Die approbierte Originalversion dieser Dissertation ist an der Hauptbibliothek der Technischen Universität Wien aufgestellt (http://www.ub.tuwien.ac.at).

The approved original version of this thesis is available at the main library of the Vienna University of Technology (http://www.ub.tuwien.ac.at/englweb/).

DISSERTATION

Consistency Conditions for Topological Open Strings

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines
Doktor der technischen Wissenschaften

unter der Leitung von

Ao. Univ. Prof. DI. Dr. techn. Maximilian Kreuzer
E136
Institut für Theoretische Physik

eingereicht an der Technischen Universität Wien
Fakultät für Physik

von

DI. Manfred Herbst
Tanngrabenstr. 16
A–3300 Winklarn
herbst@hep.itp.tuwien.ac.at

Wien, am 3. August 2004

für meine Eltern Theresia und Josef

Acknowledgements

This thesis comprises work that was done during the period March 2002 – July 2004 at the Theory Division, CERN, Switzerland and at the Institute for Theoretical Physics, Vienna University of Technology, Austria. It was financed by the Austrian Bundesministerium für Bildung, Wissenschaft und Kultur.

First and foremost I would like to thank my supervisor at CERN, Wolfgang Lerche, for introducing me to and guiding me through the subject of this thesis. I am most grateful for numerous inspiring discussions and his eagerness to share his experience.

I am equally thankful to Maximilian Kreuzer for supervising my PhD studies during the early stages at the Vienna University of Technology and for his continuing support and his interest in my work during my stay at CERN.

I would like to make special mention of Ilka Brunner, who let me benefit from her broad knowledge in string theory, thanks for all the support. Special thanks also to Calin-Iuliu Lazaroiu for fruitful collaborations and for sharing his expertise in string field theory and homotopy algebras. I particularly thank Bernhard Scheuner for many endless discussions on our research topic.

I am grateful to my fellow students Andreas Bichl, Bernhard Scheuner, Emmanuel Kohlprath, Harald Nieder, Javier Lopez–Albacete, Marc Tuckmantel, Peter Fischer and Tristan Maillard for many enlightening discussions, not only about physics. Equally, I would like to thank all the other students, who stayed for a short-term visit in the 'dungeons' of the Theory Division.

My appreciative thanks to all my colleagues and friends at the Institute for Theoretical Physics at the Vienna University of Technology for numerous debates starting on physics and ending on politics during the coffee breaks. I would like to make special mention of Alex Kling, Erwin Riegler, Emanuel Scheidegger and Robert Wimmer.

Thank you to Andi, Claudia, Daniel, Daniela, Gerolf, Harald, Helene, Lotte, Manfred, Peter, Sabine, Stefan, Tristan, Walter and all the other friends in Geneva for enjoyable times at skiing, playing badminton, 'Siedeln in Catan', singing at BBQs on the Salève and on many other occasions.

I am especially grateful to my parents, Theresia and Josef, for enabling me to pursue my professional dreams. Thank you to them and my brother Thomas for cheerful moments and unconditional support.

Most of all I would like to thank Angelika for wonderful times and for her attempts to distract me from physics, — it's a tough job.

Kurzfassung

Die besondere Bedeutung von Membranen für Superstringtheorie wurde zuerst 1995 in ihrer Rolle im Netz von nicht-perturbativen Dualitäten zwischen *a priori* unabhängigen, perturbativen Superstringtheorien erkannt. Seither hatten insbesondere *D-branes*, deren Quantenfluktuationen offene Strings sind, Auswirkungen auf zahlreiche andere Bereiche der Stringtheorie, aber auch der Quantenfeldtheorie. So initiierten sie, zum Beispiel, eine intensivere Untersuchung von nicht-kommutativer Geometrie und nicht-kommutativer Feldtheorien. Da die niederenergetischen Freiheitsgrade auf *D-branes* Eichfelder sind, spielen sie auch eine wichtige Rolle in der Konstruktion von Stringtheoriemodellen mit einer Einbettung des Standardmodells der Teilchenphysik.

Diese Arbeit konzentriert sich auf einen anderen Aspekt der Physik von Membranen, der mit der Tatsache zusammenhängt, dass *D-branes* ein *effektives Superpotential* für $\mathcal{N}=1$ supersymmetrische Modelle verursachen, dessen chirale Superfelder von den Eichfeldern auf der Membran und von der Position der Membran im Einbettungsraum herrühren. Das effektive Superpotential kann im Prinzip mit Hilfe von topologischer Stringtheorie berechnet werden, welche einen Teilsektor der vollen Superstringtheorie beschreibt. Zwar konnten im Rahmen von topologischer Stringtheorie zahlreiche exakte (nicht-perturbative) Ergebnisse erzielt werden, doch ist die Miteinbindung von *D-branes* bis dato nicht ausreichend verstanden, weshalb keine effektive Methode bekannt ist, das Superpotential (außer in wenigen einfachen Beispielen) zu berechnen.

In dieser Arbeit wird die mathematische Struktur von topologischen Stringamplituden untersucht, die zu effektiven Superpotentialtermen führen. Wie gezeigt wird, erfüllen die Amplituden Konsistenzrelationen, die Ward–Identitäten und Faktorisierungseigenschaften in der topologischen Stringtheorie entsprechen. Diese Gleichungen stellen eine Erweiterung von Differenzialgleichungen dar, die auf R. Dijkgraaf, E. Verlinde und H. Verlinde, sowie E. Witten zurückgehen, und vervollständigen jene zu einer abzählbaren Menge von sowohl algebraischen Relationen als auch Differentialgleichungen. Die Konsistenzrelationen umfassen die folgenden mathematischen Strukturen: *(i)* eine A_∞-Algebra, *(ii)* eine topologische Version der Cardy-Relation, welche eine Dualität zwischen offenen und geschlossenen Strings implementiert, und schließlich *(iii)* eine Kreuzungssymmetrie, die die Kopplung von offenen und geschlossenen Strings kontrolliert.

Ein Großteil der Analyse dieser Arbeit konzentriert sich auf die Herleitung der A_∞-Algebra, welche bereits 1963 von J. Stasheff als Verallgemeinerung von assozia-

tiven Algebren entwickelt wurden. Ihre Rolle in Stringtheorie wurde zunächst im Rahmen von Stringfeldtheorie, dann in topologischer Stringtheorie, erkannt. Auch die Mirrorsymmetrievermutung für Calabi–Yau Mannigfaltigkeiten mit *D-branes* beruht wesentlich auf A_∞-Algebren.

Die Cardy-Relation und die Kreuzungssymmetrie bewirken — so wie die A_∞-Algebra — starke Einschränkungen auf die topologischen Amplituden. Dies wird anhand einer einfachen Klasse von Modellen, den topologischen Landau–Ginzburg Modellen der A-Serie, demonstriert. Durch Lösen der Gleichungen mit Hilfe von Mathematica konnten in den einfachsten Fällen alle topologischen Amplituden eindeutig bestimmt werden. Die Strukture der Ergebnisse deutet überdies auf eine geschlossene Formel für das effektive Superpotential aller möglichen Konfigurationen von *D-branes* in diesen Modellen hin. Obwohl ein rigoroser Beweis dieser Formel bis dato fehlt, kann sie durch nicht-triviale, unabhängige Kontrollen aus der Deformationstheorie von *D-branes* verifiziert werden.

Abstract

The importance of membranes for superstring theory was first recognised in 1995 in their special role for non-perturbative dualities between seemingly different perturbative superstring theories. D-branes are a particular class of such extended objects, whose quantum fluctuations are open strings. Apart from dualities, D-branes had numerous effects on various fields in string theory as well as quantum field theory. They lead, for instance, to an enhanced investigation of non-commutative geometry and non-commutative field theories. They play moreover an important role in the search for 'realistic' superstring models with an embedding of the standard model of particle physics, which can be traced back to the fact that the low-energy degrees of freedom on D-branes are given by gauge fields.

This work concentrates on a further aspect of D-brane physics, which is related to the fact that they give rise to an *effective superpotential* for $\mathcal{N} = 1$ supersymmetric field theories; the chiral superfields descend from gauge fields on the D-brane and the position of the D-brane in the embedding space. The effective superpotential can be computed, in principle, through topological string theory, which describes a subsector of the full superstring theory. In fact, the framework of topological string theory already allowed to derive several exact (non-perturbative) results; however, the inclusion of D-branes is not sufficiently well understood, so that there exists no effective method to compute the superpotential (apart from specific examples).

In this work the mathematical structure of topological string amplitudes, which lead to terms in the effective superpotential, is investigated: It is shown that the amplitudes satisfy consistency conditions, which correspond to Ward identities and factorisation properties in topological string theory. These relations provide an extention of differential equations, found by R. Dijkgraaf, E. Verlinde and H. Verlinde as well as E. Witten, and complete the latter to a countable set of algebraic and differential equations. The consistency conditions comprise *(i)* an A_∞ algebra, *(ii)* a topological version of the Cardy condition, i.e., the open-closed string duality, and finally *(iii)* a bulk-boundary crossing symmetry, which controls the coupling of open and closed topological strings.

A main part of this work concentrates on the derivation of the A_∞ algebra, which was developed by J. Stasheff as a generalisation of associative algebras already in 1963. Its role in string theory was first discovered within string field theory and later within topological string theory. Moreover, the homological mirror symmetry conjecture relies on A_∞ algebras in an essential way.

The Cardy relation and the bulk-boundary crossing symmetry, as well as the

A_∞ algebra, impose strong restrictions on the topological amplitudes. This fact is demonstrated in a simple series of models, the A-series of topological Landau–Ginzburg models. Solving the consistency conditions with Mathematica uniquely determines all topological amplitudes in the simplest examples. Moreover, the structure of the results suggests a closed formula for the effective superpotential for all possible D-brane configurations in these models. Although this formula still lacks a rigorous proof, it stands an independent, non-trivial test from deformation theory of D-branes.

Contents

1 Introduction **1**

2 Motivation and Summary **5**
2.1 Compactification and D-branes 5
2.2 Obstructions and A_∞ algebras 8
2.3 Consistency conditions and the effective superpotential 9
2.4 Overview 11

3 $\mathcal{N} = (2, 2)$ superconformal field theories **12**
3.1 The algebra and representation theory 13
3.2 Boundary conditions and boundary states 19
3.3 A-type and B-type D-branes 22
3.4 Non-linear sigma models 23
3.5 Landau–Ginzburg models 30

4 Topological conformal field theories **34**
4.1 Definition of topological field theories 34
4.2 Topological conformal algebra 35
4.3 Physical operators and descendants 37
4.4 The effects of a boundary 38

5 Topological twisting **42**
5.1 Twisting of the closed string sector 42
5.2 Compatibility with boundary conditions 43
5.3 What do topological amplitudes compute in $\mathcal{N} = (2, 2)$ theories? . . . 44
5.4 B-twisted non-linear sigma models 45
5.5 A-twisted non-linear sigma models 49
5.6 B-twisted Landau–Ginzburg models 53

6 The WDVV equations for the prepotential **65**

7 Disk amplitudes and the superpotential **68**
7.1 The regularised amplitudes 69
7.2 Equivalence of the two types of amplitudes 71

7.3 Two point correlation functions are not deformed 73
7.4 Independence of the positions of unintegrated insertions 74
7.5 Independence of the world-sheet metric 75
7.6 Cyclicity and bulk permutation invariance 76
7.7 Deformed amplitudes and the boundary metric 77
7.8 The formal generating function and the effective superpotential . . . 79

8 Consistency conditions for disk amplitudes 81
8.1 Minimal A_∞ constraints on boundary amplitudes 82
8.2 Weak A_∞ constraints for deformed amplitudes 86
8.3 Interpretation in terms of open string field theory 91
8.4 Bulk-boundary crossing symmetry 95
8.5 Cardy conditions 98

9 Landau–Ginzburg minimal models 100
9.1 Experimental evidence 100
9.2 The effective potential for a single B-brane 104
9.3 The effective potential for general B-branes 107
9.4 The superpotential as action for a holomorphic matrix model 109

Chapter 1

Introduction

Our current understanding of nature at the most fundamental level is based on two main ideas, quantum mechanics and general relativity. The former was developed in order to describe physics at microscopic scales, whereas the latter is a theory for gravitational interaction at macroscopic distances.

The concept of quantisation was successfully applied to the three fundamental forces, electro-magnetism, weak and strong interactions, and unified them to a single quantum field theory, the *standard model* (SM) of particle physics. The interactions in the standard model are realised in terms of non-Abelian gauge theories, more precisely, in terms of the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. The first factor is responsible for the strong interaction; the corresponding gauge particles comprise eight gluons. The $SU(2)_L \times U(1)_Y$ part describes the electro-weak theory, with the photon as particle of $U(1)_{em}$ and the weak $W^{\pm}$ and Z bosons. The standard model agrees with collider experiments to astonishingly high accuracy, although it leaves elementary questions unexplained, for instance, the origin of the particle spectrum. There were several attempts to explain it in terms of grand unification theories (GUT), where the SM gauge group is embedded in a larger simple gauge group, such as $SU(5)$, $SO(10)$ or E_6. This approach was suggested by the fact that the coupling constants run with energy and meet approximately, but not quite, at the scale $M_{GUT} \sim 10^{16} GeV$. Another extension of the standard model is provided by supersymmetry, which associates to each particle of the SM a partner with opposite fermion number. However, we did not so far observe this symmetry in any experiment, which means that it must be broken at a scale above the current experimental limits. One of the great successes of supersymmetry is the unification of the gauge couplings in one point at the GUT scale.

Neither the standard model nor its extension by supersymmetry provide explanations for the values of particle masses, mixing angles and coupling constants. Likewise, the large difference between electro-weak scale $M_Z \sim 10^2 GeV$, grand unification scale $M_{GUT} \sim 10^{16} GeV$ and the Planck scale $M_{pl} \sim 10^{19} GeV$ remains a puzzle, which is known as the hierarchy problem.

The basic idea behind *general relativity* is the unification of space and time to a curved, dynamical geometry. This theory, too, is well-established by high accuracy measurements. Quite recently, several cosmological observations suggested that the universe has a non-vanishing cosmological constant of incredibly small value, $\Lambda_{grav} \sim 10^{-120}(M_{pl})^4$, which is another ingredient in the hierarchy problem. In contrast to the interactions of the standard model the gravitational field, i.e., the metric, is not quantised within general relativity, and, thus, in view of the quantum nature of the other three forces it is natural to ask for a quantum theory of gravity. A significant feature of such a theory would be the existence of a spin 2 particle, the graviton. However, the usual perturbative approach to quantum gravity suffers from non-renormalisability, which is due to the mass-dimension of Newton's constant.

One candidate for such a theory is superstring theory, which was originally developed in the course of describing strong interactions, but first overrun by the success of non-Abelian Yang–Mills theories. The idea is that the most fundamental objects are not point-like as in ordinary quantum field theory, but 1-dimensional strings; the basic concept of quantum theory is, however, retained within string theory. Since the interaction of strings is smeared out over spacetime the problem of short-distance singularities is resolved. Finally, the facts that the superstring can consistently be quantised, is anomaly-free and contains a massless spin 2 particle in the spectrum initiated the first string revolution in 1984/85.

A distinct feature of superstring theory comes from conformal invariance which requires 10 spacetime dimensions. The additional 6 dimensions are believed to be curled up on a compact space whose size is of the order of the Planck scale[1] and are, therefore, undetectable by current experiments. These extra dimensions gave even room for a quite old idea of a unification of gauge degrees of freedom and gravity in terms of Kaluza–Klein reduction, so that superstring theory seemed to be a promising candidate for a unique quantum theory of all fundamental interactions.

However, in the course of investigations it turned out that there are five different perturbative string theories: four models for oriented closed strings - type IIA, type IIB, heterotic $E_8 \times E_8$ and $SO(32)$, as well as one unoriented open string - type I. String theory did not seem to fulfil the dream of a unique consistent unified theory.

But this idea was revived in the second string revolution in 1995 when it was realized that non-perturbative effects connect the five superstring theories. Even before, T-duality between type IIA and IIB theory compactified on a circle or, more generally, mirror symmetry for compactifications on Calabi–Yau manifolds provided a perturbative duality upon two string theories. But then S-duality, a weak-strong coupling duality completed T-duality to a web of dualities between all five superstring theories. Moreover, the strong coupling limit of type IIA theory turned out to be dual to a weakly coupled theory, called M-theory, whose low energy theory is 11-dimensional supergravity, suggesting that all string theories and M-theory are limits in the moduli space of a single underlying theory, which remains unknown so far.

[1]With the investigation of D-branes it turned out that the scale of the extra dimensions could be as large as the millimetre scale.

The basic idea of S-duality is the fact that a theory in the strong coupling regime is dual to a theory in the weak coupling regime. Non-perturbative BPS states in the weakly coupled regime of one theory can become perturbative states of the dual theory in the strong coupling limit. The most important non-perturbative objects of such kind in string theory are D-branes, whose quantum excitations are open strings; they will play a central role in this thesis.

Since the discovery of the web of dualities D-branes have influenced the development not only of string theory but also of quantum field theory in numerous examples: They triggered, for instance, an enhanced investigation of non-commutative geometry and non-commutative field theories. They provide important insight into the information paradox of black hole physics. Moreover, the fact that the low-energy degrees of freedom on D-branes are given by gauge fields lead to the search for superstring models with an embedding of the standard model of particle physics.

Despite all the successes of string theory in the unification of the fundamental forces and its contributions to the understanding of strong coupling effects, a long list of open questions still remains. For instance, the particular spectrum of particles in the Standard Model as well as the coupling constants remain so far unexplained within string theory. On the one hand, string theory provides the intriguing feature that it has no fundamental constants, and all coupling constants of a low-energy effective theory descend from vacuum expectation values of scalar fields. On the other hand, no mechanism (if it exists at all) is known, which selects a *particular* set of values.

The issue of fixing vacuum expectation values currently gives rise to increased investigations and, once again, D-branes play an essential role, which is due to the fact that they entail an *effective superpotential* for $\mathcal{N} = 1$ supersymmetric field theories. The chiral superfields for the potential come from gauge fields on D-branes as well as the positions of D-branes in the embedding space.

The effective superpotential of the low-energy effective field theory can be computed in terms of topological string theory. The latter describes a subsector of the full superstring theory that leads to numerous exact (non-perturbative) results. However, the inclusion of D-branes in topological string theory is not sufficiently well understood, so that an effective method to compute the superpotential has not been found so far.

In this work the mathematical structure of topological open string amplitudes in the presence of D-branes is investigated and the relation to the effective superpotential is pointed out. The main result comprises consistency conditions for topological string amplitudes, which correspond to Ward identities and factorisation properties. These relations provide an extention of differential equations, found by R. Dijkgraaf, E. Verlinde and H. Verlinde as well as E. Witten, and complete the latter to a countable set of algebraic and differential equations. The consistency conditions include the following structures:

- A_∞ *algebra*: It was developed by J. Stasheff as a generalisation of associative algebras already in 1963. Its role in string theory was first discovered within string field theory and later within topological string theory. Moreover, the homological mirror symmetry conjecture relies on A_∞ algebras in an essential way.

- *Topological Cardy condition*: It is the topological string theory counterpart of the open-closed string duality of conformal field theory.
- *Bulk-boundary crossing symmetry*: It controls the coupling of open and closed topological strings.

All consistency conditions together imposes strong restrictions on the topological amplitudes. This fact is demonstrated in a simple series of models, the A-series of topological Landau–Ginzburg models. In this particular example the consistency conditions are strong enough to determine all topological amplitudes uniquely. The explicit calculations were done with Mathematica, and the results suggest a closed formula for the effective superpotential for all possible D-brane configurations in these models. Although this formula still lacks a rigorous proof, it stands an independent, non-trivial test from deformation theory of D-branes. Moreover, it can be interpreted as classical action for a holomorphic matrix model, which provides further evidence for its correctness.

Chapter 2

Motivation and Summary

This chapter provides a more detailed motivation for the subject of this thesis. It contains a concise summary of the main results, which include the effective superpotential as well as the consistency conditions. Finally, we give a brief outline of the thesis.

2.1 Compactification and D-branes

Closed strings

As already mentioned, the contact to physics on a 4-dimensional spacetime is realised as compactification of the superstring on an 'internal' space, i.e., on a (complex) 3-dimensional Calabi–Yau manifolds $\mathcal{X}$, which is by definition a Ricci flat Kähler manifold. The compactification of type IIA/B superstring, for instance, although not of phenomenological interest, leads to deep insights in the moduli space and the non-perturbative behaviour of $\mathcal{N} = 2$ supersymmetric field theory in four dimensions.

From the string world sheet perspective, a type IIA/B Calabi–Yau compactification is described by a $\mathcal{N} = (2,2)$ superconformal field theory (SCFT) [1–3], i.e., a $\mathcal{N} = (2,2)$ non-linear sigma model on $\mathcal{X}$. The massless string excitations in such a model are in one-to-one correspondence with Kähler and complex structure deformations of the Calabi–Yau 3-fold and can be used to deform the $\mathcal{N} = (2,2)$ SCFT. One of the most intriguing features of these SCFTs is the fact that the Kähler and complex structure deformation of the Calabi–Yau manifold are captured by two independent topological subsectors. The restriction to one of the subsectors is a well-defined procedure, known as topological twisting [4] of the $\mathcal{N} = (2,2)$ SCFT to a topological conformal field theory (TCFT). In fact, the two independent topological subsectors correspond to two ways of twisting the theory: The A-twist restricts to a TCFT, which describes the moduli space of Kähler deformations; and the B-twist restricts to complex structure deformations. Moreover, in each of the topological subsectors all tree-level closed string amplitudes are encoded in a single (holomorphic) generating function, the prepotential $\mathcal{F}$ [5, 6].

The decoupling of Kähler and complex structure deformations immediately implies that the total moduli space $\mathcal{M}$ of the $\mathcal{N} = (2, 2)$ SCFT splits into two factors, $\mathcal{M} = \mathcal{M}_K \times \mathcal{M}_C$. In [7] each factor was found to be equipped with a Kähler metric, and the geometry of $\mathcal{M}_K$ and $\mathcal{M}_C$ is completely governed by the prepotential $\mathcal{F}$. This kind of geometry is called special Kähler geometry, and the connection to 4-dimensional $\mathcal{N} = 2$ effective field theory is given by the fact that the prepotential $\mathcal{F}$ serves as effective Lagrangian and describes the spaces of vacuum expectation values (moduli) of massless scalar fields in vector and hypermultiplets [6, 8].

A priori the moduli spaces $\mathcal{M}_K$ and $\mathcal{M}_C$ differ profoundly in their structure; the prepotential for complex structure deformations can be described in purely geometric terms, whereas the prepotential for Kähler deformations gets non-perturbative world sheet instanton corrections [9, 10]. More precisely, it counts the number of possible embeddings of holomorphic sphere instantons into the Calabi–Yau manifold $\mathcal{X}$ [11, 12]. However, a $\mathbb{Z}_2$ automorphism of the $\mathcal{N} = (2, 2)$ superconformal algebra gives rise to a powerful interrelation between the moduli spaces of pairs of Calabi–Yau manifolds, $\mathcal{X}$ and $\mathcal{X}'$, which is known as mirror symmetry [13–15]. In fact, the prepotential on $\mathcal{M}_K$ ($\mathcal{M}_C$) on $\mathcal{X}$ is equal to the prepotential on $\mathcal{M}_C$ ($\mathcal{M}_K$) on the mirror $\mathcal{X}'$, upon an appropriate change of parametrisation. In terms of the topologically twisted theories this amounts to the statement that the topological A-model on $\mathcal{X}$ is equivalent or mirror to the topological B-model on $\mathcal{X}'$ and vice versa [4]. The power of mirror symmetry lies in the capability of counting of holomorphic spheres on $\mathcal{X}$ ($\mathcal{X}'$) through purely geometric terms on $\mathcal{X}'$ ($\mathcal{X}$). In other words, one can compute non-perturbative results in terms of purely perturbative calculations.

There is one more curiosity with the moduli space $\mathcal{M}$, which can best be seen from the fact that the Kähler form serves as a volume form on $\mathcal{X}$, so that changing the Kähler moduli basically means changing the volume of certain cycles within the Calabi–Yau manifold. However, in string theory nothing prevents us from extending the Kähler parameter to negative values. In other words, in string theory it is possible to change the topology of compactified dimensions smoothly [16]. In [17] the appropriate framework for modelling such processes was shown to be gauged linear sigma models, where the topology change is realised as phase transition between different vacuum configurations. Quite generically, it turned out that, apart from non-linear sigma models on Calabi–Yau manifolds, there appear even non-geometric phases, such as Landau–Ginzburg orbifold phases, which are governed by a superpotential term in the $\mathcal{N} = (2, 2)$ world sheet action. Moreover, besides providing a more complete description of the moduli space of the compactified theory the gauged linear sigma models are the appropriate physical framework for describing mirror symmetry on the level of a Lagrangian field theory [15, 18].

Open strings, D-branes

We know already that in weak-strong dualities D-branes play a quite essential role. A natural question that arises is how D-branes show up in the compactification of type

II superstring, i.e. in $\mathcal{N} = (2, 2)$ SCFT such as non-linear sigma models on Calabi–Yau manifolds or Landau–Ginzburg models. From the world sheet perspective, D-branes are specified as boundary conditions for open strings. This implies that they break world sheet supersymmetry and in turn spacetime supersymmetry. A rather special class, the BPS branes, preserves half of the spacetime supersymmetry, that is $\mathcal{N} = 1$ SUSY, which implies breaking half of world sheet supersymmetry to $\mathcal{N} = 2$. These BPS branes come in two families: called A-branes and B-branes [19–21].

As for closed strings we can perform the topological A- and B-twist for $\mathcal{N} = 2$ SCFT of open strings. A-branes are naturally described within the A-twisted topological model, whereas B-branes appear in the B-twisted model; hence the names. Although the topological twist discards lots of information, the open string TCFT computes important contributions to the 4 dimensional $\mathcal{N} = 1$ effective field theory, namely the F-terms [6].[1] In this work we shall especially be interested in the tree-level contribution from topological disk amplitudes, which gives rise to the effective superpotential for the $\mathcal{N} = 1$ spacetime theory.

In the large volume limit of a Calabi–Yau 3-fold A-branes turn out to be (real) 3-dimensional special Lagrangian submanifolds (see [22] and references therein) with flat non-Abelian gauge connection, whereas B-branes show up as holomorphic submanifolds with a non-Abelian gauge field that defines a complex structure [19, 20]. However, in the non-geometric phase there is no obvious interpretation in terms of submanifolds [23], and even for Calabi–Yau manifolds the proper description for B-branes turns out to be within algebraic geometry in terms of sheaves [24]. The problem with the picture of holomorphic submanifolds is that it breaks down when we move around in the moduli space of the Calabi–Yau manifold. Starting with a B-brane of given dimension at large volume it gets transformed into a bound state of B-branes of different dimensions. In general, such an object cannot be described as bundle over a holomorphic submanifold; the language of sheaves in algebraic geometry provides an appropriate framework (see [25, 26] and references therein).

In [27] Kontsevich put forward a conjecture for mirror symmetry of Calabi–Yau manifolds including D-branes, which is known as the homological mirror symmetry conjecture. It is formulated in the language of triangulated categories [28, 29]; see [26] for an introduction on the latter subject. For short, a category consists of objects $A, B, \ldots$ and morphisms $f, g, \ldots \in Hom(A, B)$ between pairs of objects, equipped with an associative composition of morphisms and an identity morphism in $Hom(A, A)$ for any object A. This is the most general structure for describing D-branes (as objects) and open strings (as morphisms), where the (associative) composition of morphisms corresponds to the binding of two open strings to a single one.

The term 'triangulated' basically corresponds to the following two statements for D-branes in physical terms:
(i) Bound state formation: If one turns on an open string $f \in Hom(A, B)$ between any two D-branes, then A and B can potentially form a bound state C. The objects

[1] This interpretation is, of course, only true if the central charge of the associated SCFT is $c = 9$.

A and B together with C can be arranged in a triangle diagram with morphisms in between, hence the name triangulated. Whether the binding process occurs or not depends on whether f is tachyonic or not; in other words, when f is tachyonic, the triangle describes a tachyon condensation process [30]. Notice that the triangle shows just the possibility of bindings; the direction of decay is dictated by stability [31, 32].[2]

(ii) Anti-brane: For any D-brane A there exists an anti D-brane $\bar{A}$, so that they bind or better annihilate to the closed string vacuum.

The homological mirror symmetry of [27] then conjectures an equivalence of the derived category of the Fukaya category for A-branes on $\mathcal{X}$ and the derived category of coherent sheaves for B-branes on the mirror $\mathcal{X}'$.

2.2 Obstructions and A_∞ algebras

Now, we come closer to the actual subject of this work, which is related to a fundamental difference between the closed string moduli space of Kähler and complex structure deformations and the open string moduli space. Again, in non-geometric phases there is no obvious interpretation in this manner. The important point is that the combined open-closed string moduli are *obstructed* [33–35]; opposed to the pure closed string theory, the moduli are subject to a potential, which turns out to be the effective superpotential mentioned previously.

An example from mathematics, where such obstructions occur, is the deformation theory of holomorphic vector bundles (see [36] and references therein): There, a complex structure is defined by a gauge connection with vanishing $(0, 2)$- and $(2, 0)$-part of the gauge field strength, i.e., $F^{0,2} = (\bar{\partial} + A)^2 = \bar{\partial}A + A^2 = 0$, where A is the $(0, 1)$-form part of the gauge connection. An equation of this form appears quite generically in deformation theory and is called Maurer–Cartan equation. An infinitesimal deformation δa of the connection is called obstructed if it cannot be integrated to a finite deformation, and it is a well-known fact that the deformations of the complex structure on holomorphic vector bundles are, in general, obstructed. The question is, how one can take control over these obstructions? In [35] it was shown that the obstructions of the above problem are encoded in the potential of an associated field theory for which the Maurer–Cartan equation serves as equation of motion. In the case at hand the field theory turned out to be holomorphic Chern–Simons theory [37]. By summing all tree-level Feynman diagrams of n in-coming legs one derives n-linear products which form an algebraic structure, known as A_∞ algebra [38, 39]. These products can then be combined into an effective potential, which encodes the obstructions of the deformations [35].

Physically, the above setup is realized as a B-brane, which wraps the whole Calabi–Yau manifold and supports a holomorphic vector bundles. The gauge con-

[2] However, this issue is not captured within the topological twisted theory and, therefore, not within the scope of this work.

nection on this vector bundle corresponds to an open string modulus. As it was shown in [37] the open string field theory of the topological subsector that describes the gauge fields is actually holomorphic Chern–Simons theory, described above. Moreover, in the same work the open string field theory for A-branes on special Lagrangian cycles M of the non-compact Calabi–Yau 3-folds T^*M was identified as Chern–Simons theory on M, i.e. the topological field theory of flat gauge connections. As A_∞ algebras play an essential role in open string field theory [40, 41], it should not surprise that they also occur in these topological models [35, 42].

If we consider several D-branes and therefore include open strings spanned in-between, we have to switch again to the categorical language in the sense of the last subsection, obtaining an A_∞ category (see [43] and references therein). This means that in addition to the bilinear composition of open strings we have now higher n-linear compositions, which are restricted to the A_∞-algebra relations. Indeed, the derived category of the Fukaya category of the topological A-model is an A_∞ category [44], which has, however, a quite complicated structure due to the appearance of holomorphic disk instantons. On the other hand, in the topological B-model the derived category of coherent sheaves can also be enhanced to an A_∞ category. Polishchuk suggested in [45] a refinement of the homological mirror symmetry conjecture on the level of A_∞ categories (see also [43]).

2.3 Consistency conditions and the effective superpotential

We have seen that the open string field theory approach to topological conformal string theories is naturally equipped with an A_∞ algebra, more precisely a minimal one, which means that it contains n-linear products for $n \geq 2$. This fact leads quite naturally to the question how this structure can be derived through a conformal field theory approach to TCFT without referring to string field theory. This thesis is devoted to this question and presents results that were mainly published in [46].

To begin with, we show that topological open string amplitudes, $B_{a_0 \dots a_n}$ for $n \geq 2$, on the disk are equipped with a minimal A_∞ algebra, thereby reproducing the string field theory results. The indices come from a basis of open string physical operators that are inserted in cyclic order on the disk.

One can go even further and include closed string physical operators in the amplitudes. It turns out that closed string insertions can be integrated to deformed open string amplitudes: $\mathcal{F}_{a_0 \dots a_n}(t^i)$ for $n \geq 0$, where t^i are the closed string moduli. This leads to a deformation from a minimal A_∞ algebra to a weak A_∞ algebra, i.e., for $m \geq 0$:

$$\sum_{\substack{k,\, j\, =\, 0 \\ k\, \leq\, j}}^{m} (-1)^{\tilde{a}_1 + \dots + \tilde{a}_k} \ \mathcal{F}^{a_0}{}_{a_1 \dots a_k c a_{j+1} \dots a_m}(t) \ \mathcal{F}^{c}{}_{a_{k+1} \dots a_j}(t) \ = \ 0 \quad . \tag{2.1}$$

A weak A_∞ algebra has additional n-linear products with $n = 0$ and $n = 1$, which can be interpreted as tadpoles resp. masses for open strings in the effective superpotential, which reads:

$$\mathcal{W}(s,t) = \sum_{m=0}^{\infty} \frac{1}{m+1} s_{a_m} \dots s_{a_0} \mathcal{F}_{a_0 \dots a_m}(t) \quad . \tag{2.2}$$

Here, s_a denote the open string moduli. The disk amplitudes $\mathcal{F}_{a_0\dots a_m}(t)$ are only cyclic symmetric in the open string indices, which causes sever complications: we are, in general, not able to integrate the disk amplitudes to a single function and express the A_∞ relations and the subsequent constraints in terms of this function; the effective superpotential contains only the totally symmetrised disk amplitudes, which means that it does not encode the full information. This fact has to be opposed to the closed topological string, where the prepotential $\mathcal{F}(t)$ contains the full information on sphere amplitudes.

Besides the A_∞ algebra we are able to derive two further sets of equations that put constraints on the interactions between open and closed topological strings. The first is the bulk-boundary crossing symmetry:[3]

$$\begin{aligned} &\partial_i \partial_j \partial_k \mathcal{F}(t)\ \eta^{kl}\ \partial_l \mathcal{F}_{a_0 a_1 \dots a_m}(t) = \\ &= \sum_{0 \le m_1 \le \dots m_4 \le m} (-1)^s \mathcal{F}_{a_0 \dots a_{m_1} b a_{m_2+1} \dots a_{m_3} c a_{m_4+1} \dots a_m}(t)\ \partial_i \mathcal{F}^b{}_{a_{m_1+1} \dots a_{m_2}}(t)\ \partial_j \mathcal{F}^c{}_{a_{m_3+1} \dots a_{m_4}}(t) \quad , \end{aligned} \tag{2.3}$$

and the second is a generalisation of the topological Cardy constraint:

$$\begin{aligned} &\partial_i \mathcal{F}_{a_0 \dots a_n} \eta^{ij} \partial_j \mathcal{F}_{b_0 \dots b_m} = \\ &= \sum_{\substack{0 \le n_1 \le n_2 \le n \\ 0 \le m_1 \le m_2 \le m}} (-1)^s\ \omega^{c_1 d_1}\ \omega^{c_2 d_2}\ \mathcal{F}_{a_0 \dots a_{n_1} d_1 b_{m_1+1} \dots b_{m_2} c_2 a_{n_2+1} \dots a_n}\ \mathcal{F}_{b_0 \dots b_{m_1} c_1 a_{n_1+1} \dots a_{n_2} d_2 b_{m_2+1} \dots b_m} \ . \end{aligned} \tag{2.4}$$

Here, η^{ij} and ω^{ab} are the inverse of the bulk resp. boundary topological metric. Note that relation (2.3) contains the prepotential $\mathcal{F}(t)$ of the closed topological string and, therefore, mixes open and closed topological string quantities.

These constraint equations represent the open string counterpart of the WDVV equations of closed TCFT [5], which we present for completeness:

$$\partial_i \partial_j \partial_m \mathcal{F}\ \eta^{mn}\ \partial_n \partial_k \partial_l \mathcal{F} = \partial_i \partial_k \partial_m \mathcal{F}\ \eta^{mn}\ \partial_n \partial_j \partial_l \mathcal{F} \quad . \tag{2.5}$$

They reflect associativity of the topological bulk operator product.

In order to demonstrate the power of the constraint equations (2.1), (2.3) and (2.4) we computed the disk amplitudes in the simplest examples of minimal Landau–Ginzburg models [47] by solving these equations in [46]. We were even able to propose a closed form for the effective superpotential for any configurations of D-branes in these models [48].

[3]The explicit expressions for the signs in (2.3) and (2.4) are given in chapters 8 and 9.

2.4 Overview

Here, we provide only a rough outline of this thesis. More detailed summaries can be found at the beginning of each chapter.

In chapter 3 we review well-known facts about $\mathcal{N} = (2,2)$ SCFT and their boundary conditions and boundary states. We give several explicit examples of $\mathcal{N} = (2,2)$ SCFT models. We present the basic definitions of open and closed topological conformal field theories in chapter 4 and explain the topological twisting in chapter 5. Here, we meet again the SCFT models in their topologically twisted version, where special emphasise is attached to Landau–Ginzburg minimal models.

After presenting the basic idea behind the derivation of the closed string WDVV equation (2.5) in chapter 6, we prepare for the computation of the open string constraint equations by properly defining the disk amplitudes in chapter 7. The effective superpotential is also defined there. Finally, the A_∞ algebra (2.1) as well as the bulk-boundary crossing relation (2.3) and the Cardy constraint (2.4) are derived in chapter 8.

In the last chapter we determine the superpotential for Landau–Ginzburg minimal models and investigate its relation to deformations of D-branes in these models.

Chapter 3

$\mathcal{N} = (2, 2)$ superconformal field theories

$\mathcal{N} = (2, 2)$ superconformal field theories became important for type II superstring compactifications because exactly this amount of supersymmetry is a necessary condition on the conformal field theory of the internal sector in order to obtain $\mathcal{N} = 2$ supersymmetry in four dimensions (cf. [49, 50]).

In section 3.1 we review well-known properties of $\mathcal{N} = (2, 2)$ SCFTs following [2, 3]. We start with defining the operator product algebra and explain the bosonisation of the $U(1)$ current, which plays an important role for the definition of the spectral flow operator (see [1, 2]). A special choice of the latter acts as an isomorphism between the Neveu–Schwarz and the Ramond sector and is, therefore, the internal part of the world sheet current that generates spacetime supersymmetry. (In chapter 5 the spectral flow operator will turn out essential for the explanation of the question, what topological amplitudes are computing.) In fact, we were a bit imprecise with this statement; in order to have a well-defined spacetime supersymmetric theory we have to require two additional conditions: single-valuedness of the operator product between any vertex operators of the theory and modular invariance of the torus partition function. These requirements are implemented by the Gliozzi–Scherk–Olive (GSO) projection [51].

We review the different sectors of chiral and twisted chiral primary operators [1], which span topological subsectors of the $\mathcal{N} = (2, 2)$ SCFT, i.e., the (twisted) chiral primary operators show up as physical operators in a topologically twisted theory, which we describe in subsequent chapters. After introducing the automorphism algebra $U(1) \times \mathbb{Z}_2$ of the $\mathcal{N} = (2, 2)$ superconformal algebra, we show that the $\mathbb{Z}_2$ part, called mirror automorphism, exchanges the chiral and twisted chiral sectors; a very simple property on the level of SCFT with profound consequences in concrete models.

In section 3.2 we briefly review a general description of boundary conditions in conformal field theories, where we closely follow [21, 52]; and we explain the introduction of boundary states through open-closed string duality [53, 54]. Thereafter, we discuss boundary conditions and the corresponding boundary states in $\mathcal{N}=(2,2)$ superconformal field theory in section 3.3. There appear two types, called A-type D-branes and B-type D-branes [20, 55].

In the remainder of this chapter we consider explicit realisations of $\mathcal{N}=(2,2)$ SCFTs in non-linear sigma models of Calabi–Yau manifolds and Landau–Ginzburg models. In the former, A-type and B-type boundary conditions were first studied in [20] and identified as special Lagrangian submanifolds and holomorphic submanifolds, respectively. The analysis was extended to Landau–Ginzburg models in [23, 56, 57]. Quite recently [47, 58, 59], Landau–Ginzburg B-branes were understood as matrix factorisations referring to earlier work by Warner [55]. In the discussion of boundary conditions for non-linear sigma models and Landau–Ginzburg models we will closely follow [23] and [47].

3.1 The algebra and representation theory

The symmetry algebra of a $\mathcal{N}=(2,2)$ superconformal field theory is generated by the stress-energy tensor $T(z)$ of conformal weight $h=2$, along with two fermionic currents $G(z)$ and $\bar{G}(z)$ with $h=3/2$ and a $U(1)$ R-current $J(z)$ of $h=1$. In this section we consider the superconformal algebra on the complex plane, thus describing closed strings. The algebra in operator product form is given by:

$$\begin{aligned}
T(z)\;T(0) &\sim \frac{c}{2z^4}+\frac{2}{z^2}T(0)+\frac{1}{z}\partial T(0)\;,\\
T(z)\;G(0) &\sim \frac{3}{2z^2}G(0)+\frac{1}{z}\partial G(0)\;,\\
T(z)\;\bar{G}(0) &\sim \frac{3}{2z^2}\bar{G}(0)+\frac{1}{z}\partial\bar{G}(0)\;,\\
T(z)\;J(0) &\sim \frac{1}{z^2}J(0)+\frac{1}{z}\partial J(0)\;,\\
G(z)\;\bar{G}(0) &\sim \frac{c}{3z^3}+\frac{1}{z^2}J(0)+\frac{1}{z}T(0)+\frac{1}{2z}\partial J(0)\;,\\
G(z)\;G(0) &\sim \bar{G}(z)\bar{G}(0)\sim 0\;,\\
J(z)\;G(0) &\sim +\frac{1}{z}G(0)\;,\\
J(z)\;\bar{G}(0) &\sim -\frac{1}{z}\bar{G}(0)\;,\\
J(z)\;J(0) &\sim \frac{c}{3z^2}\;.
\end{aligned}\tag{3.1}$$

The relations (3.1) comprises the holomorphic part of the algebra and there exists a whole copy for the right-moving part as well. For completeness we write the currents in terms of their mode expansions:

$$\begin{aligned} T(z) &= \sum_{n=-\infty}^{\infty} L_n\, z^{-n-2}\ , \qquad & G(z) &= \sum_{n=-\infty}^{\infty} G_{n+a}\, z^{-(n+a)-3/2}\ , \\ J(z) &= \sum_{n=-\infty}^{\infty} J_n\, z^{-n-1}\ , \qquad & \bar{G}(z) &= \sum_{n=-\infty}^{\infty} \bar{G}_{n-a}\, z^{-(n-a)-3/2}\ . \end{aligned} \tag{3.2}$$

The labels for the fermionic currents involve a number $a \in \mathbb{R}$, which causes branch-cuts in the complex plane. In the Ramond (R) sector this number is $a \in \mathbb{Z}$ and in the Neveu–Schwarz (NS) sector it is $a \in \mathbb{Z}+1/2$, so that the fermionic currents are single valued in the NS sector, but carry a branch cut in the R sector. In view of (3.1) and (3.2) the commutator algebra for the generators of the $\mathcal{N}=2$ superconformal algebra becomes:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\ , \\ [L_n, G_{m+a}] &= \left(\frac{n}{2}-(m+a)\right) G_{n+m+a}\ , \\ [L_n, \bar{G}_{m-a}] &= \left(\frac{n}{2}-(m-a)\right) \bar{G}_{n+m-a}\ , \\ [L_n, J_m] &= -mJ_{m+n}\ , \end{aligned} \tag{3.3}$$

$$\begin{aligned} [G_{n+a}, \bar{G}_{m-a}] &= L_{m+n} + \frac{1}{2}(n-m+2a)J_{m+n} + \frac{c}{6}\left((n+a)^2-\frac{1}{4}\right)\delta_{m+n,0}\ , \\ [J_n, G_{m+a}] &= +G_{n+m+a}\ , \\ [J_n, \bar{G}_{m-a}] &= -\bar{G}_{n+m-a}\ , \\ [J_n, J_m] &= \frac{c}{3}n\delta_{m+n,0}\ . \end{aligned} \tag{3.4}$$

Here, $[\cdot,\cdot]$ denotes the graded commutator.

Bosonisation of the $U(1)$ current

The last line of (3.1) suggests that $J(z)$ can be bosonised in terms of a free boson $H(z)$ with operator product expansion

$$H(z)H(0) = -\log z\ .$$

The explicit identification is

$$J(z) = i\sqrt{\frac{c}{3}}\partial H(z)\ . \tag{3.5}$$

The $U(1)$ charge q of any operator $\mathcal{O}$ is then uniquely encoded in a factor containing the boson $H(z)$, so that the operator can be written as

$$\mathcal{O} = \mathcal{O}_0 e^{iq\sqrt{\frac{3}{c}}H}\ , \tag{3.6}$$

where $\mathcal{O}_0$ denotes the contribution of vanishing charge. From the operator expansion (3.1) of the fermionic currents with the $U(1)$ current, we find, in particular, that G and $\bar{G}$ contain a factor $e^{i\sqrt{\frac{3}{c}}H}$ and $e^{-i\sqrt{\frac{3}{c}}H}$, respectively.

Spectral flow operator

An operator of special importance in $\mathcal{N} = (2,2)$ SCFT is the spectral flow operator

$$\Sigma_\eta(z) := e^{i\eta\sqrt{\frac{c}{3}}H} \ . \tag{3.7}$$

It can change the periodicity of the fermionic currents G and $\bar{G}$ with respect to any vertex operator, which can be seen from the operator product expansion:

$$e^{\pm i\sqrt{\frac{3}{c}}H(z)}\ \Sigma_\eta(0) \sim z^{\pm\eta} \ .$$

Therefore, the spectral flow operator Σ_η can change a representation of the $\mathcal{N} = (2,2)$ algebra to another representation with periodicity shifted by η. The explicit form of the spectral flow operator $\Sigma_\eta(z)$ shows that the charge and the conformal weight of any operator changes as:

$$\begin{aligned} q &\rightarrow q + \eta\frac{c}{3} \ , \\ h &\rightarrow h + q\eta + \eta^2\frac{c}{6} \ . \end{aligned}$$

The spectral flow operator with $\eta = \pm 1/2$, i.e.,

$$\Sigma_\pm := \Sigma_{\pm 1/2} = e^{\pm i\sqrt{\frac{c}{12}}H} \ , \tag{3.8}$$

turns out to be of special interest, because it implements a bijection between representations in the NS sector and the R sector. In particular, it maps the ground state in the NS sector (which corresponds to the unit operator) to a R sector ground state, which is represented by the spectral flow operator Σ_+ or Σ_- itself.

In the context of type II string compactification to 4 spacetime dimensions ($c = 9$) the spectral flow operator $\Sigma_\pm$ is part of the current, which implements spacetime supersymmetry. The operators corresponding to R ground states in the ghost and 4-dimensional Minkowski part are given by the spin fields $e^{-\phi/2}$ and

$$S_\alpha := e^{i\frac{\alpha}{2}(H^0+H^1)} \tag{3.9}$$

$$S_{\dot\alpha} := e^{i\frac{\dot\alpha}{2}(H^0-H^1)} \ , \tag{3.10}$$

respectively (cf. [50]). Here, the fields H^a are the bosonisations of the free fermions ψ^μ in flat space, i.e.,

$$\begin{aligned} e^{\pm iH^0} &\cong \frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1) \\ e^{\pm iH^1} &\cong \frac{1}{\sqrt{2}}(\psi^2 \pm i\psi^3) \ , \end{aligned}$$

and ϕ is part of the bosonisation of the (β, γ) ghost system.

The spectral flow operator and the spin fields form the currents

$$\begin{aligned} Q_\alpha &:= e^{-\phi/2} S_\alpha \Sigma_+ \ , & Q_{\dot\alpha} &:= e^{-\phi/2} S_{\dot\alpha} \Sigma_- \ , \\ \tilde{Q}_\alpha &:= e^{-\tilde\phi/2} \tilde{S}_\alpha \tilde\Sigma_\mp \ , & \tilde{Q}_{\dot\alpha} &:= e^{-\tilde\phi/2} \tilde{S}_{\dot\alpha} \tilde\Sigma_\pm \ , \end{aligned} \tag{3.11}$$

which generate the spacetime supersymmetry of type IIA or IIB superstring compactified to 4 dimensions (see [50]).

Strictly speaking we have to require an additional condition in order to realize spacetime supersymmetry: The operator product of the supersymmetry currents (3.11) has to be single valued with respect to all operators. A general vertex operator of the full theory contains a factor

$$e^{l\phi + is_0 H^0 + is_1 H^1 + iQ/\sqrt{3}H} \ . \tag{3.12}$$

From the operator product of the supersymmetry currents (3.11) with (3.12) we find that the condition for single-valuedness of the operator product expansion is[1]

$$l + s_0 + s_1 + Q \in 2\mathbb{Z} \ . \tag{3.13}$$

The projection to such operators is called the Gliozzi-Scherk-Olive (GSO) projection. It can be recast into the requirement that any vertex operator has integer charge with respect to the $U(1)$ current

$$J_{GSO} = \frac{1}{2}\partial(-\phi + iH_0 + iH_1 + i\sqrt{3}H). \tag{3.14}$$

Besides single-valuedness of the operator product expansion the GSO projection ensures modular invariance of the torus partition function.

Chiral primary states

A primary operator is defined by the operator products with the currents of the algebra (3.1) through:

$$T(z)\Psi(0) \sim \frac{h}{z^2}\Psi(0) + \frac{1}{z}\partial\Psi(0) \ , \tag{3.15}$$

$$J(z)\Psi(0) \sim \frac{q}{z}\Psi(0) \ , \tag{3.16}$$

$$G(z)\Psi(0) \sim \frac{1}{z}\Lambda \ , \tag{3.17}$$

$$\bar{G}(z)\Psi(0) \sim \frac{1}{z}\bar\Lambda \ , \tag{3.18}$$

where Λ and $\bar\Lambda$ denote the superpartners of Ψ.

[1] This is true in the type IIB theory. In type IIA the GSO projection in the R sector projects to odd integers.

A particular class of primary operators in the NS sector is the set $\mathcal{H}_c$ of chiral primary operators Φ,[2] which correspond to BPS states of the superconformal algebra. They are defined by the additional requirement $\Lambda = 0$, so that

$$G(z)\Phi(0) \sim 0 \ . \tag{3.19}$$

The saturated BPS condition for a chiral primary operator fixes the conformal weight to

$$h = \frac{q}{2} \ .$$

An upper bound for the conformal weight of chiral primary fields is given by the fact that, in view of equation (3.6), an operator of charge q provides the contribution $3q^2/2c$ to the conformal weight, so that

$$\frac{3q^2}{2c} \leq h = \frac{q}{2} \ .$$

Unitarity provides a lower bound, $h \geq 0$, and all-in-all we find for chiral primary operators:

$$0 \leq q \leq \frac{c}{3} \ . \tag{3.20}$$

In particular, the chiral primary carrying the maximal charge, $q = c/3$, is unique and can be represented by the spectral flow operator with $\eta = \pm 1$:

$$\Sigma_{\pm 1} = e^{i\sqrt{\frac{c}{3}}H} \ . \tag{3.21}$$

Upon acting with the spectral flow operator Σ_- on the chiral operators we obtain the space $\mathcal{H}_o$ of ground states of the R sector with $h = c/12$ and $-c/6 \leq q \leq c/6$. And acting twice with Σ_- we get the space $\mathcal{H}_a$ of antichiral primary operators (again in the NS sector), which are defined by

$$\bar{G}(z)\Phi(0) \sim 0 \ , \tag{3.22}$$

and carry conformal weight

$$h = -\frac{q}{2} \ .$$

Note that the identity operator (with $h = q = 0$) is both chiral and antichiral.

The chiral and the antichiral operators share the property that there operator product expansion is regular and the leading (constant) term incorporates only chiral resp. antichiral operators: Let $\phi_i \in \mathcal{H}_{c(a)}$, for $i = 1, \ldots, h_{c(a)} := \dim(\mathcal{H}c(a))$, be a basis of (anti)chiral operators, then

$$\phi_i(z)\phi_j(0) \sim C_{ij}{}^k \phi_k(0) \ . \tag{3.23}$$

[2] Strictly speaking $\mathcal{H}_c$ is the Hilbert space of chiral primary states. Due to the state-operator isomorphism of conformal field theory we will loosely regard $\mathcal{H}$ as space of chiral primary operators.

Since there are no operator product singularities between these fields, one can safely take the limit $z \to 0$. The product defined in this limit equips the (anti)chiral operators with a ring structure, which is called the (anti)chiral ring. The regular nature of the operator product (3.23) implies that the chiral operators span a topological subsector of the $\mathcal{N} = (2,2)$ superconformal field theory. This statement will become clearer later when we study the twist to topological field theories.

Until now all our considerations were restricted to one side of the $\mathcal{N} = (2,2)$ algebra. Taking into account both, the left- and the right-moving sector, the Hilbert space can be decided into 4 sectors: NS-NS, NS-R, R-NS and R-R. In particular, the chiral (c) and antichiral (a) rings combine in the NS-NS sector to c-c (a-a) operators, called (anti)chiral, and to c-a (a-c) operators, called twisted (anti)chiral.

Mirror and $U(1)$ automorphisms

The algebra (3.1) admits the automorphism group $U(1) \times \mathbb{Z}_2$. The second factor is the mirror automorphism given by

$$\begin{array}{rcl} m(T) & = & T \, , \\ m(G) & = & \bar{G} \, , \\ m(\bar{G}) & = & G \, , \\ m(J) & = & -J \, , \end{array} \tag{3.24}$$

which looks pretty harmless on the level of the superconformal algebra. It has, however, profound consequences in concrete realizations in Lagrangian field theories — it lead to the mirror symmetry conjecture for Calabi–Yau manifolds. From the first relation in (3.24) we infer that the conformal weight of operators does not change under the mirror automorphism. The last line, however, implies that the charge is inverted, which means, in particular, that chiral and antichiral operators are exchanged.

When we act with the mirror automorphism on just one side of the algebra, say the antiholomorphic part, the various (twisted) (anti)chiral operators transform as follows:

$$\begin{array}{rcl} c - c & \leftrightarrow & c - a \, , \\ a - a & \leftrightarrow & a - c \, , \\ c - a & \leftrightarrow & c - c \, , \\ a - c & \leftrightarrow & a - a \, . \end{array}$$

The twisted and the untwisted sectors are exchanged.

Besides the mirror automorphism, the algebra (3.1) admits a $U(1)$ automorphism, which acts as phase on the fermionic currents:

$$G \to e^{i\alpha} G \qquad \text{and} \quad \bar{G} \to e^{-i\alpha} \bar{G} \, . \tag{3.25}$$

Both automorphisms will turn out essential when we subsequently study boundary conditions for the $\mathcal{N} = (2,2)$ algebra.

3.2 Boundary conditions and boundary states

We will start in this section with a general discussion of boundary conditions and the associated boundary states in a general CFT and thereafter apply this technique on $\mathcal{N} = (2,2)$ SCFT.

Let us consider a general CFT on the upper half plane, which contains the holomorphic symmetry algebra $\mathcal{A}_L$ generated by the stress-energy tensor $T(z)$ and some other currents:

$$W(z) = \sum_{n\in\mathbb{Z}} \frac{W_n}{z^{n+h_w}} \ .$$

We choose the corresponding antiholomorphic symmetry algebra $\mathcal{A}_R$ to be identical to $\mathcal{A}_L$ and set $\mathcal{A} := \mathcal{A}_L = \mathcal{A}_R$. Therefore, the antiholomorphic sector contains $\bar{T}(\bar{z})$ and $\bar{W}(\bar{z})$, so that the total symmetry algebra is $\mathcal{A} \otimes \mathcal{A}$. In the presence of a boundary one has to set conditions, which relate the two sector. In order to avoid energy density flowing out of the boundary we have to demand

$$T(z) = \tilde{T}(\bar{z}) \qquad \text{for} \quad z = \bar{z} \ , \tag{3.26}$$

which shows that the total symmetry algebra $\mathcal{A} \otimes \mathcal{A}$ is broken by the boundary conditions and the best we can hope for is to preserve one copy of $\mathcal{A}$. Given an automorphism $\Omega : \mathcal{A} \to \mathcal{A}$, which acts trivially on $T(z)$, we can maintain $\mathcal{A}$ by the boundary condition [21]:

$$W(z) = \Omega(\tilde{W})(\bar{z}) \qquad \text{for} \quad z = \bar{z} \ . \tag{3.27}$$

Now we want to find a (coherent) state $|\alpha\rangle$ in the closed string Hilbert space, so that we can rewrite correlation functions on the upper half plane with boundary conditions α in terms of correlation functions on the complex plane in the presence of $|\alpha\rangle$. The map can be constructed by taking advantage of the open-closed string duality. Let us start with coordinates $(z, \bar{z})$ on the upper half plane. A correlation function of primary operators, which satisfy

$$\phi(w, \bar{w}) = \left(\frac{\partial z}{\partial w}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{h}} \phi(z, \bar{z}) \ ,$$

and are inserted in the upper half plane, describes closed strings coupled to a D-brane, which is characterized by the boundary condition α. The explicit expression for the amplitude is

$$\big\langle \phi_1(z_1, \bar{z}_1) \dots \phi_N(z_N, \bar{z}_N) \big\rangle_\alpha \ . \tag{3.28}$$

Now let us perform the map to the complex plane with coordinates $(w, \bar{w})$. This is done by the transformation [52]

$$w = e^{-2\pi i/\beta_o z} \ , \quad \bar{w} = e^{2\pi i/\beta_o \bar{z}} \ , \tag{3.29}$$

where the upper half plane is mapped outside of the unit circle $|w|=1$. Writting $w=e^{t'+i\sigma'}$, we see that the role of space and time was exchanged, i.e. $t'=\frac{2\pi}{\beta_o}\sigma\in[0,2\pi^2/\beta_o]$ and $\sigma'=-\frac{2\pi}{\beta_o}t\sim\sigma'+2\pi$, so that we clearly describe a closed string process in these coordinates. The upper half plane is mapped outside of the unit circle $|w|=1$.

Since we want to describe the correlation function (3.29) purely in closed string terms we have to implement the boundary conditions (3.26) and (3.27) on some state $|\alpha\rangle\rangle$ in the closed string Hilbert space. For that we perform the conformal transformation (3.29) of the current $W(z)$ and expand the current in terms of its oscillator modes, so that we obtain

$$\left(w^{h_w}\sum_{n\in\mathbb{Z}}\frac{W_n}{w^{n+h_w}}-(-\bar{w})^{h_w}\sum_{n\in\mathbb{Z}}\frac{\Omega(\tilde{W}_n)}{\bar{w}^{n+h_w}}\right)|\alpha\rangle\rangle=0 \qquad \text{for} \quad |w|=1\ .$$

Since this equation must be true for all w subject to the condition $|w|=1$ we infer the *Ishibashi conditions* [53]:

$$\begin{aligned} W_n-(-)^{h_w}\Omega(\tilde{W}_{-n})\big|\alpha\big\rangle_\Omega &= 0\ , \\ L_n-\tilde{L}_{-n}\big|\alpha\big\rangle_\Omega &= 0\ . \end{aligned} \tag{3.30}$$

Comparing the relations for boundary conditions (3.27) and boundary states (3.30) we recognize an additional sign factor $(-)^{h_w}$ in (3.30). This sign has a natural generization for operators with half integer conformal weight, i.e., $(-)^{h_w}\to e^{\pm i\pi h_w}$. The additional sign corresponds to a choice of spin structure.

In view of the state $\big|\,\alpha\,\big\rangle$ the correlation function (3.28) can be written as:

$$\big\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N)\big\rangle_\alpha=\big\langle\phi_1(w_1,\bar{w}_1)\ldots\phi_N(w_N,\bar{w}_N)\big|\,\alpha\,\big\rangle\ .$$

Given an irreducible highest weight representation $\mathcal{H}_i$ of $\mathcal{A}$, Ishibashi found in [53] an (up to an overall factor) unique solution for the condition (3.30). Let $\{\big|\,i,N\,\big\rangle\}_{N\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}_i$, then the corresponding Ishibashi state

$$\big|\,i\,\big\rangle\!\big\rangle_\Omega:=\sum_N\big|\,i,N\,\big\rangle\otimes UV_\Omega\overline{\big|\,i,N\,\big\rangle} \tag{3.31}$$

solves the Ishibashi conditions (3.30). U is an anti-unitary operator, which acts on $\tilde{W}_n$ through

$$U\tilde{W}_n=(-)^{h_w}\tilde{W}_{-n}U\ , \tag{3.32}$$

and $V_\Omega:\mathcal{H}_i\to\mathcal{H}_{\omega(i)}$ is an isomorphism, which is induced by the automorphism Ω, and commutes with U. Relation (3.32) means that U maps to the conjugate representation $U:\mathcal{H}_i\to\mathcal{H}_{i^\dagger}$ and therefore $\big|\,i\,\big\rangle\!\big\rangle$ couples to the closed Hilbert space $\mathcal{H}_i\otimes\mathcal{H}_{\omega(i)^\dagger}$. In a concrete theory an Ishibashi state can only appear if $\mathcal{H}_i\otimes\mathcal{H}_{\omega(i)^\dagger}$ is part of the closed string Hilbert space $\mathcal{H}=\bigoplus_{i,\bar{\imath}}N_{i\bar{\imath}}\,\mathcal{H}_i\otimes\mathcal{H}_{\bar{\imath}}$.

A priori we could take all possible linear combinations of Ishibashi states in order to form boundary states $\big|\,\alpha\,\big\rangle$, i.e.,

$$\big|\,\alpha\,\big\rangle := \sum_i B_\alpha{}^i \big|\,i\,\big\rangle\!\big\rangle \ . \tag{3.33}$$

However, besides the constraints (3.30), which implement the boundary conditions, Cardy has found a quite powerful constraint on the possible linear combinations (3.33), which stems from the modular invariance of the cylinder amplitude [54]: The amplitude of a closed string exchange between two boundary states $\big|\,\alpha\,\big\rangle$ and $\big|\,\beta\,\big\rangle$ should be the same as an open string one-loop amplitude

$$Z_{\alpha\beta}(q_o) := Tr_{\mathcal{H}_{\alpha\beta}} q_o^{L_0^{(o)}-c/24} = \sum_i n^i_{\alpha\beta}\chi_i(q_o) \ ,$$

where $q_o = e^{-2\pi t}$. In the last expression $Z_{\alpha\beta}(q_o)$ was expanded into the characters $\chi_i(q_o) = Tr_{\mathcal{H}_i} q_o^{L_0^{(o)}-c/24}$ of irreducible representations of $\mathcal{A}$. $n^i_{\alpha\beta}$ are positive integers counting the multiplicity of the representation $\mathcal{H}_i$ in the Hilbert space of open strings stretched between D-branes characterized through boundary conditions α and β. The Cardy relation can be written as

$$Z_{\alpha\beta}(q_o) = \big\langle\,\alpha\,\big|q_c^{1/2(L_0^{(c)}+\bar{L}_0^{(c)})-c/12}\big|\,\beta\,\big\rangle \ , \tag{3.34}$$

where $q_c = e^{-2\pi/t}$.

Inserting a complete set of Ishibashi states on the right-hand side of (3.34) and using the orthonormality relation

$$\big\langle\!\big\langle\,i\,\big|q_c^{1/2(L_0^{(c)}+\bar{L}_0^{(c)})-c/12}\big|\,j\,\big\rangle\!\big\rangle = \delta_{ij}\chi_i(q_c) \ ,$$

as well as the expansion (3.33), the right-hand side of (3.34) can be recast as:

$$\sum_i (B^i{}_\alpha)^* B_\beta{}^i \chi_i(q_c) \ .$$

As final manipulation we use the modular transformation of the character $\chi_i(q_c)$ under $t \longrightarrow \frac{1}{t}$, i.e.,

$$\chi_i(q_c) = \sum_j S_i^{\ j}\chi_j(q_o) \ ,$$

and obtain the *Cardy relation* in its final form [54]:

$$\sum_i (B^i{}_\alpha)^* B_\beta{}^i S_i^{\ j} = n^j_{\alpha\beta} \ . \tag{3.35}$$

The coefficients $B_\beta{}^i$ of a boundary states must provide a solution to this equation, so that $n^j_{\alpha\beta}$ are positive integers, which count representations in the open string sector.

3.3 A-type and B-type D-branes

After these general preparations we study BPS branes of $\mathcal{N}=(2,2)$ SCFT, which preserve one copy of the superconformal algebra (3.1). They come in two different types: A-branes and B-branes. The notation, A and B, was introduced in [20] and is related to the topological twisting of the $\mathcal{N}=(2,2)$ SCFT.

We start with the A-branes, which satisfy boundary conditions twisted by the mirror automorphism (3.24), i.e., the A-type boundary conditions:

$$\begin{array}{rclcl} T(z) & = & m(\tilde{T}(\bar{z})) & = & \tilde{T}(\bar{z}) \ , \\ G(z) & = & m(\tilde{G}(\bar{z})) & = & \kappa^* \tilde{\bar{G}}(\bar{z}) \ , \\ \bar{G}(z) & = & m(\tilde{\bar{G}}(\bar{z})) & = & \kappa \, \tilde{G}(\bar{z}) \ , \\ J(z) & = & m(\tilde{J}(\bar{z})) & = & -\tilde{J}(\bar{z}) \ , \end{array} \qquad \text{for} \quad z=\bar{z} \ . \tag{3.36}$$

The phase factor $\kappa=e^{i\varphi}$ accounts for the additional $U(1)$ automorphism group acting as a phase on the fermionic currents of (3.1) and leaves the freedom of choosing a spin structure [21]. Single-valuedness of the correlation functions requires that the difference of phases is always an integer and therefore we can fix

$$\kappa=(-)^s \qquad \text{for} \quad s\in\mathbb{Z} \ .$$

Let us investigate the spectrum of chiral primary operators, which are consistent with (3.36). The equations for the stress-energy tensor and the $U(1)$ current descend to $h=\tilde{h}$ resp. $q=-\tilde{q}$ on states. This implies, in particular, that $c-a$ and $a-c$ ring elements are compatible with the A-type boundary conditions in the sense that they exist on the boundary as (anti)chiral operators with conformal weight $h_b=h+\tilde{h}=2h$ and charge $q_b=q-\tilde{q}=\pm 2h_b$, whereas $c-c$ and $a-a$ elements do not.

For the associated A-type Ishibashi states the boundary conditions (3.36) imply the following relations:

$$\begin{array}{rcl} \left(L_n-\tilde{L}_{-n}\right)\big|\, i\,\big\rangle\!\big\rangle_A & = & 0 \ , \\ \left(G_n+(-)^s i\tilde{\bar{G}}_{-n}\right)\big|\, i\,\big\rangle\!\big\rangle_A & = & 0 \ , \\ \left(\bar{G}_n+(-)^s i\tilde{G}_{-n}\right)\big|\, i\,\big\rangle\!\big\rangle_A & = & 0 \ , \\ \left(J_n-\tilde{J}_{-n}\right)\big|\, i\,\big\rangle\!\big\rangle_A & = & 0 \ . \end{array} \tag{3.37}$$

Compared to the boundary conditions (3.36) we have a sign-flip in the condition (3.37) for the $U(1)$ current. Therefore, the A-type Ishibashi states incorporate Ishibashi states associated to $c-c$ and $a-a$ representations of $\mathcal{H}$ but not $c-a$ and $a-c$ representations. Given the set of Ishibashi states $\{\big|\, i\,\big\rangle\!\big\rangle_A\}$ one can then form the A-type boundary states

$$\big|\,\alpha\,\big\rangle_A=\sum_i B_\alpha^{(A)i}\big|\, i\,\big\rangle\!\big\rangle_A \ , \tag{3.38}$$

		c-c, a-a	c-a, a-c
A	boundary condition		$\times$
	Ishibashi state	$\times$	
B	boundary condition	$\times$	
	Ishibashi state		$\times$

Table 3.1: Compatibility of elements in the (anti)chiral and twisted (anti)chiral ring with boundary conditions and boundary states.

where the coefficients $B_\alpha^{(A)i}$ are subject to the Cardy relation (3.35).

The second class of BPS branes, the B-branes, is associated to the trivial automorphism (up to the sign for the fermionic currents):

$$\begin{array}{rcl} T(z) & = & \tilde{T}(\bar{z}) \ , \\ G(z) & = & (-)^s \tilde{G}(\bar{z}) \ , \\ \bar{G}(z) & = & (-)^s \tilde{\bar{G}}(\bar{z}) \ , \\ J(z) & = & \tilde{J}(\bar{z}) \ , \end{array} \qquad \text{for} \quad z = \bar{z} \ . \tag{3.39}$$

The associated B-type Ishibashi states are subject to the constraints:

$$\begin{array}{rcl} \left(L_n - \tilde{L}_{-n}\right) \left| \, i \, \right\rangle\!\!\rangle_B & = & 0 \ , \\ \left(G_n + (-)^s i \tilde{G}_{-n}\right) \left| \, i \, \right\rangle\!\!\rangle_B & = & 0 \ , \\ \left(\bar{G}_n + (-)^s i \tilde{\bar{G}}_{-n}\right) \left| \, i \, \right\rangle\!\!\rangle_B & = & 0 \ , \\ \left(J_n + \tilde{J}_{-n}\right) \left| \, i \, \right\rangle\!\!\rangle_B & = & 0 \ . \end{array} \tag{3.40}$$

The relations to the (anti)chiral and twisted (anti)chiral ring elements of the bulk theory are now exactly the other way round: The boundary conditions (3.39) admit $c-c$ and $a-a$ elements on the boundary and the B-type Ishibashi states include the representations $c-a$ and $a-c$, but not $c-c$ and $a-a$. The B-type boundary states can be written as:

$$\left| \, \alpha \, \right\rangle_B = \sum_i B_\alpha^{(B)i} \left| \, i \, \right\rangle\!\!\rangle_B \ . \tag{3.41}$$

Table (3.1) provides a collection of all the relations between BPS sectors and boundary conditions or boundary states.

3.4 Non-linear sigma models

We turn now to the discussion of explicit Lagrangian realizations of $\mathcal{N} = (2,2)$ superconformal algebras and the associated D-branes. A quite important class of theories are non-linear sigma models on Calabi-Yau 3-folds. In the following we will introduce the field content of such theories in terms of the $(2,2)$-superspace formalism in two dimensions.

The superspace is spanned by two bosonic coordinates (σ^0, σ^1) (or in complex coordinates:[3] $z = \sigma^0 + i\sigma^1$) and four fermionic coordinates $\theta^\pm, \bar{\theta}^\pm$ (with $(\theta^\pm)^\dagger = \bar{\theta}^\pm$). We define our theory on the upper half plane Σ with $\sigma^0 \in \mathbb{R}$ and $\sigma^1 \in [0, \infty)$. The world sheet metric has Euclidean signature and is flat. The supercharges and covariant derivatives are represented by

$$Q_\pm = \frac{\partial}{\partial \theta^\pm} + \bar{\theta}^\pm \frac{\partial}{\partial z^\pm} , \qquad \bar{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - \theta^\pm \frac{\partial}{\partial z^\pm} , \tag{3.42}$$

and

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - \bar{\theta}^\pm \frac{\partial}{\partial z^\pm} , \qquad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + \theta^\pm \frac{\partial}{\partial z^\pm} . \tag{3.43}$$

They satisfy the supersymmetry algebra

$$\begin{aligned} \{Q_+, \bar{Q}_+\} &= -2\partial \quad , \qquad \{D_+, \bar{D}_+\} = 2\partial \, , \\ \{Q_-, \bar{Q}_-\} &= -2\bar{\partial} \quad , \qquad \{D_-, \bar{D}_-\} = 2\bar{\partial} \, , \end{aligned} \tag{3.44}$$

In the non-linear sigma model we introduce a chiral and an antichiral superfield Φ and $\bar{\Phi}$, i.e., $\bar{D}_\pm \Phi = 0$ and $D_\pm \bar{\Phi} = 0$. The expansion in component fields reads

$$\Phi(y^\pm, \theta^\pm) = \phi(y^\pm) + \theta^+ \psi_+(y^\pm) + \theta^- \psi_-(y^\pm) + \theta^+ \theta^- F(y^\pm) \, ,$$

where $y^\pm = z^\pm - \theta^\pm \bar{\theta}^\pm$. If we set

$$\delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+ \, , \tag{3.45}$$

the variations of the fields take the form

$$\begin{aligned} \delta\phi \ &= +\epsilon_+ \psi_- - \epsilon_- \psi_+ \, , & \delta\bar{\phi} \ &= -\bar{\epsilon}_+ \bar{\psi}_- + \bar{\epsilon}_- \bar{\psi}_+ \, , \\ \delta\psi_+ &= +2\bar{\epsilon}_- \partial\phi + \epsilon_+ F \, , & \delta\bar{\psi}_+ &= -2\epsilon_- \partial\bar{\phi} + \bar{\epsilon}_+ \bar{F} \, , \\ \delta\psi_- &= -2\bar{\epsilon}_+ \bar{\partial}\phi + \epsilon_- F \, , & \delta\bar{\psi}_- &= +2\epsilon_+ \bar{\partial}\bar{\phi} + \bar{\epsilon}_- \bar{F} \, . \end{aligned} \tag{3.46}$$

In terms of the chiral and antichiral superfields one can build two supersymmetric contributions for the action. The D-term is a superspace integral of the Kähler potential of the Calabi–Yau, $K(\Phi, \bar{\Phi})$:

$$\int_\Sigma d^2 z d^4 \theta K(\Phi, \bar{\Phi}) \, . \tag{3.47}$$

The F-term, or superpotential term, can be written only in terms of chiral fields. However, non-linear sigma models do not incorporate an F-term. Performing the

[3] We occasionally use $z^\pm = \sigma^0 \pm i\sigma^1$ instead of z and $\bar{z}$.

integration over the Grassmann variables we obtain the action in terms of component fields:

$$S_{kin} = \int_\Sigma d^2z \, \Big\{ -g_{i\bar{\jmath}}(\partial\phi^i\bar{\partial}\phi^{\bar{\jmath}} + \bar{\partial}\phi^i\partial\phi^{\bar{\jmath}}) + \frac{1}{2}g_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}_- \overleftrightarrow{D}_z\psi^i_- + \\ + \frac{1}{2}g_{i\bar{\jmath}}\bar{\psi}^{\bar{\jmath}}_+ \overleftrightarrow{D}_{\bar{z}}\psi^i_+ + R_{i\bar{\imath}j\bar{\jmath}}\psi^i_+\psi^j_-\bar{\psi}^{\bar{\imath}}_-\bar{\psi}^{\bar{\jmath}}_+ \Big\} \ , \tag{3.48}$$

where we used the algebraic equation of motion

$$F^i = \Gamma^i{}_{jk}\psi^j_+\psi^k_- \ .$$

The derivative D_z is the pull-back of the covariant derivative in spacetime to the world sheet, explicitly it reads

$$D_z\psi^i_- = \partial\psi^i_- + \partial\phi^j\Gamma^i{}_{jk}\psi^k_- \ .$$

Upon variation of the action the fermionic fields obtain a first order differential equation as equation of motion and, therefore, require half the number of boundary conditions as compared to bosonic fields. Therefore, it will turn out convenient to introduce an additional boundary term:

$$S_b = \frac{i}{2}\int_{\partial\Sigma} d\tau \, g_{i\bar{\jmath}}(\bar{\psi}^{\bar{\jmath}}_-\psi^i_+ - \bar{\psi}^{\bar{\jmath}}_+\psi^i_-) \ . \tag{3.49}$$

In order to provide a closed $\mathcal{N} = (2,2)$ supersymmetric algebra the field theory (3.47) has to be invariant under left- and right-moving $U(1)$ R-symmetries. On superspace the R-symmetry acts on the superspace coordinates in the following way:

$$\begin{array}{ll} \theta^+ \to e^{i\alpha_+}\theta^+ & \theta^+ \to \theta^+ \\ \bar{\theta}^+ \to e^{-i\alpha_+}\bar{\theta}^+ & \bar{\theta}^+ \to \bar{\theta}^+ \\ \theta^- \to \theta^- & \theta^- \to e^{i\alpha_-}\theta^- \\ \bar{\theta}^- \to \bar{\theta}^- & \bar{\theta}^- \to e^{-i\alpha_-}\bar{\theta}^- \end{array} \tag{3.50}$$

Note that the integration measure $d^2zd^4\theta$ is invariant under these transformations. If we associate vanishing R-charges to the superfield Φ via

$$\Phi(z,\bar{z},\theta^\pm,\bar{\theta}^\pm) \to \Phi(z,\bar{z},e^{i\alpha_\pm}(\theta^\pm),e^{-i\alpha_\pm}(\bar{\theta}^\pm))) = \Phi(z,\bar{z},\theta^\pm,\bar{\theta}^\pm) \ ,$$

we obtain the R-charges for the component fields listed in table (3.2), where the $U(1)$ action on an arbitrary field, which carrys charges q_+ and q_-, is given by

$$R_{\alpha_\pm}(\mathcal{O}) = e^{-i\alpha_\pm q_\pm}\mathcal{O} \ .$$

The vector and axial R-charge is defined by $q_V = q_+ + q_-$ and $q_A = q_+ - q_-$, respectively.

	q_+	q_-	q_V	q_A
ψ_+^i	1	0	1	1
$\bar{\psi}_+^{\bar{\jmath}}$	−1	0	−1	−1
ψ_-^i	0	1	1	−1
$\bar{\psi}_-^{\bar{\jmath}}$	0	−1	−1	1

Table 3.2: The $U(1)$ R-charges of the components fields in the non-linear sigma model. The vector and axial charges are given by $q_V = q_+ + q_-$ and $q_A = q_+ - q_-$.

Before we turn to the analyses of the boundary conditions we state the explicit form of the $\mathcal{N} = (2,2)$ currents, which fulfil the operator product algebra (3.1). These can be evaluated by means of the Noether procedure. If there was no boundary, the left and right $U(1)$ currents corresponding to the R-symmetries are

$$J = g_{i\bar{\jmath}}\psi_+^i\bar{\psi}_+^{\bar{\jmath}} \ , \qquad \tilde{J} = g_{i\bar{\jmath}}\psi_-^i\bar{\psi}_-^{\bar{\jmath}} \ . \tag{3.51}$$

The supercurrents are give by the expressions

$$\begin{aligned} G &= g_{i\bar{\jmath}}\psi_+^i\partial\bar{\phi}^{\bar{\jmath}} \ , \qquad &\bar{G} &= g_{i\bar{\jmath}}\bar{\psi}_+^{\bar{\jmath}}\partial\phi^i \ , \\ \tilde{G} &= g_{i\bar{\jmath}}\psi_-^i\bar{\partial}\bar{\phi}^{\bar{\jmath}} \ , \qquad &\bar{\tilde{G}} &= g_{i\bar{\jmath}}\bar{\psi}_-^{\bar{\jmath}}\bar{\partial}\phi^i \ , \end{aligned} \tag{3.52}$$

and the stress-energy tensor reads

$$\begin{aligned} T &= -g_{i\bar{\jmath}}\big(\partial\phi^i\partial\bar{\phi}^{\bar{\jmath}} - \tfrac{1}{2}\psi_+^i\overleftrightarrow{D}_z\bar{\psi}_+^{\bar{\jmath}}\big) \ , \\ \tilde{T} &= -g_{i\bar{\jmath}}\big(\bar{\partial}\phi^i\bar{\partial}\bar{\phi}^{\bar{\jmath}} - \tfrac{1}{2}\psi_-^i\overleftrightarrow{D}_{\bar{z}}\bar{\psi}_-^{\bar{\jmath}}\big) \ . \end{aligned} \tag{3.53}$$

Boundary conditions

Now we turn to the investigation of supersymmetric boundary conditions of (3.48) and (3.49). The variation of the action gives rise to the constraints

$$g_{i\bar{\jmath}}\Big(\delta\phi^i\partial_1\bar{\phi}^{\bar{\jmath}} + \delta\bar{\phi}^{\bar{\jmath}}\partial_1\phi^i\Big) = 0 \ , \tag{3.54}$$

$$g_{i\bar{\jmath}}\Big((\bar{\psi}_+^{\bar{\jmath}} - \bar{\psi}_-^{\bar{\jmath}})(\delta\psi_+^i + \delta\psi_-^i) - (\delta\bar{\psi}_+^{\bar{\jmath}} + \delta\bar{\psi}_-^{\bar{\jmath}})(\psi_+^i - \psi_-^i)\Big) = 0 \ , \tag{3.55}$$

on the boundary, which can be solved in different ways.

If we wish to formulate our theory on a world sheet with boundary, one recognises first that the translation symmetry normal to the boundary is broken and, therefore, at least half of the supersymmetry is broken [20,55], so that the four supersymmetry parameters of (3.46) reduce to two parameters. In order to find out which supersymmetries remain unbroken, we perform a supersymmetry transformation of the action and find the boundary term:

$$\begin{aligned} \delta(S_{kin} + S_b) = -\frac{i}{2}\int d\tau\Big\{ & -(\epsilon_+ + \epsilon_-)g_{i\bar{\jmath}}\partial\bar{\phi}^{\bar{\jmath}}\psi_-^i - (\epsilon_- + \epsilon_+)g_{i\bar{\jmath}}\bar{\partial}\bar{\phi}^{\bar{\jmath}}\psi_+^i + \\ & +(\bar{\epsilon}_+ + \bar{\epsilon}_-)g_{i\bar{\jmath}}\partial\phi^i\bar{\psi}_-^{\bar{\jmath}} + (\bar{\epsilon}_- + \bar{\epsilon}_+)g_{i\bar{\jmath}}\bar{\partial}\phi^i\bar{\psi}_+^{\bar{\jmath}}\Big\} \ . \end{aligned} \tag{3.56}$$

From (3.56) it is apparent that there are two types of supersymmetries which can be maintained in the presence of a boundary:

(i) The A-type boundary condition relates the SUSY parameters such that

$$\begin{aligned} \epsilon &:= \epsilon_- = \bar{\epsilon}_+ \ , \\ \bar{\epsilon} &:= \bar{\epsilon}_- = \epsilon_+ \ . \end{aligned} \tag{3.57}$$

From (3.45) we find that the unbroken supersymmetries are then given by

$$Q_A = Q_+ + \bar{Q}_- \ , \qquad \text{and} \quad \bar{Q}_A = \bar{Q}_+ + Q_- \ . \tag{3.58}$$

and the transformations (3.46) become

$$\begin{aligned} \delta\phi^i &= +\bar{\epsilon}\psi_-^i - \epsilon\psi_+^i \ , & \delta\bar{\phi}^{\bar{\jmath}} &= -\epsilon\bar{\psi}_-^{\bar{\jmath}} + \bar{\epsilon}\bar{\psi}_+^{\bar{\jmath}} \ , \\ \delta\psi_+^i &= +2\bar{\epsilon}\partial\phi^i + \bar{\epsilon}F^i \ , & \delta\bar{\psi}_+^{\bar{\jmath}} &= -2\epsilon\partial\bar{\phi}^{\bar{\jmath}} + \epsilon\bar{F}^{\bar{\jmath}} \ , \\ \delta\psi_-^i &= -2\epsilon\bar{\partial}\phi^i + \epsilon F^i \ , & \delta\bar{\psi}_-^{\bar{\jmath}} &= +2\bar{\epsilon}\bar{\partial}\bar{\phi}^{\bar{\jmath}} + \bar{\epsilon}\bar{F}^{\bar{\jmath}} \ . \end{aligned} \tag{3.59}$$

The dependence on ϵ and $\bar{\epsilon}$ forces us to relate the fermionic fields ψ_-^i and $\bar{\psi}_+^{\bar{\jmath}}$ (as well as $\bar{\psi}_-^{\bar{\jmath}}$ and ψ_+^i). By supersymmetry transformation the conditions on the bosonic fields are also determined. We obtain:

$$\begin{aligned} \partial_0\phi^i - R^i{}_{\bar{\jmath}}\partial_0\phi^{\bar{\jmath}} &= 0 \ , \\ \partial_1\phi^i + R^i{}_{\bar{\jmath}}\partial_1\phi^{\bar{\jmath}} &= 0 \ , \\ \psi_-^i - R^i{}_{\bar{\jmath}}\bar{\psi}_+^{\bar{\jmath}} &= 0 \ , \\ \psi_+^i - R^i{}_{\bar{\jmath}}\bar{\psi}_-^{\bar{\jmath}} &= 0 \ , \end{aligned} \tag{3.60}$$

where ∂_0 and ∂_1 are tangent and normal to the boundary of the world sheet, respectively. The requirement that (3.56) vanishes when we insert ϵ and $\bar{\epsilon}$ gives:

$$g_{i\bar{\jmath}}R^i{}_{\bar{l}}(R^*)^{\bar{\jmath}}{}_m = g_{m\bar{l}} \ . \tag{3.61}$$

$R^i{}_{\bar{\jmath}}$ satisfies, moreover, $R^i{}_{\bar{\jmath}}(R^*)^{\bar{\jmath}}{}_l = \delta^i{}_l$, which tells us that $R^i{}_{\bar{\jmath}}$ has full rank.

If we take the boundary conditions (3.60) and plug them into the currents (3.51–3.53), we find that they indeed satisfy the A-type boundary conditions (3.36).

In order to find out what kind of submanifold γ_A the relations (3.60) define, it is convenient to combine the fields in the following way: $\phi^I := (\phi^i, \phi^{\bar{\jmath}})^T$, $\psi^I_{-,\pm} := (\psi_-^i, \pm\bar{\psi}_+^{\bar{\jmath}})^T$, $\psi^I_{+,\pm} := (\psi_+^i, \pm\bar{\psi}_-^{\bar{\jmath}})^T$; and we define:

$$\mathcal{R}^I{}_J := \begin{pmatrix} 0 & R^i{}_{\bar{\jmath}} \\ (R^*)^{\bar{l}}{}_m & 0 \end{pmatrix} \ .$$

This matrix satisfies $\mathcal{R}^{-1} = \mathcal{R}$ and, therefore, has only eigenvalues ± 1. In defining the projection operator,

$$P_\pm := \frac{1}{2}(\mathbb{1} \pm \mathcal{R}) \ , \tag{3.62}$$

the boundary conditions (3.60) can be rewritten as:

$$\begin{aligned} (P_-\partial_0\phi)^J &= 0 , \\ (P_+\partial_1\phi)^J &= 0 , \\ (P_\pm\psi_{\cdot,\mp})^J &= 0 . \end{aligned} \tag{3.63}$$

This means that P_- projects on the normal bundle to the submanifold γ_A and P_+ projects to the tangent bundle. Since $R^i{}_{\bar{\jmath}}$ has full rank we find that the supersymmetric cycle γ_A is middle-dimensional, i.e., on a Calabi-Yau 3-fold the cycle is (real) 3-dimensional; and condition (3.61) implies that the pull-back of the Kähler form ω_{IJ} to the submanifold vanishes, i.e., in terms of the projectors (3.63) we have:

$$\omega_{IJ}\ P^I_{+M}\ P^J_{+N} = 0 , \tag{3.64}$$

which tells us that γ_A is a middle-dimensional Lagrangian submanifold.

What happens if we include a B-field, i.e., the term

$$S_B = -i \int_\Sigma B_{i\bar{\jmath}}\partial\phi^i\bar{\partial}\phi^{\bar{\jmath}} . \tag{3.65}$$

$\mathcal{N} = (2,2)$ supersymmetry in the bulk requires that B is a closed form. The effect of the boundary is an additional contribution to the variation terms (3.54) and (3.55) and the supersymmetric variations (3.59) give rise to the boundary term:

$$\frac{i}{2}\int_{\partial\Sigma} d\sigma^0 \left(\bar{\epsilon}\psi^I_{-,+} - \epsilon\psi^I_{+,+}\right) B_{IJ}\partial_0\phi^J . \tag{3.66}$$

Going through the above procedure, i.e., checking compatibility of the boundary conditions with supersymmetry, we obtain the result that the B-field must vanish on γ_A. Similarly to (3.64) this can be written as:

$$B_{IJ}\ P^I_{+M}\ P^J_{+N} = 0 . \tag{3.67}$$

In view of the fact that the B-field is nothing else but the complexification of the Kähler form on the Calabi-Yau manifold this result is not very surprising.

An inclusion of a $U(N)$ gauge field A on γ_A is taken into account by the Wilson line:

$$P_{\partial\Sigma}\exp\left(-\int_{\partial\Sigma} d\sigma^0\partial_0\phi^I A_I(\phi)\right) . \tag{3.68}$$

For the $U(1)$ factor the field strengh combines with the pull-back of the B field to the gauge-invariant quantity $\mathcal{F} = B|_{\gamma_A} + F_{U(1)}$. This implies that not $B|_{\gamma_A}$ alone must vanish, but rather:

$$\mathcal{F} = 0 . \tag{3.69}$$

The condition on the non-abelian $SU(N)$ part is derived by varying (3.68) and gives rise to a term similar to (3.66). Therefore, we get the condition that the non-abelian gauge field $A_{SU(N)}$ on γ_A is flat, i.e.,

$$F_{SU(N)} = 0 . \tag{3.70}$$

Subsequently, we will refer to the non-abelian part of the gauge field strengh as F, understanding that the abelian part is contained in $\mathcal{F}$. All-in-all we found the following important result [23]:

> A supersymmetric A-cycle on a Calabi-Yau manifold $\mathcal{X}$ is a middle-dimensional Lagrangian submanifold γ_A with a flat non-Abelian gauge bundle, $F = 0$, and $\mathcal{F} = B|_{\gamma_A} + F_{U(1)} = 0$.

(ii) The B-type boundary condition has the parameters

$$\begin{aligned} \epsilon &:= \epsilon_- = -\epsilon_+ \ , \\ \bar{\epsilon} &:= \bar{\epsilon}_- = -\bar{\epsilon}_+ \ , \end{aligned} \tag{3.71}$$

so that the unbroken supercharges are given by

$$Q_B = Q_+ + Q_- \ , \qquad \text{and} \quad \bar{Q}_B = \bar{Q}_+ + \bar{Q}_- \ . \tag{3.72}$$

Because of our choice of ϵ and $\bar{\epsilon}$, equation (3.56) vanishes immediately and we do not have to care about it anymore. The B-type supersymmetry transformations become

$$\begin{aligned} \delta\phi^i &= -\epsilon\eta^i \ , & \delta\bar{\phi}^{\bar{\jmath}} &= \bar{\epsilon}\bar{\eta}^{\bar{\jmath}} \ , \\ \delta\eta^i &= 2\bar{\epsilon}\partial_0\phi^i \ , & \delta\bar{\eta}^{\bar{\jmath}} &= 2\epsilon\partial_0\bar{\phi}^{\bar{\jmath}} \ , \\ \delta\theta^i &= 2i\bar{\epsilon}\partial_1\phi^i - 2\epsilon F^i \ , & \delta\bar{\theta}^{\bar{\jmath}} &= 2i\epsilon\partial_1\bar{\phi}^{\bar{\jmath}} - 2\bar{\epsilon}\bar{F}^{\bar{\jmath}} \ , \end{aligned} \tag{3.73}$$

where $\eta^i = \psi^i_+ + \psi^i_-$ and $\theta^i = \psi^i_+ - \psi^i_-$. In order to formulate the boundary conditions we introduce two projectors on the holomorphic tangent bundle that divide the latter into a direct sum of subbundles, i.e., for a Calabi-Yau 3-fold, let P_m and P_{3-m} be the projectors such that $(P^{(t)})^i{}_j + (P^{(n)})^i{}_j = \delta^i{}_j$ and $P^{(t)}$ projects onto a m-dimensional subvector space on the fibre. Then the boundary terms (3.54) and (3.55) as well as consistency with the supersymmetry transformations (3.73) requires:

$$\begin{aligned} (P^{(n)}\partial_0\phi)^j = (P^{*(n)}\partial_0\bar{\phi})^{\bar{\jmath}} &= 0 \ , \\ (P^{(n)}\eta)^j = (P^{*(n)}\bar{\eta})^{\bar{\jmath}} &= 0 \ , \\ (P^{(t)}\partial_1\phi)^j = (P^{*(t)}\partial_1\bar{\phi})^{\bar{\jmath}} &= 0 \ , \\ (P^{(t)}\theta)^j = (P^{*(t)}\bar{\theta})^{\bar{\jmath}} &= 0 \ . \end{aligned} \tag{3.74}$$

The projector $P^{(t)}$ maps to the tangent direction, whereas $P^{(n)}$ maps to the normal bundle. In order to satisfy (3.55) we require also

$$g_{i\bar{\jmath}}(P^{(t)})^i{}_m(P^{*(n)})^{\bar{\jmath}}{}_{\bar{l}} = g_{i\bar{\jmath}}(P^{(n)})^i{}_m(P^{*(t)})^{\bar{\jmath}}{}_{\bar{l}} = 0 \tag{3.75}$$

These boundary conditions define a (complex) m-dimensional holomorphic submanifold γ_B.

If we take the boundary conditions (3.74) and plug them into the currents (3.51–3.53), we find that they satisfy the B-type boundary conditions (3.39).

Including the B field as well as the gauge field on the holomorphic submanifold is straight forward; in particular, since the B-field is the complexification of the Kähler form, we have a similar relation to (3.75). The gauge field strength on the boundary defines a complex structure, i.e., $F^{(2,0)} = F^{(0,2)} = 0$. And the $U(1)$ part combines with B to the gauge-invariant quantity $\mathcal{F} = B|_{\gamma_B} + F_{U(1)}$, where $\mathcal{F}^{(2,0)} = \mathcal{F}^{(0,2)} = 0$. We have established the following result [23]:

A supersymmetric B-cycle on a Calabi-Yau manifold $\mathcal{X}$ is a holomorphic submanifold γ_B with a holomorphic structure defined by the non-Abelian gauge field A, i.e., $F^{(2,0)} = 0$, and the $U(1)$ part satisfies $\mathcal{F}^{(2,0)} = 0$.

3.5 Landau–Ginzburg models

In Landau–Ginzburg models the D-term (3.47) is accompanied by the F-term

$$\int_\Sigma d^2z d^2\theta W(\Phi) + \text{c.c.} \ . \tag{3.76}$$

In order to have a nontrivial holomorphic potential we choose the target space to be a non-compact Calabi–Yau manifold. From the transformation of the measure $d^2zd^2\theta$ under the $U(1)$ transformations (3.50) we find that the Landau–Ginzburg theory is invariant under R-symmetry only if the superpotential $W(\Phi^i)$ is quasi-homogeneous, i.e.,

$$W(\lambda^{\omega_i}\Phi^i) = \lambda W(\Phi^i) \ , \tag{3.77}$$

where ω_i are some fractional numbers. The theory is invariant under R-symmetry if the superfields transform in the following way:

$$\Phi^i(z,\bar{z},\theta^\pm,\bar{\theta}^\pm) \to e^{-i\alpha_\pm\omega_i}\Phi^i(z,\bar{z},e^{i\alpha_\pm}(\theta^\pm),e^{-i\alpha_\pm}(\bar{\theta}^\pm))) = e^{-i\alpha_\pm\omega_i}\Phi^i(z,\bar{z},\theta^\pm,\bar{\theta}^\pm) \ .$$

The component fields have therefore the charges, which are listed in table (3.3). Additionally, we can associate scaling dimensions $d = h + \bar{h}$ and spins $s = h - \bar{h}$ to the fields, where h and $\bar{h}$ are the left- and right-moving conformal weights, when the theory is conformal invariant. The conformal weights $(h,\bar{h})$ of z and $\bar{z}$ are $(-1,0)$ and $(0,-1)$, respectively. Therefore, θ^+, $\bar{\theta}^+$ have $(-1/2,0)$ and θ^-, $\bar{\theta}^-$ have $(0,-1/2)$. Then the conformal dimensions of the superfields Φ^i are determined by a quasi-homogeneous superpotential (3.77) and are given by $(\omega_i/2,\omega_i/2)$. The conformal weights of the component fields are also listed in table (3.3).

The additional superpotential term (3.76) does not alter the off-shell supersymmetry transformations (3.46); however, the algebraic equation of motion for the

	$(h\ ,\ q_+)$	$(\bar{h}\ ,\ q_-)$	$(d\ ,\ q_V)$	$(s\ ,\ q_A)$
ϕ^i	$(\frac{\omega_i}{2}\ ,\ \omega_i)$	$(\frac{\omega_i}{2}\ ,\ \omega_i)$	$(\omega_i\ ,\ 2\omega_i)$	$(0\ ,\ 0)$
$\bar{\phi}^{\bar{\jmath}}$	$(\frac{\omega_i}{2}\ ,\ -\omega_j)$	$(\frac{\omega_i}{2}\ ,\ -\omega_j)$	$(\omega_i\ ,\ -2\omega_j)$	$(0\ ,\ 0)$
ψ^i_+	$(\frac{\omega_i+1}{2}\ ,\ \omega_i+1)$	$(\frac{\omega_i}{2}\ ,\ \omega_i)$	$(\omega_i+\frac{1}{2}\ ,\ 2\omega_i+1)$	$(\frac{1}{2}\ ,\ 1)$
$\bar{\psi}^{\bar{\jmath}}_+$	$(\frac{\omega_i+1}{2}\ ,\ -\omega_j-1)$	$(\frac{\omega_i}{2}\ ,\ -\omega_j)$	$(\omega_i+\frac{1}{2}\ ,\ -2\omega_j-1)$	$(\frac{1}{2}\ ,\ -1)$
ψ^i_-	$(\frac{\omega_i}{2}\ ,\ \omega_i)$	$(\frac{\omega_i+1}{2}\ ,\ \omega_i+1)$	$(\omega_i+\frac{1}{2}\ ,\ 2\omega_i+1)$	$(-\frac{1}{2}\ ,\ -1)$
$\bar{\psi}^{\bar{\jmath}}_-$	$(\frac{\omega_i}{2}\ ,\ -\omega_j)$	$(\frac{\omega_i+1}{2}\ ,\ -\omega_j-1)$	$(\omega_i+\frac{1}{2}\ ,\ -2\omega_j-1)$	$(-\frac{1}{2}\ ,\ 1)$

Table 3.3: The $U(1)$ R-charges and conformal weights of the components fields in Landau–Ginzburg models ($d = h + \bar{h}$ and $s = h - \bar{h}$).

auxiliary field F^i becomes

$$F^i = \Gamma^i{}_{jk}\psi^j_+\psi^k_- - \frac{1}{2}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\bar{W}\ , \tag{3.78}$$

and the action (3.48) obtains the additional term

$$S_{pot} = \int_\Sigma d^2z \left\{ -\frac{1}{4}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\bar{W}\partial_i W - \frac{1}{2}\psi^i_+\psi^j_- D_i\partial_j W - \frac{1}{2}\bar{\psi}^{\bar{\imath}}_+\bar{\psi}^{\bar{\jmath}}_- D_{\bar{\imath}}\partial_{\bar{\jmath}}\bar{W} \right\} \tag{3.79}$$

A $\mathcal{N} = (2,2)$ supersymmetry variation of (3.79) together with (3.48) and (3.49) gives, in addition to (3.56), the boundary term

$$-\frac{i}{4}\int_\Sigma d\sigma^0 \left\{ (\epsilon_- - \epsilon_+)(\bar{\psi}^{\bar{\jmath}}_- + \bar{\psi}^{\bar{\jmath}}_+)\partial_{\bar{\jmath}}\bar{W} + (\bar{\epsilon}_- - \bar{\epsilon}_+)(\psi^i_- + \psi^i_+)\partial_i W \right\}\ . \tag{3.80}$$

Let us examine what conditions we get for the Landau–Ginzburg potential W when we apply the A- and B-type boundary conditions.

(i) For the A-type boundary conditions with the supersymmetry parameters defined in (3.57) the boundary term (3.80) becomes

$$-\frac{i}{4}\int_\Sigma d\sigma^0(\epsilon - \bar{\epsilon}) \left\{ (\bar{\psi}^{\bar{\jmath}}_- + \bar{\psi}^{\bar{\jmath}}_+)\partial_{\bar{\jmath}}\bar{W} + (\psi^i_- + \psi^i_+)\partial_i W \right\}\ . \tag{3.81}$$

Using the A-type boundary conditions (3.60) we find the restriction $\partial_{\bar{\jmath}}\bar{W} = R^i{}_{\bar{\jmath}}\partial_i W$, which can be expressed in terms of the projector P_+ as

$$P_+{}^J{}_I\partial_J(W - \bar{W}) = 0\ . \tag{3.82}$$

Since P_+ is a projector into the direction tangent to the Lagrangian submanifold γ_A, this means that ImW is constant along γ_A, or otherwise stated [23]:[4]

[4]One may ask whether it is possible to introduce a boundary term containing bulk fields, which cancels (3.81); a comparison with the A-type transformations (3.59) shows immediately that this is not possible. One could try to introduce new boundary degrees of freedom in order to get rid of (3.81), but this was not considered in the literature so far.

$\boxed{\text{Given a non-compact Calabi–Yau } n\text{-fold } \mathcal{X} \text{ and a Landau–Ginzburg potential } W, \text{ the image of a Lagrangian submanifold } \gamma_A \text{ in the } W\text{-plane has to be a straight line.}}$

(ii) For the B-type boundary conditions of Landau–Ginzburg models we will find a very rich structure which is due to the appearance of additional degrees of freedom on the boundary of the world-sheet. In view of the parameters (3.71) the boundary term (3.80) becomes

$$-\frac{i}{2}\int_{\Sigma} d\sigma^0 \left\{\bar{\epsilon}\eta^i\partial_i W + \epsilon\bar{\eta}^{\bar{j}}\partial_{\bar{j}}\bar{W}\right\} \ . \tag{3.83}$$

Using the B-type boundary conditions (3.74) one is lead to the conclusion that W must be constant along the holomorphic submanifold γ_B, i.e.,

$$(P^{(t)})_i{}^j\partial_j W = 0 \ . \tag{3.84}$$

This is due to the fact that we cannot introduce an appropriate counter term containing only bulk fields on the boundary.

A quite elegant way to come around condition (3.84) was introduced in [55] and further investigated in [47, 58, 60]. It amounts to introducing additional fermionic supermultiplets on the boundary that are capable of compensating the term (3.83). In order to construct these multiplets we use superspace notation. We investigate first how the superspace and the bulk multiplet (3.45) are altered in view of supersymmetry breaking.

As a result of the B-type boundary conditions the unbroken superspace is spanned by $\theta^0 = 1/2(\theta^- + \theta^+)$ and $\bar{\theta}^0 = 1/2(\bar{\theta}^- + \bar{\theta}^+)$, so that B-type supercharges (3.72) can be represented by

$$Q_B = \frac{\partial}{\partial\theta^0} + i\bar{\theta}^0\frac{\partial}{\partial\sigma^0} \qquad \text{and} \quad \bar{Q}_B = -\frac{\partial}{\partial\bar{\theta}^0} - i\theta^0\frac{\partial}{\partial\sigma^0} \ . \tag{3.85}$$

If we try to encode the B-type supersymmetry transformations (3.73) in terms of superfields we see that the chiral multiplet Φ^i of the bulk theory rearrange into a bosonic and a fermionic multiplet Φ'^i resp. Θ'^i. The bosonic superfield Φ'^i containing ϕ^i and η^i turns out to be chiral, i.e., $\bar{D}_B\,\Phi'^i = 0$. The fields θ^i and F^i do not form a chiral multiplet, but rather combine into the fermionic superfield Θ'^i which satisfies $\bar{D}_B\,\Theta'^i = 2i\partial_1\Phi'^i$. In components we have:[5]

$$\begin{aligned}\Phi'^i(y^0,\theta^0) &= \phi^i(y^0) + \theta^0\eta^i(y^0),\\ \Theta'^i(y^0,\theta^0,\bar{\theta}^0) &= \theta(y^0) - 2\theta^0F^i(y^0) - 2i\,\bar{\theta}^0\big[\partial_1\phi^i(y^0) + \theta^0\partial_1\eta^i(y^0)\big],\end{aligned} \tag{3.86}$$

where $y^0 = \sigma^0 - \theta^0\bar{\theta}^0$.

[5]The splitting of the bulk multiplet occurs of course in any $\mathcal{N} = (2,2)$ supersymmetric theory; for example in non-linear sigma models.

Now we are prepared to solve the problem of finding a compensation for expression (3.83) on the boundary; this issue was raised in [55] and is known as Warner problem. The solution goes as follows: We introduce a collection of boundary fermionic superfield Π^A for $I = 1 \dots N$, which are not chiral but rather satisfies $\bar{D}_B\,\Pi^A = E^A(\Phi')$. This multiplet has the expansion

$$\Pi^A(y^0,\theta^0,\bar{\theta}^0) = \pi^A(y^0) + \theta^0\, l^A(y^0) - \bar{\theta}^0\left[E^A(\phi) + \theta^0\eta^l(y^0)\partial_l E^A(\phi)\right] . \tag{3.87}$$

Its component fields transform as:

$$\begin{aligned} \delta\pi^A &= \epsilon l^A - \bar{\epsilon}E^A(\phi)\ , & \delta\bar{\pi}^A &= \bar{\epsilon}\bar{l}^A - \epsilon\bar{E}^A(\bar{\phi})\ , \\ \delta l^A &= -2\bar{\epsilon}\partial_0\pi^A + \bar{\epsilon}\eta^l\partial_l E^A(\phi), & \delta\bar{l}^A &= -2\epsilon\partial_0\bar{\pi}^A - \epsilon\bar{\eta}^{\bar{l}}\bar{\partial}_{\bar{l}}\bar{E}^A(\bar{\phi})\ . \end{aligned} \tag{3.88}$$

Similar to the bulk theory we can build two terms for the action, i.e.,

$$S_{\partial\Sigma} = -\frac{i}{2}\int d\sigma^0 d^2\theta^0\; \bar{\Pi}^A\,\Pi^A + \frac{1}{2}\int_{\partial\Sigma} d\sigma^0 d\theta^0\; \Pi^A J^A(\Phi')\,|_{\bar{\theta}^0=0} + \text{c.c.}\ . \tag{3.89}$$

Using the algebraic equation of motion $l^A = -i\bar{J}^A$, the boundary action reads

$$\begin{aligned} S_{\partial\Sigma} = \frac{1}{2}\int dx^0 \Big\{ & -i\bar{\pi}^A \overleftrightarrow{D}_0 \pi^A \\ & - J^A\bar{J}^A + i\pi^A\eta^l\partial_l J^A + i\bar{\pi}^A\bar{\eta}^{\bar{l}}\bar{\partial}_{\bar{l}}\bar{J}^A + \\ & + E^A\bar{E}^A + \bar{\pi}^A\eta^l\partial_l E^A - \pi^A\bar{\eta}^{\bar{l}}\bar{D}_{\bar{l}}\bar{E}^A \Big\}\ , \end{aligned} \tag{3.90}$$

and the variation of the boundary fermion π reduces to

$$\begin{aligned} \delta\pi^A &= -i\epsilon\bar{J}^A(\bar{\phi}) - \bar{\epsilon}E^A(\phi)\ , \\ \delta\bar{\pi}^A &= +i\bar{\epsilon}J^A(\phi) - \epsilon\bar{E}^A(\bar{\phi})\ . \end{aligned} \tag{3.91}$$

The kinetic term in (3.89) is supersymmetric by construction, whereas the potential term containing J^A is not, which is due to the non-chirality of Π^A. Rather, the transformation of (3.90) generates

$$\delta S_{\partial\Sigma} = \frac{i}{2}\int_{\partial\Sigma} d\sigma^0\ \left\{\epsilon\bar{\eta}^{\bar{l}}\partial_{\bar{l}}(\bar{E}^A\bar{J}^A) + \bar{\epsilon}\eta^l\partial_l(E^AJ^A)\right\}\ . \tag{3.92}$$

This is exactly what we need in order to compensate (3.83). We come to the conclusion that the whole action is invariant under supersymmetry iff [60]

$$W = E^AJ^A + \text{const.}\ . \tag{3.93}$$

This equation will play an essential role when we try to deform the Landau–Ginzburg theory in terms of chiral ring elements; equation (3.93) relates the moduli in the bulk superpotential $W(\phi)$ to the moduli in the boundary potentials $J^A(\phi)$ and $E^A(\phi)$.

Chapter 4

Topological conformal field theories

Topological field theories [61] serve as a bridge between physics and mathematics in numerous examples (see [62] and references therein). For instance, topological sigma models were used to compute invariants of complex manifolds [63]; or correlation functions of Wilson line operators in Chern–Simons theory encode Jones polynomials of knot theory [64].

In this chapter we will introduce the most basic definitions of 2-dimensional topological conformal field theories with special emphasise to effects coming from the presence of a boundary. The content will serve as basis for the subsequent chapters 5 – 8.

After introducing (general) topological field theories in section 4.1, we specialise in section 4.2 to topological *conformal* field theory in 2 dimensions. The latter are then the main topic for the remainder of this thesis. In sections 4.3 we define physical operators and their descendants for the bulk (closed string) theory [61, 63]. Thereafter, in section 4.4 we apply the knowledge from section 3.2 and study the possible boundary conditions. Like in the bulk one can introduce physical operators and descendants on the boundary. In particular, we will find two types of physical boundary operators [54]: boundary condition preserving operators, which correspond to open strings ending on a single D-brane, and boundary condition changing operators, which correspond to open strings stretched between two D-branes.

4.1 Definition of topological field theories

A topological quantum field theory (TQFT) defined on a manifold $\mathcal{M}$ is characterised by the requirement that the correlation functions of a certain class of operators, which we call *physical operators* does not depend on the metric on $\mathcal{M}$. Specifying a basis of physical operators, $\mathcal{O}_i \in \mathcal{H}_{phys}$, the correlation functions

$$\left\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \right\rangle_\Sigma$$

are constants depending only on the labels, $i_1, \ldots, i_n$, and on the topology of Σ. However, for a realization of a TQFT in terms of a Lagrangian we need to specify a metric g_{ab} on Σ, which leads to a non-vanishing stress-energy tensor T_{ab}. In order to implement the independence of correlation functions of g_{ab}, the stress-energy tensor must decouple by some symmetry mechanism.

As it was shown in [61] the symmetry that is responsible for the decoupling of T_{ab} is generated by a nilpotent fermionic operator Q, i.e.,[1]

$$Q^2 = 0 \; . \tag{4.1}$$

The space $\mathcal{H}_{phys}$ of physical operators is defined as the cohomology of Q:

$$\mathcal{H}_{phys} = \frac{\ker Q}{\mathrm{im} Q} \; . \tag{4.2}$$

The crucial point in a TQFT is the property that the stress-energy tensor is Q-exact,

$$T_{ab} = [Q, G_{ab}] \; . \tag{4.3}$$

Since correlation functions of physical operators do not depend on Q-exact terms, the stress-energy tensor (4.3) decouples, which implies the independence of the metric g_{ab}.

There are two kinds of TQFTs and we will discuss examples of both types [62].
(i) A topological field theory of *Schwarz type* has the total action

$$S(\Phi, g) = S_0(\Phi) + [Q, V(\Phi, g)] \; ,$$

where Φ denotes the field content.
(ii) The second kind is referred to as TQFT of *Witten type*. In a sense this is a special case of the Schwarz type, where $S_0(\Phi) = 0$, so that

$$S(\Phi, g) = [Q, V(\Phi, g)] \; .$$

Since S_0 is independent of the metric g_{ab}, it follows immediately that in both cases the stress-energy tensor is given by

$$T_{ab} = [Q, \frac{1}{\sqrt{g}} \frac{\delta V}{\delta g^{ab}}] \; . \tag{4.4}$$

4.2 Topological conformal algebra

We specialise now to topological *conformal* field theories (TCFT) in 2 dimensions, which amounts to the requirement that the stress-energy tensor is traceless, $T^a{}_a = 0$,

[1] In general, it is sufficient to require that Q squares to a global symmetry of the TQFT. We will, however, be concerned only with TQFTs, which satisfy (4.1).

so that the symmetry algebra splits into a left- and right-moving part. In particular, the Q-symmetry charge splits into a left- and right-moving part Q_0 and $\tilde{Q}_0$, respectively. The stress-energy tensor can be written as

$$T(z) = [Q_0, G(z)] \ ;$$

a similar relation exists for the right-moving sector. Moreover, Q_0 is the zero mode of a fermionic current $Q(z)$ of conformal weight 1. Closure of the symmetry algebra, which is generated by $T(z)$, $G(z)$ and $Q(z)$, requires a $U(1)$ current $J(z)$. The mode expansions of the currents read

$$\begin{array}{rclrcl} T(z) &=& \sum_{n=-\infty}^{\infty} L_n \ z^{-n-2} \ , & G(z) &=& \sum_{n=-\infty}^{\infty} G_n \ z^{-n-2} \ , \\ J(z) &=& \sum_{n=-\infty}^{\infty} J_n \ z^{-n-1} \ , & Q(z) &=& \sum_{n=-\infty}^{\infty} Q_n \ z^{-n-1} \ . \end{array} \tag{4.5}$$

The topological operator product expansions of the symmetry currents (4.5) is given by

$$\begin{aligned} T(z)\ T(0) &\sim \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) \ , \\ T(z)\ G(0) &\sim \frac{2}{z^2}G(0) + \frac{1}{z}\partial G(0) \ , \\ T(z)\ Q(0) &\sim \frac{1}{z^2}Q(0) + \frac{1}{z}\partial Q(0) \ , \\ T(z)\ J(0) &\sim -\frac{\hat{c}}{z^3} + \frac{1}{z^2}J(0) + \frac{1}{z}\partial J(0) \ , \\ Q(z)\ G(0) &\sim \frac{\hat{c}}{z^3} + \frac{1}{z^2}J(0) + \frac{1}{z}T(0) \ , \\ J(z)\ Q(0) &\sim +\frac{1}{z}Q(0) \ , \\ J(z)\ G(0) &\sim -\frac{1}{z}G(0) \ , \\ J(z)\ J(0) &\sim \frac{\hat{c}}{z^2} \ , \end{aligned} \tag{4.6}$$

and the associated mode expansions are:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} \ , \\ [L_n, Q_m] &= -mQ_{n+m} \ , \\ [L_n, G_m] &= (n-m)G_{n+ma} \ , \\ [L_n, J_m] &= -mJ_{m+n} + \frac{\hat{c}}{2}n(n+1)\delta_{m+n,0} \ , \end{aligned} \tag{4.7}$$

$$\begin{aligned} [G_n, Q_m] &= L_{m+n} + mJ_{m+n} + \frac{\hat{c}}{2}n(n+1)\delta_{m+n,0} \ , \\ [J_n, G_m] &= -G_{n+m} \ , \\ [J_n, Q_m] &= +Q_{n+m} \ , \\ [J_n, J_m] &= \hat{c}n\delta_{m+n,0} \ . \end{aligned} \tag{4.8}$$

Note in particular that the TCFT algebra does not have a central charge and therefore has no conformal anomaly. However, the $L - J$ commutator relation tells us that we have an anomalous background charge equal to $\hat{c}$.

4.3 Physical operators and descendants

The physical operators in the bulk are defined as cohomology classes with respect to Q_0, i.e.,

$$\begin{aligned} &[Q_0, \phi_i] = 0 \ , \\ &\phi_i \sim \phi_i + [Q_0, \chi] \ . \end{aligned} \tag{4.9}$$

Similar relations hold for $\tilde{Q}_0$. We denote the space of physical operators of the bulk theory by H_c. For simplicity, we shall also assume that each ϕ_i is Grassmann even. This simplifies certain sign prefactors in later sections and suffices for our applications. The hermitian conjugate to Q_0 is G_0 and in view of $[G_0, Q_0] = L_0$ we can make a 'Hodge decomposition' of the space of operators. For the physical operators this implies that a representative of the cohomology class can be fixed by the requirement [5]

$$[G_0, \phi_i] = 0 \ . \tag{4.10}$$

The condition

$$[L_0, \phi_i] = 0 \ , \tag{4.11}$$

is equivalent to the definitions (4.9) and (4.10) of a physical operator and tells us that a physical operator has conformal weight $h = 0$.

Since L_0 and J_0 commute we can diagonalise the Hilbert space in eigenstates of both generators. In particular, for physical operators with

$$[J_0, \phi_i] = q_i \phi_i \ ,$$

we can infer from unitarity and the topological algebra (4.5) that the charge is restricted to the range

$$0 \leq q \leq \hat{c} \ . \tag{4.12}$$

Vanishing conformal weight and charge conservation imply already that the operator product of physical operators is regular, where the constant leading term involves again only physical operators:

$$\phi_i(z)\phi_j(0) \sim C_{ij}{}^k \phi_k(0) \ . \tag{4.13}$$

This is consistent with the observation of section 4.1 that correlation functions of physical operators are constant.

Given physical operators, one can construct descendants by using relations:

$$[Q_0, G_{-1}] = L_{-1} \qquad \text{and} \qquad [\tilde{Q}_0, \tilde{G}_{-1}] = \tilde{L}_{-1} \ . \tag{4.14}$$

Since the commutators with L_{-1} and $\tilde{L}_{-1}$ act as $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, respectively. We find that the operators

$$\begin{aligned}
\phi_i^{(1,0)} &= [G_{-1}, \phi_i] dz \ , \\
\phi_i^{(0,1)} &= [\tilde{G}_{-1}, \phi_i] d\bar{z} \ , \\
\phi_i^{(1,1)} &= [G_{-1}, [\tilde{G}_{-1}, \phi_i]] dz \wedge d\bar{z} = [\tilde{G}_{-1}, [G_{-1}, \phi_i]] d\bar{z} \wedge dz \ ,
\end{aligned}$$

satisfy the descent equations:

$$\begin{aligned}
[Q_0, \phi_i^{(1,0)}] = \partial\phi_i \quad &, \quad [\tilde{Q}_0, \phi_i^{(1,0)}] = 0 \\
[Q_0, \phi_i^{(0,1)}] = 0 \quad &, \quad [\tilde{Q}_0, \phi_i^{(0,1)}] = \bar{\partial}\phi_i \\
[Q_0, \phi_i^{(1,1)}] = \partial\phi_i^{(0,1)} = d\phi_i^{(0,1)} \quad &, \quad [\tilde{Q}_0, \phi_i^{(1,1)}] = \bar{\partial}\phi_i^{(1,0)} = d\phi_i^{(1,0)} \ .
\end{aligned} \tag{4.15}$$

Notice that $\phi_i^{(1,0)}$ and $\phi_i^{(0,1)}$ are operator-valued sections of the canonical and anti-canonical line bundles over $\mathbb{P}^1$, while $\phi_i^{(1,1)}$ is an operator-valued two-form. From the descent equation we see that the integrated operators

$$\int_{S^2} \phi_i^{(1,1)} \tag{4.16}$$

are Q-closed and, therefore, can be inserted along with local physical operators in topological correlation functions on the sphere. One could furthermore consider loop integrals of the one-form descendants $\phi_i^{(0,1)} + \phi_i^{(1,0)}$, but those observables do not play a role for what follows.

4.4 The effects of a boundary

Since the topological conformal field theory is a special type of CFT, the implementation of boundary conditions that maintain one copy of the topological symmetry algebra goes through as in the general description in section 3.2. Subsequently we take the world-sheet Σ to be the upper half plane.

Other than in $\mathcal{N} = (2,2)$ SCFT the topological algebra (4.6) does not provide a mirror automorphism, because the fermionic currents $Q(z)$ and $G(z)$ have different conformal weights. However, a $U(1)$ automorphism acting as a phase on $Q(z)$ and $G(z)$ is still present, so that the boundary conditions on the currents are given by

$$\left.\begin{aligned}
T(z) &= \tilde{T}(\bar{z}) \\
G(z) &= (-)^s \, \tilde{G}(\bar{z}) \\
Q(z) &= (-)^s \, \tilde{Q}(\bar{z}) \\
J(z) &= \tilde{J}(\bar{z})
\end{aligned}\right. \qquad \text{for} \quad z = \bar{z} \ , \tag{4.17}$$

where the sign comes from the $U(1)$ automorphism.

Let us define the operators on the upper half plane by adding the left- and right-moving contributions; we obtain, for instance,

$$Q_n + (-)^s \tilde{Q}_n \to Q_n \ . \tag{4.18}$$

In particular, we denote the zero mode by $Q = Q_0 + (-)^s \tilde{Q}_0$. Then $Q^2 = 0$ and the commutator relation (4.14) responsible for the descent equations becomes

$$[Q, G_{-1}] = d \ , \tag{4.19}$$

where $d = \partial + \bar{\partial}$ is the exterior derivative along the boundary on the upper half plane. The descent equations (4.15) for bulk operators must now be written with respect to Q and split into descent equations of two multiplets. Let us rewrite the one-form operators into $\phi_i^{(1)} = \phi_i^{(1,0)} + \phi_i^{(0,1)}$, and $\phi_i^{(n)} = \phi_i^{(1,0)} - \phi_i^{(0,1)}$. Then the descent equations for the physical operator ϕ^i read

$$[Q, \phi_i] \ = \ 0 \ , \tag{4.20}$$

$$[Q, \phi_i^{(1)}] \ = \ d\phi_i \ . \tag{4.21}$$

The second multiplet for $\phi_i^{(n)}$, which is however not physical, becomes

$$[Q, \phi_i^{(n)}] \ = \ d_n \phi_i \ , \tag{4.22}$$

$$[Q, \phi_i^{(1,1)}] \ = \ d\phi_i^{(1)} \ . \tag{4.23}$$

where $d_n = \partial - \bar{\partial}$ is the normal derivative at the boundary of the upper half plane. The last equation implies:

$$[Q, \int\limits_{\Sigma} \phi_i^{(1,1)}] = \int\limits_{\partial\Sigma} \phi_i^{(1)} \quad . \tag{4.24}$$

As opposed to the sphere, the integrated descendant is not Q-closed, which is due to the presence of a boundary term.

In addition to the physical operators (4.9) of the bulk theory we can introduce physical operators on the boundary. But here we have a subtlety, which arises from the fact that in string theory there appear in addition to open strings attached to a single D-brane (or a stack of D-branes), strings which are stretched between two different D-branes. It is well-known [54] that in conformal field theories the former correspond to *boundary condition preserving operators* (BPO), $\psi_a^{(AA)}$, whereas the latter correspond to *boundary condition changing operators* (BCO), $\psi_a^{(AB)}$. The latter switch between two boundary conditions labelled by A and B. Of course, all operators of the topological conformal algebra (4.7) are BPOs, since they are related by a single condition on the boundary. The action of the charges on BCOs can be

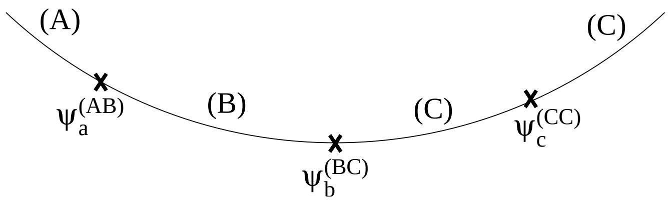

Figure 4.1: Some boundary condition changing and preserving operators inserted at the boundary of the disk.

written in a form which is very similar to that relevant for the boundary condition preserving sector. For example:

$$[G, \psi_a^{(AB)}] = G^{(AA)}\ \psi_a^{(AB)} - (-1)^{|a|}\psi_a^{(AB)}\ G^{(BB)} := \oint (G(z)\ \psi_a^{(AB)}) \quad , \tag{4.25}$$

where — using the doubling trick — the left- and right-moving currents are joined according to the boundary conditions A and B on the respective side of $\psi_a^{(AB)}$. This allows us to treat BPOs and BCOs on equal footing.

The physical operator condition becomes

$$\begin{aligned} [Q, \psi_a^{(AB)}] &= Q^{(A)}\psi_a^{(AB)} - (-)^{|a|}\psi_a^{(AB)}Q^{(B)} = 0 \ , \\ [G_0, \psi_a^{(AB)}] &= G_0^{(A)}\psi_a^{(AB)} - (-)^{|a|}\psi_a^{(AB)}G_0^{(B)} = 0 \ , \end{aligned} \tag{4.26}$$

or equivalently,

$$[L_0, \psi_a^{(AB)}] = 0 \ ,$$

and we denote the space of physical operators on the boundary by H_o.

In the presence of boundary condition changing sectors, the various algebraic structures extracted in this work are promoted to their category-theoretic counterparts. This follows in standard manner by viewing D-branes (i.e., the boundary conditions) as objects of a category and identifying BPOs and BCOs with endomorphisms and morphisms between distinct objects.

Having said all this, we will subsequently leave out the boundary condition labels in order to keep notation simple. In particular, all relations derived in this work are true for both boundary condition preserving and changing sectors. Note that for each boundary component, the labels must be "cyclically closed" in correlation functions; for example, correlators such as $\langle \psi_{a_1} \dots \psi_{a_n} \rangle$ should be expanded to $\langle \psi_{a_1}^{A_1 A_2} \dots \psi_{a_n}^{A_n A_1} \rangle$ when restoring boundary labels.

Defining:

$$\psi_a^{(1)} := [G, \psi_a] d\tau$$

and using relation (4.19), we also find the boundary descent equations:

$$[Q, \psi_a] = 0 \ , \tag{4.27}$$

$$[Q, \psi_a^{(1)}] = d_\tau \psi_a \ , \tag{4.28}$$

where d_τ is the exterior derivative along the boundary. Since operators will be inserted on the boundary in cyclic order, the typical integral of a descendant $\psi_a^{(1)}$ runs from the insertion to its left to the insertion to its right:

$$\int_{\tau_l}^{\tau_r} \psi_a^{(1)} \quad . \tag{4.29}$$

Here 'left' and 'right' should be understood in the sense of the cyclic order on the boundary of the disk, which is determined by the orientation on the boundary induced from the orientation of the interior. As a consequence, we find that the Q-variation of (4.29) need not vanish:

$$[Q, \int_{\tau_l}^{\tau_r} \psi_a^{(1)}] \;=\; \psi_a \Big|_{\tau_l}^{\tau_r} \quad . \tag{4.30}$$

Notice that the Grassmann degree of $\psi_a^{(1)}$ is opposite to that of ψ_a. It is convenient to take this into account by introducing a new grading on the boundary algebra H_o. Given an operator ψ_a, we denote its (usual) Grassmann grade by $|a|$ and define a shifted, or "suspended" grade of ψ_a by

$$\tilde{a} := |a| + 1 \pmod 2 \ . \tag{4.31}$$

When we subsequently study consistency relations of disk amplitudes, we will find that the suspended grading is more natural than the usual Grassmann grading.

Chapter 5

Topological twisting

Topological twisting is a powerful tool that relates the BPS subsectors of a $\mathcal{N}=2$ supersymmetric theory to a topological field theory, which is in some cases exactly solvable, when supplemented by geometric methods. It lead, for instance, to important insights for mirror symmetry [4].

The main focus of this work is topological *conformal* field theories and, therefore, we will start in section 5.1 with the description of the A- and B-twist on the level of the $\mathcal{N}=(2,2)$ superconformal algebra (3.1) [5, 65]. In section 5.2 we consider the compatibility of A-type and B-type D-branes of $\mathcal{N}=(2,2)$ SCFT with the two types of twisting. Thereafter, in section 5.3 we give a brief review of the question [6, 8]: What are topological amplitudes computing in $\mathcal{N}=(2,2)$ SCFTs and superstring compactifications?

In the remainder of this chapter we consider three examples of topological twisted models: Following [66] and [4] we present the topological closed string for the A- and B-twisted non-linear sigma models on Calabi–Yau 3-folds. We briefly describe D-branes as boundary conditions in these models [37] (see also [25,26] and references therein) and point out the role of boundary states of $\mathcal{N}=(2,2)$ SCFT, which are related to period integrals in the topologically twisted models [20]. The third class of examples are topological Landau–Ginzburg models. After reviewing the bulk theory [67] we investigate the physical boundary operators [47, 58, 59] and outline the relation to the triangulated category of [68,69]. As a special case we analyse the B-branes and their boundary spectrum in the A-series of Landau–Ginzburg minimal models and compare the results to $\mathcal{N}=(2,2)$ minimal models [47].

5.1 Twisting of the closed string sector

Comparing the space of the physical operators in TCFT and the space of chiral primaries in SCFT, we find many similarities. Indeed, there is a relation between these two theories, which can be made precise by the procedure of topological twisting of an $\mathcal{N} = (2,2)$ SCFT. This twist is implemented by the replacement

$$\begin{array}{rcl} T(z) & \rightarrow & T(z) + \varepsilon\frac{1}{2}\partial J(z) \ , \\ J(z) & \rightarrow & \varepsilon J(z) \ , \end{array} \qquad \varepsilon = \pm 1 \ , \tag{5.1}$$

in the superconformal algebra (3.1), which leads directly to the topological algebra (4.6). A quite important fact of the twist is the 'transformation' of the central charge c of SCFT into the background charge $\hat{c}$ of TCFT; the precise relation is

$$\hat{c} = \frac{c}{3} \ . \tag{5.2}$$

In terms of operator modes the twist (5.1) reads:

$$\begin{array}{rcl} L_n & \rightarrow & L_n - \varepsilon\frac{n+1}{2} J_n \ , \\ J_n & \rightarrow & \varepsilon J_n \ . \end{array} \tag{5.3}$$

In particular, the conformal weight h of operators with non-vanishing charge q is shifted as

$$h \rightarrow h - \varepsilon\frac{q}{2} \tag{5.4}$$

Then the fermionic SCFT currents $G(z)$ and $\bar{G}(z)$, both having conformal weight $3/2$, turn into currents with conformal weights $h = 3/2 - \varepsilon/2$ and $h = 3/2 + \varepsilon/2$, respectively. Depending on the sign ε one of these currents becomes the TCFT current $Q(z)$ with $h = 1$ and the other becomes $G(z)$ with $h = 2$.

From the map (5.4) we see immediately that either the chiral or antichiral primary fields of SCFT (again depending on the sign ε) are in one-to-one correspondence with the physical operators of TCFT. We can choose the signs ε and $\tilde{\varepsilon}$ for twisting of the left- resp. the right-moving sector independently, so that depending on this choice one of the four sectors is described by the TCFT. Following the convention of [4] we call a twist with different signs, i.e., $(\varepsilon, \tilde{\varepsilon}) = (\pm 1, \mp 1)$, an A-twist. The twist with equal signs, $(\varepsilon, \tilde{\varepsilon}) = (\pm 1, \pm 1)$, is refereed to as B-twist. In table (5.1) we give a list of the correspondence between twisting and chiral primary operators that appear as physical operators in the topological theory. We see that the A-twist projects on the twisted (anti)chiral sector and the B-twist on the (anti)chiral sector.

5.2 Compatibility with boundary conditions

The type of topological twisting puts, moreover, restrictions on the possible boundary conditions that we can consider in the $\mathcal{N} = (2,2)$ theory. Let us first consider the B-twist. Assume that we have A-type boundary conditions (3.36), then $G(z)$ is related to $\tilde{\bar{G}}(\bar{z})$. However, by the topological B-twist with $(+1,+1)$ these currents are mapped to $Q(z)$ resp. $\tilde{G}(\bar{z})$, which have different conformal weights and, therefore, cannot be related anymore. On the other hand, it is easy to see that the

	$(\varepsilon, \tilde{\varepsilon})$	phys. operators	bound. cond.	bound. states
$A-$twist	$(+1,-1)$	$c-a$	A	B
	$(-1,+1)$	$a-c$	A	B
$B-$twist	$(+1,+1)$	$c-c$	B	A
	$(-1,-1)$	$a-a$	B	A

Table 5.1: In view of the topological twisting one of the four sectors of (twisted) (anti)chiral operators is described by the topological field theory.

B-twist respects B-type boundary conditions. The same argument goes through for A- and B-type boundary conditions and the A-twist.

In section 3.3 we realized that A-brane boundary states contain Ishibashi states associated to the representations of $c-c$ or $a-a$ ring elements, and that B-brane boundary states contain Ishibashi states for $c-a$ and $a-c$ elements. Therefore, A-brane boundary states couple naturally to the B-twisted theory and vice versa. The considerations of this paragraph lead to the following conclusions (see table (5.1)):

A topological conformal field theory, arising from an $A(B)$-twist of a $\mathcal{N}=(2,2)$ SCFT, contains information about $A(B)$-branes in terms of boundary conditions and the spectrum of (anti)chiral boundary operators; and it provides information about $B(A)$-brane in terms of boundary states.

5.3 What do topological amplitudes compute in $\mathcal{N}=(2,2)$ theories?

In the following we sketch the effect of topological twisting on 3-point correlation functions on the sphere. The generalisation to arbitrary Riemann surfaces (possibly with boundaries) is worked out in [6, 8].

In section 5.1 we found that the topological twist is formally performed by adding a $U(1)$-current term to the stress-energy tensor:

$$T \to T + \varepsilon \frac{1}{2} \partial J \ .$$

Thinking in terms of the action we have to add a term, which couples the spin connection $\omega = \partial \log(\sqrt{g})$ on the world-sheet to the $U(1)$-current J, i.e.

$$\frac{\varepsilon}{8\pi} \int d^2 z \ (\omega \tilde{J} + \tilde{\omega} J) \ . \tag{5.5}$$

Upon variation of the metric this expression gives rise to the additional term for the stress-energy tensor. If we write J in terms of its bosonisation, $J = i\sqrt{c/3}\partial H$, and the Riemann tensor in terms of the spin connection, $R = \partial\bar{\partial}\log(\sqrt{g})$, expression (5.5) becomes:

$$-\frac{i\varepsilon}{4\pi}\sqrt{\frac{c}{12}}\int d^2z\sqrt{g}RH + c.c. \; . \tag{5.6}$$

By conformal invariance we can always choose a metric on the sphere such that the curvature is concentrated on two points, i.e. $R = 4\pi[\delta(z-z_1) + \delta(z-z_2)]$. Then, (5.6) gives rise to the insertion of the spectral flow operators

$$\Sigma_{-\varepsilon}(z_1)\Sigma_{-\varepsilon}(z_2) = e^{-\varepsilon i\sqrt{\frac{c}{12}}H(z_1)}e^{-\varepsilon i\sqrt{\frac{c}{12}}H(z_2)}$$

in $\mathcal{N} = (2,2)$ SCFT correlation functions. The topological correlation function can be expressed in terms of the SCFT correlation functions as follows:

$$\big\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\big\rangle_{TCFT} = \big\langle\phi_i(z_1)\Sigma_{-\varepsilon}(z_1)\;\phi_j(z_2)\Sigma_{-\varepsilon}(z_2)\;\phi_k(z_3)\big\rangle_{SCFT} \; . \tag{5.7}$$

Since the spectral flow operator is part of the generator of spacetime supersymmetry we see that topological amplitudes (5.7) compute Yukawa couplings between two fermions and a boson.[1]

Notice that for type II superstring compactifications to four dimensions we have to taken into account the external matter sector and the ghost sector in (5.7). A careful analysis was done in [6, 8] and basically shows that all the additional contributions cancel mutually, so that topological amplitudes indeed compute terms for the effective action.

5.4 B-twisted non-linear sigma models

We start with the non-linear sigma model on the sphere, so that we do not have any boundary effects. We perform the B-twist of the action (3.48), where the left- and right-moving sector is twisted with equal signs ε and $\tilde{\varepsilon}$. It is convenient to combine the fermionic fields in the following way [4]:

$$\eta^{\bar{\jmath}} = \psi_+^{\bar{\jmath}} + \psi_-^{\bar{\jmath}} \qquad \text{and} \qquad \theta_i = g_{i\bar{\jmath}}\left(\psi_+^{\bar{\jmath}} - \psi_-^{\bar{\jmath}}\right) \; .$$

The topological twist has the effect that $\eta^{\bar{\jmath}}$ is a section of the pull-back of the antiholomorphic cotangent bundle, $\Phi^*(T^{0,1}\mathcal{X})$, on the Calabi–Yau 3-fold, and θ_i is a section of $\Phi^*(T^{*1,0}(\mathcal{X}))$. On the other hand, $\rho^i_{\bar{z}} = \psi^i_-$ and $\rho^i_z = \psi^i_+$ are sections of $\bar{K}\otimes\Phi^*(T^{1,0}(\mathcal{X}))$ and $K\otimes\Phi^*(T^{1,0}(\mathcal{X}))$, respectively. K and $\bar{K}$ are the canonical and anticanonical line bundles on Σ.

[1] The term Yukawa coupling comes from heterotic string compactifications and is actually not applicable in type II superstring, because we have to take into account both left- and right-moving spectral flow.

The Q-symmetry is given by

$$Q = \bar{Q}_B = \bar{Q}_+ + \bar{Q}_- \ , \tag{5.8}$$

and the variations of the fields read

$$\begin{aligned} Q\phi^i &= 0 \ , & Q\phi^{\bar{\jmath}} &= \eta^{\bar{\jmath}} \ , \\ Q\rho^i &= d\phi^i \ , & Q\eta^{\bar{\jmath}} &= Q\theta_i = 0 \ . \end{aligned} \tag{5.9}$$

These transformations suggest the following map to geometrical objects:

$$\begin{aligned} Q &\rightarrow \bar{\partial} \ , \\ \eta^{\bar{\jmath}} &\rightarrow dZ^{\bar{\jmath}} \ , \\ \theta_i &\rightarrow \frac{\partial}{\partial Z^i} \ , \\ \phi^i &\rightarrow Z^i \ , \\ \phi^{\bar{\jmath}} &\rightarrow Z^{\bar{\jmath}} \ . \end{aligned} \tag{5.10}$$

Let us take the vector-valued form

$$\omega = \omega_{\bar{\jmath}_1 \ldots \bar{\jmath}_n}{}^{i_1 \ldots i_m} dZ^{\bar{\jmath}_1} \ldots dZ^{\bar{\jmath}_n} \frac{\partial}{\partial Z^{i_1}} \cdots \frac{\partial}{\partial Z^{i_m}} \ , \tag{5.11}$$

and associate to ω the operator

$$\mathcal{O}_\omega = \omega_{\bar{\jmath}_1 \ldots \bar{\jmath}_n}{}^{i_1 \ldots i_m} \eta^{\bar{\jmath}_1} \ldots \eta^{\bar{\jmath}_n} \theta_{i_1} \ldots \theta_{i_m} \ . \tag{5.12}$$

The Q-symmetry acts as

$$Q\mathcal{O}_\omega = \mathcal{O}_{\bar{\partial}\omega} \ .$$

The physical operators of the B-model are then in one-to-one correspondence with the Dolbeault cohomology of antiholomorphic n-forms taking values in $\bigwedge^m T^{1,0}$, i.e.,

$$\mathcal{H}_B^{(cl)} = \bigoplus_{n,m=0}^{3} H_{\bar{\partial}}^{0,n}(\wedge^m T^{1,0}) \ , \tag{5.13}$$

where the unique holomorphic 3-form on the Calabi–Yau 3-fold provides an isomorphism

$$H_{\bar{\partial}}^{0,n}(\wedge^m T^{1,0}) \cong H_{\bar{\partial}}^{3-m,n}(\mathcal{X}) \ .$$

A special cohomology class is provided by $H_{\bar{\partial}}^{0,1}(T^{1,0}) \cong H_{\bar{\partial}}^{2,1}(\mathcal{X})$, which corresponds to marginal deformations of the $\mathcal{N} = (2,2)$ SCFT. In geometric terms they give rise to complex structure deformations on $\mathcal{X}$; given a basis $\chi^a = \chi_{a;\bar{\jmath}}{}^i dZ^{\bar{\jmath}} \frac{\partial}{dlZ^i}$ of $H_{\bar{\partial}}^{0,1}(T^{1,0})$, the metric is deformed according to:

$$\delta g_{\bar{\imath}\bar{\jmath}} = \sum_{a=1}^{h^{2,1}} t^a \chi_{a;\bar{\imath}}{}^i g_{i\bar{\jmath}} \ . \tag{5.14}$$

We now turn to the action of the topological B-model. Writing (3.48) in terms of the twisted fields we obtain

$$S_B = \int_\Sigma d^2z \Big\{ -g_{i\bar{\jmath}}(\partial\phi^i\bar\partial\phi^{\bar{\jmath}} + \bar\partial\phi^i\partial\phi^{\bar{\jmath}}) + \\ + \frac{1}{2}g_{i\bar{\jmath}}\eta^{\bar{\jmath}}\overleftrightarrow{D}_{(z}\psi^i_{\bar{z})} - \frac{1}{2}g_{i\bar{\jmath}}\theta^{\bar{\jmath}}\overleftrightarrow{D}_{[z}\psi^i_{\bar{z}]} + \frac{1}{2}R_{i\bar{\imath}j\bar{\jmath}}\rho^i_z\rho^j_{\bar{z}}\eta^{\bar{\imath}}\theta_l g^{l\bar{\jmath}}\Big\} \ . \tag{5.15}$$

In fact, the whole action can be written as Q-exact term [66],

$$S_B = \int_\Sigma d^2z[Q,V] \ .$$

Since the topological theory is independent of Q-exact contributions in the action, we can introduce a parameter λ, so that $S_B \to \lambda \int [Q,V]$. From the bosonic part of the action (5.15) we find that in the limit of large λ the main contribution to the path integral comes from field configurations with

$$d\phi^i = 0 \ ,$$

so that the whole world-sheet is mapped to a point in the Calabi–Yau manifold and we do not get instanton corrections in the B-model. All computations descend to classical geometry on the Calabi–Yau 3-fold. In particular, the large volume expression,

$$\kappa_{abc} = \int_{\mathcal{X}} \Omega \wedge \chi^i_a \wedge \chi^j_b \wedge \chi^k_c \ \Omega_{ijk} \ , \tag{5.16}$$

for the Yukawa couplings of the B-model [7] is exact. Here, $\Omega \in H^{3,0}(\mathcal{X})$ denotes the unique covariantly constant holomorphic 3-form on $\mathcal{X}$.

On the boundary the action of the gauge field becomes

$$\int_{\partial\Sigma} d\phi^I A_I = \int_{\partial\Sigma} d\phi^{\bar{\jmath}} A_{\bar{\jmath}} + [Q, \int_{\partial\Sigma} \rho^i A_i] \ , \tag{5.17}$$

and, therefore, the $(0,1)$ part of the gauge field, $\bar{A} = A_{\bar{\jmath}}dZ^{\bar{\jmath}}$, twists the antiholomorphic Dolbeault operator:

$$\bar\nabla_{\bar{A}} = \bar\partial + \bar{A} \ . \tag{5.18}$$

In section 3.4 we found that the $(0,2)$ part of the field strength vanishes so that the twisted Dolbeault operator $\bar\nabla_{\bar{A}}$ defines a complex structure on a vector bundle over the holomorphic submanifold γ_B.

By the B-type boundary conditions (3.74) we find that the physical fields on the boundary lie in

$$\mathcal{H}^{(op)}_B = \bigoplus_{n=0}^{d_B} \bigoplus_{m=0}^{3-d_B} H^{0,n}_{\bar\nabla_{\bar{A}}}(\wedge^m N^{1,0}_{\gamma_B/\mathcal{X}}) \ ,$$

where d_B is the dimension of the holomorphic submanifold γ_B. The operator associated to an element ω in $H^{0,n}_{\bar\nabla_{\bar A}}(\wedge^m N^{1,0}_{\gamma_B/\mathcal{X}})$ is given by

$$\mathcal{O}_\omega = \omega_{\bar\jmath_1\ldots\bar\jmath_n}{}^{i_1\ldots i_m}\eta^{\bar\jmath_1}\ldots\eta^{\bar\jmath_n}\theta_{i_1}\ldots\theta_{i_m} \qquad \text{for} \quad 0\leq n\leq d_B,\ 0\leq m\leq 3-d_B.$$

There appear two types of physical fields, which correspond to marginal (but in general not exactly marginal) deformations in the $\mathcal{N}=2$ SCFT. A physical field $\bar a \in H^{0,1}_{\bar\nabla_{\bar A}}(\gamma_B)$ gives rise to deformations of the complex structure $\bar\nabla_{\bar A}$ of the holomorphic vector bundle over γ_B, whereas $X\in H^{0,0}_{\bar\nabla_{\bar A}}(N^{1,0}(\gamma_B))$ changes the position of the B-brane γ_B.

Let us consider a stack of N $D6$-branes wrapping the whole Calabi–Yau, which amounts to consider a holomorphic N-dimensional vector bundle over $\mathcal{X}$; we have only deformations of the complex structure on the bundle. The requirement that the deformed connection defines again a complex structure, i.e. $(\bar\nabla_{\bar A+\bar a})^2=0$, can be expressed in terms of the Maurer–Cartan equation:

$$\bar\nabla_{\bar A}\bar a+\bar a\wedge\bar a=0\ . \tag{5.19}$$

The string field theory for the open topological A-model with equation of motion (5.19) is described by the holomorphic Chern–Simons theory [37]:

$$S_{hCS}=\int_M\Omega\wedge\left(\frac{1}{2}\bar A\wedge\bar\partial\bar A+\frac{1}{3}\bar A\wedge\bar A\wedge\bar A\right)\ . \tag{5.20}$$

The generalisation to D-branes in lower-dimensional holomorphic submanifolds is done by dimensional reduction of the holomorphic Chern–Simons theory (5.20) [37, 70].

From the discussion of section 5.2 we know that despite of B-type D-branes described in terms of boundary conditions we have also A-type D-branes realized as boundary states. In the topological twisted model the physical bulk fields in $\mathcal{H}_B^{(cl)}$ couple to the RR sector Ishibashi states of the boundary state $\left|\,\alpha\,\right\rangle_B$, which correspond to $c-c$ (or $a-a$) ring elements via spectral flow. We write the RR part of the boundary state as

$$\left|\,\alpha_A\,\right\rangle_{RR}=\sum_i B^i(\alpha_A)\,\left|\,i\,\right\rangle\!\rangle_{RR}\ .$$

Picking a cohomology class $\omega_i\in H^{3-n,n}_{\bar\partial}(\mathcal{X})$, the coefficients for the RR sector Ishibashi states that correspond to $c-c$ elements are give by the amplitudes:

$$B^i(\alpha_A)=\eta^{ij}\left\langle\mathcal{O}_{\omega_i}\right|\left.\alpha_A\,\right\rangle_{RR}\ . \tag{5.21}$$

As shown in [20] using contour deformation arguments, these amplitudes depend on complex structure moduli but not on Kähler moduli. This implies that we can take the large volume limit and write the coefficients as period integrals over middle-dimensional Lagrangian submanifolds:

$$B_i(\alpha_A) = \int_{\alpha_A} \omega_i \ , \tag{5.22}$$

where ω_i for $i = 1, \ldots h^{2,1}$ is a basis for $H^{2,1}(\mathcal{X})$ and $\omega_0 \in H^{3,0}(\mathcal{X})$ is the unique holomorphic 3-form Ω. In particular, these periods do not obtain any corrections from world-sheet instantons. The coefficients (5.21) of the boundary state can be integrated with respect to t^i to a single function $B_0(\alpha_A)$:

$$B_i(\alpha_A) = \frac{\partial}{\partial t^i} B_0(\alpha_A) \ , \tag{5.23}$$

where we assumed that the t^i's are flat coordinates. Therefore, all the information of the chiral primary part of the boundary state (5.22) is encoded in the period

$$B_0(\alpha_A) = \int_{\alpha_A} \Omega \ , \tag{5.24}$$

where $\Omega \in H^{3,0}(\mathcal{X})$ is the unique holomorphic 3-form on the Calabi–Yau 3-fold.

5.5 A-twisted non-linear sigma models

We perform the A-twist of the action (3.48), where the left- and right-moving sectors are twisted with opposite signs ε and $\tilde{\varepsilon}$. The fermions ψ^i_+ and $\bar{\psi}^{\bar{\jmath}}_-$ become sections of the pull-back of the holomorphic and antiholomorphic tangent bundle of the Calabi–Yau 3-fold, $\Phi^*(T^{(1,0)})$ and $\Phi^*(T^{(0,1)})$, respectively. Following [4] we shall denote the twisted fields by $\chi^i := \psi^i_+$ and $\chi^{\bar{\jmath}} := \bar{\psi}^{\bar{\jmath}}_-$. The other fermions become one-forms on the world-sheet Σ and we denote them by $\psi^i_{\bar{z}} = \psi^i_- \in \Phi^*(T^{(1,0)}) \otimes \bar{K}$ and $\psi^{\bar{\jmath}}_z = \psi^{\bar{\jmath}}_+ \in \Phi^*(T^{(0,1)}) \otimes K$.

The Q-symmetry is given by the combination (3.58):

$$Q = Q_A = Q_+ + \bar{Q}_- \ , \tag{5.25}$$

and transformations of the fields under Q can easily be read off from (3.59). We obtain

$$\begin{aligned} Q\ \phi^i &= \chi^i \ , & Q\ \bar{\phi}^{\bar{\jmath}} &= \chi^{\bar{\jmath}} \ , \\ Q\ \chi^i &= 0 \ , & Q\ \psi^{\bar{\jmath}}_z &= 2\partial\bar{\phi}^{\bar{\jmath}} - \bar{\Gamma}^{\bar{\jmath}}{}_{\bar{l}\bar{m}}\chi^{\bar{l}}\psi^{\bar{m}}_z \ , \\ Q\ \psi^i_{\bar{z}} &= 2\bar{\partial}\phi^i - \Gamma^i{}_{jk}\chi^j\psi^k_{\bar{z}} \ , & Q\ \chi^{\bar{\jmath}} &= 0 \ . \end{aligned} \tag{5.26}$$

These transformation properties suggest the following map:

$$\begin{aligned} Q &\rightarrow d \ , \\ \chi^i &\rightarrow dZ^i \ , \\ \chi^{\bar{\jmath}} &\rightarrow dZ^{\bar{\jmath}} \ , \\ \phi^i &\rightarrow Z^i \ , \\ \phi^{\bar{\jmath}} &\rightarrow Z^{\bar{\jmath}} \ . \end{aligned} \tag{5.27}$$

Given an (n, m)-form,

$$\omega = \omega_{i_1 \ldots i_n \bar{\jmath}_1 \ldots \bar{\jmath}_m}(Z^i, Z^{\bar{\jmath}})\ dZ^{i_1} \ldots dZ^{i_n}\ dZ^{\bar{\jmath}_1} \ldots dZ^{\bar{\jmath}_n}\ ,$$

we find that Q acts on the associated local operator

$$\mathcal{O}_\omega = \omega_{i_1 \ldots i_n \bar{\jmath}_1 \ldots \bar{\jmath}_m}(\phi^i, \phi^{\bar{\jmath}})\ \chi^{i_1} \ldots \chi^{i_n}\ \chi^{\bar{\jmath}_1} \ldots \chi^{\bar{\jmath}_n}\ ,$$

as de Rham operator, i.e.,

$$[Q, \mathcal{O}_\omega] = \mathcal{O}_{d\omega}\ . \tag{5.28}$$

In particular, the map (5.27) gives a natural isomorphism between the de Rham cohomology on $\mathcal{X}$ and the space of physical fields of the topological model,

$$\mathcal{H}_A^{(cl)} \cong \bigoplus_{n,m=0}^{d} H_d^{n,m}(\mathcal{X})\ . \tag{5.29}$$

The degrees n and m of ω correspond to the left-, right-$U(1)$ charges of the operator $\mathcal{O}_\omega$.

A special case of cohomology classes is given by the Kähler class $H^{1,1}(\mathcal{X})$, which corresponds to marginal perturbations of the associated $\mathcal{N} = (2, 2)$ SCFT. On a Calabi–Yau 3-fold, $H^{1,1}(\mathcal{X})$ characterises the Kähler deformations of the Kähler metric, i.e.,

$$\delta g_{i\bar{\jmath}} = \sum_{a=1}^{h^{1,1}} t^a \omega_{a;i\bar{\jmath}}\ . \tag{5.30}$$

In terms of the twisted fields the Lagrangian (3.48) can be written as

$$\begin{aligned} S_A = \int_\Sigma d^2z \Big\{ &- g_{i\bar{\jmath}}(\partial\phi^i \bar{\partial}\phi^{\bar{\jmath}} + \bar{\partial}\phi^i \partial\phi^{\bar{\jmath}}) + \\ &+ \frac{1}{2} g_{i\bar{\jmath}} \chi^{\bar{\jmath}} \overleftrightarrow{D}_z \psi^i_{\bar{z}} + \frac{1}{2} g_{i\bar{\jmath}} \psi^{\bar{\jmath}}_z \overleftrightarrow{D}_{\bar{z}} \chi^i - R_{i\bar{\imath}j\bar{\jmath}} \chi^i \chi^{\bar{\imath}} \psi^j_{\bar{z}} \psi^{\bar{\jmath}}_z \Big\}\ . \end{aligned} \tag{5.31}$$

A key feature of the A-twisted non-linear sigma model is that the action cannot totally be written as Q-exact term, rather we have [4, 12]

$$S_A = \int_\Sigma d^2z [Q, V] + \int_\Sigma \Phi^*(\omega)\ , \tag{5.32}$$

where

$$\int_\Sigma \Phi^*(\omega) = \frac{1}{2} \int_\Sigma d^2z\ g_{i\bar{\jmath}}\ (\partial\phi^i \bar{\partial}\phi^{\bar{\jmath}} - \bar{\partial}\phi^i \partial\phi^{\bar{\jmath}}) \tag{5.33}$$

is the integral of the pull-back of the Kähler form to the world-sheet Σ. This integral depends only on the cohomology class of the Kähler form and on the homotopy class of the map Φ. The Hodge number $h^{1,1}$ on a Calabi–Yau 3-fold gives the number of Kähler deformations $t_i \in \mathbb{R}$ of the metric g and upon proper normalisation, the contribution from (5.33) is given by

$$\int_\Sigma \Phi^*(\omega) = 2\pi \sum_{i=1}^{h^{1,1}} n_i t_i \tag{5.34}$$

where the integers n_i are the instanton numbers of the map Φ, which label the homotopy class. An inclusion of the B-field leads merely to a complexification of the Kähler parameters, so that $t^i \in \mathbb{C}$. In correlation functions corresponding to the instanton numbers $\{n_i\}$ the term (5.34) gives rise to

$$\frac{e^{-2\pi \sum_i n_i t_i}}{1 - e^{-2\pi \sum_i n_i t_i}} , \tag{5.35}$$

where the geometric series comes from multiple instanton contributions.

A key point in the evaluation of the path integral is the fact that in correlation functions nothing depends on Q-exact terms and we can multiply the Q-exact term by a parameter λ, i.e., $S_A \to \lambda[Q, V] + \int \Phi^*(\omega)$, and take the limit $\lambda \to \infty$. A comparison of the action (5.31) with the splitting (5.32) shows that the bosonic part of the Q-exact term is given by $\int ||\bar{\partial}\phi||^2$ and, therefore, the path integral localises at the moduli space $\mathcal{M}_{n_i}$ of holomorphic disk instantons, i.e. on maps obeying

$$\partial \phi^{\bar{\jmath}} = \bar{\partial} \phi^i = 0 , \tag{5.36}$$

and carrying instanton numbers $\{n_i\}$. In the large volume limit $t_i \to \infty$ only the instanton with $n_i = 0$ contributes, as we see from the instanton sum (5.35). This instanton is homotopy equivalent to the point and, therefore, the main contribution in the large radius limit comes from the sphere mapped to a point in the Calabi–Yau 3-fold.

The Yukawa coupling in the large radius limit is given as intersection number of homology classes [7] that are associated to the Kähler deformations ω_a; explicity the Yukawa coupling reads

$$\kappa_{abc}^{\{0\}} = \int_{\mathcal{X}} \omega_a \wedge \omega_b \wedge \omega_c . \tag{5.37}$$

And the total coupling including instanton corrections becomes [11, 12]

$$\kappa_{abc} = \kappa_{abc}^{\{0\}} + \sum_{n_1,\ldots,n_{h^{1,1}}=0}^{\infty} n_a n_b n_c \kappa^{\{n_i\}} \frac{e^{-2\pi \sum_i n_i t_i}}{1 - e^{-2\pi \sum_i n_i t_i}} , \tag{5.38}$$

where the leading term for large t_i is (5.37) and the numbers $\kappa^{\{n_i\}}$ are integers and count the number of holomorphic instantons of degree $\{n_i\}$ [12, 71].

The boundary theory of the topological A-model is by far less well understood and we will give only some results on the space of physical boundary operators,

$\mathcal{H}_A^{(op)}$, and on the topological field theory that describes the A-model on Lagrangian submanifolds γ_A.

We found in section 3.4 that the A-type boundary conditions require a flat gauge connection on the boundary. According to [37] the Q-symmetry is then twisted by this gauge potential so that it can be realized as

$$\nabla_A = d + A \ . \tag{5.39}$$

Choosing coordinates so that χ^I points in the tangent direction of the Lagrangian submanifold the physical fields can be written as

$$a = a_{I_1 \dots I_n} \chi^{I_1} \dots \chi^{I_n} \qquad \text{for} \quad 0 \leq n \leq 3 \ , \tag{5.40}$$

and are in one-to-one correspondence with the cohomology of ∇_A, i.e.,

$$\mathcal{H}_A^{(op)} = \bigoplus_{n=0}^{3} H^m_{\nabla_A}(\gamma_A) \ . \tag{5.41}$$

A special type of physical fields is given by the cohomology $H^1_{\nabla_A}(\gamma_A)$, which corresponds to infinitesimal deformations of the gauge connection ∇_A. These deformations correspond to marginal but, in general, *not exactly* marginal deformations of the related $\mathcal{N} = 2$ SCFT. In order to provide a finite deformation of the flat connection ∇_A, the deformed connection ∇_{A+a} must again be flat:

$$(\nabla_{A+a})^2 = 0 \ .$$

Using the flatness of the original connection we derive the Maurer–Cartan equation

$$\nabla_A a + a \wedge a = 0 \ , \tag{5.42}$$

which obstructs the finite perturbations of the gauge connection.

In [37] Witten constructed the open string field theory for the flat gauge connection on a specific class of non-compact Calabi–Yau 3-folds. Taking a real 3-dimensional manifold M, the Calabi–Yau $\mathcal{X}$ is given by M together with its cotangent bundle T^*M. Then M can be shown to be a Lagrangian submanifold of $\mathcal{X}$. The string field theory for the $U(N)$ open string mode $A = A_I dx^I$, where x are real coordinates on M, turns out to be Chern-Simons gauge theory on M:

$$S_{CS} = \frac{1}{g_s} \int_M \left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right) \ . \tag{5.43}$$

On a general Calabi–Yau 3-fold $\mathcal{X}$ with several Lagrangian cycles M_i the whole open string sector of the A-model is, however, much more complicated [37], which is due to the existence of world-sheet instantons that can *(i)* wrap 2-cycles of $\mathcal{X}$ and *(ii)* end on different Lagrangian cycles M_i.

As mentioned in section 5.2 the A-twisted topological model contains information about B-type boundary states (3.41). The physical bulk fields in $\mathcal{H}_A^{(cl)}$ couple to RR sector Ishibashi states of the boundary state $|\,\alpha\,\rangle_B$, which correspond to $c-a$ (or $a-c$) ring elements via spectral flow. We write

$$|\,\alpha_B\,\rangle_{RR} = \sum_i B^i(\alpha_B)\;|\,i\,\rangle\!\rangle_{RR}\ .$$

Let $\omega_i \in H_d^{n,m}(\mathcal{X})$, then the coefficients for the Ishibashi states that correspond to $c-a$ elements are give by the amplitudes:

$$B^i(\alpha_B) = \eta^{ij}\langle \mathcal{O}_{\omega_i}|\,\alpha_B\,\rangle_{RR}\ . \tag{5.44}$$

As shown in [20] using contour deformation arguments, these amplitudes depend on the Kähler moduli but not on the complex structure moduli.

We found in section 3.4 that in the large volume limit B-type D-branes on Calabi–Yau 3-folds are given by holomophic submanifolds of complex dimension d_B. Therefore, like for the A-type boundary states discussed in the previous section we can write $B_0(\alpha_B)$ as period integral [20],[2]

$$B_0(\alpha_B) = \int_{\alpha_B}\omega^{d_B} + \mathcal{O}(e^{2\pi i t})\ , \tag{5.45}$$

where d_B is the dimension of the cycle α_B. But this time $B_0(\alpha_B)$ receives world-sheet instanton corrections. In (5.45) $\omega = \sum_i t^i\omega_i$ is the Kähler form on $\mathcal{X}$ and the ω_i's form a basis for $H^{1,1}(\mathcal{X})$. Finally, all the coefficients $B_i(\alpha_B)$ can be obtained by differentiation:

$$B_i(\alpha_B) = \frac{\partial}{\partial t^i}B_0(\alpha_B)\ ,$$

where the t^i's are again flat coordinates.

5.6 B-twisted Landau–Ginzburg models

Now, we consider the last example for a topological twisted SCFT, the topological Landau–Ginzburg models, which come from twisting the Landau–Ginzburg models of section 3.5. The field content of the bulk theory is the same as in the B-twisted non-linear sigma model. The difference is encoded in the transformation properties of the fields. From (3.73) and (3.78) we obtain

$$\begin{aligned} Q\phi^i &= 0\ , \\ Q\rho^i &= d\phi^i\ , \end{aligned} \qquad \begin{aligned} Q\phi^{\bar\jmath} &= \eta^{\bar\jmath}\ , \\ Q\eta^{\bar\jmath} &= 0\ , \\ Q\theta_i &= \partial_i W\ . \end{aligned} \tag{5.46}$$

[2] Note that this expression is only true for a vanishing gauge field on the B-brane.

These transformations suggest the identifications (5.10) with the exception that the Q-operator is twisted by a potential term:

$$Q \to \bar{\partial} + i_{\partial W} \ . \tag{5.47}$$

Because of the relations $\bar{\partial}^2 = \{\bar{\partial}, i_{\partial W}\} = i_{\partial W}^2 = 0$ the physical fields are determined by the double cohomology:

$$\mathcal{H}_{LG}^{(cl)} = \bigoplus_{n,m} H_{\bar{\partial}}(H_{i_{\partial W}}(\Omega^{0,n} \otimes \Lambda^m T^{1,0})) \ . \tag{5.48}$$

In the following we are interested in theories, where the only non-trivial part of $\mathcal{H}_{LG}^{(cl)}$ comes from $(n,m) = (0,0)$. The space of physical operators turns out to be the polynomial ring $\mathbb{C}[\phi^i]$ modulo $\partial_i W = 0$ for $i = 1, \dots, N$, i.e.:

$$\mathcal{H}_{LG}^{(cl)} = \frac{\mathbb{C}[\phi^i]}{\{\partial_i W\}} \ . \tag{5.49}$$

Analogous to the 3-point functions in the twisted non-linear sigma models we can ask now the question how to compute correlation functions of physical fields in this theory. Taking a basis $p_j(\phi) \in \mathcal{H}_{LG}^{(cl)}$ one can show [67] by localisation of the path integral that the correlation function of physical bulk fields is given by the residue formula:

$$C_{ijk} = \langle p_i(\phi)p_j(\phi)p_k(\phi)\rangle = \frac{1}{(2\pi i)^N} \oint_C d^N\phi \, \frac{p_i(\phi)p_j(\phi)p_k(\phi)}{\partial_1 W \dots \partial_N W} \ . \tag{5.50}$$

We have seen in section 5.6 that there are additional boundary degrees of freedom from the boundary fermions Π^A. In the following we investigate which physical operators can be associated with them. The supersymmetry transformations (3.91) give rise to the Q-symmetry transformations:

$$\begin{aligned} Q\pi^A &= -E^A(\phi) \ , \\ Q\bar{\pi}^A &= iJ^A(\phi) \ . \end{aligned} \tag{5.51}$$

Canonical quantisation of the boundary fermions gives the anticommutation relations:

$$\begin{aligned} \{\pi^A, \bar{\pi}^B\} &= \delta^{AB} \ , \\ \{\pi^A, \pi^B\} &= 0 = \{\bar{\pi}^A, \bar{\pi}^B\} \ . \end{aligned} \tag{5.52}$$

This algebra can be realized as 2^N-dimensional Clifford algebra representation. In this representation the Q-symmetry acting on the boundary fields is expressed through the matrix

$$Q_B = \sum_A \left(iJ^A\pi^A - E^A\bar{\pi}^A\right) \ , \tag{5.53}$$

which, in view of (3.93), satisfies the relation[3]

$$(Q_B)^2 = -iW \ . \tag{5.54}$$

[3]Here and in the following we set the integration constant in (3.93) to zero.

We write an element in the space of boundary operators $\mathcal{H}^{(op)}$ as polynomials in π and $\bar{\pi}$ with coefficients in $\mathbb{C}[\phi^i]$, i.e.:

$$\Psi(\phi, \pi, \bar{\pi}) = \sum_{i,j=0}^{N} f(\phi)_{A_1 \dots A_i B_1 \dots B_j} \pi^{A_1} \dots \pi^{A_i} \bar{\pi}^{B_1} \dots \bar{\pi}^{B_j} \ . \tag{5.55}$$

Then the Q-symmetry acts in terms of Q_B through

$$Q\Psi = [Q_B, \Psi] \ ,$$

where $[\cdot, \cdot]$ is the graded commutator. The space of boundary operators inherits a natural $\mathbb{Z}_2$ grading from the boundary fermions. The physical fields at the boundary are then given by the cohomology $H_{Q_B}(\mathcal{H}^{(op)})$.

As in the bulk theory we can use a localisation argument [58, 72] in order to derive a formula for the correlation functions of physical fields. Picking a basis of bulk and boundary physical fields, $p_i(\phi)$ resp. $\psi_a(\phi, \pi, \bar{\pi})$, the result is:

$$B_{ia} = \langle p_i \psi_a \rangle = \frac{1}{(2\pi i)^N} \oint_C d^N\phi \ \frac{p_i(\phi) \ STr((\partial Q_B)^{\wedge N} \ \psi_a)}{\partial_1 W \dots \partial_N W} \ , \tag{5.56}$$

where STr is a supertrace and

$$(\partial Q_B)^{\wedge N} := \frac{1}{n!} \sum_{\sigma \in S_N} (-)^{|\sigma|} \partial_{\sigma(1)} Q_B \dots \partial_{\sigma(N)} Q_B \ .$$

The representation (5.53) of the Q-symmetry plays also a quite important role for the expression of the topologically twisted action. Namely, the action reads

$$S_{LG} = -\frac{1}{2} \left(\int_\Sigma (W)^{(1,1)} + \int_{\partial\Sigma} (Q_B)^{(1)} \right) + [Q, \cdot] \ , \tag{5.57}$$

where we inherited the notation for the topological descendants from chapter 4. From this action we see immediately that a perturbation by a boundary physical field Ψ, i.e.,

$$\delta S = \int_{\partial\Sigma} \Psi^{(1)} \ ,$$

is an infinitesimal deformation $\delta Q_B = \Psi$ of the boundary Q-symmetry. If we consider finite perturbations, the deformed Q-symmetry has to satisfy the relation

$$(Q_B + \Psi)^2 = -iW \ .$$

Using (5.54) it can be rewritten as Maurer–Cartan equation:

$$[Q_B, \Psi] + \Psi^2 = 0 \ . \tag{5.58}$$

In the following we introduce a slight generalisation of the above construction for the boundary theory and give a categorical description thereof. Rather than considering a matrix representations of boundary operators, which descends from boundary fermions π^A and thus has rank 2^N, we consider general matrix factorisations[4]

$$W\mathbb{1} = (Q_B)^2 \ , \tag{5.59}$$

where Q_B is now a matrix of rank $2r$, and Q_B has only off-diagonal entries, i.e.,

$$Q_B = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} \ .$$

Here, J and E are rank r matrices. Such an off-diagonal representation for Q_B can always be found because of the existence of a linear involution, $\sigma : \mathcal{H}^{(op)} \to \mathcal{H}^{(op)}$, $\sigma^2 = 1$. Its eigenvalues $+1$ and -1 correspond to the $\mathbb{Z}_2$ grading of $\mathcal{H}^{(op)}$, which implies that it anticommutes with Q_B. In the diagonal representation,

$$\sigma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \ ,$$

Q_B is therefore off-diagonal.

Triangulated category

Let us summerise the structure of D-branes in topological Landau–Ginzburg models in terms of a triangulated additive category DG_W, which has the following data [47, 59, 60, 69].

The objects of DG_W, which represent D-branes, are given by matrix factorisations (5.59) of W, viewed as pairs

$$\underline{P} := (\ P_1 \underset{J_P}{\overset{E_P}{\rightleftarrows}} P_0\ )$$

of free $\mathbb{C}[\phi^i]$-modules. We denote the associated Q-symmetry by

$$D_P = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} \ .$$

The space of morphisms of DG_W is defined by

$$Hom_{DG_W}(\underline{M}, \underline{N}) := Hom(M_1 \oplus M_0, N_1 \oplus N_0) = \bigoplus_{i,j=0,1} Hom(M_i, N_j) \ , \tag{5.60}$$

where $Hom(M_i, N_j)$ are spaces of definite grading $i - j \bmod 2$. Morphisms in $Hom_{even}(\underline{M}, \underline{N}) = Hom(M_1, N_1) \oplus Hom(M_0, N_0)$ of even grade and morphisms in

[4] In the following we leave out the factor $(-i)$ in front of W in equation (5.54), which is just a matter of convention.

$Hom_{odd}(\underline{M},\underline{N}) = Hom(M_1, N_0) \oplus Hom(M_0, N_1)$ of odd grade can be resembled into matrices

$$f^{MN} := \begin{pmatrix} f_0 & 0 \\ 0 & f_1 \end{pmatrix} , \qquad \text{and} \quad g^{MN} := \begin{pmatrix} 0 & g_0 \\ g_1 & 0 \end{pmatrix} , \tag{5.61}$$

respectively. The differential of DG_W acts then through

$$D(f^{MN}) := D_M \circ f^{MN} - (-)^{|f|} f^{MN} \circ D_N \ , \tag{5.62}$$

where $|f|$ denotes the $\mathbb{Z}_2$-grade of f^{MN}.

The *derived category* (of physical fields) DB_W has the same objects as DG_W, but the morphism are elements of

$$Hom_{DB_W}(\underline{M},\underline{N}) = H_D(Hom_{DG_W}(\underline{M},\underline{N})).$$

The proof that this category is triangulated was provided in [69]. In fact, we have the following properties: DG_W provides a shift functor $\underline{P} \to \underline{P}[1]$, which maps D-branes to their anti-branes:

$$\underline{P}[1] := (\ P_0 \ \underset{-E_P}{\overset{-J_P}{\rightleftarrows}} \ P_1 \ ) \ . \tag{5.63}$$

Every odd morphism $\underline{M} \xrightarrow{f} \underline{N}$ can be completed to a distinguished triangle:

$$\underline{M} \xrightarrow{f} \underline{N} \to Cone(f) \to \underline{M}[1] \ , \tag{5.64}$$

where the cone of the triangle is defined by

$$Cone(f) := (\ M_1 \oplus N_1 \ \underset{J}{\overset{E}{\rightleftarrows}} \ M_0 \oplus N_0 \ ) \tag{5.65}$$

$$J = \begin{pmatrix} J_M & f_0 \\ 0 & J_N \end{pmatrix} \qquad \text{and} \quad E = \begin{pmatrix} E_M & f_1 \\ 0 & E_N \end{pmatrix} \ . \tag{5.66}$$

The physical meaning of this triangle is that upon turning on a tachyonic field (morphism) f between any two D-branes (objects) in such a triangle, these two D-branes can form a bound state, which is represented by the third object. For example, if we turn on the tachyonic field in the triangle (5.64) the D-branes $\underline{M}$ and $\underline{N}$ form a bound state isomorphic to $Cone(f)$.

A-series Landau–Ginzburg minimal model

Landau–Ginzburg minimal models of the A-series are described in terms of a superpotential $W(\phi)$ of degree $k+2$ in one chiral superfield ϕ. The topological D-branes are determined by the possible factorizations (3.93) of the bulk superpotential. In the following, we will use the symbol (ℓ) to label the various possible choices for $J(\phi)$ and $E(\phi)$, and study for any given such choice the topological open string spectrum. Here, ℓ is related to the polynomial degree of $J(\phi)$ and $E(\phi)$, more precisely: $\deg(J) = \ell + 1$ and $\deg(E) = k - \ell + 1$. We will determine both the spectrum of boundary preserving and boundary changing operators of a generically perturbed LG model with one variable. For the special case of the unperturbed, i.e. superconformal A-series minimal models, we will compare the spectrum obtained from the Landau-Ginzburg formulation with the spectrum one gets using BCFT techniques [73].

Recall that the chiral ring $\mathcal{R}$ of the bulk theory (5.49) is determined [1] in terms of the superpotential $W(\phi)$. The ring may be represented by polynomials Φ_i in ϕ with degrees $\deg \Phi_i = i$ equal to or less than k:

$$\{\Phi_i\} = \{1, \Phi_1(\phi), ..., \Phi_k(\phi)\} \ . \tag{5.67}$$

The Q-cohomology classes of the topological sector on the boundary can be extracted from the transformation properties

$$Q\pi \ = \ E^{(\ell)}(\phi) \, , \qquad Q\bar{\pi} \ = \ -iJ^{(\ell)}(\phi) \, . \tag{5.68}$$

Therefore, on the boundary the chiral ring $\mathcal{R}_B^{(\ell)}$ is truncated earlier than in the bulk, since it consists of polynomials modulo $J^{(\ell)}(\phi)$ and $E^{(\ell)}(\phi)$. In the generic case, when $J^{(\ell)}(\phi)$ and $E^{(\ell)}(\phi)$ have no common divisor, the Q-cohomology is empty and all topological boundary amplitudes vanish. The interesting case is when the boundary potentials have a common factor, so that we can write

$$J^{(\ell)}(\phi) \ = \ q^{(\ell)}(\phi)G^{(\ell)}(\phi) \, , \qquad E^{(\ell)}(\phi) \ = \ p^{(\ell)}(\phi)G^{(\ell)}(\phi) \, . \tag{5.69}$$

Here $G^{(\ell)}(\phi)$ is the greatest common divisor of $J^{(\ell)}(\phi)$ and $E^{(\ell)}(\phi)$; if it is non-trivial, the bosonic part of the boundary ring is given by the polynomials in ϕ modulo truncation by $G^{(\ell)}(\phi)$.

In contrast to the bulk, the chiral ring at the boundary also contains fermionic fields, since we can construct the following Q-closed field out of the boundary fermions π and $\bar{\pi}$:

$$\omega^{(\ell\ell)} \ = \ \sqrt{i}(q^{(\ell)}(\phi)\pi - ip^{(\ell)}(\phi)\bar{\pi}) \ . \tag{5.70}$$

Here the labels indicate that $\omega^{(\ell\ell)}$ is a boundary preserving operator, but we will sometimes omit these labels for notational simplicity. $\omega^{(\ell\ell)}$ satisfies an algebraic relation, which is determined by the canonical anticommutation relations (5.52). One immediately obtains [47]:

$$\left(\omega^{(\ell\ell)}\right)^2 = p^{(\ell)}(\phi)q^{(\ell)}(\phi) \, . \tag{5.71}$$

The chiral ring $\mathcal{R}_B^{(\ell)}$ in the boundary sector (ℓ) is thus given by the polynomial ring generated by ϕ and $\omega^{(\ell\ell)}$, modulo $G^{(\ell)}(\phi)$:

$$\mathcal{R}_B^{(\ell)} \ = \ \frac{\mathbb{C}[\phi, \omega^{(\ell\ell)}]}{\{G^{(\ell)}(\phi), (\omega^{(\ell\ell)})^2 - p^{(\ell)}(\phi)q^{(\ell)}(\phi)\}} \ . \tag{5.72}$$

The number of elements of the open string chiral ring is controlled by the polynomial degree $d_\ell = \deg(G^{(\ell)})$. In total we have d_ℓ bosonic fields and d_ℓ fermionic fields in the boundary preserving sector. In order to fix notation, let us denote these fields by $\Psi_\alpha^{(\ell\ell)}$, where α labels bosonic and fermionic sub-sectors in an obvious manner: $\alpha \equiv (a, \sigma)$ and $a = 0, 1, \ldots, d_\ell - 1$, $\sigma = 0, 1$. We can thus write a basis of $\mathcal{R}_B^{(\ell)}$ as

$$\begin{aligned} \{\Psi_{(a,0)}^{(\ell\ell)}\} &= \{1, \Psi_1(\phi), ..., \Psi_{d_\ell-1}(\phi)\} \ , \\ \{\Psi_{(a,1)}^{(\ell\ell)}\} &= \left\{\omega^{(\ell\ell)}, \omega^{(\ell\ell)}\Psi_1(\phi), ..., \omega^{(\ell\ell)}\Psi_{d_\ell-1}(\phi)\right\} \ , \end{aligned} \tag{5.73}$$

where $\Psi_a(\phi)$ are polynomials in ϕ of degree $\deg(\Psi_a) = a$, which will in general be different from the bulk ring polynomials $\Phi_i(\phi)$ in (5.67).

In order to determine the spectrum and the chiral ring $\mathcal{R}_B^{(\ell_1\ell_2)}$ for boundary changing fields, we can proceed in a similar way as above. First, the action of the supercharge Q_B on the boundary fields in the sector $(\ell_1\ell_2)$ can consistently be defined as

$$[Q_B, \Psi_\alpha^{(\ell_1\ell_2)}] \equiv Q_B^{(\ell_1)}\ \Psi_\alpha^{(\ell_1\ell_2)} - (-)^{|\alpha|}\ \Psi_\alpha^{(\ell_1\ell_2)}\ Q_B^{(\ell_2)} \ . \tag{5.74}$$

Then we realize that the canonical commutation relations (5.52) for π and $\bar\pi$ are universal for all boundary conditions, i.e., they do not depend on the polynomials $J^{(\ell)}$ and $E^{(\ell)}$. In fact, the supercharge

$$Q_B^{(\ell)} = iJ^{(\ell)}\pi - E^{(\ell)}\bar\pi \ ,$$

contains all the information on the boundary condition (ℓ). This implies that we can use the universality of (5.52) to construct the boundary changing operators in terms of polynomials of ϕ, π and $\bar\pi$.

We thus make the ansatz $\omega^{(\ell\ell')} = \rho(\phi)\pi + \sigma(\phi)\bar\pi$ for the fermionic boundary changing operators and determine the Q-cohomology using (5.74). In order to do so, it is convenient to define the following factorisations

$$\begin{aligned} E^{(\ell_1)} &= \hat p^{(\ell_1)} \cdot \gcd\{J^{(\ell_2)}, E^{(\ell_1)}\} \ , & E^{(\ell_2)} &= \hat p^{(\ell_2)} \cdot \gcd\{J^{(\ell_1)}, E^{(\ell_2)}\} \ , \\ J^{(\ell_2)} &= \hat q^{(\ell_2)} \cdot \gcd\{J^{(\ell_2)}, E^{(\ell_1)}\} \ , & J^{(\ell_1)} &= \hat q^{(\ell_1)} \cdot \gcd\{J^{(\ell_1)}, E^{(\ell_2)}\} \ . \end{aligned} \tag{5.75}$$

When computing the Q-cohomology we observe that $E^{(\ell_1)}J^{(\ell_1)} = E^{(\ell_2)}J^{(\ell_2)}$, which implies that in order to obtain nontrivial cohomology classes, the constant in (3.93) must be the same for the boundary sectors (ℓ_1) and (ℓ_2). Moreover, from (5.75) we find that $\hat p^{(\ell_1)}\hat q^{(\ell_1)} = \hat p^{(\ell_2)}\hat q^{(\ell_2)}$. It turns out that there occur two kinds of fermionic solutions for the Q-cohomology classes, i.e.,

$$\begin{aligned} \omega_{qp}^{(\ell_1\ell_2)}\ \Psi_{qp} &= \sqrt{i}(\hat q^{(\ell_1)}\pi - i\hat p^{(\ell_2)}\bar\pi)\ \Psi_{qp} \ , \\ \omega_{pq}^{(\ell_1\ell_2)}\ \Psi_{pq} &= \sqrt{i}(\hat q^{(\ell_2)}\pi - i\hat p^{(\ell_1)}\bar\pi)\ \Psi_{pq} \ , \end{aligned} \tag{5.76}$$

where Ψ_{qp} and Ψ_{pq} are polynomials of ϕ modulo $\gcd\{J^{(\ell_1)}, E^{(\ell_2)}\}$ and $\gcd\{J^{(\ell_2)}, E^{(\ell_1)}\}$, respectively. The solutions (5.76) are not completely independent but rather satisfy the relations

$$\begin{aligned} \hat p^{(\ell_1)}\ \omega_{qp}^{(\ell_1\ell_2)} &= \hat p^{(\ell_2)}\ \omega_{pq}^{(\ell_1\ell_2)} \ , \\ \hat q^{(\ell_2)}\ \omega_{qp}^{(\ell_1\ell_2)} &= \hat q^{(\ell_1)}\ \omega_{pq}^{(\ell_1\ell_2)} \ , \end{aligned} \tag{5.77}$$

where it is clear that common divisors could be divided out. In a similar way we make the ansatz $\beta^{(\ell\ell')} = \rho(\phi)\pi\bar{\pi} + \sigma(\phi)\bar{\pi}\pi$ for the bosonic boundary changing operators. We define the following factorisations appropriate for this case:

$$\begin{aligned} E^{(\ell_1)} &= e^{(\ell_1)} \cdot \gcd\{E^{(\ell_1)}, E^{(\ell_2)}\}\,, & J^{(\ell_2)} &= j^{(\ell_2)} \cdot \gcd\{J^{(\ell_1)}, J^{(\ell_2)}\}\,, \\ E^{(\ell_2)} &= e^{(\ell_2)} \cdot \gcd\{E^{(\ell_1)}, E^{(\ell_2)}\}\,, & J^{(\ell_1)} &= j^{(\ell_1)} \cdot \gcd\{J^{(\ell_1)}, J^{(\ell_2)}\}\,, \end{aligned} \tag{5.78}$$

which imply $e^{(\ell_1)}j^{(\ell_1)} = e^{(\ell_2)}j^{(\ell_2)}$. Likewise, there exist two kinds of solutions for the boundary changing bosons, which can be written as

$$\begin{aligned} \beta_j^{(\ell_1\ell_2)}\,\Psi_j &= \left(j^{(\ell_1)}\pi\bar{\pi} + j^{(\ell_2)}\bar{\pi}\pi\right)\Psi_j\,, \\ \beta_e^{(\ell_1\ell_2)}\,\Psi_e &= \left(e^{(\ell_2)}\pi\bar{\pi} + e^{(\ell_1)}\bar{\pi}\pi\right)\Psi_e\,, \end{aligned} \tag{5.79}$$

Ψ_j and Ψ_e being polynomials modulo $\gcd\{J^{(\ell_1)}, J^{(\ell_2)}\}$ and $\gcd\{E^{(\ell_1)}, E^{(\ell_2)}\}$, respectively. We have again relations between the solutions (5.79), namely

$$\begin{aligned} e^{(\ell_1)}\,\beta_j^{(\ell_1\ell_2)} &= j^{(\ell_2)}\,\beta_e^{(\ell_1\ell_2)}\,, \\ e^{(\ell_2)}\,\beta_j^{(\ell_1\ell_2)} &= j^{(\ell_1)}\,\beta_e^{(\ell_1\ell_2)}\,. \end{aligned} \tag{5.80}$$

Summarising, what we have found is, in contrast to the boundary preserving sector, that the spectrum “doubles” into two sets of bosonic and two sets of fermionic fields (at least for sufficiently generic perturbations). For a given sector $(\ell_1\ell_2)$ we can represent it in the following manner, modulo the relations (5.77) and (5.80):

$$\begin{aligned} \Psi_{(a,0)}^{(\ell_1\ell_2)} &= \left\{\beta_j^{(\ell_1\ell_2)}, \beta_j^{(\ell_1\ell_2)}\Psi_1(\phi), ..., \beta_j^{(\ell_1\ell_2)}\Psi_{d_j-1}(\phi)\right\}, \\ \Psi_{(a,2)}^{(\ell_1\ell_2)} &= \left\{\beta_e^{(\ell_1\ell_2)}, \beta_e^{(\ell_1\ell_2)}\Psi_1(\phi), ..., \beta_e^{(\ell_1\ell_2)}\Psi_{d_e-1}(\phi)\right\}, \\ \Psi_{(a,1)}^{(\ell_1\ell_2)} &= \left\{\omega_{qp}^{(\ell_1\ell_2)}, \omega_{qp}^{(\ell_1\ell_2)}\Psi_1(\phi), ..., \omega_{qp}^{(\ell_1\ell_2)}\Psi_{d_{qp}-1}(\phi)\right\}, \\ \Psi_{(a,3)}^{(\ell_1\ell_2)} &= \left\{\omega_{pq}^{(\ell_1\ell_2)}, \omega_{pq}^{(\ell_1\ell_2)}\Psi_1(\phi), ..., \omega_{pq}^{(\ell_1\ell_2)}\Psi_{d_{pq}-1}(\phi)\right\}, \end{aligned} \tag{5.81}$$

where the d's are the polynomial degrees of the respective divisors. In (5.81) we have extended the set of possible values of the index σ in the boundary changing sectors to $\{0, 1, 2, 3\}$, in order to account for the enlarged spectrum. Note that the actual spectrum for a given pair of factorisations is governed by which subsets of roots are common to which factors, and under specific circumstances, an example for which we will discuss momentarily, the basis (5.81) may collapse to a smaller one.

For the remainder of this section, let us discuss the unperturbed theory, which corresponds to the twisted $N = 2$ minimal model with homogenous superpotential of singularity type A_{k+1}:

$$W(\phi) = \frac{1}{k+2}\,\phi^{k+2}. \tag{5.82}$$

This theory has an unbroken $U(1)$ R-symmetry, and in order to maintain it on the boundary, we require $J(\phi)$ and $E(\phi)$ to be homogenous as well. Equation (3.93)

bosons	$q\,(k+2)$	fermions	$q\,(k+2)$
$\Psi^{(\ell\ell)}_{(0,0)} = 1$	0	$\Psi^{(\ell\ell)}_{(0,1)} = \omega$	$k-2\ell$
$\Psi^{(\ell\ell)}_{(1,0)} = \phi$	2	$\Psi^{(\ell\ell)}_{(1,1)} = \omega\phi$	$k-2\ell+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Psi^{(\ell\ell)}_{(n,0)} = \phi^n$	$2n$	$\Psi^{(\ell\ell)}_{(n,1)} = \omega\phi^n$	$k-2\ell+2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Psi^{(\ell\ell)}_{(\ell,0)} = \phi^\ell$	2ℓ	$\Psi^{(\ell\ell)}_{(\ell,1)} = \omega\phi^\ell$	k

Table 5.2: Elements of the boundary preserving chiral ring and their charges. They match precisely the open string states obtained from BCFT.

restricts the degrees of $J(\phi)$ and $E(\phi)$ to certain possibilities, and by an exchange of $\{\pi, E\}$ and $\{\bar{\pi}, -iJ\}$ we can always choose $deg(J) \leq deg(E)$. All-in-all we have the following possibilities:[5]

$$J^{(\ell)}(\phi) = \phi^{\ell+1}\ , \quad E^{(\ell)}(\phi) = \frac{1}{k+2}\phi^{k+1-\ell}\ , \quad \text{for } \ell \in \{0, 1, ..., [k/2]\}\ . \tag{5.83}$$

This indeed reproduces the set of B-type boundary labels in the rational boundary CFT of type A_{k+1}, as reviewed below. Moreover, we can also precisely match the spectrum of boundary fields for any given such boundary condition labelled by (ℓ). For this, recall that the charge of the bulk field ϕ is determined from the bulk potential, whereas the charge of the boundary fermion π follows from the boundary potential in (3.89), i.e., $q_\phi = -q_{\bar{\phi}} = \frac{2}{k+2}$ and $q_\pi = -q_{\bar{\pi}} = \frac{k-2\ell}{k+2}$ (we used here the fact that on the boundary the $U(1)$-charge is the sum of left and right charges in the bulk). Furthermore, the Q-closed fermion $\omega^{(\ell\ell)}$ takes the form

$$\omega^{(\ell\ell)} = \sqrt{i}(\pi - \frac{i}{k+2}\,\phi^{k-2\ell}\bar{\pi}), \tag{5.84}$$

and it has the same charge as π; it obviously satisfies the relation (5.71): $[\omega^{(\ell\ell)}]^2 = \frac{1}{k+2}\phi^{k-2\ell}$. Together with ϕ it generates the boundary chiral ring, and from $U(1)$ conservation we get that the natural basis is very simple: $\Psi_a(\phi) = \phi^a$, i.e.

$$\begin{aligned} \{\Psi^{(\ell\ell)}_{(a,0)}\} &= \left\{1, \phi, ..., \phi^\ell\right\}\ , \\ \{\Psi^{(\ell\ell)}_{(a,1)}\} &= \left\{\omega^{(\ell\ell)}, \omega^{(\ell\ell)}\phi, ..., \omega^{(\ell\ell)}\phi^\ell\right\}\ . \end{aligned} \tag{5.85}$$

In the boundary changing sector $(\ell_1\ell_2)$, the generators of the algebra read

[5] The choice $J(\phi) = 1$ and $E(\phi) = \frac{1}{k+2}\phi^{k+2}$ was excluded, because in that case a constant would already be Q-exact and all topological correlators would vanish.

bosons	$q\,(k+2)$	fermions	$q\,(k+2)$
$\Psi^{(\ell_1\ell_2)}_{(\Delta,0)} = \beta^{(\ell_1\ell_2)}$	2Δ	$\Psi^{(\ell_1\ell_2)}_{(\Delta,1)} = \omega^{(\ell_1\ell_2)}$	$k-2\,\bar{\ell}$
$\Psi^{(\ell_1\ell_2)}_{(\Delta+1,0)} = \beta^{(\ell_1\ell_2)}\phi$	$2(\Delta+1)$	$\Psi^{(\ell_1\ell_2)}_{(\Delta+1,1)} = \omega^{(\ell_1\ell_2)}\phi$	$k-2(\bar{\ell}-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Psi^{(\ell_1\ell_2)}_{(\Delta+n,0)} = \beta^{(\ell_1\ell_2)}\phi^n$	$2(\Delta+n)$	$\Psi^{(\ell_1\ell_2)}_{(\Delta+n,1)} = \omega^{(\ell_1\ell_2)}\phi^n$	$k-2(\bar{\ell}-n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Psi^{(\ell_1\ell_2)}_{(\bar{\ell},0)} = \beta^{(\ell_1\ell_2)}\phi^{\ell_<}$	$2\,\bar{\ell}$	$\Psi^{(\ell_1\ell_2)}_{(\bar{\ell},1)} = \omega^{(\ell_1\ell_2)}\phi^{\ell_<}$	$k-2\Delta$

Table 5.3: Elements of the boundary changing chiral rings and their charges ($\Delta = \frac{1}{2}|\ell_1-\ell_2|$, $\bar{\ell} = \frac{1}{2}(\ell_1+\ell_2)$ and $\ell_< = \min\{\ell_1,\ell_2\}$). These match as well the results from BCFT.

$$\begin{aligned} \beta^{(\ell_1\ell_2)} &= \begin{cases} \phi^{\ell_1-\ell_2}\pi\bar{\pi} + \bar{\pi}\pi & : \quad \ell_2 \leq \ell_1 \\ \pi\bar{\pi} + \phi^{\ell_2-\ell_1}\bar{\pi}\pi & : \quad \text{otherwise} \end{cases} , \\ \omega^{(\ell_1\ell_2)} &= \sqrt{i}\left(\pi - \frac{i}{k+2}\phi^{k-\ell_1-\ell_2}\bar{\pi}\right) . \end{aligned} \tag{5.86}$$

From (5.86) we find the intriguing feature that in this degenerate situation, the two sorts of each bosonic and fermionic fields (5.81) reduce to only one kind of bosons and fermions, respectively; in other words, the basis collapses to

$$\begin{aligned} \Psi^{(\ell_1\ell_2)}_{(a,0)} &= \left\{\beta^{(\ell_1\ell_2)}, \beta^{(\ell_1\ell_2)}\phi, ..., \beta^{(\ell_1\ell_2)}\phi^{\ell_<}\right\} , \\ \Psi^{(\ell_1\ell_2)}_{(a,1)} &= \left\{\omega^{(\ell_1\ell_2)}, \omega^{(\ell_1\ell_2)}\phi, ..., \omega^{(\ell_1\ell_2)}\phi^{\ell_<}\right\} , \end{aligned} \tag{5.87}$$

where $\ell_< = \min\{\ell_1,\ell_2\}$. At first sight the charge of boundary changing operators is not obvious, because π and $\bar{\pi}$ do not have a well defined charge in that case. However, taking advantage of charge conservation and the operator product $\beta^{(\ell_1\ell_2)}\beta^{(\ell_2\ell_1)} = \phi^{|\ell_1-\ell_2|} \bmod \phi^{\ell_<+1}$ as well as $\omega^{(\ell_1\ell_2)}\omega^{(\ell_2\ell_1)} = \phi^{k-\ell_1-\ell_2} \bmod \phi^{\ell_<+1}$, we conclude that $q(\beta^{(\ell_1\ell_2)}) = \frac{|\ell_1-\ell_2|}{k+2}$ and $q(\omega^{(\ell_1\ell_2)}) = \frac{k-\ell_1-\ell_2}{k+2}$.

The R-charges for the boundary fields in the basis (5.85) and (5.87) are listed in Tables 5.2 and 5.3, respectively. In the following we will show that they perfectly coincide with the charges of the boundary chiral ring on B-type D-branes in A-series superconformal minimal models, which is known from the BCFT approach.

The $\mathcal{N} = (2,2)$ minimal model can be realized as an $SU(2)$ WZW model and a Dirac fermion, coupled through a $U(1)$ gauge field. The symmetry group is $\mathbb{Z}_{2k+4} \times \mathbb{Z}_2$, where $\mathbb{Z}_{2k+4}$ is an axial R-rotation whose generator is denoted by a and $\mathbb{Z}_2$ is the fermion number $(-1)^F$.[6] Taking the orbifold by $(-1)^F$ (a non-chiral GSO-projection) one obtains the rational conformal field theory $SU(2)_k \times U(1)_2/U(1)_{k+2}$.

[6] More precisely, $a = e^{\pi i J_0}$, where J_0 is the zero-mode of the $U(1)$ R-current.

Its D-branes can be studied using standard BCFT techniques; their relation to geometry has been studied in [23, 73, 74].

In order to compare with the results obtained from the LG model, we are interested to obtain the spectrum on B-type D-branes in the *unprojected* theory, including the statistics of the boundary operators. Starting from the B-type boundary states of the rational model, one first has to undo the GSO projection to obtain the boundary states in the unprojected theory. One can then identify the action of a and $(-1)^F$ in the open string sector; the latter in particular determines the statistics. These steps have been performed in [75], and we refer to that paper for a detailed discussion. For completeness, we summarise the main steps and the result.

The primary fields of the rational model are labelled by the triple (l, m, s) where $l \in \{0, 1, 2, ..., k\}$, m is an integer modulo $2k + 4$, and s is an integer modulo 4. The NS sectors are defined by $s = 0, 2$ and the R sectors by $s = -1, 1$. We also have the identification $(l, m, s) \sim (k - l, m + k + 2, s + 2)$ and the selection rule $l + m + s = 0 \bmod 2$. The chiral primary (antichiral primary) states in the NS sector are labelled by $(l, l, 0)$ ($(l, -l, 0)$) if we use the identification in order to set $s = 0$. The symmetry group of the model is $\mathbb{Z}_{4k+8}$ (generated by the simple current $(0, 1, 1)$) for k odd and $\mathbb{Z}_{2k+4} \times \mathbb{Z}_2$ (generated by $(0, 1, 1)$ and $(0, 0, 2)$) for k even. The current $(0, 0, 2)$ distinguishes the R and NS sectors of the theory and can be viewed as the quantum symmetry of $(-1)^F$.

The Cardy states (A-type boundary states) $|L, M, S\rangle_C$ are labelled by the same set (L, M, S) as the primary states. B-type boundary states can be constructed using the fact that one can obtain the diagonal form of the charge conjugation modular invariant by taking a $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ orbifold. Hence, taking $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ orbits of A-type states plus an application of the "mirror map" (charge conjugation on the left-movers) leads to B-type boundary states. The $\mathbb{Z}_{k+2}$ acts on the Cardy states by shifting M by 2 and the $\mathbb{Z}_2$ acts by shifting S by 2. We therefore label B-type states by the orbit labels $L = \{0, 1, 2, ..., [\frac{k}{2}]\}$, $M = 0$ and $S = 0, 1$. All of these states are purely in the NSNS sector, and these branes are unoriented. A special case arises for k even and $L = \frac{k}{2}$ (this observation traces back to [76]). In this case the orbit boundary state is not elementary but can be decomposed further: There are altogether four states $|B, \frac{k}{2}, \hat{S}\rangle$ with $\hat{S} = -1, 0, 1, 2$, which are linear combination of an "orbit" NSNS part $|B, \frac{k}{2}, S\rangle$ (where S is the mod 2 reduction of $\hat{S}$) and an extra RR piece. In particular, these branes are oriented. We refer to [73] for details of the construction.

The task is now to resolve the GSO projection to obtain the branes of the unprojected theory. As explained in [75], the unoriented branes remain the same in the projected and unprojected theory. On the other hand, the oriented (short orbit) branes get re-decomposed into a NSNS and an RR part. In this work, we have developed a LG formulation of the unoriented orbit-type branes, and we point out that a LG interpretation of the oriented B-type branes has been proposed in [15, 77].

The open string spectrum between the unoriented branes can be obtained as

$$\mathcal{H}_{(L,S)(L',S')} = \bigoplus_{l+m+s\,\text{even}} N_{LL'}^{l} \mathscr{H}^{N=2}_{l,m,S-S'}, \tag{5.88}$$

where $N_{LL'}^{l}$ are the $SU(2)_k$ fusion rule coefficients. The spaces $\mathscr{H}^{N=2}_{l,m,[s]}$ are the modules of the unprojected $N = 2$ theory, which can be written in terms of the GSO-projected modules as $\mathscr{H}^{N=2}_{l,m,[s]} = \mathscr{H}_{l,m,s} + \mathscr{H}_{l,m,s+2}$. $[s]$ denotes the mod 2 reduction of s and distinguishes NS and R sectors. (Note that S and S' in (5.88) were only defined mod 2, therefore $[S-S'] = S-S'$ and the bracket can be omitted.)

Since these boundary states are purely in the NSNS sector, it is clear from the closed string sector that the Witten index between them vanishes. For the R-ground states in the open string sector this means that their contributions to $\mathrm{tr}(-1)^F$ cancel out, in other words, half of the supersymmetric R ground states are bosonic, and half of them are fermionic. More precisely, one can see that on a $(L, S)(L', S+1)$-brane pair the ground states from $\mathscr{H}^{N=2}_{l,l+1,1}$ and $\mathscr{H}^{N=2}_{l,-l-1,1}$ (which is an element of the Hilbert space $\mathscr{H}_{l,-l-1,-1}$ of the GSO-projected theory) contribute with opposite sign [75].

By spectral flow (0,–1,–1) these representations are related to $\mathscr{H}^{N=2}_{l,l,0}$. Note however that the spectral flow operator is not part of the spectrum of a single brane: RR ground states only propagate if $S - S' = 1$ mod 2 and NSNS states only if $S - S' = 0$ mod 2. In particular, there are never RR states on a single brane. It is natural to assume that the NSNS chiral primaries split up into a set of bosonic and fermions just as their RR counter parts, which propagate between branes with appropriately shifted label S.

To be explicit, the chiral ring consists of elements with charges ($\tilde{q} = q\,(k+2)$)

$$\begin{array}{llll} \tilde{q} = l & \in \{|L-L'|,\ |L-L'|+2,\ \ldots,\ (L+L')\} & \text{in} & \mathscr{H}^{N=2}_{l,l,0}\,, \\ \tilde{q} = k-l & \in \{k-(L+L'),\ k-(L+L')+2,\ \ldots,\ k-|L-L'|\} & \text{in} & \mathscr{H}^{N=2}_{l,-l-2,2}\,, \end{array} \tag{5.89}$$

where the states of $\mathscr{H}^{N=2}_{l,l,0}$ have opposite fermion number parity as compared with the states of $\mathscr{H}^{N=2}_{l,-l-2,2}$. This spectrum coincides precisely with the one listed in Tables (5.2) and (5.3), as obtained from the unperturbed Landau-Ginzburg theory; the label L of the BCFT formulation corresponds to ℓ in the LG formulation.

Chapter 6

The WDVV equations for the prepotential

We start now to investigate the main goal of this work; we want to find relations between correlation functions in topological conformal field theories taking advantage of Ward identities. In this chapter we restrict to closed string tree-level amplitudes, i.e., amplitudes on the sphere, and review the results of [5], including the WDVV equations of closed TCFT (cf. also [78, 79]).

We found in section 4.1 that Q-exact terms decouple from topological correlation functions of physical operators. Given a basis ϕ_i for H_c this implies that a correlator

$$\left\langle \phi_{i_1} \ldots \phi_{i_n} \right\rangle_{S^2} , \tag{6.1}$$

has two major properties: *(i)* The correlators are independent of the insertion point. This fact can be shown by virtue of the descent relations (4.15). *(ii)* The correlators are independent of the world-sheet metric, which follows from (4.4). In particular, the two point correlation function

$$\eta_{ij} = \left\langle \phi_i \phi_j \right\rangle_{S^2} .$$

plays the distinct role of a topological metric on H_c.

(i) and *(ii)* are very powerful statements. They imply that the correlation functions are invariant under permutations of the insertions, and they give rise to the factorisation of any n-point correlation function into 3-point correlators: We can deform the sphere with n insertions so that we get two spheres, which are connected by an infinitely long throat. The latter can be replaced by a complete system of physical operators, $\mathbb{1} = \sum \left| \phi_i \right\rangle \eta^{ij} \left\langle \phi_j \right|$, where η^{ij} is the inverse topological metric. In such a way we obtain two spheres with n_1 resp. $n - n_1 + 2$ insertions. By a recursive application of this procedure we have shown the desired factorisation into

$$C_{ijk} = \left\langle \phi_i \phi_j \phi_k \right\rangle_{S^2} . \tag{6.2}$$

In this respect 3-point correlation functions C_{ijk} (and the topological metric $\eta_{ij} = C_{ij0}$) can be viewed as fundamental objects, which are related to the operator product coefficients of relation (4.13) by $C_{ijk} = C_{ij}{}^{l}\eta_{lk}$. Given a n-point correlation function the factorisations are by far not unique [80]; we have the choice of different channels. In particular, from the factorisation of the 4-point correlator we obtain the associativity relation:

$$C_{ijk}\eta^{kl}C_{lmn} = C_{ink}\eta^{kl}C_{lmj} \ , \tag{6.3}$$

which ensures already the consistency of factorisations of arbitrary n-point functions in different channels [80].

We next consider the inclusion of integrated insertions of descendant 2-form operators (4.16). As compared to the correlators (6.1) we shall use the term *amplitudes* for correlators containing integrated insertions of descendants. These have the form:

$$C_{i_1\ldots i_n} := \big\langle \phi_{i_1}\ldots\phi_{i_3}\int \phi_{i_4}^{(1,1)}\ldots\int \phi_{i_n}^{(1,1)}\big\rangle_{S^2} \ . \tag{6.4}$$

By decoupling of Q-exact terms the amplitudes (6.4) have the same properties as the topological correlators (6.1). Note that we could have picked an arbitrary number of fixed insertions in (6.4). The distinct role of the amplitudes (6.4) with 3 fixed insertions is two-fold. First, all amplitudes can be reduced to this type through factorisation, as discussed above. Second, from the point of view of string theory we have to integrate the insertions in an amplitude over the whole moduli space of the n-punctured sphere, i.e., we have to consider exclusively integrated insertions of the type (4.16) in string amplitudes. However, the superconformal Möbius transformations of the topological conformal field theory on the sphere tell us that we can fix the position of exactly three operators, leading to (6.4). Note, in particular, that we are free to choose, which of the operators are fixed. Therefore, it is quite natural that the amplitude is totally symmetric under arbitrary permutations of all fields. This statement is true for topological *conformal* field theories, but not for a general TQFT, and it can be made precise in terms of Ward identities[1] of the amplitudes (6.4) including the 1-form charges G_{-1} and $\tilde{G}_{-1}$ [78]. A similar Ward identity shows that

$$C_{0i_2\ldots i_n} = 0 \qquad \text{for} \quad n \geq 4. \tag{6.5}$$

Let us define perturbed string amplitudes by the expression:

$$C_{i_1\ldots i_n}(t) = \langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\int_{S^2}\phi_{i_4}^{(1,1)}\ldots\int_{S^2}\phi_{i_n}^{(1,1)} e^{\sum_{p=0}^{h_c-1} t_p \int_{S^2}\phi_p^{(1,1)}}\rangle \ ,$$

which is understood as the formal power series:

$$C_{i_1\ldots i_n}(t) = \sum_{N_0\ldots N_{h_c-1}=0}^{\infty}\prod_{p=0}^{h_c-1}\frac{t_p^{N_p}}{N_p!}\langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\int_{S^2}\phi_{i_4}^{(1,1)}\ldots\int_{S^2}\phi_{i_n}^{(1,1)}\prod_{p=0}^{h_c-1}\left[\int_{S^2}\phi_p^{(1,1)}\right]^{N_p}\rangle \ .$$

[1]We do not elaborate on this point here, since we will work out such identities in great detail for disk amplitudes.

Here $t = (t_0 \dots t_{h_c-1})$ is a collection of complex-valued parameters, the closed string moduli. We can express all deformed amplitudes on the sphere with at least four insertions as partial derivatives of the deformed three-point function:

$$C_{i_1 \dots i_n}(t) = \partial_{i_4} \dots \partial_{i_n} C_{i_1 i_2 i_3}(t) \quad \text{for } n \geq 3 \quad .$$

Here and below we use the notation $\partial_i := \frac{\partial}{\partial t_i}$.

Property (6.5) then shows that the perturbed topological metric $\eta_{ij}(t) := C_{0ij}(t)$ is independent of the parameters t. And by the symmetry of the amplitudes (6.4) under all permutations of indices we infer the integrability condition:

$$\partial_i C_{jkl}(t) = \partial_j C_{ikl}(t) \ ,$$

which allows us to write the deformed three-point correlator as a triple derivative of a function $\mathcal{F}(t)$:

$$C_{ijk}(t) = \partial_i \partial_j \partial_k \mathcal{F}(t) \quad . \tag{6.6}$$

$\mathcal{F}$ is known as prepotential and is the generating function for all sphere amplitudes (6.4). In the appropriate geometric set-up, it can be interpreted as the prepotential of the effective spacetime theory associated with an $N = 2$ Calabi-Yau compactification of type II superstring.

The associativity condition as it stands in (6.3) is true for the unperturbed theory. From the flatness of the metric η_{ij} and including integrated insertions to obtain the deformed 3-point amplitudes $C_{ijk}(t)$ one can show [5] that this constraint is also realized on the generally deformed theory, i.e.:

$$C_{ijm}(t)\eta^{mn}C_{nkl}(t) = C_{ikm}(t)\eta^{mn}C_{njl}(t) \quad . \tag{6.7}$$

Using relation (6.6), this gives a system of second order, quadratic partial differential equations for the prepotential:

$$\partial_i \partial_j \partial_m \mathcal{F} \ \eta^{mn} \ \partial_n \partial_k \partial_l \mathcal{F} = \partial_i \partial_k \partial_m \mathcal{F} \ \eta^{mn} \ \partial_n \partial_j \partial_l \mathcal{F} \quad . \tag{6.8}$$

These are the well-known associativity, or WDVV relations [5, 81].

Chapter 7

Disk amplitudes and the effective superpotential

In this chapter we discuss the most basic properties of open-closed amplitudes on the disk. As on the sphere we can consider topological field theory correlation functions:

$$\left\langle \phi_{i_1} \ldots \phi_{i_n} \psi_{a_1} \ldots \psi_{a_m} \right\rangle_{D^2} , \tag{7.1}$$

with arbitrary number of physical bulk and boundary operator insertions. These correlation functions are constants, just as their counterparts (6.1) on the sphere. Therefore, we can use similar arguments to show factorisation into the fundamental correlators

$$\left\langle \phi_i \psi_a \right\rangle_{D^2} , \qquad \left\langle \psi_a \psi_b \psi_c \right\rangle_{D^2} , \tag{7.2}$$

and the sphere 3-point correlator (6.2). The different channels of factorisation give constraint equations on the fundamental correlators (7.2) and (6.2), which are worked out in detail in [82, 83].

Our main goal is, however, to investigate deformed amplitud that include integrated bulk operators (4.16), as well as integrated boundary operators (4.29). As compared to the bulk theory on the sphere, we pick up several complications, when we try to consider amplitudes with arbitrary numbers of fixed and integrated insertions. These complication are due to the presence of the boundary of the world-sheet and can be traced back to the fact that the integrated operators, (4.16) and (4.29), are no longer Q-closed. Therefore, the amplitudes are, in general, not independent of the position of the fixed insertions and not independent of the world-sheet metric.

There is, however, one exception, which occurs in topological *conformal* field theories. The string amplitudes, which come from fixing integrated insertions by virtue of the superconformal Möbius group on the disk, come in two basic forms: one has either one fixed bulk and one fixed boundary operator or solely three fixed boundary operators. It is exactly these amplitudes, which can be shown to have the usual property of topological field theories that they are constant. In contrast to the amplitudes (6.4) on the sphere, this property requires *conformal invariance* of topological field theory.

For the analysis of consistency conditions in the next chapter it is essential to introduce a proper regularisation of the disk amplitudes, which is done in section 7.1. Thereafter, we show in section 7.2 that the two basic forms of amplitudes are equal up to sign; and after some technical preparations we show in sections 7.4 and 7.5 that the amplitudes are independent of the positions of fixed operators and, moreover, invariant under variations of the world-sheet metric.

In section 7.6 we prove that disk amplitudes are invariant under cyclic permutations of boundary operators and invariant under all permutations of bulk operators. In particular, the latter condition allows us to integrate bulk operators to deformed cyclic disk amplitudes in section 7.7. Finally, we introduce a formal generating function $\mathcal{W}(s,t)$ of deformed disk amplitudes in section 7.8. Here, s and t are a collection of open resp. closed string moduli. We refer to $\mathcal{W}(s,t)$ as effective superpotential, because it coincides with the spacetime effective superpotential [6], whenever such an interpretation is possible.

7.1 The regularised amplitudes

Since disk amplitudes with integrated boundary descendants are affected by contact divergences, the conformal field theory arguments of later sections will require a regulator. We shall use a version of point-splitting for integrated bulk operators approaching the boundary of the disk and for integrated boundary operators approaching each other. This regularisation is essential only for the arguments of sections 8.1 and 8.2.

Given bulk descendants $\phi_{i_k}^{(2)}$ with $k = 1 \dots n$, we will choose their integration domain as follows:

$$\mathbb{H}_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \mathrm{Im}(z_k) \in (k\epsilon, \infty) \text{ for all } k = 1 \dots n\} \quad , \tag{7.3}$$

Here z_k are the insertion points of $\phi_{i_k}^{(2)}$, which of course are integrated over.

We next consider boundary insertions. Using superconformal Möbius invariance, three of them can be fixed while the others are integrated (see fig. 7.1(a)). A typical disk amplitude has the form:

$$\Big\langle\ \psi_{a_1} \psi_{a_2}\ P \int \psi_{a_3}^{(1)} \dots \int \psi_{a_{m-1}}^{(1)}\ \psi_{a_m} \int \phi_{i_1}^{(2)} \dots \int \phi_{i_n}^{(2)}\ \Big\rangle \quad , \tag{7.4}$$

where we fixed the positions of ψ_{a_1}, ψ_{a_2} and ψ_{a_m} to the points $\tau_1, \tau_2, \tau_m \in \mathbb{R}$, with the restriction $\tau_1 < \tau_2 < \tau_m$. The path-ordering symbol P means that the integral over $\tau_3 \dots \tau_{m-1}$ runs between τ_2 and τ_m with the constraint $\tau_2 < \tau_3 < \dots < \tau_{m-1} < \tau_m$. Including a regulator, the exact integration domain will be chosen as follows:

$$\mathbb{S}_m(\tau_2, \tau_m) = \big\{(\tau_3, \dots, \tau_{m-1}) \in \mathbb{R}^{m-3} | \tau_k - \tau_j > [2(k-j) - 1]\epsilon \ \text{ for } \ 2 \le j < k \le m\big\} \, . \tag{7.5}$$

Notice that we are requiring slightly increased separations for non-consecutive boundary insertions, rather than working with the naive point-splitting constraint

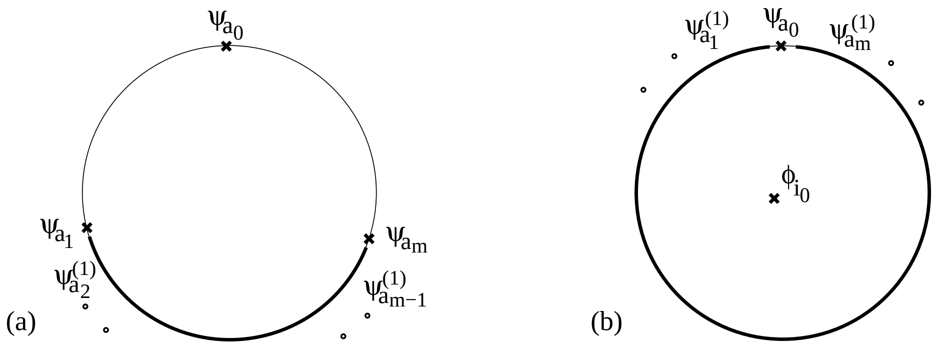

Figure 7.1: Boundary and bulk insertions for disk amplitudes. (a) Three boundary fields ψ_{a_0}, ψ_{a_1} and ψ_{a_m} are at fixed positions, the others are integrated in a path ordered way between ψ_{a_1} and ψ_{a_m}. (b) One bulk and one boundary field are fixed. In both cases additional bulk operators may be present, which are integrated over the whole disk.

$|\tau_k - \tau_j| \geq |k-j|\epsilon$. This somewhat unusual choice is made for the following reason. The factorisation procedure of the following sections makes use of the descent equation $[Q,G] = \frac{d}{d\tau}$, which implies that acting with Q on an integrated boundary insertion produces terms involving the associated zero-form operator evaluated at the boundaries of its integration interval, generally with some integrated insertions squeezed in. The increased separations chosen in (7.5) ensure the presence of non-void integration domains for the squeezed-in operators. For instance, if we consider Q acting on $\int \psi_{a_4}^{(1)}$, then our choice for the integration domain $\mathbb{S}_m(\tau_2,\tau_m)$ leads to a term of the form:

$$\psi_{a_2}(\tau_2) \int_{\tau_2+\epsilon}^{\tau_4-\epsilon} d\tau_3\ \psi_{a_3}^{(1)}(\tau_3)\ \psi_{a_4}(\tau_4)\Big|_{\tau_4=\tau_2+3\epsilon} = \psi_{a_2}(\tau_2) \int_{\tau_2+\epsilon}^{\tau_2+2\epsilon} d\tau_3\ \psi_{a_3}^{(1)}(\tau_3)\ \psi_{a_4}(\tau_2+3\epsilon)\ ,$$

which involves integration over a non-void interval. Had we used the naive condition $|\tau_k - \tau_l| > |k-l|\epsilon$, the integral in the last equation would have been $\int_{\tau_2+\epsilon}^{\tau_2+\epsilon} \psi_{a_3}^{(1)} = 0$.

Besides (7.4) one can also consider amplitudes in which PSL$(2,\mathbb{R})$-invariance is used to fix the positions of one bulk and one boundary insertion (see fig. 7.1(b)):

$$\Big\langle\ \phi_{i_1}\psi_{a_1}\ P\int \psi_{a_2}^{(1)} \ldots \int \psi_{a_m}^{(1)}\ \int \phi_{i_2}^{(2)} \ldots \int \phi_{i_n}^{(2)}\ \Big\rangle\ . \tag{7.6}$$

Naively, the integration domain is obtained from $\mathbb{S}_m(\tau_2,\tau_m)$ by replacing both τ_2 and τ_m by τ_1, where we integrate over the real line and identify $-\infty$ and ∞. However, the integrals approach ψ_{a_1} from both sides, so we have to introduce a further cut-off.

We will choose the following integration domain:

$$\begin{aligned} \mathbb{S}_m(\tau_1) = \{(\tau_2,\ldots,\tau_m)\in\mathbb{R}^{m-3} \quad | \quad & (\tau_2\ldots\tau_m) \text{ is cyclically ordered and} \\ & \tau_k-\tau_l > [2(k\text{–}l)\text{–}1]\epsilon \text{ for } \tau_k>\tau_l>\tau_1 \ , \\ & \qquad\qquad \text{or } \ \tau_1>\tau_k>\tau_l \ , \\ & \tau_k-\tau_l > [2(k\text{–}l\text{+}m)\text{–}1]\epsilon \text{ for } \tau_k>\tau_1>\tau_l\} \ . \end{aligned} \tag{7.7}$$

We will see in a moment that (after removing the regulator) the two kinds of amplitudes are equal up to sign, as has been argued before in [42].

Having defined the regularised amplitudes, we shall explore the implications of the conformal Ward identities and of the Ward identities for G. Using the doubling trick, one easily proves the relation:

$$\begin{aligned} \oint \xi(z)\big\langle G(z)\psi_{a_0}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi_{i_n}\big\rangle &= \sum_{k=0}^{m} \pm\, \xi(\tau_k)\big\langle\psi_{a_0}\ldots\psi^{(1)}_{a_k}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi_{i_n}\big\rangle \\ &\pm \sum_{k=0}^{n} \xi(w_k)\big\langle\psi_{a_0}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi^{(1,0)}_{i_k}\ldots\phi_{i_n}\big\rangle \\ &\pm \sum_{k=0}^{n} \bar\xi(\bar w_k)\big\langle\psi_{a_0}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi^{(0,1)}_{i_k}\ldots\phi_{i_n}\big\rangle \\ &= 0 \ , \end{aligned} \tag{7.8}$$

where $\xi(z) = az^2+bz+c$ with $a,b,c\in\mathbb{R}$ is a globally-defined holomorphic vector field on the upper half plane and the signs account for the grading on boundary fields. By the doubling trick, the contour integral on the left hand side encircles all fields and their images with respect to the real axis in the complex plane (which is viewed as a double cover of the upper half plane). In the right hand side we evaluated the residue at every insertion, including the images. The terms containing $\phi_i^{(1,0)}$ arise from the residue at ϕ_i, while the terms containing $\phi_i^{(0,1)}$ arise from the residues at the images of these insertions.

In the bulk sector, a similar identity implies constancy of the bulk topological metric along the moduli space and integrability of the deformed amplitudes. Below, we will study the consequences of (7.8).

7.2 Equivalence of the two types of amplitudes

We start by explaining the relation between the two kinds of disk amplitudes (7.4) and (7.6). We will show that these a priori different quantities are in fact equal up to sign factors. This was already discussed in [42] and we shall review the argument below in order to extract the correct signs for the case of boundary fields with different degrees. The derivation uses the Ward identities of G to relate integration over a bulk descendant with two integrations over boundary descendants.

As an example, consider the amplitudes $\langle \psi_a\psi_b\psi_c \int \phi_i^{(2)}\rangle$ and $\langle \phi_i\psi_a\ P\int\psi_b\int\psi_c\rangle$. We use the Ward identities:

$$\oint \xi_3 \big\langle G \oint \xi_2 G\ \psi_a(\tau_1)\ \psi_b(\tau_2)\ \psi_c(\tau_3)\ \phi_i(z,\bar z)\big\rangle = 0 \quad ,$$

and:

$$\oint \xi_3 \big\langle G\psi_a\psi_b\psi_c^{(1)}\phi_i\big\rangle = 0 \quad , \quad \oint \xi_2 \big\langle G\psi_a\psi_b^{(1)}\psi_c\phi_i\big\rangle = 0 \quad ,$$

with the following choice for the global holomorphic vector fields:

$$\xi_2(z) = (z-\tau_1)(z-\tau_3) \ \ \text{and} \ \ \xi_3(z) = (z-\tau_1)(z-\tau_2) \quad .$$

We assume the ordering $\tau_1 < \tau_2 < \tau_3$. Using equation (7.8), we obtain:

$$\frac{\xi_2(z)\bar\xi_3(\bar z) - \bar\xi_2(\bar z)\xi_3(z)}{\xi_2(\tau_2)\xi_3(\tau_3)} \ \big\langle \psi_a\psi_b\psi_c\phi_i^{(2)}\big\rangle \ = (-1)^{\tilde b}\ \big\langle \psi_a\psi_b^{(1)}\psi_c^{(1)}\phi_i\big\rangle \quad . \tag{7.9}$$

The conformal Ward identities ensure that both sides of equation (7.9) depend only on the cross-ratio $\zeta = \frac{(z-\tau_3)(\tau_2-\tau_1)}{(z-\tau_2)(\tau_3-\tau_1)}$ and its complex conjugate. Using the relations:

$$\xi_i(\tau_i)\frac{\partial\zeta}{\partial\tau_i} + \xi_i(z)\frac{\partial\zeta}{\partial z} = 0 \quad , \quad \text{for } i = 2,3 \quad ,$$

we find:

$$\frac{\xi_2(z)\bar\xi_3(\bar z) - \bar\xi_2(\bar z)\xi_3(z)}{\xi_2(\tau_2)\xi_3(\tau_3)} = \left(\frac{\partial\zeta}{\partial z}\frac{\partial\bar\zeta}{\partial\bar z}\right)^{-1}\left(\frac{\partial\zeta}{\partial\tau_2}\frac{\partial\bar\zeta}{\partial\tau_3} - \frac{\partial\zeta}{\partial\tau_3}\frac{\partial\bar\zeta}{\partial\tau_2}\right)^{-1} \quad .$$

Hence the prefactor in equation (7.9) is the Jacobian of the coordinate transformation from $(z,\bar z)$ to (τ_2,τ_3).

A priori we are free to normalise the integrations over bulk and boundary descendants independently: $\lambda_{bulk}\int\phi^{(2)}$ and $\lambda_{bound}\int\psi^{(1)}$. By integrating (7.9) we find:

$$(-1)^{\tilde b}\ \big\langle \psi_a\psi_b\psi_c \int \phi_i^{(2)}\big\rangle = -\big\langle \phi_i\ \psi_a\ P\int\psi_b^{(1)}\int\psi_c^{(1)}\big\rangle \quad , \tag{7.10}$$

where we chose the relative normalisation factor to be -1. Of course, this locks λ_{bulk} and λ_{bound} together through the relation $\lambda_{bulk} \propto \lambda_{bound}^2$.

One can easily generalise the analysis to arbitrary numbers of bulk and boundary insertions. This gives:

$$\begin{aligned} B_{a_0\ldots a_m;i_1\ldots i_n} &:= (-1)^{\tilde a_1+\ldots+\tilde a_{m-1}}\big\langle\ \psi_{a_0}\psi_{a_1}\ P\int\psi_{a_2}^{(1)}\ldots\int\psi_{a_{m-1}}^{(1)}\ \psi_{a_m}\int\phi_{i_1}^{(2)}\ldots\int\phi_{i_n}^{(2)}\ \big\rangle \\ &= -\ \big\langle\ \phi_{i_1}\psi_{a_0}\ P\int\psi_{a_1}^{(1)}\ldots\int\psi_{a_m}^{(1)}\ \int\phi_{i_2}^{(2)}\ldots\int\phi_{i_n}^{(2)}\ \big\rangle \quad . \end{aligned} \tag{7.11}$$

Thus (7.4) and (7.6) are equal up to sign, and they determine the single object $B_{a_0\ldots a_m;i_1\ldots i_n}$ defined by the expression above. Notice that for $B_{a_0a_1a_2}$ as well as $B_{a_0;i_1}$ and $B_{a_0a_1;i_1}$ such a relation does not exist for obvious reasons. As we shall see below, it is notationally convenient to define:

$$B_{a_0a_1} = B_{a_0} = B_i = 0 \quad . \tag{7.12}$$

The amplitudes (7.11) are subject to the selection rule:

$$\tilde{a}_0 + \ldots + \tilde{a}_m = \tilde{\omega} \ , \tag{7.13}$$

which is induced by the suspended grading on the space of boundary operators (4.31). $\tilde{\omega}$ is the model-dependent grade of non-vanishing amplitudes. Note that the suspended grading is more natural than the ordinary degree for physical boundary operators.

We make one final remark about equation (7.11). The first line is manifestly symmetric in the bulk indices, but this is not obvious for the second line. As in the pure bulk theory [5], there exists a Ward identity, which switches fixed and integrated bulk insertions. This can be used to show directly that the second line in (7.11) is also totally symmetric in the bulk insertions.

7.3 Two point correlation functions are not deformed

In this subsection, we show that the two-point boundary correlators are constant under bulk and boundary deformations. Let us start with the Ward identity for G in the presence of two fixed boundary insertions:

$$\oint \xi(z)\big\langle G(z)\ \psi_{a_1}(\tau_1)\psi_{a_2}(\tau_2)\psi_{a_3}(\tau_3)\big\rangle = 0 \quad .$$

Choosing $\xi(z) = (z-\tau_1)(z-\tau_2)$, we find:

$$\big\langle \psi_{a_1}\ \psi_{a_2}\ \psi_{a_3}^{(1)}\big\rangle = 0 \quad . \tag{7.14}$$

The analogous relation for a bulk perturbation:

$$\big\langle \psi_{a_1}\ \psi_{a_2}\ \phi_i^{(2)}\big\rangle = 0 \quad , \tag{7.15}$$

requires a bit more work. For this, consider the Ward identity:

$$\oint \xi_2\big\langle G \oint \xi_1 G\psi_{a_1}(\tau_1)\psi_{a_2}(\tau_2)\phi_i(w,\bar{w})\big\rangle = 0 \quad ,$$

where $\xi_1(z) = (z-\tau_1)(z-\tau_2)$ and $\xi_2(z) = (z-\tau_2)(z-\mathrm{Re}w)$. Combining this with the relation:

$$\oint \xi_1\big\langle G\psi_{a_1}\psi_{a_2}^{(1)}\phi_i\big\rangle = 0 \quad ,$$

leads to equation (7.15).

Since the supercharge G does not act on additional descendants $\int \psi_a^{(1)}$ and $\int \phi_i^{(2)}$, we easily infer the generalisation:

$$\big\langle \psi_{a_1}\, \psi_{a_2}\, P \int \psi_{a_3}^{(1)} \dots \int \psi_{a_m}^{(1)} \int \phi_{i_1}^{(2)} \dots \int \phi_{i_n}^{(2)} \big\rangle = 0 \ , \quad \text{for } m \geq 3 \text{ or } \ n \geq 1 \ . \tag{7.16}$$

In similar manner, one shows:

$$\begin{aligned}
\big\langle \phi_{i_0}^{(1,0)}\ \psi_{a_1} \ \ P \int \psi_{a_2}^{(1)} \dots \int \psi_{a_m}^{(1)} \int \phi_{i_1}^{(2)} \dots \int \phi_{i_n}^{(2)} \big\rangle &= 0 \ , \\
\big\langle \phi_{i_0}^{(0,1)}\ \psi_{a_1} \ \ P \int \psi_{a_2}^{(1)} \dots \int \psi_{a_m}^{(1)} \int \phi_{i_1}^{(2)} \dots \int \phi_{i_n}^{(2)} \big\rangle &= 0 \ ,
\end{aligned} \tag{7.17}$$

In terms of the quantities defined in equation (7.11), relation (7.16) takes the form:

$$B_{0a_1\dots a_m;i_1\dots i_n} = 0 \qquad \text{for} \ \ m \geq 3 \text{ or } n \geq 1 \ \ . \tag{7.18}$$

The identities discussed in this subsection will be important for subsequent arguments. As we shall see, they are essential for the proof that disk amplitudes are constant. Moreover, they give rise to special properties of the boundary algebra and the topological boundary metric.

7.4 Independence of the positions of unintegrated insertions

We will now show that the fundamental amplitudes (7.11) are independent of the positions of unintegrated insertions. As an example, consider the 4-point boundary amplitude. Differentiating it with respect to τ_1 and using the descent equations, we find:

$$\begin{aligned}
\frac{\partial}{\partial \tau_1} \big\langle \psi_{a_0} \psi_{a_1} \int_{\tau_1}^{\tau_3} \psi_{a_2}^{(1)} \psi_{a_3} \big\rangle &= \big\langle \psi_{a_0} [Q, \psi_{a_1}^{(1)}] \int_{\tau_1}^{\tau_3} \psi_{a_2}^{(1)} \psi_{a_3} \big\rangle - \big\langle \psi_{a_0} \psi_{a_1} \psi_{a_2}^{(1)}|_{\tau_1} \psi_{a_3} \big\rangle \\
&= (-1)^{\tilde{a}_1} \big\langle \psi_{a_0} \psi_{a_1}^{(1)} (\psi_{a_2}|_{\tau_1} - \psi_{a_2}|_{\tau_3}) \psi_{a_3} \big\rangle - \big\langle \psi_{a_0} \psi_{a_1} \psi_{a_2}^{(1)}|_{\tau_1} \psi_{a_3} \big\rangle \\
&= (-1)^{\tilde{a}_1} \Big(\big\langle \psi_{a_0} (\psi_{a_1} \psi_{a_2})^{(1)} \psi_{a_3} \big\rangle - \big\langle \psi_{a_0} \psi_{a_1}^{(1)} (\psi_{a_2} \psi_{a_3}) \big\rangle \Big) = 0 \ \ .
\end{aligned}$$

In the last line we used relation (7.16). Generalising this argument, it is not hard to show that all amplitudes (7.11) are independent on the positions of unintegrated insertions.

7.5 Independence of the world-sheet metric

Due to the nontrivial terms in the right hand side of equation (4.30), it is not immediately clear that the amplitudes (7.11) are independent of the world-sheet metric. The usual recipe of topological field theory does not work: the variation of the correlation function with respect to the metric produces an insertion of the energy-momentum tensor, which can be written as $[Q, G_{\mu\nu}]$. When pulling Q through the integrated boundary insertions, one obtains nontrivial terms induced by equation (4.30), so that one cannot immediately conclude that integrated correlators are independent of the world-sheet metric. However, conformal invariance comes to the rescue, through the conformal Ward identity:

$$
\begin{aligned}
\big\langle T(z)\ \phi_{i_1}\dots\phi_{i_n}\ \psi_{a_1}\dots\psi_{a_m}\big\rangle\ &= \\
&= \sum_{k=1}^{n}\left(\frac{h_k}{(z-z_k)^2}+\frac{1}{z-z_k}\frac{\partial}{\partial z_k}\right)\big\langle\phi_{i_1}\dots\phi_{i_n}\psi_{a_1}\dots\psi_{a_m}\big\rangle\ + \\
&+ \sum_{k=1}^{n}\left(\frac{\bar h_k}{(z-\bar z_k)^2}+\frac{1}{z-\bar z_k}\frac{\partial}{\partial \bar z_k}\right)\big\langle\phi_{i_1}\dots\phi_{i_n}\psi_{a_1}\dots\psi_{a_m}\big\rangle\ + \\
&+ \sum_{l=1}^{m}\left(\frac{h_l}{(z-\tau_l)^2}+\frac{1}{z-\tau_l}\frac{\partial}{\partial \tau_l}\right)\big\langle\phi_{i_1}\dots\phi_{i_n}\psi_{a_1}\dots\psi_{a_m}\big\rangle\ ,
\end{aligned}
\tag{7.19}
$$

where $\phi_{i_k}=\phi_{i_k}(z_k,\bar z_k)$ and $\psi_{a_l}=\psi_{a_l}(\tau_l)$ are bulk and boundary conformal primaries. In the case of interest, the conformal weights of zero-form operators are $h_l=h_k=\bar h_k=0$, while for descendants one has $h_l=h_k=\bar h_k=1$.

Let us first consider the simplest case, namely the boundary 4-point amplitude:

$$
\begin{aligned}
\int_{\tau_2}^{\tau_4} d\tau_3\ \big\langle T(z)\ \psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\big\rangle\ &=\ \sum_{l=1,2,4}\int_{\tau_2}^{\tau_4} d\tau_3\frac{1}{z-\tau_l}\frac{\partial}{\partial\tau_l}\big\langle\psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\big\rangle\ + \\
&+\ \int_{\tau_2}^{\tau_4} d\tau_3\ \frac{\partial}{\partial\tau_3}\left(\frac{1}{z-\tau_3}\big\langle\psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\big\rangle\right)\ .
\end{aligned}
$$

Using the descent relations (4.27) in the first line and evaluating the integral in the second line, we find:

$$\begin{aligned}
\int_{\tau_2}^{\tau_4} d\tau_3 \, \langle T(z) \;\; \psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\rangle \; &= \\
&= \; (-1)^{\tilde{a}_1+\tilde{a}_2}\frac{1}{z-\tau_1}\left(\langle\psi^{(1)}_{a_1}\psi_{a_2}(\psi_{a_3}\psi_{a_4})\rangle - \langle\psi^{(1)}_{a_1}(\psi_{a_2}\psi_{a_3})\psi_{a_4}\rangle\right) + \\
&+ \; (-1)^{\tilde{a}_2}\frac{1}{z-\tau_2}\left(\langle\psi_{a_1}(\psi_{a_2}\psi_{a_3})^{(1)}\psi_{a_4}\rangle - \langle\psi_{a_1}\psi^{(1)}_{a_2}(\psi_{a_3}\psi_{a_4})\rangle\right) + \\
&+ \; \frac{1}{z-\tau_3}\left(\langle\psi_{a_1}\psi_{a_2}(\psi_{a_3}\psi_{a_4})^{(1)}\rangle - \langle\psi_{a_1}(\psi_{a_2}\psi_{a_3})\psi^{(1)}_{a_4}\rangle\right) \qquad (7.20) \\
&= \; 0 \; .
\end{aligned}$$

In the last step, we used again equation (7.16). In the same manner one can show that all amplitudes (7.11) are independent of the world-sheet metric.

Remark: We have seen that taking into account the conformal invariance of the theory was essential in the last two subsections. In particular, the property of constancy does not hold for amplitudes with more fixed insertions as compared to (7.4) or (7.6). This fact will play an essential role, when we subsequently derive constraint equations, since we cannot apply simple deformation arguments as in the bulk theory on the sphere.

7.6 Cyclicity and bulk permutation invariance

We shall now prove that disk amplitudes are (graded) cyclically symmetric with respect to boundary insertions. Let us illustrate this with the boundary 4-point amplitude:

$$\oint \xi(z)\langle G(z) \;\; \psi_{a_1}(\tau_1)\psi_{a_2}(\tau_2)\psi_{a_3}(\tau_3)\psi_{a_4}(\tau_4)\rangle = 0 \quad , \qquad (7.21)$$

where $\tau_4 > \ldots > \tau_1$. Taking $\xi(z) = (z-\tau_4)(z-\tau_1)$ in equation (7.21) and using relation (7.8), we obtain $\xi(\tau_2)\langle\psi_a\psi^{(1)}_b\psi_c\psi_d\rangle = (-1)^{\tilde{b}}\xi(\tau_3)\langle\psi_a\psi_b\psi^{(1)}_c\psi_d\rangle$. From the conformal Ward identities we know that the unintegrated 4-point function depends only on the cross-ratio $\zeta = \frac{(\tau_4-\tau_3)(\tau_2-\tau_1)}{(\tau_4-\tau_2)(\tau_3-\tau_1)}$, which satisfies the relation:

$$\xi(\tau_2)\frac{\partial\zeta}{\partial\tau_2} + \xi(\tau_3)\frac{\partial\zeta}{\partial\tau_3} = 0 \quad .$$

Hence the Ward identity (7.21) implies:

$$\left(\frac{\partial\zeta}{\partial\tau_2}\right)^{-1}\langle\psi_a\psi^{(1)}_b\psi_c\psi_d\rangle = -(-1)^{\tilde{b}}\left(\frac{\partial\zeta}{\partial\tau_3}\right)^{-1}\langle\psi_a\psi_b\psi^{(1)}_c\psi_d\rangle \quad .$$

Let us integrate this equation over ζ, taking into account that on the right hand side the integration runs in the 'wrong' direction, i.e. $\int_0^1 d\zeta(\frac{\partial\zeta}{\partial\tau_2})^{-1} = \int_{\tau_1}^{\tau_3} d\tau_2$, but

$\int_0^1 d\zeta(\frac{\partial\zeta}{\partial\tau_3})^{-1} = -\int_{\tau_2}^{\tau_4} d\tau_3$. This gives the relation:

$$\langle\psi_a P \int \psi_b^{(1)}\psi_c\psi_d\rangle = (-1)^{\tilde{b}}\langle\psi_a\psi_b P \int \psi_c^{(1)}\psi_d\rangle \quad .$$

Generalising the argument to more integrated insertions, one finds the following identities:

$$\big\langle\psi_{a_0}\psi_{a_1}P\!\int\!\psi_{a_2}^{(1)}\ldots\int\!\psi_{a_{m-1}}^{(1)}\psi_{a_m}\big\rangle = (-1)^{\tilde{a}_1+\ldots+\tilde{a}_{m-2}}\big\langle\psi_{a_0}P\!\int\!\psi_{a_1}^{(1)}\ldots\int\!\psi_{a_{m-2}}^{(1)}\psi_{a_{m-1}}\psi_{a_m}\big\rangle \tag{7.22}$$

and:

$$\big\langle\phi_i\psi_{a_0}\ P\int\!\psi_{a_1}^{(1)}\ldots\int\!\psi_{a_m}^{(1)}\big\rangle = (-1)^{\tilde{a}_0+\ldots+\tilde{a}_{m-1}}\big\langle\phi_i\ P\int\!\psi_{a_0}^{(1)}\ldots\int\!\psi_{a_{m-1}}^{(1)}\psi_{a_m}\big\rangle \quad . \tag{7.23}$$

Additional bulk perturbations do not change these results.

We conclude that the fundamental disk amplitudes $B_{a_0\ldots a_m;i_1\ldots i_n}$ with $m, n \geq 0$ and $2n + m > 1$ are cyclically symmetric in the boundary indices:

$$B_{a_0\ldots a_m;i_1\ldots i_n} = (-1)^{\tilde{a}_m(\tilde{a}_0+\ldots+\tilde{a}_{m-1})}B_{a_m a_0\ldots a_{m-1};i_1\ldots i_n} \quad . \tag{7.24}$$

And as already pointed out at the end of section 7.2, all such amplitudes are totally symmetric in the bulk indices (the argument is the same as for the pure bulk case [5]).

7.7 Deformed amplitudes and the boundary metric

The very last statement of the previous section implies that we can integrate all bulk perturbations to produce deformed disk amplitudes:

$$\mathcal{F}_{a_0\ldots a_m}(t) \qquad \text{for} \quad m \geq 0 \tag{7.25}$$

which generate deformed open-closed amplitudes:

$$B_{a_0\ldots a_m;i_1\ldots i_n}(t) = \partial_{i_1}\ldots\partial_{i_n}\mathcal{F}_{a_0\ldots a_m}(t) \quad . \tag{7.26}$$

And we obtain the undeformed amplitudes (7.11) by setting $t = 0$. For $m \geq 2$, the generating functions are given by the expressions:

$$\mathcal{F}_{a_0\ldots a_m}(t) = (-1)^s\ \langle\psi_{a_0}\psi_{a_1}P\int\psi_{a_2}\ldots\int\psi_{a_{m-1}}\psi_{a_m}e^{\sum_p t_p\int_{D^2}\phi_p^{(2)}}\rangle \quad ,$$

which are understood as the formal power series:

$$\mathcal{F}_{a_0\ldots a_m}(t) = (-1)^s\sum_{N_0\ldots N_{h_c-1}=0}^{\infty}\prod_{p=0}^{h_c-1}\frac{t_p^{N_p}}{N_p!}\langle\psi_{a_0}\psi_{a_1}P\int\psi_{a_2}\ldots\int\psi_{a_{m-1}}\psi_{a_m}\Big[\int\phi_p^{(2)}\Big]^{N_p}\rangle \ ,$$

where $s = \tilde{a}_1 + \cdots + \tilde{a}_{m-1}$. The cases $m = 0$ and $m = 1$ of (7.6) are special, because one bulk operator is not integrated. However, through the Ward identity for G, such correlators are again totally symmetric in the bulk indices. Thus one can define $\mathcal{F}_a(t)$ and $\mathcal{F}_{ab}(t)$ through the relations:

$$\begin{aligned}
\partial_i \mathcal{F}_a(t) &= -\langle \phi_i \ \psi_a \ e^{\sum_p t_p \int_{D^2} \phi_p^{(2)}} \rangle \ , \\
\partial_i \mathcal{F}_{ab}(t) &= -\langle \phi_i \ \psi_a P \int \psi_b^{(1)} \ e^{\sum_p t_p \int_{D^2} \phi_p^{(2)}} \rangle \ ,
\end{aligned}$$

which determine these quantities up to t-independent terms.

Notice that $\mathcal{F}_a(t)$ and $\mathcal{F}_{ab}(t)$ need not vanish, though they must be of order at least one in t_i (cf. equations (7.12)). In particular, this means that deformations of the closed string background will generally induce tadpoles:

$$\mathcal{F}_a(t) := \langle \psi_a \rangle_t \ ,$$

where $\langle \ldots \rangle_t$ stands for the expectation value on the disk taken in the deformed theory. Such tadpoles must of course be cancelled (for example by performing a shift of the boundary topological vacuum) if the deformed theory is to be conformal (and generally a meaningful string background). This means that deformations of the bulk and boundary sectors must be locked together in order to solve the obstructions, a phenomenon well-known from joint deformation theory. We shall further discuss this phenomenon in Subsection 8.3, and exemplify it for concrete physical models in Section 9.

Cyclicity of disk amplitudes with respect to boundary insertions (equation (7.24)) implies the invariance of the deformed amplitudes under cyclic permutations of the indices:

$$\mathcal{F}_{a_0 \ldots a_m}(t) = (-1)^{\tilde{a}_m(\tilde{a}_0 + \ldots + \tilde{a}_{m-1})} \mathcal{F}_{a_m a_0 \ldots a_{m-1}}(t) \ . \tag{7.27}$$

Equations (7.18) gives rise to the restrictions:

$$\mathcal{F}_{0 a_1 \ldots a_m}(t) = 0 \quad \text{for} \quad m \neq 2 \ . \tag{7.28}$$

and

$$\omega_{a_1 a_2} = \mathcal{F}_{0 a_1 a_2}(t) = \text{ independent of } t \ . \tag{7.29}$$

ω_{ab} represents a non-degenerate topological metric for the physical boundary fields and equation (7.29) amounts to the statement that the metric is invariant under deformations of the theory. Equipped with metrics on the physical bulk and boundary operators we can define the quantities:

$$\begin{aligned}
B^a{}_{a_1 \ldots a_m; i_1 \ldots i_n} &:= \omega^{ab} B_{b a_1 \ldots a_m; i_1 \ldots i_n} \ , \\
B^i{}_{a_0 \ldots a_m; i_2 \ldots i_n} &:= \eta^{ij} B_{a_0 \ldots a_m; j i_2 \ldots i_n} \ .
\end{aligned} \tag{7.30}$$

In view of relation (7.29), the selection rule (7.13) becomes:

$$\omega_{ab} = 0 \ \text{ unless } \ \tilde{a} + \tilde{b} = \tilde{\omega} + 1 \ (\mathrm{mod}\ 2) \ . \tag{7.31}$$

Moreover, the topological boundary metric fulfils the graded symmetry property:

$$\omega_{ab} = (-1)^{\tilde{\omega}} (-1)^{\tilde{a}\tilde{b}} \omega_{ba} \ . \tag{7.32}$$

7.8 The formal generating function and the effective superpotential

It is possible to package the cyclic amplitudes $\mathcal{F}_{a_0 \ldots a_m}(t)$ defined in (7.25) into a single formal generating function as follows. Consider the free associative (but non-commutative) superalgebra $\hat{\mathbb{A}}$ generated by a set of formal variables $\hat{s}_a$ of degrees $\tilde{a} \in \mathbb{Z}_2$, where a runs from 0 to $h_o - 1$. Then we define the *formal generating function* $\hat{\mathcal{W}}(\hat{s}, t)$ through the expression:

$$\hat{\mathcal{W}}(\hat{s}, t) = \sum_{m \geq 1} \frac{1}{m} \hat{s}_{a_m} \ldots \hat{s}_{a_1} \mathcal{F}_{a_1 \ldots a_m}(t) \quad . \tag{7.33}$$

It can be viewed as an element of the associative superalgebra $\mathbb{C}[[t_0 \ldots t_{h_c-1}]] \otimes \hat{\mathbb{A}}$, where $\mathbb{C}[[t_0 \ldots t_{h_c-1}]]$ is the commutative algebra of formal power series in the variables t_i.

Since the parameters $\hat{s}_a$ are non-commuting, the quantity $\hat{\mathcal{W}}(\hat{s}, t)$ has no obvious physical interpretation, so the reader might wonder what is the use of considering non-commuting parameters in the first place. To understand this, let us introduce *(super-)commuting* parameters s_a instead, whose degrees are given by $\tilde{a}$ (such formal variables generate a free commutative superalgebra denoted by $\mathbb{A}$). Then we define a quantity $\mathcal{W}(s, t)$ by the same formula as above, but with $\hat{s}_a$ replaced by s_a:

$$\mathcal{W}(s, t) = \sum_{m \geq 1} \frac{1}{m} s_{a_m} \ldots s_{a_1} \mathcal{F}_{a_1 \ldots a_m}(t) \quad . \tag{7.34}$$

Since s_a super-commute and have the same $\mathbb{Z}_2$-degree as the boundary descendants $\psi_a^{(1)}$, they can be viewed as honest boundary deformation parameters of the world-sheet theory. In physics terms, $\mathcal{W}(s, t)$ will coincide with the spacetime effective superpotential of the untwisted $N = 2$ model, when such an interpretation of the world-sheet theory is available.

Since s_a super-commute, one finds that monomials in these variables which differ by a permutation are related through:

$$s_{a_{\sigma(m)}} \ldots s_{a_{\sigma(1)}} = \eta(\sigma; a_1 \ldots a_m) s_{a_m} \ldots s_{a_1} \quad .$$

Here σ is a permutation on n elements and $\eta(\sigma; a_1 \ldots a_m)$ is defined as the sign produced when permuting s_a to relate the left and right hand sides. Using this relation, $\mathcal{W}(s, t)$ reduces to:

$$\mathcal{W}(s, t) \;=\; \sum_{m \geq 1} \frac{1}{m!} s_{a_m} \ldots s_{a_1} \mathcal{A}_{(a_1 \ldots a_m)}(t) \quad , \tag{7.35}$$

where:[1]

$$\mathcal{A}_{a_1\ldots a_m}(t) := (m-1)!\mathcal{F}_{(a_1\ldots a_m)}(t) \;:=\; \frac{1}{m}\sum_{\sigma\in S_m}\eta(\sigma;a_1\ldots a_m)\mathcal{F}_{a_{\sigma(1)}\ldots a_{\sigma(m)}} \qquad (7.36)$$

are (super-)symmetrised combinations of the cyclic amplitudes $\mathcal{F}_{a_1\ldots a_m}$ and S_m is the group of permutations of m objects. These are the relevant, physically observable quantities, because tree-level scattering amplitudes are summed over permutations of indistinguishable incoming states. By construction these functions are integrable with respect to the boundary deformation parameters, namely they are given by partial derivatives of $\mathcal{W}$:

$$\mathcal{A}_{a_1\ldots a_m} = \partial_{a_1}\ldots\partial_{a_m}\mathcal{W}(s,t)\Big|_{s=0} \quad , \qquad (7.37)$$

where $\partial_a := \frac{\vec{\partial}}{\partial s_a}$.

It is clear that $\mathcal{W}(s,t)$ carries less information than the full set of disk amplitudes. In other words, one cannot package the entire information of the topological string theory in this quantity alone. As explained above, one way to encode tree-level world-sheet data without loosing any information is to consider the generating function $\hat{\mathcal{W}}(\hat{s},t)$ in (7.33), which is defined on a formal non-commutative parameter space.

The relation between the formal generating function $\hat{\mathcal{W}}(\hat{s},t)$ and effective superpotential $\mathcal{W}(s,t)$ can be stated more precisely as follows. Notice that the commutative superalgebra $\mathbb{A}$ can be obtained from $\hat{\mathbb{A}}$ upon dividing through the ideal J generated by the elements:

$$\hat{s}_a\hat{s}_b - (-1)^{\tilde{a}\tilde{b}}\hat{s}_b\hat{s}_a \quad .$$

In this presentation, s_a can be identified with the equivalence classes of $\hat{s}_a$ modulo this ideal. If we let $\pi : \hat{\mathbb{A}} \to \mathbb{A}$ denote the natural surjective morphism, then the precise relation between the two quantities takes the form:

$$\mathcal{W}(\pi(\hat{s}),t) = \pi(\hat{\mathcal{W}}(\hat{s},t)) \quad .$$

[1]In a general set-up, when boundary condition changing operators are included, the right hand side of (7.36) contains terms, where the boundary condition labels do not match. Equation (7.36) makes still sense when we set these terms to zero by hand.

Chapter 8

Consistency conditions for disk amplitudes

After the preparations in chapter 7 we are now ready to discuss the consistency constraints, including the A_∞ algebra, for both boundary amplitudes and mixed bulk-boundary amplitudes on the disk. A_∞ algebras were originally introduced by J. Stasheff [38,39], while A_∞ categories were first discussed by K. Fukaya [44]. The relevance of A_∞ algebras in string theory was originally pointed out in [40] in the context of open string field theory. They play a central role in the homological mirror symmetry program [27,36,44,84–87], where they arise via topological string theory [35,46].

As a warm-up and as a confirmation of the string field theory approach of [35] we derive the cyclic, unital, minimal A_∞ algebra of (pure) boundary amplitudes in section 8.1, where we take advantage of a Ward identity associated to the Q-symmetry. Moreover, we give an explicit definition of A_∞ algebras (as well as the notions: minimal, strong, weak, cyclic, unital) in the standard terminology of mathematical literature and express our results in this language.

Thereafter, in section 8.2 we derive the generalisation of the minimal A_∞ constraints for deformed boundary amplitudes (7.25). As we shall see, the relevant consistency conditions take the form of a weak, cyclic and unital A_∞ algebra, which can be viewed as an all-order deformation of the minimal A_∞ algebra of section 8.1. The strategy of the derivation is as follows: we consider first linear bulk deformations of the boundary amplitudes by inserting a single bulk physical operator on the disk and generalise afterwards to general deformations.

Rewriting the resulting weak A_∞ algebra in standard mathematical terminology allows us to interpret the topological disk amplitudes in terms of open string field theory amplitudes in section 8.3. The appearance of a weak A_∞ algebra under deformations of the closed string background generates an open string tadpole, which must be cancelled by a shift of the open string vacuum. This encodes interlocking of open and closed string deformation parameters when solving the joint deformation problem for the bulk and boundary sectors. At the end of section 8.3 we give a brief account on the algebraic interpretation of deformations of boundary amplitudes in terms of a cyclic complex. The latter is the subcomplex of the Hochschild complex for deformations of A_∞ algebras [88], which respects cyclic invariance.

Finally, in sections 8.4 and 8.5 we investigate the remaining constraints, which encode the stringy generalisation of the second bulk-boundary sewing condition and of the Cardy relation [82]. This completes the set of consistency conditions, which constrain open-closed amplitudes on the disk.

8.1 Minimal A_∞ constraints on boundary amplitudes

In this section, we discuss a countable set of algebraic constraints on tree-level boundary amplitudes on the disk, which can be viewed as the Ward identities of the Q-symmetry. These constraints arise from the relations [42]:

$$\langle [Q, \psi_{a_0}\psi_{a_1} P \int \psi^{(1)}_{a_2} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m}] \rangle = 0 \quad (m \geq 2) \quad , \tag{8.1}$$

which encode Q-invariance of the topological vacuum. They are due to equation (4.30), which induces nontrivial contributions when taking the commutator with Q on the left hand side of (8.1). From (4.30), it is clear that the resulting terms will involve amplitudes in which two boundary insertions approach each other in the limit when the regulator ϵ is removed. Therefore, the contribution on the left hand side of (8.1) is due entirely to contact singularities, and hence it can be factorised into amplitudes with lower numbers of insertions. Performing the computation, one finds that the Ward identity can be brought to a form known in the mathematics literature as a "minimal A_∞ algebra".

Acting explicitly with the operator Q on the left hand side of equation (8.1) and using the descent relation $[Q, \psi^{(1)}_{a_k}] = [Q, [G, \psi_{a_k}]] = \partial_{\tau_k} \psi_{a_k}$ produces an integration over the boundary of the moduli space of the boundary-punctured disk, i.e., where two or more punctures get together very closely. The discussion of the resulting terms involves the regularisation of section 7.1 in an essential manner.

For clarity, we first discuss the case $m = 4$. The regularised configuration space and its boundary components are shown in Figure 8.1. The left hand side of equation (8.1) becomes:

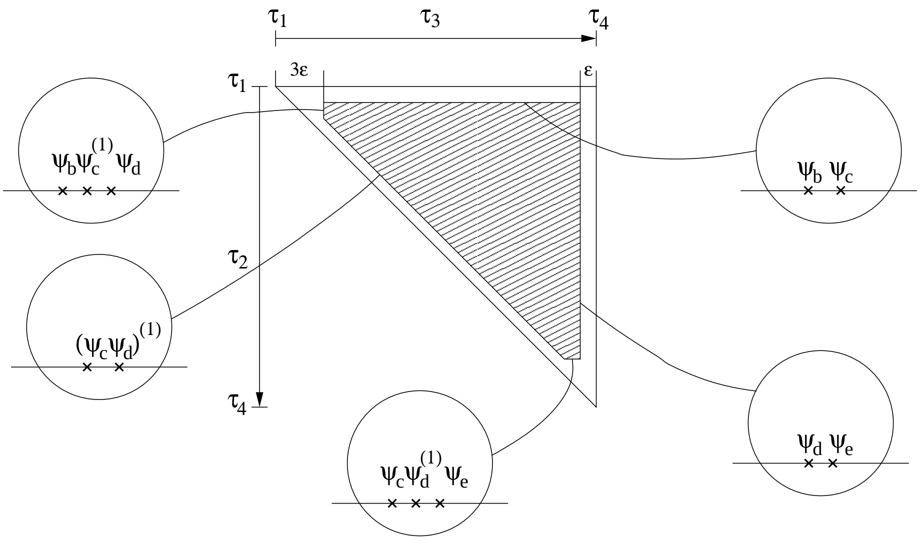

Figure 8.1: The integration domain $\mathbb{S}_5(\tau_2,\tau_5)$ and its boundary components (through a magnifying glass) for the correlation function $\big\langle\psi_a(\tau_0)\psi_b(\tau_1)\ P\!\int\psi_c^{(1)}(\tau_2)\!\int\psi_d^{(1)}(\tau_3)\ \psi_e(\tau_4)\big\rangle$.

$$
\begin{aligned}
&\sum_{k=2}^{m-1}(-1)^{s_k}\big\langle\psi_{a_0}\psi_{a_1}P\int\psi^{(1)}_{a_2}\dots\int\partial_{\tau_k}\psi_{a_k}\dots\int\psi^{(1)}_{a_{m-1}}\psi_{a_m}\big\rangle = \\
&=\sum_{k=2}^{m-1}\Bigg((-1)^{s_k}\int_{\tau_1}^{\tau_m}d\tau_k\ \partial_{\tau_k}\big\langle\psi_{a_0}\psi_{a_1}\int_{\tau_1}^{\tau_k}\psi^{(1)}_{a_2}\int_{\tau_2}^{\tau_k}\psi^{(1)}_{a_3}\dots\int_{\tau_{k-2}}^{\tau_k}\psi^{(1)}_{a_{k-1}}\psi_{a_k}\int_{\tau_k}^{\tau_{k+2}}\psi^{(1)}_{a_{k+1}}\dots\int_{\tau_k}^{\tau_m}\psi^{(1)}_{a_{m-1}}\psi_{a_m}\big\rangle \\
&\qquad-\sum_{l=2}^{k-1}\big\langle\psi_{a_0}\psi_{a_1}\int_{\tau_1}^{\tau_k}\psi^{(1)}_{a_2}\dots\Big[\psi^{(1)}_{a_l}\big|_{\tau_l\to\tau_k}\int_{\tau_l}^{\tau_k}\psi^{(1)}_{a_{l+1}}\dots\psi_{a_k}\Big]\int_{\tau_k}^{\tau_{k+2}}\psi^{(1)}_{a_{k+1}}\dots\int_{\tau_k}^{\tau_m}\psi^{(1)}_{a_{m-1}}\psi_{a_m}\big\rangle \\
&\qquad+\sum_{l=k+1}^{m-1}\big\langle\psi_{a_0}\psi_{a_1}\int_{\tau_1}^{\tau_k}\psi^{(1)}_{a_2}\dots\Big[\psi_{a_k}\dots\int_{\tau_k}^{\tau_l}\psi^{(1)}_{a_{l-1}}\psi^{(1)}_{a_l}\big|_{\tau_k\leftarrow\tau_l}\Big]\dots\int_{\tau_k}^{\tau_m}\psi^{(1)}_{a_{m-1}}\psi_{a_m}\big\rangle\Bigg) = \\
&=\sum_{k=2}^{m-1}\Bigg((-1)^{s_k}\big\langle\psi_{a_0}\psi_{a_1}P\int\psi^{(1)}_{a_1}\dots\int\psi^{(1)}_{a_{k-1}}\Big[\psi_{a_k}\big|_{\tau_k\to\tau_m}P\int\psi^{(1)}_{a_{k+1}}\dots\int\psi^{(1)}_{a_{m-1}}\psi_{a_m}\Big]\big\rangle \\
&\qquad-\big\langle\psi_{a_0}\Big[\psi_{a_1}P\int\psi^{(1)}_{a_2}\dots\int\psi^{(1)}_{a_{k-1}}\psi_{a_k}\big|_{\tau_1\leftarrow\tau_k}\Big]P\int\psi^{(1)}_{a_{k+1}}\dots\int\psi^{(1)}_{a_{m-1}}\psi_{a_m}\big\rangle \\
&\qquad-\sum_{l=2}^{k-1}\big\langle\psi_{a_0}\psi_{a_1}P\int\psi^{(1)}_{a_2}\dots\int\Big[\psi_{a_l}\big|_{\tau_l\to\tau_k}P\int\psi^{(1)}_{a_{l+1}}\dots\psi_{a_k}\Big]^{(1)}\dots\int\psi^{(1)}_{a_{m-1}}\psi_{a_m}\big\rangle\Bigg)\ ,
\end{aligned}
\tag{8.2}
$$

where the sign is given by $s_k = \tilde{a}_0 + \ldots + \tilde{a}_{k-1}$. In the second step we used the fact that the regularised configuration space is a simplex, which means that we have nested integration domains [1]. For notational simplicity, we do not indicate the cut-off ϵ in the integrals.

In the last form of (8.2), the terms in square brackets are products of boundary operators. In the limit $\epsilon \to 0$, we can factorise the result by pulling these terms out while inserting the sum $\sum_{a,b} \psi_c \omega^{cd} \psi_d$ over a basis of physical boundary operators. We conclude that the following expression must vanish:

$$\sum_{k=2}^{m-1} (-1)^{s_k} \Big(\langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_{k-1}} \psi_c \rangle \, \omega^{cd} \, \langle \psi_d \psi_{a_k} P \int \psi^{(1)}_{a_{k+1}} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle$$
$$- \langle \psi_{a_0} \psi_c P \int \psi^{(1)}_{a_{k+1}} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle \, \omega^{cd} \, \langle \psi_d \psi_{a_1} P \int \psi^{(1)}_{a_2} \ldots \int \psi^{(1)}_{a_{k-1}} \psi_{a_k} \rangle$$
$$- \sum_{l=2}^{k-1} \langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_2} \ldots \int \psi^{(1)}_{c} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \rangle \, \omega^{cd} \, \langle \psi_d \psi_{a_l} P \int \psi^{(1)}_{a_{l+1}} \ldots \psi_{a_k} \rangle \Big) \ .$$

We next rewrite this equation in terms of the quantities defined in equation (7.30). Using (7.11), we find:

$$\sum_{\substack{k,\, l\, =\, 2 \\ k-m+2\, <\, l\, \leq\, k}}^{m} (-1)^{\tilde{a}_1 + \ldots + \tilde{a}_{l-2}} B^b{}_{a_1 \ldots a_{l-2} c a_{k+1} \ldots a_m} B^c{}_{a_{l-1} \ldots a_k} = 0 \quad \text{for } m \geq 3 \ . \tag{8.3}$$

In deriving (8.3) we used the selection rules $\tilde{b} = \tilde{a}_1 + \ldots + \tilde{a}_l + 1$ for $B^b{}_{a_1 \ldots a_l}$ and $\tilde{\omega} = \tilde{a}_0 + \ldots + \tilde{a}_l$ for $B_{a_0 \ldots a_l}$. The restrictions in the sum account for the fact that the amplitudes $B_{a_0 \ldots a_l}$ are considered only for $l \geq 2$ (alternatively, one can remove these constraints and use definitions (7.12)).

The first equation in (8.3) is obtained for $m = 3$, and can be interpreted as associativity condition for the topological boundary product:

$$B^b{}_{c a_3} B^c{}_{a_1 a_2} + (-1)^{\tilde{a}_1} B^b{}_{a_1 c} B^c{}_{a_2 a_3} = 0 \ .$$

All other relations (for $m > 3$) include also boundary products $B^b{}_{a_1 \ldots a_l}$ with $l > 2$.

Algebraic description

To make contact with expressions found in the mathematics literature, let us bring (8.3) to a more familiar form. For this, we define tree-level *boundary scattering products* $r_m : H_o^{\otimes m} \to H_o$ to be the multilinear maps determined by the equations:

$$r_m(\psi_{a_1} \ldots \psi_{a_m}) = B^{a_0}{}_{a_1 \ldots a_m} \psi_{a_0} \quad , \tag{8.4}$$

where, as usual, we use implicit summation over repeated indices. The selection rule for $B^{a_0}{}_{a_1 \ldots a_m}$ gives:

$$\deg \ r_m(\psi_{a_1} \ldots \psi_{a_m}) = 1 + \sum_{j=1}^{m} \tilde{a}_j \quad ,$$

[1] For sake of easier reading, the nested integrals over τ_{k+1} to τ_{m-1} are partly written in the 'wrong' order.

so all maps r_m have degree one when H_o is endowed with the suspended grading. Equation (8.3) takes the form:

$$\sum_{\substack{k+l=m+1\\ j=0\dots k-1}}^{m} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_j} r_k(\psi_{a_1}\dots\psi_{a_j}, r_l(\psi_{a_{j+1}}\dots\psi_{a_{j+l}}), \psi_{a_{j+l+1}}\dots\psi_{a_m}) = 0 \quad , \tag{8.5}$$

where we set $r_0 = r_1 = 0$. Relations (8.5) define an A_∞ algebra [38, 39], in conventions in which all products have degree one. For reader's convenience, we summarise the standard terminology concerning such algebras:

(1) A collection of multilinear maps $r_m : H_o^{\otimes m} \rightarrow H_o$ of degree $+1$ satisfying (8.5) is called a *weak* A_∞ algebra if m is allowed to run from 0 to ∞.

(2) Such a collection is called a *strong* A_∞ algebra (or simply an A_∞ algebra) if m runs from 1 to infinity.

(3) Such a collection is a *minimal* A_∞ algebra if m runs from 2 to infinity.

Thus a (strong) A_∞ algebra is a weak A_∞ algebra for which $r_0 = 0$, while a minimal A_∞ algebra is a (strong) A_∞ algebra for which $r_1 = 0$. The algebra obtained above is a minimal A_∞ algebra. As we shall see below, bulk perturbations will generically deform this to a *weak* A_∞ algebra. This corresponds to the appearance of a tadpole induced by deformations of the closed string background.

Due to the cyclicity property (7.24) of disk amplitudes, our minimal A_∞ algebra is in fact *cyclic* with respect to the bilinear form on H_o, which is defined by the boundary topological metric:

$$\omega(\psi_a, \psi_b) = \omega_{ab} \tag{8.6}$$

Writing:

$$B_{a_0\dots a_m} = \omega(\psi_{a_0}, r_m(\psi_{a_1}\dots\psi_{a_m})) \quad ,$$

cyclicity is simply the condition (7.24) expressed in terms of string scattering products:

$$\omega(\psi_{a_0}, r_m(\psi_{a_1}\dots\psi_{a_m})) = (-1)^{\tilde{a}_m(\tilde{a}_0+\dots+\tilde{a}_{m-1})}\omega(\psi_{a_m}, r_m(\psi_{a_0}\dots\psi_{a_{m-1}})) \quad . \tag{8.7}$$

A further constraint follows from equations (7.18), which imply:

$$B^c{}_{a_1\dots a_{i-1}0a_{i+1}\dots a_m} = 0 \quad \text{for} \quad m \geq 3 \quad \text{and all } i = 1\dots m \quad ,$$

i.e.:

$$r_m(\psi_{a_1}\dots\psi_{a_{i-1}}, 1_o, \psi_{a_{i+1}}\dots\psi_{a_m}) = 0 \quad \text{for} \quad m \geq 3 \quad \text{and all } i = 1\dots m \quad . \tag{8.8}$$

On the other hand, we have:

$$r_2(\psi_a, \psi_b) = B^c{}_{ab}\psi_c \ .$$

Using the fact that 1_o is a unit for the boundary algebra, this gives:

$$r_2(1_o, \psi_a) = (-1)^{\tilde{a}} r_2(\psi_a, 1_o) = \psi_a \quad . \tag{8.9}$$

Equations (8.8) and (8.9) mean that (H_o, r_*) is *unital*, so that in total we have a unital, cyclic, minimal A_∞ algebra (see, for example, [87]).

Remark: The whole analysis of Ward identities remains true if we include boundary condition changing operators (4.26) in our considerations. In fact, the indices labelling the boundary conditions were just suppressed for notational convenience and can be recast in all amplitudes. The reason is the cyclic invariance of the boundary amplitudes. This observation is also true in subsequent sectors. When considering boundary condition changing sectors, the A_∞ algebra discussed above generalises to an A_∞ category [44].

8.2 Weak A_∞ constraints for deformed amplitudes

In the present subsection, we extend the discussion of A_∞ constraints to general open-closed amplitudes on the disk. We shall show that the A_∞ structure exhibited in Section 8.1 is promoted to a so-called *weak* A_∞ algebra, which is again cyclic and unital. For simplicity we start by discussing the case of a single boundary insertion. As we shall see below, these amplitudes can be used to define a first order deformation of the A_∞ algebra of Section 8.1, a deformation which preserves cyclicity and unitality but need not preserve minimality. We shall also discuss the general case of multiple insertions, which defines an all-order (formal) deformation in the bulk parameters t_i.

Disk amplitudes with a single bulk insertion

Insertions of bulk operators perturb the minimal A_∞ algebra extracted in Section 8.1. We first consider linear perturbations, which amount to inserting just one bulk operator in the disk amplitudes:

$$\left\langle [Q, \phi_i \psi_{a_0} P \int \psi_{a_1}^{(1)} \dots \int \psi_{a_m}^{(1)}] \right\rangle = 0 \quad . \tag{8.10}$$

As in Section 8.1, acting with Q on the integrated descendants on the left hand side produces terms in which several boundary fields approach each other. In the limit $\epsilon \to 0$, we can factorise the result by inserting complete systems of open string physical states. Notice that the integration domain for equation (8.10) differs from that of equation (8.1), because we have only one fixed boundary operator. This makes the computation more involved.

Let us illustrate this with the simplest non-trivial case, namely $m = 2$. The integration domain and its boundary components are shown in Figure 8.2. Using the descent equation (4.27), the left hand side of (8.10) becomes:

$$(-1)^{a_0} \left\langle \phi_i \psi_{a_0} \int_{\tau_0^+ + \epsilon}^{\tau_0^- - 3\epsilon} \partial_{\tau_1} \psi_{a_1} \int_{L_2(\tau_0,\tau_1)}^{R_2(\tau_0,\tau_1)} \psi_{a_2}^{(1)} \right\rangle + (-1)^{a_0+a_1+1} \left\langle \phi_i \psi_{a_0} \int_{L_1(\tau_0,\tau_2)}^{R_1(\tau_0,\tau_2)} \psi_{a_1}^{(1)} \int_{\tau_0^+ + 3\epsilon}^{\tau_0^- - \epsilon} \partial_{\tau_2} \psi_{a_2} \right\rangle \quad . \tag{8.11}$$

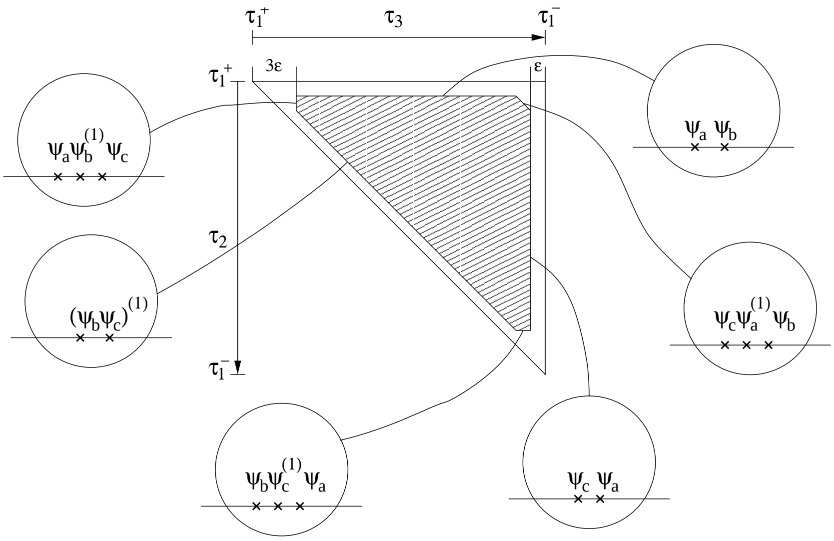

Figure 8.2: The integration domain $\mathbb{S}_3(\tau_1)$ and its boundary components (through a magnifying glass) for the correlation function $\langle \phi_i(w,\bar{w})\ \psi_a(\tau_1)\ P\int\psi_b(\tau_2)\int\psi_c(\tau_3)\rangle$. The real line, as boundary of the disk, was compactified to a circle by identifying τ_1^+ and τ_1^-.

The boundary of the integration domain can be inferred from (7.7) and is shown in Figure 8.2. Its components are given by:

$$R_2(\tau_0,\tau_1) = \begin{cases} \tau_0^- - \epsilon & \text{for} \quad \tau_1 > \tau_0^+ + 2\epsilon \\ \tau_0^- - 3\epsilon + (\tau_1 - \tau_0^+) & \text{for} \quad \tau_1 < \tau_0^+ + 2\epsilon \end{cases}$$

$$L_2(\tau_0,\tau_1) = \begin{cases} \tau_1 + \epsilon & \text{for} \quad \tau_1 > \tau_0^+ + 2\epsilon \\ \tau_0^+ + 3\epsilon & \text{for} \quad \tau_1 < \tau_0^+ + 2\epsilon \end{cases} ,$$

with similar expressions for R_1 and L_1. As in the derivation of Section 8.1, we use partial integration taking into account all boundary contributions. Compared to Section 8.1, we have an additional contribution from the upper right corner of the regularised configuration space in Figure 8.2, which comes from the boundary components R_2 for $\tau_0^+ + \epsilon < \tau_1 < \tau_0^+ + 2\epsilon$ and L_1 for $\tau_0^- - \epsilon > \tau_2 > \tau_0^- - 2\epsilon$. This contribution takes the form:

$$(-1)^{a_0+1+(a_2+1)(a_0+a_1)} \langle \phi_i \int_{\tau_0^+ + \epsilon}^{\tau_0^+ + 2\epsilon} d\tau_1 \Big(\ \psi_{a_2}^{(1)}(\tau_1 - 3\epsilon)\ \psi_{a_0}(\tau_0)\ \psi_{a_1}(\tau_1) +$$

$$+ (-1)^{a_0+a_2}\ \psi_{a_2}(\tau_1 - 3\epsilon)\ \psi_{a_0}(\tau_0)\ \psi_{a_1}^{(1)}(\tau_1)\Big)\Big\rangle$$

$$= \ -(-1)^{a_0+(a_2+1)(a_0+a_1+1)}\ \Big\langle \phi_i \Big(\psi_{a_2}(\tau_1 - 3\epsilon) \int_{\tau_1 - 2\epsilon}^{\tau_1 - \epsilon} d\tau_0 \psi_{a_0}^{(1)}(\tau_0)\ \psi_{a_1}(\tau_1)\Big)\Big\rangle \ ,$$

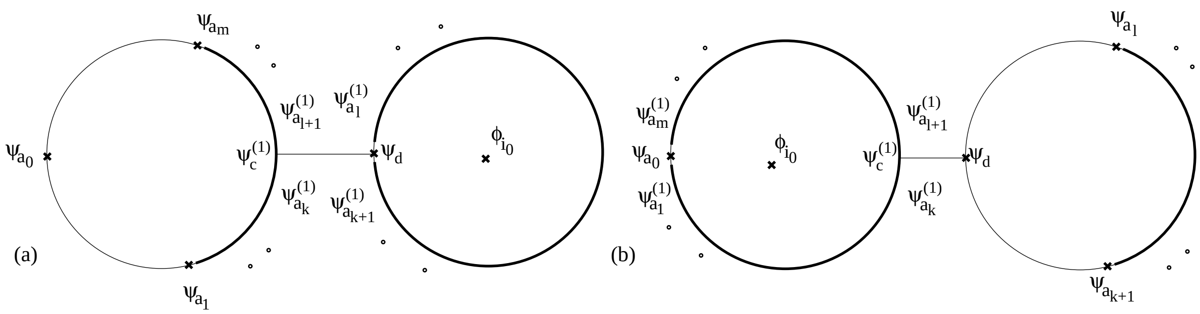

Figure 8.3: The two contributions to the factorisations leading to equation (8.14). Figure (a) shows a summand of the first term in this equation, while Figure (b) shows a summand of the second term.

where we used a Ward identity corresponding to the current G to 'move' the integral from τ_1 to τ_0.[2] Collecting all terms and factorising as in Section 8.1, we find that relation (8.10) reduces to:

$$\begin{aligned}
&(-1)^{\tilde{a}_0}\big\langle\phi_i\psi_c\int\psi^{(1)}_{a_2}\big\rangle\omega^{cd}\big\langle\psi_d\psi_{a_0}\psi_{a_1}\big\rangle\\
+\;&(-1)^{\tilde{a}_0(\tilde{a}_1+\tilde{a}_2)+\tilde{a}_1+\tilde{a}_2}\big\langle\phi_i\psi_c\big\rangle\omega^{cd}\big\langle\psi_d\psi_{a_1}\int\psi^{(1)}_{a_2}\psi_{a_0}\big\rangle\\
+\;&(-1)^{\tilde{a}_2(\tilde{a}_0+\tilde{a}_1)+\tilde{a}_2}\big\langle\phi_i\psi_c\int\psi^{(1)}_{a_1}\big\rangle\omega^{cd}\big\langle\psi_d\psi_{a_2}\psi_{a_0}\big\rangle\\
+\;&(-1)^{\tilde{a}_0+\tilde{a}_1}\big\langle\phi_i\psi_c\big\rangle\omega^{cd}\big\langle\psi_d\psi_{a_0}\int\psi^{(1)}_{a_1}\psi_{a_2}\big\rangle\\
+\;&(-1)^{\tilde{a}_0+\tilde{a}_2+\tilde{a}_2(\tilde{a}_0+\tilde{a}_1)}\big\langle\phi_i\psi_c\big\rangle\omega^{cd}\big\langle\psi_d\psi_{a_2}\int\psi^{(1)}_{a_0}\psi_{a_1}\big\rangle\\
+\;&(-1)^{\tilde{a}_0+\tilde{a}_1}\big\langle\phi_i\psi_{a_0}\int\psi^{(1)}_c\big\rangle\omega^{cd}\big\langle\psi_d\psi_{a_1}\psi_{a_2}\big\rangle\;=\;0\quad.
\end{aligned}\tag{8.12}$$

When expressed in terms of the quantities defined in (7.11), this equation takes the form:

$$\begin{aligned}
&B^{a_0}{}_{d;i}B^d{}_{a_1a_2}+B^{a_0}{}_{da_2}B^d{}_{a_1;i}+(-1)^{\tilde{a}_1}B^{a_0}{}_{a_1d}B^d{}_{a_2;i}\\
+\;&B^{a_0}{}_{da_1a_2}B^d{}_i+(-1)^{\tilde{a}_1}B^{a_0}{}_{a_1da_2}B^d{}_i+(-1)^{\tilde{a}_1+\tilde{a}_2}B^{a_0}{}_{a_1a_2d}B^d{}_i\;=\;0\quad.
\end{aligned}\tag{8.13}$$

[2] More precisely, we used the relation:

$$\oint\xi(w)\big\langle G(w)\;\phi_i(z,\bar{z})\;\psi_{a_2}(\tau_2)\;\psi_{a_0}(\tau_0)\;\psi_{a_1}(\tau_1)\big\rangle=0\;,$$

with $\xi(w)=(w-z)(w-\bar{z})$ as well as the fact that correlators depend only on the cross ratio $\zeta_l=\frac{(z-\bar{z})(\tau_l-\tau_0)}{(z-\tau_l)(\bar{z}-\tau_0)}$ for $l=1,2$. In the limit $\tau_2\to\tau_1-3\epsilon$, we have $\zeta_2=\zeta_1+\mathcal{O}(\epsilon)$ and we obtain the Jacobian $\left|\frac{\partial\tau_1}{\partial\tau_0}\right|$ up to terms $\mathcal{O}(\epsilon)$, which vanish in our limit.

The general case is a straightforward generalisation, but the computations are much more tedious. Therefore, we shall give the result without presenting the details of its proof. It is the natural generalisation of (8.13) for $m \geq 1$:

$$\begin{aligned} &\sum_{j=0}^{m}\sum_{k=0}^{j}(-1)^{\tilde{a}_1+\ldots+\tilde{a}_k}\ B^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m}\ B^{c}{}_{a_{k+1}\ldots a_j;i} \ + \\ +&\sum_{j=2}^{m}\sum_{k=0}^{j}(-1)^{\tilde{a}_1+\ldots+\tilde{a}_k}\ B^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m;i}\ B^{c}{}_{a_{k+1}\ldots a_j} \ = \ 0 \ , \end{aligned} \tag{8.14}$$

where we explicitly set $B^c{}_a = B^c = 0$, consistent with definitions (7.12). The first of equations (8.14) is obtained for $m = 1$ and coincides with the first bulk-boundary sewing constraint of TFT [82, 83]:

$$B^{a_0}{}_{ca_1}B^{c}{}_{;i} + (-1)^{\tilde{a}_1}B^{a_0}{}_{a_1c}B^{c}{}_{;i} = 0 \ . \tag{8.15}$$

General disk amplitudes

We now turn to the general case, extending the argument of the previous subsection to an arbitrary number of bulk insertions.

Consider a general disk amplitude written in the form:

$$\left\langle [Q, \phi_{i_0}\psi_{a_0}P\int\psi^{(1)}_{a_1}\ldots\int\psi^{(1)}_{a_m}\int\phi^{(2)}_{i_1}\ldots\int\phi^{(2)}_{i_n}]\right\rangle = 0 \ . \tag{8.16}$$

Acting with the Q-commutator on the integrated boundary insertions produces a sum over the terms appearing in equation (8.14). Additionally, we have all contributions from integrated bulk descendants:

$$\sum_{I\subseteq I_{0,n}}\sum_{k\leq j}^{m}(-1)^{\tilde{a}_1+\ldots+\tilde{a}_k}\ B^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m;I_{0,n}\setminus I}\ B^{c}{}_{a_{k+1}\ldots a_j;I} \ , \tag{8.17}$$

where $I_{p,q} = \{i_p, i_{p+1}, \ldots, i_{q-1}, i_q\}$. Note that the Q-variation of boundary fields does not produce terms containing $B^a{}_{I\setminus i_0}$ and $B^a{}_{b;I\setminus i_0}$. Instead, these missing terms arise from the Q-variation of the integrated bulk insertion:

$$[Q, \int\phi^{(2)}_{i_k}] = \lim_{\epsilon\to 0}\oint_{k\epsilon}\phi^{(1)}_{i_k} \ . \tag{8.18}$$

The integral runs along a loop, which follows the boundary (namely the real axis) at a distance $k\epsilon$. Using equation (8.18), we obtain contributions from a bulk operator approaching the boundary far away from any boundary operator, and from a bulk operator approaching a boundary insertion. Due to our regularisation (7.3), the loop (8.18) cuts the integration domains of the operators $\phi^{(2)}_{i_1}\ldots\phi^{(2)}_{i_{k-1}}$ into a part near the boundary and a bulk part. On the other hand, the operators $\phi^{(2)}_{i_{k+1}} \dots \phi^{(2)}_{i_n}$ are inside the loop and hence they don't produce more contact terms. In the limit $\epsilon \to 0$, factorisation proceeds by distributing the former operators in all possible ways on the two emerging disks.

As an example, consider the piece of the boundary sitting between ψ_{a_l} and $\psi_{a_{l+1}}$. A typical term generated by the process above has the form:

$$\langle \dots \int \psi_{a_l} \int_{\tau_l}^{\tau_{l+1}} \phi^{(1)}_{i_k} \int \psi_{a_{l+1}} \dots \prod_{j \neq k} \int \phi^{(2)}_{i_j} \rangle \quad .$$

Its factorisation produces the contributions $\pm \sum_{I \subseteq I_{1,k-1}} B_{\dots a_l c a_{l+1} \dots ; i_0 I_{1,n} \setminus \{i_k I\}} B^c{}_{i_k I}$, where we used the notation $\{i_k I\} = \{i_k\} \cup I$. Summing over k leads to a total contribution:

$$\pm \sum_k \sum_{I \subseteq I_{1,k-1}} B_{\dots a_l c a_{l+1} \dots ; i_0 I_{1,n} \setminus \{i_k I\}} B^c{}_{i_k I} = \pm \sum_{I \subseteq I_{1,n}} B_{\dots a_l c a_{l+1} \dots ; i_0 I_{1,n} \setminus I} B^c{}_I \quad .$$

Similarly, the factorisation of a bulk operator approaching an *integrated* boundary field gives rise to the terms:

$$\pm \sum_{I \subseteq I_{1,n}} B_{\dots a_{l-1} c a_{l+1} \dots ; i_0 I_{1,n} \setminus I} B^c{}_{a_l ; I} \quad .$$

Finally, bulk operators approaching the fixed insertion ψ_{a_0} produce:

$$\pm \sum_{I \subseteq I_{1,n}} B^{a_0}{}_{c ; I_{1,n} \setminus I} B^c{}_{a_1 \dots a_n ; i_0 I} \quad .$$

This completes the list of contributions from the boundary of the configuration space.

Gathering all terms, we find that equation (8.16) can be written in the following form:

$$\sum_{I \subseteq I_{0,n}} \sum_{\substack{k,\, j = 0 \\ k \leq j}}^{m} (-1)^{\tilde{a}_1 + \dots + \tilde{a}_k} \; B^{a_0}{}_{a_1 \dots a_k c a_{j+1} \dots a_m ; I_{0,n} \setminus I} \; B^c{}_{a_{k+1} \dots a_j ; I} \;=\; 0 \quad . \tag{8.19}$$

For $n = 0$, this reduces to equation (8.14) extracted in the previous subsection.

Notice that indices distribute in the same manner as would derivatives with respect to the formal parameters t^j. This means that we can concisely write relations (8.19) as weak A_∞ constraints for the perturbed boundary amplitude $\mathcal{F}_{a_0 \dots a_m}(t)$:

$$\sum_{\substack{k,\, j = 0 \\ k \leq j}}^{m} (-1)^{\tilde{a}_1 + \dots + \tilde{a}_k} \; \mathcal{F}^{a_0}{}_{a_1 \dots a_k c a_{j+1} \dots a_m}(t) \; \mathcal{F}^c{}_{a_{k+1} \dots a_j}(t) \;=\; 0 \quad . \tag{8.20}$$

Expanding this as a power series in t reproduces equations (8.19). The first two deformed amplitudes: $\mathcal{F}_a$ and $\mathcal{F}_{ab}$ are of order at least one in t_i, since B_a and B_{ab} vanish. The presence of these terms for $t \neq 0$ promotes the minimal A_∞ algebra of Section 8.1 to a weak A_∞ algebra, and corresponds to the generation of non-vanishing tadpoles, as discussed in section 7.7.

Algebraic formulation

Extending the discussion of Section 8.1, let us define *deformed open string scattering products* $r^t_m : H_o^{\otimes m} \to H_o$ through the relations:

$$r^t_m(\psi_{a_1} \dots \psi_{a_m}) = \mathcal{F}^a{}_{a_1 \dots a_m}(t)\psi_a \quad \text{for} \quad m \geq 1 \quad . \tag{8.21}$$

and:

$$r^t_o(1) := \mathcal{F}^a(t)\psi_a \quad ,$$

where $\mathcal{F}^a(t) = \omega^{ab}\mathcal{F}_b(t)$ and we used the fact that the product $r_o : H_o^{\otimes 0} \approx \mathbb{C} \to H_o$ is determined by its value at the complex unit $1 \in \mathbb{C}$. As in Section 8.1, equations (8.20) become:

$$\sum_{\substack{k+l=m+1 \\ j=0\dots k-1}}^{m} (-1)^{\tilde{a}_1 + \dots + \tilde{a}_j} r^t_k(\psi_{a_1} \dots \psi_{a_j}, r^t_l(\psi_{a_{j+1}} \dots \psi_{a_{j+l}}), \psi_{a_{j+l+1}} \dots \psi_{a_m}) = 0 \quad , \tag{8.22}$$

which are the standard relations defining a weak A_∞ algebra.

Remembering equation (7.27), we find that this weak A_∞ algebra is cyclic:

$$\omega(\psi_{a_0}, r^t_m(\psi_{a_1} \dots \psi_{a_m})) = (-1)^{\tilde{a}_m(\tilde{a}_0 + \dots + \tilde{a}_{m-1})}\omega(\psi_{a_m}, r^t_m(\psi_{a_0} \dots \psi_{a_{m-1}})) \quad \text{for} \quad m \geq 1 \quad . \tag{8.23}$$

Moreover, equations (7.28) and (7.29) show that (H_o, r^t_*) is unital:

$$r^t_m(\psi_{a_1} \dots \psi_{a_{i-1}}, 1_o, \psi_{a_{i+1}} \dots \psi_{a_m}) = 0 \quad \text{for} \quad (m = 1 \text{ or } m \geq 3) \quad \text{and all } i = 1 \dots m \tag{8.24}$$

and:

$$r^t_2(1_o, \psi_a) = (-1)^{\tilde{a}} r^t_2(\psi_a, 1_o) = \psi_a \quad . \tag{8.25}$$

To arrive at the last equation, we used relation (7.29) and non-degeneracy of ω.

8.3 Interpretation in terms of open string field theory

The algebraic formulation given above allows us to give an alternate description of the effective superpotential, which makes contact with open string field theory as formulated by Zwiebach (see [89] and references therein). Let us consider the object:

$$\psi := \sum_a s_a \psi_a \quad , \tag{8.26}$$

viewed as an element of the graded vector space $H_o^e := \mathbb{A} \otimes H_o$. When H_o is endowed with the suspended grading, the quantity ψ is even as an element of H_o^e. Using definition (8.21), we find the following expression for the deformed boundary amplitudes:

$$\mathcal{F}_{a_0 \dots a_m}(t) = \omega(\psi_{a_0}, r_m^t(\psi_{a_1} \dots \psi_{a_m})) \quad .$$

We would like to express this in terms of ψ. For this, consider the natural extension of ω to H_o^e, which we shall denote by the same letter. This is a bilinear form on H_o^e given as follows on decomposable elements:

$$\omega(f \otimes \psi_a, g \otimes \psi_b) = (-1)^{\tilde{a}\tilde{g}} f g \omega_{ab} \quad ,$$

where f, g are homogeneous elements of $\mathbb{A}$ of degrees $\tilde{f}$ and $\tilde{g}$. We also extend r_m^t to multilinear products on H_o^e (again denoted by the same symbol) through:

$$r_m^t(f_1 \psi_{a_1} \dots f_m \psi_{a_m}) = (-1)^{\sum_{j=2}^m (\tilde{a}_1 + \dots + \tilde{a}_{j-1})\tilde{f}_j} f_1 \dots f_m r_m^t(\psi_{a_1} \dots \psi_{a_m}) \quad .$$

With these definitions, we have:

$$r_m^t(\psi \dots \psi) = s_{a_m} \dots s_{a_1} r_m^t(\psi_{a_1} \dots \psi_{a_m})$$

and:

$$s_{a_m} \dots s_{a_0} \mathcal{F}_{a_0 \dots a_m}(t) = \omega(\psi, r_m(\psi \dots \psi)) \quad .$$

Thus equation (7.34) becomes:

$$\mathcal{W}(s,t) = \sum_{m \geq 0} \frac{1}{m+1} \omega(\psi, r_m^t(\psi^{\otimes m})) \quad .$$

This is the standard form [40] of an open string field action, though built around a background which need not satisfy the open string equations of motion (as reflected by the presence of the product r_0^t). In this interpretation, the object ψ is identified with the string field. As expected, the parameters t encode a deformation of this action, parameterised by a choice of the closed string background.

Fixing the closed string background (i.e. treating t_i as fixed parameters), the open string equations of motion take the form:

$$\partial_a \mathcal{W}(s,t) = 0 \Longleftrightarrow \sum_{m=0}^{\infty} r_m^t(\psi^{\otimes m}) = 0 \quad , \tag{8.27}$$

where $\partial_a = \frac{\partial}{\partial s^a}$. This algebraic condition is known as the Maurer-Cartan equation for a weak A_∞ algebra. The presence of r_0^t signals the fact that the reference point $s = 0$ does not satisfy this equation. Indeed, the left hand side of (8.27) at $s = 0$ equals $r_0(\psi^{\otimes 0}) := r_0(1)$.

Cancelling the tadpole

As mentioned above, deformations of the closed string background will generally produce a tadpole which must be cancelled if the deformed theory is to have a chance of being conformal. In this subsection, we explain how this can be achieved by shifting the open string vacuum, thereby making contact with previous mathematical work.

Consider a shift:

$$s_a \rightarrow s_a + \sigma_a \quad , \tag{8.28}$$

of the formal variables s_a. Here σ_a are finite variations of s_a, i.e. formal supercommuting parameters of degree $\tilde{a}$, which are taken to supercommute with s_a:

$$[\sigma_a, s_b] = 0 \quad .$$

In terms of the string field (8.26), this operation amounts to:

$$\psi \rightarrow \psi + \alpha \quad ,$$

where $\alpha := \sum_a \sigma_a \psi_a$ is an even element of H_o^e .

It is not hard to check by substitution of this shift into the string field action (8.27) that the deformed scattering products change as:

$$r_m^t \rightarrow r_m^{t;\sigma}$$

where:

$$r_m^{t;\sigma}(u_1 \dots u_m) = r^t(e^\alpha, u_1, e^\alpha, u_2, \dots, e^\alpha, u_m, e^\alpha) \tag{8.29}$$

for all $u_1 \dots u_m \in H_o^e$.

In equation (8.29), we used the notations:

$$e^\alpha := \sum_{k=0}^{\infty} \alpha^{\otimes k}$$

and:

$$r^t := \sum_{m=0}^{\infty} r_m^t \quad .$$

Notice that r^t is a map from $\oplus_{m=0}^{\infty}(H_o^e)^{\otimes m}$ to H_o^e.

In particular, the product r_0^t becomes:

$$r_0^{t;\sigma} = r^t(e^\alpha) = \sum_{m=0}^{\infty} r_m^t(\alpha \dots \alpha) \quad .$$

Hence the tadpole amplitude $\mathcal{F}_a(t) = \omega(\psi_a, r_0^t(1))$ vanishes if and only if:

$$\sum_{m=0}^{\infty} r_m^t(\alpha \dots \alpha) = 0 \Longleftrightarrow (\partial_a \mathcal{W})(\sigma, t) = 0 \quad . \tag{8.30}$$

This amounts to the well-known fact that the equations of motion for (open) string theory amount to the tadpole cancellation condition. It is not hard to check by direct computation that the products $r_m^{t;\sigma}$ with $m \geq 1$ form a *strong* A_∞ algebra provided that this equation is satisfied. Hence the Maurer-Cartan condition (8.30) describes possible transformations of a weak A_∞ algebra into a (strong) A_∞ algebra obtained by shifts of the form (8.28).

Given a solution σ of (8.30), the expansion of $\mathcal{W}$ around the new open string vacuum takes the form:

$$\mathcal{W}(s,t) = \sum_{m\geq 2} \frac{1}{m+1}\omega(\psi, r_m^{t,\sigma}(\psi^{\otimes m})) + \mathcal{W}(\sigma,t) \quad .$$

Up to the last term (which is s-independent), this is the standard form of the open string field action in the formulation of [40].

We mention that condition (8.30) plays a crucial role in the work of [44,84,86,87], where it originates in a very similar manner (see [43] for a detailed discussion).

Relation to the deformation theory of cyclic A_∞ algebras

The results deduced in this subsection are intimately related to the deformation theory of cyclic A_∞ algebras as developed in [88]. This interpretation is quite obvious, so we can be brief.

It is clear from the discussion above that insertion of bulk operators realizes an all-order deformation of the A_∞ algebra of Section 8.1, viewed as a weak A_∞ algebra which happens to be minimal for the particular value $t = 0$ of the deformation parameters. Moreover, such deformations preserve cyclicity and unitality.

To make contact with the work of [88], let us consider the case of infinitesimal deformations discussed in Subsection 8.2. This can be recovered from the more general results of the previous subsection by expanding the products r_m^t to first oder in t. Writing:

$$r_m^t = r_m + t_i\Phi_m^i + \mathcal{O}(t^2) \quad ,$$

we extract morphisms:

$$\Phi_m^i = \frac{\partial r_m^t}{\partial t_i}\Big|_{t=0} : H_o^{\otimes m} \to H_o \quad . \tag{8.31}$$

The objects $\Phi^i := \sum_{m=0}^\infty \Phi_m^i$ belong to the so-called (weak) Hochschild complex $C = \oplus_{m=0}^\infty C^m(H_o)$ of H_o, whose graded subspaces are defined through:

$$C^m(H_o) := Hom(H_o^{\otimes m}, H_o)$$

and whose differential is given by the first order variation of the weak A_∞ constraints (8.22):

$$(\delta\Phi^i)_m(\psi_{a_1}\dots\psi_{a_m}) = \frac{\partial A_m(t)}{\partial t}\Big|_{t=0} \quad , \tag{8.32}$$

where $A_m(t)$ is the left hand side of (8.22). In equation (8.32), it is understood that we replace $\left.\frac{\partial r^t_m}{\partial t_i}\right|_{t=0}$ by Φ^i_m through relation (8.31) and view the result $\delta\Phi^i$ as the action of an algebraic operator δ on Φ^i.[3] The fact that δ squares to zero follows from the A_∞ constraints.

Because our A_∞ algebras are cyclic, one has further restrictions on Φ^i which amount to the statement that they are elements of a certain subcomplex $CC(H_o)$ known as the *cyclic complex*. This can be defined as the set of elements $\Phi = \sum_m \Phi_m$ in $C(H_o)$ with the property that the quantities $\omega(\psi_{a_0}, \Phi_m(\psi_{a_1}\dots\psi_{a_m}))$ obey the cyclicity constraints:

$$\omega(\psi_{a_0}, \Phi_m(\psi_{a_1}\dots\psi_{a_m})) = (-1)^{\tilde{a}_m(\tilde{a}_0+\dots+\tilde{a}_{m-1})}\omega(\psi_{a_m}, \Phi_m(\psi_{a_1}\dots\psi_{a_{m-1}})) \quad .$$

For our maps Φ^i, these conditions follow by differentiating (8.23) with respect to t_i at $t = 0$. The Hochschild differential δ preserves the subspace $CC(H_o)$. Denoting its restriction by the same letter, one obtains the cyclic complex $(CC(H_o), \delta)$ considered in [88].[4]

Since the deformed products (8.21) obey weak A_∞ constraints for all t, differentiation of (8.22) at $t = 0$ shows that Φ^i are δ-closed:

$$\delta\Phi^i = 0 \quad .$$

Thus Φ^i define elements $[\Phi^i]$ of the cohomology of $(CC(H_o), \delta)$, known as the (weak) cyclic cohomology of the A_∞ algebra (H_o, r_*). Comparing with Subsection 8.2, it is easy to see that Φ^i can be written as:

$$\Phi^i_m(\psi_{a_1}\dots\psi_{a_m}) = B^a_{a_1\dots a_m;i}\psi_a \quad .$$

This shows that they are completely determined by the disk amplitudes $B_{a_0\dots a_m;i}$ with a single bulk insertion. Thus one has a map:

$$\phi_i \to [\Phi^i]$$

from Q-closed bulk zero-form observables to the cyclic cohomology of the A_∞ algebra (H_o, r_*). A similar statement was proposed in [42] in a particular case.

8.4 Bulk-boundary crossing symmetry

The second bulk-boundary crossing constraint of two-dimensional topological field theory [82, 83] states that the bulk-boundary map is a morphism from the bulk to

[3] Strictly speaking, this specifies δ only for elements Φ of $C(H_o)$ such that each Φ_m has degree one as a map from $H_o^{\otimes m}$ to H_o. However the definition generalises to the entire Hochschild complex.

[4] Our sign conventions differ from those of [88] by suspension. Moreover, we allow for the subspace $C^0(H_0) = CC(H_0) = \mathbb{C}$ in the Hochschild and cyclic complexes, since we consider deformations of *weak* and cyclic A_∞ algebras.

the boundary algebra [82]. In this section, we discuss the 'stringy' generalisation of this constraint.

It is clear that the relevant property arises from factorisation of the amplitude:

$$\left\langle \phi_i \phi_j \; \psi_{a_0} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_m} \right\rangle , \tag{8.33}$$

into the channel where the two bulk fields approach each other and the channel where both bulk fields approach the boundary. In contrast to the A_∞ constraints, this factorisation follows from explicit movement of the bulk operators rather than from the Ward identities of the Q-symmetry. This is similar to the mechanism leading to the WDVV equations (6.8). In the case at hand we have to deal with a subtlety which requires closer examination: we know from section 7 that only the fundamental amplitudes (7.11) are independent of the positions of unintegrated insertions. This is not the case for the amplitude (8.33), since it contains two bulk and one boundary unintegrated insertions. Therefore, it is not immediately clear that factorising (8.33) makes sense. The naive guess for the factorisation takes the form:

$$\begin{aligned}
&C^l{}_{ij} \left\langle \phi_l \; \psi_{a_0} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_m} \right\rangle = \\
&= \sum_{0 \le m_1 \le \ldots m_4 \le m} \left\langle \psi_{a_0} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_{m_1}} \; \psi_b \; P \int \psi^{(1)}_{a_{m_2+1}} \ldots \int \psi^{(1)}_{a_{m_3}} \; \psi_c \; P \int \psi^{(1)}_{a_{m_4+1}} \ldots \int \psi^{(1)}_{a_m} \right\rangle \times \\
&\quad \times \omega^{bd} \, \omega^{ce} \left\langle \phi_i \; \psi_d P \int \psi^{(1)}_{a_{m_1+1}} \ldots \int \psi^{(1)}_{a_{m_2}} \right\rangle \left\langle \phi_j \; \psi_e P \int \psi^{(1)}_{a_{m_3+1}} \ldots \int \psi^{(1)}_{a_{m_4}} \right\rangle ,
\end{aligned} \tag{8.34}$$

where $C^l{}_{ij}$ are the usual bulk ring structure constants. Since the correlation function (8.33) is not independent of the positions of the fixed insertions, we shall give an independent argument for why this relation holds. In the following, we shall denote the left hand side of (8.34) simply by (lhs), and the right hand side by (rhs).

To establish equation (8.34), we consider the amplitude (8.33) for the configurations (A) and (B) of bulk operators on the upper half plane shown in Figure 8.4. Let us denote the distance between the bulk operators by $t \in \mathbb{R}$ and assume that the two bulk operators sit on a line parallel at a distance b to the boundary. In the limit $t \to 0$, configuration (A) corresponds to the left hand side of equation (8.33), while the right hand side of this equation arises in the limit $b \to 0$ of configuration (B).

For configuration (A), we have $t = t_0$ with $|t_0| \ll 1$ and we can perform a bulk operator product expansion in t_0, so that (8.33) becomes $(lhs) + g_1(t_0, b)$, where $g_1(t_0, b) = \mathcal{O}(t_0)$ denotes contributions from higher terms in the OPE. Moving along the path p_A down toward configuration (C), the function $g_1(t_0, b)$ changes with b and becomes $g_1(t_0, b_0)$. On the other hand, configuration (B) shows the bulk operators at the distance $b = b_0 \ll 1$ from the boundary. According to the bulk-boundary operator product the amplitude (8.33) takes the form $(rhs) + g_2(t, b_0)$,

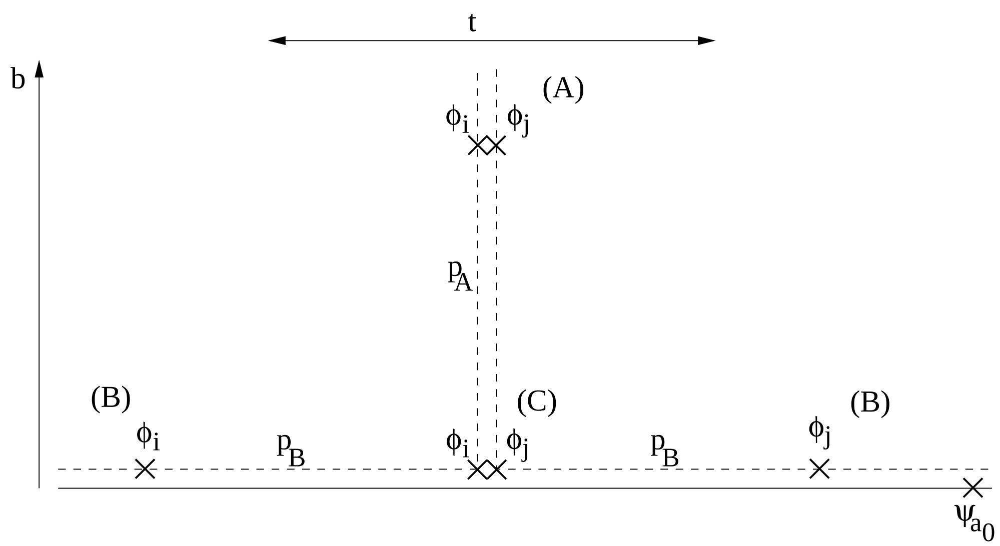

Figure 8.4: The factorisation associated with the stringy version of the second bulk-boundary crossing constraint. Configuration (A) corresponds to the topological bulk product and (B) to the factorisation at the boundary. Configuration (C) connects these channels. The quantity b is the equal distance of the bulk fields from the boundary, while t is the horizontal separation of the bulk fields.

where $g_2(t, b_0) = \mathcal{O}(b_0)$. Following the path p_B we reach again the point (C). At (C) we have $(lhs) + g_1(t_0, b_0) = (rhs) + g_2(t_0, b_0)$, which implies that $g_1(t_0, b_0)$ and $g_2(t_0, b_0)$ are non-singular for $b_0 \to 0$ and $t_0 \to 0$, respectively. Hence we can safely take the factorisation limit $t_0, b_0 \to 0$, in which g_1 and g_2 vanish, so that $(lhs) = (rhs)$. This shows that equation (8.34) holds.

Using the Ward identity (7.22) to move the integral contours, and taking into account definition (7.11), equation (8.34) gives:

$$B^{a_0}{}_{a_1 \ldots a_m; l}\, C^l{}_{ij} \quad = \tag{8.35}$$

$$= \sum_{0 \le m_1 \le \ldots m_4 \le m} (-1)^{s_{m_1 m_3}}\, B^{a_0}{}_{a_1 \ldots a_{m_1} b a_{m_2+1} \ldots a_{m_3} c a_{m_4+1} \ldots a_m} B^b{}_{a_{m_1+1} \ldots a_{m_2}; i} B^c{}_{a_{m_3+1} \ldots a_{m_4}; j} \quad ,$$

where $s_{m_1 m_3} = \tilde{a}_{m_1+1} + \ldots + \tilde{a}_{m_3}$. Note that the left hand side is manifestly symmetric in i and j whereas this symmetry is not manifest in the right hand side. This reflects the fact that one can also accomplish the factorisation of Figure 8.4 after exchanging i and j.

Additional integrated bulk insertions spread in the usual way when we factorise, so we can treat them as derivatives and combine all relations into a single equation involving the quantities $\mathcal{F}_{a_0\ldots a_m}(t)$ for $m \geq 0$ and the bulk WDVV potential $\mathcal{F}(t)$:

$$\begin{aligned}
&\partial_i\partial_j\partial_k\mathcal{F}\ \eta^{kl}\ \partial_l\mathcal{F}_{a_0a_1\ldots a_m} = \qquad (8.36)\\
&= \sum_{0\leq m_1\leq\ldots m_4\leq m} (-1)^{s_{m_1m_3}}\mathcal{F}_{a_0\ldots a_{m_1}ba_{m_2+1}\ldots a_{m_3}ca_{m_4+1}\ldots a_m}\ \partial_i\mathcal{F}^b{}_{a_{m_1+1}\ldots a_{m_2}}\ \partial_j\mathcal{F}^c{}_{a_{m_3+1}\ldots a_{m_4}}\ .
\end{aligned}$$

For $m = 0$ and $m = 1$, these equations take the form:

$$\partial_i\partial_j\partial_k\mathcal{F}\ \eta^{kl}\ \partial_l\mathcal{F}_{a_0} = \mathcal{F}_{a_0bc}\ \partial_i\mathcal{F}^b\ \partial_j\mathcal{F}^c\ , \qquad (8.37)$$

$$\begin{aligned}
\partial_i\partial_j\partial_k\mathcal{F}\ \eta^{kl}\ \partial_l\mathcal{F}_{a_0a_1} &= \mathcal{F}_{a_0bca_1}\ \partial_i\mathcal{F}^b\ \partial_j\mathcal{F}^c + \mathcal{F}_{a_0bc}\ \partial_i\mathcal{F}^b\ \partial_j\mathcal{F}^c{}_{a_1} \qquad (8.38)\\
&+ (-1)^{\tilde{a}_1}\ \mathcal{F}_{a_0ba_1c}\ \partial_i\mathcal{F}^b\ \partial_j\mathcal{F}^c + (-1)^{\tilde{a}_1}\ \mathcal{F}_{a_0bc}\ \partial_i\mathcal{F}^b{}_{a_1}\ \partial_j\mathcal{F}^c\\
&+ \mathcal{F}_{a_0a_1bc}\ \partial_i\mathcal{F}^b\ \partial_j\mathcal{F}^c\ .
\end{aligned}$$

Notice that the undeformed version of (8.37) coincides with the second bulk-boundary crossing constraint of two-dimensional TFT [82, 83].

8.5 Cardy conditions

The Cardy condition is probably the most interesting sewing constraint of 2d TFT [82, 83, 90], since it connects the exchange of closed strings between D-branes at the tree level with a one-loop open string amplitude. Allowing for insertions of both bulk and boundary fields in the corresponding cylinder amplitude leads to the following factorisation:

$$\begin{aligned}
&\left\langle\phi_i\psi_{a_0}P\int\psi_{a_1}\ldots\int\psi_{a_n}\right\rangle\eta^{ij}\left\langle\phi_j\psi_{b_0}P\int\psi_{b_1}\ldots\int\psi_{b_m}\right\rangle = \qquad (8.39)\\
&= \sum_{\substack{0\,\leq\, n_1\,\leq\, n_2\,\leq\, n\\ 0\,\leq\, m_1\,\leq\, m_2\,\leq\, m}} (-1)^s\ \omega^{c_1d_1}\omega^{c_2d_2}\\
&\quad\left\langle\psi_{a_0}\ P\int\psi_{a_1}\ldots\int\psi_{a_{n_1}}\psi_{d_1}P\int\psi_{b_{m_1+1}}\ldots\int\psi_{b_{m_2}}\psi_{c_2}P\int\psi_{a_{n_2+1}}\ldots\int\psi_{a_n}\right\rangle\\
&\quad\left\langle\psi_{b_0}\ P\int\psi_{b_1}\ldots\int\psi_{b_{m_1}}\psi_{c_1}P\int\psi_{a_{n_1+1}}\ldots\int\psi_{a_{n_2}}\psi_{d_2}P\int\psi_{b_{m_2+1}}\ldots\int\psi_{b_m}\right\rangle\ ,
\end{aligned}$$

where the sign s accounts for reshuffling of the boundary fields. The left hand side of (8.39) is the factorisation in the closed string channel, in which the cylinder becomes infinitely long. The right hand side corresponds to the generalisation of the double-twist diagram [90] of the open string channel.

Taking into account further integrated bulk insertions, equations (8.39) become:

$$\begin{aligned}
&\partial_i\mathcal{F}_{a_0\ldots a_n}\eta^{ij}\partial_j\mathcal{F}_{b_0\ldots b_m} = \qquad (8.40)\\
&= \sum_{\substack{0\,\leq\, n_1\,\leq\, n_2\,\leq\, n\\ 0\,\leq\, m_1\,\leq\, m_2\,\leq\, m}}(-1)^{s+\tilde{c}_1+\tilde{c}_2}\omega^{c_1d_1}\omega^{c_2d_2}\mathcal{F}_{a_0\ldots a_{n_1}d_1b_{m_1+1}\ldots b_{m_2}c_2a_{n_2+1}\ldots a_n}\mathcal{F}_{b_0\ldots b_{m_1}c_1a_{n_1+1}\ldots a_{n_2}d_2b_{m_2+1}\ldots b_m}\ .
\end{aligned}$$

The simplest relations in this hierarchy of constraints take the form:

$$\begin{aligned}
\partial_i \mathcal{F}_{a_0} \eta^{ij} \partial_j \mathcal{F}_{b_0} &= (-1)^{s+\tilde{c}_1+\tilde{c}_2}\, \omega^{c_1 d_1}\, \omega^{c_2 d_2}\, \mathcal{F}_{a_0 d_1 c_2}\, \mathcal{F}_{b_0 c_1 d_2} \ , \\
\partial_i \mathcal{F}_{a_0 a_1} \eta^{ij} \partial_j \mathcal{F}_{b_0} &= (-1)^{s+\tilde{c}_1+\tilde{c}_2}\, \omega^{c_1 d_1}\, \omega^{c_2 d_2}\, \mathcal{F}_{a_0 d_1 c_2 a_1}\, \mathcal{F}_{b_0 c_1 d_2} \\
&+ (-1)^{s+\tilde{c}_1+\tilde{c}_2}\, \omega^{c_1 d_1}\, \omega^{c_2 d_2}\, \mathcal{F}_{a_0 d_1 c_2}\, \mathcal{F}_{b_0 c_1 a_1 d_2} \\
&+ (-1)^{s+\tilde{c}_1+\tilde{c}_2}\, \omega^{c_1 d_1}\, \omega^{c_2 d_2}\, \mathcal{F}_{a_0 a_1 d_1 c_2}\, \mathcal{F}_{b_0 c_1 d_2} \ .
\end{aligned} \tag{8.41}$$

Taking the limit $t = 0$ in the first equation recovers the Cardy constraint of two-dimensional TFT [82,83]. Notice that the left hand side of the first equation vanishes identically if we consider insertions of the identity operator, and if the suspended degree $\tilde{\omega}$ of the symplectic structure vanishes; this reflects vanishing of the Witten index in that case.

It is worth pointing out that the arguments of chapter 7 cannot be used to show that the annulus amplitude is independent of the world-sheet metric and of the positions of unintegrated boundary insertions. In fact, experience with the bulk theory [6] suggests that there are anomalies in the Q-symmetry in open string correlators beyond tree level, so there is indeed no *a priori* reason why the annulus amplitude should be metric-independent. However, we will take the point of view that when *imposing* the Cardy condition (8.40), one focuses by definition on the topological part of the amplitude. It is not clear to us whether the Cardy relation is satisfied by the complete amplitude, which potentially involves supplementary anomalous contributions.

In the present work we will be concerned only with the topological part of the annulus amplitude.[5] To capture the full amplitude including possible holomorphic anomalies would require the analogue of $t - t^*$ equations [6] for open strings, a subject which is beyond the scope of the present work.

[5]For the Landau-Ginzburg examples described in chapter 9, we shall find that imposing the generalised Cardy condition as written above agrees with independently known results (namely, with the factorisation property of the Landau-Ginzburg potential).

Chapter 9

Application: Landau–Ginzburg minimal models

In this chapter we demonstrate the power of the consistency conditions derived in chapter 8, which include cyclicity (7.24), weak A_∞ structure (8.20), bulk-boundary sewing constraint (8.36) and Cardy relation (8.40)), by applying them to certain families of D-branes in B-type topological minimal models. In the examples considered below, we shall find that the totality of these constraints suffices to determine the effective superpotential uniquely.

In sections 9.2 and 9.3 we present the proposal of [46, 48] for the superpotential in minimal Landau–Ginzburg models on a single brane and a general composite brane, respectively. We provide some non-trivial tests for this conjecture [48]. Note that there does not exist a closed expression for the bulk prepotential of these models. The existence of a closed expression for the superpotential is therefore a quite surprising fact. Finally, in section 9.4 we point out an interpretation of the superpotential in terms of a classical action for a holomorphic matrix model [91].

9.1 Experimental evidence

According to section 5.6 the bulk sector of minimal Landau–Ginzburg models of the A-series is characterised by the level k, while D-brane boundary sectors are labelled by $\ell = 0, 1, ...[k/2]$. The bulk sector is described by a univariate polynomial $W_{LG}(\phi)$ of degree $k+2$ in the complex variable ϕ, which gives the world-sheet superpotential. On the other hand, 'non-multiple' B-type D-branes in the boundary sector ℓ correspond to factorisations of the bulk superpotential:

$$W_{LG}(\phi) \;=\; J^{(\ell)}(\phi)\, E^{(\ell)}(\phi)\ , \qquad \ell = 0, 1, ...[k/2] \tag{9.1}$$

where $J^{(\ell)}(\phi)$ is a polynomial of degree $\ell+1$ [47, 60] (cf. the end of section 5.6). The open string spectrum consists of boundary condition preserving and changing sectors and is given in tables (5.2) and (5.3) together with the $U(1)$ charges, respectively. We focus first on boundary condition preserving sectors, each of which corresponds to a degree label ℓ. For later reference we introduce the following labelling for a basis of physical boundary operators:

$$\mathcal{R}_B^{(\ell)} \equiv \{\, \psi_a \,\} = \{\phi^\alpha, \omega\phi^\alpha\}, \qquad a = 0, ..., 2\ell+1, \quad \alpha = 0, ..., \ell \; . \tag{9.2}$$

For completeness let us recall that the bulk algebra is given by the Newton ring $\mathcal{R} = \mathbb{C}[\phi]/\langle\partial_\phi W_{LG}(\phi)\rangle$, which admits the following basis when viewed as a complex vector space:

$$\mathcal{R} \equiv \{\, \phi^i \,\}, \qquad i = 0, ..., k.$$

When suitably integrated, each of the fields can be used to deform the theory. In the bulk sector we have the infinitesimal deformation

$$\delta S = \sum_{i=0}^{k} t_{k+2-i} \int d^2 z [G_{-1}, [\bar{G}_{-1}, \phi^i] \; ,$$

while in the boundary sector we have:

$$\delta S_\partial = \left(\sum_{\alpha=0}^{\ell} u_{\ell+1-\alpha} \int dx\, G(\omega\phi^\alpha) \right) + \left(\sum_{\alpha=0}^{\ell} v_{k/2+1-\alpha} \int dx\, G\phi^\alpha \right) \quad . \tag{9.3}$$

In this equation, we divided the boundary deformation parameters s_a into even and odd variables, u_α and v_α, respectively. These parameters can be formally assigned $U(1)$ charges, which can be used as labels; this is the convention employed in equation (9.3). Notice that super-integration over the moduli of boundary punctures flips the $\mathbb{Z}_2$ degree, so that odd ring elements (=topological tachyon excitations $\omega\phi^\alpha$) are associated with the bosonic deformation parameters u, and vice versa.

We are now ready to present some computations. We consider a few explicit examples and determine their effective superpotentials. As a rule, we shall find that the consistency conditions of chapter 8 lead to a unique solution, but only once *all* constraints are imposed on the open-closed amplitudes. For example, fixing $t = 0$ and imposing only the A_∞ conditions (8.3) leaves some parameters undetermined in the effective superpotential $\mathcal{W}(0, s) = \mathcal{W}(0, u, v)$. It is only after considering both open and closed deformations and imposing the bulk-boundary constraints (8.36) and Cardy conditions (8.40) that all coefficients of $\mathcal{W}(s, t)$ become uniquely determined.[1]

For the example $(k, \ell) = (3, 1)$, we find the following expressions for the per-

[1]Strictly speaking, this is true only up to choosing the normalisation of the 3-point boundary and 2-point bulk-boundary correlators. In the computations below, we normalised these correlators in a manner which is natural in the LG description. Notice that the sign in the Cardy condition (8.40) is given by: $(-1)^s = (-1)^{(\tilde{c}_1+\tilde{a}_0)(\tilde{c}_2+\tilde{b}_0)}$ in the present case.

turbed boundary correlators (7.25) on the disk:

$$
\begin{aligned}
\mathcal{F}_{021} = -\mathcal{F}_{003} = -\mathcal{F}_{012} = -\mathcal{F}_{1213} &= 1, \\
\mathcal{F}_{222} = \mathcal{F}_{2233} = \mathcal{F}_{2323} = \mathcal{F}_{23333} = \mathcal{F}_{333333} &= -1/5, \\
\mathcal{F}_{22} = \mathcal{F}_{233} = \mathcal{F}_{3333} &= t_2, \\
\mathcal{F}_{23} = \mathcal{F}_{333} &= t_3, \\
\mathcal{F}_{2} = \mathcal{F}_{33} &= t_4 - {t_2}^2, \\
\mathcal{F}_{3} &= t_5 - t_2\, t_3.
\end{aligned}
\tag{9.4}
$$

Our notation was explained after equations (9.2), namely $a = 0,1$ and $a = 2,3$ label even and odd boundary ring elements, respectively. Moreover, we listed one representative per cyclic orbit, and only the non-vanishing amplitudes. The value of $-1/5$ for the correlators which contain three unintegrated fermionic insertions arises from our normalisation, which is $\omega^2 = -\frac{\phi}{5}$. Notice that ordering of boundary indices is indeed important; for example $\mathcal{F}_{1123} = 0$ while $\mathcal{F}_{1213} = -1$. In this example, the effective superpotential takes the form:

$$
\begin{aligned}
-\mathcal{W}(t,u) \;=\;& \frac{1}{5}\Big(\frac{{u_1}^6}{6} + {u_1}^4 u_2 + \frac{3}{2}{u_1}^2{u_2}^2 + \frac{{u_2}^3}{3}\Big) + \\
&+t_2\Big(\frac{-{u_1}^4}{4} - {u_1}^2 u_2 - \frac{{u_2}^2}{2}\Big) - t_3\Big(\frac{{u_1}^3}{3} + u_1 u_2\Big) + \\
&+\Big(t_4 - {t_2}^2\Big)\Big(\frac{-{u_1}^2}{2} - u_2\Big) - \Big(t_5 - t_2 t_3\Big) u_1 \;.
\end{aligned}
\tag{9.5}
$$

Since the parameters v are odd while appearing only in anti-commutators, they drop out from the effective superpotential, even though the corresponding non-symmetrised amplitudes are non-zero and have to be taken into account when solving the constraint equations.

The effective superpotentials for some other examples are as follows. For $(k,\ell) = (4,2)$ we find:

$$
\begin{aligned}
-\mathcal{W}(t,u) \;=\;& \frac{1}{6}\Big(\frac{{u_1}^7}{7} + {u_1}^5 u_2 + 2{u_1}^3{u_2}^2 + u_1{u_2}^3 + \\
&\quad +{u_1}^4 u_3 + 3{u_1}^2 u_2 u_3 + {u_2}^2 u_3 + u_1{u_3}^2\Big) - \\
&-t_2\Big(\frac{{u_1}^5}{5} + {u_1}^3 u_2 + u_1{u_2}^2 + {u_1}^2 u_3 + u_2 u_3\Big) + \\
&+t_3\Big(\frac{-{u_1}^4}{4} - {u_1}^2 u_2 - \frac{{u_2}^2}{2} - u_1 u_3\Big) + \\
&+\Big(\frac{t_4 - 3{t_2}^2}{2}\Big)\Big(\frac{-{u_1}^3}{3} - u_1 u_2 - u_3\Big) + \Big(t_5 - 2t_2 t_3\Big)\Big(\frac{-{u_1}^2}{2} - u_2\Big) - \\
&-\Big(t_6 + \frac{{t_2}^3}{3} - \frac{{t_3}^2}{2} - t_2 t_4\Big) u_1 \;,
\end{aligned}
$$

while for $(k,\ell)=(5,2)$ we obtain:

$$
\begin{aligned}
-\mathcal{W}(t,u) = \frac{1}{7}\Big(&\frac{{u_1}^8}{8}+{u_1}^6u_2+\frac{5{u_1}^4{u_2}^2}{2}+2{u_1}^2{u_2}^3+\frac{{u_2}^4}{4}+{u_1}^5u_3+ \\
&+4{u_1}^3u_2u_3+3u_1{u_2}^2u_3+\frac{3{u_1}^2{u_3}^2}{2}+u_2{u_3}^2\Big) \\
&-t_2\Big(\frac{{u_1}^6}{6}+{u_1}^4u_2+\frac{3{u_1}^2{u_2}^2}{2}+\frac{{u_2}^3}{3}+{u_1}^3u_3+2u_1u_2u_3+\frac{{u_3}^2}{2}\Big) \\
&+t_3\Big(\frac{-{u_1}^5}{5}-{u_1}^3u_2-u_1{u_2}^2-{u_1}^2u_3-u_2u_3\Big) \\
&+\Big(t_4-2{t_2}^2\Big)\Big(\frac{-{u_1}^4}{4}-{u_1}^2u_2-\frac{{u_2}^2}{2}-u_1u_3\Big) \\
&+\Big(t_5-3t_2t_3\Big)\Big(\frac{-{u_1}^3}{3}-u_1u_2-u_3\Big) \\
&+\Big(t_6+{t_2}^3-{t_3}^2-2t_2t_4\Big)\Big(\frac{-{u_1}^2}{2}-u_2\Big) \\
&-\Big(t_7+{t_2}^2t_3-t_3t_4-t_2t_5\Big)u_1\ .
\end{aligned}
$$

Notice that $\mathcal{W}(t,u)$ has $U(1)$ charge equal to $k+3$, which is one-half of the charge of the effective prepotential $\mathcal{F}(t)$ of the bulk sector.

Finally, let us give an example of effective superpotentials for the boundary changing sector of minimal models. For simplicity we will not turn on bulk deformations. In this situation we can study the formation of D-brane composites in a fixed conformal bulk background. Let us consider the minimal model at level $k=3$ with a bound state of two D-branes, $\ell=0$ and $\ell=1$. In this setting, one has fermionic boundary operators, $(\omega^{(00)},\omega^{(01)},\omega^{(10)},\omega^{(11)},\phi\omega^{(11)})$, associated with the deformation parameters $(u_1^{(00)},u_{3/2}^{(01)},u_{3/2}^{(10)},u_2^{(11)},u_1^{(11)})$ (see section 5.6). The consistency conditions again determine all amplitudes, giving the following effective superpotential for the bosonic deformation parameters:

$$
\begin{aligned}
\mathcal{W}(t{=}0,u) \;=\; & -\frac{1}{15}\,u_2^{(11)3}-\frac{3}{10}\,u_2^{(11)2}\,u_1^{(11)2}-\frac{1}{5}\,u_2^{(11)}\,u_1^{(11)4}-\frac{1}{30}\,u_1^{(11)6}- \\
& -\frac{1}{30}u_1^{(00)6}-\frac{1}{5}\,u_1^{(00)3}\,u_{3/2}^{(01)}\,u_{3/2}^{(10)}-\frac{1}{5}\,u_1^{(00)}\,u_2^{(11)}\,u_{3/2}^{(01)}\,u_{3/2}^{(10)}- \\
& -\frac{1}{5}\,u_1^{(00)2}\,u_1^{(11)}\,u_{3/2}^{(01)}\,u_{3/2}^{(10)}-\frac{2}{5}\,u_2^{(11)}\,u_1^{(11)}\,u_{3/2}^{(01)}\,u_{3/2}^{(10)}- \\
& -\frac{1}{5}\,u_1^{(00)}\,u_1^{(11)2}\,u_{3/2}^{(01)}\,u_{3/2}^{(10)}-\frac{1}{5}\,u_1^{(11)3}\,u_{3/2}^{(01)}\,u_{3/2}^{(10)}-\frac{1}{10}\,u_{3/2}^{(01)2}\,u_{3/2}^{(10)2}\ .
\end{aligned}
\tag{9.6}
$$

9.2 The effective potential for a single B-brane

The results of the previous section, obtained by painstakingly solving the consistency conditions, suggest the following closed formula for the effective superpotential in the general boundary preserving sector labelled by (k, ℓ) [46]:

$$\mathcal{W}(t,u) \;=\; -\sum_{i=0}^{k+2} g_{k+2-i}(t)\, h_{i+1}(u) \quad , \tag{9.7}$$

where $h_i(u)$ are defined as symmetric Newton polynomials by:

$$\log\left[1-\sum_{n=1}^{\ell+1} u_n\, y^n\right] \;:=\; \sum_{i=1}^{\infty} h_i(u)\, y^i \tag{9.8}$$

and $g_{k+2-i}(t)$ are the coefficients of ϕ^i in the bulk LG superpotential:

$$W_{LG}(t) \;=\; -\sum_{i=0}^{k+2} g_{k+2-i}(t)\, \phi^i \ ,$$

whose explicit form can be found for example in [5] (here $g_0 = -1/(k+2)$ and $g_1 = 0$). Equation (9.7) implies the following expression for the deformed one-point correlators on the disk:

$$\mathcal{F}_{\ell+1-\alpha}(t) \;\equiv\; \partial_{u_\alpha}\mathcal{W}(t,u)|_{u=0} \;=\; g_{k+3-\alpha}(t)\ .$$

It is possible to cast (9.7) into a more elegant form. For this, notice that the substitution $y = 1/\phi$ reduces (9.8) (up to an u-independent logarithmic term) to:

$$\log J(\phi;u) = (l+1)\log\phi + \sum_{m=1}^{\infty} h_m\, \phi^{-m} \quad , \tag{9.9}$$

where $J(\phi;u)$ is the boundary superpotential, parametrised linearly in u, i.e.,

$$J(\phi;u) \;=\; \phi^{\ell+1} - \sum_{\alpha=0}^{\ell} u_{\ell+1-\alpha}\, \phi^{\alpha} \quad , \quad \ell = 0\ldots[k/2]\ . \tag{9.10}$$

The effective potential (9.7) can thus be written as [48]:

$$\mathcal{W}(t,u) = -\oint_{\mathcal{C}} \frac{d\phi}{2\pi i} W_{LG}(\phi;t)\log J(\phi;u) \quad , \tag{9.11}$$

where $\mathcal{C}$ is a closed counterclockwise contour encircling all $D0$-branes (i.e. all zeroes $x_i(u)$ of $J(\phi;u)$) and all cuts of the logarithm. Relation (9.11) is ambiguous due to the need of choosing appropriate branch cuts, but the ambiguity amounts to the freedom of adding an inessential constant to $\mathcal{W}(t,u)$.

From the interpretation of $\mathcal{W}(t,u)$ as a deformation potential [35,46], we expect that its u-critical set, defined by:[2]

$$\mathcal{Z}_{crit} = \{(t,u)|\partial_u \mathcal{W}(t,u) = 0\} \tag{9.12}$$

should agree *locally* with the total joint deformation locus:

$$\mathcal{Z}_{fact} = \{(u,t) \in \mathbb{C}^{\ell+1} \times \mathbb{C}^{k+1} | E_-(\phi;t,u) = 0\} \quad ,$$

where $E_-(x;t,u)$ denotes the singular part of

$$E(\phi;t,u) = \frac{W_{LG}(\phi;t)}{J(\phi;u)} \quad . \tag{9.13}$$

More precisely, in [48] it was shown that $\mathcal{Z}_{fact}$ splits into various branches of different D-brane content. Therefore, $\mathcal{Z}_{crit}$ should locally coincide with $\mathcal{Z}_{fact}$, provided that we restrict both to a small enough vicinity of a point (t,u) which lies on such a branch. Thus we are interested in polynomials $J(\phi;u)$, which lie in the vicinity of a polynomial $J_0(\phi;u(t))$ that divides $W_{LG}(\phi;t)$.

This expectation is in fact easy to check by writing $J(\phi;u) = \prod_{i=0}^{\ell}(\phi - x_i(u))$, which implies:

$$\partial_{u_\alpha}\mathcal{W}(t,u) = \oint_{\mathcal{C}} \frac{d\phi}{2\pi i}\left[W_{LG}(\phi;t)\sum_{i=0}^{\ell}\frac{\partial_{u_\alpha}x_i(u)}{\phi - x_i(u)}\right] = \sum_{i=0}^{\ell} W_{LG}(x_i(u);t)\partial_{u_\alpha}x_i(u) \quad . \tag{9.14}$$

Thus the u-critical set of $\mathcal{W}(t,u)$ is described by the linear system

$$\sum_{i=0}^{\ell}\partial_{u_\alpha}x_i(u)W_{LG}(x_i(u);t) = 0 \tag{9.15}$$

for the $\ell+1$ unknowns $W_{LG}(x_i(u))$. Now notice that the $\ell+1$ parameters u in 9.10 suffice to specify the roots of the monic degree $\ell+1$ polynomial $J(\phi;u)$. As a consequence, the discriminant:

$$\Delta(u) := \det(\partial_{u_\alpha}x_i(u)) \tag{9.16}$$

is generically non-vanishing. Hence the only solution of (9.15) is $W_{LG}(x_i(u)) = 0$ for all $i = 0\ldots\ell$. Thus each root of the polynomial $J(\phi;u)$ is also a root of $W_{LG}(\phi;t)$. Since J is close to J_0, which divides $W_{LG}(\phi;t)$, the only possibility is that the multiplicities of the roots are smaller in $J(\phi;u)$ than in $W_{LG}(\phi;t)$. Thus $J(\phi;u)$ must divide $W_{LG}(\phi;t)$, and $\mathcal{Z}_{crit}$ must coincide with the J_0-branch of $\mathcal{Z}_{fact}$ when restricted to a small enough vicinity of J_0.

[2] We treat bulk deformations as non-dynamical background fields, which is warranted at weak string coupling.

Notice that this is a purely local statement. The variety $\mathcal{Z}_{crit}$ contains components associated with polynomials J that do not divide W. However, such components do not intersect the factorisation locus $\mathcal{Z}_{fact}$, so agreement is guaranteed in the vicinity of any true solution of the factorisation problem (which, of course, is all that can be expected from the local analysis of [35, 46]).

Although the factorisation $W = JE$ persists along the u-critical set $\mathcal{Z}_{crit}$, the cohomology in the boundary sector may change along this locus. In the remainder of this subsection, we discuss the condition on $\mathcal{W}(t,u)$ which ensures the preservation of a non-trivial spectrum. On this account we differentiate equation (9.14) a second time and obtain:

$$\partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}(t,u) = \sum_{i=0}^{\ell} W_{LG}(x_i(u);t)\ \left(\partial_{u_\alpha}\partial_{u_\beta}x_i\right) + \sum_{i=0}^{\ell}\partial_{x_i}W_{LG}(x_i(u);t)\ \left(\partial_{u_\alpha}x_i\partial_{u_\beta}x_i\right)\ . \tag{9.17}$$

Suppose we stay at a point on the factorisation locus, and we require, in addition, that $J|E$, i.e., that we are on the sub-locus:

$$\mathcal{Z}_{spec} := \{(t,u)\mid W_{LG} = JE,\ J|E\} \subset \mathcal{Z}_{fact}\ . \tag{9.18}$$

Since the boundary ring $\mathcal{R}_B^{(\ell)}$, given in (5.72), is governed by the greatest common divisor of J and E, the number of odd (and even) cohomology classes takes the maximal value, $\ell+1$ along $\mathcal{Z}_{spec}$. Note that $\mathcal{Z}_{spec}$ can equivalently be described by $\mathcal{Z}_{spec} = \{(t,u)\mid J|W_{LG},\ J|W'_{LG}\}$. Therefore, we see from (9.17) that

$$\partial_{u_\alpha}\mathcal{W}(t,u) = \partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}(t,u) = 0 \quad \text{on } \mathcal{Z}_{spec}\ . \tag{9.19}$$

In order to show that (9.19) is true only on $\mathcal{Z}_{spec}$, we look at the vicinity of a point $(t_0,u_0)\in\mathcal{Z}_{spec}$, with J_0 and $E_0 = hJ_0$. Then, by the same line of argumentation as above, the non-vanishing discriminant $\Delta(u)$ ensures that

$$\mathcal{Z}_{spec} = \{(t,u)\mid \partial_{u_\alpha}\mathcal{W}(t,u) = \partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}(t,u) = 0\}\ . \tag{9.20}$$

An analogous argument can be made for the situation where J and E share a common factor G, whose degree is smaller than that of J (cf. (5.69)). Then only a corresponding subset of the cohomology survives, and this is reflected in an increased rank of $\partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}$.

In physical terms this finding can be interpreted as follows: On the factorisation locus $\mathcal{Z}_{fact}$ where $W_{LG} = JE$, the boundary preserving parameters u_α do not have tadpoles and thus the theory has a stable, supersymmetric vacuum; however a non-trivial spectrum of boundary operators is not guaranteed. Only on the sub-locus $\mathcal{Z}_{spec}\subset\mathcal{Z}_{fact}$ one has a non-trivial spectrum, and this is reflected in zero eigenvalues of the 'mass-matrix' $\partial_{u_\alpha}\partial_{u_\beta}\mathcal{W}$.

In the untwisted model, the physical interpretation is as follows. Generic bulk (t) and boundary (u) perturbations break supersymmetry, a phenomenon which can be traced back to the boundary terms (4.24) and (4.30) in the Q variation of integrated descendants. Thus t and u 'feel' a potential which represents an obstruction against such deformations. The effects of the boundary terms cancel and supersymmetry is maintained only when bulk and boundary deformations are locked together through the relation $W_{LG} = J\,E$ – this cancellation was indeed precisely why one had to introduce a boundary potential in the first place [47, 55, 60, 77, 92]. Thus it is no surprise, though a welcome check on our computations, that the critical set $\mathcal{Z}_{crit}$ of $\mathcal{W}(t,u)$ with respect to the boundary deformation parameters u corresponds to the factorisation locus of the world-sheet LG superpotential in the combined, bulk and boundary parameter space. As expected, this is precisely the locus along which the boundary data preserve half of the supersymmetry of the world-sheet action, thereby allowing for a meaningful coupling to B-type D-branes. This is similar in spirit to [93], where, in a different physical context, critical points of effective superpotentials were associated with factorisation loci in the target space geometry.

If $E(\phi; u, t_\circ)$ is generic, its greatest common denominator with $J(\phi; u)$ is trivial, hence according to (5.72) no physical open string states survive after turning on bulk and boundary deformations by allowing for general t, u. This reflects tachyon condensation of the $D2\overline{D2}$ system [60,94], leading to the trivial open string vacuum. Only upon appropriately specialising $E(\phi; u, t_\circ)$ such that it has a non-trivial common factor $G(\phi; u)$ with $J(\phi; u)$, does one find that some open string states remain in the physical spectrum. Such sub-loci of the factorisation variety $\mathcal{Z}_{fact}$ correspond to (a topological model of) tachyon condensation with non-trivial endpoint. In this version of tachyon condensation, the open string spectrum gets truncated while moving between different strata of the supersymmetry preserving locus of the effective superpotential. A very similar picture was found in [95] for the case of the open A model close to a large radius point of a Calabi-Yau compactification (see figure 2 of that paper). The topological version of tachyon condensation was discussed in detail in [95–100] in the context of open string field theory. It also plays a central role in the work of [84, 87].

9.3 The effective potential for general B-branes

Explicit computations in the spirit of section 9.1 of some low-k models with several branes suggest that we can extend the proposal (9.11) for the superpotential of a single brane to the following expression for arbitrary brane composites [48]:

$$\mathcal{W}(t,u) \;=\; -\oint_C \frac{d\phi}{2\pi i} \log\big(\det J(\phi;u)\big) W(\phi;t)\ . \tag{9.21}$$

This amounts to replacing J by $\det J$ in 9.11. For instance, for the deformation of the bound state of two D-branes of types ℓ_1 and ℓ_2 at the conformal point, the matrix J reads:

$$J^{\ell_1,\ell_2}(\phi;u) \;=\; \begin{bmatrix} \phi^{\ell_1+1} - \sum_{\alpha=0}^{\ell_1} u^{[11]}_{\ell_1+1-\alpha}\phi^\alpha & -\sum_{\gamma=0}^{\ell_{12}} u^{[12]}_{\frac12(\ell_1+\ell_2)+1-\gamma}\phi^\gamma \\ -\sum_{\gamma=0}^{\ell_{21}} u^{[21]}_{\frac12(\ell_1+\ell_2)+1-\gamma}\phi^\gamma & \phi^{\ell_2+1} - \sum_{\alpha=0}^{\ell_2} u^{[22]}_{\ell_2+1-\alpha}\phi^\alpha \end{bmatrix}\ . \tag{9.22}$$

The diagonal entries correspond to deformations by boundary condition preserving operators, whereas the off-diagonal entries correspond to deformations by boundary condition changing operators. Note that all entries in $J^{\ell_1,\ell_2}(\phi;u)$ are linear in the deformation parameters $u_\alpha^{[AB]}$. The expression for $r \times r$ factorisations is the natural generalisation of (9.22).

The proof of local agreement of the critical locus of $\mathcal{W}$ with the deformation space is similar to the case of a single brane. Let us parameterise deformations of the $r \times r$ matrix J by $u = (u_\alpha^{[AB]})$ with $\alpha = 0 \dots H$, and write $\det J(\phi;u) = \prod_{i=0}^{L} (\phi - x_i(u))$ as a monic polynomial in ϕ, where $L+1$ is the degree of $\det J(\phi)$. We also assume that $H \geq L$ and that the $H+1$ by $L+1$ matrix:

$$A(u) := (\partial_\alpha x_i(u)) \ . \tag{9.23}$$

has maximal rank. Then:

$$\partial_{u_\alpha} \mathcal{W}(t,u) = \oint_{\mathcal{C}} \frac{d\phi}{2\pi i} \left[W(\phi;t) \sum_{i=0}^{L} \frac{\partial_{u_\alpha} x_i(u)}{\phi - x_i(u)} \right] = \sum_{i=0}^{L} W(x_i(u);t) \partial_{u_\alpha} x_i(u) \ , \tag{9.24}$$

and we find that the u-critical locus $\mathcal{Z}_{crit}$ of $\mathcal{W}_{eff}$ is characterised by the condition that all roots of $\det J$ must also be roots of W. This is obviously the case along the joint deformation space

$$\mathcal{Z}_{fact} = \{(t,u) | E_-(\phi;t,u) = 0\} \ ,$$

where $E(\phi;t,u)$ is now the matrix defined by

$$E(\phi;t,u) = W(\phi;t) J^{-1}(\phi;u) \ .$$

Therefore, the inclusion $\mathcal{Z}_{fact} \subset \mathcal{Z}_{crit}$ is immediate. Local agreement of $\mathcal{Z}_{crit}$ with $\mathcal{Z}_{fact}$ after restriction to a sufficiently small vicinity of a point lying on $\mathcal{Z}_{fact}$ follows by a simple continuity argument as in the previous section. We note that the inclusion $\mathcal{Z}_{fact} \subset \mathcal{Z}_{crit}$ also follows directly from (9.21), which implies:[3]

$$\partial_{u_\alpha} \mathcal{W}(t,u) \ = \ -\oint_{\mathcal{C}} \frac{d\phi}{2\pi i} \mathrm{Tr}[E(\phi;t,u) \partial_{u_\alpha} J(\phi;u)] \ = \oint_{\mathcal{C}} \frac{d\phi}{2\pi i} \ \mathrm{Tr}[J(\phi;u) \partial_{u_\alpha} E(\phi;t,u)] \ . \tag{9.25}$$

The right hand side of this identity vanishes along $\mathcal{Z}_{fact}$, since by definition the matrix $E(\phi;t,u)$ has no singular terms there. Thus the boundary critical set of $\mathcal{W}(t,u)$ agrees with the matrix factorisation locus. This provides further evidence for our general ansatz 9.21.

[3] The sign change in the last equation reflects the fact that swapping J and E exchanges branes with antibranes.

9.4 The superpotential as action for a holomorphic matrix model

Our proposal (9.21) for the effective potential admits a matrix model interpretation. For this we integrate $W_{LG}(\phi)$ to a polynomial $V(\phi;t)$ in ϕ, i.e.:

$$\partial_\phi V(\phi;t) = W(\phi;t) \tag{9.26}$$

(clearly V is defined only up to addition of a function $c(t)$ which is independent of ϕ). Then integration by parts casts (9.11) into the form:

$$\mathcal{W}(t,u) = \oint_{\mathcal{C}} \frac{d\phi}{2\pi i} V(\phi;t) \sum_{i=0}^{L} \frac{1}{\phi - x_i(u)} = \sum_{i=0}^{L} V(x_i(u),t) \ ,$$

where $\det J(\phi;u) = \prod_{i=0}^{L}(\phi - x_i(u))$ as before. Viewing the zeros $x_i(u)$ of $\det J(\phi;u)$ as eigenvalues of a complex $(L+1)\times(L+1)$ matrix $X(u)$, we can write the effective potential as:

$$\mathcal{W}(t,u) = \mathrm{Tr}\ V(X(u),t) \ . \tag{9.27}$$

Thus $\mathcal{W}(t,u)$ coincides with the classical action[4] of a holomorphic matrix model as defined and studied in [91] (the matrix model is holomorphic rather than Hermitian because the eigenvalues $x_i(u)$ are complex). The zeroes $x_i(u)$ of $\det J$ can be viewed as the locations of $D0$-branes in the complex plane (=the target space of the Landau-Ginzburg model). Equation 9.27 shows that $\mathcal{W}(t,u)$ is the 'potential energy' of this system of $D0$ branes when the latter is placed in the external 'complex potential' $V(\phi)$. Each critical configuration of this $D0$-brane system corresponds to a deformation of the underlying Landau-Ginzburg brane.

It has been known for a long time that the generalised Kontsevich model [101] is closely related to closed string minimal models coupled to topological gravity, but the open string version of this correspondence is not well understood. A link between certain topological D-branes and the auxiliary (Miwa) matrix of the Kontsevich model was proposed in [102], in the context of a non-compact Calabi-Yau realization of the underlying closed string model.

Our Landau-Ginzburg description gives a direct relation, which differs in spirit from that proposed in [102]: in the presence of several D-branes, the bulk Landau-Ginzburg field ϕ is effectively promoted to a matrix $X(u)$. In [102], D-brane positions were mapped to the *auxiliary* matrix of the generalised Kontsevich model, so they parameterise backgrounds for the model's dynamics. In our case, D-brane positions are truly dynamical, being encoded by the matrix *variable* itself. The reason for this difference is that we study the D-brane potential (i.e. the generating function of

[4]Here, we consider only the "small phase space". It would be interesting to extend the correspondence by coupling to topological gravity and including gravitational descendants. Presumably this involves the full dynamics of the holomorphic matrix model, rather than simply its classical action.

scattering amplitudes for strings stretching between D-branes) rather than the flux superpotential (the contribution from RR flux couplings to the closed string sector) considered in [102].[5]

[5]Since D-branes carry RR charges, they can be viewed as backgrounds inducing a flux superpotential, which explains the different point of view used in [102]. Our interest, however, is in D-brane dynamics as dictated by tree-level scattering amplitudes of open strings.

Bibliography

[1] W. Lerche, C. Vafa, and N. P. Warner. Chiral Rings in $N = 2$ Superconformal Theories. *Nucl. Phys.*, B324:427, 1989.

[2] N. P. Warner. Lectures on $N = 2$ Superconformal Theories and Singularity Theory. Lectures given at Spring School on Superstrings, Trieste, Italy, Apr 3-14, 1989.

[3] B. R. Greene. String Theory on Calabi–Yau Manifolds. 1996. hep-th/9702155.

[4] E. Witten. Mirror Manifolds and Topological Field Theory. 1991. hep-th/9112056.

[5] R. Dijkgraaf, H. Verlinde, and E. Verlinde. Topological Strings in $d < 1$. *Nucl. Phys.*, B352:59–86, 1991.

[6] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Kodaira–Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes. *Commun. Math. Phys.*, 165:311–428, 1994. hep-th/9309140.

[7] P. Candelas and X. C. de la Ossa. Moduli Space of Calabi–Yau Manifolds. *Nucl. Phys.*, B355:455–481, 1991.

[8] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor. Topological Amplitudes in String Theory. *Nucl. Phys.*, B413:162–184, 1994. hep-th/9307158.

[9] M. Dine, N. Seiberg, X. G. Wen, and E. Witten. Nonperturbative Effects on the String World Sheet. *Nucl. Phys.*, B278:769, 1986.

[10] M. Dine, N. Seiberg, X. G. Wen, and E. Witten. Nonperturbative Effects on the String World Sheet. 2. *Nucl. Phys.*, B289:319, 1987.

[11] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes. A Pair of Calabi–Yau Manifolds as an Exactly Soluble Superconformal Theory. *Nucl. Phys.*, B359:21–74, 1991.

[12] P. S. Aspinwall and D. R. Morrison. Topological Field Theory and Rational Curves. *Commun. Math. Phys.*, 151:245–262, 1993. hep-th/9110048.

[13] S. T. Yau (ed.). Mirror Symmetry I. Providence, USA: AMS (1998) 444 p.

[14] B. Greene (ed.) and S. T. Yau (ed.). Mirror Symmetry II. Providence, USA: AMS (1997) 844 p. (AMS/IP studies in advanced mathematics. 1).

[15] K. Hori (ed.) et al. Mirror Symmetry. American Mathematical Society (2003), ISBN 0821829556.

[16] P. S. Aspinwall, B. R. Greene, and D. R. Morrison. Multiple Mirror Manifolds and Topology Change in String Theory. *Phys. Lett.*, B303:249–259, 1993. hep-th/9301043.

[17] E. Witten. Phases of $N = 2$ Theories in two Dimensions. *Nucl. Phys.*, B403:159–222, 1993. hep-th/9301042.

[18] K. Hori and C. Vafa. Mirror Symmetry. 2000. hep-th/0002222.

[19] K. Becker, M. Becker, and A. Strominger. Five-branes, Membranes and Non-perturbative String Theory. *Nucl. Phys.*, B456:130–152, 1995. hep-th/9507158.

[20] H. Ooguri, Y. Oz, and Z. Yin. D-branes on Calabi–Yau Spaces and their Mirrors. *Nucl. Phys.*, B477:407–430, 1996. hep-th/9606112.

[21] A. Recknagel and V. Schomerus. D-branes in Gepner Models. *Nucl. Phys.*, B531:185–225, 1998. hep-th/9712186.

[22] D. Joyce. Lectures on Calabi–Yau and Special Lagrangian Geometry. 2001. math.dg/0108088.

[23] K. Hori, A. Iqbal, and C. Vafa. D-branes and Mirror Symmetry. 2000. hep-th/0005247.

[24] J. A. Harvey and G. W. Moore. On the Algebras of BPS States. *Commun. Math. Phys.*, 197:489–519, 1998. hep-th/9609017.

[25] E. Sharpe. Lectures on D-branes and Sheaves. 2003. hep-th/0307245.

[26] P. S. Aspinwall. D-branes on Calabi–Yau Manifolds. 2004. hep-th/0403166.

[27] M. Kontsevich. Homological Algebra of Mirror Symmetry. Proceedings of the International Congress of Mathematicians (Zurich, 1994) 120–139, Birkhauser, Basel, 1995. alg-geom/9411018.

[28] M. R. Douglas. D-branes, Categories and $N = 1$ Supersymmetry. *J. Math. Phys.*, 42:2818–2843, 2001. hep-th/0011017.

[29] P. S. Aspinwall and A. E. Lawrence. Derived Categories and Zero-brane Stability. *JHEP*, 08:004, 2001. hep-th/0104147.

[30] A. Sen. Tachyon Condensation on the Brane Antibrane System. *JHEP*, 08:012, 1998. hep-th/9805170.

[31] M. R. Douglas, B. Fiol, and C. Romelsberger. Stability and BPS Branes. 2000. hep-th/0002037.

[32] T. Bridgeland. Stability Conditions on Triangulated Categories. 2002. math.ag/0212237.

[33] I. Brunner, M. R. Douglas, A. E. Lawrence, and C. Romelsberger. D-branes on the Quintic. *JHEP*, 08:015, 2000. hep-th/9906200.

[34] S. Kachru, S. Katz, A. E. Lawrence, and J. McGreevy. Open String Instantons and Superpotentials. *Phys. Rev.*, D62:026001, 2000. hep-th/9912151.

[35] C. I Lazaroiu. String Field Theory and Brane Superpotentials. *JHEP*, 10:018, 2001. hep-th/0107162.

[36] K. Fukaya. Deformation Theory, Homological Algebra and Mirror Symmetry. Prepared for 4th SIGRAV Graduate School on Contemporary Relativity and Gravitational Physics and 2001 School on Algebraic Geometry and Physics: Geometry and Physics of Branes (SAGP 2001), Como, Italy, 7-11 May 2001.

[37] E. Witten. Chern–Simons Gauge Theory as a String Theory. *Prog. Math.*, 133:637–678, 1995. hep-th/9207094.

[38] J. D. Stasheff. On the Homotopy Associativity of H-spaces, I. *Trans. Amer. Math. Soc.*, 108:275, 1963.

[39] J. D. Stasheff. On the Homotopy Associativity of H-spaces, II. *Trans. Amer. Math. Soc.*, 108:293, 1963.

[40] M. R. Gaberdiel and B. Zwiebach. Tensor Constructions of Open String Theories I: Foundations. *Nucl. Phys.*, B505:569–624, 1997. hep-th/9705038.

[41] H. Kajiura. Noncommutative Homotopy Algebras Associated with Open Strings. 2003. math.qa/0306332.

[42] C. Hofman and W. K. Ma. Deformations of Topological Open Strings. *JHEP*, 01:035, 2001. hep-th/0006120.

[43] C. I. Lazaroiu. D-brane Categories. *Int. J. Mod. Phys.*, A18:5299–5335, 2003. hep-th/0305095.

[44] K. Fukaya. Morse Homotopy, A_∞ Categories and Floer Homologies. in Proceedings of the 1993 Garc Workshop on Geometry and Topology, Lecture Notes Series, vol. **18** (Seoul National University, 1993), pp. 1–102.

[45] A. Polishchuk. Homological Mirror Symmetry with Higher Products. Proceedings of the Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), pp 247–259, AMS/IP Stud. Adv. Math. **23**, Amer. Math. Soc., Providence, RI, 2001. math.AG/9901025.

[46] M. Herbst, C. I. Lazaroiu, and W. Lerche. Superpotentials, A(infinity) Relations and WDVV Equations for Open Topological Strings. 2004. hep-th/0402110.

[47] I. Brunner, M. Herbst, W. Lerche, and B. Scheuner. Landau–Ginzburg Realization of Open String TFT. 2003. hep-th/0305133.

[48] M. Herbst, C. I. Lazaroiu, and W. Lerche. D-brane Effective Action and Tachyon Condensation in Topological Minimal Models. 2004. hep-th/0405138.

[49] T. Banks, L. J. Dixon, D. Friedan, and E. J. Martinec. Phenomenology and Conformal Field Theory or Can String Theory Predict the Weak Mixing Angle? *Nucl. Phys.*, B299:613–626, 1988.

[50] J. Polchinski. String Theory. Vol. 2: Superstring Theory and Beyond. Cambridge, UK: Univ. Pr. (1998) 531 p.

[51] F. Gliozzi, J. Scherk, and D. I. Olive. Supersymmetry, Supergravity Theories and the Dual Spinor Model. *Nucl. Phys.*, B122:253–290, 1977.

[52] M. R. Gaberdiel. D-branes from Conformal Field Theory. *Fortsch. Phys.*, 50:783–801, 2002. hep-th/0201113.

[53] N. Ishibashi. The Boundary and Crosscap States in Conformal Field Theories. *Mod. Phys. Lett.*, A4:251, 1989.

[54] J. L. Cardy. Boundary Conditions, Fusion Rules and the Verlinde Formula. *Nucl. Phys.*, B324:581, 1989.

[55] N. P. Warner. Supersymmetry in Boundary Integrable Models. *Nucl. Phys.*, B450:663–694, 1995. hep-th/9506064.

[56] S. Govindarajan, T. Jayaraman, and T. Sarkar. Worldsheet Approaches to D-branes on Supersymmetric Cycles. *Nucl. Phys.*, B580:519–547, 2000. hep-th/9907131.

[57] S. Govindarajan and T. Jayaraman. On the Landau–Ginzburg Description of Boundary CFTs and Special Lagrangian Submanifolds. *JHEP*, 07:016, 2000. hep-th/0003242.

[58] A. Kapustin and Y. Li. Topological Correlators in Landau–Ginzburg Models with Boundaries. *Adv. Theor. Math. Phys.*, 7:727–749, 2004. hep-th/0305136.

[59] S. K. Ashok, E. Dell'Aquila, and D. E. Diaconescu. Fractional Branes in Landau–Ginzburg Orbifolds. 2004. hep-th/0401135.

[60] A. Kapustin and Y. Li. D-branes in Landau–Ginzburg Models and Algebraic Geometry. *JHEP*, 12:005, 2003. hep-th/0210296.

[61] E. Witten. Topological Quantum Field Theory. *Commun. Math. Phys.*, 117:353, 1988.

[62] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson. Topological Field Theory. *Phys. Rept.*, 209:129–340, 1991.

[63] E. Witten. Topological Sigma Models. *Commun. Math. Phys.*, 118:411, 1988.

[64] E. Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351, 1989.

[65] T. Eguchi and S. K. Yang. $N = 2$ Superconformal Models as Topological Field Theories. *Mod. Phys. Lett.*, A5:1693–1701, 1990.

[66] J. M. F. Labastida and P. M. Llatas. Topological Matter in two Dimensions. *Nucl. Phys.*, B379:220–258, 1992. hep-th/9112051.

[67] C. Vafa. Topological Landau–Ginzburg Models. *Mod. Phys. Lett.*, A6:337–346, 1991.

[68] M. Kontsevich. unpublished.

[69] D. Orlov. Triangulated Categories of Singularities and D-branes in Landau–Ginzburg Models. 2003. math.ag/0302304.

[70] M. Aganagic and C. Vafa. Mirror Symmetry, D-branes and Counting Holomorphic Discs. 2000. hep-th/0012041.

[71] P. Candelas, X. C. De la Ossa, P. S. Green, and L. Parkes. An Exactly Soluble Superconformal Theory from a Mirror Pair of Calabi–Yau Manifolds. *Phys. Lett.*, B258:118–126, 1991.

[72] M. Herbst and C. I. Lazaroiu. Localization and Traces in Open-closed Topological Landau–Ginzburg Models. 2004. hep-th/0404184.

[73] J. M. Maldacena, G. W. Moore, and N. Seiberg. Geometrical Interpretation of D-branes in Gauged WZW Models. *JHEP*, 07:046, 2001. hep-th/0105038.

[74] W. Lerche, C. A. Lutken, and C. Schweigert. D-branes on ALE Spaces and the ADE Classification of Conformal Field Theories. *Nucl. Phys.*, B622:269–278, 2002. hep-th/0006247.

[75] I. Brunner and K. Hori. Orientifolds and Mirror Symmetry. 2003. hep-th/0303135.

[76] G. Pradisi, A. Sagnotti, and Y. S. Stanev. Completeness Conditions for Boundary Operators in 2d Conformal Field Theory. *Phys. Lett.*, B381:97–104, 1996. hep-th/9603097.

[77] A. Kapustin and Y. Li. D-branes in Topological Minimal Models: The Landau–Ginzburg Approach. 2003. hep-th/0306001.

[78] R. Dijkgraaf, H. Verlinde, and E. Verlinde. Notes on Topological String Theory and 2-d Quantum Gravity. Based on Lectures given at Spring School on Strings and Quantum Gravity, Trieste, Italy, Apr 24 - May 2, 1990 and at Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, Cargese, France, May 28 - Jun 1, 1990.

[79] B. Dubrovin. Geometry of 2-d Topological Field Theories. Lecture Notes in Math. **1620**, Springer, Berlin, 1996. hep-th/9407018.

[80] H. Sonoda. Sewing Conformal Field Theories. 2. *Nucl. Phys.*, B311:417, 1988.

[81] E. Witten. On the Structure of the Topological Phase of two-dimensional Gravity. *Nucl. Phys.*, B340:281–332, 1990.

[82] C. I. Lazaroiu. On the Structure of Open-closed Topological Field Theory in two Dimensions. *Nucl. Phys.*, B603:497–530, 2001. hep-th/0010269.

[83] G. W. Moore and G. B. Segal. unpublished; see http://online.kitp.ucsb.edu/online/mp01/.

[84] K. Fukaya, Y. G. Oh, H. Ohta, and K. Ono. Lagrangian Intersection Floer Theory — Anomaly and Obstruction. prnote available at http://www.kusm.kyoto-u.ac.jp/∼fukaya/fukaya.html.

[85] K. Fukaya and Y. G. Oh. Zero-loop Open Strings in the Cotangent Bundle and Morse Homotopy. *Asian J. Math.*, 1:no. 1 96, 1997.

[86] K. Fukaya. Floer Homology and Mirror Symmetry I. Proceedings of the Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, Massachusetts, 1999), pp. 15–43, AMS/IP Stud. Adv. Math. **23** (Amer. Math. Soc., Providence, RI, 2001).

[87] K. Fukaya. Floer Homology and Mirror Symmetry II. in "Minimal Surfaces, Geometric Analysis and Symplectic Geometry", Adv. Stud. Pure Math **34**, Math. Soc. Japan, Tokyo 2002.

[88] M. Penkava and A. Schwarz. A(infinity) Algebras and the Cohomology of Moduli Spaces. 1994. hep-th/9408064.

[89] B. Zwiebach. Oriented Open-closed String Theory Revisited. *Annals Phys.*, 267:193–248, 1998. hep-th/9705241.

[90] G. W. Moore. Some Comments on Branes, G-flux, and K-theory. *Int. J. Mod. Phys.*, A16:936–944, 2001. hep-th/0012007.

[91] C. I. Lazaroiu. Holomorphic Matrix Models. *JHEP*, 05:044, 2003. hep-th/0303008.

[92] C. I. Lazaroiu. On the Boundary Coupling of Topological Landau–Ginzburg Models. 2003. hep-th/0312286.

[93] F. Cachazo and C. Vafa. $N = 1$ and $N = 2$ Geometry from Fluxes. 2002. hep-th/0206017.

[94] K. Hori. Linear Models of Supersymmetric D-branes. 2000. hep-th/0012179.

[95] C. I. Lazaroiu and R. Roiban. Holomorphic Potentials for Graded D-branes. *JHEP*, 02:038, 2002. hep-th/0110288.

[96] C. I. Lazaroiu. Generalized Complexes and String Field Theory. *JHEP*, 06:052, 2001. hep-th/0102122.

[97] C. I. Lazaroiu. Unitarity, D-brane Dynamics and D-brane Categories. *JHEP*, 12:031, 2001. hep-th/0102183.

[98] C. I. Lazaroiu. Graded Lagrangians, Exotic Topological D-branes and Enhanced Triangulated Categories. *JHEP*, 06:064, 2001. hep-th/0105063.

[99] C. I. Lazaroiu, R. Roiban, and D. Vaman. Graded Chern–Simons Field Theory and Graded Topological D-branes. *JHEP*, 04:023, 2002. hep-th/0107063.

[100] C. I. Lazaroiu and R. Roiban. Gauge-fixing, Semiclassical Approximation and Potentials for Graded Chern–Simons Theories. *JHEP*, 03:022, 2002. hep-th/0112029.

[101] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, and A. Zabrodin. Unification of All String Models with $c < 1$ in the Framework of Generalized Kontsevich Model. *JETP Lett.*, 55:11–16, 1992.

[102] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa. Topological Strings and Integrable Hierarchies. 2003. hep-th/0312085.