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Robotics and Growth with Electricity and Taxation

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Abstract

The aim of this thesis is to explore the effects of robot and electricity taxes on economic growth. New technological advancements in industrial robotics, 3D printing and artificial intelligence threaten to replace human labor in more and more tasks, while using substantial amounts of electric power. Given the practical difficulties of introducing a robot tax, such as coming up with workable definitions, we explore the possibility of introducing an electricity tax instead. To this end, the model framework developed by Erling Steigum in his paper "Robotics and Growth" (2011) is expanded to include taxes and power consumption. Analytically, we show that both types of taxes have the same qualitative effect - lowering interest rates and growth, while raising wages - when robots are assumed to be more power hungry than traditional capital. In a numerical illustration, however, it is calculated that the robot tax has a much larger, long term impact on the economy, calling into question whether an electricity tax could be used interchangeably.

Contents

1 Introduction

Spurred by new technological innovations, the topic of automation has drawn increased public attention over the last few years and its economic consequences have been hotly debated by researchers. Both the public as well as the academic discourse has been particularly interested in the question whether the introduction of robots and ever more sophisticated computer programs will replace humans in the workplace and lead to unemployment.

This discussion, however, is actually not entirely new. Already in 1930, John Maynard Keynes wrote in his essay "Economic Possibilities for Our Grandchildren", laying out his vision for future economic development: "We are being afflicted with a new disease of which some readers may not yet have heard the name, but of which they will hear a great deal in the years to come—namely, technological unemployment. This means unemployment due to our discovery of means of economising the use of labour outrunning the pace at which we can find new uses for labour." Keynes was optimistic that this problem would only be temporary in nature and he predicted that living standards would increase by four to eight times during the next 100 years, as long as there were no major wars and crises.

Automation, in particular the introduction of personal computers to the workplace, did indeed affect the labor market during the second half of the 20th century, albeit not in the form of mass unemployment. Spitz-Oener (2006) documents that, rather than necessarily rendering jobs obsolete, past computerization in West Germany led to a shift within occupations from routine manual and cognitive tasks to non-routine analytical and interactive activities. This change went hand in hand with an increase in skill requirements and highly educated workers, who could gain the most from the new technological opportunities. Spitz-Oener (2006) also finds evidence of job polarization - the idea that employment in medium skill, middle class occupations has decreased, while the extremes of the distribution have grown. One explanation offered is that medium skill jobs such as office clerks involve a lot of routine tasks that computers can take over to a large extent. On the other hand, low skill occupations like waiters tend to be a part of the service sector and feature more non-routine tasks that cannot easily be automated.

Over roughly the last decade, this picture has started to shift and economists are once again concerned about the possibility of technological unemployment. They point to the rapid development of innovations which allow machines and computers to conduct non-routine tasks that were once considered safe from automation, putting even high skill jobs at risk. Beaudry et al. (2016) even argue that demand for highly educated labor has already been declining since 2000 and high-skilled workers have had to move down on the occupational ladder.

More and more emerging technologies seem to have the potential to replace human labor: Companies such as Google and Tesla have introduced self driving cars, which may put the jobs of cab and truck drivers at risk. An increasing amount of supermarkets, stores and fast food restaurants are utilizing self-checkout machines, reducing the need for staff. 3D printing technology, utilized to manufacture products like hearing aids, may cause disruptions to the international economy, as analyzed by Abeliansky et al. (2020). They find that developed countries will adopt 3D printing first, causing a loss of exports for poor countries. The International Federation of Robotics (2021) reports an ever increasing number of industrial robots - fully autonomous machines which do not need human operators - in U.S. factories (310700 in 2020).

Perhaps the greatest potential for economic disruption lies within the advancements made in artificial intelligence and machine learning, which enable the automation of non-routine cognitive tasks, once thought to be impossible. For instance, the Associated Press (in partnership with the A.I. language generation company Automated Insights) announced already in 2014 that it was going to automate most of the reporting on U.S. quarterly corporate earnings, in addition to minor sports stories. Given a patient's medical data, recent A.I. algorithms have been extremely successful in diagnosing a wide range of diseases, including diabetes, Alzheimer's and various forms of cancer (see Kumar et al. (2022)). While Frey and Osborne (2017) still considered arts to be one of the least susceptible fields, new image generation software like DALL-E 2, Midjourney and Stable Diffusion have been able to turn text prompts into artwork that has even won prices, causing uproar in the artistic community (see New York Times (2022)).

These developments have led many researchers to conclude that the current technological progress is fundamentally different to past advancements and may have severe adverse effects on the labor market. On the high end, Frey and Osborne (2017) calculate that 47% of U.S. jobs are at high risk, meaning that they could be automated over the next decade or two. Using an arguably more precise task-based approach, Arntz et al. (2016) estimate that 9% of jobs are automatable in the United States. Among OECD member states, Austria is the most vulnerable with 12% of jobs at risk according to their analysis. Acemoglu and Restrepo (2020) find that the introduction of industrial robots locally decreases both employment and wages. Lankisch et al. (2019) show that automation can contribute to increased income inequality and explain the rising skill premium that has been observed in the U.S. since the 1980s.

Given all of these potential negative effects, it makes sense to consider curbing the spread of automation, for instance by taxing the usage of robots. However, automation also brings economic benefits which have been explored in the literature.

Acemoglu and Restrepo (2021) explore the relationship between demographics & automation and find that aging societies are especially incentivized to adopt robotics, since it allows them to fill job shortages created by demographic change. Prettner (2019) uses a Solow model framework to show that robots can generate endogenous growth without the need for exogenous technological developments. In a Ramsey-style environment, Steigum (2011) comes to similar conclusions, noting that - depending on the choice of parameters - endogenous growth can even be achieved when human labor and robots are not perfect substitutes. In both models, the labor share of income shrinks continually, while the capital share increases. Acemoglu and Restrepo (2018) allow for two types of innovations - one which automates tasks and one which creates new tasks. This aims to replicate the phenomenon that technological progress has created new types of jobs over the years, such as programmers. They find that human labor and robots can coexist in the long run, as long as capital is expensive enough. Otherwise, automation will progress rapidly and labor will become redundant.

Currently there are no taxes specifically targeting robots anywhere in the world. Nevertheless, due to automation's potential adverse effects on the labor market and the revenue loss that governments would incur if tax paying workers were laid off and replaced by tax free robots, the topic is hotly debated and influential figures like Bill Gates and former New York City mayor Bill de Blasio have called for a tax to be introduced (see The Wall Street Journal (2020)). During his presidential campaign, de Blasio proposed that companies should have to pay a fee equal to five years of payroll taxes for every job they automate.

One major obstacle for the introduction of any robot tax is that it is very difficult to precisely define what a robot is in legal terms. While the International Federation of Robotics has developed a definition for industrial robots, it is hard to categorize machines in other lines of work: Should a computer be classified and taxed as a robot, since it can run sophisticated A.I. algorithms? One aim of this thesis is to explore a way of getting around this problem - by taxing electricity consumption as a proxy. The idea is, if robots use more power than traditional machines, then an electricity tax will relatively disadvantage automation and replicate the effects of a theoretical robot tax to some extent.

In recent years, the issue of robot taxes has also caught the attention of economists. To study the problem, Guerreiro et al. (2021) use an overlapping generations model (OLG) where young people may choose their desired level of education. They find that a small robot tax levied on firms is optimal during the transitory phase where lesser educated workers are not yet retired. Gasteiger and Prettner (2022) plug the production technology from Prettner (2019) into a canonical OLG model, in which young people earn wages from working and old people live off their savings. In contrast to the growth observed in the representative agent setting, they find that the economy stagnates, highlighting the importance of model selection. Additionally, they discover that a robot tax can increase wages, output per capita and overall welfare.

This thesis aims to explore the economic effects of robot and electricity taxes and to discuss to what extent they are interchangeable. To this end, the model proposed by Steigum (2011) is modified in such a way that the two types of capital - traditional (non-autonomous machines, production halls, etc.) and automation capital (industrial robots, 3D printers, etc.) - use electricity and firms have to pay taxes for power consumption and employing automation capital. The government may or may not return the tax proceeds to households in the form of transfers - both cases are examined. Along the balanced growth path, households invest in both capital stocks and endogenous growth is achievable when standard parameter values are chosen.

We find that automation capital grows the fastest, followed by traditional capital and consumption. When transfers are included, consumption is increased at every point in time, but grows at the same rate as before. The growth of traditional and automation capital is slowed by the introduction of transfers, as households are able to consume more while saving less. We find that a robot tax always lowers consumption growth and interest rates, while raising wages. If traditional capital is sufficiently power efficient relative to robots, a tax on electricity will replicate these effects qualitatively. Otherwise, the introduction of an electricity tax lowers wages, interest rates and growth.

Calibrated with U.S. data, the model predicts rapid growth for the robot stock over the next 10 years. Comparing the two types of taxes, the calculations show that the robot tax is more detrimental to growth and that electricity would have to be taxed very highly in order to be able to create a similar long term impact on the economy. Part of the reason for this is that the robot tax base (i.e. automation capital) grows faster than the electricity tax base (i.e. electricity consumption). Therefore, the quantitative results suggest that the two taxes have quite disparate effects and can not necessarily be used interchangeably.

The thesis is structured as follows: In section 2, the model is laid out and analyzed. More specifically, 2.1 explains the basic assumptions, 2.2 and 2.3 describe the household and firm problems, before the market equilibrium is derived in 2.4. 2.5 looks at the balanced growth path and the derivation of growth rates, which are then compared with the results of Steigum (2011) in 2.6. Transfers are introduced in 2.7, 2.8 deals with the impact of taxation and 2.9 describes the model dynamics.

In section 3, the model is dealt with numerically. In 3.1 the model is calibrated using data from the U.S. manufacturing sector, 3.2 lays out which variables can be observed by means of comparative statics and 3.3 describes the model predictions in the baseline scenario. The consequences of an increase in electricity prices are discussed in 3.4, 3.5 explores how an electricity tax could replicate the effects of a robot tax and 3.6 repeats this exercise with the alternate assumption that robots are less power hungry than traditional capital. Finally, section 4 summarizes the findings and gives an outlook towards possible future research avenues.

2 Model Description and Analysis

2.1 Notation and Assumptions

Consider an economy with three production inputs, human labor L , traditional physical capital $K(t)$ (machines, assembly lines, production halls, etc), and automation capital $Z(t)$ (industrial robots, 3D printers, etc). Time t evolves continuously, while the population is constant and equivalent to the workforce. There is a representative, infinitely lived individual, who maximizes their utility gained from consumption subject to a budget constraint. We assume that traditional capital and automation capital both depreciate at the same rate δ. Human labor and traditional physical capital are imperfect substitutes, whereas automation capital and human labor are perfect substitutes (Steigum, 2011; Prettner, 2019). This assumption ensures tractability of the model and is meant to present the benchmark case of full automation. The qualitative findings would not change in case of a comparatively high but imperfect substitutability between automation capital and labor (see Steigum, 2011; Gasteiger and Prettner, 2022). There is no technological progress, meaning that growth can only ever be achieved through automation and capital accumulation.

2.2 Households

Following Steigum (2011), who builds on Ramsey (1928), Cass (1965), and Koopmans (1963), the representative individual maximizes lifetime utility, which derives from the iso-elastic utility function

$$U_0 = \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1-\theta} dt, \tag{1}$$

where ρ is the time preference rate, $c(t)$ is instantaneous per capita consumption at time t, and θ determines the elasticity of intertemporal substitution. Denoting per capita assets (consisting of automation capital $Z(t)$ and traditional physical capital $K(t)$) by $m(t)$, the flow budget constraint is given by

$$\dot{m}(t) = r(t)m(t) + w(t) - c(t). \tag{2}$$

For now, the government does not transfer any of its tax revenues to households (transfers will be discussed in section 2.7). Intertemporal optimization leads to the well-known Keynes-Ramsey rule for consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho}{\theta},$$

stating that consumption expenditure growth is positive whenever the interest rate (r) overcompensates individuals for their impatience (ρ) such that individuals postpone consumption, i.e., they save.

Since $C(t) = c(t)L$, with L being the constant number of workers, the same Ramsey rule holds for aggregate consumption C:

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta}. \tag{3}$$

2.3 Production

Following Steigum (2011), output $Y(t)$ is produced according to the production function

$$Y(t) = AK(t)^{\alpha}[\beta Z(t) + L]^{1-\alpha}, \tag{4}$$

where α is the elasticity of output with respect to traditional physical capital input. Total factor productivity A is constant, since there is no technological progress as such. The parameter β is necessary to scale $Z(t)$, which is given in final goods terms, to the scale of L, which is in terms of number of workers. For instance, if $\beta = 2$, this would mean that 1 unit of automation capital could do the same work as 2 workers. Typically β will be much lower, however. Conversely, $\frac{1}{\beta}$ can be interpreted as the cost of purchasing or building enough automation capital to be able to do the same work as 1 human.

A crucial aspect that is often disregarded when analyzing the substitution of robots for workers is that the operation of robots and many other types of capital requires electricity. Their employment is thus associated with additional energy costs. We take this into account and assume that ξ_K is the electricity requirement of a unit of traditional physical capital, while ξ_Z is the electricity requirement of a unit of automation capital. In doing so, we measure all electricity requirements in kilowatt hours (kWh) per year.

Using the final good as the numéraire, the profit maximization problem of the representative firm is given by

$$\begin{aligned} \max_{K(t),L,Z(t)} \pi(t) \quad = \quad & AK(t)^{\alpha}[\beta Z(t) + L]^{1-\alpha} - w(t)L \\ & - R_K(t)K(t) - (1+\tau_Z)R_Z(t)Z(t) - (1+\tau_E)P_E[\xi_K K(t) + \xi_Z Z(t)], \end{aligned} \tag{5}$$

where $w(t)$ is the wage rate, $R_K(t)$ and $R_Z(t)$ are the rental rates for traditional physical capital and automation capital, P_E is the price for electricity, and τ_Z and τ_E are the tax rates on robot income and electricity. Note that electricity alone cannot produce any output but traditional physical capital and automation capital require electricity as a necessary input for production. Electric power is imported from an outside source. When transfers are introduced to the model in section 2.7, we will also consider the possibility that the government acts as the electricity provider and may give the proceeds from energy production to households.

In a perfectly competitive equilibrium, the wage rate, $w(t)$, the rental rate of traditional physical capital, $R_K(t)$, and the rental rate of automation capital, $R_Z(t)$, are given by the marginal products of the corresponding production factors:

$$w(t) = (1-\alpha)A\left[\frac{K(t)}{\beta Z(t) + L}\right]^{\alpha}, \tag{6}$$

$$R_Z(t) = \frac{1}{1+\tau_Z}\left\{(1-\alpha)\beta A\left[\frac{K(t)}{\beta Z(t) + L}\right]^{\alpha} - (1+\tau_E)P_E\xi_Z\right\}, \tag{7}$$

$$R_K(t) = \alpha AK(t)^{\alpha-1}[\beta Z(t) + L]^{1-\alpha} - (1+\tau_E)P_E\xi_K. \tag{8}$$

Note that the net rates of return to the investor (the interest rates) are given by $r_Z(t) = R_Z(t) - \delta$ and $r_K(t) = R_K(t) - \delta$. Other than electricity production, the economy is closed such that savings are equal to gross investment, $I(t) = S(t)$. In contrast to the Ramsey-Cass-Koopmans model, there are two investment vehicles: traditional capital and automation capital. The rational investors decide endogenously how much of their savings they would like to invest in traditional physical capital and how much in automation capital. As long as one of the two investment vehicles delivers a higher rate of return, rational investors would not invest in the other. In the market equilibrium both capital stocks are invested in and thus the following no-arbitrage condition holds:

$$R_K(t) - \delta = R_Z(t) - \delta = r(t), \tag{9}$$

where $r(t) = \frac{Z(t)r_Z(t)+K(t)r_K(t)}{Z(t)+K(t)}$ is the interest rate on household assets. Inserting from (7) and (8), defining $\beta Z(t) + L$ as *effective labor* and

$$X(t) := \frac{\beta Z(t)}{\beta Z(t) + L}$$

as the *automation share* in effective labor, we can rewrite the no-arbitrage condition as

$$\begin{aligned}
\frac{1}{1+\tau_Z}\left\{(1-\alpha)\beta A\left[\frac{K(t)}{\beta Z(t)+L}\right]^{\alpha} - (1+\tau_E)P_E\xi_Z\right\} =& \alpha A\left[\frac{K(t)}{\beta Z(t)+L}\right]^{\alpha-1} - (1+\tau_E)P_E\xi_K \\
\iff \frac{(1-\alpha)}{1+\tau_Z}\beta A\left[\frac{K(t)}{\beta Z(t)+L}\right]^{\alpha} - \alpha A\left[\frac{K(t)}{\beta Z(t)+L}\right]^{\alpha-1} =& (1+\tau_E)P_E\left(\frac{\xi_Z}{1+\tau_Z} - \xi_K\right).
\end{aligned}$$

Now, expressing the left side in terms of $X(t)$ and $Y(t)$, and multiplying both sides by $(1+\tau_Z)$ yields

$$\begin{aligned}
&\frac{(1-\alpha)X(t)Y(t)}{K(t)}\left\{\beta A\left[\frac{K(t)}{\beta Z(t)+L}\right]^{\alpha}\frac{K(t)(\beta Z(t)+L)}{\beta Z(t)AK(t)^{\alpha}(\beta Z(t)+L)^{1-\alpha}} - \right. \\
&\qquad\left. \frac{\alpha}{1-\alpha}A\left[\frac{K(t)}{\beta Z(t)+L}\right]^{\alpha-1}(1+\tau_Z)\frac{K(t)(\beta Z(t)+L)}{\beta Z(t)AK(t)^{\alpha}(\beta Z(t)+L)^{1-\alpha}}\right\} \\
=& \frac{(1-\alpha)X(t)Y(t)}{K(t)}\left\{\frac{K(t)}{Z(t)} - \frac{\alpha}{1-\alpha}\frac{1+\tau_Z}{X(t)}\right\} = (1+\tau_E)P_E(\xi_Z - (1+\tau_Z)\xi_K)).
\end{aligned}$$

Solving for the ratio of traditional physical capital to automation capital, $K(t)/Z(t)$, yields

$$\frac{K(t)}{Z(t)} = \frac{\alpha}{1-\alpha}\frac{1+\tau_Z}{X(t)} + \frac{(1+\tau_E)P_E\xi_K K(t)}{(1-\alpha)X(t)Y(t)}\left[\frac{\xi_Z}{\xi_K} - (1+\tau_Z)\right]. \tag{10}$$

Note that surging electricity costs, caused by rising prices P_E or taxes τ_E, thus lead to an increase in the ratio of traditional physical capital to automation capital if automation capital is sufficiently more energy intensive, i.e., for

$$\frac{\xi_Z}{\xi_K} > (1+\tau_Z).$$

The intuition is that, if robots are rather energy intensive, an increase in the price of electricity or in the electricity tax both imply a substitution of traditional physical capital for automation capital. The reverse holds true if automation capital is not substantially more energy intensive compared with traditional physical capital.

Substituting

$$(1-\alpha)\,X\,(t)\,Y\,(t) = \beta Z\,(t)\,\frac{(1-\alpha)\,X\,(t)\,Y(t)}{\beta Z\,(t)} = Z\,(t)\left[(1+\tau_Z)\,R_Z(t) + (1+\tau_E)P_E\xi_Z\right]$$

in (10), we obtain

$$\frac{K\,(t)}{Z\,(t)} = \frac{\alpha}{1-\alpha}\frac{1+\tau_Z}{X\,(t)} + \frac{K\,(t)}{Z\,(t)}\frac{(1+\tau_E)P_E\xi_K}{(1+\tau_Z)\,R_Z(t) + (1+\tau_E)P_E\xi_Z}\left[\frac{\xi_Z}{\xi_K} - (1+\tau_Z)\right],$$

which solves for

$$\frac{K\,(t)}{Z\,(t)} = \frac{\alpha}{1-\alpha}\frac{\Omega\,(t)}{X\,(t)} \tag{11}$$

$$\text{with} \qquad \Omega\,(t) := \frac{(1+\tau_Z)\,R_Z(t) + (1+\tau_E)P_E\xi_Z}{R_Z(t) + (1+\tau_E)P_E\xi_K}.$$

2.4 Solution

Before embarking on the analysis of the balanced growth path (BGP) of the economy, we define $k\,(t) := K\,(t)\,/[\beta Z\,(t) + L\,(t)]$ as the *traditional capital intensity* and proof that an equilibrium solution $k^* > 0$ exists and is unique.

Proposition 1. *There is a unique positive equilibrium traditional capital intensity at which the no-arbitrage relationship* (9) *is fulfilled.*

Proof. Under the no-arbitrage condition, it follows from (11) that

$$\beta Z(t) + L = \frac{\beta Z(t)}{X(t)} = \frac{(1-\alpha)\beta}{\alpha\Omega\,(t)}K\,(t) \tag{12}$$

and, thus,

$$Y(t) = AK(t)^\alpha[\beta Z(t) + L]^{1-\alpha} = A\left[\frac{(1-\alpha)\beta}{\alpha\Omega\,(t)}\right]^{1-\alpha}K(t). \tag{13}$$

Using (12), we can now get the traditional capital intensity as

$$k\,(t) = \frac{K\,(t)}{\beta Z\,(t) + L} = \frac{\alpha K(t)}{(1-\alpha)\beta K(t)}\Omega(t) = \frac{\alpha}{(1-\alpha)\beta}\Omega\,(t)\,. \tag{14}$$

Next, we can rewrite the rental rate of automation capital as

$$R_Z(t) = \frac{1}{1+\tau_Z}\left[(1-\alpha)\,A\beta k\,(t)^\alpha - (1+\tau_E)P_E\xi_Z\right].$$

This, in turn, can be used to obtain

$$\begin{aligned}\Omega\,(t) =& \frac{(1-\alpha)\beta Ak(t)^\alpha - (1+\tau_E)P_E\xi_Z + (1+\tau_E)P_E\xi_Z}{\frac{1}{1+\tau_Z}\left[(1-\alpha)\beta Ak(t)^\alpha - (1+\tau_E)P_E\xi_Z + (1+\tau_E)P_E\xi_K(1+\tau_Z)\right]}\\ =& \frac{(1-\alpha)\beta(1+\tau_Z)Ak(t)^\alpha}{(1-\alpha)\beta Ak(t)^\alpha + (1+\tau_E)P_E\left[\xi_K(1+\tau_Z) - \xi_Z\right]}.\end{aligned}$$

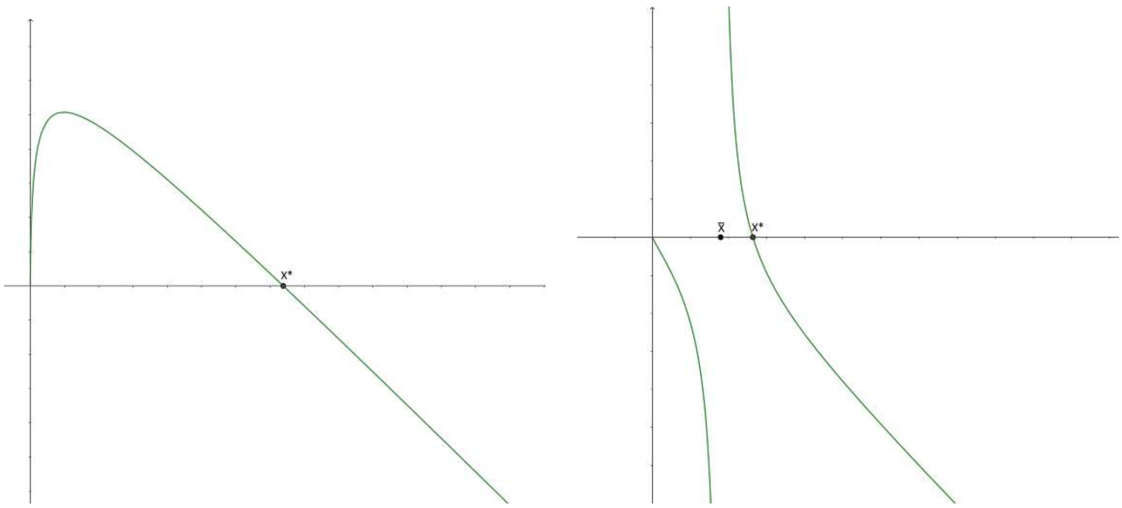

Figure 1: Proof of Auxiliary Lemma 1: Schematic representation of the form of f in both cases.

Inserting the result into (14) yields

$$k(t) = \frac{\alpha(1+\tau_Z) Ak(t)^{\alpha}}{(1-\alpha)\beta Ak(t)^{\alpha} - (1+\tau_E)P_E\xi_K\Delta_{\xi}} \tag{15}$$

with

$$\Delta_{\xi} := \frac{\xi_Z}{\xi_K} - (1+\tau_Z)$$

as an implicit expression for the stationary equilibrium value of $k(t) = k^*$. To provide a precise proof for the existence of a unique equilibrium value k^*, consider the function

$$\Sigma(k, P_E, \tau_E, \tau_Z) := \frac{\alpha(1+\tau_Z) Ak^{\alpha}}{(1-\alpha)\beta Ak^{\alpha} - (1+\tau_E)P_E\xi_K\Delta_{\xi}} - k. \tag{16}$$

We need to show that there exists a unique $k^* > 0$ such that $\Sigma(k^*, P_E, \tau_E, \tau_Z) = 0$. To this end, we prove and then apply a more general statement:

Auxiliary Lemma 1. *Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be defined as*

$$x \mapsto f(x) := \frac{ax^{\alpha}}{bx^{\alpha}+c} - x$$

with $a, b > 0$, $c \in \mathbb{R}$ and $0 < \alpha < 1$.

Then there exists a unique $x^* > 0$ such that $f(x^*) = 0$.

Proof. First, we calculate the derivative of f using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{a\alpha x^{\alpha-1}(bx^{\alpha}+c) - ax^{\alpha}\alpha bx^{\alpha-1}}{(bx^{\alpha}+c)^2} - 1 \\ &= \frac{ac\alpha x^{\alpha-1}}{(bx^{\alpha}+c)^2} - 1. \end{aligned} \tag{17}$$

If $x^* = \dfrac{ax^{*\alpha}}{bx^{*\alpha}+c}$ exists, it follows that

$$\begin{aligned} f'(x^*) =& \frac{ac\alpha x^{*\alpha-1} x^* \dfrac{bx^{*\alpha}+c}{ax^{*\alpha}}}{(bx^{*\alpha}+c)^2} - 1 \\ =& \frac{c\alpha}{bx^{*\alpha}+c} - 1. \end{aligned}$$

Case 1: $c \geq 0$

$c \geq 0$ implies both $c\alpha \geq 0$ and $bx^{*\alpha} + c > 0$. Because $0 < \alpha < 1$ it holds that

$$\begin{aligned} & bx^{*\alpha} + c > c \geq c\alpha \\ \implies & 0 \leq \frac{c\alpha}{bx^{*\alpha}+c} < 1 \\ \implies & f'(x^*) < 0 \end{aligned}$$

and since f is a continuously differentiable function on $\mathbb{R}^+$ it follows that x^*, if it exists, must be unique.

Now consider a value $0 < \tilde{x} < \min\left\{ \left(\frac{a}{b+c}\right)^{\frac{1}{1-\alpha}}, 1 \right\}$. It follows that

$$\begin{aligned} & \frac{a}{b+c} > \tilde{x}^{1-\alpha} \\ \implies & a\tilde{x}^{\alpha} > \tilde{x}(b+c) > b\tilde{x}^{\alpha+1} + c\tilde{x} \end{aligned}$$

and therefore $f(\tilde{x}) = \dfrac{a\tilde{x}^{\alpha} - b\tilde{x}^{\alpha+1} - c\tilde{x}}{b\tilde{x}^{\alpha}+c} > 0$ holds for such small values of x. On the other hand, for very large values of x we obtain via L'Hospital's rule:

$$\begin{aligned} \lim_{x\to+\infty} f(x) =& \lim_{x\to+\infty} \frac{ax^{\alpha}}{bx^{\alpha}+c} - \lim_{x\to+\infty} x \\ =& \frac{a}{b} - \infty = -\infty. \end{aligned}$$

Since f is a continuous function on $\mathbb{R}^+$ with $\lim\limits_{x\to+\infty} f(x) = -\infty$ and $f(x) > 0$ for small x, there has to be a point $x^* > 0$ where $f(x^*) = 0$.

Case 2: $c < 0$

Looking at the derivative (17), it is clear that $f'(x) < 0$ for all $x \geq 0$, i.e. that f is a strictly monotonically decreasing function on $\mathbb{R}^+$. From $f(0) = 0$, f decreases towards $-\infty$ as x approaches the pole of f at $\bar{x} = \left(\frac{-c}{b}\right)^{\frac{1}{\alpha}} > 0$. This implies that $f(x) < 0$ for all $x \in (0, \bar{x})$.

For $x \in (\bar{x}, +\infty)$, it holds that

$$\lim_{x\to\bar{x}^+} f(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} f(x) = -\infty$$

and since f is continuous and strictly monotonically decreasing in this interval, $x^* \in (\bar{x}, +\infty)$ has to exist and is unique.

$\square$

Finally, applying the Auxiliary Lemma to the function $k \mapsto \Sigma(k, P_E, \tau_E, \tau_Z)$ with $a = \alpha(1+\tau_Z)A > 0$, $b = (1-\alpha)\beta A > 0$ and $c = -(1+\tau_E)P_E\xi_K\Delta_\xi$ yields the desired result of a unique positive equilibrium k^* such that the no arbitrage condition is fulfilled. Also note that (in terms of the Auxiliary Lemma) $f'(x^*) < 0$ holds in both cases, which implies that $\Sigma_k(k^*) < 0$.

□

We can now express income as a function of the aggregate traditional capital stock and the traditional capital intensity as

$$Y(t) = A\left(\frac{\beta Z(t) + L}{K(t)}\right)^{1-\alpha} K(t) = A\left(\frac{1}{k^*}\right)^{1-\alpha} K(t). \tag{18}$$

The factor prices, in turn, become

$$\begin{aligned} R_Z^* &= \frac{1}{1+\tau_Z}\left[(1-\alpha)\beta A(k^*)^\alpha - (1+\tau_E)P_E\xi_Z\right] = r^* + \delta, && (19)\\ R_K^* &= \alpha A\left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K = R_Z^* = r^* + \delta, && (20)\\ w^* &= (1-\alpha)A(k^*)^\alpha. && (21) \end{aligned}$$

Aggregate income and the factor prices are all functions of $k^* = k^*(P_E, \tau_E, \tau_Z)$. While the production function is of the AK-type and depends on time through the aggregate capital stock, the factor prices are all constant in the market equilibrium. If growth is achieved, this means that all gains of the growing economy are paid out as interest for holding capital, while labor income stagnates.

2.5 Balanced Growth Path (BGP)

We can now derive the economic growth rates in the market equilibrium, following the steps of Steigum (2011) in his paper without electricity input. For ease of language, we will call the path where the economy satisfies the no-arbitrage condition "balanced growth path", although most variables do not grow exactly at a constant rate.

Proposition 2.

i) Along the BGP, the following growth rates are observed:

$$\frac{\dot{C}(t)}{C(t)} = g := \frac{\alpha A \left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta - \rho}{\theta}$$

$$\frac{\dot{M}(t)}{M(t)} = \frac{\dot{Z}(t)+\dot{K}(t)}{Z(t)+K(t)} = g \cdot \frac{1}{1 - \frac{w^* L \exp(-gt)}{r^*(Z_0+K_0)+w^*L}}$$

$$\frac{\dot{Z}(t)}{Z(t)} = g \cdot \frac{1}{1 - \frac{(w^*+k^*r^*)L\exp(-gt)}{(Z_0+K_0)r^*+w^*L}}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{\dot{Y}(t)}{Y(t)} = g \cdot \frac{1}{1 - \frac{(w^* - \frac{r^*}{\beta})L\exp(-gt)}{(Z_0+K_0)r^*+w^*L}}$$

ii) In case of the knife-edge condition

$$\rho = \alpha A \left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta,$$

the economy stagnates and attains a stationary level of output and consumption indefinitely.

iii) In case of

$$\rho > \alpha A \left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta,$$

the economy shrinks perpetually.

Proof. Plugging the constant $r^* = R_K^* - \delta$ into the aggregate Ramsey rule (3), we can distinguish among three cases

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta} : \begin{cases} >0, & \text{if} \quad \rho < \alpha A \left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta, \\ =0, & \text{if} \quad \rho = \alpha A \left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta, \\ <0, & \text{if} \quad \rho > \alpha A \left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta. \end{cases}$$

In the first case, consumption will perpetually grow at the rate

$$g = \frac{\alpha A \left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta - \rho}{\theta}.$$

In the second case that can only materialize by sheer coincidence for a knife-edge parameter setting, consumption stagnates indefinitely. In the third case, consumption shrinks perpetually.

In any event, aggregate consumption develops according to

$$C(t) = C(0)\exp(gt) \tag{22}$$

where $C(0)$ is the initial level of optimal aggregate consumption. Using the (aggregate) intertemporal budget constraint, $C(0)$ can be deduced. Assuming the no-Ponzi game condition for total capital $M(t) = K(t) + Z(t)$,

$$\lim_{t\to\infty} M(t)\exp(-r^*t) = 0,$$

is fulfilled, the present value of all future consumption must be equal to initial total capital ($M_0 = Z_0 + K_0$) plus the present value of future wage income:

$$\int_0^\infty C(t)\exp(-r^*t)dt = K_0 + Z_0 + \tfrac{w^*}{r^*}L.$$

Substituting in (22) yields

$$\begin{aligned}\int_0^\infty C(0)\exp((g-r^*)t)dt = \frac{C(0)}{r^*-g} = K_0 + Z_0 + \tfrac{w^*}{r^*}L\\ \Longrightarrow C(0) = (r^*-g)(K_0 + Z_0 + \tfrac{w^*}{r^*}L).\end{aligned}$$

This means that households initially choose to consume a fraction (r^*-g) of their wealth, including the present value of all future labor income. Combining this expression for $C(t)$ with the aggregate budget constraint results in an inhomogeneous linear differential equation:

$$\begin{aligned}\dot{M}(t) =& r^*M(t) - C(t) + w^*L\\ \dot{M}(t) =& r^*M(t) - (r^*-g)(Z_0 + K_0 + \tfrac{w^*}{r^*}L)\exp(gt) + w^*L.\end{aligned}$$

This type of differential equation has the general solution

$$\begin{aligned}M(t) =& \exp(r^*t)\left[B + \int \exp(-r^*t)\left(w^*L - (r^*-g)(Z_0 + K_0 + \tfrac{w^*}{r^*}L)\exp(gt)\right)dt\right]\\ =& B\exp(r^*t) - \tfrac{w^*}{r^*}L + (Z_0 + K_0 + \tfrac{w^*}{r^*}L)\exp(gt).\end{aligned}$$

Observing $M(t)$ at $t = 0$ now eliminates the integration constant B:

$$\begin{aligned}Z_0 + K_0 = M(0) = B - \tfrac{w^*}{r^*}L + Z_0 + K_0 + \tfrac{w^*}{r^*}L\\ \Longrightarrow B = 0, \quad M(t) = (Z_0 + K_0 + \tfrac{w^*}{r^*}L)\exp(gt) - \tfrac{w^*}{r^*}L\end{aligned}$$

Finally, we can derive the growth rate of $M(t) = Z(t) + K(t)$ by taking the natural logarithm and the derivative:

$$\begin{aligned}
\ln(M(t)) &= \ln(\exp(gt)) + \ln\left(Z_0 + K_0 + \tfrac{w^*}{r^*}L - \tfrac{w^*}{r^*}L\exp(-gt)\right) \\
\frac{\dot{M}(t)}{M(t)} &= g + \frac{1}{Z_0 + K_0 + \frac{w^*}{r^*}L - \frac{w^*}{r^*}L\exp(-gt)} g\frac{w^*}{r^*}L\exp(-gt) \\
&= g\left(1 + \frac{1}{Z_0 + K_0 + \frac{w^*}{r^*}L - \frac{w^*}{r^*}L\exp(-gt)}\frac{w^*}{r^*}L\exp(-gt)\right) \\
&= g\left(\frac{Z_0 + K_0 + \frac{w^*}{r^*}L - \frac{w^*}{r^*}L\exp(-gt) + \frac{w^*}{r^*}L\exp(-gt)}{Z_0 + K_0 + \frac{w^*}{r^*}L - \frac{w^*}{r^*}L\exp(-gt)}\right) \\
&= g\left(\frac{1}{1 - \frac{\frac{w^*}{r^*}L\exp(-gt)}{Z_0+K_0+\frac{w^*}{r^*}L}}\right) = g\left(\frac{1}{1 - \frac{w^*L\exp(-gt)}{(Z_0+K_0)r^*+w^*L}}\right) = g\left(\frac{1}{1 - \frac{w^*L\exp(-gt)}{Y_d(0)}}\right),
\end{aligned}$$

where $Y_d(0)$ is the (aggregate) disposable income at $t = 0$. Note that $\frac{w^*L\exp(-gt)}{(Z_0+K_0)r^*+w^*L}$ is positive under the model assumptions and therefore total capital grows faster than g, the growth rate of consumption. However, as $t \to \infty$, the growth rate of $M(t)$ converges towards g. If wage income w^*L makes up a large portion of disposable income at $t = 0$ (which implies relatively low automation capital), then the growth rate of total capital $M(t)$ will initially be significantly higher than g.

Next, we derive the growth rates of $Z(t)$ and $K(t)$. Exploiting the fact that the traditional physical capital intensity $k^* = \frac{K(t)}{\beta Z(t)+L}$ is fixed on the BGP allows us to calculate automation capital $Z(t)$ from total capital $M(t)$:

$$\begin{aligned}
K(t) &= \beta Z(t)k^* + k^*L \\
\implies Z(t) + K(t) &= Z(t) + \beta Z(t)k^* + k^*L = Z(t)(1 + k^*\beta) + k^*L \\
\implies Z(t) &= \frac{Z(t) + K(t) - k^*L}{1 + k^*\beta} = \frac{(Z_0 + K_0 + \frac{w^*}{r^*}L)\exp(gt) - \frac{w^*}{r^*}L - k^*L}{1 + k^*\beta}.
\end{aligned}$$

Again, the growth rate can be calculated by taking logarithm and derivative:

$$\begin{aligned}
\ln(Z(t)) &= \ln\left((Z_0 + K_0 + \tfrac{w^*}{r^*}L)\exp(gt) - (\tfrac{w^*}{r^*} + k^*)L\exp(-gt)\exp(gt)\right) - \ln(1 + k^*\beta) \\
\frac{\dot{Z}(t)}{Z(t)} &= g + \frac{1}{(Z_0 + K_0 + \frac{w^*}{r^*}L) - (\frac{w^*}{r^*} + k^*)L\exp(-gt)} g(\tfrac{w^*}{r^*}L + k^*)L\exp(-gt) \\
&= g\left(1 + \frac{(\frac{w^*}{r^*}L + k^*)L\exp(-gt)}{(Z_0 + K_0 + \frac{w^*}{r^*}L) - (\frac{w^*}{r^*} + k^*)L\exp(-gt)}\right) \\
&= g\left(\frac{Z_0 + K_0 + \frac{w^*}{r^*}L}{(Z_0 + K_0 + \frac{w^*}{r^*}L) - (\frac{w^*}{r^*} + k^*)L\exp(-gt)}\right) \\
&= g\left(\frac{1}{1 - \frac{(\frac{w^*}{r^*}+k^*)L\exp(-gt)}{Z_0+K_0+\frac{w^*}{r^*}L}}\right) = g\left(\frac{1}{1 - \frac{(w^*+k^*r^*)L\exp(-gt)}{(Z_0+K_0)r^*+w^*L}}\right).
\end{aligned}$$

We can observe that automation capital also grows faster than consumption, with the growth rate again converging towards g.

Since $\frac{(w^* + k^* r^*) L \exp(-gt)}{(Z_0 + K_0) r^* + w^* L} > \frac{w^* L \exp(-gt)}{(Z_0 + K_0) r^* + w^* L}$, automation capital grows faster than total capital as well, i.e. it grows faster than traditional capital.

Finally, we calculate the growth rate of traditional capital $K(t)$, which grows at the same rate as output $Y(t)$ due to the linear relationship $Y(t) = A\left(\frac{1}{k^*}\right)^{1-\alpha} K(t)$ which holds on the BGP. $K(t)$ can again be derived from $M(t)$, or alternatively from $Z(t)$, using the fact that k^* is fixed.

$$
\begin{aligned}
& K(t) = k^* \beta Z(t) + k^* L \\
\implies & K(t) = \frac{k^* \beta \left[(Z_0 + K_0 + \frac{w^*}{r^*} L) \exp(gt) - \frac{w^*}{r^*} L - k^* L\right] + (1 + k^* \beta) k^* L}{1 + k^* \beta} \\
& = \frac{k^* \beta (Z_0 + K_0 + \frac{w^*}{r^*} L) \exp(gt) - k^* \beta \frac{w^*}{r^*} L + k^* L}{1 + k^* \beta} \\
& = \frac{k^* \beta}{1 + k^* \beta} \left[(Z_0 + K_0 + \frac{w^*}{r^*} L) \exp(gt) - (\frac{w^*}{r^*} - \frac{1}{\beta}) L\right]
\end{aligned}
$$

With the familiar procedure, we obtain the growth rate:

$$
\begin{aligned}
\ln(K(t)) &= \ln\left(\frac{k^* \beta}{1 + k^* \beta}\right) + \ln(\exp(gt)) + \ln(Z_0 + K_0 + \tfrac{w^*}{r^*} L - (\tfrac{w^*}{r^*} - \tfrac{1}{\beta}) L \exp(-gt)) \\
\frac{\dot{K}(t)}{K(t)} &= g \left(1 + \frac{(\frac{w^*}{r^*} - \frac{1}{\beta}) L \exp(-gt)}{Z_0 + K_0 + \frac{w^*}{r^*} L - (\frac{w^*}{r^*} - \frac{1}{\beta}) L \exp(-gt)}\right) \\
&= g \left(\frac{Z_0 + K_0 + \frac{w^*}{r^*} L}{Z_0 + K_0 + \frac{w^*}{r^*} L - (\frac{w^*}{r^*} - \frac{1}{\beta}) L \exp(-gt)}\right) \\
&= g \left(\frac{1}{1 - \frac{(\frac{w^*}{r^*} - \frac{1}{\beta}) L \exp(-gt)}{Z_0 + K_0 + \frac{w^*}{r^*} L}}\right) = g \left(\frac{1}{1 - \frac{(w^* - \frac{r^*}{\beta}) L \exp(-gt)}{(Z_0 + K_0) r^* + w^* L}}\right)
\end{aligned}
$$

Note that $\frac{(w^* - \frac{r^*}{\beta}) L \exp(-gt)}{(Z_0 + K_0) r^* + w^* L} < \frac{w^* L \exp(-gt)}{(Z_0 + K_0) r^* + w^* L}$ and therefore traditional capital (and output) grows slower than total capital and in particular, automation capital. Whether $K(t)$ and $Y(t)$ grow faster or slower than g depends on the term $(w^* - \frac{r^*}{\beta})$. $w^* > \frac{r^*}{\beta}$ means that the wage of one worker is higher than the return on investment for automation capital that is equivalent to one worker in terms of productivity. This is generally the case in this model setup, as automation capital faces both taxation and deprecation, whereas labor does not:

$$
w^* = (1 - \alpha) A (k^*)^\alpha > \frac{1}{1 + \tau_Z} \left[(1 - \alpha) A (k^*)^\alpha - \frac{(1 + \tau_E) P_E \xi_Z}{\beta}\right] - \frac{\delta}{\beta} = \frac{r^*}{\beta},
$$

meaning that $K(t)$ and $Y(t)$ indeed grow faster than g. We can also observe that both terms in the inequality above actually become equal if there were no taxes, depreciation and electricity consumption. In this case, $K(t)$ and $Y(t)$ would grow at the constant rate g.

□

Next, we take a look at how other variables can be expressed on the BGP. We can compute the stock of automation capital as

$$\beta Z(t) = \frac{K(t)}{k^*} - L = \frac{1}{k^*} X\,(t)\, K(t). \tag{23}$$

The second expression gives a convenient scaling-relationship with the automation share $X\,(t) = \beta Z(t)/[\beta Z(t) + L]$ as an intuitive multiplier.

Note that the automation share $X\,(t)$ is an important state in its own right and we can rewrite it as a function of $K(t)$ and L

$$X\,(t) = \frac{\beta Z(t)}{\beta Z(t) + L} = 1 - k^* \frac{L}{K(t)}.$$

Finally, we compute electricity use as

$$E\,(t) = \xi_K K(t) + \xi_Z Z(t) = \left[\xi_K + \xi_Z \frac{1}{k^* \beta}\right] K(t) - \frac{\xi_Z}{\beta} L = \left[\xi_K + \xi_Z \frac{X\,(t)}{k^* \beta}\right] K(t).$$

Then the revenues collected through the robot tax are

$$\tau_Z R_Z^* Z(t) = \tau_Z R_Z^* \left[\frac{K(t)}{k^*} - L\right] \frac{1}{\beta} = \tau_Z R_Z^* \frac{X\,(t)}{k^* \beta} K(t)$$

and the ones collected through the tax on electricity become

$$\tau_E P_E [\xi_K K(t) + \xi_Z Z(t)] = \tau_E P_E \left\{ \left[\xi_K + \xi_Z \frac{1}{k^* \beta}\right] K(t) - \frac{\xi_Z}{\beta} L \right\} = \tau_E P_E \left[\xi_K + \xi_Z \frac{X\,(t)}{k^* \beta}\right] K(t).$$

Total tax revenue then becomes

$$\begin{aligned} T\,(t) &= \tau_Z R_Z^* Z(t) + \tau_E P_E [\xi_K K(t) + \xi_Z Z(t)] \\ &= \left\{ \tau_Z R_Z^* \frac{1}{k^* \beta} + \tau_E P_E \left[\xi_K + \xi_Z \frac{1}{k^* \beta}\right] \right\} K(t) - \frac{\tau_Z R_Z^* + \tau_E P_E \xi_Z}{\beta} L \\ &= \left\{ \tau_Z R_Z^* \frac{X\,(t)}{k^* \beta} + \tau_E P_E \left[\xi_K + \xi_Z \frac{X\,(t)}{k^* \beta}\right] \right\} K(t). \end{aligned}$$

Using the automation share $X(t)$, we can obtain a different way to describe the growth dynamics of automation capital:

$$g_Z(t) := \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{K}(t)}{Z(t) k^* \beta} = \frac{K(t)}{Z(t) k^* \beta} \frac{\dot{K}(t)}{K(t)} = \frac{1}{X\,(t)} \frac{\dot{K}(t)}{K(t)} = \frac{1}{X\,(t)} g_K(t).$$

Noting that $1/X(t) \geq 1$ for all $Z(t) \leq \infty$, we can again confirm that the robot stock grows in excess of traditional capital and output, but at a rate that declines with the robot share in effective labor. In the limit, where traditional labor becomes quantitatively irrelevant such that $X(t) \to 1$, the two rates converge.

For the automation share, we obtain a growth rate of

$$\dot{X}(t) = \frac{d}{dt}\left(1 - k^* \frac{L}{K(t)}\right) = \frac{k^* L}{K(t)^2}\dot{K}(t)$$
$$\Longrightarrow g_X(t) := \frac{\dot{X}(t)}{X(t)} = \frac{k^* L}{X(t)K(t)}\frac{\dot{K}(t)}{K(t)} = \frac{1 - X(t)}{X(t)}\frac{\dot{K}(t)}{K(t)} = \frac{1 - X(t)}{X(t)} g_K(t),$$

where we note that the automation share grows at a rate higher than the BGP growth rate of traditional capital and output as long as the automation share is lower than 50%:

$$\frac{\dot{X}(t)}{X(t)} \geq g_K(t) \Longleftrightarrow X(t) \leq 0.5.$$

However, the growth rate of the automation share approaches zero in the limit, when human labor becomes quantitatively irrelevant. Furthermore, we obtain the growth rate of electricity consumption as

$$\begin{aligned} g_E(t) \quad &:= \quad \frac{\dot{E}(t)}{E(t)} = \frac{\xi_K K(t)}{E(t)}\frac{\dot{K}(t)}{K(t)} + \frac{\xi_Z Z(t)}{E(t)}\frac{\dot{Z}(t)}{Z(t)} \\ &= \quad \left[\frac{\xi_K K(t)}{E(t)} + \frac{\xi_Z Z(t)}{E(t)}\frac{1}{X(t)}\right]\frac{\dot{K}(t)}{K(t)} \\ &= \quad \left[\frac{\xi_K K(t)}{E(t)} + \frac{\xi_Z Z(t)}{E(t)}\frac{1}{X(t)}\right] g_K(t). \end{aligned}$$

Upper and lower bounds for this term can be obtained, considering that $\frac{1}{X(t)} > 1$:

$$\left[\frac{\xi_K K(t)}{E(t)} + \frac{\xi_Z Z(t)}{E(t)}\frac{1}{X(t)}\right] g_K(t) < \left[\frac{\xi_K K(t)}{E(t)}\frac{1}{X(t)} + \frac{\xi_Z Z(t)}{E(t)}\frac{1}{X(t)}\right] g_K(t) = \frac{1}{X(t)} g_K(t)$$
$$\left[\frac{\xi_K K(t)}{E(t)} + \frac{\xi_Z Z(t)}{E(t)}\frac{1}{X(t)}\right] g_K(t) > \left[\frac{\xi_K K(t)}{E(t)} + \frac{\xi_Z Z(t)}{E(t)}\right] g_K(t) = g_K(t)$$

Therefore electricity consumption grows faster than the BGP growth rate g and traditional capital, but slower than automation capital:

$$\frac{\dot{E}(t)}{E(t)} \in \left(g_K(t), \underbrace{\frac{1}{X(t)} g_K(t)}_{=g_Z(t)}\right).$$

The energy shares determine the positioning within the interval. The reason why growth of electricity input is faster than BGP growth is that electricity input depends linearly on $Z(t)$ and $K(t)$, both of which grow faster than g. The growth of electricity consumption $E(t) = \xi_K K(t) + \xi_Z Z(t)$ must fall somewhere between $g_K(t)$ and $g_Z(t)$, because automation capital $Z(t)$ grows faster than traditional capital $K(t)$ and the assumption $\xi_Z > 0, \xi_K > 0$ holds.

Finally, we can calculate the growth rate of total tax revenue as

$$\begin{aligned} g_T(t) := \frac{\dot{T}(t)}{T(t)} &= \frac{\tau_Z R_Z^* Z(t)}{T(t)} \frac{\dot{Z}(t)}{Z(t)} + \frac{\tau_E P_E E(t)}{T(t)} \frac{\dot{E}(t)}{E(t)} \\ &= \frac{\tau_Z R_Z^* Z(t)}{T(t)} g_Z(t) + \frac{\tau_E P_E E(t)}{T(t)} g_E(t) \end{aligned}$$

This expression can again be bounded, using $g_K(t) < g_E(t) < g_Z(t)$:

$$\frac{\tau_Z R_Z^* Z(t)}{T(t)} g_Z(t) + \frac{\tau_E P_E E(t)}{T(t)} g_E(t) > \frac{\tau_Z R_Z^* Z(t)}{T(t)} g_K(t) + \frac{\tau_E P_E E(t)}{T(t)} g_K(t) = g_K(t)$$

$$\frac{\tau_Z R_Z^* Z(t)}{T(t)} g_Z(t) + \frac{\tau_E P_E E(t)}{T(t)} g_E(t) < \frac{\tau_Z R_Z^* Z(t)}{T(t)} g_Z(t) + \frac{\tau_E P_E E(t)}{T(t)} g_Z(t) = g_Z(t)$$

And therefore tax revenue, like electricity input, grows faster than $K(t)$, but slower than $Z(t)$.

$$\frac{\dot{T}(t)}{T(t)} \in (g_K(t), g_Z(t))$$

This is simply due to the fact that both tax bases - $Z(t)$ and $E(t)$ - grow faster than traditional capital $K(t)$, while $E(t)$ grows slower than automation capital $Z(t)$.

2.6 Comparison with Steigum (2011)

In this section, we compare the derived BGP growth rates with Steigum (2011), whose model includes population growth, but no taxes or electricity input. In the case where automation capital and labor are perfect substitutes, Steigum assumes the production function

$$Y(t) = \tilde{A}K(t)^{\alpha}\left(\nu\varepsilon Z(t) + (1-\nu)L(t)\right)^{1-\alpha},$$

where $\nu = \frac{\tilde{\beta}}{1-\alpha}$, $0 < \tilde{\beta} < 1-\alpha$ and $\varepsilon > 0$. $L(t)$ grows with rate n. Note that the weight factor $(1-\nu)$ can be factored out to obtain the parametrization used in this thesis:

$$Y(t) = \underbrace{\tilde{A}(1-\nu)^{1-\alpha}}_{=A} K(t)^{\alpha}\left(\underbrace{\frac{\nu\varepsilon}{1-\nu}}_{=\beta} Z(t) + L(t)\right)^{1-\alpha}.$$

Steigum defines permanent income per worker as

$$c_p(t) = (r^* - n)\left(m(t) + \frac{w^*}{r^* - n}\right),$$

and, following the same steps that were used to derive $\frac{\dot{M}(t)}{M(t)}$ in section 2.5, he obtains the explicit functional form and growth rate for total capital per capita:

$$m(t) = \left(m_0 + \frac{w^*}{r^* - n}\right)\exp(gt) - \frac{w^*}{r^* - n} \tag{24}$$

$$\frac{\dot{m}(t)}{m(t)} = \frac{g}{1 - (w^* \exp(-gt)/c_p(0))} = g \cdot \frac{1}{1 - \frac{w^* \exp(-gt)}{(r^*-n)\left(m_0 + \frac{w^*}{r^*-n}\right)}}.$$

Note that population growth n speeds up the growth of $m(t)$. Setting $n = 0$ yields:

$$\frac{\dot{M}(t)}{M(t)} = \frac{\dot{m}(t)}{m(t)} = \frac{g}{1 - (w^* \exp(-gt)/c_p(0))} = g \cdot \frac{1}{1 - \frac{w^* \exp(-gt)}{r^* m_0 + w^*}} = g \cdot \frac{1}{1 - \frac{w^* L \exp(-gt)}{r^* M_0 + w^* L}},$$

which is the same growth rate that we derived for $M(t)$.

While we used the relationship $K(t) = \beta Z(t)k^* + k^* L$ to obtain the growth rates for $Z(t)$, $K(t)$ and $Y(t)$, Steigum uses a different way to derive the growth rate of output per capita $y(t)$. We follow his steps in some detail to correct a typo in the formula he gives for the growth rate.

Since there are neither taxes nor electricity consumption, output per capita is divided between interest payments, depreciation and wages:

$$y(t) = (r^* + \delta)m(t) + w^*.$$

Plugging in (24), we obtain $y(t)$ in explicit functional form:

$$\begin{aligned} y(t) &= (r^* + \delta)\left(m_0 + \frac{w^*}{r^* - n}\right)\exp(gt) - (r^* + \delta)\frac{w^*}{r^* - n} + w^* \\ &= (r^* + \delta)\left(m_0 + \frac{w^*}{r^* - n}\right)\exp(gt) - w^* \frac{n + \delta}{r^* - n}. \end{aligned}$$

Applying the logarithm and differentiating according to time now yields:

$$\begin{aligned}
\ln(y(t)) &= gt + \ln\left((r^*+\delta)\left(m_0 + \frac{w^*}{r^*-n}\right) - w^*\frac{n+\delta}{r^*-n}\exp(-gt)\right)\\
\frac{\dot{y}(t)}{y(t)} &= g\left(1 + \frac{w^*\frac{n+\delta}{r^*-n}\exp(-gt)}{(r^*+\delta)\left(m_0+\frac{w^*}{r^*-n}\right) - w^*\frac{n+\delta}{r^*-n}\exp(-gt)}\right)\\
&= g\left(\frac{(r^*+\delta)\left(m_0+\frac{w^*}{r^*-n}\right)}{(r^*+\delta)\left(m_0+\frac{w^*}{r^*-n}\right) - w^*\frac{n+\delta}{r^*-n}\exp(-gt)}\right)\\
&= g\left(\frac{1}{1-\frac{w^*\frac{n+\delta}{r^*-n}\exp(-gt)}{(r^*+\delta)\left(m_0+\frac{w^*}{r^*-n}\right)}}\right) = g\left(\frac{1}{1-\frac{w^*(n+\delta)\exp(-gt)}{(r^*+\delta)(r^*-n)\left(m_0+\frac{w^*}{r^*-n}\right)}}\right)\\
&= g\left(\frac{1}{1-\frac{w^*(n+\delta)\exp(-gt)}{(r^*+\delta)c_p(0)}}\right) \neq g\left(\frac{1}{1-\frac{w^*(n+\delta)\exp(-gt)}{(r^*+\delta)(r^*-n)c_p(0)}}\right),
\end{aligned}$$

where the last term is the erroneous formula given in Steigum (2011). Proceeding with the correct expression and setting $n = 0$, we obtain further:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{y}(t)}{y(t)} = g\left(\frac{1}{1-\frac{w^*\frac{\delta}{r^*+\delta}\exp(-gt)}{r^*m_0+w^*}}\right) = g\left(\frac{1}{1-\frac{w^*L\frac{\delta}{r^*+\delta}\exp(-gt)}{r^*M_0+w^*L}}\right).$$

In order for this to be consistent with the growth rate from Proposition 2, $g\left(\frac{1}{1-\frac{(w^*-\frac{r^*}{\beta})L\exp(-gt)}{r^*M_0+w^*L}}\right)$, we need to show that

$$w^*\frac{\delta}{r^*+\delta} = w^* - \frac{r^*}{\beta}$$

holds when there are no taxes ($\tau_E = \tau_Z = 0$) and no electricity costs ($P_E = 0$). Plugging (19) and (21) into the left and then right side yields:

$$\begin{aligned}
\delta\frac{w^*}{r^*+\delta} &= \delta\frac{(1-\alpha)A\,(k^*)^\alpha}{(1-\alpha)A\beta\,(k^*)^\alpha} = \frac{\delta}{\beta}\\
w^* - \frac{r^*}{\beta} &= (1-\alpha)A\,(k^*)^\alpha - (1-\alpha)A\frac{\beta}{\beta}\,(k^*)^\alpha + \frac{\delta}{\beta} = \frac{\delta}{\beta}.
\end{aligned}$$

Therefore the growth rates we derived for $M(t)$ and $Y(t)$ are both consistent with those obtained by Steigum (2011). Note that Steigum does not derive growth rates for $K(t)$ or $Z(t)$.

2.7 Growth Rates with Transfers

We now introduce government transfers $Tr(t)$ into the model. At each point in time, the government transfers all of its tax and electricity revenues back to households. Alternatively, we could also assume that the government does not act as the electricity supplier and transfers back only the tax revenue - in this case, replace $(1 + \tau_E)$ with τ_E in the following calculations.

Given the transfers, the (aggregate) household budget constraint now looks like this along the BGP $(M(t) = K(t) + Z(t))$:

$$\dot{M}(t) = r^* M(t) - C(t) + w^* L + Tr(t)$$
$$\dot{M}(t) = r^* M(t) - C(t) + w^* L + (1 + \tau_E) P_E (\xi_K K(t) + \xi_Z Z(t)) + \tau_Z R_Z^* Z(t).$$

Note that this does not fundamentally affect the household optimization problem, meaning that consumption growth is still governed by the Ramsey rule

$$\frac{\dot{C}(t)}{C(t)} = \frac{r^* - \rho}{\theta} = g,$$

and that aggregate consumption develops according to

$$C(t) = C(0) \exp(gt) \tag{25}$$

where $C(0)$ is the initial level of optimal aggregate consumption. The no-Ponzi game condition must hold for total capital $M(t) = K(t) + Z(t)$ and, since $K(t) = \beta k^* Z(t) + k^* L$, also for the individual capital stocks $Z(t)$ and $K(t)$

$$\lim_{t\to\infty} M(t) \exp(-r^* t) = 0 \tag{26}$$
$$\lim_{t\to\infty} K(t) \exp(-r^* t) = 0 \tag{27}$$
$$\lim_{t\to\infty} Z(t) \exp(-r^* t) = 0. \tag{28}$$

According to the intertemporal budget constraint, the present value of consumption must be equal to initial total capital $(M_0 = Z_0 + K_0)$ plus the present value of wage income and the present value of all future transfers (μ):

$$\int_0^\infty C(t) \exp(-r^* t) dt = K_0 + Z_0 + \tfrac{w^*}{r^*} L + \underbrace{\int_0^\infty Tr(t) \exp(-r^* t) dt}_{\mu :=}.$$

Substituting in (25) and $K_0 = \beta k^* Z_0 + k^* L$ yields

$$\int_0^\infty C(0) \exp((g - r^*)t) dt = \frac{C(0)}{r^* - g} = \beta k^* Z_0 + Z_0 + k^* L + \tfrac{w^*}{r^*} L + \mu$$
$$\Longrightarrow C(0) = (r^* - g) \left[(1 + k^* \beta) Z_0 + (k^* + \tfrac{w^*}{r^*}) L + \mu\right]. \tag{29}$$

Using the fact that $\dot{Z}(t) = \frac{1}{1+k^*\beta}\dot{M}(t)$ and substituting in $K(t) = \beta k^* Z(t) + k^* L$ wherever possible, we obtain the following inhomogeneous linear differential equation for $Z(t)$:

$$\begin{aligned}
\dot{Z}(t) =& \frac{1}{1+k^*\beta}\left\{ r^*K(t) + r^*Z(t) - C(t) + w^*L + (1+\tau_E)P_E(\xi_K K(t) + \xi_Z Z(t)) + \tau_Z R_Z^* Z(t) \right\} \\
\dot{Z}(t) =& \frac{1}{1+k^*\beta}\Big\{ r^*\beta k^* Z(t) + r^*k^*L + r^*Z(t) - (r^*-g)\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right]\exp(gt) \\
&+ w^*L + (1+\tau_E)P_E\xi_K\beta k^* Z(t) + (1+\tau_E)P_E\xi_K k^* L + (1+\tau_E)P_E\xi_Z Z(t) + \tau_Z R_Z^* Z(t) \Big\} \\
=& \frac{1}{1+k^*\beta}\Big\{ (1+k^*\beta)r^* + \underbrace{(1+\tau_E)P_E\xi_K\beta k^* + (1+\tau_E)P_E\xi_Z + \tau_Z R_Z^*}_{\tau:=} \Big\} Z(t) \\
&+ \frac{1}{1+k^*\beta}\left\{ r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^* \right\} L \\
&- \frac{1}{1+k^*\beta}(r^*-g)\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right]\exp(gt)
\end{aligned}$$

This type of differential equation has the general solution

$$\begin{aligned}
Z(t) =& B\exp\left(\left[r^* + \frac{\tau}{1+k^*\beta}\right]t\right) \\
&- \frac{1}{(r^* + \frac{\tau}{1+k^*\beta})}\frac{1}{(1+k^*\beta)}\left\{ r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^* \right\} L \\
&+ \frac{(r^*-g)}{\left(r^* - g + \frac{\tau}{1+k^*\beta}\right)}\frac{1}{(1+k^*\beta)}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right]\exp(gt) \\
=& B\exp\left(\left[r^* + \frac{\tau}{1+k^*\beta}\right]t\right) \\
&- \frac{1}{r^*(1+k^*\beta)+\tau}\left\{ r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^* \right\} L \\
&+ \frac{(r^*-g)}{(r^*-g)(1+k^*\beta)+\tau}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right]\exp(gt).
\end{aligned}$$

At this point, one would usually consider $Z(0)$ to determine the integration constant B. However, since we have a second unknown constant μ, this approach will not work. Instead, we argue that if $B \neq 0$ the resulting $Z(t)$ would grow at a faster rate than r^*, violating the no-Ponzi game condition (28). Therefore $B = 0$ must hold. We will later check the correctness of this assumption by calculating $Z(0)$.

Next, we determine the value of μ by inserting $Z(t)$ into the definition of μ:

$$
\begin{aligned}
\mu &= \int_0^\infty Tr(t)\exp(-r^*t)dt \\
&= \int_0^\infty \left[(1+\tau_E)P_E(\xi_K K(t) + \xi_Z Z(t)) + \tau_Z R_Z^* Z(t)\right]\exp(-r^*t)dt \\
&= \int_0^\infty (1+\tau_E)P_E\xi_K k^* L\exp(-r^*t)dt \\
&\quad + \int_0^\infty \underbrace{\left[(1+\tau_E)P_E\xi_K\beta k^* + (1+\tau_E)P_E\xi_Z + \tau_Z R_Z^*\right]}_{\tau=} Z(t)\exp(-r^*t)dt \\
&= \frac{(1+\tau_E)P_E\xi_K k^* L}{r^*} - \int_0^\infty \frac{\tau}{r^*(1+k^*\beta)+\tau}\left\{r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^*\right\}L\exp(-r^*t)dt \\
&\quad + \int_0^\infty \frac{\tau(r^*-g)}{(r^*-g)(1+k^*\beta)+\tau}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right]\exp((g-r^*)t)dt \\
&= \frac{(1+\tau_E)P_E\xi_K k^* L}{r^*} - \frac{\tau}{r^*(1+k^*\beta)+\tau}\left\{k^* + \tfrac{w^*}{r^*} + \frac{(1+\tau_E)P_E\xi_K k^*}{r^*}\right\}L \\
&\quad + \frac{\tau}{(r^*-g)(1+k^*\beta)+\tau}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right] \\
&= \frac{(1+k^*\beta)(1+\tau_E)P_E\xi_K k^* L}{r^*(1+k^*\beta)+\tau} - \frac{\tau}{r^*(1+k^*\beta)+\tau}\left\{k^* + \tfrac{w^*}{r^*}\right\}L \\
&\quad + \frac{\tau}{(r^*-g)(1+k^*\beta)+\tau}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right].
\end{aligned}
$$

We can now solve for μ:

$$
\begin{aligned}
&\mu\cdot\left(1 - \frac{\tau}{(r^*-g)(1+k^*\beta)+\tau}\right) = \mu\cdot\frac{(r^*-g)(1+k^*\beta)}{(r^*-g)(1+k^*\beta)+\tau} \\
&\quad = \frac{(1+k^*\beta)(1+\tau_E)P_E\xi_K k^* L}{r^*(1+k^*\beta)+\tau} - \frac{\tau}{r^*(1+k^*\beta)+\tau}(k^* + \tfrac{w^*}{r^*})L \\
&\quad + \frac{\tau}{(r^*-g)(1+k^*\beta)+\tau}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L\right] \\
\Longrightarrow\ &\mu = \frac{(r^*-g)(1+k^*\beta)+\tau}{(r^*-g)(1+k^*\beta)}\cdot\left\{\frac{(1+k^*\beta)(1+\tau_E)P_E\xi_K k^* L}{r^*(1+k^*\beta)+\tau} - \frac{\tau}{r^*(1+k^*\beta)+\tau}(k^* + \tfrac{w^*}{r^*})L\right\} \\
&\quad + \frac{\tau}{(r^*-g)(1+k^*\beta)}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L\right].
\end{aligned}
$$

Finally, we obtain $Z(t)$:

$$\begin{aligned}
Z(t) =& - \frac{1}{r^*(1+k^*\beta)+\tau}\Big\{ r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^* \Big\} L \\
&+ \frac{(r^*-g)}{(r^*-g)(1+k^*\beta)+\tau}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L + \mu\right]\exp(gt) \\
=& - \frac{1}{r^*(1+k^*\beta)+\tau}\Big\{ r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^* \Big\} L \\
&+ \frac{(r^*-g)}{(r^*-g)(1+k^*\beta)+\tau}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L\right]\left(1 + \frac{\tau}{(r^*-g)(1+k^*\beta)}\right)\exp(gt) \\
&+ \frac{1}{1+k^*\beta}\left\{ \frac{(1+k^*\beta)(1+\tau_E)P_E\xi_K k^* L}{r^*(1+k^*\beta)+\tau} - \frac{\tau}{r^*(1+k^*\beta)+\tau}(k^* + \tfrac{w^*}{r^*})L \right\}\exp(gt) \\
=& - \frac{1}{r^*(1+k^*\beta)+\tau}\Big\{ r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^* \Big\} L \\
&+ \frac{1}{1+k^*\beta}\left[(1+k^*\beta)Z_0 + (k^* + \tfrac{w^*}{r^*})L\right]\exp(gt) \\
&+ \frac{1}{1+k^*\beta}\left\{ \frac{(1+k^*\beta)(1+\tau_E)P_E\xi_K k^* L}{r^*(1+k^*\beta)+\tau} - \frac{\tau}{r^*(1+k^*\beta)+\tau}(k^* + \tfrac{w^*}{r^*})L \right\}\exp(gt) \\
=& - \frac{1}{r^*(1+k^*\beta)+\tau}\Big\{ r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^* \Big\} L \\
&+ \left\{ Z_0 + (k^* + \tfrac{w^*}{r^*})L \cdot \frac{r^*}{r^*(1+k^*\beta)+\tau} + \frac{(1+\tau_E)P_E\xi_K k^* L}{r^*(1+k^*\beta)+\tau} \right\}\exp(gt) \\
=&(Z_0 + a)\exp(gt) - a,
\end{aligned}$$

where $a := \dfrac{(r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^*)L}{r^*(1+k^*\beta)+\tau}$. Lastly, we check if we were justified in setting the integration variable $B = 0$ by examining $Z(0)$:

$$Z(0) = Z_0 + a - a = Z_0.$$

With the validity of $Z(t)$ confirmed, we may proceed to determine the growth rate of $Z(t)$ by applying the logarithm and differentiating according to time:

$$\begin{aligned}
\ln(Z(t)) =& \ln(\exp(gt)) + \ln(Z_0 + a - a\exp(-gt)) \\
\frac{\dot{Z}(t)}{Z(t)} =& g + \frac{1}{Z_0 + a - a\exp(-gt)} ga\exp(-gt) \\
=& g \cdot \frac{Z_0 + a}{Z_0 + a - a\exp(-gt)} \\
=& g \cdot \frac{1}{1 - \frac{a\exp(-gt)}{Z_0+a}}
\end{aligned}$$

Again, we can immediately see that $Z(t)$ grows faster than g, since $a > 0$, and that the growth rate goes towards g as $t \to \infty$. To obtain a more explicit form of the growth rate, we have to consider the term $\frac{a}{Z_0+a}$:

$$\frac{a}{Z_0+a} = \frac{\frac{(r^*k^*+w^*+(1+\tau_E)P_E\xi_K k^*)L}{r^*(1+k^*\beta)+\tau}}{\frac{r^*(1+k^*\beta)Z_0+r^*k^*L+\tau Z_0+(1+\tau_E)P_E\xi_K k^*L+w^*L}{r^*(1+k^*\beta)+\tau}}$$
$$= \frac{(r^*k^*+w^*+(1+\tau_E)P_E\xi_K k^*)L}{r^*(Z_0+\underbrace{k^*\beta Z_0+k^*L}_{=K_0})+w^*L+\underbrace{\tau Z_0+(1+\tau_E)P_E\xi_K k^*L}_{=Tr(0)}}$$
$$= \frac{(r^*k^*+w^*+(1+\tau_E)P_E\xi_K k^*)L}{Y_d(0)},$$

where $Y_d(0)$ is the (aggregate) disposable income at $t=0$. Therefore, the growth rate of $Z(t)$ becomes:

$$\frac{\dot{Z}(t)}{Z(t)} = g\cdot\frac{1}{1-\frac{(r^*k^*+w^*+(1+\tau_E)P_E\xi_K k^*)L\exp(-gt)}{r^*(Z_0+K_0)+w^*L+Tr(0)}} = g\cdot\frac{1}{1-\frac{(r^*k^*+w^*+(1+\tau_E)P_E\xi_K k^*)L\exp(-gt)}{Y_d(0)}}$$

This is different than the growth rate derived in the case without transfers, which was $g\cdot\frac{1}{1-\frac{(w^*+k^*r^*)L\exp(-gt)}{(Z_0+K_0)r^*+w^*L}}$. Keep in mind that the growth rate of consumption g is the same in either case, since it was unaffected by the introduction of transfers. To find out which rate is higher, we have to consider automation capital in its simplified functional form, $Z(t)=(Z_0+a)\exp(gt)-a$. It should be noted that all of the capital stocks can be described in this form and we can observe that the growth rate

$$\frac{\dot{Z}(t)}{Z(t)} = g\cdot\frac{1}{1-\frac{a\exp(-gt)}{Z_0+a}}$$

depends positively on the term $0<\frac{a}{Z_0+a}<1$ for $a>0$, $Z_0>0$. Forming the derivative of this term with respect to a, we can see that higher values of the constant a are associated with higher growth rates:

$$\frac{d}{da}\left[\frac{a}{Z_0+a}\right] = \frac{Z_0}{(Z_0+a)^2} > 0.$$

This means we can simply look at functional form of automation capital in the case without transfers,

$$\tilde{Z}(t) = \frac{(Z_0+k^*\beta Z_0+k^*L+\frac{w^*}{r^*}L)\exp(gt)-\frac{w^*}{r^*}L-k^*L}{1+k^*\beta}$$
$$= \left(Z_0+\underbrace{\frac{(k^*+\frac{w^*}{r^*})L}{1+k^*\beta}}_{\tilde{a}:=}\right)\exp(gt)-\tilde{a},$$

and compare a with $\tilde{a}$ to see which growth rate is larger.

$$\tilde{a} = \frac{(k^*+\frac{w^*}{r^*})L}{1+k^*\beta} > \frac{(r^*k^*+w^*+(1+\tau_E)P_E\xi_K k^*)L}{r^*(1+k^*\beta)+\tau} = a$$
$$\iff r^*(1+k^*\beta)(k^*+\tfrac{w^*}{r^*})+\tau(k^*+\tfrac{w^*}{r^*}) > (1+k^*\beta)(r^*k^*+w^*+(1+\tau_E)P_E\xi_K k^*)$$
$$\iff \tau(k^*+\tfrac{w^*}{r^*}) > (1+k^*\beta)(1+\tau_E)P_E\xi_K k^*$$

Keeping in mind that $w^* > \frac{r^*}{\beta}$, as already shown in section 2.5, we can now substitute in $\tau = (1+\tau_E)P_E\xi_K\beta k^* + (1+\tau_E)P_E\xi_Z + \tau_Z R_Z^*$, which yields:

$$
\begin{aligned}
&(1+\tau_E)P_E\xi_K k^*\beta k^* + (1+\tau_E)P_E\xi_K k^* \underbrace{\frac{w^*\beta}{r^*}}_{>1} + \underbrace{\left((1+\tau_E)P_E\xi_Z + \tau_Z R_Z^*\right)\left(k^* + \tfrac{w^*}{r^*}\right)}_{\geq 0} \\
>&(1+k^*\beta)(1+\tau_E)P_E\xi_K k^*.
\end{aligned}
$$

This statement is obviously correct, proving $\tilde{a} > a$, which means that introducing transfers slows down the growth of automation capital.

Economically, this can be explained the following way: With the additional transfers, initial household disposable income $Y_d(0)$ increases, which can be used to save or consume. While the growth rate of consumption g remains unchanged, households choose a higher initial level $C(0)$ [see (29)], and therefore consumption $C(t)$ is increased at every point in time. Households therefore enjoy a higher level of utility and they choose to actually save less than in the case without transfers.

Next, we consider the growth rates of traditional capital $K(t)$ and output $Y(t)$. Using $Z_0 = \frac{K_0 - k^*L}{k^*\beta}$, we obtain

$$
\begin{aligned}
K(t) &= k^*\beta Z(t) + k^*L \\
&= k^*\beta\left((Z_0+a)\exp(gt) - a\right) + k^*L \\
&= (K_0 \underbrace{-k^*L + ak^*\beta}_{b:=})\exp(gt) + k^*L - ak^*\beta \\
&= (K_0 + b)\exp(gt) - b.
\end{aligned}
$$

Using the same procedure as before yields the following growth rate:

$$
\frac{\dot{K}(t)}{K(t)} = \frac{\dot{Y}(t)}{Y(t)} = g \cdot \frac{1}{1 - \frac{b\exp(-gt)}{K_0+b}}.
$$

Again, we need to take a closer look at the term $\frac{b}{K_0+b}$, using $\frac{K_0}{k^*\beta} = Z_0 + \frac{L}{\beta}$:

$$
\begin{aligned}
b &= \frac{\left[k^*\beta(r^*k^* + w^* + (1+\tau_E)P_E\xi_K k^*) - r^*k^*(1+k^*\beta) - \tau k^*\right]L}{r^*(1+k^*\beta)+\tau} \\
&= \frac{\left[k^*\beta(w^* + (1+\tau_E)P_E\xi_K k^*) - r^*k^* - \tau k^*\right]L}{r^*(1+k^*\beta)+\tau} \\
&= \frac{\left[w^* + (1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right]k^*\beta L}{r^*(1+k^*\beta)+\tau}
\end{aligned}
$$

$$K_0 + b = \frac{\left[r^*(1+k^*\beta)\frac{K_0}{k^*\beta} + \tau\frac{K_0}{k^*\beta} + (w^* + (1+\tau_E)P_E\xi_K k^*)L - \frac{r^*}{\beta}L - \frac{\tau}{\beta}L\right] k^*\beta}{r^*(1+k^*\beta)+\tau}$$

$$= \frac{\left[r^*(1+k^*\beta)Z_0 + \tau Z_0 + \frac{L}{\beta}r^*(1+k^*\beta-1) + \frac{\tau}{\beta}L - \frac{\tau}{\beta}L + (w^* + (1+\tau_E)P_E\xi_K k^*)L\right] k^*\beta}{r^*(1+k^*\beta)+\tau}$$

$$= \frac{\left[\underbrace{r^*(1+k^*\beta)Z_0 + r^*k^*L}_{=r^*(Z_0+K_0)} + \tau Z_0 + (w^* + (1+\tau_E)P_E\xi_K k^*)L\right] k^*\beta}{r^*(1+k^*\beta)+\tau}$$

$$\frac{b}{K_0+b} = \frac{\left[w^* + (1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right] L}{r^*(Z_0+K_0) + \tau Z_0 + (w^* + (1+\tau_E)P_E\xi_K k^*)L}$$

$$= \frac{\left[w^* + (1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right] L}{r^*(Z_0+K_0) + w^*L + Tr(0)} = \frac{\left[w^* + (1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right] L}{Y_d(0)}$$

We can now obtain the growth rate of $K(t)$ and $Y(t)$ as:

$$\frac{\dot{K}(t)}{K(t)} = \frac{\dot{Y}(t)}{Y(t)} = g \cdot \frac{1}{1 - \frac{\left(w^*+(1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right)L\exp(-gt)}{r^*(Z_0+K_0)+w^*L+Tr(0)}} = g \cdot \frac{1}{1 - \frac{\left(w^*+(1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right)L\exp(-gt)}{Y_d(0)}}.$$

Like in the case without transfers, traditional capital grows slower than automation capital. Also, $K(t)$ and $Y(t)$ again grow faster than g, since substituting in τ and r^* yields:

$$\begin{aligned}
& w^* + (1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta} \\
=& w^* + (1+\tau_E)P_E\xi_K k^* - \frac{1}{1+\tau_Z}\left[\underbrace{(1-\alpha)A(k^*)^\alpha}_{=w^*} - (1+\tau_E)P_E\xi_Z\frac{1}{\beta}\right] + \frac{\delta}{\beta} \\
& - (1+\tau_E)P_E\xi_K k^* - (1+\tau_E)P_E\xi_Z\frac{1}{\beta} - \tau_Z R_Z^*\frac{1}{\beta} \\
=& w^* - \frac{1}{1+\tau_Z}\left[w^* - (1+\tau_E)P_E\xi_Z\frac{1}{\beta}\right] + \frac{\delta}{\beta} - (1+\tau_E)P_E\xi_Z\frac{1}{\beta} \\
& - \frac{\tau_Z}{1+\tau_Z}\left[w^* - (1+\tau_E)P_E\xi_Z\frac{1}{\beta}\right] = \frac{\delta}{\beta} > 0.
\end{aligned}$$

Therefore, traditional capital and output outpace consumption and we can derive an alternate formula for the growth rate:

$$\frac{\dot{K}(t)}{K(t)} = \frac{\dot{Y}(t)}{Y(t)} = g \cdot \frac{1}{1 - \frac{\frac{\delta}{\beta}L\exp(-gt)}{r^*(Z_0+K_0)+w^*L+Tr(0)}},$$

which is similar to the rate derived by Steigum (2011), see section 2.6. It's easy to see that, if there was no deprecation, $K(t)$ and $Y(t)$ would grow at the constant rate g.

Comparing with the transfer-less growth rate of traditional capital and output,

$$g \cdot \frac{1}{1 - \frac{(w^* - \frac{r^*}{\beta})L\exp(-gt)}{(Z_0+K_0)r^*+w^*L}}$$

we can see that it is higher than the one including transfers, if and only if

$$\frac{(w^* - \frac{r^*}{\beta})L}{(Z_0 + K_0)r^* + w^*L} > \frac{\left(w^* + (1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right)L}{r^*(Z_0 + K_0) + w^*L + Tr(0)}.$$

Obviously, the denominator on the right side is larger. Therefore, if we can show that the right-hand side numerator is smaller than the one on the left, the statement is proven. Simply substituting in τ yields:

$$\begin{aligned}
&w^* - \frac{r^*}{\beta} > w^* + (1+\tau_E)P_E\xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta} \\
\iff &(1+\tau_E)P_E\xi_K k^* + (1+\tau_E)P_E\xi_Z\frac{1}{\beta} + \tau_Z R_Z^*\frac{1}{\beta} > (1+\tau_E)P_E\xi_K k^* \\
\iff &(1+\tau_E)P_E\xi_Z\frac{1}{\beta} + \tau_Z R_Z^*\frac{1}{\beta} > 0.
\end{aligned}$$

This statement is true, proving that the inclusion of transfers also slows down the growth of output and traditional capital.

Finally, we derive the growth rate of total capital $M(t)$:

$$\begin{aligned}
M(t) &= Z(t) + K(t) = Z(t)(1 + k^*\beta) + k^*L \\
&= (1 + k^*\beta)(Z_0 + a)\exp(gt) - (1 + k^*\beta)a + k^*L \\
&= (\underbrace{Z_0 + k^*\beta Z_0 + k^*L}_{=Z_0+K_0=M_0} \underbrace{-k^*L + (1+k^*\beta)a}_{c:=})\exp(gt) - (1 + k^*\beta)a + k^*L \\
&= (M_0 + c)\exp(gt) - c
\end{aligned}$$

Like before, we can calculate the growth rate:

$$\frac{\dot{M}(t)}{M(t)} = g \cdot \frac{1}{1 - \frac{c\exp(-gt)}{K_0+Z_0+c}}.$$

More explicitly, using $\tau K_0 = \tau k^* \beta Z_0 + \tau k^* L$, we obtain:

$$\begin{aligned}
c &= \frac{(r^* k^* + w^* + (1+\tau_E) P_E \xi_K k^*)(1+k^*\beta)L}{r^*(1+k^*\beta)+\tau} - k^* L \\
&= \frac{(r^* k^* - r^* k^* + w^* + (1+\tau_E) P_E \xi_K k^*)(1+k^*\beta)L - \tau k^* L}{r^*(1+k^*\beta)+\tau} \\
&= \frac{(w^* + (1+\tau_E) P_E \xi_K k^*)(1+k^*\beta)L - \tau k^* L}{r^*(1+k^*\beta)+\tau} \\
K_0 + Z_0 + c &= \frac{(w^* + (1+\tau_E) P_E \xi_K k^*)(1+k^*\beta)L - \tau k^* L + (1+k^*\beta) r^*(K_0+Z_0) + \tau K_0 + \tau Z_0}{r^*(1+k^*\beta)+\tau} \\
&= \frac{(w^* + (1+\tau_E) P_E \xi_K k^*)(1+k^*\beta)L + (1+k^*\beta) r^*(K_0+Z_0) + (1+k^*\beta)\tau Z_0}{r^*(1+k^*\beta)+\tau} \\
\frac{c}{K_0+Z_0+c} &= \frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \tau \frac{k^*}{1+k^*\beta}\right) L}{(w^* + (1+\tau_E) P_E \xi_K k^*)L + r^*(K_0+Z_0) + \tau Z_0} \\
&= \frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \tau \frac{k^*}{1+k^*\beta}\right) L}{r^*(K_0+Z_0) + w^* L + Tr(0)} = \frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \tau \frac{k^*}{1+k^*\beta}\right) L}{Y_d(0)},
\end{aligned}$$

which yields a total capital growth rate of:

$$\frac{\dot{M}(t)}{M(t)} = g \cdot \frac{1}{1 - \frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \tau \frac{k^*}{1+k^*\beta}\right) L \exp(-gt)}{r^*(K_0+Z_0) + w^* L + Tr(0)}} = g \cdot \frac{1}{1 - \frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \tau \frac{k^*}{1+k^*\beta}\right) L \exp(-gt)}{Y_d(0)}}.$$

It can easily be seen that this rate is higher than the growth rate of $K(t)$ (and therefore also higher than the one of $C(t)$), since

$$\begin{aligned}
& \tau(1+k^*\beta) > \tau k^* \beta \\
\implies & \frac{\tau}{\beta} > \tau \frac{k^*}{1+k^*\beta} \\
\implies & \frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \tau \frac{k^*}{1+k^*\beta}\right) L}{Y_d(0)} > \frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \frac{r^*}{\beta} - \frac{\tau}{\beta}\right) L}{Y_d(0)},
\end{aligned}$$

and it is lower than the growth rate of automation capital $Z(t)$, since

$$\frac{\left(w^* + (1+\tau_E) P_E \xi_K k^* - \tau \frac{k^*}{1+k^*\beta}\right) L}{Y_d(0)} < \frac{(r^* k^* + w^* + (1+\tau_E) P_E \xi_K k^*)L}{Y_d(0)}.$$

The introduction of transfers slows down the growth of $K(t)$ and $Z(t)$, and therefore the same must be true for total capital $M(t) = K(t) + Z(t)$.

2.8 Impact of Taxation

In this section, we consider the effects of both types of taxes on the economy along the balanced growth path. We prove these subsequent statements:

Proposition 3. *As long as the BGP rental rate for capital $R_Z^* = R_K^*$ is positive, the following holds:*

i) The introduction of a robot tax raises the BGP wage w^*. If automation capital is relatively power efficient, i.e. $\Delta_\xi = \frac{\xi_Z}{\xi_K} - (1+\tau_Z) < 0$, then an electricity tax lowers w^*. Otherwise, the opposite holds.

ii) Both types of taxes decrease the BGP interest rate r^*.

iii) Both types of taxes decrease the consumption growth rate g.

Proof. We first look at the impact of taxes on the BGP traditional capital intensity k^*. Keeping in mind that k^* is defined by $\Sigma(k^*(P_E, \tau_E, \tau_Z), P_E, \tau_E, \tau_Z) = 0$ [see (16)] and that $\Sigma_k(k^*) < 0$, we can form the derivative of the implicit function:

$$\begin{aligned}
\frac{dk^*}{d\tau_Z} &= -\frac{\Sigma_{\tau_Z}(k^*)}{\Sigma_k(k^*)} = \underbrace{-\frac{1}{\Sigma_k(k^*)}}_{>0} \frac{\alpha A (k^*)^\alpha \left[(1-\alpha)\beta A (k^*)^\alpha - (1+\tau_E) P_E \xi_Z\right]}{\left[(1-\alpha)\beta A (k^*)^\alpha - (1+\tau_E) P_E \xi_K \Delta_\xi\right]^2}, \\
\frac{dk^*}{d\tau_E} &= -\frac{\Sigma_{\tau_E}(k^*)}{\Sigma_k(k^*)} = \underbrace{-\frac{1}{\Sigma_k(k^*)}}_{>0} \frac{\alpha A (k^*)^\alpha (1+\tau_Z) P_E \xi_K \Delta_\xi}{\left[(1-\alpha)\beta A (k^*)^\alpha - (1+\tau_E) P_E \xi_K \Delta_\xi\right]^2}
\end{aligned}$$

where

$$\begin{aligned}
\frac{dk^*}{d\tau_Z} &> 0 \Longleftarrow (1-\alpha)\beta A (k^*)^\alpha - (1+\tau_E) P_E \xi_Z = R_Z^* (1+\tau_Z) > 0, \\
\frac{dk^*}{d\tau_E} &< 0 \Longleftrightarrow \Delta_\xi = \frac{\xi_Z}{\xi_K} - (1+\tau_Z) < 0.
\end{aligned}$$

The robot tax tends to increase the traditional capital intensity as long as the rental rate for capital is positive [see (7)]. Intuitively, this makes perfect sense, since the tax makes automation capital less attractive to investors, who shift their investments towards traditional capital instead. The energy tax, in turn, may lower the traditional capital intensity only if automation capital is equal or less energy intensive than conventional capital or otherwise if the robot tax is sufficiently high. And since the wage rate depends positively on the traditional capital intensity

$$w^* = (1-\alpha) A (k^*)^\alpha,$$

the taxes have the same effect on w^*, proving the first statement. The fact that it raises wages could be used as an argument in favor of a robot tax, considering that wages otherwise stagnate while all benefits from growth are reaped by capital. In our model, both streams of income are of course paid out to the representative household, but in the real world, where capital is unevenly distributed, this might lead to rising inequality.

Next, we consider the tax impacts on the interest rate r^*,

$$\begin{aligned} r^* &= \frac{1}{1+\tau_Z}\left[(1-\alpha)\,\beta A\,(k^*)^{\alpha}-(1+\tau_E)P_E\xi_Z\right]-\delta \\ &= \alpha A\,(k^*)^{\alpha-1}-(1+\tau_E)P_E\xi_K-\delta. \end{aligned}$$

Taking the derivative with respect to τ_Z yields

$$\frac{dr^*}{d\tau_Z} = -\alpha(1-\alpha)A(k^*)^{\alpha-2}\underbrace{\frac{dk^*}{d\tau_Z}}_{>0} < 0,$$

meaning that a robot tax decreases the interest rate. This is hardly surprising since (ceteris paribus) the tax makes it more expensive and less attractive for firms to rent automation capital. The companies partially compensate for the tax burden by reducing the amount they pay to the capital owners.

To derive the effect of the electricity tax on r^*, we need to distinguish two cases.

Case 1: $\frac{dk^*}{d\tau_E} \geq 0$

Taking the derivative of r^* with respect to τ_Z yields

$$\frac{dr^*}{d\tau_E} = -\alpha(1-\alpha)A(k^*)^{\alpha-2}\underbrace{\frac{dk^*}{d\tau_Z}}_{\geq 0} - P_E\xi_K < 0.$$

Case 2: $\frac{dk^*}{d\tau_E} < 0$

This time, we use the alternate formula for r^* to calculate the derivative:

$$\frac{dr^*}{d\tau_E} = \frac{1}{1+\tau_Z}\left[\alpha\,(1-\alpha)\,\beta A\,(k^*)^{\alpha-1}\underbrace{\frac{dk^*}{d\tau_Z}}_{<0} - P_E\xi_Z\right] < 0.$$

Again, this result is not surprising because the electricity tax makes it more expensive for firms to rent capital.

Finally, since

$$g = \frac{r^*-\rho}{\theta},$$

both types of taxes lower the consumption growth rate g along with r^*. Intuitively, this makes sense, due to the fact that the taxes disincentivize and hamper the accumulation of both capital stocks, which is the engine of economic growth in this model framework. Conversely, this means that subsidizing robots or electricity could increase growth, at least when paid for with a lump-sum tax.

□

2.9 Dynamics of the Model

It is worthwhile to take a closer look at the dynamics of the model, along and off the balanced growth path. As derived from the household problem, aggregate consumption $C(t)$ develops according to the Ramsey rule (3)

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta},$$

with the interest rate $r(t)$ being the weighted average between the two interest rates for owning automation capital $r_Z(t)$ and traditional capital $r_K(t)$

$$\begin{aligned} r_Z(t) &= R_Z(t) - \delta = \frac{1}{1+\tau_Z}\left\{(1-\alpha)\beta A\left[\frac{K(t)}{\beta Z(t) + L}\right]^{\alpha} - (1+\tau_E)P_E\xi_Z\right\} - \delta \\ r_K(t) &= R_K(t) - \delta = \alpha A K(t)^{\alpha-1}[\beta Z(t) + L]^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta \\ r(t) &= \frac{Z(t)r_Z(t) + K(t)r_K(t)}{Z(t) + K(t)}. \end{aligned}$$

Along the BGP, the no-arbitrage condition is fulfilled such that $r_Z(t) = r_K(t) = r^*$ and $C(t)$ therefore grows at a constant rate

$$\frac{\dot{C}(t)}{C(t)} = g = \frac{\alpha A\left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta - \rho}{\theta}.$$

The development of total capital, i.e. net investment, $M(t) = K(t) + Z(t)$ is governed by the aggregate household budget constraint:

$$\begin{aligned} \dot{M}(t) &= r(t)M(t) + w(t)L - C(t) + Tr(t) \\ &= Y(t) - (1+\tau_E)P_E E(t) - \tau_Z R_Z(t)Z(t) - \delta M(t) - C(t) + Tr(t), \end{aligned}$$

where government transfers, depending on the chosen model setup, could be:

i) $Tr(t) = 0$ (no transfers),

ii) $Tr(t) = \tau_E P_E E(t) + \tau_Z R_Z(t)Z(t)$ (transfer of tax revenue),

iii) $Tr(t) = (1+\tau_E)P_E E(t) + \tau_Z R_Z(t)Z(t)$ (transfer of tax revenue and proceeds from electricity production).

While off the BGP, the interest rates for both types of capital are not equal. Households will then decide to invest all of their savings into the type of capital which yields the better return, while letting the other depreciate. This will continue until $k(t)$ reaches its equilibrium value k^*, at which point the no-arbitrage condition will be fulfilled and both investments will be equally profitable. For instance, if there is relatively too little automation capital, $k(t) > k^*$, then $r_Z(t) > r_K(t)$ follows, meaning that the relatively scarce $Z(t)$ yields a higher return. Households will then invest according to

$$\begin{aligned} \dot{Z}(t) &= Y(t) - (1+\tau_E)P_E E(t) - \tau_Z R_Z(t)Z(t) - C(t) + Tr(t) - \delta Z(t) \\ \dot{K}(t) &= -\delta K, \end{aligned}$$

which lowers $k(t)$ until the equilibrium level k^* is reached. On the BGP, $Z(t)$ and $K(t)$ have to develop precisely in such a way that $k(t) = \frac{K(t)}{\beta Z(t)+L}$ does not change from k^*.

$$\begin{aligned} K(t) &= k^*\beta Z(t) + k^*L \\ \implies \dot{K}(t) &= k^*\beta\dot{Z}(t) \\ \implies \dot{M}(t) &= \dot{K}(t) + \dot{Z}(t) = (1 + k^*\beta)\dot{Z}(t) \\ \implies \dot{Z}(t) &= \frac{1}{1 + k^*\beta}\dot{M}(t) \\ \implies \dot{K}(t) &= \frac{k^*\beta}{1 + k^*\beta}\dot{M}(t) \end{aligned}$$

Note that $\frac{1}{1+k^*\beta}$ is the share of net investment going to automation capital, whereas $\frac{k^*\beta}{1+k^*\beta}$ is the share going to traditional capital. $K(t)$ will receive more investment than $Z(t)$ if

$$\begin{aligned} k^*\beta = \frac{K(t)}{Z(t) + L/\beta} &> 1 \\ \iff K(t) &> Z(t) + L/\beta, \end{aligned}$$

i.e. if the equilibrium requires more traditional capital than the sum of automation capital and workers (in final goods terms). The reverse is true if $k^*\beta < 1$.

From these considerations it also becomes clear why $Z(t)$ needs to always grow at a higher rate than $K(t)$ along the BGP: Consider the case $k^*\beta = 1$, where both capital stocks receive exactly half of net investment, which keeps the ratio k^* in equilibrium such that interest rates are equalized. $\frac{K(t)}{Z(t)+L/\beta} = 1$ implies that $Z(t) < K(t)$. Since both $Z(t)$ and $K(t)$ increase by the same amount of investment in absolute terms, automation capital's *growth rate* must therefore be higher. The growth rates will converge as $K(t)$ and $Z(t)$ go towards infinity, because the term L/β in the denominator will become irrelevant.

3 Comparative Statics

3.1 Calibration

The model is calibrated with data from the U.S. manufacturing sector, since it is a part of the economy with good data availability and also heavily affected by automation.

The Current Population Survey (2021) conducted by the Bureau of the Census for the Bureau of Labor Statistics estimates that there were 14.718 million people employed in the manufacturing industry in 2021. Taking the average of Bureau of Economic Analysis (2022b) quarterly data, the manufacturing sector contributed 2563.3 billion U.S. dollars value added to GDP during 2021. In this and subsequent sections, we assume that electricity is purchased from an external energy sector and therefore its cost needs to be subtracted from output to obtain value added, i.e. $GDP(t) = Y(t) - E(t)P_E$.

The Federal Reserve Board (2022) estimates the capital stock to be 3012.7082 billion in 2012 dollars, roughly 3555 billion in 2021 dollars. The International Federation of Robotics (2021) reports in its World Robotics 2021 Industrial Robots report that there were a total of 310700 operational industrial robots in the United States in 2020. We can combine this number with the estimate by Acemoglu and Restrepo (2020) that an industrial robot can on average perform roughly the work of 3 humans in simple tasks to obtain $R_0 := \beta Z_0 = 3 \cdot 310700 = 932100$.

We assume a conventional depreciation rate of 10% and a conventional discount rate of 5% (Bureau of Economic Analysis, 2013; Warner and Pleeter, 2001). We set the elasticity of output with respect to traditional physical capital employment at the conventional level of 1/3 and set θ such that the elasticity of intertemporal substitution is equal to 0.5 (Guvenen, 2006).

According to the U.S. Energy Information Administration (2022), the average price of industrial electricity was 7.26 Cents / kWh in 2021 and the manufacturing sector used 894476 million kWh of electricity in 2018 (U.S. EIA, 2018). The electricity usage parameters are determined such that ξ_Z is scaled up or down in relation to ξ_K, i.e. $\xi_Z = s_{\xi_Z}\xi_K$, and the model reproduces annual electricity consumption. For now, the scaling factor is chosen as $s_{\xi_Z} = 1.4$, meaning that automation capital uses 40% more electricity than traditional capital. Later on, we will also consider the opposite case. The OECD (2019) reports that there were no taxes on industrial electricity use in the U.S. in 2019.

Table 1: Parameter values

Parameter	Value	Parameter	Value
ρ	0.05	L	14.718 million
δ	0.10	α	1/3
θ	2.00	P_E	0.0726 USD/kWh
M_0	3555 billion USD	R_0	932100
E_0	894476 million kWh	s_{ξ_Z}	1.4
τ_E	0	τ_Z	0
GDP_0	2563.3 billion USD	Y_0	2628.2 billion USD

The following system of equations has to be solved in order to derive the final parameters A, β, K_0, ξ_K and ξ_Z, assuring that the model replicates the observed output and that the economy presently follows the BGP.

$$\begin{aligned}
A &= \frac{Y_0}{K_0^\alpha (R_0 + L)^{1-\alpha}} \\
M_0 &= K_0 + \frac{R_0}{\beta} \\
E_0 &= K_0 \xi_K + \frac{R_0}{\beta}\xi_Z \\
\xi_Z &= s_{\xi_Z}\xi_K \\
\beta &= \frac{\alpha(1+\tau_Z)A\left(\frac{K_0}{R_0+L}\right)^{\alpha-1} - (1+\tau_E)P_E(\xi_K(1+\tau_Z)-\xi_Z)}{(1-\alpha)A\left(\frac{K_0}{R_0+L}\right)^{\alpha}}
\end{aligned}$$

The first equation assures that total factor productivity is set such that the model reproduces the real world output Y_0. The second equation guarantees that total capital is equal to traditional capital plus automation capital. The third equation assures that electricity consumption equals the real world equivalent, while the forth simply applies the scaling factor s_{ξ_Z} to ξ_K in order to obtain ξ_Z.

The final equation makes sure that β is determined such that the economy is on the BGP at $t = 0$, i.e. that $k^* = k_0 = \frac{K_0}{R_0+L}$. It is derived from (15), the equation characterizing k^*. If $k_0 = k^*$, then it must hold that:

$$\begin{aligned}
k_0 &= \frac{\alpha\,(1+\tau_Z)\,A(k_0)^\alpha}{(1-\alpha)\,\beta A(k_0)^\alpha + (1+\tau_E)P_E(\xi_K(1+\tau_Z)-\xi_Z)} \\
\iff & (1-\alpha)\beta A(k_0)^\alpha + (1+\tau_E)P_E(\xi_K(1+\tau_Z)-\xi_Z) = \alpha\,(1+\tau_Z)\,A(k_0)^{\alpha-1} \\
\iff & (1-\alpha)\beta A(k_0)^\alpha = \alpha\,(1+\tau_Z)\,A(k_0)^{\alpha-1} - (1+\tau_E)P_E(\xi_K(1+\tau_Z)-\xi_Z) \\
\iff & \beta = \frac{\alpha\,(1+\tau_Z)\,A(k_0)^{\alpha-1} - (1+\tau_E)P_E(\xi_K(1+\tau_Z)-\xi_Z)}{(1-\alpha)A(k_0)^\alpha}.
\end{aligned}$$

Solving the equations, we obtain the final parameters. $\beta = 2.5195\times10^{-6}$ indicates that in order to purchase enough automation capital to do the work of 1 human, one would have to spend $\frac{1}{\beta} = 396908\$$. Considering the cost of industrial robots, this is on the high side, but one should keep in mind that not all human tasks are easily automatized. We also get a value for $Z_0 = \frac{R_0}{\beta} = 370$ billion USD, or 10.4% of total capital, while $K_0 = 3185$ billion USD represents the remaining 89.6%, suggesting that traditional capital still dominates in terms of quantity. The electricity usage parameters, which state how much kWh one unit of capital consumes annually, are determined to be $\xi_K = 0.2416$ and $\xi_Z = 0.3382$.

Table 2: Derived parameter values

Parameter	Value	Parameter	Value
A	2855.1	β	2.5195×10^{-6}
K_0	3185 billion USD	Z_0	370 billion USD
ξ_K	0.2416	ξ_Z	0.3382

3.2 Values Considered

For comparative statics, we are going to look at 3 categories of variables. Firstly, short term variables, meaning the value of a variable at $t = 0$, after the shock (parameter change) has occurred. These are the short term interest rates $r_Z(0)$ and $r_K(0)$, as well as short term tax revenue $T(0)$:

$$\begin{aligned} r_Z(0) &= \frac{1}{1+\tau_Z}\left\{(1-\alpha)\beta A\left[\frac{K(0)}{\beta Z(0)+L}\right]^{\alpha} - (1+\tau_E)P_E\xi_Z\right\} - \delta \\ r_K(0) &= \alpha A K(0)^{\alpha-1}[\beta Z(0)+L]^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta \\ T(0) &= \tau_Z R_Z(0)Z(0) + \tau_E P_E[\xi_K K(0) + \xi_Z Z(0)]. \end{aligned}$$

Secondly, we have variables which are constant on the balanced growth path. This is true of the consumption growth rate g, the wage w^*, the traditional capital intensity k^*, output per unit of traditional capital $\left(\frac{Y}{K}\right)^*$, and the BGP interest rates r_Z^* and r_K^*:

$$\begin{aligned} k^* =& \frac{\alpha\,(1+\tau_Z)\,A(k^*)^{\alpha}}{(1-\alpha)\,\beta A(k^*)^{\alpha} - (1+\tau_E)P_E\xi_K\Delta_\xi} \\ g =& \frac{\alpha A\left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta - \rho}{\theta} \\ r_Z^* =& \frac{1}{1+\tau_Z}\left[(1-\alpha)\,\beta A\,(k^*)^{\alpha} - (1+\tau_E)P_E\xi_Z\right] - \delta, \\ r_K^* =& \alpha A\left(\frac{1}{k^*}\right)^{1-\alpha} - (1+\tau_E)P_E\xi_K - \delta \\ w^* =& (1-\alpha)A\,(k^*)^{\alpha} \\ \left(\frac{Y}{K}\right)^* =& A(k^*)^{\alpha-1}. \end{aligned}$$

Lastly, there are long term variables which are not constant along the BGP, but converge towards a fixed value as $t \to \infty$. This applies to electricity consumption per unit of traditional capital $\frac{E}{K}$, tax revenue per unit of traditional capital $\frac{T}{K}$, electricity consumption (in kWh) per unit of output $\frac{E}{Y}$, the proportion of output used to buy electricity $\frac{EP_E}{Y}$, tax revenue per unit of output $\frac{T}{Y}$ and the proportion of tax revenue to GDP (value added):

$$\frac{E}{K}(\infty) = \lim_{t\to\infty} \frac{\xi_K K(t) + \xi_Z Z(t)}{K(t)} = \xi_K + \lim_{t\to\infty} \xi_Z \underbrace{\frac{Z(t)}{\beta Z(t) + L}}_{\to 1/\beta} \underbrace{\frac{\beta Z(t) + L}{K}}_{=1/k^*}$$

$$\frac{T}{K}(\infty) = \frac{\tau_Z R_Z^*}{\beta k^*} + \tau_E P_E \left[\xi_K + \frac{\xi_Z}{\beta k^*}\right]$$

$$\frac{E}{Y}(\infty) = \lim_{t\to\infty} \frac{\xi_K K(t) + \xi_Z Z(t)}{AK(t)(k^*)^{\alpha-1}} = \frac{\xi_K}{A(k^*)^{\alpha-1}} + \frac{\xi_Z}{A(k^*)^{\alpha}\beta}$$

$$\frac{EP_E}{Y}(\infty) = P_E \left[\frac{\xi_K}{A(k^*)^{\alpha-1}} + \frac{\xi_Z}{A(k^*)^{\alpha}\beta}\right]$$

$$\frac{T}{Y}(\infty) = \frac{\tau_Z R_Z^*}{A(k^*)^{\alpha}\beta} + \tau_E P_E \left[\frac{\xi_K}{A(k^*)^{\alpha-1}} + \frac{\xi_Z}{A(k^*)^{\alpha}\beta}\right].$$

$$\frac{T}{GDP}(\infty) = \frac{T}{Y}(\infty) \cdot \frac{1}{1 - \frac{EP_E}{Y}(\infty)}$$

3.3 The Baseline Case

Using the baseline calibration from above, we get the following results:

$T(0)$	$r_K(0)$	r_K^*	$r_Z(0)$	r_Z^*	k^*	w^*	g
0	15.752%	15.752%	15.752%	15.752%	203515	111958	5.376%

$\frac{E}{Y}(\infty)$	$\frac{T}{Y}(\infty)$	$\frac{E}{K}(\infty)$	$\frac{EP_E}{Y}(\infty)$	$\frac{T}{K}(\infty)$	$\frac{Y}{K}(\infty)$	$\frac{T}{GDP}(\infty)$
1.092	0	0.90109	7.927%	0	0.82518	0

As we can see, all short term and BGP interest rates are the same at 15.752%, as would be expected if the model was calibrated correctly. Tax revenue is zero since all tax rates were set to zero. The annual wage of a worker is 111958$, which more than the 76391$ per year reported by the U.S. Bureau of Economic Analysis (2022a) for the manufacturing sector. The consumption growth rate $g = 5.376\%$ is also higher than the 1.9% annual growth that U.S. manufacturing experienced from 2014-2019 according to the National Institute of Standards and Technology (2021). In the long term, 1.092 kWh is used to produce a final good worth 1$ and 7.927% of output is used to purchase electricity, compared to $\frac{E_0 P_E}{Y_0} = 2.47\%$ now.

Since the economy is on the BGP, we can calculate the growth rates of $M(t)$, $Z(t)$ and $K(t)$ at $t = 0$ according to Proposition 2. We assume that revenue from electricity is not transferred to households. This assumption has no consequences on later sections discussing comparative statics. The results are displayed in Table 3. As we can see, automation capital more than doubles in a year, with a growth rate of 134.64%, while total capital grows at a strong 21.20%. Traditional capital and output grow at an 8.02%

annual rate. The exceptional growth of $Z(t)$ is due to the fact that there is still relatively little automation capital relative to labor, as $\frac{R_0}{R_0+L} = 5.96\%$.

Table 3: Initial Capital Growth Rates

Growth rate	Value
$g_M(0)$	21.20%
$g_Z(0)$	134.64%
$g_K(0) = g_Y(0)$	8.02%

Furthermore, we can calculate and visualize how the growth rates will develop in the future. The results are plotted in Figure 2. All growth rates fall exponentially towards g, but it takes more than 30 years to achieve something like convergence. Automation capital expands at an extreme pace of more than 20% per year for the next half decade before settling down to a more reasonable speed.

In Figure 3, output is broken down according to the types of expenditures (left) and income (right) and plotted for the next 30 years. On the expenditure side, we can see that the consumption share falls from 55% to under 40%, as more money needs to be allocated to pay for depreciation and electricity costs. Right away, net investment in automation capital is higher than in traditional capital, at 19% of output (even though there are few robots initially), before falling slightly. Net investment in traditional capital also falls marginally. Depreciation and electricity costs are bolstered by the growing automation capital stock, which increases faster than output $Y(t)$. Depreciation grows from 13.5% to 32.5%, contributing to the fact that gross investment makes up more than 50% of output after 30 years. Likewise, industrial spending on electricity grows from 2.5% to 7.2%.

In the other plot, we can see that the labor income share collapses from 62.7% to under 10% in 30 years, since wages and population stagnate. Conversely, the interest payments to owners of automation capital explode from 2.2% all the way to 32.2%, the biggest share of the income pie (excluding depreciation), reflecting the growth of the robot capital stock. The share of income paid out as interest for traditional capital stays constant at 19%, as $K(t)$ grows at the same rate as $Y(t)$.

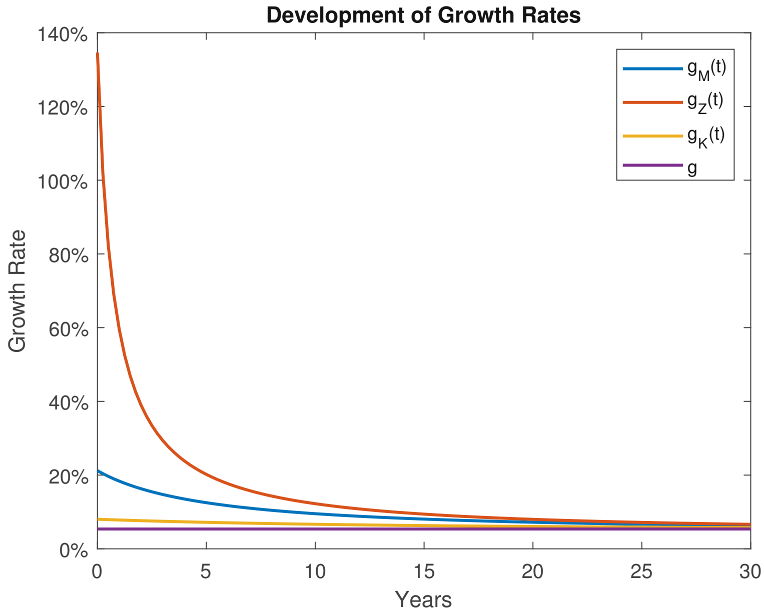

Figure 2: Various growth rates over the next 30 years for the baseline calibration.

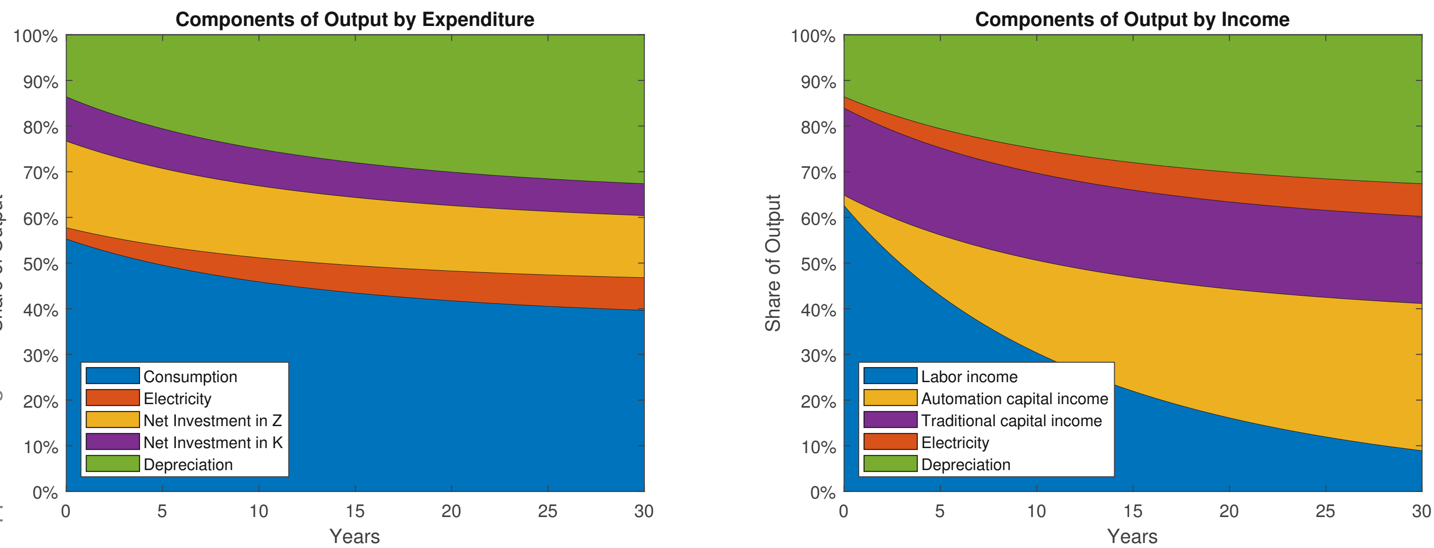

Figure 3: Development of the components of output $Y(t)$ in the baseline calibration over the next 30 years. On the left, output is broken down according to expenditures, on the right according to the types of income.

3.4 A Rise in Electricity Prices

Before discussing taxes, it might be interesting to look at a rise in electricity prices like the one observed in the present situation. According to the U.S. Energy Information Administration (2022), industrial electricity prices saw a modest 15% increase over the 2021 average to 8.35 Cents/kWh in May 2022.

Applying a comparative static analysis, we obtain the following results:

	$T(0)$	$r_K(0)$	r_K^*	$r_Z(0)$	r_Z^*	k^*	w^*
$P_E = 0.0726$	0	15.75%	15.75%	15.75%	15.75%	203,515	111,958
$\tilde{P}_E = 0.0835$	0	15.49%	15.42%	15.38%	15.42%	204,290	112,100
		-0.26pp	-0.33pp	-0.37pp	-0.33pp	+0.38%	+0.13%

	$\frac{E}{Y}(\infty)$	$\frac{T}{Y}(\infty)$	$\frac{E}{K}(\infty)$	$\frac{EP_E}{Y}(\infty)$	$\frac{T}{K}(\infty)$	$\frac{Y}{K}(\infty)$	g
$P_E = 0.0726$	1.092	0	0.90109	7.927%	0	0.82518	5.38%
$\tilde{P}_E = 0.0835$	1.0917	0	0.89859	9.116%	0	0.8231	5.21%
	-0.02%		-0.28%	+1.189pp		-0.25%	-0.17pp

An immediate effect of the price rise is that the amount of output needed to purchase electricity increases from $\frac{E_0 P_E}{Y_0} = 2.47\%$ to $\frac{E_0 \tilde{P}_E}{Y_0} = 2.84\%$, which in turn contracts the value added of the manufacturing sector $(Y_0 - E_0 \tilde{P}_E)$ by 0.38%. As we can see in the table above, the price increase leads to a short term imbalance in the interest rates. The return to automation capital $Z(t)$ sinks by 0.37 percentage points since it consumes electricity. Likewise, the return to traditional capital $K(t)$ also decreases, but only by 0.26pp since $K(t)$ is assumed to be using less electricity. This leads to investment in $K(t)$, leading to a rise in the traditional capital intensity $k(t)$. Due to the relative increase in traditional capital, the marginal product of labor rises, meaning that workers receive a slightly higher wage than before. The growth rate g drops modestly to 5.21%. In the long term, there is a slight decrease in the usage of electricity per unit of output, while the amount spent on electricity rises from 7.927% to 9.116%.

3.5 Equivalent Taxes

Using comparative statics, this section will explore the possibility of substituting a robot tax of 10% with an equivalent tax on industrial electricity consumption. However, both taxes have distinct effects and therefore we will have to examine different ways in which the taxes produce the same result:

1. Choose τ_E to equalize short term tax revenue $T(0)$
2. Choose τ_E to equalize long term tax revenue per unit of traditional capital $\frac{T}{K}(\infty)$
3. Choose τ_E to equalize the BGP traditional capital intensity k^*
4. Choose τ_E to equalize long term tax revenue per unit of output $\frac{T}{Y}(\infty)$
5. Choose τ_E to equalize the BGP growth rate of consumption g

The results are summarized in Table 4, which includes the case with a 10% robot tax, the baseline case of no taxes, as well as the 5 versions for obtaining the equivalent τ_E.

According to the model, the introduction of a 10% robot tax would immediately raise 8.66 billion USD, or 0.33% of output over a year. In the short term, the return on investment in automation capital falls by 2.34pp, while the return on traditional capital remains the same. This encourages investment in $K(t)$ and traditional capital intensity $k(t)$ rises until the new BGP level k^* is reached. This development is accompanied by a falling $r_K(t)$ and an increasing $r_Z(t)$, since $K(t)$ becomes relatively more abundant and $Z(t)$ rare, until both rates are equal again along the BGP. The new BGP real interest rate is 1.57pp lower than in the baseline case, due to taxation. Labor benefits as the abundance of traditional capital means that it can demand a 2.9% higher wage. The consumption growth rate falls by 0.79pp to 4.59% as a result of the distorting effect of the robot tax. Over the long term, electricity consumption per unit of output decreases only marginally. Tax revenue makes up 5.55% of output or 6.02% of GDP (value added), a large jump from the short term effect that can be explained by the rapid growth of the robot tax base, which makes up $\frac{R_Z^* Z}{Y}(\infty) = 55.48\%$ of output in the long run.

Compared to the robot tax, the electricity tax behaves in a similar way if $\xi_Z > \xi_K$. Here, both interest rates decrease, but $r_Z(t)$ falls more than $r_K(t)$, since automation capital uses more electricity and is therefore more affected by the tax. This again leads to an accumulation of $K(t)$ and rising wages, until the BGP is reached.

(1) Equalize short term tax revenue $T(0)$

This results in a 13.33% tax on electricity, reflecting a slightly smaller tax base $E_0 P_E$ in the short term when compared to the robot tax base $R_Z(0)Z_0$. While $T(0)$ is equalized, the longer term consequences are much less severe than the robot tax. Interest rates fall only slightly, while wages increase negligibly and growth is not as affected when comparing to the baseline. In the long term, the tax base makes up $\frac{EP_E}{Y}(\infty) = 7.93\%$ of output, which is much less than the previous robot tax base and results in tax revenues comprising only 1.06% of output and 1.15% of GDP.

(2) Equalize long term tax revenue per unit of traditional capital $\frac{T}{K}(\infty)$

(4) Equalize long term tax revenue per unit of output $\frac{T}{Y}(\infty)$

(5) Equalize the BGP growth rate of consumption g

These three approaches all yield similar tax rates of around 70% and short term tax revenues of 45 billion USD. The reason why such rates are necessary to equalize longer term effects is that the electricity tax base does not grow as quickly as the robot tax base. The long term effects are quite similar to the robot tax, with the exception that traditional capital is not as abundant and wages are therefore also lower in comparison.

(3) Equalize the BGP traditional capital intensity k^*

This approach results in by far the most extreme tax rate, 344.67%, and a short term tax revenue of 223.8 billion USD. While this has the desired result in reproducing the effects on k^*, w^*, $\frac{E}{Y}(\infty)$, $\frac{E}{K}(\infty)$, $\frac{EP_E}{Y}(\infty)$ and $\frac{Y}{K}(\infty)$, it also completely crashes all interest rates and the growth rate g. In the long term, tax revenue would make up 27.19% of output and 29.52% of GDP.

		$T(0)$	$r_K(0)$	r_K^*	$r_Z(0)$	r_Z^*	k^*	w^*	g
	$\tau_Z = 0.1$	8,661,204,642	15.75%	14.18%	13.41%	14.18%	222,326	115,306	4.59%
	$\tau_Z = \tau_E = 0$	0	15.75%	15.75%	15.75%	15.75%	203,515	111,958	5.38%
		-100%	+0pp	+1.57pp	+2.34pp	+1.57pp	-8.46%	-2.90%	+0.79pp
(1)	$\tau_E = 0.13337$	8,661,204,642	15.52%	15.46%	15.43%	15.46%	204,203	112,084	5.23%
		+0%	-0.23pp	+1.28pp	+2.01pp	+1.28pp	-8.15%	-2.79%	+0.64pp
(2)	$\tau_E = 0.66802$	43,380,606,823	14.58%	14.27%	14.11%	14.27%	206,992	112,592	4.64%
		+400.86%	-1.17pp	+0.09pp	+0.70pp	+0.09pp	-6.90%	-2.35%	+0.05pp
(3)	$\tau_E = 3.4467$	223,828,173,265	9.71%	8.13%	7.29%	8.13%	222,326	115,306	1.57%
		+2484.26%	-6.04pp	-6.04pp	-6.12pp	-6.04pp	+0%	+0%	-3.02pp
(4)	$\tau_E = 0.70065$	45,499,721,498	14.52%	14.20%	14.03%	14.20%	207,164	112,623	4.60%
		+425.33%	-1.23pp	+0.02pp	+0.62pp	+0.02pp	-6.82%	-2.33%	+0.01pp
(5)	$\tau_E = 0.7104$	46,132,622,805	14.51%	14.18%	14.01%	14.18%	207,215	112,632	4.59%
		+432.64%	-1.25pp	+0pp	+0.60pp	+0pp	-6.80%	-2.32%	+0pp

		$\frac{E}{Y}(\infty)$	$\frac{T}{Y}(\infty)$	$\frac{E}{K}(\infty)$	$\frac{EP_E}{Y}(\infty)$	$\frac{T}{K}(\infty)$	$\frac{Y}{K}(\infty)$	$\frac{T}{GDP}(\infty)$
	$\tau_Z = 0.1$	1.0866	5.55%	0.84529	7.89%	0.043164	0.77796	6.02%
	$\tau_Z = \tau_E = 0$	1.092	0%	0.90109	7.98%	0	0.82518	0%
		+0.50%	-5.55pp	+6.60%	+0.039pp	-100%	+6.07%	-6.02pp
(1)	$\tau_E = 0.13337$	1.0918	1.06%	0.89887	7.93%	0.0087038	0.82333	1.15%
		+0.48%	-4.49pp	+6.34%	+0.038pp	-79.84%	+5.83%	-4.88pp
(2)	$\tau_E = 0.66802$	1.0908	5.29%	0.89001	7.92%	0.043164	0.81592	5.75%
		+0.39%	-0.26pp	+5.29%	+0.03pp	+0%	+4.88%	-0.28pp
(3)	$\tau_E = 3.4467$	1.0866	27.19%	0.84529	7.89%	0.21152	0.77796	29.52%
		+0%	+21.64pp	+0%	+0pp	+390.04%	+0%	+23.49pp
(4)	$\tau_E = 0.70065$	1.0908	5.55%	0.88948	7.92%	0.045245	0.81547	6.03%
		+0.39%	+0pp	+5.23%	+0.03pp	+4.82%	+4.82%	+0.002pp
(5)	$\tau_E = 0.7104$	1.0907	5.63%	0.88932	7.92%	0.045866	0.81533	6.11%
		+0.39%	+0.08pp	+5.21%	+0.03pp	+6.26%	+4.80%	+0.09pp

Table 4: Resulting equivalent taxes and affected variables. Each tax regime is compared to $\tau_Z = 0.1$, with percentage or percentage point deviations given.

3.6 Comparative Statics with Electricity Efficient Robots

Zhang et al. (2022) and Wang et al. (2022) show that the introduction of robots can lower the energy intensity of an industry. If this holds also for electricity in particular, this would mean that $\xi_Z < \xi_K$, i.e. that traditional capital uses more electricity than automation capital. In this section, we set $s_{\xi_Z} = 0.6$, meaning that $Z(t)$ uses 40% less power than $K(t)$. Note that this has a minor effect on the calibration, as we can observe in Table 5.

Table 5: New derived parameter values

Parameter	Value	Parameter	Value
A	2860.5	β	2.4025×10^{-6}
K_0	3167 billion USD	Z_0	388 billion USD
ξ_K	0.2631	ξ_Z	0.1579

Again, we try to find electricity tax rates that are in some way equivalent to a robot tax of $\tau_Z = 0.1$. We consider the same approaches as before, with the exception of (3), which tries to equalize the BGP traditional capital intensity k^*. This is not possible anymore, since a robot tax will increase capital intensity, while an electricity tax will now decrease it. The results of this exercise are summarized in Table 6.

As we can see, the new calibration leads to significantly different long term power consumption. Since $Z(t)$ is now more efficient, it now takes only 0.70827 kWh to produce one good worth 1 USD and spending on electricity comprises only 5.14% of output. The introduction of the robot tax has much the same effect as before.

Imposing an electricity tax now has the effect that $r_K(t)$ falls more than $r_Z(t)$, since the inefficient traditional capital is more affected by the tax. Therefore investment goes towards automation capital, leading to a fall in traditional capital intensity $k(t)$. As $Z(t)$ becomes more abundant, $r_Z(t)$ falls and $r_K(t)$ rises, until the BGP with $k(t) = k^*$ and $r_Z(t) = r_K(t) = r^*$ is reached. The relative rarity of $K(t)$ also leads to a drop in wages, because labor's marginal product is lowered. The growth rate g is again diminished due to the tax's negative impact on capital accumulation. In the long term, electricity consumption per unit of output is decreased slightly, since the economy is forced to use relatively more efficient traditional capital.

(1) Equalize short term tax revenue $T(0)$

Similarly to the previous section, a 13.99% electricity tax is required to achieve the robot tax's short term tax revenue. Again, the effects of this tax are relatively minor and do not have nearly the same impact in the long run, since the tax base does not grow as fast. The long term tax base comprises 5.14% of output, while the robot tax base makes up 56.32%.

(2) Equalize long term tax revenue per unit of traditional capital $\frac{T}{K}(\infty)$

(4) Equalize long term tax revenue per unit of output $\frac{T}{Y}(\infty)$

(5) Equalize the BGP growth rate of consumption g

These approaches again result in similar tax rates of roughly 113%. The reason why this is significantly higher than the previous section's 70% is that, due to the lower power consumption of automation capital, the long term tax base is now only 5.12% of output. In the previous section it was 7.92%, significantly higher. Compared to the robot tax, these electricity taxes result in a similarly decreased growth rate g and wages that are roughly 3.95% lower. In the short term, tax revenues are much higher compared to the robot tax, while in the long run they are roughly the same in terms of percent of output and GDP.

		$T(0)$	$r_K(0)$	r_K^*	$r_Z(0)$	r_Z^*	k^*	w^*	g
	$\tau_Z = 0.1$	9,082,738,850	15.75%	14.15%	13.41%	14.15%	221,297	115,346	4.58%
	$\tau_Z = \tau_E = 0$	0	15.75%	15.75%	15.75%	15.75%	202,364	111,958	5.38%
		-100%	+0pp	+1.60pp	+2.34pp	+1.60pp	-8.56%	-2.94%	+0.80pp
(1)	$\tau_E = 0.1399$	9,082,738,850	15.49%	15.56%	15.59%	15.56%	201,577	111,813	5.28%
		+0%	-0.27pp	+1.41pp	+2.18pp	+1.41pp	-8.91%	-3.06%	+0.70pp
(2)	$\tau_E = 1.0474$	68,019,188,381	13.75%	14.29%	14.55%	14.29%	196,570	110,879	4.65%
		+648.88%	-2.00pp	+0.14pp	+1.14pp	+0.14pp	-11.17%	-3.87%	+0.07pp
(4)	$\tau_E = 1.1338$	73,628,936,152	13.59%	14.17%	14.45%	14.17%	196,102	110,791	4.59%
		+710.65%	-2.17pp	+0.02pp	+1.04pp	+0.02pp	-11.39%	-3.95%	+0.01pp
(5)	$\tau_E = 1.1492$	74,625,651,290	13.56%	14.15%	14.44%	14.15%	196,019	110,775	4.58%
		+721.62%	-2.20pp	+0pp	+1.02pp	+0pp	-11.42%	-3.96%	+0pp

		$\frac{E}{Y}(\infty)$	$\frac{T}{Y}(\infty)$	$\frac{E}{K}(\infty)$	$\frac{EP_E}{Y}(\infty)$	$\frac{T}{K}(\infty)$	$\frac{Y}{K}(\infty)$	$\frac{T}{GDP}(\infty)$
	$\tau_Z = 0.1$	0.71626	5.81%	0.56	5.20%	0.045425	0.78184	6.13%
	$\tau_Z = \tau_E = 0$	0.70827	0%	0.58778	5.14%	0	0.82987	0%
		-1.12%	-5.81pp	+4.96%	-0.06pp	-100%	+6.14%	-6.13pp
(1)	$\tau_E = 0.1399$	0.70796	0.72%	0.58905	5.14%	0.006%	0.83203	0.76%
		-1.16%	-5.09pp	+5.19%	-0.06pp	-86.83%	+6.42%	-5.37pp
(2)	$\tau_E = 1.0474$	0.706	5.37%	0.59735	5.13%	0.05%	0.84611	5.66%
		-1.43%	-0.44pp	+6.67%	-0.07pp	+0%	+8.22%	-0.47pp
(4)	$\tau_E = 1.1338$	0.70582	5.81%	0.59815	5.12%	0.05%	0.84745	6.12%
		-1.46%	+0pp	+6.81%	-0.08pp	+8.39%	+8.39%	-0.005pp
(5)	$\tau_E = 1.1492$	0.70579	5.89%	0.59829	5.12%	0.05%	0.84769	6.21%
		-1.46%	+0.08pp	+6.84%	-0.08pp	+9.89%	+8.42%	+0.08pp

Table 6: Resulting equivalent taxes and affected variables for the case where automation capital uses less electricity. Each tax regime is compared to $\tau_Z = 0.1$, with percentage or percentage point deviations given.

4 Conclusion

This thesis has explored the effects of power consumption, robot and electricity taxes on economic growth. Steigum's (2011) model framework was expanded such that firms additionally have to pay taxes and electricity costs when employing traditional and automation capital. Then it was shown that a unique market equilibrium exists where interest rates are equalized and households invest in both capital stocks.

Assuming reasonable parameter values, endogenous growth is achieved along this balanced growth path (BGP). If the economy is thrown off the BGP - for instance due to an external shock - households adapt by investing only in the more attractive capital stock, until the traditional capital intensity $k(t)$ reaches its new optimal level, meaning that the economy automatically returns to balanced growth.

With or without transfers, automation capital grows fastest, at a non-constant rate, followed by traditional capital and output, while consumption increases slowest, at a constant pace. All rates converge in the long term. We compared these results with Steigum (2011), finding them compatible, and corrected an error in one of his formulas. Furthermore, we showed that tax revenue and spending on electricity both fall between traditional and automation capital when it comes to growth.

The introduction of government transfers leads to an increase in consumption at every point in time, while the growth rate of $C(t)$ remains unaffected. Furthermore, the inclusion of transfers slows down the growth of automation and traditional capital. We found that robot and electricity taxes decrease interest rates and growth, suggesting that subsidies could have an opposite, pro-growth effect. Robot taxes always increase wages, while an electricity tax only does so when automation capital is relatively power hungry.

The model was then calibrated with data from the U.S. manufacturing sector. This baseline calibration predicted rapid growth for automation capital and a collapsing labor income share over the next 30 years. Next, the consequences of the currently rising electricity prices was discussed. We observed slightly lower growth, modestly higher wages and, over the long term, a larger portion of output needed to pay for electric power.

Finally, we calculated what electricity tax rates could replicate the short or long term effects of a 10% robot tax. It was found that a 13.33% electricity tax can raise the same revenue in the short term, while recreating the long term effects of the robot tax requires exorbitant rates. This is partially due to the fact that the robot tax base, i.e. automation capital, grows faster than spending on electricity. We also observed that the robot tax tends to stymie growth more than an electricity tax with the same rate. Finally, we repeated this exercise with the assumption that robots are less power hungry than traditional capital. This led to similar results, albeit with a lower proportion of output having to be spent on electricity in the long term. Overall, the numerical section implies that it is not straightforward to substitute one type of tax with the other, since the long term consequences can differ severely.

One of the leading economists of our time, Daron Acemoglu, has compared the state of robot taxation to climate change research 30 or 40 years ago, adding: "We're very much asleep at the wheel in terms of worrying, measuring, understanding this issue." (Wall Street Journal (2020)). In line with this sentiment, there are several ways in which the model framework discussed in this thesis could be expanded to address further research questions. It would be of great interest to include the environment and emissions in the model, perhaps opening up the possibility of welfare enhancing taxes and an end to unlimited growth. To explore any kind of welfare effects, more attention needs to be directed to the economy outside of the BGP and the calculation of the path towards equilibrium. Another important research question concerns obtaining better estimates for ξ_K and ξ_Z, determining which capital stock is actually more power hungry. Furthermore, population growth could be re-introduced to the model, allowing for an expanding labor force. Finally, modeling the electricity sector in a more sophisticated way might also be a useful model extension.

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