection quantifier Δ_c *by $v_{\mathcal{M}}(\Delta_c x\hat{F}(x)) = v_{\mathcal{M}}(\hat{F}(c))$.*

Note that $\Delta_c x\hat{F}(x)$ is classical (bivalent). Glöckner postulates the following desideratum for lifting Δ_c to fuzzy predicates.

Projection Quantifiers: For all formulas F: $v_{\mathcal{M}}(\mathcal{F}(\Delta_c)xF(x)) = v_{\mathcal{M}}(F(c))$.

Projection quantifiers are introduced to model quantifiers that refer to named objects, such as "John", not as a name for arbitrary people, but a particular John that one wants to single out. Linguistically, that is standard, while logically, as we will see from the following definition, such quantifiers play a special role, as they cannot be considered *quantitative* or *logical*. This property expresses that the truth value of a quantified statement should not depend on the order of domain elements, but only on quantitative aspects of the argument predicate. This can be expressed in various ways. Following [PW06], we will take the following property as the hallmark of logicality.

Definition 41 *A quantifier Q is called* quantitative *if for all bijections*[1] $\xi : D \to D$ *and all formulas F*[2]*, $v_{\mathcal{M}}(\mathsf{Q}xF(x)) = v_{\mathcal{M}^{\xi}}(\mathsf{Q}xF(x))$, where the interpretation $\mathcal{M}^{\xi}$ results from the interpretation $\mathcal{M}$ by mapping every $c \in D$ into $\xi(c)$.*

Remark 22 *The foregoing definition, and many of the following ones, do not distinguish clearly between semi-fuzzy quantifiers and fully-fuzzy quantifiers, to safe some space and make the text more readable. However, if one considers the definitions for semi-fuzzy quantifiers, one has to restrict the phrase "for all formulas F" to "for all crisp formulas $\hat{F}$", for otherwise some appearing terms are actually undefined.*

[1] Note that this corresponds to a permutation of the domain, and that we do not shuffle the set of constants, which is also called D.

[2] In case of semi-fuzzy quantifiers, we have to consider only crisp formulas. Note the following remark.

The respective desideratum for QFMs is given as:

Preservation of Quantitativity: If a semi-fuzzy quantifier Q is quantitative, then so is $\mathcal{F}(Q)$.

The following definitions capture properties that are related to extending or restricting the domain of discourse D, while not changing the interpretation of predicates over the original D.

Definition 42 *We call the interpretation $\mathcal{M}'$ a* conservative extension *of $\mathcal{M}$ and say that $\mathcal{M}'$* conservatively extends *$\mathcal{M}$, if $\mathcal{M}'$ results from $\mathcal{M}$ by (possibly) adding further elements to the domain D without changing the interpretation of predicates over D itself (i.e. $v_{\mathcal{M}'}()$ and $v_{\mathcal{M}}()$ agree on D).*

Definition 43 *A quantifier Q is called* non-decreasing under extension *if for all formulas F, $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}xF(x))$, whenever $\mathcal{M}'$ conservatively extends $\mathcal{M}$. It is called* non-increasing under extension *if, under the same condition, $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \geq v_{\mathcal{M}'}(\mathsf{Q}xF(x))$.*

Note that neither relative nor absolute quantifiers are monotonic under extension, in general. However, the classical existential quantifier and, more generally, Type I quantifiers expressing "at least k" ($k > 0$), and "at least $(q \cdot 100)\%$" ($q \in [0,1]$), are non-decreasing under extension, while the classical universal and, more generally, Type I quantifiers expressing "at most k" ($k > 0$), and "at most $(q \cdot 100)\%$" ($q \in [0,1]$), are non-increasing under extension. Although not considered by Glöckner [Glö06] or by Delgado *et al.* [DRSV14], it is not unreasonable to ask for the preservation of this property under fuzzification.

Preservation of monotonicty under conservative extension: If a semi-fuzzy quantifier Q is non-decreasing (non-increasing) under extension, then so is $\mathcal{F}(Q)$.

Expressing dualities of syllogistic reasoning, as in Aristotle's square (see, e.g., [Glö06]), requires not only the presence of a connective for negation ($\neg$), but also of antonymic quantifiers. Since we are only interested in unary quantifiers, the corresponding definition is straightforward.

Definition 44 *For any quantifier Q, its* antonym *$\mathsf{Q}^{\neg}$ is given by $v_{\mathcal{M}}(\mathsf{Q}^{\neg}xF(x)) = v_{\mathcal{M}}(\mathsf{Q}x\neg F(x))$. The* negated quantifier *$\neg\mathsf{Q}$ is given by $v_{\mathcal{M}}(\neg\mathsf{Q}xF(x)) = v_{\mathcal{M}}(\neg\mathsf{Q}xF(x))$. Moreover, the* dual quantifier *Q^{d} is defined by $v_{\mathcal{M}}(\mathsf{Q}^{d}xF(x)) = v_{\mathcal{M}}(\neg\mathsf{Q}x\neg F(x))$. In other words, the dual quantifier is the negated antonym of the given quantifier.*

Corresponding desiderata arise for a QFM $\mathcal{F}$ and semi-fuzzy quantifier Q:

Internal Negation: For all formulas F: $v_{\mathcal{M}}(\mathcal{F}(\mathsf{Q}^{\neg})xF(x)) = v_{\mathcal{M}}(\mathcal{F}(\mathsf{Q})x\neg F(x))$.

External Negation: For all formulas F: $v_{\mathcal{M}}(\mathcal{F}(^{\neg}\mathsf{Q})xF(x)) = v_{\mathcal{M}}(\neg\mathcal{F}(\mathsf{Q})xF(x))$.

Combined, these become:

Dualization: For all formulas F: $v_{\mathcal{M}}(\mathcal{F}(\mathsf{Q}^{d})xF(x)) = v_{\mathcal{M}}(\mathcal{F}(\mathsf{Q})^{d}xF(x))$.

As already indicated, dualization presupposes the existence of a unique negation operator. Glöckner tackles this problem by introducing a mechanism for deriving truth functions for propositional connectives from QFMs. While he speaks of 'canonical construction', one should emphasize that the set of truth functions preferred by Glöckner for negation, disjunction, conjunction and implication are incompatible with those of Łukasiewicz, Gödel, and Product logic[3]. More generally, Glöckner's approach to propositional connectives is incompatible with a more recent approach to deductive fuzzy logics [Háj98, CHN11, CFN15], where one starts with a (left-)continuous t-norm for conjunction, uses its residuum for implication and derives all other connectives from these in a canonical fashion.

The next definition singles out an important subclass of quantifiers.

Definition 45 *A quantifier* Q *is called* non-increasing *if, for all formulas* F, $v_{\mathcal{M}}(F(c)) \geq v_{\mathcal{M}}(F'(c))$*, for every* $c \in D$*, implies* $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{M}}(\mathsf{Q}xF'(x))$.

Note that in the case of crisp formulas $\hat{F}$ and $\hat{F}'$, the condition $v_{\mathcal{M}}(\hat{F}(c)) \geq v_{\mathcal{M}}(\hat{F}'(c))$ expresses that the extension of $\hat{F}'$ is a subset of the extension of $\hat{F}$. In this form, monotonicity of quantifiers is often discussed in linguistic literature (see, e.g., [PW06]). For example, for any constant k, the quantifier "at most k" is non-increasing, but also the vague quantifiers "nearly none" and "less than about half" are non-increasing. As to the desiderata for QFMs, we have the following:

Preservation of Monotonicity ($\geq$): If a semi-fuzzy quantifier Q is non-increasing, then so is $\mathcal{F}(\mathsf{Q})$.

The dual property is this one:

[3] Glöckner himself points out the incompatibility with Łukasiewicz logic in a footnote ([Glö06], p. 156). Clearly, the non-involutive truth function for negation, of Gödel and Product logic, is also in tension with Glöckner's axioms.

Definition 46 *A quantifier* Q *is called* non-decreasing *if, for all formulas* F, $v_{\mathcal{M}}(F(c)) \leq v_{\mathcal{M}}(F'(c))$, *for every* $c \in D$, *implies* $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{M}}(\mathsf{Q}xF'(x))$.

Accordingly, one also has the following version of Preservation of Monotonicity.

Preservation of Monotonicity ($\leq$): If a semi-fuzzy quantifier Q is non-decreasing, then so is $\mathcal{F}(\mathsf{Q})$.

The following definition reformulates monotonicity properties almost equivalently, but certainly more naturally. This is because $Prop_{\mathcal{M}}(F)$ exists for all formulas F, while the former definitions rely on particular formulas F that can be ordered with respect to the truth values they assume, when applied to elements of the domain.

Definition 47 *A quantifier* Q *is called* non-decreasing in proportion, *if for all formulas* F, G, *we have* $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \leq v_{\mathcal{M}}(\mathsf{Q}xG(x))$, *whenever* $Prop_{\mathcal{M}}F \leq Prop_{\mathcal{M}}G$. *Analogously, we call* Q non-increasing in proportion, *if* $v_{\mathcal{M}}(\mathsf{Q}xF(x)) \geq v_{\mathcal{M}}(\mathsf{Q}xG(x))$, *under the same condition.*

The following lemma follows from the fact that the condition for monotonicity in proportion is weaker than the one for ordinary monotonicity.

Lemma 2 *[BFH18] If a semi-fuzzy quantifier* Q *is non-increasing (non-decreasing) in proportion, then it is also non-increasing (non-decreasing).*

The converse direction holds for logical semi-fuzzy quantifiers; i.e. for quantitative semi-fuzzy quantifiers in the sense of Definition 41.

Lemma 3 *[BFH18] If a quantitative semi-fuzzy quantifier* Q *is non-increasing (non-decreasing), then it is also non-increasing (non-decreasing) in proportion.*

Proof:
Let $\hat{G}_1$ and $\hat{G}_2$ be two crisp formulas that fulfill $Prop_{\mathcal{M}}\hat{G}_1 \leq Prop_{\mathcal{M}}\hat{G}_2$. We define two crisp formulas $\hat{H}_1$ and $\hat{H}_2$ such that $Prop_{\mathcal{M}}\hat{H}_1 = Prop_{\mathcal{M}}\hat{G}_1$ and $Prop_{\mathcal{M}}\hat{H}_2 = Prop_{\mathcal{M}}\hat{G}_2$ and moreover $v_{\mathcal{M}}(\hat{H}_1(c)) \leq v_{\mathcal{M}}(\hat{H}_2(c))$ for all $c \in D$. Note this is always possible by introducing new monadic (crisp) predicate symbols for $\hat{H}_1$ and $\hat{H}_2$.

Since Q is quantitative and non-decreasing, it follows that $v_{\mathcal{M}}(\mathsf{Q}x\hat{H}_1(x)) \leq v_{\mathcal{M}}(\mathsf{Q}x\hat{H}_2(x))$. It therefore remains to observe, that $v_{\mathcal{M}}(\mathsf{Q}x\hat{H}_1(x)) = v_{\mathcal{M}}(\mathsf{Q}x\hat{G}_1(x))$ and $v_{\mathcal{M}}(\mathsf{Q}x\hat{H}_2(x)) = v_{\mathcal{M}}(\mathsf{Q}x\hat{G}_2(x))$, which is clear since Q is a quantitative semi-fuzzy quantifier and we have $Prop_{\mathcal{M}}\hat{H}_1 = Prop_{\mathcal{M}}\hat{G}_1$, as well as $Prop_{\mathcal{M}}\hat{H}_2 = Prop_{\mathcal{M}}\hat{G}_2$.

The case for non-increasing quantifiers works by analogy. □

Glöckner singles out QFMs that fulfill the six above mentioned desiderata ('axioms' in his terminology): Correct Generalization, Projection Quantifiers, Dualization, Preservation of Monotonicity, as well as Internal Joins and Functional Application, where the latter two are only relevant for quantifiers with more than one argument position. A QFM that satisfies these six conditions is called a *determiner fuzzification scheme (DFS)* in [Glö06]. Glöckner claims that DFSs 'capture all important aspects of systematic and coherent interpretations'. As we will see, this claim is problematic, since it neglects some features that might well be considered highly desirable, in particular from the point of view of linguistic adequateness.

Another form of preserving monotonicity is called 'Monotonicity in Quantifiers' [DHBCB04]. It is not concerned with the truth degrees of the respective argument formulas, but rather with the relative degrees of truth, resulting from different quantifiers applied to the same argument. We suggest an alternative name for the relevant property and the corresponding principle.

Definition 48 *A quantifier* Q_1 *is called* at least as strong *as a quantifier* Q_2*, in signs:* $\mathsf{Q}_1 \geq \mathsf{Q}_2$*, if for all formulas* F*, we have* $v_{\mathcal{M}}(\mathsf{Q}_1 x F(x)) \geq v_{\mathcal{M}}(\mathsf{Q}_2 x F(x))$*.*

Preservation of Quantifier Strength: If for two semi-fuzzy quantifiers $\mathsf{Q}_1, \mathsf{Q}_2$, we have $\mathsf{Q}_1 \geq \mathsf{Q}_2$, then $\mathcal{F}(\mathsf{Q}_1) \geq \mathcal{F}(\mathsf{Q}_2)$.

A further, rather natural, principle calls for a certain 'robustness' in evaluating quantified fuzzy statements. It seeks to capture the intuition that small variations, in the truth values of the (instantiated) argument formula, should only lead to small changes of the truth value of the quantified formula.

Continuity in the Argument: For any semi-fuzzy quantifier Q, the truth function of the corresponding fuzzy version $\mathcal{F}(Q)$ is *continuous*. More precisely, the following holds for all formulas F and F': for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sup_{c \in D} |v_{\mathcal{M}}(F(c)) - v_{\mathcal{M}}(F'(c))| < \delta$ gives $|v_{\mathcal{M}}(\mathcal{F}(Q) x F(x)) - v_{\mathcal{M}}(\mathcal{F}(Q) F'(x))| < \epsilon$.

Although Glöckner writes that this condition on $\mathcal{F}$ (called *arg-continuity* in [Glö06]) "is crucial to the utility" and "must be possessed by every practical model", he nevertheless does not include it in his list of axioms for determiner fuzzification schemes (DFSs), proving that he aims for a level of generality that encompasses discontinuous cases.

A further desideratum, that is called 'Coherence with Logic' in [DHBCB04], is specific to universal and existential quantification. For unary quantification, it just states that the truth functions for ∀, and for ∃, are given by the infimum and the supremum of the truth values of the argument formula. Note that, while this is obvious for crisp arguments (i.e. in ordinary classical logic), it amounts to an explicit desideratum for fuzzy (Type IV) versions of $\forall$ and $\exists$. The name 'Coherence with Logic' for this simple principle becomes understandable only if one considers binary quantifiers and additionally requires that binary universal and existential quantification be reduced to the unary case by strict analogy to classical logic, using implication and conjunction, respectively. In our context, it is better to talk of 'Supremum/Infimum Principle'.

Supremum/Infimum Principle: For all formulas F, we have:
$v_{\mathcal{M}}(\mathcal{F}(\forall)xF(x)) = \inf_{c\in D} v_{\mathcal{M}}(F(c))$ and $v_{\mathcal{M}}(\mathcal{F}(\exists)xF(x)) = \sup_{c\in D} v_{\mathcal{M}}(F(c))$.

We emphasize that the above list of desiderata for fuzzification mechanisms is not exhaustive. Here, we restrict attention to unary quantification. This renders desiderata for Glöckner's axioms for "internal joins", "functional application", and "argument insertion" [Glö06] irrelevant to our context. But even for unary quantifiers, further principles might be relevant, at least for particular application scenarios. In fact, certain more general methodological principles for the design of fuzzy quantifier models should be respected as well. Most importantly, such models should be embeddable into deductive, t-norm based fuzzy logics, as intensively studied in contemporary Mathematical Fuzzy Logic [CHN11]. In particular, the models should be compatible with Łukasiewicz logic *Ł*, which is distinguished among all the t-norm based fuzzy logic as the only one, where all truth functions of connectives and quantifiers are continuous.[4] That (a properly extended) Łukasiewicz logic should indeed be considered as a distinguished basis for modeling reasoning with vague notions, including quantifier expressions, has been argued, e.g., by Novak [Nov06]. Here, we aim at (fully-fuzzy) quantifiers that can actually be defined within the logic $Ł_\alpha(\Pi)$[5]. The QFMs for lifting semi-fuzzy quantifiers to fully-fuzzy ones, which we introduce in the next section, do not directly depend on the $\mathcal{NRG}$-game, or $Ł_\alpha(\Pi)$ respectively. However, they are partly inspired by this semantic framework.

5.2 Fuzzification mechanisms in the limelight

Quantifier fuzzification mechanisms (QFMs) are a powerful tool to arrive at fully-fuzzy quantification. One starts with a semi-fuzzy quantifier Q, that takes only crisp arguments, but, upon evaluation, may yield intermediate truth values, and associates to Q a fully-fuzzy quantifier $\mathcal{F}(\mathsf{Q})$, to be evaluated over arbitrary fuzzy argument. For our considerations of this section, we always assume the full expressive power of $Ł_\alpha(\Pi)$.

In this section, we will introduce some concrete QFMs and discuss their properties. To make a semi-fuzzy quantifier Q fit for fuzzy arguments F, we need to systematically

[4] In [MN12, MN14, Nov08] the authors introduce an alternative generalization based on fuzzy type theory, which extends Łukasiewicz logic to a higher order setting.

[5] In the remainder of this chapter, we will in particular sometimes point out if representations of quantifiers work even in less expressive settings than $Ł_\alpha(\Pi)$.

modify the way F is interpreted. One well known way to do so, is considering α-cuts. Here, the α-cut of a formula F is another (crisp) formula F'. If the truth value of F is at least α, then F' is equivalent to $\top$, and if the truth value of F is less than α, F' is equivalent to $\bot$. This method has been used for the evaluation of fuzzy quantifiers, e.g. in [BF17, DHBCB04]. It can be defined as follows:

Definition 49 *Let a formula F and $\alpha \in [0,1]$ be given. Furthermore, let $p_\alpha \in \Lambda$ be given such that $v_\mathcal{M}(p_\alpha) = \alpha$. Then the α-cut of F is defined as $F^{\geq p_\alpha} = \Delta(p_\alpha \to F)$.*

One can also define an alternative way to project F to an either fully true or fully false formula, namely by applying α-cuts to all atoms that F is built from.

Definition 50 *Let a formula F and $\alpha \in [0,1]$ be given. Furthermore, let $p_\alpha \in \Lambda$ be given such that $v_\mathcal{M}(p_\alpha) = \alpha$. Then the* at-$\alpha$*-cut of F is defined as the formula $F_{at}^{\geq p_\alpha}$ which is obtained from F by replacing each atom A in it with $\Delta(p_\alpha \to A)$.*

More generally, we introduce the following notation for later use:

Definition 51 *Let $F(x)$ and G be formulas. We denote by $F^{\geq G}(x)$ the formula $\Delta(G \to F(x))$ and by $F_{at}^{\geq G}(x)$ the formula obtained replacing any atomic formula A in $F(x)$ by $\Delta(G \to A)$. To simplify the notation, we will denote the formulas $F^{\geq F(y)}(x)$ and $F^{\geq F(c)}(x)$ by $F^{\geq y}(x)$ and $F^{\geq c}(x)$, respectively.*

Remark 23 *Note that we do not explain the lifting from Type I quantifiers to Type II quantifiers independently, as Type I quantifiers technically are also Type III quantifiers and Type II quantifiers technically are also Type IV quantifiers.*

5.2.1 Fuzzification mechanisms based on random α-cuts: $\mathcal{F}^{R_1}, \mathcal{F}^{R_2}$

Our first QFM, based on a random choice of at-α-cuts, therefore denoted as $\mathcal{F}^{R_1}$, is defined as follows:

Definition 52 *For a formula F, $p_\alpha \in \Lambda$ such that $v_\mathcal{M}(p_\alpha) = \alpha$, and a semi-fuzzy quantifier Q, we define $\mathcal{F}^{R_1}$ as follows:*

$$v_\mathcal{M}(\mathcal{F}^{R_1}(\mathsf{Q})xF(x)) = \int_0^1 v_\mathcal{M}(\mathsf{Q}xF_{at}^{\geq p_\alpha}(x))d\alpha. \tag{5.1}$$

In other words, $v_\mathcal{M}(\mathcal{F}^{R_1}(\mathsf{Q})xF(x))$ is the expected value of $v_\mathcal{M}(\mathsf{Q}xF_{at}^{\geq p_\alpha}(x))$ with respect to a uniform random choice of some $\alpha \in [0,1]$.

$\mathcal{F}^{R_1}$ might look tempting as a mechanism, but, as the following example shows, it fails to comply with the desideratum of "correct generalization", which demands that $\mathcal{F}^{R_1}(\mathsf{Q})$, applied to crisp arguments, yield the same result as the semi-fuzzy quantifier Q applied to the same crisp argument:

Example 8 *Assume* $D = \{c_1, c_2\}$ *with* $v_{\mathcal{M}}(A(c_1)) = 0.2$ *and* $v_{\mathcal{M}}(A(c_2)) = 1$ *and let* $\hat{F}(x) = \Delta A(x)$. *Now,*

$v_{\mathcal{M}}(\forall x \hat{F}(x)) = 0$. *And, for some small value* $\epsilon > 0$, *we also have:*

$v_{\mathcal{M}}(\mathcal{F}^{R_1}(\forall)x\hat{F}(x)) = 0.2 \cdot v_{\mathcal{M}}(\forall x \hat{F}_{at}^{\geq p_{0.2}}(x)) + 0.8 \cdot v_{\mathcal{M}}(\forall x \hat{F}_{at}^{\geq p_{0.2+\epsilon}}(x)) =$
$= 0.2 \cdot 1 + 0.8 \cdot 0 = 0.2 \neq 0.$

A different fuzzification mechanism, in particular related to the so called Choquet integral [DRSV14], is obtained by directly (uniformly randomly) picking α-cuts of the entire argument F. We define this second QFM, based on randomly choosing α-cuts, $\mathcal{F}^{R_2}$, as follows:

Definition 53 *For a formula* F, $p_\alpha \in \Lambda$ *such that* $v_{\mathcal{M}}(p_\alpha) = \alpha$, *and a semi-fuzzy quantifier* Q, *we define* $\mathcal{F}^{R_2}$ *as follows:*

$$v_{\mathcal{M}}(\mathcal{F}^{R_2}(\mathsf{Q})xF(x)) = \int_0^1 v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))d\alpha. \tag{5.2}$$

The same evaluation function and corresponding lifting mechanism for fuzzy quantifiers is also obtained in [DHBCB04], although motivated by a different semantics, based on voting models. The sampling of threshold values from the real unit interval, which is needed for the definition of $\mathcal{F}^{R_1}$ and $\mathcal{F}^{R_2}$, can be expressed by way of our propositional quantifier $\mathbf{\Pi}$, based on the object quantifier Π.

Theorem 37 *[BFH18] For a semi-fuzzy quantifier* Q, *and a formula* F, *we have:*

$$\mathcal{F}^{R_1}(\mathsf{Q})xF(x) \equiv \mathbf{\Pi}p\mathsf{Q}xF_{at}^{\geq p}(x), \tag{5.3}$$
$$\mathcal{F}^{R_2}(\mathsf{Q})xF(x) \equiv \mathbf{\Pi}p\mathsf{Q}xF^{\geq p}(x). \tag{5.4}$$

Proof:
Straightforward computation. □

Hence, both fuzzification mechanisms $\mathcal{F}^{R_1}$ and $\mathcal{F}^{R_2}$ are expressible in $Ł_\alpha(\Pi)$. As we assume Q to be a unary semi-fuzzy quantifier, the QFM $\mathcal{F}^{R_2}$ coincides with both, the maximum dependence model, and the independence model from [DHBCB04].

We can now examine to which extend the fuzzification mechanism $\mathcal{F}^{R_2}$ complies with the desiderata in Section 5.1.

Theorem 38 *[BFH18] The QFM $\mathcal{F}^{R_2}$ complies with* correct generalization*,* projection quantifiers*,* preservation of quantitativity*,* preservation of monotonicity under conservative extension*,* internal negation*,* external negation*,* dualization*,* preservation of monotonicity*,* preservation of quantifier strength*,* continuity in the argument*, and the* supremum/infimum principle*.*

Proof:
As we only consider unary quantifiers, in our models the distinction between maximum dependence and independence regarding [DHBCB04] collapses. Hence, we can use their results, which show correct generalization, internal negation, external negation, dualization, preservation of monotonicity, preservation of quantifier strength, continuity in the argument, and the supremum/infimum principle. Let $p_\alpha \in \Lambda$ such that $v_\mathcal{M}(p_\alpha) = \alpha$.

- Projection Quantifiers: We show that for an arbitrary formula F, and all $c \in D$, we have:

  $v_\mathcal{M}(\mathcal{F}^{R_2}(\Delta_c)xF(x)) = v_\mathcal{M}(\mathbf{\Pi}p\Delta_c F^{\geq p}(x)) = v_\mathcal{M}(\mathbf{\Pi}pF^{\geq p}(c)) = v_\mathcal{M}(F(c)).$

- Preservation of quantitativity: This is apparent.

- Preservation of monotonicity under conservative extension (non-decreasing):

  Let Q be a semi-fuzzy quantifier and assume $\mathcal{M}'$ conservatively extends $\mathcal{M}$, and that for all crisp formulas $\hat{F}$ we have that $v_\mathcal{M}(\mathsf{Q}x\hat{F}(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}x\hat{F}(x))$. We then have that $v_\mathcal{M}(\mathsf{Q}xF^{\geq p_\alpha}(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}xF^{\geq p_\alpha}(x))$, for all $\alpha \in [0,1]$. As a consequence, $v_\mathcal{M}(\mathbf{\Pi}p\mathsf{Q}xF^{\geq p}(x)) \leq v_{\mathcal{M}'}(\mathbf{\Pi}p\mathsf{Q}xF^{\geq p}(x))$.

- Preservation of monotonicity under conservative extension (non-increasing):

  Let Q be a semi-fuzzy quantifier and assume $\mathcal{M}$ conservatively extends $\mathcal{M}'$, and that for all crisp formulas $\hat{F}$ we have that $v_\mathcal{M}(\mathsf{Q}x\hat{F}(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}x\hat{F}(x))$. We then have that $v_\mathcal{M}(\mathsf{Q}xF^{\geq p_\alpha}(x)) \leq v_{\mathcal{M}'}(p)\mathsf{Q}xF^{\geq p_\alpha}(x)$, for all $\alpha \in [0,1]$. As a consequence, $v_\mathcal{M}(\mathbf{\Pi}p\mathsf{Q}xF^{\geq p}(x)) \leq v_{\mathcal{M}'}(\mathbf{\Pi}p\mathsf{Q}xF^{\geq p}(x))$.

□

This already tells us that $\mathcal{F}^{R_1}$ and $\mathcal{F}^{R_2}$ cannot coincide, as $\mathcal{F}^{R_1}$ does not comply with the desideratum of correct generalization. We provide an explicit counterexample that shows that they are different, thus correcting what has been claimed in [BF17].

Corollary 5 *[BFH18] $\mathcal{F}^{R_1}$ and $\mathcal{F}^{R_2}$ do not coincide.*

Proof:
For simplicity, we assume the domain has only one element $D = \{c\}$, and Q is either $\forall, \exists$ or Π. $A(x)$ and $B(x)$ are fuzzy atoms, with $a = v_{\mathcal{M}}(A(c)) \leq v_{\mathcal{M}}(B(c)) = b$. Then, for $p_\alpha \in \Lambda$ such that $v_{\mathcal{M}}(p_\alpha) = \alpha$, we look at $F(x)$ defined as $A(x) \oplus B(x)$, and plug it into the QFMs:

$$v_{\mathcal{M}}(\mathcal{F}^{R_1}(\mathsf{Q})xF(x)) = v_{\mathcal{M}}(\mathbf{\Pi}p(A(c)^{\geq p} \oplus B(c)^{\geq p})) =$$
$$= \int_0^1 v_{\mathcal{M}}(A(c)^{\geq p_\alpha} \oplus B(c)^{\geq p_\alpha})d\alpha = \int_0^a 1d\alpha + \int_a^b 1d\alpha + \int_b^1 0d\alpha = b.$$

$$v_{\mathcal{M}}(\mathcal{F}^{R_2}(\mathsf{Q})xF(x)) = v_{\mathcal{M}}(\mathbf{\Pi}p(A(c) \oplus B(c))^{\geq p}) =$$
$$= \int_0^1 v_{\mathcal{M}}((A(c) \oplus B(c))^{\geq p_\alpha})d\alpha = \int_0^{a+b} 1d\alpha + \int_{a+b}^1 0d\alpha = \min(1, a+b).$$

Since b and $\min(1, a+b)$ are not generally equal, the claim follows. $\square$

Example 9 *We assume a domain D such that $\mid D \mid= 4$, and consider the Type III quantifier Π. We further assume that the four objects from the domain represent balls and that there are two fully black ones, i.e., for a fuzzy predicate B, standing for the property of being black, we have $v_{\mathcal{M}}(B(c_1)) = v_{\mathcal{M}}(B(c_2)) = 1$. The other two are gray, or putting it differently, black to a certain degree. We label those two gray balls c_3 and c_4, with $v_{\mathcal{M}}(B(c_3)) = 0.9$ and $v_{\mathcal{M}}(B(c_4)) = 0.7$.*

Game semantically, the evaluation of $\mathcal{F}^{R_2}(\Pi)xB(x)$, based on a randomly sampled threshold value, can be interpreted as follows. Nature *samples a constant p_α for the propositional variable p, and the proponent **P**, has to accept the payoff associated to asserting $\Pi x B^{\geq p_\alpha}(x)$. This payoff corresponds to the (possibly) intermediate truth value, as Π is a semi-fuzzy quantifier. Then, the overall truth value is the average of the individual results. The result can be obtained straightforwardly and amounts to the truth value 0.9.*

It is worth noting that one can also immediately apply the quantifier Π to fuzzy arguments, i.e. regard it as Type IV quantifier. In this case we also get $v_{\mathcal{M}}(\Pi x B(x)) = 0.9$. It is straightforward to check that for all formulas F we have that $v_{\mathcal{M}}(\mathcal{F}^{R_2}(\Pi)xF(x)) = v_{\mathcal{M}}(\Pi x F(x))$ (telescopic sum).

5.2.2 Fuzzification mechanisms based on optimized α-cuts: $\mathcal{F}^{P_1}, \mathcal{F}^{P_2}$

We now discuss another QFM, which we call $\mathcal{F}^{P_1}$. Again, one thinks of the fuzzy argument F, of some semi-fuzzy quantifier Q, as being cut off at some level $\alpha \in [0, 1]$. Formulas with truth values of at least α are rendered true while all others are projected to false. The final truth value of the fully-fuzzy quantified statement is obtained by choosing an optimal threshold α which maximizes the truth value of the semi-fuzzy quantifier over the corresponding precisification, in conjunction with the threshold value α itself. In principle any combination of t-norm and t-conorm can be employed for the conjunction and the optimization: if we use the t-norm *min* and the t-conorm *max* we get a method based on the Sugeno integral, also called the "possibilistic method" in [DRSV14], which is why we call the QFM $\mathcal{F}^{P_1}$. Formally we define:

Definition 54 *Let F be a formula, $p_\alpha \in \Lambda$ such that $v_{\mathcal{M}}(p_\alpha) = \alpha$, and Q a semi-fuzzy quantifier. We define:*

$$v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q})xF(x)) = \sup_{\alpha\in[0,1]} \min(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))). \tag{5.5}$$

Example 10 *Let us get back to the setting of Example 9. From a game semantic perspective, we can understand the functioning of $\mathcal{F}^{P_1}$ as follows. If we evaluate $\mathcal{F}^{P_1}(\Pi)xB(x)$, the proponent **P**, has to decide how many of the gray balls are accepted as black balls. If the threshold value is chosen as 0.9, 3 of the 4 balls are accepted as black. If **P** also accepts the last ball, c_4, as black, she would have all 4 balls to qualify as black, but the conjunction in $\mathcal{F}^{P_1}$ with the threshold value 0.7 would make that a non-rational move. Hence, $v_{\mathcal{M}}(\mathcal{F}^{P_1}(\Pi)xB(x)) = 0.75$.*

We will now introduce a useful lemma, that will turn out to be helpful to express $\mathcal{F}^{P_1}$ within $Ł_\alpha(\Pi)$.

Lemma 4 *Let $\mathsf{Q}xF(x)$ be a semi-fuzzy quantified formula, $\alpha_1 \leq \alpha_2 \ldots, \leq \alpha_n$ the n (not necessarily different) truth values taken by $v_{\mathcal{M}}(F(c_i))$, $c_i \in D = \{c_1, \ldots, c_n\}$ constants (domain elements), such that $\alpha_i = v_{\mathcal{M}}(F(c_i))$. For simplicity, slightly abusing notation, we also let $F(c_0)$ stand for $\bot$ and $F(c_{n+1})$ for $\top$. Furthermore, $p_\alpha \in \Lambda$ such that $v_{\mathcal{M}}(p_\alpha) = \alpha$. Then we have:*

$$(a) \quad v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q})xF(x)) = \max_{i=1,\ldots,n+1} \min(v_{\mathcal{M}}(F(c_i)), v_{\mathcal{M}}(\mathsf{Q}xF^{\geq c_i}(x))). \tag{5.6}$$

Moreover, if Q is non-decreasing, we have:

$$(b) \quad v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q})xF(x)) = \min_{i=1,\ldots,n+1} \max(v_{\mathcal{M}}(F(c_{i-1}), v_{\mathcal{M}}(\mathsf{Q}xF^{\geq c_i}(x))), \text{ and} \tag{5.7}$$

$$(c) \quad v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q})xF(x)) = \inf_{\alpha\in[0,1]} \max(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))). \tag{5.8}$$

Proof:
By the definition of the α_i and α-cuts, it follows that, for any α in $(\alpha_{i-1}, a_i]$, we get $v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x)) = v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_{\alpha_i}}(x))$.

(a): Hence, letting $\alpha_0 = 0, \alpha_{n+1} = 1$, we obtain:

$\sup_{\alpha\in(\alpha_{i-1},a_i]} \min(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))) =$

$= \min(\sup_{\alpha \in (\alpha_{i-1}, a_i]} \alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_{\alpha_i}}(x)) \ ) =$

$= \min(\alpha_i, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_{\alpha_i}}(x)))$.

From this we get:

$v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q})xF(x)) = \sup_{\alpha \in [0,1]} \min(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))) =$

$= \max_{i=1,\ldots,n+1} \sup_{\alpha \in (\alpha_{i-1}, a_i]} \min(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))) =$

$= \max_{i=1,\ldots,n+1} \min(\alpha_i, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_{\alpha_i}}(x))) =$

$= \max_{i=1,\ldots,n+1} \min(v_{\mathcal{M}}(F(c_i)), v_{\mathcal{M}}(\mathsf{Q}xF^{\geq c_i}(x)))$.

(b): For the second claim, it suffices to show that:

$$\bigwedge_{i=1}^{n+1} (F(c_{i-1}) \vee \mathsf{Q}xF^{\geq c_i}(x)) \equiv \bigvee_{i=1}^{n+1} (F(c_i) \wedge \mathsf{Q}xF^{\geq c_i}(x)).$$

$\bigwedge_{i=1}^{n+1}(F(c_{i-1}) \vee \mathsf{Q}xF^{\geq c_i}(x)) \equiv$

$\equiv \mathsf{Q}xF^{\geq c_1}(x) \wedge (F(c_1) \vee \mathsf{Q}xF^{\geq c_2}(x)) \wedge \bigwedge_{i=3}^{n+1}(F(c_{i-1}) \vee \mathsf{Q}xF^{\geq c_i}(x)).$

We now note, since Q is non-decreasing, that:

$\mathsf{Q}xF^{\geq c_1}(x) \wedge (F(c_1) \vee \mathsf{Q}xF^{\geq c_2}(x)) \equiv (F(c_1) \wedge \mathsf{Q}xF^{\geq c_1}(x)) \vee \mathsf{Q}xF^{\geq c_2}(x).$

Since:

$(F(c_1) \wedge \mathsf{Q}xF^{\geq c_1}(x)) \wedge \bigwedge_{i=3}^{n+1}(F(c_{i-1}) \vee \mathsf{Q}xF^{\geq c_i}(x)) \equiv F(c_1) \wedge \mathsf{Q}xF^{\geq c_1}(x),$

we get:

$(\mathsf{Q}xF^{\geq c_1}(x) \wedge (F(c_1) \vee \mathsf{Q}xF^{\geq c_2}(x))) \wedge \bigwedge_{i=3}^{n+1}(F(c_{i-1}) \vee \mathsf{Q}xF^{\geq c_i}(x)) \equiv$

$\equiv ((F(c_1) \wedge \mathsf{Q}xF^{\geq c_1}(x)) \vee \mathsf{Q}xF^{\geq c_2}(x)) \wedge \bigwedge_{i=3}^{n+1}(F(c_{i-1}) \vee \mathsf{Q}xF^{\geq c_i}(x)) \equiv$

$\equiv (F(c_1) \wedge \mathsf{Q}xF^{\geq c_1}(x)) \vee (\mathsf{Q}xF^{\geq c_2}(x) \wedge \bigwedge_{i=3}^{n+1}(F(c_{i-1}) \vee \mathsf{Q}xF^{\geq c_i}(x))).$

Repeating this transformation yields the desired result.

(c): For the last claim, by analogy with (a), we have:

$\inf_{\alpha \in (\alpha_{i-1}, a_i]} \max(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))) =$

$= \max(\inf_{\alpha \in (\alpha_{i-1}, a_i]} \alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_{\alpha_i}}(x)) =$

$= \max(\alpha_{i-1}, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_{\alpha_i}}(x))).$

Hence, using (b):

$v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q})xF(x)) = \min_{i=1,\dots,n+1} \max(v_{\mathcal{M}}(F(c_{i-1}), v_{\mathcal{M}}(\mathsf{Q}xF^{\geq c_i}(x)) =$
$= \min_{i=1,\dots,n+1} \max(\alpha_{i-1}, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_{\alpha_i}}(x))) =$
$= \min_{i=1,\dots,n+1} \inf_{\alpha\in(\alpha_{i-1},a_i]} \max(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))) =$
$= \inf_{\alpha\in[0,1]} \max(\alpha, v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x))).$ □

The QFM $\mathcal{F}^{P_1}$ can be expressed in more than one way. Firstly, within $Ł_\alpha(\Pi)$, employing the propositional quantifier ∃. As an alternative, and more basic way to represent $\mathcal{F}^{P_1}$, one can even use only $Ł_\triangle$ (Łukasiewicz logic enriched with the Delta operator) at the price of a somewhat more complex representation.

Theorem 39 *[BFH18] Let* Q *be a semi-fuzzy quantifier and* F *a formula. Then we have:*

- $\mathcal{F}^{P_1}(\mathsf{Q})xF(x) \equiv \exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))$
- $\mathcal{F}^{P_1}(\mathsf{Q})xF(x) \equiv \exists y(F(y) \wedge \mathsf{Q}xF^{\geq y}(x)) \vee \mathsf{Q}x\Delta F(x)$

Proof:
The first claim follows from Definition 54, while the second is a direct consequence of Lemma 4. □

Theorem 40 *[BFH18] The QFM* $\mathcal{F}^{P_1}$ *complies with the desiderata of* correct generalization*,* projection quantifiers*,* preservation of quantitativity*,* preservation of monotonicity under conservative extension*,* preservation of monotonicity*,* preservation of quantifier strength*, and the* supremum/infimum principle*.*

Proof:

- Correct generalization works, as crisp formulas have truth values that are either always greater or equal than any threshold value, namely when they are true, or smaller or equal than any, when they are false, hence:

  $v_{\mathcal{M}}(\exists p(p \wedge \mathsf{Q}x\Delta(p \to \hat{F}(x)))) = v_{\mathcal{M}}(\mathsf{Q}x\Delta(\top \to \hat{F}(x)) = v_{\mathcal{M}}(\mathsf{Q}x\hat{F}(x)).$

- Projection Quantifiers: We show that for an arbitrary formula F, and all $c \in D$, we have:

  $v_{\mathcal{M}}(\mathcal{F}^{P_1}(\Delta_c xF(x))) = v_{\mathcal{M}}(\exists p(p \wedge \Delta(p \to F(c)))) = v_{\mathcal{M}}(F(c)).$

- Preservation of quantitativity: This is apparent.

- Preservation of monotonicity under conservative extension (non-decreasing):

  Let Q be a semi-fuzzy quantifier and assume $\mathcal{M}'$ conservatively extends $\mathcal{M}$, as well as for all crisp formulas $\hat{F}$ we have that $v_{\mathcal{M}}(\mathsf{Q}x\hat{F}(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}x\hat{F}(x))$. We then have that $v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}xF^{\geq p_\alpha}(x))$, for all $\alpha \in [0,1]$. Hence, we also have $v_{\mathcal{M}}(p_\alpha \wedge \mathsf{Q}xF^{\geq p_\alpha}(x)) \leq v_{\mathcal{M}'}(p_\alpha \wedge \mathsf{Q}xF^{\geq p_\alpha}(x))$, for all $\alpha \in [0,1]$. As a consequence, $v_{\mathcal{M}}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))) \leq v_{\mathcal{M}'}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x)))$.

- Preservation of monotonicity under conservative extension (non-increasing):

  Let Q be a semi-fuzzy quantifier and assume $\mathcal{M}$ conservatively extends $\mathcal{M}'$, as well as for all crisp formulas $\hat{F}$ we have that $v_{\mathcal{M}}(\mathsf{Q}x\hat{F}(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}x\hat{F}(x))$. We then have that $v_{\mathcal{M}}(\mathsf{Q}xF^{\geq p_\alpha}(x)) \leq v_{\mathcal{M}'}(\mathsf{Q}xF^{\geq p_\alpha}(x))$, for all $\alpha \in [0,1]$. Hence, we also have $v_{\mathcal{M}}(p_\alpha \wedge \mathsf{Q}xF^{\geq p_\alpha}(x)) \leq v_{\mathcal{M}'}(p_\alpha \wedge \mathsf{Q}xF^{\geq p_\alpha}(x))$, for all $\alpha \in [0,1]$. As a consequence, $v_{\mathcal{M}}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))) \leq v_{\mathcal{M}'}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x)))$.

- Preservation of monotonicity:

  Let Q be a non-decreasing semi-fuzzy quantifier. Furthermore, let F and G be two arbitrary formulas such that for all $c \in D$ we have that $v_{\mathcal{M}}(F(c)) \leq v_{\mathcal{M}}(G(c))$. Then we have:

  $$v_{\mathcal{M}}(\mathcal{F}(\mathsf{Q})xF(x)) = v_{\mathcal{M}}(\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x))) \leq$$
  $$\leq v_{\mathcal{M}}(\exists p(p \wedge \mathsf{Q}xG^{\geq p}(x))) = v_{\mathcal{M}}(\mathcal{F}(\mathsf{Q})xG(x)).$$

  The case where Q is non-increasing works analogously.

- Preservation of quantifier strength:

  If, for two semi-fuzzy quantifiers $\mathsf{Q}_1, \mathsf{Q}_2$, and any crisp formula $\hat{F}$, we have $v_{\mathcal{M}}(\mathsf{Q}_1x\hat{F}(x)) \leq v_{\mathcal{M}}(\mathsf{Q}_2x\hat{F}(x))$, then, for any fuzzy formula F we also have:

  $$v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q}_1)xF(x)) = v_{\mathcal{M}}(\exists p(p \wedge \mathsf{Q}_1xF^{\geq p}(x))) \leq v_{\mathcal{M}}(\exists p(p \wedge \mathsf{Q}_2xF^{\geq p}(x))) =$$
  $$= v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q}_2)xF(x)).$$

  The case where $v_{\mathcal{M}}(\mathsf{Q}_1x\hat{F}(x)) \geq v_{\mathcal{M}}(\mathsf{Q}_2x\hat{F}(x))$ works analogously.

- Supremum/Infimum principle:

  $$v_{\mathcal{M}}(\mathcal{F}(\forall)xF(x)) = \exists p(p \wedge \forall xF^{\geq p}(x)) = \inf_{c \in D} v_{\mathcal{M}}(F(c)),$$

  $$v_{\mathcal{M}}(\mathcal{F}(\exists)xF(x)) = \exists p(p \wedge \exists xF^{\geq p}(x)) = \sup_{c \in D} v_{\mathcal{M}}(F(c)).$$

□

Example 11 *Let us fix the domain D such that $\mid D \mid = 2$. Regarding Q, we use the quantifier "at least 50%", modeled as Type III quantifier, which is non-decreasing in proportion and denoted by $\mathsf{Q}_{[\geq \frac{1}{2}]}$, with $v_{\mathcal{M}}(\mathsf{Q}_{[\geq \frac{1}{2}]} x \hat{F}(x)) = \min(1, 2 \cdot Prop_{\mathcal{M}} \hat{F})$, for a crisp formula $\hat{F}$. Also, we assume that there are two fuzzy predicates A and B, with the following truth value distribution: $v_{\mathcal{M}}(A(c_1)) = 0.7$ and $v_{\mathcal{M}}(A(c_2)) = 0.1$, and $v_{\mathcal{M}}(B(c_1)) = v_{\mathcal{M}}(B(c_2)) = 0.5$. Hence, $Prop_{\mathcal{M}} A = 0.4 < 0.5 = Prop_{\mathcal{M}} B$. This means the proportion of objects fulfilling A is lower than the one of B. Still, $v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q}_{[\geq \frac{1}{2}]}) x A(x))$, which is 0.7, is greater than $v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q}_{[\geq \frac{1}{2}]}) x B(x))$, which is 0.5. That means that Lemma 3 can not be extended to fully-fuzzy quantifiers and that the property of monotonicity in proportion is not preserved by $\mathcal{F}^{P_1}$.*

A statement about the QFM $\mathcal{F}^{P_1}$, but only valid for Type I quantifiers which are non-decreasing in proportion, is that it coincides with the QFM $\mathcal{F}^{R_2}$. This is expressed by the following theorem. Hence, the compliance with many of the desiderata that were discussed in the beginning of the chapter follows also for $\mathcal{F}^{P_1}$. This is captured by Corollary 6.

Theorem 41 *[BFH18] For a formula F, and Type I quantifier Q which is non-decreasing in proportion we have:*

$$\mathcal{F}^{P_1}(\mathsf{Q}) x F(x) \equiv \mathcal{F}^{R_2}(\mathsf{Q}) x F(x). \tag{5.9}$$

Proof:
For $D = \{c_1, \ldots, c_n\}$, we assume the following truth value distribution of F:

$0 = t_0 \leq v_{\mathcal{M}}(F(c_1)) = t_1 \leq v_{\mathcal{M}}(F(c_2)) = t_2 \leq \cdots \leq v_{\mathcal{M}}(F(c_n)) = t_n \leq 1 = t_{n+1}$.

The assumption of the theorem tells us that there is a value $i_0 \in \{0, 1, \ldots, n\}$ such that for any crisp formula $\hat{F}$ we have, if $Prop_{\mathcal{M}}(\hat{F}) \geq \frac{i_0}{n}$ then we also have $v_{\mathcal{M}}(\mathsf{Q} x \hat{F}(x)) = 1$ (as long as Q is not constantly equivalent to $\perp$. However, that case is trivial.). We therefore define the set $I_{\mathsf{Q}} = \{i_0, \ldots, n\}$. Also, recall the indicator function $\mathbb{I}_{\{x \in A\}}$, which is 1, if $x \in A$ and 0 otherwise.

(1): $v_{\mathcal{M}}(\mathcal{F}^{R_2}(\mathsf{Q}) x F(x)) = \sum_{j=0}^{n} (t_{j+1} - t_j) \cdot \mathbb{I}_{\{n-j \in I_{\mathsf{Q}}\}}$, and

(2): $v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q}) x F(x)) = \max_{j=1,\ldots,n+1}(\min(t_j, \mathbb{I}_{\{n-(j-1) \in I_{\mathsf{Q}}\}}))$.

For (1), the indicator function is 1, if $n - j \geq i_0$, which is equivalent to $j \leq n - i_0$, and for (2) the indicator function is 1, if $n - (j-1) \geq i_0$, which is equivalent to $j \leq n - i_0 + 1$. Hence:

(1): $v_{\mathcal{M}}(\mathcal{F}^{R_2}(\mathsf{Q})xF(x)) = \sum_{j=0}^{n-i_0}(t_{j+1} - t_j) = t_{n-i_0+1}$, and

(2): $v_{\mathcal{M}}(\mathcal{F}^{P_1}(\mathsf{Q})xF(x)) = t_{n-i_0+1}$. $\square$

Corollary 6 *[BFH18] The QFM $\mathcal{F}^{P_1}$ complies with all desiderata mentioned in Theorem 38, for Type I quantifiers that are non-decreasing.*

Proof:
As $\mathcal{F}^{R_2}$ and $\mathcal{F}^{P_1}$ coincide for Type I quantifiers which are non-decreasing in proportion, the claim follows from Theorem 38 and Theorem 41 and Lemma 2. $\square$

The corollary cannot be extended to the general class of Type I quantifiers, as we show in the following example. Also, it shows that $\mathcal{F}^{P_1}$ does not comply with the desideratum of continuity in the argument.

Example 12 *Let us consider the Type I quantifier "none", $\forall^{\neg}$, which has the following truth function: $v_{\mathcal{M}}(\forall^{\neg}x\hat{F}(x)) = 1$, if for all constants c we have that $v_{\mathcal{M}}(\hat{F}(c)) = 0$, and $v_{\mathcal{M}}(\forall^{\neg}x\hat{F}(x)) = 0$ otherwise. Let us further assume we have $D = \{c\}$, with $v_{\mathcal{M}}(F(c)) = 0.9$. We then have:*

$v_{\mathcal{M}}(\mathcal{F}^{P_1}(\forall^{\neg})xF(x)) = 1$, *and* $v_{\mathcal{M}}(\mathcal{F}^{R_2}(\forall^{\neg})xF(x)) = 0.1 \cdot 1 + 0.9 \cdot 0 = 0.1$.

On the one side, $\mathcal{F}^{P_1}$ recognizes that there are no fully true witnesses for the argument F, which is fairly rational, especially when crisp values play a predominant role. On the other side, the truth function determined by $\mathcal{F}^{P_1}$ is not continuous in general. Indeed, even witnesses with truth value (with respect to the respective formula F) arbitrarily close to 1 are excluded. On the other hand, $\mathcal{F}^{R_2}$, as we have seen, preserves continuity in the argument instead.

We now turn to the desideratum of dualization, which is fulfilled by the QFM $\mathcal{F}^{P_1}$ for the subclass of quantifiers, which are non-decreasing. Before that, we prove a useful property of dual quantifiers.

Lemma 5 *[BFH18] If Q is a semi-fuzzy quantifier which is non-decreasing, then so is its dual Q^d.*

Proof:
We assume that Q is non-decreasing, i.e. for any two crisp formulas $\hat{F}_1, \hat{F}_1$ with $v_{\mathcal{M}}(\hat{F}_1(c)) \leq v_{\mathcal{M}}(\hat{F}_2(c))$ for all $c \in D$, we have that $v_{\mathcal{M}}(\mathsf{Q}x\hat{F}_1(x)) \leq v_{\mathcal{M}}(\mathsf{Q}x\hat{F}_2(x))$.

Now, let $\hat{G}_1, \hat{G}_2$ be two arbitrary crisp formulas with $v_{\mathcal{M}}(\hat{G}_1(c)) \leq v_{\mathcal{M}}(\hat{G}_2(c))$ for all $c \in D$. Then the following lines are all equivalent:

$v_{\mathcal{M}}(\mathsf{Q}^d x \hat{G}_1(x)) \leq v_{\mathcal{M}}(\mathsf{Q}^d x \hat{G}_2(x))$

$v_{\mathcal{M}}(\neg \mathsf{Q} x \neg \hat{G}_1(x)) \leq v_{\mathcal{M}}(\neg \mathsf{Q} x \neg \hat{G}_2(x))$

$v_{\mathcal{M}}(\mathsf{Q} x \neg \hat{G}_1(x)) \geq v_{\mathcal{M}}(\mathsf{Q} x \neg \hat{G}_2(x))$

Since we also have that $v_{\mathcal{M}}(\neg \hat{G}_1(c)) \geq v_{\mathcal{M}}(\neg \hat{G}_2(c))$ for all $c \in D$, the claim follows. □

Theorem 42 *[BFH18] The QFM $\mathcal{F}^{P_1}$ complies with the desideratum of dualization for semi-fuzzy quantifiers* Q *which are non-decreasing.*

Proof:
Recall that by Theorem 39 and the definition of dual quantifiers we have:

$\mathcal{F}^{P_1}(\mathsf{Q})^d x F(x) \equiv \neg \exists p (p \wedge \mathsf{Q} x (\neg F)^{\geq p_\alpha}(x)) \equiv \neg \exists p (p \wedge \mathsf{Q} x \Delta(p \to \neg F(x))) \equiv^{(1)} \star$

We then go on with a chain of equivalences and explain afterwards in detail why everyone of them is justified:

$\star \equiv \neg \exists p (p \wedge \mathsf{Q} x \Delta(F(x) \to \neg p)) \equiv^{(2)}$
$\equiv \forall p (\neg p \vee \neg \mathsf{Q} x \Delta(F(x) \to \neg p)) \equiv^{(3)}$
$\equiv \forall p (p \vee \neg \mathsf{Q} x \Delta(F(x) \to p)) \equiv^{(4)}$
$\equiv \forall p (p \vee \mathsf{Q}^d x \neg \Delta(F(x) \to p)) \equiv^{(5)}$
$\equiv \forall p (p \vee \mathsf{Q}^d x \Delta(p \to F(x))) \equiv^{(6)}$
$\equiv \exists p (p \wedge \mathsf{Q}^d x \Delta(p \to F(x))) \equiv^{(7)} \mathcal{F}^{P_1}(\mathsf{Q}^d) x F(x)$

(1) results from the fact that for two formulas G, H we have that $G \to H \equiv \neg H \to \neg G$.

(2) is De Morgan's law.

(3) is only a variable renaming.

(4) adds two negations after the quantifier.

(5) follows from the fact that $v_{\mathcal{M}}(\neg \Delta(F(x) \to p)) = 1$ if $v_{\mathcal{M}}(F(x)) > v_{\mathcal{M}}(p)$, and is 0 otherwise, while, on the other hand, $v_{\mathcal{M}}(\Delta(p \to F(x))) = 1$, if $v_{\mathcal{M}}(F(x)) \geq v_{\mathcal{M}}(p)$, and is 0 otherwise. Since the truth values of constants to be substituted for p range continuously in $[0, 1]$, the equivalence holds.

(6) follows from Theorem 39 and Lemma 4 point (c), as Q, and hence by Lemma 5 also Q^d, is non-deceasing.

(7) follows from Theorem 39.

□

That this result cannot be extended to quantifiers that are non-increasing, becomes clear by looking at the following example.

Example 13 *For semi-fuzzy quantifiers that are non-increasing, the desideratum of dualization fails to hold in general. As an example, we again, as in Example 12, look at the Type I quantifier "none", $\forall^\neg$. Let us assume $D = \{c\}$, and $v_\mathcal{M}(F(c)) = \frac{1}{2}$. Then:*

$v_\mathcal{M}(\forall p(p \vee \forall^\neg x F^{\geq p}(x))) = \frac{1}{2}$, *and* $v_\mathcal{M}(\exists p(p \wedge \forall^\neg x F^{\geq p}(x))) = 1$.

By the proof of Theorem 42, this means the desideratum of dualization cannot be fulfilled, as the equivalence rewriting step (6) *does not work.*

Interestingly, although $\mathcal{F}^{P_1}$ complies with the desideratum of dualization (for quantifiers that are non-decreasing), it generally does not do so for any of the two independent properties of internal and external negation. For the internal negation condition, we note that:

$$\exists p(p \wedge \mathsf{Q}x\neg(F^{\geq p}(x))) \equiv \exists p(p \wedge \mathsf{Q}x\Delta(\neg p \to \neg F(x))) \not\equiv \exists p(p \wedge \mathsf{Q}x\Delta(p \to \neg F(x))). \quad (5.10)$$

For the external negation condition, we note that for non-increasing Q, by Lemma 4 (c):

$$\neg\exists p(p \wedge \mathsf{Q}xF^{\geq p}(x)) \equiv \forall p(\neg p \vee \neg\mathsf{Q}xF^{\geq p}(x)) \not\equiv \forall p(p \vee \neg\mathsf{Q}xF^{\geq p}(x)) \equiv \exists p(p \wedge \neg\mathsf{Q}xF^{\geq p}(x)). \quad (5.11)$$

Taking inspiration from this defect, we propose the following variant of $\mathcal{F}^{P_1}$:

Definition 55 *Let F be a formula, $p_\alpha \in \Lambda$ such that $v_\mathcal{M}(p_\alpha) = \alpha$, and Q be a semi-fuzzy quantifier.*

For quantifiers Q that are non-decreasing we define:

$$v_\mathcal{M}(\mathcal{F}^{P_2}(\mathsf{Q})xF(x)) = \sup_{\alpha \in [0,1]} \min(\alpha, v_\mathcal{M}(\mathsf{Q}xF^{\geq p_\alpha}(x))). \quad (5.12)$$

For quantifiers Q that are non-increasing we define:

$$v_\mathcal{M}(\mathcal{F}^{P_2}(\mathsf{Q})xF(x)) = \sup_{\alpha \in [0,1]} \min(1 - \alpha, v_\mathcal{M}(\mathsf{Q}xF^{\geq p_\alpha}(x))). \quad (5.13)$$

Theorem 43 *[BFH18] Let* Q *be a semi-fuzzy quantifier that is non-increasing, and* F *a formula. Then we have:*

- $\mathcal{F}^{P_2}(\mathsf{Q})xF(x) \equiv \exists p(\neg p \wedge \mathsf{Q}xF^{\geq p}(x))$
- $\mathcal{F}^{P_2}(\mathsf{Q})xF(x) \equiv \exists y(\neg F(y) \wedge \mathsf{Q}xF^{\geq y}(x)) \vee \mathsf{Q}xF^{\geq \perp}(x)$

Proof:
The proof proceeds by analogy with the one of Theorem 39, i.e. actually with the one of Lemma 4 point (a). □

We then obtain the following theorem:

Theorem 44 *[BFH18]* $\mathcal{F}^{P_2}$ *complies with the desiderata of preservation of internal and external negation for semi-fuzzy quantifiers* Q *that are non-decreasing.*

Proof:
Assume Q is a semi-fuzzy quantifier that is non-decreasing. Then, as a consequence, $\mathsf{Q}^{\neg}$ is a semi-fuzzy quantifier that is non-increasing, and Q^{d} a semi-fuzzy quantifier that is non-decreasing. Hence, using Lemma 4 and Theorem 42, for $p_\alpha \in \Lambda$ such that $v_{\mathcal{M}}(p_\alpha) = \alpha$, we obtain:

$$v_{\mathcal{M}}(\mathcal{F}^{P_2}(\mathsf{Q}^{\neg})xF(x)) = \sup_{\alpha\in[0,1]} \min(1-\alpha, v_{\mathcal{M}}(\mathsf{Q}^{\neg}xF^{\geq p_\alpha}(x))) =$$
$$= 1 - \inf_{\alpha\in[0,1]} \max(\alpha, 1 - v_{\mathcal{M}}(\mathsf{Q}^{\neg}xF^{\geq p_\alpha}(x))) =$$
$$= 1 - \inf_{\alpha\in[0,1]} \max(\alpha, v_{\mathcal{M}}(\mathsf{Q}^{d}xF^{\geq p_\alpha}(x))) =$$
$$= 1 - \sup_{\alpha\in[0,1]} \min(\alpha, v_{\mathcal{M}}(\mathsf{Q}^{d}xF^{\geq p_\alpha}(x))) =$$
$$= 1 - v_{\mathcal{M}}(\mathcal{F}^{P_2}(\mathsf{Q}^{d})xF(x)) = 1 - v_{\mathcal{M}}(\mathcal{F}^{P_2}(\mathsf{Q})^{d}xF(x)) =$$
$$= v_{\mathcal{M}}(\mathcal{F}^{P_2}(\mathsf{Q})^{\neg}xF(x)) \ .$$

Preservation of external negation is a consequence of preservation of internal negation and dualization. □

While this extension of the QFM $\mathcal{F}^{P_1}$, to quantifiers that are non-increasing, might seem ad hoc, [BFH18] shows how it fits a more general framework, based on the idea of closeness measures.

CHAPTER 6

Queries and Summarizations

6.1 Querying with probabilistic quantifiers

The quantifier models $[M5]$ of Chapter 4 were dependent on the size of the domain. In this section we take up the basic idea of sampling small sets of witnesses and evaluating the quantified statements on them first, before we extrapolate truth on the full scale, based on these small scale estimations. The main difference to the former models, besides the fact that their semantics is domain size independent, is that we now interpret the vagueness of the about-hedge as follows. Let us again consider the statement “About half of all humans are male”. Instead of splitting it in two disjunctive independent statements, we consider the proportion of the scope predicate on the random sample as indicator of its real proportion with respect to the full domain of discourse. In order to do so, one has to fix a value for the desired precision of the estimate as well as a level of confidence that the estimate is true. While the precision can be understood as an acceptable tolerance margin, the confidence can be interpreted as the inherent vagueness of the statement. This chapter is based on [FHO17]. Let us briefly review the basic principles of sampling as they are taught in every good introduction course to stochastics.

6.1.1 Sampling principles

Let $Y_1, \dots, Y_s$ be independent and identically distributed Bernoulli random variables, i.e. for each $i \in \{1, \dots, s\}$ we have $Y_i \in \{0, 1\}$. Then, it is easy to see that $\frac{\sum_{i=1}^{s} Y_i}{s} = \frac{X}{s}$ is a random variable with (scaled) binomial distribution. To evaluate binary relative Type III quantifiers, we need to estimate the proportion of range elements that also fulfill the scope formula. To this end, we need to relate three parameters, namely the *sample size* s, the *confidence* $1 - \alpha \in (0, 1)$, and the *precision* of the estimate $\epsilon \in [0, 1]$. This can be expressed as follows [GW14], where $\mathbb{P}_{s,\rho}$ denotes the probability distribution for a binomial distributed random variable with parameters $s \in \mathbb{N}^+$ and $\rho \in [0,1]$:

$$\mathbb{P}_{s,\rho}(\mid \frac{X}{s} - \rho \mid \geq \epsilon) \leq \alpha. \tag{6.1}$$

The most accurate way to proceed would be to construct confidence regions, using binomial and beta quantiles, which should certainly be performed for real life applications where accuracy matters the most. Another well known and widely used approach, due to its more efficient nature, makes use of the central limit theorem [GW14], for which we have to assume that the sample size is sufficiently large, i.e. at least a few dozens. Following this way, we may calculate:

$$\mathbb{P}_{s,\rho}(\mid \tfrac{X}{s} - \rho \mid \geq \epsilon) = 1 - \mathbb{P}_{s,\rho}(\mid \tfrac{X}{s} - \rho \mid < \epsilon) = 1 - \mathbb{P}_{s,\rho}(\mid \tfrac{X - s\rho}{\sqrt{s\rho(1-\rho)}} \mid < \epsilon\sqrt{\tfrac{s}{\rho(1-\rho)}}) \approx$$

$$\approx 1 - (\Phi(\epsilon\sqrt{\tfrac{s}{\rho(1-\rho)}}) - \Phi(-\epsilon\sqrt{\tfrac{s}{\rho(1-\rho)}})) = 1 - (2\Phi(\epsilon\sqrt{\tfrac{s}{\rho(1-\rho)}}) - 1) = 2 \cdot (1 - \Phi(\epsilon\sqrt{\tfrac{s}{\rho(1-\rho)}})).$$

Note that Φ is bijective[1] and that $\rho(1-\rho) \leq \frac{1}{4}$. Hence, to make Equation 6.1 true, we need that $s \geq (\frac{\Phi^{-1}(\frac{2-\alpha}{2})}{2\epsilon})^2$.

This last inequality tells us which minimum sample size s we have to use to achieve a certain precision ϵ with confidence $1-\alpha$. To refer to this functional relation of the parameters later, we define $f : [0,1] \times (0,1) \to \mathbb{N}$ as follows:

$$f(\epsilon, \alpha) = \lceil (\frac{\Phi^{-1}(\frac{2-\alpha}{2})}{2\epsilon})^2 \rceil. \tag{6.2}$$

6.1.2 Some prose on probabilistic quantifier models

The core issue is that, unlike for small models with, say, a few hundred or thousand objects, huge models sometimes might be too big to be evaluated in total. One may think about the whole internet as a database, which ordinary computers can not completely take into account when they have a certain question, especially if the question is complex. Therefore, one often uses sampling to make (approximate) statements about states of affairs, see e.g. [CLS+10, MWA+03]. We may also think, for example, of the recent press release of the so called Panama Papers. After weeks and months of research, the media headlines often read something like “Many people from country A have done B”, or “Most people who did B come from country A”, hence the complex information from all these documents was processed, by humans, to result in condensed natural language summarizations. For our purpose, as we are dealing with vague relative quantifiers, estimations of proportions are conceptually near. This means that we can associate the player *Nature*, who samples witnessing constants for properties (uniformly) randomly, with a random-number generator, that is used when actually taking random samples from a dataset. We will define our new models for (binary) relative Type III quantifiers,

[1] As it is the distribution function of standard normally distributed random variables.

directly tied to the language from database theory. It is important to note that we restrict our attention to crisp arguments to quantifiers. The general case works via the already discussed QFMs of Chapter 5. Other approaches to querying in the presence of fuzziness directly assume fuzzy data, which practically never appears in real life though, like Pivert and Bosc [PB12], or [KZT15], who also use different quantifier models.

6.1.3 Data summarizations

It is also interesting to consider the following fact. When one answers queries by linking estimates of proportions to the semantics of (binary) relative Type III quantifiers, one actually starts with the estimate of the proportion and then checks whether it fits the semantics of the quantifier of the respective statement. That means that one can actually use the same estimate to produce natural language summarizations as well, as there is no conceptual difference at all. Hence, all theoretical considerations trivially carry over to natural language summarization, in the sense that one only needs to fix a set of quantifiers, and check whether a certain estimate of a proportion, with respect to a predicate, fits the semantics of one or more of the quantifiers from the set, to produce a number of summarizations. Of course, one can then apply certain techniques to find something like an ideal summarization, in case one produced several ones, but this, although very interesting in itself, see e.g. [DHB10], is not the topic of the present chapter.

6.2 Defining a query language

In this section, we present our concrete proposal for querying datasets, using a standard query language extended with (binary) relative Type III quantifiers.

For ease of exposition, we take a declarative logic based view of relational databases and queries over them [AHV95]. Databases are defined as finite relational structures over a given signature or *schema*. As basic query language, we consider first-order logic (FO) formulas, i.e. CL including the basic quantifiers $\forall, \exists$, over the same signature. As data values, we use constants and integer numbers, and we allow (in)equalities ($\neq, =$) of values (both constants and integers), and comparisons ($<, >, \geq, \neq$) between integers. This basic setting captures expressions of relational algebra (and thus, basic SQL) over relational databases. Moreover, significant fragments of other datamodels, and their corresponding query languages, can be viewed as special case of FO-queries over relational data as considered here. This applies in particular to the fragment of the SPARQL query language for RDF data [PAG09] which we use in Section 6.4 to illustrate our account.

6.2.1 Relational databases and FO-queries

As usual, we denote the integer numbers by $\mathbb{Z}$, and the positive integers by $\mathbb{N}^+$. We define a *(relational) schema* as comprising a set $\mathcal{R}$ of *relation names*, together with an *arity function* $ar : \mathcal{R} \rightarrow \mathbb{N}^+$, and a function *npos* that maps each $R \in \mathcal{R}$ to a (possibly empty) subset of $\{1, \ldots, ar(R)\}$ of *numeric positions* of R. [2]

Let a relational schema $(\mathcal{R}, ar, npos)$ be given. Let $\mathbf{U}$ be a set of constants or *data values*, and $\mathbf{V}$ be a countably infinite set of *variables*. A *term* is an object in $\mathbf{U} \cup \mathbb{Z} \cup \mathbf{V}$. *Atoms* take the following forms:

i: $R(t_1, \ldots, t_{ar(R)})$ with $R \in \mathcal{R}$. The t_i are terms such that $t_i \in \mathbb{Z} \cup \mathbf{V}$ if $i \in npos(R)$, and $t_i \in \mathbf{U} \cup \mathbf{V}$ if $i \notin npos(R)$;

ii: $t = t'$ and $t \neq t'$ with t, t' terms; and

iii: $t < t'$, $t > t'$, $t \leq t'$ or $t \geq t'$, for t, t' terms in $\mathbb{Z} \cup \mathbf{V}$.

An atom is called *relational* if it is of the form (i), and *ground* if all its terms are from $\mathbf{U} \cup \mathbb{Z}$. A *database instance* (or simply a *database*) is a *finite* set D of ground relational atoms. The *active domain* of a database D, denoted $ADom(D)$, is the set of constants and numbers from $\mathbf{U} \cup \mathbb{Z}$ that occur in the atoms of D.

Remark 24 *Note that, as already mentioned in the last paragraph, in this chapter, D no longer refers to the domain of an interpretation $\mathcal{M}$, but to a database with domain $ADom(D)$.*

Example 14 *Consider a schema containing, among others, the following relations:*

- *unary country and city, with no numeric positions, that is:* $npos(city) = npos(country) = \{\}$;
- *a binary city_of that relates cities and the countries they are located in, also with* $npos(city\_of) = \{\}$;
- *a binary cap_of that relates each capital city with the country it is capital of; again,* $npos(cap\_of) = \{\}$;
- *a binary has_pop with* $npos(has\_pop) = \{2\}$, *which relates countries and cities, with an integer number denoting its total population;*
- *a binary hasGDP_agr with* $npos(hasGDP\_agr) = \{2\}$, *which relates countries with an integer number (between 0 and 100) denoting the percentage of its GDP that comes from agriculture.*

A database D_1 over this schema may contain, for example, ground atoms:
$country(USA)$, $country(India)$, ... $city(Chicago)$, $city(Beijing)$, ...
$cap\_of(Beijing, China)$, $cap\_of(NewDelhi, India)$, ...
$has\_pop(China, 1385 \cdot 10^6)$, $has\_pop(Beijing, 21.6 \cdot 10^6)$, $has\_pop(Shanghai, 24.3 \cdot 10^6)$ ...

[2] The definition we use here is somewhat simplified compared to other more complex ones, which are compatible nevertheless. Such more involved ones can, e.g., also assign names and domains to properties.

A *FO-query* is a first-order formula $\psi(\vec{x})$ with free variables $\vec{x}$, built from atoms in the usual way, using the connectives $\neg$, $\wedge$, $\vee$, and the quantifiers $\exists$ and $\forall$. We refer to these variables as the *answer variables* of ψ. The *arity* of the query is the number of variables in $\vec{x}$. We call a query *Boolean* if it is 0-ary, that is, it has no free variables.

We note that a database D can be seen as a Herbrand interpretation over the predicates in the schema, with domain $ADom(D)$. An n-ary FO-query over D defines an n-ary relation over $ADom(D)$, which contains precisely the tuples for which the corresponding formula is satisfied, under the usual semantics.

Let D be a database. A *substitution* is a mapping σ from variables in $\mathbf{V}$ to values in $ADom(D)$. We write $\sigma(\vec{t})$ for the tuple that results from $\vec{t}$ by substituting each variable x with $\sigma(x)$, and we write $\sigma(\varphi)$ to denote the formula that results from φ by applying the substitution σ to all its atoms. For $x \in \mathbf{V}$, $c \in ADom$, and a substitution σ, we denote by $\sigma\{x \mapsto c\}$ the substitution σ' that has $\sigma'(x) = c$, and $\sigma'(y) = \sigma(y)$ for all remaining variables in the domain of σ. Abusing notation, we may disregard the order in tuples and treat them as sets.

The *satisfaction* in D of a formula ψ with respect to σ, in symbols $D \models_\sigma \psi$, is defined inductively in the natural way:

- For relational atoms, $D \models_\sigma R(\vec{t})$ if $R(\sigma(\vec{t})) \in D$.
- For the other atoms, $D \models_\sigma t \circledcirc t'$ if $\sigma(t) \circledcirc \sigma(t')$, where each binary predicate $\circledcirc \in \{=, \neq, \geq \dots.\}$ is interpreted as usual.
- $D \models_\sigma \psi_1 \wedge \psi_2$ if $D \models_\sigma \psi_1$ and $D \models_\sigma \psi_2$.
- $D \models_\sigma \psi_1 \vee \psi_2$ if $D \models_\sigma \psi_1$ or $D \models_\sigma \psi_2$.
- $D \models_\sigma \neg\psi$ if $D \not\models_\sigma \psi$.
- $D \models_\sigma \exists x\ \psi$ if for some $c \in ADom(D)$, we have $D \models_{\sigma\{x \mapsto c\}} \psi$.
- $D \models_\sigma \forall x\ \psi$ if for each $c \in ADom(D)$, we have $D \models_{\sigma\{x \mapsto c\}} \psi$.

Let $\psi(\vec{x})$ be a query with answer variables $\vec{x} = (x_1, \cdots, x_n)$, and let $\vec{c} = c_1, \cdots, c_n$ be a tuple of values from $\mathbf{U} \cup \mathbb{Z}$ of the same arity. Then we say that $\vec{c}$ *is an answer to* ψ *over* D if $D \models_\sigma \psi$ for the substitution σ that sends x_i to c_i for each $1 \leq i \leq n$. In this case, we write $D \models \psi(\vec{c})$.

Note that for a Boolean query ψ, there are two possibilities: The empty tuple is an answer to ψ, i.e. $D \models \psi$. In that case, we may say that ψ is *true* in D, or that its answer in D is *yes*. The other case is when the empty tuple is *not* an answer to ψ, i.e. $D \not\models \psi$. Then we say that ψ is *false*, or that its answer is *no*.

Example 15 *The following are simple examples of FO-queries over our example schema; ψ_1 is a Boolean, ψ_2 a unary, and ψ_3 a binary query.*

- ψ_1: *Is there a country with a population of more than one billion people?*
- ψ_2: *Which countries have a city with higher population than its capital?*
- ψ_3: *Which are the countries, and their capitals, such that no other city in the country has more inhabitants?*

$$\begin{aligned}
\psi_1 = \quad & \exists x, y(country(x) \wedge has_pop(x,y) \wedge (y > 1000{*}10^6)) \\
\psi_2(x) = \quad & \exists y_1, y_2, z_1, z_2(country(x) \wedge cap_of(y_1,x) \wedge city_of(y_2,x) \wedge \\
& \quad \wedge has_pop(y_1,z_1) \wedge has_pop(y_2,z_2) \wedge (z_1 < z_2)) \\
\psi_3(x,y) = \quad & \exists z(country(x) \wedge cap_of(y,x) \wedge has_pop(y,z) \wedge \\
& \quad \forall y_1, z_1((city_of(y_1,x) \wedge has_pop(y_1,z_1)) \rightarrow (z_1 \leq z)))
\end{aligned}$$

We note that $D_1 \models \psi_1$*, that is, its answer is* yes*, since the substitution* $\sigma(x) = China$*,* $\sigma(y) = 1385 \cdot 10^6$ *makes the formula true. We can also observe that the answers to* ψ_2 *contain China, and that* $(Beijing, China)$ *is* not *an answer to* ψ_3*.*

6.3 Extending FO-queries with relative quantifiers: [M6]

Assume $m \in \mathbb{N}^+$, and let $n, k \in \{0, \ldots, m\}$ with $n \neq 0$. We consider the following *(binary) relative Type III quantifiers*:

$$\text{about } \frac{k}{n} : Q^{[\approx \frac{k}{n}]} \qquad \text{at least about } \frac{k}{n} : Q^{[\gtrsim \frac{k}{n}]} \qquad \text{at most about } \frac{k}{n} : Q^{[\lesssim \frac{k}{n}]}$$

If $k = n$, then we may read both $Q^{[\approx \frac{k}{n}]}$ and $Q^{[\gtrsim \frac{k}{n}]}$ as *almost all*, and write $Q^{[\approx 1]}$.

If $k = 0$, then we may read both $Q^{[\approx \frac{k}{n}]}$ and $Q^{[\lesssim \frac{k}{n}]}$ as *nearly none* and write $Q^{[\approx 0]}$.

Note that each value of m determines a family of proportional quantifiers, while very large values of m usually do not occur in natural language. Now we define our query language, which extends FO-queries with the quantifiers syntactically just introduced above:

Definition 56 *A* query *is an expression* $q(\vec{y})$ *of the form:*

$$Qx(R(x, \vec{y'}), \psi(x, \vec{y})) \tag{6.3}$$

where $\vec{y'} \subseteq \vec{y}$*, and:*

- Q *is any of the relative quantifiers defined above, or* $\forall, \exists$*;*
- $R(x, \vec{y'})$ *is a relational atom using the variables in* $\{x\} \cup \vec{y'}$*, and whose additional terms are from* $\mathbf{U} \cup \mathbb{Z}$*, and*
- $\psi(x, \vec{y})$ *is a FO-query with answer variables* $\{x\} \cup \vec{y}$*.*

The answer variables *of* q *are* $\vec{y}$*. The* arity *of* q *is the number of variables in* $\vec{y}$*.*

Example 16 *To illustrate our language, we consider the following queries:*

q_1*: Do at least about two thirds of all countries make at most 20% of their GDP in agriculture?*

q_2*: Do about half of all cities have more than 200000 inhabitants?*
q_3*: Which countries have a capital which has more inhabitants than about half of all other capitals in the world?*
q_4*: Which countries have a capital that has more inhabitants than almost all other cities of that country?*

The first two queries are Boolean. The other two queries are unary. Queries q_3 *and* q_4 *are very similar, but they differ on the range predicate: it is unary in* q_3 *and binary in* q_4*. In our syntax, they look as follows:*

$$\begin{aligned} q_1 = &\quad Q^{[\gtrapprox \frac{2}{3}]}x\big(country(x), \exists y(hasGDP\_agr(x,y) \wedge (y \leq 20))\big) \\ q_2 = &\quad Q^{[\approx \frac{1}{2}]}x\big(city(x), \exists y(has\_pop(x,y) \wedge (y > 200000))\big) \\ q_3(y) = &\quad Q^{[\approx \frac{1}{2}]}x\big(capital(x), \exists z, z', w(cap\_of(y,w) \wedge has\_pop(w,z) \wedge \\ &\qquad\qquad \wedge\, has\_pop(x,z') \rightarrow (z > z'))\big) \\ q_4(y) = &\quad Q^{[\approx 1]}x\big(city\_of(x,y), \exists z, z', w(cap\_of(y,w) \wedge has\_pop(w,z) \wedge \\ &\qquad\qquad \wedge\, has\_pop(x,z') \rightarrow (z > z'))\big) \end{aligned}$$

Figure 6.1 shows the estimated and actual results to the queries q_1, q_2 *and* q_3*. Figure 6.2 highlights three particular answers to query* q_4*, namely China, USA, and India.*

We now define the semantics of our query language. As anticipated, it is based on *sampling*, according to the principles discussed in Section 6.1.1. We assume that values for the confidence $1-\alpha \in (0,1)$ and values for the precision $\epsilon \in [0,1]$ are given. These values then determine a minimal sample size $s = f(\epsilon, \alpha)$, cf. Equation 6.2. Then, for testing whether a given tuple of variables $\vec{c} = c_1, \cdots, c_n$ of values from $\mathbf{U} \cup \mathbb{Z}$ is among the desired answers to a query $Qx\big(R(x, \vec{y'}), \psi(x, \vec{y})\big)$, we take a random sample of size at least s of objects x that satisfy $R(x, \vec{c'})$ (where $\vec{c'}$ is the restriction of $\vec{c}$ to the positions from $\vec{y}$ that occur in $\vec{y'}$), and verify whether the proportion of those for which $\psi(x, \vec{c})$ holds (with respect to s) is within the desired range. Note that, since the sample is taken (uniformly) randomly, we may get different proportions, and hence different values, if we repeat the query evaluation. This is natural, since our semantics of the relative quantifiers defines a probability function over the possible answer tuples. As we will illustrate in the next section, the answers retrieved in this way are reliable, even for modest sample sizes.

Definition 57 *Let* D *be a database. Furthermore, let* $R(x, \vec{c'})$ *be a relational atom, and* $\psi(x, \vec{c})$ *a FO-query, such that* x *is the only free variable in both. Let* $S \subseteq ADom(D)$ *with* $S \neq \emptyset$*, then we define:*

$$Prop_D(S, \psi(x, \vec{c})) = \frac{|\{c \in S \mid D \models_{\{x \mapsto c\}} \psi(x, \vec{c})\}|}{|S|}. \qquad (6.4)$$

We now let σ be a substitution from $\mathbf{V}$ to $ADom(D)$, and let $D_R = \{c \in ADom(D) \mid R(c, \sigma(y')) \in D\}$. We define the semantics of queries as follows:

- $D \models_{\sigma,S,\epsilon} Q^{[\approx \frac{k}{n}]} x(R(x, \vec{y'}), \psi(x, \vec{y}))$ *if* $S \subseteq D_R$ *and* $Prop_D(S, \sigma(\psi(x, \vec{y}))) \in [\frac{k}{n} - \epsilon, \frac{k}{n} + \epsilon]$
- $D \models_{\sigma,S,\epsilon} Q^{[\gtrsim \frac{k}{n}]} x(R(x, \vec{y'}), \psi(x, \vec{y}))$ *if* $S \subseteq D_R$ *and* $Prop_D(S, \sigma(\psi(x, \vec{y}))) \in [\frac{k}{n} - \epsilon, 1]$
- $D \models_{\sigma,S,\epsilon} Q^{[\lesssim \frac{k}{n}]} x(R(x, \vec{y'}), \psi(x, \vec{y}))$ *if* $S \subseteq D_R$ *and* $Prop_D(S, \sigma(\psi(x, \vec{y}))) \in [0, \frac{k}{n} + \epsilon]$

Let $\epsilon \in [0, 1]$ and $\alpha \in (0, 1)$ be given. A tuple $\vec{c} = c_1, \cdots, c_n$ of values from $\mathbf{U} \cup \mathbb{Z}$ of the same arity as $\vec{y}$ is called a sampled answer to ψ over D (with precision ϵ and confidence $1 - \alpha$) *if $D \models_{\sigma,S,\epsilon} Qx(R(x, \vec{y'}), \psi(x, \vec{y}))$, where σ is the substitution that sends y_i to c_i for each $1 \leq i \leq n$, and $S \subseteq ADom(D)$ is a uniform random sample (with replacement) of size $|S| \geq f(\epsilon, \alpha)$ as described in Section 6.1.1. In this case, we may write $D \models_{\epsilon,\alpha} \psi(\vec{c})$.*

6.4 MONDIAL

To illustrate the proposed approach on real life data, we chose the MONDIAL database[3]. It is a dataset containing geographical data, that relies on open web data, such as the CIA factbook, Wikipedia, and atlases. The last major revision took place in 2015. Like most open web data, the database is not complete, and data may be somewhat imprecise. However, this is not of major concern here.

We evaluated the queries of Example 16. (In fact, the schema and queries of our running example are based on MONDIAL). We used the RDF version of MONDIAL locally and posed standard SPARQL queries, using the Java extension Apache Jena. This, together with random sampling on the list of query results, suffices to simulate the evaluation of queries in our language. In contrast to other fuzzy querying approaches, like e.g. the ones of Bosc and Pivert [PB12], we here rely on strictly classical data, and focus on their probabilistic evaluation.

Practical runs

Our goal was to test how sampling based evaluation performs for particular sample queries.

Obviously, if the amount of data in the database increases, the difference between the evaluation times for full and partial answers respectively, increases as well. However, for the present example, the MONDIAL database (16.4MB), they are still in a similar range. Some of our observations are captured in Figures 6.1 and 6.2, which show how the sample size correlates with the calculated proportions, using only a single iteration per size.

Example 17 *Figure 6.1 shows the results for the queries q_1–q_3, and Figure 6.2 for particular instances of the answer variable y in query q_4. From those results one can*

[3]MONDIAL database. (Last accessed January 30th, 2017). Retrieved from: https://www.dbis.informatik.uni-goettingen.de/Mondial/

straightforwardly evaluate the answers to the natural language queries. Taking the first one, which asks whether the proportion in question is at least about two thirds, the results show that, for almost all samples sizes, this is the case with high confidence. Similar results hold for the other queries. (Note that in Figure 6.2 just a small range of proportions is displayed). Finally, we emphasize that the graphs show the proportions obtained for one *random sample of each size. But the blue line (sampled results) converges quickly to the red line (correct proportion) if we increase the number of iterations.*

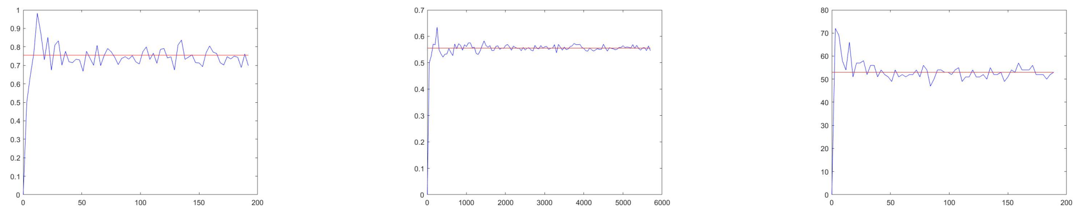

Figure 6.1: Left: query q_1; middle: query q_2; right: query q_3. The x-axis always represents possible sample sizes, i.e. the number of domain elements that fulfill the respective range predicate. For the left and the middle picture, the y-axis stands for the proportion of those range objects that also fulfill the scope predicate, while for the right picture it displays the sizes of answer sets. The blue graph shows the achieved results for samples of the sizes given by the values on the x-axis. The red graph displays the correct proportions, or answer set size respectively.

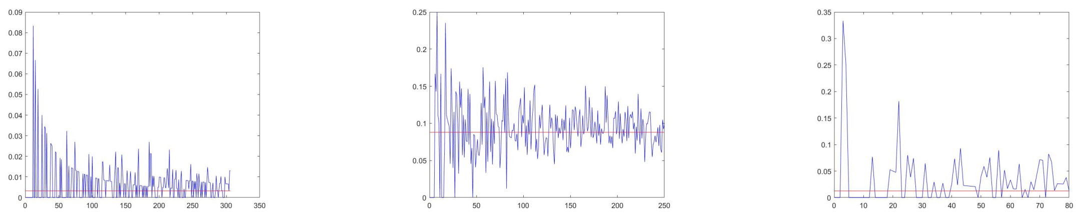

Figure 6.2: Query q_4, for: left: y = China; middle: y = USA; right: y = India. The x-axis always represents possible sample sizes, i.e. the number of domain elements that fulfill the respective range predicate. The y-axis stands for the proportion of these range objects that also fulfill the scope predicate. The blue graph shows the achieved results for samples of the sizes given by the values on the x-axis. The red graph displays the correct proportions.

Example 18 *The problem we consider now deals with a slightly different but very similar setting. The MONDIAL database stores* 5716 *cities with an average population of* 553157 *inhabitants. The question is again, how well can we approximate this average, when we only evaluate a small sample of cities.*

We employ three different sampling mechanisms[4] *[OR86]. The first one performs sampling*

[4]Note that sampling is always supposed to be performed uniformly randomly.

of witnesses with replacement, i.e. the sample can contain the same city more than only once. The other two mechanisms perform sampling without replacement, i.e. samples contain each witness at most once. For the former, we employ simple random sampling *[Sin03] and for the latter we use the algorithm proposed by Knuth [Knu97] and the one proposed by Fisher-Yates [Knu97].*

Simple random sampling works like the name suggests. Fix the size of the domain, say $n = 5716$*, and apply a random number generator with range* $\{1, \ldots, n\}$ *as many times as you want. The sample size is usually chosen smaller or equal than* n*, but not necessarily.*

The algorithm proposed by Knuth works as follows. Assume we have a list of all objects we want to select a random sample from, without replacement. In the first step, we choose any of these objects randomly and switch its position with the first element of the list. In the next step, we randomly choose one of the objects from the sublist that starts in the second position and again switch its position with the first object from the sublist. The number of steps equals the sample size. We provide the pseudo code for this procedure:

Generating a sample without replacement of size $s \leq n$ *from a list* $\{a_1, \ldots, a_n\}$*:*

Step 0: Set $i = 1$.

Step 1: Select a random number $i \leq r \leq n$.

Step 2: Swap a_i *and* a_r *in the actual list.*

Step 3: Cut out the first element, i.e a_i*, from the actual list.*

Step 4: Store the cut out element in another list, say S*, and increase* i *by* 1.

Step 5: Perform Step 1 - Step 4 another $s - 1$ *times.*

Eventually, one ends with a list S *of* s *different elements.*

The last algorithm we test is the one tracing back to Fisher and Yates and is very similar in nature to the one from Knuth, but works in a reversed manner. It essentially is a shuffling mechanism of an entire list, which a priori explains the increased performance time. Several versions are known, however, the one we implemented executes the following pseudo code:

Generating a sample of size $s \leq n$ *from a list* $\{a_1, \ldots, a_n\}$*:*

Step 0: Set $i = n$.

Step 1: Select a random number $1 \leq r \leq i$.

Step 2: Swap a_i *and* a_r *in the list.*

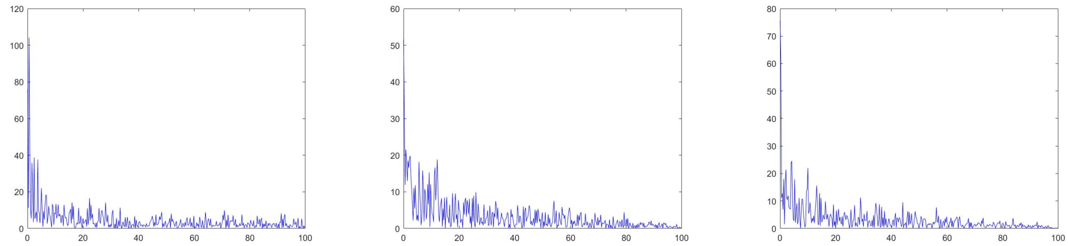

Figure 6.3: left picture: Simple Random Sampling; middle picture: Knuth's sampling algorithm; right picture: Fisher-Yates algorithm. The x-axis denotes the sample size in %, and the y-axis shows the deviation from the average in each random sample with respect to the actual average, again in %. The number of iterations is 1.

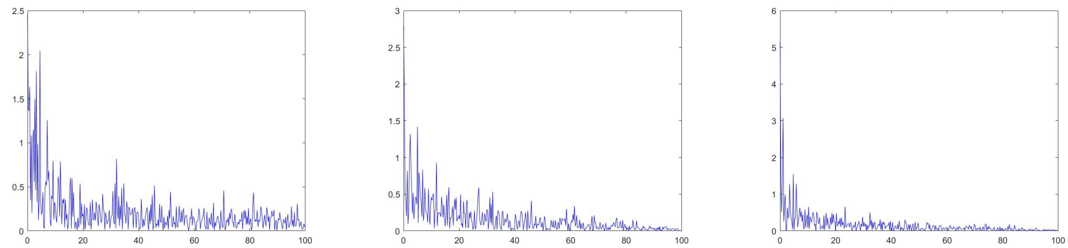

Figure 6.4: left picture: Simple Random Sampling; middle picture: Knuth's sampling algorithm; right picture: Fisher-Yates algorithm. The x-axis denotes the sample size in %, and the y-axis shows the deviation from the average in each random sample with respect to the actual average, again in %. The number of iterations is 500.

Step 3: decrease i by 1*.*

Step 4: Perform Step 1 - Step 3 another $n-2$ times.

Eventually, one ends with a shuffled version of the original list.

In the following, we give the results of our experiments. For each method, we produced a plot that shows the deviation (in %) from the true average population size (which is 553157*) dependent on the sample size. Also, again for each method, we produced a plot that displays the evaluation time that is needed, as well dependent on the sample size. To make clear how this procedures scales up, we do both for three different numbers of total iterations, first* 1*, then* 500 *and eventually* 10000*.*

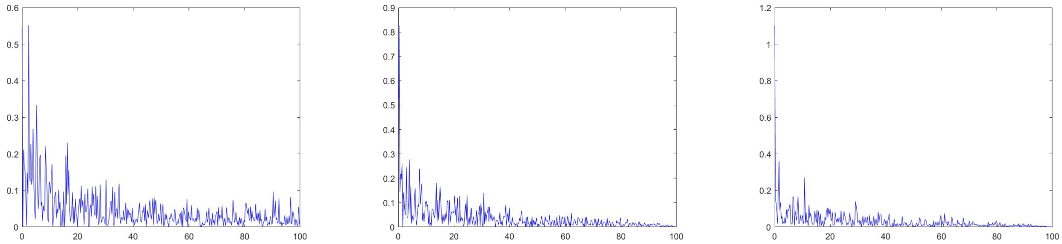

Figure 6.5: left picture: Simple Random Sampling; middle picture: Knuth's sampling algorithm; right picture: Fisher-Yates algorithm. The x-axis denotes the sample size in %, and the y-axis shows the deviation from the average in each random sample with respect to the actual average, again in %. The number of iterations is 10000.

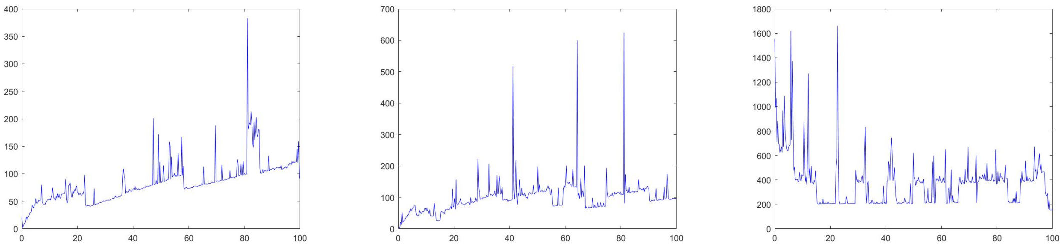

Figure 6.6: left picture: Simple Random Sampling; middle picture: Knuth's sampling algorithm; right picture: Fisher-Yates algorithm. The x-axis denotes the sample size in %, and the y-axis shows the evaluation time, needed for producing an average population size, in microseconds. The number of iterations is 1.

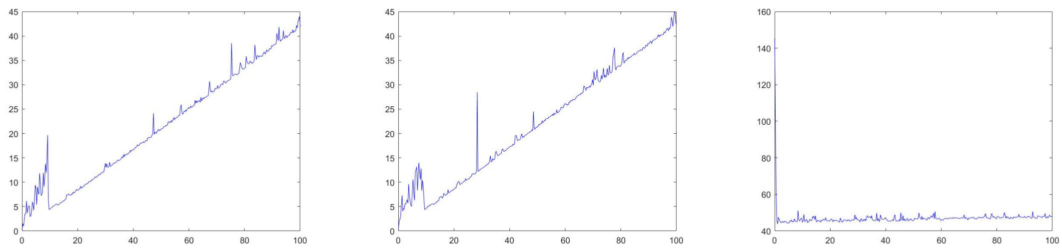

Figure 6.7: left picture: Simple Random Sampling; middle picture: Knuth's sampling algorithm; right picture: Fisher-Yates algorithm. The x-axis denotes the sample size in %, and the y-axis shows the evaluation time, needed for producing an average population size, in milliseconds. The number of iterations is 500.

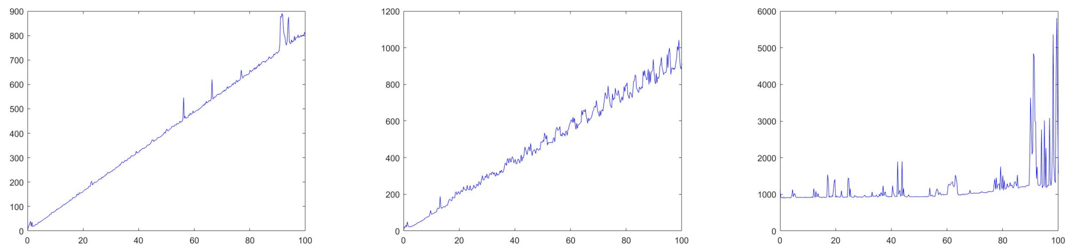

Figure 6.8: left picture: Simple Random Sampling; middle picture: Knuth's sampling algorithm; right picture: Fisher-Yates algorithm. The x-axis denotes the sample size in %, and the y-axis shows the evaluation time, needed for producing an average population size, in milliseconds. The number of iterations is 10000.

CHAPTER 7

Conclusion and Future Work

7.1 Summary

The modeling of vagueness in natural language, on the quantifier level, is subtle because it requires the adequate handling of various features and corresponding expressions. We chose to employ game semantic representations for different logics, like Hintikka's game for Classical Logic and Giles's game for Łukasiewicz logic, in order to conceptually motivate new constructs, like the Π quantifier, on firm and tangible grounds, namely by the random witness selection principle. This principle complements the principle of strategic witness selection, which is fairly standard in game theory [vB14]. This realm is what we described in Chapter 2, which follows the introduction.

The Π quantifier, based on the random witness election principle, introduced in [FR12, FR14] as an extension of Łukasiewicz logic, brings about intermediate truth values, even if applied to crisp arguments. In fact, Łukasiewicz logic is one of the three prominent t-norm based Mathematical Fuzzy Logics [CHN11], particularly the only one that has continuous truth functions for all its connectives and quantifiers. As a main contribution of this thesis, we show that the other two, Gödel logic and Product logic restricted to finite domains, can be defined within an extension of Łukasiewicz logic, again restricted to finite domains and enriched with constructs based on the already mentioned game semantic selection principles, particularly propositional quantifiers. We call this logic $Ł_\alpha(\Pi)$. As a consequence of the first definitions, all fuzzy logics that are based on finite representations of continuous t-norms, are also definable within $Ł_\alpha(\Pi)$. These results are developed in Chapter 3 [Hof18], and refer to the introduction's (C1).

The next part of the thesis, Chapters 4 and 5, referring to the introduction's (C2), is about giving an account of all the aspects that make the world of quantifiers so sparkling. Is the quantifier evaluation context dependent or independent? Do we count witnesses absolutely or relatively? Does a quantified statement refer to all objects from the domain or only to a restricted set of such? Finally, do witnesses actually crisply possess their properties or can we also admit borderline cases? The answer to any of these questions is 'both may happen'.

As to the last question, the most direct way of dealing with vague properties certainly is using fuzzy propositions. This feature is already present in fuzzy logics and is ready for use. However, in Chapter 4, quantifiers are defined for crisp arguments only. By developing our models in that way, we follow the well recognized approach of Liu and Kerre [LK98], Glöckner [Glö06], and others [DRSV14]. In Section 7.3, we will elaborate on different ways of handling vague properties.

Regarding absolute and relative quantifiers, both refer to extensional quantifiers, i.e. such that are context independent, in contrast to intensional quantifiers, that may depend on contextual information[1]. We treat a great variety of extensional quantifiers, indeed, all standard cases and combinations thereof, while the class of intensional quantifiers is basically represented by two generic quantifiers, namely "many" and "few". Nevertheless, we argue that the same readings, defined for those two, apply to other NL expressions as well. Also, we capture these readings, or models, within our framework $Ł_\alpha(\Pi)$, and enrich it by a refinement of the previously given structure [Lap00, Hof15]. When reading the material, it becomes clear that there is a lot of freedom in the interpretation of context dependent statements. As the reference situations can be chosen freely, we get non-monotonic versions of the quantifiers "many" and "few".

Although we always assume fixed domains, statements sometimes are supposed to refer to only a fraction of the objects within it. Take for example a domain of all humans and a statement like "Most boys like girls". For the evaluation of this statement, we do not need to know the whole domain, as evaluating the boys will suffice. Such quantifiers, with an additional range to the scope, are called binary, or 2-place, quantifiers. We show how quantifiers, not only based on binary $\forall$ and $\exists$, but also those based on binary Π, can be expressed in $Ł_\alpha(\Pi)$.

In Chapter 4, as already mentioned, we restrict attention to semi-fuzzy quantifiers, i.e. those that are only defined for crisp formulas, or arguments. Consequently, it remains to show how we apply systematic lifting mechanisms to semi-fuzzy quantifiers to arrive at fully-fuzzy ones. This is done in Chapter 5. We show how to express the most important versions of such mechanisms (QFMs) into $Ł_\alpha(\Pi)$, and analyze them regarding their compliance with certain new and known lifting principles [BFH18].

In Chapter 6, which refers to the introduction's (C3), we introduce yet another schema for relative semi-fuzzy quantifiers, directly tied to a query language [FHO17] that evaluates vaguely quantified statements based on a probabilistic evaluation technique. We specify the syntax and semantics of this query language and analyze its implemented routines,

[1] Note that the distinction between absolute and relative quantifiers also applies to intensional quantifiers, since extensional quantifiers can be viewed as a special case of intensional ones, namely when the contextual information is empty.

to see how well probabilistic evaluation predicts states of affairs on the full scale. Also, we explain how the same techniques can be used to produce data summarizations.

7.2 Range of the framework

The principle of random witness selection, built into a logic system on firm conceptual grounds, is one of the strengths of the perspective advocated in this thesis. Game semantically speaking, in addition to the principle of strategic reasoning, as represented by the two strategic players, we have a principle of non-strategic reasoning, represented by a third non-strategic player. By that means, we achieve an encoding of probability into the logic, and show how expressibility increases vastly. On the technical side, this allows for the definition of wide classes of t-norm based fuzzy logics. On the application side, we acquire a wide range of models for vague natural language quantifiers.

7.3 Future work

Future work certainly can go into several directions. On the logical level, one is interested in full calculi, encompassing certain properties, like standard completeness. Also, the relations of the random witness selection principle with games of imperfect information [MSS11] would be worth exploring.

On the practical side, we envision a computer software that can help managing deeper layers of processing NL texts, than the current keyword search based world provides us with. To that end, two things are attractive to consider. It would be an interesting next step to (1) investigate which particular data models, like RDF or other graph models, can store information from natural language appropriately, and apply the developed theory to data analysis, as described in Chapter 6. This, in particular, requires (2) a proper handling of vague properties, as already explained in the introduction, as well. Hence, allowing for such extensions, in a nicely parametrized fashion, also makes for a potential branch of further research.

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