COMPOSITIONS AND DECOMPOSITIONS OF BINARY RELATIONS

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Abstract. It is well known that to every binary relation on a non-void set I there can be assigned its incidence matrix, also in the case when I is infinite. We show that a certain kind of "multiplication" of such incidence matrices corresponds to the composition of the corresponding relations. Using this fact we investigate the solvability of the equation $R \circ X = S$ for given binary relations R and S on I and derive an algorithm for solving this equation by using the connections between the corresponding incidence matrices. Moreover, we describe how one can obtain the incidence matrix of a product of binary relations from the incidence matrices of its factors.

A systematic study of binary relations is a rather old task initiated in papers by J. Riguet ([7]) and R. Fraïssé, see, e.g. [4] and [5]. An algebraic approach to binary relations was introduced and developed by B. Jónsson ([6]). An approach via assigned groupoids was started by the authors in the relatively recent papers [1] and [2] and, together with P. Ševčík, in [3].

The aim of this paper is to show how the incidence matrices of given binary relations are useful for constructing relational products and decomposing a given relation into a relational product of two relations where one factor is given. As a byproduct we describe the incidence matrix of the Cartesian product of a set of given binary relations.

In the following let I be a non-empty set. Then, the Kronecker delta δ_{ij} on I is defined by

$$\delta_{ij} := \left\{ \begin{array}{ll} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{array} \right.$$

for all $i, j \in I$. Let L be a further set. By an $I \times I$-matrix $M = [a_{ij}]$ over L we mean a mapping $(i, j) \mapsto a_{ij}$ from $I \times I$ to L. If I is finite, we assume $I = \{1, \ldots, n\}$ and call the matrix an $n \times n$-matrix over L. Let $L^{I \times I}$, or $L^{n \times n}$, denote the set of all $I \times I$-matrices, or $n \times n$-matrices, respectively, over L.

To every binary relation $R \subseteq I \times I$ we assign its *incidence matrix* $M_R = [a_{ij}] \in \{0, 1\}^{I \times I}$ as follows:

$$a_{ij} := \left\{ \begin{array}{ll} 1 & \text{if } (i, j) \in R, \\ 0 & \text{otherwise.} \end{array} \right.$$

For $I \times I$-matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ over $\{0,1\}$ let $A \odot B$ denote the $I \times I$-matrix $C = [c_{ij}]$ over $\{0,1\}$ defined by

$$c_{ij} := \max_{k \in I} a_{ik} b_{kj}$$

for all $i, j \in I$, i.e. $A \odot B$ is analogously defined as the usual matrix product $A \cdot B$, only the addition operation is replaced by the maximum operation (which also works for infinite I). It means that we "multiply" the i-th row of A with the j-th column of B using this kind of "addition".

First, we will study the composition of binary relations and their incidence matrices. It was already mentioned in [7] that a certain composition of incidence matrices corresponds to the product of the corresponding relations. However, an explicit form of such a composition was not presented. We can state and prove the following elementary result.

Proposition 1. *Let $R, S \subseteq I \times I$. Then, $M_{R \circ S} = M_R \odot M_S$.*

Proof. Put $M_R = [a_{ij}]$, $M_S = [b_{ij}]$, $M_{R \circ S} = [c_{ij}]$ and $M_R \odot M_s = [d_{ij}]$ and let $k, l \in I$. Then, the following are equivalent:

$$c_{kl} = 1,$$
$$(k, l) \in R \circ S,$$
$$\text{there exists some } m \in I \text{ with } (k, m) \in R \text{ and } (m, l) \in S,$$
$$\text{there exists some } m \in I \text{ with } a_{km} = b_{ml} = 1,$$
$$\max_{m \in I} a_{km} b_{ml} = 1,$$
$$d_{kl} = 1.$$

This shows $M_{R \circ S} = M_R \odot M_S$. $\qquad\qquad\square$

For $I \times I$-matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ over $\{0,1\}$ let $A \oplus B$ denote the $I \times I$-matrix $C = [c_{ij}]$ over $\{0,1\}$ defined by

$$c_{ij} := \max(a_{ij}, b_{ij})$$

for all $i, j \in I$. Moreover, let M_0 and M_1 denote the $I \times I$-matrices $[0]$ and $[\delta_{ij}]$ over $\{0,1\}$ and put $\Delta := \{(x, x) \mid x \in I\}$.

With the knowledge of how to compose incidence matrices at hand, we can describe an algebraic structure on the set of all incidence matrices of a given dimension. Let us note that the structure of the set of binary relations on a given set with respect to relational operations (product, union, complementation etc.) was originally described by B. Jónsson, see, e.g. [6] and the references therein.

Recall that a *unitary semiring* is an algebra $(S, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the following conditions:

- $(S, +, 0)$ is a commutative monoid,
- $(S, \cdot, 1)$ is a monoid,
- $(x + y)z \approx xz + yz$ and $z(x + y) \approx zx + zy$,
- $x0 \approx 0x \approx 0$.

In the following let $2^{I \times I}$ denote the power set of $I \times I$.

Theorem 2. *Let I be a set. Then,*

(i) $(2^{I\times I}, \cup, \circ, \emptyset, \Delta)$ *is a unitary semiring,*

(ii) *the mapping* $R \mapsto M_R$ *from* $2^{I\times I}$ *to* $\{0,1\}^{I\times I}$ *is an isomorphism from* $(2^{I\times I}, \cup, \circ, \emptyset, \Delta)$ *to* $(\{0,1\}^{I\times I}, \oplus, \odot, M_0, M_1)$ *and, hence, the latter algebra is a unitary semiring, too.*

Proof. Let $R, S \subseteq I \times I$, $M_R = [a_{ij}]$, $M_S = [b_{ij}]$, $M_{R\cup S} = [c_{ij}]$, $M_R \oplus M_S = [d_{ij}]$ and $k, l \in I$.

(i) Obviously, $(2^{I\times I}, \cup, \emptyset)$ is a commutative monoid, $(2^{I\times I}, \circ, \Delta)$ a monoid and $R \circ \emptyset = \emptyset \circ R = \emptyset$. The distributivity laws can be easily verified.

(ii) The following are equivalent:

$$c_{kl} = 1,$$
$$(k,l) \in R \cup S,$$
$$(k,l) \in R \text{ or } (k,l) \in S,$$
$$a_{kl} = 1 \text{ or } b_{kl} = 1,$$
$$\max(a_{kl}, b_{kl}) = 1,$$
$$d_{kl} = 1.$$

This proves $M_{R\cup S} = M_R \oplus M_S$. From Proposition 1 we know that $M_{R\circ S} = M_R \odot M_S$. Since, finally, $M_\emptyset = M_0$ and $M_\Delta = M_1$, the mapping $R \mapsto M_R$ from $2^{I\times I}$ to $\{0,1\}^{I\times I}$ is a homomorphism from $(2^{I\times I}, \cup, \circ, \emptyset, \Delta)$ to $(\{0,1\}^{I\times I}, \oplus, \odot, M_0, M_1)$. Since this mapping is bijective, it is an isomorphism. The last assertion is clear.

$\square$

Remark 3. From Theorem 2 we conclude that $\odot$ is associative and distributive with respect to $\oplus$.

Now, we turn our attention to the decomposition of binary relations. Consider two binary relations R and S on a given set A. We ask if there exists a binary relation X on A satisfying the equation

$$R \circ X = S,$$

i.e. we ask if S can be decomposed into the given relation R and a certain (unknown) relation X. First, we present an example showing that such a relation X can be found by using the composition of incidence matrices presented above. It is a method similar to solving sets of linear equations over the two-element field but instead of the binary addition we use the binary operation max as explained before.

Example 4. Put

$$n := 3,$$
$$R := \{(1,2), (2,1), (3,2), (3,3)\},$$
$$S := \{(1,1), (1,2), (3,1), (3,2)\}.$$

We consider the equation $R \circ X = S$. This is equivalent to $M_R \odot M_X = M_S$, where

$$M_R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \text{ and } M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

We obtain immediately

$$(x_{21}, x_{22}, x_{23}, x_{11}, x_{12}, x_{13}) = (1, 1, 0, 0, 0, 0)$$

and, using our computation,

$$\max(x_{21}, x_{31}) = 1,$$
$$\max(x_{22}, x_{32}) = 1,$$
$$\max(x_{23}, x_{33}) = 0.$$

The last three equations are equivalent to

$$\max(1, x_{31}) = 1,$$
$$\max(1, x_{32}) = 1,$$
$$\max(0, x_{33}) = 0$$

and, hence, to $x_{33} = 0$. This shows that the equation $R \circ X = S$ has exactly four solutions, namely

$$X = \{(2, 1), (2, 2)\},$$
$$X = \{(2, 1), (2, 2), (3, 1)\},$$
$$X = \{(2, 1), (2, 2), (3, 2)\},$$
$$X = \{(2, 1), (2, 2), (3, 1), (3, 2)\}.$$

Next, we show how the incidence matrix of the Cartesian product of binary relations over different base sets can be derived from the incidence matrices of the corresponding factors. For this we introduce the following kind of product of relations over different base sets.

Let $(I_k)_{k \in K}$ be a family of sets, put $I := \prod_{k \in K} I_k$ and, for all $k \in K$, let $R_k \subseteq I_k \times I_k$. Define

$$\prod_{k \in K} R_k := \{((i_k)_{k \in K}, (j_k)_{k \in K}) \in I \times I \mid (i_k, j_k) \in R_k \text{ for all } k \in K\}.$$

Theorem 5. *Let $(I_k)_{k \in K}$ be a family of sets and, for all $k \in K$, let $R_k \subseteq I_k \times I_k$ and $M_{R_k} = [a_{i_k j_k}]$. Put $I := \prod_{k \in K} I_k$ and $R := \prod_{k \in K} R_k$. Then, $R \subseteq I \times I$. Let $M_R = [a_{ij}]$. Then,*

$$a_{(i_k)_{k \in K} (j_k)_{k \in K}} = \min_{k \in K} a_{i_k j_k}$$

for all $(i_k)_{k \in K}, (j_k)_{k \in K} \in I$.

Proof. Let $(l_k)_{k \in K}, (m_k)_{k \in K} \in I$. Then, the following are equivalent:

$$a_{(l_k)_{k \in K} (m_k)_{k \in K}} = 1,$$
$$((l_k)_{k \in K}, (m_k)_{k \in K}) \in R,$$
$$(l_k, m_k) \in R_k \text{ for all } k \in K,$$
$$a_{l_k m_k} = 1 \text{ for all } k \in K,$$
$$\min_{k \in K} a_{l_k m_k} = 1.$$

This shows

$$a_{(l_k)_{k \in K} (m_k)_{k \in K}} = \min_{k \in K} a_{l_k m_k}.$$

$\square$

Example 6. If

$$I := \{1, 2\},$$
$$J := \{1, 2, 3\},$$
$$R := \{(1, 1), (2, 1)\} \subseteq I \times I,$$
$$S := \{(3, 2), (3, 3)\} \subseteq J \times J,$$

then

$$M_R = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$M_S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$K := I \times J = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\},$$

$$T := R \times S = \{\big((1, 3), (1, 2)\big), \big((1, 3), (1, 3)\big), \big((2, 3), (1, 2)\big), \big((2, 3), (1, 3)\big)\} \subseteq$$
$$\subseteq K \times K,$$

$$M_T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

In the next theorem we present sufficient but not necessary conditions for solving the equation $R \circ X = S$. That these conditions are not necessary is evident from the fact that the mapping f mentioned in Theorem 7 does not exist in Example 4 although the equation $R \circ X = S$ is solvable.

Theorem 7. *Let* $R, S \subseteq I \times I$, $f \colon I \to I$ *and assume* $M_R = [\delta_{j,f(i)}]$ *and* $M_S = [b_{ij}]$.

   (i) *The equation* $R \circ X = S$ *has a solution if and only if,*

$$\textit{for all } j, k, l \in I, \textit{ we have } b_{kj} = b_{lj} \textit{ whenever } f(k) = f(l).$$

In this case, X *with* $M_X = [x_{ij}]$ *is a solution if and only if,*

$$\textit{for all } i, j \in I, \textit{ we have } x_{f(i),j} = b_{ij}.$$

   (ii) *If* f *is bijective, then the equation* $R \circ X = S$ *has exactly one solution, namely* X *with* $M_X = [b_{f^{-1}(i),j}]$.

Proof. Let $X \subseteq I \times I$ and $M_X = [x_{ij}]$.

(i) Then, the following are equivalent:

$$R \circ X = S,$$
$$M_R \odot M_X = M_S,$$
$$\max_{k \in I} \delta_{k,f(i)} x_{kj} = b_{ij} \text{ for all } i, j \in I,$$
$$x_{f(i),j} = b_{ij} \text{ for all } i, j \in I.$$

(ii) If f is bijective, then the following are equivalent:

$$x_{f(i),j} = b_{ij} \text{ for all } i, j \in I,$$
$$x_{ij} = b_{f^{-1}(i),j} \text{ for all } i, j \in I.$$

$\square$

How the mapping from Theorem 7 works is illustrated in the following example.

Example 8. Put

$$I := \{1, 2, 3\},$$
$$R := \{(1, 2), (2, 3), (3, 1)\},$$
$$S := \{(1, 1), (1, 2), (2, 3), (3, 3)\}.$$

Then, $M_R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. There is only one possibility for the mapping f, namely

$$\bigl(f(1), f(2), f(3)\bigr) = (2, 3, 1),$$
$$\bigl(f^{-1}(1), f^{-1}(2), f^{-1}(3)\bigr) = (3, 1, 2).$$

Since f is a bijection, the equation $R \circ X = S$ has the unique solution X with

$M_X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, i.e. $X = \{(1, 3), (2, 1), (2, 2), (3, 3)\}$.

We are going to show several cases in which the equation $R \circ X = S$ is not solvable.

Lemma 9. *Let $R, S \subseteq I \times I$, $M_R = [a_{ij}]$, $M_S = [b_{ij}]$ and $k, l, m, n \in I$ and assume that one of the following conditions holds:*

(i) $a_{kj} = 0$ *for all $j \in I$ and there exists some $p \in I$ with $b_{kp} \neq 0$,*
(ii) $a_{kj} = a_{lj}$ *for all $j \in I$ and there exists some $p \in I$ with $b_{kp} \neq b_{lp}$,*
(iii) $a_{kj} = \delta_{jl}$ *for all $j \in I$, $a_{nl} = b_{km} = 1$ and $b_{nm} = 0$.*

Then, the equation $R \circ X = S$ has no solution.

Proof. Assume X to be a solution with $M_X = [x_{ij}]$. Then, $M_R \odot M_X = M_S$. Now, we have

(i) $b_{kp} = \max\limits_{r \in I} a_{kr} x_{rp} = 0$, a contradiction.
(ii) $b_{kp} = \max\limits_{r \in I} a_{kr} x_{rp} = \max\limits_{r \in I} a_{lr} x_{rp} = b_{lp}$, a contradiction.

(iii) $x_{lm} = \max\limits_{j \in I} \delta_{jl} x_{jm} = \max\limits_{j \in I} a_{kj} x_{jm} = b_{km} = 1$, which implies

$$1 = a_{nl} x_{lm} \leq \max\limits_{j \in I} a_{nj} x_{jm} = b_{nm} = 0,$$

a contradiction.

$\square$

In the following proposition we present a case where the equation $R \circ X = S$ can be easily solved.

Proposition 10. *Let $R, S \subseteq I \times I$ with $R \subseteq S$ and assume R to be reflexive and S to be transitive. Then, the equation $R \circ X = S$ has a solution, namely $X = \Delta \cup (S \setminus R)$.*

Proof. Let $(a, b) \in R \circ \big(\Delta \cup (S \setminus R)\big)$. Then, there exists some $c \in I$ with $(a, c) \in R$ and $(c, b) \in \Delta \cup (S \setminus R)$. If $(c, b) \in \Delta$, then $(a, b) = (a, c) \in R \subseteq S$. If $(c, b) \in S \setminus R$, then $(a, c) \in R \subseteq S$ and $(c, b) \in S$ and, hence, $(a, b) \in S$ according to the transitivity of S. This shows $R \circ \big(\Delta \cup (S \setminus R)\big) \subseteq S$. Conversely, assume $(a, b) \in S$. If $(a, b) \in R$, then $(a, b) \in R$ and $(b, b) \in \Delta \cup (S \setminus R)$ and, hence, $(a, b) \in R \circ \big(\Delta \cup (S \setminus R)\big)$. If $(a, b) \notin R$, then $(a, a) \in R$ according to the reflexivity of R and $(a, b) \in \Delta \cup (S \setminus R)$ and, hence, $(a, b) \in R \circ \big(\Delta \cup (S \setminus R)\big)$. This shows $S \subseteq R \circ \big(\Delta \cup (S \setminus R)\big)$ completing the proof of the proposition. $\square$

Let $[a_{ij}]$ be an $I \times I$-matrix. Put

$$\vec{a}_k := (a_{ik})_{i \in I} \text{ for all } k \in I,$$
$$\max\limits_{k \in J} \vec{a}_k := (\max\limits_{k \in J} a_{ik})_{i \in I} \text{ for all } J \subseteq I.$$

(We use the convention $\max\limits_{k \in \emptyset} \vec{a}_k := (0)_{i \in I}$.)

We can formulate and prove a general result on solving the equation $R \circ X = S$ as follows.

Theorem 11. *Let $R, S \subseteq I \times I$ and put $M_R = [a_{ij}]$, $M_S = [b_{ij}]$ and $A_i := \{j \in I \mid a_{ij} = 1\}$. Then, the following are equivalent:*

(i) *The equation $R \circ X = S$ has a solution,*

(ii) *For every $k \in I$ there exists some subset X_k of I such that $\max\limits_{l \in X_k} \vec{a}_l = \vec{b}_k$ for all $k \in I$. In this case, $X = \{(i, j) \mid j \in I, i \in X_j\}$. All the solutions can be obtained in this way.*

(iii) *For every $k \in I$ there exists some subset X_k of I such that, for all $i, k \in I$, we have $A_i \cap X_k = \emptyset$ if and only if $b_{ik} = 0$.*

Proof. Let $X \subseteq I \times I$ and put $M_X = [x_{ij}]$.

(i) $\Leftrightarrow$ (ii): Put $X_k := \{j \in I \mid x_{jk} = 1\}$ for all $k \in I$. Then, the following are equivalent:

$$R \circ X = S,$$
$$M_R \odot M_X = M_S,$$
$$\max\limits_{j \in I} a_{ij} x_{jk} = b_{ik} \text{ for all } i, k \in I,$$

$$\max_{j \in X_k} a_{ij} = b_{ik} \text{ for all } i, k \in I,$$

$$\max_{j \in X_k} \vec{a}_j = \vec{b}_k \text{ for all } k \in I.$$

(i) $\Rightarrow$ (iii): Let X be a solution to the equation $R \circ X = S$ and put $X_k := \{j \in I \mid x_{jk} = 1\}$ for all $k \in I$. Then, for all $i, k \in I$, the following are equivalent:

$$A_i \cap X_k \neq \emptyset,$$

there exists some $j \in A_i \cap X_k$,

there exists some $j \in I$ satisfying $a_{ij} = x_{jk} = 1$,

$$\max_{j \in I} a_{ij} x_{jk} = 1,$$

$$b_{ik} = 1.$$

(iii) $\Rightarrow$ (i): Put

$$x_{ij} := \begin{cases} 1 & \text{if } i \in X_j \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in I$. Then, for all $i, k \in I$, we have

$$\max_{j \in I} a_{ij} x_{jk} = \max_{j \in A_i \cap X_k} 1 = b_{ik}.$$

(We have that $\max_{j \in I} a_{ij} x_{jk} = 1$ is equivalent to $A_i \cap X_k \neq \emptyset$, and $\max_{j \in \emptyset} 1$ is interpreted as 0.) This shows $M_R \odot M_X = M_S$, i.e. $R \circ X = S$. $\qquad \square$

Now, we will investigate when the incidence matrix A of a binary relation is "invertible", which means that there exists an incidence matrix B satisfying $A \odot B = B \odot A = E$, where $E := [\delta_{ij}]$. For B we will also use the notation A^{-1}. (Note that because of the associativity of $\odot$ the inverse, if it exists, is unique.)

Proposition 12. *Let n be a positive integer, put $I := \{1, \ldots, n\}$, let $A = [a_{ij}] \in \{0,1\}^{n \times n}$ and put $E := [\delta_{ij}] \in \{0,1\}^{n \times n}$. Then, the following are equivalent:*

(i) *There exists some $B \in \{0,1\}^{n \times n}$ with $A \odot B = B \odot A = E$.*
(ii) *There exists some bijection $f \colon I \to I$ satisfying $a_{ij} = \delta_{j,f(i)}$ for all $i, j \in I$.*

Proof. (i) $\Rightarrow$ (ii): For $j \in I$ let $\vec{a}_j$ and $\vec{e}_j$ denote the j-th column vector of A and E, respectively. Moreover, let $k \in I$. Since B is a solution to the equation $A \odot X = E$, according to Theorem 11 there exists some subset I_k of I such that $\max_{l \in I_k} \vec{a}_l = \vec{e}_k$. Hence, there exists some $f(k) \in I_k$ with $\vec{a}_{f(k)} = \vec{e}_k$. Clearly, $f \colon I \to I$ is injective and thus bijective and we obtain $\vec{a}_j = \vec{e}_{f^{-1}(j)}$ for all $j \in I$, i.e. $a_{ij} = \delta_{i,f^{-1}(j)} = \delta_{j,f(i)}$ for all $i, j \in I$.

(ii) $\Rightarrow$ (i): If $B = [b_{ij}] := [\delta_{j,f^{-1}(i)}]$, then

$$\max_{k \in I} a_{ik} b_{kj} = \max_{k \in I} \delta_{k,f(i)} \delta_{j,f^{-1}(k)} = \delta_{f(i),f(j)} = \delta_{ij},$$

$$\max_{k \in I} b_{ik} a_{kj} = \max_{k \in I} \delta_{k,f^{-1}(i)} \delta_{j,f(k)} = \delta_{f^{-1}(i),f^{-1}(j)} = \delta_{ij}$$

showing $A \odot B = B \odot A = E$. $\qquad \square$

Note that condition (ii) means that every row and every column of A contains exactly one 1 and that the implication (ii) $\Rightarrow$ (i) also holds for infinite I.

Remark 13. Let R and S be binary relations on a set I such that for the incidence matrix M_R of R there exists a bijection $f \colon I \to I$ as described in Theorem 7 and let $M_S = [b_{ij}]$. Put $E := [\delta_{ij}] \in \{0,1\}^{I \times I}$. It is easy to check that $A \odot E = E \odot A = A$ for every $A \in \{0,1\}^{I \times I}$. From Proposition 12 we obtain $M_R^{-1} = [\delta_{j, f^{-1}(i)}]$. Now, the following are equivalent:

$$R \circ X = S,$$
$$M_R \odot M_X = M_S,$$
$$M_X = M_R^{-1} \odot M_S$$

and, hence, $x_{ij} = \max_{k \in I} \delta_{k, f^{-1}(i)} b_{kj} = b_{f^{-1}(i), j}$ for all $i, j \in I$. Note that here we used the associativity of $\odot$.

The next theorem characterizes the solvability of the equation $R \circ X = S$ and also characterizes the corresponding solutions. From this theorem we will derive an algorithm for computing all the solutions.

Theorem 14. *Let $R, S \subseteq I \times I$, $M_R = [a_{ij}]$ and $M_S = [b_{ij}]$ and put*

$$A_i := \{j \in I \mid a_{ij} = 1\},$$
$$B_k := \{i \in I \mid b_{ik} = 0\},$$
$$C_k := \bigcup_{l \in B_k} A_l$$

for all $i, k \in I$.

   (i) *$X \subseteq I \times I$ with $M_X = [x_{ij}]$ is a solution to the equation $R \circ X = S$ if and only if the following hold:*
      (a) *$x_{jk} = 0$ for all $k \in I$ and all $j \in C_k$,*
      (b) *for every $k \in I$ and every $i \in I \setminus B_k$ there exists some $j \in A_i$ with $x_{jk} = 1$.*
  (ii) *The equation $R \circ X = S$ is solvable if and only if $A_i \neq \emptyset$ for all $k \in I$ and all $i \in I \setminus B_k$.*

Proof.
   (i) Let $X \subseteq I \times I$ and $M_X = [x_{ij}]$. Then, the following are equivalent:

$$R \circ X = S,$$
$$\max_{j \in I} a_{ij} x_{jk} = b_{ik} \text{ for all } i, k \in I,$$
$$\max_{j \in A_i} x_{jk} = b_{ik} \text{ for all } i, k \in I.$$

The last assertion is equivalent to the following one:

$$\max_{j \in A_i} x_{jk} = b_{ik} \text{ for all } k \in I \text{ and all } i \in B_k, \text{ and}$$
$$\max_{j \in A_i} x_{jk} = b_{ik} \text{ for all } k \in I \text{ and all } i \in I \setminus B_k.$$

Now, the following are equivalent:

$$\max_{j \in A_i} x_{jk} = b_{ik} \text{ for all } k \in I \text{ and all } i \in B_k,$$

$$\max_{j \in A_i} x_{jk} = 0 \text{ for all } k \in I \text{ and all } i \in B_k,$$

$$x_{jk} = 0 \text{ for all } k \in I, \text{ all } i \in B_k \text{ and all } j \in A_i,$$

$$x_{jk} = 0 \text{ for all } k \in I \text{ and all } j \in \bigcup_{i \in B_k} A_i,$$

$$x_{jk} = 0 \text{ for all } k \in I \text{ and all } j \in C_k.$$

Finally, the following are equivalent:

$$\max_{j \in A_i} x_{jk} = b_{ik} \text{ for all } k \in I \text{ and all } i \in I \setminus B_k,$$

$$\max_{j \in A_i} x_{jk} = 1 \text{ for all } k \in I \text{ and all } i \in I \setminus B_k.$$

The last assertion is equivalent to the following one:

for all $k \in I$ and all $i \in I \setminus B_k$ there exists some $j \in A_i$ with $x_{jk} = 1$.

(ii) follows from (i).

$\square$

As mentioned above, we now derive an algorithm for computing all the solutions to the equation $R \circ X = S$ provided this equation is solvable and I is finite. This algorithm consists of the following three steps (let A_i, B_k and C_k be defined as in Theorem 14):

(1) Compute A_i, B_i, C_i for all $i \in I$.
(2) Put $x_{jk} := 0$ for all $k \in I$ and all $j \in C_k$.
(3) For all $k \in I$ and $i \in I \setminus B_k$ choose some $j \in A_i$ and put $x_{jk} := 1$.
(4) Choose the remaining $x_{jk} \in \{0, 1\}$ arbitrarily.

This algorithm was already implicitly used in Example 4, see Example 15. In fact, it is similar to the method for solving linear equations. In steps (2) and (3) the algorithm reduces the possibilities for choosing the elements of M_X, whereas steps (3) and (4) determine the number of solutions.

The aforementioned algorithm will be demonstrated by the following example.

Example 15. Let us apply the algorithm to Example 4. Hence, we have

$$M_R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

We compute

$$A_1 = \{2\}, A_2 = \{1\}, A_3 = \{2, 3\},$$
$$B_1 = \{2\}, B_2 = \{2\}, B_3 = \{1, 2, 3\},$$
$$C_1 = \{1\}, C_2 = \{1\}, C_3 = \{1, 2, 3\},$$
$$I \setminus B_1 = \{1, 3\}, I \setminus B_2 = \{1, 3\}, I \setminus B_3 = \emptyset,$$
$$A_1, A_3 \neq \emptyset.$$

Hence, the equation $R \circ X = S$ is solvable and we obtain

$$x_{11} = x_{12} = x_{13} = x_{23} = x_{33} = 0,$$
$$1 \in \{x_{21}\} \cap \{x_{21}, x_{31}\} \cap \{x_{22}\} \cap \{x_{22}, x_{32}\},$$

i.e. $x_{21} = x_{22} = 1$ and $x_{31}, x_{32} \in \{0, 1\}$. Thus, we got all the four solutions derived in Example 4.

There arises the question what can be said concerning the equation $X \circ R = S$.

Remark 16. Since the equation $X \circ R = S$ is dual to the equation $R \circ X = S$, the investigation of the first equation does not bring new insights into the problem.

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