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A Review and Extension of the Health Deficit Model of Optimal Aging

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Abstract

In economic modeling chronological age is often used as a measure for aging and death. However, individuals of the same age can differ drastically in health status and recent medical advancements have had a severe impact on the human life span and health. In order to model the aging process as a life cycle model - the health deficit model - developed by Dalgaard & Strulik (2014) is presented. The model draws on recent research in the fields of biology and medicine and optimal health spending determines the speed of aging and thus the time of death. In this thesis, the life cycle model is extended to a general equilibrium model and a two dimensional fixed point problem which characterizes the steady state is formulated. The formulation of the steady state is only dependent on the steady state capital and the optimal time of death. A graphical solution of the general equilibrium model is presented and analyzed with comparative dynamics. With the help of parameter variations both models are compared and it is shown that a longer lifespan can be achieved in the general equilibrium model as long as the steady state capital is below the exogenously given capital intensity employed in the life cycle model.

Contents

1 Motivation and Introduction

Due to medical advancements the life expectancy has risen dramatically in the last century. Accompanied with a decrease in fertility rates this has lead to a dramatic increase of the median age of the world population. In Figure 1, the rise of the median age is illustrated and projections show that it is expected to rise even further.

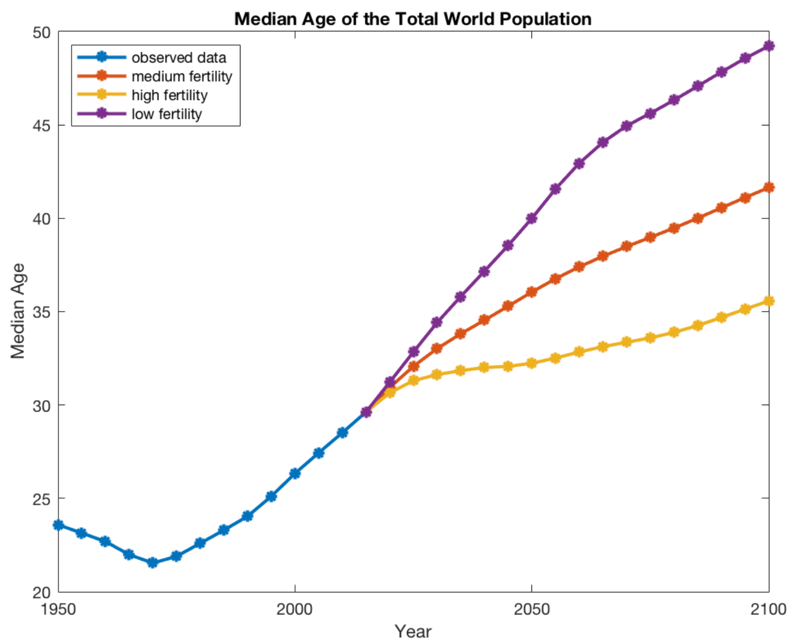

Figure 1: Median age of the total world population. Source: United Nations (2017), own illustration.

In the context of an aging population and rapid medical advancements the economic modeling of human aging has become an important undertaking. The simplest measure of the human aging process is the chronological age; on the macroeconomic level chronological age is a relatively good measure of aging and death. The Gompertz-Makeham-law of mortality shows an exponential increase of the death rate with age (Gompertz, 1825; Makeham, 1860).[1] However, individuals of the same age can dramatically differ from each other when it comes to health and aging; some 60 year-olds can be as healthy as some 40 year-olds. How to best

[1]The conditional probability $\mu(t)$ to die at age t (*force of aging*) and age t are log-linearly related: $\mu(t) = A + Re^{at}$.

incorporate this observed variance of health levels into economic models is not clear.

The task of modeling the heterogeneity of human aging was first undertaken by Grossman (1972) by drawing on the theory of human capital. In the model individuals produce their own health capital through the means of health investments. Therefore, an individual's chronological age t plays no independent role in the modeling of the aging process.

1.1 The Human Capital Model of Demand for Health

Grossman (1972) introduced the human capital model of demand for health motivated by two important aspects of human health. Firstly, he argues that the demand for health services is derived from the basic demand for good health. Therefore health should be considered an output and medical care as one of the inputs in the production of good health. Secondly, Grossman emphasizes the difference between health and other forms of human capital. Human capital in the form of knowledge raises an individuals productivity in market and non-market activities, while the stock of health affects the time that can be spent on such activities.

In the human capital model, health is introduced as a state variable similar to human capital. A person inherits an initial stock of health H_0, that depreciates with age and can be increased by investment in medical care. The evolution of health in discrete time is thus described by

$$H_{t+1} - H_t = f(I_t) - \zeta_t H_t, \qquad H_0 \text{ given}, \tag{1}$$

and in continuous time as

$$\dot{H}(t) = f(I(t)) - \zeta(t)H(t), \qquad H(0) = H_0, \tag{2}$$

where $H(t)$ describes the health capital stock. The function $f(I(t))$ (in most literature f is linear, i.e. $f(I(t)) = I(t)$) is the health production function and $I(t)$ are the investments in health at time t. The time-dependent depreciation rate is given by $\zeta(t)$.

Individuals invest in health because good health raises utility and income levels. Health investments also determine an individual's length of life and thus their time of death. Death takes place when the stock of health capital has fallen to $H_{\min}$. An important innovation of Grossman's model is that time of death T is an endogenous variable.

However over the years several authors have identified severe limitations of the human capital model. Case & Deaton (2005) argue that if the rate of biological deterioration $\zeta(t)$ is constant people will 'choose' an infinite life. Strulik (2015) also shows that the closed form solution of the human capital model with constant $\zeta(t)$ generally implies immortality.

Case & Deaton (2005) also critic the model for its inability to explain health differences of different socioeconomic groups.

Another critic of Grossman's work, brought up by Wagstaff (1986) and Breyer & Zweifel (1999) is that the model is unable to predict the strong negative correlation between health and health spending (i.e. healthy individuals go to the doctor less often).

All these criticisms are gathered and addressed by Galama (2015) by introducing two modifications. First, he argues that the costate equation for health investment should be used to determine optimal health investment and the dynamic state equation for health should be used to determine health. Interpreting these equations correctly shows that the model is able to predict the negative correlation of health and health investment. Correct use of equation (2) illustrates the dynamic nature of the health status and therefore is able to explain the widening of the health gradient between socioeconomic groups. Secondly, if the health production function $f\left(I(t)\right)$ is linear in $I(t)$ then the Hamiltonian is also linear in $I(t)$. Therefore, the control variable $I(t)$ is no longer part of the optimality condition, which is derived by taking the derivative, and its value can no longer be determined. However, without optimal investments it is no longer possible to derive paths for optimal health, wealth and consumption.

There are however two criticisms of the human capital model that are not addressed by Galama (2015). First, biologist also argue that aging is not a time-dependent but an event-dependent process (Arking, 2006). Making the depreciation rate explicitly age-dependent would however go against this assertion. Second, the dynamic equation for health (2) implies that health deteriorates faster when the health stock is relatively large, which usually means that individuals are relatively young.

In order to adress these critisisms, Dalgaard & Strulik (2014) propose a new way of modeling human aging. Motivated by research in the fields of medicine and biology they model human aging as a process of health deficit accumulation.

1.2 Health Deficits

There are four distinctive features that describe the aging process of biological species (Gavrilov & Gavrilova, 1991). First, most biological species deteriorate with age (in contrast to some primitive organisms). Second, this deterioration is best described by the Gompertz law (i.e. the mortality rate increases exponentially with age). Third, a leveling-off effect of the mortality rate in late life can be observed and the fourth feature is the so-called compensation law of mortality. The compensation law of mortality refers to the fact that relative differences in the mortality rate of different groups of the same species decrease with age. This phenomenon occurs because the higher initial death rates in the disadvantaged group are compensated by a lower 'aging rate'. This is observed in gender or country comparisons.

Gavrilov & Gavrilova (2001) show that these four fundamental aspects of the aging process can be described with the help of reliability theory. Reliability theory predicts that even systems composed of non-aging elements (constant failure rate) fail more often over time, if these systems are redundant in irreplaceable components. Aging is thus described as the loss of redundancy.

Conversely, the aging process can be characterized as one of increasing frailty. Mitnitski et al. (2001) propose an empirical measure of human frailty by describing the process of aging as an accumulation of health issues that they refer to as deficits. Health deficits can range in severity, from life threatening illnesses like cancer to issues that cause merely discomfort, like skin problems. Included in the list of possible health deficits are signs (e.g. tremors), symptoms (e.g. changes in sleep), abnormal laboratory values, disease classifications and disabilities. Deficits need to be associated with health status and generally a deficit's prevalence needs to increase with age.

A health deficit index (or frailty index) can then be defined as the proportion of the total potential deficits an individual has. The health deficit index of an individual at time t is thus given as

$$D(t) = \frac{1}{M} \sum_{a=1}^{M} \mathbb{1}_t(m)$$

where $m = 1, ..., M$ are the possible ailments a person can have (Dalgaard et al., 2018). The indicator function $\mathbb{1}_t$ assumes the value 1 if the individual suffers from condition m at time t.

In a study with 66,589 Canadians aged 15 to 79 Mitnitski et al. (2002) show that the data on deficit index $D(t)$ fits the following equation ('law of increasing frailty') very well:

$$D(t) = E + Be^{\mu t}. \tag{3}$$

The parameter μ is a physiological parameter that drives the aging process and is therefore described as the *force of aging*. While the parameter E has the same value for men and women, B and μ are gender specific. Empirical testing has shown, that men start out initially healthier, described by a lower value of B but age faster, characterized by a higher value of μ (Mitnitski et al., 2002; Strulik & Vollmer, 2013; Abeliansky & Strulik, 2018; Dalgaard et al., 2018).

Mitnitski et al. (2005) show that elderly community-dwelling individuals from Australia, Sweden and the United States accumulate deficits in a very similar way. More precisely, they estimate that health deficits increase on average by 3.5% from one year to the next. Furthermore, Abeliansky & Strulik (2018) show that the data from 10 European countries (Austria, Belgium, Denmark, France, Germany, Italy, Netherlands, Sweden and Switzerland) fits the law of increasing frailty very well. However, the projected average increase of health deficits from one birthday to the next is 7.5% for men and 5.2% for women. Mitnitski & Rockwood (2016) revised their previous estimate of 3.5% to an average of 4.5%.

Following the work of Dalgaard & Strulik (2014) a law of motion for the accumulation of deficits is derived. Differentiating the law of increasing frailty (3) with respect to age t leads to

$$\dot{D}(t) = \mu(D(t) - E), \tag{4}$$

where E decelerates the speed of deficit accumulation. In order to see that equation (4) is consistent with the law of increasing frailty (3), integrate (4) and insert the initial condition D_0 to get

$$D(t) = (D_0 - E)e^{\mu t} + E = D_0 e^{\mu t} - E(e^{\mu t} - 1).$$

Since $e^{\mu t} > 1$ for all $t > 0$, a larger value of E means less health deficits at any given age t. The autonomous parameter E is assumed to be susceptible to change by deliberate investments. Specifically, Dalgaard & Strulik (2014) propose the following law of motion for the accumulation of health deficits

$$\dot{D}(t) = \mu\left(D(t) - a - Ah(t)^{\gamma}\right) \tag{5}$$

where $D(0) = D_0$. The function h describes the health investments a person makes at time t. All environmental influences that are beyond the individual's control are contained in the parameter a. More pollution for instance implies a lower value for a. The parameters $A > 0$ and $0 < \gamma < 1$ characterize the state of health technology. A high value of A implies a very advanced health technology. The parameter γ specifies the degree of decreasing returns of health investment. The larger the value of γ, the more investments in cost-intensive highly advanced technology pay off. Note, that unlike the original formulation in the health capital model (Grossman, 1972), health investments have decreasing returns. As with the critic of Grossman's model the assumption that $\gamma < 1$ is not only plausible but necessary. If health investments had increasing or constant returns, individuals would choose a path that leads to immortality.

An individual dies when an upper bound of health deficits $\bar{D}$ is reached. Therefore, a representative agent in the model is alive as long as $D(t) < \bar{D}$. There is evidence for the existence of such an upper bound in several publications (Mitnitski et al., 2002; Rockwood & Mitnitski, 2006; Abeliansky & Strulik, 2018).

Thus, equation (5) along with the boundary condition $D(t) < \bar{D}$ supplies a complete description of aging until death, where the chronological age t plays no independent role, since the accumulation of future deficits is only dependent on already accumulated deficits and current investments.

Several empirical economic studies have taken health deficits as a descriptor of aging. Dalgaard et al. (2018) for example show that even though the chronological age of the average person in the global labor force has risen the physiological age has not.

1.3 Health Capital vs. Health Deficits

Two different ways of modeling human health have been presented: first, as a state variable similar to human capital and second, as an accumulation of health deficits. Following the work of Dalgaard & Strulik (2014) the two different approaches can be compared.

In order to examine the differences of those two ways of modeling aging consider the basic health capital accumulation equation $\dot{H}(t) = I(t) - \zeta H(t)$, where $I(t)$ describe the health investments and $H(t)$ is the stock of health capital (see e.g. Wagstaff, 1986). This equation however predicts that the loss of health and the stock of health are positively correlated, specifically that health depreciation is greater when the individual is healthy. In reality, however, the process of aging accelerates during life since both health deficits and the death rate rise exponentially with age (Mitnitski et al., 2001; Gompertz, 1825). Therefore, in practice, health loss is higher later in life when the health stock is generally low.

The standard 'fix' to this problem has been to make the depreciation rate age-dependent, so that $\zeta(t)$ increases with age ($\dot{\zeta}(t) > 0$). It is however unclear how exactly that depreciation function should be specified.

In contrast to the health capital formulation consider the basic health deficit accumulation equation $\dot{D}(t) = \mu(D(t) - E)$. By reformulating this equation into one for health accumulation the health capital model and the health deficit model can be compared and assessed. Let $\bar{H}$ denote the best attainable level of health, the current health status is $\bar{H}$ minus accumulated health deficits: $H = \bar{H} - D$ and therefore $\dot{H} = -\dot{D}$. Plugging this into equation (4) leads to:

$$\dot{H} = \mu E - \mu(\bar{H} - H(t)). \tag{6}$$

Following the terminology of Grossman (1972) the stock of health capital at which death takes place is defined as $H_{\min} = \bar{H} - \bar{D}$, where $\bar{D}$ denotes the level of deficits at which death takes place. As before, the term E can be is associated with health investments. This formulation of the accumulation of health has a crucial advantage compared to the one presented by Grossman (1972): it predicts that health loss is small for healthy individuals and larger when the health stock is already diminished, matching reality. Another advantage of the health deficit formulation is that the various parameters have clear empirical counterparts. The health deficit index has been developed and tested in the natural sciences and the parameter μ which describes the force of aging is constant over the adult life span (Mitnitski & Rockwood, 2016). Which is a clear advantage over the time-dependent, unknown, non-linear function $\zeta(t)$ describing the depreciation of health capital.

1.4 Structure

This thesis is organized as follows. In chapter 2, the health deficit model introduced by Dalgaard & Strulik (2014) is presented. In the life cycle model households maximize their lifetime utility while taking their health status into account. This framework allows us to formulate optimality conditions for human aging and death. Chapter 3 extends the life cycle model to a general equilibrium model making it possible to study the impact of physiological aging on capital accumulation. In chapter 4 the results of the life cycle model and of the general equilibrium model are compared.

2 Health Deficit Model

In this chapter, the health deficit model introduced by Dalgaard & Strulik (2014) is presented. They formulate a life cycle model with the health deficit accumulation equation (5) at its core in order to model the path of consumption and health spending of an individual over lifetime. An important aspect of the model is that with the help of health deficits the time of death can be modeled as an endogenous variable.

2.1 The Optimization Problem

Consider an individual that maximizes a discounted stream of utility over her lifetime. While it might be interesting to incorporate frailty as disutilty, in this simple model utility is derived only from consumption. The initial age is normalized to zero and the time of death T is finite and endogenous. The time preference is denoted by $\rho \geq 0$. Intertemporal utility is thus given by

$$\int_0^T u\left(c(t)\right) e^{-\rho t} dt \tag{7}$$

with the isoelastic utility functions $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ for $\sigma \neq 1$ and $u(c) = \log(c)$ for $\sigma = 1$. The measure of relative risk aversion $\sigma(c) = -cu''(c)/u'(c) = \sigma$ of the isoelastic utility function is constant and independent of c, therefore it is also called the CRRA (constant relative risk aversion) utility function. For positive σ households are risk averse, whereas $\sigma = 0$ corresponds to risk neutrality.

Individuals age and the aging process is described by health deficit accumulation. Deficits are accumulated according to equation (5).

Agents enter the labor market at time E which denotes the end of their education and retire at age R. Both E and R are fixed and exogenous.[2] Individuals receive wage income w over their working life that can be spent on consumption goods c or invested in health goods h. For simplicity w is assumed to be constant over the lifetime. The relative price of health goods is denoted with p_h. While consumption goods directly raise utility, investments in health slow down the accumulation of health deficits and thus prolong the duration of life. A longer life means more time for consumption which raises utility. Besides buying final goods c and h individuals can save or borrow money with interest rate r. The wealth accumulation equation

[2]This is an extension to Dalgaard & Strulik (2014) where education and retirement are not considered.

is thus given by

$$\dot{k}(t) = w\mathbb{1}_{[E,R]} + rk(t) - c(t) - p_h h(t). \tag{8}$$

Agents are assumed to inherit $k(0) = k_0$ and leave a bequest $k(T) = \bar{k}$.

The household optimization problem is therefore written as

$$\begin{aligned} \max_{c,h} \quad & \int_0^T u(c)\, e^{-\rho t} dt \\ \text{s.t.} \quad & \dot{D} = \mu\,(D - a - Ah^\gamma) \\ & \dot{k} = w\mathbb{1}_{[E,R]} + rk - c - p_h h \\ & D_0, \bar{D}, k_0, \bar{k} \quad \text{given.} \end{aligned}$$

It can be solved by applying optimal control theory. The state of the system at time t is described by the two state variables $D(t)$ and $k(t)$. The two control variables are $c(t)$ and $h(t)$. The initial conditions are $D(0) = D_0$ and $k(0) = k_0$ and terminal conditions are $D(T) = \bar{D}$ and $k(T) = \bar{k}$. For simplicity time indices will be suppressed in the remainder of this chapter.

The derivation of the first order conditions and the following results can be found in Appendix A.1.

2.2 Optimal Aging

From the first order conditions we obtain the following Euler equations (see Appendix A.1)

$$g_c := \frac{\dot{c}}{c} = \frac{r - \rho}{\sigma} \tag{9}$$

$$g_h := \frac{\dot{h}}{h} = \frac{r - \mu}{1 - \gamma}. \tag{10}$$

Equation (9) is the standard Euler equation for consumption, equation (10) is the counterpart for health expenditures: the health Euler equation. Both health expenditure and consumption are positively correlated with the interest rate r. Health investments decrease over time if $r < \mu$. If μ is 'large' health deficits accumulate at a higher pace in old age making late-in-life health investments relatively ineffective. Instead, individuals should invest more early in life (preventative measures). On the other hand, health investments increase over time if $r > \mu$, which means that the force of aging is relatively low making late in life investments (treatment measures) an effective way of prolonging life. The smaller γ, the more people want to smooth their health expenditures, since large one-time investments are less effective compared to smaller installments over time.

Note that it is unclear whether health investments will increase or decrease over time. However, it can be assumed that the model will predict similar investment paths for countries that are economically, technologically and culturally comparable. In similar societies the parameters will assume similar values.

The expenditure share for health ϵ_h is defined as

$$\epsilon_h := \frac{h}{h+c}. \tag{11}$$

Depending on the paths of optimal health investments h and optimal consumption c, the health expenditure share ϵ_h will fall or rise with age. The expenditure share for health will increase over the lifetime if $g_h > g_c$ or, using (9) and (10), if

$$r - \frac{1-\gamma}{\sigma}(r-\rho) > \mu. \tag{12}$$

Assuming that $r > \rho$ the condition above holds if the ratio $(1-\gamma)/\sigma$ is not too large. While a small value of σ means less incentive to smooth consumption and therefore a faster growth of consumption over time, a small value of γ increases the incentive to smooth medical expenses. The ratio $(1-\gamma)/\sigma$ is therefore a measure of the desire to smooth health investments versus the desire to smooth consumption.

As mentioned both g_c and g_h will rise with an increase of the interest rate r. But whether the expenditure share for health ϵ_h will increase depends on the desire to smooth consumption and health investments. If individuals have a greater desire to smooth consumption $(1-\gamma) < \sigma$ the health expenditure share ϵ_h will rise with an increase of r.

An increase of the force of mortality μ will incentivize individuals to invest earlier in life. The expenditure share for health will therefore decline over the life time. Health spending extends life but does not affect per period utility and is thus independent of ρ. An increase in ρ however makes consumers more impatient and therefore slows down consumption growth. Hence, the share of health expenditure will increase if agents become more impatient.

2.3 Optimal Death

The presented model is simple enough to allow for an explicit solution of the differential equations involved. The optimal time of death T can be derived from those solutions. For the boundary value problem the boundary conditions $D(0) = 0, k(0) = k_0, k(T) = \bar{k}$ and $D(T) = \bar{D}$ have to be fulfilled. Note that the terminal value T is variable. For the four dimensional dynamic system consisting of equations (5) and (8)-(10) to be fully specified the three unknowns- $c(0), h(0)$ and T - have to be determined. For simplicity's sake, we define functions $\mathcal{H}(.,.)$, $G(.,.)$ and $J(.)$ such that

$$\mathcal{H}(T,x) := \frac{\tilde{D}(T) - \bar{D}}{\mu \int_0^T e^{-\frac{\gamma x - \mu}{1-\gamma}(T-t)} dt}, \tag{13}$$

$$G(T,x) := \int_0^T e^{-xt} dt, \tag{14}$$

$$J(x) := \int_E^R e^{-xt} dt \tag{15}$$

where $\tilde{D}(t) := a + e^{\mu t}(D_0 - a)$ denotes the stock of health deficits at time t without the effect of health investments. Then integrating equations (5) and (8) and solving them for $\bar{k}$ and $\bar{D}$ respectively leads to the following set of equations for $h(0)$ and $c(0)$ (see Appendix A.2):

$$h(0) = e^{-g_h T} \left(\frac{\mathcal{H}(T,r)}{A} \right)^{\frac{1}{\gamma}} \tag{16}$$

$$c(0) = \frac{wJ(r)}{G(T, r - g_c)} - p_h e^{-g_h T} \left(\frac{\mathcal{H}(T,r)}{A} \right)^{\frac{1}{\gamma}} \frac{G(T, -g_D)}{G(T, r - g_c)} + \frac{k_0 - \bar{k} e^{-rT}}{G(T, r - g_c)}. \tag{17}$$

Solving the associated Hamiltonian for $\mathscr{H}(T) = 0$ results in (see Appendix A.3)

$$0 = u_T - \frac{e^{-\sigma g_c T}}{c(0)^\sigma} \left[\frac{\bar{D} - a}{\gamma A} p_h h(0)^{1-\gamma} e^{(1-\gamma) g_h T} - \frac{1-\gamma}{\gamma} p_h h(0) e^{g_h T} - w \mathbb{1}_{R=T} - r\bar{k} + c(0) e^{g_c T} \right] \tag{18}$$

where $u_T = \log(c(0)) + g_c T$ in case of $\sigma = 1$ and $u_T = (c(0)^{1-\sigma} \exp((1 - \sigma) g_c T) - 1)/(1 - \sigma)$ otherwise. Note that $\mathbb{1}_{R=T}$ equals 1 in case of no retirement (i.e. $R = T$) and 0 otherwise. By solving equation (18) for T the optimal time of death is obtained. The dynamic system consisting of equation (5) and (8)-(10) can therefore be solved and the optimal paths for D, k, c and h can be determined.

2.4 Implementation and Calibration

Following the work of Dalgaard & Strulik (2014) the model is calibrated to fit US data. According to Mitnitski et al. (2001) the force of aging μ is estimated to be 0.043. In order to find an estimation of the curvature parameter γ of the health production function Dalgaard & Strulik (2014) consider the health investment patterns of the United States, Canada, Australia and New Zealand. They estimate that in all those countries health spending increases annually by about 2.1%. Looking at the health Euler equation (10) it is now possible to obtain an estimate for γ. Putting $g_h = 0.021$ and $r = 0.06$ the curvature parameter can be obtained with $\gamma = 1 - (r - \mu)/g_h$.

The law of frailty (3) is not well equipped to describe the health status of individuals below the age of 20. Therefore the model is calibrated with the assumption that people are 'born' at the age of 20. The life expectancy in 2000 for an American man was 75.2, his life expectancy at age 20 is therefore 55.2 years. According to Mitnitski et al. (2002) an estimate for the initial value of the health deficit index is $D_0 = 0.0274$ and $\bar{D} = 0.1005$ at the end of life.

In order to identify the parameter a it is assumed that health technology played no significant role in restoring health status before 1900, when the life expectancy was 62 years (42 at the age of 20). Setting $A = 0$ and solving the associated differential equation $\dot{D} = \mu(D - a)$ with $D(42) = \bar{D}$ leads to $a = 0.013$.

Consumption is assumed to be constant over the life cycle, hence $\rho = r$ and the inter-temporal elasticity of substitution σ is set to unity. Therefore in the benchmark case the utility function is a logarithmic function.

In order to compare the results with Dalgaard & Strulik (2014), education and retirement are omitted in the basic calibration of the model, therefore $R = T$ and $E = 0$.

Finally, with the purpose of identifying the role of health technology and obtaining a value for A, an annual labor income of $51,335 is assumed and the relative price of health in the year 2000 is normalized to unity. Then A is calculated so that the individual dies at age 75.2 with an health deficit index of $\bar{D}$. This yields the estimate $A = 0.0139$. In table 1 the calibration of the model is summarized.

The model is calibrated to fit initial and end-of-life health deficits D_0 and $\bar{D}$ and time of death T. However the life-cycle path of health deficits is unrestricted. The same is true for wealth, where inheritance and bequest are given. The life cycle path of the health expenditure share is also unknown.

Description	Notation	Value
Capital share	α	0.33
Relative risk aversion	σ	1.0
Interest rate	r	0.06
Time preference rate	ρ	0.06
GDP per worker in 2000	y	77,003
Life expectancy at age 20 in 2000	T	55.2
Life expectancy at age 20 in 1900	T	42.0
Force of aging	μ	0.043
Health deficits at age 20	D_0	0.027
Health deficits at age 75.2	$\bar{D}$	0.10
Bequest	$k_0, \bar{k}$	0.0
Exogenous health parameter	a	0.013
Health technology (scale)	A	0.00139
Health technology (curvature)	γ	0.19
Relative price of health (normalized)	p_h	1.0
Age of retirement	R	T
Years in education	E	0

Table 1: Model calibration (see Dalgaard & Strulik, 2014, page 687, Table 1.)

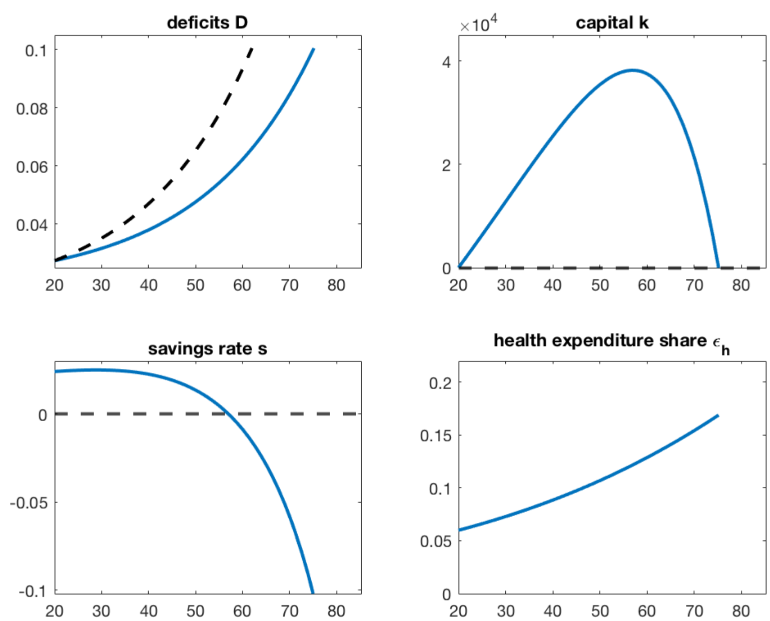

Figure 2: Optimal aging. Solid line: benchmark run with $T = 55.6$, $a = 0.013$, $A = 0.0014$, $\mu = 0.0043$, $\rho = r = 0.06$, $D_0 = 0.0274$, $\bar{D} = 0.10$, $p_h = 1$. Dashed line: no health expenditure ($A = 0$). Own simulation according to Dalgaard & Strulik (2014).

Figure 2 shows the basic run of the implemented model. The upper left panel shows the evolution of health deficits over the life time, where the dashed line describes a case without health spending and the solid line the scenario with optimal health spending as described by the model. In the upper right panel the life cycle path of an individual's wealth is displayed. It exhibits a humped shaped pattern, which is supported by the path of the savings rate $s = 1 - (c + p_h h)/(w + rk)$ presented in the lower left panel. Individuals save in early stages of life and dissave later in life. In the case of no health spending individuals do not save, since consumption is assumed to be constant ($\rho = r$) and individuals spend their entire wage w on consumption goods ($c = w$). Therefore, the dashed line in the upper right and the lower left panel coincide with the x-axis. The lower right panel shows the evolution of health expenditure $\epsilon_h = ph/(c + p_h h)$.

2.5 Comparative Dynamics

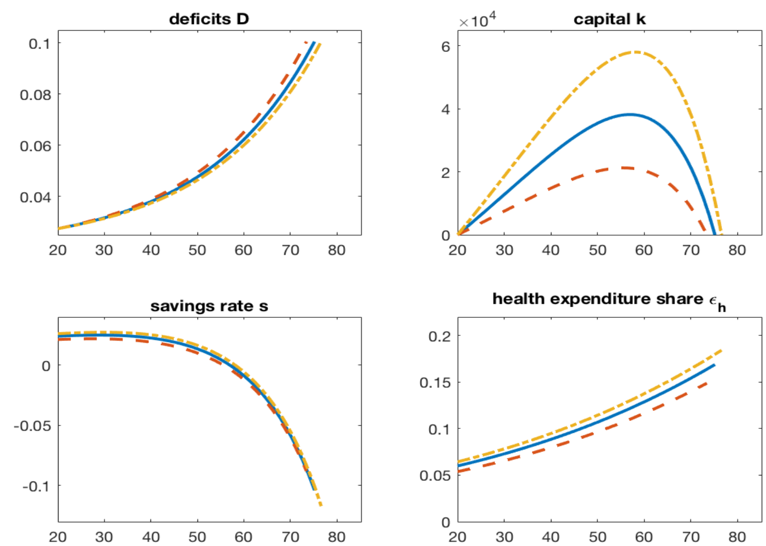

Figure 3: Variation of labor income. Blue (solid): benchmark run. Red (dashed): income decreases by $1/3$, $\Delta T = 1.73$. Yellow (dotted): income increases by $1/3$, $\Delta T = -1.43$. Own simulation according to Dalgaard & Strulik (2014).

Figure 3 shows the optimal behavior of agents when the labor income is modified. The dashed red line is associated with an income decrease by a third and the dotted yellow line describes the case where w is increased by a third. An increase in labor income increases longevity, peak wealth and the health expenditure share. Since r, ρ, μ and γ are kept constant the change in ϵ_h is brought about by a change in $h(0)/c(0)$. Since individuals prefer to smooth consumption over time, because of diminishing marginal utility of consumption, an increase in income leads to a larger increase in $h(0)$ than in $c(0)$. The life span increases by 1.73 years if wages are increased and falls by 1.43 years for a wage decrease.

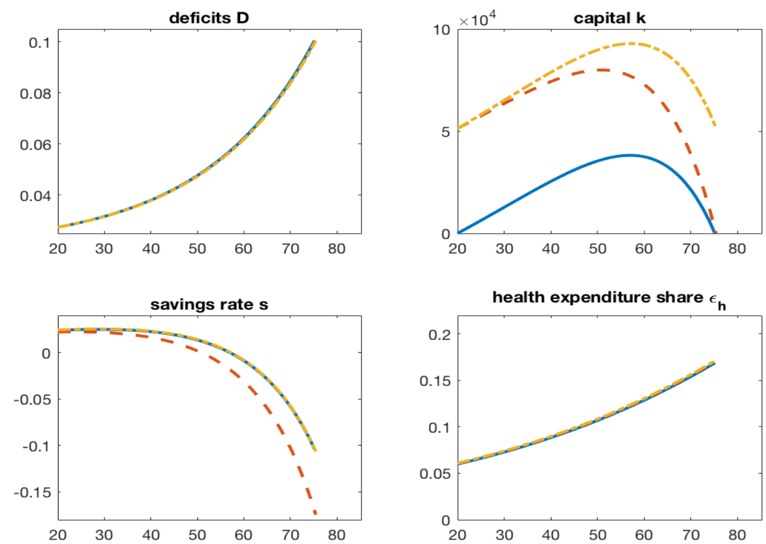

Figure 4: Variation of wealth. Blue (solid): benchmark run. Red (dashed): inheritance $k_0 = w$, bequest $\bar{k} = 0$, $\Delta T = 0.27$. Yellow (dotted): inheritance $k_0 = w$, bequest $\bar{k} = w$, $\Delta T = 0.26$. Own simulation according to Dalgaard & Strulik (2014).

Figure 4 shows how individuals behave when they receive an inheritance or have to leave a bequest. The dashed red line describes the case where agents receive an inheritance of their annual wage. The dotted yellow line characterizes the instance where agents also have to leave a bequest of one annual wage w. In both cases longevity increases by about a quarter of a year. An annual interest rate of 6% will more than double the initial bequest every twelve years. Therefore, receiving a bequest has a relatively large impact on lifetime consumption and health spending, while having to pay a bequest of the size w at the end of life has only a minor impact on the results.

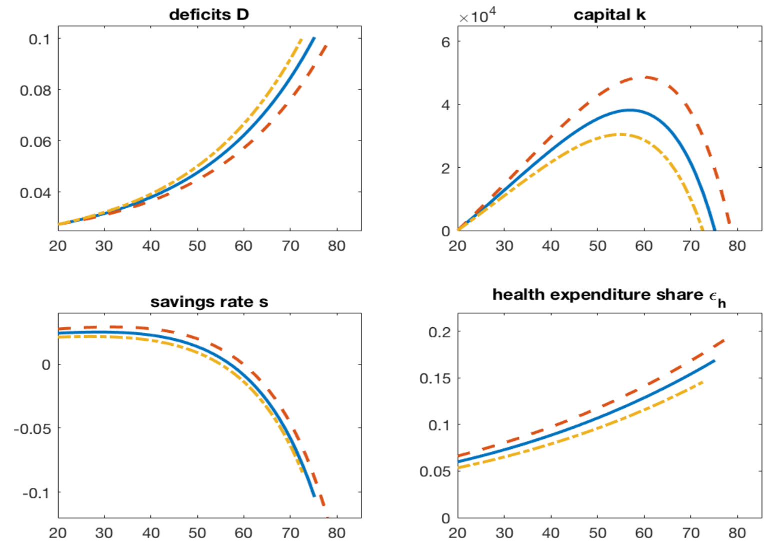

Figure 5: Variation of health prices. Blue (solid): benchmark run. Red (dashed): halving p_h, $\Delta T = 3.47$. Yellow (dotted): doubling p_h, $\Delta T = -2.59$. Own simulation according to Dalgaard & Strulik (2014).

Figure 5 shows the effect of a variation in relative health prices. The dashed red line describes the case where prices are halved and the dotted yellow line characterizes the situation when prices are doubled. When health prices increase, individuals substitute towards consumption and the health expenditure share declines. When health spending declines so do savings. An increase in relative health prices leads to a lifespan decline of 2.56 years whereas a decrease in prices prolongs life by 3.47 years.

Figure 6 illustrates the impact of health technology. The dashed red line describes the case where the health technology parameter A is decreased by a third and the yellow line shows the situation where A increases by a third. Such an increase in A prolongs life drastically by 9.39 years.

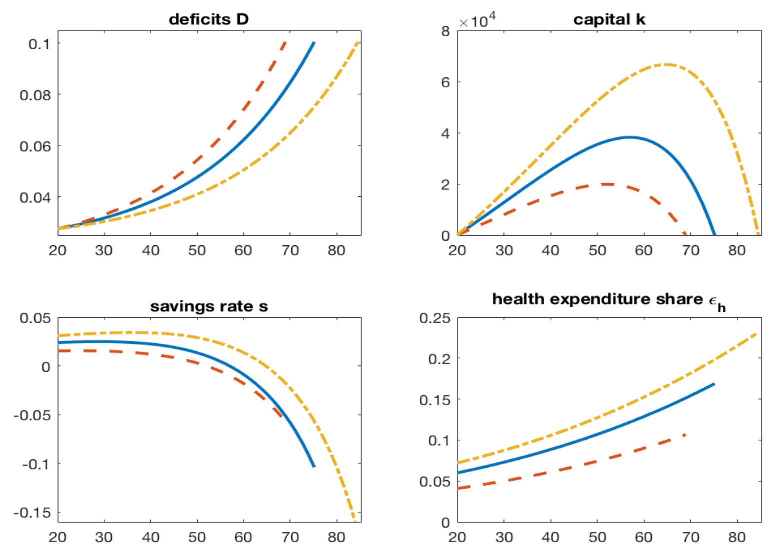

Figure 6: Variation of health technology. Blue (solid): benchmark run. Red (dashed): health technology A decreases by $1/3$, $\Delta T = -6.22$. Yellow (dotted): health technology A increases by $1/3$, $\Delta T = 9.39$. Own simulation according to Dalgaard & Strulik (2014).

Finally, Figure 7 shows the impact of retirement. The dashed red line is associated with a retirement age of 65. The upper right panel shows the need to accumulate wealth up until the age of 65 in order to finance health spending and consumption during retirement. The lower left panel illustrates the savings rate. Individuals save during their working life and dissave during retirement. Retirement does not have an impact on the health status and is therefore only associated with a lifetime wage loss. A decrease in lifetime income leads to lower health spending and therefore a shorter lifespan.

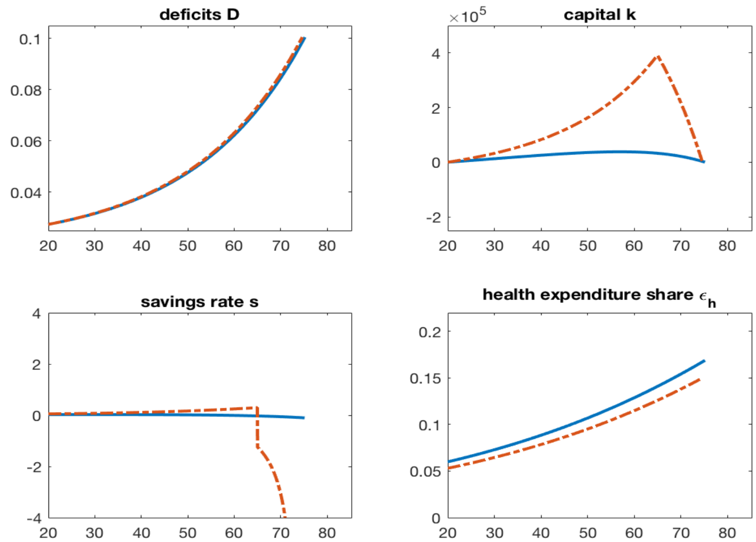

Figure 7: Variation of retirement. Blue (solid): benchmark run. Red (dashed): retirement at age 65 ($R = 45$), $\Delta T = -1.88$. Own simulation according to Dalgaard & Strulik (2014).

2.6 Extensions

There have been several extensions made to the basic health deficit model.

For instance, Schünemann et al. (2017) have looked at the effects of adapting behavior to poor health on health spending, longevity and the value of life. In this model individuals derive utility from consumption and good health. Agents measure good health by comparing their actual state of health to a reference state of health. For non-adapting individuals the reference state of health stays constant over their lifetime, while adapting agents expect their health to decline. Schünemann et al. (2017) also differentiate between naive adapting individuals, who do not take their adapting behavior into consideration when making their consumption and health spending decision, and sophisticated types that take their behavior into account. It turns out that naive adaption leads to longer and healthier life while sophisticated types spend less on health and live shorter. However, both types of adaption lead to a significantly higher value of life than not adapting at all.

Dalgaard & Strulik (2017) incorporate an optimal retirement decision into the health deficit model by assuming that aging gradually lowers wages and increases the disutility from work. The model allows to study the impact of wage growth and health technology advances on longevity and the age of retirement. The analysis of this model indicates that most of the increase in longevity in the last century can be explained by wage growth, whereas the driver behind the rising retirement age appears to be the technological progress of health care.

Dragone et al. (2018) extend the model by introducing stochastic mortality, which means individuals cannot plan when to die. They show that this scenario can be modeled by an infinite time horizon problem with a steady state of constant health, where individuals no longer accumulate health deficits and therefore no longer age in physiological terms, as a meaningful long run goal. They show that this state of 'negligible senescence' can be reached as long as the force of aging is smaller than the interest rate ($\mu < r$) and health technology is sufficiently powerful. The model is particularly well suited to study the aging process of the oldest old. Dragone et al. (2018) show that in their setup health deficits accumulate in an S-shape instead of growing exponentially over the lifetime, which is a better approximation for people over the age of 90.

A public economics model, which incorporates health deficits, is introduced by Grossmann & Strulik (2017) to study the joint design of health and retirement policy. The model tries to answer how health and retirement policy should react to the fast advances in medical and pharmaceutical research which extend the human life span. The analysis of the model shows that an increase in health technology should be accompanied by health spending to reap the benefits of more effective health care. Medical progress, however, always increases health inequality since people with a small number of health deficits profit more from the advances. Optimality also requires a lower pensions savings rate and an increase in the retirement age.

Strulik (2018) studies the link between education and health. The health status of individuals is described by their accumulated health deficits, where the accumulation can be accelerated by unhealthy behavior and slowed through health spending. Individuals differ in their return to education, with individuals endowed with a high return of education choosing more education and a healthier lifestyle. The analysis of the model shows that about half of the observed education-health gradient can be explained by differences in the return to education and that health inequality rises with advancements in health technology.

3 General Equilibrium Model with Health Deficits and a Health Care Sector

The household problem in section 2 can be seen as a partial equilibrium model where interest rate r, wage rate w and health care prices p_h are exogenously given and constant. In order to study the interplay of households and firms and to analyze how prices, $r(t)$, $w(t)$ and $p_h(t)$ will change with changes in maximum life span a general equilibrium model is formulated.

3.1 Population Structure

A stable population structure with time of death T and growth rate n is assumed. Thus, the total population at time t, denoted by $P(t)$, is

$$P(t) = B(t) \int_0^T e^{-nx} dx,$$

where $B(t)$ describes the total number of births at time t. Note that it is implicitly assumed that death always takes place after menopause. Therefore, time of death T does not affect the the total number of births.

3.2 Households

Each household consists of a representative agent. Individuals age and the aging process is described by health deficit accumulation. Deficits are accumulated according to equation (5). Individuals maximize their utility (7) through consumption and prolong their life with the help of health care goods. They face a budget constraint similar to (8). Note however that the interest rate and the wage rate are now dependent on t since in a general equilibrium model with competitive markets $r(t)$ and $w(t)$ denote the factor prices of capital and labor. The capital accumulation of a household is described by

$$\dot{k} = r(t)k(t) + w(t)\mathbb{1}_{[E,R]} - c(t) - p_h(t)h(t),$$

where $p_h(t)$ denotes the price of the health care good at time t. For simplicity, no inheritance and no bequest are assumed, hence $k(0) = \bar{k} = 0$. All results concerning the household maximization problem obtained in the previous chapter hold and are used in deriving the general equilibrium solution.

3.3 Firms

Final good sector. The production function of the final good sector is defined as

$$Y_G := F(K(t), L_g(t)) = A_g K(t)^{\alpha_g} L_g(t)^{1-\alpha_g}$$

where $K(t)$ is the capital stock and $L_g(t)$ is the labor input employed in the production of the consumption good. The production of final goods displays constant returns to scale. The total factor productivity is denoted by A_g and α_g is the capital share in the final good sector. The final good can either be saved or consumed by households.

It is assumed that the final good sector operates in a competitive market. The factors are therefore paid their marginal product. The gross interest rate, $r(t)+\delta$, is equal to the marginal product of capital and the wage rate is equal to the marginal product of labor:

$$\begin{aligned} r(t) + \delta &= \alpha_g A_g \left(\frac{K(t)}{L_g(t)} \right)^{\alpha_g - 1} \\ w(t) &= (1 - \alpha_g) A_g \left(\frac{K(t)}{L_g(t)} \right)^{\alpha_g}, \end{aligned}$$

where r denotes the net interest rate and δ the capital depreciation rate.

Health care sector. Only labor is used as an input factor in the production of health care

$$Y_H := H(L_h(t)) = A_h L_h(t)$$

where $L_h(t)$ denotes the labor input in the health care sector and A_h describes the state of technology. Health care is produced under constant returns to scale. Unlike the final good, health care goods can not be saved and hence are immediately consumed by households.

Firms in the health care sector maximize their profits

$$\Pi_H = p_h(t) H(L_h(t)) - w(t) L_h(t) = p_h(t) A_h L_h(t) - w(t) L_h(t)$$

and therefore the factor input labor is paid its real price

$$\frac{\partial H(L_h(t))}{\partial L_h(t)} = A_h = \frac{w(t)}{p_h(t)}. \tag{19}$$

Workers can choose which sector to work in and since labor is homogeneous, the wage rate in the final good sector and in the health care sector coincides.

Aggregate consumption at time t is derived by integrating per capita consumption $c(t)$ over the total population size and the goods market at time t is equal to the aggregate consumption at time t

$$C(t) = B(t) \int_0^T e^{-nx} c(x) dx.$$

The health care market is equal to the aggregate health care demand at any given time t

$$H(t) = B(t) \int_0^T e^{-nx} h(x) dx.$$

Since health care goods cannot be saved, the supply has to equal the demand at any given time t (i.e. the market has to be cleared) and therefore

$$H(t) = A_h L_h(t). \tag{20}$$

From the market clearing condition (20) we get the labor supply in the health care sector

$$L_h(t) = \frac{B(t)}{A_h} \int_0^T e^{-nx} h(x) dx. \tag{21}$$

The labor market consist of all people of working age, which either work in the health care sector or in the final good sector:

$$L_g(t) + L_h(t) = L(t) := B(t) \int_E^R e^{-nx} dx. \tag{22}$$

By plugging the market clearing condition of labor supply (21) in the health care sector into equation (22), we get

$$L_g(t) = B(t) \int_E^R e^{-nx} dx - \frac{B(t)}{A_h} \int_0^T e^{-nx} h(x) dx. \tag{23}$$

The capital market is equal to the aggregate capital at time t

$$K(t) = B(t) \int_0^T e^{-nx} k(x) dx.$$

Since there is no government and no international trade, goods can either be consumed or invested

$$Y_G(t) = I(t) + C(t).$$

Then the capital accumulation (net investment) equals gross investment minus capital depreciation

$$\dot{K}(t) = I(t) - \delta K(t) = Y_G(t) - C(t) - \delta K(t). \tag{24}$$

3.4 Equilibrium Solution

The equilibrium capital stock can be found using two different methods. First, by equating household assets with the capital employed in production. Since in equilibrium, all assets are held in terms of capital because capital is the only savings vehicle. The second method, is finding the equilibrium through break-even investment. The two methods are shown to be equivalent.

3.4.1 Equating Assets and Capital

In d'Albis (2007), the equilibrium solution is found by equating the per capita assets to the per capita capital stock employed for production, i.e. $\tilde{\kappa}(t) = K(t)/P(t)$ with $\tilde{\kappa}$ being the asset holdings.

In order to find the steady state we will take a similar approach. However since factor prices are dependent on the capital per unit of labor in the final goods sector (i.e. $K(t)/L_g(t)$), we will divide both sides of the equilibrium equation by $L_g(t)/P(t)$ in order to get

$$\kappa(t) := \tilde{\kappa}(t)\frac{1}{L_g(t)/P(t)} = \frac{K(t)}{P(t)}\frac{P(t)}{L_g(t)} = \frac{K(t)}{L_g(t)}. \tag{25}$$

By using equation (23) the ratio $L_g(t)/P(t)$ can be written as

$$\frac{L_g(t)}{P(t)} = \frac{L(t)}{P(t)} - \frac{1}{A_h}\frac{H(t)}{P(t)} = \frac{\int_E^R e^{-nx}dx}{\int_0^T e^{-nx}dx} - \frac{\int_0^T e^{-nx}h(x)dx}{A_h\int_0^T e^{-nx}dx}. \tag{26}$$

Equation (26) can be interpreted as the support ratio minus the average health care demand divided by the measure of the health care production technology A_h, which is constant over time.

Given equation (25), the steady state capital stock can be found by solving the

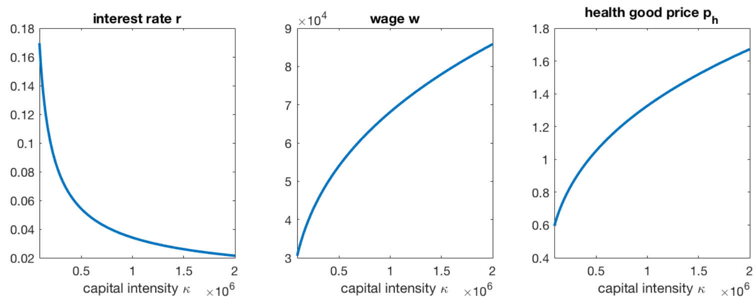

Figure 8: Rental prices r, w and p_h are functions of the capital intensity κ. Own illustration.

following equation:

$$\kappa(t) = \frac{K(t)}{L_g(t)} = \frac{B(t)\int_0^T e^{-nx}k(x)dx}{L_g(t)}.$$

Plugging in expression (51) (see appendix B.1) for $k(x)$ leads to

$$\kappa(t) = \frac{B(t)}{L_g(t)} \int_0^T e^{-nx} \int_x^T e^{-r(t-x)}[c(t) + p_h(t)h(t) - w(t)\mathbb{1}_{[E,R]}]dtdx. \quad (27)$$

The steady state is characterized by $\dot{\kappa} = 0$. At the equilibrium the rental prices are constant and functions of the capital per labor κ (see Figure 8), specifically

$$r^* = \alpha_g A_g (\kappa^*)^{\alpha_g - 1} - \delta, \quad (28)$$

$$w^* = (1-\alpha_g) A_g (\kappa^*)^{\alpha_g}, \quad (29)$$

$$p_h^* = \frac{w^*}{A_h} = \frac{(1-\alpha_g)A_g}{A_h}(\kappa^*)^{\alpha_g} \quad (30)$$

where $(^*)$ symbolizes the steady state value of a variable. For convenience we define the following steady state growth rates g_c^*, g_h^* and g_D^*, which are all dependent on the steady state interest rate r^*: $g_c^* = (r^* - \rho)/\sigma, g_h^* = (r^* - \mu)/(1-\gamma)$ and $g_D^* = (r^*\gamma - \mu)/(1-\gamma)$.

At the equilibrium, equation (27) is equal to

$$\kappa^* = \frac{B(t)}{L_g(t)} \int_0^T e^{-nx} \int_x^T e^{-r^*(t-x)}[c(t) + p_h^* h(t) - w^*\mathbb{1}_{[E,R]}]dtdx.$$

Changing the order of the integrals and substituting the consumption of final goods and health care services with the help of equations (16) and (17) leads to (see appendix B.1)

$$\kappa^* = \frac{w^*}{r^* - n}\left[\frac{G(T, n - g_c^*)}{G(T, r^* - g_c^*)}\frac{J(r^*) - \frac{G(T, g_D^*)}{A_h}\left(\frac{\mathcal{H}(T, r^*)}{A}\right)^{\frac{1}{\gamma}} e^{-g_h T}}{J(n) - \frac{G(T, n - g_h^*)}{A_h}\left(\frac{\mathcal{H}(T, r^*)}{A}\right)^{\frac{1}{\gamma}} e^{-g_h T}} - 1\right]. \quad (31)$$

By defining $N(.,.)$[3] and $U(.,.)$ such that

$$N(T, x) := \frac{J(x) - \frac{G(T, \frac{x\gamma - \mu}{1-\gamma})}{A_h}\left(\frac{\mathcal{H}(T, x)}{A}\right)^{\frac{1}{\gamma}} e^{-g_h T}}{G(T, \frac{\rho - (1-\sigma)x}{\sigma})},$$

$$U(T, x) := \frac{G(T, n - \frac{x-\rho}{\sigma})N(T, x)}{J(n) - \frac{G(T, n - \frac{x-\mu}{1-\gamma})}{A_h}\left(\frac{\mathcal{H}(T, x)}{A}\right)^{\frac{1}{\gamma}} e^{-g_h T}}$$

the steady state capital per unit of labor in the final goods sector κ^* can be written as

$$\kappa^* = \frac{w^*}{r^* - n}[U(T, r^*) - 1]. \quad (32)$$

We have therefore found an expression of the steady state capital per unit of labor in the final goods sector κ^* that is only dependent on the steady state factor prices (r^*, w^*) and the terminal age T. In order to find the steady state capital per unit of labor in the final goods sector one has to solve a fixed point problem, specifically $\phi(T, \kappa^*) = \kappa^*$, where ϕ is obtained by replacing the factor prices with their steady state values (28) and (29) in (32) such that

$$\phi(T, \kappa^*) = \frac{(1-\alpha_g)A_g(\kappa^*)^{\alpha_g}}{\alpha_g A_g(\kappa^*)^{\alpha_g - 1} - (\delta + n)}[U(T, \alpha_g A_g(\kappa^*)^{\alpha_g - 1} - \delta) - 1] = \kappa^*. \quad (33)$$

The terminal age T is given by the terminal age condition $\mathscr{H}(T) = 0$ (see eq. 18). By defining $V(.,.)$ such that

$$V(T, x) := N(T, x)^{-\sigma} e^{(\rho - x)T}\left(\frac{1}{A_h}\left[\frac{\bar{D} - a}{A\gamma}\left(\frac{\mathcal{H}(T, x)}{A}\right)^{\frac{1-\gamma}{\gamma}} - \frac{1-\gamma}{\gamma}\left(\frac{\mathcal{H}(T, x)}{A}\right)^{\frac{1}{\gamma}}\right] + N(T, x)e^{\frac{x-\rho}{\sigma}T}\right) \quad (34)$$

[3]Note that the initial consumption $c(0)$ at the steady state can then be written as $c(0) = w^* N(T, r^*)$.

the terminal age condition at the steady state can be written as (see appendix B.1)

$$\mathscr{H}(T) = u_T - (w^*)^{1-\sigma} V(T, r^*) = 0, \tag{35}$$

where $u_T = \log(c(0)) + g_c^* T$ in case of $\sigma = 1$ and $u_T = (c(0)^{1-\sigma} \exp((1-\sigma) g_c^* T) - 1)/(1-\sigma)$ otherwise. We have therefore found an expression of the terminal age condition that only depends on r^*, w^* and T. In order to make explicit that the terminal age condition depends on prices, which in turn depend on the capital-to-labor ratio κ^*, we change the notation to $\mathscr{H}(T, \kappa^*)$. At the steady state, by plugging in steady state factor prices (28) and (29), equation (35) can be rewritten as

$$\mathscr{H}(T, \kappa^*) = u_T - [(1-\alpha_g) A_g (\kappa^*)^{\alpha_g}]^{1-\sigma} V(T, \alpha_g A_g (\kappa^*)^{\alpha_g - 1} - \delta) = 0. \tag{36}$$

As a consequence, the analysis of the steady-state equilibrium turns out to be a fixed point problem consisting of equations (33) and (36) and can be written as the following system of equations:

$$\begin{pmatrix} \phi(\kappa^*, T) \\ \mathscr{H}(\kappa^*, T) \end{pmatrix} = \begin{pmatrix} \kappa^* \\ 0 \end{pmatrix}. \tag{37}$$

3.4.2 Break-even Investment

Following the work of Lau (2009) the steady solution can be found by finding the capital stock at which actual investment and break-even investment (level of current investment required to keep a constant level of capital) coincide.

As in the first method $\kappa(t)$ will denote the capital per labor $K(t)/L_g(t)$. Then the production function rewrites as

$$y_g(t) = F\left(\frac{K(t)}{L_g(t)}, 1\right) = f(\kappa(t)) = A_g \kappa(t)^{\alpha_g}.$$

Based on the overall capital accumulation (24) the dynamics of the capital per labor

$\dot{\kappa}(t)$ can be derived as follows:

$$\begin{aligned}\dot{\kappa}(t) &= \frac{\dot{K}(t)}{L_g(t)} - \kappa(t)\frac{\dot{L}_g(t)}{L_g(t)} \\ &= y_g(t) - \frac{C(t)}{L_g(t)} - (\delta + n)\kappa(t) \\ &= A_g \kappa(t)^{\alpha_g} - \frac{C(t)}{L_g(t)} - (\delta + n)\kappa(t), \end{aligned} \tag{38}$$

with n being the growth rate of L_g. [4]

The steady state is again characterized by $\dot{\kappa} = 0$ and the factor prices take on their steady state values (r^*, w^*, p^*). At the equilibrium, equation (38) is equal to

$$A_g(\kappa^*)^{\alpha_g} - \frac{C(t)}{L_g(t)} - (\delta + n)\kappa^* = 0 \tag{39}$$

which is the same as (see appendix B.1)

$$\kappa^* = \frac{1}{r^* - n}\left(\frac{C(t)}{L_g(t)} - w^*\right). \tag{40}$$

The equation above is equal to equation (52) in the appendix B.1 in the approach equating assets and capital. Hence, by proceeding in the same way as in the first approach, the fixed point equation (33) describing the steady state capital can be derived. Therefore both methods of finding the steady state are equivalent.

3.5 Implementation and Graphical Analysis

The model parameters are chosen based on the interpretation of the household problem as a partial equilibrium, since prices are fixed in the life cycle model. All variables describing human aging ($\mu, D_0, \bar{D}, a, A$ and γ) take on the same values as in section 2.4. The same is true for the time preference rate ρ and the risk aversion parameter σ (for reference, see table 1).

In order to obtain values for A_h and A_g we look at the rental price equations (28)-(29). The total factor productivities (TFP) A_h and A_g in the health sector and the final goods sector can then be obtained by rewriting and solving equations

[4] $\frac{\dot{L}_g}{L_g} = \frac{L}{L_g}\frac{\dot{L}}{L} - \frac{L_h}{L_g}\frac{\dot{L}_h}{L_h} = \frac{n}{L_g}(L - L_h) = n$

Description	Notation	Value
Capital share	α_g	0.33
Relative risk aversion	σ	1.0
Time preference rate	ρ	0.06
GDP per worker (final good sector)	y_g	77,003
Force of aging	μ	0.043
Health deficits at age 20	D_0	0.027
Health deficits at T	$\bar{D}$	0.10
Exogenous health parameter	a	0.013
Health technology (scale)	A	0.00139
Health technology (curvature)	γ	0.19
Age of retirement	R	65
Years in education	E	0
Capital depreciation rate	δ	0
Population growth rate	n	0
TFP (final good sector)	A_g	1,022
TFP (health care sector)	A_h	51,335

Table 2: Calibration of the general equilibrium model.

(28)-(29) for A_h and A_g, which leads to the following set of equations:

$$A_g = \left(\frac{r+\delta}{\alpha}\right)\left(\frac{w\alpha}{(1-\alpha)(r+\delta)}\right)^{1-\alpha}, \tag{41}$$

$$A_h = \frac{w}{p_h}. \tag{42}$$

As in section 2.4 it is assumed that the GDP per worker in the final good sector is \$77,003 and the capital share, α_g, is 1/3. The annual labor income is therefore \$51,335. We assume that there is no capital depreciation ($\delta = 0$). Then, solving equations (41) and (42) so that the interest rate r is 6%, the wage w is \$51,335 and the price p_h of the health care product is normalized to 1, we get $A_g = 1,022$ and $A_h = 51,335$. Note that the assumed values for r, w and p_h are the same as in the life cycle model in section 2.4. Finally, we assume that there is no population growth, no time spent on education after the age of 20 and people retire at the age of 65. The calibration of the model is summarized in table 2.

In order to study the equilibrium we analyze equations (33) and (36), describing the fixed point problem $(\phi(T,\kappa^*), \mathscr{H}(T,\kappa^*))' = (\kappa^*, 0)'$ which characterizes the steady state, graphically. Figure 9 depicts functions $\phi(T,\kappa)$ and $\mathscr{H}(T,\kappa)$ for different times of death T. Note that since we assumed that people are born at the age of 20, a time of death of 55 means that people die when they are 75 years

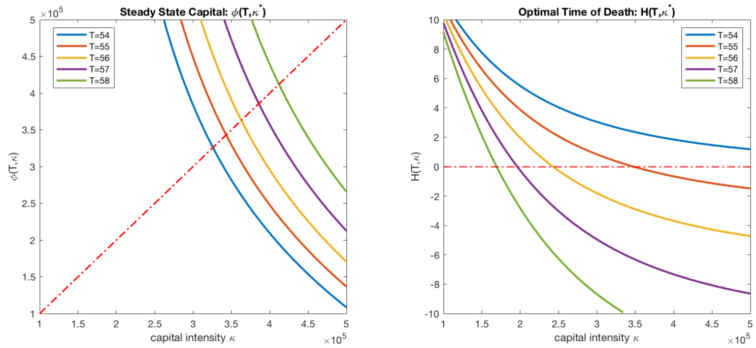

Figure 9: Left: function $\phi(T, \kappa)$ for different times of death T; Right: function $\mathscr{H}(T, \kappa)$ for different times of death T. Own simulation.

old. While the red dashed 45°-line in the left panel characterizes the values where $\phi(T, \kappa)$ equals κ, the red dashed horizontal line in the right panel illustrates where $\mathscr{H}(T, \kappa)$ equals 0.

Looking at the left panel, we can see that with a growing lifespan (larger T) the value of capital κ that solves $\phi(T, \kappa) = \kappa$ is also growing. To understand this relationship we have to look at the associated savings decisions of the households. In order to finance their longer lifespan individuals need to save more. A higher savings rate implies a higher investment rate and therefore a higher capital accumulation rate. The longer people live the higher the capital intensity κ. The right panel, on the other hand, shows a negative relationship between the values of T and κ. Function $\mathscr{H}(T, \kappa)$ captures the need for health investment in order to prolong life. Spending more on health however leaves agents with less assets to save. A lower savings rate means that less can be invested in capital accumulation. The capital intensity κ which solves $\mathscr{H}(T, \kappa) = 0$ is therefore declining as the time of death T rises.

The steady state is characterized by the pair (T^*, κ^*) that solves both $\phi(T, \kappa) = \kappa$ and $\mathscr{H}(T, \kappa) = 0$. By studying Figure 9, we can estimate where the solutions coincide and see that with the assumed calibration people die approximately at the age of 75 ($T^* = 55$) and the value of the steady state capital is close to 350,000.

Figure 10 shows the (T, κ)-plain that solve $\phi(T, \kappa) = \kappa$ and $\mathscr{H}(T, \kappa) = 0$. The intersection of the blue and red curve characterizes the steady state solution (T^*, κ^*).

Looking at the combination of (T, κ) that solve $\phi(T, \kappa) = \kappa$, it becomes apparent that a growing amount of capital per labor κ is necessary in order to finance

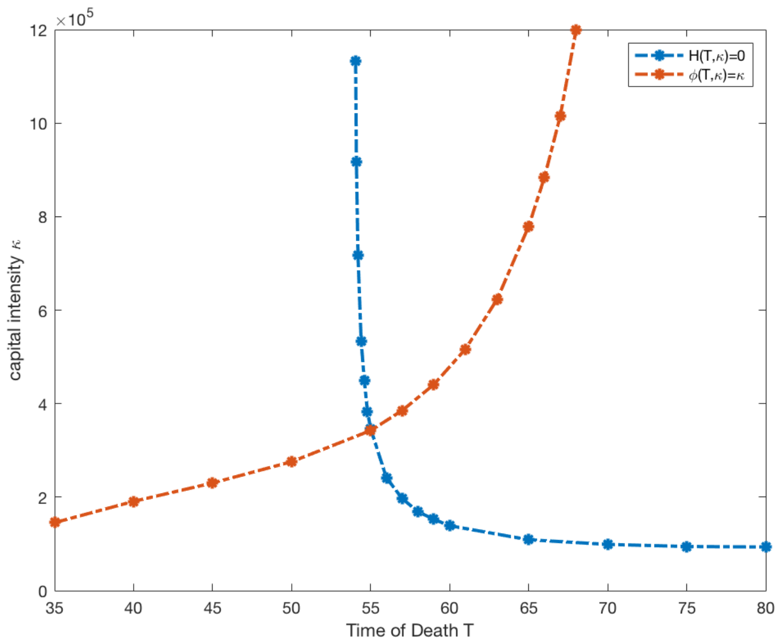

Figure 10: Solution of the fixed point problem (37). Own simulation.

additional life time. Examining the points (T,κ) that solve $\mathscr{H}(T,\kappa)=0$ we can see that for T close to 54 health spending tends towards zero. The aging process is close to a 'natural' process without medical intervention. For any T smaller than 54 the function $\mathscr{H}(T,\kappa)$ no longer intercepts the x-axis.

3.6 Comparative Dynamics

In this section the (T,κ)-plain will be used to analyze the effect of parameter changes on the pairs (T,κ) which solve $\phi(T,\kappa)=\kappa$ and $\mathscr{H}(T,\kappa)=0$ and therefore effects on the steady state solution (T^*,κ^*). In the following illustrations (Figures 11-13) points connected by a dashed line are combinations of (T,κ) which solve $\mathscr{H}(T,\kappa)=0$ and the combinations of (T,κ) connected by a dotted line solve $\phi(T,\kappa)=\kappa$.

Figure 11 depicts the solution of the fixed point problem (37) when the retirement age is modified. The blue curves are associated with the benchmark case where individuals retire at the age of 65. The red curves depict the case of early retirement at age 60 and the yellow curves characterize the solution for retirement

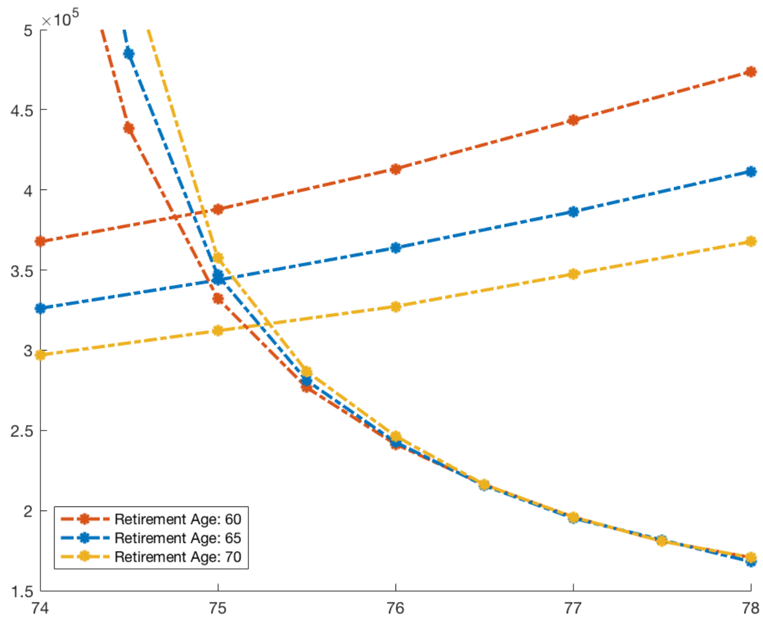

Figure 11: Solution of the fixed point problem (37) for a variation of the retirement age. Dashed lines: $\mathscr{H}(T,\kappa)=0$, dotted lines: $\phi(T,\kappa)=\kappa$.

at age 70. The intersection of two curves of the same color illustrates the steady state solution (T^*,κ^*) for each case respectively. Figure 11 demonstrates that an increase of the retirement age has a relatively large and negative effect on the capital intensity. The points (T,κ) which solve $\phi(T,\kappa)=\kappa$ shift downward for a higher retirement ages. The increase of the retirement age has only a small and positive effect on the solutions of $\mathscr{H}(T,\kappa)=0$. This can be attributed to the fact that a larger value of R has an effect on the lifetime income and hence individuals spend more on health. Therefore all changes to the time of death T can be explained by the variation of the lifetime income associated with a change of the retirement age. In summary, an increase of R marginally prolongs life and lowers κ^*.

Figure 12 illustrates the solution of the fixed point problem (37) for a variation of the health technology parameter A. The blue curves depict the benchmark case, the red curves illustrate a decrease of A by 10% and the yellow curves represent the instance where A increases by 10%. The intersection of two curves of the same color represents an approximation for the steady state solution (T^*,κ^*) for each of the cases. A modification of the health technology parameter A has a large

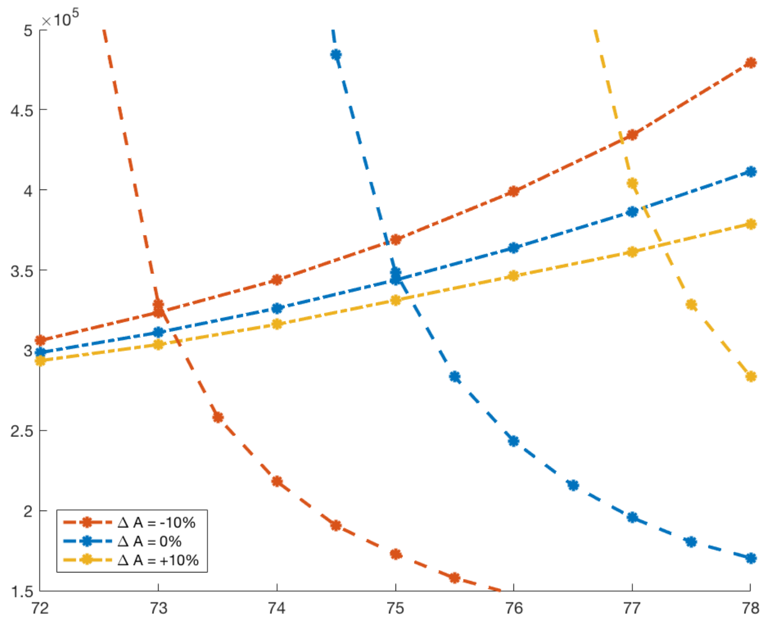

Figure 12: Solution of the fixed point problem (37) for a variation of the health technology parameter A. Dashed lines: $\mathscr{H}(T,\kappa) = 0$, dotted lines: $\phi(T,\kappa) = \kappa$.

impact on the solutions of $\mathscr{H}(T,\kappa) = 0$. In particular, given κ, an increase of the parameter A makes each investment in health more productive, which increases the time of death (i.e. shifts the points which solve $\mathscr{H}(T,\kappa) = 0$ to the right) and frees up resources which can be used to attain a higher level welfare through increases in consumption. As a consequence, savings are reduced (i.e. points which solve $\phi(T,\kappa) = \kappa$ are shifted downward as A increases). In summary, an increase in the health technology parameter A leads to a significantly longer life and increases the steady state capital stock κ^* slightly. The reverse is true for a decrease of A.

Figure 13 depicts the solution of the fixed point problem (37) for a modification of the total factor productivity (TFP) A_h of the health sector. As before, the benchmark case is depicted by the blue curves, the red curves illustrate a decrease of A_h by 10%, the yellow curves represent the instance where A_h increases by 10% and the intersection of two curves of the same color represent an approximation for the steady state solution (T^*, κ^*) of each case. An increase of the parameter A_h shifts the points which solve $\mathscr{H}(T,\kappa) = 0$ to the right and has almost no effect on the solutions of $\phi(T,\kappa) = \kappa$. The effects of an increase of the TFP A_h are similar

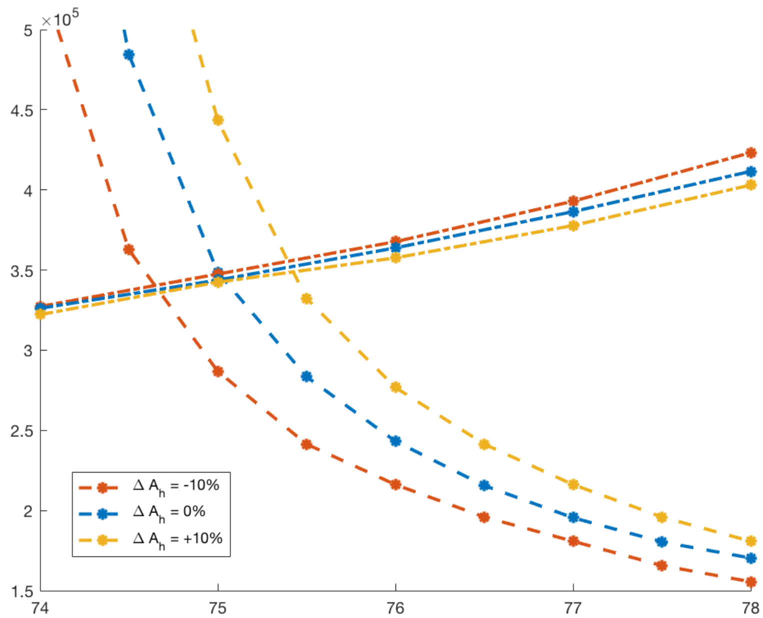

Figure 13: Solution of the fixed point problem (37) for a variation of the TFP A_h of the health sector. Dashed lines: $\mathscr{H}(T,\kappa)=0$, dotted lines: $\phi(T,\kappa)=\kappa$.

to the effects of an increase of the health technology parameter A. A larger value of A_h leads to a significantly longer life and increases the steady state capital stock κ^* marginally.

4 Comparing the Models

In this chapter, the differences of the life cycle model and the general equilibrium model will be illustrated with the help of parameter variations. As above, we look at the impact of changes of the retirement age and of health technology on the lifespan and on the steady state capital κ^* and therefore changes of the prices r^*, w^* and p_h^*. In the following illustrations the blue dashed lines always correspond to the results of the life cycle model and the red dotted lines to results of the general equilibrium model.

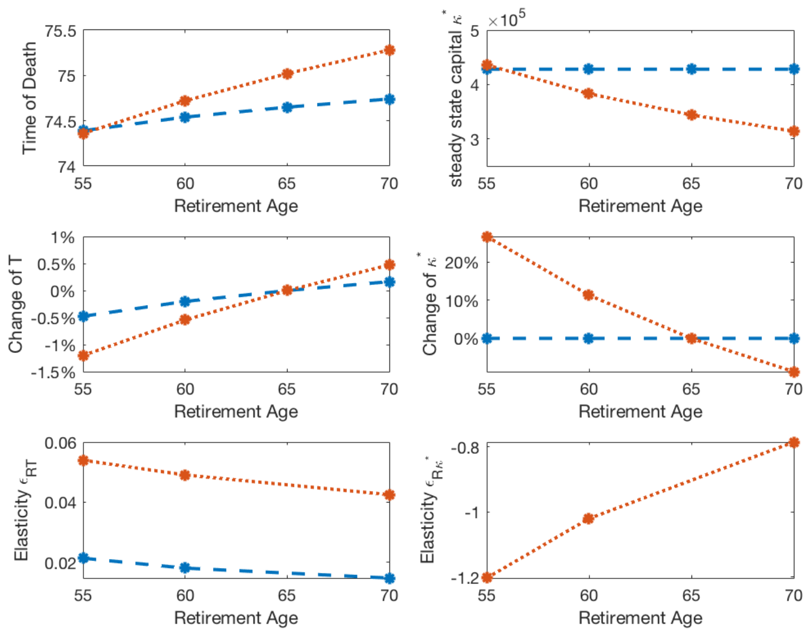

Figure 14: Variation of the retirement age. Blue (dashed): results of the life cycle model. Red (dotted): results of the general equilibrium model. Own simulation.

The effects of changes of the retirement age R on the optimal time of death and the steady state capital κ^* are illustrated in Figure 14. Looking at the top left panel we can see that in both cases an increase in the retirement age leads to a longer life. Notably, for early retirement at age 55 the life cycle model results in a longer lifespan than the general equilibrium model. However, with an increase in the retirement age the time of death predicted by the general equilibrium model rises faster and overtakes the time of death in the life cycle model. The top right panel shows the effects on the steady state capital per unit of labor κ^*. The capital intensity associated with the constant wage w and the constant interest rate r of the life cycle model is also constant at a level of approximately 428,000. The dotted red line illustrates the effects of an increase in the retirement age on the steady state capital in the general equilibrium model. For early retirement at age 55 the steady state capital is higher than the level of capital intensity in the life cycle model. An increase in the retirement age leads to a strictly decreasing steady state capital. The decrease in κ^* is accompanied by an increase in the interest rate r^* and a decrease in the steady state wage w^* and health care prices p_h^* (see Figure 8). The illustration shows that the general equilibrium model results in a longer lifetime as long as the steady state capital κ is below the level of capital employed in the life cycle model. In order to better compare the two models, the second row of panels in Figure 14 depicts the relative changes of T^* and κ^* compared to the benchmark case of each model for a variation of the retirement age R. A change of R is accompanied by a relatively small change of the time of death in both models, with the effect being more pronounced in the general equilibrium model. The left panel in the second row of Figure 14 shows a relatively large adjustment of the steady state capital in the general equilibrium model as the retirement age is varied. This dynamic is illustrated with the help of elasticities in the bottom row of panels in Figure 14. [5] With the general equilibrium model being more responsive to changes of R.

In Figure 15 the effects of changes of the health technology parameter A are illustrated. The parameter A models the influence of health technology on the accumulation of health deficits (see equation (5)). The left panel shows that with an increase in health technology the lifespan rises. The rise in lifespan is slightly larger for the general equilibrium model if the parameter A is decreased or increased up to 30%. With an increase of 40% or 50% the time of death in the life cycle model is greater than that of the general equilibrium model. Looking at the right panel we can see that the steady state capital κ^* rises as the parameter A is increased. With an increase of the health technology parameter A of 40% or 50% the level of the steady state capital κ^* is above the level of capital employed in the life cycle model. The second row of panels in Figure 15 depicts the relative changes of T and κ^* compared to the benchmark case of each model for a variation of the health technology parameter A. Varying the parameter A leads to significant changes in the length of life in both models and has a notable impact on the steady state

[5] $\epsilon_{XY} := \frac{Y-Y_0}{X-X_0}\frac{X_0}{Y_0}$ = (% change Y)/(% change X), with X_0 and Y_0 being the values of the benchmark case.

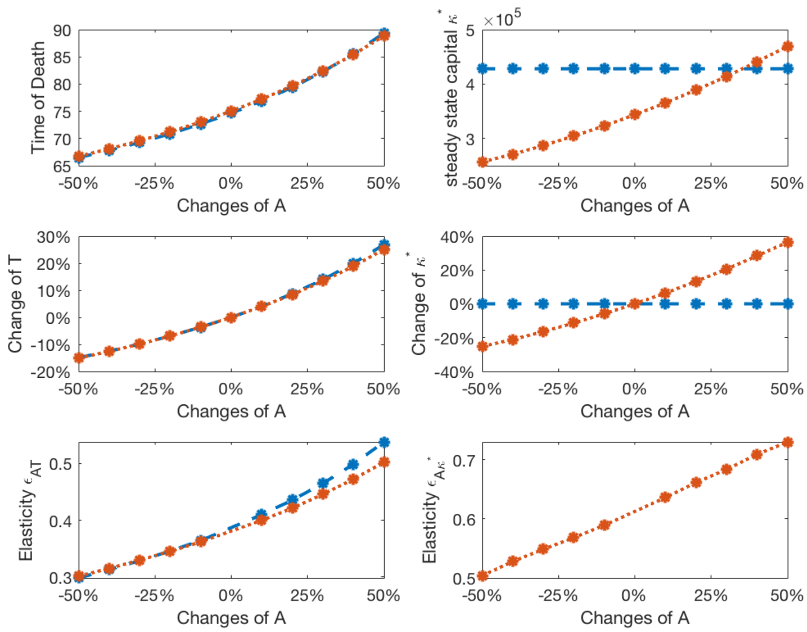

Figure 15: Variation of the health technology parameter A. Blue (dashed): results of the life cycle model. Red (dotted): results of the general equilibrium model. Own simulation.

capital κ^* in the general equilibrium model. This is emphasized by the illustrations of the elasticities ϵ_{AT} and $\epsilon_{A\kappa^*}$ in the bottom row of Figure 15.

Figure 16 shows the effects of a change of the total factor productivity (TFP) A_h in the health care sector. Since the life cycle model only considers individuals and not the production sectors, changes in A_h have no effects on the time of death in the life cycle model. However, in the general equilibrium model an increase of A_h leads to a longer lifespan. The left panel shows that if A_h is decreased by 10% the time of death in both models is approximately the same. Looking at the right panel we can see that as A_h rises the steady state capital κ^* is also increasing. It is however well below the level of capital employed in the life cycle model. The left panel in the second row of Figure 16 illustrates the percentage changes of T when the parameter is varied. A decrease of A_h of 50% shortens life by approximately 5%. This relationship is true for other changes of A_h as well and is manifested by the fact the the elasticity ϵ_{A_hT} depicted in the bottom left panel is close to 0.1. The right panel in the second row of Figure 16 depicts the percentage changes of

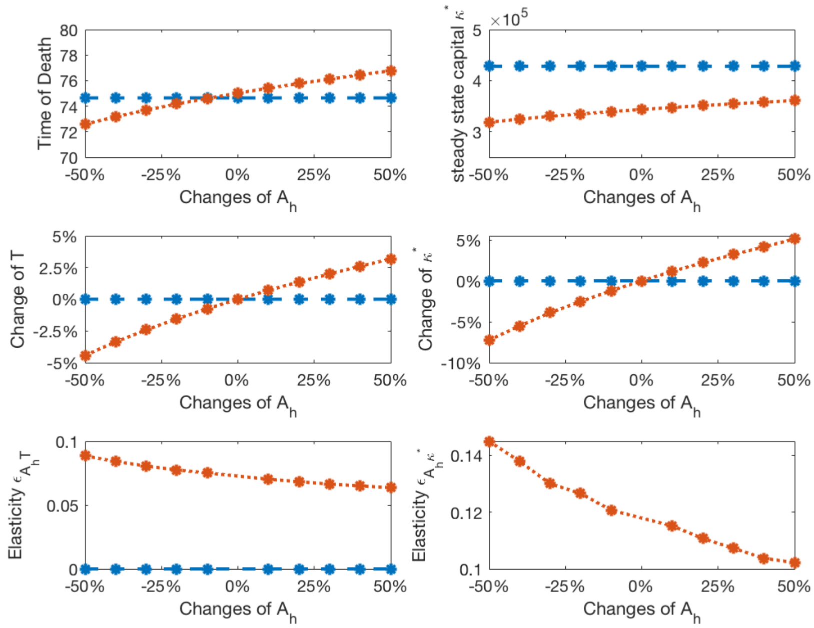

Figure 16: Variation of the TFP A_h of the health sector. Blue (dashed): results of the life cycle model. Red (dotted): results of the general equilibrium model. Own simulation.

the steady state capital κ^* for changes of A_h and the bottom right panel shows the elasticity $\epsilon_{A_h\kappa^*}$.

5 Discussion

The main motivation for this thesis is the health deficit model introduced by Dalgaard & Strulik (2014). In the framework of the model individuals maximize their lifetime utility while taking their health into consideration. The individual health status is modeled with the help of so-called health deficits, which are grounded in recent research in the fields of medicine and biology. The accumulation of such deficits can be slowed by deliberate health spending. Optimal health spending thus governs the speed of aging and therefore time of death. Hence, the health deficit model serves as a foundation for the theory of optimal aging and death based on physiology.

In this thesis, the framework of the health deficit model is used to formulate a general equilibrium model which takes physiological aging into consideration. This is achieved by having households in the general equilibrium model that face the same maximization problem as in the health deficit model. Households supply labor and capital to a final good and a health care sector and purchase consumer and health goods. The equilibrium capital stock is derived using two different methods. First, by equating assets and capital and second, through break-even investment. It is shown that the general equilibrium can be formulated as a two-dimensional fixed point problem depended on the capital per unit of labor employed in production and on the time of death. The solution of the fixed point problem is presented graphically and analyzed with the help of comparative dynamics.

The life cycle model and the general equilibrium model are compared with the help of parameter variations. It is shown graphically that allowing prices to adjust to the equilibrium results in a longer lifespan as long as the steady state capital stock is below the capital stock which is implicitly assumed by the exogenously given prices of the life cycle model.

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A Appendix: Health Deficit Model

A.1 First Order Conditions

Derivation of (9) and (10). The Hamiltonian associated with the problem of maximizing (7) subject to (5) and (8) reads as

$$\mathscr{H} = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda\left[\mu(D-a-Ah^{\gamma})\right] + \phi\left[rk+w-c-p_h h\right].$$

For $\sigma = 1$ the first term is replaced by $\log(c)$.

The co-state variables λ and ϕ can be economically interpreted in the following way (e.g. Caputo, 2005, page 86, equation 21):

$$\lambda(t) = \frac{\partial}{\partial D(t)} \int_t^T u(*)e^{-\rho s} ds$$
$$\phi(t) = \frac{\partial}{\partial k(t)} \int_t^T u(*)e^{-\rho s} ds$$

where T denotes the optimal time of death and $u(*)$ the maximized utility function. Thus, for example, $\lambda(t)$ describes the marginal loss of utility during the remaining lifetime from an additional unit of health deficits $D(t)$.

The first order conditions with regards to c and h are

$$\frac{\partial \mathscr{H}}{\partial c} = c^{-\sigma} - \phi = 0 \tag{43}$$
$$\frac{\partial \mathscr{H}}{\partial h} = -\lambda\mu A\gamma h^{\gamma-1} - p_h\phi = 0. \tag{44}$$

The co-state equations are

$$\dot{\lambda} = -\frac{\partial \mathscr{H}}{\partial D} + \lambda\rho = -\lambda\mu + \lambda\rho \tag{45}$$
$$\dot{\phi} = -\frac{\partial \mathscr{H}}{\partial k} + \phi\rho = -\phi r + \phi\rho. \tag{46}$$

Taking the logarithmic derivative of (43) and (44) leads to

$$\frac{\dot{c}}{c} = -\frac{1}{\sigma}\frac{\dot{\phi}}{\phi} \tag{47}$$
$$\frac{\dot{h}}{h} = \frac{1}{1-\gamma}\left(\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\phi}}{\phi}\right). \tag{48}$$

Equation (48) shows that growth rate of health expenditure h depends positively on the growth rate of λ/ϕ. Thus, if the shadow price of health λ grows faster than the shadow price of consumption ϕ the future health status will contribute more to the remaining lifetime utility than future consumption and therefore it is optimal to increase health spending over time, i.e. $\dot{h}/h$ is positive. By plugging (45) and (46) into the equations (47) and (48) we obtain the Euler equations for consumption (9) and for health investments (10).

A.2 Initial Conditions h(0) and c(0)

Derivation of (16) and (17). Integrating the health deficit accumulation equation (5), given $D(0) = D_0$, leads to:

$$D(t) = a + e^{\mu t}(D_0 - a) - \mu A e^{\mu t} \int_0^t e^{-\mu s} h(s)^\gamma ds. \tag{49}$$

For notational convenience, let us define the stock of health deficits at time t without the effect of health investments as $\tilde{D}(t) = a + e^{\mu t}(D_0 - a)$. With the help of the health Euler equation (10), $h(t)$ can be written as

$$h(t) = h(0)e^{g_h t} = h(T)e^{-g_h(T-t)}.$$

By substituting and evaluating equation (49) at time of death T, we get

$$\begin{aligned}\bar{D} = D(T) &= \tilde{D}(T) - \mu A e^{\mu T} \int_0^T e^{-\mu t} h(T)^\gamma e^{-g_h \gamma (T-t)} dt \\ &= \tilde{D}(T) - \mu A h(T)^\gamma \int_0^T e^{-g_D(T-t)} dt.\end{aligned}$$

Note that $\gamma g_h - \mu \equiv (\gamma r - \mu)/(1-\gamma) := g_D$. Rearranging terms leads to

$$h(T) = \left(\frac{\tilde{D}(T) - \bar{D}}{\mu A \int_0^T e^{-g_D(T-t)} dt} \right)^{\frac{1}{\gamma}}.$$

By using equation (13) this simplifies to

$$h(T) = \left(\frac{\mathcal{H}(T, r)}{A} \right)^{\frac{1}{\gamma}}$$

and therefore

$$h(t) = e^{-g_h(T-t)} \left(\frac{\mathcal{H}(T,r)}{A} \right)^{\frac{1}{\gamma}}. \tag{50}$$

By evaluating the expression above at time $t = 0$ equation (16) in the text is obtained.

Next, integrate the dynamics of capital (8) with $k(T) = \bar{k}$ to obtain

$$k(t) = \int_t^T e^{-r(x-t)}[c(x) + p_h h(x) - \mathbb{1}_{[E,R]} w]dx + \bar{k}e^{-r(T-t)}. \tag{51}$$

By evaluating at the time of birth $t = 0$ we get

$$k_0 = \int_0^T e^{-rt}[c(t) + p_h h(t)]dt - w \int_E^R e^{-rt} dt + \bar{k}e^{-rT}.$$

Rearranging the equation above leads to

$$\int_0^T e^{-rt} c(t)dt = w \int_E^R e^{-rt} dt - p_h \int_0^T e^{-rt} h(t)dt + k_0 - \bar{k}e^{-rT}$$

Using the Euler equation (9), $c(t)$ can be written as

$$c(t) = c(0)e^{g_c t} = c(T)e^{-g_c(T-t)}.$$

By substituting for $c(t)$ and plugging in equation (50) for $h(t)$ we get

$$c(0) \int_0^T e^{-rt} e^{g_c t} dt = \\ w \int_E^R e^{-rt} dt - p_h \left(\frac{\mathcal{H}(T,r)}{A} \right)^{\frac{1}{\gamma}} \int_0^T e^{-rt} e^{-g_h(T-t)} dt + k_0 - \bar{k}e^{-rT}.$$

By using (14) the equation above simplifies to equation (17) in the text.

A.3 Optimal Time of Death

Derivation of (18). Optimality requires the Hamiltonian to equal 0 ($\mathscr{H}(T) = 0$) at the time of death, otherwise living longer or dying earlier would increase utility. This condition is derived from the dynamic envelope theorem (e.g. Caputo, 2005, page 232, Theorem 9.1):

$$\frac{\partial}{\partial T}\int_t^T u(*)e^{-\rho s}\,ds = \frac{\partial}{\partial T}\int_0^T \mathscr{H}(t)\,dt = \mathscr{H}(T) = 0.$$

The equation above indicates that $\mathscr{H}(t)$ represents the marginal value of remaining lifetime. The age at which life extension no longer brings additional utility defines the optimal length of life T. Noting $D(T) = \bar{D}$ and $k(T) = \bar{k}$ the Hamiltonian reads:

$$\begin{aligned}\mathscr{H}(T) = u(c(T)) + \lambda(T)\left[\mu(\bar{D} - a - Ah(T)^{\gamma})\right]& \\ + \phi(T)\left[r\bar{k} + w(T) - c(T) - p_h h(T)\right] &= 0.\end{aligned}$$

Inserting $\lambda(T)$ and $\phi(T)$ from (45) and (46) yields

$$u_T - \frac{1}{c(T)^{\sigma}}\left[\frac{(\bar{D}-a)}{\gamma A}p_h h(T)^{1-\gamma} - \frac{1-\gamma}{\gamma}p_h h(T) - w(T) - r\bar{k} - c(T)\right] = 0,$$

where $u_T = \log(c(T))$ in case of log-utility and $u_T = (c(T)^{1-\sigma} - 1)/(1-\sigma)$ otherwise. Note that $w(T) = w$ in case of no retirement (i.e. $R = T$) and 0 otherwise. Inserting $h(T) = h(0)\exp(g_h T)$ and $c(T) = c(0)\exp(g_c T)$ in the equation above leads to equation (18) in the text.

B Appendix: General Equilibrium Model

B.1 Steady State Capital

Derivation of (31). By switching integration order and with the help of equation (14), the per capita assets can be written as

$$\begin{aligned}\kappa^* &= \frac{B(t)}{L_g(t)}\int_0^T\int_0^t e^{-x(n-r^*)}e^{-r^*t}(c(t) + p_h^* h(t) - \mathbb{1}_{[E,R]}w^*)dxdt \\ &= \frac{1}{r^*-n}\frac{B(t)}{L_g(t)}\int_0^T [e^{(r^*-n)t} - 1]e^{-r^*t}(c(t) + p_h^* h(t) - \mathbb{1}_{[E,R]}w^*)dt \\ &= \frac{1}{r^*-n}\frac{B(t)}{L_g(t)}\Bigg(\int_0^T e^{-nt}(c(t) + p_h^* h(t) - \mathbb{1}_{[E,R]}w^*)dt \\ &\qquad - \int_0^T e^{-r^*t}(c(t) + p_h^* h(t) - \mathbb{1}_{[E,R]}w^*)dt\Bigg)\end{aligned}$$

Note that the second term is equal to $k(0) = 0$, therefore

$$\begin{aligned}\kappa^* &= \frac{1}{r^*-n}\frac{B(t)}{L_g(t)}\left(\int_0^T e^{-nt}(c(t)+p_h^* h(t)dt - w^*\int_E^R e^{-nt}dt\right)\\ &= \frac{1}{r^*-n}\frac{1}{L_g(t)}\left[C(t)+p_h^* H(t) - w^* L(t)\right].\end{aligned}$$

By using equations (19) and (22) the term $w^* L(t)$ can be rewritten as $w^* L_g(t) + p_h^* A_h L_h(t)$. Using the market clearing condition (20) leads to

$$\begin{aligned}\kappa^* &= \frac{1}{r^*-n}\frac{1}{L_g(t)}\left[C(t)+p_h^* H(t) - w^* L_g(t) - p_h^* H(t)\right]\\ &= \frac{1}{r^*-n}\left(\frac{C(t)}{L_g(t)} - w^*\right)\\ &= \frac{1}{r^*-n}\left(\frac{\int_0^T e^{-nx}c(x)}{\int_E^R e^{-nx}dx - \frac{1}{A_h}\int_0^T e^{-nx}h(x)dx} - w^*\right)\end{aligned} \tag{52}$$

By substituting for $c(x)$ and $h(x)$ and using equations (13)-(15) we get

$$\kappa^* = \frac{1}{r^*-n}\left(\frac{c(0)G(T, n-g_c^*)}{J(n) - \frac{G(T,n-g_h^*)}{A_h}\left(\frac{\mathcal{H}(T,r^*)}{A}\right)^{\frac{1}{\gamma}} e^{-g_h^* T}} - w^*\right)$$

By plugging in equation (17) for $c(0)$, we get

$$\kappa^* = \frac{1}{r^*-n}\left(\frac{G(T,n-g_c^*)}{G(T,r^*-g_c^*)}\frac{w^* J(r^*) - p_h^* G(T, g_D^*)\left(\frac{\mathcal{H}(T,r^*)}{A}\right)^{\frac{1}{\gamma}} e^{-g_h^* T}}{J(n) - \frac{G(T,n-g_h^*)}{A_h}\left(\frac{\mathcal{H}(T,r^*)}{A}\right)^{\frac{1}{\gamma}} e^{-g_h^* T}} - w^*\right)$$

Using the fact that $p_h^* = w^*/A_h$ and rearranging terms leads to equation (31) in the text.

Derivation of (36). By using the fact that at the steady state $c(0) = w^* N(T, r^*)$ the terminal age condition (18) can be rewritten as

$$\begin{aligned}\mathscr{H}(T) = u_T - (w^*)^{-\sigma} N(T,r^*)^{-\sigma} e^{-\sigma g_c^* T}&\left(p_h^*\left[\frac{\bar{D}-a}{A\gamma}\left(\frac{\mathcal{H}(T,r^*)}{A}\right)^{\frac{1-\gamma}{\gamma}}\right.\right.\\ &\left.\left.-\frac{1-\gamma}{\gamma}\left(\frac{\mathcal{H}(T,r^*)}{A}\right)^{\frac{1}{\gamma}}\right] + w^* N(T,r^*)e^{g_c^* T}\right).\end{aligned}$$

Using the fact that $p_h^* = w^*/A_h$ and (34) leads to to equation (35) in the text.

Derivation of (40). By adding $\alpha_g A_g(\kappa^*)^{\alpha_g}$ to both sides of equation (39) we get

$$\alpha_g A_g(\kappa^*)^{\alpha_g} - (\delta + n)k^* = \frac{C(t)}{L_g(t)} - (A_g(\kappa^*)^{\alpha_g} - \alpha_g A_g(\kappa^*)^{\alpha_g})$$

with the term in parenthesis being equal to the steady state wage w^*. Factoring out κ leads to

$$\kappa^*[\alpha_g A_g(\kappa^*)^{\alpha_g - 1} - (\delta + n)] = \frac{C(t)}{L_g(t)} - w^*$$

and by substituting the steady state interest rate (28) we get equation (40) in the text.