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Fuzzy Logic and Supervaluationism

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Contents

Table of Contents **1**

1 Preface **2**

1.1 Mission Statement . . . 2
1.2 A Brief Overview . . . 3
1.3 Acknowledgements . . . 4

2 Fuzzy Logic **5**

2.1 Fuzzy Sets . . . 5
2.2 Fuzzy Connectives . . . 8
2.3 Fuzzy Control Systems . . . 11
2.4 T-norm Based Fuzzy Logics . . . 15
2.4.1 Logics . . . 15
2.4.2 Conjunctions – T-norms . . . 20
2.4.3 Implications and Negations – Residua . . . 22
2.4.4 Disjunctions – T-conorms and Lattices . . . 24
2.4.5 Hájek's Basic Logic (BL) . . . 26
2.4.6 Stronger Logics . . . 27

3 Supervaluationism **29**

3.1 The Basic Concept . . . 29
3.2 Precisifications . . . 29
3.3 "Truth is Supertruth" . . . 31
3.4 Penumbral Connections . . . 35
3.5 The D-Operator . . . 36

4 Sorites **37**

4.1 Sorites Paradox . . . 37
4.1.1 Conditional Sorites . . . 38
4.1.2 Mathematical Induction Sorites . . . 39
4.1.3 Modus Ponens . . . 39

4.2 Sorites and Fuzzy Logic 39
4.3 Sorites in Supervaluationism 41

5 Fuzzy Logic as a Theory of Vagueness **43**
5.1 Fifteen Points of Critique of Fuzzy Logic – The Questions, Analyses and Attempts to Answer Them 45
5.1.1 Improper Precision 45
5.1.2 Linear Ordering of Truth Values 46
5.1.3 Truth Functionality 47
5.1.4 Higher Order Vagueness 48
5.1.5 Different Truth Functions for Connectives 48
5.1.6 Worries about “(Too) Many Logics” 49
5.1.7 Hedging via Disjunctions 50
5.1.8 Sacrificed Principles of Classical Logics 52
5.1.9 Epistemic, Ontic or Pragmatic Character? 55
5.1.10 Surface Phenomena 56
5.1.11 Penumbral Connections 56
5.1.12 It is Only a Model 57
5.1.13 Relation to Natural Language 57
5.1.14 Operational Deficiency 59
5.1.15 Record of Discourse 59
5.2 Lessons Learned 61
5.2.1 Attitudes Differ 61
5.2.2 A Question of Modeling? 62
5.2.3 A Theory of Vagueness? 64

6 Supervaluationism Applied **65**
6.1 Supervaluationist Geography 65
6.2 A User-Aided Reduction of the Precisification Space 66

7 Connections between Fuzzy Logic and Supervaluationism **70**
7.1 A Shared Logical Framework 70
7.2 Creating Statistics of Precisification Spaces 74
7.2.1 Is a Compromise Needed? 74
7.2.2 A Discrete Number of Precisifications 74
7.2.3 Continuous Precisification Spaces 77
7.3 Max Black – Hazy Sets 82

8 Closing Remarks **85**

A Symbols Used **87**

List of Figures **90**

Bibliography **91**

Chapter 1

Preface

1.1 Mission Statement

Although fuzzy logic and supervaluationism are both designed to deal with the concept of vagueness, they are generally not considered to be compatible. They represent two quite different approaches to analyzing the logical structures of reasoning with vague notions and propositions. While fuzzy logic was developed as an engineering tool, supervaluationism was formulated as a theory of vagueness in the field of philosophy and logic. Whereas one approach seeks to meet the pragmatic needs of technical applications, the other aspires to find a mathematically and philosophically satisfying analysis of reasoning under vagueness. This is illustrated by the facts that although fuzzy logic is an extremely successful engineering tool, it lacks a firm basis and general acceptance as a theory of vagueness, and although supervaluationism is a successful theory of vagueness, it has yet to find many practical applications.

The author of this thesis personally experienced the rift between these two approaches, though not necessarily in direct relation to supervaluationism. At the NAFIPS 2006 conference in Montreal, Canada, those who attended the plenary talks witnessed a heated debate between Bernard de Baets and Burhan Türkşen regarding the objectives of science. While Türkşen, an engineer, asserted that practical applicability is sufficient proof for the validity of a concept and that he had little use for concepts that he cannot apply in practice, de Baets argued from a mathematical standpoint, emphasizing the futility of dealing with facts the meaning of which we do not truly understand.

This rift was again evident during the course of surveying work recently undertaken by the author, which will be discussed in a later chapter. Inspired by these experiences, the mission statement of this thesis is to examine both of these theories from a “neutral” viewpoint, seeking to find potential relationships in the field of applications, examining the question of whether these theories are really as disjoint – in application and objectives – as they seem to be at first glance and looking for possible bridges with respect to application. Basically, it is an attempt to put both of these theories into the framework they have each been avoiding – that is, evaluating fuzzy logic in the context of a theory of vagueness and supervaluationism as a practical tool in the field of engineering.

1.2 A Brief Overview

After having stated the aims of the thesis in Chapter 1, Chapter 2 begins with an exploration of one of the two basic concepts addressed by this thesis. It will be apparent that the term "fuzzy logic" is, in itself, fuzzy. It encompasses various approaches that share only one basic idea, yet all are referred to as "fuzzy logic." In the attempt to avoid being accused of having misused the term, the author has sought to cover all possible usages of the term in this chapter and has discussed the naming controversies – see Section 2.4.1 in particular.

After having explored the framework of fuzzy logic in all its manifestations, the framework of supervaluationism is investigated in Chapter 3. The objective there is to present the framework used in later sections in advance, so that the reader will not have to spend time checking definitions of terms deemed to be basic in this theory.

Chapter 4 explores the Sorites paradox, an ancient puzzle about borderline deliberations in logical reasoning. Both fuzzy logic and supervaluationism consider themselves to be able to solve the Sorites paradox in their respective framework. By illustrating how they each do this, it has been possible to illustrate the vastly different "attitudes" these two theories have towards solving problems.

As discussed in the mission statement, fuzzy logic has had more success as a technical tool than as a theory of vagueness. Chapter 5 explores this phenomenon by looking at conceptual problems logicians and philosophers see in the framework of fuzzy logic and seeks to explain the reasons for these. The chapter also tries to offer insights into the relationship between empirical science and theoretical science: is a theory valid because it works or does it work because it is valid? Or can it work without being valid?

Just as fuzzy logic has not gained acceptance as a full-fledged theory of vagueness, supervaluationism lacks practical applications. It is deemed to be too theoretical and too "rigid" to be particularly useful in engineering. Chapter 6 attempts to see whether this is really the case or whether it would not be possible to apply supervaluationism in practice. This effort rejects the notion that supervaluationism should in any way compete with fuzzy logic and, instead, tries to find applications better suited to its framework.

Chapter 7 explores the questions of whether the differences in the frameworks of the two theories do, in fact, make them opposites or whether it does not actually leave open possibilities of applying them in a complementary manner. Both logics (Section 7.1) and practical applications (Section 7.2) are considered here.

The closing remarks, found in Chapter 8, briefly recapitulate this summary in terms of results rather than aims.

As the author has experienced difficulties arising from the fact that various individuals and communities often use different symbols for one and the same concept – or, even worse, use one and the same symbol for different concepts – a concerted attempt has been made to be consistent in the use of symbols in this thesis. A table of which symbols are used for which concepts is provided in Appendix A.1.

1.3 Acknowledgements

Chapter 5 of this thesis will discuss survey work that I conducted in early 2007. As the nature of this work might suggest, I am indebted to many individuals who made contributions to the project, including Mirko Navara of the Czech Technical University; Carles Noguera and Lluís Godo Lacasa of the Spanish National Research Council; Robert Kosik and Reinhard Viertl of the Vienna University of Technology; Enric Trillas, Thomas Vetterlein and Christian Borgelt of the European Center for Soft Computing; Siegfried Gottwald of the University of Leipzig; Jerry M. Mendel of the University of Southern California; Vesa A. Niskanen of the University of Helsinki; Hans-Jürgen Zimmermann of INFORM; Vilém Novák of the University of Ostrava; and Petr Hájek, Petr Cintula and Libor Behounek of the Academy of Sciences of the Czech Republic. If I should have the opportunity to publish the results of this survey in some other form in the future, my thanks also go to anyone else who might still want to contribute to this discussion.

Special thanks go to Christian G. Fermüller of the Vienna University of Technology for the initial formulation of the questions dealt with in Chapter 5, and to Rudolf Seising of the Medical University of Vienna for extensive assistance with “social networking” related to the survey, as well as for other indispensable help and advice in the preparation of this thesis.

Thanks also go to Laura Bradley for contributing some of the artwork that went beyond my artistic skills.

Chapter 2

Fuzzy Logic

2.1 Fuzzy Sets

Fuzzy logic was derived from fuzzy set theory, a concept introduced by Lotfi A. Zadeh in 1965 [48]. The fuzzy approach attempts to deal with vagueness by assigning linguistic variables varying degrees of applicability. It has proven to be useful in a wide range of applications, primarily in the field of control theory. The term "fuzzy logic" was coined by Zadeh's Berkeley colleague, George Lakoff, a professor of linguistics, in his 1972 paper *Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts* [24].

Fuzzy set theory is an expansion of set theory, with the difference that unlike classical set theory, the boundaries of a set are not precise, but unsharp or fuzzy. In a classical set – such as the set of "odd numbers between 0 and 100" – it will always be perfectly clear if a value belongs to the set or not.

As the name suggests, a fuzzy set does not have a sharp boundary. A value can

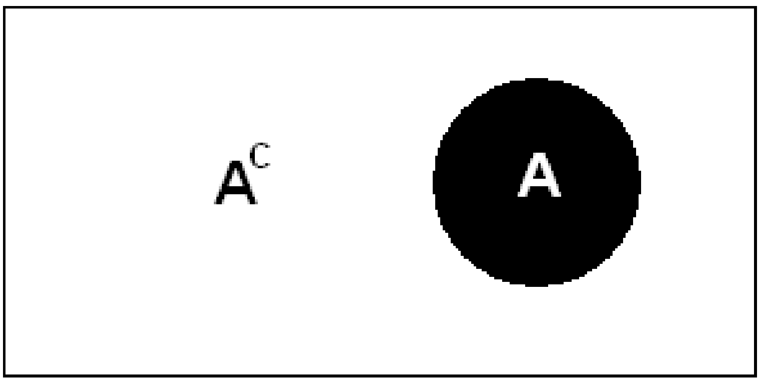

Figure 2.1: *A classical set and its complement.*

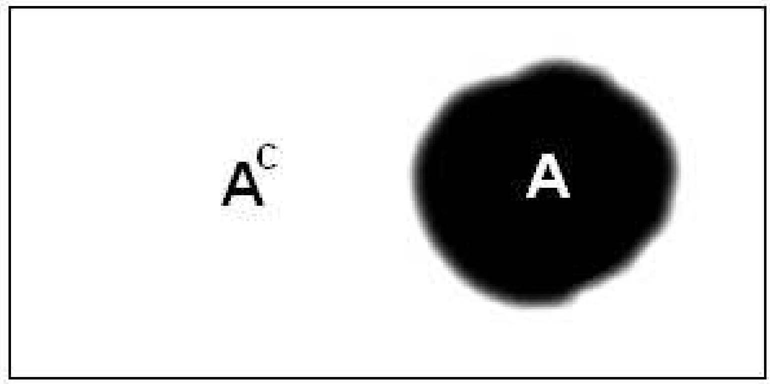

Figure 2.2: *A fuzzy set and its complement.*

clearly be in a set or it can clearly be outside the set, but it can also be "somewhere in between." Membership in a fuzzy set is signified by the so-called membership function $\mu(x)$, which moves between the values 0 and 1. If an object definitely belongs to a set, the membership function will return the value 1 for it. If it clearly does not belong to a set, the membership function will return the value 0. For anything else, an incomplete degree of membership will be returned.

An example from reality can be used to illustrate the concept. One could consider a set A called "European countries," as well as a set B called "Asian countries." One could then try to determine the degrees of membership that Switzerland, Japan and Russia would have in these sets.

The boundaries of Switzerland clearly do not extend beyond the European continent. It fully belongs to the one set and fully does not belong to the other set.

$$\mu_A(Switzerland) = 1$$
$$\mu_B(Switzerland) = 0$$

Japan, on the other hand, does not have any territory anywhere near anything that could be called Europe. It clearly is not a member of this set and fully belongs to the set of Asian countries.

$$\mu_A(Japan) = 0$$
$$\mu_B(Japan) = 1$$

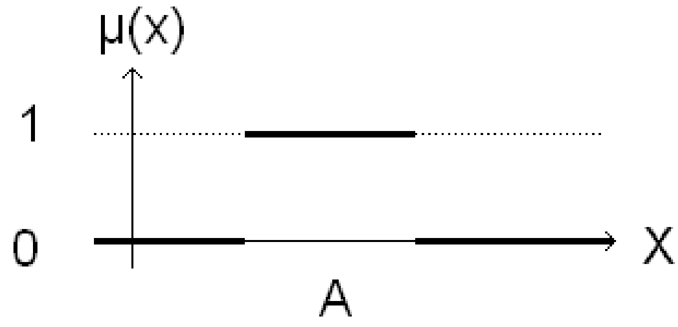

Figure 2.3: *Membership function of a classical set.*

Russia is considerably more difficult to categorize. It lies both within Europe and within Asia, and thus is not clearly either a European country or a non-European country.

If one was to regard land mass as the relevant factor – whether land mass or population would be the best factor to choose is a question of modeling – one could say that Russia belongs to Europe to 23.191%, and to Asia to 76.809%. Thus, its degree of memberships in the two sets would be:

$$\mu_A(Russia) = 0.23191$$
$$\mu_B(Russia) = 0.76809$$

It should be noted, however, that these sets only happen to be disjoint. There are no rules in fuzzy set theory dictating that the sum of all membership functions must add up to a certain value, as there are in statistics. We could also consider a third set, C, called "Alpine countries," to which Switzerland would belong to the same degree that it belongs to the set of European countries.

If one was to illustrate the membership function in a simple one-dimensional environment, sets in classical theory would look like "steps" – with one set consisting of one or several, "steps."

In a fuzzy environment, the same graph can have any possible shape, as long as the $[0, 1]$ interval is not broken.

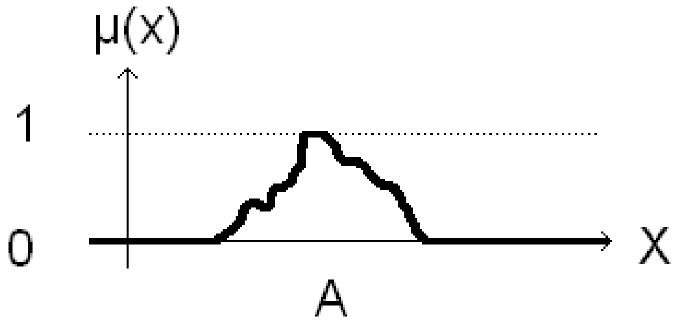

Figure 2.4: *Membership function of a fuzzy set.*

2.2 Fuzzy Connectives

As an expansion of set theory, fuzzy logic must be able to handle the same operations as classical set theory – namely, union $\cup$, intersection $\cap$ and complement A^C. More complicated connections between sets can be constructed once these building blocks have been established.

Practically, these operations and their "counterparts" in logic (disjunction $\vee$, conjunction $\wedge$ and negation $\neg$) are evaluated using a wide array of methods, through so-called t-norms, which will be addressed in Section 2.4.2.

When starting to work with fuzzy sets, Zadeh proposed relatively simple means for the evaluation of the two basic connectives and the complement. The union of two sets A and B, written as $A \cup B$, includes elements that are members of either set A or set B (or of both) in classical set theory. For example, the group consisting of "numbers that are odd or greater than 10" would be a union in classical logic of the set of odd numbers and the set of numbers larger than 10. In fuzzy set theory, the union is classically evaluated as the maximum of the two degrees of membership – the highest degree to which it belongs to one of the connected groups is the degree to which it belongs to the union of the two groups.

$$\mu_{A\cup B}(x) = Max(\mu_A(x), \mu_B(x))\forall x \in X$$

In standard set theory the intersection between two sets A and B, $A \cap B$ includes any elements that are members of both set A and set B. For example, the set "even numbers greater than 10" would be an intersection between two sets – even numbers and numbers greater than 10. In a fuzzy environment the classical evaluation of an intersection employs the minimum of the two degrees of membership.

$$\mu_{A\cap B}(x) = Min(\mu_A(x), \mu_B(x))\forall x \in X$$

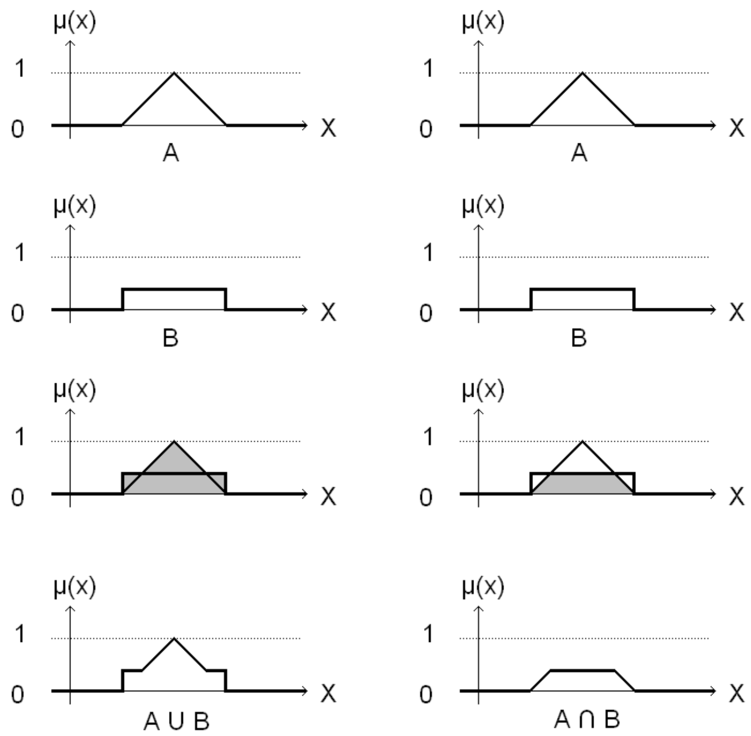

Figure 2.5: *Union and intersection of two fuzzy sets.*

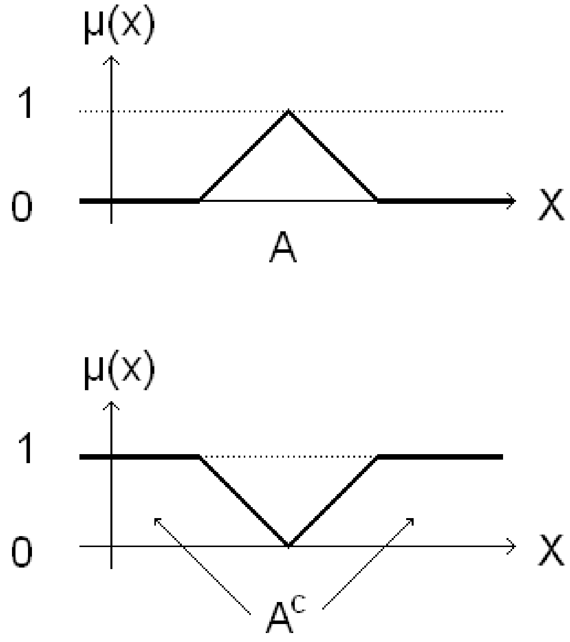

Figure 2.6: *The complement of a fuzzy set.*

The complement of a set A^C in classical set theory consists of all elements that are not members of this set. For example, odd numbers are the complement of even numbers, and vice versa.

In a fuzzy context the complement of the membership in a set is formed by subtracting its degree of applicability from 1.

$$\mu_{A^C}(x) = 1 - \mu_A(x) \forall x \in X$$

In classical set theory, set A is a subset of set B ($A \subseteq B$) if every element of B is also an element of A. For example, the set "real numbers between 10 and 20" is a subset of the set "real numbers between 5 and 25," as there is no element which is a member of the first set but not a member of the second set. Thus, $A \subseteq B$ can be defined as follows:

$$A \subseteq B \leftrightarrow A \cap B = B \quad \forall x \in X$$

This same relation can be applied in a fuzzy environment. If the degree of membership in set A is smaller than or equal to the degree of membership in set B over the entire interval considered, the maximum of the membership functions will resemble the membership function of B.

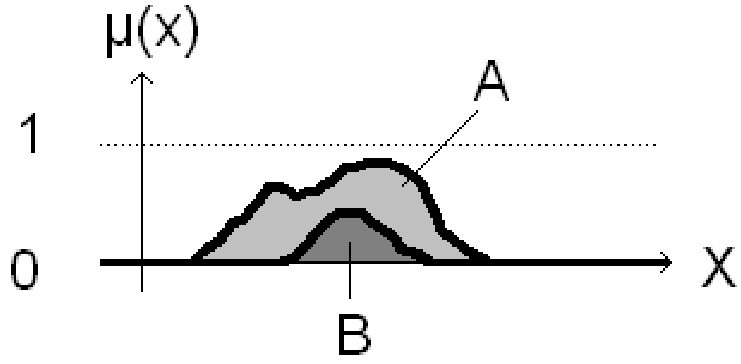

Figure 2.7: *A subset of a fuzzy set.*

$$\mu_A(x) \leq \mu_B(x) \quad \forall x \in X$$

If illustrated, this would mean that A's membership curve always remains less than or equal to B's membership curve.

2.3 Fuzzy Control Systems

Even though the term "fuzzy logic" was first introduced in the context of linguistics, its successes to date have been more numerous and more significant as an engineering tool in the field of technology. As such, it seems sensible to examine an example of how fuzzy logic is actually used in practice, in order to illustrate its real-world application.

In a control circuit fuzzy logic employs two mechanisms – fuzzification and defuzzification – to convert sharp inputs into sharp outputs in a fuzzy computation process. It is important to note that any conscientious handling of uncertainty in practical applications happens solely in the computation process, as the input to such a system will generally consist of measurements that have specific values and output parameters generally need to have a certain value of some sort in order to be used in practice.

The first process noted, fuzzification, is the process in which the respective values of the membership function over the range of possible values is determined. For example, one could try to define the meaning of the word "cold," for the purposes of a room heating system. When speaking of room temperatures, most people will consider any temperature below 10 °C to be cold. When the temperature rises above this, the term "cold" becomes less applicable – though it does, for a time, still remain applicable to some degree – which is where varying degrees of applicability come in.

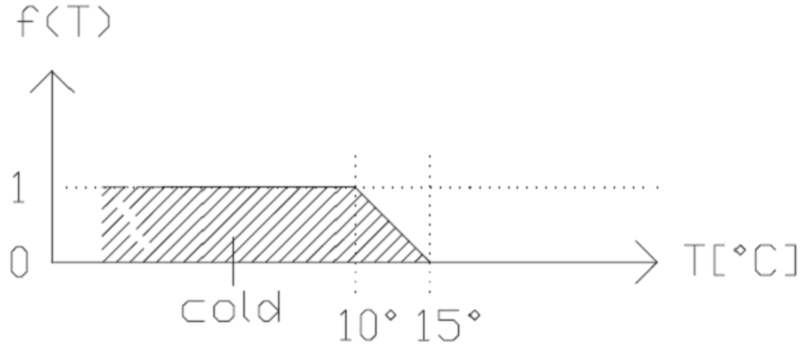

Figure 2.8: *A fuzzy set for the term "cold," in relation to the room temperature in a room in* °C.

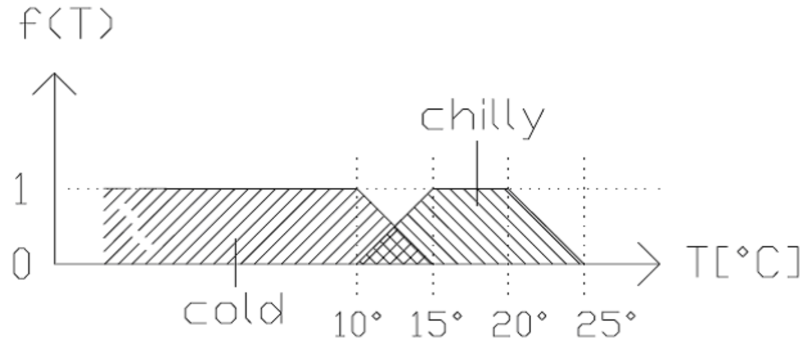

Figure 2.9: *An additional fuzzy set added to the example.*

As is evident from Figure 2.8, the form of a fuzzy set does not resemble that of statistical distributions – a fuzzy set can have any shape imaginable within the $[0, 1]$ interval. Fuzzy sets do not exclude each other, and no statements whatsoever can be made about the integral over the entire spectrum.

An example of the aforementioned varying degrees of applicability that are essential to fuzzy logic can be seen here between the point under which the temperature is definitely cold (found to be 10 °C, where the linguistic value "cold" would have the value 1) and the point over which the temperature is definitely not cold (found to be 15 °C, where the same linguistic value would be applicable to a degree of 0). For instance, in this example, at the temperature of 12 °C, the linguistic variable would be applicable to the degree of 0.6.

To elaborate the example of a room heating system, one could define a second fuzzy set "chilly" for a situation in which it is not really cold per se, but in which heating would still be desirable – to a lesser extent, since the room temperature still lies beneath the ideal room temperature, which one could place at 23 °C.

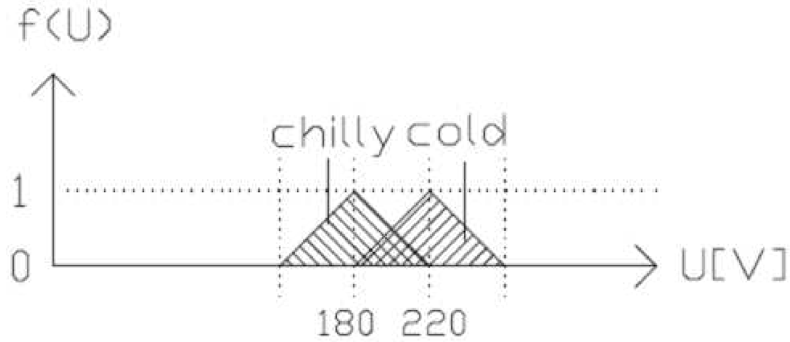

Figure 2.10: *Output functions for this example.*

It must be noted that the definition of these fuzzy sets is totally ambiguous. The nature of statements like "cold to a degree of 0.734" is as obscure to human thinking as the binary reasoning used in classical logic is. The definition of fuzzy sets is, at best, the ideal approximation of a term as it is used in matters of human thinking. One should not expect a higher exactness from a fuzzy system than the empirical data used in its definition can offer.

With these fuzzy sets defined, it is possible to determine how applicable a certain term might be in a certain situation. This information is, however, not directly usable by a heating system – it now only knows how cold it is in a given situation. But what does the system do when it is cold? Sharp data has been fuzzified, but a technical mechanism does not know how to operate with fuzzy data – it needs an exact input value to work with, such as a voltage to be induced into a circuit, or the liquid flow allowed in a coolant system.

For this purpose, it is necessary to define so-called output functions for each fuzzy set. These functions define the magnitude of a certain effect that the validity of certain terms, such as "cold" and "chilly," have on an output variable – for example, the voltage fed into a heating circuit – which is roughly indirectly proportional to the temperature in a room. The colder it is, the more one heats. If it is indisputably cold, the circuit will heat at its full capacity. For example, we could consider 220 volts as the full capacity of a system, an input into the heating system that will be desired in a truly cold situation. One could define the voltage fed into the system in a chilly situation as 180 volts.

Having defined these functions, it is possible to "defuzzify" them. By applying the so-called max-min rule, it is possible to combine several inputs for more complex rules. In this simple case, we consider only one input value, the room temperature. Depending on this, certain input sets may be more or less applicable. The value to which they are applicable functions as a "cutoff" point for the output variables –

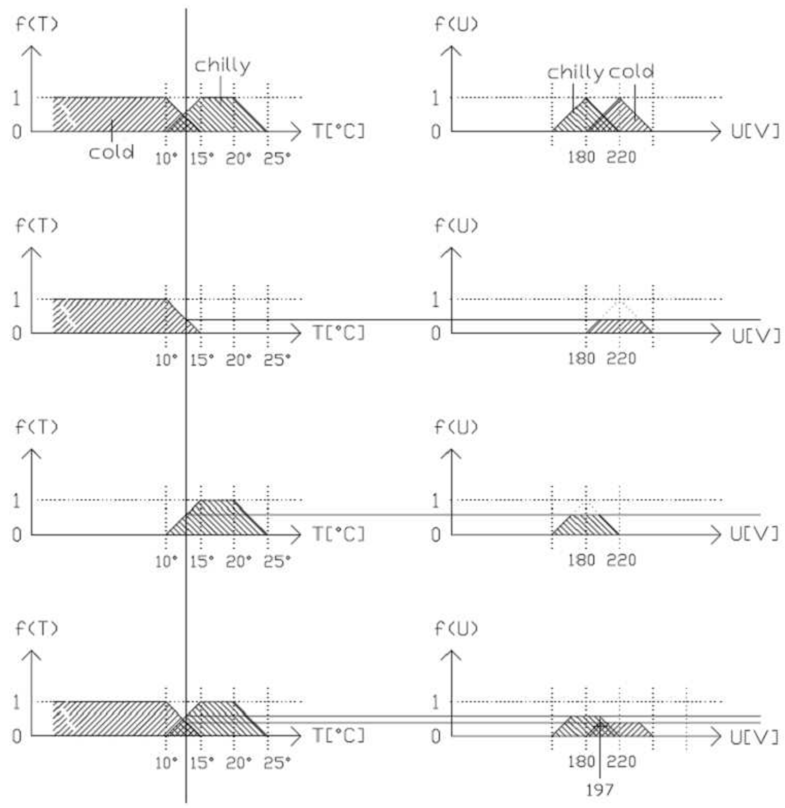

Figure 2.11: *Fuzzification and defuzzification.*

the output sets are "cut off" at the value to which the corresponding input set is applicable. Summed up for all rules – here, one for "cold" and one for "chilly" – we will get a landscape of trapezoids, of which the "center of gravity" is the output value. This is a sharp value which a technical system can deal with.

This simplified example can be imagined as follows: in cases in which neither "cold" nor "chilly" are fully applicable, the respective membership functions at the measured temperature tell us how relevant the desired output of each situation is. The weighed result will lie somewhere in between. The borderline temperature of 13 °C can function as a good example, which is illustrated in Figure 2.11

The crisp input of 13 °C results in the crisp output of $\sim$ 197 volts, a number which can be fed into the control circuit. This alters the temperature in the room, which alters the results the sensors record, which in turn alters the membership functions and thus the output variable. The circuit is complete.

Clearly, the fuzzy approach works in a technical environment. However, this does not indicate how adequate it is in other contexts, such as linguistics or as a theory of vagueness. Fuzzy logic models describe circumstances of reality in relatively simple terms – here, for example, the term "cold" was reduced to one simple trapezoid. It seems naïve to assume that anything one could say about the term "cold," in any context could be expressed with one single simple geometrical shape.

2.4 T-norm Based Fuzzy Logics

2.4.1 Logics

Logic – a term derived from the ancient Greek word $\lambda\acute{o}\gamma o\varsigma$, meaning word, thought, argument or principle – is the study of validly inferring from knowledge. The way the term "fuzzy logic" has been used thus far in this thesis is somewhat misleading, since set theory and the application of set theory in technical applications are not connected with the millennia-old quest for correct methods of reasoning. However, the principles of fuzzy set theory can be applied with these goals in mind.

In order to reduce confusion, Zadeh himself coined two terms to distinguish fuzzy logics that are truly logics from other applications to which the name "fuzzy logics" is sometimes applied [49]. He differentiated between "fuzzy logic in a broad sense" and "fuzzy logic in a narrow sense," where fuzzy logic in a broad sense can be seen as a synonym for fuzzy set theory and the scope of its technical applications. Fuzzy logic in a narrow sense is a theory of approximate reasoning based on many-valued logics.

This so-called deductive fuzzy logic is based on the study of completeness, decidability, complexity, et cetera. This approach is illustrated in Petr Hájek's 2000 contribution *Why Fuzzy Logic?* [18]. Hájek starts by setting a few design choices that he deems reasonable for all fuzzy logics:

- The real unit interval $[0, 1]$ is taken as the standard set of truth values, where 0 signifies absolute falsity and 1 absolute truth. Standard ordering of real numbers, through $\leq$, applies.

- Logic is taken to be truth functional – the truth value of a connective for example, a conjunction, is uniquely determined by the respective truth values of the element and by the chosen truth function of the connective.

- Continuous t-norms, as illustrated in Section 2.4.2, are applied as possible truth functions for the conjunction.

- The truth function of the implication is defined by the residuum, as is discussed in Section 2.4.3.

- The truth function of the negation is $x \Rightarrow 0$, where $\Rightarrow$ is the truth function for implication.

Before further exploring deductive fuzzy logics, the basic principles of logics used will be reviewed. We will only deal with so-called standard structures, where the set of truth values is identified with the real closed unit interval $[0, 1]$. More general classes of algebraic structures are taken into account in [20]. Moreover, we will restrict attention to the propositional level of logics.

Syntax

In logics and linguistics, *syntax* – from ancient Greek $\sigma\upsilon\nu$-, meaning "with," and $\tau\acute{\alpha}\xi\iota\varsigma$, meaning "arrangement" – is the study of the rules governing the structure of sentences. In order to grasp the construction of logical formulas, we must analyze the manner in which they are built up from their most basic elements, much as it is essential to know what the subject and the predicate of a sentence are in natural languages in order to linguistically analyze it.

The most basic building blocks of propositional formulas are so-called *propositional variables*. Their intended use (see 'Semantics', below) is as follows: In non-graduated logics, a propositional variable is a variable that can be either true or false. In fuzzy logics, they can hold any intermittent value between 0 (false) and 1 (true), as set in the design choices. A propositional variable represents an atomic statement, such as "the sun is shining" or "Mary is sleeping." They are generally designated by lower case letters a, b, c, et cetera. The set of all propositional variables is $PV = \{a, b, c, \ldots\}$. These propositional values are also known as *atomic formulas*, since they cannot be split up into smaller elements.

In order to construct more complex *propositional formulas* from our atoms, *operator symbols* or *logical connectives* are used. Every operator symbol has an arity j. This arity j means that the operator symbol requires j propositional variables (or formulas, as operators can be used recursively) to yield a result. Connectives with the arity 2 are referred to as binary operators.

The set of all operators is $\Omega = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_j \cup \ldots \cup \Omega_m$. In more familiar propositional calculi, the following operators are used:

- $\Omega_0 = \{\bot, \top\}$
- $\Omega_1 = \{\neg\}$
- $\Omega_2 = \{\wedge, \vee, \leftrightarrow, \rightarrow\}$

$\bot$ and $\top$ are logical constants. As such, they can be considered to be either operators with the arity 0, or simply constant values. $\bot$ is a logical contradiction, which always has a "false" truth value, while $\top$ is a tautology, which always has a "true" truth value.

The negation $\neg$ of a proposition a translates as the statement "not a."

The binary operator $\wedge$, the logical conjunction, is a logical conjunction and $a \wedge b$ reads as "a and b." The logical disjunction, $\vee$ is a logical "or" and $a \vee b$ reads as "a or b."

The biconditional $\leftrightarrow$, signifies logical equivalence. $a \leftrightarrow b$ reads as "a, if and only if b" or as "b, if and only if a."

The logical material implication, $\rightarrow$, signifies that the first term implies the second term. $a \rightarrow b$ reads as "b, if a." An example of an implication that would form a tautology would be $a \rightarrow b$, where a is the statement "it is raining" and b is the statement "the ground is wet." If it is raining, the ground must be wet. However, a and b are not logically equivalent – the ground could be wet due to having been sprinkled by a hose or due to fluids having been spilled.

Starting with propositional variables and repeatedly using the defined operators, any valid formula can be constructed. Formulas in a general sense (formulas that do not have to be atoms) will be described using Greek letters φ, ψ, χ, et cetera.

Semantics

Semantics, from the Greek $\sigma\acute{\eta}\mu\alpha$ meaning sign, deals with aspects of meaning in a natural or logical language. Whereas syntax only deals with the correct construction of sentences, semantics is concerned with the actual meaning of the elements of sentences.

In linguistics one can easily illustrate the difference between syntax and semantics by creating nonsense sentences such as "A toothbrush swims a donkey." While syntactically correct – it has a subject, a predicate and an object, all in their correct grammatical forms – the sentence does not actually mean anything. It is semantically incoherent.

In propositional logics semantics deals with the assignment of truth values to propositional variables and with setting functions for connectives, thus giving them "meaning."

In a fuzzy environment, an assignment v for a set of propositional variables PV is a mapping of all its element to the real number interval between 0 and 1. $v : PV \longrightarrow [0, 1]$. These actual values will be designated by the lower case letters x, y, z, q et cetera.

When semantically evaluating a formula, one must also choose how to evaluate connectives – a matter which will be examined in detail below. Suffice it to say that in our context a function $*$ is chosen for the conjunction and the usage of the other connectives is chosen in accordance with the respective conjunction.

Evaluation

Given a logical formula φ, its truth value under the assignment v, using truth functions based on $*$, is denoted by $e_*^v(\phi)$. The function e_*^v, which assigns truth values to formulas, is called *evaluation*. The evaluation of an atom a, $e_*^v(a)$ is taken directly from the assignment v.

In order to evaluate a more complex formula, one must "replace" the syntactic connectives with the respectively chosen semantic functions. For example, a logic could evaluate the implication $\rightarrow$ over a specific function $\Rightarrow_*$. Given this, one could split a formula as follows:

$$e_*^v(\varphi \rightarrow \psi) = e_*^v(\varphi) \Rightarrow_* e_*^v(\psi)$$

Using this methodology, it is possible to process complex formulas into calculable functions. As an example with only very few possible evaluations, let us consider the formula $\varphi = a \wedge b$. We could consider two assignments, $v_1 = \{a \longrightarrow 0.6, b \longrightarrow 0.4\}$ and $v_2 = \{a \longrightarrow 0.3, b \longrightarrow 0.6\}$. We could use a minimum for the conjunction or the product. Given these possibilities, we would get four respective evaluations.

- $e_{min}^{v_1}(a \wedge b) = min\{e_{min}^{v_1}(a), e_{min}^{v_1}(b)\} = min\{0.6, 0.4\} = 0.4$
- $e_{min}^{v_2}(a \wedge b) = min\{e_{min}^{v_2}(a), e_{min}^{v_2}(b)\} = min\{0.3, 0.6\} = 0.3$
- $e_{prod}^{v_1}(a \wedge b) = e_{prod}^{v_1}(a) \cdot e_{prod}^{v_1}(b) = 0.6 \cdot 0.4 = 0.24$
- $e_{prod}^{v_2}(a \wedge b) = e_{prod}^{v_2}(a) \cdot e_{prod}^{v_2}(b) = 0.3 \cdot 0.6 = 0.18$

Validity and Satisfiability

Given a logic L_* and a specific assignment v_x, a formula φ is *true* when $e_*^{v_x}(\varphi) = 1$, that is, when it is true in this specific case.

Validity is a stronger concept. φ is *valid* in a logic L_* when it is true under all assignments - $e_*^v(\varphi) = 1 \forall v$.

When an assignment exists under which φ is true, φ is *satisfiable* – $\exists v : e_*^v(\varphi) = 1$.

It is important not to confuse these two concepts. Just because a formula is not valid does not necessarily mean that it is not true. Validity implies truth, truth implies satisfiability. But these implications are not reversible.

Let us choose a logic L_{min}in which operators are based on Zadeh's set operators discussed in Section 2.2. The minimum doubles as the conjunction, the maximum as the disjunction.

Three examples would be:

- $\varphi = a \leftrightarrow a$
- $\psi = a$
- $\chi = \neg(a \leftrightarrow a)$

Formula φ will always be true. Thus it is valid and satisfiable. ψ is true when a is mapped to the value 1, but false when it is mapped to the value 0. It is thus satisfiable, as an assignment exists under which it is true, but not valid, as it is not true under all assignments. Since there is no assignment under which χ is true, it is neither satisfiable nor valid.

Modus Ponens

Modus ponens (Latin for "mode that affirms") is a simple form of argumentation(inference rule) commonly used in logics. It simply states the following:

$$\varphi, (\varphi \rightarrow \psi) \vdash \psi$$

If φ implies ψ, and we know φ to be true, then we also know ψ to be true. A classical example applied to propositional logics that illustrates modus ponens is:

- "If Socrates is human, Socrates is mortal"
- "Socrates is human"
- "Therefore, Socrates is mortal"

Here, φ is the statement "Socrates is human" and ψ is the statement "Socrates is mortal."

Soundness and Completeness

Formal logics should have an *adequate* proof system. Here this means: a finite set of schematic axioms and some simple inference rules, like modus ponens.

A formula is called derivable in such a system if it is either an instance of an axiom or the instance of the conclusion of an inference rule, where the premises are instances of already derived formulas. Adequateness, here, means soundness and completeness. A proof system is *sound* if all derivable formulas are tautologies. It is *complete* if all tautologies are derivable.

2.4.2 Conjunctions – T-norms

A t-norm – short for "triangular norm" – is a binary operation used in multi-valued logics and in probabilistic metric spaces, generally to compute connectives between two fuzzy sets.

The classical methods introduced by Zadeh, as discussed in 2.2, are not always suitable in practice. For example, the evaluation of the disjunction between two sets using the minimum of the two membership functions is unsuitable when dealing with disjoint groups, an issue that will be addressed in Section 5.1.7. It is desirable to expand the options given when choosing semantics for a fuzzy logic.

So what would a viable truth function for the conjunction be? In t-norm based fuzzy logic, any function satisfying certain conditions for qualifying as a t-norm is considered a possibility. It must treat both factors of the conjunction as equal and the ordering must not matter. It must increase monotonically in its two dimensions – "0.5 and 0.3" may not have a higher value than "0.6 and 0.3." The conjunction with the value 1 must preserve the second value in a conjunction. Associativity must be preserved.

To define a t-norm mathematically: A t-norm is a function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the following four properties:

- Commutativity – $T(x, y) = T(y, x)$
- Monotonicity – $((x \leq y)$ and $(z \leq q))$ implies $T(x, y) \leq T(z, q)$
- Associativity – $T(x, T(y, z)) = T(T(x, y), z)$
- 1 is the identity element – $T(x, 1) = x$

A more detailed account of the principle of t-norms can, for example, be found in [23].

We will limit ourselves to handling three important t-norms here: the minimum t-norm T_{min}, which is also known as the Gödel t-norm, the Łukasiewicz t-norm $T_{Ł}$ and the product t-norm T_{prod}. These t-norms are listed in Figure 2.12 and are used as the *strong conjunction* & in their respective logics. Logics employing the minimum t-norm are called Gödel logics **G**; logics employing the Łukasiewicz t-norm are called the Łukasiewicz logics **Ł**; and logics employing the product t-norm are called product logics Π.

Their layout can also be illustrated graphically, in three dimensions as well as with the use of contour lines.

In addition to the strong conjunction, every t-norm based fuzzy logic also possesses a so-called *weak conjunction*, or *lattice conjunction* $\wedge$. The standard semantics for the

T-norm
$T_{min}(x, y) = min\{x, y\}$
$T_{Ł}(x, y) = max\{0, x + y - 1\}$
$T_{prod}(x, y) = x \cdot y$

Figure 2.12: *Three popular t-norms*

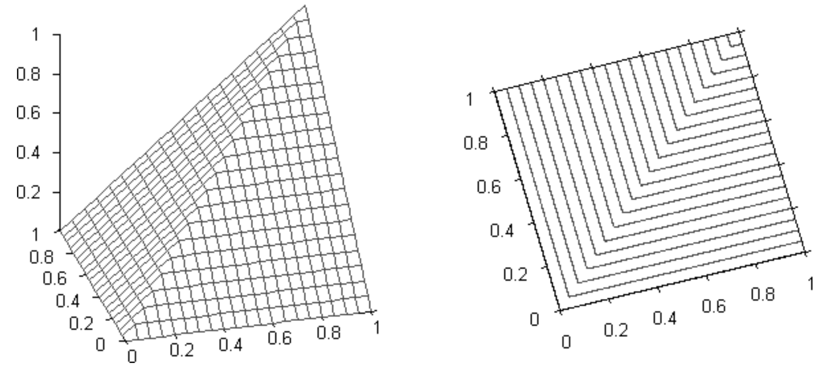

Figure 2.13: *Graph of the minimum t-norm*

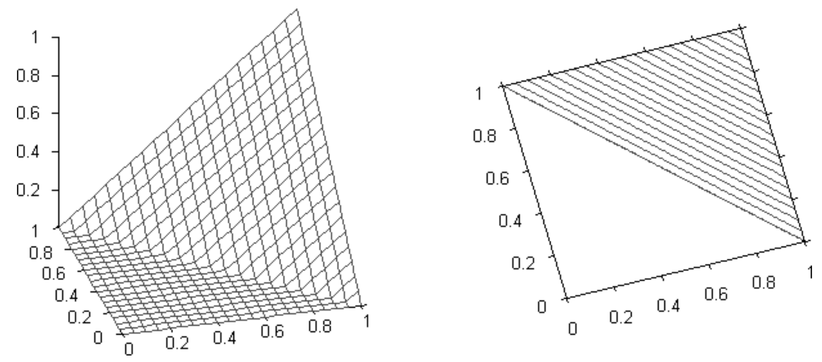

Figure 2.14: *Graph of the Łukasiewicz t-norm*

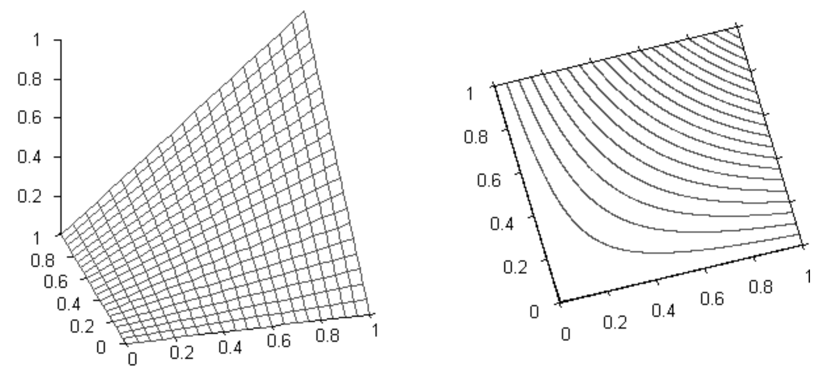

Figure 2.15: *Graph of the product t-norm*

T-norm	Residuum
$T_{min}(x, y) = min\{x, y\}$	$(x \Rightarrow_{min} y) = \begin{cases} 1, & x \leq y \\ y, & x > y \end{cases}$
$T_{Ł}(x, y) = max\{0, x + y - 1\}$	$(x \Rightarrow_{Ł} y) = min\{1, 1 - x + y\}$
$T_{prod}(x, y) = x \cdot y$	$(x \Rightarrow_{prod} y) = \begin{cases} 1, & x \leq y \\ \frac{y}{x}, & x > y \end{cases}$

Figure 2.16: *Three popular t-norms and their residua*

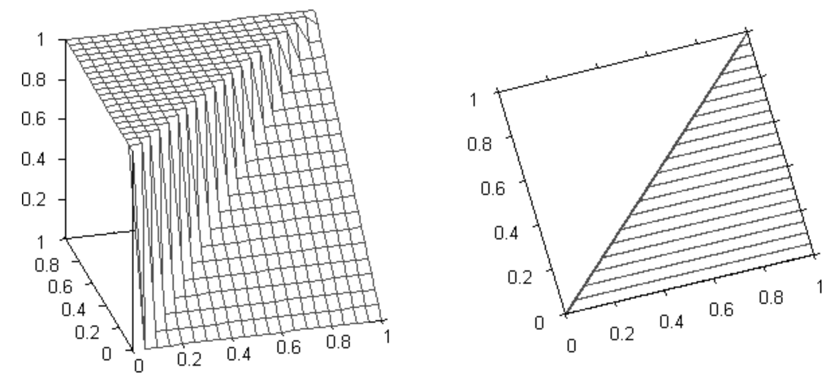

Figure 2.17: *Graph of the minimum t-norm's residuum, the standard Gödel implication*

weak conjunction is the minimum t-norm, as the pointwise largest t-norm and as the only t-norm in which the idempotence of the conjunction $\varphi \leftrightarrow (\varphi \wedge \varphi)$ holds. Thus, in Gödel logics the weak conjunction and the strong conjunction are identical.

2.4.3 Implications and Negations – Residua

For any left-continuous t-norm $*$, a unique binary operation $\Rightarrow_*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ exists, for which the following is true:

$$T(z, x) \leq y \text{ if and only if } z \leq (x \Rightarrow_* y) \text{ for all } x, y, z \in [0, 1]$$

This operation $\Rightarrow_*$ is the *residuum* of the t-norm $*$. It acts as the truth function for the implication.

As noted in the design choices in 2.4.1, the truth function of the negation is $x \Rightarrow_* 0$ (x implies falsity, using the respective truth function for the implication.) I.e., the truth function for the negation as $n(x) = x \Rightarrow_* 0$.

If the antecedent of an implication (x in the case of $x \Rightarrow_* y$) has a truth value lower than or equal to the truth value of the consequent (y in this example), the residuum

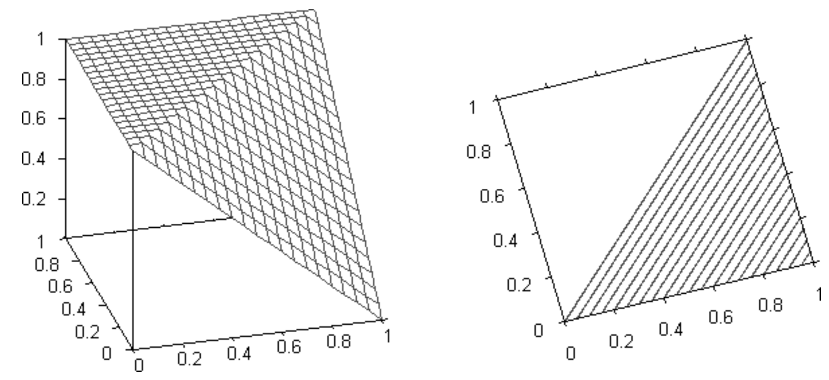

Figure 2.18: *Graph of the Łukasiewicz t-norm's residuum, the standard Łukasiewicz implication*

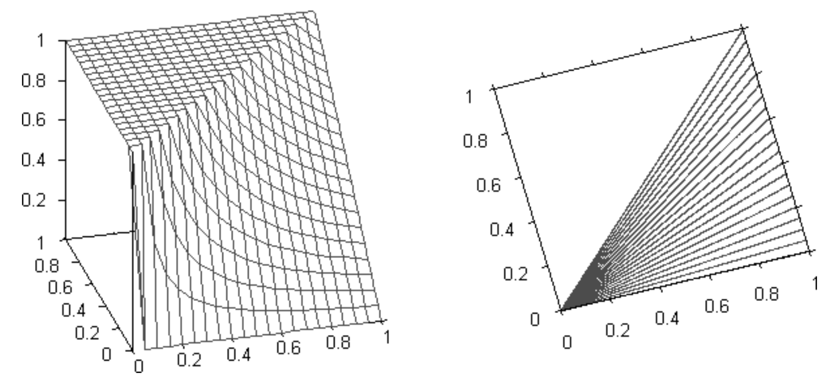

Figure 2.19: *Graph of the product t-norm's residuum, the Goguen implication*

will evaluate as 1 under any t-norm, as is evident in the graphical representations of the residua.

2.4.4 Disjunctions – T-conorms and Lattices

Every t-norm has a dual t-conorm, or s-norm. Given a t-norm, its complementary co-norm is defined by:

$$S(x, y) = 1 - T(1 - x, 1 - y)$$

As such, the resulting t-conorm satisfies the following conditions:

- Commutativity – $S(x, y) = S(y, x)$
- Monotonicity – $((x \leq y)$ and $(z \leq q))$ implies $S(x, y) \leq S(z, q)$
- Associativity – $S(x, S(y, z)) = S(S(x, y), z)$
- 0 is the identity element – $S(x, 0) = x$

The t-conorms formed from the three respective major t-norms are given their own names. The minimum t-norm's dual t-conorm is the maximum t-conorm S_{max}; the Łukasiewicz t-norm's is the bounded sum t-conorm $S_{Ł}$; and the product t-norm's dual is the probabilistic sum t-conorm S_{sum}.

A T-conorm is used to represent the *strong disjunction* $\veebar$, the dual of the strong conjunction. Much as the minimum t-norm serves as the weak conjunction (or lattice conjunction) for all t-norms, its dual, the maximum t-conorm, serves as the *weak disjunction* (or lattice disjunction) for all t-conorms.

It should be noted that while the strong conjunction is often considered to be the "standard" conjunction, the weak disjunction is generally regarded to be the "standard" disjunction. In fact, it is a matter of debate whether the strong disjunction should be used at all.

Take, for example, the bounded sum t-conorm, the Łukasiewicz t-norm's dual. It is defined as $S_{Ł}(x, y) = min\{x + y, 1\}$. Given a situation, for instance, in which one was to evaluate the statement "my glass is full *or* it is light in this room," where my glass is half full (full to a degree of 0.5) and the lights in this room are controlled by a dimmer working up to half of its capacity (making it light in the room to a degree of 0.5), the statement would evaluate to 1 – to complete truth, although neither fact considered is true to this degree. To many, this disqualifies t-conorms other than the maximum t-conorm as means for correct reasoning.

T-norm	residuum	t-conorm
$T_{min}(x,y) = min\{x,y\}$	$(x \Rightarrow_{min} y) = \begin{cases} 1, & x \leq y \\ y, & x > y \end{cases}$	$S_{max}(x,y) = max\{x,y\}$
$T_{Ł}(x,y) = max\{0, x+y-1\}$	$(x \Rightarrow_{Ł} y) = min\{1, 1-x+y\}$	$S_{Ł}(x,y) = min\{x+y, 1\}$
$T_{prod}(x,y) = x \cdot y$	$(x \Rightarrow_{prod} y) = \begin{cases} 1, & x \leq y \\ \frac{y}{x}, & x > y \end{cases}$	$S_{sum}(x,y) = x + y - x \cdot y$

Figure 2.20: *Three popular t-norms, their residua and their dual t-conorms*

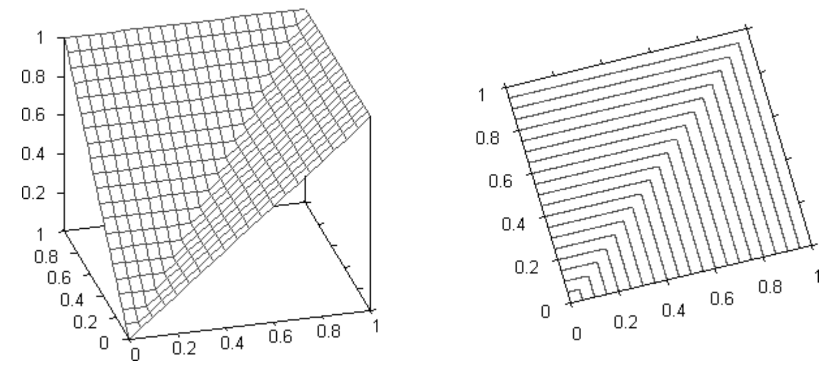

Figure 2.21: *Graph of the maximum t-conorm*

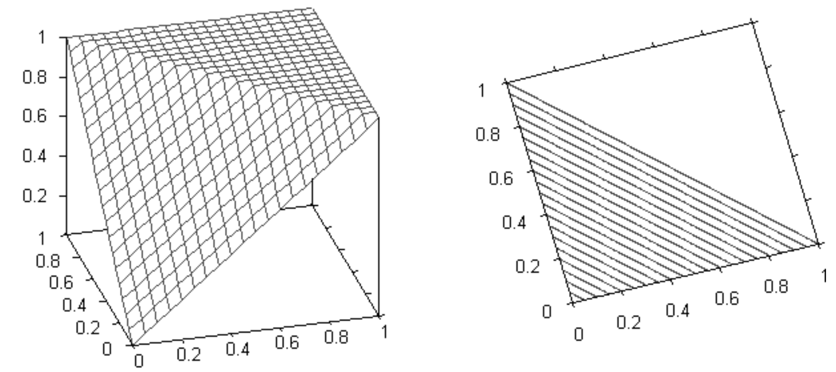

Figure 2.22: *Graph of the bounded sum t-conorm*

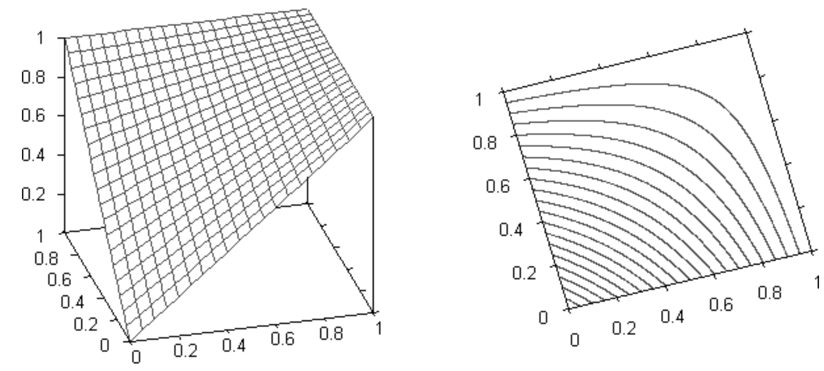

Figure 2.23: *Graph of the probabilistic sum t-conorm*

2.4.5 Hájek's Basic Logic (BL)

Hájek [20] introduced the logic of all continuous t-norms as **BL** – basic propositional fuzzy logic. Syntactically, it is based on the same connectives as the specific t-norm based logics presented above. For the semantics of these connectives, all continuous t-norms are taken into consideration.

A formula φ is only a genuine tautology in **BL**, that is, a **BL**-tautology, if it is valid under all continuous t-norms and under all assignments – for all t-norms $*$ and all assignments v, $e^v_*(\varphi) = 1$. It is a L_*-tautology if it is valid under the logic L_*, based on a specific t-norm $*$. Therefore, if for one specific $*$ and all assignments v, $e^v_*(\varphi) = 1$.

A sound and complete system is given by the following axioms, with modus ponens as the only inference rule:

- $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- $(\varphi \wedge \psi) \to \varphi$
- $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
- $(\varphi \wedge (\varphi \to \psi)) \to (\psi \wedge (\psi \to \varphi))$
- $(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$
- $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
- $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
- $\bot \to \varphi$

2.4.6 Stronger Logics

As mentioned above, **BL** is the logic of all continuous t-norms, taking all possible continuous t-norms into consideration. As such, logical principles preserved by one t-norm, but not by others, are not valid in **BL**.

One can create stronger logics by binding the framework of **BL** to specific t-norms. When this is done, the resulting logic uses one specific t-norm as the means of evaluating the (strong) conjunction and its residuum for implications (and thus, by extension, negations).

The resulting logic is axiomatized by the basic axioms of **BL**, as well as by additional axioms characterizing the specific t-norm used.

Gödel Logic G

Gödel logics **G** employ the minimum t-norm, which is also known as the Gödel t-norm.

As such, it evaluates the basic logical operators as follows:

- Conjunction: $T_{min}(x, y) = min\{x, y\}$
- Implication: $(x \Rightarrow_{min} y) = \begin{cases} 1, & x \leq y \\ y, & x > y \end{cases}$
- Negation: $(x \Rightarrow_{min} y) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$

It is axiomatized by the axioms listed in 2.4.5, as well as by the idempotence of the conjunction, $\varphi \leftrightarrow (\varphi \wedge \varphi)$.

Łukasiewicz Logic Ł

Łukasiewicz logics **Ł** employ the Łukasiewicz t-norm and its residuum.

- Strong conjunction: $T_{Ł}(x, y) = max\{0, x + y - 1\}$
- Implication: $(x \Rightarrow_{Ł} y) = min\{1, 1 - x + y\}$
- Negation: $(x \Rightarrow_{Ł} 0) = 1 - x$

The schema of double negation, $\varphi \leftrightarrow \neg\neg\varphi$, is added to the axioms.

Product Logic Π

Referring to the product t-norm one obtains the following truth functions for product logic Π."

- Strong conjunction: $T_{prod}(x,y) = x \cdot y$
- Implication: $(x \Rightarrow_{prod} y) = \begin{cases} 1, & x \leq y \\ \frac{y}{x}, & x > y \end{cases}$
- Negation: $(x \Rightarrow_{prod} 0) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$

Its proof system contains the additional axioms $(\varphi \to \neg\varphi) \to \neg\varphi$ and $\neg\neg\chi \to (((\varphi \wedge \chi) \to (\psi \wedge \chi)) \to (\varphi \to \chi))$.

Chapter 3

Supervaluationism

3.1 The Basic Concept

Unlike fuzzy logic, supervaluationism considers vagueness to be a concept related to ambiguity. Vagueness does not mean that our terms can be applicable to a certain degree, but that the usage of our terms is not precisely defined.

For example, consider a map of (mainland) France, as shown in Figure 3.1.

If one was to state, while looking at this map, that France was a hexagon, it would not be possible to say whether this statement was obviously true or obviously false. The borders and shores of France hardly form straight, accurate lines between six well-defined points (see Figure 3.2). In a purely mathematical sense, it is not a hexagon. Nevertheless, the resemblance to a hexagon is striking enough for the country to be often referred to in French as “L'hexagone.”

Similarly, the statement “the Earth is spherical” is not correct in theory – the shape of the Earth resembles that of a slightly oblate spheroid. In most everyday contexts, however, the statement “the Earth is spherical” will be considered accurate enough to be used.

Kit Fine's 1975 paper *Vagueness, Truth and Logic* [12] provides a thorough characterization of the principles of supervaluationism. Consequently, it will be used as the basis for many of the definitions that follow in this chapter.

3.2 Precisifications

Supervaluationism is based on the concept of so-called precisifications, which eliminate vagueness from a vague statement by limiting the space of admissible precisifications. Fine defines a precisification as “admissible” if it respects certain semantic constraints pertaining to competent use of language.

Figure 3.1: *France*

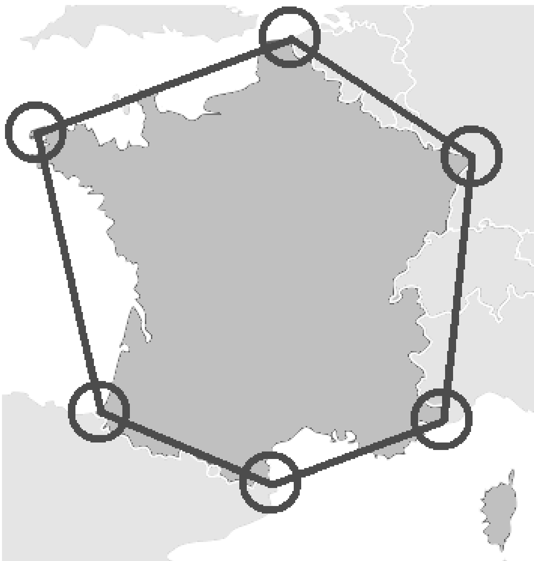

Figure 3.2: *Is France a hexagon?*

A precisification can be complete, but it does not have to be. For example, one could precisify the statement "this mountain is high by stating that it is true if "this mountain" has a height of more than 3000 meters. With this precisification, one could easily determine whether a given mountain is high or not – one look at its measurements would answer this question. This would be a complete precisification, as it removes all vagueness from a term.

However, one could also precisify the statement that one is talking about a mountain that would be considered high if it were located in the Swiss Alps. Given this, we would know that anything higher than the Dufourspitze, Switzerland's highest mountain, would definitely be considered high. We would also know that anything that might pass as a high mountain in the Netherlands would not have sufficient height. But we would still not know where the exact borderline between a high mountain and a non-high mountain lies.

We would know that a mountain with Mount Everest's dimensions, if it were in Switzerland, would be considered high. Likewise, we would know that Sugarloaf Mountain in Rio De Janeiro would not be high enough to qualify, as it barely rises above the lowest elevation found in all of Switzerland. We would not, however, know whether the Zugspitze, Germany's highest mountain (which is not nearly as high as the Dufourspitze), would qualify as high under these conditions.

Clearly, any precisification employed will be arbitrary. By defining a precisification, one states that it is *possible* to sharpen a given statement in a particular way. One does not exclude any other precisifications of a term. For example, by saying that one could set the limit between "high" and "not high at 3000 meters, one does not say that it would be impossible to set the limit at 2999 meters or 3001 meters. Practically, there will be an unlimited number of possible admissible precisifications. The conditions under which a precisification is admissible or not admissible will be explored below in Section 5.1.11, where penumbral connections will be discussed.

3.3 "Truth is Supertruth"

Given a precisification, one can say if a statement is true or not – as we have just done. If we consider a height of 3000 meters over sea level to be the borderline between "high" and "not high," we can say that Mont Blanc, with a height of $4,808$ meters, is high.

As mentioned above, however, this precisification is just one of an infinite number of precisifications. We have evaluated the statement "Mont Blanc is high" as true under one given precisification, but under other precisifications, the same statement is not evaluated as true.

This is where supervaluationism distinguishes between truth and supertruth [22]. Supervaluationism will only declare a statement to be definitely true if all admissible precisifications show the statement to be true. For example, the statement "Mount Everest is a high mountain" would be supertrue, if we restrict ourselves to mountains on the Earth. As Mount Everest is the highest mountain on the planet, there is no admissible precisification of "x is a high mountain" that does not include it.

This is also the case for false statements. A statement is only clearly false or "superfalse," if it is false under all admissible precisifications. For example, with respect to the Earth, the statement "the North Pole is close to the South Pole" is superfalse. Located at the two opposing ends of the globe, there is no geographical definition of "close" that would place these two points close to each other.

In a formal sense, the set of all admissible precisifications is called the precisification space Π. In order to be supertrue or superfalse, all individual precisifications included in the precisification space must agree on the truth value of the statement in question.

A classical example of a statement that clearly evaluates as true is found in H. Mehlberg's *The Reach of Science* [29], published in 1958. The example of a vague term given there is "Toronto," the location of Mehlberg's university. Toronto is considered to be a vague term, since there are several precisifications of what exactly belongs to Toronto and what does not. Does "Toronto" end where the administrative region ends or does it end where the metropolitan area ends? If one considers the latter, just where does one draw the line – if one leaves the city on a road, when exactly has one left Toronto?

Due to the vagueness of the term "Toronto," it would thus be impossible to evaluate a statement such as "there is an odd number of trees in Toronto" as either true or false in supervaluationism, even if one was omniscient, as the number of trees in Toronto would differ from one precisification to the next – and it is most unlikely that this number would be odd under all possible precisifications.

It is, however, valid to say that the statement "Toronto is in Canada" is supertrue, as there is no valid precisification of "Toronto" (or "Canada") by which Toronto would not fall into Canada.

This approach, differing from the approach used by fuzzy logic, enables supervaluationism to "save" some concepts of classical logic that do not hold in fuzzy logic, such as the law of the excluded middle:

$$\varphi \wedge \neg\varphi$$

An example of this is the statement "the sky is blue and not blue." In fuzzy logic, depending on the t-norm employed, this statement could evaluate as true up to a degree of 0.5 (using the three t-norms discussed in Section 2.4.2), whereas classical logic dictates that it would have to be a contradiction.

In supervaluationism, this statement evaluates as superfalse – no matter where one draws the line between "blue" and "not blue," every precisification will make one of the two premises of the conjunction false. If a specific precisification dictates that the sky is blue, the same precisification thus automatically sees "the sky is not blue" as false, leading the whole statement to evaluate as false. If a precisification sees the sky as "not blue," the same effect occurs.

Thus, the statement cannot evaluate as true and thus is superfalse. In this, it abides by an important principle of classical logic that is violated by other theories. Also, the result of a disjunction of two disjoint statements evaluates as true even when we have no knowledge of "how true" the statements are, simply because both cannot be false *at the same time.*

Another practical example of a question easily and clearly answerable in a supervaluationist environment can be illustrated using the sketch of an archery target in Figure 3.3. The target has four zones. One arrow has hit the center, while each other zone has been hit by two arrows. An example of a vague statement in regard to this sketch would be "a majority of the arrows are close to the center of the target." Since there are 7 arrows, 4 arrows would constitute a majority. It is, however, not determinable what "close to the center of the target" means. Does it mean just the bullseye? Or does it mean the inner two circles? Or the three inner circles? Or is our perspective so broad that anything that hits the target at all is considered close to its center?

There are four admissible precisification of "close to the center of the target," and the statement is only true in two of these four precisifications. It is, thus, neither supertrue nor superfalse, and supervaluationism would not allow a definite statement in this situation.

In order to put this example into the established framework, one could assign labels to these four possibilities. One could call them Π_1, Π_2, Π_3 and Π_4 respectively. Together they form the precisification space Π.

As can be seen, the values of Π_i differ. There is thus no supertruth in this statement.

However, there are statements that would evaluate as supertrue or superfalse. For example, the statement "an odd number of arrows has hit the inner circles of the target" would evaluate as supertrue, as all admissible precisifications of "the inner circles of the target" contain an odd number of arrows.

As before, there are four possible definitions of what the "inner circles" could be – it could be just the central circle, it could be the inner two circles, it could be the

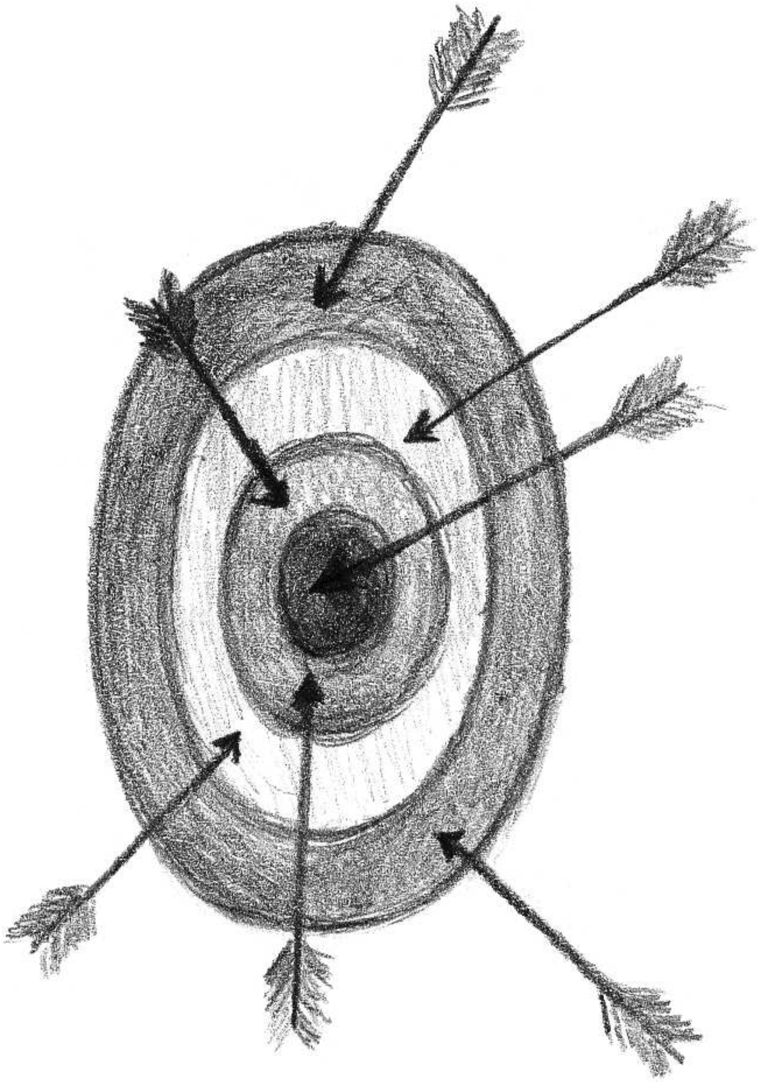

Figure 3.3: *An archery target.*

Π_i	$\|\varphi\|_{\Pi_i}$
1	0
2	0
3	1
4	1

Figure 3.4: *Evaluation over the complete precisification space.*

i	n_{Π_i}	$\|\varphi\|_{\Pi_i}$
1	1	1
2	3	1
3	5	1
4	7	1

Figure 3.5: *Supertruth.*

inner three circles or, theoretically (though not very sensible semantically), it could mean all four circles. Under each precisification of "the inner circles," n arrows have hit the area under consideration. If this value n is odd, the statement is correct under this precisification.

The truth value is identical under all precisifications – the statement is always true. Thus, this statement is supertrue. There most certainly is an odd number of arrows in the inner circles of the archery target.

This example illustrates that the question regarding the number of trees in Toronto, although not answerable, was not of a nature that is unanswerable in principle. In a admittedly more simple context, a similar question was, in fact, answerable.

3.4 Penumbral Connections

Precisifications reduce the vagueness in a given situation. They do not, however, "alter" definite knowledge already present before a precisification. In particular, this affects so-called penumbral connections. Penumbral connections occur when the applicability of a term correlates to a numerical dimensionality. For example, the term "high," when applied to mountains, generally correlates to the height of the mountain's summit above sea level.

If we are considering the statement "Mont Blanc is a high mountain," we cannot, by default, answer this question. Different precisifications would offer us different answers. If, however, we knew that the Matterhorn was considered to be a high mountain, it would be clear that Mont Blanc would also have to be regarded as a high mountain. Mont Blanc is $4,808$ meters high, whereas the Matterhorn is $4,478$ meters high. If a height of $4,478$ meters is enough to make a mountain high, $4,808$ must suffice as well.

The principle of penumbral connection dictates that given a correlation with a specific measurement, if we know that a predicate applies to item X, it applies to all items that have a greater value with respect to this measurement, however minuscule the difference might be. If we know that a mountain measuring 3000 meters is a high mountain, we know that a mountain measuring $3000,0001$ meters is a high mountain as well.

Through these penumbral connections, it becomes possible to eliminate precisifications that are not admissible. A precisification for the term "high," which notes that the Matterhorn is a high mountain, but that Mont Blanc is not, is not admissible, as it violates the penumbral connection between these two items.

Stating that the Matterhorn is a high mountain does not remove the vagueness from the term "high." We still cannot say where the boundary between "high" and "not high" lies, or if it even exists. But we do know that the boundary and any possible area of indeterminacy must lie below the Matterhorn's height.

3.5 The D-Operator

As an additional tool, Kit Fine defined a so-called "definitely" operator D. For a statement to be definitely true, it must be true before any vague terms in the statement have been sharpened and before any penumbral connections have been considered.

The significance of this operator is that in a supervaluationist environment, a precisification cannot alter truth values which were already supertrue or superfalse before the precisification. Stating that $DF(\varphi)$ is true for a statement $F(\varphi)$ implies that it can be taken as true in any context, no matter what precisifications have been applied.

Based on this operator, we can create a counterpart, the "indefinitely" operator, I, for statements which are not initially definite.

$$IF(a) \leftrightarrow \neg DF(\varphi) \wedge \neg D \neg F(\varphi)$$

Any statement that is neither definitely true nor definitely false is the base point of our consideration.

The motivation behind the creation of these operators was the handling of higher order vagueness. Higher order vagueness is present, for example, when the statement $IIF(\varphi)$ evaluates as true – this would make φ a second order borderline case of F. To put this in simpler terms, we are not even sure if φ is a borderline case or not.

Chapter 4

Sorites

4.1 Sorites Paradox

The Sorites paradox (from the Greek *σωρός* (sorōs), meaning heap, of which *σωρίτης* (sōritēs) is the adjective form) is a logical paradox that has puzzled man for more than two millennia. Sometimes also translated into English as the "paradox of the heap," it can be summarized by the question of how many grains of sand make a heap. A single grain of sand is not a heap. Two single grains of sand are not a heap. A pile of grains of sand filling a soccer field, however, is most definitely a heap.

Logically, this is not automatically consistent. It seems intuitive that a single grain of sand cannot be a heap. It also seems intuitive that the addition of a single grain of sand will not suffice to create a heap. How does one express the fact that heaps are created gradually in a logical system that has allowed a contradiction to arise even though only statements that seem to be intuitive are involved?

There is a wide array of closely related logical paradoxes that have been contemplated over the millennia. A man with a solitary hair on his head would definitely be called bald. One single hair will not make the difference between bald and not bald. However, a full head of hair, consisting of single hairs, will prevent a person from being bald.

At its birth an infant is not old. A single day will not make it old. A man in his 90s, however, is clearly old, even if his life could be quantified as the gradual addition of single days to his age.

In all of these cases we would intuitively say that it "just gradually happens." Solving such paradoxes in logical systems has been a challenge for philosophers for a very long time. Consequently, it seems prudent to explore various forms in which a paradox can be represented schematically before attempting to "solve" it.

We formalize the statement "i grains of sand are a heap" as h_i, and the statement

"i grains of sand are not a heap" as $\neg h_i$. As such, the statements $\neg h_1$ and $h_{10^{10^{100}}}$ will generally be considered to be true, as a single grain will never be regarded as a heap and a googolplex of grains will always be regarded as a heap.

4.1.1 Conditional Sorites

In essence, the Stanford Encyclopedia of Philosophy [45] formalizes a conditional Sorites as follows:

$\neg h_1$

If $\neg h_1$ then $\neg h_2$

If $\neg h_2$ then $\neg h_3$

...

If $\neg h_{i-1}$ then $\neg h_i$

-- -- -- -- -- -- -- -- -- --

$\neg h_i$ (where i can be arbitrarily large)

In natural language, this could be described as follows:

(1) A single grain of sand is not a heap.

(2) If one grain of sand is not a heap, two grains of sand are not a heap either.

(3) If two grain of sand are not a heap, three grains of sand are not a heap either.

...

(i) If $i - 1$ grains of sand do not make a heap, then i grains of sand do not make a heap either, where i can be arbitrarily large.

When dealing with this approach, it is possible to assign different truth values to different "steps" in the process. One could, for example, assign a different truth value to "If $\neg h_{1000}$ then to $\neg h_{1001}$," without questioning the truth of "If $\neg h_1$ then $\neg h_2$."

4.1.2 Mathematical Induction Sorites

Using the principle of mathematical induction, one could also formulate the statement as follows:

$\forall n(\neg h_n \rightarrow \neg h_{n+1})$

-- -- -- -- -- -- -- -- -- --

$\forall n \neg h_n$

Or, in words:

1. A single grain of sand is not a heap
2. For all i: if i grains of sand do not make a heap, then i + 1 grains of sand do not make a heap either.

This formulation might look like a shorthand version of the conditional Sorites at first glance. It is not, however, as the expansion of the heap is summarized in a single statement here. The truth of the statement can only be questioned as a whole – the step from one grain to two is logically equivalent to the step from 1000 grains to 1001.

4.1.3 Modus Ponens

Modus ponens (Latin for "mode that affirms") was already covered in Section 2.4.1. To recapitulate, it states:

$$\varphi, (\varphi \rightarrow \psi) \vdash \psi$$

Using modus ponens, it is possible to go from a single grain of sand to an arbitrarily large number of grains of sand, while the logic still dictates that what we have is not a heap. Sooner or later, therefore, modus ponens will lead us to a logical contradiction in classical logic, as some number of grains of sand, in some large dimension, will be classified as a heap – in direct contradiction to modus ponens.

4.2 Sorites and Fuzzy Logic

Petr Hájek and Vilém Novák published an account of this issue in their 2003 article *The Sorites Paradox and Fuzzy Logic* [19]. In it, they review approaches to dealing with Sorites using fuzzy logic.

The authors make use of a vague predicate "feasible" **Fe**, which becomes less applicable the larger its object becomes. With the definition of this predicate, the Sorites paradox can still be represented in the following form:

- $\mathbf{Fe}(0)$
- $(\forall x)(\mathbf{Fe}(x) \Rightarrow \mathbf{Fe}(x+1))$
- $(\exists x)\neg\mathbf{Fe}(x)$

If all of these statements were true, this would contradict the principle of induction. To solve this problem, the basic predicate logic $\mathbf{BL}\forall$ is extended by a fuzzy unary connective (hedge) "almost true" designated as At. $At(\varphi)$ reads as "it is almost true that φ."

The operator might be used as follows: If x is small, it is *almost true* that $x+1$ is small – the truth degree of "$x+1$ is small" is only a little bit smaller than the truth degree of "x is small."

We can also add the following two schemata:

1. $\varphi \rightarrow At(\varphi)$
2. $(\varphi \rightarrow \psi) \rightarrow (At(\varphi) \rightarrow At(\psi))$

If we know φ to be true, we know that it is also almost true, as almost true requires a lower truth value. If φ implies ψ, and φ is almost true, ψ is definitely also (at least) almost true.

If the operator is applied multiple times, the short notation $At^n(\varphi)$ is used for $At(At(\ldots(At(\varphi))\ldots))$ (n copies of At).

The logic being created is based on the language of Peano arithmetic [20]. The axioms of Peano arithmetic are fairly intuitive, and thus will not be discussed here. They are based on the usage of a so-called successor function symbol S, which yields the succeeding natural number of any natural number – $Sx = x+1$, if x is a natural number.

The language thus consists of:

- binary equality predicate $=$, unary predicate **Fe**
- constant $\perp$ (zero), unary function symbol S (successor)
- binary function symbols $+$, . (addition, multiplication)

The following axioms are valid:

1. $x = y \vee x \neq y$ (crispiness axiom for =)

2. all axioms of Peano arithmetic

3. $x < y \rightarrow (\mathbf{Fe}(y) \rightarrow \mathbf{Fe}(x))$

4. $\mathbf{Fe}(x) \rightarrow (At(\mathbf{Fe}(S(x))) \wedge At(\mathbf{Fe}(x + x) \wedge At(\mathbf{Fe}(x.x)))$

This extension of Peano arithmetic PA using **BL**∀, is referred to as PA_{at}.

In order to illustrate that classical logics are preserved, it must be shown that tertium non datur is valid for each formula φ in PA not including **Fe**, as PA is not part of classical logics. Since the only predicate remaining in this case is the binary equality predicate =, tertium non datur follows from the crispness axiom for atomic statements. Through induction, it can be proven that this is true for all quantified statements, a process which can be found in [19]. It follows that a formula not containing **Fe** is provable in PA using classical logics if and only if it is provable in PA_{at} over **BL**∀.

The authors proceed to prove the following theorem for PA_{at}:

$$\mathbf{Fe}(x) \wedge \mathbf{Fe}(y) \rightarrow (At(\mathbf{Fe}(x + z)) \wedge At(\mathbf{Fe}(x \cdot y)))$$

This statement is proven by reducing the statement to the axioms set previously in this section. This leads to the following corollary: For each term $t(x_1, \ldots, x_n)$ of PA with the variables indicated, there is a natural number n such that

$$PA_{at} \vdash \mathbf{Fe}(\bar{0}) \wedge \mathbf{Fe}(x_1) \wedge \ldots \wedge \mathbf{Fe}(x_n) \rightarrow At^n(\mathbf{Fe}(t(x_1, \ldots, x_n)))$$

In this case, the Sorites paradox is no longer a paradox, since $(\forall x)(\mathbf{Fe}(x) \Rightarrow \mathbf{Fe}(x + 1))$ is not considered to be true – it is only considered to be *almost* true.

4.3 Sorites in Supervaluationism

To give a fairly straightforward example of Sorites in supervaluationism, let us consider the concept of "old," which was explored in Chapter 3. As noted, different precisifications, all equally valid, will consider the borderline between "old" and "not old" to be in different places. Exactly at this borderline, the statement that a minor time unit – say, a day – will not make the difference; it will not hold. If one was to put a borderline between "old" and "not old" at a person's 65th birthday, the one day difference between the 365th day of the 65th year of the person's life and the 1st day of the 66th year of that person's life would, indeed, signify the difference between "old" and "not old."

A summary of Sorites under supervaluationism can be found in [9] and [10].

The assumption is made that one grain of sand is under no circumstances a heap, but that all competent observers would call a pile consisting of $10,000$ grains of sand – or more – a heap. In the interest of simplicity, the statement "i grains of sand are a heap" will be written as h_i. Within the given conditions, it is clear that h_1 is superfalse, while $h_{10,000}$ is supertrue.

It also seems reasonable to say that there will be more than one possible precisification in this example. Thus, there will be an h_i, with $1 < i < 10,000$, which is true under one precisification, but false under another precisification, and thus is neither supertrue nor superfalse. Thus, the logical contradiction caused by modus ponens is eliminated, since modus ponens is only allowed either when both premises are supertrue or when both premises are superfalse.

Indeed, $(\forall x)(h(x) \Rightarrow h(x+1))$ is not only not supertrue, it is in fact superfalse. Somewhere between grain number 1 and grain number $10,000$ there is a borderline between "not a heap" and "heap." Even though we do not know where it is, we know that it exists somewhere – and wherever this borderline is, the statement is false.

$$(\exists x)(h(x) = 0 \wedge h(x+1) = 1)$$

Chapter 5

Fuzzy Logic as a Theory of Vagueness

As discussed above, fuzzy logic has by now successfully established itself as an engineering tool. Though its purpose and validity in any context were highly controversial in the early years, this initial criticism was defused by the practical success of fuzzy set theory, to a large degree under the name of "fuzzy logic." This began with Assilian's and Mamdani's steam engine in the 1970s [28] and has extended over an ever-expanding range of applications, from noodle cookers to washing machines, up to the present day. The history of fuzzy set theory's birth, development and progression has been documented by Rudolf Seising in his book *The Fuzzification of Systems* [39].

The acceptance of fuzzy logic as a technical tool, however, has not necessarily led to an acceptance of fuzzy set theory as a theory of vagueness or as an instrument for handling natural language – a matter over which there is a certain rift within the fuzzy community, which will be examined later.

In one of these groups – the "engineers" – many might argue that these objectives are not, never were and never will be the intended domain of fuzzy logic. However, probability theory, which deals with a type of uncertainty diametrically opposed to fuzziness, has in its longer lifetime managed to become a principle that to a large degree is accepted as an integral part of almost every aspect of life. It has become much more than "just a tool," while dealing with a concept (degrees of probability – "with a degree of likeliness of 0.5, Austria will beat Liechtenstein in the upcoming match") no more natural than the concept on which fuzzy logic is based (degrees of applicability – "Italy played rather well in yesterday's match").

Many philosophers have argued and still argue that fuzzy logic must deal with some of these points in order to receive any recognition as a valuable theory of vagueness. This chapter represents an attempt to analyze how relevant the said points actually are and whether it would be possible to overcome them, or whether trying to solve these problems is basically an attempt to teach an elephant to fly.

This chapter will compile the insights, thoughts and answers collected during the course of a recent project the author participated in. The core of this project was formed by fifteen questions formulated by Christian G. Fermüller of the Vienna University of Technology after he was involved in the organization of and then participated in the *Prague International Colloquium: Uncertainty – Reasoning about probability and vagueness* in the Czech Republic in September 2006 [11] [25]. The context of this conference was not discussions focused specifically on fuzzy logic as a technical tool, but deliberations concerning the conceptual handling of vagueness. As such, fuzzy logic was only considered in the narrow sense of its being used as a conceptual method to handle vagueness. This distinction between "fuzzy logic in a narrow sense" and "fuzzy logic in a broad sense" has been discussed in Section 2.4.1. It should also be kept in mind that while the questions were later considered in many different contexts, only this narrow sense of fuzzy logic as a theory of vagueness was under discussion when these questions were originally compiled. The decision to approach the fuzzy community with these questions led to some interesting and unexpected results, which will be discussed in 5.2.1.

The 15 questions attempt to summarize the points of criticism and doubt encountered in and around the conference in Prague, among other places. Through international survey work and extensive participation from all over the world, it was possible to formulate some examinations of the points brought up – though not to answer all points raised to a level at which further contributions would stop being welcome.

It should be noted that this project was not meant to be an attempt to question the feasibility of fuzzy logic in any of its present-day applications or to disqualify fuzzy logic in general. Nonetheless this impression seems to have been made in some quarters, despite all the precautions taken, which is a further indication of the rift between different methods of thinking that exists in the so-called "fuzzy community." If anything, the project is an attempt to defend fuzzy logic against points brought up against its being anything more than just a "useful tool."

5.1 Fifteen Points of Critique of Fuzzy Logic – The Questions, Analyses and Attempts to Answer Them

5.1.1 Improper Precision

What do truth values like 0.5476324 *mean? How do we arrive at such values? Does FL provide any means to distinguish reasonable from unreasonable attributions of values? (A complete theory of vagueness should provide answers to such questions.)*

In a fuzzy environment one can, in fact, encounter truth values of which the interpretation can be difficult. Especially if a system uses a high granularity for its fuzzy values, one can come across a great array of decimal places that might appear to be far too detailed to serve as an approximation of vague facts.

In some simple cases, a numerical value other than 0 and 1 can have a meaning that could be considered to be straightforward. For example, when establishing a degree of applicability of the term "luminous" to a pixel in a grayscale bitmap image that allows 256 different gray values distributed at equal distances over the spectrum, it seems intuitive to assign a pixel with the luminosity value of 128, the fuzzy value of 0.5. With this same system, it is also possible to get fuzzy values with more decimal places that are still intuitive. For example, a luminosity value of 129 would, under linear mapping, lead to a fuzzy value of 0.5019608 ($\frac{128}{255}$) – a fuzzy value no less applicable than a probability value of the same sort, or the fuzzy value 0.5. in the preceding pixel.

However, in this example, the fuzzy value approximates granularity rather than vagueness. When venturing out of this very narrow realm of quite specific cases, the problem remains. What does it mean to say that Ginger loves Fred to a degree of 0.5476324? That fuzzy logic is an adequate tool for handling granularity is beyond doubt. Some might, however, consider granularity to be the opposite of vagueness.

In statistics, the answer would be quite simple – given that that one's models are robust. If an event will happen with a statistical probability of 0.5476324, this would, for example, mean that given $10,000,000$ attempts, one would expect the event to happen $5,476,324$ times. Statistical probability gives us a direct path to an expected value.

So what does it mean to say that a book is long to the degree of 0.5476324? Intuitively, colleagues agree, not much.

Some see it as an abstraction of an actual value, such as any exact number will always be – a person listing his weight as 72 kilograms will rarely weigh exactly this much. However, in the realm of real numbers, one can still determine what one is abstracting from.

This is generally considered to be a question of modeling. The luminosity example illustrated that, in some cases, fuzzy values can be confirmed as adequate in reality. In most cases, this is not the case – but is this a problem with fuzzy logic or a problem with how we try to describe reality when creating a model?

5.1.2 Linear Ordering of Truth Values

This seems to force one to judge the relative truth of intuitively incomparable statements, such as e.g., "John is tall" "Mary is rich," "Ginger loves Fred," etc. How can this be justified? (Note: it is insufficient to point out that algebraic models may also be non-linear. The deeper worry here is that this does not explicitly *reflect the "incomparability" of at least some vague propositions.)*

One element of classical logics which is actually preserved in fuzzy logics is linearity. This can be described with the following axiom:

$$(a \to b) \vee (b \to a)$$

Given that the value of a is smaller or equal to the value of b, any t-norm based fuzzy logic will compute $(a \to b)$ as 1. Thus, the content of this statement can be summarized as:

$$(val(a) \leq val(b)) \text{ or } (val(b) \leq val(a))$$

A consequence of the validity of this axiom in a fuzzy environment is that a linear ordering of fuzzy values is always legitimate.

However, a linear ordering of values might not be desired. Thus, this axiom does not seem to fit into a theory of vagueness in the eyes of many philosophers. If you cannot clearly accept or reject statements, how can you compare them? The issue is, therefore, that the mathematical frameworks allow a comparison that does not seem natural or intuitive.

The consensus here seems to be that the fact that both values in a comparison are within the same system and thus are theoretically comparable does not mean that they are necessarily comparable in context even if the logical framework would allow such a comparison. Statements such as "John is taller than Bill is fat" are not answerable in terms of crisp numbers in natural language. It is not possible to say if a body mass of 81 kilograms is larger than a body size of 181 centimeters.

However, within the realm of crisp numbers, a comparison between body height and body weight would be a comparison between two values of different dimensionalities. There is only one unit of truth in fuzzy logic. When comparing fuzzy values, both values will be truth values, noted as a value on an interval stretching from 0 to 1. While one cannot compare inches with pounds, one is able to compare two truth values, no matter where they came from.

This is possible in probability theory, when dealing with degrees of probability. Much like degrees of truth, degrees of probability are noted as a value within an interval extending from 0 to 1. If I am aware of the respective probabilities, I can evaluate statements such as "it is more likely that I will die in a plane crash than that I will win the lottery," even though the safety of my traveling has no connection whatsoever with the results of a lottery.

5.1.3 Truth Functionality

This seems to clash with many intuitions (see, e.g., D. Edgington for very explicit arguments against truth functional connectives applied to vague propositions [7]). In particular, it is forcefully argued (by many experts) that the semantic status (truth value) of ψ, given that both φ and $\varphi \rightarrow \psi$ are "true to some intermediary degree," also depends on the intentional *relation between φ and ψ, and not just on their respective truth values.*

Implications in fuzzy logic are always material implications and not intentional implications. Thus, implications are indeed truth functional. This "problem," though avoided in some modal logics, is not unique to fuzzy logic. It is quite possible to create bogus implications that will, in spite of their abstract nature, still hold true when evaluated, even in a logic that does not use incomplete truth values. Saying that the moon being made of green cheese would imply that pigs can fly, though clearly nonsense, would still evaluate as true – simply because the moon is not made of green cheese and thus it is impossible to negate this statement. Lack of consideration of conditionality is, however, a problem encountered in most logics. It is, of course, a problem relevant to fuzzy logic as well. But truth functionality is not a problem exclusive to fuzzy logics – it is not created by fuzzy logic.

However, in classical logics, truth functional connectives do not cause the kind of issues that they cause in fuzzy logics. In fuzzy logics, truth functional implications lead to the possibly undesired linear ordering of truth values, which was discussed in Section 5.1.2.

Apparently, however, there are non-functionally expressible theories of fuzzy sets, though they have not been practically applied up to now. Time will tell if fuzzy modal logics will suceed or not.

5.1.4 Higher Order Vagueness

Even if the truth values themselves are replaced by "fuzzy values" or something similar, the problem does not disappear: – at some level (order) "improper precisions" must creep in for any formal fuzzy logic – at every level (order) it remains unclear how we arrive at the corresponding "fuzzy truth value." (How should we distinguish between an artifact of the model and a "genuinely representing" property of truth values?)

This consideration might be judged to be related to Zadeh's type-2 fuzzy sets, recently presented and elaborated in *Type-2 Fuzzy Sets Made Simple* [30] by Jerry M. Mendel and Robert I. John and *Type-2 Fuzzy Sets: Some Questions and Answers* [31] by Jerry M. Mendel. This concept takes into consideration the fact that fuzzy sets themselves are, in turn, uncertain.

Dealing with higher order vagueness seems highly relevant to the "objective" of fuzzy set theory. However, the higher one goes with the vagueness considered, the more complex a model gets. And practically speaking, a certain degree of imprecision is generally accepted in order to keep the modeling simple, intuitive and comprehensible. Though type-2 fuzzy sets are superior in power, their complexity and non-intuitive nature have probably contributed to their failure as yet to become an accepted standard, even though they do not consider any vagueness above second order.

In statistics one could also model a system with various levels of overlying uncertainty. However, in practice an engineer will at some point choose to "cut off" his measurements at a certain threshold at which viewing deeper into the problem handled is no longer relevant or necessary.

5.1.5 Different Truth Functions for Connectives

Where are the criteria that allow us to pick the right or best one? There seems to be a lack of arguments from "first principles."

It is, indeed, possible to compute the same connectives in various ways in fuzzy logic, as was discussed in Section 2.4.2 – something that in classical mathematics or probability theory would seem alien. Finding various vastly different interpretations for "+" or for the joint probability of two statements happening in unison would not be acceptable.

In fuzzy logic, however, the "AND" connective can, for example, be described through an unlimited range of possibilities, which will generally offer vastly different results. The tools employed here, the so-called t-norms, were addressed in Section 2.4.2.

To recapitulate, the three t-norms chosen as significant to this paper were:

$$T_{min}(x, y) = min\{x, y\}$$
$$T_{Ł}(x, y) = max\{0, a + b - 1\}$$
$$T_{prod}(x, y) = x * y$$

In probability theory, the combined probability of two independent events can be denoted quite easily. The "OR" connective can be denoted by:

$$P(x \vee y) = P(x) + P(y) - P(x) * P(y)$$

And the "AND" connective can be denoted by:

$$P(x \wedge y) = P(x) * P(y)$$

It is also quite possible to validate these formulas through mathematical deduction or, alternatively, through empirical validation.

This brings us back to the problem of empirical validation – a perpetual millstone around fuzzy logic's neck. While it is quite often possible to validate independence of events in a statistical environment, it is quite hard to empirically validate much of anything in a fuzzy environment.

If one removes the possibility of empirical validation from statistics. however, it again loses its advantage – a topic that will be discussed in Section 5.2.2. It is quite unclear in what manner any connection between statistical values should be evaluated if their relation is not known – if it is not known whether one depends on the other or even if they are disjoint or not.

5.1.6 Worries about "(Too) Many Logics"

Correct reasoning should – like rationality in general – point to just one overall logic of which other logics can be (modal etc.) extensions or "limit cases." However (modern) FL is about an ever increasing range of logics.

This question, in particular seems to accent a split between two vastly different approaches to fuzzy logic within the fuzzy community, a phenomenon further explored in Section 5.2.1.While many people argue that fuzzy logic creates models and should thus be treated correspondingly, others are interested in the practical validity of data handled by a fuzzy system, since they do not regard "it is only a model" as a satisfying answer.

If one was to accept this answer as valid, however, the endless realm of models describing a given situation would become something not specific to fuzzy logic. It is encountered in reality as well, as one can quite easily see when looking at various vastly different maps of one and the same city (some designed for pedestrians, some for motorists, some for users of public transport, et cetera). The question remaining here is whether an ideal model of a given situation exists.

This problem is generally perceived as a problem of reality mapping, rather than of the logic itself. Mapping of reality can be critical in any theory, as the complexity of reality is impossible to capture completely with models. Some respondents even argue that this is not a problem, but rather a blessing, since the multitude of logics allows various applications from which one can pick and choose, depending on the specific needs of a given situation.

Also, it was noted that statistics suffer from the same "disease." Outside of ideal textbook examples, it is very rare for a statistical situation to possess one, and only one, stochastic model that ideally describes it. In most cases, the person doing the modeling will have several models from which to choose. The choice made in such cases can be just as ambiguous as it is in a fuzzy environment. However, here the person picking the model can at least do a statistical test on the chosen model to see how well it works.

5.1.7 Hedging via Disjunctions

[Cited here from Roy Sorenson's entry for "Vagueness" in the Stanford Encyclopedia of Philosophy [45]:] "Critics of the many-valued approach complain that it botches phenomena such as hedging. If I regard you as a borderline case of 'tall man', I cannot sincerely assert that you are tall and I cannot sincerely assert that you are of average height. But I can assert the hedged claim 'You are tall or of average height'. The many-valued rule for disjunction is to assign the whole statement the truth-value of its highest disjunct. Normally, the added disjunct in a hedged claim is not more than the other disjuncts.

Thus it cannot increase the degree of truth. Disappointingly, the proponent of many-valued logic cannot trace the increase of assertibility to an increase [in] the degree of truth."

Sorensen [obviously] intends to evaluate disjunctions through the maximum. But can "disjunction for hedging" really be explained by, e.g., Lukasiewicz, "strong disjunction"? Why should any particular truth function for disjunction adequately represent

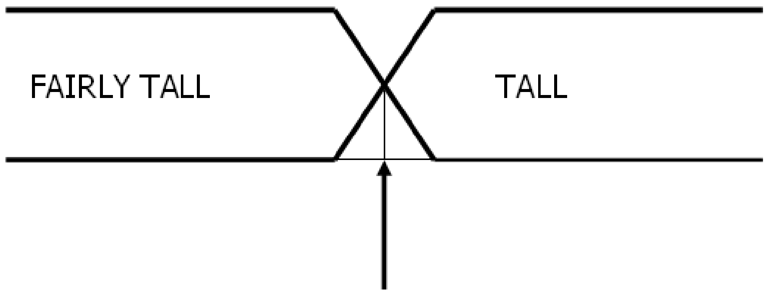

Figure 5.1: *A borderline case between fairly tall and tall*

hedging in natural languages? Granted that disjunction by minimum is also a "real disjunction," how many "real disjunctions" are there in natural language? How do we get to know them? Can fuzzy logic provide guidance for answers?

Fuzzy hedging can indeed often lead to insufficiencies, particularly in the field of natural language processing, as it does not, by default, consider the intentional relation between terms, but only their mathematical relation.

For example, if a person is a borderline case between tall and average height, he might have the fuzzy value of 0.5 for both "tall" and "average height." Intuitively, if I was to ask if the said person is "tall or of average height," the answer would have to be yes if he is a borderline case. Surely covering both possibilities must fully include him.

A disjunction can be evaluated in many ways. Classically, it is evaluated with a minimum. In t-norm based fuzzy logics, any t-norm's dual t-conorm can be theoretically used to evaluate the (controversial) "strong disjunction." As discussed in Section 2.20, the three relevant t-conorms in this context are:

- Gödel logic: $S_{max}(x, y) = max\{x, y\}$
- Łukasiewicz logic: $S_{Ł}(x, y) = min\{x + y, 1\}$
- Product logic: $S_{sum}(x, y) = x + y - x \cdot y$

However, of the three prominent t-norms, only the Łukasiewicz t-norm's dual, the bounded sum t-conorm, offers the desired result here.

- Gödel logic: $S_{max}(0.5, 0.5) = max\{0.5, 0.5\} = 0.5$
- Łukasiewicz logic: $S_{Ł}(0.5, 0.5) = min\{0.5 + 0.5, 1\} = 1$
- Product logic: $S_{sum}(x, y) = 0.5 + 0.5 - 0.5 \cdot 0.5 = 0.75$

The answer to this problem, from a practical point of view, is simple: in a practical application, when working with disjoint fuzzy sets, it would be bad modeling to use anything but bounded sums to evaluate disjunctions. Evaluating a disjunction through a maximum, regardless of the context, can be – and has been – compared to evaluating the probability of two disjoint events occurring in unison in statistics through the formula $P(x \wedge y) = P(x) * P(y)$, regardless of context.

For example, the probability of rolling an odd number with one die is 0.5. Likewise, the probability of rolling an even number is also 0.5. Of course, the chances of rolling an odd number and an even number at the same time in one single role are not very promising. If one was to use the classical statistical formula for the probability of two independent events happening in unison, one would get a statistical value of $P(odd \wedge even) = 0.5 * 0.5 = 0.25$.

Obviously, this is nonsense, since the connection between two events considered must be taken into account in statistics. Likewise, it is indispensable to consider the practical connection between various linguistic variables when deciding how to compute connectives between them.

5.1.8 Sacrificed Principles of Classical Logics

(Most, if not all) fuzzy logics sacrifice principles of classical logics that seem intuitively "correct" even from (e.g.) a constructive or "relevance" point of view (e.g., the law of contradiction $\neg(\varphi \wedge \neg\varphi)$ and idempotence of conjunction $\varphi \rightarrow \varphi \wedge \varphi$ etc.) How can such radical deviations from traditional "laws" be justified?

This issue seems to be related to issues brought up by Charles Elkan in his highly controversial 1993 paper *The Paradoxical Success of Fuzzy Logic* [8], in which he claimed that the mathematical foundations of fuzzy logic collapse under close consideration. Several responses from the fuzzy community illustrated how Elkan had based his analyses on false assumptions, using tools in certain situations that fuzzy logicians would never deem adequate. The law of contradiction is one such principle, which one can make collapse in fuzzy logic, but only by intentionally using means that are less than ideal.

Using the fuzzy connectives as they were initially proposed, as has been discussed in Section 2.2, the law of contradiction does indeed fail. If, for example, one was to consider a situation in which $val(\varphi) = 0.5$ and was to compute conjunctions as the minimum, $(\varphi \wedge \neg\varphi)$ would evaluate as 0.5. If one was to evaluate negations through the complement, $1 - val(\varphi)$, the entire statement would evaluate as 0.5.

T-norm based fuzzy logics, however, use other means that preserve the law of contradiction. The Łukasiewicz t-norm already evaluates $(\varphi \wedge \neg\varphi)$ as 0 for any φ.

The other two popular t-norms evaluate this statement to a value between 0 and 1. For example, in the given example of $val(\varphi) = 0.5$:

- $T_{min}(0.5, 0.5) = 0.5$
- $T_{Ł}(0.5, 0.5) = 0$
- $T_{prod}(0.5, 0.5) = 0.25$

However, only Łukasiewicz logics evaluate the negation of a truth value φ as $1 - val(\varphi)$. In Gödel logics and product logics, the negation of any value larger than 0 is 0. As such, the law of contradiction also holds here.

The idempotence of the conjunction, $\varphi \rightarrow \varphi \wedge \varphi$, is another matter entirely. If φ is true, φ and φ must both be true – in natural language, a trivial statement. However, only in Gödel logics is the result of $\varphi \wedge \varphi$ identical to φ. In the other two major t-norm based logics, the result of $\varphi \wedge \varphi$ will always be smaller than φ, unless $\varphi = 1$. In Łukasiewicz logics, $\varphi \wedge \varphi$ even evaluates as 0 for any φ smaller than or equal to 0.5.

Take, for example, the case of $val(\varphi) = 0.6$, in which the results would be:

- $T_{min}(0.6, 0.6) = 0.6$
- $T_{Ł}(0.6, 0.6) = 0.2$
- $T_{prod}(0.6, 0.6) = 0.36$

So only in Gödel logics can $\varphi \rightarrow \varphi \wedge \varphi$ be simplified to $\varphi \rightarrow \varphi$, which is trivially true. In other logics, we get an implication with a consequent smaller than φ, unless φ is 1. In this example:

- $(0.6 \Rightarrow_{min} 0.6) = 1$
- $(0.6 \Rightarrow_{Ł} 0.2) = 0.6$
- $(0.6 \Rightarrow_{prod} 0.36) = 0.6$

Among the three t-norm based logics considered, this principle of classical logics is thus only preserved in Gödel logics. In addition to the example given, the following derrivation illustrates how the idempotence of conjunction generally holds:

- $val(\varphi \to \psi) = (val(\varphi) \Rightarrow_{min} val(\psi)) = \begin{cases} 1, & val(\varphi) \leq val(\psi) \\ val(\psi), & val(\varphi) > val(\psi) \end{cases}$
- $val(\varphi \wedge \varphi) = T_{min}(val(\varphi), val(\varphi)) = min\{val(\varphi), val(\varphi)\} = val(\varphi)$
- $val(\varphi \to \varphi \wedge \varphi) = val(\varphi \to \varphi) = (\varphi \Rightarrow_{min} \varphi) = 1$

Another principle to be considered is the law of the excluded middle, or tertium non datur, $\varphi \vee \neg\varphi$. While Łukasiewicz logics, possibly (see Section 2.4.4) computing disjunctions through the bounded sum t-conorm and the negation of φ as $1 - val(\varphi)$, will always yield the value 1 for this statement, this is not the case in Gödel logics or in product logics. For example, consider $val(\varphi) = 0.5$.

- $(0.5 \Rightarrow_{min} 0) = 0$
- $(0.5 \Rightarrow_{Ł} 0) = 0.5$
- $(0.5 \Rightarrow_{prod} 0) = 0$

Since both Gödel logics and product logics assign 0 as the value to the negation of any value greater than 0, $\varphi \vee \neg\varphi$ will be equivalent to $\varphi \vee \bot$ for any φ greater than 0.

The evaluation of the complete example in this case, through the respective t-conorms (if one was to choose to employ them), yields the following results:

- $S_{max}(0.5, 0) = 0.5$
- $S_{Ł}(0.5, 0.5) = 1$
- $S_{sum}(0.5, 0) = 0.5$

This law only holds in Łukasiewicz logics, and here also only if t-conorms are used. To illustrate that it holds generally, and not just in this specific example, we can combine the formulas of the negation and the disjunction in the following example.

- $val(\varphi \veebar \psi) = S_{Ł}(val(\varphi), val(\psi)) = min\{val(\varphi) + val(\psi), 1\}$
- $val(\neg\varphi) = 1 - val(\varphi)$
- $val(\varphi \veebar \neg\varphi) = S_{Ł}(val(\varphi), val(\neg\varphi)) = S_{Ł}(val(\varphi), 1 - val(\varphi)) = min\{val(\varphi) + 1 - val(\varphi), 1\} = min\{1, 1\} = 1$

The preservation of this principle is regarded to be an argument supporting supervaluationism against fuzzy logic as a theory of vagueness, something that will be discussed in Section 7.1.

However, arguments are also found against the preservation of principles such as the law of the excluded middle. The statement “either you’re with us or you’re against us” has caused confusion all over the world throughout the course of history. In logics, this is equivalent to the aforementioned law, which classical logics preserves. If human thinking does not necessarily preserve it, it is questionable if a theory of vagueness, trying to approximate human thinking, should preserve it.

If one wanted to preserve all these principles, however, one would have a problem in fuzzy logics. To preserve the idempotence of the conjunction, one has to confine oneself to Gödel logics. To preserve the law of the excluded middle, one has to stick to Łukasiewicz logics. One cannot have both at the same time.

5.1.9 Epistemic, Ontic or Pragmatic Character?

It is left unclear whether the “degree of truth” has an epistemic, an “ontic” or a “purely pragmatic” character; different interpretations (Giles [13] [14] / Ruspini [38] / Mundici [6] / Behounek’s [1] resource interpretation/voting semantics, etc.) seem to imply different answers. (See, e.g., Jeff Paris [2] [35] for problems with some of these interpretations). A theory of vagueness should include clear answers to such questions.

This question is aimed at the nature of facts handled in fuzzy logic – where do these facts come from?

If they were to be of epistemic character, they would be based on our knowledge and our perception. The word “epistemic” is based on the term “epistemology,” which in turn is based on two Greek words – ἐπιστήμη(epistēme), meaning knowledge or science, and λόγος(lōgos), meaning account or explanation. An epistemic character would mean that the facts we use are based on our perception of reality and our knowledge of realty – where knowledge is defined as the area in which our truths and our beliefs coincide.

”Ontic,” on the other hand, relates to the factual physical existence of a circumstance. The word “ontic” also stems from Greek, namely, from the Greek word όντος (ōntos), which is a participle of ειναι(einai), meaning “to be.” It refers to factual circumstances in reality, not to how people perceive these on an individual basis.

”Pragmatic” would mean that the logic works on a result-driven basis. A person employing pragmatism is interested in getting a useful result and does not care how valid the model employed might be on a theoretical basis.

Most respondents seem to credit fuzzy logic with having an ontic character that may be used pragmatically – and is often used in this way in feedback control systems. As a question of semantics, the question was not particularly interesting to most respondents. It was noted, however, that a similar question has often been asked about probability theory and has never been answered in a satisfactory manner.

5.1.10 Surface Phenomena

Fuzzy logic is only an ad-hoc model for some "surface phenomena" that may be useful for engineering purposes, but does not help us (a lot) in answering "deep questions" about correct reasoning, the metaphysical or ontological status of vague predicates, epistemic and – probably most important – prescriptive (deontic) aspects of logic in general.

It is true that modeling in fuzzy logic is generally based on surface phenomena. However, most respondents seem to consider this to be an issue of modeling and not of the logical system used. Determining the metaphysical origins of knowledge is difficult in any circumstances.

Any kind of reasoning used in practical situations is hard to analyze to its deepest level. Even successful doctors are often credited for making good decisions in a hypothetical and conjectural fashion, rather than in a deductive manner.

5.1.11 Penumbral Connections

Many philosophers follow Kit Fine [12] in asserting that "penumbral connections" should be modeled directly in any logic seeking to deal with vagueness. (E.g., if it is indeterminate whether X is blue or green, it is still definitely true that it is monocolored, etc.) Can FL compete with supervaluationism in accommodating penumbral connections?

Kit Fine explains what he regards to be a penumbral connection in detail in his 1975 paper '*Vagueness, truth and logic* [12].

The concept of penumbral connections was discussed in Section 5.1.11. An example of a penumbral connection that cannot be modeled in fuzzy logics would be the following:

Consider a color f and a monocolored object x. The object is definitely monocolored, but it is not definitely of the color f. No matter what color this object is, $f(x) \vee \neg f(x)$ is true.

Technicians argue that fuzzy logic should not compete here, since penumbral connections of this sort do not lie within its modeling range.

5.1.12 It is Only a Model

FL often insists on a kind of application-oriented point of view. However, it is not enough to reply "it is only a model" to worries about a particular logic or semantic machinery. This would beg the question of whether the model is adequately representing how we should reason correctly in various situations. In general, it is doubtful whether an "engineering approach" can help us create a full-fledged theory of vagueness. Mathematical models can only be a part of *or* a tool within *a theory of vagueness.*

Many responses agreed that fuzzy logic is, indeed, only a tool for modeling that comes from the field of engineering, but that there is nothing "only" about this. Some claim that fuzzy logic never pretended to offer a foundation for theoretical understanding of vagueness, while others claim that fuzzy logic is on its way towards eventual success at this task through a process of abstraction.

Also, any practical applications of any theory of vagueness use models – probability theory as well can only hope to model occurrences in the real world. Thus, a better question here would be whether the models used are adequate, judged in terms of the respective background of a theory – a background that is present in probability theory. This problem will be discussed in the conclusions, in section 5.2.2.

Natural language is also "only a model," used by humans to share their thoughts and impressions with others. Making and using models are elementary aspects of human thinking – seeing it as a problem seems ridiculous to many respondents.

5.1.13 Relation to Natural Language

FL has an uneasy relation to natural language. On the one hand, it is often claimed that FL is "close to natural language discourse." On the other, it does not respect the fact that in natural language we do not use (concrete, linearly ordered) intermediate truth values and (different) truth functional connectives.

Fuzzy logic is a precise tool for dealing with imprecise data, while natural language is imprecise in a manner that cannot be quantified – or is based on a model so complex that it is impossible to determine the relation between possibly precise causes and human thought – and thus natural language. (On a very basic level, the human brain does function digitally – a neuron either transmits a signal or it does not). The author of this thesis has previously been a co-author of papers on this topic [40] [41] [42].

The relationship between formal logic and natural language is a problematic one in any case. The question is what additional contributions fuzzy logic can make here, having been called "the logic of natural language" by many.

There are limits to how well classical fuzzy logic can approximate natural language. It is theoretically possible that if, at some point in the future, the functioning of the human brain is better understood, it will be possible to model natural language adequately through the use of a great number of stages of fuzziness, as opposed to only one or two (as are considered in type-2 fuzzy sets – see Section 5.1.4). However, a model describing such a situation would become complex beyond comprehension – and, therefore, unusable.

Thus the fact remains that while humans would refer to a person as "rather tall" or "really tall," fuzzy logic states that a person is tall to a degree of 0.7 and 0.9.

Many respondents argue that natural language seems like a fairly abstract field in which to try to implement fuzzy logic. Though this might be true with respect to applications today, it should be noted that the term fuzzy logic was actually not coined by Lotfi Zadeh, who spoke of fuzzy sets only initially, but by his Berkeley colleague George P. Lakoff, a professor of linguistics, in his 1972 paper *Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts* [24], in which he explored the possibilities of applying fuzzy set theory to natural language.

It is not, however, true that humans do not use different truth functional connectives for the same terms in natural language. The word "and" can have quite different functional meanings in natural language, depending on the context. The meaning of the word "and" in the sentence "My car is new and red" differs functionally from that of same word in the sentence "I will buy a car and drive to Canada." In the first example, it is commutative – one could easily turn around the two elements connected by the conjunction without altering the meaning of the sentence. The sentence "My car is new and red" is equivalent to the sentence "My car is red and new." In the second sentence, however, the "and" contains a temporal element. "I will buy a car and drive to Canada." implies that I will buy a car, wherever I might be, and then drive it to Canada. If I was to turn it around, and say "I will drive to Canada and buy a car," the sentence means that I will drive to Canada *first* (in my old car?), and only then buy a car. The "and" is not commutative.

Similarly, the connective "or" can also have vastly different meanings. Compare "If you have a club card or are a pensioner, admission is free" with "follow the rules or be expelled." In the first example, admission is free if one of the conditions is met. In the second example, the two events described are disjoint – if you follow the rules, you will not be expelled. If you've been expelled, you haven't followed the rules. The two statements are connected in an exclusive manner – a manner that would employ the XOR operator in classical logics.

This, again, might give fuzzy logic an advantage over other approaches. As mentioned above – as an objection to fuzzy logic – fuzzy logic can employ a great number of functions for one and the same connective. When considering natural language, this seems like a good thing, actually, as natural language also does not have clear rules for computing its own connectives – conjunctions.

It is, however, true that natural language, unlike fuzzy logic (see 5.1.3,) is not always truth functional. Take the word "because," for example. It only models causal relationships.

One could not model the sentence "The street is wet, because it has rained" using material implications only. Consequently, as all implications in fuzzy logics are material implications, it does not seem to be appropriate to call fuzzy logic "*the* logic of natural language," as some have done, when such a basic functionality of natural language such as causal relationships cannot be modeled with the methods (currently) available to fuzzy logic.

5.1.14 Operational Deficiency

FL does not compare favorably with probability theory (PT) as a theory of (another type!) of uncertainty. Granted that FL is about degrees of truth as opposed to degrees of belief, one may be disappointed about the lack of convincing and robust models in FL as compared to PT. There is nothing like the paradigmatic application of PT to (e.g.) games of chance, where it is universally agreed that highly non-trivial, uniquely determined computations give you demonstrably *and* empirically well-corroborated *(unique) values corresponding to rational expectation. Will similarly robust, non-trivial guidelines for complex information processing ever come from FL?*

No satisfactory answer to this question was received and further opinions and would be greatly appreciated. Most respondents seem to see this as something that only time will answer.

5.1.15 Record of Discourse

Many theoreticians agree that paying attention to the specific context *("record of discourse") of an assertion (by competent speakers) is of utmost importance in understanding what's going on in a "(forced march) Sorites" situation (and probably in all situations, where vagueness is involved). FL does not pay sufficient attention to this and therefore cannot compete (in particular) with contextualist theories of vagueness (Shapiro [43] [44], Graff [15] [16] [17], Raffman [36] [37], etc.) with respect to* *questions about the best/correct way of* actual reasoning *in concrete dialogue scenarios (about Sorites, etc.).*

This issue is based on Stuart Shapiro's 2006 book *Vagueness in Context* [43]. Shapiro illustrates what he means by a "record of discourse" through a large, idealized set of so-called competent speakers, which the following example will be loosely based on.

For example, let us assume that we have 256 monocolored cards, arranged in a row. The first card on the far left end of the row will be clearly red. In the RGB color representation system, it would have the value of $(255, 0, 0)$. The last card in the row, on the far right, will be clearly yellow, with a RGB value of $(255, 255, 0)$. Every intermittent card will differ from its neighbor by exactly one point on the green scale of RGB. The second card would have a RGB value of $(255, 1, 0)$, the third one $(255, 2, 0)$ and so on.

Human vision cannot distinguish 256 different tones between red and yellow; no human, no matter how good his or her vision is, can tell the difference between $(255, 0, 0)$ and $(255, 1, 0)$.

A set of competent speakers can now be asked to look at the cards and decide what color each one is. They would all believe the card on the far left to be red and they would definitely believe the card on the far right to be yellow. Cards in the middle of the row would be regarded to be orange.

So where are the borderlines between red, orange and yellow? One must assume that somewhere along the line, starting from the red end, speakers will look at cards and not think that they are clearly red.

If one now allowed the speakers to communicate with each other, they might disagree about where the borderlines between the cards lie. They might find good arguments to explain why they believe certain cards to be red or not to be red. "We can agree that this card is red, can't we? So how can this one not be red, if you cannot see a difference between them?"

At some point, somewhere along the row of cards, this process would stop – on the left, on the right and on the edges of the "orange zone." However, discussions would still take place in the borderline areas. This record of discourse might make some people change their minds and alter the way that vague facts are interpreted in this specific situation.

This is the kind of discourse which fuzzy logic does not take into consideration. Due to the technical community's lack of familiarity with Shapiro's ideas, it was not possible to collect any theoretical solutions to this problems at this time. Possibly, uncertainty caused by this phenomenon could be included when modeling the statistical uncertainty considered in type-2 fuzzy sets. This would, however, not be sufficient and accurate modeling of the phenomenon at hand.

5.2 Lessons Learned

5.2.1 Attitudes Differ

The first conclusion is that many issues brought up regarding fuzzy logic are actually considered to be difficulties of modeling reality rather than issues pertaining specifically to the theory itself. In probability theory, it can often be easy to empirically prove the validity – or invalidity – of a chosen model. In fuzzy logic, this can be much more difficult.

The second major conclusion of the data collection project has interestingly enough not been of a technical, mathematical or philosophical nature, but of a sociological nature. There seems to be a fairly clear-cut difference in attitudes towards questions and contemplations of this nature. The author experienced this rift at the NAFIPS 2006 conference in Montreal on a personal level, as was discussed in this chapter's introduction, but was not aware of the magnitude of the division between these two "schools of thought."

Within the "technical half" of the community [3] [4] [26] [27], only a few of the issues addressed in this paper are relevant. The general maxim seems to be that fuzzy logic is a valid tool because it works – its practical successes invalidate conceptual and philosophical questions about it. If there were conceptual problems with fuzzy logic, it just would not work in practice. Trying to solve some of the points addressed here is not of interest to representatives of this community, since in their eyes fuzzy logic is not supposed to deal with these issues and has never pretended to have solved them.

On the other hand, there are mathematicians and logicians [23] [46] [32] [33] [34] who seem to regard contemplations of this sort as highly relevant and would themselves be interested in learning about possible answers to the questions raised. This is not surprising, since these questions stem from a philosophical context. Within this group, the practical successes of fuzzy logic generally imply that it is an excellent abstraction of reality, but do not necessarily imply that it is a valid representation of the many layers of vagueness encountered in reality. Fuzzy logic's successes do not imply that it is anything more than just a model or that it is a valid theory of vagueness.

For the author, who has a limited background in the fields of mathematics and philosophy, it is hard to see what there is "only" about a model, since from a technical point of view, practically all methods applied and tools used are models, abstractions and simplifications of reality. And it is not just difficult to model reality in its entirety, it is impossible to do so, as the Heisenberg uncertainty principle [21], among other theories, states.

Some of the reactions to the questions raised were of a very emotional nature. This can probably be explained by the fact that fuzzy set theory is a relatively young discipline in science that faced a great deal of unjustified criticism in its early days. These attacks were eventually discredited by the practical successes of fuzzy logic, but it is possible that some of the individuals who experienced those relatively recent times have continued to be very defensive of their theory until the present day.

Also, prior attempts to discredit fuzzy logic [8] might have led some people to believe that this project was an attempt to revive such efforts. This, however, was never the motivation behind the project, whose raison d'ztre was to explore objections *already existing* outside of the so-called fuzzy community and to analyze them. Its focus was to probe whether the questions raised are actually problems with fuzzy logic or problems related to people's understanding of fuzzy logic. Moreover, a further goal was to seek possible solutions for problems deemed to be genuine.

5.2.2 A Question of Modeling?

A number of the points raised in various questions were perceived as issues of modeling rather than as problems of the models of the logic employed. How well does my model correspond to reality and can I evaluate this in experiments? Probability theory can employ statistical tests to evaluate the adequacy of a statistical model in a given situation. It can empirically confirm its data, which fuzzy logic cannot do.

To many, this advantage in the field of modeling represents an "unfair advantage," as it is not a matter of the mathematical or logical framework used, but a matter of the empirical data collection. To best illustrate this, it seems sensible to create a situation in which probability theory also cannot employ empirical means to verify its models.

Such a situation was suggested in the form of a betting shop. For example, a betting shop could take bets regarding the physical height of presidents of various countries. The people taking bets and the people setting the odds would lack actual knowledge about the corresponding presidents.

In one game, for example, bets could be made on the question of who is taller, the president of Finland or the president of South Korea. Lacking any knowledge about the individuals in question, odds could still be quoted. Since Finns are statistically taller than South Koreans, the odds put people betting on the president of South Korea in a better position to earn more money, since, statistically, this is considered the less probable answer. The probability of the Finnish president being taller is considered to be higher.

If one was then to receive the additional information that Finland has a very strong history of emancipation, one could consider the probability of Finland having a female president to be relatively high. As women are statistically not as tall as men, this would reduce the statistical probability of the Finnish president being the taller of the two people, in view of the given knowledge.

This probability would be further altered if it was confirmed that the president of Finland is in fact female, whereas the president of South Korea is not.

If one was to receive the even further information that the prime minister of Finland, Matti Vanhanen, has the formidable body height of 1.98 meters, this would alter the odds once again in favor of the Finnish president, as it illustrates the possibility that Finnish people can be even taller than formerly believed – though the possibility must also be considered that Matti Vanhanen is an outlier in any Finnish population statistics.

A similar procedure could be continued for a very long time, giving the participants and the odds-setters an ever-increasing amount of information on both presidents' ethnic background, family background, hobbies, et cetera. And at every step of the procedure, odds could be quoted and probabilities estimated.

At some point, one could remove the blinds and reveal that the president of Finland, Tarja Halonen, is 1.72 meters tall, while the president of South Korea, Roh Moo-hyun, is 1.68 meters tall. Thus, the president of Finland, regardless of any of the prior considerations, is clearly the taller of the two individuals.

So why was the betting shop talking about the "probability of the President of Finland being taller than the President of South Korea"? Obviously, since Tarja Halonen is four centimeters taller than her Korean colleague, the probability of the statement "the President of Finland is taller than the President of South Korea" is 1, while the probability value of the statement "the President of South Korea is taller than the President of Finland" has the value 0. Any other values are simply not correct.

Nevertheless, we encountered them, even when using "clean" statistical methods, since the information available to the individuals in the betting shop a priori did not make a better modeling of the statistical situation possible.

The comprehensible, and thus comparable, nature of statistical values is generally based on the fact that modeling in statistics is generally easier than it is in a fuzzy environment, as one can often prove the validity of one's models through experiments. If one removes this "unfair advantage, "however, it becomes clear that probability theory does not necessarily fare any better than fuzzy logic does under less ideal conditions.

Type-2 Fuzzy Sets

In the technical community, there were also many individuals who considered some of the issues raised in the fifteen questions to be essential problems even in technical applications of fuzzy set, and view them as driving factors for the development of type-2 fuzzy sets to enable better modeling of the uncertainties not covered by "classical" fuzzy sets.

5.2.3 A Theory of Vagueness?

Many of the issues raised in the questions can be solved by specific fuzzy logics. One can preserve some principles of classical logics by confining oneself to t-norm x and others by confining oneself to t-norm y (see Section 5.1.8). One can preserve the idempotence of the conjunction by restricting oneself to Gödel logics and solve the problem of hedging via disjunction by sticking to Łukasiewicz logics.

Outside the realm of fuzzy logics in a narrow sense, this is sufficient. While some of these principles might be important in an application, they will rarely all be necessary. A "pick and choose" approach has proven to be quite successful.

There is no question about the fact that fuzzy logic has established itself as a valid representation of granularity. There are few issues regarding the representation of graduated concepts, as long as the graduation can be justified, which is quite commonly the case – see 5.1.1. This is not enough, however, to qualify fuzzy logic as a theory of vagueness.

When handling vagueness, fuzzy logic has shown itself to be applicable and profitable in many situations. It is not without its mathematical basis. However, to find universal acceptance as a theory of vagueness, more will be needed. A full-fledged theory of vagueness will have to be evaluated according to various criteria, including adequate representation of language usage, internal coherence and the scope of applications. These criteria are discussed in detail by Rosanna Keefe in [22].

It is still an open question whether or not fuzzy logic can be considered a full theory of vagueness in these respects. It would definitely help fuzzy logic if certain principles considered to be elemental in classical logics could be generally secured in fuzzy logics, and if one did not have to choose between various alternative fuzzy logics, some preserving certain principles of classical logics and others preserving different ones. This lends support to attempts to combine fuzzy logics with other theories of vagueness – such as supervaluationism – in attempts to utilize the advantages of both. This idea will be explored in Section 7.1.

Chapter 6

Supervaluationism Applied

6.1 Supervaluationist Geography

Supervaluationism was created as a conceptual tool to handle vagueness, and not as a technical tool. As such, the idea of creating applications based on it seems extremely problematic at first. By saying that a statement lying on the cusp of ambiguity is neither true nor false, it refuses to give a sharp answer in just the cases in which applications in the field of engineering would be interested and in which a relevant tool would be employed in the first place.

This is not to say that a supervaluationist approach has never been considered in a practical context. An example of contemplations regarding possible practical applications of supervaluationism can be found in Achille C. Varzi's 2001 article *Vagueness in Geography* [47], which explores predominantly supervaluationist approaches to handling vagueness in a geographical context. In it, he deals with countless fundamentally vague questions encountered in geography – including some of a Sorites nature.

In geographical terms, a precisification would not be something complex and abstract – it would simply be a possible border for a term. This makes the concept seem very appealing within this particular realm.

Varzi's reflections include, for example, the nature of the object commonly referred to as "Everest," specifically, where does it begin or end? In considering this question, he divides Mount Everest and its surroundings into hunks, designated as h_i. The hunk on the very top of Mount Everest would be h_0. The next, adjacent hunk would be h_1. If one was to follow a line leading away from the summit of the mountain, the next hunk would be h_2. Hunk number i, away from the top, is referred to as h_i.

In a Sorites manner, it would be possible to state that:

1. Hunk h_0 is part of Mount Everest
2. For all k: if h_k is part of Mount Everest, then so is hunk h_{k+1}.

Hunk h_0 refers to the top of Everest. The logical conundrum represented by this situation would, without further considerations, make the entire planet Everest.

In supervaluationism, one could define precisifications, for example, by means of contour lines – the natural "borders" that mountains, as geographical terms, would have. When leaving the summit of Everest, one would very soon reach a point at which there would theoretically be a precisification that did not acknowledge the ground under one's feet as part of Everest. Thus, it would not be possible to walk from the summit of Everest to central Katmandu, while still claiming to be on Everest, due to modus ponens.

While the question of whether a mountaineer's encampment near the base of the mountain is on the mountain or not would not be answerable, it would be supertrue that the summit of Everest is part of Everest and that Katmandu is not.

Everest is located on the border between Nepal and Tibet. Various precisifications might put the main bulk of the mountain in either Nepal or in Tibet. Thus, it is not possible to clearly evaluate statements such as "Everest is mostly in Tibet." It is, however, possible to clearly evaluate the statement "Either Everest is located mostly in Tibet or it is not," as any precisification would put the main bulk of Everest on one side of the border.

6.2 A User-Aided Reduction of the Precisification Space

The basic problem with applications of supervaluationism lies in the relatively small proportion of supertrue or superfalse statements when vague statements are considered. However, it would be possible to let a user in a dialogue game reduce the size of the precisification space through an interactive dialogue.

To stay in the realm of geography, the task would be to design a user interface for user inquiries, which might often include vague concepts.

- *"Is Detroit close to Chicago?"*
- *"Is Mount Fuji a high mountain?"*
- *"Is Bogotá at a high altitude?"*

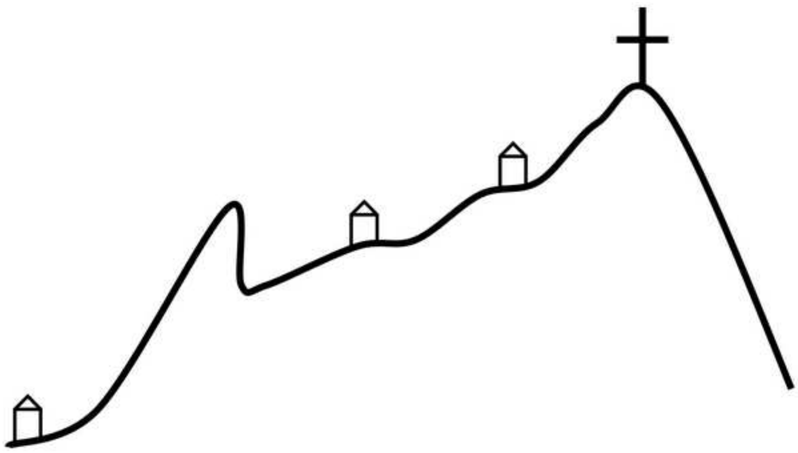

Figure 6.1: *An illustration for user-aided precisification space reduction.*

- *"How many mountains are there in the Alps?"*
- *"What is the largest city in Western Europe?"*

Such could be the nature of vague questions encountered by this system – and, naturally, a system would, lacking further context, not be able to answer most questions of this type. The system could answer questions such as:

- *"Is Mount Everest a high mountain?"*
- *"Is the North Pole close to the South Pole?"*

In the context of geography on Earth, these questions could be answered as (super)true or (super)false, respectively, as there would be no admissible precisification under which the answers dictated by common sense would be contradicted. Mount Everest is the highest mountain on Earth, thus, no user with realistic expectations could consider it to be anything but high. The North and South Poles lie at opposite ends of the planet and thus have the greatest distance between them possible without leaving the Earth.

But what about all the other statements? Since a computer would not know the context of these questions, it is would be difficult for it to answer them – much as this would also be hard for a human being. Whether Detroit is close to Chicago depends on whether the user intends to fly, drive, cycle or walk. Mount Fuji, though a formidable mountain, might seem petite to a professional alpinist who has been in the Himalayas. The application has no way of knowing the context of a query by default.

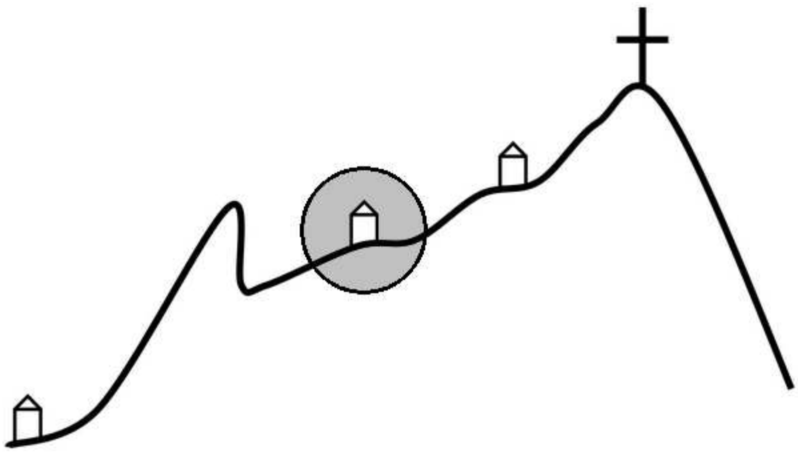

Figure 6.2: *"Is this hut close to the top of the mountain?"*

Through a dialogue procedure, however, the application could expand its knowledge of the context of a question. For example, let us consider the example shown here of three huts built in the area surrounding a mountain top, which is illustrated in Figure 6.1. The user has asked the application the following question:

"How many huts are there close to the top of the mountain?"

In order to evaluate this question, the application needs to know, for each individual hut, if it is "close to the top of the mountain." However, this is a vague predicate – though it seems clear that the cross is close to the top of the mountain and that the hut on the far left of the graph is not close to the summit in this context. The situation is not clear with respect to the two huts in the middle of the illustration.

Without further knowledge, the computer could only say that this statement is not answerable, as its evaluation runs into statements that are neither supertrue nor superfalse – namely, the truth values of "hut no. x is close to the top of the mountain." Alternatively, however, the application could highlight one of the huts and ask the user if he considers this hut to be close to the top of the mountain, as seen in Figure .

If the user is able to give a definite answer to this question, the precisification space will instantly be vastly reduced in size, thanks to the concept of penumbral connections. For example, let us assume that the user gave a definite answer regarding the proximity to the mountain's summit of the second hut on the left in the sketch

If he stated that he does indeed consider this hut to be close to the top of the mountain, this information would dictate that any object closer to the summit than the hut in question must also be close to the summit, as this would constitute a penumbral connection. In this case, we would thus have a supertrue statement at this

Figure 6.3: *"Would you consider this to be a mountain?"*

point, given this context. No definite line was set between "close to the top" and "not close to the top," but the area in which the answers to this question are considered admissible has been reduced to an area where it no longer influences the answer to our question. Hut 1 was disqualified from the beginning. The user has stated that hut 2 is close to the summit of the mountain, as he sees it. Thus, hut 3 must also be close to the summit of the mountain, as the penumbral connection dictates. The answer to the question is now 2.

If the example at hand had been more complex, further inquiries would have been necessary. For example, if the user had asked how many mountains there are in the Alps, the application might have given the user several dozen cases of borderline cases of "hill" and "mountain," before determining exactly how many objects the user would consider to be mountains lie in the Alps (See 6.3).

After the end of a session with this application, the framework created through user interaction could be saved or discarded, depending on the user's wishes – if he considers the data he gave the computer highly situation-dependent or if he tends to see it as a representation of his personal thinking and thus also relevant to future situations.

Chapter 7

Connections between Fuzzy Logic and Supervaluationism

7.1 A Shared Logical Framework

Generally, supervaluationism and fuzzy logic are not considered to be compatible concepts. While supervaluationism considers vagueness to be a variance between the way different people use certain terms, fuzzy logic lives on quantifying vagueness. While fuzzy logic does not uphold concepts such as *tertium non datur* ($\varphi \vee \neg\varphi$) in general, supervaluationism regards the preservation of these concepts to be essential. Only Łukasiewicz logics preserve them, when using strong disjunctions based on t-conorms, but even there tertium non datur fails when one uses the weak, lattice disjunction – as is considered the norm by many (see Section 2.4.4).

Fuzzy logic employs graduated degrees of applicability in regard to linguistic variables – the degree to which they are applicable can vary between 0 and 1. Supervaluationism considers a separation between true facts and false facts to be possible, while acknowledging that there are multiple possibilities for this discrimination. Which one is valid and which one is not, depends on the context.

For example, let us consider the statement "Italy is a large country." If a fuzzy logician was asked to evaluate this statement, he might look at tables of world's nations and their respective land masses, and would then attempt to quantify Italy's size with respect to international standards.

A supervaluationist would claim that this statement is, per se, not answerable. If it was necessary to give a define answer, he would ask what dimensions the person making the statement has in mind. Large compared to a country like Switzerland or large compared to a country like Canada? If he cannot gather any additional data, he accepts that he cannot say whether this statement is true or not, since the lack of context does not disqualify either precisifcations making Italy large or precisifications that do not do so.

The approaches represent two entirely different methods of reasoning. Nevertheless, there have been attempts to find common ground between these theories and to combine them in one single logic [10], as an extension on Robin Giles' "games" [13] [14].

This approach starts by building a simple logical framework, based on principles explored in Sections 2.4. It uses a t-norm $*$ to function as the strong conjunction &, its residuum $\Rightarrow_*$ as the implication $\rightarrow$ and the constant $\perp$, as the falsum to define a fuzzy logic $\mathbf{L}_{(*)}$. The missing connectives are defined by means of established formulas. The negation is defined by $\neg\varphi = \varphi \rightarrow 0$, the weak conjunction by $(\varphi \wedge B) = \varphi \& (\varphi \rightarrow \psi)$, and the disjunction by the lattice, $(\varphi \vee \psi) = ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((B \rightarrow \varphi) \rightarrow \varphi)$.

A function v^* is a valuation for $\mathbf{L}_{(*)}$, assigning a truth value from the real unit interval $[0, 1]$ to each propositional value.

- $v^*(\varphi \& \psi) = v^*(\varphi) * v^*(\psi)$
- $v^*(\varphi \rightarrow \psi) = v^*(\varphi) \Rightarrow_* v^*(\psi)$
- $v^*(\perp) = 0$

At this point, a choice of t-norm has to be made. Choosing the minimum t-norm, or Gödel t-norm, leads to a logic called **G**, choosing the Łukasiewicz t-norm would lead to a logic called **Ł** and choosing the product t-norm would lead to a logic called Π. All three of these differ in their attributes. The choice of a fitting one depends on the requirements of the specific situation. Thus, the requirements of this situation must be examined at this point.

In addition to the "obvious" design choices discussed in 2.4.5, a further design choice is selected for this situation.

- Small changes in $v^*(\varphi)$ or $v^*(\psi)$ result in, at most, small changes in $v^*(\varphi \rightarrow \psi)$.

For this to be true, the implication and thus the function $\Rightarrow_*$, must be continuous.

To recapitulate, the residua of the three "major" t-norms are:

- $(x \Rightarrow_{min} y) = \begin{cases} 1, & x \leq y \\ y, & x > y \end{cases}$
- $(x \Rightarrow_{Ł} y) = min\{1, 1 - x + y\}$
- $(x \Rightarrow_{prod} y) = \begin{cases} 1, & x \leq y \\ \frac{y}{x}, & x > y \end{cases}$

In Gödel logics, this is not the case – the implication is discontinuous at any point where $x = y < 1$.

- $\lim_{x \to y^+}(x \Rightarrow_{min} y) = y$
- $\lim_{x \to y^-}(x \Rightarrow_{min} y) = 1$

In product logics, there is one point at which the residuum is discontinuous – the point $(0, 0)$.

- $\lim_{x \to 0}(x \Rightarrow_{prod} 0) = 0$
- $\lim_{y \to 0}(0 \Rightarrow_{prod} y) = 1$

In Łukasiewicz logics, however, the residuum is defined as the minimum of two continuous functions. The minimum of an array of continuous functions is also continuous. As such, **Ł** is "suited" to satisfy this condition. In fact, **Ł** is the only fuzzy logic to satisfy this condition, as is proven in [10].

Up till now, all symbols used represent those used in t-norm based fuzzy logics. To expand upon this using supervaluationist means, a unary connective S is defined, which can be read as *"it is supertrue that ..."*. The resulting logic is referred to as **SŁ**.

As was the case in Section 3.3, a precisification space Π is defined; a specific precisification will be referred to as π. Π is endowed with a probability measure μ using the σ-algebra formed by all subsets of the precisifications in Π, where μ represents the relative plausibility of different precisifications. For example, if one was dealing with a cutoff point between the predicate "cold" and its negation "not cold," we could have a precisification π_x, putting the cutoff point at 10 °C and a precisification π_y, putting the cutoff point at 20 °C. As it is more plausible that the cutoff point will be regarded to be around π_x, $\mu(\pi_x)$ will be larger than $\mu(\pi_y)$.

SŁ has the following elements at this point:

- A set of propositional variables $p \in V = \{p_1, p_2, \dots\}$.
- The constant $\bot$.
- The connectives & and $\rightarrow$, and by extension $\neg$, $\wedge$ and $\vee$.
- The supertrue-connective S.
- A set of precisification points $\pi \in W = \{\pi_1, \pi_2, \dots\}$.
- The mapping $e : W \times V \rightarrow \{0, 1\}$.

- The probability measure μ using the σ-algebra formed by all subsets of W.
- The precisification space, formalized as a triple $\Pi = \langle W, e, \mu \rangle$

Given these, the *local truth value* $\|\varphi\|_\pi$ is inductively defined for every precisification point $\pi \in W$ as follows:

- $\|p\|_\pi = e(\pi, p)$, for $p \in V$
- $\|\bot\|_\pi = 0$
- $\|\varphi \& \psi\|_\pi = \begin{cases} 1 & \text{if } \|\varphi\|_\pi = 1 \text{ and } \|\psi\|_\pi = 1 \\ 0 & \text{otherwise} \end{cases}$
- $\|\varphi \to \psi\|_\pi = \begin{cases} 0 & \text{if } \|\varphi\|_\pi = 1 \text{ and } \|\psi\|_\pi = 0 \\ 1 & \text{otherwise} \end{cases}$
- $\|S\varphi\|_\pi = \begin{cases} 1 & \text{if } \forall \sigma \in W : \|\varphi\|_\sigma = 1 \\ 0 & \text{otherwise} \end{cases}$

As one can see, local truth values do not use t-norms; they use classical means only. Global truth values, on the other hand, are defined as follows:

- $\|p\|_\Pi = \mu(\{\pi \in W | e(\pi, p) = 1\})$, for $p \in V$
- $\|\bot\|_\Pi = 0$
- $\|\varphi \& \psi\|_\Pi = \|\varphi\|_\Pi *_{\text{Ł}} \|\psi\|_\Pi$
- $\|\varphi \to \psi\|_\Pi = \|\varphi\|_\Pi \Rightarrow_{\text{Ł}} \|\psi\|_\Pi$
- $\|S\varphi\|_\Pi = \|S\varphi\|_\pi$ for any $\pi \in W$

It should be noted that the $\|S\varphi\|_\pi$ is per se global – there is no such thing as "local supertruth." To be supertrue, a statement must be true globally.

When $\|\varphi\|_\Pi = 1$ for all precisification spaces Π, the formula φ is called *valid* in **SŁ**.

In the scope of the supertrue connective S, formulas behave as they would in classical logics, and not as they would in fuzzy logics. For example, the formula $\varphi \vee \neg\varphi$, while a tautology in classical logics (and thus also in supervaluationism), is not a tautology in fuzzy logics. As discussed in Section 5.1.8, its value can be as low as 0.5 when the weak conjunction is used. $S(\varphi \vee \neg\varphi)$, on the other hand, is true, as the scope of S is true under any precisification, as was discussed in Section 3.3.

7.2 Creating Statistics of Precisification Spaces

7.2.1 Is a Compromise Needed?

Technical systems using fuzzy logic employ various methods to gather the data on which they base their knowledge, which is represented by means of fuzzy sets. Classical statistical methods are often used. In some cases the programmer will make a guess when fitting fuzzy sets and will then refine them through trial and error.

Though not unproblematic, supervaluationism might also have some potential here. The theoretical foundations of supervaluationism would, however, make such an attempt "unclean."

But if one was willing to accept compromises, some possibilities might be explored that would retain some of the advantages of supervaluationism. In fact, it might be possible to find bridges to older proposals for dealing with vagueness in practical applications. The most straightforward compromise that comes to mind here is to meter the precisification space statistically, and thus to create "fuzzy data" formed by supervaluationist means. It should be noted that the evaluation of complex formulas in such a case, would no longer take place through supervaluationist means. Only atoms are evaluated in a supervaluationist manner.

7.2.2 A Discrete Number of Precisifications

As a simple illustration of a supervaluationist task that could be forced to give values even when a statement is neither supertrue nor superfalse, the "complete" precisification space created in Section 3.3 and the question of what the "inner circles" of an archery target might be can be revisited. The statements to be evaluated remain the same. We can start by going back to the statement that "a majority of arrows are close to the center of the target."

A majority is constituted by 4 arrows out of 7. The four innermost arrows are within the central three circles of the target. Therefore, anyone who considers "close to the center of the target" to be only the central circle or the two central circles would take this statement to be false. Anyone who considers the third circle close to the center would regard this statement to be true.

The precisification space still includes four possible precisifications, Π_1, Π_2, Π_3 and Π_4. As before, some precisifications see the statement as true and some see it as false.

There is no supertruth here. Very lazily, one could say that under a share of 0.5 of precisifications the statement is considered to be true. However, this would be just as inaccurate as it is lazy, since it considers all precisifications to be equally valid. This

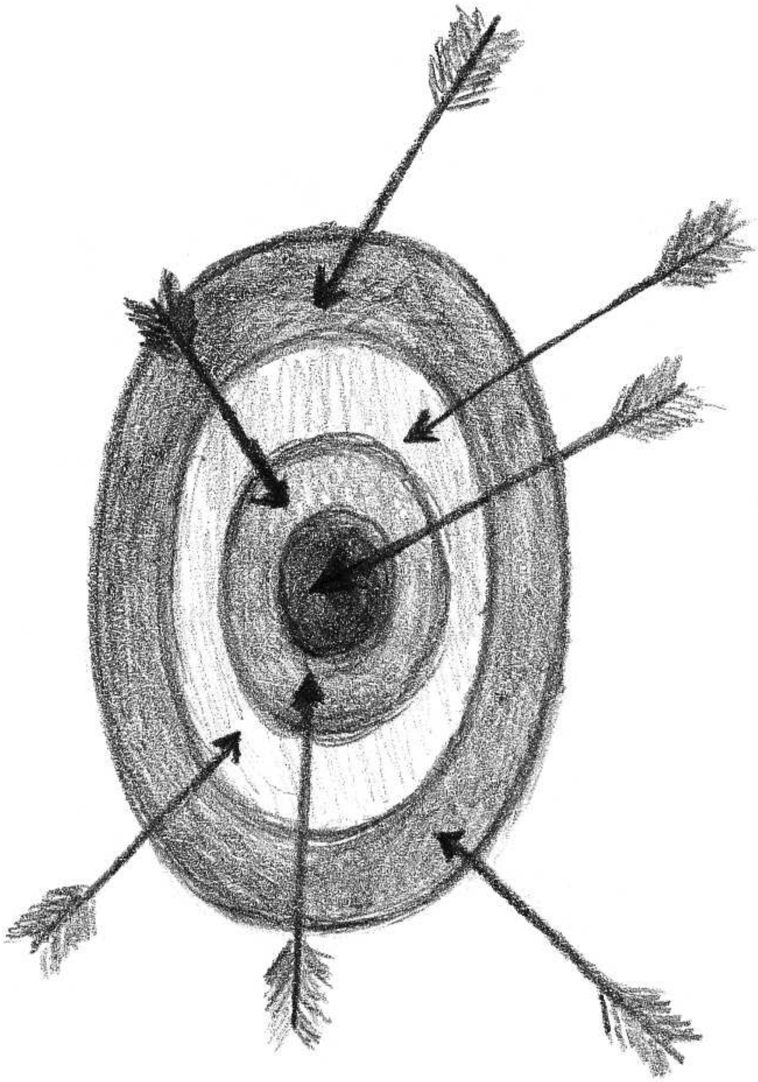

Figure 7.1: *An archery target, revisited.*

Π_i	$\|a\|_{\Pi_i}$
1	0
2	0
3	1
4	1

Figure 7.2: *Evaluation over the complete precisification space.*

Π_i	$\mu(\Pi_i)$
Π_1	0.3
Π_2	0.6
Π_3	0.1
Π_4	0

Figure 7.3: *Weights assigned to precisifications.*

Π_i	$\mu(\Pi_i)$	$\lVert\varphi\rVert_{\Pi_i}$	$\mu(\Pi_i) * \lVert\varphi\rVert_{\Pi_i}$
Π_1	0.3	0	0
Π_2	0.6	0	0
Π_3	0.1	1	0.1
Π_4	0	1	0
Π	1	1	0.1

Figure 7.4: *Creation of a weighted mean.*

will not be the case, however, since not all precisifications will be equally "popular." Just because it is possible to consider all four circles "the inner circles," it is not very likely that anyone would actually use this definition. Just referring to the central circle or to the inner two seems much more likely. One could statistically determine how many people would apply each respective precisification through surveys. This methodology will, of course, introduce some of the ambiguity of fuzzy applications into supervaluationism.

To continue with the example, statistical values adding up to 1 will be assigned to the respective elements of the precisification space. These will be used as weights when the mean value is created later on. We can refer to the statistical metric over the precisification space as μ. It should be noted that the author did not actually conduct any statistical surveys and thus the values are purely fictional.

With these weights, one could create a weighted mean over the individual evaluations.

$$\lVert\varphi\rVert = \sum_i \mu(\Pi_i) * \lVert\varphi\rVert_{\Pi_i}$$

Here, our truth value would be 0.1.

It should be noted, however, that while uncertainty has been introduced in such cases, supertrue statements remain supertrue, and superfalse statements remain superfalse. Let us reconsider the statement "an odd number of arrows has hit the inner circles of the target," which was considered supertrue. The evaluation of this statement in the framework we are now using can be seen in Figure 7.5.

Π_i	$\mu(\Pi_i)$	$\|\varphi\|_{\Pi_i}$	$\mu(\Pi_i) * \|\varphi\|_{\Pi_i}$
Π_1	0.3	1	0.3
Π_2	0.6	1	0.6
Π_3	0.1	1	0.1
Π_4	0	1	0
Π	1	1	1

Figure 7.5: *Weighted mean over a supertrue statement.*

As the weights were chosen to add up to one, and all the weights are multiplied by 1 and then summed up, it is clear that a supertrue statement cannot offer any truth value other than 1, while superfalse statements cannot offer truth values other that 0.

7.2.3 Continuous Precisification Spaces

In the previous example, it was a big help that the concentric circles on the archery target limited the number of possible answers to the statement considered.

If the question had been asked about a target without these, the task would have been much more challenging, since there would be no discrete number of possible precisifications. One would be working with a continuous precisification space.

One could consider an archery target that does not have any rings. The only marked spot on this target would be the bullseye. The target could be hit by 7 arrows. The four arrows nearest the bullseye would constitute "the majority" that would be of interest to us.

Even though the graphic is two-dimensional, only one parameter is actually important for the question at hand – the distance of each individual arrow to the center of the target. Since creating statistical models with a single dimension is a lot easier than creating one with two dimensions, one can simplify this example by measuring each arrow's distance from the center of the target and placing it on a single line representing the distance of the arrow from the center. The arrows' distances will range from 0 (the bullseye) to R (the radius of the target).

Now it is necessary to determine what "close to the center of the target actually means, much as it was necessary to determine what "the inner circles" of the archery target were in the discrete example.

Obviously, there will not be one definite answer. Even though we are only working with one dimension now, there will be an unlimited array of answers, ranging from 0 to R. Using statistical means again, one could consider it to be possible to approximate answers given by individuals to the question of what the center of the target is using

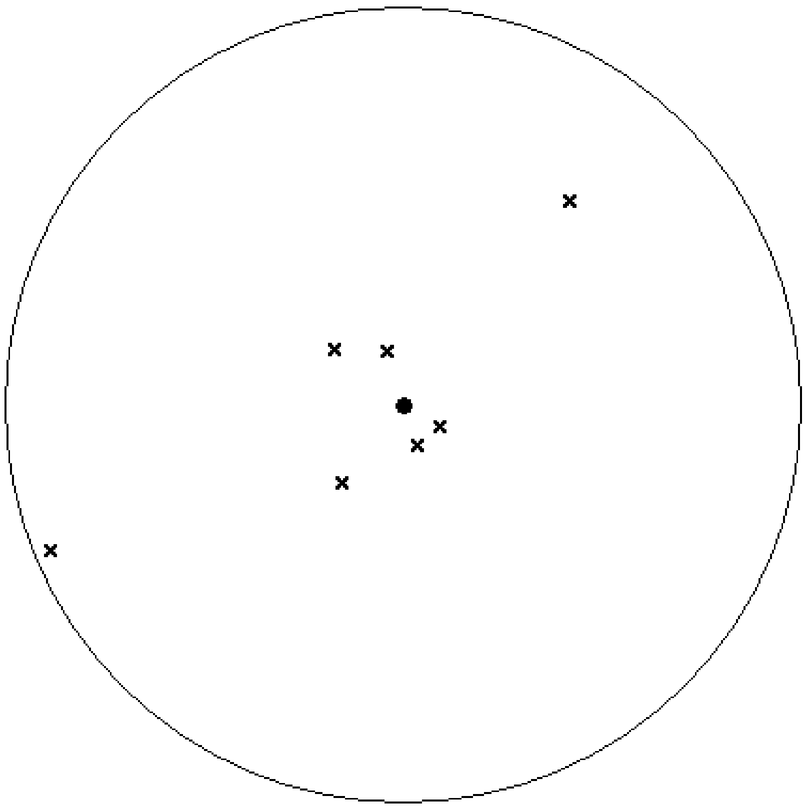

Figure 7.6: *A ringless archery target.*

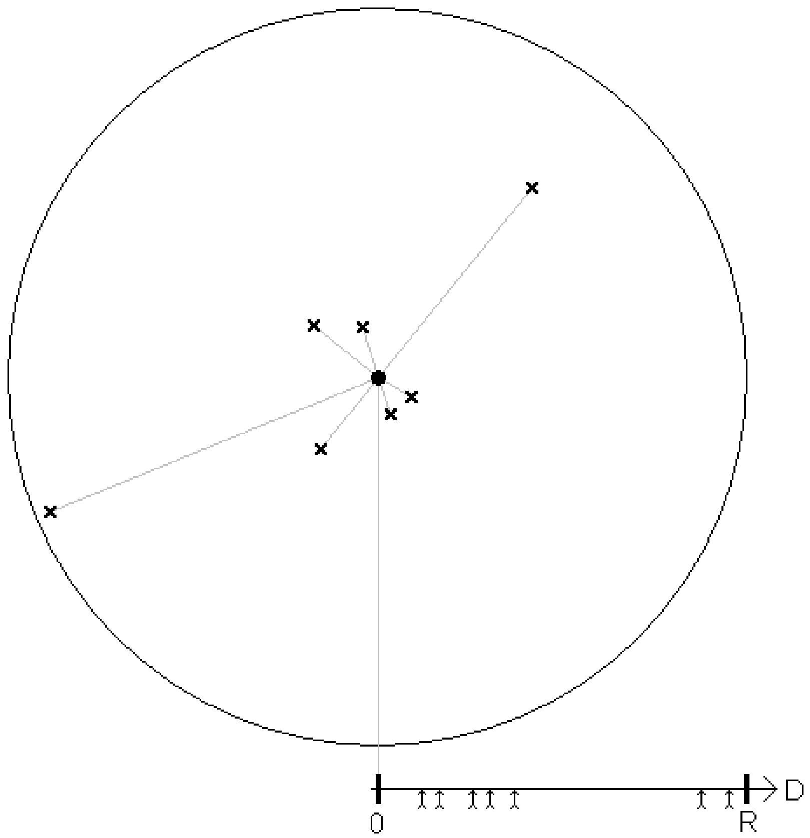

Figure 7.7: *Reducing the dimensionality of the problem at hand.*

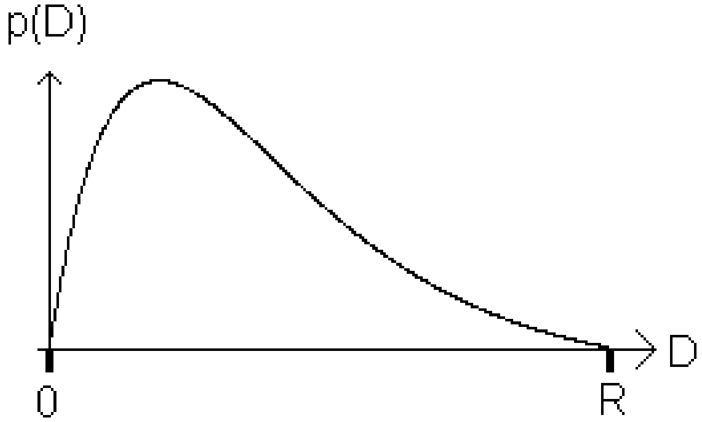

Figure 7.8: *A probability distribution of where the limit of "close to the center of the target might lie.*

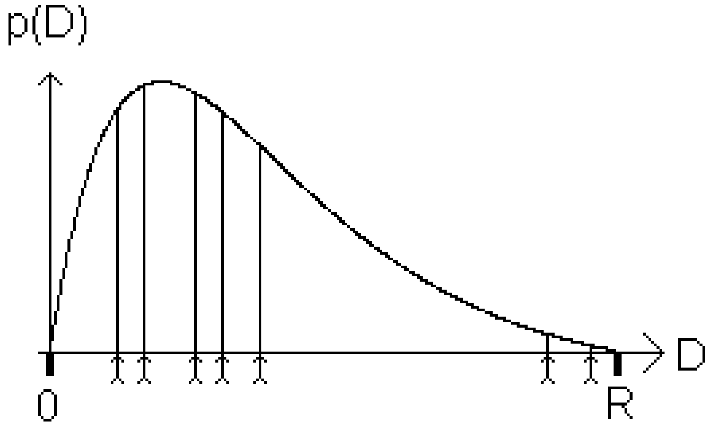

Figure 7.9: *The arrows' distances to the center of the target mapped to the distribution*

a continuous statistical model and assigning a probability density over the range of possible solutions – the integral of which, as with any probability distribution, would add up to one. The density function will be referred to as $p(D)$.

Since, once again, no statistical research was actually conducted in this research, the author chose a random χ^2 probability distribution for the question at hand that seems potentially plausible. Much as was true with the random weights assigned to the precisifications in the discrete case, the actual statistical value of this model is nil. It is only meant to be an illustration of the possibility.

If one was now to combine the data available at this point, it would already be possible to make an evaluation. One could plot the arrow's distances to the center of the target onto the graph's axis.

Anyone putting the border between "close to the center" and "not close to the center" closer to the center of the target would consider the statement to be false.

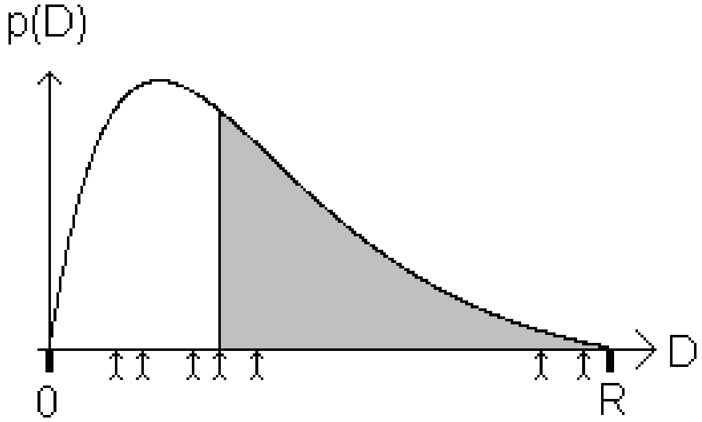

Figure 7.10: *The part of curve in which the statement at hand is true.*

Anyone placing it further out would consider it to be true.

What we now have is a probability density function. If in statistics one wants to determine the probability of a statistical value lying within a certain area in a density allocation, one forms an integral over the area under consideration. Over the entire distribution, the integral would be one – which is evident, since the probability of the value lying somewhere within the probability model is clearly one. Here, 0 will refer to a distance of zero from the center of the target and A_4 will refer to arrow number 4's distance from the center of the target.

$$\|\varphi\| = \int_0^{A_4} p(D)$$

Of course, since this is not a mathematically clean example, stating an actual value would be preposterous. However, one could approximate it to be somewhere around 0.5.

A more complex example would be the question of whether an odd number of arrows are close to the center of the target. If one follows the axis of the graph from 0 to R, it can be seen that this statement is at times correct and at times incorrect. Before one has "encountered" the first arrow from the center, the statement is false. After having encountered the first arrow from the center, it becomes true and remains true until the next arrow is encountered.

One can thus see that there are areas on the axis of the graph in which the statement would evaluate as true, if they contained the border between "close to the center of the board" and "not close to the center of the board."

Instead of forming separate integrals, one could also simply form an integral over the entire function, while including the respective truth values in the integral.

$$\|\varphi\| = \int_0^{R} p(D) * \Pi(D)$$

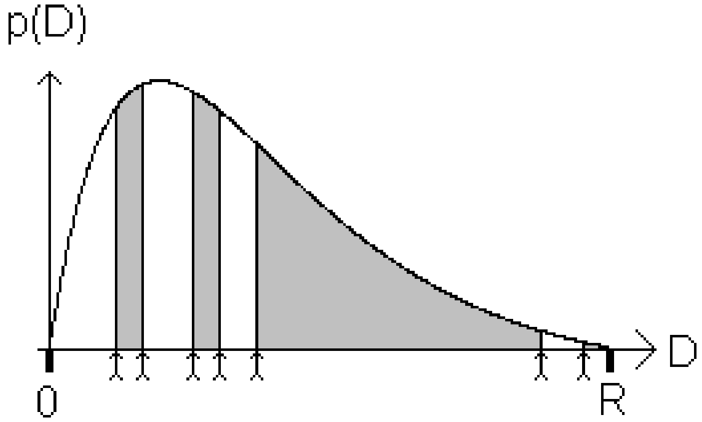

Figure 7.11: *The "areas" under the curve in which the statement at hand is true.*

Here, the value would lie somewhere around 0.7, making the statement appear to be fairly true – which, when looking at the target, seems intuitive.

This example remains simplistic, however, since it was possible to reduce the dimensionality of the problem to one dimension, making the quantification of precisifications relatively easy, even though an unlimited number of precisifications were possible. In the case of precisifying a term such as "Toronto," this would not be possible, as the boundary of a city does not have to be round and can have any shape imaginable. In such a case, quantifying the precisification space with a continuous model does not appear to be a solvable task.

7.3 Max Black – Hazy Sets

Max Black, a Cambridge philosopher and mathematician, penned one possible approach to handling vagueness in his 1937 article *Vagueness. An exercise in logical analysis* [5].

Black proposed a new method to symbolize vagueness: "a quantitative differentiation, admitting of degrees, and correlated with the indeterminacy in the divisions made by a group of observers." ([5], p. 441) He assumed that the vagueness of a word involves variations in its application by different users of a language and that these variations fulfill systematic and statistical rules when one symbol has to be discriminated from another. He defined the consistency to which a symbol x is applied in a language L as $C(L, x)$.

The consistency of an application is a statistical measurement over a sum of individuals forced to make a decision in a given situation. The consistency of the symbol x, under a language L, signifies the share of decision makers that claim the symbol

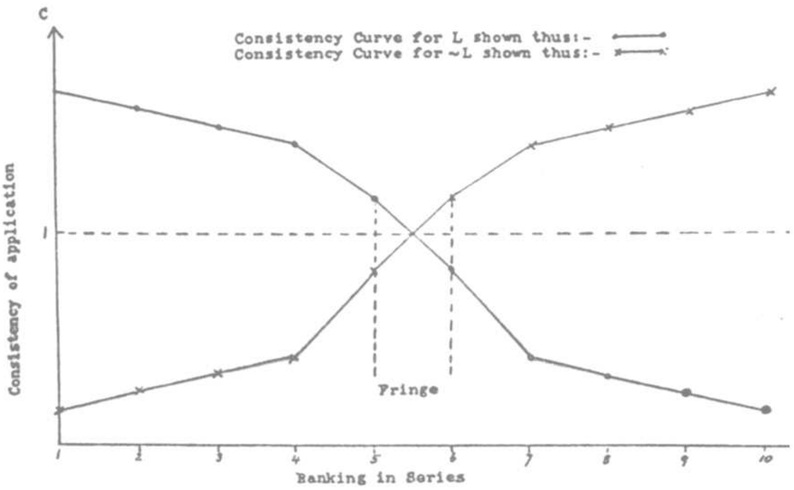

Figure 7.12: *Consistency of application of a typical vague symbol, ([5], p. 443).*

to be applicable in a given situation. If all decision makers agree that the term is applicable, its consistency is 1. If all decision makers agree that it is not applicable, the consistency is 0. If 90% of decision makers claim the symbol to be applicable, the consistency is 0.9.

Accordingly, the complement of the consistency is the consistency with $\neg L$, where $\neg L$ signifies an inversion of L.

$$C(\neg L, x) = 1 - C(L, x)$$

For example, L could be "x is a large city." In this case, $\neg L$ would be "x is not a large city." x could hold a wide array of meanings, such as "Moscow," "Reykjavík" or "Vaduz." "

If one was to chose "Reykjavík," for example, the value of $C(L, x)$ would be the percentage of people, in a questioned group, who consider "Reykjavík," with its roughly $103,000$ inhabitants, to be a large city. It should be noted that anyone questioned is forced to give a clear yes or no answer and cannot refuse to make a decision because he is uncertain. If 13% decide that they consider "Reykjavík" to be a large city, $C(L, x)$ has the value of 0.13. The value of $C(\neg L, x)$ is 0.87.

In Black's example, he arranged the various situations considered in a series of declining consistency. He called the graph created by this the "consistency curve" of the application of the language L. He depicted the consistency curve of $\neg L$ as well, which due to the nature of $C(\neg L, x)$ appears as a reflection of $C(L, x)$ along the value of 0.5.

It is possible to consider this approach as being related to supervaluationism in that one could regard the forced decisions, on which the consistency curve is based, as precisifications. One could assume that if a person was forced to make decisions in certain situations, the decision-maker would be precisifying the term in question, on a conscious or subconscious level.

Statements would only be supertrue (or superfalse) when the consistency curve reaches the maximum value (or zero), as these incidents, respectively, signify a complete agreement among all persons questioned, who are all considered to be competent speakers. In between, supervaluationism would consider the statement indeterminable, as no consensus exists. The aforementioned compromise, however, would be to accept a statistical metric of the precisifications that the consistency curve represents, as a degree of applicability. Mathematically and philosophically, this would be quite unclean, but it could be considered to be a necessary modeling step, if one was to try to apply these concepts in reality. It could thus be regarded as a simplification of supervaluationism.

An example of an application can be found in [40], that illustrates the possibility of employing Black's fringe in a multidimensional symptom space in medical philosophy.

Chapter 8

Closing Remarks

Selecting a topic that included "reviewing" fuzzy logic as a whole, as described in the mission statement found in Chapter 1, turned out to be a much more challenging task than it initially seemed. This was due not to the complex nature of the framework of fuzzy logic, but to the passionately defended divergent views regarding what fuzzy logic is or is supposed to be.

Lotfi Zadeh himself spoke of different fuzzy logics, as was discussed in Section 2.4.1. The fact that the only thing some of the concepts operating under the name of "fuzzy logic" have in common is the basic idea of graduated degrees of applicability has led to many misunderstandings and also some anger – as was discussed in Section 5.2.1. A broader consensus on the usage of terms could help prevent further unpleasantness in the future.

In this specific case, however, it was not actually a problem that the term "fuzzy logic" covers a wide array of vastly different concepts, as was discussed in Chapter 2, ranging from expansions of mathematical set theory to circuit control systems and systems of logical reasoning. All were relevant here and they made the comparisons and combinations with supervaluationism much more interesting.

It should be noted that the choice of supervaluationism as the second focal point of this thesis was not an unambiguous choice. Supervaluationism was chosen relatively randomly, as a theory of vagueness that differs vastly from fuzzy logic in both the goals set for the theory and the contexts in which it has had its successes. While fuzzy logic (in a broad sense) aims to quantify vagueness to ease problems caused by vague data in a reasoning system, supervaluationism is based on the idea that it is not possible to unambiguously quantify vagueness - which has resulted in its being excluded from a broad range of practical applications, as was discussed in Chapter 3. The "difference in attitudes" between the approaches was visible in particular when contemplating the way in which the theories respectively handle the Sorites paradox, as was discussed in Chapter 4. The problems fuzzy logic encounters due to its somewhat more "pragmatic" approach towards handling vagueness were discussed in Chapter 5.

As was asserted in Chapter 7, however, this rift in the objectives of the two theories does not necessarily mean that they are incompatible. On the contrary, it seems, in very narrow contexts, to be possible to preserve (some of) the advantages of both when combining them, be it in logical systems or in applied systems. It is ill-advised to consider these theories to be antipoles just because their orientations differ so greatly. It would be interesting to know if the same could be said about the relationship between fuzzy logic and contextualist or pragmatist theories of vagueness.

And, in general, it should be noted that supervaluationism might not be as fruitless in the field of technical applications as it might seem at first. Some ideas of applying supervaluationism as more than "just" a theory of vagueness were discussed in Chapter 7. It clearly cannot compete with fuzzy logic in fuzzy logic's domain of application – this would be futile. However, it does seem conceivable that supervaluationism might have a niche in the field of dialogue systems.

Supervaluationism only considers statements to be definitely answerable when they are either true or false, under all viable precisifications. And generally, there will be too many viable precisifications for too many vague terms to leave many questions clearly answerable. What, however, happens when one allows a system to reduce the scope of the precisification space (either permanently or for the length of one conversation) through user interaction? By probing what exactly the user might mean in a given situation, precisification spaces can be reduced until they become capable of giving some answers that the user might desire.

Appendix A

Symbols Used

It is unheard of for a concept in the field of logics or mathematics to be widely represented by one and only one symbol in all publications around the world. In order to avoid confusion, it seems prudent to supply a reference table of the symbols used in this thesis and their respective meanings.

Symbol	Meaning
$+$	arithmetic addition
$\cdot$	arithmetic multiplication
$\leq$	arithmetic inequality
$<$	strict arithmetic inequality
$\times$	Cartesian product
$\longrightarrow$	function mapping
$\forall$	universal quantification
$\exists$	existential quantification
$\wedge$	weak conjunction
$\vee$	weak disjunction
$\&$	strong conjunction
$\veebar$	strong disjunction
$\leftrightarrow$	equivalence
$\neg$	negation
$\rightarrow$	material implication
$\neg$	negation
$\vdash$	inference
$\top$	tautology
$\bot$	contradiction
$*$	t-norm
$\Rightarrow_*$	residuum of t-norm $*$
$\cap$	intersection
$\cup$	union
A^C	complement of A
$\subseteq$	subset
$\in$	set membership

Figure A.1: *Symbols Used.*

List of Figures

2.1 *A classical set and its complement.* 5
2.2 *A fuzzy set and its complement.* 6
2.3 *Membership function of a classical set.* 7
2.4 *Membership function of a fuzzy set.* 8
2.5 *Union and intersection of two fuzzy sets.* 9
2.6 *The complement of a fuzzy set.* 10
2.7 *A subset of a fuzzy set.* 11
2.8 *A fuzzy set for the term "cold," in relation to the room temperature in a room in* °C. 12
2.9 *An additional fuzzy set added to the example.* 12
2.10 *Output functions for this example.* 13
2.11 *Fuzzification and defuzzification.* 14
2.12 *Three popular t-norms* 21
2.13 *Graph of the minimum t-norm* 21
2.14 *Graph of the Łukasiewicz t-norm* 21
2.15 *Graph of the product t-norm* 21
2.16 *Three popular t-norms and their residua* 22
2.17 *Graph of the minimum t-norm's residuum, the standard Gödel implication* 22
2.18 *Graph of the Łukasiewicz t-norm's residuum, the standard Łukasiewicz implication* 23
2.19 *Graph of the product t-norm's residuum, the Goguen implication* 23
2.20 *Three popular t-norms, their residua and their dual t-conorms* 25
2.21 *Graph of the maximum t-conorm* 25
2.22 *Graph of the bounded sum t-conorm* 25
2.23 *Graph of the probabilistic sum t-conorm* 26

3.1 *France* 30
3.2 *Is France a hexagon?* 30
3.3 *An archery target.* 34

3.4 *Evaluation over the complete precisification space.* 34
3.5 *Supertruth.* 35

5.1 *A borderline case between fairly tall and tall* 51

6.1 *An illustration for user-aided precisification space reduction.* 67
6.2 *"Is this hut close to the top of the mountain?"* 68
6.3 *"Would you consider this to be a mountain?"* 69

7.1 *An archery target, revisited.* 75
7.2 *Evaluation over the complete precisification space.* 75
7.3 *Weights assigned to precisifications.* 76
7.4 *Creation of a weighted mean.* 76
7.5 *Weighted mean over a supertrue statement.* 77
7.6 *A ringless archery target.* 78
7.7 *Reducing the dimensionality of the problem at hand.* 79
7.8 *A probability distribution of where the limit of "close to the center of the target might lie.* 80
7.9 *The arrows' distances to the center of the target mapped to the distribution* 80
7.10 *The part of curve in which the statement at hand is true.* 81
7.11 *The "areas" under the curve in which the statement at hand is true.* . . . 82
7.12 *Consistency of application of a typical vague symbol, ([5], p. 443).* . . . 83

A.1 *Symbols Used.* 88

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