Dissertation

Hartman measurable sets and functions

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von

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Kurzfassung

Gegenstand dieser Arbeit ist der Begriff der Hartman Meßbarkeit von Mengen bzw. Funktionen auf einer topologischen Gruppe. Der Name leitet sich vom polnischen Mathematiker S. Hartman ab, welcher u.a. auf dem Gebiet der Gleichverteilung von Folgen auf nichtkompakten topologischen Gruppen arbeitete. Grundidee dabei ist, einer Teilmenge H der topologischen Gruppe G, welche als nichtkompakt angenommen werden kann, eine gewisse - zunächst nur für Teilmengen kompakter Gruppen definierte - Eigenschaft zuzusprechen, wenn $\iota(H)$ diese Eigenschaft besitzt und ι ein stetiger Homomorphismus von H in eine kompakte Gruppe C ist. Ähnliches läßt sich für auf G definierte Funktionen f bewerkstelligen, wenn man zu Funktionen der Gestalt $F \circ \iota$ mit einer auf der kompakten Gruppe H definierten Funktion F übergeht.

Kapitel 1: Wir beginnen mit einigen allgemeinen Bemerkungen über Gruppenkompaktifizierungen und fastperiodische Funktionen. Insbesondere führen wir die Bohrkompaktifizierung (ι_b, bG) einer topologischen Gruppe G ein und zeigen, dass eine umkehrbar eindeutige Korrespondenz zwischen fastperiodischen Funktionen auf G und stetigen Funktionen auf bG besteht. Der Inhalt dieses Kapitels orientiert sich an den entsprechenden Resultaten in [BJM] und [HR].

Kapitel 2: Wir führen den Begriff der Hartman Meßbarkeit für Teilmengen einer topologischen Gruppe G ein: Eine Menge $H \subseteq G$ ist genau dann Hartman meßbar, wenn es eine Stetigkeitsmenge $M \subseteq M$ gibt (eine Stetigkeitsmenge ist eine Menge, deren topologischer Rand eine Nullmenge bezüglich des Haarschen Maßes ist) und eine Gruppenkompaktifizierung (ι, C), sodass $H = \iota^{-1}(M)$. Das System aller Hartman meßbaren Mengen auf G ist eine Boolesche Algebra und es gibt genau ein normiertes endlich additives und translationsinvariantes Maß auf dieser Booleschen Algebra. Die Resultate dieses Kapitels orientieren sich vor allem an [FPTW] und [Win].

Kapitel 3: In Verallgemeinerung von Kapitel 2 definieren wir die Hartman Meßbarkeit für Funktionen wie folgt: Eine Funktion $f : G \to \mathbb{C}$ ist genau dann Hartman meßbar, wenn es eine Riemann integrierbare Funktion $F : C \to \mathbb{C}$ gibt (eine Funktion ist Riemann integrierbar, wenn die Menge ihrer Unstetigkeitsstellen eine Nullmenge bezüglich des Haarschen Maßes ist) und eine Gruppenkompaktifizierung (ι, C), sodass $f = F \circ \iota$. Insbesondere interessieren wir uns für die Struktur der Menge aller Gruppenkompaktifizierungen (ι, C) von G, sodass eine gegebene Hartman meßbare Funktion f als $F \circ \iota$ mit einer auf C definierten Riemann integrierbaren Funktion F dargestellt (wir sagen auch *realisiert*) werden kann. Ist G eine LCA Gruppe mit separablem Dual, so ist eine solche Realisierung stets schon auf einer metrisierbaren Gruppe C möglich (Theorem 2). In wichtigen Spezialfällen läßt sich eine Realisierung auch mit Hilfe des Fourierspektrums der Hartman meßbaren Funktion f angeben (Theorem 5). Da sich eine fastperiodische Funktion f stets als $F \circ \iota$ mit einer auf einer Gruppenkompaktifizierung (ι, C) definierten stetigen Funktion F darstellen läßt, ist jede fastperiodische

Funktion auch Hartman meßbar. Eine bekannte und gut untersuchte Verallgemeinerung der fastperiodischen Funktionen stellen die schwach fastperiodischen Funktionen dar. In Theorem 6 zeigen wir, dass es i.a. Hartman meßbare Funktionen gibt, welche nicht schwach fastperiodisch sind. Theorem 9 dagegen zeigt, dass C_0-Funktionen, i.e. Funktionen die im Unendlichen verschwinden, stets Hartman meßbar sind. Wir beschäftigen uns außerdem mit dem Zusammenhang zwischen Fouriertransformation von Maßen und Hartman Meßbarkeit. Die Resultate dieses Kapitels entstammen zum Großteil [Mar] und [MW].

Kapitel 4: Wir zeigen die Existenz eines kompakten Hausdorffraumes hG, sodass jede stetige Funktion auf hG in eindeutiger Weise einer Hartman meßbaren Funktion auf G entspricht und diskutieren das Darstellungsproblem für hG. Abschließend führen wir die Funktionenklasse der Banach fastperiodischen Funktionen (BAP) ein und zeigen Verknüpfungspunkte zu den Hartman meßbaren Funktionen auf. Die Resultate dieses Kapitels stammen zum Teil aus einer Kooperation mit M. Beiglböck, vgl. [BM].

Abstract

Main topic of this paper is the concept of Hartman measurability of sets and functions defined on some topological group G. Nomenclature is in honor of the polish mathematician S. Hartman who worked on equidistribution of sequences on noncompact groups. The main idea is the following: given a subset H of a noncompact group G, we say that H has a certain property (e.g. being equidistributed) iff $\iota(H)$ has this property, for $\iota : G \to C$ a group homomorphism, and C a compact group. Similarly we can extend this concepts to functions by considerung functions $f : G \to \mathbb{C}$ which have a representation of the form $F \circ \iota$ for a function F defined on the compact group H.

Chapter 1: We introduce group compactifications and collect some basic facts regarding almost periodic functions. We show that almost periodic functions on G are in one-one correspondence to the continuous functions defined on the Bohr compactification (ι_b, bG). The results presented in this chapter are mainly from [BJM] and [HR].

Chapter 2: We introduce the concept of Hartman measurability: Given a topological group G and a subset $H \subseteq G$, we say that H is Hartman measurable if there exists a continuity set $M \subseteq C$ (recall that a set is a continuity set iff its topological boundary is a null set with respect to the Haar measure) and a group compactification (ι, C) such that $H = \iota^{-1}(M)$. The system of all Hartman measurable sets on G is a Boolean algebra. There exists exactly one normalized finitely additive and translation invariant measure on this Boolean algebra. [FPTW] and [Win] are resources for the results of this chapter.

Chapter 3: Generalizing the results of the previous chapter we define Hartman measurability for functions: a function $f : G \to \mathbb{C}$ is Hartman measurable iff there exists a Riemann integrable function $F : C \to \mathbb{C}$ (recall that a function is Riemann integrable when its set of discontinuities is a null set with respect to the Haar measure) and a group compactification (ι, C) such that $f = F \circ \iota$. For a given Hartman measurable function f we are interested in the structure of all group compactifications (ι, C) such that there is a Riemann integrable $F : C \to \mathbb{C}$ with $f = F \circ \iota$. In this situation we say that F realizes f on C. If G is an LCA group with separable dual, we prove that exists a realization with metrizable C (Theorem 2). In some situations it is possible to use the Fourier spectrum of the Hartman measurable function f to obtain a group compactification on which f can be realized (Theorem 5). For every almost periodic function $f : G \to \mathbb{C}$ there exists a group compactification (ι, C) and a continuous Function $F : C \to \mathbb{C}$ such that $f = F \circ \iota$. Hence almost periodic functions are Hartman measurable. As the weakly almost periodic functions generalize almost periodic functions, it is a relevant question whether they are also a generalization of Hartman measurable functions. We give a negative answer in Theorem 6, showing that in general there are Hartman measurable functions which are not weakly almost periodic. Theorem 9 shows that every C_0-function, i.e. a function vanishing at infinity, is

Hartman measurable. We consider Fourier transformations of measures and give an example of a measure inducing a nontrivial Hartman measurable function. The results of this chapter are taken from [Mar] and [MW].

Chapter 4: We prove the existence of a compact Hausdorff space hG such that every continuous function on hG corresponds to a unique Hartman measurable function f on G. Every Hartman measurable function can be obtained in such a way. Finally we introduce Banach almost periodicity and connect this concept with Hartman measurability. The results of this chapter were partly obtained in cooperation with M. Beiglböck, cf. [BM].

Contents

Chapter 1

Introduction

1.1 Preliminaries and Notation

A left invariant mean (L.I.M.) is a functional m assigning to each bounded function (of a certain subspace of bounded functions) a complex number in a linear way, such that for real-valued functions f the following two assertions hold

1. $\inf |f| \leq m(|f|) \leq \sup |f|$ (hence "mean"),
2. $m(f_x) = m(f)$ for all left translations $f_x : a \mapsto f(ax)$ (hence "invariant").

On a compact[1] group G existence and uniqueness of Haar measure implies that $m : f \mapsto \int f d\mu_G$, with the Haar measure μ_G on G, is the unique L.I.M. on the space $C(G)$ of continuous functions on G. On non compact locally compact groups this direct approach is no longer possible because $\mu_G(G) = \infty$ and there is no way to normalize the Haar measure. Nevertheless, if the group G is amenable (e.g. every abelian group is) there are plenty of invariant means defined even on the larger algebra of bounded functions $B(G)$. Of particular interest are those bounded functions on which all L.I.M.s defined on $B(G)$ take the same value. They are denoted by $AC(G)$ and called *almost convergent.* In fact $AC(G)$ is a linear subspace of $B(G)$ (but in general not a subalgebra). Well known subspaces of $AC(G)$ are those of almost periodic functions $\mathcal{A}(G)$ and the weakly almost periodic functions $\mathcal{W}(G)$. As standard references we mention the monographs [Gre] and [Pat].

As we will see in section 1.2 the algebra of (weakly) almost periodic functions $\mathcal{A}(G)$ (resp. $\mathcal{W}(G)$) is isomorphic to the algebra of continuous functions $C(bG)$ (resp. $C(wG)$). From the viewpoint of universal objects, (ι_b, bG) turns out to be the maximal topological group compactification, while (ι_w, wG) turns out as

[1]Throughout this paper "compact" always includes the Hausdorff separation axiom.

the maximal semitopological semigroup compactification of G. Regarding compactifications we will stick to the notations established in [BJM] and used in [FPTW, SchSW, Win]. For a modern and extensive treatment of (weakly) almost periodicity we refer to the monograph [BJM].

The greek letter μ will denote an arbitrary Borel measure, the latin letter m a mean resp. a finitely additive measure. The measure theoretic completion of μ is denoted by $\tilde{\mu}$. The support $\mathrm{supp}(\mu)$ of the Borel measure μ on the topological space X is defined by the requirement that $X \setminus \mathrm{supp}(\mu)$ is the union of all open μ-null sets. As usual we denote the set all of continuous (complex-valued) functions defined on the topological space X by $C(X)$, the set of all bounded functions defined on X by $B(X)$ and the set of all bounded continuous functions by $C_b(X)$. Both $B(X)$ and $C_b(X)$ equipped with the norm $\|f\|_\infty := \sup_{x\in X} |f(x)|$ are Banach spaces.

The filter of all neighborhoods of a point x in the topological space X will be denoted by $\mathfrak{U}(x)$.

For two subsets $A, B \subseteq X$ of a metric space let $\mathrm{dist}(A, B) := \inf_{x\in A, y\in B} d(x, y)$ denote the distance between A and B. We will say that A and B are separated if $\mathrm{dist}(A, B) > 0$. The diameter of a set $A \subseteq X$ is defined by $\mathrm{diam}(A) := \sup_{x,y\in A} d(x, y)$, the open ball with center x and radius ε w.r.t. the metric d is defined by $K_d(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ When it causes no confusion the subscript d will be omitted.

The dual of a locally compact abelian group G will be denoted by $\hat{G}$. As a standard reference for topological groups resp. LCA groups we refer to [Arm] and [HR]. For a function $f : X \to Y$ and $A \subseteq Y$ we will occasionally use the notation $[f \in A] := \{x \in X : f(x) \in A\} = f^{-1}(A)$. The characteristic function $1\!\!1_A$ of a set A is defined by the requirement $1\!\!1_A(x) = 1$ for $x \in A$ and $1\!\!1_A(x) = 0$ otherwise. $A\Delta B$ denotes the symmetric difference of sets: $1\!\!1_{A\Delta B} = |1\!\!1_A - 1\!\!1_B|$.

1.2 Almost periodic functions

The concept of almost periodicity was first introduced by Harald Bohr in a series of papers published in *Acta Mathematatica* between 1923 and 1926.

Definition 1.2.1. *A continuous function $f : \mathbb{R} \to \mathbb{C}$ is called almost periodic if for every $\varepsilon > 0$ the set $P(f; \varepsilon) := \{x \in \mathbb{R} : \forall y \in \mathbb{R}\ |f(x + y) - f(y)| < \varepsilon\}$ of ε-almost periods has bounded gaps, i.e. there exists $M = M(\varepsilon) > 0$ such that $P(f; \varepsilon)$ intersects every interval $[a, b] \subseteq \mathbb{R}$ with length $b - a \geq M$.*

Note that for periodic f we have $P(f; \varepsilon) \supseteq p \cdot \mathbb{Z}$, where p is a primitive period of f. Hence every continuous periodic function is also almost periodic. It is also clear that every almost periodic function is bounded since

$$|f(y)| \leq |f(y) - f(x+y)| + |f(x)| \leq \varepsilon + \sup_{x \in [0,M]} |f(x)|.$$

If we denote as usual by $\tau_x : y \mapsto x + y$ the translation on $\mathbb{R}$ by $x \in \mathbb{R}$, it is not hard to check that a bounded function f is almost periodic iff the set $O(f) := \{\tau_x f : x \in \mathbb{R}\}$ has compact closure in the Banach space $(C_b(\mathbb{R}), \|.\|_\infty)$. This concept of almost periodicity easily translates to an arbitrary topological group G.

Definition 1.2.2. *Let G be a topological group. A bounded continuous function $f : G \to \mathbb{C}$ is (left) almost periodic iff $O(f) := \{\lambda_g f : g \in G\}$ has compact closure in the Banach space $(C_b(G), \|.\|_\infty)$. Here $\lambda_g : h \mapsto gh$ denotes the (left) translation on G by $g \in G$.*

In a similar fashion one can define right almost periodic functions using the right translations $\rho_g : h \mapsto hg$. Nevertheless it turns out that a function is right almost periodic iff it is left almost periodic.

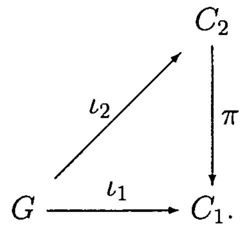
The set $\mathcal{A}(G)$ of all almost periodic functions on G has the following properties:

- $\mathcal{A}(G)$ is an algebra, i.e. $f + g$ and fg are almost periodic, whenever f and g are almost periodic. For $f \in \mathcal{A}(G)$ and $\lambda \in \mathbb{C}$ also $\lambda f \in \mathcal{A}(G)$.
- $\mathcal{A}(G)$ is closed in $C_b(G)$, i.e. $\lim_{n\to\infty} f_n$ is almost periodic, whenever every f_n is almost periodic and the limit is obtained w.r.t. the norm $\|.\|_\infty$.
- $\mathcal{A}(G)$ is translation invariant, i.e. for every $g \in G$ $\lambda_g f$ and $\rho_g g$ are almost periodic, whenever f is almost periodic
- $\mathcal{A}(G)$ admits a unique invariant mean, i.e. there is a unique L.I.M. $m : \mathcal{A}(G) \to \mathbb{C}$ such that $m(f) \geq 0$ whenever $f \geq 0$ and $m(\lambda_g f) = m(f)$ for every $g \in G$.

In particular $\mathcal{A}(G)$ is a Banach subalgebra of $C_b(G)$. Since $\mathcal{A}(G)$ contains all constant functions and is also invariant under complex conjugation, $\mathcal{A}(G)$ is in fact a unital *-subalgebra of $C_b(G)$. $\mathcal{A}(G)$ is therefore even a unital C^*-algebra, because in any closed *-subalgebra of $C_b(G)$ the C^*-axiom $\|ff^*\| = \|f^*f\| = \|f\|^2$ is valid.

1.3 Group compactifications

Let G be a topological group, C a compact group and $\iota : G \to C$ a continuous homomorphism with image $\iota(G)$ dense in C. In this case (ι, C) is called a group compactification of G. On the class of all group compactifications we can impose a preorder by defining $(\iota_1, C_1) \leq (\iota_2, C_2)$ iff there is a continuous group homomorphism $\pi : C_2 \to C_1$ such that $\pi \circ \iota_2 = \iota_1$, i.e. iff the diagram



commutes. Using compactness of C_1 and C_2 one obtains that $(\iota_1, C_1) \leq (\iota_2, C_2)$ *and* $(\iota_1, C_1) \geq (\iota_2, C_2)$ imply that π is a topological group isomorphism $C_2 \to C_1$. We call two group compactifications (ι_1, C_1) and (ι_2, C_2) equivalent iff there exists a topological group isomorphism $\pi : C_2 \to C_1$ such that $\pi \circ \iota_2 = \iota_1$. Thus $\leq$ is a partial order on group compactifications modulo equivalence.

For each given topological group G there exists a maximal group compactification w.r.t. the order $\leq$, the Bohr compactification (ι_b, bG). Each group compactification (ι, C) is equivalent to a group compactification $(\tilde{\iota}, \tilde{C}) \leq (\iota_b, bG)$ Thus the Bohr compactification of a given group G is a universal object within the class of group compactifications and up to equivalence there are at most $2^{\text{card}(bG)}$ different group compactifications of G.

The *set* of all group compactifications $\{(\iota, C) : C \subseteq bG)\}$ is a complete lattice. The join

$$\bigvee_{i \in I} (\iota_i, C_i) =: (\iota, C)$$

is equivalent to the compactification given by $\iota(g) := (\iota_i(g))_{i \in I} \in \prod_{i \in I} C_i$ and $C := \overline{\iota(G)} \subseteq \prod_{i \in I} C_i$.

One important property of group compactifications the following: Every continuous function on G that can be extended to a continuous function on a group compactification of G is almost periodic. More precise:

Proposition 1.3.1. *Let G be a topological group and (ι, X) a group compactification of G. For $F : X \to \mathbb{C}$ denote by $\iota^* F$ the function defined via $\iota^* F(g) := F \circ \iota(g)$. Then $\iota^* C(X) \subseteq \mathcal{A}(G)$.*

Proof. First note that due to compactness of X every continuous function F on X is almost periodic. Indeed $O_X(F) := \{\lambda_x F : x \in X\} \subseteq C(X)$ is the image of the compact space X under the continuous mapping $x \mapsto \lambda_x F$. Next note that $\iota^* : C(X) \to C_b(G)$ is a bounded linear mapping. Hence it sends the compact set $O_X(F) \subseteq C(X)$ onto the compact set $\iota^* O_X(F) \subseteq C_b(G)$ and

$$\iota^* O_X(F) \supseteq \{\iota^*(\lambda_{\iota(g)} F) : g \in G\} = \{\lambda_g \iota^* F : g \in G\} = O_G(\iota^* F),$$

proving that $\iota^* F$ is almost periodic on G. □

Using the Gelfand-Representation Theorem for unital abelian C^*-algebras we can show that every almost periodic function on G can be extended to a continuous function on the Bohr compactification of G. First we need some auxiliary results.

Definition 1.3.2. *Let G be a topological group $f \in C_b(G)$ and $\mu \in C_b(G)'$, the dual space of $C_b(G)$. We define the (left) introversion by μ as*

$$T_\mu f(s) := \mu(\lambda_s f);$$

a translation invariant subspace $A \subseteq C_b(G)$ is (left) introversion invariant if $T_\mu f \in A$ whenever $f \in A$ and $\mu \in A'$.

If $A \subseteq C_b(G)$ is a *-subalgebra that is both translation and introversion invariant one can define a semigroup operation on A' via $\nu * \mu := T_\mu^* \nu$, more explicitly this means

$$\nu * \mu(f) := \nu(T_\mu)f, \quad \text{for } f \in A,\ \nu, \mu \in A'.$$

This binary operation can be regarded as an extension of the group operation on G since $\delta_g * \delta_h = \delta_{gh}$, for $g, h \in G$ and δ_g the functional defined via $\delta_g(f) := f(g)$. If $A \subseteq \mathcal{A}(G)$ is a translation and introversion invariant *-subalgebra then the operation $*$ is jointly continuous, i.e. continuous as a mapping $A' \times A' \to A'$, w.r.t. the weak-*-topology. For details we refer to the monograph [BJM].

Theorem 1. *Let G be a topological group. The algebra $\mathcal{A}(G)$ of almost periodic functions on G is isometrically isomorphic to $C(bG)$, the algebra of continuous function on bG.*

Sketch of Proof. We show that $\mathcal{A}(G)$ is isometrically isomorphic to the continuous functions defined on the weak-* closure of $\{\delta_g : g \in G\} \subseteq \mathcal{A}(G)'$. Let us denote this closure by X. Note that by the Banach-Alaoglu theorem X is compact. Furthermore $\delta : G \to X$ is a continuous homomorphism w.r.t. the group operation on G resp. the operation $*$ on X. Using the joint continuity of $*$ and density of $\delta(G)$ in X standard topological arguments yield that X is a topological group, i.e. (δ, X) is group compactification of G. For given $f \in \mathcal{A}(G)$ consider the function $F \in C(X)$ defined via $F(\mu) := \mu(f)$, with $\mu \in X$. Clearly $\delta^* F = F \circ \delta = f$ and thus $\delta^* C(X) \supseteq \mathcal{A}(G)$. Since the inclusion $\delta^* C(X) \subseteq \mathcal{A}(G)$ holds trivially, we have $\delta^* C(X) = \mathcal{A}(G)$.

Since (δ, X) is a group compactification with $\delta^* C(X) = \mathcal{A}(G)$, due to maximality the same must hold for the Bohr compactification: $\iota_b^* C(bG) = \mathcal{A}(G)$. The Banach-Stone Theorem asserts that there exists a homeomorphism $\varphi : X \to bG$ such that $\varphi \circ \delta = \iota_b$. Since δ has dense image, it is easy to deduce that φ is actually a continuous group isomorphism. Thus $(\delta, X) \geq (\iota_b, bG)$. The universal property of the Bohr compactification implies $(\delta, X) \leq (\iota_b, bG)$ and thus (δ, X) and (ι_b, bG) are equivalent group compactifications. □

After we have established the isomorphy $\mathcal{A}(G) \cong C(bG)$ we can describe the invariant mean for $\mathcal{A}(G)$: Since every almost periodic function is of the form

$F \circ \iota_b$ with a unique $F \in C(bG)$ and $\iota_b(G)$ is dense in bG L.I.M.s on $\mathcal{A}(G)$ and L.I.M.s on bG are in one-one correspondence. Uniqueness of the Haar measure μ_b on bG implies that the only L.I.M. m_{bG} on $C(bG)$ is given by $F \mapsto \int_{bG} f d\mu_{bG}$. Hence the unique L.I.M. m_G on $\mathcal{A}(G)$ is given by

$$m_G(f) := \int_{bG} F d\mu_b$$

where F is the unique function from $C(bG)$ such that $f = F \circ \iota_b$.

We can mimic these constructions when replacing the category of topological groups by categories of other topological-algebraic objects, for example by semigroups with compatible topology:

Definition 1.3.3. *A topological space S with a semigroup operation $\cdot$ is semitopological if for each $s \in S$ the mappings $\lambda_s : t \mapsto st$ and $\rho_s : t \mapsto ts$ are continuous. A pair (ι, C) is a semitopological semigroup compactification of S if C is a compact semitopological semigroup and $\iota : S \longrightarrow C$ is continuous semigroup homomorphism with dense image $\iota(S)$.*

There exists a maximal semitopological semigroup compactification (which is unique up to equivalence), the so called w.a.p. compactification (ι_w, wS).

Definition 1.3.4. *Let S be a semitopological semigroup. A continuous function $f : S \longrightarrow \mathbb{C}$ is (left) w.a.p. iff the closure of the set $O(f) := \{\lambda_s f : s \in S\}$ w.r.t. the weak topology on $C_b(S)$ is compact.*

Again one can show that left and right w.a.p. functions are the same thing and that the set $\mathcal{W}(S)$, consisting of all w.a.p. functions defined on S is a *-subalgebra of $C_b(S)$. Every w.a.p. function on S can be extended to a continuous function on wS and $\mathcal{W}(S)$ coincides with the space $\iota_w^* C(wS)$. If $S = G$ is a topological group, then $(\iota_b, bG) \leq (\iota_w, wG)$ as semitopological semigroup compactification. Denoting the canonical projection $\pi : wG \longrightarrow bG$ (note that π is open), one can show that the measure μ_w, defined by the requirement $\mu_w(A) := \mu_b(\pi A)$, induces the only L.I.M. on $C(wG)$. Hence the unique L.I.M. m_G on $\mathcal{W}(G)$ is given by

$$m_G(f) := \int_{wS} F d\mu_w$$

where F is the unique function from $C(wG)$ such that $f = F \circ \iota_w$. There are positive weakly almost periodic functions with zero mean value, denoted by $\mathcal{W}(G)_0$, while every positive almost periodic function has a positive mean value.

Chapter 2

Hartman measurable sets

2.1 Definition and Basic Properties

In this section we will describe how for a topological group G the Haar measure μ_C on a group compactification (ι, C) can be used to define a finitely additive probability measure μ_G on a non trivial Boolean set algebra on G.

The idea is to consider sets $H = \iota^{-1}(M)$ and define $\mu_G(H) := \mu_C(M)$. Investigations in [FPTW] have shown that this approach works if one requires that $M \subseteq C$ is a continuity set (also called Jordan measurable), i.e. a set the topological boundary ∂M of which satisfies $\mu_C(\partial M) = 0$. We will now give the relevant arguments:

It is easy to verify that the system $S_C \subseteq \mathfrak{P}(C)$ of all μ_C-continuity sets on C is a Boolean set-algebra (due to the fact that $\partial M = \partial M^c$ and $\partial(A \cup B) \subseteq \partial A \cup \partial B$). However S_C is in general no σ-algebra.

Lemma 2.1.1. *Let G be a topological group and let (ι, C) be a group compactification. If $M_1 \subseteq C$ and $M_2 \subseteq C$ are μ_C-continuity sets with $\iota^{-1}(M_1) = \iota^{-1}(M_2)$ then $\mu_C(M_1) = \mu_C(M_2)$.*

Proof. Let $D := \iota(G)$ and note that D is dense in X. Also note that $M_1 \cap D = \iota \circ \iota^{-1}(M_1) = \iota \circ \iota^{-1}(M_2) = M_2 \cap D$. First we show that $(M_1 \setminus M_2)^\circ = \emptyset$. Suppose by contradiction that there exists $x \in (M_1 \setminus M_2)^\circ = M_1^\circ \setminus \overline{M_2}$. Since D is dense we may w.l.o.g. assume that $x \in D$. In particular $x \in M_1$ but $x \notin M_2$ contradicting $M_1 \cap D = M_2 \cap D$. Thus $(M_1 \setminus M_2)^\circ = \emptyset$ and so the μ_C-continuity set $M_1 \setminus M_2$ (remember that μ_C-continuity sets form a Boolean algebra) is a μ_C-null set because

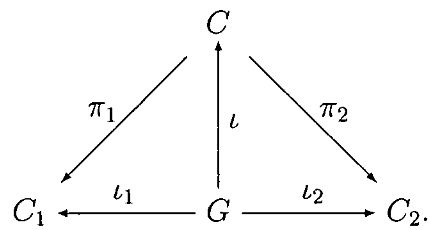
$$0 = \mu_C\left(\partial(M_1 \setminus M_2)\right) = \mu_C\left(\overline{(M_1 \setminus M_2)} \setminus (M_1 \setminus M_2)^\circ\right) = \mu_C(\overline{M_1 \setminus M_2}).$$

Thus we have $\mu_C(M_1 \cup M_2) = \mu_C(M_1) + \mu_C(M_2 \setminus M_1) = \mu_C(M_1)$. By symmetry $\mu_C(M_1) = \mu_C(M_1 \cup M_2) = \mu_C(M_2)$. □

This enables us for each group compactification (ι, C) to define a finitely additive measure on the Boolean set algebra $\iota^{-1}(S_C) \subseteq \mathfrak{P}(G)$ by letting $\mu_G : \iota^{-1}(M) \mapsto \mu_C(M)$. Sets $H \subseteq G$ arising in this way are called (ι, C)-Hartman measurable. Next we show that we can get rid of the dependence on a particular group compactification (ι, C).

Lemma 2.1.2. *Let G be a topological group and let (ι_1, C_1) resp. (ι_2, C_2) be group compactifications. If $M_1 \subseteq C_1$ and $M_2 \subseteq C_2$ are μ_{C_1} resp. μ_{C_2}-continuity sets with $\iota^{-1}(M_1) = \iota^{-1}(M_2)$ then $\mu_{C_1}(M_1) = \mu_{C_2}(M_2)$.*

Proof. Let $(\iota, C) = (\iota_1, C_1) \vee (\iota_2, C_2)$ be the supremum of the two group compactifications and $\pi_i : C \to C_i$, $i = 1, 2$ the canonical epimorphisms. Since the Haar



measure on C_i is given by $\mu_{C_i} = \pi_i \circ \mu_C$, where μ_C is the Haar measure on C, it is easy to show that $\tilde{M_1} := \pi_1^{-1}(M_1)$ and $\tilde{M_2} := \pi_2^{-1}(M_2)$ are μ_C-continuity sets on C (we will give a more detailed argument in Lemma 3.2.5). Furthermore

$$\iota^{-1}(\tilde{M_1}) = \iota^{-1} \circ \pi_1^{-1}(M_1) = \iota_1^{-1}(M_1) = \iota_2^{-1}(M_2) = \iota^{-1} \circ \pi_2^{-1}(M_2) = \iota^{-1}(\tilde{M_2}).$$

Thus Lemma 2.1.1 implies $\mu_C(\tilde{M_2}) = \mu_C(\tilde{M_1})$ and thus

$$\mu_{C_1}(M_1) = \pi_1 \circ \mu_{M_1} = \mu(\tilde{M_1}) = \mu(\tilde{M_2}) = \pi_2 \circ \mu_{M_2} = \mu_{C_2}(M_2).$$

□

Definition 2.1.3. *Let G be a topological group. A set $H \subseteq G$ is called Hartman measurable if one of the following equivalent assertions holds:*

1. *There exists a group compactification (ι_C, C) such that H is (ι, C)-Hartman measurable,*
2. *H is (ι_b, bG)-Hartman measurable.*

Here (ι_b, bG) denotes the Bohr compactification of the topological group G. The system of all Hartman measurable sets on G is denoted by $\mathfrak{H}(G)$.

For the case $G = \mathbb{Z}$ (additive group of integers) number theoretic aspects of Hartman measurable sets have been studied in [SchSW], while in [Win] the dynamical aspect has been stressed. This gets clear by the observation that $\iota : \mathbb{Z} \to C$ can be interpreted as a $\mathbb{Z}$-action on C, generated by the ergodic group rotation

$T : c \mapsto c + \iota(1)$. In particular the information about (ι, C) contained in H or, equivalently, in the binary coding sequence $1\!\!1_H$ (characteristic function) of this dynamical system has been investigated. We will extend these results in chapter 2.

From Definition 2.1.3 it is clear that $\mathfrak{H}(G) = \iota_b^{-1}(S_{bG}) \subseteq \mathfrak{P}(G)$ is a Boolean set-algebra and $\mu_G = \iota_b \circ \mu_b$ is a finitely additive invariant measure on $\mathfrak{H}(G)$. As a starting point we will restrict our investigations to the case $G = \mathbb{Z}$ and prove uniqueness of $\mu_{\mathbb{Z}}$.

Remark: Every finitely additive invariant measure on the Boolean algebra $\mathfrak{H}(G)$ can be extended to a finitely additive invariant measure on $\mathfrak{P}(G)$ whenever the group G is amenable ($\mathbb{Z}$ along with any other abelian group is amenable). However, this extension need not to be unique (cf. Theorem 10.8 in [Wag]).

Proposition 2.1.4. *Let ν be a finitely additive invariant measure on $\mathfrak{H}(\mathbb{Z})$. Then $\nu = \mu_G$.*

Proof. Let $H \in \mathfrak{H}(\mathbb{Z})$ be fixed. There exists a group compactification (ι, C) and a μ_C-continuity set M such that $H = \iota^{-1}(M)$. Suppose that C is metrizable (we will show in Theorem 2 that this is no loss of generality). Consider the dynamical system (C, T) where $T : x \mapsto x + \iota(1)$. Since T is an ergodic group rotation of the compact metrizable monothetic group C, the transformation T is uniquely ergodic (Theorem 6.20 in [Wal]). Thus Lemma 6.19 in [Wal] assures that

$$\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n \to \int_C f \mu_C \tag{2.1}$$

in the topology of uniform convergence, for every continuous function $f : C \to \mathbb{C}$. It is not hard to show that (2.1) is also valid for functions of the form $f = 1\!\!1_M$ with M a μ_C-continuity set (the relevant property is Riemann integrability, which is of major concern for chapter 2 and will there be discussed in great detail). Thus

$$\frac{1}{N}\sum_{n=0}^{N-1} 1\!\!1_M(T^n x) \to \mu_C(M)$$

uniformly for $x \in C$. Restricting this relation to $\iota(\mathbb{Z})$ yields that

$$\frac{1}{N}\sum_{n=0}^{N-1} 1\!\!1_{H-n}(g) \to \mu_{\mathbb{Z}}(H) \tag{2.2}$$

uniformly for $g \in \mathbb{Z}$. Let ν be any finitely additive translation invariant measure on $\mathfrak{H}(\mathbb{Z})$. We establish a one-one correspondence between finitely additive translation invariant measures on $\mathfrak{H}(\mathbb{Z})$ and non negative invariant linear functionals on $\operatorname{span}\mathfrak{H}(\mathbb{Z})$: For a given measure ν let $n := n_\nu$ be defined via $n_\nu(1\!\!1_H) := \nu(H)$, and for a given functional n defined a measure via $\nu = \nu_n := n(1\!\!1_H)$. Note that nonnegativity of these functionals implies boundedness w.r.t. to the topology of uniform convergence. Thus if we regard $n = n_\nu$ as a non negative invariant functional on span $\mathfrak{H}(\mathbb{Z})$ the LHS of (2.2) equals $\nu(H)$. Since the RHS of (2.2) equals $\mu_{\mathbb{Z}}(H)$ the two measures ν and $\mu_{\mathbb{Z}}$ coincide. □

Although every set $H \in \mathfrak{H}(\mathbb{Z})$ has a unique measure, Hartman sets are rather special among sets with this property. For example, the system

$$\mathfrak{AC}(\mathbb{Z}) := \{A \subseteq \mathbb{Z} : \mu(A) = \nu(A) \text{ for every two translation invariant finitely additive measures } \mu \text{ and } \nu\}$$

is not closed under forming intersections, while the system $\mathfrak{H}(\mathbb{Z})$ is.

Proposition 2.1.5. *A subset $A \subseteq \mathbb{Z}$ has a uniform density (Banach density) iff $A \in \mathfrak{AC}(\mathbb{Z})$.*

Proof. Consider the upper resp. lower Banach density $d^*(A)$ resp. $d_*(A)$ of A. Recall that

$$d^*(A) := \limsup_{|b-a|\to\infty} \frac{\#\{[a,b] \cap A\}}{\#[a,b]} \tag{2.3}$$

resp.

$$d_*(A) := \liminf_{|b-a|\to\infty} \frac{\#\{[a,b] \cap A\}}{\#[a,b]}. \tag{2.4}$$

By definition A has uniform density iff upper and lower Banach density coincide. It is important to note that

$$\frac{\#\{[a+x, b+x] \cap A\}}{\#[a,b]} = \frac{1}{\#[a,b]} \sum_{n=a}^{b} \sigma_n 1\!\!1_A(x)$$

for any interval $[a,b]$ and $x \in \mathbb{Z}$ (recall that σ is the shift operator on $\mathbb{Z}$). Thus in particular

$$d^*(A) = \limsup_{|b-a|\to\infty} \sup_{x\in\mathbb{Z}} \left(\frac{1}{\#[a,b]} \sum_{n=a}^{b} \sigma_n 1\!\!1_A(x) \right) = \sup_{\substack{F\subseteq\mathbb{Z},\\ \text{finite}}} \sup_{x\in\mathbb{Z}} \left(\frac{1}{\#F} \sum_{n\in F} \sigma_n 1\!\!1_A(x) \right).$$

It is known (cf. [Pat]) that $d^*(A)$ coincides with the quantity

$$S(A) := \sup\{\mu(A) : \mu \text{ is an invariant finitely additive measure on } \mathfrak{P}(\mathbb{Z})\}.$$

By symmetry we get that $d_*(A)$ coincides with

$$I(A) := \inf\{\mu(A) : \mu \text{ is an invariant finitely additive measure on } \mathfrak{P}(\mathbb{Z})\}.$$

Thus $A \in \mathfrak{AC}(\mathbb{Z})$ is equivalent to $S(A) = ud(A) = d^*(A) = I(A) = d_*(A)$. □

Having this characterization of $\mathfrak{AC}(\mathbb{Z})$, we can find A, B in $\mathfrak{AC}(\mathbb{Z})$ such that $A \cap B \notin \mathfrak{AC}(\mathbb{Z})$:

Let C be any set with $d^*(C) > 0$ and $d_*(C) = 0$, e.g. $C = \bigcup_{n=1}^{\infty} \pm[2^n, 2^n + n]$. Let $A := 3\mathbb{Z}$ and $B := 3\mathbb{Z} \setminus C \dot\cup ((3\mathbb{Z} \setminus C) + 1)$, then $ud(A) = ud(B) = \frac{2}{3}$ but $A \cap B = C \notin \mathfrak{AC}(\mathbb{Z})$. In fact for any given infinite set $S \subseteq \mathbb{Z}$ we can find subsets $A, B \subseteq S$ with this property.

Proposition 2.1.6. *For a subset $A \subseteq \mathbb{Z}$ the following assertions are equivalent:*

1. $A \in \mathfrak{AC}(\mathbb{Z})$
2. $(\mathbb{1}_A - \mu_G(A) \cdot \mathbb{1}_A) \in \overline{K_0(\mathbb{Z})}$ *where*

$$K_0(\mathbb{Z}) := \bigcup_{n=-\infty}^{\infty} (\sigma^n - I)B(\mathbb{Z})$$

and the closure is taken w.r.t. the topology of uniform convergence.

Proof. This is a special case of Propositions 2.1 and 2.2 in [Pat]. □

Example 2.1.7. Invariant transformations of $\mathfrak{AC}(\mathbb{Z})$: *We will construct a family of measure preserving transformations $\mathcal{T}$, called cut-and-paste operators, such that $\mathcal{T}\mathfrak{AC}(\mathbb{Z}) \subseteq \mathfrak{AC}(\mathbb{Z})$ but $\mathcal{T}\mathfrak{H}(\mathbb{Z}) \not\subseteq \mathfrak{H}(\mathbb{Z})$.*

To this end let us identify a subset $A \subseteq \mathbb{Z}$ with its characteristic function $\mathbb{1}_A \in B(\mathbb{Z})$. We are going to consider linear operators $T : B(\mathbb{Z}) \to B(\mathbb{Z})$ of the form

$$T = \sum_{i=1}^{N} \sigma^{k_i} P_{A_i}, \quad k_i \in \mathbb{Z}. \tag{2.5}$$

σ denotes again the shift operator defined by $\sigma f(k) := f(k+1)$ and P_{A_i} denotes the multiplication operator defined by $P_{A_i} f := f \mathbb{1}_{A_i}$. Observe that P_{A_i} is a projection onto the subspace $\Pi(A_i) := \{f \in B(\mathbb{Z}) : \text{supp}(f) \subseteq A_i\}$. If all the sets $A_i + k_i$ are pairwise disjoint, then it is obvious that T maps characteristic functions on characteristic functions. Thus we can write $T\mathbb{1}_A := \mathbb{1}_{\mathcal{T}A}$ for a mapping $\mathcal{T} : \mathfrak{P}(\mathbb{Z}) \to \mathfrak{P}(\mathbb{Z})$.

We claim that $\mathcal{T}\mathfrak{AC}(\mathbb{Z}) \subseteq \mathfrak{AC}(\mathbb{Z})$ whenever the following conditions on the sets A_i hold:

1. $(A_i + k_i) \cap (A_j + k_j) = \emptyset$ for $i \neq j$,
2. $A_i \Delta (A_i + n)$ is finite for each $i = 1, \ldots, N$ and $n \in \mathbb{Z}$,
3. $\mathbb{Z} \setminus (A_1 \cup \ldots \cup A_N)$ is finite.

Note that the last condition implies that $ud(\mathcal{T}\mathbb{Z}) = 1$ (this will also ensure that $\mathcal{T}$ preserves measure). Let us denote the commutator of two linear operators by $[T_1, T_2] = T_1T_2 - T_2T_1$. Elementary calculations yield that

$$T(\sigma^n - I)f = (\sigma^n - I)Tf + [T, \sigma^n]f, \quad n \in \mathbb{Z}.$$

Since $K_0(G)$ is shift invariant, we have $TK_0 \subseteq K_0 + \bigcup_{n=-\infty}^{\infty}[T, \sigma^n]K_0$. To calculate the commutator $[T, \sigma^n]$ we use that

$$\begin{aligned} \left[P_i, \sigma^k\right] f &= \sigma f \cdot (\mathbb{1}_{A_i} - \mathbb{1}_{A_i - k}) &\subseteq \Pi(A_i \Delta (A_i - n)), \\ \left[\sigma^{k_i} P_i, \sigma^n\right] f &= \sigma^{k_i}[P_i, \sigma^n]f - [\sigma^{k_i}, \sigma^n]f &\subseteq \Pi((A_i + k_i)\Delta(A_i - n)). \end{aligned}$$

Thus for each $n \in \mathbb{Z}$:

$$[T, \sigma^n] \subseteq \bigcup_{i=1}^{N} \Pi\Big((A_i + k_i)\Delta(A_i - n)\Big) \subseteq \Pi\Big(\bigcup_{i=1}^{N}(A_i + k_i)\Delta(A_i - n)\Big).$$

Since Condition 2 implies that $[T, \sigma^n]f$ is of finite support, we can conclude that $[T, \sigma^n]B(\mathbb{Z}) \subseteq \overline{K_0(\mathbb{Z})}$. This already implies invariance of $\mathfrak{AC}(\mathbb{Z})$ w.r.t. the transformation T:

$$T\mathfrak{AC}(\mathbb{Z}) = T\overline{K_0(Z)} + T\mathbb{C} \cdot \mathbb{1}_{\mathbb{Z}} \subseteq \overline{TK_0(\mathbb{Z})} + \mathbb{C} \cdot T\mathbb{1}_{\mathbb{Z}} \subseteq \overline{K_0}(Z) + \mathbb{C} \cdot \mathbb{1}_{\mathcal{T}\mathbb{Z}} \subseteq \mathfrak{AC}(\mathbb{Z}).$$

Let us have a closer look at the simplest case and consider the transformation $\mathcal{T}$ associated with the parameters $A_1 = -\mathbb{N}$, $A_2 = \mathbb{N}$, $k_1 = 0$, $k_2 = 1$. Thus $\mathcal{T}$ maps a binary sequence $(\ldots x_{-2}x_{-1}\underline{x_0}x_1x_2\ldots)$ onto the binary sequence $(\ldots x_{-2}x_{-1}\underline{0}x_2x_3\ldots)$.

We show that for this particular transformation $\mathcal{T}$ there exits a Hartman measurable set $H \in \mathfrak{H}(\mathbb{Z})$ such that $\mathcal{T}H$ is not Hartman measurable. Suppose by contradiction that both H and $\mathcal{T}H$ are Hartman measurable. Then also $H\Delta\mathcal{T}H \in \mathfrak{H}(\mathbb{Z})$; if we denote H again as a binary sequence, this means $(\ldots 00\underline{0} * * \ldots) \in \mathfrak{H}(\mathbb{Z})$. Since $H\Delta\mathcal{T}H \in \mathfrak{H}(\mathbb{Z})$ has arbitrarily long blocks of zeros Theorem 4 in [SchSW] implies that any finite subword w of $H\Delta\mathcal{T}H$ appears with asymptotic frequency $p(w) = 0$. On the other hand it is easily computed that if the subword (10) occurs with positive asymptotic frequency in H, then the subword (1) occurs with positive asymptotic frequency in $H\Delta\mathcal{T}H$. As there clearly exist Hartman measurable sets $H \subseteq \mathbb{Z}$ with the property that (10) occurs with positive asymptotic frequency this is a contradiction.

For an arbitrary transformation of the form (2.5) essentially the same reasoning as above applies once we replace the subword (10) by a subword of length k_1 resp. of length $k_i + 1$ for $i = 2, \ldots, N$ such that the chosen subwords occur with positive asymptotic frequencies within the set A_i.

2.2 Characterization results

Let G be an abelian group and τ_g the translation by $g \in G$. We introduce two mappings:

- for a Hartman measurable set M denote by $d_M : G \longrightarrow [0,1]$ the mapping $g \mapsto m_G(M \Delta \tau_g M)$,
- for a μ_C-continuity set M^* on some group compactification (ι, C) denote by $d_{M^*} : C \longrightarrow [0,1]$ the mapping $g \mapsto \mu_C(M^* \Delta \tau_g M^*)$.

Note that the mapping d_{M^*} (and similarly the mapping d_M) can be used to define a translation invariant pseudometric by letting $\rho_{M^*}(g,h) := d_{M^*}(g-h)$. The set of zeros $\{g \in C : d_{M^*}(g) = 0\}$ is always a closed subgroup. We will denote this subgroup by $\ker d_{M^*}(g)$.

Now consider sets of the form $F(M,\varepsilon) := \{g \in G : d_M(g) < \varepsilon\}$ and denote by $\mathcal{F}(M)$ the filter on G generated by $\{F(M,\varepsilon) : \varepsilon > 0\}$, i.e. the set of all $F \subseteq G$ such that there exists an $\varepsilon > 0$ with $F(M,\varepsilon) \subseteq F$. When we have a realization M^* of M on some group compactification (ι, C) we can transfer the topological data encoded in the neighborhood filter of the unit 0_C in C to the original group G by considering its preimage under ι. To be precise: Let (ι, C) be a group compactification and $\mathfrak{U}(C, 0_C)$ the filter on C generated by neighborhoods of the unit element 0_C in C. By $\mathfrak{U}_{(\iota,C)}$ we denote the filter on G generated by $\iota^{-1}(\mathfrak{U}(C, 0_C))$. Note that if the mapping ι is one-one, $\iota^{-1}(\mathfrak{U}(C, 0_C))$ is already a filter.

For the group $\mathbb{Z}$ of the integers, Theorem 2 in [Win] states that the filter $\mathfrak{U}_{(\iota,C)} \subseteq \mathfrak{P}(\mathbb{Z})$ contains much information about the filter $\mathfrak{U}(C, 0_C) \subseteq \mathfrak{P}(C)$ and hence about the group compactification (ι, C):

Theorem: Let $M \subseteq \mathbb{Z}$ be a Hartman measurable set and (ι, C) a group compactification of the integers such that M can be realized on C via the continuity set M^*, i.e. there is a μ_C-continuity set $M^* \subseteq C$ such that $\iota^{-1}(M^*) = M$. Then $H = \ker d_{M^*}$ is a closed subgroup of C and $\mathcal{F}(M) = \mathfrak{U}_{(\pi_H \circ \iota, C/H)}$, for $\pi_H : C \longrightarrow C/H$ the canonical mapping.

With the tool developed in the next chapter we will able to prove the following generalization:

Proposition 2.2.1. *Let G be an LCA group with separable dual $\hat{G}$. If $M \subseteq G$ is a Hartman measurable set and (ι, C) is a group compactification of G such that M can be realized on C via the continuity set M^* then $H = \ker d_{M^*}$ is a closed subgroup of X and $\mathcal{F}(M) = \mathfrak{U}_{(\pi_H \circ \iota, C/H)}$, for $\pi_H : C \longrightarrow C/H$ the canonical mapping.*

For the group of integers $\mathbb{Z}$ and a Hartman measurable set $M \subseteq \mathbb{Z}$ the filter $\mathcal{F} = \mathcal{F}(M)$ can also used to define a subgroup of $\mathbb{T} = \hat{\mathbb{Z}}$ consisting of all those elements α, such that the limit of $\{n\alpha\}_{n \in \mathbb{Z}}$ with respect to the filter $\mathcal{F}$ vanishes:

$$\mathrm{Sub}(M) := \mathcal{F}\text{–}\lim_n \lfloor n\alpha \rfloor = 0.$$

Recall that for some sequence $\{x_n\}_{n \in \mathbb{Z}}$ we have $\mathcal{F}\text{–}\lim_n x_n = 0$ iff for every $\varepsilon > 0$ the set $\{n \in \mathbb{Z} : |x_n| \leq \varepsilon\}$ belongs to the filter $\mathcal{F}$. In [Win] it is shown that all three objects - filter, compactification and subgroup - carry the same information regarding a fixed Hartman set M. It is worth to note that in [BSW] it proven that any subgroup of a compact abelian group G can be written as $\{g \in G : \mathcal{F}\text{–}\lim_{\chi \in \hat{G}} \chi(g) = 0_G\}$ for some filter $\mathcal{F}$ on $\hat{G}$.

Chapter 3

Hartman measurable functions

3.1 Motivation

As any set $M \subseteq G$ can be identified with the characteristic function $f := 1\!\!1_M \in B(G)$, the characterizing properties of M being Hartman measurable can be formulated in terms of the function M. It turns out that the relevant property is Riemann integrability: $M \subseteq G$ is Hartman measurable iff there exists a characteristic function $F = 1\!\!1_{M^*}$ on some group compactification (ι, C) that is Riemann integrable and such that $f = F \circ \iota$. Once such a characterization in functional terms is obtained, we can extend the concept of Hartman measurability from sets to functions, by dropping the requirement that f has to be a characteristic function and deal with arbitrary bounded complex-valued functions instead.

3.2 Riemann integrability

In the following let X be a compact (including the Hausdorff separation axiom) space and μ a positive regular Borel measure with full support $\text{supp}(\mu) = X$. A bounded function $f : X \to \mathbb{C}$ is called μ-Riemann integrable iff the set $\text{disc}(f)$ of points of discontinuity is a μ-null set. Let us denote the set of all these functions by $R_\mu(X)$.

Note that for any continuous (complex-valued) function $g : \mathbb{C} \to \mathbb{C}$ the inclusion $\text{disc}(g \circ f) \subseteq \text{disc}(f)$ holds for arbitrary $f : G \to \mathbb{C}$. Thus every continuous g is a left multiplier for $R_\mu(X)$ in the sense that $g : \mathbb{C} \to \mathbb{C}$ continuous and $f \in R_\mu(X)$ implies $g \circ f \in R_\mu(X)$.

The following characterization of Riemann integrability, a proof of which can be found in [Tal], is important for us:

Proposition 3.2.1. *For a real-valued μ-measurable function f the following as-*

sertions are equivalent:

1. *f is μ-Riemann integrable*

2. *For every $\varepsilon > 0$ there exist continuous functions g_ε and h_ε such that $g_\varepsilon \leq f \leq h_\varepsilon$ and $\int_X (h_\varepsilon - g_\varepsilon) d\mu < \varepsilon$.*

For a complex-valued μ-measurable function f consider its decomposition in real and imaginary part $f = \mathtt{Re}f + i\mathtt{Im}f$. Since $\mathrm{disc}(f) \subseteq \mathrm{disc}(\mathtt{Re}f) \cup \mathrm{disc}(\mathtt{Im}f)$ Riemann integrability of the real-valued functions $\mathtt{Re}f$ and $\mathtt{Im}f$ implies Riemann integrability of f. On the other hand the mappings $\mathtt{Re} : \mathbb{C} \longrightarrow \mathbb{R}$ and $\mathtt{Im} : \mathbb{C} \longrightarrow \mathbb{R}$ are left multipliers for $R_\mu(X)$, so Riemann integrability of f implies Riemann integrability of $\mathtt{Re}f$ and $\mathtt{Im}f$. Thus we obtain Proposition 3.2.1 also for complex-valued functions

Corollary 3.2.2. *For a complex-valued μ-measurable function f the following assertions are equivalent:*

1. *f is μ-Riemann integrable*

2. a. *For every $\varepsilon > 0$ there exist continuous functions g_ε and h_ε such that $g_\varepsilon \leq Ref \leq h_\varepsilon$ and $\int_X (h_\varepsilon - g_\varepsilon) d\mu < \varepsilon$.*

   b. *For every $\varepsilon > 0$ there exist continuous functions $\tilde{g}_\varepsilon$ and $\tilde{h}_\varepsilon$ such that $\tilde{g}_\varepsilon \leq Imf \leq \tilde{h}_\varepsilon$ and $\int_X (\tilde{h}_\varepsilon - \tilde{g}_\varepsilon) d\mu < \varepsilon$.*

Using the notation from Proposition 3.2.1 we collect some easy but important implications:

1. Let $g := \sup_{n\in\mathbb{N}} g_{\frac{1}{n}}$ and $h := \inf_{n\in\mathbb{N}} h_{\frac{1}{n}}$, then $g = h$ μ-a.e., so f coincides μ-a.e. with a μ-measurable function and is itself $\tilde{\mu}$-measurable (recall that $\tilde{\mu}$ denotes the completion of the measure μ).

2. $R_\mu(X)$ is a uniformly closed and translation invariant *-subalgebra of both $B(X)$ and $L^\infty(X; \tilde{\mu})$.

3. If X is a compact group and μ the normalized Haar measure on X, then the L.I.M. $m : f \mapsto \int_X f d\mu$ defined on $C(X)$ can be extended to a L.I.M. defined on $R_\mu(X)$ by letting

$$\tilde{m}(f) := \sup_{g\in C(X),\, g\leq f} m(f) = \inf_{h\in C(X),\, f\leq h}$$

for real-valued Riemann integrable functions f. This extension is unique.

Remark: Let f be μ-Riemann integrable on the compact space X and μ a finite regular Borel measure with $\mathrm{supp}(\mu) = X$. Let us denote the oscillation of f at x by

$$\mathrm{Os}_f(x) := \limsup_{y \to x} f(y) - \liminf_{y \to x} f(y).$$

Since Os_f is upper semicontinuous (a proof of this fact can be found e.g. in [Bou1]) the sets $A_n := [\mathrm{Os}_f \geq \frac{1}{n}] \subseteq \mathrm{disc}(f)$ are closed. Using monotonicity of μ we conclude that each A_n is a closed μ-null set and hence nowhere dense. This implies that $\mathrm{disc}(f) = \bigcup_{n \geq 0} A_n$ is both F_σ and a meager $\mu-$null set.

Corollary 3.2.3. *Let X, Y be compact topological spaces, μ a positive regular Borel measure with supp(μ) $= X$, and $\pi : X \longrightarrow Y$ a continuous mapping. Let $\nu := \pi \circ \mu$ denote the Borel measure on Y defined by $\nu(A) := \mu(\pi^{-1}(A))$. Then $f \in R_\nu(Y)$ implies $f \circ \pi \in R_\mu(X)$.*

Proof. Using the elementary fact that $\mathrm{disc}(f \circ \pi) \subseteq \pi^{-1}\mathrm{disc}(f)$, one immediately obtains that $\mu(\mathrm{disc}(f \circ \pi)) \leq \mu(\pi^{-1}\mathrm{disc}(f)) = \nu(\mathrm{disc}(f)) = 0$. □

Let $\mathcal{C}_\mu(X)$ denote the linear space generated by characteristic functions of μ-continuity sets on X. Let us call these functions *simple (μ-)continuity functions.* Clearly $\mathcal{C}_\mu(X) \subseteq R_\mu(X)$.

Proposition 3.2.4. *$\mathcal{C}_\mu(X)$ is uniformly dense in $R_\mu(X)$.*

Proof. Let f be μ-Riemann integrable and w.l.o.g. assume that f takes values in the interval $[0, 1)$. We introduce the level-sets $M_t := [0 \leq f < t]$ and the function

$$\varphi_f(t) := \tilde{\mu}(M_t).$$

Since φ_f is increasing, it has at most countably many points of discontinuity. Consider $\mu(\{x : f(x) = t\}) \leq \varphi_f(r) - \varphi_f(s)$ for $s < t < r$. If φ_f is continuous at t this implies

$$\sup_{s<t} \varphi_f(s) = f(t) = \inf_{r>t} \varphi_f(r),$$

and so $\{x : f(x) = t\}$ is a $\mu-$null set for $t \notin \mathrm{disc}(\varphi_f)$.

Now let $x \in \partial M_t$. If f is continuous at x we have $f(x) = t$. So

$$\partial M_t \subseteq \mathrm{disc}(f) \cup \{x : f(x) = t\}.$$

The first set on the right-hand side is a μ-null set since f is μ-Riemann integrable and the second one is a μ-null set at least for each continuity point t of φ_f. So for all but at most countably many t the set M_t is a μ-continuity set. In particular the set $N_f := \{t : \mu(\partial M_t) = 0\} \subseteq [0, 1]$ is dense.

Now we can approximate f uniformly by members of $\mathcal{C}_\mu(X)$: Given $\varepsilon > 0$ let $n > \frac{1}{\varepsilon}$. Pick real numbers $\{t_i\}_{i=0}^n \subseteq N_f$ with

$$t_0 = 0 < t_1 < \frac{1}{n} < \ldots < t_i < \frac{i}{n} < t_{i+1} < \ldots < \frac{n-1}{n} < t_n = \|f\|_\infty < 1.$$

Let $A_i := M_{t_i} \setminus M_{t_{i-1}}$, so $|f(x) - \frac{i-1}{n}| < \varepsilon$ on A_i for $i = 1, 2, \ldots, n$. For

$$x \in \bigcup_{i=1}^{n} A_i = M_1 \setminus M_0 = X$$

we conclude

$$\Big| \underbrace{\sum_{i=1}^{n} \frac{i}{n} \mathbb{1}_{A_i}(x)}_{\in \mathcal{C}_\mu(X)} - f(x) \Big| < \varepsilon,$$

i.e. $\| \sum_{i=1}^{n} \frac{i}{n} \mathbb{1}_{A_i} - f \|_\infty < \varepsilon$ and thus f is in the uniform closure of $\mathcal{C}_\mu(X)$. $\square$

Lemma 3.2.5. *Let X, Y be compact topological spaces, μ a positive regular Borel measure with supp$(\mu) = X$, and $\pi : X \to Y$ a continuous mapping. Let $\nu := \pi \circ \mu$ denote the Borel measure on Y defined by $\nu(A) := \mu(\pi^{-1}(A))$. If f is a simple ν-continuity function on Y, then there exists a simple μ-continuity function g of Y such that $f = g \circ \pi$.*

Proof. The mapping $f \mapsto f \circ \pi$ is linear, so it suffices to prove the assertion for characteristic functions. Let $f = \mathbb{1}_A$ with a ν-continuity set A. Since $\mu(\partial(\pi^{-1}A)) \leq (\pi \circ \mu)(A) = \nu(A) = 0$ we obtain that $B := \pi^{-1}(A)$ is a μ-continuity set. Thus we can pick $g = \mathbb{1}_B$. $\square$

Lemma 3.2.6. *Let f, g be μ-Riemann integrable functions on the compact space X and μ a positive regular Borel measure with supp$(\mu) = X$. If f and g coincide on a dense set, then they coincide on the complement of a meager μ-null set.*

Proof. It suffices to show that $[f \neq g] \subseteq \operatorname{disc}(f) \cup \operatorname{disc}(g)$. To do so, let x be a point of continuity both for f and g. For any $\varepsilon > 0$ we can pick a a neighborhood U of x such that $y \in U$ implies $|f(y) - f(x)| < \varepsilon/2$ and $|g(y) - g(x)| < \varepsilon/2$. Since $[f = g]$ is dense there exists $y_\varepsilon \in U \cap [f = g]$. We conclude

$$|f(x) - g(x)| \leq |f(x) - f(y_\varepsilon)| + |g(y_\varepsilon) - g(x)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this implies $f(x) = g(x)$. $\square$

Corollary 3.2.7. *Let f, g be simple μ-continuity functions on the compact space X and μ a positive regular Borel measure with supp$(\mu) = X$. If f and g coincide on a dense set, then they coincide on a set of full μ-measure with non-void interior.*

Proof. There are (possibly after refinement) disjoint μ-continuity sets $A_1, \ldots, A_k$ such that $f = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{A_i}$ and $g = \sum_{i=1}^{k} \beta_i \mathbb{1}_{A_i}$. From Lemma 3.2.6 we know that $[f \neq g]$ is a μ-null set. Thus $\alpha_i \neq \beta_i$ for some $i = 1, \ldots k$ implies that A_i is a μ-null set. Since the closure of any μ-continuity null set is again a null set, $\bigcup_{\alpha_i \neq \beta_i} \overline{A_i}$ is a closed μ-continuity null set containing $[f \neq g]$. $\square$

3.3 Hartman measurability

Let us now introduce the concept of Hartman measurability for functions defined on a topological group G:

Definition 3.3.1. *A function bounded function $f : G \to \mathbb{C}$ is Hartman measurable iff f can be extended to a Riemann integrable function on the Bohr compactification (ι_b, bG), i.e. denoting the normalized Haar measure on bG by μ_b, f is Hartman measurable iff there exists a Riemann integrable function $F \in R_{\mu_b}(bG)$ such that $F \circ \iota_b$. The set of all Hartman measurable functions on G is denoted by $\mathcal{H}(G)$.*

Since $\iota_b^* : F \mapsto F \circ \iota_b$ is a linear and multiplicative mapping from the *-algebra $R_{\mu_b}(bG)$ onto $\mathcal{H}(G)$, the set of Hartman measurable functions on G is also a *-algebra. Having the implications of Proposition 3.2.1 in mind, we conclude that $\mathcal{H}(G)$ is translation invariant and consists of functions with unique mean-value. If G is locally compact, then every Hartman measurable function is measurable with respect to $\tilde{\mu_G}$, the completion of the Haar measure on G.

For a Boolean set-algebra $\Xi \subseteq \mathfrak{P}(X)$ on X let us define

$$B(X, \Xi) := \overline{\operatorname{span}}\{1\!\!1_A : A \in \Xi\} \subseteq B(X)$$

where the closure is taken w.r.t. the topology of uniform convergence.

Let us denote the Boolean set-algebra of μ-continuity sets in bG by $b\Sigma$. Thus the algebra of Hartman measurable sets on G is $\iota_b^{-1}(b\Sigma)$. With this notation $B(bG, b\Sigma)$ is the space of Riemann integrable functions on bG and the space $\mathcal{H}(G)$ of Hartman measurable functions is $\iota_b^* B(bG, b\Sigma)$.

Proposition 3.3.2. *Let G be a topological group. Then $\mathcal{H}(G) = B(G, \iota_b^{-1} b\Sigma)$, in particular $\mathcal{H}(G)$ is uniformly closed.*

Proof. Obviously $\operatorname{span}\{1\!\!1_A : A \in \iota_b^{-1}(b\Sigma)\} \subseteq \mathcal{H}(G) \subseteq B(G, \iota_b^{-1}(b\Sigma))$. Since $\operatorname{span}\{1\!\!1_A : A \in \iota_b^{-1}(b\Sigma)\}$ is dense in $B(G, \iota_b^{-1} b\Sigma)$, it suffices to show that $\mathcal{H}(G) = \iota_b^* B(bG, b\Sigma)$ is uniformly closed. $\iota_b^* : F \mapsto F \circ \iota_b$ is a continuous homomorphism of the C^*-algebra $B(bG, b\Sigma)$ into the C^*-algebra $B(G, \iota_b^{-1} b\Sigma)$. By basic C^*-algebra theory its image is closed (Theorem I.5.5 in [Dav]). □

Remark: Application of Proposition 3.2.1 yields that if a real-valued function f is Hartman measurable, then for every $\varepsilon > 0$ there are almost periodic functions g_ε and h_ε with $g_\varepsilon \leq f \leq h_\varepsilon$ such that $m(h_\varepsilon - g_\varepsilon) < \varepsilon$ for the unique invariant mean m on $\mathcal{A}(G)$.

At least for so called *maximally almost periodic* (MAP) groups we can reverse this assertion. Recall that a topological group G is MAP iff for the Bohr compactification (ι_b, bG) the morphism $\iota_b : G \to bG$ is one-one or, equivalently, iff the almost periodic functions $\mathcal{A}(G)$ separate the points of G. Since continuous characters of LCA groups separate points and are almost periodic, every LCA group is MAP.

Proposition 3.3.3. *Let G be a MAP group. For a real-valued function f on G the following assertions are equivalent:*

1. *f is Hartman measurable.*
2. *For every $\varepsilon > 0$ there are almost periodic functions g_ε and h_ε with $g_\varepsilon \leq f \leq h_\varepsilon$ such that $m(h_\varepsilon - g_\varepsilon) < \varepsilon$ for the unique invariant mean m on $\mathcal{A}(G)$.*

Proof. Since the implication (1) $\Rightarrow$ (2) follows immediately from the definition of Hartman measurability and Proposition 3.2.1, we will only prove (2) $\Rightarrow$ (1). Suppose for every $n \in \mathbb{N}$ there exist $g_n, h_n \in \mathcal{A}(G)$ with $g_n \leq f \leq h_n$ and $m(h_n - g_n) < \frac{1}{n}$. Let (ι, C) be any group compactification on which all g_n and h_n may be realized and such that $\iota : G \to C$ is one-one; w.l.o.g. we may take C to be the Bohr compactification bG of G.

Denote the unique continuous extensions of g_n, h_n to bG by $\tilde{g}_n$ resp. $\tilde{h}_n$. According to Proposition 3.2.1 every function F on bG satisfying $\sup_n \tilde{g}_n \leq F \leq \inf_n \tilde{h}_n$ is Riemann integrable with respect to the Haar measure on bG and any two such functions differ only on (a subset of) a μ_b-null set. Now pick F subject to the conditions

1. $\sup_n \tilde{g}_n \leq F \leq \inf_n \tilde{h}_n$ and
2. $F \circ \iota_b(g) = f(g)$ for every $g \in G$.

Note that $g_n = \tilde{g}_n \circ \iota_b \leq f \leq \tilde{h}_n \circ \iota_b = h_n$ ensures that Condition 1 and Condition 2 are compatible. Since ι_b is one-one such a function F exists. □

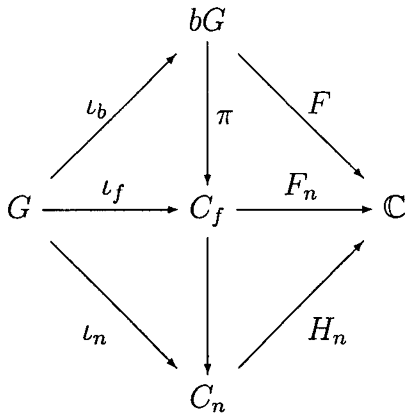
Let us turn now towards the *realizability* of Hartman measurable functions. Given a Hartman measurable function f, we say that F *realizes* f, if F is a Riemann integrable function defined on a group compactification (ι, C) such that $f = F \circ \iota$. In this situation we also say that f *can be realized on* (ι, C). On the other hand, if we are given a Riemann integrable function F on some group compactification (ι, C) we say that F *induces the Hartman measurable function* $f = F \circ \iota$.

Proposition 3.3.4. *Every Hartman measurable function f defined on the group of integers $\mathbb{Z}$ can be realized on a* metrizable *group compactification (ι_f, C_f), i.e. there exists a Riemann integrable function $F \in R_{\mu_{C_f}}(C_f)$ such that $f = F \circ \iota_f$.*

Proof. If a characteristic function f is Hartman measurable, the set $A := [f = 1]$ must be a Hartman set in the sense of [FPTW]. By Theorem 4 of [Win] there exists a metrizable group compactification (ι_f, C_f) and a continuity set $B \subseteq C_f$ such that $A = \iota_f^{-1}(B)$, so $1\!\!1_A = 1\!\!1_B \circ \iota_f$. The same applies if f is a finite linear combination of such characteristic functions; in this case we have to replace the metrizable group compactification for A by the supremum (=subdirect product) of all finitely many involved group compactifications (cf. [BJM]). This supremum is metrizable because it is a subspace of a finite product of metrizable spaces.

If $f = F \circ \iota_b$ is an arbitrary Hartman measurable function (w.l.o.g. induced by a Riemann integrable function defined on the Bohr-compactification bG) there exists a sequence of simple continuity functions F_n uniformly approximating F, according to Proposition 3.2.4. Let (ι_n, C_n) be a metrizable group compactification on which the function $f_n := F_n \circ \iota_b$ can be realized. Denote by (ι_f, C_f) the supremum of the group compactifications (ι_n, C_n), $n \in \mathbb{N}$. C_f is a subdirect product of at most countable many metrizable factors and hence also metrizable.

Let π, π_n be the canonical projections $bG \to C_f$ resp. $bG \to C_n$, i.e. $\pi \circ \iota_b = \iota_f$ and $\pi_n \circ \iota_b = \iota_n$ Since $(\iota_n, C_n) \leq (\iota, C_f)$ each f_n can be realized on C_f by a simple μ_f-continuity function H_n (Lemma 3.2.5). So $H_n \circ \iota_f = f_n = F_n \circ \iota_b$. Next



we prove that the functions $\pi^* H_n$ and F_n coincide on the dense set $\iota_b G$: $g \in G$ implies that

$$\pi^*(H_n \circ \iota_b)(g) = H_n \circ \pi \circ \iota_b(g) = H_n \circ \iota_f(g) = f_n(g) = F_n \circ \iota_b(g).$$

Since $\pi^* H_n, F_n \in \mathcal{C}_{\mu_b}(bG)$, the set of simple μ_b-continuity functions, we may invoke Corollary 3.2.7 to conclude that $\pi^* H_n$ and F_n coincide on an open set $U \subseteq bG$ with $\mu_b(U) = 1$. This is the same as to say that the sequence $\{H_n\}_{n=1}^{\infty}$ converges uniformly on the set $V := \pi U$. Since π is an open mapping, V is open and has measure $\mu_f(\pi U) = \mu_b(\pi^{-1}(\pi U)) \geq \mu_b(U) = 1$.

Thus the sequence $\{H_n\}_{n=1}^{\infty}$ converges on $V \subseteq C_f$ to a function H which satisfies $H \circ \iota_f = F \circ \iota_b = f$. Since V is open with full measure we may extend H as we wish without adding discontinuities on a set of more than zero μ_f-measure. Since $\mu_f(\mathrm{disc}(H|_V)) = \mu_b(\mathrm{disc}(F|_U)) = 0$, the function H is Riemann integrable. □

Note that we can adopt this proof for an arbitrary topological group G once we know that every Hartman *set* can be realized on a metrizable group compactification. Careful consideration of the proofs of the relevant parts in Theorem 2 in [FPTW] and Theorem 4 in [Win] show that the arguments given there apply not only to $\mathbb{Z}$ but to any LCA group with separable dual. We will give now the details of these arguments.

There is a unified approach to the theory of Bohr compactifications of LCA groups (cf. [HR] Theorem 26.12). Bohr compactifications bG of an LCA group G may either be regarded as the Pontryagin dual of $\hat{G}_d$, the Pontryagin dual of $\hat{G}$ endowed with the discrete topology, or as a certain subgroup of the product space $\prod_{\chi\in\hat{G}} \overline{\chi(G)}$. Note that $\overline{\chi(G)}$ is topologically isomorphic to $\mathbb{R}/\mathbb{Z}$ or to $\mathbb{Z}/n\mathbb{Z}$.

A group compactification (ι, C) of G is called finite resp. countable dimensional iff (ι, C) is equivalent to a supremum (=subdirect product) of finitely resp. countably many group compactifications of the type $(\chi, \overline{\chi(G)})$. Since each of the spaces $\overline{\chi(G)} \in \{\mathbb{R}/\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \dots\}$ is metrizable, we may use the term countable dimensional group compactification synonymously to the term metrizable group compactification. Furthermore we call a group compactification (ι, C) injective iff ι is one-one.

Lemma 3.3.5. *Let G be an LCA group with separable dual. Then each metrizable group compactification (ι, C) is covered by an injective and metrizable group compactification $(\tilde{\iota}, \tilde{C})$.*

Proof. First we construct a metrizable group compactification (ι_m, C_m) such that ι_m is one-one. Let A be a dense countable subgroup of $\hat{G}$. Let

$$C_m := \prod_{\chi\in A} \overline{\chi(G)}.$$

We claim that the continuous homomorphism $\iota_m : G \to C_m$ defined by $g \mapsto (\chi(g))_{\chi\in A}$ is one-one. Suppose $g_1 \neq g_2$. The continuous characters separate points on G, so there exists $\chi_0 \in G$ such that $|\chi_0(g_1) - \chi(g_2)| = \varepsilon > 0$. Pick $\tilde{\chi} \in A$ such that $|\tilde{\chi}(g_i) - \tilde{\chi}_0(g_i)| < \varepsilon/2$ for $i = 1, 2$. Thus $|\tilde{\chi}(g_1) - \tilde{\chi}(g_2)| \geq \varepsilon/2 > 0$, in particular $\tilde{\chi}(g_1) \neq \tilde{\chi}(g_2)$. Finally we conclude

$$\iota_m(g_1) = (\chi(g_1))_{\chi\in A} \neq (\chi(g_2))_{\chi\in A} = \iota_m(g_2).$$

This proves that ι_m is one-one. Let $(\tilde{\iota}, \tilde{C})$ be the supremum of (ι, C) and (ι_m, C_m) Clearly $(\tilde{\iota}, \tilde{C}) \geq (\iota, C)$ and $\tilde{C}$ is metrizable. Furthermore $\tilde{\iota}(g) = \tilde{\iota}(h)$ implies both $\iota_m(g) = \iota_m(h)$ and $\iota(g) = \iota(h)$. Thus $\tilde{\iota}$ is one-one. □

Lemma 3.3.6. *Let G be an LCA group and $T \subseteq G$ a Hartman measurable set. For every $\varepsilon > 0$ there are Hartman measurable sets T_ε and T^ε, realized on a finite dimensional (and hence metrizable) group compactification, such that $T_\varepsilon \subseteq T \subseteq T^\varepsilon$ and $m(T^\varepsilon \setminus T_\varepsilon) < \varepsilon$, where m denotes the invariant finitely additive measure defined on Hartman sets.*

Proof. Theorem 2 in [SchSW] tells us how to approximate an arbitrary Hartman measurable set by a finite dimensional one: Let $M \subseteq bG$ be a μ_b-continuity-set realizing T, i.e. $T = \iota_b^{-1}(M)$. Use the inner regularity of the Haar measure on bG to find an inner approximation $K \subseteq M$ and $\mu_b(M \setminus K) < \varepsilon$, K compact.

Since the Bohr compactification bG can be regarded as subspace of $\prod_{\chi \in \hat{G}} \overline{\chi(G)}$, a topological base $(B_i)_{i \in I}$ of bG is obtained by restricting the standard topological base $(\tilde{B}_i)_{i \in I}$ of the product space $\prod_{\chi \in \hat{G}} \overline{\chi(G)}$ to bG. The sets B_i can be chosen to be finite intersections of sets of the form

$$D_{\chi_0;a,b} := \{(\alpha_\chi)_{\chi \in \hat{G}} : \alpha_{\chi_0} \in (a,b)\}.$$

This implies that the base $(\tilde{B}_i)_{i \in I}$ consists of μ_b-continuity sets.

We can cover K by finitely many sets of the form $B_i \cap M$, $i = 1, \dots n$ with $(B_i)_{i \in I}$. Each set $B_i \cap M$ is a μ_b-continuity set and induces a Hartman measurable set on G that may be realized on a finite dimensional group compactification (ι, C). The same is true for the finite union $K \subseteq \bigcup_{l=1}^n (B_i \cap M) \subseteq M$, so let $T_\varepsilon := \iota^{-1}(\bigcup_{l=1}^n B_i \cap M)$. In a similar fashion one constructs T^ε using outer regularity of the Haar measure μ_b. □

Lemma 3.3.7. *Any Hartman measurable set T on an LCA group G with separable dual $\hat{G}$ can be realized on a metrizable group compactification of G, i.e. there exists a group compactification (ι, C) with C metrizable and a μ_C-continuity set $M \subseteq C$, such that $T = \iota^{-1}(M)$.*

Proof. We follow the lines of Theorem 4 in [Win]. So let T be a Hartman measurable set and $\{T_{1/n}\}_{n=1}^\infty, \{T^{1/n}\}_{n=1}^\infty$ sequences of finite dimensional Hartman measurable sets, approximating T from inside resp. outside. Let (ι, C) be the supremum of all involved, at most countably many, compactifications. This implies that C is metrizable. By Lemma 3.3.5 we can assume w.l.o.g. that (ι, C) is injective as well.

Denote by M_n resp. M^n the μ_C-continuity sets on C that realize the Hartman measurable sets $T_{1/n}$ resp. $T^{1/n}$. Thus $M_\infty := \bigcup_{n=1}^\infty M_n^\circ$ is open, $M^\infty := \bigcap_{n=1}^\infty \overline{M^n}$ is closed and $\iota^{-1}(M_\infty) \subseteq T \subseteq \iota^{-1}(M^\infty)$. Let $M := M_\infty \cup \iota_C(T)$. Since ι_C is one-one the preimage of M under ι coincides with the given Hartman measurable set T. Furthermore

$$\mu_C(\partial M) \leq \mu(M^\infty \setminus M_\infty) = \lim_{n \to \infty} \mu(M^n \setminus M_n) = 0$$

shows that M is actually a μ_C-continuity set. □

We can now generalize Proposition 3.3.4 and obtain the following

Theorem 2. *Any Hartman measurable function f on an LCA group G with separable dual $\hat{G}$ can be realized on a metrizable group compactification.*

Proof. Apply Proposition 3.3.4 and Lemma 3.3.7. □

Remark: Note that the class of all LCA groups with countable dual contains all countable and 1^{st} countable LCA groups.

Since Hartman measurable functions on the group G correspond to the Riemann integrable functions on the compact group bG, Theorem 2 essentially states that Riemann integrable functions on LCA groups with separable dual always factor through metrizable groups.

Next we will show that there exists a LCA group G with non-separable dual $\hat{G}$, such that is a Hartman measurable set $T \subseteq G$ that cannot be obtained via a metrizable group compactification of G.

Example 3.3.8. Let $G = 2^{\mathfrak{c}}$. The dual of the compact group G is the discrete group $\hat{G} = 2^{\mathfrak{c}*}$. We may interpret G as the group of all functions $f : \mathbb{R} \to \{0, 1\}$ and $\hat{G}$ as the group of all functions with finite support $f : \mathbb{R} \to \{0, 1\}$; the group operation is pointwise addition of functions modulo 2. In particular note that $\hat{G}$ is a discrete uncountable group, hence non-separable. Consider the subgroups $G_{\mathbb{N}}$ and $G_{\mathbb{R}\setminus\mathbb{N}}$ consisting of all those functions the support of which is a subset of $\mathbb{N}$ resp. $\mathbb{R} \setminus \mathbb{N}$. Clearly $G = G_{\mathbb{N}} \oplus G_{\mathbb{R}\setminus\mathbb{N}}$. Note that both $G_{\mathbb{N}}$ and $G_{\mathbb{R}\setminus\mathbb{N}}$ are closed μ_G-null sets.

Let us construct a set $A \subseteq G_{\mathbb{R}\setminus\mathbb{N}}$ with the property that $\forall g \in G_{\mathbb{R}\setminus\mathbb{N}}, g \neq 0$ $\exists a \in A, b \in G_{\mathbb{R}\setminus\mathbb{N}} \setminus A$, $a, b \neq 0$ such that $g = a + b$. To find such a set A pick a well-ordering $\{x_\alpha : \alpha < 2^{\mathfrak{c}}\}$ of $G_{\mathbb{R}\setminus\mathbb{N}}$.

Transfinite induction: Suppose that for all ordinals $\alpha < \alpha_0$ we have already defined a function $\beta : \alpha_0 \to 2^{\mathfrak{c}}$ such that

$$\{x_\alpha + x_{\beta(\alpha)} : \alpha < \alpha_0\} \cap \{x_{\beta(\alpha)} : \alpha < \alpha_0\} = \emptyset.$$

We want to find a $\beta(\alpha_0)$ such that $x_{\beta(\alpha_0)} \notin \{x_\alpha + x_{\beta(\alpha)} : \alpha < \alpha_0\}$ and $x_{\beta(\alpha_0)} \neq x_{\alpha_0} + x_{\beta(\alpha_0)}$. The "exceptional" set $D_{\alpha_0} := \{\delta : x_\delta = x_\alpha + x_{\beta(\alpha)} \text{ for an } \alpha < \alpha_0\}$ has a most α_0 many elements. Since $|\alpha_0| < 2^{\mathfrak{c}}$ the set $\{x_\alpha : \alpha < 2^{\mathfrak{c}}\} \setminus D_{\alpha_0}$ is not empty. Define $\beta(\alpha_0) := \min(\{x_\alpha : \alpha < 2^{\mathfrak{c}}\} \setminus D_{\alpha_0})$. The set $\{x_{\beta(\alpha)} : \alpha < 2^{\mathfrak{c}}\}$ has the desired property.

Since $G_{\mathbb{R}\setminus\mathbb{N}}$ is a closed μ_G-null set, such a set A is a μ_G-continuity set and $f := \mathbb{1}_A$ is a Hartman measurable function.

Claim: The Hartman measurable function f cannot be realized on any metrizable compactification of G.

Proof. Suppose by contradiction $f = \tilde{f} \circ \pi$ with a Riemann integrable function $\tilde{f}$ defined on the compactification (ι, C) and the canonical epimorphism $\pi : G \to C$. $\pi(G) = C$ implies that we must have $\tilde{f} = \mathbb{1}_{\pi A}$ and thus $\mathbb{1}_A(x) = \mathbb{1}_{\pi A}(\pi x)$. Since C is metrizable, but G is not, π cannot be one-one. Hence $\ker \pi$ must contain some non trivial $g \in G$. Since $G = G_{\mathbb{N}} \oplus G_{\mathbb{R}\setminus\mathbb{N}}$ and $G_{\mathbb{N}}$ is metrizable $\ker \pi \nsubseteq G_{\mathbb{N}}$ and hence $\ker \pi$ must contain an element $g = g_1 \oplus g_2$ with $g_2 \neq 0$.

Let $g = g_1 + a + b$ with $g_1 \in G_{\mathbb{N}}, a \in A, b \in G_{\mathbb{R}\setminus\mathbb{N}} \setminus A$ and $a, b \neq 0$.

Case 1: If $g_1 = 0$ we obtain the following contradiction:

$$1 = \mathbb{1}_A(a) = \mathbb{1}_A(g + b) = \mathbb{1}_{\pi A}(\pi(g + b)) = \mathbb{1}_{\pi A}(\pi(b)) = \mathbb{1}_A(b) = 0.$$

Case 2: If $g_1 \neq 0$, then the support of $g_1 + b$ is not contained in $\mathbb{R} \setminus \mathbb{N}$ and hence not contained in A. Thus we obtain the following contradiction:

$$0 = \mathbb{1}_A(g_1 + b) = \mathbb{1}_{\pi A}(\pi(g_1 + b)) = \mathbb{1}_{\pi A}(\pi(g + g_1 + b)) = \mathbb{1}_{\pi A}(\pi a) = \mathbb{1}_A(a) = 1.$$

□

3.4 Filters associated with Hartman measurable functions

By definition every Hartman measurable function φ on the (abelian) group G has a realization on the Bohr compactification bG by a Riemann integrable function $\varphi^* \in R_{\mu_b}(bG)$. The mapping

$$d_{\varphi^*} : x \mapsto \|\varphi^* - \tau_x\varphi^*\|_1 = \int_{bG} |\varphi^* - \tau_x\varphi^*| d\mu, \quad x \in bG$$

is continuous on bG (cf. [Els], Corollary 2.32). This implies that $d_\varphi := d_{\varphi^*} \circ \iota_b$ is an almost periodic function on G. Denoting the unique invariant mean on $\mathcal{H}(G)$ by m_G, we can also write $d_\varphi(g) = m_G(|\varphi - \tau_g\varphi|)$. It is then tempting to define $F(\varphi, \varepsilon) := \{g \in G : d_\varphi(\tau_g\varphi) < \varepsilon\}$ and denote by $\mathcal{F}(\varphi)$ the filter on G generated by $\{F(\varphi, \varepsilon) : \varepsilon > 0\}$.

In the LCA setting, we can apply the tools developed in [Win] to conclude a functional analogue of Theorem 2 in [Win].

Definition 3.4.1. *Let $\varphi \in \mathcal{H}(G)$ be realized by φ^* on the group compactification (ι, C). φ^* is called an aperiodic realization of φ iff* $\ker d_{\varphi^*} := \{x \in C : \|\varphi^* - \tau_x\varphi^*\|_1 = 0\} = \{0_C\}$.

Theorem 3. *Let $\varphi \in \mathcal{H}(G)$ be realized by φ^* on the group compactification (ι, C). Then $\mathcal{F}(\varphi) \subseteq \mathfrak{U}_{(\iota,C)}$. Furthermore $\mathcal{F}(\varphi) = \mathfrak{U}_{(\iota,C)}$ if φ^* is an aperiodic realization.*

Proof. Suppose $\varphi = \varphi^* \circ \iota$ with $\varphi^* \in R_{\mu_C}(C)$ for a group compactification (ι, C). For any set $A \in \mathcal{F}(\varphi)$ there exists $\varepsilon > 0$ such that $d_\varphi(x) < \varepsilon$ implies $x \in A$. Using almost periodicity of d_φ, i.e. continuity of d_{φ^*}, we find a neighborhood

$U \in \mathfrak{U}(C, 0_C)$ such that $d_{\varphi^*}(U) \subseteq [0, \varepsilon)$. For every $x \in \iota^{-1}(U) \in \mathcal{F}_C$ we have $d_\varphi(x) < \varepsilon$. Consequently $\iota^{-1}(U) \subseteq A \in \mathfrak{U}_{(\iota,C)}$ and hence $\mathcal{F}(\varphi) \subseteq \mathfrak{U}_{(\iota,C)}$.

Suppose that $\varphi^* \in R_{\mu_C}(C)$ is aperiodic, i.e. $d_{\varphi^*}(x) = 0$ iff $x = 0_C$, the unit in C. Let $A \in \mathfrak{U}_{(\iota,C)}$ be arbitrary; w.l.o.g. we can assume $A \supseteq \iota^{-1}(U)$ for an open neighborhood $U \in \mathfrak{U}(C, 0_C)$. Due to the continuity of d_{φ^*} and compactness of C we have $d_{\varphi^*}(x) \geq \varepsilon > 0$ for $x \in C \setminus U^\circ$. This implies $\iota(\{g \in G : d_\varphi(g) < \varepsilon\}) \subseteq U$ and hence $\{g \in G : d_\varphi(g) < \varepsilon\} \subseteq \iota^{-1}(U) \subseteq A \in \mathcal{F}(\varphi)$. Thus $\mathfrak{U}_{(\iota,C)} \subseteq \mathcal{F}(\varphi)$ and consequently $\mathfrak{U}_{(\iota,C)} = \mathcal{F}(\varphi)$. □

Definition 3.4.2. *Let $\varphi \in \mathcal{H}(G)$ and let (ι, C) be a group compactification of G. A function $\psi^* \in R_{\mu_C}(C)$ is called an almost realization of φ iff $m_G(|\varphi - \psi|) = 0$ for $\psi := \psi^* \circ \iota$ and m_G the unique invariant mean on $\mathcal{H}(G)$.*

Theorem 4. *Every $\varphi \in \mathcal{H}(G)$ has an aperiodic almost realization on some group compactification (ι, X). If $\varphi^* : X \to \mathbb{C}$ is an aperiodic almost realization of φ then $\mathcal{F}(\varphi) = \mathfrak{U}_{(\iota,X)}$.*

Proof. We only have to prove that an aperiodic almost realization exists, the rest follows from Theorem 3. Let φ^* be a realization of φ on X. The reader will easily check that $H := \ker d_{\varphi^*} = \{x \in X : d_{\varphi^*}(x) = 0\}$ is a closed subgroup of the compact abelian group X.

Weil's formula for continuous functions on quotients (Theorem 3.22 in [Els]) states that there exists $\alpha > 0$ such that for every $f \in C(X)$

$$\int_{X/H} \underbrace{\Big(\int_H f(s+t) d\mu_H(t) \Big)}_{=^\flat f(s)} d\mu_{X/H}(s) \;=\; \alpha \int_X f(u) d\mu_X(u) \tag{3.1}$$

holds. This implies that the canonical mapping $^\flat : C(X) \to C(X/H)$, $f \to {}^\flat f$ defined by ${}^\flat f(s+H) = \int_H f(s+t) d\mu_H(t)$ satisfies $\|{}^\flat f\|_1 \leq \alpha \|f\|_1$. We rescale the Haar measure on H such that $\alpha = 1$. Thus we can extend $^\flat$ to a continuous linear operator $L^1(X) \to L^1(X/H)$. Furthermore positivity of $^\flat$ enables us to extend $^\flat$ to a mapping defined on $R_{\mu_X}(X)$ in the following way:

According to Proposition 3.2.1 $f \in R_{\mu_X}(X)$ implies that there are $g_n, h_n \in C(X)$ such that $g_n \leq f \leq h_n$ and $\|h_n - g_n\|_1 \to 0$ as $n \to \infty$. Thus every function $\tilde{f}$ on X/H satisfying

$$f_\bullet := \sup_{n \geq 0} {}^\flat g_n \leq \tilde{f} \leq \inf_{n \geq 0} {}^\flat h_n =: f^\bullet$$

is in $R_{\mu_{X/H}}(X/H)$. Note that $f_\bullet$ and $f^\bullet$ are μ_H-measurable and coincide μ_H-a.e.. To define ${}^\flat f$ we pick any function $\tilde{f}$ satisfying $f_\bullet \leq \tilde{f} \leq f^\bullet$. Then Weil's formula (3.1) will still be valid, regardless of the particular choice of the functions g_n, h_n and ${}^\flat f$.

Since φ^* is Riemann integrable on X, there exist functions $\varphi_n \in C(X)$ such that $\|\varphi^* - \varphi_n\|_1 \to 0$. Note that the convergence $d_{\varphi_n} \to d_{\varphi^*}$ is even uniform on X:

$$|d_{\varphi_n}(s) - d_{\varphi^*}(s)| = \Big| \|\tau_s\varphi_n - \varphi_n\|_1 - \|\tau_s\varphi^* - \varphi^*\|_1 \Big| \leq 2\|\varphi_n - \varphi^*\|_1 \to 0.$$

Using the continuity of $^\flat$ as a mapping on $L^1(X)$ the same argument also shows that $|d_{^\flat\varphi^*}(s+H) - d_{^\flat\varphi_n}(s+H)| \leq 2\|\varphi_n - \varphi^*\|_1 \to 0$ uniformly on X/H. Now suppose $d^*_{^\flat\varphi}(s+H) = 0$. Thus

$$d_{\varphi^*}(s) = \lim_{n\to\infty} d_{\varphi_n}(s) = \lim_{n\to\infty} d_{^\flat\varphi_n}(s+H) = 0$$

implies $s \in H$, i.e. $s + H = 0_X + H \in X/H$. So $^\flat\varphi^*$ is aperiodic.

We show that φ^* being a realization of φ implies that $^\flat\varphi^*$ is an almost realization of φ. By definition $t \in H$ iff $A_t := \{s \in X : \varphi^*(s+t) = \varphi^*(s)\}$ has μ_X-measure 1. Applying Weil's formula (3.1) to the function $f = 1\!\!1_{A_t} \in L^1(X)$ gives

$$\int_{X/H} {^\flat f}\, d\mu_{X/H} = \int_{X/H} {^\flat 1\!\!1_{A_t}}(s+H)\, d\mu_{X/H}(s+H) = \int_X f\, d\mu_X = 1. \tag{3.2}$$

Plugging the definition of $^\flat$ into (3.2) we get $\mu_{X/H}$-a.e. the identity

$${^\flat 1\!\!1_{A_t}}(s+H) = \int_H 1\!\!1_{A_t}(s+u)\, d\mu_H(u) = 1.$$

So for every $t \in H$ and $\mu_{X/H}$-a.e. $s + H$ we know that the set $\{u \in H : \varphi^*(s+t+u) \neq \varphi^*(s+u)\}$ is a μ_H-null set. This means

$$\tau_t(\tau_s {\varphi^*}_{|H}) = \tau_s {\varphi^*}_{|H} \quad \mu_H\text{-a.e.}$$

Thus $\tau_s\varphi^*$ is constant μ_H-a.e. on H and for $\mu_{X/H}$ almost all $s + H$ we have

$${^\flat\varphi^*}(s+H) \;=\; \int_H \tau_s\varphi^*(t)\, d\mu_H(t) = \int_H \varphi(s)^*\, d\mu_H(t) = \varphi^*(s).$$

Let $\pi_H : X \to X/H$ be the quotient mapping onto the group compactification $(\tilde{\iota}, X/H)$. Let $\psi^* := {^\flat\varphi^*} \circ \pi_H$. Since $^\flat\varphi^*$ is Riemann integrable on X/H it follows by Lemma 3.2.5 that ψ^* is Riemann integrable on X. Weil's formula (3.1), together with the fact that the Haar measure on the quotient X/H is given by $\mu_{X/H} = \pi_H^{-1} \circ \mu_X$, implies $\psi^* = \varphi^*$ μ_X-a.e. Thus the function ψ defined by

$$\psi := \psi^* \circ \iota = {^\flat\varphi^*} \circ \tilde{\iota}$$

satisfies $m_G(|\varphi - \psi|) = \|\varphi^* - \psi^*\|_1 = 0$ for the unique invariant mean m_G. Thus ψ^* is the required almost realization of φ. □

Corollary 3.4.3. *Every $\varphi \in \mathcal{A}(G)$ has an aperiodic realization on some group compactification (ι, X).*

Proof. We use the notation from Theorem 4. If φ is almost periodic then φ^* is continuous. Consequently ${}^b\varphi^*$ and $\psi^* := {}^b\varphi^* \circ \pi$ are also continuous. Since these functions coincide μ_X-a.e. they coincide everywhere on X. This implies that φ^* is constant on H-cosets and ${}^b\varphi^*(s + H) = \varphi^*(s)$ for all $s + H \in X/H$. So φ^* is even a realization of φ, not only an almost realization. □

This Corollary is a special case of Følner's "Main Theorem for Almost Periodic Functions" in [Føl].

Remark: Note that for any given realization of a Hartman measurable function $\varphi \in \mathcal{H}(G)$ on a group compactification (ι, C) we can w.l.o.g. assume that there exists an aperiodic almost realization of φ on a group compactification $(\tilde{\iota}, \tilde{C})$ with $(\tilde{\iota}, \tilde{C}) \leq (\iota, C)$. Since we have shown in Theorem 2 that every Hartman measurable function on an LCA group with separable dual has a realization on a metrizable group compactification, every Hartman measurable function on such a group has an aperiodic almost realization on a metrizable group compactification.

Definition 3.4.4. *Let G be an LCA group and Γ a subgroup of $\hat{G}$. The induced group compactification (ι_Γ, X_Γ) is defined via*

$$X_\Gamma := \overline{\{(\chi(g))_{\chi\in\Gamma} : g \in G\}} \leq \prod_{\chi\in\Gamma(\varphi)} C_\chi, \quad C_\chi \cong \begin{cases} \mathbb{T} & \text{if } ord(\chi) = \infty, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } ord(\chi) = n. \end{cases}$$

Lemma 3.4.5. *Let G be an LCA group and let (ι, C) a group compactification. Then there exists a unique subgroup $\Gamma \leq \hat{G}$ such that (ι_Γ, X_Γ) and (ι, C) are equivalent. Furthermore (ι, C) is the supremum of all group compactifications (ι_γ, X_γ) such that $(\iota_\gamma, X_\gamma) \leq (\iota, C)$ (writing in short (ι_γ, X_γ) for $(\iota_{\langle\gamma\rangle}, X_{\langle\gamma\rangle})$).*

The mapping $(\iota, C) \mapsto \Gamma$ is a bijection between equivalence classes of group compactifications of G and subgroups of $\hat{G}$.

Proof. See Theorem 26.13 in [HR]. □

Corollary 3.4.6. *Let $\varphi \in \mathcal{H}(G)$. Any two group compactifications (ι_1, C_1) and (ι_2, C_2) on which φ has an aperiodic almost realization are equivalent.*

Proof. By Theorem 3 we have $\mathfrak{U}_{(\iota_1,C_1)} = \mathcal{F}(\varphi) = \mathfrak{U}_{(\iota_2,C_2)}$. A straight forward adaption of Theorem 1 in [Win] implies that the mapping

$$\Phi : \hat{G} \geq \Gamma \mapsto (\iota_\Gamma, X_\Gamma)$$

coincides with the composition of the mappings

$$\begin{aligned} \Sigma : (\iota_b, bG) \geq (\iota, C) &\mapsto \mathfrak{U}_{(\iota,C)}, \\ \Psi : \mathfrak{P}(G) \supseteq \mathcal{F} &\mapsto \{\gamma \in \hat{G} : \gamma(\mathcal{F}) \to 0\}. \end{aligned}$$

Since Lemma 3.4.5 states that $\Phi = \Psi \circ \Sigma$ is invertible, Σ must be one-one. In particular $\mathfrak{U}_{(\iota_1, C_1)} = \mathfrak{U}_{(\iota_2, C_2)}$ implies that (ι_1, C_1) and (ι_2, C_2) are equivalent group compactifications. □

For the rest of this section assume that G is an LCA group with separable dual.

Corollary 3.4.7. *Every filter $\mathcal{F}(\varphi)$ with $\varphi \in \mathcal{H}(G)$ coincides with a filter $\mathfrak{U}_{(\iota,C)}$ for a metrizable group compactification (ι, C). If φ^* is an arbitrary realization of φ, say on the Bohr compactification bG, we can take $X \cong bG/\ker d_{\varphi^*}$.*

Corollary 3.4.8. *Hartman measurable functions induce exactly the filters coming from metrizable group compactifications.*

Proof. In Theorem 3 in [Win] for every metrizable group compactification (ι, C) of the integers $G = \mathbb{Z}$, an aperiodic Hartman periodic function of the form $f = 1\!\!1_A$ is constructed. The same construction can be done in an arbitrary LCA group G as long as the dual $\hat{G}$ contains a countable and dense subset. This shows that any $\mathfrak{U}_{(\iota,C)}$ with metrizable C can be obtained already by an Hartman measurable set, i.e. by a filter $\mathcal{F}(\varphi)$ with $\varphi = 1\!\!1_A$. As we have shown in Theorem 2 any Hartman measurable function on G can be realized on a metrizable group compactification. Thus Theorem 3 implies that no filter $\mathcal{F}(\varphi)$ can coincide with $\mathfrak{U}_{(\iota,C)}$ for a non metrizable group compactification (ι, C). □

3.5 Subgroups associated with Hartman measurable functions

For Hartman measurable φ let us denote by $\Gamma(\varphi)$ the (countable) subgroup of $\hat{G}$ generated by the set

$$\operatorname{spec} \varphi := \{\chi \in \hat{G} : m_G(\varphi \cdot \overline{\chi}) \neq 0\}$$

of all characters with non vanishing Fourier coefficients. We will prove that $\Gamma = \Gamma(\varphi)$ determines a group compactification (ι_Γ, X_Γ) such that φ can be realized aperiodically on X_Γ. First we deal with almost periodic functions:

Proposition 3.5.1. *Let (ι, X) be a group compactification of the LCA group G and let Γ be a subgroup of $\hat{G}$. If every character $\chi \in \Gamma$ has a representation $\chi = \eta \circ \iota$ with a continuous character $\eta \in \hat{X}$, then every function $f \in \overline{\operatorname{span}}\, \Gamma \subseteq \mathcal{A}(G)$ has a realization on (ι, X).*

Proof. This is essentially a reformulation of Theorem 5.7 in [Bur]. In fact the Stone-Weierstrass Theorem implies that $\overline{\operatorname{span}}\, \Gamma = \iota^* C(X)$. □

If $\Gamma = \Gamma(\varphi)$ for some $\varphi \in \mathcal{A}(G)$, we want to show that $\varphi \in \overline{\operatorname{span}}\, \Gamma(\varphi)$, i.e. that the almost periodic function φ can be realized on X.

Proposition 3.5.2. *Let $\varphi \in \mathcal{A}(G)$ and (ι_Γ, X_Γ) the group compactification of G induced by the subgroup $\Gamma = \Gamma(\varphi) \leq \hat{G}$. Then for every continuous character $\chi \in \Gamma(\varphi)$ there exists a continuous character $\eta \in \hat{X}_\Gamma$ such that $\chi = \eta \circ \iota_\Gamma$.*

Proof. The compact group X_Γ is by definition topologically isomorphic with

$$\overline{\{(\chi(g))_{\chi\in\Gamma(\varphi)} : g \in G\}} \leq \prod_{\chi\in\Gamma} C_\chi, \quad C_\chi \cong \begin{cases} \mathbb{T} & \text{if } \operatorname{ord}(\chi) = \infty, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } \operatorname{ord}(\chi) = n. \end{cases}$$

For each $\chi_0 \in \Gamma(\varphi)$ the restriction of the projection

$$\pi_{\chi_0} : \prod_{\chi\in\Gamma(\varphi)} C_\chi \to C_{\chi_0}$$

to X_Γ is a continuous character. Thus $\pi_{\chi_0}|_{X_\Gamma}$ is an element of $\widehat{X_\Gamma}$ and $\chi_0 = \pi_{\chi_0}|_{X_\Gamma} \circ \iota_\Gamma$. □

Proposition 3.5.3. *Let $\varphi \in \mathcal{A}(G)$ and let (ι_Γ, X_Γ) be a group compactification of G such that φ can be realized by a continuous function $\varphi^* : X_\Gamma \to \mathbb{C}$. Then each continuous character $\chi \in \Gamma(\varphi)$ has a representation $\chi = \eta \circ \iota_\Gamma$ with $\eta \in \hat{X}_\Gamma$.*

Proof. Obviously it is enough to prove the assertion for a generating subset of $\Gamma(\varphi)$. Let $\chi \in \hat{G}$ be such that $m_G(\varphi \cdot \overline{\chi}) \neq 0$. Define a linear functional $m_\chi : C(X_\Gamma) \to \mathbb{C}$ via $\psi \mapsto m_\chi(\psi) = m_G((\psi \circ \iota_\Gamma) \cdot \overline{\chi})$ for $\psi \in C(X_\Gamma)$. It is routine to check that m_χ is bounded, in fact $\|m_\chi\| \leq 1$. Since X_Γ is compact the complex-valued mapping $\tilde{\eta} : X_\Gamma \mapsto m_\chi(\tau_x\varphi^*)$ is continuous on X_Γ (the mapping $x \mapsto \tau_x\varphi$ is continuous). For $g \in G$ we compute

$$\begin{aligned} \tilde{\eta} \circ \iota_\Gamma(g) &= m_G((\tau_{\iota_\Gamma(g)}\varphi^* \circ \iota_\Gamma) \cdot \overline{\chi}) = m_G(\tau_g(\varphi^* \circ \iota_\Gamma) \cdot \overline{\chi}) \\ &= m_G((\varphi^* \circ \iota_\Gamma) \cdot \tau_{-g}\overline{\chi}) = m_G((\varphi^* \circ \iota_\Gamma) \cdot \chi(g)\overline{\chi}) \\ &= \chi(g) m_\chi(\varphi^*) = \chi(g)\tilde{\eta}(0). \end{aligned}$$

Since $\tilde{\eta}(0) = m_\chi(\varphi^*) = m_G(\varphi \cdot \overline{\chi}) \neq 0$ we can define $\eta := \tilde{\eta}(0)^{-1}\tilde{\eta}$. The mapping $\eta : C_\Gamma \to \mathbb{T}$ is continuous and satisfies the functional equation

$$\eta(\iota_\Gamma(g) + \iota_\Gamma(h)) = \tilde{\eta}(0)^{-1}\tilde{\eta}(\iota_\Gamma(g) + \iota_\Gamma(h)) = \chi(g)\chi(h) = \eta(\iota_\Gamma(g))\eta(\iota_\Gamma(h))$$

on the dense set $\iota_\Gamma(G)$. Hence η is a bounded character and $\eta \circ \iota_\Gamma = \chi$. □

Corollary 3.5.4. *Let $\varphi \in \mathcal{H}(G)$ be realized by φ^* on the group compactification (ι_Γ, X_Γ). Then each $\chi \in \Gamma(\varphi)$ has a representation $\chi = \eta \circ \iota_\Gamma$ with $\eta \in \hat{X}_\Gamma$.*

Proof. For every $\chi \in \hat{G}$ with $m_G(\varphi \cdot \overline{\chi}) = \alpha \neq 0$ we can pick a continuous function $\psi^* : C_\Gamma \to \mathbb{C}$ such that $\|\psi^* - \varphi\|_1 < |\alpha|/2$. Then $\psi := \psi^* \circ \iota_\Gamma$ satisfies

$$|m_G(\varphi \cdot \overline{\chi}) - m_G(\psi \cdot \overline{\chi})| \leq m_G(|\varphi - \psi|) \leq \|\psi^* - \varphi^*\|_1 < |\alpha|/2.$$

In particular $m_G(\psi \cdot \overline{\chi}) \neq 0$. Applying Proposition 3.5.3 to the function $\psi \in \mathcal{A}(G)$ yields that the character χ can be realized on X_Γ. □

For an almost periodic function φ the subgroup $\Gamma(\varphi)$ contains all the relevant information not only to reconstruct φ from its Fourier-data but also describes a minimal group compactification on which a realization of φ exists. It is not obvious how to obtain similar results for non almost periodic Hartman measurable functions. The following example illustrates how the straight forward approach breaks down.

Example 3.5.5. Let $\varphi_n(k) := \prod_{j=1}^{n} \cos^2\left(2\pi \frac{k}{3^j}\right)$ on $G = \mathbb{Z}$. As each factor in this product is periodic (hence almost periodic) and $\mathcal{A}(G)$ is closed under multiplication, each φ_n is an almost periodic function. As we will show in Proposition 3.7.4 $\lim_{n\to\infty} \varphi_n(x) =: \varphi(x)$ exists and defines a non negative Hartman measurable function with $m_{\mathbb{Z}}(\varphi) = 0$. Since $\Gamma(\varphi_n) \cong \mathbb{Z}/3^n\mathbb{Z}$

$$\lim_{n\to\infty} \Gamma(\varphi_n) = \bigcup \Gamma(\varphi_n) \cong \mathbb{Z}_3^\infty,$$

for the Prüfer 3-group $\mathbb{Z}_3^\infty$ (i.e. the subgroup of all complex 3^n-th roots of unity for $n \in \mathbb{N}$). On the other hand

$$\Gamma(\lim_{n\to\infty} \varphi_n) = \{0\}.$$

Thus $\Gamma(\lim_{n\to\infty} \varphi_n) \neq \lim_{n\to\infty} \Gamma(\varphi_n)$.

Proposition 3.5.6. *Let $\{K_n\}_{n=1}^\infty$ denote the family of Fejér kernels on $\mathbb{T}^k$. The convolution operators on $L^1(\mathbb{T}^k)$ defined by*

$$\sigma_n : \varphi \mapsto K_n * \varphi$$

are non negative, their norm is uniformly bounded by $\|\sigma_n\| = 1$ and $\sigma_n\varphi(x) \to \varphi(x)$ a.e. for every $\varphi \in L^1(\mathbb{T}^k)$. Furthermore $\sigma_n\varphi \in$ span $\Gamma(\varphi)$ for every $n \in \mathbb{N}$.

Proof. This is a reformulation of the results form 44.51 in [HR]. □

Let f be Riemann integrable on $X = \mathbb{T}^k$, w.l.o.g. real-valued, and $\varphi_i, \psi_i \in C(X)$ such that $\varphi_i \geq f \geq \psi_i$ and $\|\varphi_i - \psi_i\|_1 < \varepsilon_i$ for a sequence $\{\varepsilon_i\}_{i=1}^\infty$ of positive real numbers, tending monotonically to 0. We know that $\sigma_n f(x) \to f(x)$ for a.e. $x \in X$. Thus we have

$$\varphi_n^* := \sigma_n\varphi_n \geq \sigma_n f \geq \sigma_n\psi_n = \psi_n^*$$

and

$$\|\varphi_n^* - \psi_n^*\|_1 \leq \|\sigma_n(\varphi_n^* - \psi_n^*)\|_1 \leq \|\sigma_n\| \, \|\varphi_n - \psi_n\|_1 \leq \varepsilon_n.$$

Let $\varphi^* := \inf_{n\in\mathbb{N}} \varphi_n$ and $\psi^* := \sup_{n\in\mathbb{N}} \psi_n$. If we assume w.l.o.g. ψ_n to increase and φ_n to decrease as $n \to \infty$, the same will hold for ψ_n^* and φ_n^*. This implies that in the inequality

$$\varphi^*(x) = \lim_{n\to\infty} \varphi_n^*(x) \geq \limsup_{n\to\infty} \sigma_n f \geq \liminf_{n\to\infty} \sigma_n f \geq \lim_{n\to\infty} \psi_n^*(x) = \psi^*(x)$$

equality holds μ_X-a.e. on X. Thus we can apply Proposition 3.2.1 and conclude that any function f^* with $\varphi^* \geq f^* \geq \psi^*$ is Riemann integrable (and coincides μ_X-a.e. with f). In particular $f^\bullet := \limsup_{n\to\infty} \sigma_n f$ and $f_\bullet := \liminf_{n\to\infty} \sigma_n f$ are (lower resp. upper semicontinuous) Riemann integrable functions that coincide μ_X-a.e. with f.

We have use countable dimensional group compactifications already in chapter 2. Let us call a group compactification (ι, C) finite dimensional iff C is topologically isomorphic to a closed subgroup of $\mathbb{T}^n$ for some $n \in \mathbb{N}$. Note that if (ι, C) is finite dimensional, then every group compactification covered by (ι, C) is finite dimensional as well. A Hartman measurable function $\varphi \in \mathcal{H}(G)$ can be realized finite dimensionally iff there exists a realization of φ on some finite dimensional group compactification.

Proposition 3.5.7. *For a compact LCA group C the following assertions are equivalent:*

1. *C is finite dimensional,*
2. *$\hat{C}$ is finitely generated,*
3. *C is topological isomorphic to $\mathbb{T}^k \times F$ for $k \in \mathbb{N}$ and a finite group F of the form*
$$F = \prod_{i=1}^{N} (\mathbb{Z}/n_i\mathbb{Z})^{p_i}, \quad p_i \text{ prime.}$$

Proof. (1) $\Rightarrow$ (2): Suppose C is a closed subgroup of $\mathbb{T}^n$. By Pontryagin's duality Theorem $\hat{C} \cong \widehat{\mathbb{T}^n}/\mathrm{Ann}(C) \cong \mathbb{Z}^n/\mathrm{Ann}(C)$. Since $\mathbb{Z}^n$ is a finitely generated abelian group and the class of finitely generated abelian groups is closed under taking quotients, this implies (2).

(2) $\Rightarrow$ (3): Theorem A.27 in [HR] implies that $\hat{C}$ is of the form $\mathbb{Z}^k \times F$, with $k \in \mathbb{N}$ and F finite of the postulated form. Invoking again Pontryagin duality, we conclude $C \cong (\mathbb{Z}^k \times F)\hat{} \cong \mathbb{T}^k \times F$.

(3) $\Rightarrow$ (1): Let $m := \sum_{i=1}^{N} p_i$. Clearly $\mathbb{T}^k \times \prod_{i=1}^{N} (\mathbb{Z}/n_i\mathbb{Z})^{p_i}$ is topologically isomorphic to a closed subgroup of T^{k+m}. □

Proposition 3.5.8. *Let $\varphi \in \mathcal{H}(G)$. If φ can be realized finite dimensionally, then there is an almost realization of φ on the (finite dimensional) compactification induced by $\Gamma := \Gamma(\varphi)$.*

Proof. Let φ be realized finite dimensionally on some group compactification (ι, C). Since there exists a group compactification covered by (ι, C), on which φ can be almost realized aperiodically (cf. Theorem 4), we can assume w.l.o.g. that φ can realized aperiodically already on (ι, C). We have to show that (ι, C) and (ι_Γ, X_Γ) are equivalent.

Let ψ^* be an aperiodic almost realization of φ on $X_\Gamma \cong \mathbb{T}^k \times F$ with $k \in \mathbb{N}$ and F finite. Let us denote the elements of $T^k \times F$ by tupels $(\vec{\alpha}, x)$. For every fixed $\vec{\alpha} \in \mathbb{T}^k$ define a mapping $\psi_{\vec{\alpha}} : F \longrightarrow \mathbb{R}$ via

$$\psi^*_{\vec{\alpha}}(x) := \psi^*(\vec{\alpha}, x).$$

For each $\overline{\chi} \in \hat{F}$, the dual of the finite group F, define the F-Fourier coefficient of $\psi^*_{\vec{\alpha}}$ as

$$c_{\overline{\chi}}(\vec{\alpha}) := \int_F \psi^*_{\vec{\alpha}}(x)\overline{\chi}(x)dx = \frac{1}{\#F}\sum_{x\in F}\psi^*(\vec{\alpha}; x)\overline{\chi}(x) \in \mathbb{C}.$$

is a Riemann integrable function defined on $\mathbb{T}^k$:

For every $x \in F$ the mapping $\gamma_x : \mathbb{T}^k \longrightarrow X$ defined via $\vec{\alpha} \mapsto (\vec{\alpha}; x)$ is continuous and measure-preserving. ψ^* is, by definition, Riemann integrable. Thus for each $x \in F$ the mapping $\psi^* \circ \gamma_x : \mathbb{T}^k \longrightarrow \mathbb{C}$ is Riemann integrable (cf. [Els]). Note that

$$c_{\overline{\chi}}(\vec{\alpha}) = \sum_{x\in F}(\psi^* \circ \gamma_x)(\vec{\alpha})\overline{\chi}(x).$$

Hence for each fixed character $\chi \in \hat{F}$ the mapping $c_{\overline{\chi}} : \mathbb{T}^k \longrightarrow \mathbb{C}$ defined via $c_{\overline{\chi}}(\vec{\alpha}) \mapsto \sum_{x\in F}(\psi^* \circ \gamma_x)(\vec{\alpha})\overline{\chi}(x)$ is Riemann integrable on $\mathbb{T}^k$. Thus Proposition 3.5.6 implies $\sigma_n c_\chi(\vec{\alpha}) \longrightarrow c_\chi(\vec{\alpha})$ a.e. on $\mathbb{T}^k$. Taking into account that the Haar measure on F is the normalized counting measure, we get

$$\psi^*_n(\vec{\alpha}; x) := \sum_{\overline{\chi}\in\hat{F}}\left(\sigma_n c_{\overline{\chi}}(\vec{\alpha})\right)\overline{\chi}(x) \rightarrow \sum_{\overline{\chi}\in\hat{F}} c_{\overline{\chi}}(\vec{\alpha})\overline{\chi}(x) = \psi^*_{\vec{\alpha}}(x) = \psi^*(\vec{\alpha}; x)$$

for almost every $\vec{\alpha} \in \mathbb{T}^k$ and every $x \in F$, as $n \longrightarrow \infty$. Since Haar measure μ_C on C is the product measure of the Haar measures on $\mathbb{T}^k$ and F, (3.5) holds μ_C-a.e. on C. μ_X-a.e. on X. Thus we conclude that any function majorizing $\liminf_{n\to\infty}\psi^*_n$ and minorizing $\limsup_{n\to\infty}\psi^*_n$ is an almost realization of φ. Note that according to the properties of the Fejér kernels on $\mathbb{T}^k$ (see 44.51 in [HR]) for each character $(\eta \times \chi)(\vec{\alpha}; x) := \eta(\vec{\alpha})\chi(x)$, $\eta \in \hat{\mathbb{T}^k}$ and $\chi \in \hat{F}$, there exists an $n_0 \in N$ such that for $n \geq n_0$ in the Fourier expansion of ψ^*_n the X-Fourier coefficient associated with the character does not vanish iff the $\mathbb{T}^k$-Fourier coefficient

$$c_\eta(c_\chi) = \int_{\mathbb{T}^k} c_{\overline{\chi}}(\vec{\alpha})\overline{\eta}(\vec{\alpha})d\vec{\alpha}$$

does not vanish. A simple computation shows that the $X-$Fourier coefficients of ψ^* are given by

$$\begin{aligned} c_{\eta\times\chi}(\psi^*) &= \int_{\mathbb{T}^k}\int_F \psi^*(\vec{\alpha}, x)\overline{\eta}(\vec{\alpha})\overline{\chi}(x)d\vec{\alpha}dx \\ &= \int_{\mathbb{T}^k} c_\chi(\vec{\alpha})\overline{\eta}(\vec{\alpha})d\vec{\alpha} = c_\eta(c_\chi) \end{aligned}$$

So the character $\eta \times \chi$ contributes to the X-Fourier expansion of ψ^* if and only if $c_{\eta\times\chi}(\psi) \neq 0$. Thus $\psi^*_n \in \operatorname{span}\Gamma(\varphi)$ for every $n \in \mathbb{N}$, implying that there exist almost realizations of φ on the group compactification induced by $\Gamma(\varphi)$, e.g. $\liminf_{n\to\infty}\psi^*_n$ or $\limsup_{n\to\infty}\psi^*_n$. □

Combining this result with the results of the previous section we obtain

Theorem 5. *Let $\varphi \in \mathcal{H}(G)$ and $\Gamma = \Gamma(\varphi) \leq \hat{G}$. The following assertions hold:*

1. $(\iota_\Gamma, X_\Gamma) \leq (\iota, C)$ *for every compactification (ι, C) on which φ can be realized. In particular $\mathcal{F}(\varphi) \subseteq \mathfrak{U}_{(\iota_\Gamma, X_\Gamma)}$.*

2. *Assume that $\varphi \in \mathcal{A}(G)$ or that φ can be realized finite dimensionally. Then φ can be realized aperiodically on X_Γ. In particular $\mathcal{F}(\varphi) = \mathfrak{U}_{(\iota_\Gamma, X_\Gamma)}$.*

We strongly conjecture that the second assertion in Theorem 5 holds for *any* Hartman measurable function, at least on LCA groups G with separable dual $\hat{G}$. A proof of this might utilize more general summation methods (in the flavour of Theorems 44.43 and 44.47 in [HR]) than the Fejér summation presented here.

In [Win] it is shown that for any Hartman measurable set $M \subseteq G = \mathbb{Z}$ and the induced filter $\mathcal{F} = \mathcal{F}(M)$ there is an aperiodic realization of $\varphi_M = 1\!\!1_M$ on the compactification determined by the subgroup $\mathrm{Sub}(M) = \{\alpha : \mathcal{F}\text{–}\lim_n \lfloor n\alpha \rfloor = 0\}$ or, equivalently, $\mathrm{Sub}(M) = \{\alpha : \mathcal{F}\text{–}\lim_n e^{2\pi i n \alpha} = 1\}$.

Together with Theorem 5 this implies that for Hartman sets M with finite dimensional realization both the group compactifications of $\mathbb{Z}$ induced by the subgroups $\Gamma(\varphi_M)$ and $\mathrm{Sub}(M)$ admit aperiodic realizations of φ_M. Hence uniqueness (up to equivalence) of the minimal compactification admitting an aperiodic almost realization of φ (Corollary 3.4.6) implies that in this situation $\Gamma(\varphi_M) = \mathrm{Sub}(M)$ holds. In the general situation we can prove the following

Proposition 3.5.9. *For a Hartman measurable function $\varphi \in \mathcal{H}(G)$ let $\mathcal{F} = \mathcal{F}(\varphi)$, $\Gamma = \Gamma(\varphi)$ and $Sub(\varphi) = \{\chi \in \hat{G} : \mathcal{F}\text{-}\lim_g \chi(g) = 1_\mathbb{C}\}$. Then $\Gamma(\varphi) \leq Sub(\varphi)$.*

Proof. Suppose $\chi \in \Gamma(\varphi)$. To prove $\mathcal{F}\text{–}\lim_g \chi(g) = 1_\mathbb{C}$ (unit element of the multiplicative group of complex numbers) we have to show that for every $\varepsilon > 0$ the set $\{g \in G : |1 - \chi(g)| < \varepsilon\}$ belongs to the filter $\mathcal{F}(\varphi)$, i.e. that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\{g \in G : m_G(|\tau_g\varphi - \varphi|) < \delta\} \subseteq \{g \in G : |1 - \chi(g)| < \varepsilon\} \in \mathcal{F}(\varphi). \tag{3.3}$$

Using the fact that m_G is an invariant mean and that χ is a homomorphism, we can do the following calculation:

$$\chi(g)\, m_G(\varphi \cdot \overline{\chi}) = m_G(\tau_g\varphi \cdot \overline{\chi}) = m_G((\tau_g\varphi - \varphi) \cdot \overline{\chi}) + m_G(\varphi \cdot \overline{\chi}).$$

Using $\|\chi\|_\infty = 1$ this further implies

$$|1 - \chi(g)| \cdot |m_G(\varphi \cdot \overline{\chi})| = |m_G((\tau_g\varphi - \varphi) \cdot \overline{\chi})| \leq m_G(|\tau_g\varphi - \varphi|).$$

Since $m_G(\varphi \cdot \overline{\chi}) \neq 0$ we can define $\delta := \varepsilon \cdot \frac{m_G(|\tau_g\varphi - \varphi|)}{|m_G(\varphi \cdot \overline{\chi})|} > 0$. With this choice of δ indeed $m_G(|\tau_g\varphi - \varphi|) < \delta$ implies $|1 - \chi(g)| < \varepsilon$, i.e. (3.3) holds. □

3.6 Generalized jump discontinuities

The first part of this section is devoted to the study of a certain type of discontinuities and its connection to Riemann integrability.

3.6.1 Riemann integrable functions with(out) a g.j.d.

Definition 3.6.1. *A function $f : X \to \mathbb{C}$ on a topological space X has a generalized jump discontinuity (g.j.d.) at $x \in X$ iff there exist open sets O_1 and O_2, such that $x \in \overline{O_1} \cap \overline{O_2}$ but $\overline{f(O_1)} \cap \overline{f(O_2)} = \emptyset$.*

Note that this is the same as to say that the sets $f(O_1)$ and $f(O_2)$ are separated or, equivalent, $\operatorname{dist}(f(O_1), f(O_2)) > 0$ for the standard metric on $\mathbb{R}$.

A few examples concerning generalized jump discontinuities:

1. The function $f_1(x) = \mathbb{1}_{[0,\frac{1}{2})}(x)$ on $X = [0,1]$ has a g.j.d. at $x = \frac{1}{2}$. On the other hand the function $f_2(x) = \mathbb{1}_{\{\frac{1}{2}\}}(x)$ has no g.j.d.
2. Generalizing the previous example: The characteristic function of a continuity set A such that both A and A^c have non-void interior has g.j.d.s at the common boundary of the interiors.
3. The function
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x > 0, \\ 0 & x = 0 \end{cases}$$
defined on $X = [0,1]$ has a g.j.d. at 0. To see this, consider the open sets $O_1 := f^{-1}((1/2, 1])$ and $O_2 := f^{-1}([-1, -1/2))$.
4. If f is constant μ-a.e. on X and $\operatorname{supp}(\mu) = X$, then f has no g.j.d. in X. The argument goes as follows: For $c \in \mathbb{C}$ the fact $\mu([f \neq c]) = 0$ implies that set $[f = c]$ is dense in X. Thus $c \in \overline{f(O_1)} \cap \overline{f(O_2)}$ for any non-void open sets O_1 and O_2.

Proposition 3.6.2. *Let X be a topological space. The set $J(X)$ of all functions having a g.j.d. is open in $B(X)$ w.r.t. the topology of uniform convergence.*

Proof. Let $f \in J(X)$ have a g.j.d. at $x \in X$. Thus there exist disjoint non-void open sets O_1 and O_2 with $\partial O_1 \cap \partial O_2 \neq \emptyset$ but $\overline{f(O_1)} \cap \overline{f(O_2)} = \emptyset$. Let $\varepsilon := \operatorname{dist}(f(O_1), f(O_2)) > 0$ and suppose $g \in B(X)$ is such that $\|f - g\|_\infty \leq \varepsilon/8$. Then
$$|\operatorname{dist}(g(O_1), g(O_2)) - \operatorname{dist}(f(O_1), f(O_2))| \leq \varepsilon/2.$$
This implies $\operatorname{dist}(g(O_1), g(O_2)) \geq \varepsilon/2 > 0$, i.e. g has a g.j.d. at $x \in X$. □

Lemma 3.6.3. *Let f, g be μ-Riemann integrable functions on the compact space X and μ a positive regular Borel measure with supp$(\mu) = X$. If f has a g.j.d. at x and $f = g$ μ-a.e. on X, then g has a g.j.d. at x.*

Proof. Choose O_1 and O_2 according to the definition of a g.j.d. for f at x. Since f and g coincide μ-a.e. they coincide on a dense set of continuity points. Thus for every open neighborhood U of x we can pick $x_i^U \subseteq U \cap O_i$, which is both a point of continuity for f and g and such that $f(x_i^U) = g(x_i^U)$, $i = 1, 2$. Now pick open neighborhoods O_i^U of x_i^U such that $O_i^U \subseteq U \cap O_i$ and

$$\operatorname{diam}(g(O_i^U)) < \frac{1}{3} \operatorname{dist}(f(O_1), f(O_2)), \ i = 1, 2.$$

Let $U_i := \bigcup_{U \in \mathfrak{U}(x)} O_i^U$, for $i = 1, 2$. The U_1, U_2 are open sets with

$$\operatorname{dist}(g(U_1), g(U_1)) \geq d(f(O_1), f(O_1)) > 0.$$

Furthermore we have $x_i^U \in U_i$ for all $U \in \mathfrak{U}(x)$,$i = 1, 2$. Thus $x \in \overline{U_1} \cap \overline{U_2}$. Hence x is a g.j.d. for g. □

Proposition 3.6.4. *Let f be a μ-Riemann integrable function on the compact metric space X and μ a positive regular Borel measure with supp$(\mu) = X$. If f has no g.j.d., then there is a unique continuous function f_r, the regularization of f, such that f and f_r coincide on $X \setminus$ disc(f).*

Proof. Note that the regularization f_r is uniquely determined by the requirement that f_r is continuous and $f_r(x) = f(x)$ for $x \notin \operatorname{disc}(f)$.

Let d be compatible metric on X. Let $x \in \operatorname{disc}(f)$. For each $y \in X \setminus \operatorname{disc}(f)$ there is $\varepsilon_y > 0$ such that $f(K(y, \varepsilon_y))$ has diameter less than $d(x, y)$. Let $V_n := K(x, \frac{1}{n}) \setminus \operatorname{disc}(f)$. So $x \in \overline{O_n}$ for each of the open sets $O_n := \bigcup_{y \in V_n} K(y, \varepsilon_y)$, $n \in \mathbb{N}$. We are going to show that $A := \bigcap_{n=1}^{\infty} \overline{f(O_n)} \subseteq \mathbb{C}$ consists of exactly one point.

A is non-void thanks to the finite intersection property of the compact sets $\overline{f(O_n)}$, $n \in \mathbb{N}$. Suppose by contradiction that A contains two distinct points $\lambda_1 \neq \lambda_2$. Thus there are sequences $\{x_n^{(i)}\}_{n=1}^{\infty} \subseteq X$ such that $x_n^{(i)} \in O_n$ and

$$\lim_{n \to \infty} x_n^{(i)} = x,$$

$$\lim_{n \to \infty} f(x_n^{(i)}) = \lambda_i, \quad \text{for } i = 1, 2.$$

For arbitrary $\eta > 0$ and $x' \in K(x_n^{(i)}, \varepsilon_{x_n^{(i)}})$ we have

$$|f(x') - \lambda_i| \leq |f(x') - f(x_n^{(i)})| + |f(x_n^{(i)}) - \lambda_i| < d(x, x_n^{(i)}) + |f(x_n^{(i)}) - \lambda_i| < 2\eta$$

whenever $n = n(\eta)$ is large enough $i = 1, 2$. Let $N := n(|\lambda_1 - \lambda_2|/2)$ and

$$O^{(i)} = \bigcup_{n \geq N} K(x_n^{(i)}, \varepsilon_{x_n^{(i)}}), \quad \text{for } i = 1, 2.$$

O_1 and O_2 are disjoint open sets with $x \in \overline{O^{(1)}} \cap \overline{O^{(2)}}$ and $\overline{f(O^{(1)})} \cap \overline{f(O^{(2)})} = \emptyset$. Hence x is a g.j.d. for f; contradiction.

Define $f_r(x)$ to be the unique point in the intersection $\bigcap_{n=1}^{\infty} \overline{f(O_n)}$. We claim that f_r is continuous. Suppose again by contradiction that x is a point of discontinuity for f_r. Since X is metrizable there are sequences $\{x_n^{(i)}\}_{n=1}^{\infty} \subseteq X$ such that

$$\lim_{n\to\infty} x_n^{(i)} = x,$$

$$\lim_{n\to\infty} f(x_n^{(i)}) = \zeta_i$$

for $i = 1, 2$ and $\zeta_1 \neq \zeta_2$. W.l.o.g we can assume $x_n^{(i)} \in O_n$, hence we can repeat the argument given above and $x \in X$ would be a g.j.d. for f; contradiction. □

Remark: Note that for a function f without g.j.d.s we have

$$\|f_r\|_\infty = \sup_{x\in X} |f_r(x)| = \sup_{x\in X\setminus \mathrm{disc}(f)} |f_r(x)| = \sup_{x\in X\setminus \mathrm{disc}(f)} |f(x)| \leq \sup_{x\in X} |f(x)| = \|f\|_\infty.$$

Thus the mapping $f \mapsto f_r$ is continuous (w.r.t. the topology of uniform convergence) on its domain of definition.

Corollary 3.6.5. *Let f be a μ-Riemann integrable function on the compact metric space X and μ a positive regular Borel measure with supp$(\mu) = X$. The following assertions are equivalent:*

1. *There exists a function $g \in C(X)$ such that $[f \neq g]$ is a meager μ-null set.*
2. *f has no g.j.d.*

Proof. (1) ⇒ (2): Suppose f has a g.j.d. at x. Pick open sets O_1 and O_2 according to the definition of a g.j.d. and pick two sequences $\{x_n^{(i)}\}_{n=1}^{\infty} \subseteq O_i \cap [f = g]$ such that $\lim_{n\to\infty} x_n^{(i)} = x$, for $i = 1, 2$. Thus we obtain the contradiction

$$\lim_{n\to\infty} f(x_n^{(1)}) = \lim_{n\to\infty} g(x_n^{(1)}) = g(x) = \lim_{n\to\infty} g(x_n^{(2)}) \lim_{n\to\infty} f(x_n^{(2)}) \in \overline{f(O_1)} \cap \overline{f(O_2)}.$$

(2) ⇒ (1):Apply Proposition 3.6.4 and Lemma 3.6.3. □

3.6.2 Hartman measurable functions with(out) a g.j.d.

As an application of the previous section we use the concept of generalized jump discontinuities to discuss the connections between Hartman measurability and weakly almost periodicity.

For our next main result we need the (easy implication of) Grothendieck's Double Limit Criterion for weakly almost periodic functions (cf. [BJM]).

Recall that (ι_w, wG) denotes the w.a.p.-compactification of G. The mapping $\iota_w^* : C(wG) \to C(G)$ defined via $\iota_w^* : f \mapsto f \circ \iota_w$ is an isometry onto $\mathcal{W}(G)$, the space of weakly almost periodic (w.a.p.) functions on G. This definition of weakly almost periodicity is equivalent to the classical notion of Eberlein. By Grothendieck's Double Limit Criterion a bounded continuous function $f : G \to \mathbb{C}$ is weakly almost periodic iff

$$\lim_{n\to\infty}\lim_{m\to\infty} f(x_n y_m) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_n y_m) \tag{3.4}$$

for all sequences $\{x_n\}_{n=1}^\infty, \{y_m\}_{m=1}^\infty \subseteq G$ such that the two invoked limits exist.

Claim: $f \in \mathcal{W}(G)$ implies that (3.4) holds.

Proof. Since f is w.a.p. the translation orbit $O(f) = \{\tau_g f : g \in G\}$ is conditionally compact w.r.t. the weak topology on $C_b(G)$. We need (a) Eberlein's Theorem which states that in the Banach space $C_b(G)$ weak compactness is equivalent to weak sequential compactness and (b) the Banach-Alaoglu Theorem which states that the unit ball in $C_b(G)'$ is weak-*-compact. Since all the relevant limits exists we can assume w.l.o.g. that $\tau_{x_n} f \to g$ weakly for a function $g \in C_b(G)$ and that $\delta_{y_m} \to \varphi$ weak-* for a functional $\varphi \in C_b(G)'$. As usual $\tau_{x_n} f$ denotes the function defined by $\tau_x f : y \mapsto f(xy)$ and δ_{y_m} denotes the functional defined by $\delta_y : f \mapsto f(y)$. Thus

$$\begin{aligned} \text{LHS of (3.4)} &= \lim_{n\to\infty}\lim_{m\to\infty} \delta_{y_m}(\tau_{x_n} f) = \lim_{n\to\infty} \varphi(\tau_{x_n} f) = \varphi(g) \\ &= \lim_{m\to\infty} \delta_{y_m}(g) = \lim_{m\to\infty}\lim_{n\to\infty} \delta_{y_m}(\tau_{x_n} f) = \text{RHS of (3.4)}. \end{aligned}$$

□

Theorem 6. *Let G be a topological group and let $f \in \mathcal{H}(G)$ be a Hartman measurable function on G that can be realized on a* metrizable *group compactification (ι, C). If there exists a realization $F \in R_{\mu_C}(C)$ of f in C, such that disc(F) contains a g.j.d., then f is not weakly almost periodic.*

Note that according to Theorem 2 for an LCA group with separable dual every Hartman measurable function has a realization on a metrizable group compactification.

Proof. Let $d(.,.)$ be a translation-invariant metric on C (such a metric exists, Theorem 8.6 in [HR]). Assume that $\xi \in C$ is a g.j.d. for F and let O_1 and O_2 be as in the definition of a g.j.d. Since $\iota(G)$ is dense we find a sequence $\{\iota y_k\}_{k=1}^\infty \subseteq O_2$ converging to ξ. Since O_2 is open we can find a decreasing sequence of positive real numbers $\{\alpha_k\}_{k=1}^\infty$, such that the open ball with radius α_k and center ιy_k still is in O_2. Now choose a sequence $\{\iota x_n\}_{n=1}^\infty \subseteq O_1$ converging to ξ in such a way that $d(\iota x_n, \xi) \le \alpha_m/3$. So for $n > k$ we have

$$d(\iota x_n, \iota x_k) \le 2/3 \cdot \alpha_k < \alpha_k.$$

This construction guarantees that $\iota y_k \cdot \iota x_n \cdot \iota x_k^{-1} \in O_2$ for $n > k$. Now let $z_k := x_k^{-1} \cdot y_k$. Since ιz_k tends to zero for $k \to \infty$, for each fixed $x \in O_1$ the sequence $\{x \cdot \iota z_k\}_{k=1}^\infty$ stays eventually in O_1. We may pass over to any subsequence $\{z_{k_l}\}_{l=1}^\infty$ without losing this property.

On the other hand for each k the sequence $\{\iota x_n \cdot \iota z_k\}_{n=1}^\infty = \{\iota x_n \cdot \iota x_k^{-1}\}_{n=1}^\infty \cdot \iota y_k$ is in O_2 whenever $n > k$. Thus for each k the sequence of real numbers $\{f(x_n \cdot z_k)\}_{n=1}^\infty$ has an accumulation point in $\overline{f(O_2)}$. By a routine diagonalization argument we find a subsequence such that $\{f(x_{n_m} \cdot z_k)\}_{m=1}^\infty$ converges for each fixed k. Doing the same again, but now for the other index, we find that

$$\lim_{l\to\infty} \lim_{m\to\infty} f(x_{n_m} \cdot z_{k_l}) \in \overline{f(O_2)}. \tag{3.5}$$

By passing over to convergent subsequences again, we not only preserve convergence of the iterated limit (3.5) but also get symmetrically

$$\lim_{r\to\infty} \lim_{s\to\infty} f(x_{n_{m_r}} \cdot z_{k_{l_s}}) \in \overline{f(O_1)}. \tag{3.6}$$

Suppose by contradiction that f is weakly almost periodic. Grothendieck's Double Limit Criterion implies that both the iterated limits (3.5) and (3.6) coincide. Since $f(O_1)$ is separated from $f(O_2)$, this cannot be true. Thus $f \notin \mathcal{W}(G)$. □

Let us call a LCA group $\mathcal{H}$-rich if there is a function defined on G which is Hartman measurable but not weakly almost periodic.

Corollary 3.6.6. *Each LCA group with separable dual is $\mathcal{H}$-rich.*

Proof. Let (ι, C) be an injective and metrizable group compactification of G. By Theorem 6 it suffices to find two disjoint μ_C-continuity sets O_1 and O_2 with non-void interior and a common boundary-point ξ and consider the function $f = 1\!\!1_{O_i}$, $i = 1, 2$. Then ξ is a g.j.d. for f and thus by Theorem 6 $f \in \mathcal{H}(G) \setminus \mathcal{W}(G)$.

Let $d(.,.)$ be a compatible metric on C. In the following we use the fact that for every $x \in C$ there are open balls $K(x, r)$ with center x and arbitrarily small radius $r > 0$ that are μ_C-continuity sets (cf. Example 1.3, 175 p. in [KN], or an argument similar to the proof of our Proposition 3.2.4).

Let $x \in C$ be arbitrary. We define two sequences of disjoint open μ_C-continuity sets $\{O_n^{(1)}\}_{n=1}^\infty$ and $\{O_n^{(2)}\}_{n=1}^\infty$ by induction: Suppose we have already defined $O_1^{(1)}, \ldots, O_n^{(1)}$ and $O_1^{(2)}, \ldots, O_n^{(2)}$ such that

$$\mu\left(\bigcup_{j=1}^n O_j^{(i)}\right) < \frac{1}{2}\left(1 - \frac{1}{2^n}\right) \quad \text{and} \quad 0 < \operatorname{dist}\left(\bigcup_{j=1}^n O_j^{(i)}, x\right) < \frac{1}{2^n}, \; i = 1, 2.$$

Let $r < \min_{i,j} \left\{\operatorname{dist}(O_j^{(i)}, x)\right\}$ and take $x_1, x_2 \in K(x, r)$. Pick $\rho < \min\{\frac{r}{2}, \frac{1}{2^{n+1}}\}$ and such that $O_{n+1}^{(i)} := U_\rho(x_i)$ are μ_C-continuity sets of μ_C-measure less than $\frac{1}{2^{n+1}}$.

Letting $O_i := \bigcup_{j=1}^{\infty} O_j^{(i)}$, $i = 1, 2$ we obtain two disjoint open sets. It is easy to check that $\partial O_i = \{x\} \cup \bigcup_{n=1}^{\infty} \partial O_n^{(i)}$, $i = 1, 2$. So O_1 and O_2 are μ_C-continuity sets with $x \in \partial O_1 \cap \partial O_2$. Thus x is a g.j.d. of $f = 1\!\!1_{O_1}$ (or $f = 1\!\!1_{O_2}$) and $f \in \mathcal{H}(G) \setminus \mathcal{W}(G)$. □

For the sake of completeness we present a more direct approach to construct examples of functions $f \in \mathcal{H}(G) \setminus \mathcal{W}(G)$ for an LCA group G. We will heavily use LCA structure theory (as a standard reference we refer to the monograph [Arm]). Note that if a group H is $\mathcal{H}$-rich and there is a continuous homomorphism from G onto H, then G is also $\mathcal{H}$-rich. As in Corollary 3.6.6 this essentially boils down to finding two disjoint continuity sets with non-void interior and a common boundary point.

Example 3.6.7. If G admits a continuous character χ with infinite range - a moment's reflections shows that this is equivalent to $\hat{G}$ being not a torsion group - we can take two disjoint continuity sets with non-void interior and a common boundary point on $\mathbb{T}$ (which certainly exist) and transfer them via χ^{-1} on G.

Example 3.6.8. Let $Z(p)$ denote the factor group $\mathbb{Z}/p\mathbb{Z}$. We consider the (discrete) weak direct product

$$\prod_{i \in I}^{*} Z(p).$$

The character group of G is topologically isomorphic to the (compact) direct product

$$\hat{G} = \prod_{i \in I} Z(n).$$

Obviously every $\chi(g) \in \mathbb{T}$ with $g \in G, \chi \in \hat{G}$ is a n-th root of unity. Therefore $\chi(G)$ is finite and the construction of the previous example won't work here.

Let p=2. We can identify every member of G with a function $x : \omega \to \{0, 1\}$ having finite support, i.e. $G \cong 2^{<\omega}$. Equivalently we can identify every member of G with a finite 0-1 sequence.

Now consider the group compactification of G given by the natural embedding $\iota : 2^{<\omega} \to 2^{\omega}$. Let

$$M := \{x \in 2^{\omega} : \exists k \in \omega \text{ such that } x_k = x_{k+1} = 1\}.$$

M is the union of clopen sets $A_k := \{x \in 2^{\omega} : x_k = x_{k+1} = 1\}$ and hence is open itself. Since M is invariant under the unilateral shift, which is an ergodic transformation on 2^{ω}, and since M is of positive measure we have $\mu(M) = 1$. Thus M is a continuity set.

$A_0 := \{x : x(0) = 0\}$ and $A_1 := \{x : x(0) = 1\}$ forms a partition of $\hat{G}$ into clopen sets and $\iota^{-1}(A_0 \cap M)$, $\iota^{-1}(A_1 \cap M)$ are the desired disjoint continuity sets with non-void interior and common boundary point on G.

For $p > 2$ the same ideas apply to the set

$$\{x : \exists k \in \omega, \exists j \neq 0 \text{ such that } x_k = x_{k+1} = j\}.$$

In a similar fashion one can construct examples when the underlying group is a weak direct product of the form $\prod_{i \in I}^{*} Z(n_i)$ with bounded factors. To do so, note that one of the sets $I(n) = \{i \in I : n_i = n\}$ is infinite. Hence G can be decomposed into a product

$$\prod_{i \in I(2)}^{*} Z(2) \times \ldots \times \prod_{i \in I(N)}^{*} Z(N)$$

with one factor isomorphic to a weak direct product for which we have already constructed suitable sets.

Lemma 3.6.9. *Each non-compact discrete LCA group G is $\mathcal{H}$-rich.*

Proof. If the dual $\hat{G}$ is not a torsion group, then Example 3.6.7 shows that G is $\mathcal{H}$-rich. If the dual $\hat{G}$ is a torsion group, $\hat{G}$ is even a compact torsion group and hence of the form $\prod_{i \in I} Z(n_i)$ with bounded n_i and infinite index set I. By duality G is topologically isomorphic to the corresponding weak direct product and hence (the last part of) Example 3.6.8 shows that G is $\mathcal{H}$-rich. □

Theorem 7. *Each non-compact LCA group G is $\mathcal{H}$-rich.*

Proof. Suppose G (or one of its subgroups) is non-compact but compactly generated. Then $G \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$ where K is compact and $n, m \in \mathbb{N}$ are not both 0. In either case there is a subgroup admitting a character with infinite range, extending this character to G we can use Example 3.6.7.

Suppose neither G nor one of its non-compact subgroups is compactly generated. This implies that for any compact set $C \subseteq G$ the enveloping closed subgroup $\overline{\langle C \rangle}$ is again compact and distinct from G. Let us call such a group *hereditarily non-compactly generated.* We will show that every hereditarily non-compactly generated group is "large enough" to be $\mathcal{H}$-rich.

By Theorem 5.14 in [HR] there exists a compact open subgroup $H \subseteq G$, which is in our situation necessarily distinct from G. So the LCA group G/H is discrete. If this quotient is non-compact we are done, because Lemma 3.6.9 tells us that any such LCA group is $\mathcal{H}$-rich. If on the other hand this quotient is compact it must be finite. So G is the union of finitely many compact residue classes and therefore also compact, which is a contradiction to our assumption. □

The converse problem, namely to find weakly almost periodic functions that are not Hartman measurable seems to be harder. So we content ourselves with the special case $G = \mathbb{Z}$. The key ingredient for our example are ergodic sequences.

These sequences were extensively studied by Rosenblatt and Wierdl in their paper [RW]. Also in the context of Hartman measurability ergodic sequences were already mentioned in [SchSW].

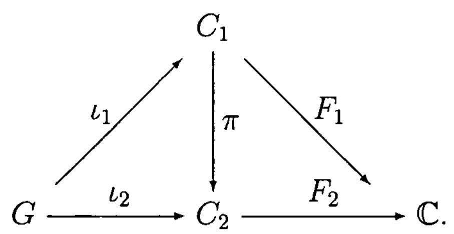
Example 3.6.10. Ergodic sequences: It is known (Theorem 11 and Examples in [SchSW]) that ergodic sequences such as $(|k|\log|k|)_{k\in\mathbb{Z}}$ cannot be Hartman measurable. On the other hand 0-1 sequences with the property that the length between consecutive 1s tends to infinity while the length of consecutive 0s stays bounded are weakly almost periodic (Theorem 4.2 in [BlH]). Thus $\mathcal{E}\subseteq\mathcal{W}(\mathbb{Z})\setminus\mathcal{H}(\mathbb{Z})$ for the class $\mathcal{E}$ of characteristic functions of ergodic sequences on $\mathbb{Z}$.

Question 1: Can this example be modified for general LCA groups?

Theorem 6 motivates us to further investigate Hartman measurable functions without g.j.d.s. First we show that the property of having g.j.d.s does not depend on the special realization of a Hartman measurable function.

Proposition 3.6.11. *Let f be a Hartman measurable function on G, realized both by F_1 and F_2 on the group compactifications (ι_1, C_1) resp. (ι_2, C_2). If F_1 has a g.j.d., then F_2 also has a g.j.d.*

Proof. Let x be a g.j.d. for F_1. First suppose $(\iota_1, C_1)\geq(\iota_2, C_2)$, i.e. there is a continuous epimorphism $\pi: C_1\to C_2$ with $\iota_2=\pi\circ\iota_1$ and $f=F_1\circ\iota_1=F_2\circ\iota_2$.



Thus F_1 and $F_2\circ\pi$ coincide on $\iota_1(G)$. Hence Lemma 3.6.3.1 implies that $F_1=F_2\circ\pi$ μ_1-a.e. and Lemma 3.6.3.2 implies that $F_2\circ\pi$ has a g.j.d. at x whenever F_1 has a g.j.d. at x. Pick disjoint open sets $O_1, O_2\subseteq C_1$ according to the definition of a g.j.d. for $F_2\circ\pi$ at x, i.e. $x\in\overline{O_1}\cap\overline{O_2}$ but $\overline{F_2\circ\pi(O_1)}\cap\overline{F_2\circ\pi(O_2)}=\emptyset$.

Since π is a continuous and open mapping, $\pi(O_1)$ and $\pi(O_2)$ are open sets such that $\pi(x)\in\overline{\pi(O_1)}\cap\overline{\pi(O_2)}$. We claim that $\pi(O_1)$ and $\pi(O_2)$ are disjoint: Suppose by contradiction $y\in\pi(O_1)\cap\pi(O_2)$. Since $\pi(O_1)\cap\pi(O_2)$ is open it has non-void intersection with the dense set $\iota_2(G)$. Thus we may w.l.o.g. assume $y\in\iota(G)$, i.e. there exists $g\in G$ such that $y\in\iota_2(g)=\pi\circ\iota_1(g)$. Consequently there exist $x_i\in O_i$ such that $\pi(x_i)=y$, $i=1,2$. Thus

$$f(g)=F_2\circ\iota_2(g)=F_2(y)=F_2\circ\pi(x_i)\in F_2\circ\pi(O_1)\cap F_2\circ\pi(O_2)$$

contradicts the choice of O_1 and O_2.

So $\pi(O_1),\pi(O_2)\subseteq C_2$ are disjoint open sets such that $\pi(x)\in\overline{\pi(O_1)}\cap\overline{\pi(O_2)}$. Also, per definition $\overline{F_2(\pi(O_1))}\cap\overline{F_2(\pi(O_2))}=\emptyset$. Thus x is a g.j.d. for F_2.

In the general case let π be the canonical epimorphism $bG \to C_1$ and define $F_b := F_1 \circ \pi$. It is easy to check that if F_1 has a g.j.d. at x, F_b has a g.j.d. at every point of $\pi^{-1}\{x\}$. It is also clear that F_b, F_1 and F_2 induce the same Hartman measurable function f on G. Now apply the first part of this prove to the two functions F_b and F_2. □

This result shows that "being realized by a function having a g.j.d." is an inert property of the Hartman measurable function and does not depend on the particular realization. In particular if $f = F \circ \iota$ with F having a g.j.d. on an arbitrary compactification, then any realization on a metric compactification has a g.j.d. as well. In virtue of this proposition we can consider the set of all Hartman measurable functions such that one (and hence all) realizations lack a g.j.d.

Definition 3.6.12. *Let G be a topological group. Let*

$$\begin{aligned}\mathcal{H}_c(G) &:= \{f \in \mathcal{H}(G) : \forall(\iota, C)\forall x \in C \quad f = F \circ \iota \text{ with } F \in R_{\mu_C}(C) \\ &\qquad \text{implies that } F \text{ has no g.j.d. at } x\} \\ &= \{f \in \mathcal{H}(G) : \exists(\iota, C)\forall x \in C \quad f = F \circ \iota \text{ with } F \in R_{\mu_C}(C) \\ &\qquad \text{implies that } F \text{ has no g.j.d. at } x\}.\end{aligned}$$

Let us turn now to another property regarding realizability of Hartman measurable functions, similar to the one we have just discussed.

Proposition 3.6.13. *Let f be a Hartman measurable function on G, realized both by F_1 and F_2 via the compactifications (ι_1, C_1) resp. (ι_2, C_2). If $[F_1 \neq 0]$ is a meager μ_1-null set, then $[F_2 \neq 0]$ is a meager μ_2-null set.*

Proof. First suppose $(\iota_1, C_1) \geq (\iota_2, C_2)$, i.e. $f = F_2 \circ \iota_2 = F_1 \circ \iota_1 \in \mathcal{H}(G)$. We know that $[F_1 \neq 0]$ is a meager μ_1-null set. Lemma 3.6.3.1 implies that $F_2 = F_1 \circ \pi$ on $\iota_2(G)$. Thus $\pi^{-1}([F_1 \neq 0])$ and $[F_2 \neq 0]$ differ at most on a meager μ_2-null set. Note that $\mu_2(\pi^{-1}([F_1 \neq 0])) = \mu_1([F_1 \neq 0]) = 0$ implies that $\pi^{-1}([F_1 \neq 0])$ is a μ_2-null set.

π is an open continuous surjection of compact spaces. It is easy to verify that π preserves Baire-categorical properties, i.e. preimages of 1^{st}-category sets are 1^{st}-category sets and preimages of 2^{nd}-category sets are 2^{nd}-category sets. Thus if $[F_1 \neq 0]$ is meager in C_1 then $\pi^{-1}([F_1 \neq 0])$ is meager in C_2.

In the general case let π be the canonical epimorphism $bG \to C_1$ and define $F_b := F_1 \circ \pi$. It is easy to check that if $[F_1 \neq 0]$ is a meager μ_1-null set, then $[F_b \neq 0]$ is a meager μ_b-null set. Again, F_b, F_1 and F_2 induce the same Hartman measurable function f on G. Now apply the first part of this prove to the two functions F_b and F_2. □

Similar to the situation of g.j.d.s for different realizations of a fixed Hartman measurable function, cf. Proposition 3.6.11, also the property of vanishing outside a meager null set does not depend on the special choice of the realization. Thus we define the set of all those Hartman measurable functions such that all realizations vanish outside a meager null set.

Definition 3.6.14. *Let G be a topological group. Let*

$$\begin{aligned}\mathcal{H}_0(G) &:= \{f \in \mathcal{H}(G) : \forall(\iota, C) \quad f = F \circ \iota \text{ with } F \in R_{\mu_C}(C) \\ &\qquad \text{implies that } [F \neq 0] \text{ is a meager } \mu_C\text{-null set }\} \\ &= \{f \in \mathcal{H}(G) : \exists(\iota, C) \quad f = F \circ \iota \text{ with } F \in R_{\mu_C}(C) \\ &\qquad \text{implies that } [F \neq 0] \text{ is a meager } \mu_C\text{-null set }\}.\end{aligned}$$

Proposition 3.6.15. *$\mathcal{H}_0(G)$ and $\mathcal{H}_c(G)$ are translation invariant and uniformly closed *-algebras.*

Proof. From the definition resp. Corollary 3.6.5 it is clear that both $\mathcal{H}_0(G)$ and $\mathcal{H}_c(G)$ are *-algebras and invariant under translations.

1. Let $R_0(G) := \{f \in R_\mu(bG) : [f \neq 0] \text{ is a meager null set}\}$. Note that $R_0(G)$ is a closed subalgebra of $R_\mu(bG)$ (due to the fact that a countable union of meager null sets is again a meager null set). Since ι_b^* is a continuous homomorphism of C^*-algebras and $\iota_b^* R_0(G) = \mathcal{H}_0(G)$ (Definition 3.6.14), $\mathcal{H}_0(G)$ is closed (Theorem I.5.5 in [Dav]).

2. $J(bG)$, the set of all bounded functions on bG having a g.j.d., is open w.r.t. the topology of uniform convergence (Proposition 3.6.2). Thus $C(bG) \oplus R_0(G)$, the set of all bounded functions on bG without a g.j.d., is closed (cf. Corollary 3.6.5). Since $C(bG) \oplus A$ is a closed algebra and $\mathcal{H}_c(G) = \iota_b^*(C(bG) \oplus A)$, again Theorem I.5.5 in [Dav] yields that $\mathcal{H}_c(G)$ is closed

□

Remark: It is worthwhile to note that $\mathcal{A}(G) \cap \mathcal{H}_0(G) = \{0\}$. This is due to the fact that $f \in \mathcal{H}_0(G)$ implies $m(|f|) = 0$, which is impossible for almost periodic functions different from 0.

An easy way to obtain functions from $\mathcal{H}_0(G)$ is to consider compact μ_b-null sets: $\mu_b(\overline{K}) = 0$ implies $K^\circ = \emptyset$, in particular K is a meager μ_b-continuity set. Thus $f := 1\!\!1_{\iota_b^{-1}(K)} \in H_0(G)$ and $m(f) = 0$ for the unique invariant mean defined on $\mathcal{H}(G)$.

1. f is almost periodic iff $K \cap \iota_b(G) = \emptyset$,

2. $f \in C_0(G)$ iff $\iota_b^{-1}(K)$ is compact.

Lemma 3.6.16. *For any topological group G the inclusion $\mathcal{H}_0(G) \subseteq \mathcal{H}_c(G)$ holds.*

Proof. It suffices to show that $f \in \mathcal{H}_0(G)$ implies $F(O_1) \cap F(O_2) \neq \emptyset$ for every realization F of f and any pair of disjoint open subsets O_1, O_2. This follows from the fact that $[F \neq 0]$ is a null set w.r.t. the Haar measure, implying that the set $[F = 0]$ is dense. □

In the last part of this section want to identify those topological groups G for which $\mathcal{H}_c(G) = \mathcal{H}_0(G)$ and shed some light on the structure of $\mathcal{H}_c(G) \setminus \mathcal{H}_0(G)$.

Proposition 3.6.17. *For every $f \in \mathcal{H}_c(G)$ there exists a unique almost periodic function f_a and a unique function $f_0 \in \mathcal{H}_0(G)$ such that $f := f_a + f_0$.*

Proof. Existence: Let F_1 be a realization of f on a group compactification (ι, C). Using Proposition 3.6.4 we can decompose $F = F^r + (F - F^r)$, the first summand being continuous and the second one having support on a meager μ_C-null set (cf. the Remark on page 17). Thus $f_a := F \circ \iota \in \mathcal{A}(G)$, $f_0 := (F - F^r) \circ \iota \in H_0(G)$ and, obviously, $f = f_a + f_0$.

Uniqueness: Suppose $f = f_a^{(1)} + f_0^{(1)} = f_a^{(2)} + f_0^{(2)}$ with $f_a^{(1)}, f_a^{(2)} \in \mathcal{A}(G)$ and $f_0^{(1)}, f_0^{(2)} \in \mathcal{H}_0(G)$. This implies

$$f_a^{(1)} - f_a^{(2)} = f_0^{(1)} - f_0^{(2)} \in \mathcal{A}(G) \cap \mathcal{H}_0(G) = \emptyset$$

and hence $f_a^{(1)} = f_a^{(2)}$ an $f_0^{(1)} = f_0^{(2)}$. □

A direct consequence is the following

Theorem 8. *Let G be a topological group. The mapping $P : \mathcal{H}_c \to \mathcal{A}(G)$ defined via $f \mapsto f_a$ is bounded projection with $\|P\| = 1$. Furthermore $m(Pf) = m(f)$ for the unique invariant mean m on $\mathcal{H}(G)$.*

Theorem 8 together with the fact $\mathcal{A}(G) \cap \mathcal{H}_0(G)$ imply $\mathcal{H}_c(G) = \mathcal{A}(G) \oplus \mathcal{H}_0(G)$.

Corollary 3.6.18. *Let G be a topological group. $\mathcal{H}_c(G) = \mathcal{H}_0(G)$ iff G is minimally almost periodic (map).*

Question 2: How are $\mathcal{H}_c$ and $\mathcal{W}$ related? We know that $\mathcal{H}_c \supseteq \mathcal{H} \cap \mathcal{W}$. Is this inclusion strict? We conjecture that $\mathcal{H}_c \setminus \mathcal{W}$ is "large".

In the last part of this section we try to relate the spaces C_0, of functions vanishing at infinity, and $\mathcal{H}_0$. Recall that every LCA group is maximally almost periodic (MAP).

Lemma 3.6.19. *Let G be a non-compact topological group and let (ι, C) be a group compactification.*

1. *If G is a MAP group then $\mu_C(\iota(K)) = 0$ for every σ-compact $K \subseteq G$.*

2. If G is a LCA group then and $\iota(G)$ is μ_C-measurable then $\mu_C(\iota(G)) = 0$.

Proof. To prove (1) first suppose that K is compact. We claim that there is a sequence $(g_i)_{i\in\mathbb{N}}$ such that $g_iK \cap g_jK = \emptyset$ for $i \neq j$: Let us construct such a sequence inductively. Suppose that $(g_iK)_{i=1}^n$ is a family of pairwise disjoint sets; we prove that there exists $g_{n+1} \in G$ such that $(g_iK)_{i=1}^n$ is also a family of pairwise disjoint sets. Suppose by contradiction that for every $g \in G$ there is a j such that $g_jK \cap gK \neq \emptyset$. Then $g \in g_jKK^{-1}$. So $G = \bigcup_{j=1}^n g_jKK^{-1}$ is compact, contradiction. Thus there exists $g_{n+1} \in G$ such that $(g_iK)_{i=1}^{n+1}$ is a pairwise disjoint family.

Since G is MAP, ι is one-one. The sets $\iota(g_iK)$ form an infinite collection of pairwise disjoint translates of the compact subset $\iota(K)$. If $\mu_C(\iota(K)) > 0$ would hold, this would imply

$$1 = \mu_C(\iota(K)) \geq \sum_{i=1}^{\infty} \mu_C(\iota(g_iK)) = \sum_{i=1}^{\infty} \mu_C(\iota(K)) = \infty.$$

Consequently $\mu_C(\iota(K)) = 0$.

If K is not compact but σ-compact the assertion follows clearly from the fact that μ_C is σ-additive.

To prove (2) we use a result of Glicksberg (cf. Theorem 1.2 in [Gli]) which states that for an LCA group G and a group compactification (ι, C) a subset $A \subseteq \iota^{(G)}$ is compact in C iff $\iota^{-1}(A)$ is compact in G.

Suppose $\iota(G)$ has positive Haar measure. Due to inner regularity of μ_C there is a compact subset $A \subseteq \iota(G) \subseteq C$ which also has positive Haar measure. Consider $K := \iota^{-1}(A)$ which is a compact subset of G due to Glicksberg's Theorem. Since ι is one-one we conclude $A = \iota(\iota^{-1}(A)) = \iota(K)$. According to assertion (1) A must be a null set w.r.t. Haar measure on C, a contradiction. □

Theorem 9. *Let G be a MAP group. Then $C_0(G) \subseteq \mathcal{H}(G)$. If G is not compact then we even have $C_0(G) \subseteq \mathcal{H}_0(G)$.*

Proof. In the first step we show $C_0(G) \subseteq \mathcal{H}(G)$. If G is compact there is nothing to prove, so suppose that G is not compact. Let $f \in C_0(G)$ be a function on G vanishing at infinity. Define F on bG by

$$F(x) := \begin{cases} f(g) & \text{if } x = \iota_b(g),\ g \in G \\ 0 & \text{else,} \end{cases}$$

then $f = F \circ \iota_b$. W.l.o.g. assume $f \geq 0$ and $\|f\|_\infty \leq 1$. For every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq G$ such that $|f(x)| < \varepsilon$ for $x \in G \setminus K_\varepsilon$. By Lemma 3.6.19 we have $\mu_b(\iota_b(K_\varepsilon)) = 0$, thus regularity of the Haar measure implies that we can find an open set O containing the compact set $A := \iota_b(K_\varepsilon)$ such that

$\mu_b(O) < \varepsilon$. Let h be an Urysohn function separating A and $bG \setminus O$. Consider the continuous function $g_\varepsilon := h + \varepsilon \mathbb{1}_{bG}$. Since $0 \leq F \leq g_\varepsilon$ and

$$\int_{bG} g_\varepsilon d\mu_b \leq \mu_b([h > 0]) + \varepsilon \leq 2\varepsilon,$$

Lemma 3.2.1 implies $F \in R_\mu(bG)$. Hence f is Hartman measurable. It remains to show that $[F \neq 0]$ is a meager μ_b-null set. For each $n \in \mathbb{N}$ the set $[|f| \geq 1/n]$ is compact. Hence $\iota_b([|f| \geq 1/n]) = [|F| \geq 1/n]$ is a compact μ_b-null set, in particular $[|F| > 1/n] \subseteq [|F| \geq 1/n]$ is nowhere dense. Since we have

$$\bigcup_{n=1}^{\infty} [|F| > 1/n] = [F \neq 0] \subseteq \iota_b(G),$$

Lemma 3.6.19 implies that $[F \neq 0]$ is a meager μ_b-null set. Thus $f \in \mathcal{H}_0(G)$. □

The following example shows that for $G = \mathbb{Z}$ the space $C_0(G)$ of functions vanishing at infinity is a proper subspace of $\mathcal{H}_0(G)$.

Example 3.6.20. *Let $T = \{t_n : n \in \mathbb{N}$ be a lacunar set of positive integers, i.e. $t_1 < t_2 < t_3 < \ldots$ with $\limsup_{n\to\infty} t_n/t_{n+1} = \varepsilon < 1$. Then $\mathbb{1}_T \in \mathcal{H}_0(\mathbb{Z}) \setminus C_0(\mathbb{Z})$.*

Proof. The proof of Theorem 9 in [SchSW] tells us that for each $n \in \mathbb{N}$ there exists an n-dimensional compactification C_n and a compact continuity set K_n with $\mu_n(K_n) \leq 4n\varepsilon^n$ such that $\iota_n^{-1}(K_n) \supseteq T$. Furthermore we can arrange $(\iota_n, C_n) \leq (\iota_{n+1}, C_{n+1})$ and $\pi_{n+1,n}^{-1}(K_n) \supseteq K_{n+1}$, where $\pi_{n+1,n} : C_{n+1} \to C_n$ is the canonical projection. Let (ι, C) be the supremum of the compactifications $\{(\iota_n, C_n) : n \in \mathbb{N}\}$, and let $\pi_n : C \to C_n$ be the canonical projection onto C_n. Thus $K := \bigcap_{n=1}^{\infty} \pi_n^{-1}(K_n)$ is a compact μ-null set (hence a μ_b-continuity set) with $\iota^{-1}(K) \supseteq T$. □

Question 3: Can this example be modified for more general LCA groups?

3.7 Hartman measurability and Fourier Transformation

For any topological group G the set C_0 of functions vanishing at infinity provides a particularly easy example of functions that are weakly almost periodic but not almost periodic. Thus $\mathcal{A} \oplus C_0 \subseteq \mathcal{W}$. However, the "interesting" weakly almost periodic functions are the members of $\mathcal{W} \setminus (\mathcal{A} \oplus C_0)$.

There are locally compact groups with the property that $\mathcal{W} = \mathcal{A} \oplus C_0$, namely the so called minimally w.a.p. groups. For these groups Theorem 9 implies

$\mathcal{W} = \mathcal{A} \oplus C_0 \subsetneq \mathcal{H}$. However no non-compact LCA group can be minimally w.a.p. (cf. [Chou]).

Question 4: Do there exist LCA groups or non minimally w.a.p. groups with $\mathcal{W} \subsetneq \mathcal{H}$?

In this section we will explicitly construct a function $f \in \mathcal{W}(\mathbb{Z}) \cap \mathcal{H}_c(\mathbb{Z}) \setminus (\mathcal{A}(\mathbb{Z}) \oplus C_0(\mathbb{Z}))$. Let us first recall some facts about the Fourier transformation of measures on LCA groups, since our example will heavily rely on these facts. Let G be an LCA group. By $\mathcal{M}(G)$ we denote the set of all finite complex Borel measures on G. Recall that $\mathcal{M}(G)$ can be regarded as the dual of the Banach space $C_0(G)$ via the mapping:

$$\langle f, \mu \rangle := \int_G f(x) d\mu(x).$$

Also recall that we convolute two measures $\mu, \nu \in \mathcal{M}(G)$ according to the formula

$$\langle f, \mu * \nu \rangle = \int_{G \times G} f(x+y) d(\mu \otimes \nu)(x, y).$$

The Fourier-Stieltjes transform $\mu \to \hat{\mu}$ assigns to each $\mu \in \mathcal{M}(G)$ a uniformly continuous function on $\hat{G}$:

$$\hat{\mu}(\chi) := \int_G \chi(x) d\mu(x).$$

The mapping $\mu \to \hat{\mu}$ is a continuous homomorphism of the convolution algebra $(\mathcal{M}(G), *)$ into the algebra $(UC_b(\hat{G}), \cdot)$ of uniformly continuous functions on G, equipped with pointwise multiplication. The set $\{\hat{\mu} : \mu \in \mathcal{M}(G)\}$ of all Fourier-Stieltjes transforms is called Fourier-Stieltjes algebra of $\hat{G}$ and denoted by $\mathcal{B}(\hat{G})$. Bochner's Theorem implies that for every $\mu \in \mathcal{M}(G)$ the Fourier-Stieltjes transform $\hat{\mu}$ is a linear combination of positive definite functions. Since every positive definite function is weakly almost periodic, every Fourier-Stieltjes transformation is weakly almost periodic. It is well known that in general (i.e. for non compact G) the inclusions $\mathcal{A}(G) \subsetneq \overline{\mathcal{B}(G)} \subsetneq \mathcal{W}(G)$ are strict.

Proposition 3.7.1. *Let G be a locally compact group. The following assertions hold*

1. *if μ is discrete, then $\hat{\mu} \in \mathcal{A}(\hat{G})$,*
2. *if μ is absolute continuous with respect to the Haar measure λ_G on G, then $\hat{\mu} \in C_0(\hat{G})$ (for $G = \mathbb{T}$ this is the Riemann-Lebesgue Lemma),*
3. *$m_{\hat{G}}(\hat{\mu}) = \mu(\{0_G\})$ for the unique invariant mean $m_{\hat{G}}$ on $\hat{G}$; in particular: if μ is continuous, then $\hat{\mu}$ has zero mean-value.*

Proof. See section 1.3 in [Rud] □

Before we present our example we need the following simple lemma. Note that for LCA groups $G \cong (\hat{G})\hat{}$ via Pontryagin's Duality Theorem.

Lemma 3.7.2. *Let G be a discrete LCA group and $\{\nu_n\}_{n=1}^{\infty}$ a norm-bounded sequence of discrete measures in $\mathcal{M}(\hat{G})$. Then the following assertions are equivalent:*

1. *The sequence $\{\hat{\nu_n}\}_{n=1}^{\infty}$ of almost periodic functions on G converges pointwise to a function f.*

2. *The sequence $\{\nu_n\}_{n=1}^{\infty}$ of discrete measures on $\hat{G}$ converges weak-* to a measure μ.*

In either case f is the Fourier-Stieltjes transform of $\nu :=$ weak--$\lim_{n\to\infty} \nu_n$.*

Proof. (1) $\Rightarrow$ (2): Let $f_n := \hat{\nu_n}$. By weak-*-compactness of the unit ball in $C_0(\hat{G})' = \mathcal{M}(\hat{G})$ we can find a weak-*-limit-point μ of the set $\{\nu_n : n \in \mathbb{N}\}$. So for every $x \in G$ and $\varepsilon > 0$ there is a subsequence n_k (depending of course on x) such that for $k \geq k_0$

$$|\hat{\mu}(x) - f_{n_k}(x)| = |\int_{\hat{G}} \chi(x)d\mu - \int_{\hat{G}} \chi(x)d\nu_{n_k}| = |\int_{\hat{G}} \hat{\hat{x}}d(\mu - \nu_{n_k})| \leq \varepsilon.$$

Here we needed the compactness of $\hat{G}$ (this implies that the function $\hat{\hat{x}} : \chi \mapsto \chi(x)$ defined on $\hat{G}$ vanishes at infinity). Now we use the that $f_n(x) \to f(x)$ pointwise:

$$|\hat{\mu}(x) - f(x)| \leq |\hat{\mu}(x) - f_{n_k}(x)| + |f_{n_k}(x) - f(x)| \leq 2\varepsilon,$$

thus weak-*-$\lim_{n\to\infty} \nu_n = \mu$ and $\lim_{n\to\infty} f_n(x) = \hat{\mu}(x)$. (2) $\Rightarrow$ (1) is trivial since $\mu \mapsto \hat{\mu}(x) = \int_{\hat{G}} \hat{\hat{x}}d\mu$ is a weak-*-continuous functional for every $x \in G$. □

Lemma 3.7.3. *Let $f \in \mathcal{A}(\mathbb{Z})$, $f \geq 0$ and $f \neq 0$. Then $\limsup_{|k|\to\infty} f(k) > 0$, i.e. there exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$*

$$\sup_{|k|\geq N} f(k) \geq \varepsilon > 0.$$

Proof. $f \in \mathcal{A}(\mathbb{Z})$ implies that the set $P_f(\varepsilon) := \{k : \|f - \tau_k f\|_\infty < \varepsilon\}$ of ε-almost periods is relatively dense in $\mathbb{Z}$, i.e. there exists $K_\varepsilon \in \mathbb{N}$ such that $[a,b] \cap P_f(\varepsilon) \neq \emptyset$ whenever $|b-a| \geq K_\varepsilon$ (Theorem 4.10.3 in [BJM]). Since $f \neq 0$ there is some $k_0 \in \mathbb{Z}$ such that $f(k_0) > 0$, w.l.o.g. assume $k_0 = 0$. Let $\varepsilon := f(0)/2 > 0$. Relative density of $P_f(\varepsilon)$ implies that for each $N \in \mathbb{N}$ there exists $k_\varepsilon \geq N$ such that $\|f - \tau_{k_\varepsilon} f\|_\infty < \varepsilon$. In particular $|f(0) - f(k_\varepsilon)| < \varepsilon$ and hence $f(k_\varepsilon) \geq \varepsilon$. □

We will construct a function on $\mathbb{Z}$ using discrete measures on $\hat{\mathbb{Z}} = \mathbb{T}$. Denote by δ_{χ_α} the Dirac measure on $\hat{\mathbb{Z}} \cong [0,1]$ which is concentrated on the character $\chi_\alpha : k \mapsto \exp(2\pi i k\alpha)$. The discrete measures ν_n defined recursively by $\nu_0 := \delta_{\chi_{1/2}}$ and

$$\nu_n := \nu_{n-1} * \frac{1}{2}\left(\delta_{\chi_{-1/3^n}} + \delta_{\chi_{1/3^n}}\right)$$

are all probability measures, so their norm is bounded by 1. Using the fact $\widehat{\nu_n * \nu_{n-1}} = \hat{\nu_n}\hat{\nu_{n-1}}$ one easily computes

$$\tilde{f}_n(k) := \hat{\nu_n}(k) = \prod_{j=1}^{n} \cos\left(2\pi\frac{k}{3^j}\right).$$

Proposition 3.7.4. *Let μ be the Cantor measure on the ternary Cantor set $C \subseteq [0,1]$. Then the Fourier-Stieltjes transform $f := \widehat{\mu * \mu}$ is a member of the set $(\mathcal{H}(\mathbb{Z}) \cap \mathcal{W}_0(\mathbb{Z})) \setminus (\mathcal{A}(\mathbb{Z}) \oplus C_0(\mathbb{Z}))$.*

Proof. We use the notation from above and break the proof into three steps:

1. $f \in \mathcal{W}_0(\mathbb{Z})$: Every $\tilde{f}_n$ is almost periodic. We show now that the $\tilde{f}_n$ converge pointwise: Since $\lim_{j\to\infty} \cos\left(2\pi\frac{k}{3^j}\right) \to 1$ for each fixed k, and since all terms of this sequence are non negative whenever $j \geq \log_3 2k =: j(k)$. We see that $\{\tilde{f}_{j(k)+n}(k)/\tilde{f}_{j(k)}(k)\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of non negative real numbers. Hence $\lim_{n\to\infty} \tilde{f}_n(k)$ exists. So we can use Lemma 3.7.2 to conclude that

$$\tilde{f}(k) = \prod_{j=1}^{\infty} \cos\left(2\pi\frac{k}{3^j}\right)$$

is a Fourier-Stieltjes transform. Lemma 3.7.2 also implies that $\tilde{f}$ is the Fourier-Stieltjes transform of the canonical singular measure μ on the ternary Cantor set[1] $C \subseteq [0,1] \simeq \hat{\mathbb{Z}}$. This is due to the fact that the probability measures $\nu_n \to \mu$ in the weak-*-topology. Thus not only $f \in \mathcal{W}(\mathbb{Z})$ but $f \in \mathcal{W}_0(\mathbb{Z})$. The same considerations apply verbatim to the discrete measures $\nu_n * \nu_n$ and the function

$$f(k) := \tilde{f}^2(k) = \prod_{j=1}^{\infty} \cos^2\left(2\pi\frac{k}{3^j}\right) \geq 0.$$

2. $f \in \mathcal{H}(\mathbb{Z})$: If we let $f_n := \tilde{f}_n^2 \in \mathcal{A}(\mathbb{Z})$ we get that $0 \leq f \leq f_n$ for each $n \in \mathbb{N}$. Via the Fourier-Stieltjes transform we can compute the invariant mean value of f_n as follows (cf. [Rud]):

$$m(f_n) = m(\widehat{\nu_n * \nu_n}) = \nu_n * \nu_n(\{0\}) = \sum_{\gamma\in\hat{\mathbb{Z}}} |\nu_n(\{\gamma\})|^2 = \frac{1}{2^n} \to 0.$$

Thus Proposition 3.3.3 implies $f \in \mathcal{H}(\mathbb{Z})$.

[1] if t denotes the Cantor-Lebesgue ternary function then $\mu([0,x)) := t(x)$

3. $f \notin \mathcal{A}(\mathbb{Z}) \oplus C_0(\mathbb{Z})$: Since $\tilde{f}$ satisfies the functional equation $\tilde{f}(3k) = \tilde{f}(k)$, $k \in \mathbb{Z}$, we have $\tilde{f}(3^k) = \tilde{f}(0) \neq 0$. Thus $\tilde{f} \notin C_0(\mathbb{Z})$ and $f = \tilde{f}^2 \notin C_0(\mathbb{Z})$.

   Suppose there exists a representation $f = f_a + f_0 \geq 0$ with $f_a \in \mathcal{A}(\mathbb{Z})$ and $f_0 \in C_0(\mathbb{Z})$. Furthermore let

$$f_a = \underbrace{(f_a \vee 0)}_{:=f_a^+ \geq 0} + \underbrace{(f_a \wedge 0)}_{:=f_a^- \leq 0}.$$

   $m(f) = m(f_0)$ implies $m(f_a) = 0$ and hence $m(f_a^+) = -m(f_a^-)$. Since we already know $f \notin C_0(\mathbb{Z})$ necessarily $f_a \neq 0$. Thus $m(-f_a^-) > 0$. $-f_a^-$ is a non negative almost periodic function with non zero mean-value. Hence $-f_a^- \neq 0$ and we can apply Lemma 3.7.3. We obtain that there exists $\varepsilon > 0$ such that

$$\inf_{|k| \geq N} f_a(k) = \inf_{|k| \geq N} f_a^-(k) \leq -\varepsilon < 0.$$

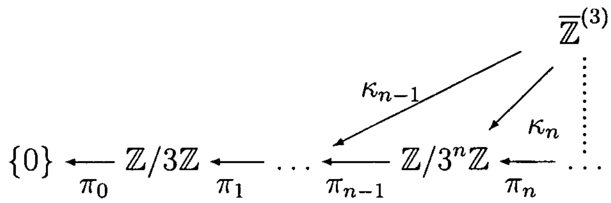
   Note that $f_a^-(k_0) \neq 0$ implies $f_a^+(k_0) = 0$. Let N_0 be such that $|f_0(k)| < \varepsilon/2$ for $|k| \geq N_0$. Thus there exists $k_0 \geq N_0$ such that

$$f(k_0) = f_a(k_0) + f_0(k_0) = f_a^-(k_0) + f_0(k_0) \leq -\varepsilon + \varepsilon/2 = -\varepsilon/2 < 0.$$

   Since $f \geq 0$ this is a contradiction.

□

Finally we want to construct a realization of f without g.j.d. and thus prove that $f \in \mathcal{H}_c(\mathbb{Z})$. Consider the compact group of the 3-adic integers $\overline{\mathbb{Z}}^{(3)}$ realized as projective limit of the cyclic groups $\mathbb{Z}/3^n\mathbb{Z}$



Regarding $\mathbb{Z}/3^n Z$ as $\{0, 1/3^n, \ldots, 1 - 1/3^n\}$ with addition modulo 1, we can interpret the projective limit $\overline{\mathbb{Z}}^{(3)}$ as set of certain sequences $(a_n)_{n\in\mathbb{N}} \subseteq [0,1]^{\mathbb{N}}$ with $a_n \in \{0, 1/3^n, \ldots, 1 - 1/3^n\}$. One easily checks that for each integer $k \in \mathbb{Z}$ the sequence $\iota_3(k) = (k/3^n)_{n\in\mathbb{N}}$ defines an element of the projective limit $\overline{\mathbb{Z}}^{(3)}$ and that the mapping $\iota_3 : \mathbb{Z} \to \overline{\mathbb{Z}}^{(3)}$ is a continuous homomorphism. Hence $(\iota_3, \overline{\mathbb{Z}}^{(3)})$ is group compactification of $\mathbb{Z}$. One can also realize the projective limit $\overline{\mathbb{Z}}^{(3)}$ as set of formal sums $\sum_{n=0}^{\infty} a_n 3^n$ with $a_n \in \{0,1,2\}$. Mapping the formal sum $\sum_{n=0}^{\infty} a_n 3^n$ to the sequence $(\sum_{k=0}^{n-1} a_k/3^n)_{n\in\mathbb{N}}$ establishes a one-one correspondence.

Observe that

$$\frac{k}{3^n} = \frac{\kappa_{n-1} \circ \iota_3(k)}{3^n} \mod 1$$

where κ_n denotes the canonical projections from the projective limit onto the cyclic group $\mathbb{Z}/3^n\mathbb{Z} \cong \{0, 1/3^n, \ldots, 1 - 1/3^n\} \subseteq [0,1]$. Thus for each n the continuous function

$$F_n(x) = \cos^2\left(2\pi\frac{\kappa_{n-1}(x)}{3^n}\right), \quad x \in \overline{\mathbb{Z}}^{(3)}$$

satisfies $F_n \circ \iota_3 = f_n$. Let $F(x) = \prod_{n=1}^{\infty} F_n(x)$, so $f = F \circ \iota_3$. F, however, cannot be continuous since this would imply that f is almost periodic.

A necessary condition for $F(k) \neq 0$ is that $\kappa_{n-1}(x)/3^n \mod 1$ tends to zero (so that $F_n(x)$ tends to 1). If x corresponds to the formal sum $\sum_{k=0}^{\infty} a_k 3^k$ with n^{th}-partial sum s_n this means

$$s_n/3^{n+1} = (a_n/3 + a_{n-1}/3^2 + \cdots + a_0/3^{n+1}) \to 0 \mod 1.$$

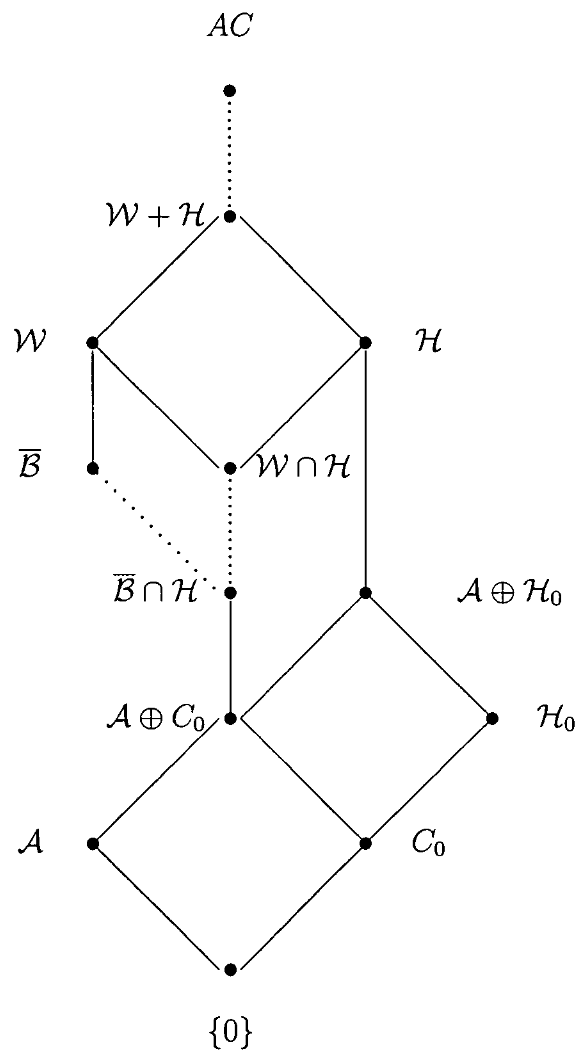
Since $a_{n-1}/3^2+\cdots+a_0/3^{n+1} \leq 1/3-1/3^{n+2}$ this is only possible if $a_j = 0$ finally or $a_j = 2$ finally (otherwise $3^{-n}\kappa_{n-1}x \mod 1 \in (1/3, 2/3)$ infinitely often). Formal sums with this property correspond to the elements of $\iota_3(\mathbb{Z})$. So this implies $F(x) = 0$ for every $x \in \overline{\mathbb{Z}}^{(3)} \setminus \iota_3(\mathbb{Z})$ and $\mathrm{disc}(F) = [F \neq 0] = \iota_3(\mathbb{Z})$, which is both meager and a μ_3-null set. Thus $f \in \mathcal{H}_c(\mathbb{Z})$.

Loosely spoken, F has a similar behavior with respect to points of continuity as the classical example of a Riemann integrable function on $[0,1]$ with infinitely many points of discontinuity:

$$f(x) = \begin{cases} \frac{1}{q} & \text{for } x = \frac{p}{q},\ \gcd(p,q) = 1 \\ 0 & \text{for } x \text{ irrational.} \end{cases}$$

3.8 Relation to other spaces of functions

The part of the lattice of subalgebra of $B(\mathbb{Z})$ drawn on page 53, summarizes some of our results concerning the space $\mathcal{H}(\mathbb{Z})$ of Hartman measurable functions on the integers.



B	...	bounded functions
AC	...	almost convergent functions, cf. Section 1.1
$\mathcal{W}$	...	weakly almost periodic functions, cf. Section 1.1
$\mathcal{H}$	...	Hartman measurable functions, cf. Definition 3.3.1
$\mathcal{A}$	...	almost periodic functions, cf. Section 1.1
$\overline{B}$	...	Fourier-Stieltjes algebra, cf. Section 3.7
$\mathcal{H}_c$	...	Hartman measurable functions without g.j.d., cf. Definition 3.6.12
$\mathcal{H}_0$	...	Hartman measurable functions with support on a meager null set, cf. Definition 3.6.14
C_0	...	functions vanishing at infinity, cf. Section 1.1

Inclusions indicated by | are proper, spaces connected by ⋮ could possibly coincides (although we think this is very unlikely).

- For $\mathcal{W}(\mathbb{Z})\backslash\mathcal{H}(\mathbb{Z}) \neq \emptyset$ and $\mathcal{H}(\mathbb{Z})\backslash\mathcal{W}(\mathbb{Z}) \neq \emptyset$ cf. Corollary 3.6.6 and Example 3.6.10.

- For $\mathcal{H} \neq \mathcal{A} \oplus \mathcal{H}_0$ cf. Corollary 3.6.6.
- For any non compact LCA group $\mathcal{W} \setminus \overline{\mathcal{B}} \neq \emptyset$, cf. Theorem 4.9 in [Bur].
- For $(\overline{\mathcal{B}} \cap \mathcal{H}) \setminus (\mathcal{A} \oplus C_0) \neq \emptyset$ and $G = \mathbb{Z}$, cf. Section 3.7. We conjecture that this holds for any non compact LCA group as well.
- For $\mathcal{H}_m \setminus C_0 \neq \emptyset$ and $G = \mathbb{Z}$, cf. Example 3.6.20.

Question 5: For which topological groups G do exist functions $f \in \overline{\mathcal{B}} \setminus \mathcal{H}$ and functions $f \in \mathcal{H} \setminus \overline{\mathcal{B}}$.

Question 6: How are $\overline{\mathcal{B}}$ and $\mathcal{W} \cap \mathcal{H}$ related: Is there a simple condition on functions in $\overline{\mathcal{B}}$ that implies Hartman measurability?

Chapter 4

The Hartman compactification

4.1 Gelfand Theory

When dealing with the space of Hartman measurable functions two aspects are important: First $\mathcal{H}(G)$ is a unital *-subalgebra of $B(G)$ and, second, $\mathcal{H}(G) = B(\Sigma, G)$ where Σ is the Boolean algebra of Hartman measurable sets (Proposition 3.3.2).

Regarding the first aspect we recall again the Gelfand Representation Theorem and the Banach-Stone Theorem:

Theorem 10. *Every commutative unital C^*-algebra A is isometrically isomorphic to the algebra of continuous functions $C(X)$ of some compact Hausdorff space X. The space X is called the structure space of A, or, synonymously the Gelfand compactum of A.*

Theorem 11. *Let X_1 and X_2 be compact Hausdorff spaces. $C(X_1)$ and $C(X_2)$ are isometrically isomorphic as unital Banach algebras iff X_1 and X_2 are homeomorphic as topological spaces.*

The proofs of both the Gelfand Representation Theorem and the Banach-Stone Theorem can be found in the books [Dav] resp. [DS]. The Banach-Stone Theorem implies that the structure space of a commutative unital C^*-algebra is unique up to homeomorphisms.

Definition 4.1.1. *For a topological group G the structure space of $\mathcal{H}(G)$, the space of Hartman measurable functions, is denoted by hG.*

Remark: The structure space of $\mathcal{A}(G)$, the space of almost periodic functions is bG, the Bohr compactification. The structure space of $\mathcal{W}(G)$, the space of weakly almost periodic functions is wG, the w.a.p. compactification.

There are various models for the structure space of a commutative unital C^*-algebra, for instance the space of maximal ideals. For *-subalgebras of $B(G)$ (such as $\mathcal{A}(G), \mathcal{H}(G)$ or $\mathcal{W}(G)$) we can construct the Gelfand compactum in the following way (for details we refer to [BJM]): Consider $\delta(G) = \{\delta_g : g \in G\}$ the set of all linear functionals of the form $\delta_g(f) := f(g)$ with $g \in G$ and $f \in \mathcal{H}(G)$. Cleary $\delta(G)$ is a bounded subset of the dual space $\mathcal{H}(G)'$. The weak-* closure of $\delta(G)$ in $\mathcal{H}(G)'$ is a compact Hausdorff space (Banach-Alaoglu Theorem). Indeed $\overline{\delta(G)}^{w^*}$ is homeomorphic to hG (Theorem 2.1.8 in [BJM]). Note that $\overline{\delta(G)}^{w^*}$ equals $MM(\mathcal{H}(G))$, the set of all multiplicative means on $\mathcal{H}(G)$.

This shows in particular that $\delta : G \to hG$ is a continuous mapping with dense image, i.e. (δ, hG) is a compactification of the topological space G (it is in general not a group compactification!).

Recall that a right-topological semigroup is a topological space S that is algebraically a semigroup such that for each $s \in S$ the mapping $\rho_s : t \mapsto ts$ is continuous.

Proposition 4.1.2. *hG is a compact right-topological semigroup and $\delta : G \to hG$ is a continuous homomorphism of right topological semigroups.*

Proof. Using the representation $hG \cong \overline{\delta(G)}^{w^*}$ the semigroup operation $*$ is given by

$$\mu * \nu(f) := \mu(T_\nu f),$$

where T_ν is the so-called introversion operator given by

$$T_\nu(f)(s) := \nu(\tau_s f).$$

We have to show that the binary operation $*$ is well defined, i.e. we have to show that $T_\nu f \in \mathcal{H}(G)$ whenever $\nu \in hG$ and $f \in \mathcal{H}(G)$. We use the fact that

$$\{T_\nu f : \nu \in hG\} = \overline{\mathrm{co}}^{(p)}\{\tau_g f : g \in G\},$$

where the closure is taken w.r.t. the topology of pointwise convergence (cf. Proposition 2.2.3 in [BJM]) and *co* denotes the convex hull. Let (ι, C) be a group compactification of G, such that there exists a Riemann integrable function $F : C \to \mathbb{C}$ that realizes f. Theorem 8 in [Tal] implies that $\overline{\mathrm{co}}^{(p)}\{\tau_{\iota(g)}F : g \in G\}$ consists entirely of Riemann integrable functions. Continuity of $\iota : G \to C$ implies continuity of the mapping $\iota^* : F \mapsto F \circ \iota$ w.r.t. the topology of pointwise convergence on C resp. G. Thus

$$\iota^* \left(\overline{\mathrm{co}}^{(p)}\{\tau_{\iota(g)}F : g \in G\}\right) = \overline{\mathrm{co}}^{(p)}\{\tau_g f : g \in G\} = \{T_\nu f : \nu \in hG\} \subseteq \mathcal{H}(G).$$

It is then easy to check that hG equipped with the operation $*$ is indeed right-topological (cf. Theorem 2.2.11 in [BJM]). □

For a discrete semigroup S it is possible to endow the Stone-Cech compactification βS with such a semigroup operation, that $(\iota_b, \beta S)$ is maximal among all right-topological semigroup compactifications of S. This means that for any right-topological semigroup compactification (ι, C) there exists a continuous semigroup homomorphism $\pi : \beta S \rightarrow C$ such that $\pi \circ \iota_b = \iota$. For all it's worth we also note that for LCA groups G with separable dual the Hartman compactification hG is strictly right-topological - and not semitopological - because this would imply $\mathcal{H}(G) \subseteq \mathcal{W}(G)$ in contradiction with Corollary 3.6.6.

4.2 Representation of Riemann integrable functions

The Stone-Representation Theorem is an analogue of the Gelfand-Representation Theorem for Boolean algebras.

Theorem 12. *Every Boolean algebra B is isomorphic to the algebra of clopen sets $Cl(X)$ of some totally disconnected compact Hausdorff space X. X is called the Stone space of A.*

Theorem 13. *Let X_1 and X_2 be totally disconnected compact Hausdorff space. $Cl(X_1)$ and $Cl(X_2)$ are isomorphic as Boolean algebras iff X_1 and X_2 are homeomorphic as topological spaces.*

This implies that the structure space of a Boolean algebra is unique up to homeomorphisms. The proofs of both Theorems can be found in [Vla].

When we deal with algebras of the form $B(G, \Sigma)$ with a Boolean algebra $\Sigma \subseteq \mathfrak{P}(G)$ the concepts of Gelfand compactum and Stone space nicely fit together.

Proposition 4.2.1. *The Stone space of a unital Boolean algebra $\Sigma \subseteq \mathfrak{P}(G)$ is homeomorphic to the Gelfand compactum of the C^*-algebra $B(G, \Sigma)$.*

Proof. Let X be the Gelfand compactum of $B(G, \Sigma)$. We know that X consists of all multiplicative means $m \in B(G, \Sigma)'$ equipped with the weak-*-topology. Each such mean m defines a finitely additive measure on Σ via $\mu_m(A) := m(\mathbb{1}_A)$, $A \in \Sigma$. Note that due to multiplicativity μ_m takes only the values 0 and 1 and hence is a Boolean homomorphism of Σ onto $\{0, 1\}$. On the other hand each multiplicative mean on $B(G, \Sigma)$ is uniquely determined by its values on the set $\{\mathbb{1}_A : A \in \Sigma\}$. Thus the mapping $\mathrm{MM}(B(G, \Sigma)) \rightarrow \mathrm{Hom}(\Sigma, \{0, 1\})$ that sends a multiplicative mean m to the Boolean homomorphism induced by μ_m is one-one and onto. Since the Stone space of Σ is homeomorphic to the set of Boolean homomorphisms $\mathrm{Hom}(\Sigma, \{0, 1\})$ equipped with the topology inherited form the product topology on $\{0, 1\}^\Sigma$ it only remains to show that the mapping $m \rightarrow \mu_m$

is continuous. But this is obvious since for $\varepsilon < 1/2$ a basic weak-* neighborhood of m_0

$$U_{A_1,\ldots,A_n;\varepsilon}(m_0) = \{m : \max_{i=1,\ldots n} |m_0(\mathbb{I}_{A_i}) - m(\mathbb{I}_{A_i})| \leq \varepsilon\}$$

is mapped onto the basic neighborhood of μ_0

$$V_{A_1,\ldots,A_n}(\mu_0) = \{\mu : \mu(A_i) = \mu_0(A_i),\ i = 1,\ldots n\}.$$

□

Let $\Sigma \subseteq \mathfrak{P}(X)$ be a Boolean set-algebra. A Σ-ultrafilter is an element $p \in \mathfrak{P}(\mathfrak{P}(\Sigma))$ such that the following conditions are fulfilled

1. $\emptyset \notin p$
2. $A \in p, B \in p \Rightarrow A \cap B \in p$
3. $A \in p, B \in \Sigma, A \subseteq B \Rightarrow B \in p$
4. $\forall A \in \Sigma : A \in p \vee X \setminus A \in p.$

Lemma 4.2.2. *There is a one-one correspondence between Σ-ultrafilters and multiplicative means of $B(\Sigma, X)$.*

Proof. For p a Σ-ultrafilter consider the canonical extension of the mapping $m_p(1_A) = 1_p(A)$ to $B(\Sigma, X)$ and for m a multiplicative functional consider the set $p_m := \{A \in \Sigma : m(A) = 1\}$. □

Conclusion: determining the Gelfand space of the space of Hartman measurable functions is equivalent to determine the Stone space of the Boolean algebra of Hartman measurable sets.

The same considerations apply for $R[0, 1]$, the space of Riemann integrable functions on the unit interval, and $\mathfrak{Jor}$, the Boolean Algebra of Jordan measurable sets. However the problem of giving a satisfactory description of the structure space is still unsolved.

Example 4.2.3 (Regulated functions).

A bounded function $f : [0, 1] \to \mathbb{C}$ is called regulated iff for each $t \in (0, 1)$ both $\delta_t^+(f) := \lim_{x\to t^+} f(x)$ and $\delta_t^-(f) := \lim_{x\to t^-} f(x)$ exists and furthermore $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ exist. It is easily seen that $\mathrm{Rg}[0, 1]$, the space of regulated functions, is a unital Banach algebra, when endowed with the usual norm $\|.\|_\infty$. Furthermore

$$\mathrm{Rg}[0, 1] = B([0, 1], \mathfrak{I}) \subseteq B([0, 1], \mathfrak{Jor}) = R[0, 1],$$

where $\mathfrak{J}$ denotes the Boolean algebra generated by all (open and closed) subintervals of $[0,1]$ (cf. [Bou2]). Let p be a $\mathfrak{J}$-ultrafilter with $\{0\} \notin p$ and $\{1\} \notin p$. Define $\alpha_p := \inf\{t : [0,t) \in p\}$ and $\beta_p := \sup\{s : (s,1] \in p\}$. It is easy to see that $\alpha_p = \beta_p$ and that there are only three possible cases

1. $[0,\alpha) \in p$, $(\alpha,1] \notin p$
2. $[0,\alpha) \notin p$, $(\alpha,1] \in p$
3. $[0,\alpha) \notin p$, $(\alpha,1] \notin p$

Ultrafilters satisfying 1 correspond to the measure concentrated on the interval $[0,\alpha]$ (as element of the Boolean algebra $\mathfrak{J}$), while ultrafilters satisfying 2 correspond to the measure concentrated on the interval $[\alpha,1]$ and ultrafilters satisfying 3 correspond to the measure concentrated on the singleton $\{\alpha\}$. Speaking in terms of functionals, the interval $[0,\alpha]$ corresponds to δ_α^-, the interval $[\alpha,1]$ corresponds to δ_α^+ and the singleton $\{\alpha\}$ corresponds to δ_α. The ultrafilters generated by $\{0\}$ resp. $\{1\}$ correspond to the functionals δ_0^- resp. δ_1^+.

Thus the structure space of $\mathrm{Rg}[0,1]$ is homeomorphic to the space

$$X = \{\delta_t^+ : t \in (0,1]\} \,\dot\cup\, \{\delta : t \in [0,1]\} \,\dot\cup\, \{\delta_t^- : t \in [0,1)\}$$

endowed with the following topology:

1. basic neighborhoods of δ_t^- are given by

$$\begin{aligned} U_n(t^-) &:= \{\mu : \mu([0, t+\frac{1}{n})] = 1\} \\ &= [\delta_t^-, \delta_{t+\frac{1}{n}}^-) \,\dot\cup\, (\delta_t^+, \delta_{t+\frac{1}{n}}^+) \,\dot\cup\, (\delta_t, \delta_{t+\frac{1}{n}}), \end{aligned}$$

2. basic neighborhoods of δ_t^+ are given by

$$\begin{aligned} U_n(t^+) &:= \{\mu : \mu((t-\frac{1}{n}, 1] = 1\} \\ &= [\delta_{t-\frac{1}{n}}^-, \delta_t^-) \,\dot\cup\, (\delta_{t-\frac{1}{n}}^+, \delta_t^+] \,\dot\cup\, (\delta_{t-\frac{1}{n}}, \delta_t), \end{aligned}$$

3. a basic neighborhood of δ_t is given by

$$U(t) := \{\mu : \mu([0,t]) = \mu([t,1]) = 1\} = \{\delta_t\},$$

i.e. each element δ_t is isolated.

A moment's reflection shows that this topology coincides with the order topology induced by the total order defined via $\delta_t^+ < \delta_t < \delta_t^-$ and $\delta_s^- < \delta_t^+$ whenever $s < t$.

4.3 Banach almost periodic functions

The results of this section were obtained in joint work with Mathias Beiglböck, cf. [BM].

Definition 4.3.1. *A subset $A \subseteq \mathbb{Z}$ is called relatively dense, or syndetic, iff there exists a finite set $F \subseteq \mathbb{Z}$ such that $A + F = \mathbb{Z}$.*

Recall the classical definition of Bohr's almost periodic functions: a bounded function $f : \mathbb{Z} \to \mathbb{C}$ is almost periodic iff for each $\varepsilon > 0$ the set $\{t : \|f - f_t\| < \varepsilon\}$ is syndetic. We want to replace the norm $\|.\|_\infty$ by $m_\mathbb{Z}(|.|)$ for the unique invariant mean on $AC(\mathbb{Z})$. Note that for any $f \in AC(\mathbb{Z})$ the L.I.M. $\mu_\mathbb{Z}(f)$ coincides with $d^*(f)$, the upper Banach density.

Definition 4.3.2. *A bounded function $f : \mathbb{Z} \to \mathbb{C}$ is called Banach almost periodic (BAP) iff for every $\varepsilon > 0$ there exists a syndetic set $S \subseteq \mathbb{Z}$ such that for all $t \in S$, $d^*(|f - f_t|) \leq \varepsilon$ holds uniformly in t. More precisely this means: f is Banach almost periodic iff for all $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a syndetic set S such that for all $a, b \in \mathbb{N}, b - a \geq n$ and $t \in S$ we have $d_{[a,b]}(|f - f_t|) \leq \varepsilon$. The set of all Banach almost periodic functions on $\mathbb{Z}$ is denoted by $BAP(\mathbb{Z})$.*

We collect some basic properties of BAP functions:

Proposition 4.3.3. *Let $f \in BAP(\mathbb{Z}))$. Then the following assertions hold*

1. *Definition 4.3.2 yields the same class of functions if 'syndetic' is replaced by 'IP*' or 'positive upper Banach density'.*
2. *The inclusion $\mathcal{H}(\mathbb{Z}) \subsetneq BAP(\mathbb{Z})$ holds and is proper.*
3. *$BAP(\mathbb{Z})$ is a C^*-subalgebra of $B(\mathbb{Z})$, the bounded functions defined on $\mathbb{Z}$.*
4. *Every $1\!\!1_A \in BAP(\mathbb{Z}) \cap 2^\mathbb{Z}$ has uniform density in all block lengths. More precisely, this means: for every finite word $w \in 2^{<\omega}$ there exists a number $d(w; A) \in [0, 1]$, called the asymptotic frequency of w in A, such that $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that*
$$\left| \frac{S_{w;A}([a, b])}{b - a} - d(w; A) \right| \leq \varepsilon,$$
whenever $b - a \geq N$. Here $S_{w;A}([a, b])$ denotes the number of occurrences of w within $1\!\!1_A \restriction_{[a,b]}$.
5. *$f \in BAP(\mathbb{Z})$ implies $\overline{co}^{(p)} O(f) \subseteq BAP(\mathbb{Z})$, where co denotes the convex hull and the closure is taken w.r.t. the topology of pointwise convergence.*
6. *$f \in BAP(\mathbb{Z})$ and $e \in E(\beta\mathbb{Z})$ imply $d^*(|f - e\!-\!\lim_t f_t|) = 0$. Here $E(\beta\mathbb{Z})$ denotes the set of all idempotent ultrafilters on $\mathbb{Z}$, and $e\!-\!\lim$ denotes the filter limit w.r.t. e.*

Proof:

1. It is clear that BAP(IP*) $\Rightarrow$ BAP(syndetic) $\Rightarrow$ BAP(density). Pick $f \in$ BAP(density). For $\varepsilon > 0$ pick $n_0 \in \mathbb{N}$ and $E \subseteq \mathbb{Z}$ such that $d^*(E) > 0$ and for all $a, b \in \mathbb{Z}, b - a \geq n_0$, $t \in E$ we have $d_{[a,b]}(|f - f_t|) < \varepsilon/2$. Let $t_1, t_2 \in E$ and $a, b \in \mathbb{Z}, b - a > n_0$. Then
$$d_{[a,b]}(|f - f_{t_1 - t_2}|) = d_{[a-t_2, b-t_2]}(|f_{t_2} - f_{t_1}|)$$
$$\leq d_{[a-t_2,b-t_2]}(|f - f_{t_1}|) + d_{[a-t_2,b-t_2]}(|f_{t_2} - f|) < \varepsilon.$$
Since $E - E$ is IP*, we see that $f \in$ BAP(IP*). Thus BAP(density) $\Rightarrow$ BAP(IP*).

2. Let $f \in \mathcal{H}(Z)$. For every $\varepsilon > 0$ there exists $g \in AP(\mathbb{Z}) \subsetneq$ BAP($\mathbb{Z}$) such that and $\mu_{\mathbb{Z}}(|g - f|) = d^*(|g - f|) \leq \varepsilon$. Since we have $d^*(|f - f_t|) \leq d^*(|f - g|) + d^*(|g - g_t|) + d(|g_t - f_t|) \leq 3\varepsilon$ $g \in$ BAP($\mathbb{Z}$) implies $f \in$ BAP($\mathbb{Z}$).

3. BAP($\mathbb{Z}$) is uniformly closed because $\|f - g\|_\infty \leq \varepsilon$ implies $\left|d_{[a,b]}(f - f_t) - d_{[a,b]}(g - g_t)\right| \leq 2\varepsilon$.
Next we show that along with $f, g \in$ BAP($\mathbb{Z}$) also $f + g \in$ BAP($\mathbb{Z}$) resp. $fg \in$ BAP($\mathbb{Z}$). Let $S = S(f, \varepsilon)$ and $T = T(g, \varepsilon)$ be IP* sets. Note that $S \cap T$ is an IP* set and for $t \in S \cap T$ we have
$$d_{[a,b]}(|(f + g) - (f + g)_t|) \leq d_{[a,b]}(|f - f_t|) + d_{[a,b]}(|g - g_t|) \leq 2\varepsilon$$
resp.
$$\begin{aligned} d_{[a,b]}(|(fg) - (fg)_t|) &\leq d_{[a,b]}(|(f - f_t)g|) + d_{[a,b]}(|f_t(g - g_t)|) \\ &\leq \|g\|_\infty d_{[a,b]}(|f - f_t|) + \|f_t\|_\infty d_{[a,b]}(|g - g_t|) \\ &\leq 2\max\{\|f\|_\infty, \|g\|_\infty\}\varepsilon \end{aligned}$$
Thus $S \cap T$ is good for $f + g$ and 2ε resp. for fg and $2\max\{\|f\|_\infty, \|g\|_\infty\}\,\varepsilon$.

4. Fix $w \in 2^{<\omega}$ and suppose by contradiction that $\beta = d_*(w; A) < d^*(w; A) =: \alpha$. For any $\varepsilon > 0$ we can find arbitrarily long intervals $[a_1, b_1]$ and $[a_2, b_2]$ (denoting $n_1 = b_1 - a_1$ and $n_2 = b_2 - a_2$) such that $\frac{S_{w;A}([a_1,b_1])}{n_1} > \alpha - \varepsilon$ and $\frac{S_{w;A}([a_2,b_2])}{n_2} < \beta + \varepsilon$ and $\varepsilon := \frac{\alpha - \beta}{4}$. Subsequently we will use the following
Fact: Suppose $d^*_{[a,b]}(|f - g|) < \varepsilon$ with $f = \mathbb{I}_A$ and $g = \mathbb{I}_B$ then (for large $n = b - a$) we have
$$|S_{w;A}([a, b] - S_{w;B}([a, b])| < \varepsilon|w|.$$
Proof: $d_{[a,b]}(|f - g|) < \varepsilon$ implies that $f \restriction [a, b]$ and $g \restriction [a, b]$ differ at most at εn positions. Each variation can alter at most $|w|$ occurrences of w within A resp. B.

Let S be a syndetic set that is good for f and $\varepsilon_k > 0$ and let $M_k \in \mathbb{N}$ be such that $S + [0, M_k] = \mathbb{Z}$. W.l.o.g. we may suppose $n_1/M_k > k$ resp. $n_2/n_1 > k$ and that n_1 is large enough that

$$d_{|[a_1,b_1]}(f - f_t) < \varepsilon_k, t \in M_k$$

Since M_k is syndetic there exist pairwise distinct elements $s_1, \dots s_l \in M_k$ such that $[a_1 + s_i, b_1 + s_i] \subseteq [a_2, b_2]$ for $i = 1, \dots, l$. Note that the set $[a_2, b_2] \setminus \left(\bigcup_{i=1}^{l} [a_1 + s_i, b_1 + s_i] \right)$ has $o(k^{-1})$ elements and that also $|\frac{l}{n_2} - \frac{1}{n_1}|$ is $o(k^{-1})$ We want to give a lower bound for $S_{w;A}([a_2, b_2]$: Since in each translate of $[a_1, b_1]$ by an element of M_k we have at least $S_{w;A}([a_1, b_1] - \varepsilon_k |w| n_1$ occurrences of w this implies

$$\begin{aligned} \frac{S_{w;A}([a_2, b_2]}{n_2} &\geq \frac{l}{n_2} (S_{w;A}([a_1, b_1] - \varepsilon_k |w| n_1) && (4.1) \\ &\geq \frac{S_{w;A}([a_1, b_1]}{n_1} - \eta && (4.2) \end{aligned}$$

for arbitrary $\eta > 0$ whenever k is large enough. Picking an $\eta < \frac{\alpha - \beta}{8}$ yields a contradiction.

5. For given $\varepsilon > 0$ let $S = S(f, \varepsilon) \subseteq \mathbb{Z}$ and $N \in \mathbb{N}$ be such that S is syndetic and $d_{[a,b]}(|f - f_t|) < \varepsilon$ whenever $t \in S$ and $b - a \geq N$. We claim that $d_{[a,b]}(|g - g_t|) < 2\varepsilon$ whenever $t \in S$ and $b - a \geq N$. Since $g \in \overline{\text{co}}^{(p)} O(f)$, for each fixed $t \in S$ there exists a convex combination

$$h = \sum_{i=1}^{k} \lambda_i f_{m_i} \in \overline{\text{co}}^{(p)} O(f)$$

such that $|h - g| \upharpoonright [a, b] < \varepsilon/2$ and $|h_t - g_t| \upharpoonright [a, b] = |h - g| \upharpoonright [a + t, b + t] < \varepsilon/2$. Thus

$$\begin{aligned} d_{[a,b]}(|g - g_t|) &\leq d_{[a,b]}(|g - h|) + d_{[a,b]}(|h - h_t|) + d_{[a,b]}(|h_t - g_t|) \\ &\leq \varepsilon + d_{[a,b]}(|h - h_t|). \end{aligned}$$

Plugging in the definition of h yields

$$\begin{aligned} d_{[a,b]}(|h - h_t|) &= d_{[a,b]}\left(\left| \sum_{i=1}^{k} \lambda_i (f_{m_i} - f_{m_i + t}) \right| \right) \\ &\leq \sum_{i=1}^{k} \lambda_i d_{[a+m_i, b+m_i]}(|f - f_t|) \\ &\leq \sum_{i=1}^{k} \lambda_i \varepsilon = \varepsilon. \end{aligned}$$

Thus $S(f, \varepsilon) = S(g, 2\varepsilon)$.

6. Let $g = e\text{-}\lim_t f_t$. For $\varepsilon > 0$ choose an IP* set S and $n_0 \in \mathbb{N}$ such that for all $a, b \in \mathbb{Z}, b - a > n_0$ and $t \in S$ one has $d_{[a,b]}(|f - f_t|) < \varepsilon$. By the definition of the filter limit

$$B = \{t \in \mathbb{Z} : \max_{a \le s \le b} |g(s) - f_t(s)| \le \varepsilon\} \in e.$$

Since $S \in e$, $B \cap S$ is nonempty, so pick $t \in B \cap S$. Then

$$d_{[a,b]}(|f - g|) \le d_{[a,b]}(|f - f_t|) + d_{[a,b]}(|f_t - g|) < 2\varepsilon.$$

Since ε was arbitrary, the claim follows.

Remark: Since BAP($\mathbb{Z}$) is an abelian unital C^*-algebra BAP($\mathbb{Z}$) $\cong C(ba\mathbb{Z})$ for some compact space $ba\mathbb{Z}$, the structure space or Gelfand compactum of BAP($\mathbb{Z}$). According to the Gelfand theory for semigroups we can find a compatible right-topological semigroup structure on $ba\mathbb{Z}$.

Lemma 4.3.4. *Let (X, T) be a dynamical system and let e be a minimal idempotent in $\beta\mathbb{N}$ and let $x \in X$. Then $e\text{-}\lim_n T^n(x)$ is uniformly recurrent.*

Proof. See 6.9. in [BeH]. □

Proposition 4.3.5. *Let $g_1 = 1\!\!I_{A_1}$ and $g_2 = 1\!\!I_{A_2}$ be members of UDB($\mathbb{Z}$). Then $\overline{O}^p(g_1) = \overline{O}^p(g_2)$ iff there exists $f \in 2^\omega$ such that $\overline{O}^p(g_i) \subseteq \overline{O}^p(f)$.*

Proof. Suppose that $\overline{O}^p(g_i) \subseteq \overline{O}^p(f)$. We will prove that $\overline{O}^p(g_1) = \overline{O}^p(g_2)$.

First we show that for any finite word $w \in 2^{<\omega}$ the asymptotic densities $d(w; g_1)$ and $d(w; g_2)$ coincide. For each $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that for $b - a \ge n$

$$\frac{S_{w;A_i}([a,b])}{b-a} - d(w; A)| < \varepsilon/4,$$

where $S_{w;A_i}([a,b])$ denotes the number of occurrences of w within $1\!\!I_{A_i} \restriction_{[a,b]}$. There exist $k_1, k_2 \in \mathbb{Z}$ such that $\tau_{k_1} g_1$ and $\tau_{k_2} g_2$ coincide on $[a, b]$ (and equal f). Thus

$$\left| \frac{S_{w;A_1}([a - k_1, b_1 - k_1])}{b-a} - \frac{S_{w;A_2}([a - k_2, b_2 - k_2])}{b-a} \right| < \varepsilon/2$$

differ at most at $\varepsilon/2$ resp. $d(w; g_1)$ and $d(w; g_2)$ differ at most at ε. Since $\varepsilon > 0$ was arbitrary $d(w; g_1) = d(w; g_2)$.

Note that there are only finitely many words of a given length that contain a prescribed subword w. Thus for every w with $S_{w;A_i} > 0$ we can find $w' > w$ with $S_{w';A_i} > 0$. Take a sequence $w_1 < w_2 < \ldots < w_n < \ldots$ of finite words with $|w_n| \to \infty$ and $S_{w;A_i} > 0$. Then there exist $l_n, k_n \in \mathbb{Z}$ such that $\tau_{l_n} f \restriction_{[0,|w_n|]} = \tau_{k_n} g \restriction_{[0,|w_n|]} = w_n$. This implies dist $\left(\overline{O}^p(g_1), \overline{O}^p(g_2)\right) = 0$, i.e. $\overline{O}^p(g_1) \cap \overline{O}^p(g_2) \neq \emptyset$. Thus $\overline{O}^p(f) = \overline{O}^p(g)$.

□

Corollary 4.3.6. *Let $f \in BAP(\mathbb{Z})$. There exists a uniformly recurrent $g \in \overline{O}^p(f)$ such that $d^*(|f - g|) = 0$. Furthermore for every such two uniformly recurrent functions g_1 and g_2 the orbits closures $\overline{O}^p(g_1)$ and $\overline{O}^p(g_2)$ coincide. Let us denote this orbit closure by $M(f)$.*

Proof. We can assume w.l.o.g. that f is real-valued. Fix $\alpha, \beta \in \mathbb{R}$ such that $f \in [\alpha, \beta]^{\mathbb{Z}}$. Pick a minimal idempotent $e \in \beta\mathbb{N}$. Then $g = e\text{–}\lim_t f_t$ is uniformly recurrent in the dynamical system $([\alpha, \beta]^{\mathbb{Z}}, \sigma)$ by Lemma 4.3.4. By Proposition 4.3.3 we have $d^*(|f - g|) = 0$. Note that any minimal subsystem of $\overline{O}^p(f)$ is generated by an uniformly recurrent function g belonging to UDB($\mathbb{Z}$). Now apply Proposition 4.3.5 to yield uniqueness of $M(f)$. □

Corollary 4.3.7. *For Hartman measurable f every uniformly recurrent member g in $M(f)$ is also Hartman measurable.*

Proof. Since $M(f) \subseteq \overline{O}^p(f) \subseteq \mathcal{H}(\mathbb{Z})$ this is obvious, cf. Proposition 4.1.2. □

Let $\mathcal{N}(\mathbb{Z})$ be the set of all functions $f \in B(\mathbb{Z})$ such that $d^*(f) = 0$. It follows from the statements above that BAP($\mathbb{Z}$) $\supseteq \mathcal{H}(\mathbb{Z}) + \mathcal{N}(\mathbb{Z})$. Whether this inclusion can be reversed is an open problem.

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