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Graphs with Two Moplexes☆

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Abstract

Moplexes are natural graph structures that arise when lifting Dirac's classical theorem from chordal graphs to general graphs. The notion is known to be closely related to lexicographic searches in graphs as well as to asteroidal triples, and has been applied in several algorithms related to graph classes such as interval graphs, claw-free, and diamond-free graphs. However, while every non-complete graph has at least two moplexes, little is known about structural properties of graphs with a bounded number of moplexes. The study of these graphs is, among others, motivated by the parallel between moplexes in general graphs and simplicial modules in chordal graphs: unlike in the moplex setting, properties of chordal graphs with a bounded number of simplicial modules are well understood. For instance, chordal graphs having at most two simplicial modules are interval.

In this work we initiate an investigation of k-moplex graphs, which are defined as graphs containing at most k moplexes. Of particular interest is the smallest nontrivial case, $k = 2$, which forms a counterpart to the class of interval graphs. As our main structural result, we show that the class of connected 2-moplex graphs is sandwiched between the classes of proper interval graphs and cocomparability graphs; moreover, both inclusions are tight for hereditary classes. From a complexity theoretic viewpoint, this leads to the natural question of whether the presence of at most two moplexes guarantees a sufficient amount of structure to efficiently solve problems that are known to be intractable on cocomparability graphs, but not on proper interval graphs. We develop new reductions that answer this question negatively for two prominent problems fitting this profile, namely Graph Isomorphism and Max-Cut. Furthermore, for graphs with a higher number of moplexes, we lift the previously known result that graphs without asteroidal triples have at most two moplexes to the more general setting of larger asteroidal sets. We also discuss sufficient conditions for the existence of Hamiltonian paths in 2-moplex graphs as well as connections with avoidable vertices.

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Keywords: moplex; simplicial vertex; avoidable vertex; asteroidal number; cocomparability graph; Hamiltonian path

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1. Introduction

The fundamental class of *chordal graphs*, i.e., graphs where every cycle of length greater than three has a chord, has been extensively studied in the literature. A celebrated result by Dirac states that every non-complete chordal graph has at least two non-adjacent simplicial vertices [20]. Equivalently, every non-complete chordal graph contains at least two *simplicial modules*, that is, maximal clique modules containing a simplicial vertex [18]. Moreover, since simplicial modules are present in every chordal graph, one could classify chordal graphs into "slices" based on how many simplicial modules they have—for instance, it is easy to observe that chordal graphs with at most two simplicial modules are interval graphs (due to the well-known connection between simplicial vertices and *leafage* of chordal graphs and a characterisation of interval graphs via leafage [29]).

On the other hand, there are many general graphs which do not contain any simplicial vertices, and attempting to use simplicial modules to slice up the class of all graphs in a similar manner would not be meaningful. But while simplicial modules do not have the same fundamental role in general graphs as in chordal graphs, the notion of *moplexes* introduced by Berry and Bordat [5] promises to hold precisely that role. In particular, a moplex is an inclusion-maximal set of vertices such that (1) they form a clique, (2) they form a module, and (3) their neighbourhood, if non-empty, is a minimal separator, and Berry and Bordat lifted Dirac's Theorem to general graphs by showing that every non-complete graph has at least two non-adjacent moplexes [5]. Subsequent works have pointed to connections between moplexes and asteroidal triples [6, 32] as well as lexicographic searches [7, 39, 35]. Moplexes have also been used in various algorithms, e.g., for computing a minimal completion to an interval graph [34], for computing minimal triangulations of claw-free graphs [9], and for recognising diamond-free graphs without induced cycles of length at least five [8].

But in spite of these fundamental connections and useful applications, the interconnection between the structural properties of graphs and their moplexes is still not well understood. In this work we initiate an investigation into what happens if one uses moplexes to slice up the class of general graphs. Do graphs containing a bounded number of moplexes have useful structural or algorithmic properties? And if chordal graphs with at most two simplicial modules form a natural subclass of the fundamental class of interval graphs, what can we say about graphs with at most two moplexes?

For a positive integer k, a *k-moplex graph* is a graph that contains at most k moplexes, and moreover we let the *moplex number* of a graph be the number of moplexes it contains. Our first, introductory result provides a link between the moplex number and the *asteroidal number* of a graph [25, 19], generalising an earlier result of Berry and Bordat for graphs with at most two moplexes [6].

Theorem 1.1. *The asteroidal number is a lower bound on the moplex number.*

Theorem 1.1 immediately implies that the nice algorithmic features of graphs with bounded asteroidal number also hold for graphs with a bounded number of moplexes. This includes polynomial-time algorithms for various algorithmic problems [22, 17, 27, 28], existence of a spanning tree approximating vertex distances up to a constant additive term [25], a constant factor approximation algorithm for treewidth [15], and an upper bound on the treewidth in terms of the maximum degree [13]. We remark that while computing the asteroidal number of a graph is NP-hard [24], the moplex number of a graph is polynomial-time computable [6].

A graph class is *hereditary* if it is closed under vertex deletion. The class of 1-moplex graphs is hereditary, but not of particular interest, as it is precisely the class of complete graphs. However, as one can verify using the family of paths, the class of k-moplex graphs is not hereditary for any $k \geq 2$. This makes understanding the structure of k-moplex graphs more challenging; in fact, even the structure of 2-moplex graphs is not yet fully understood. Berry and Bordat [6] showed that 2-moplex graphs are AT-free and that all connected induced subgraphs of a graph G are 2-moplex if and only if a G is a proper interval graph. We strengthen the former result and complement the latter one by proving further results relating the class of 2-moplex graphs to the hierarchy of hereditary graph classes. More precisely, a graph class $\mathcal{G}$ can be naturally mapped to the following four nested hereditary graph classes, two contained in $\mathcal{G}$ and two containing $\mathcal{G}$:

(i) the largest hereditary graph class contained in $\mathcal{G}$,

(ii) the class of graphs for which every connected induced subgraph belongs to $\mathcal{G}$,

(iii) the class of all induced subgraphs of connected graphs in $\mathcal{G}$, and

(iv) the smallest hereditary graph class containing $\mathcal{G}$.

When $\mathcal{G}$ is the class of 2-moplex graphs, a result of Berry and Bordat shows that the class from (ii) is the class of proper interval graphs. We determine the other three hereditary classes related to 2-moplex graphs. We show that the class from (i) is the class of cochain graphs, while the classes from (iii) and (iv) both coincide with the class of cocomparability graphs.

Theorem 1.2. *Let G be a graph. Then, every induced subgraph of G is a* 2*-moplex graph if and only if G is a cochain graph.*

Theorem 1.3. *The smallest hereditary graph class containing the class of* 2*-moplex graphs is the class of cocomparability graphs.*

We then consider the question of whether the structure of 2-moplex graphs can be used to develop efficient algorithms for problems that are known to be intractable on cocomparability graphs, but not on proper interval graphs. We develop reductions showing that this is not the case for two prominent examples of such problems, namely Max-Cut and Graph Isomorphism, both of which remain as hard on 2-moplex graphs as they are on cocomparability graphs. For proper interval graphs, the complexity of Max-Cut is still open, while Graph Isomorphism is solvable in linear time [31].

Theorem 1.4. *When restricted to the class of cobipartite* 2*-moplex graphs,* Max-Cut *is* NP*-complete and* Graph Isomorphism *is* GI*-complete.*

Theorem 1.4 provides some indication that the class of 2-moplex graphs is a significant generalisation of the class of connected proper interval graphs. Nevertheless, we believe that 2-moplex graphs share the well-known structural property of proper interval graphs that connectedness is a necessary and sufficient condition for the existence of a Hamiltonian path [10].

Conjecture 1.5. *Every connected* 2*-moplex graph has a Hamiltonian path.*

Using Theorem 1.3, we give partial support for the conjecture by verifying it in the special case when the graph has only two avoidable modules. In order to explain the result, we need to introduce some terminology. Let v be a vertex in a graph G. An *extension* of v in G is an induced P_3 in G having v as midpoint. An extension of v is *failing* if it is not contained in any induced cycle, and vertex v is *avoidable* in G if it has no failing extension. Note that every simplicial vertex is avoidable, and a vertex in a chordal graph is avoidable if and only if it is simplicial. The concept of avoidable vertices goes back to the work of Ohtsuki, Cheung, and Fujisawa [33] who proved that every graph has an avoidable vertex. Vertices having this property were later dubbed *OCF-vertices* (see, e.g., [23, 4]), and very recently the notion has reemerged under the name of avoidable vertices [3, 14]. Since every graph has an avoidable vertex, every graph has an *avoidable module*, that is, an inclusion-maximal clique module containing an avoidable vertex. Thus, for a positive integer k we say that a graph is *k-avoidable* if it contains at most k avoidable modules.

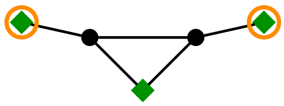
Since a moplex in a graph can only contain avoidable vertices [4], the number of moplexes in a graph G is upper bounded by the number of avoidable modules. In other words, for each positive integer k, every k-avoidable graph is a k-moplex graph. Thus, the following result gives some evidence for Conjecture 1.5.

Theorem 1.6. *Every connected* 2*-avoidable graph contains a Hamiltonian path.*

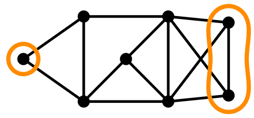
Already Berry et al. [4] observed that there exist graphs with avoidable vertices that do not belong to a moplex; indeed, while every k-avoidable graph is a k-moplex graph, the converse implication fails already for $k = 2$ (see Figure 1a). As our final contribution, we show that there is a simple transformation mapping any k-moplex graph to a k-avoidable induced subgraph.

Proposition 1.7. *Let G be a k-moplex graph and let $A \subset V(G)$ be the set of avoidable vertices of G that do not belong to any moplex. Then the graph $G - A$ is k-avoidable.*

Several proofs are omitted or merely sketched due to space constraints.



(a) The bull graph has exactly two moplexes but three avoidable vertices (the green diamonds).



(b) A graph with exactly two moplexes.

Fig. 1: Graphs containing two moplexes (circled in orange).

2. Preliminaries

All graphs in this paper are undirected and simple. We assume familiarity with basic concepts in graph theory as used, e.g., by West [38]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The closed and open neighbourhood of a vertex v in G are denoted as $N[v]$ and $N(v)$, respectively. These concepts are naturally extended to sets $X \subseteq V(G)$ so that $N[X]$ is defined as the union of all closed neighbourhoods of vertices in X, and $N(X)$ is defined as the set $N[X] \setminus X$. A *cutset* in G is a set of vertices $S \subseteq V(G)$ such that $G - S$ has more connected components than G. For two vertices $u, v \in V(G)$ a cutset $S \subseteq V(G) \setminus \{u, v\}$ is a *u,v-separator* if u and v are contained in different components of $G - S$, and is a *minimal u,v-separator* if no proper subset of S is a u,v-separator. A *minimal separator* is a minimal u,v-separator for some non-adjacent vertex pair u and v. A vertex is *simplicial* if its neighbourhood forms a clique. A vertex set M is a *module* if each vertex $v \in V(G) \setminus M$ is either adjacent to every vertex in M or not adjacent to any vertex in M. A *clique module* is a module that is a clique. A *simplicial module* is an inclusion-maximal clique module containing a simplicial vertex. Note that all vertices in a simplicial module are simplicial.

We now give a formal definition of *moplexes*, which play a central role in this paper. For purely technical reasons we use the definition from [32], which extends the one in [5] so that the vertex set of any complete graph is also a moplex. An illustration of the notion is provided in Figure 1b.

Definition 2.1. *A* moplex *in a graph G is an inclusion-maximal clique module $X \subseteq V(G)$ such that $N(X)$ is either empty or a minimal separator in G. A moplex X is* simplicial *if $N(X)$ is a clique.*

Besides the class of chordal graphs, several other hereditary graph classes play an important role in our study. A graph G is *cobipartite* if its complement is bipartite, that is, if the vertex set of G can be partitioned into two cliques. Furthermore, G is said to be a *cochain graph* if we can write $V(G) = X \cup Y$ where X and Y are disjoint cliques with $X = \{x_1, \ldots, x_k\}$ such that $N[x_i] \subseteq N[x_j]$ for all $1 \leq i < j \leq k$. A graph $G = (V, E)$ is an *interval graph* if it has an *interval representation*, that is, if its vertices can be put in a one-to-one correspondence with a family of closed intervals on the real line such that two distinct vertices are adjacent if and only if the corresponding intervals intersect. If G has an interval representation in which no interval contains another interval then G is said to be a *proper interval graph*. A vertex set A in a graph G is an *asteroidal set* if for each $a \in A$, the vertices in $A \setminus \{a\}$ are all contained in a single connected component of $G - N[a]$, see [37]. Asteroidal sets of cardinality three are called *asteroidal triples*, and graphs not containing any asteroidal triples are called *AT-free*. A prominent subclass of AT-free graphs is the class of *cocomparability graphs*, which are graphs whose complements have a transitive orientation of their edges.

For further background on graph classes, we refer to [16]. We denote by $3K_1$ the edgeless graph with three vertices, and, for $k \geq 3$, by C_k the cycle of length k.

3. Structural Properties

In this section, we focus on establishing structural properties of k-moplex graphs for a fixed k, with a special focus on the smallest nontrivial graph class defined in this way, namely the class of 2-moplex graphs. We begin by recalling a result on minimal separators and moplexes.

Theorem 3.1 (Berry and Bordat [6])**.** *For every minimal separator S in a graph G, each component of $G - S$ contains a moplex of G.*

Using this theorem, we can already obtain some preliminary results on the structure of 2-moplex graphs.

Lemma 3.2. *Let G be a non-complete 2-moplex graph and denote by U and W its two moplexes. Then the following holds:*

1. *U and W are disjoint simplicial moplexes, and*
2. *for every minimal separator S in G, the graph $G - S$ contains exactly two connected components, one of which contains U and the other one W.*

We now generalise a result by Berry and Bordat stating that a graph has an asteroidal triple of vertices if and only if it has an asteroidal triple of moplexes [6]. The *asteroidal number* of a graph G is defined as the maximum size of an asteroidal set (see, e.g., [29, 25, 2]). An *asteroidal set of moplexes* in a graph G is a set $\{X_1, \ldots, X_k\}$ of pairwise disjoint moplexes in G such that for each $i \in \{1, \ldots, k\}$, the graph $G - N[X_i]$ contains all moplexes X_j, $j \neq i$, in the same connected component. The result of Berry and Bordat [6] corresponds to the case $k = 3$ of the following more general statement.

Theorem 3.3. *A graph has an asteroidal set of vertices of size k if and only if it has an asteroidal set of moplexes of size k.*

Theorem 1.1 is an immediate consequence of Theorem 3.3. On the other hand, the gap between the moplex and asteroidal numbers can be arbitrarily large (consider, e.g., the class of stars). Theorem 1.1 implies that the asteroidal number is computable in polynomial time in any class of graphs with bounded moplex number. Together with results from [22], Theorem 1.1 also implies that the minimum dominating set problem and the minimum total dominating set problem can be solved in polynomial time in classes of graphs of bounded moplex number. The same holds for the maximum independent set problem, the minimum independent dominating set problem, and the minimum efficient dominating set problem, along with their weighted variants [17], for the k-colouring problem, for any fixed k [27], and for the minimum weight feedback vertex set problem [28]. Furthermore, for graphs of bounded moplex number bounded degree implies bounded treewidth, as a consequence of the fact that graphs with asteroidal number at most k have chordality at most $2k + 1$ and a result of Bodlaender and Thilikos [13].

For reference we state explicitly the previous result of Berry and Bordat on 2-moplex graphs [6] (which is now an immediate corollary of Theorem 1.1).

Corollary 3.4. *Every 2-moplex graph is AT-free.*

Next, we address various relations between the class of 2-moplex graphs and hereditary graph classes. Berry and Bordat [6] characterised the graphs in which every connected induced subgraph has at most two moplexes as exactly the proper interval graphs. We provide an analogous characterisation, but without requiring the induced subgraphs to be connected.

Theorem 1.2. *Let G be a graph. Then, every induced subgraph of G is a 2-moplex graph if and only if G is a cochain graph.*

Proof sketch. Suppose first that each induced subgraph of G has at most two moplexes. Since for every $k \geq 4$ the cycle C_k is a connected graph in which every vertex forms a moplex, G must be a chordal graph. Furthermore, since the graph $3K_1$ has three moplexes, G has independence number at most two. As G is a chordal graph, it is also perfect and hence, the vertex set of G can be covered with two disjoint cliques X and Y (see [30]), that is, G is cobipartite. Furthermore, any two vertices in X have comparable neighbourhoods in Y, since otherwise G would contain an induced 4-cycle. We conclude that G is a cochain graph.

Sufficiency follows from a result of Berry and Bordat stating that every connected proper interval graph is a 2-moplex graph [6], once we notice that every cochain graph is a proper interval graph and that every cochain graph is either connected or the disjoint union of two complete graphs. □

We next show that the smallest hereditary graph class containing the class of 2-moplex graphs is the class of cocomparability graphs. In order to prove that every 2-moplex graph G is a cocomparability graph, we identify a property common to all minimal x,y-separators, for any two non-adjacent vertices x and y of G. We then exploit this property to orient the edges of the complement of G in a transitive way.

Fix a 2-moplex graph G that is not a complete graph, with the two moplexes $U, W \subseteq V(G)$. For two non-adjacent vertices x and y we denote by $\mathcal{S}(x, y)$ the set of all minimal x,y-separators in G. Given $S \in \mathcal{S}(x, y)$, we say that U *prefers x to y with respect to S* if U and x lie in the same connected component of $G - S$. By Lemma 3.2, either U prefers x to y with respect to S or U prefers y to x with respect to S. As the following key lemma shows, which of these two cases occurs is actually independent of the choice of S.

Lemma 3.5. *For every two non-adjacent vertices x and y exactly one of the following conditions holds:*

- *U prefers x to y with respect to all $S \in \mathcal{S}(x, y)$, or*
- *U prefers y to x with respect to all $S \in \mathcal{S}(x, y)$.*

Proof sketch. Suppose towards a contradiction that there exist two minimal x,y-separators S and S' such that U and x lie in the same component of $G - S$, and U and y lie in the same component of $G - S'$. Then, due to Lemma 3.2, W and y lie in the same component of $G - S$, and W and x lie in the same component of $G - S'$. Furthermore, $x \notin U$, as x and U lie in different components of $G - S'$. In a similar way we conclude that neither x nor y can belong to $U \cup W$. Now fix any $u \in U$ and $w \in W$. To complete the proof, it suffices to show that $\{x, u, w\}$ is an asteroidal triple in G and use Corollary 3.4 to derive a contradiction. □

Lemma 3.5 could also be stated with W in place of U, which means that if for *some* minimal x,y-separator vertices x and y belong to components containing moplexes U and W, respectively, then in fact for *every* minimal x,y-separator x and y belong to components containing moplexes U and W, respectively. If U prefers x to y with respect to all $S \in \mathcal{S}(x, y)$, we will say that *U prefers x to y*. We now define binary relations R_U and R_W over $V(G)$ as follows: $xR_Uy \iff U$ prefers x to y and $xR_Wy \iff W$ prefers x to y. By Lemma 3.5 both relations are well-defined. Furthermore, either U prefers x to y or U prefers y to x (in which case W prefers x to y). We thus have $xR_Uy \iff yR_Wx$, that is, $R_W = R_U^{-1}$. Note also that by definition, xR_Uy or xR_Wy implies that x and y are distinct and non-adjacent.

Lemma 3.6. *Relations R_U and R_W are transitive.*

We remark that R_U is a strict partial order on the vertices of G. Furthermore, since R_U is an orientation of the edges of the complement of G, Lemma 3.6 implies the main result of this section.

Proposition 3.7. *Every* 2*-moplex graph is a cocomparability graph.*

Note that Proposition 3.7 is a strengthening of Corollary 3.4. As we show below, this result is best possible.

Proposition 3.8. *Every cocomparability graph is an induced subgraph of some connected* 2*-moplex graph.*

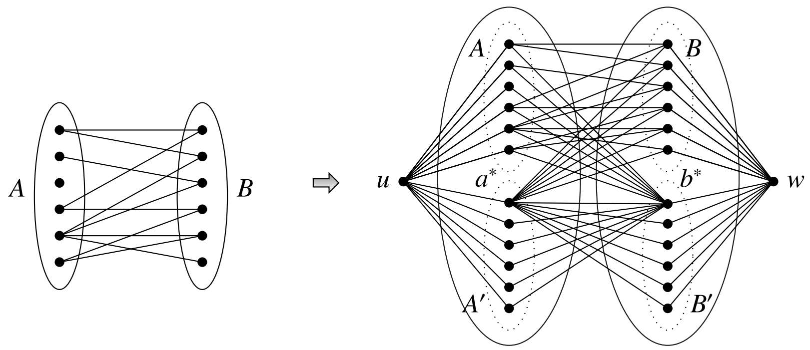
Propositions 3.7 and 3.8 together directly imply Theorem 1.3.

4. Hardness Results

In this section, we present lower bounds for two classical problems, Max-Cut and Graph Isomorphism, for 2-moplex graphs. In the Max-Cut problem, we are given a graph G and a positive integer k and the question is whether G contains a bipartite subgraph with at least k edges. In the Graph Isomorphism problem, we are given two graphs and are asked whether they are isomorphic.

Recall that the class of connected 2-moplex graphs is sandwiched between the classes of connected proper interval and cocomparability graphs. It is known that Max-Cut is NP-complete on cobipartite graphs [12], and thus on cocomparability graphs. Interestingly, the complexity of Max-Cut is still open on proper interval graphs [11, 26, 1, 21]. The Graph Isomorphism problem is solvable in linear time in the class of interval graphs [31]. Since the problem is GI-complete on bipartite graphs [36], it is also GI-complete on cobipartite graphs and thus on cocomparability graphs. We show that both problems remain hard on cobipartite graphs with at most two moplexes.

The hardness reduction for Max-Cut is based on the following construction (see also Figure 2).



G is cobipartite $\quad$ G' is a cobipartite 2-moplex graph

Fig. 2: An example of Construction 1 used to prove Theorem 1.4. The ellipses represent cliques.

Construction 1. *Let $G = (A \cup B, E)$ be a cobipartite graph such that A and B are disjoint cliques. We define the graph G' obtained from G as follows:*

- *add a set A' containing $|A|$ vertices and a vertex u such that $\{u\} \cup A \cup A'$ is a clique;*
- *add a set B' containing $|B|$ vertices and a vertex w such that $\{w\} \cup B \cup B'$ is a clique;*
- *fix a vertex $a^* \in A'$ and connect it to every vertex in $B \cup B'$;*
- *fix a vertex $b^* \in B'$ and connect it to every vertex in $A \cup A'$.*

Theorem 1.4. *When restricted to the class of cobipartite 2-moplex graphs,* Max-Cut *is* NP*-complete and* Graph Isomorphism *is* GI*-complete.*

Proof sketch for Max-Cut. It is well known that the problem is in NP. To prove NP-hardness, we make a reduction from Max-Cut on cobipartite graphs, which is NP-hard [12]. Let $G = (A \cup B, E)$ be a cobipartite graph such that A and B are disjoint cliques. Let G' be the graph obtained from G through Construction 1. Note that G' can be computed in polynomial time. Furthermore, G' is a cobipartite graph with exactly two moplexes. To complete the proof, it suffices to show that there exists a cut of size at least k in G if and only if there exists a cut of size at least $(|A| + 1)^2 + (|B| + 1)^2 + k$ in G'.
Proof sketch for Graph Isomorphism. We reduce from the isomorphism problem on connected bipartite graphs. Let G_1, G_2 be connected bipartite graphs with colour classes A_1 and B_1, A_2 and B_2 respectively. We construct two graphs G'_1 and G'_2 as follows. The vertex set of G'_i for $i \in \{1, 2\}$ consists of the vertex set of G_i together with two extra vertices u_i and w_i. For $i \in \{1, 2\}$, each of the sets $A_i \cup \{u_i\}$ and $B_i \cup \{w_i\}$ forms a clique in G'_i. Furthermore, we add an edge to G'_i between $x \in A_i$ and $y \in B_i$ if and only if x and y are adjacent in G_i. The obtained graphs are cobipartite and one can verify that they are 2-moplex graphs. It now suffices to show that there is an isomorphism $f : G_1 \to G_2$ if and only if there is an isomorphism $f' : G'_1 \to G'_2$. □

We note that the proof of Theorem 1.4 also implies the stronger statement that Max-Cut is NP-complete and Graph Isomorphism is GI-complete even when restricted to the class of cobipartite 2-avoidable graphs.

5. Remarks on Hamiltonian Paths and Avoidable Modules

Bertossi proved that a proper interval graph has a Hamiltonian path if and only if it is connected [10]. As stated in Conjecture 1.5, we believe that this result is a consequence of the analogous statement for the larger class of 2-moplex graphs. Using Proposition 3.7 and the stronger structural restrictions guaranteed by bounding the number of avoidable modules we can derive the following nontrivial result, partially settling Conjecture 1.5.

Theorem 1.6. *Every connected* 2*-avoidable graph contains a Hamiltonian path.*

The proof is based on partitioning the graph into layers, using the strict partial order on the vertices given by a transitive orientation of the complement obtained from Proposition 3.7 and then constructing the Hamiltonian path from one avoidable module to the other, visiting these layers in a monotone fashion.

An indication that Theorem 1.6 may give some nontrivial support towards Conjecture 1.5 is given by the fact that there are several connections between moplexes and avoidable vertices (or avoidable modules). For example, as noticed in Lemma 3.2, if a graph contains exactly two moplexes, then these moplexes are simplicial. Thus, in the class of 2-moplex graphs, the fact that avoidable modules form a common generalisation of both simplicial modules and moplexes can be strengthened to the chain of implications "moplex ⇒ simplicial module ⇒ avoidable module." Furthermore, the following general transformation maps any k-moplex graph to a k-avoidable induced subgraph.

Proposition 1.7. *Let G be a k-moplex graph and let $A \subset V(G)$ be the set of avoidable vertices of G that do not belong to any moplex. Then the graph $G - A$ is k-avoidable.*

Proof sketch. It is enough to prove that the removal of any vertex from A cannot cause a previously non-avoidable vertex to become avoidable. As this property holds for any vertex contained in an avoidable module of size at least two, we will assume by contradiction that $x \in A$ is an avoidable vertex in G such that $N[v] \neq N[x]$ for all $v \neq x$, and after the removal of x some vertex y becomes avoidable in the graph $G' = G - x$. First, observe that y must admit a failing extension in G of the form xyz for some $z \in V(G) \setminus N[x]$. In particular, y must be a neighbour of x. Denote by C_z the connected component containing z in $G - N[x]$. Furthermore, notice that in G we must have $N(z) \cap N(x) \subseteq N(y)$, for otherwise xyz would not be a failing extension. This implies that for every vertex $x' \in N(y) \setminus (N[C_z] \cup \{x\})$, path $x'yz$ is a failing extension of y in G'. To conclude the proof, it remains to show that such a vertex x' exists. Thus, y is non-avoidable in G'. □

6. Concluding Remarks

Moplexes provide a tool that has the potential to lift the beneficial structural properties of simplicial modules in chordal graphs to the setting of all graphs, see, e.g., [4, 6, 7]. We thus believe that graphs with a bounded number of moplexes form interesting graph classes which were well overdue for further study. We introduce the moplex number of a graph, focusing our study on properties of graphs with moplex number 2, the smallest non-trivial class in the moplex-number hierarchy.

There are several natural questions related to moplexes that still remain open. It would be interesting to fully understand Hamiltonian and other structural properties of 2-moplex graphs. In particular, while it is easy to identify some operations (such as adding a universal vertex or a true twin) that preserve the moplex number, we do not know whether the class of 2-moplex graphs can be characterised by a composition result—notably by identifying a set of operations and a base class that can be used to generate every 2-moplex graph. For $k > 2$, how do the classes of k-moplex graphs relate with the hierarchy of hereditary graph classes? Analogous questions can also be asked about k-avoidable graphs.

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