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Fuzzy Logics in the Context of Theories of Vagueness

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unter der Anleitung von

Ao.Prof.Dipl.-Ing.Dr.techn.Christian Georg Fermüller

durch

Margit Schütz
Gentzgasse 150/4/22
1180 Wien

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I dedicate this thesis to my parents Erich and Roswitha, my sister Gudrun and my 'tesoro' Marco and thank my professor Christian Fermüller for his advice and his patience.

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Introduction

Präzision ist nicht Wahrheit.

Henri Matisse

The simple statement 'Precision is not truth' of the great French painter Henri Matisse could be the central theme of this master thesis, that deals with vagueness - a concept, that at first sight seems to be the contrary of precision, a property usually very appreciated by scientists and engineers. Observing the world in which we are living, one notices quickly, that it is not precise at all, a circumstance, that does not cause any problem for us in our everyday life, but many if we want to understand the phenomenon of vagueness and even more if we want to understand the way humans reason in its presence.

Reasoning is a mental activity that usually takes place in a natural language and even if the available information is vague or imprecise humans can use it to come to a conclusion. Logic is the systematic study of valid inference, where the terms of reasoning, deduction or thinking could be used as synonyms for inference, and is both, a branch of philosophy and a branch of mathematics and consequently also a branch of computer science. Hence, the question of how reasoning works in the presence of vagueness is a central theme for philosophers as well as for scientists working in the field of logic.

Thus, on the one hand, there is a lively discourse on vagueness in analytic philosophy, which aims at understanding the principles of reasoning in the presence of vagueness.

On the other hand, the aim to understand the principles of reasoning in the presence of vagueness and of reasoning in general is an issue, that concerns also mathematicians, engineers and computer scientists. Although reasoning seems to be a purely human ability, in this time of rapidly advancing technology the wish to create machines which imitate human behavior has become very exigent. Therefore it is necessary to formalize the human ability of reasoning in the presence of vagueness for the purpose of automation in expert systems, computer vision, control engineering or pattern recognition, to mention just a few examples. To this end, formal languages, deductive systems and model-theoretic semantics have to be developed. However, most of the scientists and engineers working in the field of vagueness are convinced that the truth comes in degrees and support so-called degree theories. Often they consider fuzzy logics, in particular logics based on t-norms, as *the* logics of vagueness as these formalisms are well-understood and have had a lot of success in the past.

Altogether, these two fields of research seem to be closely related and the challenge of formalizing reasoning in the presence of vagueness seems to be met best by an interdisciplinary team of philosophers, scientists, researchers and engineers. Surprisingly, these apparently closely related fields of research almost ignore each other. Many philosophers dismiss the formalisms of many-valued logics as inadequate, while according to many engineers and computer scientists the theories of vagueness supported by philosophers often lack concrete applications. The reasons for the reciprocal disregards are surely manifold: probably not only ignorance and prejudices, but also differences in methodology and research objectives may make their contributions to it.

The curious fact, that philosophers and computer scientists are mainly working separately from each other, shall be the point of departure for this diploma thesis, which is intended to present both points of view, pointing out the strong points and problems of every theory as well as showing possible ways of combing philosophical approaches with fuzzy logic. That is also the reason for the choice of the title 'Fuzzy logics in the context of theories of vagueness'.

Concretely, the aims of the thesis are:

- to provide an overall view of the ongoing discourse in analytic philosophy and to discuss the most important theories of vagueness, giving a useful classification;

- to discuss fuzzy logics, the approach mostly supported by computer scientists;

- showing different possibilities of how to derive fuzzy logic from the first principles of approximate reasoning.

Point of departure for my diploma thesis was Rosanna Keefe's book *Theories of Vagueness* [Kee00] upon the advice of my supervisor, Christian G.Fermüller, in March 2006. Shortly after, Steward Shapiro's book *Vagueness in Context* [Sha06] (see also subsection 3.5.5) was published, another book recommended by my supervisor as an introductive literature to enter the philosophical debate on vagueness, but at that time I had already begun my work, so that it would have been time-consuming to change the structure of my thesis, which is as follows:

Chapter 1 addresses the problems arising in the presence of vagueness. After a short discussion about vagueness in our everyday life, I will explain what vagueness is for my purpose, describing its most important characteristics proposed by Keefe [Kee00]. Afterwards a short definition of multi-dimensional and higher-order vagueness is given, followed by a list of related concepts from which vagueness has to be distinguished. I will conclude the chapter with some basic considerations about reasoning in the presence of vagueness.

Chapter 2 is supposed to summarize the most important criteria that should be met by any theory of vagueness. Hereby I will mainly follow Keefe [KSe02], [KSe02] and Fermüller [Fer03].

The aim of chapter 3 is to present the philosophical discussion on vagueness. For this purpose, I will present the main ideas of at least one defender of the epistemic approach [Wil94], [Wil], of the gap theories [Kle52], of the degree theories [Mac], of supervaluationism [Kee00], [KSe02] and of the pragmatic approach [Lew83], [Lew69], explaining the point of view of at least one defender of each theory of vagueness. This classification for theories of vagueness is obviously not the only possible one. The plurality of the various contributions to the on-going discourse on vagueness allow a lot of different classifications and it is a big challenge to find a useful classification.

Chapter 4 deals with fuzzy logic. This degree theoretic approach of approximating reasoning in the presence of vagueness has produced many different formalisms. I will discuss Petr Hájek's basic logic BL (see [Háj98], [Háj]) and three of its most important extensions: Łukasiewicz logic (Ł), Gödel logic (G) and Product logic ($\prod$).

Chapter 5 explain different approaches which try to derive fuzzy logics from the first principles of approximate reasoning. I will mainly concentrate on Christian Fermüller's approaches described in [Fer03], [Fer04], [FP03], [FK] and [Fer07].

Chapter 6 summarizes very briefly the results of this paper.

Chapter 1

Basic Concepts

Einerseits ist klar, dass jeder Satz unserer Sprache 'in Ordnung ist, wie er ist.' D.h., dass wir nicht ein Ideal anstreben: Als hätten unsere gewöhnlichen, vagen Sätze noch keinen ganz untadelhaften Sinn und eine vollkommene Sprache wäre erst von uns zu konstruieren. -Anderseits scheint es klar: Wo Sinn ist, muss vollkommene Ordnung sein. -Also muss die vollkommene Ordnung auch im vagsten Satze stecken.

Ludwig Wittgenstein [Wit77, p.32]

Introduction

Does one grain of sand make a heap? Do two grains of sand make a heap? Do three grains of sand make a heap? Do ten thousand grains of sand make a heap? Everyone will agree that a single grain of sand does not make a heap while ten thousand grains of sand do. However, how many grains of sand are necessary so that we call a certain quantity of grains of sand a heap?

Is a man with one hair on his head bald? Is a man with two hairs on his head bald? Is a man with three hairs on his head bald? Is a man with hundred thousand hairs on is head bald? Obviously a man with one hair on his head is bald just as a man with two or three hairs on his head, while a man with hundred thousand of hairs distributed properly is definitely not bald. But there are some cases for which it can be difficult for us to decide whether a man is bald or not. In our everyday language we would typically define him as neither clearly bald nor clearly not bald. But how many hairs must a man have to be considered as (clearly) bald or as (clearly) not bald? What is the difference between men who are clearly bald and those who are not?

Is a man measuring 1.60m tall? Is it a man measuring 1.75m? What about a man measuring 1.90m? If we consider the term 'tall', we would all agree that persons measuring 1.60m are short while people measuring 1.90 are tall, but a man of 1.75m height is neither clearly tall nor clearly short. That means that there are people of a certain height for whom it is not easy to decide whether they are tall or not. But where is the border between short people, people that are neither clearly short nor clearly tall and those who are tall? Can we define a precise height that indicates the minimal height of a tall person? Does the information that someone is neither tall nor not tall have a meaning to us?

The questions mentioned above seem nearly impossible to be answered because they involve vague predicates. But these examples are not special cases, on the contrary, most expressions in our language are vague, a fact, that is illustrated very well by the cartoon shown in Fig.1.1, in which a woman describes suspect using the vague predicates 'tall', 'young' and 'thin'. Despite the great amount of vague predicates our communication works and our vague expressions make sense, an observation that was made also by the famous Austrian philosopher Ludwig Wittgenstein, as the citation at the beginning of the chapter shows.

Fig. 1.1: Vague expressions are frequently used in our natural language.[1]

Often, our expressions are vague even if they pertain to concrete observable properties of physical objects. Let us consider, for instance, the color 'red'. It is perfectly obvious, since colors form a continuum, that there are shades of colors for

[1] The cartoon is taken from the website http://www.cartoonstock.com.

which it is difficult for us to call them simply red or not red, while for others we can undoubtedly say whether they are red or not. All these categorizing problems do not arise because we are ignorant of the meaning of the words, but because the meaning of these words are vague. As the precedent example demonstrates, graduality seems to be a feature which is closely connected with vagueness, but it has be pointed out that these are two different concepts.

But what is vagueness? How can we define it? What are the characteristics of vague predicates?

In this chapter I try to give an answer to these questions. For this purpose, I will discuss three main characteristics of vague expressions proposed by Rosanna Keefe [Kee00], illustrating them by clear examples. Afterwards a short definition of multi-dimensional and higher-order vagueness is given, followed by a list of other phenomena from which vagueness has to be distinguished. At the end of the chapter I will reflect on reasoning in the presence of vagueness.

1.1 Vagueness

There are a lot of different ideas of how to treat vague predicates and of how to formalize reasoning in the presence of this type of predicates, but before entering this debate, I want to give a definition of vagueness, which is useful for the purpose of this thesis. The definitions of vagueness suggested in literature are numerous and this shows that the intuitions about vagueness differ greatly and that the notion of the term 'vagueness' is not perfectly clear.

I decided to mention three main characteristics proposed by Rosanna Keefe [Kee00] to define vague predicates. These characteristics are shared by all vague predicates and are closely connected with their vagueness. The advantage of this definition is that most of the various parties debating about vagueness largely agree about it. However, one has to bear in mind that these characteristics are not completely independent from each other, on the contrary, one characteristic implicates the other and vice versa.

1.1.1 Borderline cases

A first characteristic of vague predicates is that they admit borderline cases. Borderline cases are cases to which the predicate in question neither (clearly) applies nor (clearly) does not apply.

Paradigm examples of vague concepts that admit borderline cases are color predicates. Of course, there are situations in which all objects have a definite color, but in most contexts no definite color can be identified. Consider the sun at dawn that often is neither (clearly) red nor (clearly) not red. Often its color seems to be a mixture of red and orange. It is not possible to find the exact point at which the change of color begins.

Fig. 1.2 shows the different greyscales, starting with the color 'black' and ending with the color 'white'. But where is the exact point that separates these two colors from each other?

Fig. 1.2: Where is the exact point that separates the color 'black' from the color 'white'?

Baldness is a typical vague concept, too. There are some men who are definitely not bald, others who are definitely bald, while between them there are men who seem impossible to be called bald or not bald.

As mentioned above, also 'tall' is a vague predicate, as some people are borderline tall, which means that they are neither (clearly) tall nor (clearly) not tall.

There are many different ideas concerning the origin of borderline cases and how they have to be treated. To give a brief outlook, I will resume here the point of view of the main approaches, that will be discussed in detail in chapter 3. The epistemic approach described in section 3.1 claims that the indeterminateness if a certain predicate (clearly) applies to a special instance or not is an epistemic problem. Defenders of this theory argue that borderline case predications are always *true* or *false* and that it is our ignorance which makes it impossible for us to make a decision whether a certain predicate applies to a special instance or not. Defenders of gap theories (see section 3.2) on the other hand claim that vague statements apparently lack proper truth values, as they seem to be neither *true* nor *false*. On this approach all borderline cases receive the same truth value *indefinite*. Degree theories (see section 3.3) assign truth values from the unit interval [0,1] to borderline cases where 0 stands for absolute (classical) falsity and 1 for absolute (classical) truth. According to supervaluationism borderline cases can be either *true* or *false*, as discussed in section 3.4. Pragmatic approaches (see section 3.5) define borderline cases as sentences over which there are disagreements among the languages in the clusters.

1.1.2 Fuzzy boundaries

A second, related characteristic of vague predicates is, that they apparently lack well-defined extensions. That means, that there are no sharp boundaries between e.g. the set of people that are (clearly) tall and those who are (clearly) not tall, nor is there an exact point in the color spectrum that separates the color 'red' from 'orange'. Turning back to Fig.1.2 it can be easily noticed that the boundary between the colors 'white' and 'black', or if you want the boundaries between the colors 'white', 'grey' and 'black' are fuzzy, i.e. not sharp.

More generally, the lack of well-defined extensions means, that, if spatial closeness indicates similarity, no sharp line can be drawn between the candidates satisfying some vague predicate and those who do not. Indeed, vague predicates are described as having fuzzy boundaries, while predicates according to classical logic and semantics have to have well-defined extensions.

To illustrate the idea of fuzzy boundaries, let us consider the example of the word 'chair' proposed by Max Black [Bla]. Consulting a monolingual English dictionary[2], a chair can be defined as 'a piece of furniture for one person to sit on.' The definition is very general and there is indeed an extraordinary variety of objects to which the same word 'chair' is applied. There are armchairs, reading chairs, dining-room chairs, kitchen chairs, chairs of different materials, height and size. There are chairs

[2] Collins Cobuild New Student's Dictionary, Harper Collins Publishers, 1997, p.96.

with backs and some without backs. The variety of the application results from the fact that the definition of chair, namely that a chair is a single seat, is compatible with a lot of objects varying in form, size and material. But there can also be found objects, like a lump of wood, for which it is difficult to say whether they are chairs or not. Obviously, there are a lot of objects that are definitely no chairs.

Other examples of vague predicates that do not have sharp boundaries can be found in geography, as pointed out by Achille Varzi [Var01]. Many geographical names and descriptions are vague, for instance, we do not know where the Sahara desert exactly begins and where it ends. It is not clear where the Missouri exactly enters the Mississippi and what the extensions of a city like Rio de Janeiro exactly cover. According to Varzi, these problems do not only arise in geography, but also in history. The baroque era and the Renaissance are periods that most people can distinguish, even though their temporal boundaries may be indeterminate.

A 'more philosophical' and 'more artistic' example for borderline cases and fuzzy boundaries can be found in the famous drawing *Sky & Water I* (see Fig.1.3) of the Dutch graphic artist Maurits Cornelis Escher in which light plays on shadow to morph fish in water into swans in the sky.

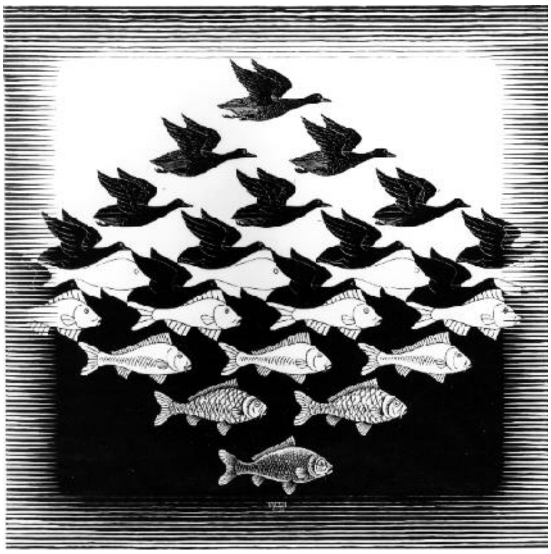

Fig. 1.3: M.C.Escher's painting *Sky & Water I.*[3]

Considering only the white parts of the drawing, at the top the color white serves only as sky, while at the bottom fish can be recognized. In the middle of the

[3] The painting is taken from M.C.Escher's official website http://www.mcescher.com.

drawing it is not clearly recognizable whether it still serves as sky or if there are already fish. Considering only the black parts of the drawing, at the bottom the color black serves only as water, while at the top swans can be recognized. In the middle of the drawing again it is not clearly recognizable whether it still serves as water or if there are already swans.

Thus, speaking in terms of borderline cases, the white parts of the drawing which cannot be definitely recognized either as sky or as fish as well as the black parts which cannot be definitely recognized as water or swans can be interpreted as borderline cases. The boundary between the sky and the fish as well as the boundary between the water and the swans is not are sharp, but fuzzy.

1.1.3 Sorites paradoxes

A third characteristic of vague predicates is that they are susceptible to sorites paradoxes. The name of the paradox is derived from the Greek word *soros* (meaning 'heap') and as explained by Dominic Hyde [Hyd05] it originally referred to a puzzle known as *The Heap*: Would you describe a single grain of sand as a heap? No. Would you describe two grains of sand as a heap? No. Sooner or later the presence of a heap has to be admitted, but where can be drawn the line?

This puzzle of antiquity is now more usually described as a paradox that can be presented as a formal argument, having the following logical structure:

> 1 grain of sand does not make a heap.
> If 1 grain of sand does not make a heap then 2 grains of sand do not.
> If 2 grains of sand do not make a heap then 3 grains do not.
> ...
> If 9999 grains of sand do not make a heap then 10 000 do not
>
> ---
>
> 10 000 grains of sand do not make a heap.

Formally, two versions of the sorites paradox can be distinguished which are explained in the following paragraphs.

Sorites paradox with a series of conditionals

The first possibility is to describe the sorites paradox as a series of conditionals. Let F be the soritical predicate (i.e. 'does not make a heap') and x_n a subject expression in the series with regard to which F is soritical (i.e. 'n grains of sand'), then the sorites paradox can be represented schematically as follows:

Fx_1
If Fx_1 then Fx_2
If Fx_2 then Fx_3
...
If Fx_{n-1} then Fx_n

Fx_n, where n can be arbitrarily large

Sorites paradox with a universally quantified premise

The second variant replaces the set of the conditional premises by a universally quantified premise, the so-called inductive premise. Let n be a variable ranging over the natural numbers, let $\forall n$ (...n...) assert that every natural number n satisfies the condition ...n... and let $\forall n\ (Fx_{n-1} \wedge Fx_n)$ represent a claim of the form $\forall\ n$(if Fx_{n-1} then Fx_n). Then the sorites paradox can be represented by mathematical induction:

Fx_1
$\forall\ n\ (Fx_{n-1} \wedge Fx_n)$

$\forall\ n\ (Fx_n)$

Many different versions of the sorites paradox exist. For instance, we can inverse the heap-paradox, starting with the assumption that 10 000 grains of sand do make a heap. If you take away one single grain, obviously, the result would be a heap too, because intuitively, one single grain cannot make a difference whether or not a number of grains are a heap. But if we now took 10 000 grains of sand and took away grains one by one, we would come to the conclusion that even 1 grain of sand does form a heap.

Wang paradox

A special version of the ancient Greek paradox is the Wang paradox [Dum] defined as follows:

(1) 0 is small;

(2) If n is small, $n + 1$ is small too.

(3) Therefore, every number is small.

In this case as well, the premise (1), the induction basis, is clearly true, while the conclusion (3) is clearly false.

Sorites paradox in our everyday life

However, sorites paradoxes are not mere curiosities. Arguments with a sorites structure are part of our everyday life and can be found everywhere.

It is enough to think of the height of persons. If we argue that someone who measures 1.80m is tall, then a person being a hundredth of an centimeter shorter is tall too. But if you imagine a line of persons, starting with someone who is 1.80m of height, and each of them a hundredth of a centimeter shorter than the previous one you might inevitably come to the conclusion, that even a person that is 1.40m is tall.

The same phenomenon happens with our (natural) language abilities. Immediately after birth a baby is not able to speak a language, but there will be a time when the child does have the ability to speak a language. If we now serially order the life of the child by seconds, starting from the first breath, we all agree that the child cannot speak a language at the first second nor a second later. If we apply this rule arbitrary many times, we come to the conclusion, that the person cannot speak a language even at the age of 50 years.

Another example is the so-called ethical 'slippery slope'. We all agree that it is wrong to abort a baby nine months after conception. I am sure everyone would consent with the following principle: if it is wrong to abort a baby nine months after conception then it is wrong to abort a baby nine months minus one second after the conception. Continuing this reasoning we would come to the conclusion, that abortion is wrong even immediately after conception. But, on the other hand there are a lot of persons who consider abortion as acceptable up to a certain moment after conception, because for them up to this specific moment, the future baby is 'only' an accumulation of cells and not a human being.

In the field of jurisdiction vagueness is also a relevant aspect. On the one hand, there are expressions like 'minor' or 'is of responsible age' that draw lines arbitrarily. On the other hand, there are expressions like 'can be aborted' that have to be combined with vague concepts like 'is a person'.

Possible ways of responding to the sorites paradox will be presented in the next chapter.

1.1.4 Comparatives

Apart from the predicates mentioned in the examples above, comparatives can be vague as well. At first sight this might be surprising. Considering the natural numbers, a unique ordering is possible and the comparative 'is a smaller number than' will not cause any problem. For each natural number we can clearly say whether it is smaller than another given natural number or not. Considering the comparative 'older than' the case becomes already more difficult because there can be borderline instances of the comparative due to the fact that we might not know what to count as the instance of birth of someone. It would be possible to count as birth the moment of the first breath, the moment of the first crying or the moment when the baby comes out of the womb. As a consequence we cannot always say whether the birth of someone is before or after the birth of someone else.

1.1.5 The 'definitely' operator

When dealing with vagueness, it seems to be useful to have a possibility to express formally that a given predication is or is not of borderline status. Crispin Wright [Wri, p.229] actually claims that 'when dealing with vagueness, it is essential to have the expressive resources afforded by an operator expressing *definiteness* or *determinacy*' and Rosanna Keefe [Kee00, p.28] adds that 'we will fail to fulfil the central tasks of a theory of vagueness unless we introduce the D operator.' This operator, also called 'definitely' or 'determinately' operator, is defined as follows: Dp holds when p is determinately or indeterminately *true*. The I operator holds when p is indeterminate or borderline and is equivalent to $(\neg Dp \ \& \ \neg D\neg p)$.

As I will discuss in subsection 1.3, the D operator can also be used to treat the phenomenon of higher-order vagueness.

1.2 Multi-dimensional vagueness

The height of a person is measured by a single continuous parameter, but many, if not most vague predicates are influenced by more than one parameter. If we use the term 'big' to describe a person, it depends at least on two parameters, namely on height and volume, and is therefore (at least) two-dimensional. Nevertheless, the term 'big' seems to be a quite simple example. There are other predicates like 'nice', 'ugly' or 'dangerous' for which it is definitely more difficult to say on which and on how many parameters they depend on.

However, the three features mentioned in section 1.1 which characterize vague predicates are also valid for multi-dimensionally vague ones.

- Multi-dimensionally vague predicates admit borderline cases. Let us consider the term 'friendly'. Some persons scoring well in some relevant aspects but not in others are neither (clearly) friendly nor (clearly) unfriendly.

- Multi-dimensionally vague predicates do not have sharp boundaries because no uniquely appropriate ordering can be found so that boundary-marking points can be placed.

- Finally, multi-dimensionally vague predicates are also susceptible to sorites paradoxes.

I want to illustrate the validity of the three features of vague predicates (see section 1.1) by a two-dimensional example. We could characterize the 'intelligence' of persons by the number of exercises solved correctly within a determinate time, disregarding other features relevant to being intelligent[4]. In this case the intelligence of a person would depend on two parameters: the number of exercises solved and the time used for solving these exercises. Consider a series of persons differing gradually in this respect, starting with a person who solved all exercises within the minimum of time and ending with a person who was not able to solve a single exercise within the maximum of time. Without doubt the first person who solved all exercises correctly within a minimum time is intelligent, as well as the person who solved all the exercises but one within the minimum of time and the person who solved

[4] Even subjectiveness and context-dependence are eliminated.

all the exercises within the minimum of time plus one second. If you carry on this reasoning you come to the conclusion that every person, even the person who was not able to solve a single exercise within the maximum of time, is intelligent. This result is obviously false according to the established maxims of time and solutions.

1.2.1 Incomparability of multi-dimensionally vague propositions

Another phenomenon closely related to multi-dimensional vagueness is the frequent incomparability of statements that contain vague predicates. Let us consider the multi-dimensional vague predicate 'nice'. Comparing persons to each other with regard to this characteristic, one will certainly find pairs of persons about who there is no doubt about which one is 'nicer' or whether they are equally 'nice'. But there will also be pairs of persons for who it is nearly impossible to decide who is 'nicer', especially if we compare persons to each other who are 'nice' in different ways. If we compare two vague statements to each other, it is even more difficult to decide which statement is 'more true' than the other. It is nearly impossible to decide whether the statement 'the sky is blue' is 'more true' than 'Anna is a friendly person'.

1.3 Higher-order vagueness

As I have mentioned above, vague predicates are characterized by having borderline cases, i.e. cases to which the predicate in question neither clearly applies nor clearly does not apply. The lack of a sharp boundary between the positive and negative extensions of a vague predicate is responsible for the appearance of these borderline cases. Usually, also the set of borderline cases of vague predicates is not sharply bounded, because the existence of a sharp border of borderline cases is as implausible as the existence of a sharp border for the appliance of the predicate. This means, that judging an instance to be a borderline case is not a matter of definite truth or definite falsity. This nonexistence of sharp boundaries of a predicate between its borderline cases and its positive and negative extensions leads to the conclusion that there are borderline borderline cases, so-called second-order borderline cases. Obviously, this argument can be repeated indefinitely, resulting in an unlimited hierarchy of possible borderline cases of different orders. This phenomenon can be called unlimited higher-order vagueness. However, Kit Fine [Fin, p.150] says about higher-order vagueness that 'our intuitions seem to run out after the second or third orders of vagueness. Perhaps this is because our understanding of vague language is, to a large extend, confused. One sees blurred boundaries, not clear boundaries to boundaries.'

Timothy Williamson [Wil94] explains the concept of higher-order vagueness by a concrete example: He claims that the difficulties presented by the question 'When did Rembrandt become old?' are also presented by the question 'When did Rembrandt become clearly old?'. That means that at some time it was unclear whether it was unclear whether Rembrandt was old. The same difficulties arise from the question 'When did Rembrandt become clearly clearly old?'. This point reiterates ad infinitum.

Rosanna Keefe [Kee00, p.27] suggests to use an iterated application of the D and the I operator defined in subsection 1.1.5 to express higher-order vagueness: the lack of sharp boundaries to the borderline cases of the predicate F can be expressed by $(\neg DFx \ \& \ \neg D\neg Fx)$. Second-order vagueness, speaking in terms of the D operator this means that there are borderline cases of 'definitely' F, can be expressed with the statement $\neg DDFx \ \& \ \neg D\neg DFx$.

Summarizing, we could say that 'vague' itself is a vague concept because of the lack of sharp boundaries between the negative and positive extensions of a vague predicate and its borderline cases, between its borderline cases and its second-order borderline cases and so on, creating in this way higher-order vagueness.

A predicate with borderline cases does not necessarily have to lack sharp boundaries between its positive and negative extensions and its borderline cases. However, there is a discussion going on between the scientists whether predicates with sharply bounded borderline cases should actually be considered as vague.

1.4 Related concepts

In ordinary language the term 'vague' is often used in an imprecise and ambiguous way. Consulting an English thesaurus[5] one can find numerous synonyms of 'vague': amorphous, blurred, dim, doubtful, fuzzy, generalized, hazy, ill-defined, imprecise, indefinite, indeterminate, indistinct, lax, loose, nebulous, obscure, shadowy, uncertain, unclear, unknown, unspecified, woolly.

As these synonyms demonstrate, the adjective 'vague' often seems to refer to the large sphere of the 'unspecific'. Therefore it is important to emphasize that for my purposes vagueness should not be mixed up with the following concepts:

1. Underspecifity
2. Ambiguity
3. Context dependence
4. Probability
5. Generality

Underspecifity

Vagueness, as understood here, is not underspecifity. If someone replies to the question 'How many bottles of wine do we have in our cellar?' with 'More than five', the answer is indeterminate but not vague in a philosophical relevant sense. Although 'more than five' is only a vague hint at the number of bottles in the cellar, the statement fixes an exact boundary and it is not susceptible to sorites paradoxes. According to that the apparent vagueness in this case depends only on the lack of precise information.

Ambiguity

Vagueness should not be mixed up with ambiguity. A word or a phrase that is ambiguous has at least two specific meanings. If we hear the sentence 'Jack went to the bank', it can mean that Jack went to a river bank to relax or that he went

[5] Pons Collins English Thesaurus in A - Z Form, Harper Collins Publishers, Glasgow, 1992, p.773.

to a bank to take his money there. In this case, the English language denominates two different concepts (a bank to sit down and a bank to take the money there) with the identical sequence of letters and the same sequence of sounds. Only from the context in which a person says this sentence we can understand which bank is meant. In other natural languages these two different concepts can be denominated with two different expressions, like it is in Italian. The bank to sit down is called 'panchina', while the bank to bring the money there is called 'banca'.

However, for us vagueness and ambiguity are two different phenomena. The contrast between vagueness and ambiguity is obscured by the fact that most words are both, vague and ambiguous. The word bank - in both indicated senses - is a vague term: it is not only ambiguous, but also a vague term. Analogically to the example of the 'chair' (see subsection 1.1.2), it is not well-defined which objects can be called a bank to sit down as it is not clear what the precise extensions of a money bank are.

Context dependence

Vagueness cannot be understood by only concentrating on context-dependence, which often helps to eliminate ambiguity as the example of the ambiguous expression 'bank' demonstrated.

Obviously, a lot of terms have different extensions or meanings in different contexts. The extension of the term 'tall' is supposed to vary from context to context. Speaking about the height of persons, a person of 1.90m is tall, but considering the height of mountains a mountain of 1.90m is negligibly small. On the other hand, even if one fixes the context, the term 'tall' remains vague, with borderline cases and fuzzy boundaries and is still susceptible to sorites paradoxes.

Similarly, the word 'now' is not vague only because its reference depends on the time of utterance. Conversely, vagueness remains even when the context is fixed.

Probability

Vagueness should not be mixed up with probability. For instance, if the weather forecast says that it is going to rain in the afternoon with a probability of 50%, this probability of 50% does not indicate vagueness.

To emphasize the difference between these two concepts we can use the example of a dice. If we consider the statement 'Next time Anna dices a number higher than three', we can say, that this will happen with a probability of 50%. If we consider on the other hand the statement 'Most of the time Anna dices a high number', we cannot assign any probability to this statement, because it involves at least two vague predicates: 'most of the time' and 'high number'.

Generality

Furthermore, vagueness should not be identified with (excessive) generality. As mentioned above, 'chair' is a vague term (see subsection 1.1.2). As it is a general word, the expression 'chair' can be (clearly) applied to a lot of objects differing in size, material or shape, as there are a lot of objects like a lamp which are definitely no chairs. But we can also find objects for which it is difficult to say whether they are chairs or not. This incapability to decide whether a certain object is a chair or not arises because the term 'chair' is a vague term and therefore lacks well-defined boundaries. It does not arise because 'chair' is a general word.

A general term does not necessarily have to be a vague term. E.g. the mathematical concept of 'prime numbers' show thats a term can be general without being vague. The set of prime numbers is exactly defined as it is given by all natural numbers which have exactly two divisors: 1 and itself. Also expressions of pragmatic generality like 'more than once' are not necessarily vague, while in contrast 'many times' is a vague expression.

1.5 Reasoning in presence of vagueness

Without doubt, vagueness forms part of our life: It is present in nature, in geography, in our language, in short, it is present in every part of the world that surrounds us. Usually, in our daily life vagueness does not cause any problems for us, as vague predicates do have a certain meaning for us that everybody can understand which is satisfactory to us.

However, for computer scientists and logicians, who usually aim to automate information processing, vagueness is a problem because they have to find a way to formalize reasoning in the presence of vague predicates. For this purpose they have to create a theory of vagueness which means, that they have to identify a formal logic, including a semantics for a vague language. But what is the best semantic treatment for vague terms? What is the best logic involving vague expressions?

The simplest approach would certainly be retaining classical logic and semantics. At the core of classical logic and semantics is the principle of bivalence. That means, that in classical logic there are only two truth values, namely *true* and *false*. Therefore every statement is either *true* or *false* and also borderline case predications assume only one of these two values. Retaining classical logic and semantics has a lot of advantages. The syntax and semantics of classical logic is quite simple, but nevertheless powerful. Classical logic has had a lot of success in the past and has been integrated in theories of other domains. But retaining classical logic and semantics carries its own problem when facing vague predicates. Once we are ready to leave classical logic there are a lot of non-classical options.

Before entering the debate on the different theories dealing with vagueness, I will discuss in chapter 2 the most important general criteria which have to be met by any theory of vagueness. Chapter 3 addresses the discourse going on in analytical philosophy, discussing the most important theories, which try to model the phenomenon of vagueness, both inside and outside classical logic. Chapter 4 on the other hand is dedicated to fuzzy logics for which usually opt engineers and computer scientists.

Chapter 2

General considerations concerning theories of vagueness

La ciencia consiste en sustituir el saber que parecía seguro por una teoría, o sea, por algo problemático.

José Ortega y Gasset [OyG05, p.25]

Introduction

According to the famous Spanish philosopher José Ortega y Gasset science consists in substituting knowledge, which seems to be reliable and assured, by a theory, i.e. something problematic. In my opinion this statement mirrors perfectly the difficulties and problems of the attempt to comprehend natural phenomena.

Nevertheless, for the purpose of technical innovation and progress, natural phenomena have to be studied and analyzed in order to create theories based on facts which help to find algorithms that allow machines to mimic human behavior. Therefore this short chapter addresses the explanation of the most important criteria mentioned in literature that a theory of vagueness should meet. Some of these criteria like simplicity or comprehensiveness are very general and are valid for any theory about any phenomenon, while others, like the exigence of finding an answer to the sorites paradoxes, are very specific for theories that deal with vagueness.

2.1 Important criteria

It is nearly impossible to decide objectively and without prejudices which of the competing theories of vagueness described in chapter 3 is the best one because it is not easy to compare them to each other. One of the reasons why comparing different theories of vagueness is so difficult is the fact that the defenders of the various theories have different points of view concerning the origin of vagueness. In some theories, vagueness is a purely linguistic phenomenon, in others it is an epistemic or an ontological phenomenon. An further reason why comparing the various theories causes problems is the disagreement among the researchers whether classical logic and semantics should be preserved at all costs or could be abandoned, generating a lot of non-classical options.

However, in literature different criteria have been discussed for evaluating theories of vagueness. In this section I shall examine the most important ones. Obviously, this list will not be complete and one could add other criteria as the problem of vagueness represents also a philosophical problem and is therefore partly a question of beliefs. Some of the mentioned criteria will be very general and suitable for any theory about any phenomenon; others are very specific for theories that deal with vagueness. Besides, it has to be considered that not all of the criteria are completely independent from each other and sometimes a trade-off has to be made. Moreover, certain criteria can be more important in some theories of vagueness while in other they are less important.

2.1.1 Pre-philosophical judgements and intuitions

We all have a lot of pre-philosophical judgements and intuitions that influence our everyday reasoning in the presence of vague statements. A theory of vagueness should preserve as many of these pre-philosophical judgements and intuitions as possible. This criterion is obviously a very general one.

It can be possible that a certain theory has to give up some of these widely hold opinions, for instance, for the sake of theoretical benefits like simplicity. It is too quick to assume that a theory of vagueness that contradicts to some of our intuitions has to be rejected. On the other hand, a theory that preserves all our intuitions and pre-philosophical judgements has not to be defended conclusively. Sometimes our intuitions conflict with one another or we cannot trust them in problematic cases.

Besides, the body of relevant opinions and pre-philosophical judgements is obviously nothing predefined and there can be a dispute about which are the opinions that we should save in any case and which are less important.

However, if a theory judges our strong, seemingly upon agreed opinions and intuitions as false or mistaken, the theory should at least explain why so many people judge this situation in a different way.

2.1.2 Compatibility with natural language-use

Another very general criterion for a theory of vagueness arises from the idea that vague expressions have a determined meaning for the group of speakers of a certain language. Sometimes, the meaning of a vague expression can change, depending on e.g. the situation. For human speakers of the same language there is no problem to understand different meanings of an expression in different situations. Considering the term 'tall', everyone will agree that 1.80m is tall for a woman, but rather small for a professional basketball player.

If we want to develop a theory of vagueness, we always have to keep in mind what the different meanings of the vague expressions are and how the meaning of vague expressions can change according to different situations. A theory of vagueness should be compatible with the different meanings of a vague expression and with the way those meanings sometimes change. Furthermore, no theory of vagueness should confer meanings on vague expressions which are contrary to the meanings that a vague expression has for the speakers.

2.1.3 Simplicity and clarity

Adequate implicity and clarity are very important criteria. That seems to be evident and should be met by any theory about any phenomenon. Proceeding this thought, a theory of vagueness should be as simple and as clear as possible without compromising its explanatory value. The explanation of the main ideas of a theory should be understandable and plausible to other persons.

Certainly, it is not always easy to evaluate theories with respect to these criteria.

2.1.4 Comprehensiveness

A theory of vagueness should explain all phenomena regarding reasoning in the presence of vagueness in a plausible way. However, it can happen that a phenomenon usually treated by theories of vagueness is judged by a certain theory as not relevant for reasoning in the case of vague statements. In such a case, the defenders of this theory should at least explain, why the phenomenon is not concerned with vagueness from their point of view and why this phenomenon seems to be relevant for rival researchers, when reasoning in the presence of vagueness.

2.1.5 Relation to classical logic

Classical logic with its two truth values *true* and *false* has been adopted successfully in various domains. Indeed, for many researchers of vagueness it is problematic to abandon classical logic and semantics when creating a theory of vagueness because of its simplicity, power, former success and the integration into theories of other domains.

Although it seems desirable to preserve classical logic, it is not obligatory to retain classical syntax and semantics in its entirety. There are theories that adopt a certain non-classical logic in order to formalize reasoning in the presence of vagueness. However, as classical logic is successful to such an extent, theories that opt for non-classical syntax and semantics should explain why classical-logic was abandoned and does not work for this theory.

Summarizing, one can say that classical logic might be revised, but any deviation from it must be well motivated.

2.1.6 Borderline cases

In the previous chapter (see subsection 1.1.1) vague predicates were characterized by having borderline cases. Borderline cases are instances to which the predicate in question neither (clearly) applies nor (clearly) does not apply.

Obviously, we might expect a theory of vagueness to deliver an answer the question of how borderline cases should be treated. The possibilities are numerous.

Retaining classical logic, borderline cases are either *true* or *false*. This is exactly what defenders of the epistemic approach, discussed in section 3.1 claim. However, according to this theory sometimes we do not or cannot know, if the borderline cases are *true* or *false*, because we are ignorant of the fact where the boundaries are.

Similarly, also in the pragmatic account (see section 3.5) each precise language of the cluster of similar languages used by the community of speakers classifies borderline cases as being *true* or *false*, whereas different languages of the cluster can classify borderline cases of a certain predicate differently.

Also according to the supervaluational approach (see section 3.4), by making precise the predicate in question borderline cases of this predicate are either *true* or *false*. It is a characteristic feature of borderline cases that they can be classified differently by different precisifications.

But, as mentioned above, classical logic does not have to be preserved at all costs and there can be a lot of non-classical solutions to the problem of borderline cases. There are theories in which a predication for a borderline case is both *true and false*. They admit so-called truth value gluts. In other theories borderline predications are *neither true nor false*. They admit so-called truth value gaps. Alternatively, a third truth value can be introduced which allows to classify borderline cases as *neutral*, *indeterminate* or *indefinite*, leading to a three-valued logic, like Michael Tye's [Tye] logic described in section 3.2. Degree theories, that will be discussed in section 3.3, admit truth values from the real unit interval [0,1]. Borderline cases assume some real truth value between 0 and 1. Also fuzzy logics ,which will be discussed in chapter 4, admit truth values from the real unit interval [0,1], where 0 means absolute falsity and 1 absolute truth, while the values in between are assigned to borderline cases. From this point of view, fuzzy logic can be considered a degree theory.

2.1.7 Solution to sorites paradoxes

One important characteristic of vague predicates is their susceptibility to sorites paradoxes (see subsection 1.1.3). Every serious contribution to the discourse of vagueness should provide a solution to this kind of paradox.

As I already have discussed in the previous chapter, sorites paradoxes have - considering the version of the sorites paradox with the universally quantified premise - the following structure:

(1) Fx_1

(2) For all n, if Fx_{n-1} then Fx_n

(3) Fx_n

where n is a variable ranging over the natural numbers, F is the soritical predicate and x_n is a subject expression in the sequence with regard to which F is soritical. The two premises (1) and (2) of a sorites paradox seem both *true*, but nevertheless for a suitably large n the putative conclusion (3) appears *false*.

Remembering the example of the heap of sand, we said, that

(1) one grain of sand does not make a heap,

(2) for all i, if i grains of sand do not make a heap, also $i + 1$ grains do not make a heap,

(3) therefore, even 10000 grains of sand, (distributed properly), do not make a heap of sand.

Intuitively in this example the premises (1) and (2) are *true* as well, while the conclusion (3) is obviously *false*.

There are various responses to soritical reasoning. I will discuss four particulary important possibilities, considering the version of the sorites paradox with the universally quantified premise:

- One could deny the validity of the argument and claim that the conclusion does not necessarily follow from premise (1) and the universally quantified premise (2). At first sight this seems to implicate that absolutely fundamental rules of inference have to be given up, as *modus ponens*[1] is denied, the only rule of inference that is used for this argument. But there is another possibility: alternatively, considering the sorites paradox with the series of conditional, we could accept each step on its own, but claim, that too many steps of inference can put into question the truth of the conclusion.

- It is possible to put into question the strict truth of the inductive premise (2). As a consequence, since classical logic requires sharp boundaries of predications, within a classical framework, there has to be an n so that '$F_n \wedge \neg F_{n+1}$'. In a nonclassical framework there are a lot of other ways of questioning the

[1] *Modus ponens* is a rule of inference which, from a given conditional sentence and its antecedent, allows you to conclude its consequent.

strict truth of the inductive premise. These possibilities will be mentioned when discussing the various particular theories in detail.

- We can accept the validity of the argument and the truth of the inductive premise (2), but contest the truth of premise (1) or the falsity of the conclusion (3).

  Let us consider the sorites paradox (H+) with the two premises 'One single grain of sand is not a heap' and 'adding one grain of sand will not turn it into a heap'. Accepting the argument and the two premises leads to the conclusion that there are no heaps. But accepting the conclusion of every sorites paradox creates problems. This can be illustrated by the example of the sorites paradox (H-) with the two premises 'ten thousand grains of sand make a heap' and 'removing one grain of sand does not change anything about the fact that it is a heap'. Accepting the argument and these two premises implicate that even the solitary last grain of sand is a heap. This conclusion is incompatible with the conclusion of (H+).

  A solution could be to deny the initial premise of (H-). This implicates that there are no heaps at all as (H+) shows us. As a consequence it is not true that ten thousand grains of sand make a heap.

  If we argue this way, at least one problem remains: How can we decide which of a pair of sorites paradoxes is valid? Why there are no heaps rather than everything being a heap? The necessity to decide this in turn leads to the fact that vague predicates do not have any serious applications because they apply either to nothing ('is a heap') or to everything ('is not a heap').

- We can accept premise (1) and the inductive premise (2) and deny the conclusion (3). Accepting the validity of the argument and the two premises but denying the conclusion implicates that there cannot be any coherent logic governing vague language, because semantic rules dictate the two premises to be true and they dictate the conclusion to be false. In this case sorites paradoxes reveal the incoherence of rules governing vague language.

2.1.8 Higher-order vagueness

As mentioned before, higher-order vagueness arises from the fact that 'vague' itself is a vague concept. As a consequence it lacks sharp boundaries between its positive extensions and negative extensions and the borderline cases. This leads to the conclusion that there are 'borderline borderline cases', so-called 'second-order borderline cases'. This reasoning can be repeated infinitely, resulting in an unlimited hierarchy of borderline cases of different order, a phenomenon called 'higher-order vagueness'.

The observation that 'vague' itself is a vague concept seems to be obvious, but on the other hand one could claim that 'vague' is not a vague concept, arguing that it is only ambiguous, applied inadequately or too general. Such a point of view has to be proved by incontestable arguments.

In any case, any serious theory of vagueness should address the issue of higher-order vagueness explicitly.

2.1.9 Degree of formalization

The criterion of the degree of formalization and mathematization of a theory is certainly of different importance for philosophers, logicians, mathematicians, scientists and engineers. While mathematicians, logicians and engineers tend to search for an elegant, non-trivial and coherent body of mathematical definitions and theorems for the analysis for certain concepts, some philosophers often ignore the need of an adequate mathematical model.

2.1.10 Applicability

Vagueness is a wide field of research and there are a lot of different scientists working on it: philosophers, logicians, mathematicians, engineers and computer scientists. Obviously, not all of them have the same motivations and objectives of research. While some of them may try to find out the reasons why vagueness arises, others would place emphasis on the practical aspects of a theory of vagueness like applicability. When evaluating a theory of vagueness, one should always keep in mind these different motivations.

Especially for computer scientists the practical benefit of a theory of vagueness, its possible applications, may be the most important goal of their research as they want to find solutions for existing problems. Therefore it is useful to evaluate contributions to the vagueness discourse also with regard to their applicability.

2.2 Conclusions

As explained in section 2.1, the criteria for evaluating a theory of vagueness are numerous and manifold and probably it is not possible that one single theory matches all of them perfectly. Therefore one has always to keep in mind the purpose of the theory of vagueness and concentrate on the criteria that are important to reach this goal. The importance of the different criteria may change according to the aim to be reached.

Summarizing, perhaps we could say that theorists who are trying to elaborate a theory of vagueness should preserve as many of our judgements or opinions of various kind as possible and should meet theoretical requirements like simplicity and practical benefit like applicability.

Chapter 3

Theories of vagueness

La filosofia sembra che si occupi solo della verità, ma forse dice solo fantasie, e la letteratura sembra che si occupi solo di fantasie, ma forse dice la verità.

Antonio Tabucchi [Tab94, p.30]

Introduction

Citing the great Italian writer Antonio Tabucchi, philosophy is a discipline which seems to deal with truth only, but perhaps it is about fantasy, while literature is a discipline which seems to deal with fantasy only, but perhaps it is about truth. This statement, whether it is true or not, addresses one big challenge that has to be faced by philosophers: they cannot prove their theories by experiments or facts, because their theories are often at least partly a question of beliefs. On the one hand, this fact makes philosophy so fascinating for many people, but on the other hand, it is a reason why many people do not take it serious.

However, in this chapter I will present the main theories discussed in analytic philosophy which are challenging the phenomenon of vagueness. For this purpose I will discuss the different theories of vagueness in detail, presenting the point of view of at least one defender of each approach, and I will try to give an answer to the question if and in which way these different theories meet the criteria mentioned in section 2.1.

There are various approaches trying to model the phenomenon of vagueness both inside and outside of classical logic. The issue of classifying these theories has been addressed by many philosophers, logicians, psychologists and computer scientists. The plurality of the various contributions to the on-going discourse on vagueness allows a lot of different classifications. Therefore, one of the biggest challenges is to find a useful classification of the theories which points out similarities and the essential differences between the different contributions.

According to Christian G. Fermüller [Fer03] one possible classification of theories of vagueness is the following division into the following five groups:

1. Epistemic account
2. Gap theories
3. Degree theories
4. Supervaluationism
5. Pragmatic account

This classification considers mainly two parameters, namely on the one hand the number of truth values introduced to handle borderline cases and on the other hand the idea of the origin of vagueness. Perhaps it would also be possible to classify with regards to their answer to sorites paradoxes or the degree to which they retain classical logic and semantics.

In the *epistemic approach*, which will be discussed in section 3.1, vagueness is seen as a type of ignorance and fuzzy boundaries occur because we are ignorant about the extensions of vague predicates. As a consequence borderline case predications are always *true* or *false*. It is our ignorance which makes it impossible for us to make a decision whether a certain predicate applies to a special instance or not. The epistemic view retains classical logic and semantics.

Supporters of *gap theories* described in section 3.2 are convinced that classical logic is the only 'correct' logic. As the semantics of classical logic cannot be applied to vague predicates, for researchers supporting this theory no correct reasoning in the presence of vagueness is possible. On a less radical view, vague statements lack proper truth values and in addition to the two classical truth values *true* and *false* there is a third truth value *indefinite* denoting the 'truth value gap'.

Researches supporting *degree theories* discussed in section 3.3 usually are convinced that truth comes in degrees and assign to borderline case predications truth values from some algebraic structure. Most degree theories agree that possible degrees of truth can be identified by real numbers of the closed interval [0,1]. An example for degree theories are fuzzy logics, described in chapter 4.

The *supervaluational approach* presented in section 3.4 retains classical logic but adopts non-classical semantics. Defenders of this theory do not believe that vague predicates actually do have crisp and fixed boundaries. To evaluate a statement correctly, supervaluationists think that we have to consider all possible precisifications of the meaning of the components of a statement. Therefore a statement is *true* if and only if it is *true* on all possible precisifications. As a consequence there can be borderline cases for which it is impossible to decide whether a certain predicate (clearly) applies to it or (clearly) does not apply.

Also the *pragmatic approach*, which will be discussed in section 3.5, retains classical logic and semantics in its entirety. Philosophers supporting this theory, consider vagueness as a pragmatic phenomenon caused by the relation between the use of a language and the language itself. Language itself is precise on this account.

3.1 The epistemic approach

According to the epistemic approach vagueness is a form of ignorance. Vague predicates have sharp boundaries like exact ones, it is only our ignorance which makes it impossible for us to decide whether a vague predicate (clearly) applies to a certain instance or (clearly) does not apply. As a consequence vague predicates can be handled in the same way as exact ones and classical logic and semantics can be retained in its entirety.

One of the most elaborated and sophisticated defences of the epistemic approach can be found in Timothy Williamson's[Wil94] book *Vagueness*. A further very elaborated defence can be found in Roy Sorensen's[Sor01] book *Vagueness and Contradiction*. My discussion of the epistemic approach is mainly based on the ideas of Timothy Williamson.

3.1.1 Timothy Williamson's view of vagueness

Although epistemic theorists are convinced that vague predicates have sharp boundaries, they do not deny that vagueness exists. As mentioned above, they claim that vagueness arises because we humans are ignorant about something that would make us able to understand where the limits of a predicate's extensions fall.

To clarify the concept of ignorance, Timothy Williamson [Wil94, p.185] gives the following example:

> No one knows whether I am thin. I am not clearly thin; I am not clearly not thin. The word 'thin' is too vague to enable an utterance of 'TW is thin' to be recognized as *true* or as *false*, however accurately my waist is measured and the result compared with vital statistics for the rest of the population. I am a borderline case for 'thin'. [...] Then since we do not know that TW is thin and do not know that TW is not thin, we are ignorant of something. Either 'TW is thin' expresses an unknown truth, or 'TW is not thin' does. We do not even have an idea how to find out whether TW is thin, given my actual measurements and those of the rest of the population. Arguably, we cannot know the circumstances that TW is thin or that TW is not thin; in that sense, we are necessarily ignorant of something.

3.1.2 Borderline cases

If the epistemic theory of vagueness was true, each proposition would have a single truth value, *true* or *false.* As a consequence, on the epistemic view of vagueness, a vague predicate cannot be characterized by having borderline cases, if a borderline case is defined as an object that falls neither in the positive nor in the negative extension of a predicate, but in a region, so that vague utterances about these borderline cases are *neither true nor false.* Kit Fine [Fin] uses the expression 'penumbra', derived from the Latin *umbra* (shadow), to indicate the area in which borderline cases fall.

Therefore, in the epistemic view, vagueness and borderline cases have to be characterized epistemically. Accordingly, vagueness does not mean that a vague predicate lacks sharp and well-defined boundaries. Quite the contrary, from the epistemic point of view even vague predicates have sharp boundaries, but we do not - and perhaps even cannot - know where the limits of the positive and negative extension of the vague predicate fall. In other words, also vague utterances in borderline cases are *true* or *false*, but we humans have no idea how to find out which.

Summarizing, Timothy Williamson holds the view that borderline cases have sharp boundaries, but that we do not know where they fall. Many philosophers, logicians, researchers and computer scientists working in the field of vagueness consider this thesis to be unworthy of any serious consideration and therefore reject the epistemic theory of vagueness, arguing that a term is vague if and only if it admits borderline cases for which we do not know whether a predicate (clearly) applies or (clearly) does not apply and a case is said to be a borderline case if our ability to decide it does not depend on our ignorance.

For Timothy Williamson [Wil] supporting this very technical definition of borderline cases means committing a *petitio principii* to the epistemic view of vagueness. According to him an utterance like 'TW is thin' would be typically called a borderline case, but one should not assume automatically without argument that our inability to decide the matter does not depend on our ignorance. But what is the fact of which we are ignorant? For Williamson [Wil, p.271] the answer is obvious: 'we are ignorant either of the fact that TW is thin or the fact that TW is not thin (our ignorance prevents us from knowing it).'

3.1.3 Classical logic and semantics, principle of bivalence

The most obvious argument for the epistemic view of vagueness is probably the fact that classical logic and semantics can be retained and do not have to be revised. As Timothy Williamson [Wil94, p.186] claims, 'classical semantics and logic are vastly superior to the alternatives in simplicity, power, past success and integration with theories in other domains'.

As a consequence, bivalence holds as well. Abandoning bivalence for vague utterances one pays a high price: classical truth-conditional semantics cannot longer be applied and probably not even classical logic. But how is the principle of bivalence to be understood from the epistemic point of view?

The principle does not say that everything is either *true* or *false*. It only claims truth or falsity if and only if something has been said to be the case. Thus, the principle of bivalence says that each proposition has a single truth value, *true* or *false*.

For Timothy Williamson [Wil] every theory of vagueness should respect the principle of bivalence, because according to him it cannot be denied consistently. His argumentation for demanding that the principle of bivalence has to be respected is the following: suppose that Anna is a borderline case of ‘tall’ and ‘Anna is tall’ is neither *true* nor *false*. As ‘Anna is tall’ is not *true*, Anna cannot be tall - if Anna were tall, ‘Anna is tall’ would be *true*. As Anna is not tall, ‘Anna is short’ has to be *true* and ‘Anna is tall’ should to be *false*, which is contradicting the assumption that ‘Anna is tall’ is neither *true* nor *false*.

The *margin of error principle*

It is certainly very advantageous that the epistemic approach of vagueness retains classical logic and semantics and the principle of bivalence.

However, can one seriously claim that there is a last second of our childhood followed by the first second of our adulthood? Is it not absurd to think that there is a precise point on the spectrum where red turns into orange, that the loss of one single hair can turn someone bald or that one millimeter can make the difference between the group of the short and the group of the tall people?

Let us suppose that such precise and sharp boundaries exist also for vague predicates. Why are we not able to find out where they fall? Why are we ignorant of the correct classification of borderline cases?

According to Rosanna Keefe [Kee00] there are a lot of very general and unsatisfying responses to these questions like suggesting that ignorance is our default state and that this lack of knowledge has no need to be explained.

Timothy Williamson [Wil94], by contrast, offers a very specific and detailed explanation of the source of our ignorance. He is convinced that knowledge from perceptual sources is typically inexact because our apparatus of perception has limited sensitivity and cannot guarantee complete accuracy. The concept of the so-called ‘inexact knowledge’ can be made clear by the following examples: our sense of sight can give us an idea about the number of people in a football stadium, but it is almost impossible to find out the exact number of the people in this stadium only by looking at the crowd, without obtaining any other information. Our sense of hearing can give us an idea about the loudness of a noise, but we would not know the exact volume in decibel only by listening. Our sense of taste can give us an idea about the ingredients of some food, but in order to find out the exact consistency we have to use other methods.

Therefore, a certain margin of error is necessary when perceptual sources are used to acquire information. More explicitly, where our knowledge is inexact, our beliefs are reliable only if we leave room for some error. For given cognitive capacities are, reliability increases with the width of the margin. The more accurate the cognitive capacities are, the narrower is the margin needed to achieve a given level of reliability.

This idea can be illustrated very well by the following example. Let us suppose that Anna is borderline 'tall' on the tall-side of the boundary, but only just. Our (true) belief that Anna is tall should not count as real knowledge, because it should not be true just by luck. If our use of 'tall' was only a little bit different, so that Anna is just on the short-side of the boundary, we would still believe that 'Anna is tall', but in this case our persuasion that 'Anna is tall' would be wrong.

According to Williamson this reasoning leads to the following *margin of error principle*:

Margin of error principle 1. *'A' is* true *in all cases similar to cases in which 'It is known that A' is* true*.*

That means, if x and y differ only marginally regarding one characteristic and if we know, that x has one feature that depends on this characteristic, also y has this feature. The following example illustrates this idea very clearly: we know that x has one hair less on his head than y. If we know that x is a bald, than y is bald as well.

The *margin of error principle* should explain our ignorance. Suppose that x consists of n grains of sand and still makes a heap, while y consists of $n-1$ grains of sand and does not make a heap anymore. If we want to know where the positive and negative extensions of the predicate 'heap' fall, we should also know that 'x is a heap and y is not a heap'. But if we know that 'x is a heap' the *margin of error principle* implicates that also y is heap. Thus, we cannot know that y is not a heap. It is impossible for us to know a conjunction of the form 'n grains of sand make a heap and $n-1$ grains do not make a heap'. Summarizing, we should know something that we cannot know if we want to understand where the extensions of a vague predicate are.

3.1.4 Sorites paradoxes

In the epistemic view of vagueness, the argument of the sorites paradoxes is valid, but the strict truth of the inductive premise is put into question (see the explanation of version of the sorites paradox with a universally quantified premise in subsection 1.1.3).

Roy Sorensen [Sor01, p.1] for instance argues that 'the argument is valid by mathematical induction. The first premise is obviously true. The conclusion is obviously false. Therefore, my only recourse is to reject the induction step.'

That means, that vague terms are sensitive to arbitrarily small differences: there will be a millisecond that makes the difference between a child and an adult and there will be a grain that turns a 'non-heap' into a heap.

However, Sorenson [Sor01, p.32] makes another observation regarding the argument of sorites paradoxes:

> Many vague phrases are too complex to ever be thought about. Iterating 'the mother of' a thousand times yields a grammatical predicate of English. No one will have a gestalt switch concerning 'grand^{998}-mother'. Yet we know there are sorites arguments that use 'grand^{998}-mother' as the inductive predicate. The infinite class of sorites argument that are beyond our limits of memory and attention cannot owe their existence to a human penchant. [...] Human beings cannot [...] even grasp those arguments.

In other words, Sorenson claims that sorites paradoxes often come to conclusions that are too long and too complex to be judged by anyone. Reproducing Stephan Shapiro's [Sha06, p.41] example 'such and such a scene is a very interestingly sort of weird ... funny ...' it seems to be obvious that no human being is able to understand this complex sentence.

3.1.5 Higher-order vagueness

Not only borderline cases, but also higher-order vagueness is characterized epistemically by the epistemic approach of vagueness: there are sharp boundaries between the cases to which the vague predicate (clearly) applies and the borderline cases of this vague predicate, as there are sharp boundaries between the cases to which the predicate (clearly) does not apply and its borderline cases. The boundaries only appear fuzzy to us because we are ignorant of the exact boundaries and thus we do not know whether the predicate (clearly) applies or not.

All different orders of vagueness are treated in the same way, i.e. as a matter of ignorance, with classical logic applicable at every stage.

3.1.6 Compatibility with natural language-use

The epistemic approach of vagueness is often criticized because it is not compatible with natural language use as far as it does not preserve a necessary connection between meaning and use. If nature does not draw a border between the negative and the positive extensions of a predicate, there will be a border only if we draw it by the use of this predicate. In the case of vague predicates there is no precise border, because our use only draws a border zone (also called penumbra as explained in subsection 3.1.2) instead of a real border. In other words, a term means what it means because it is used in the way it is used. The epistemic view of vagueness however postulates sharp boundaries even in borderline cases and thus draws lines where our use does not.

However, this view of language is controversial. Defenders of the epistemic approach are convinced, as described in subsection 3.1.2, that even vague predicates have sharp boundaries, but we do not know where the limits of the positive and the negative extensions of the vague predicates fall.

The compatibility with the natural langue-use, as Sven Walter [We05] argues, is fundamental. The reliability of a mechanism often depends just on the fact that it does not react neither in a positive nor in a negative way in the event of borderline cases. The price that we have to pay for a clearly positive or clearly negative reaction is very high, because there would also be false reactions. Sometimes mechanisms that do not react are safer than mechanisms that react in a wrong way.

3.2 Gap theories

As mentioned in the introductory section of this chapter, some defenders of gap theories regard classical logic as the only correct logic. If classical logic cannot be applied to a certain phenomenon, it will be classified as inconsistent with correct reasoning. In other words, correct reasoning in the presence of vagueness seems to be impossible for them.

Less radical defenders of the gap theory defend another approach based on a simple observation: vague statements apparently lack proper truth values, as they seem to be neither *true* nor *false*.

In practice, calculi and formal semantics of these theories correspond to a three-valued logic. In addition to the two classical truth values *true* and *false*, there is a third one, namely *indefinite*, the so-called truth value gap.

There are many different contributions to gap theories from different researchers. One of the most important three-valued logic is the one of Stephen Cole Kleene [Kle52]. Based on the ideas of Kleene, Michael Tye [Tye] developed his three-valued logic that I am going to present in the next paragraphs because it is, as I think, one of the most elaborated ones.

3.2.1 Michael Tye's use of Kleene's three-valued logic

As vague statements in the event of borderline cases seem to be neither *true* nor *false*, Michael Tye introduced a third truth value which is different from *true* and *false* (namely *indefinite*). Strictly speaking, this third value is not a real truth value, but rather a truth value gap. According to Tye there are two reasons from which truth value gaps can result:

- there are gaps due to vagueness when something is said which is neither *true* nor *false* and
- gaps due to failure of reference or presupposition.

The connectives

The introduction of the third truth value *indefinite* for vague propositions that is different from *true* and *false* is not sufficient to solve the problem. Also, the truth values of complex statements have to be determined. Corresponding to the two-valued connectives, Michael Tye [Tye, p.282] proposes the following truth tables for the the three-valued connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\equiv$ (equivalence), where I is denoting *indefinite*, T *true* and F *false*:

p	$\neg p$
T	F
I	I
F	T

Tab. 3.1: Tye's truth table for the connective $\neg$.

	$p \wedge q$		
p \ q	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

Tab. 3.2: Tye's truth table for the connective $\wedge$.

	$p \vee q$		
p \ q	T	I	F
T	T	T	T
I	T	I	I
F	T	I	F

Tab. 3.3: Tye's truth table for the connective $\vee$.

	$p \rightarrow q$		
p \ q	T	I	F
T	T	I	F
I	T	I	I
F	T	T	T

Tab. 3.4: Tye's truth table for the connective $\rightarrow$.

	$p \equiv q$		
p \ q	T	I	F
T	T	I	F
I	I	I	I
F	F	I	T

Tab. 3.5: Tye's truth table for the connective $\equiv$.

Stephen Cole Kleene [Kle52, p.335] maintains that 'these strong tables are uniquely determined as the strongest possible regular extensions of the classical two-valued tables, i.e. they are regular, and have a *true* or a *false* in each position where any regular extension of the two-valued tables can have a *true* or a *false*.'

The rules for the construction of these truth tables follow the rules of Kleene [Kle52] and are comparatively simple:

1. If the truth value of a statement is *true*, then its negation is *false* and vice versa.
2. A conjunction is *true* if both its conjuncts are *true*, and *false* if at least one conjunct is *false*; otherwise it is *indefinite*.
3. A disjunction is *true* if at least one disjunct is *true*, and *false* if both its disjuncts are *false*; otherwise it is *indefinite*.
4. The truth value of $p \rightarrow q$ is the same as the truth value of $\neg p \vee q$.
5. The truth value of $p \equiv q$ is the same as the truth value of $(p \rightarrow q) \wedge (q \rightarrow p)$.

As far as only the two truth values *true* and *false* are involved, the tables correspond to the usual two-valued truth tables, described e.g. in [Háj98].

In Kleene's and hence also in Tye's logic there are no tautologies[1], since two-valued tautologies can take the value *indefinite* in Tye's three-valued logic. For instance, the classical tautologies $p \vee \neg p$, $\neg(p \wedge \neg p)$ or $p \rightarrow p$ are *indefinite*, if p is *indefinite*. At least, classical tautologies cannot be *false* and classical contradictions[2] cannot be *true*; at most they can be *indefinite*. Michael Tye introduced the term 'quasi-tautology' for statements that have no false substitution instance and 'quasi-contradiction' for statements that have no true substitution instance. Like in classical logic, a conclusion is *true* if all premises are *true*.

[1] A tautology is a statement that is necessarily *true* because, by virtue of its logical form, it cannot be used to make a false assertion.

[2] A contradiction is a statement that is necessarily *false* because, by virtue of its logical form, it cannot be used to make a true assertion.

Interpretation of predicates

For the purposes of the formal semantics, Michael Tye suggests a special treatment for vague predicates that he explains by means of monadic vague predicates[3]. The generalization to n-place predicates is straight-forward.

Let F be a monadic vague predicate and D a non-empty domain. To F there are assigned an extension S (the set of objects of which F is *true*) and an anti-extension S' (the set of objects of which F is *false*). One should pay attention to the fact that the sets S and S' are not classical sets, but vague sets.

The concept of vague sets can be explained in the following way: Let x be a borderline F, if there is no determinate matter of fact about whether x is an F or not. A set S is vague, if and only if it has borderline members and there is no determinate matter of fact about whether there are objects that are members, borderline members nor non-members.

With the introduction of vague sets, Tye enunciates the truth conditions for vague proposition as follows: Let x be an individual constant and i_x the object in D assigned to x. Then Fx is *true* if and only if $i_x \in S$, Fx is *false* if and only if $i_x \in S'$ and Fx is *indefinite* if and only if there is no determinate matter of fact about whether i_x belongs to S (or to S').

Interpretation of quantifiers

In Tye's three-valued logic the quantifiers are introduced as follows:

An existential quantification $(\exists x)$ Fx is *true* if Fx is *true* for some assignment of an object of D to x and *false* if Fx is *false* for all assignments; $(\exists x)Fx$ is *indefinite*, if Fx is *indefinite* at least for one assignment and for all remaining assignments.

A universal quantification $(\forall x)Fx$ is *true*, if Fx is *true* for all assignments of objects of D to x and *false* if Fx is *false* for at least one assignment; otherwise it is *indefinite*.

3.2.2 Classical logic and semantics

As mentioned above, classical logic and semantics cannot be retained. A three-valued logic is adopted with a third truth value *indefinite*.

[3] A monadic predicate is a predicate that takes only one argument.

3.2.3 Borderline cases

All borderline cases receive the same truth value *indefinite*, an idea that can create some problems in connection with higher-order vagueness, as discussed in the next paragraphs.

3.2.4 Higher-order vagueness

As I have stated before, Michael Tye eliminated the sharp boundary between the positive and the negative extensions of a vague predicate F by introducing a third category of objects for which it is indefinite whether F (clearly) applies or (clearly) does not apply. These objects receive the third truth-value *indefinite*.

However, difficulties still arise in connection with higher-order vagueness. Even if there is no last object x_i so that 'Fx_i is *true* and Fx_{i+1} is *false*', it seems that there is a last object x_i for which 'Fx_i is *true* and Fx_{i+1} is *indefinite*'. In other words, it seems that the three-valued logic replaces the classical sharp boundary between the positive and the negative cases by two sharp boundaries: one sharp boundary between the positive cases and the indefinite ones and a second sharp boundary between the indefinite cases and the negative ones. Consequently, there still seems to be a single hair that makes the difference between a man who is bald and a man who is borderline bald. This seems to be as implausible as the idea that the removal of one hair makes the difference between a man who is bald and a man who is definitively not bald. However, there is still another problem: the fact that the borderline cases are sharply bounded implies that there is no room for borderline borderline cases and thus for second-order vagueness. In fact, this contradiction is one of the major reasons which prompt a lot of logicians to move from a three-valued logic to many-valued or even infinite-valued logics, which are described in section 3.3. Tye on the other hand adheres to his three-valued logic.

So, how does Tye [Tye, p.290] address the problem of higher-order vagueness? First of all he tries to avoid the sharp boundary between the positive cases and the borderline cases as well as the sharp boundary between the borderline cases and the negative cases, claiming 'that it is not true that there are sharp transitions between the true and the indefinite statements and the indefinite and the false statements', as for him this idea is purely intuitive: according to him the most competent language users will not precisely agree upon where the boundaries are to be drawn in the sequence between the *true*, the *indefinite*, and the *false* statements. Of course this people may specify a precise point if they are forced to assign either *true* or *false* or *neither* to each statement $M_o, M_1, ...M_{100000}$ one after the other. But probably, even one and the same person will not pick up exactly the same points twice.

These considerations are closely connected to David Lewis' pragmatic approach of vagueness which will be described in detail in section 3.5. Lewis claims that there is a cluster of similar precise languages from which the speaker of a language can choose. The different languages of a cluster can classify borderline cases of a certain predicate differently and the same speaker can choose different languages from the cluster in different situations, depending on his beliefs, habits and intentions.

However, Tye is convinced that neither the transition from the *true* to the *indefinite* statements nor the transition from the *indefinite* to the *false* statements is sharp. Thus, for Tye there are statements, for which it is not clear which truth value has to be assigned to them - the semantics does not require that they have to be *true*, *false* or *indefinite*. If there was a sharp boundary between the *positive* and the *indefinite* as well as between the *negative* and the *indefinite* cases, then every statement would either have to be *true*, *false* or *indefinite*, but as this is *indefinite*, there are no sharp transitions.

3.2.5 Sorites paradoxes

Michael Tye's view of higher-order vagueness leads us directly to his response on the phenomenon of sorites paradoxes: for him, both versions of the sorites paradox described in subsection 1.1.3 are valid, but the argument is not coherent. If we consider the version of the sorites paradox with the series of conditionals according to Tye at least one of these conditionals is not *true*. In the case of the sorites paradox with the inductive premise Michael Tye puts into question the strict truth of the universally quantified premise. According to him Fx_i and Fx_{i+1} are *indefinite*, if x_i and x_{i+1} are borderline F. Hence, also the inductive premise 'For every n: If Fx_n, then Fx_{n+1}' and the conditional premises 'If Fx_i, then Fx_{i+1}' are *indefinite*.

3.2.6 Prephilosophical judgements and intuitions

Many critics of Michael Tye's three-valued logic claim that the truth values assigned to some statements are contradictory to our intuitions. Leaving aside the 'quasi tautologies' and the 'quasi contradictories', we come across truth values which seem to not coincide with our intuitions especially in the case of complex statements, like the following example of Sven Walter [We05, p.12] demonstrates: suppose that Anna and Maria are both borderline tall, but that Maria is taller than Anna. In this case, according to Tye's logic, the statement 'Anna is tall and Maria is not tall' is *indefinite* just like the statement 'Anna is tall and Maria is tall'. Intuitively, we would say that the first proposition should be *false*, because nobody who is taller than someone who is tall can be not tall. Statements like 'Everybody who is taller than someone who is tall' are intuitionally *true*, while for Tye they are *indefinite*. He does not provide any satisfactory explanation why our intuitions in this case are completely false.

These ideas remember Kit Fine's penumbral connections (see subsection 3.4.2), which according to supervaluationism have to be respected when a predicate is made precise. Also fuzzy logic requests that semantic constraints and meaning postulates are part of a theory of vagueness.

Another arguable aspect is, that Tye's three-valued logic assigns the same truth value to all borderline cases, even if intuitively 50 grains of sand form more plausibly a heap than 10 grains of sand. As mentioned before, this problem is the reason why many researchers are in favor of a many-valued logic. However, Tye argues that the introduction of a many-valued or even infinite-valued logic does not solve the problem and defends his logic: according to him, the introduction of more truth values, even if it seems to have sense, still implicates that there have to be sharp boundaries between the last conditional that assumes the truth value 0 and the first conditional that assumes a truth value unequal 0 and this is utterly implausible for Michael Tye. It is a feature of vague propositions that there is no determinate matter of fact, at which exact point the truth value changes between a man without hair and a man with one million of hairs on his head. A theory with gradual truth values cannot represent this feature, regardless of the number of truth values that are introduced.

However, Michael Tye [Tye, p.293] claims that the advantage of his theory is due to the fact that

> [...] unlike other prominent semantics, it concedes that the world is, in certain respects, intrinsically and robustly vague; and it avoids, at all levels, a commitment to sharp dividing lines. This position is, I suggest, consonant with both our ordinary, commonsense view of what there is and our pre-theoretical intuitions about vagueness.

3.3 Degree theories

In this section I am going to consider degree theories, i.e. theories of vagueness that introduce, apart from the two classic truth values *true* and *false*, new truth values and therefore adopt a many-valued or even indefinite-valued logic. Rosanna Keefe[Kee00] defines a many-valued logic as a logic which has more than two truth values and which is truth-functional. A logic is truth-functional if the truth value of a compound sentence is determined by the values of its compounds.

This definition is very general and applies to a lot of theories. In this sense, fuzzy logic described in chapter 4 can be seen as a many-valued logic, i.e. as a degree theory. In section 4.4.1 I will discuss this idea in more detail.

Also Michael Tye's three-valued logic described in section 3.2 can be seen as a special case of a many-valued logic, because it introduces a truth value gap that, strictly speaking, represents a third truth value. As Tye's three-valued logic has some difficulties in connection with higher-order vagueness, the introduction of logics with a high or even infinite number of truth values seems to be the logical consequence.

The theory on which I am going to focus in the next paragraphs is the infinite-valued logic of Kenton Machina [Mac] who makes use of Łukasiewicz logic. In this context I will also refer to Petr Hájek's basic fuzzy propositional logic BL (see section 4.2) and his design choices (see subsection 4.2.1), drawing comparisons.

3.3.1 Kenton Machina's use of Łukasiewicz logic

In his contribution Kenton Machina introduces a many-valued logic, taking as truth values the real numbers from the unit interval [0,1], where 0 represents complete (classical) falsehood and 1 complete (classical) truth. The values in between are used to characterize borderline cases. The higher the real number, the 'truer' the proposition. In other words, the truth value of the proposition 'Anna is bald' is growing, if the number of Anna's hairs is declining.

As I have discussed before, Tye's three-valued logic has been criticized several times, because it assigns the same (non-classical) truth value *indefinite* to all borderline cases, even if intuitively a man with 10 hairs on his head is balder than a man with 100 hairs. Machina infers from that observation, that there is a sort of continuum of borderline cases and tries to give consideration to this intuition by suggesting a continuum of truth values, with an ordering relation defined on it. As mentioned before he uses the unit interval [0,1].

However, for Machina there is no unique solution for the assignment of truth values to the borderline cases, but he offers several reasonable possibilities: as the assignments of truth values should not be completely arbitrary, Machina suggests to use empirical investigations to find out some patterns of the common man's classification of borderline cases which could help us to assign truth values to propositions. In this way we can receive an assignment that is neither completely determined by the empirical data, nor completely arbitrary, but - as Machina [Mac, p.188] claims - 'fortunately, the assignment of exact values usually does not matter much for deciding on logical relations between vague propositions; what is of importance instead is the ordering relation between the values of various propositions.'

The connectives

In the next paragraphs I am going to discuss the definition of the connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\equiv$ (equivalence) proposed by Kenton Machina. His system was originally proposed by Jan Łukasiewicz, but Łukasiewicz did not see it as a logic of vagueness and interpreted the values as probabilities.

In the following explanations $|p|$ stands for the truth value of the proposition p.

- **Negation $\neg$**

  As it seems natural for Machina that as p gets truer, $\neg p$ gets falser and vice versa, he defined the negation $\neg$ in the usual way:

$$|\neg p| = 1 - |p| \tag{3.1}$$

- **Conjunction** $\wedge$

  Machina sees the conjunction $\wedge$ in a fairly classical way; i.e. if one conjunct is *false*, the whole conjunction becomes *false*, no matter how *true* the other conjunct is. Therefore he requires:

$$|p \wedge q| = \min(|p|, |q|) \tag{3.2}$$

  If someone allows $|p \wedge q| > \min(|p|, |q|)$, then the conjunction of the premises could be *truer* than the *falsest* premise in the argument.

- **Disjunction** $\vee$

  Similarly, the disjunction $\vee$ is defined in a fairly classical way:

$$|p \vee q| = \max(|p|, |q|) \tag{3.3}$$

  This definition, however, puts into the question the law of the excluded middle as $p \vee \neg p$ is not necessarily always completely *true*. In most of the cases the value is an intermediate (i.e. non-classical) one. However, as the disjunction cannot become more than half-way *false*, Machina claims that we have a 'law of the more or less excluded middle'.

- **Implication** $\rightarrow$

  The implication $\rightarrow$ is defined in the following way:

$$\begin{aligned} |p \rightarrow q| = \quad & 1 & \text{if } |q| > |p| \\ & 1 - |p| + |q| & \text{if } |q| \leq |p| \end{aligned} \tag{3.4}$$

  Kenton Machina chose this definition for the implication instead of the classical definition $|p \rightarrow q| = |\neg p \vee q|$ because according to him, the classical definition could lead to cases that are contradictory to our intuitions, as the following example demonstrates: intuitively, the formula $p \rightarrow p$ should always be *absolutely true*, but, adopting the classical definition of the implication, if $|p| = \frac{1}{2}$, then also $|p \rightarrow p| = \frac{1}{2}$.

But what are Machina's reasons to define the implication in this way?

There are mainly two considerations which led Machina to adopt this definition:

1. First of all, the classical connection between $\rightarrow$ and logical inference should be preserved. That means, that an argument should be completely *true*, if its logical form requires truth-preservation, i.e. if it instantiates an argument scheme which has the property that its conclusion must be at least as *true* as the *falsest* premise. Accordingly, Machina's definition of the implication coincides with the classical definition of the implication, as far as only the classical truth values 0 and 1 are concerned. Generalizing this classical principle, a conditional has to be completely *true*, if the consequent is at least as *true* as the antecedent.

   These conditions leads to the following restriction:

$$\text{if } |p| < |q| \text{ and } |r \rightarrow p| = 1, \text{ then } |r \rightarrow q| = 1 \tag{3.5}$$

2. Secondly, Machina defines that

$$\text{if } |q| < |p| \text{ and } |q| \neq |r|, \text{ then } |p \rightarrow q| \neq |p \rightarrow r|. \tag{3.6}$$

   Alternatively, one could claim that $|p \rightarrow q|$ is uniformly 0 for all values of $|q|$ such that $|q| < |p|$. The disadvantage of this definition is that even the many-valued system cannot make any distinction between arguments which are nearly truth-preserving and those which are not truth-preserving at all and that is the reason why Machina rejects the second alternative. To make a distinction between arguments which are nearly truth-preserving and those which are not truth-preserving at all, he suggests the following conditions:

$$\text{if } |p| < |q| \leq |r|, \text{ then } |r \rightarrow p| < |r \rightarrow q| \tag{3.7}$$

$$\text{if } |r| \leq |p| < |q|, \text{ then } |q \rightarrow r| < |p \rightarrow r| \tag{3.8}$$

Considering the conditions 3.5, 3.6, 3.7 and 3.8, the definition of $|r \rightarrow p|$ given in 3.4 can be easily found. We only know that the conditional has the value 1, if $|r| \leq |p|$. In case of $|p| \leq |r|$ the value of the conditional has to increase if $|p|$ increases and to decrease if $|p|$ decreases, while $|r|$ is kept fixed at some intermediate value.

- **Equivalence** $\equiv$

  The equivalence $\equiv$ is defined in the classical way:

$$|p \equiv q| = |(p \rightarrow q) \wedge (q \rightarrow p)| \tag{3.9}$$

In this context I want to refer to Petr Hájek's design choices for his basic fuzzy propositional logic described in subsection 4.2.1. As we will see, the motivation for his choices are quite similar and therefore also his definition of the connectives.

Kinds of vagueness

In order to add quantifiers to his logic, Kenton Machina employs a generalized set theory described by Lofti Zadeh and Joseph Goguen which differs from an ordinary set theory by the fact that the membership relation is a gradual relation. Formally, this is achieved by mapping an ordinary set into an index set. The degree of the membership of an element of the domain is indicated by the element of the index set to which the element is mapped.

According to Machina, this set theory has to be modified to meet the requirements of vagueness. Both, the reasons for the modifications and the modifications themselves are going to be discussed in the next paragraphs.

Machina claims that in a natural language there are at least three different kinds of vagueness:

1. **Conflict vagueness**, which occurs if the application of a predicate is governed by contradictory semantical rules.

2. **Gap vagueness**, which occurs if the semantic rules fail to say anything about whether certain objects fall into the positive or the negative extension of a predicate.

3. **Weighting vagueness**, which occurs if the relevant properties of an object indicate only to a limited extend whether the object can be placed in the extension of the predicate or not.

Machina's conclusions from these observations are the following:

1. To represent conflict vagueness, Machina allows a given predicate letter to be assigned more than one partial extension. Each extension is intended to represent the extension determined by one nonconflicting set of criteria for the application of the predicate denoted by the predicate letter.

2. To represent gap vagueness it has to be possible that the function which assigns extensions to predicate letters, do not necessarily need to indicate whether an element of the domain falls into the fuzzy extension of a predicate letter.

3. To represent weighting vagueness, it is necessary that the predicate letter can have fuzzy extension so that some members of the domain fall in the extension of a given predicate letter only to a limited extend.

However, as long as we only want to find out the truth values of vague utterances, these definitions are not important. But if we want to reveal semantic relationships, the different kinds of vagueness are an important aspect.

Predicates and quantifiers

In detail, Machina defines that the interpretation, **M**, of his language consists of the following:

- A non-empty set **D**, the domain of the interpretation **M**.
- The unit interval **I**, the index of the interpretation **M**.
- The set **E** of possible extensions, i.e. the set of all ordered pairs, consisting of an n-place predicate letter ($n \geq 1$) and an n-tuple of elements of **D**, where the number of places in the predicate equals the number of places in the n-tuple.
- A finite set **F** of predicate interpretation functions. Each member of this set is a function having a subset of **E** as domain and a subset of **I** as range. For each predicate letter at least one element of **F** has in its domain an element of **E**, which has the predicate letter as first member.
- A denotation function **d** which assigns to each individual constant an element of **D**.
- A valuation function **v** such that
  - **v** assigns to each sentence letter a value in [0,1]
  - **v** assigns to each n-place predicate letter Φ followed by n individual constants $a_1, a_2, ..., a_n$ a value in [0,1] according to the following conditions
    * If no element **f** of **F** interprets Φ for $\langle a_1, a_2, ..., a_n\rangle$, then $\mathbf{v}(\Phi(a_1, a_2, ..., a_n)) = 0.5$
    * If only one element **f** of **F** interprets Φ for $\langle a_1, a_2, ..., a_n\rangle$, then $\mathbf{v}(\Phi(a_1, a_2, ..., a_n)) = \mathbf{f}(\langle\Phi\langle a_1, a_2, ..., a_n\rangle\rangle)$
    * If more than one element of **F** interprete Φ for $\langle a_1, a_2, ..., a_n\rangle$, then $\mathbf{v}(\Phi(a_1, a_2, ..., a_n))$ should be chosen so that it is somewhere within the range of values given to $\langle\Phi\langle a_1, a_2, ..., a_n\rangle\rangle$ by these elements of **F**
  - If a predicate letter contains variables in any of its argument places, a value of **D** has to be assigned to them; afterwards the evaluation is done in the usual way.
  - As mentioned before, for every wellformed formulas A and B and assignments of values to variables the following coherences have to be conserved:
    * $\mathbf{v}(\neg A) = 1 - \mathbf{v}(A)$;
    * $\mathbf{v}(A \wedge B) = \min(\mathbf{v}(A), \mathbf{v}(B))$;

  * $\mathbf{v}(A \vee B) = \max(\mathbf{v}(A), \mathbf{v}(B))$;
  * $\mathbf{v}(A \rightarrow B) = 1 - \mathbf{v}(A) + \mathbf{v}(B)$ if $\mathbf{v}(B) \leq \mathbf{v}(A)$ and $\mathbf{v}(A \rightarrow B) = 1$ otherwise.

- For any well-formed formula A, $\mathbf{v}((\forall)A))$ is, relative to an assignment of values to variables, the greatest lower bound of the values of $\mathbf{v}(A)$ relative to all possible assignments that differ from each other at most with respect to the value assigned to x, denoted by $\mathrm{glb}_x(\mathbf{v}(A))$.

  $\mathbf{v}((\exists x)A)$ is, relative to an assignment of values to variables, the least upper bound of the values of $\mathbf{v}(A)$ relative to all possible assignments that differ from each other at most with respect to the value assigned to x, denoted by $\mathrm{lub}_x(\mathbf{v}(A))$.

At this point, I would like to refer once more to Petr Hájek's definition of his basic fuzzy propositional calculus which will be described in subsection 4.2.2.

3.3.2 Sorites paradoxes

In classical logic, a conditional with a *true* antecedent cannot be *true*, if its consequent is *false*; Machina generalized this classical principle and defined that in his logic a conditional cannot be *absolutely true*, if its consequent is *'more' false* than its antecedent.

Machina's idea seems to coincide with our intuitions: In the case of borderline predictions our intuitions say that anyone who has one more hair on his head than someone bald has to be bald as well, as it seems implausible that one hair makes the difference between the group of the bald people and the group of the people who are not bald. Nevertheless, anyone who has one more hair on his head than a bald person seems to be a little bit 'less' bald. At a certain point, when n is sufficiently high, a proposition like 'Anyone with n hairs on his head is bald' does not seem to be *absolutely true* anymore. For Machina this means that the truth values of the proposition has to decrease. Hence there are conditional premises to which a value $\neq 1$ is assigned.

More precisely, suppose that the proposition 'Someone with n hairs on his head is bald' receives the truth value 0.99 and the proposition 'Someone with $n+1$ hairs on his head is bald' receives the truth value 0.98, as it seems a little bit *'more' false* that the person with one more hair is bald than the same claim about the other person. The conditional 'If a person with n hairs is bald, a person with $n+1$ hairs is bald as well' receives the truth value 0.99. This implicates that *modus ponens* is not valid, because the conclusion with 0.98 is *'more' false* than the *falsest* premise which has the truth value 0.99. *Modus ponens*, the rule of inference, is quasi valid, but with each repetition the guarantee of truth diminishes, so that plenty of repetitions can lead to an *absolutely false* premise.

Reassuming, the inductive premise of the argument is interpreted as being *quite true*, because this seems to be plausible. The argument form has some validity and therefore it preserves truth quite well for many steps when the initial premise is *quite true*. There is no point in the argument chain where we lose the guarantee of truth all in one big jump. As Machina [Mac, p.201] argues, this '[...] is, what the common man wants: he is convinced that such 'slippery slope' arguments are fine if they are not carried too far.'

3.3.3 Pre-philosophical judgements and intuitions

Also Machina's degree theory seems to contradict some of our pre-philosophical judgements and intuitions.

As mentioned above, Machina uses the unit interval [0,1] as truth values for his logic, where 0 stands for absolute falsity and 1 for absolute truth, while the values in between are used to characterize borderline cases. This is advantageous, because in this way there are different truth values which can be assigned to borderline cases in order to satisfy our intuitions that some borderline cases may be *'more' true* than others. Nevertheless, the assignment of a completely exact truth value like 0.453 to a proposition like 'This man is bald' is problematic. What should express a truth value like this?

There is also a further problem which cannot be resolved by the introduction of an infinite number of truth values: it still seems that there are sharp boundaries between the propositions which are *completely true*, the propositions which are *completely false* and the rest of the propositions which are neither clearly *true* nor clearly *false*. There is still one last conditional which receives the value 1, followed by a first conditional which receives a value $\neq 1$, as well as there is a last conditional which receives a value $\neg\, 0$, followed by the first conditional which receives the value 0.

Another arguable aspect is Machina's evaluation of complex propositions. In case $|p| = 0.5$, classical tautologies like $|p \vee \neg p|$ and classical contradictions like $|p \wedge \neg p|$ are 0.5 uniformly. Suppose, Marco and Anna are borderline tall, but Marco is a little bit taller than Anna. Therefore the statement 'Marco is tall' has the truth value 0.5, while the proposition 'Anna is tall' has the truth value 0.4. According to Machina's definition of the conjunction, the proposition 'Anna is tall and Marco is not tall' receives the truth value 0.4, as well as 'Anna is tall and Marco is tall'. This does not seem to be plausible, because the first statement should be *completely false*, as nobody who is taller than someone tall can be not tall. In the latter case, however, a truth value of 0.4 seems adequate, as Marco and Anna are both borderline tall.

3.4 Supervaluationism

Michael Tye and Kenton Machina try to imitate our intuitions which seems to tell us that propositions about borderlines are *neither true nor false* by introducing new truth values. However, these approaches can be problematic, as discussed before, because they abandon classical logic.

Nevertheless, it seems that an ideal theory should retain classical logic, but at the same time provide the possibility that propositions about borderline cases can be classified as *neither true nor false* without implicating the exigence to draw sharp boundaries for vague predicates.

In this section I will discuss supervaluationism, a theory of vagueness which foregoes the semantic 'principle of bivalence' in order to keep the semantic 'principle of the law of the excluded middle' and the rest of classical logic.

3.4.1 Rosanna Keefe's supervaluationism

One of the most vehement defenders of supervaluationism is Rosanna Keefe[Kee00] whose theory I will explain in the following paragraphs. She [KSe02, p.153] wants a theory of vagueness to meet the following three conditions:

1. borderline cases should be classified as *neither true nor false* without the

exigence that they have to take some specific numerical value;

2. sharp boundaries for vague predicates should be avoided, because according to Keefe there is nothing in our language or the world that determines particular locations for such hidden boundaries;

3. the logic should be tried and tested and surprising and counter-intuitive consequences concerning principles or inferences - classical logic would be ideal.

The core piece of her approach is the idea to come to a classical valuation of a vague language by so-called complete precisifications or sharpenings. The principle of precisification will be explained in the next paragraph.

Precisifications

If Tek is neither clearly tall nor clearly not tall but borderline tall, he is neither a member of the positive nor of the negative extension of the predicate 'tall', but a member of a so-called penumbra. Surely, the predicate 'tall' could be made precise by fixing a height boundary among the heights of the borderline tall people, such that anyone above it counts as tall, but this boundary would be arbitrary. However, as long as these boundaries are drawn in the penumbra, we do not make a mistake. It is a characteristic of borderline cases that they can classified as either *true* or *false* by different precisifications.

Nevertheless, the instance of the law of the excluded middle formed by a borderline case is *true* and its negation is necessarily *false*. This is possible because a statement is *true* only if it is *true* under all admissible precisifications, *neither true nor false* in case it is *true* under some admissible precisifications and *false* under others and it is *false* in case it is *false* under every admissible precisification. In this way, a statement of the form $p \vee \neg p$ is guaranteed to be *true* even when, being borderline, neither p nor $\neg p$ is *true*, since in every admissible precisification one of the statements p or $\neg p$ will be *true*.

Admissible precisifications

Citing Stephan Schiffer [Sch], an 'admissible precisification of a statement is [...] the assignment to it of a precise bivalent interpretation under which the statement may be either *true* or *false* if it is borderline, but will be *true* under the interpretation if it is *determinately true*, and *false* under the interpretation if it is *determinately false*'.

Fig. 3.1 illustrates a possible precisification of the vague predicate 'heap'. As long as the border for the precise predicate 'new heap' is drawn in the penumbra of the predicate 'heap', the precisification is admissible.

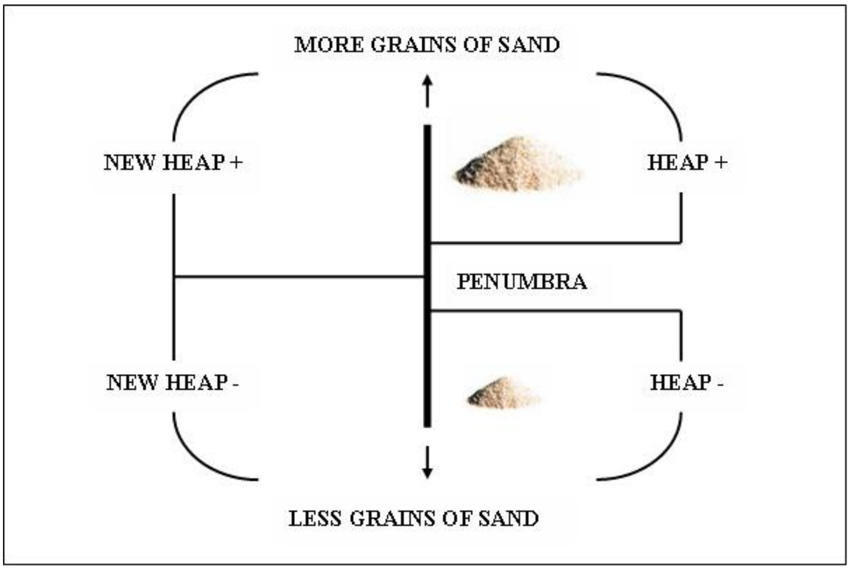

Fig. 3.1: Precisification of the predicate 'heap'.[4]

Complete precisifications

By drawing a sharp boundary for every vague predicate one receives a complete precisification, i.e. a classical valuation of a vague language without any type of vagueness: every predicate has its positive and its negative extensions which are sharply bounded and complimentary to each other.

3.4.2 Uncontroversial truths and penumbral connections

We have to consider mainly two aspects, when we try to make precise a predicate.

First of all, we have to preserve uncontroversial truths. If Tek is 1.90 meter tall he is clearly tall and the proposition 'Tek is tall' has to be *true* on all precisifications.

[4] A similar figure can be found in [Böh95, p.13].

If Tek measures 1.60 meter he is clearly short and the proposition ‘Tek is tall’ has to be *false* on all precisifications. If Tek is 1.75 meter tall he is neither clearly tall nor clearly short and some precisifications will place him in the positive extension of the predicate ‘tall’, while others will place him in the negative extension of the predicate ‘tall’.

Secondly, valid precisifications have to respect certain semantic constraints. Anyone taller than a tall person has to be tall as well. These so-called penumbral truths or penumbral connections can be internal or external. Internal penumbral truths concern the instances of the same predicate. There is no acceptable precisification of ‘tall’ according to which people who measure 1.80 meter are tall, but those of 1.85 are not. External penumbral truths concern related predicates. The predicate *tall* cannot be sharpened so that someone measuring 1.75 meter is tall, while simultaneously the predicate *short* is sharpened in order that someone measuring 1.75 meter is short.

3.4.3 Classical logic and semantics

Supervaluationism retains classical logic by introducing an extended semantics for vague languages which is a result of the so-called supervaluation: our natural language use does not determine precise extension for vague predicates, but a (vague) range within which the precise extensions would fall, if there were precise extensions.

Super-truth and super-falsity

Substituting vagueness for exactness, hence making precise a predicate, would involve fixing a sharp boundary between the predicate’s positive and negative extensions and as a consequence every borderline would fall either in the positive or in the negative extension of the predicate. Vague predicates have a lot of admissible precisifications with different positive and negative extensions. A proposition is *super-true* if and only if it is *true* on all complete precisifications and false if and only if it is false on all complete precisification. Otherwise it is neither *super-true* nor *super-false*. That means, that some cases are undetermined, but otherwise the demands of classical logic are still satisfied.

Rosanna Keefe uses the definitely operator discussed in subsection 1.1.5 to describe her point of view concerning vagueness. According to the supervaluational approach Dp is true just in case p is *true* on all possible precisifications and is *false* otherwise. Keefe [Kee00, p.28] claims that the introduction of this operator is necessary because otherwise 'we will fail to fulfil the central tasks of a theory of vagueness [...]. It is only when we have that operator that we can state that borderline cases occupy a gap between definite truth and definite falsity without committing ourselves to a gap between truth and falsity.'

Classical rules of inference and classical principles

However, the introduction of the D-operator causes some problems for classical logic. A number of critics like Kenton Machina, Timothy Williamson or Kit Fine (see [Kee00, p.176]) have emphasized how supervaluationist logic fails to preserve a series of classical rules of inference and classical principles, like *reductio ad absurdum*, contraposition, deduction theorem and disjunction elimination, when the D-operator comes into play, as the following examples show:

1. The *reductio ad absurdum*[5] cannot be retained, as Kenton Machina [Mac, p.178] argues.

2. As far as the contraposition[6] is concerned, in classical logic holds:

$$\text{if } A \models_{CL} B, \text{ then } \neg B \models_{CL} \neg A. \tag{3.10}$$

   With supervaluationism on the other hand it is not always the case that if $A \models_{SV} B$ then $\neg B \models_{SV} \neg A$. More precisely, if we consider the example of $A \models_{SV} DA$, it is not always the case that $DA \models_{SV} A$, namely if A is *true* only on some but not on all precisifications. In this cases $\neg DA$ is *super-true*, while A is not.

[5] The *reductio ad adsurdum* is a type of logical argument where one assumes a claim for the sake of argument, derives an absurd or ridiculous outcome, and then concludes that the original assumption must have been wrong as it led to an absurd result.

[6] Contraposition is a form of immediate inference in which from a given proposition another is inferred having for its subject the contradictory of the original predicate.

3. According to the classical deduction theorem[7] we can infer that

$$\text{if } A \models_{CL} B, \text{ then } \models_{CL} A \rightarrow B. \tag{3.11}$$

   If we have $A \models_{SV} DA$, it is not usually the case that $\models_{SV} A \rightarrow DA$; namely for borderline A there can be complete specifications on which A is *true*, while DA is *false*. In this case also $\models_{SV} A \rightarrow DA$ is *false*.

4. Classically, according to the argument by cases we can infer that

$$\text{if } A \models_{CL} C \text{ and } B \models_{CL} C, \text{ then } A \vee B \models_{CL} C. \tag{3.12}$$

   With supervaluationism and the D-operator we have $A \models_{SV} DA$ and $\neg A \models_{SV} D\neg A$. If we introduce the connective $\vee$, it is the case that $A \models_{SV} DA \vee D\neg A$ and $\neg A \models_{SV} DA \vee D\neg A$, but this does not implicate that $A \vee \neg A \models_{SV} DA \vee D\neg A$. If A is borderline, the premise is *true* and the conclusion *false*.

New supervaluational principles

Because of the problems mentioned above, Rosanna Keefe suggested the following new principles, if the D-operator is involved:

1. *The reductio ad absurdum:* If from A and B derives a contradiction A and B cannot be both *true*; so when B is *true*, A is *not true*, i.e.

$$\text{from } A, B \models_{SV} C \wedge \neg C \text{ infer } B \models_{SV} \neg DA. \tag{3.13}$$

2. *The contraposition:* Suppose $A \models C$, which guarantees that it is not possible that A is *true* and C *false*. This seems to be compatible with the possibility that C is *false*, while A is *neither true nor false*. According to Keefe, from the falsity of C we should infer that A is *not true*, i.e.

$$\text{from } A \models_{SV} C \text{ infer } \neg C \models_{SV} \neg DA. \tag{3.14}$$

[7] The deduction theorem states that if a formula B is deducible from A then the implication $\models_{CL} A \rightarrow B$ is demonstrable.

3. *The deduction theorem:* Suppose $A \models_{SV} C$, where A or C contains the D-operator. If A is *true* then C, which can be captured as $\models_{SV} DA \rightarrow C$. Accordingly,

$$\text{from } A, B \models_{SV} C \text{ infer } B \models_{SV} DA \rightarrow C. \tag{3.15}$$

4. *The disjunction elimination:* If C derives from A and also from B, the truth of A guarantees the truth of C as does the truth of B. Keefe suggests the following new rule:

$$\text{from } A \models_{SV} C \text{ and } B \models_{SV} \text{ infer } DA \vee DB \models_{SV} C. \tag{3.16}$$

Disjunction and conjunction

As mentioned above, the semantics of supervaluationism is not a classical semantics. A proposition of the form $(p \vee \neg p)$ like 'Tek is bald' or 'Tek is not bald' is *super-true*, even if Tek is borderline bald. Independently from where we draw the boundary between the bald people and the people who are not bald the proposition is *true*, because one disjunctive is always *true*, while the other disjunctive is *false*. In other words, a disjunction can be *super-true*, even if none of its disjunctives is *super-true*.

Equally, a conjunction can be *super-false*, even if none of its conjunctives is *super-false*. A proposition like 'Tek is bald' and 'Tek is not bald' can be *super-false* if Tek is borderline bald, even if neither 'Tek is bald' nor 'Tek is not bald' is *super-false*.

Existential and universal quantification

Furthermore, an existential quantification can be *super-true*, even if no concrete instance is *super-true*. This can be shown best by the following example: on each precisification there is a n so that 'A man with n hairs is bald and a man with $n+1$ hairs is not bald' is *true*; but there is no concrete n for which the statement 'A man with n hairs on his head is bald and a man with $n+1$ hairs is not bald' is *super-true*, because for each n this proposition is *false* on some precisifications.

Similarly, also a universal quantification can be *super-false*, even if no concrete instance is *super-false*. A proposition like 'For each n: if a man with n hairs on his head is bald, then also a man with $n+1$ hairs is bald' is *false* on each precisification and therefore *super-false*, even if there is no concrete n for which the proposition 'If man with n hairs on his head is bald, then also a man with $n+1$ hairs is bald' is *super-false*.

Principle of bivalence

Also the principle of bivalence does not hold, because propositions about borderline cases can be neither *super-true* nor *super-false*. Rosanna Keefe [Kee00, p.154] argues that the failure of the principle of bivalence 'fits our intuitions about borderline cases, our attitudes towards them and our associated linguistic behavior (in particular, the typical hesitancy and disagreement over borderline cases).'

3.4.4 Borderline cases

The supervaluational approach of vagueness comes up to the characteristic features of borderline cases by admitting truth value gaps. As explained before, supervaluationism claims that all possible precisifications have to be considered to be able to classify a proposition, so that a proposition is *super-true* if and only if it is *true* on all possible precisifications and *super-false* if and only if it is *false* on all possible precisifications; otherwise it is neither *super-true* nor *super-false*.

3.4.5 Sorites paradoxes

Turning back to the problem of a sorites paradox described in subsection 1.1.3, I will discuss the answer of supervaluationism by means of the example of the heap: supervaluationism can easily meet at the same time both our intuition that 1 grain of sand does not make a heap, as Fx_1 is *super-false*, and our intuition that 10 000 grains do make a heap, as Fx_{10000} is *super-true*. On the other hand, there is at least one statement Fx_i, where $1 < i < 10\ 000$, which is neither *super-true* nor *super-false*, as there are admissible precisifications such that Fx_i evaluates to 0 and others such that Fx_i evaluates to 1. As a result we obtain that there is an i such that $Fx_i then Fx_{i+1}$ is *not true* on all admissible precisification and hence neither *super-true* nor *super-false*.

More generally, the universally quantified premise of a sorites argument

$$\text{for all } i,\ \neg(Fx_i \wedge \neg Fx_{i+1}) \tag{3.17}$$

seems to be very plausible according to our intuitions and intuitively we are inclined to agree to it. Nevertheless supervaluationism claims that this premise is *false*, while its negation

$$\text{for some } i,\ Fx_i \wedge \neg(Fx_{i+1}) \tag{3.18}$$

is considered to be *true*, even if this seems - at least at first sight - to be contrary to our intuitions.

Even if the treatment of sorites paradoxes is one of the least appealing facets of the supervaluational approach, Rosanna Keefe [KSe02, p.183ff.] defends the answer of the supervaluational approach and argues in the following way:

- firstly, we must accept something counter-intuitive, if we want to find an answer to sorites paradoxes;
- secondly, it is not always the case that we would assent to the inductive premise of a sorites argument. Taking the color spectrum as an example, everyone would agree that somewhere me must stop to call a color 'red' and begin to call it 'orange', even if there is no particular point in the spectrum which separates the color 'red' from 'orange';
- thirdly, our intuitions about 3.17 and 3.18 are mistaken: our belief that there is no *true* instance of the universal quantification 3.17 gets confused with the belief that the universally quantified statement is *not true*; our belief that no instance of the existential quantification 3.18 is *false* gets confused with the belief that 3.18 is *not false*.

3.4.6 Higher-order vagueness

At first sight, higher-order vagueness does not seem to create any problem for supervaluationism. It seems that sharp boundaries would exist only if there was a concrete n so that a proposition like 'A man with n hairs is bald, while a man with $n+1$ hairs is not bald' would be *true*. Such a concrete n does not exists, as every precisification draws the boundary between bald men and men who are not bald at a different point. But there are still three sharpen-bounded sets: propositions that are *super-true*, proposition that are *super-false* and propositions that are *neither super-true nor super-false*. Accordingly, a statement is either *true* or *false* on all possible precisifications; a third possibility, that would allow borderline borderline cases does not exist.

3.5 The pragmatic approach

'Languages themselves are free of vagueness but [...] the linguistic conventions of a population, or the linguistic habits of a person, select not a point but a fuzzy region in the space of precise languages.' explains David Lewis [Lew83, p.228]. This is the point of view of researchers who treat vagueness as a pragmatic phenomenon which arises from the way we humans use our language.

In this section I will consider the pragmatic account of vagueness, presenting mainly the point of view of David Lewis [Lew83, Lew69].

3.5.1 David Lewis' philosophy of language

According to David Lewis, languages are represented by set-theoretic entities which assign meanings to the strings of symbols. Sentences receive their meaning only relative to a context; more precisely, the meaning of a sentence in a language is a function from the specification of a context to a set of possible worlds in which the sentence would be *true*, if stated in a given context.

For David Lewis languages themselves are precise. Citing Rosanna Keefe [Kee00, p.141] for defenders of the pragmatic approach of vagueness 'there is no vagueness in languages themselves, nor need there be any vagueness in the world; instead it arises as a feature of the relation between language-users and languages.' Vagueness, on Lewis' approach, depends on the fact which precise language is used by a community of speakers. The beliefs, intentions and habits of language users do not determine a convention of one single language, but a cluster of similar languages among which the speakers can choose. Different languages of the cluster can classify borderline cases of a certain predicate differently; some languages will draw the boundary of 'tall' at 1.75m, some at 1.76m and so forth. Also the proposition 'Italy is boot-shaped' is a good example of a sentence that is precise enough, hence *true*, for many contexts, but *not true enough* for many others.

3.5.2 Borderline cases

From the supervaluational point of view, borderline case predications are sentences over which there are disagreements about their meaning among the different precise languages of the cluster.

3.5.3 Higher-order vagueness

If the clusters of languages were sharply bounded, then the borderline cases would be sharply bounded as well and hence there would be no room for higher-order vagueness. However, Lewis recognizes the importance of treating the phenomenon of higher-order vagueness. Therefore he claims that there is a fuzzy region in the space of precise languages from which we chose one language, providing in that way a possibility for higher-order vagueness. However, Keefe[Kee00] objects that simply asserting this is not enough and asks for a detailed explanation of what a cluster with fuzzy boundaries would be.

3.5.4 Relations to other theories of vagueness

At first sight the pragmatic account of vagueness may seem independent from other theories of vagueness, but in fact it is closely related to the epistemic account and to supervaluationism, as I will show in the next paragraphs.

The relation to the epistemic account can be explained in the following way: as vagueness is a pragmatic matter according to the pragmatic approach, the languages of a cluster are not vague themselves. Vagueness arises from the relation between language-users and the languages, i.e. the people do not use one single language, but chose among a cluster of similar languages. If we consistently sticked to one single of these non-vague languages, the pragmatic view would collapse into the epistemic view: vague predicates would have unique and sharp meanings, even if our linguistic behavior suggested otherwise. One may argue that it is only the lack of knowledge that impedes us to use the correct exact language. Therefore, to distinguish the pragmatic from the epistemic approach, a cluster of precise languages must be involved.

One reason Lewis gives for denying that there is only one single common language, is, that otherwise language learning could not be explained. For someone who wants to learn the meaning of a predicate, it would be too difficult to identify the uniquely relevant extension of a communal language on the basis of the limited experience of other people's use of the predicate. It is easier to grasp a cluster of extensions which correspond to a cluster of languages, because thus the learner does not need to narrow the options down to one.

As mentioned above, the pragmatic account of vagueness is also closely related to supervaluationism as the following considerations show: from the supervaluational point of view, we can make all our vague predicates precise by a complete precisification of the language. In other words, by means of a complete precisification we get a completely precise language since each predicate has precise extensions. These complete precisifications of the language correspond one to one to the Lewisian precise languages of the appropriate cluster, so that the pragmatic account coincides exactly with supervaluationism; the only difference is the denotation: one theory uses the expression 'precisification of the vague language', while the other adopts the expression 'precise language of a cluster' to label the same entities. In fact, it is very difficult to distinguish between the concept of the precisification and the choice of a specific exact language from the cluster of all possible languages. But - as claimed by Rosanna Keefe [Kee00, p.143] - 'the pragmatic account *could* be a substantially different theory were it to give the precise languages a significantly different role from what supervaluationism attributes to precisifications.'

Defenders of a pragmatic theory of vagueness often argue that it is - saying it with the words of Christian Fermüller [Fer03] - 'more appropriate to base supervaluationism or epistemic accounts of vagueness on pragmatic principles than to reduce the pragmatic account to supervaluationism or epistemology.

3.5.5 Steward Shapiro's *Vagueness in Context*

A very recent publication is Stewart Shapiro's [Sha06] book *Vagueness in Context*. Also Shapiro is convinced that vagueness is a linguistic phenomenon due to the kinds of languages that humans speak. In his book he tries to develop both a philosophical and a formal, model-theoretic account of the meaning, function, and logic of vague terms in an idealized version of a natural language like English. As he writes in the preface of his book, it is a commonplace that the extensions of vague terms vary with contextual factors. For instance, a person can be tall with respect to the average man, but not tall (or maybe even short) with respect to professional basketball players. The main feature of Shapiro's account is the extensions of vague terms also vary during a conversation. There are situations in which a competent speaker of the language con go either way in the borderline area of a vague predicate without sinning against the meaning of the words. Shapiro calls this *open-texture*. The technical model theory has a similar structure to the supervaluational approach, even if the notion of super-truth does not play a central role in the development of validity.

3.6 Comparison of the different theories

As discussed in chapter 2, it is not easy to compare the different theories of vagueness to each other, because there are so many different criteria (see section 2.1) that can be used to evaluate a theory of vagueness. However, it is not only the high number of criteria that makes the comparison so difficult, but also the fact that not every criterion is equally important in all theories.

However, in table 3.6 I try to give a brief overview over the most important theories of vagueness, summing up the main statements of one defender of each theory described in this chapter. Hereby, I concentrate on the most important criteria that should be met by a theory of vagueness, disregarding other general criteria as simplicity or clearness which should be valid for any theory about any phenomenon; in my opinion it is necessary that a theory of vagueness delivers an answer to the question of how to treat borderline cases and the sorites paradox (the third characteristic feature, namely that vague predicates have fuzzy boundaries, described in section 1.1, is closely related with the two other features and will not be treated separately here). Apart from these three characteristics also higher-order vagueness is a phenomenon that has to be considered by any theory of vagueness. Furthermore, I consider classical logic and semantics to be a very important aspect. Syntax and semantics of classical logic are comparatively simple, but nevertheless powerful. Classical logic and semantics have been very successful in the past and have been integrated in theories of other domains.

The epistemic and the pragmatic approaches are the only theories which retain classical logic and semantics in their entirety. For both accounts language itself is free from vagueness. In the former case it is some type of ignorance which prevents us from knowing the exact extensions of vague predicates; in the latter case vagueness is caused by our use of the language. Supervaluationism is a middle course. It retains classical logic, but adopts non-classical semantics. Gap theories like Michael Tye's approach normally adopt a three-valued logic, while degree theories adopt a many-valued or even infinite-valued logic normally with truth-values from the real unit interval [0,1].

As discussed before, borderline cases are cases for which it is unclear whether a vague predicate (clearly) applies or (clearly) does not apply (see subsection 1.1.1). From the epistemic point of view, borderline cases in this classical sense do not exits, because also vague predicates have sharp and unique boundaries, even if we humans cannot find out where they fall. The pragmatic account defines borderline cases as those sentences over which there are disagreements among the languages of a cluster. Gap theories usually assign the same truth value *indefinite* to all borderline cases, while degree theories normally assign to them truth from the real unit interval [0,1]. On the other hand, the supervaluational approach admits truth value gaps for borderline cases which are considered to be *neither super-true super-false*.

Vague predicates are susceptible to sorites paradoxes (see subsection 1.1.3). The answers to this phenomenon are quite different: considering the version of the sorites paradox with the inductive premise, the epistemic approach judges the argument as valid by mathematical induction, but contests the truth of the inductive premise, because the first premise is obviously *true* and the conclusion is obviously *false*. That means that vague predicates are sensitive to arbitrarily small differences and that they have sharp boundaries: there is a grain of sand that makes the difference between a heap and a non-heap. According to the theory of Michael Tye, considering the sorites paradox with the conditional premises, at least one conditional premise is *not true*, while Kenton Machina argues that with each repetition of the inductive premise the guarantee of truth diminishes so that it is possible to come to an *absolute false* conclusion after many steps. According to Rosanna Keefe's supervaluationism, for some i, $(Fx_i \wedge \neg Fx_{i+1})$ is *true*, i.e. there is a point where one hair makes the difference between someone being bald and someone not being bald. However, the precisifications of a vague predicate can draw the border between the positive and the negative extensions differently. This is the main difference to the way in which the epistemic approach responds to sorites paradoxes, as the latter account says that there is one special grain of sand that discriminates a heap from a non-heap.

The last column of table 1.1 provides information about the answers of the different theories to the phenomenon of higher-order vagueness. On the epistemic account higher-order vagueness does not exist, because according to this theory there are sharp boundaries between borderline cases and the positive or negative cases. Also Michael Tye's gap theory does not model higher-order vagueness. The idea of defenders of this theory is that there are fuzzy boundaries between the positive or negative and the borderline cases. Also in Kenton Machina's infinite-valued logic higher-order vagueness is not modeled. Defenders of supervaluationism claim that the admissibility of the precisifications itself is subject of further precisification. David Lewis claims that the cluster from which the speakers of a language choose it not sharply bounded.

	Classical logic & semantics	**Borderline cases**	**Sorites paradox**	**Higher-order** vagueness
Timothy Williamson's epistemic approach	Classical logic and semantics are retained. Principle of bivalence holds.	Borderline cases do not exist.	The argument is valid, but the truth of the inductive premise is contested.	Does not exist in this account.
Michael Tye's gap theory	Adopts a three-valued logic, with a third truth value *indefinite*.	All borderline cases receive the same truth value *indefinite*.	The classical sorites is valid, but at least one conditional premise is not *true*.	Is not modeled. Idea: fuzzy boundaries
Kenton Machina's degree theory	Adopts an infinite-valued logic with truth values from the unit interval [0,1].	Borderline cases receive values between 0 (classical) falsehood and 1 (classical) truth.	With each repetition the guarantee of truth diminishes until to an absolute false conclusion.	It is not modeled in standard accounts.
Rosanna Keefe's supervaluationism	Retains classical logic, adopts non classical semantics.	There are truth value gaps for borderline cases that are neither super-true nor super-false.	For some i $Fx_i \wedge \neg Fx_{i+1}$ is *true*.	Admissability of precisifications itself is subject of further precisification.
David Lewis' pragmatic approach	Retains classical logic and semantics in its entirety.	Sentences over which there are disagreements among the languages in the clusters.	Sorites series hint at a 'switch of languages'.	The cluster from which a language is selected is not sharply bounded.

Tab. 3.6: Comparison of the different theories of vagueness.

Chapter 4

Fuzzy logics

Fuzzy logic in the narrow sense is a beautiful logic, but it is also important for applications: it offers foundations.

Petr Hájek [Háj98, p.5]

Introduction

Vagueness is not merely a philosophical problem. It has to be faced by logicians, engineers and computer scientists as well. In this time of rapidly advancing technology, the dream of producing machines which mimic human reasoning, that is often based on uncertain and imprecise information, has become one of the main challenges for many scientists. Therefore it is necessary to formalize reasoning in the presence of vague information. Fuzzy concepts have to be modeled mathematically, as Hung T.Nguyen and Elbert A. Walker [NW99] write in the preface of their book *A First Course in Fuzzy Logic*, 'for the purpose of automation in expert systems, computer vision, control engineering and pattern recognition'.

Fuzzy set theory provides a machinery for imitating reasoning if the available information is imprecise or vague. It is now one of the leading and most successful methodologies for the treatment of the phenomenon of vagueness and it is a well-established sound formal system with numerous applications in the field of automatic control and experts system, as Petr Hájek's statement at the beginning of the chapter emphasizes. In this connection, it is fascinating to see that precision seems to be the most important prerequisite to understand the imprecise.

Indeed, logicians, engineers and computer scientists are often accused of treating language as though it were precise and ignoring its vagueness, as Timothy Williams [Wil94] states. According to some critics the standards of computer scientists regarding valid and invalid reasoning are good enough for artificial precise languages, but when they are applied to natural vague languages in which humans reason about the world which they experience they nearly break down. For Williamson a perfectly precise language for such type of human reasoning is an idealization which cannot be realized. Although we can make our language less vague, we cannot make it perfectly precise. Even by stipulating what our words are to mean, we cannot reach a perfectly precise language, because our stipulations would be made in not perfectly precise terms and so also the reformed language would inherit some of that vagueness.

However, it seems true that computer scientists and engineers in a way remain prisoners of their work which has to be precise and probably sometimes they envy Henri Matisse for who - like for nature - the precise did not matter, because for the French painter precision did not necessarily mean truth or reality. Probably he foresaw, like Bernd Demant [Dem93, p.1] writes in his book *Fuzzy-Theorie oder Die Faszination des Vagen*, that no technical apparatus, even if it was equipped with the best electronic system and the best software, could imitate the grace and ease with which a cat runs on a garden fence, stops in the middle of it, turns around and goes back the same way with the same secureness.

Nevertheless, fuzzy logic is the most successful machinery for imitating reasoning in case of imprecise or vague information and therefore this chapter addresses fuzzy logic, explaining the main concepts regarding fuzzy logic. A very elaborated presentation of fuzzy logic can be found in Petr Hájek's [Háj98] book *Metamathematics of Fuzzy Logic*. The most important aspects of this book were reassumed by Hájek [Háj] in his article *Why fuzzy logic?*. This chapter is mainly based on these elaborations.

4.1 Lofti Zadeh's fuzzy logic

Citing Vilém Nóvak [Nov05] the main idea for motivationg the development of fuzzy logic can be formulated as follows: 'Fuzzy logic is a special many-valued logic addressing the vagueness phenomenon and developing tools for its modeling via truth degrees taken from an ordered scale. It is expected to preserve as many properties of classical logic as possible.'

As mentioned in the citation of Vilém Nóvak, fuzzy logic is a many-valued logic, which allows intermediate values between the conventional values 0/1, black/white, yes/no, etc, like Fig.4.1 shows in an amusing way.

Fig. 4.1: Fuzzy logic.[1]

The term *fuzzy logic* emerged for the first time in 1965 in connection with the development of the theory of fuzzy sets, when Lofti Zadeh [Zad65] presented his mathematical modeling of fuzzy concepts. Zadeh's main idea was that meaning in natural language is a matter of degree: if we consider a proposition like 'x is a long river' it is not always possible for us to assert that it is either *true* or *false*. For this reason, according to Zadeh, the ordinary indicator function, or characteristic function, χ_A of a subset A of a set X which specifies whether or not an element is in A, is not sufficient in case of vague concepts, because there are only two values that the characteristic function χ_A can take:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \tag{4.1}$$

To suit vague concepts, Zadeh generalized this notion by allowing images of elements to be in real unit interval [0,1] rather than being restricted to the two element set {0,1} which leads to his definition of fuzzy subsets (usually, fuzzy subsets are referred to as fuzzy sets):

[1] The figure is taken from the website of the Association for the Advancement of Artificial Intelligence www.aaai.org.

Definition 4.1. *A fuzzy subset A of a (crisp) set X is determined by a function $f_A(x)$: $X \mapsto [0,1]$.*

In other words, a fuzzy subset A of a (crisp) set X is characterized by a membership function $f_A(x)$ which associates with each point in X a real number in the unit interval [0,1]. The value of $f_A(x)$ represents the degree of membership of x in A. The nearer the value of $f_A(x)$ to 1, the higher is the grade of membership of x in A; the nearer the value of $f_A(x)$ to 0, the lesser is grade of membership of x in A.

The functions whose images are contained in the two element set $\{0, 1\}$ correspond to ordinary subsets of X. In this case $f_A(x)$ would be reduced to the familiar indicator or characteristic function χ_A of a set A. This implicates that ordinary subsets are special cases of fuzzy subsets.

For a fuzzy concept, different membership functions can be considered. The choice which function should be adopted is subjective and context dependent.

The membership function can be represented graphically. Fig.4.2 shows two examples of possible membership functions $f_A(x)$:

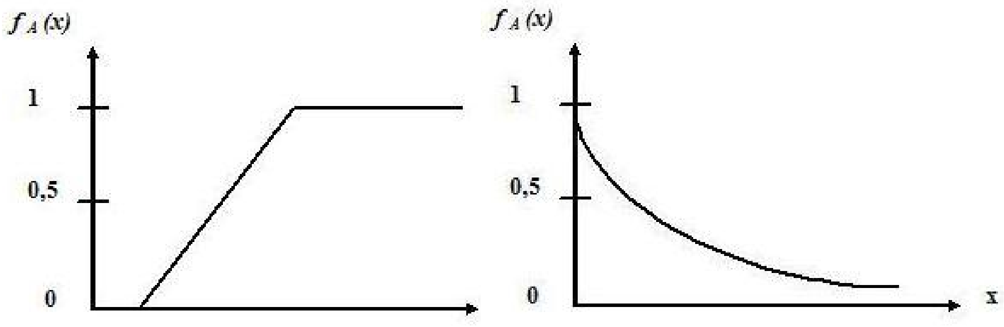

Fig. 4.2: Examples of possible membership functions $f_A(x)$.

Two main directions in fuzzy logic suggested by Lofti Zadeh have to be distinguished: *fuzzy logic in narrow sense* and *fuzzy logic in broad sense*, as illustrated in Fig. 4.3.

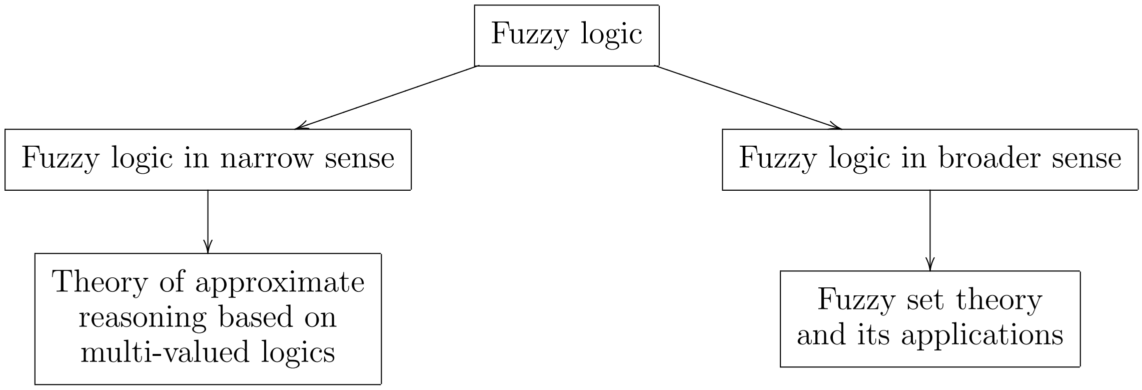

Fig. 4.3: Fuzzy logic in narrow and broader sense.

4.1.1 Fuzzy logic in narrow sense

Fuzzy logic in narrow sense is an attempt to define formal apparatus to define an adequate notion of approximate reasoning in presence of vague information based on many-valued logic. It is a truth functional logical system, i.e. a truth value of a compound formula can be computed from truth values of its subformulas using truth functions of connectives. Quoting Petr Hájek [Háj98, p.2] fuzzy logic is 'a logic with a comparative notion of truth: sentences may be compared according to their truth values.' In narrow sense, fuzzy logic can be seen as an extension of traditional multi-valued logics.

4.1.2 Fuzzy logic in broad sense

Fuzzy logic in broad sense is an extension of fuzzy logic in narrow sense and serves mainly as apparatus for fuzzy control, analysis of vagueness in natural language and several other application domains. I quote once more Petr Hájek [Háj, p.2], who explains that 'in broad sense, the term *fuzzy logic* has been used as anonymous with *fuzzy set theory and its applications*'. Fuzzy logic in broad sense can be considered as one of the techniques of soft-computing, i.e. computational methods tolerant to suboptimality and impreciseness (vagueness) which give quick, simple and sufficiently good solutions.

For my purpose, the practical applications of fuzzy logic are only of marginal interest and therefore I will concentrate on fuzzy logic in narrow sense.

4.2 Basic fuzzy propositional logic

In logic and mathematics, a propositional calculus (or a sentential calculus) is a formal system in which formula representing propositions can be formed by combining atomic propositions using logical connectives, and a system of formal proof rules allows certain formula to be established as 'theorems' of the formal system.

There are various systems of fuzzy logic, not just one. There is one basic logic (BL) and there are three of its most important extensions: Łukasiewicz logic (Ł), Gödel logic (G) and the Product logic ($\prod$), which will be discussed in subsection 4.2.3.

The basic fuzzy propositional logic, that I present in this chapter, was developed by Petr Hájek [Háj98] [Háj] and the calculus he describes is the result of his design choices, which I will sketch in the next paragraph. Obviously, these design choices are not obligatory, but they seem to be rather reasonable.

4.2.1 Petr Hájek's design choices

1. One of the most important properties of vagueness is its continuity, as Antonín Dvořák and Vilém Novák [DN] claim. This means that for similar objects the extent in which they have a certain property should also be similar. Therefore it seems natural to require the continuity of the truth functions of the logical connectives. Hájek takes the real unit interval [0,1] to be the standard set of truth values, where 1 stands for absolute (classical) truth and 0 for absolute (classical) falsity. The natural ordering $\leq$ of reals serves as comparison of the truth values, as the logic should be a logic with a comparative notion of truth.

2. The constructed logic is truth functional. A logic is truth functional if the truth value of a compound sentence can be computed from the truth values of its subformulas by using the truth functions of the connectives. Thus, e.g., the truth value of the conjunction $\varphi \mathbin{\&} \psi$ is uniquely determined by the truth value of φ, of ψ and by the chosen truth function for &.

3. Continuous t-norms are taken as possible truth function of the conjunction. Intuitionally, a high truth value for the conjunction $\varphi \mathbin{\&} \psi$ should indicate

that the truth values of φ and ψ are high, reason why the truth function of the conjunction should be non-decreasing in both arguments. 1 should be the unit element, while 0 its zero element. All these requirements are met by continuous t-norms.

Definition 4.2. *A t-norm is a binary operation $*$ defined on [0,1] (t: [0,1] $*$ [0,1] $\mapsto$ [0,1]) which satisfies the following conditions:*

- *$*$ is associative, i.e. for all x, y, $z \in$ [0,1]: $(x * (y * z) = (x * y) * z)$*
- *$*$ is commutative, i.e. for all x, $y \in$ [0,1]: $(x * y = y * z)$*
- *$*$ is non-decreasing in each argument (if $x \leq x'$ then $x * y \leq x' * y$ and if $y \leq y'$ then $x * y \leq x * y'$)*
- *1 is the unit element ($1 * x = x$).*

Definition 4.3. *A t-norm $*$ is a continuous t-norm if it is continuous as a real function.*

The three most important continuous t-norms are:

- $x * y = \max(0,\ x + y - 1)$ (Łukasiewicz t-norm)
- $x * y = \min(x,\ y)$ (Gödel t-norm)
- $x * y = x \cdot y$ (Product t-norm)

These continuous t-norms are fundamental because any other continuous t-norm is an ordinal sum construction of these three.

4. The truth value of the implication in the classical two-valued logic is defined in the following way:

$$(\varphi \rightarrow \psi) = (\neg\varphi \vee \psi) = \max(1 - \varphi, \psi) \tag{4.2}$$

This definition can be problematic if fuzzy concepts are involved, as the following example shows: consider the 'fuzzy' version of the proposition 'When it rains, the street is wet'. Suppose that it is drizzling, thus to the proposition 'it is raining' is assigned a truth value of 0.5. As the street is not wet, but only wettish, the proposition 'the street is wet' has a truth value of 0.6. Using the classical definition of the implication, 'When it rains, the street is wet' receives a truth value of 0.6 and is only *'half' true*, which is an absurd conclusion.

The idea of the classical definition of the implication is, that $\varphi \to \psi$ is *true* iff the truth value of ψ is at least as high as the truth value of φ, otherwise it is *false*.

Generalizing this classical principle, Hajék proposes to require that a high truth value should indicate that the truth value of ψ is not much lesser than the truth value of ψ. Thus, the truth function of $x \Rightarrow y$ should be non-increasing in x and non-decreasing in y. Furthermore, it should be possibile to compute a lower bound of the truth degree y of ψ, if the truth degree x of φ and the truth degree $x \Rightarrow y$ of $(\varphi \to \psi)$ is known. The operation computing the lower bound for y should be non-decreasing in both arguments. 1 should be the unit element and 0 the neutral element. Even if it may be difficult to justify commutativity or associativity, Hájek proposes to take a t-norm $*$ which requires:

$$\text{if } a \leq x \text{ and } b \leq x \Rightarrow y \text{ then } a * b \leq y. \tag{4.3}$$

Substituting in 4.3 a for x and b for z, we get:

$$\text{if } z \leq x \Rightarrow y \text{ then } x * z \leq y. \tag{4.4}$$

On the other hand, to make the rule powerful, Hájek requests that

$$\text{if } x * y \leq y \text{ then } z \leq x \Rightarrow y \tag{4.5}$$

and hence

$$x * z \leq y \text{ iff } z \leq (x \Rightarrow y). \tag{4.6}$$

Then it follows that $x \Rightarrow y$ is the *maximal* z satisfying $x * y \leq z$. In other words, if $*$ is our continuous t-norm, then its residuum is the operation $\Rightarrow$, defined as follows:

$$x \Rightarrow y = \max\{z \mid x * z \leq y\}. \tag{4.7}$$

Iff $x \leq y$, then $x \Rightarrow y = 1$; if $x > y$, the residua of the above t-norms are:

- $x \Rightarrow y = 1 - x + y$ (residuum of the Łukasiewicz t-norm)
- $x \Rightarrow y = y$ (residuum of the Gödel t-norm)
- $x \Rightarrow y = \frac{y}{x}$ (residuum of Product t-norm)

These implications are also called R-implications, where R stands for residuum.

5. The truth function of the negation is (-)$x = x \Rightarrow 0$, i.e. x implies falsity.

4.2.2 The basic fuzzy propositional calculus

The result of Hájek design choices is the basic fuzzy propositional logic (BL). BL has classical syntax and hence all notions known from classical logic are identical with the corresponding classical ones. It differs from the classical boolean logic only by the set of axioms. By fixing a continuous t-norm $*$, also a propositional calculus $PC(*)$ is fixed.

Definition 4.4. *$PC(*)$ is a propositional calculus with the propositional variables $p_1, p_2, \ldots$, the connectives & and $\rightarrow$ and the truth constant $\bar{0}$ denoting falsity. Each propositional variable is a formula; also $\bar{0}$ is a formula. If φ and ψ are formulas, then also φ & ψ and $\varphi \rightarrow \psi$ are formulas. Each formula with the connectives $\wedge$, $\vee$ and $\equiv$ or the negation $\neg$ is semantically equivalent to a formula built with the constant $\bar{0}$ and the connectives & and $\rightarrow$:*

$$\begin{aligned}
\varphi \wedge \psi &\quad \textit{is} \quad \varphi \,\&\, (\varphi \rightarrow \psi) \\
\varphi \vee \psi &\quad \textit{is} \quad ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\
\neg\varphi &\quad \textit{is} \quad \varphi \rightarrow \bar{0} \\
\varphi \equiv \psi &\quad \textit{is} \quad (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)
\end{aligned}$$

The conjunction ($\wedge$) and the disjunction ($\vee$) can be also expressed as follows:

$$\begin{aligned}\varphi \wedge \psi &= \min\,(\varphi, \psi)\\ \varphi \vee \psi &= \max(\varphi, \psi)\end{aligned}$$

Evaluation of propositional variables

Definition 4.5. *An evaluation of the propositional variable p is a mapping e that assigns to the propositional variable p a truth value e(p) $\in$ [0,1].*

All formulas are evaluated as follows:

$$\begin{aligned}e(\bar{0}) &= 0\\ e(\varphi \to \psi) &= (e(\varphi) \Rightarrow e(\psi)) \quad \text{For any formulas } \varphi,\, \psi\\ e(\varphi \;\&\; \psi\;) &= (e(\varphi) * e(\psi)).\end{aligned}$$

$$\begin{aligned}e(\varphi \wedge \psi) &= \min(e(\varphi),\, e(\psi))\\ e(\varphi \vee \psi) &= \max(e(\varphi),\, e(\psi)).\end{aligned}$$

Hence, also the conjunction ($\wedge$) and the disjunction ($\vee$) can be expressed in terms of $*$ and $\Rightarrow_*$:

$$\begin{aligned}\min(x, y) &= x * (x \Rightarrow_* y)\\ \max(x, y) &= \min((x \Rightarrow_* y) \Rightarrow_* y, (y \Rightarrow_* x) \Rightarrow_* x).\end{aligned}$$

Tautologies

Definition 4.6. *A formula φ is a $*$-tautology of the propositional calculus PC($*$) if e(φ) = 1 for each evaluation e.*

In other words, a formula φ is a $*$-tautology if it is *absolutely true* under each evaluation. For different t-norms t_1, t_2, the set of the $*$-tautologies of $PC(t_1)$ can be different from the set of the $*$-tautologies of $PC(t_2)$.

Definition 4.7. *A formula φ is a 1-tautology if it is $*$-tautology for each continuous t-norm $*$.*

Axioms of the basic fuzzy propositional logic

The axioms of the basic logic BL are formulas that are 1-tautologies, i.e. they are $*$-tautologies in each $PC(*)$ ($*$ is a continuous t-norm).

The following formulas are taken as axioms of the basic fuzzy propositional logic BL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $(\varphi \,\&\, \psi) \to \varphi$

(A3) $(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$

(A4) $(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))$

(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$

(A5b) $((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\varphi \to \psi) \to \chi) \to \chi)$

(A7) $\bar{0} \to \varphi$

Axiom (A1) expresses the transitivity of the implication. (A2) says that the &-conjunction implies its first element, while (A3) guarantees the commutativity of the &-conjunction. (A4) expresses the commutativity of the $\wedge$-conjunction and (A5) the residuation. (A6) says that: if χ follows from $\varphi \to \psi$ then if χ also follows from $\psi \to \varphi$ then χ. (A7) says that $\bar{0}$ implies everything.

Deduction rule

The deduction rule of the basic logic BL is *modus ponens*, i.e. from φ and $(\varphi \to \psi)$ we infer ψ.

Completeness

The basic fuzzy propositional logic proves φ iff φ is a 1-tautology. Making use of *modus ponens*, we can follow that if φ and $\varphi \to \psi$ are 1-tautologies, also ψ is a 1-tautology.

4.2.3 Łukasiewicz (Ł), Gödel (G) and Product logic ($\prod$)

There are three well-known logics given by the three important t-norms defined above that are stronger than the basic logic: the Łukasiewicz (Ł), Gödel (G) and Product ($\prod$) logic.

1. Łukasiewicz logic Ł

The propositional calculus PC($*_{\text{Ł}}$) is determined by the Łukasiewicz t-norm which is taken as the truth function of the conjunction:

$$x * z = \max(0, x + y - 1). \tag{4.8}$$

For a complete axiomatization of the Łukasiewicz logic it is enough to add to the axioms of the basic logic the axiom ($\neg\neg$) of the double negation:

$$\neg\neg\varphi \to \varphi. \tag{4.9}$$

Rational Pavelka Logic

We receive Pavelka logic when we add to Łukasiewicz's logic a truth constant $\bar{r}$ for each *rational* $r \in [0,\ 1]$, postulating that $e_{\text{Ł}}(\bar{r}) = \text{r}$. The key observation is that, for any evaluation e, if $e(\varphi) = r$, then for any formula ψ, $e(\psi) \geq r$ iff $e(\varphi \to \psi)$. More formally, $e_{\text{Ł}}(\bar{r} \to \varphi) = 1$ iff $e_{\text{Ł}}(\varphi) \geq r$ and $e_{\text{Ł}}(\varphi \to \bar{r}) = 1$ iff $e_{\text{Ł}}(\varphi) \leq r$. This provides us with the possibility to express estimates of the truth degree of a formula.

2. Gödel logic G

The propositional calculus PC($*_G$) is determined by the Gödel t-norm which is taken as the truth-function of the conjunction:

$$x * y = \min(x, y). \tag{4.10}$$

Gödel logic G is the basic logic plus the idempotence of the conjunction:

$$\varphi \equiv (\varphi \& \varphi). \tag{4.11}$$

3. Product logic $\prod$

The propositional calculus PC($*_{\prod}$) is determined by the Product t-norm which is taken as the truth-function of the conjunction:

$$x * y = x \cdot y. \tag{4.12}$$

Product logic $\prod$ is the basic logic plus two additional axioms:

$$(\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi \tag{4.13}$$

$$\neg\neg\chi \rightarrow (((\varphi\&\chi) \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)). \tag{4.14}$$

Remark: the truth-function of the negation in Ł is (-)x = 1 - x, while in G logic and $\prod$ logic the truth-function of the negation is (-)0 = 1 and (-)x = 1 for x 0. Hence, BL in general has no dual disjunction. Only Łukasiewicz logic has a dual disjunction, because in Ł (-)(-)x = x.

4.2.4 BL-algebras

For each t-norm $*$, the unit interval [0,1] endowed with the truth functions of the connectives is a linearly ordered BL-algebra. For detailed definitions and proofs I refer to Petr Hájek's book *Metamathematics of Fuzzy Logic* [Háj98, p.46ff].

4.3 Basic fuzzy predicate logic

In this section the extension of the propositional calculus to a predicate calculus will be described. Basic fuzzy predicate logic has the same formulas as classical predicate logic.

4.3.1 The basic fuzzy predicate calculus

Extending the propositional calculus from subsection 4.2.2 we come to the following predicate calculus:

Definition 4.8. *Let $P_1, P_2, ...$ be predicates, each having its arity (unary, binary, ...), $c, d, ...$ constants, $x, y, ...$ object variables, & and $\rightarrow$ connectives, $\bar{0}$ and $\bar{1}$ the truth constants and $\forall$ the universal and $\exists$ the existential quantifier. The other connectives ($\wedge, \vee, \neg, \equiv$) are defined as described in subsection 4.2.2. Terms are object constants and object variables.*

$P(t_1, ..., t_n)$ is an atomic formula where P is a predicate of the arity n and $t_1, ..., t_n$ are terms. The truth constants $\bar{0}$ and $\bar{1}$ are formulas. If φ and ψ are formulas and x is an object variable, then $\varphi \rightarrow \psi$, φ & ψ, $\forall(x)\varphi$ and $\exists(x)\varphi$ are formulas. By using this rule arbitrarily many times, each formula results from atomic formulas.

Evaluation of object variables

Definition 4.9. *An interpretation of $P_1, ..., P_n$ is a structure $\boldsymbol{M} = \langle M,\ (r_P)_{P \text{ predicate}} \rangle$ where M is a non-empty set (domain) and for each predicate P of the arity n, r_p is an n-ary fuzzy relation on M, i.e. a mapping that associates with each n-tuple $(a_1, ..., a_n)$ of elements of M a truth degree $r_p(a_1, ..., a_n) \in [0,1]$.*

The truth value of a formula φ in $\mathbf{M}$ depends on the semantics of the connectives, i.e. the chosen t-norm $*$ and on the evaluation e of the object variables by the elements of M. The truth value $\| \varphi \|^*_{\mathbf{M},e}$ of a formula φ is defined inductively as follows:

$$
\begin{aligned}
\| P(x_1, ..., x_n) \|^*_{\mathbf{M},e} &= r_p(e(x_1), ..., e(x_n)) \\
\| \varphi \& \psi \|^*_{\mathbf{M},e} &= \| \varphi \|^*_{\mathbf{M},e} * \| \psi \|^*_{\mathbf{M},e} \\
\| \varphi \rightarrow \psi \|^*_{\mathbf{M},e} &= \| \varphi \|^*_{\mathbf{M},e} \Rightarrow \| \psi \|^*_{\mathbf{M},e} \\
\| (\forall x)\varphi \|^*_{\mathbf{M},e} &= \inf\, e_x \| \varphi \|^*_{\mathbf{M},e} \\
\| (\exists x)\varphi \|^*_{\mathbf{M},e} &= \sup\, e_x \| \varphi \|^*_{\mathbf{M},e}
\end{aligned}
$$

where e_x runs over all evaluations.

Generalizing this from t-norms $*$ to BL-algebras, r_p is a mapping into the domain of the algebra. For the quantified formulas $\| \varphi \|^{\mathbf{L}}_{\mathbf{M}}, e$ ($\mathbf{L}$ being an BL-algebra, see i.e. [Háj98, p.46ff]) has to be defined if the infimum/supremum does not exist in $\mathbf{L}$.

Tautologies

Definition 4.10. *A formula φ is a $*$-tautology of PC($*$) ($*$ is a continuous t-norm) if $\| \varphi \|^{*}_{\boldsymbol{M},e} = 1$ for each interpretation $\boldsymbol{M}$ and M-evaluation e.*

Definition 4.11. *A formula φ is a 1-tautology if it is a $*$-tautology for each continuous t-norm $*$.*

Axioms of the basic fuzzy predicate logic BL∀

The axioms of the basic fuzzy predicate logic BL∀ are 1-tautologies of the basic fuzzy predicate logic and consist of the axioms of BL described in subsection 4.2.2 plus the following logical axioms for the quantifiers:

(∀1)	$(\forall x)\varphi(x) \to \varphi(t)$	(t substitutable for x in $\varphi(x)$)
(∃1)	$\varphi(t) \to (\exists x)\varphi(x)$	(t substitutable for x in $\varphi(x)$)
(∀2)	$(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$	(x not free in ν)
(∃2)	$(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$	(x not free in ν)
(∀3)	$(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$	(x not free in ν)

Deduction rules

The deduction rules of the basic fuzzy predicate calculus are two:

1. modus ponens (from φ, $\varphi \to \psi$ infer ψ)

2. generalization (from φ infer $(\forall x)\varphi$).

4.4 Fuzzy logic as a theory of vagueness

Fuzzy logic is one possible methodology for the treatment of vagueness. As discussed in chapter 2 each theory of vagueness should meet certain criteria. But what is the response of fuzzy logic to the criteria mentioned in section 2.1? In this section I will try to answer this question, discussing briefly, corresponding to table 3.6 in chapter 3, the relation of fuzzy logic to classical logic, how fuzzy logic treats borderline cases, its solution to sorites paradoxes and how it deals with higher-order vagueness.

4.4.1 Classical logic and semantics

As mentioned above, defenders of fuzzy logic are convinced that truth comes in degrees. To come up to this idea, the logical consequence is to introduce new truth values apart from the classical ones 0 (absolute falsehood) and 1 (absolute truth). Fuzzy logic uses as the set of truth values the real numbers from the unit interval [0,1]. The higher the real number, the *truer* the proposition. In section 3.3, when talking about degree theories, I cited Rosanna Keefe saying that degree theories are many valued logics (i.e. that they have more than the classical two truth values 0 and 1) which are truth functional. According to this definition, fuzzy logic can be seen as a degree theory, because it has an infinite number of truth values and it is truth functional.

4.4.2 Higher-order vagueness, sorites paradox and borderline cases

One of the arguments against the degree-theoretical approach of treating vagueness, and therefore also against fuzzy logic, is its alleged inability to deal with higher-order vagueness (see Tab. 3.6 in chapter 3). The usual argument of the defenders of the degree-theoretical approach, namely that higher-order vagueness can be modeled as borderline cases of a vague predicate receive an intermediate value between 0 and 1, seems to fail , because in a sorites series $Fx_1, ..., Fx_n$ there must still be an n such that x_n is determinately F and x_{n+1} is not. This implicates that there is still a sharp boundary between the set of

In literature there can be found some approaches that try to come up to the phenomenon of higher-order vagueness.

Libor Behounek [Beh06] is convinced that a propositional many-valued logic ‘is just a very simple and crude degree-theoretical model of gradual vagueness, insufficient to capture its subtiler features.’ According to him, more sophisticated models based on degree-based approaches are provided by first-order and higher-order many-valued logics of vagueness, which has been developed in the past few years by advancing propositional and first-order fuzzy logic (see i.e. [Nov04]). Summing up briefly the main idea of his approach, the multiple use of an imperfectly *true* premise decreases the guaranteed truth value of the conclusion. For the sorites paradox this means, that the repeated usage of the inductive premise gradually decreases the validity of the conclusion and therefore a *determinately false* conclusion Fx_n can follow from a *determinately true* premise Fx_1, an idea that has already been proposed by Kenton Machina (see subsection 3.3.2). Very large truth values are practically indistinguishable from 1 in ordinary arguments and only millions of repetitions of the inductive premise can lead to the invalidity of the conclusion of a sorites argument. To represent higher-order vagueness Behounek introduces the fuzzy predicate $True(\alpha)$, saying that the truth value from α is indistinguishable from 1 in ordinary arguments and thus represents (practical) truth. The more times α can be used in an argument without degrading it to falsity, the more α is indistinguishable from 1. Borderline cases of $True$ are the truth values of borderline borderline cases of the fuzzy predicate F. For a more detailed and more formal explanation I refer to Libor Behounek's paper 'A model of higher-order vagueness in higher-order fuzzy logic' [Beh06].

4.5 Summary

Fuzzy logics are motivated by the idea that in the presence of vague notions and propositions truth comes in degrees. This degree theoretic approach of approximating reasoning in the presence of vagueness has produced many different formalisms. In this chapter I discussed Petr Hájek's basic logic BL and described his 'design choices':

1. The unit interval [0,1] is taken as the set of truth values where 0 represents absolute falsity and 1 absolute truth. The natural ordering $\leq$ of reals serves as comparison of the truth values.

2. The logic is truth functional: The truth value of the compound statements depends only on the truth values of its subformulas.

3. Continuous t-norms $*$ are taken as the truth function of the conjunction.

4. The residuum $\Rightarrow_*$ of the t-norm $*$ serves as the truth function for the implication. $\Rightarrow_*$: $[0,1]\times[0,1] \rightarrow [0,1]$ is the unique function satisfying $x \Rightarrow_* y = max\{z|x * z \leq y\}$.

5. The truth function of the negation is $(-)x = x \Rightarrow_* 0$, i.e. x implies falsity.

Table 4.1 gives an overview about the three most important continuous t-norms and their residua:

	t-norm	**associated residuum**
Łukasiewicz	$x *_{Ł} y = \max(0, x+y-1)$	$x \Rightarrow_{Ł} y = \min(1, 1-x+y)$
Gödel	$x *_G y = \min(x, y)$	$x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Product	$x *_P y = x \cdot y$	$x \Rightarrow_P y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$

Tab. 4.1: Łukasiewicz, Gödel and Product t-norms and their residua.

These three continuous t-norms are fundamental because any other continuous t-norm is a combination of them (see [Háj98, p.32]).

Consequently, the following definition of propositional logics associated with a continuous t-norm $*$ was defined:

Definition 4.12. *For a continuous t-norm $*$ with the associated residuum $\Rightarrow_*$ a logic $\boldsymbol{L}_*$ was fixed based on a language with the constant $\bar{0}$, the binary connectives $\rightarrow$, & and the defined connectives $\varphi \wedge \psi =_{def} \varphi$ & $(\varphi \rightarrow \psi)$, $\varphi \vee \psi =_{def} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, $\neg\varphi =_{def} \varphi \rightarrow \bar{0}$, $\varphi \equiv \psi =_{def} (\varphi \rightarrow \psi)$ & $(\psi \rightarrow \varphi)$.*
An evaluation of a propositional variable p is a mapping e that assigns to p a truth value from the real unit interval [0,1]. Formulas are evaluated in the following way: $e(\bar{0}) = 0$, $e(\varphi \rightarrow \psi) = (e(\varphi) \Rightarrow e(\psi))$, $e(\varphi \mathbin{\&} \psi) = (e(\varphi) * e(\psi))$.
A formula φ is valid in $\boldsymbol{L}_*$ iff $e(\varphi) = 1$ for all evaluations e for each continuous t-norm $*$.

The logics determined by the t-norms $*_{Ł}$, $*_G$ and $*_{\prod}$ with the residua $\Rightarrow_{Ł}$, $\Rightarrow_G$ and $\Rightarrow_{\prod}$ are called Łukasiewicz logic (Ł), Gödel logic (G) and Product logic ($\prod$).

Chapter 5

Fuzzy logics and models of approximate reasoning

People are inclined to think there is a world of facts as opposed to a world of words which describe these facts. I am not too happy about that.
What rebels in us against such a suggestion is the feeling that the fact is there objectively no matter in which way we render it. I perceive something that exists and put it into words. From this, it seems to follow up that something exists independent of, and prior to the language; language merely serves at the end of communication. What we are liable to overlook here is the way we see a fact - i.e., what we emphasize and what we disregard - is our work.

Friedrich Waismann [Sha06, p.190]

Introduction

This chapter is supposed to show different approaches that try to combine fuzzy logics with ideas from the philosophical discourse on vagueness. More explicitly, I will sketch approaches that establish a relation between truth-functional t-norm based fuzzy logics and competing models of approximate reasoning and contributions that demonstrate a way how to derive a fuzzy logic from the first principles of approximate reasoning.

In 1974 Robin Giles tried to derive logics from fundamental reasoning principles and presented a strategic two-person game as a formal model of reasoning. Giles' analysis was originally referred to the phenomenon of dispersion in the context of the physical quantum theory. Only later he tried to apply the same concept of providing tangible meanings to logically complex fuzzy propositions. In this connection he discovered that the propositions which can be asserted initially in his game without having to expect a loss of money on average coincide with those that are valid in

Łukasiewicz logic Ł.

This approach was fundamental, as Robin Giles addressed a big philosophical challenge: the problem of how a fuzzy logic can be derived from the principles of approximate reasoning. Giles' approach allows to relate two at least at first sight two very different theories of vagueness, namely degree based fuzzy logic with supervaluationism with respect to admissible precisification defined in subsection 3.4.1, a theory introduced by Kit Fine and defended vehemently by Rosanna Keefe (see section 3.4) which is very popular among philosophers.

Robin Giles' idea was resumed by Christian Fermüller et al. (see [Fer04], [FP03], [FK] and [Fer07]). Defining a new logic SŁ which extends Łukasiwiecz logic, they relate supervaluationism and degree based reasoning. Furthermore, they provide a game based characterization of SŁ which shall also be discussed in this chapter.

5.1 Combining supervaluationism and degree based reasoning

In 2006, Fermüller and Kosik [FK] presented an approach which tries to combine supervaluationism and t-norm based fuzzy logics. Even if at first sight the two theories seem to be incompatible, according to the authors there is a common ground between them: they claim that t-norm based fuzzy logics can be interpreted as referring to classical precisifications (see subsection 3.4.1) at different levels of formula evaluation.

Fermüller and Kosik base their argumentation on the answer of the two theories to the sorites paradox (for supervaluationism see subsection 3.4.5, for fuzzy logic see subsection 3.3.2 and 4.4.2).

As mentioned before, considering once more the example of the heap, supervaluationism comes up to our intuitions that 'one grain of sand does not make a heap' and that '10 000 grains of sand distributed properly do make a heap', as the first statement is *super-false*, while the latter is *super-true*, but it does not accommodate our intuition that 'the removal of one single grain from a heap cannot turn it into a non-heap'. Fuzzy logics in narrow sense, on the other hand, offers the possibility to classify this last intuition as *almost true* assigning to it a truth value close to 1.

However, as far as the role of the logical connectives in vague statements is concerned, supervaluationism and degree theories are quite similar: a proposition of the form $(\varphi \& \neg\varphi)$ ('Anna is tall and is not tall' could be an example) is *super-false* according to supervaluationism, because either φ or $\neg\varphi$ is evaluated to 0. Also according to Kit Fine such a proposition is *definitely false*, because a person cannot be tall and not tall at the same moment (see penumbral connections described in subsection 3.4.2), even if both statements per se are neither *definitely true* nor *definitely false*.[1] This is the crucial point for Fermüller and Kosik: the truth value of an atomic proposition p assigned by fuzzy logics can be related to the density of the admissible precisifications which classify p as *true*.

5.1.1 Precisification spaces

To formalize their idea Fermüller and Kosik defined the following precisification space $\prod$, i.e. a space which contains only admissible complete precisifications (see subsection 3.4.1):

Definition 5.1. *Let Fx_i be the proposition saying that 'i grains of sand distributed properly make a heap'. $\prod$ consists of all classical interpretations I, which fulfill the following conditions which model the penumbral connections proposed by Kit Kine [Fin] (see also subsection 3.4.2):*

1. *$I(Fx_1) = 0$ and $I(Fx_{10000}) = 1$*

2. *$i \leq j$ implies $I(Fx_i) \leq I(Fx_j)$ for all $i, j \in 1,\ldots,10\,000$*

The first condition simply says that Fx_1 is *super-false*, while Fx_{10000} is *super-true*. The second condition imitates Fines penumbral connections as it says that if a precisification declares i grains of sand to be a heap, for all $j \geq i$, the same precisification has to declare j grains of sand to be a heap as well. I(if Fx_i, then Fx_{i-1}) $= 1$ in all but one interpretations I and thus intuitively $\prod$ respects our intuition that 'the removal of one grain of sand from a heap cannot turn it into a non-heap'. Fermüller and Kosik propose to accept the idea that truth comes in degrees, as defenders of fuzzy logics claim, because this allows to make information

[1] By asserting $(\varphi \& \neg\varphi)$ one may intend to express that both component statements are *partly true* and under this reading the statement may receive an intermediate value between 0 and 1.

explicit which is implicit in precisification spaces but not used in supervaluationism. More in detail, they suggest to define a *global truth value* of Fx_i with respect to $\prod$. For the example of the heap they chose $\frac{i-1}{9999}$ as *global truth value.*

Furthermore, they introduce an probability measure μ in $\prod$ which should represent a probability measure. The idea for the introduction of this value, is the simple consideration, that it is more likely that the border between an accumulation of grains of sand which form a heap and which do 'non heap' is near to 100 grains than near to 9999.

5.1.2 SŁ

They follow Hájeks design choices described in subsection 4.2.1 and add a further design choice, requiring that:

1. the truth function $\Rightarrow_*$ for the is continuous, as small changes in $e^*(A)$ and $e^*(B)$ should result in, at most, small changes in $e^*(A \rightarrow B)$.

Hence, their logic **SŁ** extends Łukasiewicz logic (see 4.2.3) and incorporates also classical logic as far as classical vocabulary is adopted. The innovation of their approach is the introduction of the unary connective S which is intended to make the concept of super-truth explicit in their logic.

Definition 5.2. *Let $p \in V = \{p_1, p_2, \ldots\}$ be proposition variables, $\bar{0}$ the truth constant denoting falsity and & and $\rightarrow$ connectives. A precisification space $\prod$ is the triple $\langle\ W, e, \mu\ \rangle$ where $W = \{\pi_1, \pi_2, \ldots\}$ is a non-empty set whose elements π_i are called precisification points, e is a mapping $W \times V \mapsto \{0,1\}$ and μ is a probability measure on the σ-algebra*[2] *formed by all subsets of W. Given a precisification space π a local truth value $\|A\|_\pi$ is defined for every formula A and every precisification point $\pi \in W$ inductively by:*

$$\|p\|_\pi = e(\pi, p), \text{for } p \in V \tag{5.1}$$

$$\|\bar{0}\|_\pi = 0 \tag{5.2}$$

[2] A σ-algebra over a set X is a nonempty collection S of subsets of X that is closed under complementation and countable unions of its members. It is a boolean algebra, completed to include countably infinite operations.

$$\|A\&B\|_\pi = \begin{cases} 1 & \textit{if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

$$\|A \to B\|_\pi = \begin{cases} 1 & \text{if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

$$\|SA\|_\pi = \begin{cases} 1 & \text{if } \forall \sigma \in W : \|A\|_\sigma = 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

The global truth value $\|A\|_\pi$ depends on the underlying Łukasiewicz t-norm$*_Ł$ and is defined as follows:

$$\|p\|_\prod = \mu(\{\pi \in W \| e(\pi, p) = 1\}),\ \text{for } p \in V \tag{5.6}$$

$$\|\bar{0}\|_\prod = 0 \tag{5.7}$$

$$\|A\&B\|_\prod = \|A\|_\prod *_Ł \|B\|_\prod \tag{5.8}$$

$$\|A \to B\|_\prod = \|A\|_\prod \Rightarrow_Ł \|B\|_\prod \tag{5.9}$$

$$\|SA\|_\prod = \|SA\|_\pi \text{ for any } \pi \in W \tag{5.10}$$

Definition 5.3. *A formula F is valid in SŁ iff F is valid in all precisification spaces $\langle\, W, e, \mu\rangle$ where W is finite.*

Axioms of SŁ

The axioms of SŁ are the axioms described in subsection 4.2.2 plus the following axioms for the unary connective S:

(A1$_S$) $S(A \vee \neg A)$

(A2$_S$) $SA \vee \neg SA$

(A3$_S$) $S(A \to B) \to (SA \to SB)$

(A4$_S$) $S\ (SA \to A)$

(A5$_S$) $SA \to SSA$

(A6$_S$) $\neg SA \to S\neg SA$

5.2 Dialogue games as a foundation for fuzzy logics

In this section the concept of dialogue games as well as Robin Giles's game for Ł, which is the basis for the game of SŁ, will be explained.

5.2.1 Robin Giles' dialogue game for Ł

Robin Giles' dialogue game for Łukasiewicz logic Ł is based on Paul Lorenz' attempt to provide a dialogical foundation for logic in general. The main ideas of Paul Lorenz will be sketched in the next paragraph. For a more detailed explanation see e.g. chapter 3-5 of [BK82].

Paul Lorenz's dialogic foundation for logic

As indicated above, Paul Lorenz introduced the dialectical or dialogical logic in modern formal logic. His preference for a dialectical logic is already apparent in [Lor60] where he puts classical logic on a par with a logic of cooperative debates (dialectics) and intuitionistic logic with a logic of competitive debates (eristics derived form the Greek word *eris* which means discord).

For Paul Lorenz the logical constants can be defined in terms of their role in rational debates or critical dialogues between to parties: on the one hand the proponent who defends a thesis and on the other hand an opponent who opposes it. According to Else M. Barth and Erik C. W. Krabbe [BK82, p.12] this social definition of logical constants can be regarded as a theoretical elaboration of Wittgenstein's notion of language games (as explained, e.g., in [Wit77]). In Wittgenstein's and Lorenz's opinion critical debates between two parties shall constitute the fundamental objects for a logical study. The self-critical case, the case in which reasoning is carried out by one person only, i.e. in which the two parties coincide in one person, should be regarded and studied as an important special case.

Robin Giles's game

Robin Giles combined Paul Lorenz's approach with a risk based evaluation of atomic propositions. This is specific to the context of vagueness if vagueness is understood as a phenomenon implying dispersion - Giles's analysis was originally referred to the phenomenon of dispersion in quantum theory.

For this purpose he introduced a game that consisting of two independent components (see i.e. [Fer04]):

1. Betting for positive results of experiments
2. A dialogue game for the reduction of compound formulas

These components will be described in detail in the following paragraphs.

1. Betting for positive results of experiments

The following components and definitions are needed:

- *Two players*

  The two players of the game - the *proponent* who defends a thesis and the *opponent* who opposes it - agree to pay 1 € to the other player in case the statement they assert is *false*.

- *The elementary state of the game* $[p_1, ..., p_m || q_1, ..., qn]$

  One player asserts each p_i in the multiset of the atomic statements $\{p_1, ..., p_m\}$, while the other player asserts each q_i in the multiset of the atomic statements $\{q_1, ..., qn\}$.

  Each propositional variable q refers to an experiment E_q with a binary (yes/no) result. Consequently, each statement q can be interpreted as 'E_q yields a positive result'. When repeated, the experiments E_q may yield different results.

- *A fixed risk value* $\langle q\rangle^r \in [0,1]$ *for each assertion* q

  The risk value $\langle q\rangle^r$ denotes the probability that E_q yields a negativ result.

The binary yes/no-experiments are not completely arbitrary, even if may show different outcomes when repeated.

The risk for the special atomic formula $\perp$ (falsum) is $\langle\perp\rangle^r = 1$, the risk for the multiset $\{q_1, ..., qn\}$ of atomic formulas is defined as $\langle q_1, ..., q_n\rangle^r = \sum_{i=1}^{n}\langle q_i\rangle^r$ and the risk for the empty multiset is defined as $\langle\rangle^r = 0$.

The condition $\langle p_1, ..., p_m\rangle^r \geq \langle q_1, ..., q_n\rangle^r$ expresses that first player does not have to expect any loss (on the contrary, he can possibly expect some gain) when betting on the truth of atomic statements.

2. A dialogue game for the reduction of compound formulas

Robin Giles defined the meaning of the logic connectives using rules of a dialogue game which reduces the arguments of compound formulas to arguments of their subformulas. Following Fermüller [Fer04], the main rule of this game can be stated as follows:

Implication

Rule 1. *If player 1 asserts $A \to B$ then, whenever player 2 chooses to attack this statement by asserting A, player 1 has to assert also B and vice versa.*

Hence, the meaning of the implication is determined by the principle that if a player asserts 'if A, then B' ($A \to B$), he is obligated to assert B, in case A is granted.

In this context it is useful to remember that all formulas in Łukasiewicz logic can be built with propositional variables, the connectives & and $\to$ and the truth constant $\bar{0}$ denoting falsity, as all formulas containing other connectives are semantically equivalent to formulas built only with these two connectives (see subsection 4.2.2). Moreover & can be defined as follows: $(\psi \ \& \ \varphi)$ is $\neg(\psi \to \neg\, \varphi)$. By substituting $\neg\varphi$ for $\varphi \to \bar{0}$, $(\neg(\psi \to \neg\, \varphi)$ is $(\psi \to (\varphi \to \bar{0}))\!\to \bar{0}$, which in turn means that all formulas can be build with propositional variables, the connective $\to$ and the truth constant $\bar{0}$ (see i.e. [Háj98, p.65]).

Nevertheless it is useful to see that the meaning of all relevant connectives can be specified by plausible dialogue rules. In the following paragraphs I will briefly summarize the rules as described by Fermüller (see [Fer07]):

Weak conjunction $\wedge$

The weak conjunction $\wedge$ can be interpreted as follows:

Rule 2. *If player 1 asserts $A \wedge B$, player 1 has to assert also*

- *B if player 2 attacks A and*
- *A if player 2 attacks B.*

This rule can be generalized easily to a rule for a universal quantification, saying that if player 1 asserts $A_1 \wedge ... \wedge A_i$, $i \in \mathbb{N}$, player 1 has to assert A_i for any $i \in \mathbb{N}$ player 2 may choose.

Weak disjunction $\vee$

The weak disjunction $\vee$ can be interpreted as follows:

Rule 3. *If player 1 asserts $A \vee B$, player 1 has to assert A or B that he may choose himself.*

As described in subsection 4.2.2, in Łukasiewicz logic all formulas containing $\wedge$ or $\vee$ are evaluated as follows: For any formula ψ, φ $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$ and $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$.

Strong conjunction &

The strong conjunction & can be interpreted as follows:

Rule 4. *If player 1 asserts $A\&B$, then player 1 has to assert either A and B or $\bar{0}$.*

Analogically, the rules for negation, strong conjunction and equivalence can be defined.

No special rules are necessary to regulate the succession of the moves in this dialogue game, but it is required that each assertion is attacked at most once. As soon as player 1 attacks by asserting A or indicates that he will not attack $A \to B$ at all, $A \to B$ is removed from the multiset of all formulas asserted by player 2. Every run of the game ends in an elementary state $[p_1, ..., p_m || q_1, ..., q_m]$. Given an assignment of $\langle . \rangle_r$ of risk values to all p_i and q_i, the condition $\langle p_1, ..., p_m \rangle^r \geq \langle q_1, ..., q_n \rangle^r$ expresses that first player does not have to expect any loss.

To illustrate a move of the game, Fermüller [FK] gives the following example: Suppose that player 1 asserts $p \rightarrow q$ for some atomic formulas p and q. In this case the initial state of the game is $[||p \rightarrow q]$. Player 2 has the possibility either to assert p which would force player 1 to assert q or to refuse to attack $p \rightarrow q$. In the first case the game ends in the elementary state $[p||q]$, while in the latter case it ends in the state $[||]$. If an assignment $\langle . \rangle^r$ of risk values gives $\langle p \rangle^r \geq \langle q \rangle^r$ player 1 will win the game, whatever move player 2 chooses. In this case player 1 has a winning strategy for $p \rightarrow q$ in all assignments of risk values where $\langle p \rangle^r \geq \langle q \rangle^r$.

The interesting point of Giles' theory is the connection that he established between the sketched game and Łukasiewicz logic. He discovered that the propositions that can be asserted by a player at the beginning of the game without having to expect a loss of money on average coincide with these that are valid in Łukasiewicz logic.

Theorem 5.1. *Every assignment $\langle . \rangle^r$ of risk values to the atomic formulas occurring in a formula F induces an evaluation $e\langle . \rangle^r$ for Łukasiewicz logic Ł such that $e\langle . \rangle^r(F) = 1$, iff the player has a winning strategy for F in the sketched game.*

In other words, if a player has a winning strategy for the game presented above, every assignment of risk values to the atomic formulas of a formula F implies that the evaluation e of the risk values assigned to the atomic formulas of F is 1. Consequently, F is valid in Ł, iff the player has a winning strategy for all assignments of risk values to the atomic formulas of F.

5.2.2 Fermüller's and Kosik's extension of Giles's game for SŁ

Fermüller and Kosik present an extension of Giles's game for their logic SŁ. The essential idea is to replace the 'dispersive elementary experiments' by 'indeterministic evaluations' over precisification spaces.

As mentioned in subsection 5.2.1, Giles's game consists of two independent components which are modified in the following way:

1. Betting for positive results of experiments

- The two players of the game agree to pay 1 € to the opponent player for each atomic statement if it is *false* according to a randomly chosen admissible precisification.
- Given a precisification space $\prod = \langle W, e, \mu \rangle$, the risk value $\langle p \rangle_{\prod}$, denoting the probability of a negative result, associated with the proposition p is defined as follows: $\langle p \rangle_{\prod} = \mu(\{\pi \in W | e(\pi, p) = 0\})$; the risk value for the atomic formula $\perp$ is $\langle \perp \rangle_{\prod} = 0$; the risk value for the multiset $\{p_1, ..., p_m\}$ is $\langle p_1, ..., p_m \rangle = \sum_{i=1}^{m} \langle p_i \rangle_{\prod}$, while for the empty multiset the risk value $\langle \rangle_{\prod} = 0$;

2. A dialogue game for the reduction of compound formulas

Formulas in SŁ are built up from propositional variables, $\perp$ and the connectives $\rightarrow$ and S which was introduced for SŁ (all formulas containing other connectives are equivalent to formulas built with these connectives as described in subsection 5.2.1). Consequently, also the rules of the dialogue game had to be extended as follows:

Rule 5. *If player 1 asserts SA then player 1 has to assert that A holds at every precisification point π which player 2 may choose and vice versa.*

The introduction of this new rule for the connective S makes it necessary to add the information of the precisification point in question for the formula A. The notion A^{π} indicates that 'A holds at the precisification point π' (by contrast, A^{ε} indicates that A is not referring to a particular precisification point). In consequence, also the rules defined before have to be adapted. Considering Rule 1 for the implication $\rightarrow$ we have to stipulate that in applying the rule the precisification point index of $A \rightarrow B$ is conferred also to the subformulas A and B. If Rule 5 for the connective S is applied to an already indexed formula, the index is overwritten by an index chosen by the opponent. Hence, also the definition of the risk values was augmented by $\langle p^{\pi} \rangle_{\prod} = 1 - e(\pi, p)$ with respect to the precisification space $\prod = \langle W, e, \mu \rangle$.

Theorem 5.2. *A formula F is valid in SŁ iff I have a winning strategy for the game starting with my assertion of F for every precisification space $\prod$.*

A proof of the adequacy of the game as well as for the theorem can be found in [FK].

5.3 Open questions and future work

The logic $S\text{Ł}$ is a first approach to combine supervaluationism with degree-based reasoning. The introduction of the unary operator S allows to formalize the concept of super-truth. However, there are still a lot of open questions regarding this logic:

- can there be introduced other modal operators which seem to be relevant in modeling propositional attitudes arising in contexts of vagueness?
- how can quantifiers be defined?
- is there a possibility to model higher-order vagueness?

Chapter 6

Conclusions

In conclusion, I want to represent the results of my work with respect to the three objectives made in the introductory chapter of this master thesis:

- Vague terms, such as 'tall', 'red' or 'bald', are part of our everyday life and are characterized by admitting borderline cases, having fuzzy boundaries and being susceptible to sorites paradoxes. Vagueness poses a fundamental challenge for classical logic and semantics which classifies all propositions as either *true* or *false*.

- In analytic philosophy there is a lively discourse going on about vagueness, producing different approaches inside and outside classical logic. The plurality of contributions allows a lot of different classifications and one of the biggest challenges is to find a useful one. The classification which I chose was proposed by Fermüller [Fer03] and divides the different approaches into 5 groups: epistemic approaches, gap theories, degree theories, supervaluationism and pragmatic approaches.

- The phenomenon of vagueness poses a great challenge also to computer scientists when they want to mimic human behavior and reasoning which often is based on vague information. Fuzzy logics based on t-norms are for most people working in this field *the* methodology for the treatment of vagueness. Fuzzy logic is a special type of many-valued logic with a comparative notion of truth, taking truth degrees from the real unit interval [0,1]. In the last decades it has become a well-established sound formal system with numerous applications in the field of automatic control and expert systems.

- In literature hardly any contributions to the discourse on vagueness can be found that consider the philosophic discussion as well as fuzzy logics and try to demonstrate a way to derive a fuzzy logic from the first principles of approximate reasoning. In the 1970s Robin Giles tried to derive logics form fundamental reasoning principles and presented a strategic two-person dialogue game as a formal model for reasoning, which has been resumed and extended by other researchers. One of these extensions discussed in this thesis is the logic $S\text{Ł}$ which combines supervaluationism and degree-based reasoning and which is the set of those formulas that can be asserted by a player in a dialogue game over an arbitrary precisification space without having to expect a loss of money. There are also other approaches trying to characterize t-norm based fuzzy logics by dialogue and betting games as dialogue games cover a wide range of topics relevant for approximate reasoning, see [Fer07].

- Traditional approaches developed for fundamental problems in logic can help to explain fuzzy logics and to derive mathematical structures used in fuzzy logics from the first assumptions about correct reasoning - reason enough why the research in this field should be of great interest also for computer scientists.

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