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ORIGINAL PAPER

Fractional characteristic functions, and a fractional calculus approach for moments of random variables

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Abstract

In this paper we introduce a fractional variant of the characteristic function of a random variable. It exists on the whole real line, and is uniformly continuous. We show that fractional moments can be expressed in terms of Riemann–Liouville integrals and derivatives of the fractional characteristic function. The fractional moments are of interest in particular for distributions whose integer moments do not exist. Some illustrative examples for particular distributions are also presented.

Keywords Fractional calculus (primary) · Characteristic function · Mittag–Leffler function · Fractional moments · Mellin transform

Mathematics Subject Classification 60E10 · 26A33 · 33E12

1 Introduction and preliminaries

In this section we present the definition of our notion of the fractional characteristic function and some of its basic properties. For a real random variable X with probability density (pdf) $p(x)$, the (classical) characteristic function of X is defined by

Dedicated to Prof. V. Kiryakova, on the occasion of her 70th birthday.

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$$\varphi(t) = \mathbb{E}(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} p(x)dx,$$

or via Taylor expansion

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \int_{-\infty}^{\infty} x^k p(x)dx = \sum_{k=0}^{\infty} (\mathbb{E}X^k) \frac{(it)^k}{k!},$$

where $\mathbb{E}X^k = \varphi^{(k)}(0)$ are the integer moments.

Definition 1 We define the fractional characteristic function (fractional CHF) $\varphi_\alpha(t)$, $0 < \alpha < 1, t \in \mathbb{R}$, of the random variable X via the Mittag–Leffler transform of $p(x)$, i.e.

$$\varphi_\alpha(t) = \mathbb{E}(E_\alpha(i(tX)^\alpha)) = \int_{-\infty}^{\infty} E_\alpha(i(tx)^\alpha)p(x)dx.$$

Here $E_\alpha(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{\Gamma(\alpha k + 1)}$ denotes the one-parameter Mittag–Leffler function, for details see Appendix. Note that variants of the Mittag–Leffler transform other than the one used here have been considered in the literature on integral transforms, see e.g., [1]. Since $E_\alpha(0) = 1$, it is clear that $\varphi_\alpha(0) = 1$. For $\alpha = 1$, it follows from $E_1(t) = \exp t$ that $\varphi_1(t) = \varphi(t)$.

Definition 2 The fractional moment generating function (fractional MGF) $M_\alpha(t), 0 < \alpha < 1, t \in \mathbb{R}$, of the random variable X is defined by

$$M_\alpha(t) = \mathbb{E}(E_\alpha((tX)^\alpha)) = \int_{-\infty}^{\infty} E_\alpha((tx)^\alpha)p(x)dx.$$

If X is not clear from the context, we write $\phi_{\alpha,X}$ and $M_{\alpha,X}$. In general, the fractional MGF has complex values. For non-negative random variables, it is real-valued on $\mathbb{R}^+$. If $\frac{1}{2} < \alpha \leq 1$, then the fractional CHF and the fractional MGF are related by

$$M_\alpha(t) = \varphi_\alpha(e^{-i\pi/(2\alpha)}t), \quad t \geq 0.$$

This follows from the equality

$$(te^{-i\pi/(2\alpha)})^\alpha = t^\alpha e^{-i\pi/2} = -it^\alpha, \tag{1.1}$$

which requires $\arg(t) > \pi(\frac{1}{2\alpha} - 1)$. For $X \geq 0$, the estimate

$$|\varphi_\alpha(t)| \leq \mathbb{E}\big[E_\alpha(|(tX)^\alpha|)\big] \leq M_\alpha(|t|), \quad t \in \mathbb{R},$$

follows from the triangle inequality. Below, we will find explicit expressions for the fractional CHF of several distributions. For some, this generalizes the well-known expressions of the CHF, but not always, as in some cases α may be profitably matched with a parameter of the distribution (see Example 7).

2 Some properties of the fractional CHF, and first examples

After the basic properties of the fractional CHF mentioned at the end of the preceding section, we present some further results, partially based on the asymptotic behavior of the Mittag-Leffler function. We also give evaluations of the fractional CHF for some concrete examples in this section.

Proposition 1 *The fractional CHF $\varphi_\alpha(t)$ exists for all $t \in \mathbb{R}$.*

Proof For $t = 0$ this is clear, and for $t \neq 0$, by (5.2), $E_\alpha(i(tx)^\alpha) = O(|x|^{-\alpha})$ for $x \to \pm\infty$. □

Proposition 2 *The fractional MGF exists exactly for those $t \in \mathbb{R}$ for which the classical MGF exists.*

Proof Indeed, by (5.1), for $t > 0$ we have

$$E_\alpha((tx)^\alpha) \sim \begin{cases} \frac{1}{\alpha}e^{tx}, & x \to \infty, \\ -\frac{(-tx)^{-\alpha}e^{-\alpha i\pi}}{\Gamma(1-\alpha)} & x \to -\infty, \end{cases}$$

and for $t < 0$

$$E_\alpha((tx)^\alpha) \sim \begin{cases} -\frac{(-tx)^{-\alpha}e^{-\alpha i\pi}}{\Gamma(1-\alpha)}, & x \to \infty, \\ \frac{1}{\alpha}e^{tx}, & x \to -\infty. \end{cases}$$

From these estimates it is clear that $E_\alpha((tx)^\alpha)$ can be replaced by e^{tx} when assessing existence of the fractional MGF. □

Due to the branch cut of the power function, we presume the fractional CHF and MGF to be most useful for non-negative random variables, and thus focus on these in our examples. In the following example, we find the fractional CHF of the half-Cauchy distribution. By the two preceding propositions, it exists on $\mathbb{R}$, but the fractional MGF exists only for $t \leq 0$.

Example 1 Let X have the half-Cauchy distribution with parameter $\beta > 0$, i.e. the pdf $p(x) = \frac{2}{\pi}\frac{\beta}{x^2+\beta^2}, x > 0$. Let $p^*(s)$ and $g^*(s)$ be the Mellin transforms of the functions $p(x)$ and $g(x) = E_\alpha(ix^\alpha)$, respectively. Since

$$E_\alpha(i(tx)^\alpha) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha}\frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(1-s)}(e^{-i\frac{\pi}{2}}t^\alpha)^{-\frac{s}{\alpha}}$$

and (see [2], p. 322, integral 3.241 (4))

$$\frac{1}{x^2+\beta^2} \overset{\mathcal{M}}{\longleftrightarrow} \frac{\beta^{s-2}}{2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right), \quad 0<s<2,\ \beta \neq 0,$$

by application of Parseval's convolution equality [3],

$$\int_0^{+\infty} p(x)g(xt)dx = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} p^*(1-s)g^*(s)t^{-s}ds,$$

we obtain

$$\begin{aligned}
\varphi_\alpha(t) &= \frac{2\beta}{\pi}\int_0^\infty \frac{E_\alpha(i(tx)^\alpha)}{x^2+\beta^2}dx \\
&= \frac{1}{\alpha\beta\pi}\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{2})\Gamma(\frac{s}{2})}{\Gamma(1-s)}(\beta e^{i\frac{\pi}{2\alpha}}t^{-2})^s ds \\
&= \frac{1}{\alpha\beta\pi}H_{3,3}^{2,2}\left[\frac{\beta e^{i\frac{\pi}{2\alpha}}}{t^2}\middle| \begin{matrix} (1,\frac{1}{\alpha}) & (1,\frac{1}{2}) & (1,0) \\ (1,\frac{1}{\alpha}) & (1,\frac{1}{2}) & (0,-1) \end{matrix}\right],
\end{aligned}$$

where $H_{3,3}^{2,2}$ is the Fox H-function. Conversely,

$$M_\alpha(t) = \frac{2\beta}{\pi}\int_0^\infty \frac{E_\alpha((tx)^\alpha)}{x^2+\beta^2}dx = \infty, \quad t>0,$$

see Proposition 2, because $x \mapsto e^{tx}/x^2$ is not integrable for $t>0$.

For $t \geq 0$, we have

$$E_\alpha(i(tx)^\alpha) = \cos_\alpha((tx)^\alpha) + i\sin_\alpha((tx)^\alpha),$$

where

$$\cos_\alpha((tx)^\alpha) = \sum_{n=0}^\infty \frac{(-1)^n(tx)^{2n\alpha}}{\Gamma(2n\alpha+1)}$$

and

$$\sin_\alpha((tx)^\alpha) = \sum_{n=0}^\infty \frac{(-1)^n(tx)^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}$$

are fractional cosine and sine trigonometric functions, respectively. This implies

$$\varphi_\alpha(t) = \int_{-\infty}^{\infty} \cos_\alpha((tx)^\alpha)p(x)dx + i \int_{-\infty}^{\infty} \sin_\alpha((tx)^\alpha)p(x)dx.$$

Since

$$|E_\alpha(i(tx)^\alpha)| = \sqrt{\sin_\alpha^2((tx)^\alpha) + \cos_\alpha^2((tx)^\alpha)},$$

we obtain the following bound for the fractional CHF:

$$|\varphi_\alpha(t)| \leq \int_{-\infty}^{\infty} \sqrt{\sin_\alpha^2((tx)^\alpha) + \cos_\alpha^2((tx)^\alpha)}\,p(x)dx, \quad t \geq 0.$$

The fractional CHF is always continuous on $\mathbb{R}$, by the bound (5.3) (see in Appendix) and the dominated convergence theorem. In fact, we have:

Proposition 3 *The fractional CHF of any random variable is uniformly continuous on* $\mathbb{R}$.

Proof By Theorem 4.3 in [4], we have

$$E_{\alpha,\alpha}(iu^\alpha) = O(|u|^{-\alpha}), \quad u \to \pm\infty \text{ in } \mathbb{R},$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.1}$$

denotes the two-parameter Mittag-Leffler function. Indeed, the exponential term $\exp(z^{1/\alpha})$ in (4.4.16) of [4] decays for $z = iu^\alpha$, and is negligible compared to the algebraic asymptotic expansion. It follows that

$$\frac{d}{du} E_\alpha(iu^\alpha) = iu^{\alpha-1} E_{\alpha,\alpha}(iu^\alpha)$$

is bounded for $u \in \mathbb{R} \setminus [-1, 1]$. As E_α is an entire function, this implies that $u \mapsto E_\alpha(iu^\alpha)$ is uniformly continuous on $\mathbb{R}$. Now let $\varepsilon > 0$. By (5.3) (Appendix), we have

$$\sup_{u\in\mathbb{R}} |E_\alpha(iu^\alpha)| < \infty.$$

We can thus choose $A > 0$ such that

$$\int_{|x|\geq A} |E_\alpha(i((t+h)x)^\alpha) - E_\alpha(i(tx)^\alpha)|p(x)dx \leq \varepsilon, \quad t, h \in \mathbb{R}.$$

Now consider the integral

$$\int_{-A}^{A} |E_\alpha(i((t+h)x)^\alpha) - E_\alpha(i(tx)^\alpha)|p(x)dx, \quad t \in \mathbb{R}. \tag{2.2}$$

We have $(t+h)x - tx = hx$, and x is bounded. Therefore, by uniform continuity of $E_\alpha(iu^\alpha)$, we can make (2.2) smaller than ε by choosing h small enough. □

We now comment on the related paper [5], which defines the fractional Laplace transform of a function f by

$$\alpha \lim_{b\to\infty} \int_0^b (b-x)^{\alpha-1} E_\alpha(-s^\alpha x^\alpha) f(x)dx.$$

For $0 < \alpha < 1$, any function with bounded support, such as the density of a bounded random variable, has a fractional Laplace transform that is identically zero under this definition, which raises doubts about its usefulness. Moreover, the inversion formula presented in [5] rests on the identity

$$E_\alpha((x+y)^\alpha) \stackrel{?}{=} E_\alpha(x^\alpha)E_\alpha(y^\alpha), \tag{2.3}$$

which is wrong for $\alpha \neq 1$, see [6] for a discussion. The second formula used in the proof of (3.9) of [5] is also incorrect. It was recently established in [7] that $\leq$ holds in (2.3) for $0 < \alpha < 1$. This implies the following inequality for the fractional MGF of the sum of two independent random variables. (Recall that we assume $0 < \alpha < 1$ throughout.)

Proposition 4 *Let X, Y be non-negative independent random variables and $t \geq 0$. Then we have*

$$M_{\alpha,X+Y}(t) \leq M_{\alpha,X}(t)M_{\alpha,Y}(t).$$

Proof We use the inequality from [7] we just mentioned, and compute

$$\begin{aligned} M_{\alpha,X+Y} &= \mathbb{E}\big[E_\alpha\big((t^{1/\alpha}X + t^{1/\alpha}Y)^\alpha\big)\big] \\ &\leq \mathbb{E}\big[E_\alpha(tX^\alpha)E_\alpha(tY^\alpha)\big] \\ &= M_{\alpha,X}(t)M_{\alpha,Y}(t). \end{aligned}$$

□

The Mellin transform and special functions like Mittag–Leffler, Fox's H-functions and Meijer's G-function, have found a large number of applications in probability theory. These functions are representable as Mellin–Barnes integrals of the product of gamma functions and are therefore suited to represent statistics of products and quotients of independent random variables whose fractional moments are expressible as gamma or related functions. Applications of the Mellin transform and special functions to statistics and probability theory can be found, for example in [8–12] and the references therein.

Example 2 (Space-time fractional diffusion model) Fractional moments are very useful in dealing with random variables with power-law distributions, $F(x) \sim |x|^{\mu}$, $\mu > 0$, where $F(x)$ is the distribution function. Indeed, in such cases, moments $\mathbb{E}X^q$ exist only if $q < \mu$ and integer order moments greater than μ diverge. Distributions of this type are encountered in a wide variety of contexts, see the extensive literature in [13] and [14] where power-law statistics appear in the framework of anomalous diffusion in many fields of applied science. Fractional partial differential equations are a useful tool for modelling of various anomalous diffusion in complex systems exhibiting pronounced deviations from Brownian diffusion, which is normally described by the standard diffusion equation. Let $L_{\alpha}^{\theta}(x)$, $0 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, $x \in \mathbb{R}$ is the class of the α-Lévy stable probability densities, where α denotes the index of stability (or Lévy index) and θ is a real-parameter related to the asymmetry, improperly referred to as the skewness. The fundamental solution of space-time fractional diffusion model with space Riesz-Feller fractional derivative of order α is given in [15–18] by

$$L_{\alpha}^{\theta}(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1}\left[\frac{(\eta t)^{\frac{1}{\alpha}}}{|x|} \middle| \begin{matrix} (1,1) & (c,c) \\ (\frac{1}{\alpha}, \frac{1}{\alpha}) & (c,c) \end{matrix} \right] du, \quad 0 < \alpha < 1, |\theta| \leq \alpha.$$

Another form of this density is given by

$$L_{\alpha}^{\theta}(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{\frac{1}{\alpha}}} \middle| \begin{matrix} (1-\frac{1}{\alpha}, \frac{1}{\alpha}) & (1-c,c) \\ (0,1) & (1-c,c) \end{matrix} \right] du, \quad 1 < \alpha < 2, |\theta| \leq 2-\alpha.$$

In both cases $c = \frac{\alpha-\theta}{2\alpha} \neq 0$ and η is a diffusion constant. The Mellin transform of $L_{\alpha}^{\theta}(x)$ can be found in [16]:

$$L_{\alpha}^{\theta}(x) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha} \frac{\Gamma(s)\Gamma(\frac{1-s}{\alpha})}{\Gamma(c-cs)\Gamma(1-c+cs)}.$$

From this we can compute the "upper" fractional CHF φ_{α}^{+}, where we only integrate over $\mathbb{R}^{+}$. Assuming $0 < \rho < 1$ (see [16] for details), we have

$$\begin{aligned} \varphi_{\alpha}^{+}(t) &= \int_0^{\infty} E_{\alpha}(i(tx)^{\alpha}) L_{\alpha}^{\theta}(x) dx \\ &= \frac{1}{2\pi i \alpha^2} \int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(cs)\Gamma(1-cs)} (e^{i\frac{\pi}{2\alpha}} t^{-2})^s ds \\ &= \frac{1}{\alpha^2} H_{3,2}^{1,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{t^2} \middle| \begin{matrix} (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,c) \\ & (1,\frac{1}{\alpha}) & (0,-c) \end{matrix} \right]. \end{aligned} \tag{2.4}$$

Using (1.1), we can express the upper fractional MGF M_{α}^{+} for $\frac{1}{2} < \alpha < 1$ and $t \geq 0$, we just have to replace the argument of $H_{3,2}^{1,2}$ in (2.4) by $e^{3i\pi/(2\alpha)} t^{-2}$.

3 Fractional moments

Fractional moments have been investigated by many authors. For example, in [19], fractional moments are used to compute densities of univariate and bivariate random variables numerically. As already mentioned in Section 2, fractional moments are indeed important when the density of the random variable has inverse power-law tails and, consequently, it lacks integer order moments. Fractional moments of a non-negative random variable are expressible by the Mellin transform of the density and this fact has been widely used in the literature, in particular in research on algebra of random variables. That is, the Mellin transform is the principal mathematical tool to handle problems involving products and quotients of independent random variables. If the fractional moments $\mathbb{E}((iX^\alpha)^k)$ do not grow too fast, namely

$$|\mathbb{E}((iX^\alpha)^k)| \le c^k \Gamma(\alpha k + 1)$$

for some $c > 0$, then they appear in the fractional power series expansion of the fractional CHF:

$$\varphi_\alpha(t) = \sum_{k=0}^{\infty} \frac{(it^\alpha)^k}{\Gamma(\alpha k + 1)} \int_{-\infty}^{\infty} x^{\alpha k} p(x) dx = \sum_{k=0}^{\infty} \mathbb{E}((iX^\alpha)^k) \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}. \quad (3.1)$$

This is correct for small $|t|$, under the assumption that $t \ge 0$ or that $X \ge 0$. If neither of these is satisfied, then

$$(i(tx)^\alpha)^k = (it^\alpha)^k x^{\alpha k}$$

does not hold in general.[1] The fractional moments in (3.1) can be expressed via fractional derivatives of the fractional CHF at $t = 0$. In this part we will show how the fractional calculus operators of RL (Riemann-Liouville) type can be used to calculate the fractional moments of random variables, via its fractional CHF. The fractional RL integral $(I_\pm^\gamma f)(x)$, resp. derivative $(D_\pm^\gamma f)(x)$ are defined by

$$(I_\pm^\gamma f)(x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty u^{\gamma-1} f(x \mp u)\, du, \quad \gamma > 0,$$

$$(D_\pm^\gamma f)(x) = \frac{d}{dx} (I_\pm^{1-\gamma} f)(x)$$
$$= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_0^\infty u^{-\gamma} f(x \mp u)\, du, \quad 0 < \gamma < 1.$$

Hilfer [20] gave the following representation of the RL fractional derivative:

$$(D_\pm^\gamma f)(x) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \left[f'(x \mp u) \int_u^\infty \frac{dv}{v^{1+\gamma}} \right] du.$$

[1] Note that, for complex u, v and $\beta > 0$, the numbers $(uv)^\beta = |uv|^\beta \exp(\beta i \arg(uv))$ and $u^\beta v^\beta = |uv|^\beta \exp(\beta i (\arg u + \arg v))$ do not agree in general; they do for $\arg u + \arg v \in (-\pi, \pi]$.

Interchanging the order of integration, we get

$$(D_{\pm}^{\gamma} f)(x) = \frac{1}{\Gamma(-\gamma)} \int_0^{\infty} \frac{f(x \mp u) - f(x)}{u^{1+\gamma}} du. \tag{3.2}$$

In the notation of [21] (see p. 70), our D_+^{γ} is the operator D_{a+}^{γ} with $\gamma \in (0, 1)$ and $a = -\infty$. The operator D_{0+}^{γ} with $\gamma > 0$ is defined in the standard way, as in [21]. By (2.1.17) in [21], we have

$$D_{0+}^{\alpha k} t^{\alpha j} = \frac{\Gamma(\alpha j + 1)}{\Gamma(\alpha(j-k)+1)} t^{\alpha(j-k)},$$

and so the fractional moment in (3.1) can be expressed by the recursive relation

$$D_{0+}^{\alpha k} \left(\varphi_\alpha(t) - \sum_{j=0}^{k-1} \mathbb{E}[(iX^\alpha)^j] \frac{t^{\alpha j}}{\Gamma(\alpha j + 1)} \right) \Bigg|_{t=0}$$
$$= \sum_{j=k}^{\infty} \mathbb{E}[(iX^\alpha)^j] \frac{t^{\alpha(j-k)}}{\Gamma(\alpha(j-k)+1)} \Bigg|_{t=0} = \mathbb{E}[(iX^\alpha)^k].$$

We now calculate the fractional integral of the fractional CHF,

$$(I_{\pm}^{\gamma} \varphi_\alpha)(x) = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} u^{\gamma-1} \varphi_\alpha(x \mp u)\, du$$
$$= \frac{1}{\Gamma(\gamma)} \int_0^{\infty} u^{\gamma-1} \int_{-\infty}^{\infty} E_\alpha(i(k(x \mp u))^\alpha) p(k) dk\, du. \tag{3.3}$$

In particular,

$$(I_{\pm}^{\gamma} \varphi_\alpha)(0) = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{\infty} p(k) dk \int_0^{\infty} u^{\gamma-1} E_\alpha(i(\mp ku)^\alpha) du. \tag{3.4}$$

Using the well-known integral formula (see, e.g., p. 313 in [4])

$$\int_0^{\infty} z^{s-1} E_\alpha(-wz) dz = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)} w^{-s}, \quad 0 < \Re(s) < 1,$$

we conclude that

$$\int_0^{\infty} u^{\gamma-1} E_\alpha(i(\mp ku)^\alpha) du = \frac{\Gamma(\frac{\gamma}{\alpha})\Gamma(1-\frac{\gamma}{\alpha})}{\alpha \Gamma(1-\gamma)} (-i(\mp k)^\alpha)^{-\frac{\gamma}{\alpha}}, \quad 0 < \gamma < \alpha,$$

and thus

$$(I^{\gamma}_{\pm}\varphi_{\alpha})(0) = \frac{\sin(\pi\gamma)e^{i\frac{\pi\gamma}{2\alpha}}}{\alpha\sin(\frac{\pi}{\alpha}\gamma)} \int_{-\infty}^{\infty} (\mp k)^{-\gamma} p(k)dk.$$

Analogously,

$$(D^{\gamma}_{\pm}\varphi_{\alpha})(0) = \frac{\sin(\pi\gamma)e^{-i\frac{\pi\gamma}{2\alpha}}}{\alpha\sin(\frac{\pi}{\alpha}\gamma)} \int_{-\infty}^{\infty} (\mp k)^{\gamma} p(k)dk.$$

We have shown the following result:

Proposition 5 *If* $0 < \gamma < \alpha$ *and the double integral in* (3.4) *is absolutely convergent, then*

$$(I^{\gamma}_{\pm}\varphi_{\alpha})(0) = \frac{\sin(\pi\gamma)e^{i\frac{\pi\gamma}{2\alpha}}}{\alpha\sin(\frac{\pi}{\alpha}\gamma)}\mathbb{E}((\mp X)^{-\gamma}) \tag{3.5}$$

and

$$(D^{\gamma}_{\pm}\varphi_{\alpha})(0) = \frac{\sin(\pi\gamma)e^{i\frac{\pi(-\gamma)}{2\alpha}}}{\alpha\sin(\frac{\pi}{\alpha}\gamma)}\mathbb{E}((\mp X)^{\gamma}). \tag{3.6}$$

For $\alpha = 1$ we get the result given by Cottone et al. [19]:

$$(I^{\gamma}_{\pm}\varphi_{1})(0) = (I^{\gamma}_{\pm}\varphi)(0) = e^{i\frac{\pi}{2}\gamma}\mathbb{E}((\mp X)^{-\gamma}),$$
$$(D^{\gamma}_{\pm}\varphi_{1})(0) = (D^{\gamma}_{\pm}\varphi)(0) = e^{-i\frac{\pi}{2}\gamma}\mathbb{E}((\mp X)^{\gamma}).$$

It is well-known that RL fractional integrals and derivatives can be defined for complex order γ with $\Re(\gamma) > 0$ (see [21]). Clearly, Proposition 5 extends to $0 < \Re(\gamma) < \alpha$. The following formulas generalize (11a)–(12b) of [19], and are derived analogously. From (3.3) and (3.2), we obtain the connections of $(I^{\gamma}_{\pm}\varphi_{\alpha})(0)$ and $(D^{\gamma}_{\pm}\varphi_{\alpha})(0)$ with the Mellin transform of the fractional CHF:

$$\Gamma(\gamma)(I^{\gamma}_{\pm}\varphi_{\alpha})(0) = \mathcal{M}\{\varphi_{\alpha}(\mp u); \gamma\}$$
$$\Gamma(-\gamma)(D^{\gamma}_{\pm}\varphi_{\alpha})(0) = \mathcal{M}\{(\varphi_{\alpha}(\mp u) - \varphi_{\alpha}(0)); -\gamma\}$$

Applying the inverse Mellin transform and recalling $\varphi_{\alpha}(0) = 1$, we obtain two representations of the fractional CHF:

$$\varphi_{\alpha}(-u) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} u^{-\gamma}\Gamma(\gamma)(I^{\gamma}_{+}\varphi_{\alpha})(0)d\gamma \tag{3.7}$$

$$= \frac{1}{2\pi i\alpha}\int_{\rho-i\infty}^{\rho+i\infty} u^{-\gamma}\frac{\Gamma(\gamma)}{\sin(\pi\gamma)}\sin(\frac{\pi}{\alpha}\gamma)e^{i\frac{\pi}{2\alpha}\gamma}\mathbb{E}((-X)^{-\gamma})d\gamma$$

and

$$\varphi_\alpha(-u) = 1 + \frac{1}{2\pi i\alpha}\int_{\rho-i\infty}^{\rho+i\infty} u^{\gamma}\frac{\Gamma(-\gamma)}{\sin(\pi\gamma)}\sin(\frac{\pi}{\alpha}\gamma)e^{-i\frac{\pi}{2\alpha}\gamma}\mathbb{E}((-X)^{\gamma})d\gamma,$$

where $u > 0$ and the integrals are performed vertically with fixed real part ρ, belonging to the so-called fundamental strip of the Mellin transform of the function φ_α. The latter equations are integral extensions of (3.1). The density function is restored in [19] from (3.7), with $\alpha = 1$, by using inverse Fourier transform:

$$\begin{aligned} p(x) &= \frac{1}{(2\pi)^2}\int_{\rho-i\infty}^{\rho+i\infty} \Gamma(\gamma)\Gamma(1-\gamma)\big\{\mathbb{E}((-iX)^{-\gamma})(ix)^{\gamma-1} \\ &\quad + \mathbb{E}((iX)^{-\gamma})(-ix)^{\gamma-1}\big\}d\gamma. \end{aligned}$$

4 Illustrative examples

Example 3 Let X be uniformly distributed, i.e. $X \sim U(a, b)$, with $0 \le a < b$. Then

$$\begin{aligned} \varphi_\alpha(t) &= \frac{1}{b-a}\sum_{k=0}^{\infty}\frac{(it^\alpha)^k}{\Gamma(\alpha k+1)}\int_a^b x^{\alpha k}dx \\ &= \frac{b}{b-a}E_{\alpha,2}(i(tb)^\alpha) - \frac{a}{b-a}E_{\alpha,2}(i(ta)^\alpha), \end{aligned}$$

and

$$M_\alpha(t) = \frac{b}{b-a}E_{\alpha,2}((tb)^\alpha) - \frac{a}{b-a}E_{\alpha,2}((ta)^\alpha),$$

where $E_{\alpha,2}$ is the two-parameter Mittag–Leffler function (2.1). To investigate the fractional moments, we compute

$$\begin{aligned} (I_-^\gamma\varphi_\alpha)(0) &= \frac{1}{\Gamma(\gamma)}\int_0^\infty u^{\gamma-1}\varphi_\alpha(u)\,du \\ &= \frac{1}{\Gamma(\gamma)}\left\{\frac{b}{b-a}\int_0^\infty u^{\gamma-1}E_{\alpha,2}(i(ub)^\alpha)du\right. \end{aligned}$$

$$-\frac{a}{b-a}\int\limits_0^\infty u^{\gamma-1}E_{\alpha,2}(i(ua)^\alpha)du\Big\}$$
$$=\frac{\Gamma(\frac{\gamma}{\alpha})\Gamma(1-\frac{\gamma}{\alpha})}{\alpha\Gamma(\gamma)\Gamma(2-\gamma)}\left\{\frac{b}{b-a}(-ib^\alpha)^{-\frac{\gamma}{\alpha}}-\frac{a}{b-a}(-ia^\alpha)^{-\frac{\gamma}{\alpha}}\right\}$$
$$=\frac{\sin(\pi\gamma)e^{i\frac{\pi\gamma}{2\alpha}}}{\alpha(1-\gamma)(b-a)\sin(\frac{\pi}{\alpha}\gamma)}(b^{1-\gamma}-a^{1-\gamma}).$$

By (3.5) from Proposition 5, we finally get

$$\mathbb{E}X^{-\gamma}=\frac{b^{1-\gamma}-a^{1-\gamma}}{(1-\gamma)(b-a)}.$$

This is just an illustration for Proposition 5; of course, the fractional moments can be computed directly as well. The same applies to the examples below: We find an expression for the fractional CHF, and for some examples we also illustrate our fractional calculus approach to get the fractional moments.

Example 4 Let X be exponentially distributed, i.e. $X\sim\mathcal{E}(\lambda)$, $\lambda>0$. Then

$$\varphi_\alpha(t)=\lambda\sum_{k=0}^\infty\frac{(it^\alpha)^k}{\Gamma(\alpha k+1)}\int\limits_0^\infty e^{-\lambda x}x^{\alpha k}dx$$
$$=\lambda\sum_{k=0}^\infty\frac{(it^\alpha)^k}{\Gamma(\alpha k+1)}\frac{\Gamma(\alpha k+1)}{\lambda^{\alpha k+1}}$$
$$=\sum_{k=0}^\infty\left[i\left(\frac{t}{\lambda}\right)^\alpha\right]^k=\frac{\lambda^\alpha}{\lambda^\alpha-it^\alpha},\quad |t|<\lambda,$$

and

$$M_\alpha(t)=\frac{\lambda^\alpha}{\lambda^\alpha-t^\alpha},\quad |t|<\lambda.$$

Example 5 Let X be a random variable with $p(x)=c_pe^{-x^p}$, $x>0$, $p>0$, where $c_p=1/\Gamma(1+1/p)$ is the normalization constant. Then,

$$\varphi_\alpha(t)=c_p\int\limits_0^\infty E_\alpha(i(tx)^\alpha)e^{-x^p}dx=c_p\sum_{k=0}^\infty\frac{(it^\alpha)^k}{\Gamma(\alpha k+1)}\int\limits_0^\infty e^{-x^p}x^{\alpha k}dx$$
$$=c_p\sum_{k=0}^\infty\frac{(it^\alpha)^k}{p\Gamma(\alpha k+1)}\Gamma\Big(\frac{\alpha k+1}{p}\Big)=\frac{c_p}{p}\,{}_1\Psi_1\left[\begin{matrix}(\frac{1}{p},\frac{\alpha}{p})\\(1,\alpha)\end{matrix}\middle|it^\alpha\right],$$

where ${}_1\Psi_1$ is the Fox-Wright function. We note that, if $p \in \mathbb{N}$, then using the Legendre multiplication formula for the Gamma function, the last result can be expressed via the multi-index Mittag–Leffler function, defined by Kiryakova [22]. For $p = 2$, using the Legendre duplication formula for the gamma function, we get

$$\varphi_\alpha(t) = \frac{\pi}{4} E_{\frac{\alpha}{2}}\Big(i\Big(\frac{t}{2}\Big)^\alpha\Big).$$

Note that the fractional moments are

$$\mathbb{E}X^{-\gamma} = c_p \int_0^\infty x^{-\gamma} e^{-x^p} dx = \frac{\Gamma\Big(\frac{1-\gamma}{p}\Big)}{p\Gamma\Big(1+\frac{1}{p}\Big)}, \quad 0 < \gamma < 1.$$

Especially, for $p = 2$, i.e. the half-normal distribution (up to scaling), we have

$$\mathbb{E}X^{-\gamma} = \frac{1}{\sqrt{\pi}} \Gamma\Big(\frac{1-\gamma}{2}\Big), \quad 0 < \gamma < 1.$$

Finally, for $p = 1$, i.e. $X \sim \mathcal{E}(1)$, we obtain $\mathbb{E}X^{-\gamma} = \Gamma(1-\gamma)$.

The preceding example is not so amenable for illustrating Proposition 5, as we did in Example 3, because the fractional integral of the fractional CHF is less easy to compute.

Example 6 We will calculate the fractional CHF and MGF for a random variable with pdf $p(x) = E_\alpha(-x^\alpha)$, $0 < \alpha < 1$, defined by Pollard [23] and Mainardi [24]. Let $p^*(s)$ and $g^*(s)$ be the Mellin transforms of the functions $p(x) = E_\alpha(-x^\alpha)$, $x > 0$ and $g(x) = E_\alpha(ix^\alpha)$. By application of Parseval's convolution equality [3],

$$\int_0^{+\infty} p(x)g(xt)dx = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} p^*(1-s)g^*(s)t^{-s} ds,$$

we obtain

$$\begin{aligned}\varphi_\alpha(t) &= \int_0^\infty E_\alpha(i(tx)^\alpha)E_\alpha(-x^\alpha)dx \\ &= \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \frac{1}{\alpha} \frac{\Gamma(\frac{1-s}{\alpha})\Gamma(1-\frac{1-s}{\alpha})}{\Gamma(s)} \frac{1}{\alpha} \frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-s)} (-it^\alpha)^{\frac{-s}{\alpha}} t^{-s} ds\end{aligned}$$

$$= \frac{1}{2\pi i \alpha^2} \int_{\rho - i\infty}^{\rho + i\infty} \frac{\Gamma(\frac{1-s}{\alpha})\Gamma(1 - \frac{1-s}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(1 - \frac{s}{\alpha})}{\Gamma(s)\Gamma(1-s)} (e^{i\frac{\pi}{2\alpha}} t^{-2})^s ds$$

$$= \frac{1}{\alpha^2} H_{3,3}^{2,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{t^2} \middle| \begin{matrix} (\frac{1}{\alpha}, -\frac{1}{\alpha}) & (0, -\frac{1}{\alpha}) & (0, 1) \\ (\frac{1}{\alpha}, \frac{1}{\alpha}) & (1, \frac{1}{\alpha}) & (0, -1) \end{matrix}\right],$$

where $H_{3,3}^{2,2}$ is the Fox H-function. To investigate the fractional moments, we first find the fractional integral of CHF, and use Proposition 5 and the Mellin transform formula for the H-function [9]:

$$(I_{-}^{\gamma}\varphi_\alpha)(0) = \frac{1}{\alpha^2 \Gamma(\gamma)} \int_0^\infty u^{\gamma - 1} H_{3,3}^{2,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{u^2} \middle| \begin{matrix} (\frac{1}{\alpha}, -\frac{1}{\alpha}) & (0, -\frac{1}{\alpha}) & (0, 1) \\ (\frac{1}{\alpha}, \frac{1}{\alpha}) & (1, \frac{1}{\alpha}) & (0, -1) \end{matrix}\right] du$$

$$= \frac{e^{-i\frac{\pi\gamma}{4\alpha}}}{2\alpha^2} \frac{\Gamma(\frac{1}{\alpha} - \frac{\gamma}{2\alpha})\Gamma(1 - \frac{1}{\alpha} - \frac{\gamma}{2\alpha})\Gamma^2(1 - \frac{\gamma}{2\alpha})}{\Gamma(\gamma)\Gamma(-\frac{\gamma}{2})\Gamma(1 - \frac{\gamma}{2})}.$$

Hence, by Proposition 5 we get

$$\mathbb{E}X^{-\gamma} = \frac{e^{-i\frac{3\pi\gamma}{4\alpha}} \sin(\frac{\pi\gamma}{\alpha})}{\alpha\pi} \frac{\Gamma(-\gamma)\Gamma(\frac{1}{\alpha} - \frac{\gamma}{2\alpha})\Gamma(1 - \frac{1}{\alpha} - \frac{\gamma}{2\alpha})\Gamma^2(1 - \frac{\gamma}{2\alpha})}{\Gamma^2(-\frac{\gamma}{2})}.$$

Example 7 We consider the Mittag–Leffler waiting time density

$$p(x) = x^{\alpha - 1} E_{\alpha,\alpha}(-x^\alpha), \ x > 0,$$

as in [25]. Since

$$x^{\alpha - 1} E_{\alpha,\alpha}(-x^\alpha) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha} \frac{\Gamma(\frac{s+\alpha-1}{\alpha})\Gamma(1 - \frac{s+\alpha-1}{\alpha})}{\Gamma(1-s)},$$

we obtain

$$\varphi_\alpha(t) = \int_0^\infty E_\alpha(i(tx)^\alpha) x^{\alpha-1} E_{\alpha,\alpha}(-x^\alpha) dx$$

$$= \frac{1}{2\pi i \alpha^2} \int_{\rho - i\infty}^{\rho + i\infty} \frac{\Gamma(1 - \frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(1-s)} \frac{\Gamma(1 - \frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(s)} (e^{i\frac{\pi}{2\alpha}} t^{-2})^s ds$$

$$= \frac{1}{\alpha^2} H_{3,3}^{2,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{t^2} \middle| \begin{matrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{\alpha}) & (0, 1) \\ (1, \frac{1}{\alpha}) & (1, \frac{1}{\alpha}) & (0, -1) \end{matrix}\right].$$

Then,

$$(I_{-}^{\gamma}\varphi_{\alpha})(0) = \frac{1}{\alpha^2\Gamma(\gamma)}\int_0^{\infty} u^{\gamma-1} H_{3,3}^{2,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{u^2}\middle| \begin{matrix} (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,1) \\ (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,-1) \end{matrix}\right] du$$
$$= -e^{\frac{-i\pi\gamma}{4\alpha}} \frac{\pi^2}{\gamma\alpha^2\Gamma^2(-\frac{\gamma}{2})\sin^2\frac{\pi\gamma}{2\alpha}},$$

and hence,

$$\mathbb{E}X^{-\gamma} = -e^{-i\frac{3\pi\gamma}{4\alpha}} \frac{\pi^2\cot(\frac{\pi\gamma}{2\alpha})}{\gamma\alpha\Gamma^2(-\frac{\gamma}{2})\sin(\pi\gamma)}.$$

The computation of the fractional CHF and fractional moments for the Mathai waiting time density [26], $p(x) = x^{\alpha\beta-1}E^{\beta}_{\alpha\beta,\alpha}(-x^{\alpha})$, $\alpha, \beta > 0$, $x > 0$, is similar, and we leave it for the reader.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix: The Mittag–Leffler function

The (one-parameter) Mittag–Leffler function is an entire function, defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

In this paper, we only consider the case $0 < \alpha < 1$. By (5.1.26) in [27], the expansion of E_α at infinity is

$$E_\alpha(z) \sim \begin{cases} \frac{1}{\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, & |\arg z| < \frac{3}{2}\pi\alpha, \\ -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, & |\arg(-z)| < \frac{1}{2}\pi(2-\alpha). \end{cases} \tag{5.1}$$

We specialize this to $z = iu^\alpha$ with $u \in \mathbb{R}$. We have

$$(iu^\alpha)^{1/\alpha} = \begin{cases} |u| \exp\left(\frac{i\pi}{2\alpha}\right), & u \geq 0, \\ |u| \exp\left(i\left(1 + \frac{1}{2\alpha}\right)\pi\right), & u < 0,\ 0 < \alpha \leq \frac{1}{2}, \\ |u| \exp\left(i\left(1 - \frac{3}{2\alpha}\right)\pi\right), & u < 0,\ \frac{1}{2} < \alpha < 1. \end{cases}$$

From this it follows that the conditions $\Re(z^{1/\alpha}) \geq 0$ and $|\arg z| < \frac{3}{2}\pi\alpha$ cannot be simultaneously satisfied for $z = iu^\alpha$ with $u \in \mathbb{R}$. Therefore, the exponential term $e^{z^{1/\alpha}}$ in (5.1) is negligible, and the algebraic expansion yields the first order asymptotics

$$E_\alpha(iu^\alpha) \sim \frac{i}{\Gamma(1-\alpha)u^\alpha}, \quad u \to \pm\infty \text{ in } \mathbb{R}. \tag{5.2}$$

In particular, since E_α is an entire function,

$$\sup_{u \in \mathbb{R}} |E_\alpha(iu^\alpha)| < \infty. \tag{5.3}$$

For similar results and details on the two-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$, (2.1), we refer to [4].

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