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Choice logics and their computational properties

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ABSTRACT

Qualitative Choice Logic (QCL) and Conjunctive Choice Logic (CCL) are formalisms for preference handling, with especially QCL being well established in the field of AI. So far, analyses of these logics need to be done on a case-by-case basis, albeit they share several common features. This calls for a more general choice logic framework, with QCL and CCL as well as some of their derivatives being particular instantiations. We provide such a framework, which allows us, on the one hand, to easily define new choice logics and, on the other hand, to examine properties of different choice logics in a uniform setting. In particular, we investigate strong equivalence, a core concept in non-classical logics for understanding formula simplification, and computational complexity. Our analysis also yields new results for QCL and CCL. For example, we show that the main reasoning task regarding preferred models of choice logic formulas is Θ_2^{P}-complete for QCL and CCL, while being Δ_2^{P}-complete for a newly introduced choice logic. The complexity of preferred model entailment for choice logic theories ranges from coNP to Π_2^{P}.

1. Introduction

Representing preferences and reasoning about them is a key challenge in many areas of AI research. One of the most fruitful approaches to preference representation has been the use of logic-based formalisms [19,33]. Two closely related examples from the literature are Qualitative Choice Logic (QCL) [11] and Conjunctive Choice Logic (CCL) [10]. Especially QCL has proven to be a useful preference formalism, with applications ranging from logic programming [12] to alert correlation [3] to database querying [31]. However, several key computational properties of QCL and CCL have not been studied yet. This includes strong equivalence, a tool to understand formula simplification, and the computational complexity of main reasoning tasks.

QCL extends classical propositional logic with a non-classical connective $\vec{\times}$ called ordered disjunction. Intuitively, $F \vec{\times} G$ means that it is preferable to satisfy F but, if that is not possible, satisfying G is also acceptable. More specifically, interpretations ascribe a number, called satisfaction degree, to QCL-formulas. The preferred models are those interpretations that satisfy the formula to the least degree. Preferences or soft constraints that can be modeled by QCL and similar logics naturally occur in many knowledge representation contexts. Consider for example a product configuration system for a car [37,24]. Now assume a user of the system is interested in the sports version of the car and wants an automatic transmission. Moreover, the user would like to have either cruise control or a lane assistant, but cruise control is more important to them. Such a query could be formalized in QCL as follows:

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$$sport \wedge automatic \wedge (cruise \vec{\times} lane).$$

Without other information, this formula has the following preferred models $\{sport, automatic, cruise\}$ and $\{sport, automatic, cruise, lane\}$, both resulting in a satisfaction degree of 1 under QCL-semantics. On the other hand, the interpretation $\{sport, automatic, lane\}$ results in a degree of 2, meaning that it also satisfies the formula, but not optimally. But what if the knowledge base of the system contains the information that it is not possible to have the sport version of the car with cruise control? In that case, $\{sport, automatic, lane\}$ becomes the preferred model since the interpretations that previously resulted in a degree of 1 no longer satisfy the formula together with the knowledge base. In practice, such configuration specifications can be much larger. It is evident that choice logics can serve as a formal tool to analyze such specifications, e.g. to determine which are the common elements in all models that meet the user's requirements best.

Similarly to ordered disjunction, CCL introduces ordered conjunction ($\vec{\odot}$), where the intended meaning of $F\vec{\odot}G$ is that it is preferable to satisfy both F and G, but satisfying only F is also acceptable. However, the two types of preferences expressed by QCL and CCL are certainly not the only ones. One could, for example, think of a more fine-grained choice connective: given $F \circ G$, it would be best to satisfy both F and G, second best to satisfy only F, and third best to satisfy only G. This natural preference cannot be succinctly represented in QCL and CCL. One could also desire a connective with the same basic behavior as $\vec{\times}$ in QCL, but in which satisfaction degrees are handled in a different way. There is a multitude of interesting logics related to QCL and CCL that have yet to be defined, and which may very well prove to be as useful as QCL.

In this paper, we propose a general framework for choice logics that, on the one hand, makes it easy to define new choice logics by specifying one or more choice connectives and, on the other hand, allows us to settle open questions regarding the computational properties of QCL and CCL in a uniform way. In detail, our main contributions are as follows:

- We formally define a framework that captures both QCL and CCL, as well as infinitely many new related logics. To showcase the versatility of our framework we explicitly introduce two such new logics called Lexicographic Choice Logic (LCL) and Simple Conjunctive Choice Logic (SCCL).
- We characterize strong equivalence via simpler equivalence notions for large classes of choice logics. This further enables us to analyze properties related to strong equivalence more easily, and also provides valuable insights into the nature of choice logics.
- We analyze the computational complexity of choice logics in detail.[1] For example, we show that the complexity of the main decision problem regarding preferred models of choice logic formulas ranges from NP- to Δ_2^{P}-completeness with QCL and CCL being located in between (Θ_2^{P}-complete). Depending on the choice logic and on how preferred models of theories are defined, the complexity of preferred model entailment for theories ranges from coNP to Π_2^{P}, with some problems being complete for Θ_2^{P}, Δ_2^{P}, and even $\Delta_2^{\mathsf{P}}[\mathcal{O}(\log^2 n)]$. The complexity of checking for strong equivalence follows from our characterization via simpler equivalence notions for large classes of choice logics.

Related work QCL and CCL are not the only logic-based preference representation formalisms. Other prominent examples include the preference logics introduced by von Wright [39] and van Benthem et al. [5]; for a more complete overview, see the surveys by Domshlak et al. [19] and Bienvenu et al. [9]. Most of these formalisms differ from choice logics in that they only represent preferences, while QCL and CCL integrate the representation of truth and preference in one formalism. One exception is recent work by Charalambidis et al. [15], which is conceptually closely related to our LCL, but differs in that formulas are assigned lists of truth values instead of satisfaction degrees.

From a technical standpoint, many non-monotonic logics are closely related to QCL, as they are inherently connected to preferences [36]. This is particularly true for propositional circumscription and possibilistic logic [11]. However, unlike choice logics, these formalisms are not primarily designed to represent preferences and often rely on constructs outside of the logical language (circumscription policy, possibility distribution) to represent knowledge. It is also worth noting that choice logics are technically very different from traditional infinite-valued logics [25] as, for example, choice logics use classical interpretations that make atoms either true or false.

There are also systems that are based on QCL's ordered disjunction, but do not fit into our framework. For instance, Jiang et al. [26] introduce a modal logic that contains a binary connective with a similar meaning to ordered disjunction, while Zhang & Thielscher [40] state that they took inspiration from QCL for their prioritized disjunction, which is used to reason about game strategies.

The original semantics of ordered disjunction in so-called Logic Programs with Ordered Disjunction (LPODs) [12] is based directly on the semantics of QCL. Strong equivalence in this context was investigated [22]. Furthermore, an alternative semantics for LPODs has been laid out recently [16].

The remainder of the paper is structured as follows: Section 2 contains the definition of our framework, examples for logics that can be defined within it, and a result on the expressiveness of choice logics. The notion of strong equivalence is examined in Section 3, and the complexity of choice logic formulas is analyzed in Section 4. In Section 5 we investigate the notion of preferred model entailment with respect to logical properties and computational complexity. Finally, Section 7

[1] The complexity of some decision problems pertaining to QCL was conjectured by Lang [29] but never formally investigated.

contains a summary of our results and pointers to future work. This paper is an extended version of [8], containing full proofs for all propositions as well as new results on choice logic theories (Section 5).

2. Choice logic framework

In this section, we introduce a framework for choice logics which generalizes QCL and CCL. We then give examples of logics belonging to this framework and provide a result on the expressiveness of choice logics. In what follows, PL stands for classical propositional logic, $\mathcal{U}$ denotes the alphabet of propositional variables, and an interpretation $\mathcal{I}$ is defined as a set of propositional variables such that $a \in \mathcal{I}$ if and only if a is set to true by $\mathcal{I}$. If $\mathcal{I}$ satisfies a classical formula F, we write $\mathcal{I} \models F$.

2.1. Syntax and semantics

A choice logic has two types of connectives: classical connectives (here we use $\neg$, $\wedge$, and $\vee$), and binary choice connectives, with which preferences can be expressed.

Definition 1. The set of choice connectives $\mathcal{C}_{\mathcal{L}}$ of a choice logic $\mathcal{L}$ is a finite set of symbols such that $\mathcal{C}_{\mathcal{L}} \cap \{\neg, \wedge, \vee\} = \emptyset$. The set $\mathcal{F}_{\mathcal{L}}$ of formulas of $\mathcal{L}$ is defined inductively as follows:

1. $a \in \mathcal{F}_{\mathcal{L}}$ for all $a \in \mathcal{U}$;
2. if $F \in \mathcal{F}_{\mathcal{L}}$, then $(\neg F) \in \mathcal{F}_{\mathcal{L}}$;
3. if $F, G \in \mathcal{F}_{\mathcal{L}}$, then $(F \circ G) \in \mathcal{F}_{\mathcal{L}}$ for $\circ \in (\{\wedge, \vee\} \cup \mathcal{C}_{\mathcal{L}})$.

$var(F)$ denotes the set of all variables in a formula $F \in \mathcal{F}_{\mathcal{L}}$.

For example, $\mathcal{C}_{\text{QCL}} = \{\vec{\times}\}$, $\mathcal{C}_{\text{CCL}} = \{\vec{\odot}\}$, and $\mathcal{C}_{\text{PL}} = \emptyset$. Formulas that do not contain a choice connective are classical formulas. The semantics of a choice logic is given by two functions, satisfaction degree and optionality. The satisfaction degree of a formula given an interpretation is either a natural number or ∞. The lower this degree, the more preferable the interpretation. The optionality of a formula describes the maximum finite satisfaction degree that this formula can be ascribed. As we will see in Section 2.2, optionality is used to penalize less preferable interpretations.

Definition 2. The optionality of a choice connective $\circ \in \mathcal{C}_{\mathcal{L}}$ in a choice logic $\mathcal{L}$ is given by a function $opt^{\circ}_{\mathcal{L}} \colon \mathbb{N}^2 \to \mathbb{N}$ such that $opt^{\circ}_{\mathcal{L}}(k, \ell) \leq (k+1) \cdot (\ell+1)$ for all $k, \ell \in \mathbb{N}$. The optionality of an $\mathcal{L}$-formula is given via $opt_{\mathcal{L}} \colon \mathcal{F}_{\mathcal{L}} \to \mathbb{N}$ with

1. $opt_{\mathcal{L}}(a) = 1$, for every $a \in \mathcal{U}$;
2. $opt_{\mathcal{L}}(\neg F) = 1$;
3. $opt_{\mathcal{L}}(F \wedge G) = max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$;
4. $opt_{\mathcal{L}}(F \vee G) = max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$;
5. $opt_{\mathcal{L}}(F \circ G) = opt^{\circ}_{\mathcal{L}}(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$, $\circ \in \mathcal{C}_{\mathcal{L}}$.

Classical formulas have an optionality of 1, meaning that they will always be satisfied to a degree of either 1 or ∞. For any choice connective $\circ$, the optionality of $F \circ G$ is bounded such that $opt_{\mathcal{L}}(F \circ G) \leq (opt_{\mathcal{L}}(F)+1) \cdot (opt_{\mathcal{L}}(G)+1)$. The reason for this is that there are $opt_{\mathcal{L}}(F)$ many finite degrees that could be ascribed to F, plus the infinite degree ∞. Likewise for G. Thus, there are at most $(opt_{\mathcal{L}}(F)+1) \cdot (opt_{\mathcal{L}}(G)+1)$ possibilities when combining the degrees of F and G. Next, we define the satisfaction degree of choice logic formulas given an interpretation. In the following, we write $\overline{\mathbb{N}}$ for $(\mathbb{N} \cup \{\infty\})$.

Definition 3. The satisfaction degree of a choice connective $\circ \in \mathcal{C}_{\mathcal{L}}$ in a choice logic $\mathcal{L}$ is given by a function $deg^{\circ}_{\mathcal{L}} \colon \mathbb{N}^2 \times \overline{\mathbb{N}}^2 \to \overline{\mathbb{N}}$ where either $deg^{\circ}_{\mathcal{L}}(k, \ell, m, n) \leq opt^{\circ}_{\mathcal{L}}(k, \ell)$ or $deg^{\circ}_{\mathcal{L}}(k, \ell, m, n) = \infty$ holds for all $k, \ell \in \mathbb{N}$ and all $m, n \in \overline{\mathbb{N}}$. The satisfaction degree of an $\mathcal{L}$-formula under an interpretation is given via $deg_{\mathcal{L}} \colon 2^{\mathcal{U}} \times \mathcal{F}_{\mathcal{L}} \to \overline{\mathbb{N}}$ with

1. $deg_{\mathcal{L}}(\mathcal{I}, a) = \begin{cases} 1 & \text{if } a \in \mathcal{I}, \\ \infty & \text{otherwise,} \end{cases}$ for every $a \in \mathcal{U}$;
2. $deg_{\mathcal{L}}(\mathcal{I}, \neg F) = \begin{cases} 1 & \text{if } deg_{\mathcal{L}}(\mathcal{I}, F) = \infty, \\ \infty & \text{otherwise;} \end{cases}$
3. $deg_{\mathcal{L}}(\mathcal{I}, F \wedge G) = max(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$;
4. $deg_{\mathcal{L}}(\mathcal{I}, F \vee G) = min(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$;
5. $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg^{\circ}_{\mathcal{L}}(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G), deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$, $\circ \in \mathcal{C}_{\mathcal{L}}$.

We also write $\mathcal{I} \models_m^{\mathcal{L}} F$ for $deg_{\mathcal{L}}(\mathcal{I}, F) = m$. If $m < \infty$, we say that $\mathcal{I}$ satisfies F (to a finite degree), and if $m = \infty$, then $\mathcal{I}$ does not satisfy F. If F is a classical formula, then $\mathcal{I} \models_1^{\mathcal{L}} F \iff \mathcal{I} \models F$ and $\mathcal{I} \models_\infty^{\mathcal{L}} F \iff \mathcal{I} \not\models F$. The symbols $\top$ and $\bot$ are shorthand for the formulas $(a \vee \neg a)$ and $(a \wedge \neg a)$, where a can be any variable. We have $opt_{\mathcal{L}}(\top) = opt_{\mathcal{L}}(\bot) = 1$, $deg_{\mathcal{L}}(\mathcal{I}, \top) = 1$ and $deg_{\mathcal{L}}(\mathcal{I}, \bot) = \infty$ for any interpretation $\mathcal{I}$ in every choice logic $\mathcal{L}$.

The semantics of the classical connectives are fixed and are the same as for QCL and CCL. $F \wedge G$ is assigned the maximum degree of F and G because both formulas need to be satisfied. Conversely, we use the minimum degree for $F \vee G$ since satisfying either option suffices, and we do not need to concern ourselves with the less preferable option. Observe that it is still necessary to define $opt_{\mathcal{L}}(F \vee G) = max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$, as the case that either option is not satisfied has to be allowed for. Regarding negation, note that $\neg F$ can only assume the degrees 1 or ∞. Therefore, $\neg$ can be seen as classical negation when applied to a classical formula, and as a tool to neutralize satisfaction degrees otherwise. In order to define a form of negation that results in different degrees of satisfaction, we would need to keep track of degrees of dissatisfaction. We observe that the semantics of the classical connectives used here are not the only possible ones. See also the brief discussion on PQCL and QCL+ in Section 2.2.

A classical connective that is not considered in choice logics is that of implication ($\to$). One could of course, as in classical logic, see the formula $A \to B$ as shorthand for $\neg A \vee B$, with the result that if A is not satisfied then $\neg A \vee B$ is satisfied to a degree of 1, and if A is satisfied (to some finite degree) then $\neg A \vee B$ is satisfied to the same degree as B. In our opinion, this is a sensible interpretation of $A \to B$ in the context of choice logics. If A is true, then B must also be satisfied (to some finite degree). Whether A is satisfied optimally is not relevant in this case. However, if one so wishes, alternative semantics for implication, possibly in the spirit of certain many-valued logics, can be added to logics of our framework via additional choice connectives.

From Definitions 2 and 3 it follows that the satisfaction degree of a choice logic formula is bounded by its optionality, as intended:

Lemma 1. *Let $\mathcal{L}$ be a choice logic. For all interpretations $\mathcal{I}$ and all $\mathcal{L}$-formulas F, either $deg_{\mathcal{L}}(\mathcal{I}, F) \leq opt_{\mathcal{L}}(F)$ or $deg_{\mathcal{L}}(\mathcal{I}, F) = \infty$.*

Proof. By structural induction. Let $\mathcal{L}$ be a choice logic and $\mathcal{I}$ an interpretation.

Base case: if $a \in \mathcal{U}$, then either $deg_{\mathcal{L}}(\mathcal{I}, a) = 1 = opt_{\mathcal{L}}(a)$ or $deg_{\mathcal{L}}(\mathcal{I}, a) = \infty$.

Step case: let F and G be $\mathcal{L}$-formulas with $opt_{\mathcal{L}}(F) = k$, $opt_{\mathcal{L}}(G) = \ell$, $deg_{\mathcal{L}}(\mathcal{I}, F) = m$, and $deg_{\mathcal{L}}(\mathcal{I}, G) = n$. By the I.H., either $m \leq k$ or $m = \infty$. Likewise, $n \leq \ell$ or $n = \infty$. The case for $\neg F$ is analogous to the base case. For $F \wedge G$, either $deg_{\mathcal{L}}(\mathcal{I}, F \wedge G) = max(m, n) \leq max(k, \ell) = opt_{\mathcal{L}}(F \wedge G)$ or $deg_{\mathcal{L}}(\mathcal{I}, F \wedge G) = \infty$. Analogous for $F \vee G$. For any choice connective $\circ \in \mathcal{C}_{\mathcal{L}}$, we have that $opt_{\mathcal{L}}(F \circ G) = opt_{\mathcal{L}}^{\circ}(k, \ell)$ and $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg_{\mathcal{L}}^{\circ}(k, \ell, m, n)$. By Definition 3, either $deg_{\mathcal{L}}^{\circ}(k, \ell, m, n) \leq opt_{\mathcal{L}}^{\circ}(k, \ell)$ or $deg_{\mathcal{L}}^{\circ}(k, \ell, m, n) = \infty$. □

Moreover, note that only those variables that actually occur in a formula F can influence the optionality and satisfaction degree of F, meaning that we can assume $\mathcal{I} \subseteq var(F)$ for any interpretation $\mathcal{I}$:

Lemma 2. *Let $\mathcal{L}$ be a choice logic, $\mathcal{I}$ an interpretation, and F an $\mathcal{L}$-formula. Let $\mathcal{J} = \mathcal{I} \cap var(F)$. Then $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{J}, F)$.*

Proof. Follows directly from the fact that the semantics of all connectives in a choice logic are given by functions over the optionalities and satisfaction degrees of their operands. □

An interpretation that satisfies a formula to a finite degree will be referred to as a *model* of that formula. However, often we are more interested in the most preferable models of a formula, i.e., the models with the lowest satisfaction degree. We call these the preferred models of a fomula.

Definition 4. Let $\mathcal{L}$ be a choice logic, F an $\mathcal{L}$-formula, and $\mathcal{I}$ an interpretation. $\mathcal{I}$ is a model of F, written as $\mathcal{I} \in Mod_{\mathcal{L}}(F)$, if $deg_{\mathcal{L}}(\mathcal{I}, F) < \infty$. $\mathcal{I}$ is a preferred model of F, written as $\mathcal{I} \in Prf_{\mathcal{L}}(F)$, if $\mathcal{I} \in Mod_{\mathcal{L}}(F)$ and, for all interpretations $\mathcal{J}$, $deg_{\mathcal{L}}(\mathcal{I}, F) \leq deg_{\mathcal{L}}(\mathcal{J}, F)$.

By the above definition, $Prf_{\mathcal{L}}(F) = \emptyset$ if and only if $deg_{\mathcal{L}}(\mathcal{I}, F) = \infty$ for all interpretations $\mathcal{I}$. We observe that, while the model relation is monotonic, the preferred model relation is generally non-monotonic (see e.g. Example 1).

2.2. Examples of choice logics

To define a choice logic $\mathcal{L}$, it suffices to specify the choice connectives $\mathcal{C}_{\mathcal{L}}$ of that logic, and to provide the optionality- and satisfaction degree functions of every $\circ \in \mathcal{C}_{\mathcal{L}}$. The simplest example for a choice logic is classical propositional logic, with $\mathcal{C}_{\mathrm{PL}} = \emptyset$. Since there are no choice connectives in PL, all formulas are built using $\neg$, $\wedge$, and $\vee$. From the way the classical connectives handle satisfaction degrees and optionality (cf. Definitions 2 and 3), we can conclude that, for all PL-formulas F, $\mathcal{I} \models_1^{\mathrm{PL}} F \iff \mathcal{I} \models F$ and $\mathcal{I} \models_\infty^{\mathrm{PL}} F \iff \mathcal{I} \not\models F$. Thus, $\mathcal{I} \in Prf_{\mathrm{PL}}(F) \iff \mathcal{I} \models F$.

QCL can be expressed in our framework as follows:

Table 1
The classical connectives $\wedge$, $\vee$ and the choice connectives $\vec{\times}$ (QCL), $\vec{\odot}$ (CCL), $\vec{\diamond}$ (LCL), and $\vec{\circledast}$ (SCCL) applied to atoms.

$\mathcal{I}$	$a \wedge b$	$a \vee b$	$a \vec{\times} b$	$a \vec{\odot} b$	$a \vec{\diamond} b$	$a \vec{\circledast} b$
$\emptyset$	∞	∞	∞	∞	∞	∞
$\{b\}$	∞	1	2	∞	3	∞
$\{a\}$	∞	1	1	2	2	2
$\{a, b\}$	1	1	1	1	1	1

Definition 5. QCL is the choice logic such that $\mathcal{C}_{\text{QCL}} = \{\vec{\times}\}$ and, if $k = opt_{\text{QCL}}(F)$, $\ell = opt_{\text{QCL}}(G)$, $m = deg_{\text{QCL}}(\mathcal{I}, F)$, and $n = deg_{\text{QCL}}(\mathcal{I}, G)$, then

$$opt_{\text{QCL}}(F \vec{\times} G) = opt^{\vec{\times}}_{\text{QCL}}(k, \ell) = k + \ell, \text{ and}$$

$$deg_{\text{QCL}}(\mathcal{I}, F \vec{\times} G) = deg^{\vec{\times}}_{\text{QCL}}(k, \ell, m, n) = \begin{cases} m & \text{if } m < \infty; \\ n + k & \text{if } m = \infty, n < \infty; \\ \infty & \text{otherwise.} \end{cases}$$

In the above definition we can see how optionality is used to penalize non-satisfaction: if F is satisfied, then $deg_{\text{QCL}}(\mathcal{I}, F \vec{\times} G) = deg_{\text{QCL}}(\mathcal{I}, F) \le opt_{\mathcal{L}}(F)$; if F is not satisfied, but G is, then $deg_{\text{QCL}}(\mathcal{I}, F \vec{\times} G) = deg_{\text{QCL}}(\mathcal{I}, G) + opt_{\mathcal{L}}(F) > opt_{\mathcal{L}}(F)$. Thus, any interpretation which satisfies F is automatically more preferable than one that does not.

Consider the simple formula $a \vec{\times} b$. Then, $\{a\}$ and $\{a, b\}$ are assigned degree 1, while $\{b\}$ is assigned degree 2 and $\emptyset$ is assigned degree ∞ (see also Table 1). From there, we can infer that $\{a\}$ and $\{a, b\}$ are preferred models of $a \vec{\times} b$, while $\emptyset$ and $\{b\}$ are not. Let us now consider a somewhat more involved example.

Example 1. Let $F = (a \vec{\times} c) \wedge (b \vec{\times} c)$. Consider the interpretation $\{a, b\}$. Then $deg_{\text{QCL}}(\{a, b\}, a \vec{\times} c) = deg_{\text{QCL}}(\{a, b\}, b \vec{\times} c) = 1$. Hence, $deg_{\text{QCL}}(\{a, b\}, F) = max\big(deg_{\text{QCL}}(\{a, b\}, a \vec{\times} c), deg_{\text{QCL}}(\{a, b\}, b \vec{\times} c)\big) = 1$.

Analogously, $deg_{\text{QCL}}(\{a, b, c\}, F) = 1$. For $\{a, c\}$ we have $deg_{\text{QCL}}(\{a, c\}, a \vec{\times} c) = 1$ and $deg_{\text{QCL}}(\{a, c\}, b \vec{\times} c) = 2$. Therefore $deg_{\text{QCL}}(\{a, c\}, F) = 2$. Similarly, we have $deg_{\text{QCL}}(\mathcal{I}', F) = 2$ for $\mathcal{I}' \in \{\{c\}, \{b, c\}\}$. Finally, $deg_{\text{QCL}}(\mathcal{I}'', F) = \infty$ for $\mathcal{I}'' \in \{\emptyset, \{a\}, \{b\}\}$. Thus, $\{a, b\}, \{a, b, c\} \in Prf_{\text{QCL}}(F)$.

Consider now that we obtain the additional information that a and b can not be satisfied simultaneously, i.e., $F' = (a \vec{\times} c) \wedge (b \vec{\times} c) \wedge \neg(a \wedge b)$. Then $deg_{\text{QCL}}(\mathcal{J}, F') = 2$ for $\mathcal{J} \in \{\{c\}, \{a, c\}, \{b, c\}\}$ and $deg_{\text{QCL}}(\mathcal{J}', F') = \infty$ for $\mathcal{J}' \in \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. In fact, F' can not be satisfied to a degree of 1, and we have that $\{c\}, \{a, c\}, \{b, c\} \in Prf_{\text{QCL}}(F')$.

From Brewka et al. [11] it is known that $\vec{\times}$ is associative, meaning that given arbitrary QCL-formulas A, B, and C the formulas $((A \vec{\times} B) \vec{\times} C)$ and $(A \vec{\times} (B \vec{\times} C))$ always have the same optionality and satisfaction degrees. We can therefore write $F_1 \vec{\times} F_2 \ldots \vec{\times} F_n$ to express that we prefer F_1 to F_2, F_2 to F_3, and so on. For variables $a_1, \ldots, a_n$ with $a_i \neq a_j$ for all $i \neq j$, we have that $\{a_i\} \models^{\text{QCL}}_i (a_1 \vec{\times} a_2 \ldots \vec{\times} a_n)$.

Next, we formally define CCL. However, our function for the satisfaction degree of $\vec{\odot}$ differs from the one given by Boudjelida & Benferhat [10]. In fact, the original definition of CCL, although it can be expressed in our framework if desired, fails to capture the intended meaning of ordered conjunction.[2] From now on, when defining a choice logic, we will write $opt_{\mathcal{L}}(F \circ G) = f(k, \ell)$ instead of $opt_{\mathcal{L}}(F \circ G) = opt^{\circ}_{\mathcal{L}}(k, \ell) = f(k, \ell)$ for the sake of brevity. Analogously for $deg_{\mathcal{L}}(F \circ G)$.

Definition 6. CCL is the choice logic such that $\mathcal{C}_{\text{CCL}} = \{\vec{\odot}\}$ and, if $k = opt_{\text{CCL}}(F)$, $\ell = opt_{\text{CCL}}(G)$, $m = deg_{\text{CCL}}(\mathcal{I}, F)$, and $n = deg_{\text{CCL}}(\mathcal{I}, G)$, then

$$opt_{\text{CCL}}(F \vec{\odot} G) = k + \ell, \text{ and}$$

$$deg_{\text{CCL}}(\mathcal{I}, F \vec{\odot} G) = \begin{cases} n & \text{if } m = 1, n < \infty; \\ m + \ell & \text{if } m < \infty \text{ and } (m > 1 \text{ or } n = \infty); \\ \infty & \text{otherwise.} \end{cases}$$

Intuitively, given $F \vec{\odot} G$, it is best to satisfy both F and G, while satisfying only F is less preferable, but still acceptable. As intended, $\vec{\odot}$ is associative under this new semantics for CCL. This will be shown formally in Section 3 (see Lemma 11). For a series of distinct propositional variables, we have that $\{a_1, \ldots, a_{(n-i+1)}\} \models^{\text{CCL}}_i (a_1 \vec{\odot} a_2 \ldots \vec{\odot} a_n)$.

[2] Under the semantics described in Definition 8 of the original CCL-paper [10], a formula $F \vec{\odot} G$ will always be ascribed a degree of either ∞ or 1: if $deg_{\text{CCL}}(\mathcal{I}, F) = \infty$ then $deg_{\text{CCL}}(\mathcal{I}, F \vec{\odot} G) = \infty$; if $deg_{\text{CCL}}(\mathcal{I}, F) = 1$ then $deg_{\text{CCL}}(\mathcal{I}, F \vec{\odot} G) = deg_{\text{CCL}}(\mathcal{I}, G)$; if $1 < deg_{\text{CCL}}(\mathcal{I}, F) < \infty$ then $deg_{\text{CCL}}(\mathcal{I}, F \vec{\odot} G) = deg_{\text{CCL}}(\mathcal{I}, F) + opt_{\text{CCL}}(G)$. However, the last case never applies since classical formulas can only assume degrees ∞ or 1.

Example 2. Let $F = (a\vec{\odot}c) \wedge (b\vec{\odot}c)$. Consider the interpretation $\{a, b, c\}$. Then $deg_{\mathrm{CCL}}(\{a, b, c\}, a\vec{\odot}c) = deg_{\mathrm{CCL}}(\{a, b, c\}, b\vec{\odot}c) = 1$. It follows that $deg_{\mathrm{CCL}}(\{a, b, c\}, F) = max\left(deg_{\mathrm{CCL}}(\{a, b, c\}, a\vec{\odot}c), deg_{\mathrm{CCL}}(\{a, b, c\}, b\vec{\odot}c)\right) = 1$.

Now, for $\{a, b\}$ we have $deg_{\mathrm{CCL}}(\{a, b\}, a\vec{\odot}c) = 2$ and $deg_{\mathrm{CCL}}(\{a, b\}, b\vec{\odot}c) = 2$. Hence, $deg_{\mathrm{CCL}}(\{a, b\}, F) = 2$. For $\{a, c\}$ we have $deg_{\mathrm{CCL}}(\{a, c\}, a\vec{\odot}c) = 1$ and $deg_{\mathrm{CCL}}(\{a, c\}, b\vec{\odot}c) = \infty$. Therefore $deg_{\mathrm{CCL}}(\{a, c\}, F) = \infty$. Similarly, it can be verified that $deg_{\mathrm{CCL}}(\mathcal{I}, F) = \infty$ for $\mathcal{I} \in \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. Thus, $\{a, b, c\} \in Prf_{\mathrm{CCL}}(F)$.

While QCL and CCL feature only a single choice connective, our framework now easily allows to define a choice logic with the choice connectives of both QCL and CCL, i.e. $\mathcal{C}_{\mathrm{QCCL}} = \{\vec{\times}, \vec{\odot}\}$, $opt^{\vec{\times}}_{\mathrm{QCCL}} = opt^{\vec{\times}}_{\mathrm{QCL}}$, $deg^{\vec{\times}}_{\mathrm{QCCL}} = deg^{\vec{\times}}_{\mathrm{QCL}}$, and likewise for CCL and $\vec{\odot}$. In this way, a choice logic with several choice connectives can always be seen as the combination of other choice logics.

We now introduce a new logic, called Lexicographic Choice Logic (LCL), whose choice connective allows us to deal with satisfaction degrees in a more fine-grained manner.

Definition 7. LCL is the choice logic such that $\mathcal{C}_{\mathrm{LCL}} = \{\vec{\diamond}\}$ and, if $k = opt_{\mathrm{LCL}}(F)$, $\ell = opt_{\mathrm{LCL}}(G)$, $m = deg_{\mathrm{LCL}}(\mathcal{I}, F)$, and $n = deg_{\mathrm{LCL}}(\mathcal{I}, G)$, then

$$opt_{\mathrm{LCL}}(F\vec{\diamond}G) = (k+1)\cdot(\ell+1) - 1, \text{ and}$$

$$deg_{\mathrm{LCL}}(\mathcal{I}, F\vec{\diamond}G) = \begin{cases} (m-1)\cdot\ell + n & \text{if } m < \infty, n < \infty; \\ k\cdot\ell + m & \text{if } m < \infty, n = \infty; \\ k\cdot\ell + k + n & \text{if } m = \infty, n < \infty; \\ \infty & \text{otherwise.} \end{cases}$$

Given $F\vec{\diamond}G$ it is best to satisfy both F and G, second best to satisfy only F, and third best to satisfy only G. Satisfying neither F nor G is not acceptable.

Example 3. Let $F = (a\vec{\diamond}(b\vec{\diamond}c))$. The only interpretation that ascribes a degree of ∞ to F is $\emptyset$. The remaining 7 interpretations applicable to F each result in a different degree, ranging from 1 to 7: $\{a, b, c\} \models^{\mathrm{LCL}}_1 F$, $\{a, b\} \models^{\mathrm{LCL}}_2 F$, $\{a, c\} \models^{\mathrm{LCL}}_3 F$, $\{a\} \models^{\mathrm{LCL}}_4 F$, $\{b, c\} \models^{\mathrm{LCL}}_5 F$, $\{b\} \models^{\mathrm{LCL}}_6 F$, and $\{c\} \models^{\mathrm{LCL}}_7 F$.

In QCL, the same information must be encoded by explicitly ranking the individual interpretations, i.e., by the formula $F' = (a \wedge b \wedge c)\vec{\times}(a \wedge b)\vec{\times}(a \wedge c)\vec{\times}a\vec{\times}(b \wedge c)\vec{\times}b\vec{\times}c$.

As exemplified above, LCL, in contrast to QCL (or CCL), enables us to succinctly encode lexicographic orderings over variables. This will be important when analyzing the computational complexity of LCL in Sections 4 and 5.

It is also possible to define a choice logic that does not use optionality. We call the following new logic Simple Conjunctive Choice Logic (SCCL).

Definition 8. SCCL is the choice logic such that $\mathcal{C}_{\mathrm{SCCL}} = \{\vec{\circledast}\}$ and, if $k = opt_{\mathrm{SCCL}}(F)$, $m = deg_{\mathrm{SCCL}}(\mathcal{I}, F)$, and $n = deg_{\mathrm{SCCL}}(\mathcal{I}, G)$, then

$$opt_{\mathrm{SCCL}}(F\vec{\circledast}G) = k + 1, \text{ and}$$

$$deg_{\mathrm{SCCL}}(\mathcal{I}, F\vec{\circledast}G) = \begin{cases} m & \text{if } m < \infty, n < \infty; \\ m+1 & \text{if } m < \infty, n = \infty; \\ \infty & \text{otherwise.} \end{cases}$$

SCCL is similar to CCL in that the intended meaning of $F\vec{\circledast}G$ is the same as that of $F\vec{\odot}G$, i.e., satisfying F and G is most preferable, but satisfying only F is also acceptable. However, SCCL does not use optionality to penalize less preferable interpretations. Instead, the degree of such interpretations is simply incremented by 1. As we will see in the next example, this can lead to questionable behavior. In fact, we define SCCL mainly for technical reasons, namely to highlight a choice logic in which optionality is not used.

Example 4. Consider $F = (a\vec{\circledast}(b\vec{\circledast}c))$. Then both $\{a, b\}$ and $\{a, b, c\}$ ascribe a degree of 1 to F. The fact that $(b\vec{\circledast}c)$ is not optimally satisfied by $\{a, b\}$ is irrelevant in this case. This is in contrast to CCL, where $(a\vec{\odot}(b\vec{\odot}c))$ would be satisfied to a degree of 2 by $\{a, b\}$. Note that $\vec{\circledast}$ is not associative, since $\{a, b\}$ ascribes a degree of 1 to F but a degree of 2 to $((a\vec{\circledast}b)\vec{\circledast}c)$.

Lastly, we want to discuss two variants of QCL introduced by Benferhat & Sedki [4] called PQCL and QCL+. Both of these logics define $\vec{\times}$ in the same way as standard QCL, but differ in how classical connectives deal with satisfaction degrees.

The alternative conjunctions and disjunctions featured in PQCL and QCL+ can be implemented in our framework as choice connectives, if desired. In fact, some behave more like choice connectives than classical connectives, as, for example, the conjunction of PQCL favors interpretations that ascribe a lower satisfaction degree to the first operand rather than the second. The semantics of $\vec{\diamond}$ in LCL is built on this principle, and is actually an extension of $\wedge$ in PQCL. Now, in contrast to the semantics of QCL, negation in PQCL and QCL+ is decomposable, e.g., the formula $\neg(a \vec{\times} b)$ is defined to be equivalent to $\neg a \vec{\times} \neg b$. This behavior cannot directly be simulated in our framework. However, by restricting ourselves to formulas where negations appear only in front of atoms PQCL and QCL+ can be captured by our framework as fragments.

2.3. Expressiveness

It can be shown that any logic defined in our framework is powerful enough to express arbitrary assignments of satisfaction degrees to interpretations by a formula as long as the degrees are obtainable in the following sense:

Definition 9. A degree $m \in \overline{\mathbb{N}}$ is called obtainable in a choice logic $\mathcal{L}$ if there exists an interpretation $\mathcal{I}$ and an $\mathcal{L}$-formula F such that $deg_{\mathcal{L}}(\mathcal{I}, F) = m$. By $\mathcal{D}_{\mathcal{L}}$ we denote the set of all degrees obtainable in a choice logic $\mathcal{L}$.

For example, $\mathcal{D}_{\text{PL}} = \{1, \infty\}$ and $\mathcal{D}_{\mathcal{L}} = \overline{\mathbb{N}}$ for $\mathcal{L} \in \{\text{QCL}, \text{CCL}, \text{LCL}, \text{SCCL}\}$. As soon as a degree m is obtainable, any interpretation can be assigned this degree via a suitable formula.

Lemma 3. *Let $m \in \mathcal{D}_{\mathcal{L}}$ for some choice logic $\mathcal{L}$. Then there is an $\mathcal{L}$-formula F such that $deg_{\mathcal{L}}(\mathcal{I}, F) = m$ for every interpretation $\mathcal{I}$.*

Proof. Let $\mathcal{L}$ be a choice logic, and let $\mathcal{I}$ be any interpretation. Let $m \in \mathcal{D}_{\mathcal{L}}$. Since m is obtainable in $\mathcal{L}$, there is a formula G such that $deg_{\mathcal{L}}(\mathcal{J}, G) = m$ for some interpretation $\mathcal{J}$. We obtain $F = T(G)$ by transforming G as follows:

1. $T(a) = \begin{cases} \top & \text{if } a \in \mathcal{J} \\ \bot & \text{otherwise} \end{cases}$
2. $T(\neg A) = \neg T(A)$
3. $T(A \circ B) = T(A) \circ T(B)$, where $\circ \in \{\wedge, \vee\} \cup \mathcal{C}_{\mathcal{L}}$.

The above transformation takes G and replaces every variable a that is contained in $\mathcal{J}$ by $\top$. If a is not contained in $\mathcal{J}$, then it will be replaced by $\bot$. It is easy to see that $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{J}, G) = m$ since

$$\begin{aligned} deg_{\mathcal{L}}(\mathcal{I}, T(a)) &= \begin{cases} deg_{\mathcal{L}}(\mathcal{I}, \top) & \text{if } a \in \mathcal{J} \\ deg_{\mathcal{L}}(\mathcal{I}, \bot) & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } a \in \mathcal{J} \\ \infty & \text{otherwise} \end{cases} \\ &= deg_{\mathcal{L}}(\mathcal{J}, a). \quad \square \end{aligned}$$

With the above lemma we can tackle the issue of formula synthesis:

Proposition 4. *Let $\mathcal{L}$ be a choice logic. Let V be a finite set of propositional variables, and let s be a function $s\colon 2^V \to \mathcal{D}_{\mathcal{L}}$. Then there is an $\mathcal{L}$-formula F such that for every $\mathcal{I} \subseteq V$, $deg_{\mathcal{L}}(\mathcal{I}, F) = s(\mathcal{I})$.*

Proof. Let $G_{\mathcal{J}}$ be the classical formula that characterizes some interpretation $\mathcal{J} \subseteq V$, i.e.

$$G_{\mathcal{J}} = \Big(\bigwedge_{a \in \mathcal{J}} a\Big) \wedge \Big(\bigwedge_{a \in (V \setminus \mathcal{J})} \neg a\Big).$$

Observe that $\mathcal{J} \models G_{\mathcal{J}}$, but $\mathcal{J}' \not\models G_{\mathcal{J}}$ for all $\mathcal{J}' \subseteq V$ such that $\mathcal{J}' \neq \mathcal{J}$. From Lemma 3, we know that for every $\mathcal{J} \subseteq V$, there is an $\mathcal{L}$-formula $S_{\mathcal{J}}$ such that $deg_{\mathcal{L}}(\mathcal{J}, S_{\mathcal{J}}) = s(\mathcal{J})$. Furthermore, let

$$F = \bigvee_{\mathcal{J} \subseteq V} (G_{\mathcal{J}} \wedge S_{\mathcal{J}}).$$

Let $\mathcal{I} \subseteq V$ be an arbitrary interpretation, and let C be an arbitrary clause in F, i.e. $C = (G_{\mathcal{J}} \ \wedge \ S_{\mathcal{J}})$ for some $\mathcal{J} \subseteq V$. We distinguish two cases:

1. $\mathcal{I} = \mathcal{J}$. Then $deg_{\mathcal{L}}(\mathcal{I}, G_{\mathcal{J}}) = 1$ and $deg_{\mathcal{L}}(\mathcal{I}, S_{\mathcal{J}}) = s(\mathcal{I})$, which implies that $deg_{\mathcal{L}}(\mathcal{I}, C) = s(\mathcal{I})$.
2. $\mathcal{I} \neq \mathcal{J}$. Then $deg_{\mathcal{L}}(\mathcal{I}, G_{\mathcal{J}}) = \infty$ and therefore $deg_{\mathcal{L}}(\mathcal{I}, C) = \infty$.

By construction, there is exactly one clause in F such that $\mathcal{I} = \mathcal{J}$. Since this clause is satisfied with degree $s(\mathcal{I})$ by $\mathcal{I}$, and all other clauses are ascribed a degree of ∞ by $\mathcal{I}$, we have that $deg_{\mathcal{L}}(\mathcal{I}, F) = s(\mathcal{I})$. $\square$

Observe that Proposition 4 also speaks about interpretations $\mathcal{I}$ that are no subsets of V: since $var(F) \subseteq V$, $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I} \cap V, F) = s(\mathcal{I} \cap V)$.

3. Strong equivalence

In this section we investigate strong equivalence, a concept which has received extensive attention in logic programming [32,21,22] and knowledge representation in general [23,2]. Two choice logic formulas are strongly equivalent if they can be substituted for each other in a larger context without affecting preferred models. In the following, $F[A/B]$ denotes the substitution of an occurrence of A in F by B.

Definition 10. Two formulas A and B of a choice logic $\mathcal{L}$ are strongly equivalent, written as $A \equiv_s^{\mathcal{L}} B$, if $Prf_{\mathcal{L}}(F) = Prf_{\mathcal{L}}(F[A/B])$ for all $\mathcal{L}$-formulas F.

Strong equivalence is a crucial notion towards formula simplification, as it allows us to replace subformulas by simpler but strongly equivalent formulas. Unfortunately, to check whether two formulas are strongly equivalent using the definition above requires us to go through infinitely many formulas. Thus a characterization of strong equivalence that avoids this is needed. We show that strong equivalence often can be decided via simpler equivalence notions.

Definition 11. Two formulas A and B of a choice logic $\mathcal{L}$ are degree-equivalent, written as $A \equiv_d^{\mathcal{L}} B$, if $deg_{\mathcal{L}}(\mathcal{I}, A) = deg_{\mathcal{L}}(\mathcal{I}, B)$ for all interpretations $\mathcal{I}$.

Lemma 5. *Let $\mathcal{L}$ be a choice logic. If $A \equiv_d^{\mathcal{L}} B$, then $Prf_{\mathcal{L}}(A) = Prf_{\mathcal{L}}(B)$.*

Proof. Assume $A \equiv_d^{\mathcal{L}} B$. Then

$$\begin{aligned}\mathcal{I} \in Prf_{\mathcal{L}}(A) &\iff deg_{\mathcal{L}}(\mathcal{I}, A) \neq \infty \text{ and } deg_{\mathcal{L}}(\mathcal{I}, A) \leq deg_{\mathcal{L}}(\mathcal{J}, A) \text{ for all } \mathcal{J} \\ &\iff deg_{\mathcal{L}}(\mathcal{I}, B) \neq \infty \text{ and } deg_{\mathcal{L}}(\mathcal{I}, B) \leq deg_{\mathcal{L}}(\mathcal{J}, B) \text{ for all } \mathcal{J} \\ &\iff \mathcal{I} \in Prf_{\mathcal{L}}(B). \quad \square\end{aligned}$$

The converse of Lemma 5 is not true: a and $(a \vec{\times} b)$ have the same preferred models in QCL, but $\{b\} \models_{\infty}^{\text{QCL}} a$, while $\{b\} \models_{2}^{\text{QCL}} (a \vec{\times} b)$. Since the satisfaction degree of a formula might also depend on optionality, the notion of degree-equivalence is not strong enough in many cases; we thus also consider the following notion, which actually appears under the name of strong equivalence in the original QCL-paper [11].

Definition 12. Two $\mathcal{L}$-formulas A and B of a choice logic $\mathcal{L}$ are fully equivalent, written as $A \equiv_f^{\mathcal{L}} B$, if $A \equiv_d^{\mathcal{L}} B$ and $opt_{\mathcal{L}}(A) = opt_{\mathcal{L}}(B)$.

Lemma 6. *Let $\mathcal{L}$ be a choice logic. Then $A \equiv_f^{\mathcal{L}} B$ if and only if $F \equiv_f^{\mathcal{L}} F[A/B]$ for all $\mathcal{L}$-formulas F.*

Proof. Assume that $A \equiv_f^{\mathcal{L}} B$. Then, since the optionality- and degree-functions of any connective in a choice logic are functions over the optionalities and satisfaction degrees of the immediate subformulas, the fact that $F \equiv_f^{\mathcal{L}} F[A/B]$ holds for any $\mathcal{L}$-formula F can be shown easily by structural induction.

For the converse direction, assume that $A \not\equiv_f^{\mathcal{L}} B$. Choose $F = A$. Then $F[A/B] = B$, and therefore $F \not\equiv_f^{\mathcal{L}} F[A/B]$. $\square$

The above lemma states that two formulas are fully equivalent if and only if they can be substituted for each other without affecting satisfaction degree or optionality. This is even stronger than strong equivalence which only demands that substitution does not affect preferred models.

Proposition 7. *$A \equiv_f^{\mathcal{L}} B \implies A \equiv_s^{\mathcal{L}} B \implies A \equiv_d^{\mathcal{L}} B$ for any choice logic $\mathcal{L}$ and all $\mathcal{L}$-formulas A, B.*

Proof. $A \equiv_f^{\mathcal{L}} B \implies A \equiv_s^{\mathcal{L}} B$ follows from Lemma 6. It remains to show that $A \equiv_s^{\mathcal{L}} B \implies A \equiv_d^{\mathcal{L}} B$: assume $A \not\equiv_d^{\mathcal{L}} B$. We want to show that there is a formula F such that $Prf_{\mathcal{L}}(F) \neq Prf_{\mathcal{L}}(F[A/B])$.

Since $A \not\equiv_d^{\mathcal{L}} B$, there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models_m^{\mathcal{L}} A$ and $\mathcal{I} \models_n^{\mathcal{L}} B$ with $m \neq n$. Let $k = min(m, n)$. Due to Proposition 4, we know that there are formulas G and H such that the minimum degree that satisfies G and H is k. Thus, there are interpretations $\mathcal{I}_G, \mathcal{I}_H$ such that $\mathcal{I}_G \models_k^{\mathcal{L}} G$ and $\mathcal{I}_H \models_k^{\mathcal{L}} H$. We can assume that G and H are variable disjoint from each other as well as from A and B, since we can always rename variables if necessary. Thus, by Lemma 2, we can assume $\mathcal{I} \cap \mathcal{I}_G = \emptyset$, $\mathcal{I} \cap \mathcal{I}_H = \emptyset$, and $\mathcal{I}_G \cap \mathcal{I}_H = \emptyset$. We now construct

$$F = (A \wedge G) \vee (a \wedge H),$$

where a is a fresh variable that does not occur in A, B, G or H. We can therefore also assume that a is not contained in $\mathcal{I}$, $\mathcal{I}_G$, or $\mathcal{I}_H$.

Observe that the minimal degree with which F (or $F[A/B]$) can possibly be satisfied is k, as either G or H need to be satisfied. Furthermore, $\mathcal{I}_H \cup \{a\} \models_k^{\mathcal{L}} F$ and $\mathcal{I}_H \cup \{a\} \models_k^{\mathcal{L}} F[A/B]$. This means that any preferred model of F must satisfy F with a degree of k. The same is true for preferred models of $F[A/B]$. Also observe that since a is not contained in $\mathcal{I}$ or $\mathcal{I}_G$, $\mathcal{I} \cup \mathcal{I}_G \models_\infty^{\mathcal{L}} (a \wedge H)$. We distinguish two cases:

1. $k = m$. Then $\mathcal{I} \models_k^{\mathcal{L}} A$, and therefore $\mathcal{I} \cup \mathcal{I}_G \models_k^{\mathcal{L}} (A \wedge G)$. This implies $\mathcal{I} \cup \mathcal{I}_G \models_k^{\mathcal{L}} F$, i.e. $\mathcal{I} \cup \mathcal{I}_G \in Prf_{\mathcal{L}}(F)$. Analogously, since $\mathcal{I} \models_n^{\mathcal{L}} B$, we have that $\mathcal{I} \cup \mathcal{I}_G \models_n^{\mathcal{L}} (B \wedge G)$. Therefore, $\mathcal{I} \cup \mathcal{I}_G \models_n^{\mathcal{L}} F[A/B]$. Since $n > k$, we have $\mathcal{I} \cup \mathcal{I}_G \notin Prf_{\mathcal{L}}(F[A/B])$.
2. $k = n$. Analogous to (1), but with $\mathcal{I} \cup \mathcal{I}_G \notin Prf_{\mathcal{L}}(F)$ and $\mathcal{I} \cup \mathcal{I}_G \in Prf_{\mathcal{L}}(F[A/B])$.

It can be concluded that $Prf_{\mathcal{L}}(F) \neq Prf_{\mathcal{L}}(F[A/B])$, i.e. $A \not\equiv_s^{\mathcal{L}} B$. □

In general, all three equivalence notions are different. For example, in SCCL, a and $(a \vec{\circledast} a)$ are not fully equivalent, since they differ in optionality, but they are strongly equivalent, since optionality does not impact satisfaction degrees in SCCL. On the other hand, in QCL, a and $(a \vec{\times} a)$ are degree-equivalent and $F = (((a \vec{\times} b) \vee (c \vec{\times} d)) \wedge \neg a \wedge \neg c)$ has $\{b\}$ as a preferred model, but replacing the first occurrence of a in F by $a \vec{\times} a$ means $\{b\}$ is no longer preferred. However, for natural classes of choice logics strong equivalence coincides with either degree-equivalence or full equivalence.

Definition 13. A choice logic $\mathcal{L}$ is called optionality-ignoring if for all $\circ \in \mathcal{C}_{\mathcal{L}}$, $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg_{\mathcal{L}}(\mathcal{I}, F' \circ G')$ whenever $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I}, F')$ and $deg_{\mathcal{L}}(\mathcal{I}, G) = deg_{\mathcal{L}}(\mathcal{I}, G')$.

It is easy to see that PL and SCCL are optionality-ignoring, while QCL, CCL, and LCL are not.

Proposition 8. *Let $\mathcal{L}$ be an optionality-ignoring choice logic. For all $\mathcal{L}$-formulas A, B we have that $A \equiv_s^{\mathcal{L}} B$ iff $A \equiv_d^{\mathcal{L}} B$.*

Proof. The only-if-direction follows directly from Proposition 7. For the if-direction, we can prove that if $A \equiv_d^{\mathcal{L}} B$, then $F \equiv_d^{\mathcal{L}} F[A/B]$ for all $\mathcal{L}$-formulas F, which implies that $Prf_{\mathcal{L}}(F) = Prf_{\mathcal{L}}(F[A/B])$ for all $\mathcal{L}$-formulas F. This can be done by structural induction, analogous to Lemma 6. The critical difference here is that $\mathcal{L}$ is optionality-ignoring, and therefore optionality can not influence degrees when substituting B for A. □

Similar to the above result, we can characterize strong equivalence by full equivalence in choice logics where, as soon as two formulas differ in optionality, they can no longer be safely substituted for each other without affecting satisfaction degrees.

Definition 14. A choice logic $\mathcal{L}$ is called optionality-differentiating if for all $\mathcal{L}$-formulas A and B with $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$, there is an $\mathcal{L}$-formula F such that $F \not\equiv_d^{\mathcal{L}} F[A/B]$.

PL, QCL, CCL, and LCL are optionality-differentiating (while SCCL is not). We show this for QCL: Consider any A, B such that $opt_{\text{QCL}}(A) \neq opt_{\text{QCL}}(B)$. Then $F = ((A \wedge \bot) \vec{\times} a)$ has the desired property: $(A \wedge \bot)$ can never be satisfied, and since $opt_{\text{QCL}}(A \wedge \bot) = opt_{\text{QCL}}(A)$, F is satisfied by $\{a\}$ with a degree of $opt_{\text{QCL}}(A) + 1$. Likewise, $F[A/B]$ is satisfied by $\{a\}$ with a degree of $opt_{\text{QCL}}(B) + 1$. The cases of CCL and LCL are similar.

On the other hand, SCCL is clearly not optionality-differentiating: while $opt_{\text{SCCL}}(a) \neq opt_{\text{SCCL}}(a \vec{\circledast} a)$ we have that $F \equiv_d^{\text{SCCL}} F[a/(a \vec{\circledast} a)]$ for all SCCL-formulas F, since optionality has no impact on satisfaction degree in SCCL.

Proposition 9. *Let $\mathcal{L}$ be an optionality-differentiating choice logic. For all $\mathcal{L}$-formulas A, B we have that $A \equiv_s^{\mathcal{L}} B$ iff $A \equiv_f^{\mathcal{L}} B$.*

Proof. The if-direction follows directly from Proposition 7. For the only-if-direction, assume that $A \equiv_s^{\mathcal{L}} B$. Then, from Proposition 7, we know that $A \equiv_d^{\mathcal{L}} B$. It remains to show that $opt_{\mathcal{L}}(A) = opt_{\mathcal{L}}(B)$.

Towards a contradiction, assume that $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$. This means that, since $\mathcal{L}$ is optionality-differentiating, there exists an $\mathcal{L}$-formula A' such that $A' \not\equiv_d^{\mathcal{L}} A'[A/B]$. Observe that therefore A must occur in A'. Let $B' = A'[A/B]$. Then $A' \not\equiv_d^{\mathcal{L}} B'$. By the contrapositive of Proposition 7, there exists a formula F such that $Prf_{\mathcal{L}}(F) \neq Prf_{\mathcal{L}}(F[A'/B'])$. By the construction of F in the proof for Proposition 7, we can assume that A' occurs only once in F, and that A only occurs in A'. Therefore, replacing A' by $A'[A/B]$ in F is the same as simply replacing A by B in F, i.e. $F[A'/B'] = F[A'/A'[A/B]] = F[A/B]$. Thus, $Prf_{\mathcal{L}}(F) \neq Prf_{\mathcal{L}}(F[A/B])$. But then $A \not\equiv_s^{\mathcal{L}} B$. Contradiction. □

In fact, strong equivalence and full equivalence are interchangeable *only* for optionality-differentiating choice logics.

Proposition 10. *Let $\mathcal{L}$ be choice logic that is not optionality-differentiating. Then there are $\mathcal{L}$-formulas A and B such that $A \equiv_s^{\mathcal{L}} B$ but $A \not\equiv_f^{\mathcal{L}} B$.*

Proof. If $\mathcal{L}$ is not optionality-differentiating then, by definition, there are $\mathcal{L}$-formulas A and B such that $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$ while $F \equiv_d^{\mathcal{L}} F[A/B]$ holds for all $\mathcal{L}$-formulas F. Clearly, $A \not\equiv_f^{\mathcal{L}} B$. However, $F \equiv_d^{\mathcal{L}} F[A/B]$ implies $Prf_{\mathcal{L}}(F) = Prf_{\mathcal{L}}(F[A/B])$, i.e., $A \equiv_s^{\mathcal{L}} B$. □

In Section 2.2, we discussed associativity of $\vec{\times}$ in QCL. A matching strong equivalence result can now be achieved for CCL as well, and it is easily proven using Proposition 9.

Corollary 11. *The choice connective $\vec{\odot} \in \mathcal{C}_{\text{CCL}}$ is associative, meaning that $((F\vec{\odot}G)\vec{\odot}H) \equiv_s^{\text{CCL}} (F\vec{\odot}(G\vec{\odot}H))$ for any $F, G, H \in \mathcal{F}_{\text{CCL}}$.*

Proof. For readability, we write opt instead of opt_{CCL} and deg instead of deg_{CCL} in this proof. First of all, by Definition 6, $opt((F\vec{\odot}G)\vec{\odot}H) = opt(F\vec{\odot}(G\vec{\odot}H))$. We further show that $((F\vec{\odot}G)\vec{\odot}H) \equiv_d^{\text{CCL}} (F\vec{\odot}(G\vec{\odot}H))$. Let $\mathcal{I}$ be an arbitrary interpretation. Let $d_A = deg(\mathcal{I}, A)$ for $A \in \{F, G, H\}$. We distinguish the following cases:

- $d_F = 1$, $d_G = 1$, and $d_H < \infty$. Then $deg(\mathcal{I}, F\vec{\odot}G) = 1$, $deg(\mathcal{I}, G\vec{\odot}H) = d_H$. This implies $deg(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = d_H = deg(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H))$.
- $d_F = 1$, $d_G = 1$, and $d_H = \infty$. Then $deg(\mathcal{I}, F\vec{\odot}G) = 1$, $deg(\mathcal{I}, G\vec{\odot}H) = 1 + opt(H)$. We can conclude that $deg(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = 1 + opt(H) = deg(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H))$.
- $d_F = 1$ and $1 < d_G < \infty$. Then $deg(\mathcal{I}, F\vec{\odot}G) = d_G$, $deg(\mathcal{I}, G\vec{\odot}H) = d_G + opt(H)$. This entails that $deg(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = d_G + opt(H) = deg(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H))$.
- $d_F = 1$ and $d_G = \infty$. Then $deg(\mathcal{I}, F\vec{\odot}G) = 1 + opt(G)$, $deg(\mathcal{I}, G\vec{\odot}H) = \infty$. This further implies $deg(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = 1 + opt(G) + opt(H) = deg(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H))$.
- $1 < d_F < \infty$. Then $deg(\mathcal{I}, F\vec{\odot}G) = d_F + opt(G)$, which entails that $deg(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = d_F + opt(G) + opt(H) = deg(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H))$.
- $d_F = \infty$. Then $deg(\mathcal{I}, F\vec{\odot}G) = \infty$, which implies $deg(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = \infty = deg(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H))$. □

To conclude, knowing whether a choice logic is optionality-ignoring (e.g. SCCL) or optionality-differentiating (e.g. QCL, CCL, LCL) is useful, since it allows to decide strong equivalence via degree- or full equivalence. However, note that choice logics which are neither optionality-ignoring nor -differentiating do exist. One could, for example, define a choice connective $\circ$ where the satisfaction degree of $F \circ G$ depends on the optionality of G, but only in some cases, e.g. when $opt_{\mathcal{L}}(G) = 3$.

4. Computational complexity for choice logic formulas

Next, we examine the computational complexity of some decision problems pertaining to choice logic formulas. We assume familiarity with basic complexity theoretical concepts such as the classes P, coNP, and NP, as well as computation using NP-oracles. Additionally, we will encounter the following complexity classes in this paper:

Definition 15. Let Q be a decision problem.

- Q is in Δ_2^{P} if it can be decided in polynomial time by an algorithm with access to an NP-oracle [28].
- Q is in Θ_2^{P}, also called $\Delta_2^{\mathsf{P}}[\mathcal{O}(\log n)]$, if it can be decided in polynomial time by an algorithm which is allowed $\mathcal{O}(\log n)$ number of calls to an NP-oracle, where n is the size of the input [38].
- Q is in $\Delta_2^{\mathsf{P}}[\mathcal{O}(\log^2 n)]$ if it can be decided in polynomial time by an algorithm which is allowed $\mathcal{O}(\log^2 n)$ number of calls to an NP-oracle, where n is the size of the input [13].
- Q is in Σ_2^{P} if it can be decided in nondeterministic polynomial time by an algorithm with access to an NP-oracle [1].
- Q is in Π_2^{P} if the complement of Q is in Σ_2^{P}.

So far we have imposed only few restrictions on optionality- and degree functions, which means that there are choice logics whose semantics are given by computationally expensive or even undecidable functions. For reasons of practicality, we focus on so-called tractable choice logics.

Definition 16. A choice logic $\mathcal{L}$ is called tractable if the optionality- and degree functions of every choice connective in $\mathcal{L}$ are polynomial-time computable.

All logics presented in this paper are tractable in the above sense. In this section, we consider the following problems:

Definition 17. Given a choice logic $\mathcal{L}$ we define the following decision problems:

- $\mathcal{L}$-DegreeChecking: given an $\mathcal{L}$-formula F, an interpretation $\mathcal{I}$, and a satisfaction degree $k \in \overline{\mathbb{N}}$, does $deg_{\mathcal{L}}(\mathcal{I}, F) \le k$ hold?
- $\mathcal{L}$-DegreeSat: given an $\mathcal{L}$-formula F and a satisfaction degree k, is there an interpretation $\mathcal{I}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) \le k$ holds?
- $\mathcal{L}$-PMChecking: given an $\mathcal{L}$-formula F and an interpretation $\mathcal{I}$, does $\mathcal{I} \in Prf_{\mathcal{L}}(F)$ hold?
- $\mathcal{L}$-PMContainment: given an $\mathcal{L}$-formula F and a propositional variable x, is there an interpretation $\mathcal{I} \in Prf_{\mathcal{L}}(F)$ such that $x \in \mathcal{I}$?
- $\mathcal{L}$-DegreeEquiv (resp. $\mathcal{L}$-FullEquiv): given two $\mathcal{L}$-formulas A and B, does $A \equiv_d^{\mathcal{L}} B$ (resp. $A \equiv_f^{\mathcal{L}} B$) hold?

$\mathcal{L}$-DegreeChecking and $\mathcal{L}$-DegreeSat are concerned with satisfaction degrees. Indeed, they are straightforward generalizations of model checking and satisfiability for classical propositional logic. Reformulating the problems in terms of preferred models yields $\mathcal{L}$-PMChecking and $\mathcal{L}$-PMContainment. Note that $\mathcal{L}$-PMContainment lets us analyze the complexity of finding a specific preferred model: given a formula F over variables $x_1, \dots, x_n$ we can first ask whether x_1 is in a preferred model of F. If yes (resp. no), we obtain F' by replacing all occurrences of x_1 by $\top$ (resp. $\bot$) and then ask whether x_2 is contained in a preferred model of F'. Thus, we can find a preferred model of F by solving $\mathcal{L}$-PMContainment exactly n times.

Note that we can define the problem $\mathcal{L}$-StrongEquiv analogously to $\mathcal{L}$-DegreeEquiv and $\mathcal{L}$-FullEquiv. By our characterizations from Section 3 we can then obtain the complexity of $\mathcal{L}$-StrongEquiv via results for $\mathcal{L}$-FullEquiv (resp. $\mathcal{L}$-DegreeEquiv) when dealing with optionality-differentiating (resp. optionality-ignoring) choice logics.

We now investigate the complexity of the above problems. A summary of the results can be found at the end of this section in Table 2.

Proposition 12. *$\mathcal{L}$-DegreeChecking is in* P *for any tractable choice logic $\mathcal{L}$.*

Proof. The degree of F under $\mathcal{I}$ can be computed by applying $deg_{\mathcal{L}}$ to $\mathcal{I}$ and F (see Definition 3). For tractable $\mathcal{L}$ this procedure clearly runs in polynomial time. Now it remains to simply check whether $deg_{\mathcal{L}}(\mathcal{I}, F) \le k$. □

Proposition 13. *$\mathcal{L}$-DegreeSat is* NP*-complete for any tractable choice logic $\mathcal{L}$.*

Proof. Regarding NP-membership, given an $\mathcal{L}$-formula F and a satisfaction degree k, it suffices to guess an interpretation $\mathcal{I} \subseteq var(F)$ and check whether $deg_{\mathcal{L}}(\mathcal{I}, F) \le k$. Since $\mathcal{L}$-DegreeChecking is in P (cf. Proposition 12) this check can be done in polynomial time.

For NP-hardness, we provide a reduction from Sat. Let F be an arbitrary classical formula. Then, for any interpretation $\mathcal{I}$ it holds that $\mathcal{I} \models F \iff \mathcal{I} \models_1^{\mathcal{L}} F \iff deg_{\mathcal{L}}(\mathcal{I}, F) \le 1$. Thus, there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models F$ if and only if there is an interpretation $\mathcal{I}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) \le 1$. □

Let us now consider $\mathcal{L}$-PMChecking, which is, in general, harder than $\mathcal{L}$-DegreeChecking.

Proposition 14. *$\mathcal{L}$-PMChecking is in* coNP *for any tractable choice logic $\mathcal{L}$.*

Proof. We show that the complementary problem is in NP: given an $\mathcal{L}$-formula F and an interpretation $\mathcal{I}$, guess an interpretation $\mathcal{J} \subseteq var(F)$ and check whether $deg_{\mathcal{L}}(\mathcal{J}, F) < deg_{\mathcal{L}}(\mathcal{I}, F)$. If yes, then $\mathcal{I} \notin Prf_{\mathcal{L}}(F)$, i.e., $(F, \mathcal{I})$ is a yes-instance of co-$\mathcal{L}$-PMChecking. Note that $deg_{\mathcal{L}}(\mathcal{J}, F) < deg_{\mathcal{L}}(\mathcal{I}, F)$ can be checked in polynomial time since $\mathcal{L}$ is tractable. □

We observe that PL-PMChecking and PL-DegreeChecking are identical, since $\mathcal{I} \in Prf_{\mathrm{PL}}(F) \iff \mathcal{I} \models_1^{\mathrm{PL}} F$. Hence, we cannot expect coNP-hardness in general, but we show the result for all logics where degrees other than 1 and ∞ are obtainable, and thus for QCL, CCL, SCCL, and LCL.

Proposition 15. *$\mathcal{L}$-PMChecking is* coNP*-complete for any tractable choice logic $\mathcal{L}$ for which $\mathcal{D}_{\mathcal{L}} \neq \{1, \infty\}$.*

Proof. coNP-membership is by Proposition 14. Regarding hardness, recall that Unsat is the problem of deciding whether a given classical formula F is unsatisfiable. We provide a reduction from an arbitrary instance F of Unsat. Let $m \in \mathcal{D}_{\mathcal{L}} \setminus \{1, \infty\}$, and let a be a variable that does not occur in F. By Proposition 4, there exists an $\mathcal{L}$-formula G such that $\mathcal{I} \models_m^{\mathcal{L}} G$ if $a \in \mathcal{I}$ and $\mathcal{I} \models_\infty^{\mathcal{L}} G$ if $a \notin \mathcal{I}$ for all interpretations $\mathcal{I}$. Note that the size of G is constant with respect to the size of F. We now construct an instance $(F', \{a\})$ of $\mathcal{L}$-PMChecking, where

$$F' = (F \vee G) \wedge \neg(F \wedge G).$$

It remains to prove that F is a yes-instance of Unsat if and only if $(F', \{a\})$ is a yes-instance of $\mathcal{L}$-PMChecking:

"$\Longrightarrow$": Assume F is a yes-instance of Unsat. Then there is no interpretation $\mathcal{J}$ such that $\mathcal{J} \models F$, i.e. $deg_{\mathcal{L}}(\mathcal{J}, F) = \infty$ for all $\mathcal{J}$. Since $deg_{\mathcal{L}}(\{a\}, G) = m$, we have that $deg_{\mathcal{L}}(\{a\}, F') = m$. Indeed, F' can not be satisfied to a degree lower than m since F is unsatisfiable and since m is the lowest degree with which G can be satisfied. Thus, $\{a\} \in Prf_{\mathcal{L}}(F')$.

"$\Longleftarrow$": We proceed by contrapositive. Assume F is a no-instance of Unsat. Then there is an interpretation $\mathcal{J}$ such that $\mathcal{J} \models F$. Because a does not occur in F, we can assume that $a \notin \mathcal{J}$, and therefore $\mathcal{J} \models^{\mathcal{L}}_{\infty} G$. Thus, $\mathcal{J} \models^{\mathcal{L}}_{1} (F \vee G)$ and $\mathcal{J} \models^{\mathcal{L}}_{1} \neg(F \wedge G)$, which implies that $\mathcal{J} \models^{\mathcal{L}}_{1} F'$. Recall that $\{a\} \models^{\mathcal{L}}_{m} G$. We distinguish two cases:

1. $\{a\} \models F$. Then $\{a\} \models^{\mathcal{L}}_{1} (F \vee G)$, $\{a\} \models^{\mathcal{L}}_{\infty} \neg(F \wedge G)$, and therefore $\{a\} \models^{\mathcal{L}}_{\infty} F'$.
2. $\{a\} \not\models F$. Then $\{a\} \models^{\mathcal{L}}_{m} (F \vee H)$, $\{a\} \models^{\mathcal{L}}_{1} \neg(F \wedge H)$, and therefore $\{a\} \models^{\mathcal{L}}_{m} F'$.

In any case, $deg_{\mathcal{L}}(\{a\}, F') > deg_{\mathcal{L}}(\mathcal{J}, F')$ which implies $\{a\} \notin Prf_{\mathcal{L}}(F')$. □

We now turn to $\mathcal{L}$-PMContainment. In order to show Δ^{P}_2 containment for this problem, we first need to prove the following lemma which gives an upper bound for the optionality of choice logic formulas relative to their size. In the following, $|F|$ denotes the total number of variables occurrences in F, e.g. $|(x \wedge x \wedge y)| = 3$.

Lemma 16. *Let $\mathcal{L}$ be a choice logic. Then, for every $\mathcal{L}$-formula F it holds that $opt_{\mathcal{L}}(F) < 2^{(|F|^2)}$.*

Proof. By structural induction over $\mathcal{F}_{\mathcal{L}}$.

- Base case. $F = a$, where a is a propositional variable. Then $|F| = 1$ and $opt_{\mathcal{L}}(F) = 1 < 2^{(|F|^2)}$.
- Step case. As our I.H., let G and H be $\mathcal{L}$-formulas such that $opt_{\mathcal{L}}(G) < 2^{(|G|^2)}$ and $opt_{\mathcal{L}}(H) < 2^{(|H|^2)}$. We distinguish the following cases:
  1. $F = (\neg G)$. Then $|F| = |G| \geq 1$ and $opt_{\mathcal{L}}(F) = 1 < 2^{(|F|^2)}$.
  2. $F = (G \wedge H)$ or $F = (G \vee H)$. Then $|F| = |G| + |H|$ and
  $$opt_{\mathcal{L}}(F) = max(opt_{\mathcal{L}}(G), opt_{\mathcal{L}}(H)) < max(2^{(|G|^2)}, 2^{(|H|^2)}) < 2^{(|F|^2)}.$$
  3. $F = (G \circ H)$, where $\circ \in \mathcal{C}_{\mathcal{L}}$. Then $|F| = |G| + |H|$. Of course, $opt_{\mathcal{L}}(G) < 2^{(|G|^2)}$ is the same as $opt_{\mathcal{L}}(G) \leq 2^{(|G|^2)} - 1$. Likewise for H. Thus,
  $$\begin{aligned} opt_{\mathcal{L}}(F) &\leq (opt_{\mathcal{L}}(G) + 1) \cdot (opt_{\mathcal{L}}(H) + 1) \\ &\leq ((2^{(|G|^2)} - 1) + 1) \cdot ((2^{(|H|^2)} - 1) + 1) = 2^{(|G|^2)} \cdot 2^{(|H|^2)} \\ &= 2^{(|G|^2) + (|H|^2)} < 2^{((|G|+|H|)^2)} = 2^{(|F|^2)}. \quad \square \end{aligned}$$

The above upper bound for the optionality of a formula is likely not a tight bound, but it is good enough for our purposes.

Proposition 17. *$\mathcal{L}$-PMContainment is in Δ^{P}_2 and NP-hard for any tractable choice logic $\mathcal{L}$.*

Proof. NP-hardness follows from Sat. Δ^{P}_2-membership: let (F, x) be an instance of $\mathcal{L}$-PMContainment. We provide a decision procedure which runs in polynomial time with respect to $|F|$, makes $\mathcal{O}(|F|^2)$ calls to an NP-oracle, and determines whether (F, x) is a yes-instance of $\mathcal{L}$-PMContainment:

Conduct a binary search over $(1, \dots, opt_{\mathcal{L}}(F), \infty)$. In each step of the binary search, we call an NP-oracle that decides $\mathcal{L}$-DegreeSat to check whether there is an interpretation $\mathcal{I}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) \leq k$, where k is the mid-point of the current step in the binary search. In the end, we will find the minimum k such that $deg_{\mathcal{L}}(\mathcal{J}, F) = k$ for some $\mathcal{J}$, i.e., every preferred model of F must satisfy F with degree k. By Lemma 16, $opt_{\mathcal{L}}(F) < 2^{(|F|^2)}$. Since binary search runs in logarithmic time, we require at most $\mathcal{O}(\log(opt_{\mathcal{L}}(F))) = \mathcal{O}(\log(2^{(|F|^2)})) = \mathcal{O}(|F|^2)$ oracle calls for this procedure.

Finally, we conduct a last NP-oracle call to guess an interpretation $\mathcal{I}$ and check whether $deg_{\mathcal{L}}(\mathcal{I}, F) = k$ and $\mathcal{I} \models x$. □

The complexity results of Proposition 17 are tight bounds in the sense that there are choice logics for which $\mathcal{L}$-PMContainment is Δ^{P}_2-complete (Proposition 21) and choice logics for which it is NP-complete (just take $\mathcal{L} = \mathrm{PL}$). However, there are also choice logics for which the complexity lies between these two classes. The key is to restrict optionality.

Proposition 18. *$\mathcal{L}$-PMContainment is in Θ^{P}_2 for tractable choice logics $\mathcal{L}$ in which $opt_{\mathcal{L}}(F) \in \mathcal{O}(|F|^c)$ holds for some constant c and all $\mathcal{L}$-formulas F.*

Proof. Analogous to the proof of Proposition 17, except that we require at most $\mathcal{O}(\log(opt_{\mathcal{L}}(F))) = \mathcal{O}(\log(|F|^c)) = \mathcal{O}(\log(|F|))$ oracle calls. $\square$

Recall that the optionality of $A \vec{\times} B$ in QCL is defined as $opt_{\text{QCL}}(A) + opt_{\text{QCL}}(B)$. Now, since a formula F has at most $|F| - 1$ occurrences of $\vec{\times}$, $opt_{\text{QCL}}(F) \leq |F|$ for all QCL-formulas F. The same argument can be made for CCL, where the optionality of $A \vec{\odot} B$ is given by $opt_{\text{CCL}}(A) + opt_{\text{CCL}}(B)$, and SCCL, where the optionality of $A \vec{\circledast} B$ is given by $opt_{\text{SCCL}}(A) + 1$. Together with the above proposition this implies that $\mathcal{L}$-PMContainment is in Θ_2^{P} for $\mathcal{L} \in \{\text{QCL, CCL, SCCL}\}$. In fact, these logics are Θ_2^{P}-complete. We show this by a reduction from the following Θ_2^{P}-complete problem [18]:

Definition 18. LogLexMaxSat is the decision problem where, given a PL-formula F and an ordering $x_1 > \cdots > x_n$ over n of the variables in F with $n \leq \log(|F|)$, we ask whether x_n is true in the lexicographically largest interpretation $\mathcal{J} \subseteq \{x_1, \ldots, x_n\}$ that can be extended to a model of F.

Proposition 19. *$\mathcal{L}$-PMContainment is Θ_2^{P}-complete for $\mathcal{L} \in \{\text{QCL, CCL, SCCL}\}$.*

Proof. Θ_2^{P}-membership is established in Proposition 18. Regarding hardness, consider an arbitrary instance $(F, (x_1, \ldots, x_n))$ of LogLexMaxSat. We construct an instance (F', x_n) of QCL-PMContainment as follows: Let $\mathcal{J}_i$ be the lexicographically i-th largest interpretation over $x_1 > \cdots > x_n$. For example, $\mathcal{J}_1 = \{x_1, \ldots, x_n\}$, $\mathcal{J}_2 = \{x_1, \ldots, x_{n-1}\}$, and $\mathcal{J}_{(2^n)} = \emptyset$. We characterize each of those interpretations by a formula, namely

$$A_i = \Big(\bigwedge_{x \in \mathcal{J}_i} x \Big) \wedge \Big(\bigwedge_{x \in \{x_1, \ldots, x_n\} \setminus \mathcal{J}_i} \neg x \Big).$$

Then, for any interpretation $\mathcal{I}$, we have that $\mathcal{I} \models A_i \iff \mathcal{I} \cap \{x_1, \ldots, x_n\} = \mathcal{J}_i$. Now let

$$F' = F \wedge (A_1 \vec{\times} A_2 \vec{\times} \cdots \vec{\times} A_{(2^n)}).$$

Observe that this construction is polynomial in $|F|$, as $n \leq \log(|F|)$, and therefore $2^n \leq |F|$. It remains to show that $(F, (x_1, \ldots, x_n))$ is a yes-instance of LogLexMaxSat iff (F', x_n) is a yes-instance of QCL-PMContainment.

"$\Longrightarrow$": Let $(F, (x_1, \ldots, x_n))$ be a yes-instance of LogLexMaxSat. Then there exists an interpretation $\mathcal{I}$ such that $x_n \in \mathcal{I}$, $\mathcal{I} \models F$, and such that $\mathcal{J}_k = \mathcal{I} \cap \{x_1, \ldots, x_n\}$ is the lexicographically largest interpretation over $x_1 > \cdots > x_n$ that can be extended to a model of F. Observe that $\mathcal{I} \models A_k$, but $\mathcal{I} \not\models A_r$ for any $r \neq k$. Therefore, by the semantics of ordered disjunction in QCL, $deg_{\text{QCL}}(\mathcal{I}, F') = k$. Let $\mathcal{I}'$ be any interpretation other than $\mathcal{I}$. If $\mathcal{I}' \not\models F$, then $deg_{\text{QCL}}(\mathcal{I}', F') = \infty$. If $\mathcal{I}' \models F$, then it can not be that $\mathcal{J}_{k'} = \mathcal{I}' \cap \{x_1, \ldots, x_n\}$ is lexicographically larger than $\mathcal{J}_k$ with respect to $x_1 > \cdots > x_n$. Thus, $k \leq k'$. By the same reasoning as above, $deg_{\text{QCL}}(\mathcal{I}', F') = k'$. This means that there is no interpretation that satisfies F' to a smaller degree than $\mathcal{I}$, i.e. $\mathcal{I} \in Prf_{\text{QCL}}(F')$. Since also $x_n \in \mathcal{I}$, we can conclude that (F', x_n) is a yes-instance of QCL-PMContainment.

"$\Longleftarrow$": Let (F', x_n) be a yes-instance of QCL-PMContainment. Then there is an interpretation $\mathcal{I}$ such that $x_n \in \mathcal{I}$ and $\mathcal{I} \in Prf_{\text{QCL}}(F')$. By the construction of F', we have that $\mathcal{I} \models F$. Towards a contradiction, assume there is an interpretation $\mathcal{I}'$ such that $\mathcal{I}' \models F$, and such that $\mathcal{J}_{k'} = \mathcal{I}' \cap \{x_1, \ldots, x_n\}$ is lexicographically larger than $\mathcal{J}_k = \mathcal{I} \cap \{x_1, \ldots, x_n\}$ with respect to $x_1 > \cdots > x_n$. Then $k' < k$. But by the same argument as in the only-if-direction, we can conclude that $deg_{\text{QCL}}(\mathcal{I}, F') = k$ and $deg_{\text{QCL}}(\mathcal{I}', F') = k'$, i.e. $deg_{\text{QCL}}(\mathcal{I}, F') < deg_{\text{QCL}}(\mathcal{I}', F')$. But then $\mathcal{I}$ is not a preferred model of F'. Contradiction. This means that $\mathcal{J}_k$ is the lexicographically largest interpretation with respect to $(x_1, \ldots, x_n)$ that can be extended to a model of F. Therefore, $(F, (x_1, \ldots, x_n))$ is a yes-instance of LogLexMaxSat. $\square$

In contrast, the optionality of LCL can grow exponentially with the formula size. Therefore, we can not use the argument above to show that LCL-PMContainment is in Θ_2^{P}. Indeed, it turns out that LCL-PMContainment is Δ_2^{P}-complete. In order to prove this, we observe that LCL allows us to represent the lexicographic order with an exponentially smaller formula than QCL, CCL, and SCCL.

Lemma 20. *Let $x_1 > \cdots > x_n$ be an ordering over n propositional variables. Let $\mathcal{I}_k \subseteq \{x_1, \ldots, x_n\}$ be the lexicographically k-th largest interpretation over this ordering, and let $F_n = (x_1 \vec{\diamond} (x_2 \vec{\diamond} (\cdots (x_{n-1} \vec{\diamond} x_n))))$. Then*

$$deg_{\text{LCL}}(\mathcal{I}_k, F_n) = \begin{cases} k & \text{if } k < 2^n \\ \infty & \text{if } k = 2^n \end{cases}$$

Proof. First, we show $opt_{\text{LCL}}(F_n) = 2^n - 1$ by induction over n:

- Base case: $n = 1$. Then $opt_{\text{LCL}}(F_n) = opt_{\text{LCL}}(x_1) = 1 = 2^n - 1$.

- Step case: $n > 1$. Then $F_n = (x_1 \vec{\diamond} A)$, where $A = (x_2 \vec{\diamond}(\cdots(x_{n-1} \vec{\diamond} x_n)))$. By our I.H., $opt_{\mathrm{LCL}}(A) = 2^{n-1} - 1$, and thus

$$\begin{aligned} opt_{\mathrm{LCL}}(F_n) &= opt_{\mathrm{LCL}}(x_1 \vec{\diamond} A) \\ &= (opt_{\mathrm{LCL}}(x_1) + 1) \cdot (opt_{\mathrm{LCL}}(A) + 1) - 1 \\ &= (1+1) \cdot ((2^{n-1} - 1) + 1) - 1 \\ &= (2 \cdot 2^{n-1}) - 1 = 2^n - 1. \end{aligned}$$

Now we proceed with the main proof, again by induction over n:

- Base case: $n = 1$. Then $F_n = x_1$, $\mathcal{I}_1 = \{x_1\}$, and $\mathcal{I}_2 = \emptyset$. Clearly, $deg_{\mathrm{LCL}}(\mathcal{I}_1, F_n) = 1$ and $deg_{\mathrm{LCL}}(\mathcal{I}_2, F_n) = \infty$, as required.
- Step case: $n > 1$. Then $F_n = (x_1 \vec{\diamond} A)$, where $A = (x_2 \vec{\diamond}(\cdots(x_{n-1} \vec{\diamond} x_n)))$. Let $\mathcal{J}_\ell$ be the lexicographically ℓ-th largest interpretation over $x_2 > \cdots > x_n$. By our I.H., $deg_{\mathrm{LCL}}(\mathcal{J}_\ell, A) = \ell$ if $\ell < 2^{n-1}$, and $deg_{\mathrm{LCL}}(\mathcal{J}_\ell, A) = \infty$ if $\ell = 2^{n-1}$. We can obtain the k-th largest interpretation $\mathcal{I}_k$ over $x_1 > \cdots > x_n$ as follows:
  1. If $k \le 2^{n-1}$, then $\mathcal{I}_k = \mathcal{J}_k \cup \{x_1\}$. Therefore, $deg_{\mathrm{LCL}}(\mathcal{I}_k, x_1) = 1$. There are two cases:
     (a) $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) < \infty$. Then $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = deg_{\mathcal{L}}(\mathcal{J}_k, A) = k$.
     (b) $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) = \infty$. Then $k = 2^{n-1}$. Thus, $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = opt_{\mathrm{LCL}}(A) + 1 = (2^{n-1} - 1) + 1 = 2^{n-1} = k$.
  2. If $k > 2^{n-1}$, then $\mathcal{I}_k = \mathcal{J}_{(k-2^{n-1})}$. Therefore, $deg_{\mathrm{LCL}}(\mathcal{I}_k, x_1) = \infty$. There are two cases:
     (a) $deg_{\mathrm{LCL}}(\mathcal{J}_{(k-2^{n-1})}, A) < \infty$. Then $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = opt_{\mathrm{LCL}}(A) + 1 + deg_{\mathcal{L}}(\mathcal{J}_{(k-2^{n-1})}, A) = (2^{n-1} - 1) + 1 + (k - 2^{n-1}) = k$.
     (b) $deg_{\mathrm{LCL}}(\mathcal{J}_{(k-2^{n-1})}, A) = \infty$. Then $k - 2^{n-1} = 2^{n-1}$, and therefore $k = 2^n$. Furthermore, $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = \infty$, as required. □

For LCL, we can thus drop the condition $n \le \log(|F|)$ in the construction above (Proposition 19), yielding a reduction from the Δ_2^{P}-complete problem LexMaxSat [18].

Definition 19. LexMaxSat is the decision problem where, given a PL-formula F and an ordering $x_1 > \cdots > x_n$ over all variables in F, we ask whether x_n is true in the lexicographically largest model of F.

Proposition 21. LCL-PMContainment *is* Δ_2^{P}*-complete.*

Proof. Δ_2^{P}-membership is established in Proposition 17. Regarding hardness, let $(F, (x_1, \dots, x_n))$ be an arbitrary instance of LexMaxSat. We construct an instance (F', x_n) of LCL-PMContainment, where

$$F' = F \wedge (x_1 \vec{\diamond}(x_2 \vec{\diamond}(\cdots(x_{n-1} \vec{\diamond} x_n)))).$$

It remains to show that $(F, (x_1, \dots, x_n))$ is a yes-instance of LexMaxSat if and only if (F', x_n) is a yes-instance of LCL-PMContainment.

"$\Longrightarrow$": Let $(F, (x_1, \dots, x_n))$ be a yes-instance of LexMaxSat. Then there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models F$, $x_n \in \mathcal{I}$, and $\mathcal{I}$ is the lexicographically largest model of F with respect to the ordering $x_1 > \cdots > x_n$. Let $\mathcal{J}$ be any interpretation other than $\mathcal{I}$. If $\mathcal{J} \not\models F$, then $deg_{\mathrm{LCL}}(\mathcal{J}, F') = \infty$, and $\mathcal{J}$ is not a preferred model of F'. If $\mathcal{J} \models F$, then $\mathcal{J}$ must be lexicographically smaller than $\mathcal{I}$. By Lemma 20, we can directly infer that $deg_{\mathrm{LCL}}(\mathcal{I}, x_1 \vec{\diamond}(\cdots(x_{n-1} \vec{\diamond} x_n))) < deg_{\mathrm{LCL}}(\mathcal{J}, x_1 \vec{\diamond}(\cdots(x_{n-1} \vec{\diamond} x_n)))$, and therefore $deg_{\mathrm{LCL}}(\mathcal{I}, F') < deg_{\mathrm{LCL}}(\mathcal{J}, F')$. This means that $\mathcal{I} \in Prf_{\mathrm{LCL}}(F')$. Since also $x_n \in \mathcal{I}$, we have that (F', x_n) is a yes-instance of LCL-PMContainment.

"$\Longleftarrow$": Let (F', x_n) be a yes-instance of LCL-PMContainment. Then there is an interpretation $\mathcal{I}$ such that $x_n \in \mathcal{I}$ and $\mathcal{I} \in Prf_{\mathrm{LCL}}(F')$. Towards a contradiction, assume that there is an interpretation $\mathcal{J}$ such that $\mathcal{J} \models F$, and such that $\mathcal{J}$ is lexicographically larger than $\mathcal{I}$ with respect to $x_1 > \cdots > x_n$. But, by Lemma 20, this means that $deg_{\mathrm{LCL}}(\mathcal{J}, F') < deg_{\mathrm{LCL}}(\mathcal{I}, F')$, which implies that $\mathcal{I} \notin Prf_{\mathrm{LCL}}(F')$. Contradiction. Thus, $\mathcal{I}$ is the lexicographically largest model of F. Since $x_n \in \mathcal{I}$, we have that $(F, (x_1, \dots, x_n))$ is a yes-instance of LexMaxSat. □

Finally, we consider $\mathcal{L}$-FullEquiv, the problem of deciding whether $A \equiv_f^{\mathcal{L}} B$ holds for given $\mathcal{L}$-formulas A and B, as well as the analogously defined problems $\mathcal{L}$-DegreeEquiv and $\mathcal{L}$-StrongEquiv.

Proposition 22. *For any tractable choice logic $\mathcal{L}$, (1) $\mathcal{L}$-FullEquiv and $\mathcal{L}$-DegreeEquiv are* coNP*-complete; (2) $\mathcal{L}$-StrongEquiv is* coNP*-complete if $\mathcal{L}$ is optionality-ignoring or -differentiating.*

Proof. We prove NP-membership of co-$\mathcal{L}$-FullEquiv: given two $\mathcal{L}$-formulas A and B, guess an interpretation $\mathcal{I} \subseteq var(A) \cup var(B)$ and check whether $deg_{\mathcal{L}}(\mathcal{I}, A) \neq deg_{\mathcal{L}}(\mathcal{I}, B)$ or $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$. These checks can be done in polynomial time for tractable $\mathcal{L}$.

Table 2
Summary of complexity results for choice logic formulas.

	Tract.	PL	QCL/CCL/SCCL	LCL
$\mathcal{L}$-DegreeChecking	in P	in P	in P	in P
$\mathcal{L}$-DegreeSat	NP-c	NP-c	NP-c	NP-c
$\mathcal{L}$-PMChecking	in coNP	in P	coNP-c	coNP-c
$\mathcal{L}$-PMContainment	NP-h/in Δ_2^{P}	NP-c	Θ_2^{P}-c	Δ_2^{P}-c
$\mathcal{L}$-DegreeEquiv	coNP-c	coNP-c	coNP-c	coNP-c
$\mathcal{L}$-FullEquiv	coNP-c	coNP-c	coNP-c	coNP-c

As for coNP-hardness of $\mathcal{L}$-FullEquiv, let F be an arbitrary instance of Unsat. We construct an instance (A, B) of $\mathcal{L}$-FullEquiv with $A = F$ and $B = \bot$. Observe that $opt_{\mathcal{L}}(A) = opt_{\mathcal{L}}(B) = 1$, since F is a classical formula. Additionally, A and B are degree-equivalent if and only if $\mathcal{I} \not\models A$ for all $\mathcal{I}$, which is the case if and only if $F = A$ is a yes-instance of Unsat.

The proof for $\mathcal{L}$-DegreeEquiv is analogous. Furthermore, the results for $\mathcal{L}$-StrongEquiv follow from our characterizations in Section 3 (Propositions 8 and 9). $\square$

Table 2 summarizes our complexity results for formulas of tractable choice logics, thus including all specific logics studied so far. The most interesting results are arguably those for $\mathcal{L}$-PMContainment: for an arbitrary tractable choice logic, we have a lower bound in the form of NP-hardness and an upper bound in the form of Δ_2^{P}-membership. Our results for individual choice logics show that our framework allows for instantiations located at the lower bound (e.g. PL being NP-complete), the upper bound (e.g. LCL being Δ_2^{P}-complete), and in-between (e.g. QCL being Θ_2^{P}-complete). Note that a complexity analysis of non-tractable logics may be conducted in a similar way by using oracles for the optionality- and satisfaction degree functions. Observe that the complexity of $\mathcal{L}$-StrongEquiv was not considered for choice logics that are neither optionality-ignoring nor -differentiating.

5. Preferred model entailment

So far, we considered the properties of single choice logic formulas. However, a comprehensive analysis requires us to study choice logic theories as well. Given a choice logic $\mathcal{L}$, an $\mathcal{L}$-theory T is simply a set of $\mathcal{L}$-formulas. Especially interesting in this regard is the notion of preferred model entailment, where a theory T entails a classical formula F, written as $T \mathrel{|\!\sim} F$, if and only if F is true in all preferred models of T. The preferred models of choice logic theories can be determined in numerous ways, with Brewka et al. [11] most notably introducing lexicographic and inclusion-based semantics.

In this section, we will investigate preferred model entailment with regards to logical properties such as cautious monotony (Section 5.1) and computational complexity (Section 5.2). Our analysis will be conducted for large classes of choice logics, and several preferred model semantics, including the lexicographic and inclusion-based approaches as well as newly introduced semantics. First, we introduce a simple yet previously not considered approach where a finite $\mathcal{L}$-theory $T = \{A_1, \ldots, A_n\}$ can intuitively be seen as the formula $A_1 \wedge \cdots \wedge A_n$.

Definition 20. Let $\mathcal{L}$ be a choice logic, $\mathcal{I}$ an interpretation, and T an $\mathcal{L}$-theory. $\mathcal{I}$ is a model of T, written as $\mathcal{I} \in Mod_{\mathcal{L}}(T)$, if $\mathcal{I} \in Mod_{\mathcal{L}}(F)$ for all $F \in T$. $\mathcal{I}$ is a minmax preferred model of T, written as $\mathcal{I} \in Prf^{mm}_{\mathcal{L}}(T)$, if $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and $max\{deg_{\mathcal{L}}(\mathcal{I}, F) \mid F \in T\} \leq max\{deg_{\mathcal{L}}(\mathcal{J}, F) \mid F \in T\}$ for all other interpretations $\mathcal{J}$.

Lexicographic and inclusion-based semantics are defined as follows:

Definition 21. Let $\mathcal{L}$ be a choice logic, $\mathcal{I}$ an interpretation, and T an $\mathcal{L}$-theory. $\mathcal{I}^k_{\mathcal{L}}(T)$ denotes the set of formulas in T satisfied to a degree of k by $\mathcal{I}$, i.e., $\mathcal{I}^k_{\mathcal{L}}(T) = \{F \in T \mid deg_{\mathcal{L}}(\mathcal{I}, F) = k\}$.

- $\mathcal{I}$ is a lexicographically preferred model of T, written as $\mathcal{I} \in Prf^{lex}_{\mathcal{L}}(T)$, if $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and if there is no $\mathcal{J} \in Mod_{\mathcal{L}}(T)$ such that, for some $k \in \mathbb{N}$ and all $l < k$, $|\mathcal{I}^k_{\mathcal{L}}(T)| < |\mathcal{J}^k_{\mathcal{L}}(T)|$ and $|\mathcal{I}^l_{\mathcal{L}}(T)| = |\mathcal{J}^l_{\mathcal{L}}(T)|$ holds.
- $\mathcal{I}$ is an inclusion-based preferred model of T, written as $\mathcal{I} \in Prf^{inc}_{\mathcal{L}}(T)$, if $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and if there is no $\mathcal{J} \in Mod_{\mathcal{L}}(T)$ such that, for some $k \in \mathbb{N}$ and all $l < k$, $\mathcal{I}^k_{\mathcal{L}}(T) \subset \mathcal{J}^k_{\mathcal{L}}(T)$ and $\mathcal{I}^l_{\mathcal{L}}(T) = \mathcal{J}^l_{\mathcal{L}}(T)$ holds.

In contrast to the minmax approach, the lexicographic approach considers satisfaction degrees in a bottom-up way, i.e., it is most important to maximize the number of formulas satisfied to a degree of 1, second-most important to maximize the number of formulas satisfied to a degree of 2, and so on. Similarly for the inclusion-based approach. Let us provide an example for the various preferred model semantics.

Example 5. Consider the QCL-theory $T = \{a \vec{\times} c, b \vec{\times} c, \neg(a \wedge b)\}$, and note that if we see T as the conjunction of it's formulas we get $F' = (a \vec{\times} c) \wedge (b \vec{\times} c) \wedge \neg(a \wedge b)$ from Example 1. From there, we can infer that $\{c\}, \{a, c\}, \{b, c\} \in Prf^{mm}_{\mathrm{QCL}}(T)$. Now observe that

$$\{c\}^1_{\mathrm{QCL}}(T) = \{\neg(a \wedge b)\} \text{ and } \{c\}^2_{\mathrm{QCL}}(T) = \{a \vec{\times} c, b \vec{\times} c\},$$
$$\{a, c\}^1_{\mathrm{QCL}}(T) = \{a \vec{\times} c, \neg(a \wedge b)\} \text{ and } \{a, c\}^2_{\mathrm{QCL}}(T) = \{b \vec{\times} c\},$$
$$\{b, c\}^1_{\mathrm{QCL}}(T) = \{b \vec{\times} c, \neg(a \wedge b)\} \text{ and } \{b, c\}^2_{\mathrm{QCL}}(T) = \{a \vec{\times} c\}.$$

Thus, $\{c\}$ satisfies less formulas to a degree of 1 than $\{a, c\}$ or $\{b, c\}$. Formally, $|\{c\}^1_{\mathrm{QCL}}(T)| = 1$, $|\{c\}^2_{\mathrm{QCL}}(T)| = 2$, $|\{a, c\}^1_{\mathrm{QCL}}(T)| = |\{b, c\}^1_{\mathrm{QCL}}(T)| = 2$, and $|\{a, c\}^2_{\mathrm{QCL}}(T)| = |\{b, c\}^1_{\mathrm{QCL}}(T)| = 1$. Thus, $\{a, c\}, \{b, c\} \in Prf^{lex}_{\mathrm{QCL}}(T)$ but $\{c\} \notin Prf^{lex}_{\mathrm{QCL}}(T)$. For the inclusion-based semantics it can be checked in a similar way that $\{a, c\}, \{b, c\} \in Prf^{inc}_{\mathrm{QCL}}(T)$ but $\{c\} \notin Prf^{inc}_{\mathrm{QCL}}(T)$.

Note that under the minmax semantics the ranking of an interpretation depends only on a single formula, namely that with the highest degree under the given interpretation. In contrast, under the lexicographic and inclusion-based semantics all formulas in a theory may influence an interpretation's ranking. We will now propose a preferred model semantics that constitutes a middle ground in this matter: given an $\mathcal{L}$-theory $T = \{A_1, \ldots, A_n\}$ and an interpretation $\mathcal{I}$, $f(n)$-many formulas, with $1 \leq f(n) \leq n$, may influence the ranking of $\mathcal{I}$. For example, if we set $f(n) = 3$, then the three lowest (or highest) degrees obtained in T under $\mathcal{I}$ could be lexicographically compared to the three lowest (or highest) degrees obtained by another interpretation. For our complexity analysis, considering $f(n) = \log(n)$ is especially interesting.

Definition 22. Let $\mathcal{L}$ be a choice logic and $T = \{A_1, \ldots, A_n\}$ a finite $\mathcal{L}$-theory. The log-worst formulas of T relative to an interpretation $\mathcal{I}$ are given by a set $L^{\mathcal{I}}_{\mathcal{L}}(T)$ such that $|L^{\mathcal{I}}_{\mathcal{L}}(T)| = \lceil \log(n) \rceil$ and such that, for all $A \in T \setminus L^{\mathcal{I}}_{\mathcal{L}}(T)$, $deg_{\mathcal{L}}(\mathcal{I}, A) \leq min\{deg_{\mathcal{L}}(\mathcal{I}, B) \mid B \in L^{\mathcal{I}}_{\mathcal{L}}(T)\}$. An interpretation $\mathcal{I}$ is a log-lexicographically preferred model of T, written as $\mathcal{I} \in Prf^{log}_{\mathcal{L}}(T)$, if $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and if there is no $\mathcal{J} \in Mod_{\mathcal{L}}(T)$ such that, for some $k \in \mathbb{N}$ and all $l > k$, $|\mathcal{I}^k_{\mathcal{L}}(L^{\mathcal{I}}_{\mathcal{L}}(T))| > |\mathcal{J}^k_{\mathcal{L}}(L^{\mathcal{J}}_{\mathcal{L}}(T))|$ and $|\mathcal{I}^l_{\mathcal{L}}(L^{\mathcal{I}}_{\mathcal{L}}(T))| = |\mathcal{J}^l_{\mathcal{L}}(L^{\mathcal{J}}_{\mathcal{L}}(T))|$ holds.

In the log-lexicographic approach, satisfaction degrees are considered in a top-down manner, i.e., we strive to minimize the number of formulas satisfied to high degrees. As a result, $\mathcal{I} \in Prf^{log}_{\mathcal{L}}(T)$ implies $\mathcal{I} \in Prf^{mm}_{\mathcal{L}}(T)$, meaning that the log-lexicographic semantics is a refinement of the minmax semantics.

Moreover, note that $L^{\mathcal{I}}_{\mathcal{L}}(T)$ from Definition 22 is not necessarily unique for a given T. However, for the log-lexicographic semantics it is of no importance which exact $L^{\mathcal{I}}_{\mathcal{L}}(T)$ is considered, as we only care about how many formulas are satisfied to certain degrees, not *which* formulas.

Example 6. Let $T = \{a \vec{\times} c, b \vec{\times} c, \neg(a \wedge b)\}$, just as in Example 5. Note that $\lceil \log_2(3) \rceil = 2$, i.e., given an interpretation, using the log-lexicographic semantics we are only interested in those two formulas that are satisfied to a maximal degree. For $\{c\}$ this is $L^{\{c\}}_{\mathrm{QCL}}(T) = \{a \vec{\times} c, b \vec{\times} c\}$ with both formulas satisfied to a degree of 2. For $\{a, c\}$ this can be $L^{\{a,c\}}_{\mathrm{QCL}}(T) = \{\neg(a \wedge b), b \vec{\times} c\}$ or $L^{\{a,c\}}_{\mathrm{QCL}}(T) = \{a \vec{\times} c, b \vec{\times} c\}$ with $b \vec{\times} c$ satisfied to a degree of 2 but $\neg(a \wedge b)$ and $a \vec{\times} b$ satisfied to a degree of 1. Analogously for $L^{\{b,c\}}_{\mathrm{QCL}}(T)$. Thus, $\{a, c\}, \{b, c\} \in Prf^{log}_{\mathrm{QCL}}(T)$ but $\{c\} \notin Prf^{log}_{\mathrm{QCL}}(T)$.

Now that we introduced some ways to determine the preferred models of a choice logic theory, we formally define the notion of preferred model entailment:

Definition 23. Let $\mathcal{L}$ be a choice logic, T an $\mathcal{L}$-theory, F a classical formula, and σ a preferred model semantics, e.g., $\sigma \in \{mm, lex, inc, log\}$. Then $T \mathrel{|\!\!\sim}^{\sigma}_{\mathcal{L}} F$ if and only if $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T)$ implies $\mathcal{I} \models F$.

5.1. Logical properties of preferred model entailment

First, we show that preferred model entailment is non-monotonic for all preferred model semantics considered here (mm, lex, inc, log), and for all choice logics in which more than two satisfaction degrees can be obtained (which of course includes QCL, CCL, LCL, and SCCL).

Proposition 23. *Let $\mathcal{L}$ be a choice logic such that $\mathcal{D}_{\mathcal{L}} \neq \{1, \infty\}$ and let $\sigma \in \{mm, lex, inc, log\}$. The preferred model entailment $\mathrel{|\!\!\sim}^{\sigma}_{\mathcal{L}}$ is non-monotonic, i.e., $T \mathrel{|\!\!\sim}^{\sigma}_{\mathcal{L}} B$ does not necessarily imply $T \cup \{A\} \mathrel{|\!\!\sim}^{\sigma}_{\mathcal{L}} B$.*

Proof. Let $k \in \mathcal{D}_{\mathcal{L}} \setminus \{1, \infty\}$. Let a be a propositional variable. By Proposition 4 we know that there is an $\mathcal{L}$-formula F such that $deg_{\mathcal{L}}(\mathcal{I}, F) = 1$ if $a \in \mathcal{I}$ and $deg_{\mathcal{L}}(\mathcal{I}, F) = k$ if $a \notin \mathcal{I}$. Then under all considered semantics $\mathcal{I}$ is a preferred model of F if and only if it contains a. Therefore, we have $\{F\} \mathrel{|\!\!\sim}^{\sigma}_{\mathcal{L}} a$. Now consider $\{F\} \cup \{\neg a\}$. We observe that $\mathcal{I} \in Mod_{\mathcal{L}}(\{F\} \cup \{\neg a\})$ if and only if $a \notin \mathcal{I}$. It follows that for all preferred model semantics $\{F\} \cup \{\neg a\} \mathrel{|\!\!\not\sim}^{\sigma}_{\mathcal{L}} a$. $\square$

Non-monotonic inference relations of numerous formalisms have been studied previously, and desirable properties have been laid out by Kraus et al. [27]. One such property is that of cautious monotony, where $T \mid\!\sim A$ and $T \mid\!\sim B$ implies $T \cup \{A\} \mid\!\sim B$. In the original QCL-paper [11] it was shown that $\mid\!\sim^{lex}_{\mathrm{QCL}}$ satisfies this property. In fact, cautious monotony is satisfied by all choice logics under all preferred model semantics considered in this paper:

Proposition 24. *Let $\mathcal{L}$ be a choice logic and $\sigma \in \{mm, lex, inc, log\}$. The inference relation $\mid\!\sim^{\sigma}_{\mathcal{L}}$ satisfies cautious monotony for finite theories, i.e., $T \mid\!\sim^{\sigma}_{\mathcal{L}} A$ and $T \mid\!\sim^{\sigma}_{\mathcal{L}} B$ implies $T \cup \{A\} \mid\!\sim^{\sigma}_{\mathcal{L}} B$ for all finite $\mathcal{L}$-theories T and all classical formulas A, B.*

Proof. Assume $T \mid\!\sim^{\sigma}_{\mathcal{L}} A$ and $T \mid\!\sim^{\sigma}_{\mathcal{L}} B$. Note that A and B are classical formulas. Let $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T \cup \{A\})$. Then $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and $deg_{\mathcal{L}}(\mathcal{I}, A) = 1$. Towards a contradiction, assume $\mathcal{I} \notin Prf^{\sigma}_{\mathcal{L}}(T)$. Since $\mathcal{I} \in Mod_{\mathcal{L}}(T)$, and since T is finite, there must be $\mathcal{J} \in Prf^{\sigma}_{\mathcal{L}}(T)$ that is more preferable than $\mathcal{I}$ with respect to T. By $T \mid\!\sim^{\sigma}_{\mathcal{L}} A$ also $deg_{\mathcal{L}}(\mathcal{J}, A) = 1$. We claim that then $\mathcal{J}$ is more preferable than $\mathcal{I}$ for $T \cup \{A\}$ for all considered semantics, which is a contradiction:

- For the minmax semantics, we observe that $max\{deg_{\mathcal{L}}(\mathcal{I}, F) \mid F \in T\} = max\{deg_{\mathcal{L}}(\mathcal{I}, F) \mid F \in T \cup \{A\}\}$. The same holds for $\mathcal{J}$.
- For the lexicographic and the inclusion-based semantics, observe that for $l \neq 1$ we have $\mathcal{I}^l_{\mathcal{L}}(T \cup \{A\}) = \mathcal{I}^l_{\mathcal{L}}(T)$ and for $l = 1$ we have $\mathcal{I}^1_{\mathcal{L}}(T \cup \{A\}) = \mathcal{I}^1_{\mathcal{L}}(T) \cup \{A\}$. The same holds for $\mathcal{J}$.
- For the log-lexicographic semantics there exists a $k \in \mathbb{N}$ such that for all $l > k$, $|\mathcal{I}^k_{\mathcal{L}}(L^{\mathcal{I}}_{\mathcal{L}}(T))| > |\mathcal{J}^k_{\mathcal{L}}(L^{\mathcal{J}}_{\mathcal{L}}(T))|$ and $|\mathcal{I}^l_{\mathcal{L}}(L^{\mathcal{I}}_{\mathcal{L}}(T))| = |\mathcal{J}^l_{\mathcal{L}}(L^{\mathcal{J}}_{\mathcal{L}}(T))|$. As $|L^{\mathcal{I}}_{\mathcal{L}}(T)| = |L^{\mathcal{J}}_{\mathcal{L}}(T)|$ it is not possible that $k = 1$. Therefore, we must have $|\mathcal{I}^k_{\mathcal{L}}(L^{\mathcal{I}}_{\mathcal{L}}(T \cup \{A\}))| > |\mathcal{J}^k_{\mathcal{L}}(L^{\mathcal{J}}_{\mathcal{L}}(T \cup \{A\}))|$ and $|\mathcal{I}^l_{\mathcal{L}}(L^{\mathcal{I}}_{\mathcal{L}}(T \cup \{A\}))| = |\mathcal{J}^l_{\mathcal{L}}(L^{\mathcal{J}}_{\mathcal{L}}(T \cup \{A\}))|$ for all $l > k$.

Thus, $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T)$ and by $T \mid\!\sim^{\sigma}_{\mathcal{L}} B$ also $\mathcal{I} \models B$. □

Note that we only considered finite theories in Proposition 24. Brewka et al. [11] did not explicitly make this assumption when investigating $\mid\!\sim^{lex}_{\mathrm{QCL}}$, but we believe it was implicitly assumed. In fact, $\mid\!\sim^{lex}_{\mathrm{QCL}}$ does not satisfy cautious monotony if infinite theories are allowed. The following result can likely be generalized for other preferred model semantics and choice logics, but we do not consider this here.

Proposition 25. *The inference relation $\mid\!\sim^{lex}_{\mathrm{QCL}}$ does not satisfy cautious monotony for infinite theories, i.e., there is an infinite* QCL-*theory T and classical formulas A, B such that $T \mid\!\sim^{lex}_{\mathrm{QCL}} A$, $T \mid\!\sim^{lex}_{\mathrm{QCL}} B$, but $T \cup \{A\} \not\mid\!\sim^{lex}_{\mathrm{QCL}} B$.*

Proof. In the following, by $\top^i$ we denote i occurrences of $\top$ connected by $\vec{\times}$. For example, $\top^1 = \top$, $\top^2 = \top \vec{\times} \top$, and $\top^3 = \top \vec{\times} \top \vec{\times} \top$. For $i \in \mathbb{N}$, let

$$A_i = \{a_i \vec{\times} \top^1, a_i \vec{\times} \top^2, \dots, a_i \vec{\times} \top^i\}.$$

The additional occurrences of $\top$ are added simply to duplicate the formula $a_i \vec{\times} \top$. Observe that $\{a_i\}$ satisfies all i formulas in A_i to a degree of 1, and that an interpretation that sets a_i to false satisfies all i formulas in A_i to a degree of 2. Furthermore, we define

$$R = \{\bot \vec{\times} \neg(a_i \wedge a_j) \mid i, j \in \mathbb{N}, i \neq j\}$$

and finally

$$T = R \cup \bigcup_{i \in \mathbb{N}} A_i.$$

The formulas in R can at best be satisfied to a degree of 2. Furthermore, R enforces that, if $i \neq j$, then a_i and a_j can not be set to true by the same interpretation. For any $i \in \mathbb{N}$, the interpretation $\{a_i\}$ satisfies all formulas in T to some finite degree. Moreover, $|\{a_i\}^1_{\mathrm{QCL}}(T)| = i$. This means that, for any $i \in \mathbb{N}$, $\{a_i\}$ is not a preferred model of T since there is always $j > i$ such that $|\{a_i\}^1_{\mathrm{QCL}}(T)| < |\{a_j\}^1_{\mathrm{QCL}}(T)|$. In fact, T has no preferred models, since if some $\mathcal{I}$ does not satisfy any a_i, then $|\mathcal{I}^1_{\mathrm{QCL}}(T)| = 0$. Thus, $T \mid\!\sim^{lex}_{\mathrm{QCL}} a_1$ and $T \mid\!\sim^{lex}_{\mathrm{QCL}} a_2$ hold vacuously.

Now we consider the extension of T by a_1, i.e. $T \cup \{a_1\}$. To satisfy $T \cup \{a_1\}$, a_1 must be set to true, but a_1 can not be set to true at the same time as any other a_i with $i \neq 1$. Thus, $\{a_1\}$ is the only model of $T \cup \{a_1\}$ (without loss of generality, we can assume $\mathcal{I} \subseteq \{a_1, a_2, \dots\}$ for all interpretations $\mathcal{I}$ we are dealing with). Therefore, $\{a_1\}$ is the single preferred model of $T \cup \{a_1\}$ which means that $T \cup \{a_1\} \not\mid\!\sim^{lex}_{\mathrm{QCL}} a_2$. □

Another important property is that of cumulative transitivity, which is known to hold for $\mid\!\sim^{lex}_{\mathrm{QCL}}$. Again, we can generalize this result. Note that the following proposition holds for infinite theories, except for the log-lexicographic semantics which are only defined for finite theories.

Proposition 26. *Let $\mathcal{L}$ be a choice logic and $\sigma \in \{mm, lex, inc, log\}$. The inference relation $\mid\!\sim^{\sigma}_{\mathcal{L}}$ satisfies cumulative transitivity, i.e., $T \mid\!\sim^{\sigma}_{\mathcal{L}} A$ and $T \cup \{A\} \mid\!\sim^{\sigma}_{\mathcal{L}} B$ implies $T \mid\!\sim^{\sigma}_{\mathcal{L}} B$ for all $\mathcal{L}$-theories T and all classical formulas A, B.*

Proof. Assume $T \mid\!\sim^{\sigma}_{\mathcal{L}} A$ and $T \cup \{A\} \mid\!\sim^{\sigma}_{\mathcal{L}} B$. Note that A and B are classical formulas. Let $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T)$. Then $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ and, by $T \mid\!\sim^{\sigma}_{\mathcal{L}} A$, $deg_{\mathcal{L}}(\mathcal{I}, A) = 1$. It is easy to see that $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T \cup \{A\})$ which, by $T \cup \{A\} \mid\!\sim^{\sigma}_{\mathcal{L}} B$, implies $\mathcal{I} \models B$. □

Lastly, Brewka et al. [11] also considered the property of rational monotony [30] and showed that it is satisfied by $\mid\!\sim^{lex}_{\mathrm{QCL}}$. The result is achieved by transforming all formulas in a given QCL-theory into a normal form, and to then further transform the theory into a stratified knowledge base. We now show that rational monotony is satisfied in all choice logics and under all considered preferred model semantics except the inclusion-based approach.

Proposition 27. *Let $\mathcal{L}$ be a choice logic and $\sigma \in \{mm, lex, log\}$. The inference relation $\mid\!\sim^{\sigma}_{\mathcal{L}}$ satisfies rational monotony, i.e., $T \cup \{A\} \not\mid\!\sim^{\sigma}_{\mathcal{L}} B$ and $T \not\mid\!\sim^{\sigma}_{\mathcal{L}} \neg A$ implies $T \not\mid\!\sim^{\sigma}_{\mathcal{L}} B$ for all $\mathcal{L}$-theories T and all classical formulas A, B.*

Proof. Assume $T \cup \{A\} \not\mid\!\sim^{\sigma}_{\mathcal{L}} B$ and $T \not\mid\!\sim^{\sigma}_{\mathcal{L}} \neg A$. Then there is $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T \cup \{A\})$ such that $\mathcal{I} \not\models B$ and $\mathcal{J} \in Prf^{\sigma}_{\mathcal{L}}(T)$ such that $\mathcal{J} \models A$. Note that $deg_{\mathcal{L}}(\mathcal{I}, A) = deg_{\mathcal{L}}(\mathcal{J}, A) = 1$. Towards a contradiction, assume that $\mathcal{I} \notin Prf^{\sigma}_{\mathcal{L}}(T)$. Then $\mathcal{J}$ is more preferable than $\mathcal{I}$ for T, which also means that $\mathcal{J}$ is more preferable than $\mathcal{I}$ for $T \cup \{A\}$, i.e., $\mathcal{I} \notin Prf^{\sigma}_{\mathcal{L}}(T \cup \{A\})$. Contradiction. Thus, $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T)$. Since $\mathcal{I} \not\models B$ we have $T \not\mid\!\sim^{\sigma}_{\mathcal{L}} B$. □

Proposition 28. *Let $\mathcal{L}$ be a choice logic such that $\mathcal{D}_{\mathcal{L}} \neq \{1, \infty\}$. The inference relation $\mid\!\sim^{inc}_{\mathcal{L}}$ does not satisfy rational monotony, i.e., there is an $\mathcal{L}$-theory T and classical formulas A, B such that $T \cup \{A\} \not\mid\!\sim^{\sigma}_{\mathcal{L}} B$, $T \not\mid\!\sim^{\sigma}_{\mathcal{L}} \neg A$, but $T \mid\!\sim^{\sigma}_{\mathcal{L}} B$.*

Proof. Let $k \in \mathcal{D}_{\mathcal{L}} \setminus \{1, \infty\}$. Let a, b, c, be propositional variables. By Proposition 4 we know that there are $\mathcal{L}$-formulas F, G, H over $\{a, b, c\}$ such that

$$deg_{\mathcal{L}}(\{a\}, F) = 1, deg_{\mathcal{L}}(\{a\}, G) = k, deg_{\mathcal{L}}(\{a\}, H) = k$$
$$deg_{\mathcal{L}}(\{b\}, F) = k, deg_{\mathcal{L}}(\{b\}, G) = 1, deg_{\mathcal{L}}(\{b\}, H) = 1$$
$$deg_{\mathcal{L}}(\{c\}, F) = k, deg_{\mathcal{L}}(\{c\}, G) = k, deg_{\mathcal{L}}(\{c\}, H) = 1,$$

and $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I}, G) = deg_{\mathcal{L}}(\mathcal{I}, H) = \infty$ for all other $\mathcal{I} \subseteq \{a, b, c\}$. Let $T = \{F, G, H\}$, $A = \neg b$, and $B = a \vee b$. It can be verified that $Prf^{inc}_{\mathcal{L}}(T) = \{\{a\}, \{b\}\}$ and $Prf^{inc}_{\mathcal{L}}(T \cup \{A\}) = \{\{a\}, \{c\}\}$. Thus, $T \cup \{A\} \not\mid\!\sim^{\sigma}_{\mathcal{L}} B$, $T \not\mid\!\sim^{\sigma}_{\mathcal{L}} \neg A$, but $T \mid\!\sim^{\sigma}_{\mathcal{L}} B$. □

5.2. Complexity of preferred model entailment

We will now analyze the notion of preferred model entailment with respect to its computational complexity. To do so, we must also consider the problem of preferred model checking for choice logic theories.

Definition 24. Let $\mathcal{L}$ be a choice logic and σ a preferred-model semantics for $\mathcal{L}$-theories, e.g., $\sigma \in \{mm, lex, inc, log\}$. We define the decision problems

- $\mathcal{L}$-PMChecking[σ]: Given a finite $\mathcal{L}$-theory T and an interpretation $\mathcal{I}$, decide whether $\mathcal{I} \in Prf^{\sigma}_{\mathcal{L}}(T)$;
- $\mathcal{L}$-Entailment[σ]: Given a finite $\mathcal{L}$-theory T and a classical formula F, decide whether $T \mid\!\sim^{\sigma}_{\mathcal{L}} F$.

A summary of results for the above problems can be found at the end of this section in Table 3. Regarding $\mathcal{L}$-PMChecking[σ] for choice logic theories, we see no complexity increase compared to $\mathcal{L}$-PMChecking for single formulas.

Proposition 29. *For all $\sigma \in \{mm, lex, inc, log\}$, $\mathcal{L}$-PMChecking[σ] is in P for $\mathcal{L} = \mathrm{PL}$ and in coNP for any tractable choice logic $\mathcal{L}$. $\mathcal{L}$-PMChecking[σ] is coNP-complete for any tractable choice logic $\mathcal{L}$ for which $\mathcal{D}_{\mathcal{L}} \neq \{1, \infty\}$ holds.*

Proof. P-membership for PL is straightforward. Furthermore, coNP-hardness for tractable choice logics with $\mathcal{D}_{\mathcal{L}} \neq \{1, \infty\}$ follows from results for single choice logic formulas (see Proposition 15). As for coNP-membership of tractable choice logics, it is easy to see that the complementary problem is in NP: given an $\mathcal{L}$-theory T and an interpretation $\mathcal{I}$, guess an interpretation $\mathcal{J}$ and check whether $\mathcal{J}$ is more preferred than $\mathcal{I}$. This can be done in polynomial time for all preferred-model semantics we are concerned with ($\sigma \in \{mm, lex, inc, log\}$). □

Note that in classical propositional logic the problems PL-Entailment[σ] with $\sigma \in \{mm, lex, inc, log\}$ are simply the problem of classical entailment. Thus, these problems are coNP-complete by well-known properties of PL. We now turn our attention to $\mathcal{L}$-Entailment[mm] in the general case. For convenience, we write $opt_{\mathcal{L}}(T)$ for $max\{opt_{\mathcal{L}}(F) \mid F \in T\}$.

Proposition 30. *$\mathcal{L}$-ENTAILMENT[mm] is in Δ_2^P and coNP-hard for all tractable choice logics. $\mathcal{L}$-ENTAILMENT[mm] is in Θ_2^P for a tractable choice logic $\mathcal{L}$ if for some constant c and all $\mathcal{L}$-formulas F it holds that $opt_{\mathcal{L}}(F) \in \mathcal{O}(|F|^c)$.*

Proof. coNP-hardness follows from coNP-hardness of PL. We show membership for the complementary problem, where we ask whether a classical formula F evaluates to false under some minmax preferred model of a given theory $T = \{A_1, \dots, A_n\}$: first, we conduct a binary search over $(1, \dots, opt_{\mathcal{L}}(T))$, in each step using an NP-oracle to answer whether there is an interpretation $\mathcal{I}$ such that $max\{deg_{\mathcal{L}}(\mathcal{I}, A) \mid A \in T\} \leq k$, where k is the current position of the binary search. In this way, we find the minimum m such that $max\{deg_{\mathcal{L}}(\mathcal{J}, A) \mid A \in T\} = m$ for any interpretation $\mathcal{J}$. If $m = \infty$, we already have a no-instance since $Prf_{\mathcal{L}}^{mm}(T) = \emptyset$, i.e., F is true in all preferred models of T. If $m < \infty$, we conduct a final NP-oracle call in which we guess an interpretation $\mathcal{I}$ and ask whether $max\{deg_{\mathcal{L}}(\mathcal{I}, A) \mid A \in T\} = m$ and whether $\mathcal{I} \not\models F$. This procedure runs in polynomial time, making use of $\mathcal{O}(\log(opt_{\mathcal{L}}(T)))$ NP-oracle calls. □

Completeness results for specific choice logics follow directly by the above proposition as well as from known results for single formulas (cf. Table 2).

Proposition 31. *$\mathcal{L}$-ENTAILMENT[mm] is Θ_2^P-complete for $\mathcal{L} \in \{$QCL, CCL, SCCL$\}$ and Δ_2^P-complete for $\mathcal{L} =$ LCL.*

Proof. Membership results follow from Proposition 30, hardness results from $\mathcal{L}$-PMCONTAINMENT (cf. Table 2): an instance (F, x) of $\mathcal{L}$-PMCONTAINMENT can be reduced to an instance $(\{F\}, \neg a)$ of co-$\mathcal{L}$-ENTAILMENT[*mm*]. Then x is true under some preferred model of F if and only if $\neg a$ is false under some (minmax) preferred model of $\{F\}$. □

As the minmax semantics is equivalent to taking the conjunction of all formulas in the theory, it is not surprising that we see the same complexity landscape as for $\mathcal{L}$-PMCONTAINMENT. This is different for $\mathcal{L}$-ENTAILMENT[*lex*], which can not be represented in terms of single formulas.

Proposition 32. *$\mathcal{L}$-ENTAILMENT[lex] is in Δ_2^P and coNP-hard for every tractable choice logic $\mathcal{L}$.*

Proof. coNP-hardness follows from coNP-hardness of PL. Δ_2^P-Membership of the complementary problem: given a theory $T = \{A_1, \dots, A_n\}$ and a classical formula F, we first conduct a binary search over $(1, \dots, opt_{\mathcal{L}}(T))$ to find the smallest k_1 such that some formula $A \in T$ is satisfied by a model of T to a degree of k_1. We then conduct a binary search over $(1, \dots, n)$ to find the maximum number m_1 such that $|\mathcal{I}_{\mathcal{L}}^{k_1}(T)| = m_1$ for some $\mathcal{I} \in Mod_{\mathcal{L}}(T)$. Observe that any $\mathcal{J} \in Prf_{\mathcal{L}}^{lex}(T)$ must satisfy exactly m_1 formulas in T to a degree of k_1. The above procedure makes $O(\log(opt_{\mathcal{L}}(T)) + \log(n))$ NP-oracle calls.

We proceed inductively: for $i > 1$, we conduct a binary search over $(k_{i-1}+1, \dots, opt_{\mathcal{L}}(T))$ to find the minimum degree k_i with $k_i > k_{i-1}$ such that $deg_{\mathcal{L}}(\mathcal{I}, A) = k_i$ for some $A \in T$ and $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ with $|\mathcal{I}_{\mathcal{L}}^{k_j}(T)| = m_j$ for all $j < i$. Then we conduct a binary search over $(1, \dots, n - \sum_{j=1}^{i-1} m_j)$ to find the maximum number m_i such that $|\mathcal{I}_{\mathcal{L}}^{k_i}(T)| = m_i$ for some $\mathcal{I} \in Mod_{\mathcal{L}}(T)$ with $|\mathcal{I}_{\mathcal{L}}^{k_j}(T)| = m_j$ for all $j < i$. Again, this requires $O(\log(opt_{\mathcal{L}}(T)) + \log(n))$ NP-oracle calls.

The above procedure has to be executed at most n times to find the 'degree-profile' for preferred models of T, i.e., every preferred model of T must satisfy exactly m_j formulas in T to a degree of k_j. Thus, $O(n \cdot (\log(opt_{\mathcal{L}}(T)) + \log(n)))$ NP-oracle calls are required so far.

Lastly, we make a final NP-oracle call to guess an interpretation $\mathcal{I}$ and, using k_j and m_j, check whether $\mathcal{I} \in Prf_{\mathcal{L}}^{lex}(T)$ and $\mathcal{I} \not\models F$. □

Proposition 33. *$\mathcal{L}$-ENTAILMENT[lex] is Δ_2^P-complete for $\mathcal{L} \in \{$QCL, CCL, SCCL, LCL$\}$.*

Proof. Membership for each $\mathcal{L} \in \{$QCL, CCL, SCCL, LCL$\}$ follows from Proposition 32. Hardness for LCL follows from LCL-PMCONTAINMENT (cf. Table 2 and the proof of Proposition 31). We show hardness for the complementary problem and $\mathcal{L} =$ QCL: let $(F, (a_1 \dots, a_n))$ be an instance of LEXMAXSAT. For every $1 \leq i \leq n$ we construct an $\mathcal{L}$-formula A_i such that, for any interpretation $\mathcal{I}$, $deg_{\mathcal{L}}(\mathcal{I}, A_i) = i$ if $a_i \in \mathcal{I}$ and $deg_{\mathcal{L}}(\mathcal{I}, a_i) = n + 1$ if $a_i \notin \mathcal{I}$.:

$$A_i = \bot \vec{\times} \dots \vec{\times} \bot \vec{\times} a_i \vec{\times} \bot \vec{\times} \dots \vec{\times} \bot \vec{\times} \top,$$

where $\bot$ occurs $i - 1$ times before a_i and $n - i$ times after a_i. For example, if $n = 4$, then $A_1 = a_1 \vec{\times} \bot \vec{\times} \bot \vec{\times} \bot \vec{\times} \top$, $A_2 = \bot \vec{\times} a_2 \vec{\times} \bot \vec{\times} \bot \vec{\times} \top$, and so on. We now construct an instance $(T, \neg a_n)$ of co-$\mathcal{L}$-ENTAILMENT[*lex*] with $T = \{F, A_1, \dots, A_n\}$. It remains to show that a_n is true in the lexicographically maximal model of F with respect to $a_1 > \dots > a_n$ if and only if $\neg a_n$ is false in some lexicographically preferred model of T.

Assume a_n is true in the lexicographically maximal model $\mathcal{I}$ of F with respect to $a_1 > \dots > a_n$. Since each A_i is always satisfied to some degree, and since $\mathcal{I} \models F$, we have $\mathcal{I} \in Mod_{\mathcal{L}}(T)$. Moreover, there can be no other $\mathcal{J} \in Mod_{\mathcal{L}}(T)$ such that, for some k, $|\mathcal{I}_{\mathcal{L}}^k(T)| < |\mathcal{J}_{\mathcal{L}}^k(T)|$ and, for all $l < k$, $|\mathcal{I}_{\mathcal{L}}^l(T)| = |\mathcal{J}_{\mathcal{L}}^l(T)|$. Otherwise, there would be some variable a_k such that

$\mathcal{J} \models a_k$, $\mathcal{I} \not\models a_k$, and $\mathcal{J} \models a_l \iff \mathcal{I} \models a_l$ for all $l < k$. But then $\mathcal{I}$ would not be the lexicographically maximal model of F. Thus, $\mathcal{I} \in Prf^{lex}_{\mathcal{L}}(T)$. Furthermore, $\mathcal{I} \not\models \neg a_n$.

Assume a_n is not true in the lexicographically maximal model of F. If F is not satisfiable, then neither is T, i.e., T has no preferred model and we have a no-instance of co-$\mathcal{L}$-ENTAILMENT[*lex*]. If F is satisfiable, then, by the same argument as above, $\mathcal{I} \in Prf^{lex}_{\mathcal{L}}(T)$ for the lexicographically largest model $\mathcal{I}$ of F with respect to $a_1 > \cdots > a_n$. In fact, $\mathcal{I}$ is the unique lexicographically preferred model of T. Furthermore, $\mathcal{I} \models \neg a_n$. □

The proofs for log-lexicographic semantics are similar to those of regular lexicographic semantics. However, the complexity for choice logics with polynomially-bounded optionality is actually located *inbetween* Θ_2^{P} and Δ_2^{P}, namely in $\Delta_2^{\mathrm{P}}[\mathcal{O}(\log^2 n)]$. The only truly natural $\Delta_2^{\mathrm{P}}[\mathcal{O}(\log^2 n)]$-complete problem we are aware of is model checking for a specific temporal logic [34]. A less natural, but useful $\Delta_2^{\mathrm{P}}[\mathcal{O}(\log^2 n)]$-complete problem is LOG²LEXMAXSAT, which is defined analogously to LOGLEXMAXSAT (cf. Definition 18) except that we are given a lexicographic order over $\log^2(n)$ variables [35].

Proposition 34. *$\mathcal{L}$-ENTAILMENT[log] is in Δ_2^{P} and* coNP*-hard for all tractable choice logics. $\mathcal{L}$-ENTAILMENT[log] is in $\Delta_2^{\mathrm{P}}[\mathcal{O}(\log^2 n)]$ for tractable $\mathcal{L}$ if for some constant c and all $\mathcal{L}$-formulas F it holds that $opt_{\mathcal{L}}(F) \in \mathcal{O}(|F|^c)$.*

Proof. coNP-hardness follows from coNP-hardness of PL. Regarding membership, we can determine the degree-profile of T's log-lexicographically preferred models analogously to the proof of Proposition 32, except that we do not need to execute the binary searches n times. In fact, by the definition of the log-lexicographic preferred model semantics, we need to execute the binary searches only $\mathcal{O}(\log(n))$ times, giving us a decision procedure that in total makes use of $\mathcal{O}(\log(n) \cdot (\log(opt_{\mathcal{L}}(T)) + \log(n)))$ NP-oracle calls. □

Proposition 35. *$\mathcal{L}$-ENTAILMENT[log] is $\Delta_2^{\mathrm{P}}[\mathcal{O}(\log^2 n)]$-complete for $\mathcal{L} \in \{\mathrm{QCL}, \mathrm{CCL}, \mathrm{SCCL}\}$ and Δ_2^{P}-complete for $\mathcal{L} = \mathrm{LCL}$.*

Proof. Membership for each $\mathcal{L} \in \{\mathrm{QCL}, \mathrm{CCL}, \mathrm{SCCL}, \mathrm{LCL}\}$ follows from Proposition 32. Hardness for LCL follows from LCL-PMCONTAINMENT (cf. Table 2 and the proof of Proposition 31). We show hardness for co-$\mathcal{L}$-ENTAILMENT[*log*] and $\mathcal{L} = \mathrm{QCL}$ by a reduction from LOG²LEXMAXSAT, where we are given a classical formula F over variables $X = \{x_1, \ldots, x_n\}$ and an ordering $x_1 > \ldots > x_l$ over $l = \log^2(n)$ variables. For every $1 \leq i \leq \log(n)$ we construct a formula A_i such that $deg_{\mathcal{L}}(\mathcal{I}, A_i) = k$ iff $\mathcal{I}$ is the lexicographically k-th largest interpretation with respect to the ordering $x_{(i-1)\cdot\log(n)+1} > \cdots > x_{i\cdot\log(n)}$. Recall that we already know how to construct such a formula A_i from the proof of Proposition 19, and that $|A_i| \in \mathcal{O}(\log(n) \cdot 2^{\log(n)}) = \mathcal{O}(\log(n) \cdot n)$. Now, for every $1 \leq i \leq \log(n)$, we construct $B_i = \bot \vec{\times} \cdots \vec{\times} \bot \vec{\times} A_i$, where $\bot$ appears $(\log(n) - i) \cdot 2^{\log(n)}$ times before A_i. Observe that $|B_i| \in \mathcal{O}(\log(n) \cdot 2^{\log(n)} + |A_i|) = \mathcal{O}(\log(n) \cdot n + |A_i|)$ for all B_i. Lastly, we construct $n - \log(n) - 1$ formulas C_j such that $deg_{\mathcal{L}}(\mathcal{J}, C_j) = 1$ for all interpretations $\mathcal{J}$. We can now define our $\mathcal{L}$-theory

$$T = \{F, B_1, \ldots, B_{\log(n)}, C_1, \ldots, C_{n-\log(n)-1}\}.$$

Observe that T contains exactly n formulas. Moreover, $B_{\log(n)}$ can be satisfied to a degree between 1 and $2^{\log(n)}$, $B_{\log(n)-1}$ to a degree between $2^{\log(n)} + 1$ and $2 \cdot 2^{\log(n)}$, and so on until finally B_1 can be satisfied to a degree between $(\log(n) - 1) \cdot 2^{\log(n)} + 1$ and $\log(n) \cdot 2^{\log(n)}$. The formulas C_j are always satisfied to a degree of 1 and therefore do not influence the log-lexicographically preferred models of T. Furthermore, the lexicographically largest model $\mathcal{I}$ of F with respect to the ordering $x_1 > \cdots > x_l$ satisfies B_1 optimally among the models of F, and is therefore preferred in T to all models of F that are lexicographically smaller regarding $x_1 > \cdots > x_{\log(n)}$. $\mathcal{I}$ also satisfies B_2 optimally among those models of F that satisfy B_1 optimally. By extension, we can conclude that an interpretation $\mathcal{J}$ is a lexicographically maximal model of F with respect to $x_1 > \cdots > x_l$ if and only if $\mathcal{J} \in Prf^{log}_{\mathcal{L}}(T)$. □

Lastly, we examine inclusion-based semantics. Here, we can not show Δ_2^{P} containment in general. Indeed, as it turns out $\mathcal{L}$-ENTAILMENT[*inc*] is Π_2^{P}-complete for all studied choice logic except PL.

Proposition 36. *$\mathcal{L}$-ENTAILMENT[inc] is in Π_2^{P} and* coNP*-hard for every tractable choice logic $\mathcal{L}$.*

Proof. coNP-hardness follows from coNP-hardness of PL. It is fairly easy to see that the complementary problem of $\mathcal{L}$-ENTAILMENT[*inc*] is in Σ_2^{P}: given a theory $T = \{A_1, \ldots, A_n\}$ and a classical formula F, we guess an interpretation $\mathcal{I}$ and, in coNP, check whether $\mathcal{I} \in Prf^{inc}_{\mathcal{L}}(T)$ and $\mathcal{I} \not\models F$. □

To show Π_2^{P}-completeness for specific choice logics, we can make use of an already existing translation from propositional circumscription to QCL [11, Proposition 10]. In fact, this existing translation starts from *prioritized* circumscription. Our construction, however, considers unprioritized circumscription and is therefore slightly simpler. Note that entailment for propositional circumscription is known to be Π_2^{P}-complete [20].

Table 3
Complexity of choice logic theories ($\sigma \in \{mm, lex, inc, log\}$).

	Tract.	PL	QCL/CCL/SCCL	LCL
$\mathcal{L}$-PMChecking[σ]	in coNP	in P	coNP-c	coNP-c
$\mathcal{L}$-Entailment[mm]	coNP-h/in Δ_2^{P}	coNP-c	Θ_2^{P}-c	Δ_2^{P}-c
$\mathcal{L}$-Entailment[log]	coNP-h/in Δ_2^{P}	coNP-c	$\Delta_2^{\mathsf{P}}[\mathcal{O}(\log^2 n)]$-c	Δ_2^{P}-c
$\mathcal{L}$-Entailment[lex]	coNP-h/in Δ_2^{P}	coNP-c	Δ_2^{P}-c	Δ_2^{P}-c
$\mathcal{L}$-Entailment[inc]	coNP-h/in Π_2^{P}	coNP-c	Π_2^{P}-c	Π_2^{P}-c

Definition 25. Let T be a classical propositional theory, F a classical formula, and $(P; R)$ a circumscription policy, where P are atoms to be minimized, and R are fixed atoms with $P \cap R = \emptyset$. A model $\mathcal{I}$ of T is $(P; R)$-minimal for T if there is no other model $\mathcal{J}$ of T such that $\mathcal{I} \cap R = \mathcal{J} \cap R$ and $\mathcal{J} \cap P \subset \mathcal{I} \cap P$.

CircEntailment is the decision problem where, given a classical theory T, a classical formula F, and a circumscription policy $(P; R)$, we ask whether F is true in all $(P; R)$-minimal models of T.

Proposition 37. *$\mathcal{L}$-Entailment[inc] is Π_2^{P}-complete for $\mathcal{L} \in \{\text{QCL, CCL, SCCL, LCL}\}$.*

Proof. Membership for each $\mathcal{L} \in \{\text{QCL, CCL, SCCL, LCL}\}$ follows from Proposition 36. We show hardness for $\mathcal{L} = \text{QCL}$. Consider an arbitrary instance $(T, F, (P; R))$ of CircEntailment. Let

$$T' = T \cup \{\neg p \vec{\times} p \mid p \in P\} \cup \{r \vec{\times} \neg r, \neg r \vec{\times} r \mid r \in R\}.$$

It remains to show that F is true in all $(P; R)$-minimal models of T if and only if F is true in all $Prf_{\mathcal{L}}^{inc}(T')$. We show this by proving that $\mathcal{I}$ is a $(P; R)$-minimal model of T if and only if $\mathcal{I} \in Prf_{\mathcal{L}}^{inc}(T')$.

Assume $\mathcal{I}$ is a $(P; R)$-minimal model of T. Then definitely $\mathcal{I} \in Mod_{\text{QCL}}(T')$, since all formulas in $T' \setminus T$ are always satisfied, either to a degree of 1 or 2. Let $\mathcal{J}$ be any other model of T'. For any $r \in R$, if $r \in \mathcal{I}$ but $r \notin \mathcal{J}$, then $\mathcal{I}$ and $\mathcal{J}$ are incomparable with respect to the *inc*-semantics since $deg_{\text{QCL}}(\mathcal{I}, r \vec{\times} \neg r) = 1$, $deg_{\text{QCL}}(\mathcal{J}, r \vec{\times} \neg r) = 2$, $deg_{\text{QCL}}(\mathcal{I}, \neg r \vec{\times} r) = 2$, and $deg_{\text{QCL}}(\mathcal{J}, \neg r \vec{\times} r) = 1$. Likewise if $r \notin \mathcal{I}$ but $r \in \mathcal{J}$. Thus, assume $r \in \mathcal{I} \iff r \in \mathcal{J}$ for all $r \in R$. Then $\mathcal{I} \cap P \subseteq \mathcal{J} \cap P$, since $\mathcal{I}$ is $(P; R)$-minimal. Therefore, for all $p \in P$, $deg_{\text{QCL}}(\mathcal{I}, \neg p \vec{\times} p) \leq deg_{\text{QCL}}(\mathcal{J}, \neg p \vec{\times} p)$. We can conclude that $\mathcal{I} \in Prf_{\text{QCL}}^{inc}(T')$.

Assume $\mathcal{I}$ is not a $(P; R)$-minimal model of T. If T is not satisfiable, then neither is T', and we are done. If T is satisfiable, then there must be an interpretation $\mathcal{J}$ with $\mathcal{J} \cap R = \mathcal{I} \cap R$ and $\mathcal{J} \cap P \subset \mathcal{I} \cap P$. By the same argument as above, for all $p \in P$, $deg_{\text{QCL}}(\mathcal{J}, \neg p \vec{\times} p) \leq deg_{\text{QCL}}(\mathcal{I}, \neg p \vec{\times} p)$. Furthermore, for at least one $q \in P$, $deg_{\text{QCL}}(\mathcal{J}, \neg q \vec{\times} q) < deg_{\text{QCL}}(\mathcal{I}, \neg q \vec{\times} q)$. Thus, $\mathcal{I} \notin Prf_{\text{QCL}}^{inc}(T')$. $\square$

Table 3 summarizes our complexity results for choice logic theories. Maybe the most interesting point here is that in entailment for QCL, CCL, and SCCL the complexity rises when going from minmax to (log-)lexicographic semantics. However, for LCL, all three problems are equally hard. Thus, the additional expressiveness of the (log-)lexicographic semantics makes entailment harder for choice logics with polynomially bounded optionality. For inclusion-based semantics we see an additional jump in complexity to Π_2^{P}-completeness for all considered choice logics (except PL). We observe that there are two ways in which the complexity of $\mathcal{L}$-Entailment[σ] is determined: on the one hand by the choice logic (e.g. QCL vs. LCL), and on the other hand by the preferred model semantics (e.g. minmax vs. lexicographic vs. inclusion-based). Furthermore, with $\mathcal{L}$-Entailment[log] ($\mathcal{L} \in \{\text{QCL, CCL, SCCL}\}$) we introduced fairly natural $\Delta_2^{\mathsf{P}}[\mathcal{O}(\log^2 n)]$-complete problems.[3]

6. Discussion

This section features some additional discussion on our choice logic framework. Firstly, as already pointed out, the framework was defined in a very general way, with little restrictions on the optionality and degree functions of choice connectives. Our motivation for this is that we are interested in the computational properties of choice logics, meaning logics in the spirit of QCL and CCL. Investigating these computational properties does not require all logics of our framework to be well-behaved. When investigating the framework with respect to logical properties, it could be interesting to impose additional restrictions. However, one needs to be careful how this is implemented. For instance, since choice connectives should express a preference, it might appear reasonable to require all choice connectives to be asymmetric and, given $A \circ B$, prefer interpretations satisfying just A to those satisfying just B. The logics considered in this paper fulfill this condition. But this approach would exclude some interesting connectives. For example, one could interpret $A \circ B$ as "preferably A and B, but if not, then A and B are equally acceptable".

[3] We consider $\mathcal{L}$-Entailment[log] to be at least as natural as the Θ_2^{P}-complete LogLexMaxSat and certainly more natural than Log^{2}LexMaxSat.

Moreover, we defined choice connectives as binary. In this way, our framework generalizes QCL and CCL in a straightforward way. Since QCL and CCL are associative, their connectives can also act as n-ary operators, expressing a choice over n options. Similarly, LCL allows us to encode a lexicographic ordering over n options (cf. Proposition 20), but brackets have to be chosen with care since the choice connective of LCL is not associative. It must be noted that not all choices over n options can be reduced to a choice over two options, and thus to a binary choice connective.

Another aspect that needs to be highlighted is an implicit assumption of commensurability when computing optionality. Our framework allows for logics using several choice connectives, as exemplified via QCCL which combines ordered disjunction ($\vec{\times}$) from QCL with ordered conjunction ($\vec{\odot}$) from CCL. For example, both the formulas $a\vec{\times}b$ and $c\vec{\odot}d$ have an optionality of 2. Thus, it is implicitly assumed that the optionality of $a\vec{\times}b$ is commensurable to that of $c\vec{\odot}d$, i.e., that the two possible degrees of satisfaction in $a\vec{\times}b$ are comparable to the two possible degrees in $c\vec{\odot}d$. This is especially important when combining two choice connectives, e.g., in the formula $(a\vec{\times}b) \wedge (c\vec{\odot}d)$. However, note that this assumption is already made when only a single choice connective is used, since the optionalities of $a\vec{\times}b$ and $c\vec{\times}d$ are also assumed to be commensurable even though they express preferences between different options. See Chang [14] for a more in-depth discussion of the question when and if preferences can be considered commensurable.

Lastly, we will discuss the mapping of choice logic formulas into other formalisms by the elimination of choice connectives. Brewka et al. [11] shows several such transformations, including to stratified knowledge bases and propositional circumscription. Another example is work by Confalonieri & Nieves [17] who showed that, in the context of LPODs, ordered disjunction can be eliminated by transforming $A\vec{\times}B$ into $A \vee ((not\, A) \wedge B)$, where *not* denotes default negation. However, the resulting formula is not classical. Indeed, a purely syntactic yet concise transformation from $A\vec{\times}B$ into a classical formula F may not be possible since it must be determined whether A is satisfiable in order to determine the actual best option.

7. Conclusion

We defined and investigated a general framework for choice logics that captures both QCL and CCL. Moreover, the framework allows us to easily define new logics as examplified via SCCL and LCL. We have shown that strong equivalence is interchangeable with degree-equivalence for optionality-ignoring choice logics (e.g. SCCL) and with full equivalence for optionality-differentiating choice logics (e.g. QCL, CCL, LCL). By definition, both degree and full equivalence require a similar behavior with respect to a finite number of models, while strong equivalence requires a similar behavior with respect to infinitely many formulas. Therefore, our characterization results provide an important tool for checking strong equivalence in practice.

Moreover, the computational complexity of tractable choice logics was investigated in detail. Here the complexity of $\mathcal{L}$-PMContainment turned out to be of particular interest, as it depends on the expressiveness of the considered choice logic. Finally, for choice logic theories, we observed that the interaction between the expressiveness of the choice logic and the complexity of the preferred model semantics determines the complexity of entailment.

An initial definition of our framework and some further results regarding choice logics can be found in the master thesis of the first author [6]. This includes another choice logic not considered here called Exclusive Disjunctive Choice Logic (XCL), which is equipped to deal with preferences over mutually exclusive options. Moreover, encodings in Answer Set Programming that allow specifying and experimenting with new choice logics in a straightforward fashion have been provided by the authors [7].

Regarding future work, new choice logics may be defined explicitly with concrete use cases in mind. Furthermore, some properties of our framework have yet to be investigated. This includes a characterization of associativity and general concepts towards normal forms (as examined for QCL and CCL by Brewka et al. [11] and Boudjelida & Benferhat [10]). Moreover, it could be studied in more detail how preferred model entailment for choice logics can fit into the framework of Kraus et al. [27] (examined for QCL by Brewka et al. [11] and to some extent in this paper in Section 5.1). An additional possibility for future work is to further restrict the semantics of choice connectives to ensure desirable behavior, and to extend our framework to allow n-ary choice connectives (cf. Section 6). Lastly, the complexity of checking for strong equivalence for choice logics that are neither optionality-ignoring nor optionality-differentiating remains open.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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