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# On Constructing Assertional and Complementary Sequent Calculi for Non-Deterministic Finite-Valued Logics

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# Abstract

*Non-deterministic finite-valued logics*, proposed by Avron and Lev, are a generalisation of traditional finite-valued logics where the evaluation of formulas represents in general a non-functional relation. In this thesis, we introduce systematic methods for constructing both assertional as well as complementary sequent-style proof systems for any given non-deterministic finite-valued logic. In contrast to traditional, assertional proof systems, whose aim are to axiomatise the valid formulas of a logic, a *complementary system*, also referred to as a *rejection system*, is a proof system which axiomatises the *invalid formulas* of a logic. Rejection systems therefore introduce a purely syntactic way of determining non-validity without having to consider countermodels, which can be useful in procedures for automated deduction and proof search. The method of axiomatic rejection was first introduced by Jan Łukasiewicz in the 1930s and subsequently rejection systems for many well-known logics have been proposed. Our method is based on so-called *many-sided sequents*, which are a natural generalisation of standard two-sided sequents to the finite-valued case. Furthermore, the systematic constructions we introduce in this thesis are generalisations of similar methods for standard finite-valued logics as proposed by Zach for assertional sequent-type calculi and by Bogojeski and Tompits for the complementary case. As special instances of our method, we provide concrete calculi for specific paraconsistent logics which can be elegantly expressed in terms of non-deterministic two- and three-valued semantics, respectively.

# Kurzfassung

*Nichtdeterministische endlichwertige Logiken*, eingeführt von Avron und Lev, sind eine Verallgemeinerung von klassischen endlichwertigen Logiken in der die Auswertung von Formeln im Allgemeinen eine nicht-funktionale Relation darstellt. In dieser Arbeit führen wir systematische Methoden zur Konstruktion von sowohl assertiven als auch komplementären Sequenzenkalkülen für jede gegebene nichtdeterministische endlichwertige Logik ein. Im Gegensatz zu traditionellen Kalkülen, welche die gültigen Formeln einer Logik axiomatisieren, formalisieren *komplementäre Kalküle* die *ungültigen Formeln* einer Logik. Somit erlauben komplementäre Kalküle eine rein syntaktische Charakterisierung von Ungültigkeit ohne die Notwendigkeit der Verwendung von Gegenmodellen, was besonders im Bereich des automatischen Schließens von Vorteil sein kann. Die ersten komplementären Kalküle wurden von Jan Łukasiewicz in den 1930er Jahren eingeführt und in weiterer Folge wurden zahlreiche andere Kalküle dieser Art für unterschiedliche bekannte Logiken entwickelt. Unsere eingeführten Methoden basieren auf sogenannten *mehrseitigen Sequenten*, welche eine natürliche Verallgemeinerung von klassischen zweiseitigen Sequenten für mehrwertige Logiken darstellen. Weiters sind die systematischen Konstruktionen, die in dieser Arbeit eingeführt werden, Verallgemeinerungen von existierenden analogen Methoden für traditionelle mehrwertige Logiken, und zwar der Konstruktionen von Zach für den assertiven Fall und von Bogojeski und Tompits für den Komplementären. Desweiteren führen wir durch Anwendung unserer Methode konkrete Kalküle für ausgewählte parakonsistente Logiken ein, welche in eleganter Weise durch nichtdeterministische zwei- und dreiwertige Semantiken charakterisiert werden können.

# Contents

CHAPTER 1

# Introduction

*Non-deterministic finite-valued logics* are a special case of non-deterministic many-valued logics which were introduced by Avron and Lev [7] as a generalisation of traditional many-valued logics [53, 31, 32]. The latter logics in turn are non-classical ones that allow to have any number of different truth degrees, as opposed to the two classical truth values. In the finite case, the number of truth degrees is restricted to be finite. The idea of a logic with more than two truth values was first developed and investigated in modern logic independently by Łukasiewicz [28] and Post [37].

While in the general study of many-valued logics over a propositional language $\mathcal{L}$, their semantics are often specified via a *matrix* [53], i.e., by a triple consisting of a set $\mathcal{V}$ of truth values, a set $\mathcal{D}$ of designated truth values, and a function assigning to each connective of $\mathcal{L}$ a truth table, Avron and Lev [7] generalised the idea of matrices in their approach to what they call *non-deterministic matrices*, or *Nmatrices*. The only difference lies in the interpretation of the connectives of $\mathcal{L}$: in this new framework, the entries of a truth table for a connective can be any non-empty set of truth values. The intended meaning is that all of the values in a set are options that can be chosen non-deterministically.

In this thesis, we deal with proof systems for non-deterministic finite-valued logics. Traditionally, the aim of a proof system is to give an axiomatic characterisation of the class of valid formulas of a logic, such systems are called *assertional*. However, one can also have an opposite point of view and aim for a proof theory which axiomatises the complementary class of formulas, i.e., the class of all invalid formulas, giving rise to the notion of a *complementary calculus*, or *rejection system*. Such systems therefore introduce a purely syntactic way of rejecting formulas without having to consider countermodels, which can be useful in procedures for automated deduction and proof search. In general, rejection systems constitute an interesting yet lesser known branch of logic. The latter observation may be drawn from the fact that a certain restriction of this complementary approach to build a proof theory is that sound and complete rejection systems can only be realised for logics whose invalid formulas are recursively enumerable. Hence, most complementary calculi deal with a decidable, propositional language only.

Historically, rejection calculi were first studied by Jan Łukasiewicz [29, 30] in his analysis of Aristotelian syllogistic and subsequently rejection systems for many well-known logics have been proposed, like, e.g., for classical logic [52, 14, 25], modal logics [25, 50], intuitionistic logic [22, 47, 49], description logics [11], and finite-valued logics [16, 48, 36, 13].[1]

The specific contribution of this thesis is to develop systematic methods for generating both assertional as well as complementary systems for any given finite Nmatrix. In particular, our methods generalise corresponding approaches for standard finite-valued logics as introduced by Zach [54] for the assertional case and by Bogojeski and Tompits [13] for the complementary one, where the latter method is in turn an adaption of the method by Zach. Both of these approaches are based on sequents which are *many-sided*, constituting a natural generalisation of standard two-sided sequents to the finite-valued case, as independently proposed by Schröter [45] and Rousseau [44]. In the approach of Rousseau, sequents of the form $\Gamma_1 \mid \cdots \mid \Gamma_m$ are used for axiomatising the valid formulas of an $m$-valued logic, where each $\Gamma_i$ $(1 \leq i \leq m)$, referred to as a *component* of the sequent, is a finite set of formulas with the intuitive meaning that each component represents a truth value. To wit, $\Gamma_1 \mid \cdots \mid \Gamma_m$ is true under an interpretation $I$ iff there is at least one $i$ such that, for some formula $A \in \Gamma_i$, $A$ is assigned the $i$-th truth value under $I$. Consequently, a classical sequent $\Gamma \vdash \Delta$ corresponds to $\Gamma \mid \Delta$, where the first component represents the truth value $\mathbf{f}$ while the second the truth value $\mathbf{t}$. Accordingly, following the method of Bogojeski and Tompits [13], we employ *many-sided anti-sequents*, which are tuples of the form $\Gamma_1 \nmid \cdots \nmid \Gamma_m$ with a complementary semantics, i.e., an interpretation $I$ satisfies an anti-sequent $\Gamma_1 \nmid \cdots \nmid \Gamma_m$ (or *refutes* the anti-sequent) exactly in case the corresponding sequent $\Gamma_1 \mid \cdots \mid \Gamma_m$ is not true under $I$. Although Avron and Konikowska [5] already presented a systematic approach to construct assertional proof systems for Nmatrices, their approach is based on *Rasiowa-Sikorski systems* [41], which employ signed formulas and which can be viewed as a dual tableau method [27]. In basing both of our methods on many-sided sequents, however, we obtain a uniform proof-theoretical account for axiomatising validity and non-validity for Nmatrices.

We note in passing that, similar to traditional proof systems, different types of rejection calculi have been developed in the literature besides sequent systems, mostly based on a Hilbert-style axiomatisation—indeed, following the seminal approach of Łukasiewicz [29, 30]—but also on a natural deduction method [51].

Concerning non-deterministic many-valued logics in general, their semantics are modified so that valuations must not comply with the condition of truth-functionality, which is the property that the truth value of a complex formula is uniquely determined by the truth value of its immediate subformulas. Instead, this condition is replaced by a similar, but weaker, condition stating that the value of a complex formula must be an *element of* the

[1] For an excellent recent survey about rejection systems, cf. the paper by Goranko, Pulcini, and Skura [26].

value of the formulas it consists of. It is also important at which point in the valuation of a formula the values are selected. This gives rise to two different semantics: (i) in the *dynamic semantics*, values are chosen for each tuple independently, so the choice is made at the lowest possible point of computation, and (ii) in the *static semantics*, which has first been introduced and studied by Avron and Konikowska [5], the values are chosen *globally*, i.e., they are fixed before any computation begins.

These non-deterministic logics not only have interesting theoretical properties but can be applied for various practical topics too, e.g., they allow to model the inherent non-deterministic behaviour of electrical circuits [8], the modeling of reasoning in inconsistent databases [7, 4], or reasoning about computation errors [6].

However, arguably the most prominent examples of logics for which the semantics can be expressed in terms of Nmatrices come from the area of *paraconsistent logics*. We will consider several paraconsistent logics introduced in the literature and derive concrete proof and rejection systems for these logics based on our general framework. More specifically, as examples for the dynamic semantics, we present proof and rejection systems for the paraconsistent logics **CLuN** [10] and **Cio** [18], and, as an example for the static semantics, a rejection system for **CAR** [21], also known as *Carnot's logic*.

The thesis is organised as follows. In Chapter 2 we introduce the necessary background required for our subsequent elaborations. We introduce the concept of non-deterministic matrices following Avron and Lev [7] as well as the paraconsistent logics for which we introduce assertional and complementary calculi. In Chapter 3 we will generalise the concept of a (complementary) partial normal form and formally introduce the notion of many-sided sequents while providing some important preparatory results. Chapters 4 and 5 constitute the main part of our thesis, where we lay down procedures to generate assertional and complementary calculi for any finite Nmatrix, respectively, for both the dynamic as well as for the static semantics, and provide concrete calculi for the paraconsistent logics **CLuN**, **Cio**, and **CAR**. Finally, Chapter 6 concludes the thesis with a short summary and an outlook on possible future work.

CHAPTER 2

# Background

## 2.1 Non-Deterministic Many-Valued Logics

Avron and Lev [7] introduced the notion of a *non-deterministic many-valued logic* as a generalisation of a standard many-valued logic, where in the evaluation of a formula non-deterministic choices can be made. This implies that the valuations in these logics can violate the principle of *truth functionality*, which expresses the property that the truth value of a complex formula is uniquely determined by the truth values of its immediate subformulas.

### 2.1.1 Syntax

We base our framework on propositional languages whose syntax is defined in the following manner.

By an *alphabet*, $\mathcal{A}$, we understand a set $PROP \cup CON \cup AUX$, where $PROP$, $CON$, and $AUX$ are disjoint sets of symbols such that

(i) $PROP$ is a countably infinite set of *propositional variables*,

(ii) $CON$ is a non-empty, finite set of *connectives* of arbitrary arity, and

(iii) $AUX$ is a set of auxiliary symbols, usually containing symbols for parentheses and interpunctation.

We refer to an alphabet $\mathcal{A}$ whose set of connectives is $CON$ also as an alphabet *for* $CON$.

For an alphabet $\mathcal{A} = PROP \cup CON \cup AUX$, a *propositional language over* $\mathcal{A}$ is a tuple $\mathcal{L}_{\mathcal{A}} = (\mathcal{A}, FORM)$, where $FORM$ is the smallest set such that

(i) $PROP \subseteq FORM$, and

(ii) if $A_1, \ldots, A_n \in FORM$ and $\circ \in CON$ is an $n$-ary connective, then $\circ(A_1, \ldots, A_n) \in FORM$.

The elements of $FORM$ are *well-formed formulas* of $\mathcal{L}_\mathcal{A}$, or simply *formulas*, and the elements of $PROP$ are also referred to as *atomic formulas*, or *atoms*.

For unary connectives, we usually omit the brackets, and, for binary connectives, we use infix instead of prefix notation and omit brackets when they are clear from the context. We usually denote formulas by uppercase Latin letters $A$, $B$, $C$, …, possibly with indices, atomic formulas by uppercase Latin letters $P$, $Q$, $R$, …, also possibly with indices, and sets of formulas by uppercase Greek letters $\Gamma$, $\Delta$, $\Pi$, … .

### 2.1.2 Semantics

Let $\mathcal{A} = PROP \cup CON \cup AUX$ be an alphabet and $\mathcal{L}_\mathcal{A}$ a language over $\mathcal{A}$. Then, a *non-deterministic matrix*, or *Nmatrix*, for $\mathcal{L}_\mathcal{A}$ is a triple $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$, where

(i) $\mathcal{V}$ is a non-empty set whose elements are called *truth values*,

(ii) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated truth values*, and

(iii) $\mathcal{O}$ includes, for every $n$-ary connective $\circ \in CON$, an $n$-ary semantic function $def_\circ : \mathcal{V}^n \to \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}$.

Nmatrices are a direct generalisation of the matrices used to specify standard many-valued logics, for example as discussed by Urquhart [53] or Malinowski [31, 32]. The only difference lies in the interpretation function of the connectives: their range is extended to all non-empty subsets of truth values instead of only a single value. Note that we will often not mention the specific language of an Nmatrix whenever it is clear from the context.

A connective $\circ$ is *deterministic* in an Nmatrix if the range of its semantic function $def_\circ$ only contains singleton sets. An Nmatrix for which all connectives are deterministic will be called a *deterministic* Nmatrix. Clearly, deterministic Nmatrices are in one-to-one correspondence with standard matrices. If the set of truth values of an Nmatrix is finite, we will speak of a *finite* Nmatrix.

Note that we implicitly assume that the truth values in an Nmatrix have an underlying ordering. In the case of finite Nmatrices, we denote the $i$-th truth value (with respect to the implicit ordering) by $\mathsf{v}_i$.

Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix. A *dynamic valuation*, or simply a *valuation*, *in* $\mathcal{M}$ is a function $I : FORM \to \mathcal{V}$ such that, for each $n$-ary connective $\circ \in CON$, the following holds, for all $A_1, \ldots, A_n \in FORM$:

$(SCL)$ $I(\circ(A_1,\ldots,A_n)) \in \mathit{def}_\circ(I(A_1),\ldots,I(A_n))$.

Furthermore, a *static valuation* in $\mathcal{M}$ is a function $I : \mathit{FORM} \to \mathcal{V}$ which satisfies condition $(SCL)$ together with the following *compositionality principle*: for each $\circ \in \mathit{CON}$ and for every $A_1,\ldots,A_n,B_1,\ldots,B_n \in \mathit{FORM}$,

$(CMP)$ if $I(A_i) = I(B_i)$, for every $i \in \{1,\ldots,n\}$, then

$$I(\circ(A_1,\ldots,A_n)) = I(\circ(B_1,\ldots,B_n)).$$

As we can see, the condition of truth functionality of valuations is replaced by the similar but weaker condition $(SCL)$. Dynamic valuations correspond to the intuition of making a non-deterministic choice in the evaluation of a formula at the lowest possible point of computation. On the other hand, if we consider static valuations, the additional condition $(CMP)$ implies that the value of a formula is indeed uniquely determined by the value of its immediate subformulas. This corresponds to the intuition that all non-deterministic choices are made *ahead* of the evaluation of the formula. We will later see that each static valuation in an Nmatrix is closely related to a deterministic version of that Nmatrix.

Let $I$ be a valuation in $\mathcal{M}$, $A \in \mathit{FORM}$, and $\Gamma \subseteq \mathit{FORM}$. We say that $I$ *satisfies* $A$, in symbols $I \models A$, if $I(A) \in \mathcal{D}$, i.e., $I(A)$ is a designated truth value. Moreover, $I$ is a *model* of $\Gamma$, in symbols $I \models \Gamma$, if it satisfies all formulas in $\Gamma$. If $I$ is additionally a static valuation, then $I$ is a *static model* of $\Gamma$. We call $A$ (*dynamically*) *valid* in $\mathcal{M}$, in symbols $\models_\mathcal{M} A$, if $I \models A$ for each valuation $I$ in $\mathcal{M}$. As well, $A$ is *statically valid* in $\mathcal{M}$, in symbols $\models^s_\mathcal{M} A$, if $I \models A$ for each static valuation $I$ in $\mathcal{M}$. Furthermore, given some $\Delta \subseteq \mathit{FORM}$, $\Gamma$ (*dynamically*) *entails* $\Delta$ *in* $\mathcal{M}$, or $\Delta$ (*dynamically*) *follows from* $\Gamma$ *in* $\mathcal{M}$, in symbols $\Gamma \models_\mathcal{M} \Delta$, if, for every model $I$ of $\Gamma$ in $\mathcal{M}$, we have $I \models B$ for some $B \in \Delta$. Likewise, $\Gamma$ *statically entails* $\Delta$ *in* $\mathcal{M}$, or $\Delta$ *statically follows from* $\Gamma$ *in* $\mathcal{M}$, in symbols $\Gamma \models^s_\mathcal{M} \Delta$, if, for every static model $I$ of $\Gamma$ in $\mathcal{M}$, we have $I \models B$ for some $B \in \Delta$.

Note that we will usually omit the subscript of the above entailment relations if the Nmatrix in question is clear from the context.

The following two observations are immediate from the above definitions:

(i) for every Nmatrix $\mathcal{M}$ it holds that $\models_\mathcal{M} \subseteq \models^s_\mathcal{M}$, and

(ii) for every deterministic Nmatrix $\mathcal{M}$ it holds that $\models_\mathcal{M} = \models^s_\mathcal{M}$.

For the purpose of this thesis, we will identify *logics* with *proof systems*, i.e., sets of rules and axioms in a fixed propositional language. With this in mind, we come to the following definition: Let $\mathcal{M}$ be an Nmatrix and $\mathcal{L}$ a logic induced by a proof system $\vdash_\mathcal{L}$. Then, $\mathcal{M}$ is *characteristic* for $\mathcal{L}$ iff $\models_\mathcal{M} = \vdash_\mathcal{L}$. Furthermore, $\mathcal{M}$ is *statically characteristic* for $\mathcal{L}$ iff $\models^s_\mathcal{M} = \vdash_\mathcal{L}$.

A very important property of logics that have a characteristic finite Nmatrix is the following result due to Avron and Lev [7]:

**Proposition 2.1.** A logic which has a finite characteristic Nmatrix is decidable.

## 2.2 Paraconsistent Logics and Nmatrices

We next introduce some paraconsistent logics which can be elegantly characterised in terms of Nmatrices and which we will use to construct concrete instantiations of our general framework, i.e., providing specific rejection calculi for these logics.

### 2.2.1 The Logics CLuN and CAR

The logic **CLuN**, introduced by Batens, De Clercq, and Kurtonina [10], is a weakening of classical propositional logic dropping the consistency requirement of the negation, i.e., if a formula evaluates to true, then it is not necessary that its negation evaluates to false.

*Carnot's logic*, or **CAR** for short, has been formally introduced by da Costa and Béziau [21]. Motivation for their investigations were texts written by Lazare Carnot, a French 18th century mathematician and scientist, suggesting a logic in which the law of double negation does not hold, but all other properties of classical negation are preserved.

Syntactically, both **CAR** and **CLuN** use a language which is formed over an alphabet using the connectives $\{\wedge, \vee, \supset, \sim\}$. In the following, let $\mathbf{HCL}^+$ be a Hilbert-style axiomatisation of the positive fragment of classical propositional logic.

A Hilbert-style axiomatisation of **CLuN** is obtained by adding the axiom

$$(A \supset \sim A) \supset \sim A, \tag{2.1}$$

to $\mathbf{HCL}^+$, whilst an axiomatisation of **CAR** is obtained by adding

$$(A \supset B) \supset ((A \supset \sim B) \supset \sim A). \tag{2.2}$$

Note that by instantiating the formula $B$ with $A$ in (2.2) we obtain a formula equivalent to (2.1). Therefore, if a formula is valid in **CLuN**, then also in **CAR**.

Now let $\mathcal{M}_1 = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix with $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}\}$, along with the semantic definitions of the connectives in $\mathcal{O}$ as given by the truth tables depicted in Figure 2.1. Then, as shown by Avron and Konikowska [5], the following holds:

**Proposition 2.2.** The Nmatrix $\mathcal{M}_1$ is characteristic for **CLuN** and statically characteristic for **CAR**.

**Example 2.1.** Let $I$ be any valuation in $\mathcal{M}_1$ and consider the formula $\sim A \vee A$. There are two possibilities:

(i) if $I(A) = \mathbf{t}$, then $I(\sim A \vee A) = \mathbf{t}$, and

| $\sim$ | |
|---|---|
| $\mathbf{f}$ | $\mathbf{t}$ |
| $\mathbf{t}$ | $\{\mathbf{f}, \mathbf{t}\}$ |

| $\vee$ | $\mathbf{f}$ | $\mathbf{t}$ |
|---|---|---|
| $\mathbf{f}$ | $\mathbf{f}$ | $\mathbf{t}$ |
| $\mathbf{t}$ | $\mathbf{t}$ | $\mathbf{t}$ |

| $\wedge$ | $\mathbf{f}$ | $\mathbf{t}$ |
|---|---|---|
| $\mathbf{f}$ | $\mathbf{f}$ | $\mathbf{f}$ |
| $\mathbf{t}$ | $\mathbf{f}$ | $\mathbf{t}$ |

| $\supset$ | $\mathbf{f}$ | $\mathbf{t}$ |
|---|---|---|
| $\mathbf{f}$ | $\mathbf{t}$ | $\mathbf{t}$ |
| $\mathbf{t}$ | $\mathbf{f}$ | $\mathbf{t}$ |

Figure 2.1: Truth tables for the connectives of $\mathcal{M}_1$.

(ii) if $I(A) = \mathbf{f}$, then $I({\sim}A) = \mathbf{t}$, and thus $I({\sim}A \vee A) = \mathbf{t}$.

Hence, since $I$ was chosen arbitrarily, ${\sim}A \vee A$ is valid in $\mathcal{M}_1$. △

**Example 2.2.** Consider now the formula $A \supset {\sim}{\sim}A$, where $A$ is an atom, and define the valuation $I$ with $I(A) = \mathbf{t}$, $I({\sim}A) = \mathbf{t}$, and $I({\sim}{\sim}A) = \mathbf{f}$. Then, clearly $I(A \supset {\sim}{\sim}A) = \mathbf{f}$. Thus, $A \supset {\sim}{\sim}A$ is not dynamically valid. On the other hand, assuming $A$ now being an arbitrary formula, let $I$ be any static valuation in $\mathcal{M}_1$ with $I(A) = \mathbf{t}$ (if $I(A) = \mathbf{f}$, the implication is trivially true). We distinguish two cases:

(i) if $I({\sim}A) = \mathbf{t}$, then, since $I$ is a static valuation, $I({\sim}{\sim}A) = \mathbf{t}$, and

(ii) if $I({\sim}A) = \mathbf{f}$, then $I({\sim}{\sim}A) = \mathbf{t}$.

Since $I$ was an arbitrarily chosen static valuation in $\mathcal{M}_1$, it holds that $A \supset {\sim}{\sim}A$ is statically valid in $\mathcal{M}_1$. △

### 2.2.2 The Logic Cio

The logic **Cio**, first studied by Carnielli and Marcos [18], is one of the many logics of formal inconsistency (LFI) [17], as championed by da Costa [19]. The idea behind LFIs is to separate the notions of contradictoriness and inconsistency by introducing new operators in the language.

More specifically, **Cio** is part of the so-called **C**-Systems [20], a subclass of LFIs based on a positive fragment of a consistent logic. In our case, this will again be the positive fragment of classical propositional logic, axiomatised by the Hilbert system $\mathbf{HCL}^+$. **Cio** has a special unary logical connective, $\sharp$, where the intended meaning of a formula $\sharp A$ is that $A$ is consistent.

Formally, the language of **Cio** is defined over an alphabet using the connectives

$$\{\wedge, \vee, \supset, \neg, \sharp\},$$

where $\sharp$ and $\neg$ are unary connectives and the others binary. We get a Hilbert-style axiomatisation of **Cio** by adding the following axioms to $\mathbf{HCL}^+$:

$$\neg A \vee A, \tag{$\mathbf{t}$}$$

$$\sharp A \supset (A \wedge \neg A \supset B), \tag{$\mathbf{p}$}$$

$$\neg\neg A \supset A, \tag{$\mathbf{c}$}$$

| $\sharp$ | |
|---|---|
| **f** | **t** |
| **i** | **f** |
| **t** | **t** |

| $\neg$ | |
|---|---|
| **f** | **t** |
| **i** | $\mathcal{D}$ |
| **t** | **f** |

| $\wedge$ | **f** | **i** | **t** |
|---|---|---|---|
| **f** | **f** | **f** | **f** |
| **i** | **f** | $\mathcal{D}$ | **t** |
| **t** | **f** | **t** | **t** |

| $\vee$ | **f** | **i** | **t** |
|---|---|---|---|
| **f** | **f** | **t** | **t** |
| **i** | **t** | $\mathcal{D}$ | **t** |
| **t** | **t** | **t** | **t** |

| $\supset$ | **f** | **i** | **t** |
|---|---|---|---|
| **f** | **t** | **t** | **t** |
| **i** | **f** | $\mathcal{D}$ | **t** |
| **t** | **f** | **t** | **t** |

Figure 2.2: Truth tables for the connectives of $\mathcal{M}_2$.

$$\neg\sharp A \supset A \wedge \neg A, \tag{\textbf{i}}$$

$$(\sharp A \vee \sharp B) \supset (\sharp(A \wedge B) \wedge \sharp(A \vee B) \wedge \sharp(A \supset B)). \tag{\textbf{o}}$$

Avron [2] introduced the following Nmatrix for the semantics of **Cio**: define $\mathcal{M}_2 = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ by setting $\mathcal{V} = \{\mathbf{f}, \mathbf{i}, \mathbf{t}\}$, $\mathcal{D} = \{\mathbf{i}, \mathbf{t}\}$, and $\mathcal{O}$ as given by the truth tables in Figure 2.2. Moreover, he showed the following result:

**Proposition 2.3.** The dynamic semantics of $\mathcal{M}_2$ are characteristic for **Cio**.

**Example 2.3.** The general *principle of explosion*, i.e., the validity of $\bot \supset B$ for every formula $B$ (or, equivalently, the validity of $A \wedge \neg A \supset B$) is not valid in **Cio**. Just consider formulas $A$ and $B$ together with a valuation $I$ such that $I(B) = \mathbf{f}$, $I(A) = \mathbf{i}$, $I(\neg A) = \mathbf{i}$, and $I(A \wedge \neg A) = \mathbf{t}$. Then, $I$ is a dynamic valuation and $I(A \wedge \neg A \supset B) = \mathbf{f}$. △

CHAPTER 3

# Preparatory Concepts

In this section, we generalise the concept of a *partial normal form*, as introduced by Rosser [43], as a device to encode the semantic functions of classical matrices into two-valued logic. Afterwards, we introduce *many-sided sequents* and *many-sided anti-sequents*, following Rousseau [44] and Bogojeski and Tompits [13], respectively, constituting the underlying syntactic elements of our subsequent construction methods of proof calculi for Nmatrices.

## 3.1 Generalised Partial Normal Forms

The basic idea of an *i-th partial normal form* for a connective $\circ$ is to encode the situations in which the truth function for $\circ$ takes the $i$-th truth value. This idea has been adapted by Bogojeski and Tompits [13] to what they call *complementary partial normal forms*. Where an $i$-th complementary partial normal form for a connective $\circ$ encodes the situations in which the truth function for $\circ$ does not take the $i$-th truth value.

It is suggestive to generalise the idea of a (complementary) partial normal form to any subset of truth values of a given finite Nmatrix. In this sense, a *generalised partial normal form* for a connective $\circ$ with respect to a set $S$ of truth values will provide an encoding of the situations where the truth function for $\circ$ takes any of the elements in $S$, whereas the respective *generalised complementary partial normal form* describes the situations where the truth function for $\circ$ does not take any of the elements in $S$.

Following the method of Zach [54] for systematically generating assertional sequent-style calculi for finite-valued logics and the method of Bogojeski and Tompits [13], which is an adaption of Zach's approach to the rejection case, we introduce the concepts of *signed formula expressions* and establish some basic properties of them which will be elemental for our elaborations in Chapters 4 and 5.

### 3.1.1 Signed Formula Expressions

The idea behind signed formula expressions is to *sign* each formula of a many-valued logic with truth values. These new expressions will be considered as the atoms of a new propositional language. Their intended meaning is that a formula $A$ signed with a truth value $v$ will be true under a valuation iff $A$ takes the value $v$ under the valuation. This means that the many-valued semantics are reduced to a two-valued semantics.

**Definition 3.1.** Let $\mathcal{L}_\mathcal{A} = (\mathcal{A}, FORM)$ be a propositional language over some alphabet $\mathcal{A}$ and $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ a finite Nmatrix for $\mathcal{L}_\mathcal{A}$. Then, the set $EXP_\mathcal{M}$ of *signed formula expressions* (SFEs) *in* $\mathcal{M}$ is the smallest set such that

(i) $\dot{\top}, \dot{\bot} \in EXP_\mathcal{M}$,

(ii) $\{A^w \mid A \in FORM, w \in \mathcal{V}\} \subseteq EXP_\mathcal{M}$,

(iii) if $A \in EXP_\mathcal{M}$, then $\dot{\neg} A \in EXP_\mathcal{M}$,

(iv) if $A_1, A_2 \in EXP_\mathcal{M}$, then $A_1 \mathbin{\dot{\wedge}} A_2 \in EXP_\mathcal{M}$, and

(v) if $A_1, A_2 \in EXP_\mathcal{M}$, then $A_1 \mathbin{\dot{\vee}} A_2 \in EXP_\mathcal{M}$.

□

A signed formula expression of the form $A^w$ is called a *signed atom*, whilst a *signed literal* is a signed atom possibly preceded by the operator $\dot{\neg}$. An SFE $F$ is called *positive* if all of its signed literals have the form $A^w$, and $F$ is *negative* if every signed literal in $F$ has the form $\dot{\neg} A^w$.

We also make use of the following notational conventions: If $F_1, \ldots, F_n$ are SFEs, then $\dot{\bigvee}_{i=1}^n F_i$ is defined as the SFE

$$F_1 \mathbin{\dot{\vee}} (F_2 \mathbin{\dot{\vee}} \ldots \, (F_{n-1} \mathbin{\dot{\vee}} F_n) \ldots)$$

and $\dot{\bigwedge}_{i=1}^n F_i$ as the SFE

$$F_1 \mathbin{\dot{\wedge}} (F_2 \mathbin{\dot{\wedge}} \ldots (F_{n-1} \mathbin{\dot{\wedge}} F_n) \ldots).$$

If $n = 0$, then $\dot{\bigvee}_{i=1}^n F_i := \dot{\bot}$, called the *empty disjunction*, and $\dot{\bigwedge}_{i=1}^n F_i := \dot{\top}$, called the *empty conjunction*. Additionally, for a formula $A$ and a set $W \subseteq \mathcal{V}$ of truth values, we use $A^W$ to denote $\dot{\bigwedge}_{w \in W} A^w$. Furthermore, for a truth value $v \in \mathcal{V}$, we define $\overline{v}$ as $\mathcal{V} \setminus \{v\}$.

**Definition 3.2.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix for a language $\mathcal{L}_\mathcal{A} = (\mathcal{A}, FORM)$. Then, the *satisfaction relation*, $\models_{SFE}^{\mathcal{M}}$, between an valuation $I$ and a signed formula expression $F$ relative to $\mathcal{M}$ is inductively defined as follows:

(i) if $F = \dot{\top}$, then $I \models_{SFE}^{\mathcal{M}} F$,

(ii) if $F = \dot{\bot}$, then $I \not\models_{SFE}^{\mathcal{M}} F$,

(iii) if $F = A^w$ with $A \in FORM$ and $w \in \mathcal{V}$, then $I \models^{\mathcal{M}}_{SFE} F$ iff $I(A) = w$,

(iv) if $F = \dot{\neg} F'$, then $I \models^{\mathcal{M}}_{SFE} F$ iff $I \not\models^{\mathcal{M}}_{SFE} F'$,

(v) if $F = F_1 \dot{\wedge} F_2$, then $I \models^{\mathcal{M}}_{SFE} F$ iff $I \models^{\mathcal{M}}_{SFE} F_1$ and $I \models^{\mathcal{M}}_{SFE} F_2$, and

(vi) if $F = F_1 \dot{\vee} F_2$, then $I \models^{\mathcal{M}}_{SFE} F$ iff $I \models^{\mathcal{M}}_{SFE} F_1$ or $I \models^{\mathcal{M}}_{SFE} F_2$.

Whenever $I \models^{\mathcal{M}}_{SFE} F$ holds, we say that $I$ *satisfies* $F$. An *SFE* $F$ is called *satisfiable in* $\mathcal{M}$ iff there is some valuation $I$ in $\mathcal{M}$ such that $I \models^{\mathcal{M}}_{SFE} F$, and $F$ is a *tautology* if $I \models^{\mathcal{M}}_{SFE} F$, for every valuation $I$. Finally, we write $F_1 \equiv^{\mathcal{M}}_{SFE} F_2$ if, for every valuation $I$ in $\mathcal{M}$, $I \models^{\mathcal{M}}_{SFE} F_1$ precisely when $I \models^{\mathcal{M}}_{SFE} F_2$. □

Whenever an Nmatrix $\mathcal{M}$ is clear from the context, we will omit the superscript in the satisfaction relation. Note that, due to the close correspondence of signed formula expressions and boolean propositional logic, the associative, commutative, distributive, and de Morgan's laws clearly hold for SFEs.

The following lemma will prove useful in the next section. It corresponds to the intuition that whenever a formula takes a certain value under a valuation, then this formula cannot take any other value under the same valuation.

**Lemma 3.1.** Let $\mathcal{M}$ be a finite Nmatrix and $w_1, \ldots, w_n \in \mathcal{V}$. Then,

$$\dot{\bigwedge}_{i=1}^{n} \dot{\neg} A_i^{\overline{w}_i} \equiv_{SFE} \dot{\bigwedge}_{i=1}^{n} A_i^{w_i}.$$

*Proof.* It is clearly sufficient to show $\dot{\neg} A_i^{\overline{w}_i} \equiv_{SFE} A_i^{w_i}$, for all $i \in \{1, \ldots, n\}$. So, let $I$ be a valuation in $\mathcal{M}$. Then, $I \models_{SFE} \dot{\neg} A_i^{\overline{w}_i}$ iff

$$I \models_{SFE} \dot{\neg} A_i^{v}, \text{ for every } v \in \mathcal{V} \setminus \{w_i\}, \tag{3.1}$$

by definition of $\overline{w}_i$. But (3.1) is equivalent to

$$I(A_i) \neq v, \text{ for all } v \in \mathcal{V} \setminus \{w_i\}, \tag{3.2}$$

by definition of $\models_{SFE}$. Now, (3.2) holds exactly when $I(A_i) = w_i$ is the case, which in turn is equivalent to $I \models_{SFE} A_i^{w_i}$, again by definition of $\models_{SFE}$. ■

### 3.1.2 Definition and Basic Properties of Generalised Partial Normal Forms

**Definition 3.3.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be a finite Nmatrix, $\circ(A_1, \ldots, A_n) \in FORM$, $S$ a non-empty strict subset of $\mathcal{V}$, and

$$W_S^{\circ} = \{(w_1, \ldots, w_n) \mid \text{ there is some } x \in def_{\circ}(w_1, \ldots, w_n) \text{ such that } x \notin S\}.$$

(i) The *generalised partial normal form for* $\circ(A_1, \ldots, A_n)$ *with respect to* $S$, denoted by $gpnf_S[\circ(A_1, \ldots, A_n)]$, is defined as

$$gpnf_S[\circ(A_1, \ldots, A_n)] = \dot{\bigwedge}_{(w_1,\ldots,w_n) \in W_S^\circ} \left( \dot{\bigvee}_{i=1}^{n} \dot{\bigvee}_{w \in \overline{w_i}} A_i^w \right).$$

(ii) The *generalised complementary partial normal form for* $\circ(A_1, \ldots, A_n)$ *with respect to* $S$, denoted by $gcpnf_S[\circ(A_1, \ldots, A_n)]$, is defined as

$$gcpnf_S[\circ(A_1, \ldots, A_n)] = \dot{\bigvee}_{(w_1,\ldots,w_n) \in W_S^\circ} \left( \dot{\bigwedge}_{i=1}^{n} \dot{\bigwedge}_{w \in \overline{w}_i} \dot{\neg} A_i^w \right).$$

□

If $S = \{w\}$ is a singleton set, then we may also simply write $gpnf_w$ and $gcpnf_w$ instead of $gpnf_{\{w\}}$ and $gcpnf_{\{w\}}$ respectively.

Note that the generalised partial normal form only contains signed atoms and is in conjunctive normal form while the generalised complementary partial normal form is always negative and in disjunctive normal form.

As an immediate consequence of De Morgan's laws, we have the following lemma:

**Lemma 3.2.** Let $\mathcal{M}$ be a finite Nmatrix, $\circ(A_1, \ldots, A_n) \in FORM$, and $S$ a non-empty strict subset of $\mathcal{V}$. Then,

$$gpnf_S[\circ(A_1, \ldots, A_n)] \equiv_{SFE} \dot{\neg} gcpnf_S[\circ(A_1, \ldots, A_n)].$$

This lemma shows a natural symmetry between partial normal forms and their complementary counterpart. We can exploit this fact and only establish properties of the generalised complementary partial normal forms and transfer these properties with minor modifications to the generalised partial normal forms. The reason we chose to do it this way is that, in our opinion, the proofs in the complementary case are more natural.

The following theorems express the central properties of our normal forms:

**Theorem 3.1.** Let $\mathcal{M}$ be a finite Nmatrix, $S$ a non-empty strict subset of $\mathcal{V}$, and $I$ a valuation in $\mathcal{M}$. Then, the following two statements hold:

1. If $I \models_{SFE} gcpnf_S[\circ(A_1, \ldots, A_n)]$, then there exists some $x \notin S$ such that the dynamic valuation

$$I'(\varphi) := \begin{cases} x, & \text{if } \varphi = \circ(A_1, \ldots, A_n), \\ I(\varphi), & \text{otherwise} \end{cases}$$

   is well-defined and satisfies $I' \models_{SFE} \dot{\neg}(\circ(A_1, \ldots, A_n))^S$.

2. If $I \models_{SFE} \dot{\neg}(\circ(A_1, \ldots, A_n))^S$, then $I \models_{SFE} gcpnf_S[\circ(A_1, \ldots, A_n)]$.

*Proof.* For the proof of the first part of the theorem, suppose

$$I \models_{SFE} gcpnf_S[\circ(A_1,\ldots,A_n)].$$

Then, there is some $(w_1,\ldots,w_n) \in W_S$ such that $I \models_{SFE} \dot{\bigwedge}_{i=1}^n \dot{\neg} A_i^{\overline{w_i}}$. This means, however, that there is some $x \in def_\circ(w_1,\ldots,w_n)$ with $x \notin S$. Furthermore, by Lemma 3.1, we get $I \models_{SFE} \dot{\bigwedge}_{i=1}^n A_i^{w_i}$, which in turn means that $I(A_i) = w_i$, for all $i \in \{1,\ldots,n\}$. Now, defining $I'$ as above, we can see that

$$I'(\circ(A_1,\ldots,A_n)) = x \in def_\circ(w_1,\ldots,w_n) = def_\circ(I'(A_1),\ldots,I'(A_n)).$$

Therefore, $I'$ is a well-defined dynamic valuation in $\mathcal{M}$. Moreover, it clearly holds that $I' \models_{SFE} \dot{\neg}(\circ(A_1,\ldots,A_n))^S$.

Let us now turn to the second part of the theorem. Suppose

$$I \models_{SFE} \dot{\neg}(\circ(A_1,\ldots,A_n))^S.$$

Furthermore, let

$$x := I(\circ(A_1,\ldots,A_n)) \in def_\circ(I(A_1),\ldots,I(A_n))$$

and $w_i := I(A_i)$, for all $i \in \{1,\ldots,n\}$. Then, $x \notin S$ (otherwise it would be a contradiction to the assumption) and therefore $(w_1,\ldots,w_n) \in W_S$. By our definitions it is also clear that $I \models_{SFE} \dot{\bigwedge}_{i=1}^n A_i^{w_i}$ and thus, by Lemma 3.1, we have that $I \models_{SFE} \dot{\bigwedge}_{i=1}^n \dot{\neg} A_i^{\overline{w_i}}$. Since this is one of the disjuncts in $gcpnf_S[\circ(A_1,\ldots,A_n)]$, we can conclude that $I \models_{SFE} gcpnf_S[\circ(A_1,\ldots,A_n)]$ must hold. ■

**Theorem 3.2.** Let $\mathcal{M}$ be a finite Nmatrix, $S$ a non-empty strict subset of $\mathcal{V}$, and $I$ a valuation in $\mathcal{M}$. Then, the following two statements hold:

1. If $I \not\models_{SFE} gpnf_S[\circ(A_1,\ldots,A_n)]$, then there exists some $x \notin S$ so that the dynamic valuation
$$I'(\varphi) := \begin{cases} x, & \text{if } \varphi = \circ(A_1,\ldots,A_n) \\ I(\varphi), & \text{otherwise} \end{cases}$$
is well-defined and satisfies $I' \not\models_{SFE} \circ(A_1,\ldots,A_n)^S$.

2. If $I \models_{SFE} gpnf_S[\circ(A_1,\ldots,A_n)]$, then $I \models_{SFE} \circ(A_1,\ldots,A_n)^S$.

*Proof.* Follows immediately from Theorem 3.1 and Lemma 3.2. ■

An immediate consequence of the above theorems are the following special cases:

**Corollary 3.1.** Let $\mathcal{M}$ be a finite Nmatrix, $S = \{v_i\}$ a singleton subset of $\mathcal{V}$, and $\circ$ a deterministic connective in $\mathcal{M}$. Then,

$$gcpnf_S[\circ(A_1,\ldots,A_n)] \equiv_{SFE}^{\mathcal{M}} \dot{\neg}(\circ(A_1,\ldots,A_n))^S.$$

| | |
|---|---|
| $gpnf_{\mathbf{f}}[\sim A] = \dot{\bot}$; | $gcpnf_{\mathbf{f}}[\sim A] \equiv_{SFE} \dot{\top}$; |
| $gpnf_{\mathbf{t}}[\sim A] = A^{\mathbf{f}}$; | $gcpnf_{\mathbf{t}}[\sim A] \equiv_{SFE} \dot{\neg} A^{\mathbf{f}}$; |
| $gpnf_{\mathbf{f}}[A_1 \vee A_2] \equiv_{SFE} A_1^{\mathbf{f}} \dot{\wedge} A_2^{\mathbf{f}}$; | $gcpnf_{\mathbf{f}}[A_1 \vee A_2] \equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \dot{\vee} \dot{\neg} A_2^{\mathbf{f}}$; |
| $gpnf_{\mathbf{t}}[A_1 \vee A_2] \equiv_{SFE} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{t}}$; | $gcpnf_{\mathbf{t}}[A_1 \vee A_2] \equiv_{SFE} \dot{\neg} A_1^{\mathbf{t}} \dot{\wedge} \dot{\neg} A_2^{\mathbf{t}}$; |
| $gpnf_{\mathbf{f}}[A_1 \wedge A_2] \equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{f}}$; | $gcpnf_{\mathbf{f}}[A_1 \wedge A_2] \equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \dot{\wedge} \dot{\neg} A_2^{\mathbf{f}}$; |
| $gpnf_{\mathbf{t}}[A_1 \wedge A_2] \equiv_{SFE} A_1^{\mathbf{t}} \dot{\wedge} A_2^{\mathbf{t}}$; | $gcpnf_{\mathbf{t}}[A_1 \wedge A_2] \equiv_{SFE} \dot{\neg} A_1^{\mathbf{t}} \dot{\vee} \dot{\neg} A_2^{\mathbf{t}}$; |
| $gpnf_{\mathbf{f}}[A_1 \supset A_2] \equiv_{SFE} A_1^{\mathbf{t}} \dot{\wedge} A_2^{\mathbf{f}}$; | $gcpnf_{\mathbf{f}}[A_1 \supset A_2] \equiv_{SFE} \dot{\neg} A_1^{\mathbf{t}} \dot{\vee} \dot{\neg} A_2^{\mathbf{f}}$; |
| $gpnf_{\mathbf{t}}[A_1 \supset A_2] \equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{t}}$. | $gcpnf_{\mathbf{t}}[A_1 \supset A_2] \equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \dot{\wedge} \dot{\neg} A_2^{\mathbf{t}}$. |

Figure 3.1: Minimal SFEs equivalent to the generalised partial normal forms as well as to the generalised complementary partial normal forms for the connectives in $\mathcal{M}_1$.

*Proof.* The result follows from Theorem 3.1 and the fact that for a deterministic connective it must hold that $I = I'$. ■

**Corollary 3.2.** Let $\mathcal{M}$ be a finite Nmatrix, $S = \{v_i\}$ a singleton subset of $\mathcal{V}$, and $\circ$ a deterministic connective in $\mathcal{M}$. Then,

$$gpnf_S[\circ(A_1, \dots, A_n)] \equiv_{SFE}^{\mathcal{M}} \circ(A_1, \dots, A_n)^S.$$

*Proof.* The result is shown similarly as Corollary 3.1, but invoking Theorem 3.2 instead of Theorem 3.1. ■

These corollaries show that the notions of a generalised partial normal form and of a generalised complementary partial normal form indeed properly extend the corresponding definitions of a partial normal form, as introduced by Zach [54], and of a complementary partial normal form, as defined by Bogojeski and Tompits [13], for the deterministic case.

**Example 3.4.** Recall the Nmatrix $\mathcal{M}_1$ defined in Subsection 2.2.1 that can be used to characterise the logics **CLuN** and **CAR** via its dynamic and static semantics, respectively. Figure 3.1 depicts shortest SFEs which are equivalent to the generalised partial normal forms as well as to the generalised complementary partial normal forms for the singleton subsets of the truth values of $\mathcal{M}_1$, i.e., for the sets $\{\mathbf{f}\}$ and $\{\mathbf{t}\}$. △

**Example 3.5.** Consider now the logic **Cio** introduced in Subsection 2.2.2 and its characteristic Nmatrix $\mathcal{M}_2$. There are exactly six non-empty strict subsets of the set $\{\mathbf{f}, \mathbf{i}, \mathbf{t}\}$ of truth values of $\mathcal{M}_2$. This means that, for each connective of **Cio**, there are six generalised complementary partial normal forms. Figures 3.2 and 3.3 depict shortest SFEs which are equivalent to the respective generalised partial normal forms and to the respective generalised complementary partial normal forms. △

$$
\begin{aligned}
gpnf_{\mathbf{f}}[\sharp A] &\equiv_{SFE} A^{\mathbf{i}}; & gpnf_{\mathbf{f}}[\neg A] &\equiv_{SFE} A^{\mathbf{t}};\\
gpnf_{\mathbf{i}}[\sharp A] &\equiv_{SFE} \dot{\bot}; & gpnf_{\mathbf{i}}[\neg A] &\equiv_{SFE} \dot{\bot};\\
gpnf_{\mathbf{t}}[\sharp A] &\equiv_{SFE} A^{\mathbf{f}} \dot{\vee} A^{\mathbf{t}}; & gpnf_{\mathbf{t}}[\neg A] &\equiv_{SFE} A^{\mathbf{f}};\\
gpnf_{\{\mathbf{f},\mathbf{i}\}}[\sharp A] &\equiv_{SFE} A^{\mathbf{i}}; & gpnf_{\{\mathbf{f},\mathbf{i}\}}[\neg A] &\equiv_{SFE} A^{\mathbf{t}};\\
gpnf_{\{\mathbf{f},\mathbf{t}\}}[\sharp A] &\equiv_{SFE} \dot{\top}; & gpnf_{\{\mathbf{f},\mathbf{t}\}}[\neg A] &\equiv_{SFE} A^{\mathbf{f}} \dot{\vee} A^{\mathbf{t}};\\
gpnf_{\{\mathbf{i},\mathbf{t}\}}[\sharp A] &\equiv_{SFE} A^{\mathbf{f}} \dot{\vee} A^{\mathbf{t}}; & gcpnf_{\{\mathbf{i},\mathbf{t}\}}[\neg A] &\equiv_{SFE} A^{\mathbf{f}} \dot{\vee} A^{\mathbf{i}};
\end{aligned}
$$

$$
\begin{aligned}
gpnf_{\mathbf{f}}[A_1 \vee A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\wedge} A_2^{\mathbf{f}};\\
gpnf_{\mathbf{i}}[A_1 \vee A_2] &\equiv_{SFE} \dot{\bot};\\
gpnf_{\mathbf{t}}[A_1 \vee A_2] &\equiv_{SFE} (A_1^{\mathbf{i}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{t}}) \dot{\wedge} (A_1^{\mathbf{f}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{t}});\\
gpnf_{\{\mathbf{f},\mathbf{i}\}}[A_1 \vee A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\wedge} A_2^{\mathbf{f}};\\
gpnf_{\{\mathbf{f},\mathbf{t}\}}[A_1 \vee A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{t}};\\
gpnf_{\{\mathbf{i},\mathbf{t}\}}[A_1 \vee A_2] &\equiv_{SFE} A_1^{\mathbf{i}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{i}} \dot{\vee} A_2^{\mathbf{t}};
\end{aligned}
$$

$$
\begin{aligned}
gpnf_{\mathbf{f}}[A_1 \supset A_2] &\equiv_{SFE} (A_1^{\mathbf{i}} \dot{\vee} A_1^{\mathbf{t}}) \dot{\wedge} A_2^{\mathbf{f}};\\
gpnf_{\mathbf{i}}[A_1 \supset A_2] &\equiv_{SFE} \dot{\bot};\\
gpnf_{\mathbf{t}}[A_1 \supset A_2] &\equiv_{SFE} (A_1^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{i}} \dot{\vee} A_2^{\mathbf{t}}) \dot{\wedge} (A_1^{\mathbf{f}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{t}});\\
gpnf_{\{\mathbf{f},\mathbf{i}\}}[A_1 \supset A_2] &\equiv_{SFE} (A_1^{\mathbf{i}} \dot{\vee} A_1^{\mathbf{t}}) \dot{\wedge} A_2^{\mathbf{f}};\\
gpnf_{\{\mathbf{f},\mathbf{t}\}}[A_1 \supset A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{t}};\\
gpnf_{\{\mathbf{i},\mathbf{t}\}}[A_1 \supset A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{i}} \dot{\vee} A_2^{\mathbf{t}};
\end{aligned}
$$

$$
\begin{aligned}
gpnf_{\mathbf{f}}[A_1 \wedge A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{f}};\\
gpnf_{\mathbf{i}}[A_1 \wedge A_2] &\equiv_{SFE} \dot{\bot};\\
gpnf_{\mathbf{t}}[A_1 \wedge A_2] &\equiv_{SFE} (A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{t}}) \dot{\wedge} (A_1^{\mathbf{i}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{i}}) \dot{\wedge} (A_1^{\mathbf{f}} \dot{\vee} A_1^{\mathbf{i}} \dot{\vee} A_2^{\mathbf{i}} \dot{\vee} A_2^{\mathbf{t}});\\
gpnf_{\{\mathbf{f},\mathbf{i}\}}[A_1 \wedge A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{f}};\\
gpnf_{\{\mathbf{f},\mathbf{t}\}}[A_1 \wedge A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \dot{\vee} A_1^{\mathbf{t}} \dot{\vee} A_2^{\mathbf{f}} \dot{\vee} A_2^{\mathbf{t}};\\
gpnf_{\{\mathbf{i},\mathbf{t}\}}[A_1 \wedge A_2] &\equiv_{SFE} (A_1^{\mathbf{i}} \dot{\vee} A_1^{\mathbf{t}}) \dot{\wedge} (A_2^{\mathbf{i}} \dot{\vee} A_2^{\mathbf{t}});
\end{aligned}
$$

Figure 3.2: Minimal SFEs equivalent to the generalised partial normal forms for the connectives of $\mathcal{M}_2$.

## 3.2 Many-Sided Sequents

In this section, we introduce the notions of *many-sided sequents* and *many-sided anti-sequents* that will form the basis of the proof and rejection systems introduced in the following chapters. Many-sided sequents were independently introduced by Schröter [45]

$$
\begin{aligned}
gcpnf_{\mathbf{f}}[\sharp A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{i}}; & gcpnf_{\mathbf{f}}[\neg A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{t}};\\
gcpnf_{\mathbf{i}}[\sharp A] &\equiv_{SFE} \dot{\top}; & gcpnf_{\mathbf{i}}[\neg A] &\equiv_{SFE} \dot{\top};\\
gcpnf_{\mathbf{t}}[\sharp A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A^{\mathbf{t}}; & gcpnf_{\mathbf{t}}[\neg A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{f}};\\
gcpnf_{\{\mathbf{f},\mathbf{i}\}}[\sharp A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{i}}; & gcpnf_{\{\mathbf{f},\mathbf{i}\}}[\neg A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{t}};\\
gcpnf_{\{\mathbf{f},\mathbf{t}\}}[\sharp A] &\equiv_{SFE} \dot{\bot}; & gcpnf_{\{\mathbf{f},\mathbf{t}\}}[\neg A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A^{\mathbf{t}};\\
gcpnf_{\{\mathbf{i},\mathbf{t}\}}[\sharp A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A^{\mathbf{t}}; & gcpnf_{\{\mathbf{i},\mathbf{t}\}}[\neg A] &\equiv_{SFE} \dot{\neg} A^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A^{\mathbf{i}};
\end{aligned}
$$

$$
\begin{aligned}
gcpnf_{\mathbf{f}}[A_1 \vee A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\vee}} \dot{\neg} A_2^{\mathbf{f}};\\
gcpnf_{\mathbf{i}}[A_1 \vee A_2] &\equiv_{SFE} \dot{\top};\\
gcpnf_{\mathbf{t}}[A_1 \vee A_2] &\equiv_{SFE} (\dot{\neg} A_1^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}}) \mathbin{\dot{\vee}} (\dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}});\\
gcpnf_{\{\mathbf{f},\mathbf{i}\}}[A_1 \vee A_2] &\equiv_{SFE} A_1^{\mathbf{f}} \mathbin{\dot{\vee}} \dot{\neg} A_2^{\mathbf{f}};\\
gcpnf_{\{\mathbf{f},\mathbf{t}\}}[A_1 \vee A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}};\\
gcpnf_{\{\mathbf{i},\mathbf{t}\}}[A_1 \vee A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}};
\end{aligned}
$$

$$
\begin{aligned}
gcpnf_{\mathbf{f}}[A_1 \supset A_2] &\equiv_{SFE} (\dot{\neg} A_1^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}}) \mathbin{\dot{\vee}} \dot{\neg} A_2^{\mathbf{f}};\\
gcpnf_{\mathbf{i}}[A_1 \supset A_2] &\equiv_{SFE} \dot{\top};\\
gcpnf_{\mathbf{t}}[A_1 \supset A_2] &\equiv_{SFE} (\dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}}) \mathbin{\dot{\vee}} (\dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}});\\
gcpnf_{\{\mathbf{f},\mathbf{i}\}}[A_1 \supset A_2] &\equiv_{SFE} (\dot{\neg} A_1^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}}) \mathbin{\dot{\vee}} \dot{\neg} A_2^{\mathbf{f}};\\
gcpnf_{\{\mathbf{f},\mathbf{t}\}}[A_1 \supset A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg}; A_2^{\mathbf{t}}\\
gcpnf_{\{\mathbf{i},\mathbf{t}\}}[A_1 \supset A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}};
\end{aligned}
$$

$$
\begin{aligned}
gcpnf_{\mathbf{f}}[A_1 \wedge A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}};\\
gcpnf_{\mathbf{i}}[A_1 \wedge A_2] &\equiv_{SFE} \dot{\top};\\
gcpnf_{\mathbf{t}}[A_1 \wedge A_2] &\equiv_{SFE} (\dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}}) \mathbin{\dot{\vee}} (\dot{\neg} A_1^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{i}})\\
&\qquad \mathbin{\dot{\vee}} (\dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}});\\
gcpnf_{\{\mathbf{f},\mathbf{i}\}}[A_1 \wedge A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}};\\
gcpnf_{\{\mathbf{f},\mathbf{t}\}}[A_1 \wedge A_2] &\equiv_{SFE} \dot{\neg} A_1^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{f}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}};\\
gcpnf_{\{\mathbf{i},\mathbf{t}\}}[A_1 \wedge A_2] &\equiv_{SFE} (\dot{\neg} A_1^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_1^{\mathbf{t}}) \mathbin{\dot{\vee}} (\dot{\neg} A_2^{\mathbf{i}} \mathbin{\dot{\wedge}} \dot{\neg} A_2^{\mathbf{t}});
\end{aligned}
$$

Figure 3.3: Minimal SFEs equivalent to the generalised complementary partial normal forms for the connectives in $\mathcal{M}_2$.

and by Rousseau [44] and are a generalisation of the classical two-sided sequents of Gentzen [24] for many-valued logics. Here, we follow the notation as used by Rousseau.

We will study the close relationship of many-sided sequents and anti-sequents with signed formula expressions of a special form. This will allow us to transform the generalised partial normal forms of the previous chapter into many-sided sequents as well as the generalised complementary partial normal forms into many-sided anti-sequents that will be important in the definition of the proof and rejection systems as discussed later.

### 3.2.1 Syntax of Many-Sided Sequents and Anti-Sequents

**Definition 3.6.** Let $\mathcal{M}$ be an $m$-valued Nmatrix for a language $\mathcal{L}_{\mathcal{A}} = (\mathcal{A}, \mathit{FORM})$. Then,

(i) a *many-sided sequent*, or simply a *sequent*, is an ordered tuple of the form

$$\mathfrak{P} = \Gamma_1 \mid \cdots \mid \Gamma_m,$$

and

(ii) a *many-sided anti-sequent*, or simply an *anti-sequent*, is an ordered tuple of the form

$$\mathfrak{R} = \Gamma_1 \nmid \cdots \nmid \Gamma_m,$$

where each $\Gamma_i$, for $1 \leq i \leq m$, is a finite set of formulas of $\mathit{FORM}$, called the $i$*-th component* of $\mathfrak{P}$ and $\mathfrak{R}$, respectively. □

Intuitively, the $i$-th component of a sequent and of an anti-sequent for an Nmatrix $\mathcal{M}$ corresponds to the $i$-th truth value of $\mathcal{M}$ that is given by the implicit ordering underlying the truth values.

In what follows, we introduce some conventions for dealing with many-sided sequents and anti-sequents more easily.

To begin with, following custom for sequent systems, given a component $\Gamma$ and some finite set $\Delta = \{A_1, \ldots, A_n\}$ of formulas, we may write "$\Gamma, \Delta$" instead of "$\Gamma \cup \Delta$", and "$\Gamma, A_1, \ldots, A_n$" instead of "$\Gamma \cup \{A_1, \ldots, A_n\}$".

Furthermore, by the *empty* $m$*-component many-sided sequent*, $\mathfrak{F}_m$, we understand the sequent $\emptyset \mid \cdots \mid \emptyset$, and the *empty* $m$*-component many-sided anti-sequent*, $\mathfrak{T}_m$, refers to the anti-sequent $\emptyset \nmid \cdots \nmid \emptyset$.

For two $m$-component sequents $\mathfrak{P}_1 = \Gamma_1 \mid \cdots \mid \Gamma_m$ and $\mathfrak{P}_2 = \Delta_1 \mid \cdots \mid \Delta_m$, we define the combination of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ by

$$\mathfrak{P}_1, \mathfrak{P}_2 = \Gamma_1, \Delta_1 \mid \cdots \mid \Gamma_m, \Delta_m.$$

The combination of two anti-sequents is defined similarly.

For a sequent $\mathfrak{P} = \Gamma_1 \mid \cdots \mid \Gamma_m$ and a set $\Delta$ of formulas,

$$\mathfrak{P}, [i : \Delta]$$

denotes the many-sided sequent that has the same components as $\mathfrak{P}$ but additionally contains $\Delta$ in its $i$-th component, i.e.,

$$\mathfrak{P}, [i : \Delta] = \Gamma_1 \mid \cdots \mid \Gamma_i, \Delta \mid \cdots \mid \Gamma_m.$$

This notation can also be applied repeatedly to a sequent in the following manner: Let $\mathfrak{P} = \Gamma_1 \mid \cdots \mid \Gamma_m$, then

$$\mathfrak{P}, [i_1 : \Delta_1], \ldots, [i_n : \Delta_n] = \Gamma_1 \mid \cdots \mid \Gamma_{i_1}, \Delta_1 \mid \cdots \mid \Gamma_{i_n}, \Delta_n \mid \cdots \mid \Gamma_m.$$

Given a sequent $\mathfrak{P}$, a set of formulas $\Delta$, and a set $M \subseteq \{1, \ldots, m\}$, we define

$$\mathfrak{P}, [M : \Delta] = \mathfrak{P}, [i_1 : \Delta], \ldots, [i_n : \Delta],$$

where $M = \{i_1, \ldots, i_n\}$.

Similar definitions apply for anti-sequents as well.

We now define the semantics of many-sided sequents and anti-sequents.

**Definition 3.7.** Let $\mathcal{M}$ be an $m$-valued Nmatrix, $\mathfrak{P} = \Gamma_1 \mid \cdots \mid \Gamma_m$ an $m$-component sequent, and $I$ a valuation in $\mathcal{M}$. We say that $I$ *satisfies* $\mathfrak{P}$ if there is some $i$ $(1 \leq i \leq m)$ and some formula $A \in \Gamma_i$ such that $I(A) = \mathsf{v}_i$, where $\mathsf{v}_i$ denotes the $i$-th truth value of $\mathcal{M}$. In this case, $I$ is said to be a *model* of $\mathfrak{P}$. □

We say that a many-sided sequent $\mathfrak{P}$ is *valid* in $\mathcal{M}$ if every valuation in $\mathcal{M}$ is a model of $\mathfrak{P}$. As well, $\mathfrak{P}$ is *statically valid* in $\mathcal{M}$ if every static valuation in $\mathcal{M}$ is a model of $\mathfrak{P}$.

**Definition 3.8.** Let $\mathcal{M}$ be an $m$-valued Nmatrix, $\mathfrak{R} = \Gamma_1 \nmid \cdots \nmid \Gamma_m$ an $m$-component anti-sequent, and $I$ a valuation in $\mathcal{M}$. We say that $I$ *refutes* $\mathfrak{R}$ if, for every $i$ $(1 \leq i \leq m)$ and every formula $A \in \Gamma_i$, $I(A) \neq \mathsf{v}_i$, where $\mathsf{v}_i$ is the $i$-th truth value of $\mathcal{M}$. If $I$ refutes $\mathfrak{R}$, then $I$ is said to be a *countermodel* of $\mathfrak{R}$. If $I$ is additionally a static valuation, we say it is a *static countermodel*. □

An anti-sequent $\mathfrak{R}$ is *refutable* if it has at least one countermodel and *statically refutable* if it has at least one static countermodel. Furthermore, $\mathfrak{R}$ is *unsatisfiable* if every valuation refutes $\mathfrak{R}$. Note that the empty anti-sequent $\mathfrak{T}_m$ is refuted by every valuation.

We are now able to establish the close relationship between many-sided sequents and anti-sequents.

**Lemma 3.3.** A many-sided sequent $\mathfrak{P} = \Gamma_1 \mid \cdots \mid \Gamma_m$ of an Nmatrix $\mathcal{M}$ is valid iff the corresponding anti-sequent $\mathfrak{R} = \Gamma_1 \nmid \cdots \nmid \Gamma_m$ is not refutable.

*Proof.* Suppose $\mathfrak{P}$ is not valid. Then, there is some interpretation $I$ that does not satisfy $\mathfrak{P}$, i.e., for all $i \in 1, \ldots, n$ and every formula $A \in \Gamma_i$, it holds that $I(A) \neq \mathsf{v}_i$, where $\mathsf{v}_i$ is the $i$-th truth value of $\mathcal{M}$. But this means that $I$ refutes $\mathfrak{R}$. The other direction of the proof follows a similar argument. ■

The above lemma will allow us to establish properties of anti-sequents and transfer them (with minor modifications) to properties of sequents.

### 3.2.2 Properties of Many-Sided Sequents and Anti-Sequents

The following theorems establish the close relationship between many-sided sequents and anti-sequents to signed formula expressions.

**Theorem 3.3.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix and $F$ a negative conjunctive SFE of the form $\dot{\neg} A_1^{w_1} \dot{\wedge} \cdots \dot{\wedge} \dot{\neg} A_n^{w_n}$ in $EXP_{\mathcal{M}}$, where $w_1, \ldots, w_n \in \mathcal{V}$. Furthermore, for each $i \in \{1, \ldots, m\}$, let

$$\Gamma_i = \{A \mid \dot{\neg} A^w \in \mathit{literals}(F), w = \mathsf{v}_i\},$$

where $\mathit{literals}(F)$ denotes the set of all signed literals that are used in $F$.

Then, for any valuation $I$ in $\mathcal{M}$, $I \models_{SFE} F$ iff $I$ refutes $\mathfrak{R} = \Gamma_1 \nmid \cdots \nmid \Gamma_m$.

*Proof.* For the direction from left to right, suppose that $I$ is a valuation in $\mathcal{M}$ with $I \models F$. Hence, $I \models \dot{\neg} A_i^{w_i}$, for every $i \in \{1, \ldots, n\}$. Now let $A$ be an arbitrary formula in some $i$-th component $\Gamma_i$ of $\mathfrak{R}$. By definition, we have $\dot{\neg} A^{\mathsf{v}_i} \in \mathit{literals}(F)$ and thus $I \models \dot{\neg} A^{\mathsf{v}_i}$, which is equivalent to $I(A) \neq \mathsf{v}_i$. Since $A$ was chosen arbitrarily, we conclude that $I$ refutes $\mathfrak{R}$.

For the other direction, suppose that $I$ is a valuation that refutes $\mathfrak{R}$ and let $\dot{\neg} A^w \in \mathit{literals}(F)$. Then, clearly, there is some $i \in \{1, \ldots, m\}$ such that $w = \mathsf{v}_i$, and thus $A \in \Gamma_i$. Since $I$ refutes $\mathfrak{R}$, we have $I(A) \neq \mathsf{v}_i$ which is equivalent to $I \models \dot{\neg} A^w$. Since $\dot{\neg} A^w$ was an arbitrary literal of $F$, given that $F$ is a conjunction of literals, we conclude that $I \models_{SFE} F$. ■

**Theorem 3.4.** Let $F$ be a positive disjunctive SFE of the form $A_1^{w_1} \dot{\vee} \cdots \dot{\vee} A_n^{w_n}$, where $A_1, \ldots, A_n$ are formulas of the $m$-valued Nmatrix $\mathcal{M}$ and $w_1, \ldots, w_n \in \mathcal{V}$. Furthermore, for each $i \in \{1, \ldots, m\}$, let $\Gamma_i = \{A \mid A^w \in \mathit{atoms}(F), w = \mathsf{v}_i\}$, where $\mathit{atoms}(F)$ denotes the set of all signed atoms that are used in $F$.

Then, for any valuation $I$ in $\mathcal{M}$, $I \models_{SFE} F$ iff $I$ satisfies $\mathfrak{P} = \Gamma_1 \mid \cdots \mid \Gamma_m$.

*Proof.* The result is an immediate consequence of Theorem 3.3 and Lemma 3.3, and by observing that the negation of $F$ is equivalent to a negative conjunctive SFE. ■

While two-sided sequents encode entailment directly, this is not the case for many-sided sequents. However, the next theorems show that entailment can nevertheless be encoded into many-sided sequents, by adding the set of conclusions of an entailment relation to the components that correspond to a designated truth value and the set of hypotheses to all other components.

**Theorem 3.5.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix for $\mathcal{L}_{\mathcal{A}} = (\mathcal{A}, FORM)$ and $\Gamma, \Delta \subseteq FORM$ finite sets of formulas of $\mathcal{L}_{\mathcal{A}}$. Additionally, let $M^+ = \{i \mid \mathsf{v}_i \in \mathcal{D}\}$ and $M^- = \{i \mid \mathsf{v}_i \in \mathcal{V} \setminus \mathcal{D}\}$. Then,

$$\Gamma \not\models \Delta \text{ iff the many-sided anti-sequent } \mathfrak{T}_m, [M^- : \Gamma], [M^+ : \Delta] \text{ is refutable.}$$

*Proof.* The result is established in view of the following chain of equivalences:

$$\begin{aligned}
\Gamma \not\models \Delta \text{ iff } & \text{there is some } I \text{ such that } I \models \Gamma \text{ and } I \not\models \psi, \text{ for all } \psi \in \Delta,\\
\text{iff } & \text{there is some } I \text{ such that } I \models \varphi, \text{ for all } \varphi \in \Gamma,\\
& \quad \text{and } I \not\models \psi, \text{ for all } \psi \in \Delta,\\
\text{iff } & \text{there is some } I \text{ such that } I(\varphi) \in \mathcal{D}, \text{ for all } \varphi \in \Gamma,\\
& \quad \text{and } I(\psi) \in \mathcal{V} \setminus \mathcal{D}, \text{ for all } \psi \in \Delta,\\
\text{iff } & \text{there is some } I \text{ which refutes both } \mathfrak{T}_m, [M^- : \Gamma] \text{ and } \mathfrak{T}_m, [M^+ : \Delta],\\
\text{iff } & \mathfrak{T}_m, [M^- : \Gamma], [M^+ : \Delta] \text{ is refutable.}
\end{aligned}$$

■

**Theorem 3.6.** Under the circumstances of Theorem 3.5, it holds that

$$\Gamma \models \Delta \text{ iff the many-sided anti-sequent } \mathfrak{F}_m, [M^- : \Gamma], [M^+ : \Delta] \text{ is valid.}$$

*Proof.* This is an immediate consequence of Theorem 3.5 and Lemma 3.3. ■

CHAPTER 4

# Generating Sequent Calculi for Non-Deterministic Matrices

In this chapter, we introduce a systematic method to generate many-sided proof systems for any given finite Nmatrix. While Avron and Konikowska [5] already provided such a method based on Rasiowa-Sikorski systems [41], the procedure described here generalises the approach of Zach [54] and, together with the rejection systems discussed in Chapter 5, will provide a more uniform account of axiomatising validity and non-validity for Nmatrices.

As in general the static and dynamic semantics of Nmatrices differ from each other, we also discuss a way to adapt our original method for the static case to generate many-sided anti-sequent calculi that are sound and complete with respect to the static semantics. Furthermore, we apply our method to obtain specific calculi for the paraconsistent logics **CLuN**, **Cio**, and **CAR**.

## 4.1 Sequent Calculi for the Dynamic Semantics

### 4.1.1 Postulates of the Calculi

**Definition 4.1.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix and $S$ a non-empty strict subset of $\mathcal{V}$. Furthermore, let the generalised partial normal form for $\circ(A_1, \ldots, A_n)$ with respect to $S$ be of the form

$$gpnf_S[\circ(A_1, \ldots A_n)] = \dot{\bigwedge}_{j=1}^{k_i} \delta^j_{\circ:S}(A_1, \ldots, A_n),$$

where $k_i$ is the number of conjuncts in the generalised partial normal form and the SFE $\delta^j_{\circ:S}(A_1, \ldots, A_n)$ is the $j$-th conjunct of it, which is a disjunction of signed atoms. Moreover, let $\hat{\delta}^j_{\circ:S}(A_1, \dots A_n)$ be the sequent constructed from $\delta^j_{\circ:S}(A_1, \dots A_n)$ as described in Theorem 3.4.

Then, the inference rule $(\circ : S)$ is given as follows:

$$\frac{\mathfrak{P}, \hat{\delta}^1_{\circ:S}(A_1, \dots A_n) \qquad \cdots \qquad \mathfrak{P}, \hat{\delta}^{k_i}_{\circ:S}(A_1, \dots A_n)}{\mathfrak{P}, [S : \circ(A_1, \dots, A_n)]} \ (\circ : S),$$

where $\mathfrak{P} = \Gamma_1 \mid \cdots \mid \Gamma_m$ is an arbitrary $m$-component sequent subject to the following proviso:

$(RES)_S$ For all $i \in \{j \mid \mathsf{v}_j \in S\}$, $\circ(A_1, \dots, A_n) \notin \Gamma_i$.

□

Some remarks are in order:

1. It is important to note that one can use *any* positive SFE in conjunctive normal form equivalent to the generalised partial normal form to introduce the above rules. It is therefore beneficial to minimise the normal form with a method like the well-known Quine-McCluskey procedure [39, 40, 34].
2. Since $\dot{\top}$ is a positive SFE in conjunctive normal form (as it is an empty conjunction), it can happen that it occurs as minimal generalised partial normal form. In this case, according to the above definition, a rule without premisses will be introduced.
3. We can see that the above definition is a very natural generalisation of the rules defined by Zach [54]. The crucial difference is the proviso $(RES)_S$, which plays an important role in the soundness and completeness of the calculi.
4. In case the generalised partial normal form is equivalent to $\dot{\bot}$, no rule has to be introduced.

**Definition 4.2.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix with $\mathcal{V} = \{\mathsf{v}_1, \dots, \mathsf{v}_m\}$ being its set of truth values. Then, the many-sided proof system $\mathsf{MSP}_\mathcal{M}$ consists of

(i) axiom schemata of the form

$$\mathfrak{F}_m, [\{1, \dots, m\} : A],$$

where $\mathfrak{F}_m$ is the empty $m$-component many-sided sequent and $A$ is any SFE;

(ii) the inference rules $(\circ : S)$ as given in Definition 4.1, for every connective $\circ$ of $\mathcal{M}$ and every non-empty strict subset $S$ of $\mathcal{V}$; and

(iii) weakening rules for every $i$-th component of the form:

$$\frac{\mathfrak{P}}{\mathfrak{P}, [i : A]} (w : i) .$$

□

As usual, a *proof* in $\mathsf{MSP}_{\mathcal{M}}$ is a finite tree whose nodes are sequents and such that the top-most sequents are axioms of $\mathsf{MSP}_{\mathcal{M}}$ and the parent nodes of each node $\mathfrak{P}$ are the premisses of some inference rule of $\mathsf{MSP}_{\mathcal{M}}$ whose conclusion is $\mathfrak{P}$. We refer to a proof also as a proof *of* its root element. A sequent $\mathfrak{P}$ is *provable* in $\mathsf{MSP}_{\mathcal{M}}$ if there is a proof of $\mathfrak{P}$ in $\mathsf{MSP}_{\mathcal{M}}$.

### 4.1.2 Adequacy of the Calculi

**Theorem 4.1.** If a sequent is provable in $\mathsf{MSP}_{\mathcal{M}}$, then it is valid in $\mathcal{M}$.

*Proof.* Let $\mathfrak{P}$ be a sequent which is provable in $\mathsf{MSP}_{\mathcal{M}}$. We show by induction on the size of a proof of $\mathfrak{P}$ that $\mathfrak{P}$ is valid in $\mathcal{M}$.

Suppose first that the proof consists only of $\mathfrak{P}$. Then, $\mathfrak{P}$ is an axiom and it is of the form $\mathfrak{F}_m, [\{1, \ldots, m\} : A]$ for some SFE $A$. Consider an arbitrary valuation $I$ and let $v := I(A)$. Then, $I$ clearly satisfies $\mathfrak{P}$ since $A$ is in the component of $\mathfrak{P}$ corresponding to $v$.

Now suppose that $\mathfrak{P}$ has been derived by some weakening rule $(w : i)$ with a valid premiss. Then, $\mathfrak{P}$ is of the form $\mathfrak{P}', [i : A]$ for some SFE $A$, where $\mathfrak{P}'$ is the premiss of the application of $(w : i)$. Consider a valuation $I$. Since the premiss $\mathfrak{P}'$ is valid, $I$ satisfies $\mathfrak{P}'$, but then $I$ clearly also satisfies $\mathfrak{P}', [i : A]$. Thus, $\mathfrak{P}$ is also valid.

Lastly, suppose $\mathfrak{P}$ has been derived by a rule $(\circ : S)$ with valid premisses of the form $\mathfrak{P}', \hat{\delta}^j_{\circ:S}(A_1, \ldots, A_n)$, where $1 \leq j \leq k_i$. Then, $\mathfrak{P} = \mathfrak{P}', [S : \circ(A_1, \ldots, A_n)]$. Consider again an arbitrary valuation $I$. If $I$ satisfies the sequent $\mathfrak{P}'$, we are done. So, suppose this is not the case. Then, $I$ has to satisfy all of the sequents $\hat{\delta}^j_{\circ:S}(A_1, \ldots, A_n)$. By Theorem 3.4, we have that $I$ satisfies all of the corresponding SFEs $\delta^j_{\circ:S}(A_1, \ldots, A_n)$ which are all of the conjunctions of the generalised partial normal form. This means that $I$ satisfies $gpnf_S[\circ(A_1, \ldots, A_n)]$. By Theorem 3.2, we conclude that $I \models_{SFE} \circ(A_1, \ldots, A_n)^S$ which immediately implies that $I$ satisfies the consequent $\mathfrak{P}$ of the rule. ■

The proof of the completeness theorem proceeds similar to the one given by Zach [54] for the standard $m$-valued case and is based on the method of a *reduction tree*, as originally used by Schütte [46].

**Theorem 4.2.** If a sequent is valid in $\mathcal{M}$, then it is provable in $\mathsf{MSP}_{\mathcal{M}}$.

*Proof.* We will show that every $m$-component many-sided sequent $\mathfrak{P}$ is either provable in $\mathsf{MSP}$ or has a countermodel in $\mathcal{M}$.

A reduction tree, $T_{\mathfrak{P}}$, for $\mathfrak{P}$ is an upward rooted tree of many-sided sequents constructed from $\mathfrak{P}$ in stages as follows:

Stage 0: Write $\mathfrak{P}$ at the root of the reduction tree.

Stage $s+1$: If the many-sided sequent $\mathfrak{P}^s$ of a branch $B$ at stage $s$ contains only propositional constants or if there is some formula $A$ that occurs in every component of $\mathfrak{P}^s$, then this branch is called *closed* and the reduction on this branch is stopped. A branch that does not have this property is called *open*. For every open branch $B$ at stage $s$, choose a non-atomic formula $A$ contained in the topmost many-sided anti-sequent $\mathfrak{P}^s$ of $B$ and let $S$ be the set of truth values so that $A$ appears exactly in those components of $\mathfrak{P}$ with indices in $\{i \mid \mathsf{v}_i \in S\}$. Then, $A$ has the form $\circ(A_1, \ldots, A_n)$ and

$$\mathfrak{P}^s = \mathfrak{P}', [S : \circ(A_1, \ldots, A_n)],$$

where $\circ$ is some $n$-ary connective, $A_1, \ldots, A_n$ are propositional formulas, and $\mathfrak{P}'$ is some many-sided sequent that does not contain $A$. Note that $S \neq \mathcal{V}$ since otherwise the branch would be closed. Replace $\mathfrak{P}^s$ in the reduction tree by

$$\frac{\mathfrak{P}', \hat{\delta}^1_{\circ:S}(A_1, \ldots, A_n) \quad \cdots \quad \mathfrak{P}', \hat{\delta}^k_{\circ:S}(A_1, \ldots, A_n)}{\mathfrak{P}', [S : \circ(A_1, \ldots, A_n)]},$$

where $\hat{\delta}^j_{\circ:S}$ is the $j$-th conjunct of $gpnf_S[\circ(A_1, \ldots, A_n)]$. Each of the upper many-sided sequents $\mathfrak{P}', \hat{\delta}^j_{\circ:S}(A_1, \ldots, A_n)$ now represents the many-sided sequent of a new branch in stage $s+1$, obtained from the branch $B$ in stage $s$.

A reduction tree $T_{\mathfrak{P}}$ constructed in this way will always be finite, since at some stage $s$, every formula with a connective will be reduced to atomic formulas or appears in every component of a sequent. Let $C$ be the set of all closed branches in $T_{\mathfrak{P}}$. We distinguish two cases based in the branches in $C$:

Case 1: For every branch $B$ in $C$, there exists some formula $A$ that is contained in every component of the branches topmost many-sided sequent $\mathfrak{P}^B$, i.e., $\mathfrak{P}^B$ is of the form

$$\mathfrak{P}', [\{1, \ldots, m\} : A],$$

where $A$ is some formula. In this case, we can prove every topmost many-sided sequent $\mathfrak{P}^B$ in $\mathsf{MSP}$ by starting with

$$\mathfrak{F}_m, [\{1, \ldots, m\} : A]$$

as the axiom and adding the rest of the formulas in $\mathfrak{P}'$ via the weakening rules. From here we can easily construct a proof for $\mathfrak{P}$ in $\mathsf{MSP}$ by following each branch $B$ from the topmost many-sided sequent to the root of the reduction tree $T_{\mathfrak{P}}$.

Case 2: There is at least one closed branch $B$ where there is no formula that is contained in every component of the topmost many-sided sequent $\mathfrak{P}^B$ of $B$. Note that by the definition of a closed branch, this automatically implies that the topmost sequent contains only propositional constants. We will show that $\mathfrak{P}^B$ is refutable and that if a sequent on branch $B$ is refutable, then the sequent directly underneath it on $B$ is also refutable. Combining these two facts will imply that the root sequent $\mathfrak{P}$ is refutable.

A countermodel of $\mathfrak{P}^B$ can be constructed as follows: For every propositional constant $P$ in $\mathfrak{P}^B$, define $I(P) = \mathsf{v}_i$, where $\mathsf{v}_i$ is the corresponding truth value for some $i$-th component $\Gamma_i$ from $\mathfrak{P}^B$, such that $P \notin \Gamma_i$.

Now let $\mathfrak{P}'$ some refutable sequent on $B$ that has a predecessor of the form

$$\mathfrak{P}'', [S : \circ(A_1, \ldots, A_n)].$$

Then, $\mathfrak{P}'$ is of the form

$$\mathfrak{P}'', \hat{\delta}^j_{\circ:S}(A_1, \ldots, A_n),$$

for some $1 \leq j \leq k$, where $\mathfrak{P}''$ does not contain $\circ(A_1, \ldots, A_n)$. Let $I$ be a valuation that refutes $\mathfrak{P}'$. Then, $I$ refutes both $\mathfrak{P}''$ and $\hat{\delta}^j_{\circ:S}(A_1, \ldots, A_n)$. Therefore, $I$ also refutes the corresponding SFE $\delta^j_{\circ:S}(A_1, \ldots, A_n)$ which is one of the conjuncts of the generalised partial normal form. We conclude that $I$ refutes $gpnf_S[\circ(A_1, \ldots, A_n)]$. Thus, by Theorem 3.2, we have that there is some valuation $I'$ that refutes $\circ(A_1, \ldots, A_n)^S$, which implies that $I'$ also refutes the corresponding sequent $[S : \circ(A_1, \ldots, A_n)]$. Since $I'$ and $I$ agree on every formula except for $\circ(A_1, \ldots, A_n)$, and $\circ(A_1, \ldots, A_n)$ does not appear in $\mathfrak{P}''$, $I'$ also refutes $\mathfrak{P}''$ and thus the whole sequent $\mathfrak{P}'', [S : \circ(A_1, \ldots, A_n)]$.

With this we have proven that if the reduction tree $T_{\mathfrak{P}}$ has a closed branch $B$ such that there is no formula that is contained in every component of the topmost many-sided sequent $\mathfrak{P}^B$ of $B$, then there is a countermodel for the many-sided sequent at the root $\mathfrak{P}$, i.e., $\mathfrak{P}$ is not valid. ■

### 4.1.3 Sequent Calculi for the Paraconsistent Logics CLuN and Cio

In this section we will exemplify the method introduced above by constructing specific many-sided sequent calculi for the logics **CLuN** and **Cio** as special instances of our general framework.

Let us first consider **CLuN**. We already computed minimal forms of the generalised partial normal forms for $\mathcal{M}_1$ in Figure 3.1. We can immediately devise the calculus $\mathsf{MSP}_{\mathcal{M}_1}$ from this. The resultant rules are depicted in Figure 4.1.

It is important to note that, due to the fact that the connectives $\vee$, $\wedge$, and $\supset$ are all deterministic and their semantics correspond to the usual semantics of classical logic, the rules generated by our method correspond to the standard rules for the two-sided Gentzen-style proof system of classical propositional logic. Indeed, the only difference of this system is the lack of an introduction rule for the negation in the left side of the sequent.

**Example 4.3.** Since the formula $(A \supset {\sim}A) \supset {\sim}A$ is an axiom of **CLuN**, it clearly must be valid in $\mathcal{M}_1$. The following proof in $\mathsf{MSP}_{\mathcal{M}_1}$ establishes this fact in a purely syntactic manner:

$$\frac{\Gamma, A \mid \Delta}{\Gamma \mid \Delta, {\sim}A}\ ({\sim} : \mathbf{t})$$

$$\frac{\Gamma, A \mid \Delta \qquad \Gamma, B \mid \Delta}{\Gamma, A \vee B \mid \Delta}\ (\vee : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta, A, B}{\Gamma \mid \Delta, A \vee B}\ (\vee : \mathbf{t})$$

$$\frac{\Gamma, A, B \mid \Delta}{\Gamma, A \wedge B \mid \Delta}\ (\wedge : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta, A \qquad \Gamma \mid \Delta, B}{\Gamma \mid \Delta, A \wedge B}\ (\wedge : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta, A \qquad \Gamma, B \mid \Delta}{\Gamma, A \supset B \mid \Delta}\ (\supset : \mathbf{f}) \qquad \frac{\Gamma, A \mid \Delta, B}{\Gamma \mid \Delta, A \supset B}\ (\supset : \mathbf{t})$$

Figure 4.1: Inference rules of the proof system $\mathsf{MSP}_{\mathcal{M}_1}$ for **CLuN**.

$$\dfrac{\dfrac{\dfrac{A \mid A}{\emptyset \mid A, {\sim}A}\ ({\sim} : \mathbf{t}) \qquad {\sim}A \mid {\sim}A}{A \supset {\sim}A \mid {\sim}A}\ (\supset : \mathbf{f})}{\emptyset \mid (A \supset {\sim}A) \supset {\sim}A}\ (\supset : \mathbf{t}) \ .$$

△

Now let us consider the logic **Cio**. The generalised partial normal forms for **Cio** are given in Figure 3.2. From them we can devise the inference rules for the resulting calculus $\mathsf{MSP}_{\mathcal{M}_2}$ for **Cio** as depicted in Figure 4.2.

**Example 4.4.** Recall the axioms

$$\sharp A \supset (A \wedge \neg A \supset B), \tag{$\mathbf{p}$}$$

$$\neg\sharp A \supset A \wedge \neg A, \tag{$\mathbf{i}$}$$

of the logic **Cio**. Clearly, they must be valid in **Cio**. We can arrive at this fact syntactically using the following proofs in $\mathsf{MSP}_{\mathcal{M}_2}$:

(i) Proof of axiom (**p**):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \mid B, A \mid B, A}{\sharp A, A \mid B \mid B, A}\ (\sharp : \mathbf{f})}{\sharp A, A, \neg A \mid B \mid B}\ (\neg : \mathbf{f})}{\sharp A, A \wedge \neg A \mid B \mid B}\ (\wedge : \mathbf{f})}{\sharp A \mid A \wedge \neg A \supset B \mid A \wedge \neg A \supset B}\ (\supset : \{\mathbf{i}, \mathbf{t}\})}{\emptyset \mid \sharp A \supset (A \wedge \neg A \supset B) \mid \sharp A \supset (A \wedge \neg A \supset B)}\ (\supset : \{\mathbf{i}, \mathbf{t}\})$$

$$\frac{\Gamma \mid \Delta, A \mid \Pi}{\Gamma, \sharp A \mid \Delta \mid \Pi}(\sharp : \mathbf{f}) \qquad \frac{\Gamma, A \mid \Delta \mid \Pi, A}{\Gamma \mid \Delta \mid \Pi, \sharp A}(\sharp : \mathbf{t}) \qquad \frac{\Gamma \mid \Delta, A \mid \Pi}{\Gamma, \sharp A \mid \Delta, \sharp A \mid \Pi}(\sharp : \{\mathbf{f}, \mathbf{i}\})$$

$$\frac{}{\Gamma, \sharp A \mid \Delta \mid \Pi, \sharp A}(\sharp : \{\mathbf{f}, \mathbf{t}\}) \qquad \frac{\Gamma, A \mid \Delta \mid \Pi, A}{\Gamma \mid \Delta, \sharp A \mid \Pi, \sharp A}(\sharp : \{\mathbf{i}, \mathbf{t}\})$$

$$\frac{\Gamma \mid \Delta \mid \Pi, A}{\Gamma, \neg A \mid \Delta \mid \Pi}(\neg : \mathbf{f}) \qquad \frac{\Gamma, A \mid \Delta \mid \Pi}{\Gamma \mid \Delta \mid \Pi, \neg A}(\neg : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta \mid \Pi, A}{\Gamma, \neg A \mid \Delta, \neg A \mid \Pi}(\neg : \{\mathbf{f}, \mathbf{i}\})$$

$$\frac{\Gamma, A \mid \Delta \mid \Pi, A}{\Gamma, \neg A \mid \Delta \mid \Pi, \neg A}(\neg : \{\mathbf{f}, \mathbf{t}\}) \qquad \frac{\Gamma, A \mid \Delta, A \mid \Pi}{\Gamma \mid \Delta, \neg A \mid \Pi, \neg A}(\neg : \{\mathbf{i}, \mathbf{t}\})$$

$$\frac{\Gamma, A \mid \Delta \mid \Pi \qquad \Gamma, B \mid \Delta \mid \Pi}{\Gamma, A \vee B \mid \Delta \mid \Pi}(\vee : \mathbf{f})$$

$$\frac{\Gamma, B \mid \Delta, A \mid \Pi, A, B \qquad \Gamma, A, B \mid \Delta \mid \Pi, A, B}{\Gamma \mid \Delta \mid \Pi, A \vee B}(\vee : \mathbf{t})$$

$$\frac{\Gamma, A \mid \Delta \mid \Pi \qquad \Gamma, B \mid \Delta \mid \Pi}{\Gamma, A \vee B \mid \Delta, A \vee B \mid \Pi}(\vee : \{\mathbf{f}, \mathbf{i}\})$$

$$\frac{\Gamma, A, B \mid \Delta \mid \Pi, A, B}{\Gamma, A \vee B \mid \Delta \mid \Pi, A \vee B}(\vee : \{\mathbf{f}, \mathbf{t}\}) \qquad \frac{\Gamma \mid \Delta, A, B \mid \Pi, A, B}{\Gamma \mid \Delta, A \vee B \mid \Pi, A \vee B}(\vee : \{\mathbf{i}, \mathbf{t}\})$$

$$\frac{\Gamma \mid \Delta, A \mid \Pi, A \qquad \Gamma, B \mid \Delta \mid \Pi}{\Gamma, A \supset B \mid \Delta \mid \Pi}(\supset : \mathbf{f})$$

$$\frac{\Gamma, A \mid \Delta, B \mid \Pi, B \qquad \Gamma, A, B \mid \Delta \mid \Pi, A, B}{\Gamma \mid \Delta \mid \Pi, A \supset B}(\supset : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta, A \mid \Pi, A \qquad \Gamma, B \mid \Delta \mid \Pi}{\Gamma, A \supset B \mid \Delta, A \supset B \mid \Pi}(\supset : \{\mathbf{f}, \mathbf{i}\})$$

$$\frac{\Gamma, A, B \mid \Delta \mid \Pi, A, B}{\Gamma, A \supset B \mid \Delta \mid \Pi, A \supset B}(\supset : \{\mathbf{f}, \mathbf{t}\}) \qquad \frac{\Gamma, A \mid \Delta, B \mid \Pi, B}{\Gamma \mid \Delta, A \supset B \mid \Pi, A \supset B}(\supset : \{\mathbf{i}, \mathbf{t}\})$$

$$\frac{\Gamma \mid \Delta \mid \Pi, A, B \qquad \Gamma, B \mid \Delta, A, B \mid \Pi, A \qquad \Gamma, A \mid \Delta, A, B \mid \Pi; B}{\Gamma \mid \Delta \mid \Pi, A \wedge B}(\wedge : \mathbf{t})$$

$$\frac{\Gamma, A, B \mid \Delta \mid \Pi}{\Gamma, A \wedge B \mid \Delta \mid \Pi}(\wedge : \mathbf{f}) \qquad \frac{\Gamma, A, B \mid \Delta \mid \Pi}{\Gamma, A \wedge B \mid \Delta, A \wedge B \mid \Pi}(\wedge : \{\mathbf{f}, \mathbf{i}\})$$

$$\frac{\Gamma, A, B \mid \Delta \mid \Pi, A, B}{\Gamma, A \wedge B \mid \Delta \mid \Pi, A \wedge B}(\wedge : \{\mathbf{f}, \mathbf{t}\}) \qquad \frac{\Gamma \mid \Delta, A \mid \Pi, A \qquad \Gamma \mid \Delta, B \mid \Pi, B}{\Gamma \mid \Delta, A \wedge B \mid \Pi, A \wedge B}(\wedge : \{\mathbf{i}, \mathbf{t}\})$$

Figure 4.2: Inference rules of the proof system $\mathsf{MSP}_{\mathcal{M}_2}$ for **Cio**.

(ii) Proof of axiom (**i**):

$$\dfrac{\dfrac{\dfrac{\dfrac{A \mid A \mid A, A}{\emptyset \mid A \mid A, \sharp A}\,(\sharp : \mathbf{t}) \qquad \dfrac{\dfrac{A, A \mid A \mid A}{A \mid \neg A \mid \neg A, A}\,(\neg : \{\mathbf{i},\mathbf{t}\})}{\emptyset \mid \neg A \mid \neg A, \sharp A}\,(\sharp : \mathbf{t})}{\emptyset \mid A \wedge \neg A \mid A \wedge \neg A, \sharp A}\,(\wedge : \{\mathbf{i},\mathbf{t}\})}{\neg \sharp A \mid A \wedge \neg A \mid A \wedge \neg A}\,(\neg : \mathbf{f})}{\emptyset \mid \neg \sharp A \supset A \wedge \neg A \mid \neg \sharp A \supset A \wedge \neg A}\,(\supset : \{\mathbf{i},\mathbf{t}\})$$

△

## 4.2 Sequent Calculi for the Static Semantics

We now will modify the previously introduced systematic method of generating sequent calculi for the dynamic semantics to obtain sequent calculi which are sound and complete with respect to the static semantics. Since every static valuation is also a dynamic valuation, it clearly holds that if a formula is dynamically valid, then it is also statically valid. By contraposition, this means that if a formula is statically refutable, then it is also dynamically refutable. In Example 2.2 we saw that it can be the case that a formula is dynamically refutable yet not statically. As we will see, it is sufficient to only consider all calculi that arise from all *deterministic versions* of a given Nmatrix. Before we come to the definition of the calculi and the proofs of soundness and completeness, we have to elaborate what we mean by deterministic versions of a given Nmatrix. This is the purpose of the following subsection.

### 4.2.1 Deterministic Casts

**Definition 4.5.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix for a language $\mathcal{L}_{\mathcal{A}}$, where $\mathcal{A} = PROP \cup CON \cup AUX$, and let $\circ \in CON$ be an $n$-ary connective with semantic function $def_{\circ} \in \mathcal{O}$. Then, a function $def^{c}_{\circ} : \mathcal{V}^n \to \mathcal{P}(\mathcal{V}) \setminus \emptyset$ is a (*deterministic*) *cast* of $def_{\circ}$ if, for every $(w_1, \ldots, w_n) \in \mathcal{V}^n$, there is some $\mathsf{v} \in def_{\circ}(w_1, \ldots, w_n)$ such that $def^{c}_{\circ}(w_1, \ldots, w_n) = \{\mathsf{v}\}$. □

Note that every $n$-ary connective has at least one and at most $m^{n^2}$ deterministic casts, where $m$ is the number of truth values of a given Nmatrix.

We extend the notion of casts to whole Nmatrices in a natural way as follows:

**Definition 4.6.** Let $\mathcal{M}$ be as in Definition 4.5. An Nmatrix

$$\mathcal{M}^c = (\mathcal{V}, \mathcal{D}, \{def^{c}_{\circ} \mid \circ \in CON\})$$

is a (*deterministic*) *cast* of $\mathcal{M}$ if each function $def^{c}_{\circ}$ is a cast of the corresponding function $def_{\circ}$. □

Every Nmatrix has a cast and there are potentially $\prod_{i=1}^{k} m^{n_i^2}$ many casts of a given Nmatrix, where $m$ is the number of truth values, $k$ is the number of connectives, and $n_1, \ldots, n_k$ are the arities of the connectives. It is important to note that this maximum amount of casts is very unlikely to appear in any reasonable logic. In practice, the amount of casts will be much lower.

**Lemma 4.1.** Let $\mathcal{M}$ be a finite Nmatrix and $\mathcal{M}^c$ a deterministic cast of $\mathcal{M}$. If $I$ is a (static) valuation in $\mathcal{M}^c$, then $I$ is a (static) valuation in $\mathcal{M}$.

*Proof.* Since the set of truth values of $\mathcal{M}$ and $\mathcal{M}^c$ coincide, we only need to show that $(SCL)$ (and $(CMP)$) hold with respect to $\mathcal{M}$. By condition $(SCL)$ with respect to $\mathcal{M}^c$ and the definition of casts, we have

$$I(\circ(A_1, \ldots, A_n)) \in \mathit{def}_\circ^c(I(A_1), \ldots, I(A_n)) \subseteq \mathit{def}_\circ(I(A_1), \ldots, I(A_n)).$$

Clearly, if $(CMP)$ holds for $I$ in $\mathcal{M}^c$, then also in $\mathcal{M}$, since it is not dependent on the matrix. ∎

An immediate consequence of the above lemma is that whenever a formula is refutable in a cast $\mathcal{M}^c$ of an Nmatrix $\mathcal{M}$, then it is also refutable in $\mathcal{M}$. The converse need not be true in general, but there is a close relationship between static valuations in an Nmatrix and a cast that *corresponds* to that valuation. This fact is formalised as follows:

**Lemma 4.2.** Let $\mathcal{M}$ be an Nmatrix for a language $\mathcal{L}_\mathcal{A} = (\mathcal{A}, \mathit{FORM})$, $\circ \in \mathit{CON}$ an $n$-ary connective, and $I$ a static valuation in $\mathcal{M}$. Then, there exists a cast $\mathit{def}_\circ^c$ of $\mathit{def}_\circ$ such that, for all $A_1, \ldots, A_n$,

$$\mathit{def}_\circ^c(I(A_1), \ldots, I(A_n)) = \{I(\circ(A_1, \ldots, A_n))\}. \tag{4.1}$$

*Proof.* We first define a function $f : \mathrm{ran}(I)^n \to \mathcal{P}(\mathcal{V}) \setminus \emptyset$, where $\mathrm{ran}(I)$ is the range of $I$ defined as $\{I(A) \mid A \in \mathit{FORM}\}$. Consider now $I(A_1), \ldots, I(A_n) \in \mathrm{ran}(I)$ and define

$$f(I(A_1), \ldots, I(A_n)) := \{I(\circ(A_1, \ldots, A_n))\}.$$

We need to show that $f$ is well-defined. So, consider $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathit{FORM}$ with $I(A_i) = I(B_i)$, for all $i \in \{1, \ldots, n\}$. Since $I$ is a static valuation, we know that condition $(CMP)$ holds and thus $I(\circ(A_1, \ldots, A_n)) = I(\circ(B_1, \ldots, B_n))$. Hence, $f$ is indeed well-defined.

We now extend $f$ to $\mathcal{V}^n$ as follows: Define

$$\mathit{def}_\circ^c(w_1, \ldots, w_n) := \begin{cases} f(w_1, \ldots, w_n) & \text{if } (w_1, \ldots, w_n) \in \mathrm{ran}(I)^n, \\ \{\mathsf{v}(w_1, \ldots, w_n)\} & \text{otherwise,} \end{cases}$$

where $\mathsf{v}(w_1, \ldots, w_n) \in \mathit{def}_\circ(w_1, \ldots, w_n)$ chooses an element out of $\mathit{def}_\circ(w_1, \ldots, w_n)$. Since $I(\circ(A_1, \ldots, A_n)) \in \mathit{def}_\circ(w_1, \ldots, w_n)$, $\mathit{def}_\circ^c$ is clearly a cast for $\mathit{def}_\circ$ and satisfies the desired property by construction. ∎

Given an Nmatrix $\mathcal{M}$ and a static valuation $I$ in $\mathcal{M}$, we call a cast $\mathcal{M}^c$ *canonical* with respect to $I$ iff equation (4.1) holds for all connectives. It is an immediate consequence of Lemma 4.2 that there is always exactly one canonical cast of an Nmatrix with respect to a static valuation. Furthermore, for canonical casts, the converse of Lemma 4.1 is true: If $I$ is a static valuation in $\mathcal{M}$, then it is a static valuation in the canonical cast $\mathcal{M}^c$. As a consequence, if a formula is refuted by a static valuation $I$ in $\mathcal{M}$, then it is also refuted by that valuation in the canonical cast $\mathcal{M}^c$ of $\mathcal{M}$ with respect to $I$.

### 4.2.2 Postulates of the Static Calculi

With the definitions of the previous subsection at hand, we are already in a position to provide the definition of provability in a sequent calculus for the static case.

**Definition 4.7.** Let $\mathcal{M}$ be an $m$-valued Nmatrix. We say that an $m$-component sequent $\mathfrak{P}$ is *provable in the static many-sided proof calculus* $\mathsf{MSP}^s_{\mathcal{M}}$ if, for every deterministic cast $\mathcal{M}^c$ of $\mathcal{M}$, it holds that $\mathfrak{P}$ is provable in $\mathsf{MSP}_{\mathcal{M}^c}$. □

Note that the above definition gives rise to a well-defined proof system since we already saw that the number of casts of a given Nmatrix is always a finite and greater than zero. Moreover, the important restriction is that the rules of the different casted calculi cannot be mixed in a proof.

Given the preparatory work of the previous subsection, it is straightforward to prove soundness and completeness of the calculi.

**Theorem 4.3.** If a sequent is provable in $\mathsf{MSP}^s_{\mathcal{M}}$, then it is statically valid in $\mathcal{M}$.

*Proof.* Suppose $\mathfrak{P}$ is provable in $\mathsf{MSP}^s_{\mathcal{M}}$. By definition, this means that $\mathfrak{P}$ is provable in $\mathsf{MSP}_{\mathcal{M}^c}$, for every deterministic cast $\mathcal{M}^c$ of $\mathcal{M}$. By Theorem 4.1, we have that $\mathfrak{P}$ is valid in every $\mathcal{M}^c$. Since all matrices $\mathcal{M}^c$ are deterministic, we have that $\mathfrak{P}$ is statically valid in $\mathcal{M}^c$. Now suppose that $\mathfrak{P}$ is not statically valid in $\mathcal{M}$, then there is some static valuation $I$ that does not satisfy $\mathfrak{P}$. Let $\mathcal{M}^c$ be the canonical cast of $\mathcal{M}$ with respect to $I$. By definition, $I$ is a static valuation in $\mathcal{M}^c$ that does not satisfy $\mathfrak{P}$. This is a contradiction to our observation that $\mathfrak{P}$ is statically valid in all deterministic casts of $\mathcal{M}$. Thus, it must hold that $\mathfrak{P}$ is statically valid in $\mathcal{M}$. ■

**Theorem 4.4.** If a sequent is statically valid in $\mathcal{M}$, then it is provable in $\mathsf{MSP}^s_{\mathcal{M}}$.

*Proof.* Suppose $\mathfrak{P}$ is statically valid in $\mathcal{M}$. Let $\mathcal{M}^c$ be an arbitrary deterministic cast of $\mathcal{M}$ and $I$ an arbitrary valuation in $\mathcal{M}^c$. By Lemma 4.1, it holds that $I$ is a static valuation in $\mathcal{M}$. By our assumption, it holds that $I$ is a model of $\mathfrak{P}$. Since $I$ was chosen arbitrarily, $\mathfrak{P}$ must be statically valid in $\mathcal{M}^c$, and thus also dynamically valid in $\mathcal{M}^c$, since $\mathcal{M}^c$ is deterministic. By Theorem 4.2, we conclude that $\mathfrak{P}$ is provable in $\mathsf{MSP}_{\mathcal{M}^c}$. Since the choice of $\mathcal{M}^c$ was arbitrary, we see that $\mathfrak{P}$ is provable in $\mathsf{MSP}^s_{\mathcal{M}}$. ■

### 4.2.3 A Sequent Calculus for the Logic CAR

We already observed that the matrix $\mathcal{M}_1$ characteristic for the logic **CAR** has only two deterministic casts, $\mathcal{M}_1^{c1}$ and $\mathcal{M}_1^{c2}$, which differ only in their interpretation of negation.

$$\frac{\Gamma, A \mid \Delta \qquad \Gamma, B \mid \Delta}{\Gamma, A \vee B \mid \Delta} (\vee : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta, A, B}{\Gamma \mid \Delta, A \vee B} (\vee : \mathbf{t})$$

$$\frac{\Gamma, A, B \mid \Delta}{\Gamma, A \wedge B \mid \Delta} (\wedge : \mathbf{f}) \qquad \frac{\Gamma \mid \Delta, A \qquad \Gamma \mid \Delta, B}{\Gamma \mid \Delta, A \wedge B} (\wedge : \mathbf{t})$$

$$\frac{\Gamma \mid \Delta, A \qquad \Gamma, B \mid \Delta}{\Gamma, A \supset B \mid \Delta} (\supset : \mathbf{f}) \qquad \frac{\Gamma, A \mid \Delta, B}{\Gamma \mid \Delta, A \supset B} (\supset : \mathbf{t})$$

Figure 4.3: Common inference rules of the proof systems $\mathsf{MSP}_{\mathcal{M}^{c_1}}$ and $\mathsf{MSP}_{\mathcal{M}^{c_2}}$.

For the proof system $\mathsf{MSP}^s_{\mathcal{M}_1}$, we need to compute the systems $\mathsf{MSP}_{\mathcal{M}_1^{c_1}}$ and $\mathsf{MSP}_{\mathcal{M}_1^{c_2}}$. The generalised partial normal forms with respect to the casts also only differ from the ones given in Figure 3.1 in the case of $\sim$, which are as follows:

$$gpnf_{\mathbf{f}}^{\mathcal{M}_1^{c_1}}[\sim A] \equiv_{SFE} A^{\mathbf{t}}; \qquad gpnf_{\mathbf{f}}^{\mathcal{M}_1^{c_2}}[\sim A] \equiv_{SFE} \dot{\bot};$$
$$gpnf_{\mathbf{t}}^{\mathcal{M}_1^{c_1}}[\sim A] \equiv_{SFE} A^{\mathbf{f}}; \qquad gpnf_{\mathbf{t}}^{\mathcal{M}_1^{c_2}}[\sim A] \equiv_{SFE} \dot{\top}.$$

The rules for $\mathsf{MSP}_{\mathcal{M}^{c_1}}$ and $\mathsf{MSP}_{\mathcal{M}^{c_2}}$ are as follows: while both $\mathsf{MSP}_{\mathcal{M}^{c_1}}$ and $\mathsf{MSP}_{\mathcal{M}^{c_2}}$ have the common rules depicted in Figure 4.3, $\mathsf{MSP}_{\mathcal{M}^{c_1}}$ uses additionally the rules

$$\frac{\Gamma \mid \Delta, A}{\Gamma, \sim A \mid \Delta} (\sim : \mathbf{f}) \quad \text{and} \quad \frac{\Gamma, A \mid \Delta}{\Gamma \mid \Delta, \sim A} (\sim : \mathbf{t})_1,$$

whereas $\mathsf{MSP}_{\mathcal{M}^{c_2}}$, on the other hand, uses instead the rule

$$\frac{}{\Gamma \mid \Delta, \sim A} (\sim : \mathbf{t})_2.$$

Note that a proof in $\mathsf{MSP}^s_{\mathcal{M}_1}$ is only considered as valid if it consists of two proofs, where one of them only uses rules $(\sim : \mathbf{f})$ and $(\sim : \mathbf{t})_1$ and the other exclusively uses the rule $(\sim : \mathbf{t})_2$.

**Example 4.8.** Recall the axiom

$$(A \supset B) \supset ((A \supset \sim B) \supset \sim A) \tag{4.2}$$

of the logic **CAR**. We can derive a proof of it in the calculus $\mathsf{MSP}^s_{\mathcal{M}_1}$ as follows:

(i) A proof of axiom (4.2) in $\mathsf{MSP}_{\mathcal{M}^{c_1}}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{A, A \supset {\sim}B \mid A}{A \supset {\sim}B \mid A, {\sim}A}\,(\sim : \mathbf{t}) \qquad \dfrac{\dfrac{B, A \mid A}{B \mid A, {\sim}A}\,(\sim : \mathbf{t})_1 \qquad \dfrac{B \mid B, {\sim}A}{{\sim}B, B \mid {\sim}A}\,(\sim : \mathbf{f})}{B, A \supset {\sim}B \mid {\sim}A}\,(\supset : \mathbf{f})}{A \supset B, A \supset {\sim}B \mid {\sim}A}\,(\supset : \mathbf{f})}{A \supset B \mid (A \supset {\sim}B) \supset {\sim}A}\,(\supset : \mathbf{t})}{\emptyset \mid (A \supset B) \supset ((A \supset {\sim}B) \supset {\sim}A)}\,(\supset : \mathbf{t})
$$

(ii) A proof of axiom (4.2) in $\mathsf{MSP}_{\mathcal{M}^{c_2}}$:

$$
\dfrac{\dfrac{\dfrac{}{A \supset B, A \supset {\sim}B \mid {\sim}A}\,(\sim : \mathbf{t})_2}{A \supset B \mid (A \supset {\sim}B) \supset {\sim}A}\,(\supset : \mathbf{t})}{\emptyset \mid (A \supset B) \supset ((A \supset {\sim}B) \supset {\sim}A)}\,(\supset : \mathbf{t})
$$

△

CHAPTER 5

# Generating Anti-Sequent Calculi for Non-Deterministic Matrices

We now introduce a systematic method for generating sequent-style rejection systems for any given finite Nmatrix. Similarly to the work of Bogojeski and Tompits [13], they are directly constructed from the generalised complementary partial normal forms introduced in Section 3.1 and are based on *many-sided anti-sequents*. Moreover, we show soundness and completeness of the introduced calculi.

Since the number of rules in the resultant calculi might be exponential in the number of truth values of a given Nmatrix, we will discuss two ways of reducing the amount of rules in some special cases, yielding optimised versions of the calculi. Furthermore, similar to the assertional case, we also discuss a way to adapt our original method to obtain calculi for the static semantics.

## 5.1 Anti-Sequent Calculi for the Dynamic Semantics

In contrast to the rules for the assertional case, where there is only a single introduction rule for a fixed connective and a strict non-empty subset of truth values, but with as many premisses as there are conjuncts in the corresponding generalised partial normal form, here we have as many introduction rules as there are disjuncts in the corresponding generalised complementary partial normal form—but all of the rules have exactly one premiss.

### 5.1.1 Postulates of the Calculi

**Definition 5.1.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix and $S$ a non-empty strict subset of $\mathcal{V}$. Furthermore, let the generalised complementary partial normal form for

$\circ(A_1, \ldots, A_n)$ with respect to $S$ be of the form

$$gcpnf_S[\circ(A_1, \ldots, A_n)] = \dot{\bigvee}_{j=1}^{k_i} \sigma^j_{\circ:S}(A_1, \ldots, A_n),$$

where $k_i$ is the number of disjuncts in the generalised complementary partial normal form and $\sigma^j_{\circ:S}(A_1, \ldots A_n)$ is the $j$-th disjunct of it, which is a conjunction of negated signed atoms. Moreover, let $\hat{\sigma}^j_{\circ:S}(A_1, \ldots, A_n)$ be the anti-sequent constructed from $\sigma^j_{\circ:S}(A_1, \ldots, A_n)$ as described in Theorem 3.3.

Then, for every $j \in \{1, \ldots, k_i\}$, the inference rule $(\circ : S)_j$ is given as follows:

$$\frac{\mathfrak{R}, \hat{\sigma}^j_{\circ:S}(A_1, \ldots A_n)}{\mathfrak{R}, [S : \circ(A_1, \ldots, A_n)]} \; (\circ : S)_j,$$

where $\mathfrak{R} = \Gamma_1 \nmid \cdots \nmid \Gamma_m$ is an arbitrary $m$-component anti-sequent subject to the following proviso:

$(RES)_S$ For all $i \in \{j \mid \mathsf{v}_j \in S\}$, $\circ(A_1, \ldots, A_n) \notin \Gamma_i$.

□

As for Definition 4.1, some remarks are in order:

1. Like for the assertive case, since one can use any negative SFE equivalent to the generalised complementary partial normal form to introduce the above rules, it is beneficial to minimise the normal form.

2. Since $\dot{\bot}$ is a negative SFE in disjunctive normal form (viz. an empty disjunction), it may occur as a minimal generalised complementary partial normal form, and so no rule will be introduced.

3. The above definition is clearly a natural generalisation of the rules defined by Bogojeski and Tompits [13] with the difference that the proviso $(RES)_S$ is imposed in our case, which plays an important role in keeping the rules sound. This condition is also the reason for introducing generalised complementary partial normal forms for all strict non-empty subsets of truth values instead of only singletons: otherwise, the completeness of the calculi would be violated.

We are now ready to introduce the anti-sequent calculi for every given $m$-valued Nmatrix.

**Definition 5.2.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix with $\mathcal{V} = \{\mathsf{v}_1, \ldots, \mathsf{v}_m\}$ being its set of truth values. Then, the many-sided rejection calculus $\mathsf{MSR}_{\mathcal{M}}$ consists of

(i) axiom schemata of the form

$$\mathfrak{T}_m, [M_1 : P_1], \ldots, [M_k : P_k],$$

where $\mathfrak{T}_m$ is the empty anti-sequent, $P_1, \ldots, P_k$ are distinct propositional variables, and $M_1, \ldots, M_k \subset \{1, \ldots, m\}$; and

(ii) the inference rules $(\circ : S)_1, \ldots, (\circ : S)_{k_i}$ as given in Definition 5.1, for every connective $\circ$ of $\mathcal{M}$ and every non-empty strict subset $S$ of $\mathcal{V}$.

□

Since all inference rules of $\mathsf{MSR}_{\mathcal{M}}$ are unary, a proof in $\mathsf{MSR}_{\mathcal{M}}$ is a *sequence* of anti-sequents having a single axiom at its beginning. More specifically, a proof in $\mathsf{MSR}_{\mathcal{M}}$ is a finite sequence $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$ of anti-sequents such that $\mathfrak{R}_1$ is an axiom of $\mathsf{MSR}_{\mathcal{M}}$ and each $\mathfrak{R}_i$ is the conclusion of an inference rule of $\mathsf{MSR}_{\mathcal{M}}$ with premiss $\mathfrak{R}_{i-i}$, for $1 < i \leq k$. As usual, a proof $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$ is also referred to as a proof *of* its last element $\mathfrak{R}_k$. Finally, an anti-sequent $\mathfrak{R}$ is *provable* in $\mathsf{MSR}_{\mathcal{M}}$ if there is a proof of $\mathfrak{R}$ in in $\mathsf{MSR}_{\mathcal{M}}$.

Note that the axioms of $\mathsf{MSR}_{\mathcal{M}}$ are anti-sequents whose components are sets of propositional variables such that no variable appears in all components.

### 5.1.2 Adequacy of the Calculi

Concerning the adequacy of our calculi, we start as usual with the proof of their soundness.

**Theorem 5.1.** If an anti-sequent is provable in $\mathsf{MSR}_{\mathcal{M}}$, then it is refutable in $\mathcal{M}$.

*Proof.* Let $\mathfrak{R}$ be an anti-sequent which is provable in $\mathsf{MSR}_{\mathcal{M}}$. We show by induction on the length of a proof of $\mathfrak{R}$ that $\mathfrak{R}$ is refutable in $\mathcal{M}$.

So, suppose first that $\mathfrak{R}$ is an axiom. Then, $\mathfrak{R}$ is of the form

$$\mathfrak{T}_m, [M_1 : P_1], \ldots, [M_k : P_k],$$

where $P_1, \ldots, P_k$ are different propositional variables and $M_1, \ldots, M_k \subset \{1, \ldots, m\}$. We construct a valuation $I$ as follows: For each propositional variable $P_j$, choose one $i \in \{1, \ldots, m\} \setminus M_j$ and define $I(P_j) = \mathsf{v}_i$. For this valuation $I$, whenever a propositional variable is contained in an $i$-th component of $\mathfrak{R}$, $I \neq \mathsf{v}_i$ holds, which means that $I$ refutes the anti-sequent $\mathfrak{R}$.

Suppose now that $\mathfrak{R}$ has been derived by some rule $(\circ : S)_j$ with a refutable premiss. Then, $\mathfrak{R}$ must be of the form $\mathfrak{R}', [S : \circ(A_1, \ldots, A_n)]$ and the premiss is of the form $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}$, where $\hat{\sigma}^j_{\circ:S}$ is as in Definition 5.2. Let $I$ be a valuation refuting $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}$. Then, $I$ refutes both $\mathfrak{R}'$ and $\hat{\sigma}^j_{\circ:S}$. Thus, by Theorem 3.3, $I \models_{SFE} \sigma^j_{\circ:S}$, and since this is just one of the disjuncts of $gcpnf_S[\circ(A_1, \ldots, A_n)]$, we have $I \models_{SFE} gcpnf_S[\circ(A_1, \ldots, A_n)]$. By Theorem 3.1, it follows that there is some $x \notin S$ such that the valuation

$$I'(\varphi) := \begin{cases} x, & \text{if } \varphi = \circ(A_1, \ldots, A_n), \\ I(\varphi), & \text{otherwise} \end{cases}$$

is well defined and it holds that $I' \models_{SFE} \dot{\neg}(\circ(A_1,\ldots,A_n))^S$. This means that for all $s \in S$, $I'(\circ(A_1,\ldots,A_n)) \neq s$ and therefore $I'$ is a countermodel of the anti-sequent $\mathfrak{T}_m, [S : \circ(A_1,\ldots,A_n)]$. Furthermore, because of the proviso $(RES)_S$, we have that $\mathfrak{R}'$ contains the formula $\circ(A_1,\ldots,A_n)$ only in components that correspond to a value in $S$. Since $I'$ only differs from $I$ in the value of $\circ(A_1,\ldots,A_n)$, and $I$ refutes $\mathfrak{R}'$, it must also hold that $I'$ refutes $\mathfrak{R}'$. Altogether we have showed that $I'$ refutes $\mathfrak{R}', [S : \circ(A_1,\ldots,A_n)]$. ■

The proof of the completeness theorem proceeds basically in the same manner as the one given by Bogojeski and Tompits [13] for the standard $m$-valued case and is based on the method of a reduction tree, similar to the proof of Theorem 4.2. The crucial step in our case, however, is the observation that even though every rule has the proviso $(RES)_S$, we can still generate a complete reduction tree because there are rules for each non-empty strict subset of truth values.

**Theorem 5.2.** If an anti-sequent is refutable in $\mathcal{M}$, then it is provable in $\mathsf{MSR}_{\mathcal{M}}$.

*Proof.* Assume that the Nmatrix $\mathcal{M}$ is $m$-valued and let $\mathfrak{R}$ be a many-sided anti-sequent for $\mathcal{M}$. We show that $\mathfrak{R}$ is either provable in $\mathsf{MSR}_{\mathcal{M}}$ or irrefutable in $\mathcal{M}$ (i.e., there is no valuation $I$ that refutes $\mathfrak{R}$).

Let us define a reduction tree, $T_{\mathfrak{R}}$, for $\mathfrak{R}$ as an upward rooted tree of many-sided anti-sequents constructed from $\mathfrak{R}$ in stages as follows:

Stage 0: Write $\mathfrak{R}$ at the root of the reduction tree.

Stage $s+1$: If the topmost many-sided anti-sequent of a branch at stage $s$ contains only propositional variables, then this branch is called *closed* and the reduction on this branch is stopped. A branch that does not have this property is called *open*.

For every open branch $B$ at stage $s$, choose a non-atomic formula $A$ contained in exactly those components of the topmost many-sided anti-sequent $\mathfrak{R}^s$ of $B$ with indices in $\{i \mid \mathsf{v}_i \in S\}$. Then, $A$ has the form $\circ(A_1,\ldots,A_n)$ and $\mathfrak{R}^s = \mathfrak{R}', [S : \circ(A_1,\ldots,A_n)]$, where $\circ$ is some $n$-ary connective, $A_1,\ldots,A_n$ are formulas, and $\mathfrak{R}'$ is some anti-sequent not containing $\circ(A_1,\ldots,A_n)$. Replace $\mathfrak{R}^s$ in the reduction tree by the figure

$$\frac{\mathfrak{R}', \hat{\sigma}^1_{\circ:S}(A_1,\ldots,A_n) \quad \cdots \quad \mathfrak{R}', \hat{\sigma}^{k_i}_{\circ:S}(A_1,\ldots,A_n)}{\mathfrak{R}', [S : \circ(A_1,\ldots,A_n)]}\,,$$

where every anti-sequent $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}(A_1,\ldots,A_n)$, $1 \leq j \leq k_i$, is the premiss of an instance of the inference rule $(\circ : S)_j$ of $\mathsf{MSR}$ constructed as in Definition 5.1. It is important to note that $\mathfrak{R}'$ satisfies the proviso $(RES)_S$. This concludes the construction of stage $s+1$.

Now, a reduction tree $T_{\mathfrak{R}}$ constructed in this way will always be finite since, at some stage $s$, every formula with a connective will be reduced to atomic formulas. Let $C^{T_{\mathfrak{R}}}$ be the set of all closed branches in $T_{\mathfrak{R}}$. We distinguish two cases based on the branches in $C^{T_{\mathfrak{R}}}$:

Case 1: For at least one branch $B$ in $C^{T_{\mathfrak{R}}}$, its topmost many-sided anti-sequent, $\mathfrak{R}^B$, is an axiom of $\mathsf{MSR}_{\mathcal{M}}$, i.e., there is no propositional variable that appears in every component of $\mathfrak{R}^B$. With $\mathfrak{R}^B$ as an axiom, we can easily construct a proof for $\mathfrak{R}$ in $\mathsf{MSR}_{\mathcal{M}}$ by following the branch $B$ to the root of the reduction tree $T_{\mathfrak{R}}$. Again note that all provisos of the form $(RES)_S$ are satisfied in the constructed proof.

Case 2: There is no closed branch $B$ in $C^{T_{\mathfrak{R}}}$ such that its topmost many-sided anti-sequent $\mathfrak{R}^B$ is an axiom in $\mathsf{MSR}_{\mathcal{M}}$. This means that for every branch $B$ in $C^{T_{\mathfrak{R}}}$, there is no valuation $I$ which refutes its topmost many-sided anti-sequent $\mathfrak{R}^B$.

We will show by induction on the depth of a many-sided anti-sequent in the tree $T_{\mathfrak{R}}$ that every anti-sequent in the reduction tree $T_{\mathfrak{R}}$, including the root anti-sequent $\mathfrak{R}$, is irrefutable, where the depth is defined as follows: Let $\mathfrak{R}^s$ be the topmost anti-sequent of a branch introduced at stage $s$ of the construction of $T_{\mathfrak{R}}$ and let $k$ be the lowest stage at which every outgoing branch from $\mathfrak{R}^s$ has been closed. Then, the depth $d$ of $\mathfrak{R}^s$ is defined as $m - s$.

Base case: We show that any many-sided anti-sequent at depth 0 is irrefutable. Let $\mathfrak{R}^s$ be an arbitrary many-sided anti-sequent at some stage $s$ at depth 0. This means that $\mathfrak{R}^s$ is the topmost many-sided anti-sequent of some closed branch $B$, i.e., $\mathfrak{R}^s = \mathfrak{R}^B$. Then, it follows directly from the conditions of Case 2 that there is no valuation $I$ that refutes $\mathfrak{R}^s$.

Induction hypothesis: Assume that every many-sided anti-sequent at any depth $0 \leq n \leq d$ in $T_{\mathfrak{R}}$ is irrefutable.

Induction step: We want to show that any many-sided anti-sequent at depth $d+1$ is also irrefutable. Let $I$ be an arbitrary valuation and $\mathfrak{R}^s$ an arbitrary many-sided anti-sequent at some stage $s$ at depth $d+1$. Then, $\mathfrak{R}^s$ is reduced at stage $s+1$ to many-sided anti-sequents of the form $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}$ for every $j \in \{1, \ldots, k_i\}$. $\mathfrak{R}^s$ itself has the form

$$\mathfrak{R}', [S : \circ(A_1, \ldots, A_n)]$$

for some $n$-ary connective $\circ$ and some formulae $A_1, \ldots, A_n$. Since all the many-sided anti-sequents resulting from the reduction of $\mathfrak{R}^s$ have a depth smaller or equal to $d$, from the induction hypothesis it follows that every one of them is irrefutable, thus they are also not refuted by $I$. This means that there are two cases, either $I$ does not refute $\mathfrak{R}'$ or $I$ does not refute all the many-sided anti-sequents of the form $\hat{\sigma}^j_{\circ:S}$ where $j \in \{1, \ldots, k_i\}$. If $I$ does not refute $\mathfrak{R}'$, then it does not refute $\mathfrak{R}^s$ either, since $\mathfrak{R}'$ is contained in it. Otherwise $I$ does not refute any of the many-sided anti-sequents $\hat{\sigma}^j_{\circ:S}$ where $j \in \{1, \ldots, k_i\}$. From Theorem 3.3 it follows that $I$ does not satisfy the corresponding SFEs of the form $\sigma^j_{\circ:S}$ for every $j \in \{1, \ldots, k_i\}$ and consequently the SFE $\dot{\bigvee}^k_{j=1} \sigma^j_{\circ:S}$, which is an instance of the $gcpnf_S[\circ(A_1, \ldots, A_n)]$. From Theorem 3.1 we can conclude that $I$ does not satisfy the SFE $\dot{\neg}(\circ(A_1, \ldots, A_n))^S$ either, which is semantically equivalent to the many-sided anti-sequent $\mathfrak{T}_m, [S : \circ(A_1, \ldots, A_n)]$. Since $I$ does not refute

$$\mathfrak{T}_m, [S : \circ(A_1, \ldots, A_n)],$$

which is contained in $\mathfrak{R}^s$, it does not refute $\mathfrak{R}^s$. Because the valuation $I$ was chosen arbitrarily, we can conclude $\mathfrak{R}$ is not refuted by any valuation, making it irrefutable.

From this, it follows that if the reduction tree $T_{\mathfrak{R}}$ does not have a closed branch $B$, where the topmost many-sided anti-sequent $\mathfrak{R}^B$ is an axiom of $\mathsf{MSR}_{\mathcal{M}}$, then the root of $T_{\mathfrak{R}}$, given by $\mathfrak{R}$, is irrefutable. ∎

### 5.1.3 Optimisations

There are two significant and natural available optimisations of the previously introduced calculi. Firstly, if a generalised complementary partial normal form is tautological, i.e., evaluates to true under all valuations, we can simply omit the anti-sequent resulting from the generalised complementary partial normal form as described by Theorem 3.3 in the premiss of the rule.

More formally, let us introduce the following modified inference rule:

**Definition 5.3.** Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an $m$-valued Nmatrix, $S$ some non-empty strict subset of $\mathcal{V}$, and $\circ$ a connective in $\mathcal{M}$. Then, the inference rule $(\circ : S)$ is given as follows:

$$\frac{\mathfrak{R}}{\mathfrak{R}, [S : \circ(A_1, \ldots, A_n)]}\ (\circ : S),$$

where $\mathfrak{R}$ is an arbitrary $m$-component anti-sequent subject to the proviso $(RES)_S$ as in Definition 5.1. □

Secondly, if a connective is deterministic, we can clearly restrict the subsets of truth values for which their generalised complementary partial normal forms are considered to singletons. With this, for the case of deterministic matrices, the calculi generated by the optimised procedure and the calculi generated by the method of Bogojeski and Tompits [13] coincide.

Applying these two optimisations, we can define the following calculi:

**Definition 5.4.** Let $\mathcal{M}$ be an $m$-valued Nmatrix and $\mathcal{V} = \{\mathsf{v}_1, \ldots, \mathsf{v}_m\}$ its set of truth values. Then, the *optimised* many-sided rejection calculus $\mathsf{MSR}'_{\mathcal{M}}$ consists of the following items:

(i) the axiom schemata of $\mathsf{MSR}_{\mathcal{M}}$,

(ii) for every *non-deterministic* connective $\circ$ of $\mathcal{M}$ and every non-empty, strict subset $S$ of $\mathcal{V}$,

  (a) the inference rule $(\circ : S)$ from Definition 5.3, providing $gcpnf_S[\circ(A_1, \ldots, A_n)]$ is a tautology, and otherwise

  (b) the inference rules $(\circ : S)_1, \ldots, (\circ : S)_{k_i}$ as given by Definition 5.1,

(iii) for every *deterministic* connective $\circ$ of $\mathcal{M}$ and every *singleton subset* $S$ of $\mathcal{V}$, the inference rules as in items ((a)) and ((b)) but *without the restriction* $(RES)_S$, and

(iv) *weakening rules* for every $i$-th component of the form

$$\frac{\mathfrak{R}, [i : A]}{\mathfrak{R}} \ (w : i).$$

□

Notice that we also added weakening rules for every component in $\mathsf{MSR}'_{\mathcal{M}}$, which are required for achieving completeness of the calculi. Moreover, the notion of a proof and provability in $\mathsf{MSR}'_{\mathcal{M}}$ is defined similar as for $\mathsf{MSR}_{\mathcal{M}}$.

The proof of soundness for the optimised calculi proceeds just like for the unoptimised ones.

**Theorem 5.3.** If an anti-sequent is provable in $\mathsf{MSR}'_{\mathcal{M}}$, then it is refutable in $\mathcal{M}$.

*Proof.* Since the axioms of $\mathsf{MSR}'_{\mathcal{M}}$ and $\mathsf{MSR}_{\mathcal{M}}$ coincide, we only need to treat the cases for rules which are in $\mathsf{MSR}'$ but not in $\mathsf{MSR}$.

First of all, it obviously holds that if the premiss of a weakening rule is refutable, then so is its conclusion.

So, let us consider now an anti-sequent $\mathfrak{R}$ and suppose that it has been derived by some rule $(\circ : S)$ with a refutable premiss. Then, $\mathfrak{R}$ must be of the form

$$\mathfrak{R}', [S : \circ(A_1, \ldots, A_n)],$$

where $\mathfrak{R}'$ is the premiss of $(\circ : S)$ and it holds that $gcpnf_S[\circ(A_1, \ldots, A_n)]$ is a tautology. Now, let $I$ be a valuation that refutes $\mathfrak{R}'$. Clearly, $I \models_{SFE} gcpnf_S[\circ(A_1, \ldots, A_n)]$. As in the proof of Theorem 5.1, it follows then that $\mathfrak{R}$ is refutable.

Suppose now that $\mathfrak{R}$ has been derived by a rule $(\circ : S)_j$ with a refutable premiss, where $\circ$ is a deterministic connective and $S$ is a singleton subset of $\mathcal{V}$. Then, $\mathfrak{R}$ must be of the form $\mathfrak{R}', [S : \circ(A_1, \ldots, A_n)]$ and the premiss is of the form $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}$. Let $I$ be a valuation refuting $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}$. It follows that $I$ refutes both $\mathfrak{R}'$ and $\hat{\sigma}^j_{\circ:S}$. Thus, by Theorem 3.3, $I \models_{SFE} \sigma^j_{\circ:S}$, and since this is just one of the disjuncts of $gcpnf_S[\circ(A_1, \ldots, A_n)]$, we have $I \models_{SFE} gcpnf_S[\circ(A_1, \ldots, A_n)]$. By Corollary 3.1, we obtain $I \models_{SFE} \dot{\neg}(\circ(A_1, \ldots, A_n))^S$, which means that for all $s \in S$, $I(\circ(A_1, \ldots, A_n)) \neq s$. Therefore, $I$ is a countermodel of the anti-sequent $\mathfrak{T}_m, [S : \circ(A_1, \ldots, A_n)]$. Thus, $I$ refutes $\mathfrak{R}$. ■

We next turn to establish completeness of the optimised calculi. We first show two preparatory results.

**Lemma 5.1.** Let $\mathcal{M}$ be a finite Nmatrix. Suppose $S$ is a strict, non-empty subset of $\mathcal{V}$ and $S'$ a non-empty subset of $S$. Then, it holds that

$$gcpnf_S[\circ(A_1, \ldots, A_n)] \models_{SFE} gcpnf_{S'}[\circ(A_1, \ldots, A_n)].$$

*Proof.* Since $S' \subseteq S$, it clearly holds that $W_S^\circ \subseteq W_{S'}^\circ$. Now let $I$ be a valuation in $\mathcal{M}$ that satisfies $gcpnf_S[\circ(A_1, \ldots, A_n)]$. Then, $I$ must satisfy one of its disjuncts. But because of $W_S^\circ \subseteq W_{S'}^\circ$ this disjunct must also be part of $gcpnf_{S'}[\circ(A_1, \ldots, A_n)]$, and thus $I$ must satisfy $gcpnf_{S'}[\circ(A_1, \ldots, A_n)]$ as well. ∎

Instead of proving completeness directly, we will prove that the optimised calculus can simulate any proof in the standard calculus.

**Lemma 5.2.** If an anti-sequent is provable in $\mathsf{MSR}_\mathcal{M}$, then it is provable in $\mathsf{MSR}'_\mathcal{M}$.

*Proof.* We proceed by induction on the length $n$ of a proof in $\mathsf{MSR}_\mathcal{M}$.

Induction Base: Let $\mathfrak{R}$ be provable in $\mathsf{MSR}_\mathcal{M}$ and assume $n = 1$. Then clearly $\mathfrak{R}$ can only be an axiom and therefore it is also an axiom of $\mathsf{MSR}'_\mathcal{M}$.

Induction Step: Let $\mathfrak{R}$ now be provable in $\mathsf{MSR}_\mathcal{M}$ with a proof of length $n > 0$ and assume that the statement holds for all anti-sequents having a proof of length less than $n$. Then there must be some rule $(\circ : S)_j$ that was applied last in this proof. That means that $\mathfrak{R} = \mathfrak{R}', [S : \circ(A_1, \ldots, A_n)]$ and the premiss $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}$ has a proof of length $n - 1$. By the induction hypothesis, $\mathfrak{R}', \hat{\sigma}^j_{\circ:S}$ has a proof in $\mathsf{MSR}'$. We distinguish the following cases:

Case 1: Suppose $\circ$ is a deterministic connective and $gcpnf_S[\circ(A_1, \ldots, A_n)]$ is not a tautology. Assume $S = \{w_1, \ldots, w_k\}$ and let $j_i$ so that $\sigma^{j_i}_{\circ:w_i} = \sigma^j_{\circ:S}$, for each $i \in \{1, \ldots, k\}$. Note that such a $j_i$ exists since $W_S \subseteq W_{w_i}$. We can thus duplicate the anti-sequent $\hat{\sigma}^j_{\circ:S}$ as follows:

$$\mathfrak{R}', \hat{\sigma}^j_{\circ:S} = \mathfrak{R}', \hat{\sigma}^{j_1}_{\circ:w_1}, \ldots, \hat{\sigma}^{j_k}_{\circ:w_k}.$$

Now we successively apply the rules $(\circ : w_1)_{j_1}, \ldots, (\circ : w_k)_{j_k}$ to arrive at the anti-sequent $\mathfrak{R}$. Note that for these rules the proviso $(RES)_S$ is not stipulated and thus they can be applied one after the other.

Case 2: Suppose $\circ$ is a non-deterministic connective and $gcpnf_S[\circ(A_1, \ldots, A_n)]$ is a tautology. Then, we can first successively apply the weakening rule to get a proof of $\mathfrak{R}$ in $\mathsf{MSR}'_\mathcal{M}$, and afterwards we apply rule $(\circ : S)$.

Case 3: Suppose $\circ$ is a deterministic connective and $gcpnf_S[\circ(A_1, \ldots, A_n)]$ is a tautology. This case is a combination of Cases 1 and 2. What is important to notice is that from Lemma 5.1 it follows that $gcpnf_{w_i}[\circ(A_1, \ldots, A_n)]$ is a tautology for every $i \in \{1, \ldots, k\}$.

Case 4: Suppose $\circ$ is a non-deterministic connective and $gcpnf_S[\circ(A_1, \ldots, A_n)]$ is not a tautology. Then, the rule $(\circ : S)_j$ is part of $\mathsf{MSR}'_\mathcal{M}$ and can therefore be applied to obtain $\mathfrak{R}$. ∎

**Theorem 5.4.** If an anti-sequent is refutable in $\mathcal{M}$, then it is provable in $\mathsf{MSR}'_\mathcal{M}$.

*Proof.* This follows directly from the completeness of $\mathsf{MSR}_\mathcal{M}$ and Lemma 5.2. ∎

$$\frac{\Gamma \nmid \Delta}{\Gamma, {\sim}A \nmid \Delta}\,(\sim : \mathbf{f}) \qquad \frac{\Gamma, A \nmid \Delta}{\Gamma \nmid \Delta, {\sim}A}\,(\sim : \mathbf{t})$$

$$\frac{\Gamma, A \nmid \Delta}{\Gamma, A \vee B \nmid \Delta}\,(\vee : \mathbf{f})_1 \qquad \frac{\Gamma, B \nmid \Delta}{\Gamma, A \vee B \nmid \Delta}\,(\vee : \mathbf{f})_2 \qquad \frac{\Gamma \nmid \Delta, A, B}{\Gamma \nmid \Delta, A \vee B}\,(\vee : \mathbf{t})$$

$$\frac{\Gamma, A, B \nmid \Delta}{\Gamma, A \wedge B \nmid \Delta}\,(\wedge : \mathbf{f}) \qquad \frac{\Gamma \nmid \Delta, A}{\Gamma \nmid \Delta, A \wedge B}\,(\wedge : \mathbf{t})_1 \qquad \frac{\Gamma \nmid \Delta, B}{\Gamma \nmid \Delta, A \wedge B}\,(\wedge : \mathbf{t})_2$$

$$\frac{\Gamma \nmid \Delta, A}{\Gamma, A \supset B \nmid \Delta}\,(\supset : \mathbf{f})_1 \qquad \frac{\Gamma, B \nmid \Delta}{\Gamma, A \supset B \nmid \Delta}\,(\supset : \mathbf{f})_2 \qquad \frac{\Gamma, A \nmid \Delta, B}{\Gamma \nmid \Delta, A \supset B}\,(\supset : \mathbf{t})$$

Figure 5.1: Inference rules of the optimised rejection calculus $\mathsf{MSR}'_{\mathcal{M}_1}$ for **CLuN**.

### 5.1.4 Anti-Sequent Calculi for the Paraconsistent Logics CLuN and Cio

Analogous to Section 4.1.3, we now construct specific many-sided anti-sequent calculi for the paraconsistent logics **CLuN** and **Cio** as special instances of our general framework.

Let us consider **CLuN** first. We already determined minimised forms of the generalised complementary partial normal forms for $\mathcal{M}_1$ in Figure 3.1. We can immediately devise the optimised calculus $\mathsf{MSR}'_{\mathcal{M}_1}$ from this. The resultant rules are depicted in Figure 5.1.

It is important to note that, due to the fact that the connectives $\vee$, $\wedge$, and $\supset$ are all deterministic and their semantics correspond to the usual semantics of classical logic, the rules generated by our method correspond to the rules for the two-sided Gentzen-style rejection system of classical propositional logic as described by Bonatti [14] and Goranko [25], except for the rule $(\sim : \mathbf{f})$. Indeed, instead of having as premiss a sequent with the principal formula in its second component, there is no mention of the principal formula in the premiss of $(\sim : \mathbf{f})$ at all. This corresponds to the intuition that in **CLuN** the negation of a formula can always chosen to be true, so the statement that a negation of a formula is false can always be rejected.

**Example 5.5.** We showed in Example 2.2 that the formula $A \supset {\sim}{\sim}A$ is not valid in **CLuN**, where $A$ is an atom. The following proof in $\mathsf{MSR}'_{\mathcal{M}_1}$ establishes this property in a purely syntactical manner:

$$\cfrac{\cfrac{\cfrac{A \nmid \emptyset}{A, {\sim}A \nmid \emptyset}\,(\sim : \mathbf{f})}{A \nmid {\sim}{\sim}A}\,(\sim : \mathbf{t})}{\emptyset \nmid A \supset {\sim}{\sim}A}\,(\supset : \mathbf{t})$$

Note that from the above proof we can immediately construct a countermodel: To refute the axiom, a valuation needs to make $A$ true. The use of rule $({\sim} : \mathbf{f})$ implies that ${\sim}A$ also needs to be true, and the application of $(\supset: \mathbf{t})$ enforces ${\sim}{\sim}A$ to be evaluated as false. △

Now let us turn to the logic **Cio**. For this instance, we obtain a three-sided rejection calculus that is induced by the Nmatrix $\mathcal{M}_2$ from Figure 2.2 and its corresponding generalised complementary partial normal form as given in Figure 3.3. Since this logic is three-valued, the number of rules in our rejection calculus is larger than than that in the previous example. The inference rules for the resulting calculus $\mathsf{MSR}'_{\mathcal{M}_2}$ for **Cio** are depicted in Figure 5.2. It is important to note that each rule $(\circ : S)_i$ and $(\circ : S)$ in Figure 5.2 is subject to the respective proviso $(RES)_S$.

**Example 5.6.** As shown in Example 2.3, the principle of explosion, $A \wedge \neg A \supset B$, is not valid in **Cio**, in particular for atoms $A$ and $B$. We can now also arrive at this observation by means of the following proof in $\mathsf{MSR}'_{\mathcal{M}_2}$:

$$\dfrac{\dfrac{\dfrac{A \nmid B \nmid B, A}{A, \neg A \nmid B \nmid B}\,(\neg : \mathbf{f})}{A \wedge \neg A \nmid B \nmid B}\,(\wedge : \mathbf{f})}{\emptyset \nmid A \wedge \neg A \supset B \nmid A \wedge \neg A \supset B}\,(\supset: \{\mathbf{i}, \mathbf{t}\})$$

△

## 5.2 Anti-sequent Calculi for the Static Semantics

In this section, similarly to the assertional case, we will modify the previously introduced systematic method of generating anti-sequent calculi for the dynamic semantics to get anti-sequent calculi which are sound and complete with respect to the static semantics.

### 5.2.1 Postulates of the Static Calculi

Recall the definitions introduced in Subsection 4.2.1. With them at hand, we are already in a position to provide the definition of provability in an anti-sequent calculus for the static case.

**Definition 5.7.** Let $\mathcal{M}$ be an $m$-valued Nmatrix. A sequence $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$ of $m$-component anti-sequents is a *proof in* $\mathsf{MSR}^s_{\mathcal{M}}$, where $\mathsf{MSR}^s_{\mathcal{M}}$ is the *static* version of $\mathsf{MSR}_{\mathcal{M}}$, if there is a deterministic cast $\mathcal{M}^c$ of $\mathcal{M}$ such that $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$ is a proof in $\mathsf{MSR}'_{\mathcal{M}^c}$. Moreover, an $m$-component anti-sequent $\mathfrak{R}$ is *provable in* $\mathsf{MSR}^s_{\mathcal{M}}$ if there is a proof in $\mathsf{MSR}^s_{\mathcal{M}}$ whose last element is $\mathfrak{R}$. □

Similarly to the assertional case, the above definition gives rise to a well-defined proof system.

$$\frac{\Gamma \nmid \Delta, A \nmid \Pi}{\Gamma, \sharp A \nmid \Delta \nmid \Pi}\,(\sharp{:}\mathbf{f}) \qquad \frac{\Gamma \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta, \sharp A \nmid \Pi}\,(\sharp{:}\mathbf{i}) \qquad \frac{\Gamma, A \nmid \Delta \nmid \Pi, A}{\Gamma \nmid \Delta \nmid \Pi, \sharp A}\,(\sharp{:}\mathbf{t}) \qquad \frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma, \neg A \nmid \Delta \nmid \Pi}\,(\neg{:}\mathbf{f})$$

$$\frac{\Gamma \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta, \neg A \nmid \Pi}\,(\neg{:}\mathbf{i}) \qquad \frac{\Gamma, A \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta \nmid \Pi, \neg A}\,(\neg{:}\mathbf{t}) \qquad \frac{\Gamma \nmid \Delta \nmid \Pi, A}{\Gamma, \neg A \nmid \Delta, \neg A \nmid \Pi}\,(\neg{:}\{\mathbf{f},\mathbf{i}\})$$

$$\frac{\Gamma, A \nmid \Delta \nmid \Pi, A}{\Gamma, \neg A \nmid \Delta \nmid \Pi, \neg A}\,(\neg{:}\{\mathbf{f},\mathbf{t}\}) \qquad \frac{\Gamma, A \nmid \Delta, A \nmid \Pi}{\Gamma \nmid \Delta, \neg A \nmid \Pi, \neg A}\,(\neg{:}\{\mathbf{i},\mathbf{t}\})$$

$$\frac{\Gamma, A \nmid \Delta \nmid \Pi}{\Gamma, A \vee B \nmid \Delta \nmid \Pi}\,(\vee{:}\mathbf{f})_1 \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi}{\Gamma, A \vee B \nmid \Delta \nmid \Pi}\,(\vee{:}\mathbf{f})_2 \qquad \frac{\Gamma \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta, A \vee B \nmid \Pi}\,(\vee{:}\mathbf{i})$$

$$\frac{\Gamma, B \nmid \Delta, A \nmid \Pi, A, B}{\Gamma \nmid \Delta \nmid \Pi, A \vee B}\,(\vee{:}\mathbf{t})_1 \qquad \frac{\Gamma, A, B \nmid \Delta \nmid \Pi, A, B}{\Gamma \nmid \Delta \nmid \Pi, A \vee B}\,(\vee{:}\mathbf{t})_2$$

$$\frac{\Gamma, A \nmid \Delta \nmid \Pi}{\Gamma, A \vee B \nmid \Delta, A \vee B \nmid \Pi}\,(\vee{:}\{\mathbf{f},\mathbf{i}\})_1 \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi}{\Gamma, A \vee B \nmid \Delta, A \vee B \nmid \Pi}\,(\vee{:}\{\mathbf{f},\mathbf{i}\})_2$$

$$\frac{\Gamma, A, B \nmid \Delta \nmid \Pi, A, B}{\Gamma, A \vee B \nmid \Delta \nmid \Pi, A \vee B}\,(\vee{:}\{\mathbf{f},\mathbf{t}\}) \qquad \frac{\Gamma \nmid \Delta, A, B \nmid \Pi, A, B}{\Gamma \nmid \Delta, A \vee B \nmid \Pi, A \vee B}\,(\vee{:}\{\mathbf{i},\mathbf{t}\})$$

$$\frac{\Gamma \nmid \Delta, A \nmid \Pi, A}{\Gamma, A \supset B \nmid \Delta \nmid \Pi}\,(\supset{:}\mathbf{f})_1 \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi}{\Gamma, A \supset B \nmid \Delta \nmid \Pi}\,(\supset{:}\mathbf{f})_2 \qquad \frac{\Gamma \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta, A \supset B \nmid \Pi}\,(\supset{:}\mathbf{i})$$

$$\frac{\Gamma, A \nmid \Delta, B \nmid \Pi, B}{\Gamma \nmid \Delta \nmid \Pi, A \supset B}\,(\supset{:}\mathbf{t})_1 \qquad \frac{\Gamma, A, B \nmid \Delta \nmid \Pi, A, B}{\Gamma \nmid \Delta \nmid \Pi, A \supset B}\,(\supset{:}\mathbf{t})_2$$

$$\frac{\Gamma \nmid \Delta, A \nmid \Pi, A}{\Gamma, A \supset B \nmid \Delta, A \supset B \nmid \Pi}\,(\supset{:}\{\mathbf{f},\mathbf{i}\})_1 \qquad \frac{\Gamma, B \nmid \Delta \nmid \Pi}{\Gamma, A \supset B \nmid \Delta, A \supset B \nmid \Pi}\,(\supset{:}\{\mathbf{f},\mathbf{i}\})_2$$

$$\frac{\Gamma, A, B \nmid \Delta \nmid \Pi, A, B}{\Gamma, A \supset B \nmid \Delta \nmid \Pi, A \supset B}\,(\supset{:}\{\mathbf{f},\mathbf{t}\}) \qquad \frac{\Gamma, A \nmid \Delta, B \nmid \Pi, B}{\Gamma \nmid \Delta, A \supset B \nmid \Pi, A \supset B}\,(\supset{:}\{\mathbf{i},\mathbf{t}\})$$

$$\frac{\Gamma, A, B \nmid \Delta \nmid \Pi}{\Gamma, A \wedge B \nmid \Delta \nmid \Pi}\,(\wedge{:}\mathbf{f}) \qquad \frac{\Gamma \nmid \Delta \nmid \Pi}{\Gamma \nmid \Delta, A \wedge B \nmid \Pi}\,(\wedge{:}\mathbf{i}) \qquad \frac{\Gamma \nmid \Delta \nmid \Pi, A, B}{\Gamma \nmid \Delta \nmid \Pi, A \wedge B}\,(\wedge{:}\mathbf{t})_1$$

$$\frac{\Gamma, B \nmid \Delta, A, B \nmid \Pi, A}{\Gamma \nmid \Delta \nmid \Pi, A \wedge B}\,(\wedge{:}\mathbf{t})_2 \qquad \frac{\Gamma, A \nmid \Delta, A, B \nmid \Pi; B}{\Gamma \nmid \Delta \nmid \Pi, A \wedge B}\,(\wedge{:}\mathbf{t})_3$$

$$\frac{\Gamma, A, B \nmid \Delta \nmid \Pi}{\Gamma, A \wedge B \nmid \Delta, A \wedge B \nmid \Pi}\,(\wedge{:}\{\mathbf{f},\mathbf{i}\}) \qquad \frac{\Gamma, A, B \nmid \Delta \nmid \Pi, A, B}{\Gamma, A \wedge B \nmid \Delta \nmid \Pi, A \wedge B}\,(\wedge{:}\{\mathbf{f},\mathbf{t}\})$$

$$\frac{\Gamma \nmid \Delta, A \nmid \Pi, A}{\Gamma \nmid \Delta, A \wedge B \nmid \Pi, A \wedge B}\,(\wedge{:}\{\mathbf{i},\mathbf{t}\})_1 \qquad \frac{\Gamma \nmid \Delta, B \nmid \Pi, B}{\Gamma \nmid \Delta, A \wedge B \nmid \Pi, A \wedge B}\,(\wedge{:}\{\mathbf{i},\mathbf{t}\})_2$$

Figure 5.2: Inference rules of the optimised rejection calculus $\mathsf{MSR}'_{\mathcal{M}_2}$ for **Cio**.

With Lemmata 4.1 and 4.2 at hand, it is straightforward to prove soundness and completeness of the static calculus by reducing it to the soundness and completeness of the underlying calculi.

**Theorem 5.5.** If an anti-sequent is provable in $\mathsf{MSR}^s_{\mathcal{M}}$, then it is statically refutable in $\mathcal{M}$.

*Proof.* Suppose $\mathfrak{R}$ is provable in $\mathsf{MSR}^s_{\mathcal{M}}$. By definition, there is some deterministic cast $\mathcal{M}^c$ of $\mathcal{M}$ so that $\mathfrak{R}$ is provable in $\mathsf{MSR}'_{\mathcal{M}^c}$. By the soundness of $\mathsf{MSR}'_{\mathcal{M}^c}$, we know that $\mathfrak{R}$ is refutable in $\mathcal{M}^c$, and since for deterministic matrices the static and dynamic semantics coincide, we have that $\mathfrak{R}$ is statically refutable in $\mathcal{M}^c$. Therefore, there is some static valuation $I$ in $\mathcal{M}^c$ that refutes $\mathfrak{R}$. By Lemma 4.1, $I$ is also a static valuation in $\mathcal{M}$. Consequently, $\mathfrak{R}$ is statically refutable in $\mathcal{M}$. ■

**Theorem 5.6.** If an anti-sequent is statically refutable in $\mathcal{M}$, then it is provable in $\mathsf{MSR}^c_{\mathcal{M}}$.

*Proof.* Suppose $I$ is a static valuation refuting an anti-sequent $\mathfrak{R}$ in $\mathcal{M}$. Let $\mathcal{M}^c$ be the canonical cast of $\mathcal{M}$ with respect to $I$. Clearly, $I$ is also a valuation in $\mathcal{M}^c$ and thus refutes $\mathfrak{R}$ in $\mathcal{M}^c$. By completeness of $\mathsf{MSR}'_{\mathcal{M}^c}$, $\mathfrak{R}$ is provable in $\mathsf{MSR}'_{\mathcal{M}^c}$, and thus, by definition, $\mathfrak{R}$ is provable in $\mathsf{MSR}^s_{\mathcal{M}}$. ■

### 5.2.2 An Anti-Sequent Calculus for CAR

As for the dynamic case, we now instantiate our method to obtain a calculus for the paraconsistent logic **CAR**, which is induced by the static semantics of the Nmatrix $\mathcal{M}_1$ as given in Figure 2.1. Since the only non-deterministic connective of $\mathcal{M}_1$ is $\sim$, and $def_{\sim}$ only has two deterministic casts, $\mathcal{M}_1$ accordingly also has only two deterministic casts, viz. $\mathcal{M}_1^{c1}$ with truth table

| $\sim$ | |
|---|---|
| **f** | **t** |
| **t** | **f** |

and $\mathcal{M}_1^{c2}$ with truth table

| $\sim$ | |
|---|---|
| **f** | **t** |
| **t** | **t** |

The generalised complementary partial normal forms with respect to the casts also differ from the ones given in Figure 3.1 only in the case of $\sim$, which are as follows:

$$gcpnf_{\mathbf{f}}^{\mathcal{M}_1^{c1}}[\sim A] \equiv_{SFE} \dot{\neg} A^{\mathbf{t}}; \qquad gcpnf_{\mathbf{f}}^{\mathcal{M}_1^{c2}}[\sim A] \equiv_{SFE} \dot{\top};$$
$$gcpnf_{\mathbf{t}}^{\mathcal{M}_1^{c1}}[\sim A] \equiv_{SFE} \dot{\neg} A^{\mathbf{f}}; \qquad gcpnf_{\mathbf{t}}^{\mathcal{M}_1^{c2}}[\sim A] \equiv_{SFE} \dot{\bot}.$$

The rules for $\mathsf{MSR}'_{\mathcal{M}^{c_1}}$ and $\mathsf{MSR}'_{\mathcal{M}^{c_2}}$ are as follows: while both $\mathsf{MSR}'_{\mathcal{M}^{c_1}}$ and $\mathsf{MSR}'_{\mathcal{M}^{c_2}}$ have the common rules depicted in Figure 5.3, $\mathsf{MSR}'_{\mathcal{M}^{c_1}}$ uses additionally the rules

$$\frac{\Gamma, A \nmid \Delta}{\Gamma, A \vee B \nmid \Delta}(\vee : \mathbf{f})_1 \qquad \frac{\Gamma, B \nmid \Delta}{\Gamma, A \vee B \nmid \Delta}(\vee : \mathbf{f})_2 \qquad \frac{\Gamma \nmid \Delta, A, B}{\Gamma \nmid \Delta, A \vee B}(\vee : \mathbf{t})$$

$$\frac{\Gamma, A, B \nmid \Delta}{\Gamma, A \wedge B \nmid \Delta}(\wedge : \mathbf{f}) \qquad \frac{\Gamma \nmid \Delta, A}{\Gamma \nmid \Delta, A \wedge B}(\wedge : \mathbf{t})_1 \qquad \frac{\Gamma \nmid \Delta, B}{\Gamma \nmid \Delta, A \wedge B}(\wedge : \mathbf{t})_2$$

$$\frac{\Gamma \nmid \Delta, A}{\Gamma, A \supset B \nmid \Delta}(\supset : \mathbf{f})_1 \qquad \frac{\Gamma, B \nmid \Delta}{\Gamma, A \supset B \nmid \Delta}(\supset : \mathbf{f})_2 \qquad \frac{\Gamma, A \nmid \Delta, B}{\Gamma \nmid \Delta, A \supset B}(\supset : \mathbf{t})$$

Figure 5.3: Common inference rules of the rejection calculi $\mathsf{MSR}'_{\mathcal{M}_1^{c1}}$ and $\mathsf{MSR}'_{\mathcal{M}_1^{c2}}$ for **CAR**.

$$\frac{\Gamma \nmid \Delta, A}{\Gamma, \sim A \nmid \Delta}(\sim : \mathbf{f})_1 \quad \text{and} \quad \frac{\Gamma, A \nmid \Delta}{\Gamma \nmid \Delta, \sim A}(\sim : \mathbf{t})_1,$$

whereas $\mathsf{MSR}'_{\mathcal{M}^{c_2}}$, on the other hand, uses instead the rule

$$\frac{\Gamma \nmid \Delta}{\Gamma, \sim A \nmid \Delta}(\sim : \mathbf{f})_2.$$

Recall that a proof in $\mathsf{MSR}^s_{\mathcal{M}_1}$ is only considered valid if either rules $(\sim : \mathbf{f})_1$ and $(\sim : \mathbf{t})_1$ or else $(\sim : \mathbf{f})_2$ are used in the proof exclusively.

**Example 5.8.** As shown by Da Costa and Béziau [21], the following formulas are not valid in **CAR**:

$$(\sim P \supset \sim Q) \supset (Q \supset P), \tag{5.1}$$
$$Q \supset (\sim Q \supset P), \text{ and} \tag{5.2}$$
$$\sim\sim P \supset P. \tag{5.3}$$

The following proofs show that these formulas are provable in $\mathsf{MSR}^s_{\mathcal{M}_1}$:

(i) Proof of (5.1):

$$\cfrac{\cfrac{\cfrac{\cfrac{Q \nmid P}{\sim Q, Q \nmid P}(\sim : \mathbf{f})_2}{\sim P \supset \sim Q, Q \nmid P}(\supset : \mathbf{f})_2}{\sim P \supset \sim Q \nmid Q \supset P}(\supset : \mathbf{t})}{\emptyset \nmid (\sim P \supset \sim Q) \supset (Q \supset P)}(\supset : \mathbf{t})$$

(ii) Proof of (5.2):

$$\cfrac{\cfrac{\cfrac{Q \nmid P}{Q, {\sim}Q \nmid P}\,({\sim} : \mathbf{f})_2}{Q \nmid {\sim}Q \supset P}\,(\supset : \mathbf{t})}{\emptyset \nmid Q \supset ({\sim}Q \supset P)}\,(\supset : \mathbf{t})$$

(iii) Proof of (5.3):

$$\cfrac{\cfrac{\emptyset \nmid P}{{\sim}{\sim}P \nmid P}\,({\sim} : \mathbf{f})_2}{\emptyset \nmid {\sim}{\sim}P \supset P}\,(\supset : \mathbf{t})$$

△

CHAPTER 6

# Conclusion

In this thesis, we presented generalisations of the methods introduced by Zach [54] and by Bogojeski and Tompits [13] to systematically generate sequent-style assertional and complementary calculi for finite-valued logics. These new methods are applicable to every logic that can be defined using the generalised framework of non-deterministic matrices developed by Avron and Lev [7]. We proved soundness and completeness of the generated calculi and discussed optimisations to reduce the number of generated rules. Since there are two natural semantics for every Nmatrix, we also discussed how to adapt our method of the dynamic semantics also for the static case.

As special instances of our systematic approach, we constructed specific sequent and anti-sequent calculi for some paraconsistent logics, viz. for **CLuN** [10], **Cio** [18], and **CAR** [21]. To the best of our knowledge, no sequent-style rejection calculi have been introduced for these logics previously.

Even though a procedure for the assertional case has been previously described by Avron and Konikowska [5], based on Rasiowa-Sikorski systems [41] and using signed formulas, our method generates different calculi and constitutes a uniform approach for dealing with assertional and complementary calculi.

An interesting approach for future work would be to also construct two-sided anti-sequent calculi for **Cio**. To this end, one could adapt a method introduced by Avron, Ben-Naim, and Konikowska [3] to the refutational case which allows to transform many-sided sequent calculi to two-sided ones. Due to the systematic nature of our procedure, it could be implemented as a possible extension of MULTLOG [9] that currently only generates proof systems for a given finite classical matrix.

Rejection systems are particularly useful to build proof systems for nonmonotonic reasoning. In particular, Bonatti and Olivetti [15] pioneered the use of rejection systems for building sequent-style axiomatisations for the main nonmonotonic reasoning formalisms, viz. for default logic [42], autoepistemic logic [35], and circumscription [33]. More recently, following the method of Bonatti and Olivetti [15], Geibinger and Tompits [23] introduced a general method for building sequent-style calculi for various nonmonotonic paraconsistent logics based on minimal entailment over finite-valued logics, including the well-known approaches by Priest [38] and Arieli and Avron [1]. Similar systems could also be realised, e.g., for the approaches by Besnard and Schaub [12].

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