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Sharp Laplace asymptotics in infinite dimensions

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Introduction

We consider measures on infinite dimensional spaces of the form

$$e^{-\beta U(\varphi)}\gamma_\beta(d\varphi) \tag{1}$$

in the limit $\beta \to \infty$, where U is a suitable Potential and γ_β is a centered Gaussian measure with covariance $\frac{1}{\beta}(m-\Delta)^{-1}$ for a mass $m > 0$. The measure γ_β will be realized on suitable spaces of functions/distributions over the $d-$dimensional Torus $\mathbb{T}^d$. In particular we are interested in

$$U(\varphi) = \int_{\mathbb{T}^d} V(\varphi(x))\,dx \tag{2}$$

for $d = 1, 2$, where $V : \mathbb{R} \to \mathbb{R}$ is a suitable function.

Motivation

Such measures arise e.g. as invariant measures of stochastic Reaction-Diffusion equations. These are stochastic perturbations of parabolic semilinear equations of the form

$$\frac{\partial}{\partial t}\varphi = \Delta\varphi - m\varphi - V'(\varphi) + \sqrt{\tfrac{2}{\beta}}\,\xi, \tag{3}$$

where ξ denotes space-time white noise. That is, ξ is a centered Gaussian process with variance $\mathbb{E}[\xi(x,t)\xi(y,s)] = \delta(x-y)\delta(t-s)$.

Formally, equation (3) can be written as gradient diffusion

$$\frac{\partial}{\partial t}\varphi = -DF(\varphi) + \sqrt{\tfrac{2}{\beta}}\,\xi \tag{4}$$

with

$$F(\varphi) = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi(x)|^2\,dx + \frac{m}{2}\int_{\mathbb{T}^d} |\varphi(x)|^2\,dx + \int_{\mathbb{T}^d} V(\varphi(x))\,dx.$$

Motivated by finite dimensions, the invariant measure of equation (4) should then be

$$\text{“}\,e^{-\beta F(\varphi)}\,d\varphi\,\text{”}.$$

However, this expression is not meaningful since Lebesgue measure does not exist in infinite dimensions.

Luckily, if we split “$e^{-\beta F(\varphi)}\,d\varphi$” into

$$\text{“}\,e^{-\beta\int_{\mathbb{T}^d} V(\varphi(x))\,dx}\, e^{-\beta\left(\frac{1}{2}\int_{\mathbb{T}^d}|\nabla\varphi(x)|^2\,dx + \frac{m}{2}\int_{\mathbb{T}^d}|\varphi(x)|^2\,dx\right)}\,d\varphi\,\text{”},$$

one can make sense of $\gamma_\beta(d\varphi) :=$ “$e^{-\beta\left(\frac{1}{2}\int_{\mathbb{T}^d}|\nabla\varphi(x)|^2\,dx + \frac{m}{2}\int_{\mathbb{T}^d}|\varphi(x)|^2\,dx\right)}\,d\varphi$” as a Gaussian measure on suitable Hilbert spaces (dependent on the space dimension d) and we arrive at (1) with U given in (2).

Indeed, this computation can be made rigorous and the measure given in (1) with U from (2) is the invariant measure of equation (3), see e.g. [8] and [14].

Thus, when considering the long time behavior of solutions to equation (3) for large β, it is interesting to study the asymptotic behavior of (1) in the limit $\beta \to \infty$.

Equation (3) arises for example in statistical mechanics, where β is the inverse absolute temperature. The limit $\beta \to \infty$ then describes the low temperature behavior of the system. For more background on this, see e.g. [12].

Difficulties and Results

One difficulty that arises immediately is that $\varphi(x)$ is not well defined if φ is a distribution. Hence $V(\varphi(x))$ does not make sense. On the one dimensional torus $\mathbb{T}$ this problem does not occur, since in that case the measure γ_β exists on sufficiently smooth function spaces.

However, on the two dimensional torus $\mathbb{T}^2$, the measure γ_β only can be defined on spaces of distributions. We will treat this by some renormalization procedure, see e.g. [16] for further details.

To get an intuition what we can expect to be the asymptotic behavior of (1), we consider the finite dimensional analogue. Let $W \in C^2(\mathbb{R}^d;\mathbb{R})$ with global minimum $M := \min_{y\in\mathbb{R}^d} W(y)$ and assume there exist $x_1,\dots,x_n \in \mathbb{R}^d$ with $W(x_i) = M$ and $\det(D^2W(x_i)) > 0$ for $i = 1,\dots,n$. Furthermore assume that $W(x) - M \geq \alpha|x|^2$ for $x \in B_R^c$ for some $\alpha, R > 0$. Then

$$\lim_{\beta\to\infty} \left(\int_{\mathbb{R}^d} e^{-\beta W(x)}\, dx \left(\sum_{i=1}^n e^{-\beta M} \left(\tfrac{2\pi}{\beta}\right)^{d/2} \left(\det D^2W(x_i)\right)^{-1/2} \right)^{-1} \right) = 1.$$

This result goes back to Laplace in 1774, see [17]. It tells us, that for large β we have

$$\int_{\mathbb{R}^d} e^{-\beta W(x)}\, dx \approx C_\beta\, e^{-\beta M}.$$

Hence the value of the integral is mainly determined by the global minimum of W. Note that we are interested in sharp asymptotics, i.e. in the precise value of C_β, not just asymptotics on a logarithmic scale of the form

$$-\lim_{\beta\to\infty} \frac{1}{\beta} \log \int_{\mathbb{R}^d} e^{-\beta W(x)}\, dx = M.$$

Statements of this type are well studied in so called Large Deviations Theory, see e.g. [9].

Motivated by the finite dimensional analogue, we expect for $e^{-\beta F(\varphi)}\, d\varphi$ in the limit $\beta \to \infty$ to concentrate at the local minima of F. Indeed, on the one dimensional torus, this is what we observe, see Theorem 2.11. However, when renormalizing F on the two dimensional torus to $:\!F\!:$, we change the local minima. So $e^{-\beta :F:(\varphi)}\, d\varphi$ should concentrate on the minima of $:\!F\!:$. But in fact, $:\!F\!:$ is not even bounded below! Nevertheless, it turns out that $e^{-\beta :F:(\varphi)}\, d\varphi$ concentrates at the local minima of F, see Theorem 6.2.

This thesis is structured as follows: In Section 1 we give a quick introduction to Gaussian measures on Hilbert spaces and fractional Sobolev spaces. We conclude this section with Theorem 1.29, which is an existence and uniqueness result for Gaussian measures on fractional Sobolev spaces. More details concerning Gaussian measures on Hilbert spaces can be found e.g. in [6].

In Section 2, we present some results on Laplace asymptotics for measures of the form $e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi)$, where U is a sufficiently smooth (i.e. at least C^2) function with finitely many minima. The main result, Theorem 2.11, gives the global asymptotic behavior in the case of multiple global minima.

In Section 3 we construct for $d = 1$ the perturbation of the Gaussian measure we are interested in. More precisely, we define $\int_{\mathbb{T}} V(\varphi(x))\, dx$ for γ_β-a.e. $\varphi \in L^2(\mathbb{T})$, where V is a polynomial with even degree and positive leading coefficient. The key ingredient is the Sobolev embedding, Theorem 3.2.

In Section 4 we apply the results of Section 2 to the perturbation constructed in Section 3.

In Section 5 we are interested in defining a perturbation for $d = 2$. As described earlier, problems arise since φ are no functions anymore but distributions. For that reason, we cannot define $\int_{\mathbb{T}^2} V(\varphi(x))\,dx$ for polynomials V. We remedy this by introducing Wick polynomials $:V:$, which is called renormalization. However, when renormalizing the polynomial V, we loose the boundedness from below, also for polynomials with even degree. Hence it is not straight forward to show that $e^{-\int_{\mathbb{T}^2} :V:(\varphi(x))\,dx}$ is integrable w.r.t. γ_β. Following [7], we use the hypercontractivity of the Ornstein-Uhlenbeck semigroup to handle this.

In Section 6 we finally do asymptotics for the perturbation constructed in Section 5. As described earlier, the perturbation of the local minima through the renormalization does not count, see Theorem 6.2. We mainly follow [2], where we give the details of some aspects of the proof.

For more background on infinite dimensional Laplace asymptotics, see e.g. [11], [10], [3] and references therein. More details on fractional Sobolev spaces and Fourier transform on the torus can be found e.g. in [1], [13].

1 Basics

1.1 Gaussian measures on Hilbert spaces

1.1 Definition. Let I be an index set with $|I| = |\mathbb{N}|$. We call a function $w : I \to \mathbb{R}$ a *weight* and define the weighted space $\ell^2_w(I)$ by

$$\ell^2_w(I) = \{u \in \mathbb{R}^I \,:\, \|u\|^2_{\ell^2_w(I)} := \sum_{k\in I} w_k^2\, u_k^2 < \infty\}.$$

1.2 Remark. With $\mathcal{B}(\mathbb{R}^I)$ we denote the Borel sets on $\mathbb{R}^I$, i.e. the sigma algebra generated by the projections from $\mathbb{R}^I$ to $\mathbb{R}$. //

1.3 Lemma. *Let w be a weight on I. Then $\ell^2_w(I) \in \mathcal{B}(\mathbb{R}^I)$.*

Proof. The projections $\pi_k : \mathbb{R}^I \to \mathbb{R}$, defined by $\pi_k(x) := x_k$ are Borel measurable for every $k \in I$ by definition. Since multiplication and countable sums of measurable functions remain measurable, also $f(x) := \sum_{k\in I} w_k^2\, x_k^2$ is Borel measurable. Finally it is enough to note that $\ell^2_w(I) = f^{-1}([0,\infty))$. ❑

1.4 Definition. For $a, c \in \mathbb{R}$, $c > 0$ the Gaussian measure $N_{a,c}$ is defined on the Borel sets $\mathcal{B}(\mathbb{R})$ by

$$N_{a,c}(B) = \frac{1}{\sqrt{2\pi c^2}} \int_B e^{-\frac{(x-a)^2}{2c^2}}\, dx, \quad B \in \mathcal{B}(\mathbb{R}).$$

Instead of $N_{0,c}$ we shall also write N_c.

1.5 Lemma. *Let $a, c, h \in \mathbb{R}$, $c > 0$. Then*

$$\int_{\mathbb{R}} x\, N_{a,c}(dx) = a,$$
$$\int_{\mathbb{R}} (x-a)^2\, N_{a,c}(dx) = c^2,$$
$$\int_{\mathbb{R}} e^{hx}\, N_{a,c}(dx) = e^{ah+h^2c^2/2},$$
$$\int_{\mathbb{R}} e^{ihx}\, N_{a,c}(dx) = e^{iah-h^2c^2/2}.$$

1.6 Definition. Let a, c be weights on I with $c > 0$. Then, in abuse of notation, a probability measure $N_{a,c}$ is defined on $(\mathbb{R}^I, \mathcal{B}(\mathbb{R}^I))$ as the unique extension of the product measure $\prod_{k\in I} N_{a_k,c_k}$ defined on the cylindrical subsets of $\mathbb{R}^I$. Instead of $N_{0,c}$ we shall write N_c.

1.7 Lemma. *Let a, c, w be weights on I, where $c > 0$ and $a, c \in \ell^2_w(I)$. Then $N_{a,c}(\ell^2_w(I)) = 1$.*

Proof. By the monotone convergence theorem we have

$$\int_{\mathbb{R}^I} \|x\|^2_{\ell^2_w(I)}\, N_{a,c}(dx) = \int_{\mathbb{R}^I} \sum_{k\in I} w_k^2\, x_k^2\, N_{a,c}(dx) = \sum_{k\in I} \int_{\mathbb{R}} w_k^2\, x_k^2\, N_{a_k,c_k}(dx_k) = \sum_{k\in I} w_k^2\,(a_k^2 + c_k^2),$$

which is finite since $a, c \in \ell^2_w(I)$. But then $\{x \in \mathbb{R}^I : \|x\|_{\ell^2_w(I)} = \infty\}$ is a set of $N_{a,c}$ measure 0 and equivalently $\ell^2_w(I) = \{x \in \mathbb{R}^I : \|x\|_{\ell^2_w(I)} < \infty\}$ has measure 1. ❑

1.8 Lemma. *Let a, c be weights on I with $c > 0$ and $h, g \in (\mathbb{R}^I)'$, where*

$$(\mathbb{R}^I)' = \{x \in \mathbb{R}^I : x_k \neq 0 \text{ for at most finitely many } k \in I\}$$

is the topological dual space of $\mathbb{R}^I$. Then

$$\int_{\mathbb{R}^I} h(x)\, N_{a,c}(dx) = \sum_{k \in I} a_k h_k,$$

$$\int_{\mathbb{R}^I} h(x)g(x)\, N_{a,c}(dx) = \sum_{k \in I} h_k g_k (a_k^2 + c_k^2),$$

$$\int_{\mathbb{R}^I} e^{h(x)}\, N_{a,c}(dx) = e^{\sum_{k \in I} a_k h_k + h_k^2 c_k^2/2},$$

$$\int_{\mathbb{R}^I} e^{ih(x)}\, N_{a,c}(dx) = e^{\sum_{k \in I} i a_k h_k - h_k^2 c_k^2/2}.$$

Proof. We only show the last equation, the other equalities can be proved in the same way. Let $h \in \mathbb{R}^I_0$, then there exist $k_1, \ldots k_n \in I$ s.t. $h_{k_i} \neq 0$ for $i = 1, \ldots, n$ and $h_k = 0$ for $k \in I \setminus \{k_1, \ldots, k_n\}$. Thus

$$\begin{aligned}
\int_{\mathbb{R}^I} e^{ih(x)}\, N_{a,c}(dx) &= \int_{\mathbb{R}^I} e^{i \sum_{k \in I} h_k x_k}\, N_{a,c}(dx) \\
&= \int_{\mathbb{R}} e^{i h_{k_1} x_{k_1}}\, N_{a_{k_1}, c_{k_1}}(dx_{k_1}) \cdots \int_{\mathbb{R}} e^{i h_{k_n} x_{k_n}}\, N_{a_{k_n}, c_{k_n}}(dx_{k_n}) \\
&= e^{i a_{k_1} h_{k_1} - h_{k_1}^2 c_{k_1}^2/2} \cdots e^{i a_{k_n} h_{k_n} - h_{k_n}^2 c_{k_n}^2/2} \\
&= e^{\sum_{k \in I} i a_k h_k - h_k^2 c_k^2/2}.
\end{aligned}$$

❑

1.9 Remark. The topological dual space $\ell^2_w(I)'$ of $\ell^2_w(I)$ is identified with $\ell^2_{1/w}(I)$ via $f(x) = \sum_{k \in I} f_k x_k$ for $f \in \ell^2_{1/w}(I)$ and $x \in \ell^2_w(I)$. //

1.10 Corollary. *Let a, c, w be weights on I with $c > 0$ and $a, c \in \ell^2_w(I)$. Then for $h, g \in \ell^2_{1/w}(I) \simeq (\ell^2_w(I))'$ we have*

$$\int_{\ell^2_w(I)} h(x)\, N_{a,c}(dx) = \sum_{k \in I} a_k h_k,$$

$$\int_{\ell^2_w(I)} h(x)g(x)\, N_{a,c}(dx) = \sum_{k \in I} h_k g_k (a_k^2 + c_k^2),$$

$$\int_{\ell^2_w(I)} e^{h(x)}\, N_{a,c}(dx) = e^{\sum_{k \in I} a_k h_k + h_k^2 c_k^2/2},$$

$$\int_{\ell^2_w(I)} e^{ih(x)}\, N_{a,c}(dx) = e^{\sum_{k \in I} i a_k h_k - h_k^2 c_k^2/2}.$$

1.11 Remark. The series in Corollary 1.10 are finite. For example, for $h \in \ell^2_{1/w}(I)$ we have that $(h_k^2/w_k^2)_{k\in I}$ is bounded. Thus

$$\sum_{k\in I} h_k^2 c_k^2 = \sum_{k\in I} h_k^2/w_k^2 c_k^2 w_k^2 \leq C \sum_{k\in I} c_k^2 w_k^2 < \infty$$

for $c \in \ell^2_w(I)$. //

1.12 Lemma. *Let c, w be weights with $c > 0$ and $c \in \ell^2_w(I)$. Furthermore let $\theta \in \mathbb{R}$ and define $T : \ell^2_w(I) \times \ell^2_w(I) \to \ell^2_w(I)$ by $T(x,y) := x\sin(\theta) + y\cos(\theta)$. Then the measure $T\#(N_c \otimes N_c)$ coincides with N_c.*

Proof. We show that the Fourier transforms of $\mu := T\#(N_c \otimes N_c)$ and N_c coincide on $\ell^2_w(I)$. Let $h \in \ell^2_{1/w}(I)$, then by the Transformation Theorem

$$\begin{aligned}
\widehat{\mu}(h) &= \int_{\ell^2_w(I)} e^{ih(x)}\, T\#(N_c \otimes N_c)(dx) \\
&= \int_{\ell^2_w(I)\times\ell^2_w(I)} e^{ih(T(x,y))}\, (N_c \otimes N_c)(dxdy) \\
&= \int_{\ell^2_w(I)\times\ell^2_w(I)} e^{i\sin(\theta)h(x)}\, e^{i\cos(\theta)h(y)}\, (N_c \otimes N_c)(dxdy) \\
&= \int_{\ell^2_w(I)} e^{i\sin(\theta)h(x)}\, N_c(dx) \int_{\ell^2_w(I)} e^{i\cos(\theta)h(y)}\, N_c(dy) \\
&= e^{-1/2\,\sin^2(\theta)\sum_{k\in I} h_k^2 c_k^2}\, e^{-1/2\,\cos^2(\theta)\sum_{k\in I} h_k^2 c_k^2} \\
&= e^{-1/2\sum_{k\in I} h_k^2 c_k^2} \\
&= \widehat{N_c}(h).
\end{aligned}$$

❑

1.13 Lemma. *Let c, w be weights on I with $c > 0$ and $c \in \ell^2_w(I)$. Then*

$$\int_{\ell^2_w(I)} h(x)^{2n}\, N_c(dx) = \Big(\sum_{k\in I} h_k^2 c_k^2\Big)^n (2n-1)!!$$

for all $h \in \ell^2_{1/w}(I)$ and $n \in \mathbb{N}$, where $(2n-1)!! := (2n-1)(2n-3)\cdots 1$. Note that the series is finite by Remark 1.11.

Proof. From Corollary 1.10 we get

$$\int_{\ell^2_w(I)} e^{\lambda h(x)}\, N_c(dx) = e^{\lambda^2/2\sum_{k\in I} h_k^2 c_k^2}$$

for every $\lambda \in \mathbb{R}$. Differentiating this equation $2n$ times with respect to λ and evaluating at

$\lambda = 0$ yields with the notation $S := \sum_{k\in I} h_k^2 c_k^2$

$$\begin{aligned}\int_{\ell_w^2(I)} h(x)^{2n}\, N_c(dx) &= \frac{\partial^{2n}}{\partial\lambda^{2n}}\left(\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{k!\,2^k}S^k\right)\Bigg|_{\lambda=0} = \sum_{k=0}^{\infty}\frac{\partial^{2n}}{\partial\lambda^{2n}}\frac{\lambda^{2k}}{k!\,2^k}S^k\Bigg|_{\lambda=0}\\ &= \sum_{k=n}^{\infty}\frac{\lambda^{2k-2n}}{k!\,2^k}S^k(2k)(2k-1)\cdots(2k-2n+1)\Bigg|_{\lambda=0}\\ &= \frac{1}{n!\,2^n}S^n(2n)!\\ &= S^n\,(2n-1)!!,\end{aligned}$$

where differentiation and integration can be changed since there is an integrable majorant and differentiation and summation can be changed since the series converges uniformly. ❑

1.14 Lemma. *Let a, c, w be weights on I with $c, w > 0$ and $a, c \in \ell_w^2(I)$ and let $0 < \epsilon < \frac{1}{c_k^2 w_k^2}$ for every $k \in I$. Then*

$$\int_{\ell_w^2(I)} e^{\frac{\epsilon}{2}\|x\|^2_{\ell_w^2(I)}}\, N_{a,c}(dx) = e^{\sum_{k\in I}\frac{\epsilon a_k^2 w_k^2}{2(1-\epsilon c_k^2 w_k^2)}}\prod_{k\in I}(1-\epsilon c_k^2 w_k^2)^{-1/2},$$

where $\sum_{k\in I}\frac{\epsilon a_k^2 w_k^2}{2(1-\epsilon c_k^2 w_k^2)} < \infty$ and $0 < \prod_{k\in I}(1-\epsilon c_k^2 w_k^2)^{-1/2} < \infty$.

Proof. W.l.o.g assume $I = \mathbb{N}$. Then by the monotone convergence theorem we have

$$\int_{\ell_w^2(\mathbb{N})} e^{\frac{\epsilon}{2}\|x\|^2_{\ell_w^2(\mathbb{N})}}\, N_{a,c}(dx) = \int_{\ell_w^2(\mathbb{N})}\lim_{n\to\infty} e^{\frac{\epsilon}{2}\sum_{k=1}^n x_k^2 w_k^2}\, N_{a,c}(dx) = \lim_{n\to\infty}\int_{\ell_w^2(\mathbb{N})} e^{\frac{\epsilon}{2}\sum_{k=1}^n x_k^2 w_k^2}\, N_{a,c}(dx).$$

Using $\epsilon c_k^2 w_k^2 < 1$, an elementary computation yields

$$\begin{aligned}\int_{\ell_w^2(\mathbb{N})} e^{\frac{\epsilon}{2}\sum_{k=1}^n x_k^2 w_k^2}\, N_{a,c}(dx) &= \prod_{k=1}^n \int_{\mathbb{R}} e^{\frac{\epsilon}{2}x_k^2 w_k^2}\, N_{a_k,c_k}(dx_k)\\ &= \prod_{k=1}^n (1-\epsilon c_k^2 w_k^2)^{-1/2} e^{\frac{\epsilon a_k^2 w_k^2}{2(1-\epsilon c_k^2 w_k^2)}}\\ &= e^{\sum_{k=1}^n \frac{\epsilon a_k^2 w_k^2}{2(1-\epsilon c_k^2 w_k^2)}}\prod_{k=1}^n(1-\epsilon c_k^2 w_k^2)^{-1/2}.\end{aligned}$$

Since $c \in \ell_w^2(\mathbb{N})$ and $\epsilon c_k^2 w_k^2 < 1$, we have

$$0 < \lim_{n\to\infty}\prod_{k=1}^n (1-\epsilon c_k^2 w_k^2)^{-1/2} < \infty.$$

Furthermore, $c \in \ell_w^2(\mathbb{N})$ implies that $c_k^2 w_k^2 \to 0$ for $k \to \infty$, hence there exists $\lambda > 0$ such that $\epsilon c_k^2 w_k^2 \le \lambda < 1$ for every $k \in \mathbb{N}$. Thus

$$\lim_{n\to\infty}\sum_{k=1}^n \frac{\epsilon a_k^2 w_k^2}{2(1-\epsilon c_k^2 w_k^2)} \le \lim_{n\to\infty}\sum_{k=1}^n \frac{\epsilon a_k^2 w_k^2}{2(1-\lambda)} < \infty$$

since $a \in \ell_w^2(\mathbb{N})$. ❑

1.2 Fourier Transform on $\mathbb{T}^d$

For $d \in \mathbb{N}$ consider the d-dimensional Torus $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$. With $L^2(\mathbb{T}^d)$ we denote the set (equivalence classes) of real valued square integrable functions on $\mathbb{T}^d$ and with $\ell^2(\mathbb{Z}^d)$ the set of real valued square summable sequences on $\mathbb{Z}^d$.

Endowed with

$$(f,g)_{L^2(\mathbb{T}^d)} := \int_{\mathbb{T}^d} f(x)g(x)\,dx$$

and

$$(u,v)_{\ell^2(\mathbb{Z}^d)} := \sum_{k\in\mathbb{Z}^d} u_k v_k$$

the spaces $(L^2(\mathbb{T}^d), (.,.)_{L^2(\mathbb{T}^d)})$ and $(\ell^2(\mathbb{Z}^d), (.,.)_{\ell^2(\mathbb{Z}^d)})$ are Hilbert spaces.

1.15 Lemma. *Let $S \subseteq \mathbb{Z}^d$ be a set with the property that $k \in S \Leftrightarrow -k \notin S$. Then $\{1, \sqrt{2}\cos(2\pi k\cdot), \sqrt{2}\sin(2\pi k\cdot) : k \in S\}$ is an ONB of $(L^2(\mathbb{T}^d), (.,.)_{L^2(\mathbb{T}^d)})$.*

We fix a set $S \subseteq \mathbb{Z}^d$ with the property of Lemma 1.15 and define

$$e_k(x) := \begin{cases} 1, & k = 0, \\ \sqrt{2}\cos(2\pi k\cdot x), & k \in S, \\ \sqrt{2}\sin(2\pi k\cdot x), & k \in -S. \end{cases}$$

Then $(e_k)_{k\in\mathbb{Z}^d}$ is an ONB of $(L^2(\mathbb{T}^d), (.,.)_{L^2(\mathbb{T}^d)})$ and $e_k(x)^2 + e_{-k}(x)^2 = 2$ for all $k \in \mathbb{Z}^d$ and $x \in \mathbb{T}^d$. Note that e_k is not only an element of $L^2(\mathbb{T}^d)$ but also of $C^\infty(\mathbb{T}^d)$.

1.16 Remark. Let $(\lambda_k)_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ with $\lambda_k = \lambda_{-k}$ for all $k \in \mathbb{Z}^d$. Then for arbitrary $x \in \mathbb{T}^d$

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \lambda_k\, e_k(x)^2 &= \lambda_0 + \sum_{k\in S} \lambda_k\, e_k(x)^2 + \sum_{k\in -S} \lambda_k\, e_k(x)^2 \\ &= \lambda_0 + \sum_{k\in S} \lambda_k\, e_k(x)^2 + \sum_{k\in S} \lambda_{-k}\, e_{-k}(x)^2 \\ &= \lambda_0 + \sum_{k\in S} \lambda_k\, \underbrace{(e_k(x)^2 + e_{-k}(x)^2)}_{=2} \\ &= \lambda_0 + 2\sum_{k\in S}\lambda_k \\ &= \sum_{k\in\mathbb{Z}^d}\lambda_k.\end{aligned}$$

//

1.17 Definition. For $f = \sum_{k\in\mathbb{Z}^d} f_k\, e_k \in L^2(\mathbb{T}^d)$ its *Fourier transform* $\widehat{f} : \mathbb{Z}^d \to \mathbb{R}$ is defined by

$$\widehat{f}(k) := f_k.$$

1.18 Lemma. *For $f, g \in L^2(\mathbb{T}^d)$ we have $(f,g)_{L^2(\mathbb{T}^d)} = (\widehat{f}, \widehat{g})_{\ell^2(\mathbb{Z}^d)}$.*

Proof. Let $f, g \in L^2(\mathbb{T}^d)$, then by the continuity of the scalar product we have

$$(f,g)_{L^2(\mathbb{T}^d)} = \Big(\sum_{k\in\mathbb{Z}^d} \widehat{f}(k)\, e_k, \sum_{j\in\mathbb{Z}^d} \widehat{g}(j)\, e_j\Big)_{L^2(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d} \widehat{f}(k)\,\widehat{g}(k)\, \underbrace{(e_k, e_j)_{L^2(\mathbb{T}^d)}}_{=\delta_{kj}} = (\widehat{f},\widehat{g})_{\ell^2(\mathbb{Z}^d)}.$$

❑

We denote by $\mathcal{D}(\mathbb{T}^d) := C^\infty(\mathbb{T}^d)\ \big(= C_c^\infty(\mathbb{T}^d) = \mathcal{S}(\mathbb{T}^d)\big)$ the space of real valued *testfunctions*.

1.19 Definition. A functional $F : \mathcal{D}(\mathbb{T}^d) \to \mathbb{R}$ is *bounded*, if there exist $C > 0$ and $n \in \mathbb{N}_0$ s.t.

$$|F(\varphi)| \le C|\varphi|_{C^n} \quad \text{for every } \varphi \in \mathcal{D}(\mathbb{T}^d),$$

where $|\varphi|_{C^n} := \sum_{|\alpha|\le n} \sup_{x\in\mathbb{T}^d} |D^\alpha \varphi(x)|$. Instead of $F(\varphi)$ we shall write $\langle F, \varphi\rangle$. With $\mathcal{D}'(\mathbb{T}^d)$ we denote the space of all real valued bounded and linear functionals on $\mathcal{D}(\mathbb{T}^d)$ and call them *distributions*.

1.20 Remark. Any $f \in L^1(\mathbb{T}^d)$ can be identified with a distribution $F_f \in \mathcal{D}'(\mathbb{T}^d)$ via

$$F_f(\varphi) = \int_{\mathbb{T}^d} f(x)\varphi(x)\, dx.$$

//

1.21 Definition. Let $f \in \mathcal{D}'(\mathbb{T}^d)$, then

(i) the *distributional derivative* $\partial^\alpha f \in \mathcal{D}'(\mathbb{T}^d)$ is for any multi-index $\alpha \in \mathbb{N}_0^d$ defined by

$$\langle \partial^\alpha f, \varphi\rangle := (-1)^{|\alpha|}\, \langle f, \partial^\alpha \varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{T}^d),$$

(ii) the *product* with a testfunction $\psi \in \mathcal{D}(\mathbb{T}^d)$ is defined by

$$\langle \psi f, \varphi\rangle := \langle f, \psi\varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{T}^d),$$

(iii) the *Fourier transform* $\widehat{f} : \mathbb{Z}^d \to \mathbb{R}$ is defined by

$$\widehat{f}(k) := \langle f, e_k\rangle.$$

1.22 Remark. Definition 1.21 (iii) is consistent with Definition 1.17, since for $f \in L^2(\mathbb{T}^d) \subseteq L^1(\mathbb{T}^d)$ Remark 1.20 applies and therefore

$$\begin{aligned}\widehat{F_f}(k) = \langle F_f, e_k\rangle = \int_{\mathbb{T}^d} f(x)\, e_k(x)\, dx &= \int_{\mathbb{T}^d} \sum_{j\in\mathbb{Z}^d} \widehat{f}(j)\, e_j(x)\, e_k(x)\, dx \\ &= \sum_{j\in\mathbb{Z}^d} \widehat{f}(j) \int_{\mathbb{T}^d} e_j(x)\, e_k(x)\, dx = \widehat{f}(k).\end{aligned}$$

Furthermore, the distributional derivative coincides with the classical dervative if it exists. This can be shown using partial integration and the fundamental lemma of calculus of variation. //

1.23 Fakta. Let $f, g \in \mathcal{D}'(\mathbb{T}^d)$, $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{T}^d)$, then

(i) $\widehat{f+g} = \widehat{f} + \widehat{g}$,

(ii) $\widehat{\lambda f} = \lambda\,\widehat{f}$,

(iii) $f(\varphi) = (\widehat{f}, \widehat{\varphi})_{\ell^2(\mathbb{Z}^d)}$,

Proof. (i) and (ii) follow immediately from the definition. To show (iii), note that by the continuity of f we have

$$f(\varphi) = f(\sum_{k\in\mathbb{Z}^d} \widehat{\varphi}(k)\, e_k) = \sum_{k\in\mathbb{Z}^d} \widehat{\varphi}(k)\, f(e_k) = \sum_{k\in\mathbb{Z}^d} \widehat{\varphi}(k)\, \widehat{f}(k) = (\widehat{f}, \widehat{\varphi})_{\ell^2(\mathbb{Z}^d)}.$$

❑

1.3 Fractional Sobolev spaces

1.24 Definition. Let $s \in \mathbb{R}$ and $m > 0$. The *fractional Sobolev space* $H^s_m(\mathbb{T}^d)$ is defined by

$$H^s_m(\mathbb{T}^d) := \{f \in \mathcal{D}'(\mathbb{T}^d) \,:\, \sum_{k\in\mathbb{Z}^d} (m + 4\pi^2|k|^2)^s |\widehat{f}(k)|^2 < \infty\}.$$

Together with the scalar product

$$(f, g)_{H^s_m(\mathbb{T}^d)} := \sum_{k\in\mathbb{Z}^d} (m + 4\pi^2|k|^2)^s \widehat{f}(k)\overline{\widehat{g}(k)}$$

the space $H^s_m(\mathbb{T}^d)$ is a Hilbert space. Instead of $H^s_1(\mathbb{T}^d)$ we shall also write $H^s(\mathbb{T}^d)$.

1.25 Remark. An immediate consequence of this definition is, that $H^s_m(\mathbb{T}^d)$ is isometric isomorph to $\ell^2_w(\mathbb{Z}^d)$, where $w_k := (m + 4\pi^2|k|^2)^{s/2}$ for $k \in \mathbb{Z}^d$. Its dual space $(H^s_m(\mathbb{T}^d))' \simeq (\ell^2_w(\mathbb{Z}^d))' \simeq \ell^2_{1/w}(\mathbb{Z}^d)$ is therefore identified with $H^{-s}_m(\mathbb{T}^d)$ and with the same argumentation as in the proof of Fakta 1.23 (iii) we see that

$$f(\varphi) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k)\widehat{\varphi}(k)$$

for $f \in H^{-s}_m(\mathbb{T}^d)$ and $\varphi \in H^s_m(\mathbb{T}^d)$. //

1.26 Remark. One can show, that for $s \in \mathbb{N}_0$ the space $H^s(\mathbb{T}^d)$ is the usual Sobolev space, where derivatives up to order s are L^2 functions. //

1.27 Example. Consider $\delta_x : \mathcal{D}(\mathbb{T}^d) \to \mathbb{R}$, $\delta_x(\varphi) := \varphi(x)$ for $x \in \mathbb{T}^d$. Then δ_x satisfies $|\delta_x(\varphi)| \le C|\varphi|_{C^n}$ with $C = 1$ and $n = 0$, hence δ_x belongs to $\mathcal{D}'(\mathbb{T}^d)$. Furthermore $|\widehat{\delta_x}(k)|^2 = |\delta_x(e_k)|^2 = |e_k(x)|^2 \le 2$ for every $k \in \mathbb{Z}^d$. Thus δ_x belongs to $H^s(\mathbb{T}^d)$ if $\sum_{k\in\mathbb{Z}^d}(1 + 4\pi^2|k|^2)^s < \infty$, which is equivalent to $s < -d/2$ according to Lemma 1.28. //

1.4 Gaussian measures on $H^s(\mathbb{T}^d)$

1.28 Lemma. *For $m>0$ and $s\in\mathbb{R}$, the series*

$$\sum_{k\in\mathbb{Z}^d}(m+4\pi^2|k|^2)^s$$

is finite, if and only if $s<-d/2$.

Proof. If $s<-d/2$, then $(m+4\pi^2|k|^2)^s<(4\pi^2|k|^2)^s$ for $k\neq 0$. Since $|.|$ is an equivalent norm to $\|.\|_\infty$, it follows that

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}}(m+4\pi^2|k|^2)^s\leq C\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\|k\|_\infty^{2s}.$$

Let $n\in\mathbb{N}$, then $\|k\|_\infty=n$ if and only if $k_i\in\{-n,\dots,n\}$ for $i=1,\dots,d$ and not $k_i\in\{-(n-1),\dots,(n-1)\}$ for $i=1,\dots,d$. Hence there are $(2n+1)^d-(2n-1)^d$ vectors $k\in\mathbb{Z}^d$ satisfying $\|k\|_\infty=n$. Thus

$$C\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\|k\|_\infty^{2s}=C\sum_{n\in\mathbb{N}}\Big((2n+1)^d-(2n-1)^d\Big)\,n^{2s}.$$

Furthermore there exists $N\in\mathbb{N}$ and $C>0$, such that $(2n+1)^d-(2n-1)^d\leq C\,(2n)^{d-1}$ for every $n\geq N$. Therefore

$$C\sum_{n\geq N}\Big((2n+1)^d-(2n-1)^d\Big)\,n^{2s}\leq C\sum_{n\geq N}n^{d-1+2s},$$

where the last series is finite if $d+2s<0$, or equivalently, $s<-d/2$.
For $s\geq -d/2$ one can show in an analogue way, that the series does not converge. ❑

1.29 Theorem. *Let $s,\sigma\in\mathbb{R}$ and $d\in\mathbb{N}$ with $s<\sigma-d/2$. Furthermore let $a\in H^s(\mathbb{T}^d)$ and $\beta>0$. Then there exists a unique probability measure $\gamma_\beta=\gamma_{\beta,a,\sigma,m}$ on $H^s(\mathbb{T}^d)$ such that for $h\in H^{-s}(\mathbb{T}^d)$*

$$\widehat{\gamma_\beta}(h)=\int_{H^s(\mathbb{T}^d)}e^{ih(\varphi)}\,\gamma_\beta(d\varphi)=e^{ih(a)-\frac{1}{2\beta}\|h\|^2_{H_m^{-\sigma}(\mathbb{T}^d)}}.$$

Moreover, γ_β concentrates on $H^{s'}(\mathbb{T}^d)$ for every $s\leq s'<\sigma-d/2$ with $a\in H^{s'}(\mathbb{T}^d)$, i.e. $\gamma_\beta(H^{s'}(\mathbb{T}^d))=1$. The measure γ_β is called Gaussian measure *on $H^s(\mathbb{T}^d)$ with mean a, mass m and covariance operator $\frac{1}{\beta}(m-\Delta)^{-\sigma}$.*

Proof. Let $\widetilde{a},c,w$ be the weights on $\mathbb{Z}^d$ given by $\widetilde{a}_k:=\widehat{a}(k)$, $w_k:=(1+4\pi^2|k|^2)^{s/2}$ and $c_k:=\beta^{-1/2}(m+4\pi^2|k|^2)^{-\sigma/2}$. Then $H^s(\mathbb{T}^d)\simeq\ell^2_w(\mathbb{Z}^d)$ and let $T:\ell^2_w(\mathbb{Z}^d)\to H^s(\mathbb{T}^d)$ be the corresponding isomorphism. Note that for $h\in H^{-s}(\mathbb{T}^d)$ we have $h\circ T\in(\ell^2_w(\mathbb{Z}^d))'=\ell^2_{1/w}(\mathbb{Z}^d)$. For $\gamma_\beta:=T\#N_{\widetilde{a},c}$ we conclude together with Corollary 1.10 that

$$\begin{aligned}\widehat{\gamma_\beta}(h)&=\int_{H^s(\mathbb{T}^d)}e^{ih(\varphi)}\,T\#N_{\widetilde{a},c}(d\varphi)=\int_{\ell^2_w(\mathbb{Z}^d)}e^{ih(Tx)}\,N_{\widetilde{a},c}(dx)=e^{\sum_{k\in\mathbb{Z}^d}i\widetilde{a}_k(h\circ T)_k-(h\circ T)_k^2c_k^2/(2\beta)}\\&=e^{ih\circ T(\widetilde{a})-\frac{1}{2\beta}\,\|h\circ T\|^2_{\ell^2_c(\mathbb{Z}^d)}}=e^{ih(a)-\frac{1}{2\beta}\,\|h\|^2_{H_m^{-\sigma}(\mathbb{T}^d)}},\end{aligned}$$

if $\widetilde{a}, c \in \ell^2_w(\mathbb{Z}^d)$. But $\widetilde{a} \in \ell^2_w(\mathbb{Z}^d)$ is equivalent to $a \in H^s(\mathbb{T}^d)$ and this is true by assumption. Furthermore,

$$\sum_{k\in\mathbb{Z}^d} w_k^2 c_k^2 = \beta^{-1} \sum_{k\in\mathbb{Z}^d} (1+4\pi^2|k|^2)^s (m+4\pi^2|k|^2)^{-\sigma} \leq \beta^{-1} \sum_{k\in\mathbb{Z}^d} (\widetilde{m}+4\pi^2|k|^2)^{s-\sigma}$$

for some $\widetilde{m} > 0$, which is finite if $s - \sigma < -d/2$ according to Lemma 1.28. Thus we also have $c \in \ell^2_w(\mathbb{Z}^d)$.

The measure γ_β is indeed a probability measure, since

$$\gamma_\beta(H^s(\mathbb{T}^d)) = N_{\widetilde{a},c}(T^{-1}(H^s(\mathbb{T}^d))) = N_{\widetilde{a},c}(\ell^2_w(\mathbb{Z}^d)) = 1,$$

by Lemma 1.7. For $s < s' < \sigma - d/2$ with $a \in H^{s'}(\mathbb{T}^d)$ define $w'_k := (1+4\pi^2|k|^2)^{s'/2}$. Then we have

$$\gamma_\beta(H^{s'}(\mathbb{T}^d)) = N_{\widetilde{a},c}(T^{-1}(H^{s'}(\mathbb{T}^d))) = N_{\widetilde{a},c}(\ell^2_{w'}(\mathbb{Z}^d)) = 1,$$

again by Lemma 1.7, since we also have $\widetilde{a}, c \in \ell^2_{w'}(\mathbb{Z}^d)$.

The uniqueness of γ_β follows from the uniqueness of the Fourier transform. ❑

2 Laplace Aysmptotics on $H^s(\mathbb{T}^d)$

In this section we consider the Gaussian free field (GFF) with mass $m > 0$ on $H^s(\mathbb{T}^d)$. Let $s < 1 - d/2$, then according to Theorem 1.29 there exists a unique centered Gaussian measure γ_β on $H^s(\mathbb{T}^d)$ satisfying

$$\widehat{\gamma_\beta}(h) = e^{-\frac{1}{2\beta}\|h\|^2_{H_m^{-1}(\mathbb{T}^d)}}$$

for every $h \in H^{-s}(\mathbb{T}^d)$. The measure $\gamma := \gamma_1$ is called GFF.

The aim is to determine the sharp asymptotic behavior for $\beta \to \infty$ of

$$\int_{H^s(\mathbb{T}^d)} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi)$$

for potentials $U \in C^2(H^s(\mathbb{T}^d); \mathbb{R})$ such that

$$U(\varphi) + \frac{m}{2}\int_{\mathbb{T}^d} |\varphi(x)|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\varphi(x)|^2\,dx$$

has finitely many non-degenerate minima.

2.1 Asymptotics at the origin

2.1 Remark. Note that

$$\int_{H^s(\mathbb{T}^d)} e^{\delta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\,\gamma(d\varphi) < \infty$$

if and only if $\delta < m/2$ by Lemma 1.14. //

2.2 Lemma. *For a measurable function f on $\Omega \subseteq H^s(\mathbb{T}^d)$ we have*

$$\int_\Omega f(\varphi)\,\gamma_\beta(d\varphi) = \int_{\sqrt{\beta}\Omega} f(\varphi/\sqrt{\beta})\,\gamma(d\varphi).$$

Proof. With $X : \sqrt{\beta}\Omega \to \Omega$ defined by $X(\varphi) = \varphi/\sqrt{\beta}$, we have by the Transformation Theorem

$$\int_{\sqrt{\beta}\Omega} f(\varphi/\sqrt{\beta})\,\gamma(d\varphi) = \int_{X^{-1}(\Omega)} f(X(\varphi))\,\gamma(d\varphi) = \int_\Omega f(\varphi)\,X\#\gamma(d\varphi).$$

It remains to show that $X\#\gamma = \gamma_\beta$. This is done by using again the Transformation Theorem and uniqueness of the Fourier transform,

$$\begin{aligned}\int_{H^s(\mathbb{T}^d)} e^{ih(\varphi)}\,X\#\gamma(d\varphi) &= \int_{H^s(\mathbb{T}^d)} e^{ih(X(\varphi))}\,\gamma(d\varphi)\\ &= \int_{H^s(\mathbb{T}^d)} e^{ih(\varphi/\sqrt{\beta})}\,\gamma(d\varphi)\\ &= e^{-\frac{1}{2}\left\|h/\sqrt{\beta}\right\|^2_{H_m^{-1}(\mathbb{T}^d)}}\\ &= e^{-\frac{1}{2\beta}\|h\|^2_{H_m^{-1}(\mathbb{T}^d)}}\\ &= \int_{H^s(\mathbb{T}^d)} e^{ih(\varphi)}\,\gamma_\beta(d\varphi).\end{aligned}$$

❑

2.3 Remark. For a function $f : H^s(\mathbb{T}^d) \to \mathbb{R}$ we denote its Fréchet derivative by $Df : H^s(\mathbb{T}^d) \to L(H^s(\mathbb{T}^d); \mathbb{R})$. Its second order derivative we denote by $D^2 f : H^s(\mathbb{T}^d) \to L(H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d); \mathbb{R})$ //

2.4 Theorem (Local Asymptotics at the origin)**.** *Let $U \in C^2(H^s(\mathbb{T}^d); \mathbb{R})$ and assume that*

- $U(0) = 0$,
- $DU(0)(\xi) = 0$ *for all* $\xi \in H^s(\mathbb{T}^d)$,
- *there exists* $\rho < m$ *s.t.* $D^2U(0)(\xi, \xi) \geq -\rho \|\xi\|^2_{H^s(\mathbb{T}^d)}$ *for all* $\xi \in H^s(\mathbb{T}^d)$.

Then there exists $\alpha > 0$ s.t.

$$\lim_{\beta\to\infty} \int_{B_\alpha} e^{-\beta U(\varphi)} \gamma_\beta(d\varphi) = \int_{H^s(\mathbb{T}^d)} e^{-1/2 D^2 U(0)(\varphi,\varphi)} \gamma(d\varphi),$$

where $B_\alpha = \left\{\varphi \in H^s(\mathbb{T}^d) : \|\varphi\|_{H^s(\mathbb{T}^d)} < \alpha\right\}$.

Proof. Since $U \in C^2(H^s(\mathbb{T}^d); \mathbb{R})$ we have by Taylor expansion

$$U(\varphi/\sqrt{\beta}) = U(0) + DU(0)(\varphi/\sqrt{\beta}) + 1/2 D^2 U(0)(\varphi/\sqrt{\beta}, \varphi/\sqrt{\beta}) + R(\varphi/\sqrt{\beta}),$$

where $|R(\psi)| \leq C(\psi) \|\psi\|^2_{H^s(\mathbb{T}^d)}$ with $C(\psi) \to 0$ for $\psi \to 0$. Thus by Lemma 2.2 we have

$$\int_{B_\alpha} e^{-\beta U(\varphi)} \gamma_\beta(d\varphi) = \int_{B_{\sqrt{\beta}\alpha}} e^{-\beta U(\varphi/\sqrt{\beta})} \gamma(d\varphi) = \int_{B_{\sqrt{\beta}\alpha}} e^{-1/2 D^2 U(0)(\varphi,\varphi) - \beta R(\varphi/\sqrt{\beta})} \gamma(d\varphi).$$

By assumption we have

$$\begin{aligned} -1/2 D^2 U(0)(\varphi, \varphi) - \beta R(\varphi/\sqrt{\beta}) &\leq \rho/2 \|\varphi\|^2_{H^s(\mathbb{T}^d)} + \beta C(\varphi/\sqrt{\beta}) \left\|\varphi/\sqrt{\beta}\right\|^2_{H^s(\mathbb{T}^d)} \\ &\leq \rho/2 \|\varphi\|^2_{H^s(\mathbb{T}^d)} + C(\varphi/\sqrt{\beta}) \|\varphi\|^2_{H^s(\mathbb{T}^d)}. \end{aligned}$$

For $\varphi \in B_{\sqrt{\beta}\alpha}$ we have $\left\|\varphi/\sqrt{\beta}\right\|_{H^s(\mathbb{T}^d)} < \alpha$, hence we can choose α sufficiently small such that

$$\rho/2 + C(\varphi/\sqrt{\beta}) \leq C < m/2$$

for $\varphi \in B_{\sqrt{\beta}\alpha}$ and for all $\beta > 0$. Since $e^{C\|\varphi\|^2_{H^s(\mathbb{T}^d)}}$ is integrable for $C < m/2$ according to Remark 2.1, we have by the dominated convergence Theorem

$$\lim_{\beta\to\infty} \int_{B_\alpha} e^{-\beta U(\varphi)} \gamma_\beta(d\varphi) = \int_{H^s(\mathbb{T}^d)} \lim_{\beta\to\infty} \mathbb{1}_{B_{\sqrt{\beta}\alpha}}(\varphi)\, e^{-1/2 D^2 U(0)(\varphi,\varphi) - \beta R(\varphi/\sqrt{\beta})} \gamma(d\varphi)$$

and the conclusion follows with

$$\beta|R(\varphi/\sqrt{\beta})| \leq \beta C(\varphi/\sqrt{\beta}) \left\|\varphi/\sqrt{\beta}\right\|^2_{H^s(\mathbb{T}^d)} = C(\varphi/\sqrt{\beta}) \|\varphi\|^2_{H^s(\mathbb{T}^d)} \overset{\beta\to\infty}{\longrightarrow} 0.$$

❑

2.5 Theorem (Global Asymptotics, unique minimum at the origin)**.** *Let $U \in C^2(H^s(\mathbb{T}^d);\mathbb{R})$ with the assumptions from the previous Lemma. In addition, assume that there exists $\eta < m/2$ such that $U(\varphi) \geq -\eta \,\|\varphi\|^2_{H^s(\mathbb{T}^d)}$. Then*

$$\lim_{\beta\to\infty} \int_{H^s(\mathbb{T}^d)} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) = \int_{H^s(\mathbb{T}^d)} e^{-1/2D^2U(0)(\varphi,\varphi)}\, \gamma(d\varphi).$$

Proof. For $\alpha > 0$ we have

$$\int_{H^s(\mathbb{T}^d)} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) = \underbrace{\int_{B_\alpha} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi)}_{=:I_1(\beta)} + \underbrace{\int_{B^c_\alpha} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi)}_{=:I_2(\beta)}.$$

For sufficiently small $\alpha > 0$ we already know that $I_1(\beta)$ gives the right limit, hence it suffices to show that we can neglect $I_2(\beta)$. Using Lemma 2.2 and our assumptions, we get

$$I_2(\beta) = \int_{B^c_{\sqrt{\beta}\alpha}} e^{-\beta U(\varphi/\sqrt{\beta})}\, \gamma(d\varphi) \leq \int_{B^c_{\sqrt{\beta}\alpha}} e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi).$$

We define a probability measure ν by

$$\nu(A) = Z^{-1} \int_A e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi),$$

where $Z = \int_{H^s(\mathbb{T}^d)} e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi)$. Note that $Z < \infty$ by Remark 2.1 since $\eta < m/2$. Hence we have by Markov's Inequality for $\epsilon > 0$

$$\begin{aligned} I_2(\beta) &\leq Z\nu(B^c_{\sqrt{\beta}\alpha}) = Z\nu([\|\varphi\|_{H^s(\mathbb{T}^d)} \geq \sqrt{\beta}\alpha]) = Z\nu([e^{\epsilon\|\varphi\|^2_{H^s(\mathbb{T}^d)}} \geq e^{\epsilon\beta\alpha^2}]) \\ &\leq Ze^{-\epsilon\beta\alpha^2} \int_{H^s(\mathbb{T}^d)} e^{\epsilon\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \nu(d\varphi) \\ &= e^{-\epsilon\beta\alpha^2} \int_{H^s(\mathbb{T}^d)} e^{\epsilon\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi). \end{aligned}$$

Choosing $\epsilon > 0$ such that $\epsilon + \eta < m/2$, the last integral is finite and we have that $I_2(\beta)$ is exponentially decreasing in β. ❑

2.2 Asymptotics with multiple global minima

Remember that we denote the Gaussian measure centered in $\psi \in H^s(\mathbb{T}^d)$ with covariance operator $\frac{1}{\beta}(m-\Delta)^{-1}$ by $\gamma_{\psi,\beta}$, i.e. $\widehat{\gamma_{\psi,\beta}}(h) = e^{ih(\psi)}\, e^{-1/(2\beta)\|h\|^2_{H^{-1}_m(\mathbb{T}^d)}}$. Instead of $\gamma_{0,\beta}$ we write γ_β.

2.6 Lemma. *Let $\psi \in H^s(\mathbb{T}^d)$ and f be a measurable function on $\Omega \subseteq H^s(\mathbb{T}^d)$. Then*

$$\int_\Omega f(\varphi)\, \gamma_\beta(d\varphi) = \int_{\Omega-\psi} f(\varphi+\psi)\, \gamma_{-\psi,\beta}(d\varphi).$$

Proof. With $X : \Omega \to \Omega + \psi$ defined by $X(\varphi) = \varphi + \psi$, we have by the Transformation Theorem

$$\int_\Omega f(\varphi)\,\gamma_\beta(d\varphi) = \int_{X^{-1}(\Omega)} f(X(\varphi))\,X^{-1}\#\gamma_\beta(d\varphi).$$

It remains to show that $X^{-1}\#\gamma_\beta = \gamma_{-\psi,\beta}$. This is done by using again the Transformation Theorem and uniqueness of the Fourier transform,

$$\begin{aligned}\int_{H^s(\mathbb{T}^d)} e^{ih(\varphi)}\,X^{-1}\#\gamma_\beta(d\varphi) &= \int_{H^s(\mathbb{T}^d)} e^{ih(X^{-1}(\varphi))}\,\gamma_\beta(d\varphi)\\ &= \int_{H^s(\mathbb{T}^d)} e^{ih(\varphi-\psi)}\,\gamma_\beta(d\varphi)\\ &= e^{ih(-\psi)}\,e^{-\frac{1}{2\beta}\|h\|^2_{H^{-1}_m(\mathbb{T}^d)}}\\ &= \int_{H^s(\mathbb{T}^d)} e^{ih(\varphi)}\,\gamma_{-\psi,\beta}(d\varphi).\end{aligned}$$

❑

2.7 Remark. Remember that for $a, \lambda \in \mathbb{R}$ we have $N_{a,\lambda}(B) = (2\pi\lambda^2)^{-1/2}\int_B e^{-(x-a)^2/(2\lambda^2)}\,dx$ for $B \in \mathcal{B}(\mathbb{R})$. Thus

$$\frac{dN_{a,\lambda}}{dN_\lambda}(x) = e^{-\frac{a^2}{2\lambda^2}+\frac{ax}{\lambda^2}}.$$

//

2.8 Theorem (Kakutani)**.** *Let a, c be weights on I. If $\sum_{k\in I} a_k^2/c_k^2 < \infty$, then we have*

$$\frac{dN_{a,c}}{dN_c}(x) = \prod_{k\in I}\frac{d\gamma_{a_k,c_k}}{d\gamma_{c_k}}(x_k) = \prod_{k\in I} e^{-\frac{a_k^2}{2c_k^2}+\frac{a_kx_k}{c_k^2}}.$$

Proof. See [6, Theorem 2.7]. ❑

2.9 Corollary. *Let $\psi \in H^1(\mathbb{T}^d)$. Then*

$$\frac{d\gamma_{\psi,\beta}}{d\gamma_\beta}(\varphi) = e^{-\beta/2\|\psi\|^2_{H^1_m(\mathbb{T}^d)}+\beta(\psi,\varphi)_{H^1_m(\mathbb{T}^d)}}.$$

Proof. We can apply Theorem 2.8 with $I = \mathbb{Z}^d$, $a_k = \widehat{\psi}(k)$ and $c_k^2 = 1/\beta(m+4\pi^2|k|^2)^{-1}$. Since $\psi \in H^1(\mathbb{T}^d)$ we have $\sum_{k\in\mathbb{Z}^d} a_k^2/\lambda_k^2 = \beta\sum_{k\in\mathbb{Z}^d}(m+4\pi^2|k|^2)\widehat{\psi}(k)^2 = \beta\,\|\psi\|^2_{H^1_m(\mathbb{T}^d)} < \infty$. ❑

In the following we will do asymptotics away from the origin. For simplicity, we restrict to minima that are constant functions. This can be generalized to functions in the Cameron

Martin space of the measure γ. For more details, see [6].

2.10 Theorem (Local Asymptotics away from the origin). *Let $U \in C^2(H^s(\mathbb{T}^d);\mathbb{R})$ and assume there exists $\varphi_0 \in \mathbb{R}$ such that*

- $U(\varphi_0) = -m\varphi_0^2/2$,
- $DU(\varphi_0)\xi = -m\varphi_0 \int_{\mathbb{T}^d} \xi(x)\,dx$ *for all* $\xi \in H^s(\mathbb{T}^d)$,
- *there exists* $\rho < m$ *s.t.* $D^2U(\varphi_0)(\xi,\xi) \geq -\rho\,\|\xi\|^2_{H^s(\mathbb{T}^d)}$ *for all* $\xi \in H^s(\mathbb{T}^d)$.

Then there exists $\alpha > 0$ s.t.

$$\lim_{\beta\to\infty} \int_{B_\alpha(\varphi_0)} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi) = \int_{H^s(\mathbb{T}^d)} e^{-1/2 D^2U(\varphi_0)(\varphi,\varphi)}\,\gamma(d\varphi),$$

where $B_\alpha(\varphi_0) = \{\varphi \in H^s(\mathbb{T}^d) : \|\varphi - \varphi_0\|_{H^s(\mathbb{T}^d)} < \alpha\}$.

Proof. Let $\alpha > 0$. By Lemma 2.6 we have

$$\int_{B_\alpha(\varphi_0)} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi) = \int_{B_\alpha(0)} e^{-\beta U(\varphi+\varphi_0)}\,\gamma_{-\varphi_0,\beta}(d\varphi).$$

Since φ_0 is constant, we have for $\varphi \in H^s(\mathbb{T}^d)$ that $\|\varphi_0\|_{H^1_m(\mathbb{T}^d)} = \sqrt{m}\varphi_0$ and $(\varphi_0,\varphi)_{H^1_m(\mathbb{T}^d)} = m\varphi_0\widehat{\varphi}(0) = m\varphi_0\int_{\mathbb{T}^d}\varphi(x)\,dx$. Hence Corollary 2.9 yields

$$\int_{B_\alpha(0)} e^{-\beta U(\varphi+\varphi_0)}\,\gamma_{-\varphi_0,\beta}(d\varphi) = \int_{B_\alpha(0)} e^{-\beta U(\varphi+\varphi_0)}\,e^{-\beta m\varphi_0^2/2-\beta m\varphi_0\int_{\mathbb{T}^d}\varphi(x)\,dx}\,\gamma_\beta(d\varphi).$$

The function $\widetilde{U}$, defined by $\widetilde{U}(\varphi) := U(\varphi+\varphi_0) + m\varphi_0^2/2 + m\varphi_0\int_{\mathbb{T}^d}\varphi(x)\,dx$ satisfies

- $\widetilde{U}(0) = U(\varphi_0) + m\varphi_0^2/2 = 0$,
- $D\widetilde{U}(0)\xi = DU(\varphi_0)\xi + m\varphi_0\int_{\mathbb{T}^d}\xi(x)\,dx = 0$ for all $\xi \in H^s(\mathbb{T}^d)$,
- $D^2\widetilde{U}(0)(\xi,\xi) = D^2U(\varphi_0)(\xi,\xi) \geq -\rho\,\|\xi\|^2_{H^s(\mathbb{T}^d)}$ for all $\xi \in H^s(\mathbb{T}^d)$.

Therefore by Theorem 2.4 we get

$$\lim_{\beta\to\infty}\int_{B_\alpha(0)} e^{-\beta\widetilde{U}(\varphi)}\,\gamma_\beta(d\varphi) = \int_{H^s(\mathbb{T}^d)} e^{-1/2D^2\widetilde{U}(0)(\varphi,\varphi)}\,\gamma(d\varphi) = \int_{H^s(\mathbb{T}^d)} e^{-1/2D^2U(\varphi_0)(\varphi,\varphi)}\,\gamma(d\varphi)$$

for some $\alpha > 0$ sufficiently small. ❑

2.11 Theorem (Global Asymptotics, multiple global minima). *Let $U \in C^2(H^s(\mathbb{T}^d);\mathbb{R})$ and assume there exists $\varphi_i \in \mathbb{R}$, $i = 1,\dots,n$ such that*

- $U(\varphi_i) = -m\varphi_i^2/2$ *for* $i = 1,\dots,n$,
- $DU(\varphi_i)\xi = -m\varphi_i\int_{\mathbb{T}^d}\xi(x)\,dx$ *for all* $\xi \in H^s(\mathbb{T}^d)$ *and* $i = 1,\dots,n$,
- *there exists* $\rho < m$ *s.t.* $D^2U(\varphi_i)(\xi,\xi) \geq -\rho\,\|\xi\|^2_{H^s(\mathbb{T}^d)}$ *for all* $\xi \in H^s(\mathbb{T}^d)$ *and* $i = 1,\dots,n$,

- *there exists* $\eta < m/2$ *s.t.* $U(\varphi) \geq -\eta \|\varphi\|^2_{H^s(\mathbb{T}^d)}$ *for* $\varphi \in (\bigcup_{i=1}^n B_{\alpha_i}(\varphi_i))^c$ *and* α_i *sufficiently small.*

Then

$$\lim_{\beta\to\infty} \int_{H^s(\mathbb{T}^d)} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) = \sum_{i=1}^n \int_{H^s(\mathbb{T}^d)} e^{-1/2 D^2 U(\varphi_i)(\varphi,\varphi)}\, \gamma(d\varphi).$$

Proof. Let $\alpha_1, \dots, \alpha_n > 0$ and define $B := \bigcup_{i=1}^n B_{\alpha_i}(\varphi_i)$. Then

$$\int_{H^s(\mathbb{T}^d)} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) = \underbrace{\sum_{i=1}^n \int_{B_{\alpha_i}(\varphi_i)} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi)}_{=:I_1(\beta)} + \underbrace{\int_{B^c} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi)}_{=:I_2(\beta)}.$$

From Theorem 2.10 we already know that $I_1(\beta)$ gives the right limit for α_i sufficiently small, hence we have to show that we can neglect $I_2(\beta)$.

Using Lemma 2.2 and our assumptions, we get

$$I_2(\beta) \leq \int_{B^c} e^{\beta\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma_\beta(d\varphi) = \int_{\sqrt{\beta}(B^c)} e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi).$$

We define a probability measure ν by

$$\nu(A) = Z^{-1} \int_A e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi),$$

where $Z = \int_{H^s(\mathbb{T}^d)} e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi)$. Note that $Z < \infty$ by Remark 2.1 since $\eta < m/2$. Hence we have by Markov's Inequality for $\epsilon > 0$

$$\begin{aligned} I_2(\beta) &\leq Z\nu(\sqrt{\beta}(B^c)) \leq Z\nu([\|\varphi - \varphi_1\|_{H^s(\mathbb{T}^d)} \geq \sqrt{\beta}\alpha_1]) = Z\nu([e^{\epsilon\|\varphi-\varphi_1\|^2_{H^s(\mathbb{T}^d)}} \geq e^{\epsilon\beta\alpha_1^2}]) \\ &\leq Ze^{-\epsilon\beta\alpha_1^2} \int_{H^s(\mathbb{T}^d)} e^{\epsilon\|\varphi-\varphi_1\|^2_{H^s(\mathbb{T}^d)}}\, \nu(d\varphi) \\ &\leq Ze^{-\epsilon\beta\alpha_1^2} \int_{H^s(\mathbb{T}^d)} e^{2\epsilon(\|\varphi\|^2_{H^s(\mathbb{T}^d)} + \|\varphi_1\|^2_{H^s(\mathbb{T}^d)})}\, \nu(d\varphi) \\ &= e^{-\epsilon\beta\alpha_1^2} e^{2\epsilon\|\varphi_1\|^2_{H^s(\mathbb{T}^d)}} \int_{H^s(\mathbb{T}^d)} e^{2\epsilon\|\varphi\|^2_{H^s(\mathbb{T}^d)}} e^{\eta\|\varphi\|^2_{H^s(\mathbb{T}^d)}}\, \gamma(d\varphi). \end{aligned}$$

Choosing $\epsilon > 0$ such that $2\epsilon + \eta < m/2$, the last integral is finite and we have that $I_2(\beta)$ is exponentially decreasing in β. ❏

3 Perturbation of the Gaussian Free Field on $L^2(\mathbb{T})$

In this section we consider the Gaussian Free Field with mass $m > 0$ on $L^2(\mathbb{T})$. Since $0 < 1 - 1/2$, according to Theorem 1.29 there exists a unique centered Gaussian measure γ_β on $L^2(\mathbb{T})$ satisfying

$$\widehat{\gamma_\beta}(h) = e^{-\frac{1}{2\beta}\|h\|^2_{H_m^{-1}(\mathbb{T})}}$$

for every $h \in L^2(\mathbb{T})$.

Given $n \in \mathbb{N}$ and $a_0, \dots, a_{2n} \in \mathbb{R}$ with $a_{2n} > 0$, the aim is to give a meaning to

$$U(\varphi) := \sum_{k=0}^{2n} a_k \int_{\mathbb{T}} \varphi^k(x)\, dx$$

for γ_β-a.e. $\varphi \in L^2(\mathbb{T})$ and define the measure

$$d\nu_\beta := Z_\beta^{-1} e^{-U} d\gamma_\beta,$$

where $Z_\beta := \int_{L^2(\mathbb{T})} e^{-U(\varphi)}\, \gamma_\beta(d\varphi)$ is a normalization constant. The measure ν_β is called *Gibbs measure*.

3.1 Lemma. *Let $d \in \mathbb{N}$, $p \in [2, +\infty]$ and $(\lambda_k)_{k\in\mathbb{Z}^d} \in \ell^q(\mathbb{Z}^d)$, where $1/p + 1/q = 1$. Then*

$$\Big\| \sum_{k\in\mathbb{Z}^d} \lambda_k\, e_k \Big\|_{L^p(\mathbb{T}^d)} \le \sqrt{2}^{\,1-2/p}\, \|(\lambda_k)_{k\in\mathbb{Z}^d}\|_{\ell^q(\mathbb{Z}^d)}\,.$$

Proof. Consider the mapping $T : \lambda = (\lambda_k)_{k\in\mathbb{Z}^d} \mapsto \sum_{k\in\mathbb{Z}^d} \lambda_k\, e_k$. Then T maps from $\ell^1(\mathbb{Z}^d)$ into $L^\infty(\mathbb{T}^d)$ with operatornorm $\|T\|_{\ell^1(\mathbb{Z}^d)\to L^\infty(\mathbb{T}^d)} = \sqrt{2}$, since

$$\|T(\lambda)\|_{L^\infty(\mathbb{T}^d)} = \Big\| \sum_{k\in\mathbb{Z}^d} \lambda_k\, e_k \Big\|_{L^\infty(\mathbb{T}^d)} \le \sum_{k\in\mathbb{Z}^d} \|\lambda_k\, e_k\|_{L^\infty(\mathbb{T}^d)} \le \sqrt{2} \sum_{k\in\mathbb{Z}^d} |\lambda_k| = \sqrt{2}\, \|\lambda\|_{\ell^1(\mathbb{Z}^d)}\,.$$

In addition, we have equality for some λ.

Furthermore, T maps from $\ell^2(\mathbb{Z}^d)$ into $L^2(\mathbb{T}^d)$ with operatornorm $\|T\|_{\ell^2(\mathbb{Z}^d)\to L^2(\mathbb{T}^d)} = 1$, since

$$\begin{aligned}
\|T(\lambda)\|_{L^2(\mathbb{T}^d)} &= \Big\| \sum_{k\in\mathbb{Z}^d} \lambda_k\, e_k \Big\|_{L^2(\mathbb{T}^d)} = \Big(\int_{\mathbb{T}^d} \Big| \sum_{k\in\mathbb{Z}^d} \lambda_k\, e_k(x) \Big|^2 dx \Big)^{1/2} \\
&= \Big(\int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \lambda_k\, \lambda_j\, e_k(x)\, e_j(x)\, dx \Big)^{1/2} \\
&= \Big(\sum_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \lambda_k\, \lambda_j \int_{\mathbb{T}^2} e_k(x)\, e_j(x)\, dx \Big)^{1/2} \\
&= \Big(\sum_{k\in\mathbb{Z}^d} \lambda_k^2 \Big)^{1/2} = \|\lambda\|_{\ell^2(\mathbb{Z}^d)}\,.
\end{aligned}$$

The Riesz-Thorin interpolation Theorem, see [15, Theorem IX.17], then states that T maps from $\ell^{p_t}(\mathbb{Z}^d)$ into $L^{q_t}(\mathbb{T}^d)$ with operatornorm $\|T\|_{\ell^{p_t}(\mathbb{Z}^d)\to L^{q_t}(\mathbb{T}^d)} = \sqrt{2}^{\,1-t}$ for every $t \in [0,1]$, where $1/p_t = 1 - t/2$ and $1/q_t = t/2$. Choosing $t = 2/p \in [0,1]$ yields $p_t = p/(p-1)$ and $q_t = p$ and hence

$$\|T(\lambda)\|_{L^p(\mathbb{T}^d)} \le \sqrt{2}^{\,1-2/p}\, \|\lambda\|_{\ell^{p/(p-1)}(\mathbb{Z}^d)}$$

for $\lambda \in \ell^{p/(p-1)}(\mathbb{Z}^d)$, which is the desired estimate. ❑

3.2 Theorem (Sobolev embedding)**.** *Let $p \geq 2$ and $s/d > 1/2 - 1/p$. Then there exists $C > 0$ such that*

$$\|f\|_{L^p(\mathbb{T}^d)} \leq C\,\|f\|_{H^s(\mathbb{T}^d)}$$

for every $f \in H^s(\mathbb{T}^d)$.

Proof. Let $q \in [1,2]$ with $1/p + 1/q = 1$ and $f \in H^s(\mathbb{T})$.

First step: We show that there exists a constant $C > 0$ such that $\|\widehat{f}\|_{\ell^q(\mathbb{Z})} \leq C\,\|f\|_{H^s(\mathbb{T})}$. Using Hölder's inequality with $r = 2/q \geq 1$ and $r' = 2/(2-q) \geq 2$ yields

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^q &= \sum_{k\in\mathbb{Z}^d} (1+4\pi^2|k|^2)^{sq/2}(1+4\pi^2|k|^2)^{-sq/2}|\widehat{f}(k)|^q \\ &\leq \left(\sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^s|\widehat{f}(k)|^2\right)^{q/2}\left(\sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-sq/(2-q)}\right)^{(2-q)/q}.\end{aligned}$$

Thus it remains to show that $\sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-sq/(2-q)} < \infty$. By Lemma 1.28 this is equivalent to $-sq/(2-q) < -d/2$, which we can rewrite to $s/d > (2-q)/(2q)$. Since $q = p/(p-1)$, we have

$$\frac{2-q}{2q} = \frac{2-p/(p-1)}{2p/(p-1)} = \frac{2(p-1)-p}{2p} = \frac{p-2}{2p} = \frac{1}{2} - \frac{1}{p}$$

and thus $s/d > (2-q)/(2q)$ by assumption.

Second step: By step one we have $\widehat{f} \in \ell^q(\mathbb{Z}^d)$, hence we can apply Lemma 3.1 with $\lambda_k = \widehat{f}(k)$ to see that $\|f\|_{L^p(\mathbb{T}^d)} \leq \sqrt{2}\|\widehat{f}\|_{\ell^q(\mathbb{Z}^d)}$, which yields together with step one the desired estimate. ❑

3.3 Lemma. *Let $p \geq 1$. Then for γ_β-a.e. $\varphi \in L^2(\mathbb{T})$ we have $\varphi \in L^p(\mathbb{T})$.*

Proof. For $1 \leq p \leq 2$ the statement holds true by Hölder's inequality. If $p > 2$, choose $1/2 > s' > 1/2 - 1/p > 0$. By Theorem 1.29, γ_β concentrates on $H^{s'}(\mathbb{T})$. Hence γ_β-a.e. $\varphi \in L^2(\mathbb{T})$ even satisfies $\varphi \in H^{s'}(\mathbb{T})$ and thus $\varphi \in L^p(\mathbb{T})$ by Theorem 3.2. ❑

3.4 Definition. Let $k \in \mathbb{N}_0$. Then for γ_β-a.e. $\varphi \in L^2(\mathbb{T})$ define

$$U_k(\varphi) := \int_{\mathbb{T}} \varphi(x)^k\,dx.$$

3.5 Lemma. *Define $U := \sum_{k=0}^{2n} a_k U_k$, where $a_i \in \mathbb{R}$ for $i = 0,\dots,2n$ and $a_{2n} > 0$. Then $e^{-U} \in L^p(L^2(\mathbb{T}),\gamma_\beta)$ for $p \geq 1$.*

Proof. We have

$$U(\varphi) = \int_{\mathbb{T}} Q(\varphi(x))\,dx$$

for γ_β-a.e. $\varphi \in L^2(\mathbb{T})$ and a polynomial Q with even degree $2n$ and positive leading coefficient a_{2n}. Hence U is bounded from below and thus $e^{-U} \in L^p(L^2(\mathbb{T}), \gamma_\beta)$. ❑

3.6 Remark. Alternatively, instead of using the Sobolev embedding, Theorem 3.2, one can follow the same strategy as in Section 5, see also [7]. Note that there is no renormalization necessary in the case of $d = 1$, since $T_{x,N}$ is already a Cauchy sequence in that case, compare Lemma 5.1. //

4 Laplace Asymptotics on $L^2(\mathbb{T})$

The goal is to study the asymptotic behavior of

$$\int_{L^2(\mathbb{T})} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi)$$

in the limit $\beta \to \infty$ for potentials U as defined in Lemma 3.5. To that end, we will apply the results of Section 2.

Let $n \in \mathbb{N}$, $n \geq 2$ and $s \in \mathbb{R}$ with $1/2 > s > 1/2 - 1/n$. Consider

$$U_n(\varphi) = \int_{\mathbb{T}} \varphi(x)^n\,dx$$

from Definition 3.4.

4.1 Lemma. *For $n \in \mathbb{N}$, $n \geq 2$ and $s \in \mathbb{R}$ with $1/2 > s > 1/2 - 1/n$ we have $U_n \in C^2(H^s(\mathbb{T});\mathbb{R})$ with*

$$DU_n(\varphi)\xi = n \int_{\mathbb{T}} \varphi(x)^{n-1}\xi(x)\,dx$$

and

$$D^2U_n(\varphi)(\xi,\eta) = n(n-1)\int_{\mathbb{T}} \varphi(x)^{n-2}\xi(x)\eta(x)\,dx.$$

Proof. **Step 1:** We show that $U_n \in C^1(H^s(\mathbb{T});\mathbb{R})$. To that end, define

$$l_\varphi(\xi) := n \int_{\mathbb{T}} \varphi(x)^{n-1}\xi(x)\,dx$$

for $\varphi, \xi \in H^s(\mathbb{T})$. Then l_φ is linear and bounded, since by Hölder's inequality and the Sobolev embedding, Theorem 3.2, we have

$$\begin{aligned}
\|l_\varphi\|_{H^s(\mathbb{T})\to\mathbb{R}} &= \sup_{\xi\in H^s(\mathbb{T})\setminus\{0\}} \frac{|l_\varphi(\xi)|}{\|\xi\|_{H^s(\mathbb{T})}} \\
&= \sup_{\xi\in H^s(\mathbb{T})\setminus\{0\}} \frac{|n\int_{\mathbb{T}} \varphi(x)^{n-1}\xi(x)\,dx|}{\|\xi\|_{H^s(\mathbb{T})}} \\
&\leq \sup_{\xi\in H^s(\mathbb{T})\setminus\{0\}} \frac{n(\int_{\mathbb{T}} |\varphi(x)|^n\,dx)^{(n-1)/n}(\int_{\mathbb{T}} |\xi(x)|^n\,dx)^{1/n}}{\|\xi\|_{H^s(\mathbb{T})}} \\
&= \sup_{\xi\in H^s(\mathbb{T})\setminus\{0\}} \frac{n\,\|\varphi\|_{L^n(\mathbb{T})}^{n-1}\,\|\xi\|_{L^n(\mathbb{T})}}{\|\xi\|_{H^s(\mathbb{T})}} \\
&\leq \sup_{\xi\in H^s(\mathbb{T})\setminus\{0\}} \frac{nC_n^n\,\|\varphi\|_{H^s(\mathbb{T})}^{n-1}\,\|\xi\|_{H^s(\mathbb{T})}}{\|\xi\|_{H^s(\mathbb{T})}} \\
&= nC_n^n\,\|\varphi\|_{H^s(\mathbb{T})}^{n-1}.
\end{aligned}$$

In addition, using Hölder's inequality and the Sobolev embedding, we have

$$\begin{aligned}
|U_n(\varphi+\xi)-U_n(\varphi)-l_\varphi(\xi)| &= \left|\int_{\mathbb{T}}\sum_{k=0}^{n}\binom{n}{k}\varphi(x)^{n-k}\xi(x)^k\,dx-\int_{\mathbb{T}}\varphi(x)^n\,dx-n\int_{\mathbb{T}}\varphi(x)^{n-1}\xi(x)\,dx\right| \\
&= \left|\int_{\mathbb{T}}\sum_{k=2}^{n}\binom{n}{k}\varphi(x)^{n-k}\xi(x)^k\,dx\right| \\
&\le \sum_{k=2}^{n}\binom{n}{k}\int_{\mathbb{T}}|\varphi(x)|^{n-k}|\xi(x)|^k\,dx \\
&\le \sum_{k=2}^{n}\binom{n}{k}\left(\int_{\mathbb{T}}|\varphi(x)|^n\,dx\right)^{(n-k)/n}\left(\int_{\mathbb{T}}|\xi(x)|^n\,dx\right)^{k/n} \\
&= \sum_{k=2}^{n}\binom{n}{k}\|\varphi\|_{L^n(\mathbb{T})}^{n-k}\|\xi\|_{L^n(\mathbb{T})}^{k} \\
&\le \sum_{k=2}^{n}\binom{n}{k}C_n^n\|\varphi\|_{H^s(\mathbb{T})}^{n-k}\|\xi\|_{H^s(\mathbb{T})}^{k}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{\xi\to 0}\frac{|U_n(\varphi+\xi)-U_n(\varphi)-l_\varphi(\xi)|}{\|\xi\|_{H^s(\mathbb{T})}} &\le \lim_{\xi\to 0}\frac{\sum_{k=2}^{n}\binom{n}{k}C_n^n\|\varphi\|_{H^s(\mathbb{T})}^{n-k}\|\xi\|_{H^s(\mathbb{T})}^{k}}{\|\xi\|_{H^s(\mathbb{T})}} \\
&= \lim_{\xi\to 0}\sum_{k=2}^{n}\binom{n}{k}C_n^n\|\varphi\|_{H^s(\mathbb{T})}^{n-k}\|\xi\|_{H^s(\mathbb{T})}^{k-1} \\
&= 0.
\end{aligned}$$

Hence $U_n\in C^1(H^s(\mathbb{T});\mathbb{R})$ with $DU_n(\varphi)\xi=l_\varphi(\xi)$.

Step 2: We show that $U_n\in C^2(H^s(\mathbb{T});\mathbb{R})$. To that end, define

$$k_\varphi(\xi,\eta):=n(n-1)\int_{\mathbb{T}}\varphi(x)^{n-2}\xi(x)\eta(x)\,dx$$

for $\varphi,\xi,\eta\in H^s(\mathbb{T})$. Then $k_\varphi(\xi,.)$ is linear and bounded, since by Hölder's inequality and the Sobolev embedding, Theorem 3.2, we have

$$\begin{aligned}
\|k_\varphi(\xi,.)\|_{H^s(\mathbb{T})\to\mathbb{R}} &= \sup_{\eta\in H^s(\mathbb{T})\setminus\{0\}}\frac{|k_\varphi(\xi,\eta)|}{\|\eta\|_{H^s(\mathbb{T})}} \\
&= \sup_{\eta\in H^s(\mathbb{T})\setminus\{0\}}\frac{|n(n-1)\int_{\mathbb{T}}\varphi(x)^{n-2}\xi(x)\eta(x)\,dx|}{\|\eta\|_{H^s(\mathbb{T})}} \\
&\le \sup_{\eta\in H^s(\mathbb{T})\setminus\{0\}}\frac{n(n-1)\|\varphi\|_{L^n(\mathbb{T})}^{n-2}\|\xi\|_{L^n(\mathbb{T})}\|\eta\|_{L^n(\mathbb{T})}}{\|\eta\|_{H^s(\mathbb{T})}} \\
&\le \sup_{\eta\in H^s(\mathbb{T})\setminus\{0\}}\frac{n(n-1)C_n^n\|\varphi\|_{H^s(\mathbb{T})}^{n-2}\|\xi\|_{H^s(\mathbb{T})}\|\eta\|_{H^s(\mathbb{T})}}{\|\eta\|_{H^s(\mathbb{T})}} \\
&= n(n-1)C_n^n\|\varphi\|_{H^s(\mathbb{T})}^{n-2}\|\xi\|_{H^s(\mathbb{T})}.
\end{aligned}$$

Furthermore $k_\varphi \in L(H^s(\mathbb{T}), L(H^s(\mathbb{T});\mathbb{R}))$, since

$$\begin{aligned}
\|k_\varphi\|_{H^s(\mathbb{T})\to L(H^s(\mathbb{T});\mathbb{R})} &= \sup_{\xi\in H^s(\mathbb{T})\setminus\{0\}} \frac{\|k_\varphi(\xi,.)\|_{H^s(\mathbb{T})\to\mathbb{R}}}{\|\xi\|_{H^s(\mathbb{T})}} \\
&\leq \sup_{\xi\in H^s(\mathbb{T})\setminus\{0\}} \frac{n(n-1)C_n^n \|\varphi\|^{n-2}_{H^s(\mathbb{T})} \|\xi\|_{H^s(\mathbb{T})}}{\|\xi\|_{H^s(\mathbb{T})}} \\
&= n(n-1)C_n^n \|\varphi\|^{n-2}_{H^s(\mathbb{T})}.
\end{aligned}$$

Using the convention, that the empty sum equals zero, we have again by using Hölder and Sobolev, that

$$\begin{aligned}
|DU_n(\varphi+\xi)\eta - DU_n(\varphi)\eta - k_\varphi(\xi,\eta)| &= \Bigg| n\int_{\mathbb{T}} \sum_{k=0}^{n-1} \binom{n-1}{k} \varphi(x)^{n-1-k}\xi(x)^k\eta(x)\,dx \\
&\quad - n\int_{\mathbb{T}} \varphi(x)^{n-1}\eta(x)\,dx - n(n-1)\int_{\mathbb{T}} \varphi(x)^{n-2}\xi(x)\eta(x)\,dx \Bigg| \\
&= \left| n\int_{\mathbb{T}} \sum_{k=2}^{n-1} \binom{n-1}{k} \varphi(x)^{n-1-k}\xi(x)^k\eta(x)\,dx \right| \\
&\leq \sum_{k=2}^{n-1} \binom{n-1}{k} \|\varphi\|^{n-1-k}_{L^n(\mathbb{T})} \|\xi\|^k_{L^n(\mathbb{T})} \|\eta\|_{L^n(\mathbb{T})} \\
&\leq \sum_{k=2}^{n-1} \binom{n-1}{k} C_n^n \|\varphi\|^{n-1-k}_{H^s(\mathbb{T})} \|\xi\|^k_{H^s(\mathbb{T})} \|\eta\|_{H^s(\mathbb{T})}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{\xi\to 0} \frac{|DU_n(\varphi+\xi)\eta - DU_n(\varphi)\eta - k_\varphi(\xi,\eta)|}{\|\xi\|_{H^s(\mathbb{T})}} &\leq \lim_{\xi\to 0} \frac{\sum_{k=2}^{n-1}\binom{n-1}{k} C_n^n \|\varphi\|^{n-1-k}_{H^s(\mathbb{T})} \|\xi\|^k_{H^s(\mathbb{T})} \|\eta\|_{H^s(\mathbb{T})}}{\|\xi\|_{H^s(\mathbb{T})}} \\
&= \lim_{\xi\to 0} \sum_{k=2}^{n-1} \binom{n-1}{k} C_n^n \|\varphi\|^{n-1-k}_{H^s(\mathbb{T})} \|\xi\|^{k-1}_{H^s(\mathbb{T})} \|\eta\|_{H^s(\mathbb{T})} \\
&= 0.
\end{aligned}$$

Hence $U_n \in C^2(H^s(\mathbb{T});\mathbb{R})$ with $D^2U_n(\varphi)(\xi,\eta) = k_\varphi(\xi,\eta)$. ❑

4.2 Theorem. *Let $n \in \mathbb{N}$, $n \geq 2$ and $a_k \in \mathbb{R}$, $k = 2,\dots,2n$ with $a_{2n} > 0$ and $a_2 > -m/2$. Furthermore assume that there exists $\eta < m/2$, such that $\sum_{k=2}^{2n} a_k z^k \geq -\eta z^2$ for $z \in \mathbb{R}$. For γ_β-a.e. $\varphi \in L^2(\mathbb{T})$ consider*

$$U(\varphi) = \sum_{k=2}^{2n} a_k U_k(\varphi) = \sum_{k=2}^{2n} a_k \int_{\mathbb{T}} \varphi(x)^k\,dx$$

from Lemma 3.5. Then

$$\begin{aligned}
\lim_{\beta\to\infty} \int_{L^2(\mathbb{T})} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi) &= \int_{L^2(\mathbb{T})} e^{-a_2\|\varphi\|^2_{L^2(\mathbb{T})}}\,\gamma(d\varphi). \\
&= \prod_{k\in\mathbb{Z}} (1 + 2a_2(m+4\pi^2|k|^2)^{-1})^{-1/2}.
\end{aligned}$$

Proof. Let $s \in \mathbb{R}$ with $1/2 > s > 1/2 - 1/(2n)$. Then γ concentrates on $H^s(\mathbb{T}) \subseteq L^2(\mathbb{T})$, hence

$$\int_{L^2(\mathbb{T})} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) = \int_{H^s(\mathbb{T})} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi).$$

We check that the assumptions of Theorem 2.5 are satisfied. We have $U(0) = 0$ and by Lemma 4.1 we have $U \in C^2(H^s(\mathbb{T}); \mathbb{R})$ with

- $DU(0)\xi = 0$ for $\xi \in H^s(\mathbb{T})$ and
- $D^2U(0)(\xi, \xi) = 2a_2 \|\xi\|^2_{L^2(\mathbb{T})}$ for $\xi \in H^s(\mathbb{T})$. Hence if $a_2 \geq 0$ we have $D^2U(0)(\xi, \xi) \geq 0$ (choose $\rho = 0$) and for $a_2 \in (-m/2, 0)$ we have $D^2U(0)(\xi, \xi) \geq -2|a_2|\, \|\xi\|^2_{H^s(\mathbb{T})}$ (choose $\rho = 2|a_2| < m$).

By assumption, we also have $U(\varphi) \geq -\eta \|\varphi\|^2_{L^2(\mathbb{T})}$ for some $\eta < m/2$. Thus if $\eta \leq 0$, we have $U(\varphi) \geq 0$ and for $\eta \in (0, m/2)$ we have $U(\varphi) \geq -\eta \|\varphi\|^2_{H^s(\mathbb{T})}$.

We therefore can apply Theorem 2.5 to get

$$\lim_{\beta\to\infty} \int_{H^s(\mathbb{T})} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) = \int_{H^s(\mathbb{T})} e^{-a_2\|\varphi\|^2_{L^2(\mathbb{T})}}\, \gamma(d\varphi) = \int_{L^2(\mathbb{T})} e^{-a_2\|\varphi\|^2_{L^2(\mathbb{T})}}\, \gamma(d\varphi).$$

By Lemma 1.14 we have

$$\int_{L^2(\mathbb{T})} e^{-a_2\|\varphi\|^2_{L^2(\mathbb{T})}}\, \gamma(d\varphi) = \prod_{k\in\mathbb{Z}} (1 + 2a_2(m + 4\pi^2|k|^2)^{-1})^{-1/2}.$$

❏

4.3 Corollary. *Let $a, b, c \in \mathbb{R}$ with $a > 0$, $c > -m/2$ and $b^2 < 4a(c + m/2)$. For γ_β-a.e. $\varphi \in L^2(\mathbb{T})$ we define*

$$U(\varphi) := a \int_{\mathbb{T}} \varphi(x)^4\, dx + b \int_{\mathbb{T}} \varphi(x)^3\, dx + c \int_{\mathbb{T}} \varphi(x)^2\, dx.$$

Then

$$\begin{aligned}\lim_{\beta\to\infty} \int_{L^2(\mathbb{T})} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) &= \int_{L^2(\mathbb{T})} e^{-c\|\varphi\|^2_{L^2(\mathbb{T})}}\, \gamma(d\varphi).\\ &= \prod_{k\in\mathbb{Z}} (1 + 2c(m + 4\pi^2|k|^2)^{-1})^{-1/2}.\end{aligned}$$

Proof. To apply Theorem 4.2 it remains to show that there exists $\eta < m/2$ such that $az^4 + bz^3 + cz^2 \geq -\eta z^2$ for $z \in \mathbb{R}$.

From $b^2 < 4a(c + m/2)$ we see that $az^2 + bz + (c + m/2) > 0$ for $z \in \mathbb{R}$ and hence $az^2 + bz + c > -m/2$. Since $a > 0$, the parabola $az^2 + bz + c$ takes its minimum value and we even have $az^2 + bz + c \geq -\eta$ for some $\eta < m/2$. But this implies $az^4 + bz^3 + cz^2 \geq -\eta z^2$. ❏

4.4 Remark. For $m = 1$ and $c > -1/2$ we have

$$\int_{L^2(\mathbb{T})} e^{-c\|\varphi\|^2_{L^2(\mathbb{T})}}\, \gamma(d\varphi) = \prod_{k\in\mathbb{Z}} (1 + 2c(1 + 4\pi^2|k|^2)^{-1})^{-1/2} = \frac{\sinh(1/2)}{\sinh(\sqrt{1 + 2c}/2)}.$$

//

4.5 Example. We can apply Corollary 4.3 to $U(\varphi) = \int_{\mathbb{T}} p(\varphi(x))\,dx$ with

$$p(z) = z^2(z-2)^2 = z^4 - 4z^3 + 4z^2,$$

see Figure 1. The limit for $m = 1$ is $\frac{\sinh(1/2)}{\sinh(3/2)} = \frac{e}{1+e+e^2}$.

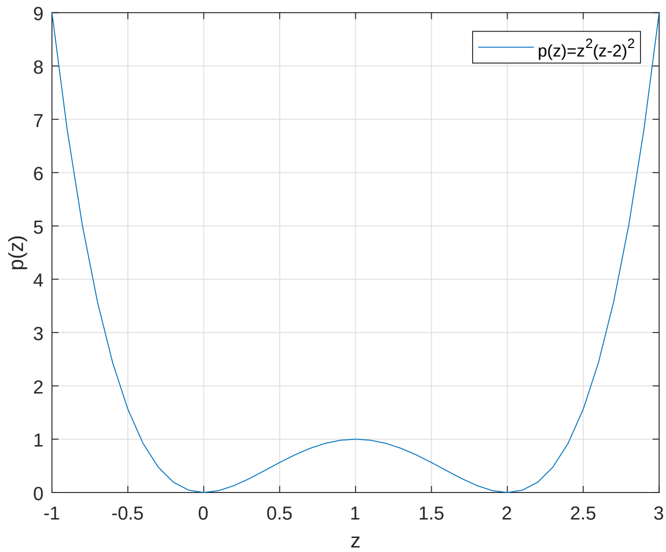

Figure 1

//

4.6 Example. We cannot apply Corollary 4.3 to $U(\varphi) = \int_{\mathbb{T}} p(\varphi(x))\,dx$ with

$$p(z) = z^2(z-2)^2 - \frac{m}{2}z^2 = z^4 - 4z^3 + (4 - \frac{m}{2})z^2,$$

since $b^2 = 4a(c + m/2)$. //

4.7 Example. Again consider U from Example 4.6, i.e. $U(\varphi) = \int_{\mathbb{T}} p(\varphi(x))\,dx$ with

$$p(z) = z^2(z-2)^2 - \frac{m}{2}z^2 = z^4 - 4z^3 + (4 - \frac{m}{2})z^2.$$

Then $U \in C^2(H^s(\mathbb{T}); \mathbb{R})$ for $1/2 > s > 1/2 - 1/4$ and

$$DU(\varphi)\xi = 4\int_{\mathbb{T}} \varphi(x)^3\xi(x)\,dx - 12\int_{\mathbb{T}} \varphi(x)^2\xi(x)\,dx + (8-m)\int_{\mathbb{T}} \varphi(x)\xi(x)\,dx,$$

$$D^2U(\varphi)(\xi,\eta) = 12\int_{\mathbb{T}} \varphi(x)^2\xi(x)\eta(x)\,dx - 24\int_{\mathbb{T}} \varphi(x)\xi(x)\eta(x)\,dx + (8-m)\int_{\mathbb{T}} \xi(x)\eta(x)\,dx.$$

Hence for $\varphi_1 = 0$, $\varphi_2 = 2$ we have

- $U(\varphi_i) = -m\varphi_i^2/2$,

- $DU(\varphi_i)\xi = -m\varphi_i \int_{\mathbb{T}} \xi(x)\, dx$ for $\xi \in H^s(\mathbb{T})$,
- $D^2U(\varphi_i)(\xi,\xi) = (8-m)\, \|\xi\|^2_{L^2(\mathbb{T})} \geq 0$ (choose $\rho = 0$),
- One has to show that for $\varphi \in H^s(\mathbb{T})$ with $\|\varphi\|_{H^s(\mathbb{T})} \geq \alpha_1$ and $\|\varphi - 2\|_{H^s(\mathbb{T})} \geq \alpha_2$ we have $U(\varphi) \geq -\eta\, \|\varphi\|^2_{H^s(\mathbb{T})}$ for some $\eta < m/2$. It is enough to show that $U(\varphi) \geq -\eta\, \|\varphi\|^2_{L^2(\mathbb{T})}$, since for $\eta \leq 0$ we have $U(\varphi) \geq 0$ and for $\eta \in (0, m/2)$ we have $U(\varphi) \geq -\eta\, \|\varphi\|^2_{H^s(\mathbb{T})}$.

Then one can apply Lemma 2.11 to see

$$\begin{aligned}
\lim_{\beta\to\infty} \int_{L^2(\mathbb{T})} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) &= \lim_{\beta\to\infty} \int_{H^s(\mathbb{T})} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) \\
&= \int_{H^s(\mathbb{T})} e^{-1/2 D^2U(0)(\varphi,\varphi)}\, \gamma(d\varphi) + \int_{H^s(\mathbb{T})} e^{-1/2 D^2U(2)(\varphi,\varphi)}\, \gamma(d\varphi) \\
&= 2 \int_{H^s(\mathbb{T})} e^{-(8-m)/2 \|\varphi\|^2_{L^2(\mathbb{T})}}\, \gamma(d\varphi) \\
&= 2 \int_{L^2(\mathbb{T})} e^{-(8-m)/2 \|\varphi\|^2_{L^2(\mathbb{T})}}\, \gamma(d\varphi) \\
&= 2 \prod_{k\in\mathbb{Z}} (1 + (8-m)(m + 4\pi^2 |k|^2)^{-1})^{-1/2}.
\end{aligned}$$

For $m = 1$ we have

$$\lim_{\beta\to\infty} \int_{L^2(\mathbb{T})} e^{-\beta U(\varphi)}\, \gamma_\beta(d\varphi) = \frac{2\sinh(1/2)}{\sinh(\sqrt{2})}.$$

//

5 Perturbation of the Gaussian Free Field on $H^{-\epsilon}(\mathbb{T}^2)$

Next we consider $d = 2$ and $\sigma = 1$. However, since $\sigma - d/2 = 0$, the GFF with mass $m > 0$ does not exist on $L^2(\mathbb{T}^2)$ anymore. Hence we have to enlarge the Hilbert space to get a Gaussian measure.

Fix $\epsilon > 0$. Then $s := -\epsilon$ satisfies $s < \sigma - d/2$, hence according to Theorem 1.29 there exists a unique centered Gaussian measure γ_β on $H^{-\epsilon}(\mathbb{T}^2)$ s.t.

$$\widehat{\gamma_\beta}(h) = e^{-\frac{1}{2\beta}\|h\|^2_{H_m^{-1}(\mathbb{T}^2)}}$$

for every $h \in H^{-\epsilon}(\mathbb{T}^2)$.

Given $n \in \mathbb{N}$ and $a_0, \ldots, a_{2n} \in \mathbb{R}$ with $a_{2n} > 0$, the aim is – analogue to Section 3 – to give a meaning to

$$U(\varphi) := \sum_{k=0}^{2n} a_k \int_{\mathbb{T}} \varphi^k(x)\, dx$$

for $\varphi \in H^{-\epsilon}(\mathbb{T}^2)$ and define the measure

$$d\nu_\beta := Z_\beta^{-1} e^{-U} d\gamma_\beta,$$

where Z_β is a normalization constant.

For $N \in \mathbb{N}$ and $x \in \mathbb{T}^2$ define a functional $T_{x,N} : H^{-\epsilon}(\mathbb{T}^2) \to \mathbb{R}$ by

$$T_{x,N}(\varphi) := \sum_{|k| \leq N} \widehat{\varphi}(k) e_k(x).$$

Note that the summation is over $k \in \mathbb{Z}^2$ and $|k| = \sqrt{k_1^2 + k_2^2}$. For the sake of simplicity, we shall also write $\varphi_N(x)$ instead of $T_{x,N}(\varphi)$.

5.1 Lemma.

(i) Let $N \in \mathbb{N}$, then the map $(x, \varphi) \mapsto T_{x,N}(\varphi)$ is measurable.

(ii) Let $x \in \mathbb{T}^2$ and $N \in \mathbb{N}$, then the map $\varphi \mapsto T_{x,N}(\varphi)$ is linear and bounded.

(iii) Let $x \in \mathbb{T}^2$ and $N \in \mathbb{N}$, then $T_{x,N}$ is a Gaussian random variable with

$$\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{iT_{x,N}(\varphi)}\, \gamma_\beta(d\varphi) = e^{-\frac{1}{2\beta}\sum_{|k|\leq N}(m+4\pi^2|k|^2)^{-1}}.$$

(iv) Let $x \in \mathbb{T}^2$ and $N \in \mathbb{N}$, then $T_{x,N} \in L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$ with

$$\|T_{x,N}\|^2_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma_\beta)} = \frac{1}{\beta} \sum_{|k|\leq N} (m + 4\pi^2|k|^2)^{-1}.$$

(v) However, $(T_{x,N})_{N\in\mathbb{N}}$ is NOT a Cauchy sequence in $L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$.

Proof.

(i) As composition of measurable functions.

(ii) For $\varphi \in H^{-\epsilon}(\mathbb{T}^2)$ we have

$$\begin{aligned}|T_{x,N}(\varphi)| &= \Big|\sum_{|k|\le N} \widehat{\varphi}(k)\, e_k(x)\Big| = \Big|\sum_{|k|\le N} (1+4\pi^2|k|^2)^{\epsilon/2}\, e_k(x)\, (1+4\pi^2|k|^2)^{-\epsilon/2}\, \widehat{\varphi}(k)\Big| \\ &\le \sqrt{\sum_{|k|\le N} (1+4\pi^2|k|^2)^{\epsilon}\, e_k(x)^2}\sqrt{\sum_{|k|\le N} (1+4\pi^2|k|^2)^{-\epsilon}\, \widehat{\varphi}(k)^2} \\ &\le \sqrt{\sum_{|k|\le N} (1+4\pi^2|k|^2)^{\epsilon}}\ \|\varphi\|_{H^{-\epsilon}(\mathbb{T}^2)}\,.\end{aligned}$$

(iii) Since $T_{x,N}$ is linear and bounded, it is a Gaussian random variable. Thus

$$\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{iT_{x,N}(\varphi)}\, \gamma_\beta(d\varphi) = e^{-\frac{1}{2\beta}\|T_{x,N}\|^2_{H_m^{-1}(\mathbb{T}^2)}},$$

where

$$\begin{aligned}\|T_{x,N}\|^2_{H_m^{-1}(\mathbb{T}^2)} &= \sum_{k\in\mathbb{Z}^2} (m+4\pi^2|k|^2)^{-1}|\widehat{T_{x,N}}(k)|^2 = \sum_{k\in\mathbb{Z}^2} (m+4\pi^2|k|^2)^{-1}|T_{x,N}(e_k)|^2 \\ &= \sum_{k\in\mathbb{Z}^2} (m+4\pi^2|k|^2)^{-1}\Big|\sum_{|j|\le N} \widehat{e_k}(j)\, e_j(x)\Big|^2 = \sum_{|k|\le N} (m+4\pi^2|k|^2)^{-1}\, e_k(x)^2 \\ &= \sum_{|k|\le N} (m+4\pi^2|k|^2)^{-1}.\end{aligned}$$

(iv) Follows from (iii) together with Corollary 1.10.

(v) Let $M, N \in \mathbb{N}$ and w.l.o.g. assume $M \le N$. Then

$$\begin{aligned}\int_{H^{-\epsilon}(\mathbb{T}^2)} |T_{x,N}(\varphi) - T_{x,M}(\varphi)|^2\, \gamma_\beta(d\varphi) &= \frac{1}{\beta}\ \|T_{x,N} - T_{x,M}\|^2_{H_m^{-1}(\mathbb{T}^2)} \\ &= \frac{1}{\beta}\sum_{k\in\mathbb{Z}^2} (m+4\pi^2|k|^2)^{-1}\Big|\sum_{|j|\le N} \widehat{e_k}(j) e_j(x) - \sum_{|l|\le M} \widehat{e_k}(l)\, e_l(x)\Big|^2 \\ &= \frac{1}{\beta}\sum_{M<|k|\le N} (m+4\pi^2|k|^2)^{-1}\, e_k(x)^2 \\ &= \frac{1}{\beta}\sum_{M<|k|\le N} (m+4\pi^2|k|^2)^{-1},\end{aligned}$$

which does not become arbitrary small for $M, N \to \infty$ due to Lemma 1.28.

❑

In view of applications the "right" object to consider is not $T_{x,N}$, but a suitable recentering thereof. This procedure is called renormalization, see [16] for more background. Here we follow [7]. To that end, it is convenient to introduce Hermite-Polynomials.

5.2 Definition. For $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$ the n-th *Hermite-Polynomial* H_n is defined by

$$H_n(x) := \frac{(-1)^n}{\sqrt{n!}}\, e^{x^2/2}\, \frac{d^n}{dx^n}(e^{-x^2/2}).$$

Furthermore, for $t, x \in \mathbb{R}$ the *generating function* G is defined by $G(t,x) := e^{-t^2/2+tx}$.

5.3 Lemma. *For $t, x \in \mathbb{R}$ we have*

$$G(t,x) = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(x).$$

Proof. We first show by induction that

$$\frac{d^n}{dx^n}(e^{-x^2/2}) = (-1)^n \frac{\partial^n}{\partial t^n}(e^{-(t-x)^2/2})|_{t=0} \tag{5}$$

for $n \in \mathbb{N}_0$. Indeed the statement is true for $n = 0$. Assume now that (5) is true for fixed $n \in \mathbb{N}$. Then

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2/2}) = \frac{d}{dx}\frac{d^n}{dx^n}(e^{-x^2/2}) &= \frac{d}{dx}(-1)^n \frac{\partial^n}{\partial t^n}(e^{-(t-x)^2/2})|_{t=0} \\ &= (-1)^n \frac{\partial^n}{\partial t^n}\Big((t-x)\, e^{-(t-x)^2/2}\Big)\Big|_{t=0} \\ &= (-1)^{n+1} \frac{\partial^{n+1}}{\partial t^{n+1}}\Big(e^{-(t-x)^2/2}\Big)\Big|_{t=0},\end{aligned}$$

which concludes the induction.

Note that $e^{-(t-x)^2/2}$ is even analytic w.r.t. t, hence its Taylor series converges on $\mathbb{R}$. Using this together with (5) we get

$$\begin{aligned}G(t,x) = e^{-t^2/2+tx} &= e^{x^2/2}\, e^{-(t-x)^2/2} \\ &= e^{x^2/2} \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{\partial^n}{\partial t^n}(e^{-(t-x)^2/2})|_{t=0} \\ &= e^{x^2/2} \sum_{n=0}^{\infty} \frac{t^n}{n!}(-1)^n \frac{d^n}{dx^n}(e^{-x^2/2}) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(x).\end{aligned}$$

❑

5.4 Lemma. *Let $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$, then*

(i) $xH_{n+1}(x) = \sqrt{n+2}\, H_{n+2}(x) + \sqrt{n+1}\, H_n(x)$,

(ii) $\frac{d}{dx} H_{n+1}(x) = \sqrt{n+1}\, H_n(x)$,

(iii) $\frac{d^2}{dx^2} H_n(x) - x\frac{d}{dx} H_n(x) = -nH_n(x)$.

Proof. (*i*). For the generating function G we have

$$\frac{\partial}{\partial t} G(t,x) = e^{-t^2/2+tx}(-t+x) = (x-t)G(t,x)$$

and hence by Lemma 5.3

$$(x-t)G(t,x) = \frac{\partial}{\partial t}G(t,x) = \frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{t^n}{\sqrt{n!}}H_n(x) = \sum_{n=1}^{\infty}\frac{nt^{n-1}}{\sqrt{n!}}H_n(x) = \sum_{n=1}^{\infty}\frac{nt^{n-1}}{\sqrt{n!}}H_n(x)$$
$$= \sum_{n=0}^{\infty}\frac{(n+1)t^n}{\sqrt{(n+1)!}}H_{n+1}(x) = H_1(x) + \sum_{n=1}^{\infty}\frac{(n+1)t^n}{\sqrt{(n+1)!}}H_{n+1}(x).$$

On the other hand we have by Lemma 5.3

$$\begin{aligned}(x-t)G(t,x) &= (x-t)\sum_{n=0}^{\infty}\frac{t^n}{\sqrt{n!}}H_n(x)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{\sqrt{n!}}xH_n(x) - \sum_{n=0}^{\infty}\frac{t^{n+1}}{\sqrt{n!}}H_n(x)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{\sqrt{n!}}xH_n(x) - \sum_{n=1}^{\infty}\frac{t^n}{\sqrt{(n-1)!}}H_{n-1}(x)\\
&= \underbrace{xH_0(x)}_{=H_1(x)} + \sum_{n=1}^{\infty}\frac{t^n}{\sqrt{n!}}xH_n(x) - \sum_{n=1}^{\infty}\frac{t^n}{\sqrt{(n-1)!}}H_{n-1}(x).\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty}\frac{(n+1)t^n}{\sqrt{(n+1)!}}H_{n+1}(x) = \sum_{n=1}^{\infty}\frac{t^n}{\sqrt{n!}}xH_n(x) - \sum_{n=1}^{\infty}\frac{t^n}{\sqrt{(n-1)!}}H_{n-1}(x)$$

and comparing coefficients yields

$$\frac{(n+1)}{\sqrt{(n+1)!}}H_{n+1}(x) = \frac{1}{\sqrt{n!}}xH_n(x) - \frac{1}{\sqrt{(n-1)!}}H_{n-1}(x)$$

for $n \geq 1$, which concludes the proof of (i).

(ii). By definition we have

$$\begin{aligned}\frac{d}{dx}H_{n+1}(x) &= \frac{d}{dx}\frac{(-1)^{n+1}}{\sqrt{(n+1)!}}e^{x^2/2}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2/2})\\
&= \frac{(-1)^{n+1}}{\sqrt{(n+1)!}}\left(xe^{x^2/2}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2/2}) + e^{x^2/2}\frac{d^{n+2}}{dx^{n+2}}(e^{-x^2/2})\right)\\
&= xH_{n+1}(x) - \sqrt{n+2}\,H_{n+2}(x),\end{aligned}$$

which coincides with $\sqrt{n+1}H_n(x)$ according to (i).

(iii). By (i) and (ii) we have

$$\begin{aligned}\frac{d^2}{dx^2}H_n(x) &= \frac{d}{dx}\sqrt{n}\,H_{n-1}(x)\\
&= \frac{d}{dx}\left(xH_n(x) - \sqrt{n+1}\,H_{n+1}(x)\right)\\
&= H_n(x) + x\frac{d}{dx}H_n(x) - \sqrt{n+1}\frac{d}{dx}H_{n+1}(x)\\
&= H_n(x) + x\frac{d}{dx}H_n(x) - \sqrt{n+1}\sqrt{n+1}\,H_n(x)\\
&= x\frac{d}{dx}H_n(x) - nH_n(x).\end{aligned}$$

□

5.5 Remark. From Lemma 5.4 it follows that the n-th Hermite-Polynomial is indeed a polynomial of degree n. The first Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$$
$$H_3(x) = \frac{1}{\sqrt{6}}(x^3 - 3x), \quad H_4(x) = \frac{1}{2\sqrt{6}}(x^4 - 6x^2 + 3).$$

//

5.6 Definition. For $n, N \in \mathbb{N}$, $x \in \mathbb{T}^2$ and $\varphi \in H^{-\epsilon}(\mathbb{T}^2)$ we define

$$\rho_N^2 := \rho_N(\varphi, x)^2 := \int_{H^{-\epsilon}(\mathbb{T}^2)} \varphi_N(x)^2 \, \gamma(d\varphi) = \sum_{|k| \le N} (m + 4\pi^2 |k|^2)^{-1}$$

and the *Wick renormalization* $:\varphi_N^n:$ of φ_N^n by

$$:\varphi_N^n:(x) := \sqrt{n!}\, \rho_N^n \, H_n\Big(\frac{\varphi_N(x)}{\rho_N}\Big).$$

Note that $\rho_N(\alpha\varphi) = \alpha\rho_N(\varphi)$ and hence $:(\alpha\varphi)_N^n:(x) = \alpha^n :\varphi_N^n:(x)$ for $\alpha \ge 0$.

The first renormalized powers are

$$:\varphi_N^1:(x) = \varphi_N(x),$$
$$:\varphi_N^2:(x) = \varphi_N^2(x) - \rho_N^2,$$
$$:\varphi_N^3:(x) = \varphi_N^3(x) - 3\,\varphi_N(x)\,\rho_N^2,$$
$$:\varphi_N^4:(x) = \varphi_N^4(x) - 6\,\varphi_N^2(x)\,\rho_N^2 + 3\,\rho_N^4.$$

5.7 Lemma. *Let $x, y \in \mathbb{T}^2$ and $l, n, M, N \in \mathbb{N}$. Then*

$$\int_{H^{-\epsilon}(\mathbb{T}^2)} :\varphi_N^n:(x) :\varphi_M^l:(y) \, \gamma(d\varphi) = \delta_{ln} \, n! \Big(\sum_{|k| \le M \wedge N} (m + 4\pi^2 |k|^2)^{-1} e_k(x) \, e_k(y) \Big)^n.$$

Proof. Using the definition of the generating function G together with Corollary 1.10, it follows for $s, t \in \mathbb{R}$ that

$$\begin{aligned} I &:= \int_{H^{-\epsilon}(\mathbb{T}^2)} G(t, \varphi_N(x)/\rho_N) \, G(s, \varphi_M(y)/\rho_M) \, \gamma(d\varphi) \\ &= \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-t^2/2 + t\,\varphi_N(x)/\rho_N} \, e^{-s^2/2 + s\,\varphi_M(y)/\rho_M} \, \gamma(d\varphi) \\ &= e^{-(t^2+s^2)/2} \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{t/\rho_N \, T_{x,N}(\varphi) + s/\rho_M \, T_{y,M}(\varphi)} \, \gamma(d\varphi) \\ &= e^{-(t^2+s^2)/2} \, e^{\frac{1}{2} \left\| t/\rho_N \, T_{x,N} + s/\rho_M \, T_{y,M} \right\|^2_{H_m^{-1}(\mathbb{T}^2)}}. \end{aligned}$$

With

$$\begin{aligned}
&\|t/\rho_N\, T_{x,N} \;+\; s/\rho_M\, T_{y,M}\|^2_{H_m^{-1}(\mathbb{T}^2)} \\
&= \|t/\rho_N\, T_{x,N}\|^2_{H_m^{-1}(\mathbb{T}^2)} + \|s/\rho_M\, T_{y,M}\|^2_{H_m^{-1}(\mathbb{T}^2)} + 2\,(t/\rho_N\, T_{x,N}\,,\; s/\rho_M\, T_{y,M})_{H_m^{-1}(\mathbb{T}^2)} \\
&= t^2 + s^2 + \frac{2st}{\rho_N\rho_M} \sum_{k\in\mathbb{Z}^2} (m + 4\pi^2|k|^2)^{-1}\, \widehat{T_{x,N}}(k)\, \widehat{T_{y,M}}(k) \\
&= t^2 + s^2 + \frac{2st}{\rho_N\rho_M} \sum_{k\in\mathbb{Z}^2} (m + 4\pi^2|k|^2)^{-1} \Big(\sum_{|j|\le N} \widehat{e_k}(j)\, e_j(x) \Big) \Big(\sum_{|l|\le M} \widehat{e_k}(l)\, e_l(y) \Big) \\
&= t^2 + s^2 + \frac{2st}{\rho_N\rho_M} \sum_{|k|\le M\wedge N} (m + 4\pi^2|k|^2)^{-1}\, e_k(x)\, e_k(y).
\end{aligned}$$

we get

$$I = e^{\frac{st}{\rho_N\rho_M} \sum_{|k|\le M\wedge N} (m+4\pi^2|k|^2)^{-1}\, e_k(x)\, e_k(y)}.$$

On the other hand, using Lemma 5.3 yields

$$\begin{aligned}
I &= \int_{H^{-\epsilon}(\mathbb{T}^2)} \Big(\sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(\varphi_N(x)/\rho_N) \Big) \Big(\sum_{l=0}^{\infty} \frac{s^l}{\sqrt{l!}} H_l(\varphi_M(y)/\rho_M) \Big) \gamma(d\varphi) \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^n s^l}{\sqrt{n!}\sqrt{l!}} \int_{H^{-\epsilon}(\mathbb{T}^2)} H_n(\varphi_N(x)/\rho_N) H_l(\varphi_M(y)/\rho_M)\, \gamma(d\varphi) \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^n s^l}{\sqrt{n!}\sqrt{l!}} \frac{1}{\sqrt{n!}\rho_N^n} \frac{1}{\sqrt{l!}\rho_M^l} \int_{H^{-\epsilon}(\mathbb{T}^2)} {:}\varphi_N^n{:}(x)\, {:}\varphi_M^l{:}(y)\, \gamma(d\varphi),
\end{aligned}$$

where integration and summation can be changed since the series converge uniformly.

Comparing coefficients in the two representations of I yields the result. ❑

5.8 Lemma. *For $\varphi \in H^{-\epsilon}(\mathbb{T}^2)$ and $n, N \in \mathbb{N}$ define*

$$U_{n,N}(\varphi) := \int_{\mathbb{T}^2} {:}\varphi_N^n{:}(x)\, dx. \tag{6}$$

Then $U_{n,N} \in L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma)$.

Proof. Using Fubini and Lemma 5.7 we get

$$\begin{aligned}
\int_{H^{-\epsilon}(\mathbb{T}^2)} U_{n,N}(\varphi)^2\, \gamma(d\varphi) &= \int_{H^{-\epsilon}(\mathbb{T}^2)} \Big(\int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} {:}\varphi_N^n{:}(x)\, {:}\varphi_N^n{:}(y)\, dx \Big) dy \Big) \gamma(d\varphi) \\
&= \int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} \Big(\int_{H^{-\epsilon}(\mathbb{T}^2)} {:}\varphi_N^n{:}(x)\, {:}\varphi_N^n{:}(y)\, \gamma(d\varphi) \Big) dx \Big) dy \\
&= \int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} n! \Big(\sum_{|k|\le N} (m + 4\pi^2|k|^2)^{-1}\, e_k(x)\, e_k(y) \Big)^n dx \Big) dy.
\end{aligned}$$

If $n = 1$, we have $\|U_{n,N}\|_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)} = m^{-1}$. Using Lemma 3.1 with $\lambda_k := (m + 4\pi^2|k|^2)^{-1}\, e_k(y)$, we see that $\|U_{n,N}\|_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)}$ is bounded by some constant only dependent on n, if $n \ge 2$. ❑

5.9 Theorem. *Let $n \in \mathbb{N}$. Then $(U_{n,N})_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)$.*

Proof. Fix $n \in \mathbb{N}$ and let $M, N \in \mathbb{N}$. Using Fubini we get

$$\begin{aligned} I &:= \int_{H^{-\epsilon}(\mathbb{T}^2)} |U_{n,N}(\varphi) - U_{n,M}(\varphi)|^2 \, \gamma(d\varphi) \\ &= \int_{H^{-\epsilon}(\mathbb{T}^2)} \Big(\int_{\mathbb{T}^2} {:}\varphi_N^n{:}(x) - {:}\varphi_M^n{:}(x) \, dx \Big)^2 \gamma(d\varphi) \\ &= \int_{H^{-\epsilon}(\mathbb{T}^2)} \Big(\int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} ({:}\varphi_N^n{:}(x) - {:}\varphi_M^n{:}(x))({:}\varphi_N^n{:}(y) - {:}\varphi_M^n{:}(y)) \, dx \Big) dy \Big) \gamma(d\varphi) \\ &= \int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} \Big(\int_{H^{-\epsilon}(\mathbb{T}^2)} ({:}\varphi_N^n{:}(x) - {:}\varphi_M^n{:}(x))({:}\varphi_N^n{:}(y) - {:}\varphi_M^n{:}(y)) \, \gamma(d\varphi) \Big) dx \Big) dy. \end{aligned}$$

W.l.o.g. assume $M \le N$, then Lemma 5.7 implies

$$I = n! \int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} \Big(\Big(\underbrace{\sum_{|k|\le N} (m + 4\pi^2|k|^2)^{-1} \, e_k(x) \, e_k(y)}_{=:S_N(x,y)} \Big)^n - \Big(\underbrace{\sum_{|k|\le M} (m + 4\pi^2|k|^2)^{-1} \, e_k(x) \, e_k(y)}_{=:S_M(x,y)} \Big)^n \Big) dx \Big) dy.$$

The identity $a^n - b^n = (a-b) \sum_{j=0}^{n-1} a^j \, b^{n-1-j}$ yields

$$\begin{aligned} I &= n! \int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} (S_N(x,y) - S_M(x,y)) \sum_{j=0}^{n-1} S_N(x,y)^j \, S_M(x,y)^{n-1-j} \, dx \Big) dy \\ &= n! \sum_{j=0}^{n-1} \int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} \big(S_N(x,y) - S_M(x,y)\big) \, S_N(x,y)^j \, S_M(x,y)^{n-1-j} \, dx \Big) dy, \end{aligned}$$

and applying Hölder's inequality to the inner integral gives

$$\begin{aligned} I \le n! \sum_{j=0}^{n-1} \int_{\mathbb{T}^2} \Big(\int_{\mathbb{T}^2} \big(S_N(x,y) - S_M(x,y)\big)^2 \, dx \Big)^{1/2} \cdot \\ \cdot \underbrace{\Big(\int_{\mathbb{T}^2} S_N(x,y)^{4j} \, dx \Big)^{1/4}}_{=:II} \underbrace{\Big(\int_{\mathbb{T}^2} S_M(x,y)^{4(n-1-j)} \, dx \Big)^{1/4}}_{=:III} dy. \end{aligned}$$

By Lemma 3.1, II and III are bounded by some constant C only dependent on n. Together with

$$\begin{aligned}\int_{\mathbb{T}^2} \big(S_N(x,y) - S_M(x,y)\big)^2\,dx &= \int_{\mathbb{T}^2} \Big(\sum_{M<|k|\le N} (m+4\pi^2|k|^2)^{-1}\, e_k(x)\, e_k(y)\Big)^2\,dx \\ &= \int_{\mathbb{T}^2} \Big(\sum_{M<|k|\le N} (m+4\pi^2|k|^2)^{-1}\, e_k(x)\, e_k(y)\Big)\Big(\sum_{M<|l|\le N} (m+4\pi^2|l|^2)^{-1}\, e_l(x)\, e_l(y)\Big)\,dx \\ &= \sum_{M<|k|\le N}\ \sum_{M<|l|\le N} (m+4\pi^2|k|^2)^{-1}\,(m+4\pi^2|l|^2)^{-1}\, e_k(y)\, e_l(y) \int_{\mathbb{T}^2} e_k(x)\, e_l(x)\,dx \\ &= \sum_{M<|k|\le N} (m+4\pi^2|k|^2)^{-2}\, e_k(y)^2 \\ &= \sum_{M<|k|\le N} (m+4\pi^2|k|^2)^{-2}\end{aligned}$$

we conclude that

$$I \le C^2\, n\, n!\Big(\sum_{M<|k|\le N} (m+4\pi^2|k|^2)^{-2}\Big)^{1/2}.$$

Thanks to Lemma 1.28 the last expression becomes arbitrary small for $M, N \to \infty$. ❑

Since $L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma)$ is complete, there exists

$$U_n := \lim_{N\to\infty} U_{n,N} \in L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma). \tag{7}$$

5.10 Remark. We also have $U_n \in L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$, since for $N \in \mathbb{N}$ we have

$$\begin{aligned}\|U_{n,N}\|^2_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma_\beta)} &= \int_{H^{-\epsilon}(\mathbb{T}^2)} U_{n,N}(\varphi)^2\,\gamma_\beta(d\varphi) = \int_{H^{-\epsilon}(\mathbb{T}^2)} U_{n,N}(\varphi/\sqrt{\beta})^2\,\gamma(d\varphi) \\ &= \frac{1}{\beta^n}\int_{H^{-\epsilon}(\mathbb{T}^2)} U_{n,N}(\varphi)^2\,\gamma(d\varphi) = \frac{1}{\beta^n}\,\|U_{n,N}\|^2_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)} < \infty.\end{aligned}$$

//

The next step is to show that for $n \in 2\mathbb{N}$ we have $e^{-U_n} \in L^p(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$ for every $p \ge 1$. To that end, following [5], we first give a proof of the Wiener chaos estimate, Corollary 5.21.

5.1 Ornstein-Uhlenbeck semigroup

From now on let c, w be weights on an index set I with $c > 0$, $c \in \ell^2_w(I)$ and let N_c be the measure on $\ell^2_w(I)$ from Definition 1.6.

5.11 Definition. For $t \ge 0$ and $f \in L^1(\ell^2_w(I), N_c)$ the *Ornstein-Uhlenbeck* semigroup T_t is defined by

$$T_t f(x) := \int_{\ell^2_w(I)} f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\, N_c(dy).$$

The *generator* L of the Ornstein-Uhlenbeck semigroup T_t is defined by

$$Lf := \lim_{t\to 0}\frac{T_t f - f}{t}$$

for all f where the limit exists in $L^2(\ell^2_w(I), N_c)$.

5.12 Lemma. *Let $s, t \geq 0$, $p \geq 1$ and $f, g \in L^p(\ell^2_w(I), N_c)$. Then*

(i) $\|T_t f\|_{L^p(\ell^2_w(I),N_c)} \leq \|f\|_{L^p(\ell^2_w(I),N_c)}$,

(ii) $\int_{\ell^2_w(I)} T_t f(x)\, N_c(dx) = \int_{\ell^2_w(I)} f(x)\, N_c(dx)$,

(iii) $(T_t(fg))^2 \leq T_t(f^2)\, T_t(g^2)$,

(iv) $T_s f \overset{s\to\infty}{\longrightarrow} \int_{\ell^2_w(I)} f(x)\, N_c(dx)$ *in* $L^p(\ell^2_w(I), N_c)$,

(v) $T_s T_t f = T_{s+t} f$.

Proof. For the sake of simplicity we denote $H := \ell^2_w(I)$.

(i). With $\theta := \arcsin(e^{-t})$ we have $\sin(\theta) = e^{-t}$ and $\cos(\theta) = \sqrt{1 - e^{-2t}}$. Hence using $X(x, y) := x \sin(\theta) + y \cos(\theta)$ we have by Lemma 1.12 and Hölder's inequality

$$\begin{aligned}
\|T_t f\|^p_{L^p} &= \int_H |T_t f(x)|^p\, N_c(dx) \\
&= \int_H \left| \int_H f(X(x,y))\, N_c(dy) \right|^p N_c(dx) \\
&\leq \int_H \left| \Big(\int_H |f(X(x,y))|^p\, N_c(dy) \Big)^{1/p} \Big(\int_H 1^q\, N_c(dy) \Big)^{1/q} \right|^p N_c(dx) \\
&= \int_H \int_H |f(X(x,y))|^p\, N_c(dy)\, N_c(dx) \\
&= \int_H |f(x)|^p\, N_c(dx) \\
&= \|f\|^p_{L^p}.
\end{aligned}$$

(ii). Using again X from (i) and Lemma 1.12 yields

$$\begin{aligned}
\int_H T_t f(x)\, N_c(dx) &= \int_H \int_H f(X(x,y))\, N_c(dy)\, N_c(dx) = \int_H f(x)\, X\#(N_c \otimes N_c)(dx) \\
&= \int_H f(x)\, N_c(dx).
\end{aligned}$$

(iii). By the Cauchy-Schwartz inequality we have

$$\begin{aligned}
(T_t(fg))^2(x) &= \Big(\int_H f(X(x,y)) g(X(x,y))\, N_c(dy) \Big)^2 \\
&\leq \int_H f(X(x,y))^2\, N_c(dy) \int_H g(X(x,y))\, N_c(dy) \\
&= T_t(f^2)(x)\, T_t(g^2)(x).
\end{aligned}$$

(iv). For bounded and continuous functions f, we have by the dominated convergence theorem

$$\begin{aligned}\lim_{s\to\infty}\left\|T_sf-\int_H f dN_c\right\|_{L^p}^p &= \lim_{s\to\infty}\int_H\left|T_sf(x)-\int_H f(y)\,N_c(dy)\right|^p N_c(dx)\\ &= \int_H\left|\lim_{s\to\infty}T_sf(x)-\int_H f(y)\,N_c(dy)\right|^p N_c(dx)\\ &= \int_H\left|\int_H f(y)\,N_c(dy)-\int_H f(y)\,N_c(dy)\right|^p N_c(dx)\\ &= 0.\end{aligned}$$

For $f\in L^p(H,N_c)$ let f_n be a sequence of bounded and continuous functions with $\|f_n-f\|_{L^p}\to 0$ for $n\to\infty$. Let $\epsilon>0$ and $n\in\mathbb{N}$ with $\|f_n-f\|_{L^p}\leq\epsilon/2$. Then

$$\begin{aligned}\left\|T_sf-\int_H f\,dN_c\right\|_{L^p} &\leq \left\|T_sf-T_sf_n\right\|_{L^p}+\left\|T_sf_n-\int_H f_n\,dN_c\right\|_{L^p}+\left\|\int_H f_n\,dN_c-\int_H f\,dN_c\right\|_{L^p}\\ &\leq \|f_n-f\|_{L^p}+\left\|T_sf_n-\int_H f_n\,dN_c\right\|_{L^p}+\|f_n-f\|_{L^p}\\ &\leq \epsilon+\left\|T_sf_n-\int_H f_n\,dN_c\right\|_{L^p}\xrightarrow{s\to\infty}\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, this concludes the proof of (iv).
(v). With $\theta:=\arccos\sqrt{\frac{1-e^{-2t}}{1-e^{-2(s+t)}}}$ we have $\cos(\theta)=\sqrt{\frac{1-e^{-2t}}{1-e^{-2(s+t)}}}$ and $\sin(\theta)=e^{-t}\sqrt{\frac{1-e^{-2s}}{1-e^{-2(s+t)}}}$. Thus with $F(z):=f(e^{-(s+t)}x+\sqrt{1-e^{-2(s+t)}}z)$ we have

$$\begin{aligned}T_sT_tf(x) &= \int_H T_tf(xe^{-s}+y\sqrt{1-e^{-2s}})\,N_c(dy)\\ &= \int_H\int_H f(xe^{-(s+t)}+ye^{-t}\sqrt{1-e^{-2s}}+z\sqrt{1-e^{-2t}})\,N_c(dz)\,N_c(dy)\\ &= \int_H\int_H f(xe^{-(s+t)}+y\sin(\theta)\sqrt{1-e^{-2(s+t)}}+z\cos(\theta)\sqrt{1-e^{-2(s+t)}})\,N_c(dz)\,N_c(dy)\\ &= \int_H\int_H F(y\sin(\theta)+z\cos(\theta))\,N_c(dz)\,N_c(dy)\\ &= \int_H F(y)\,N_c(dy)\\ &= \int_H f(e^{-(s+t)}x+\sqrt{1-e^{-2(s+t)}}y)\,N_c(dy)\\ &= T_{s+t}f(x).\end{aligned}$$

❑

5.13 Definition. Consider the ONB $(b_k)_{k\in I}$ of $\ell^2_w(I)$, where $b_k=(w_k^{-1}\delta_{jk})_{j\in I}$. Then the projections $\pi_j:\ell^2_w(I)\to\mathbb{R}$ are defined by $x=\sum_{k\in I}x_kb_k\mapsto x_j$ for $j\in I$.

Furthermore we define the space of real valued testfunctions by

$$\mathcal{F}C_c^\infty(\ell^2_w(I);\mathbb{R}):=\big\{f:\ell^2_w(I)\to\mathbb{R}\;:\;\exists n\in\mathbb{N},u_f\in C_c^\infty(\mathbb{R}^n;\mathbb{R}),k_1,\dots,k_n\in I:\\ f(x)=u_f(\pi_{k_1}(x),\dots,\pi_{k_n}(x))\big\}.$$

According to [5, Corollary A.3.13], $\mathcal{F}C_c^\infty(\ell^2_w(I);\mathbb{R})$ is a dense subset of $L^p(\ell^2_w(I);\mathbb{R})$ for $p\geq 1$.

For $f \in \mathcal{F}C_c^\infty(\ell_w^2(I);\mathbb{R})$ and $k \in I$ define $D_k f : \ell_w^2(I) \to \mathbb{R}$ by

$$D_k f(x) := \lim_{\epsilon \to 0} \frac{f(x + \epsilon b_k) - f(x)}{\epsilon}.$$

In addition, we define Df on $\ell_w^2(I)$ by

$$Df(x) := \sum_{k \in I} D_k f(x)\, b_k.$$

Note that this in fact is a finite sum.

5.14 Lemma. *Let $f, g \in \mathcal{F}C_c^\infty(\ell_w^2(I);\mathbb{R})$ and $t \geq 0$. Then*

(i) $\frac{d}{dt} T_t f = L T_t f$, *in particular $T_t f$ is in the domain of L,*

(ii) $\int_{\ell_w^2(I)} (Df(x), Dg(x))_{\ell_c^2(I)}\, N_c(dx) = -\int_{\ell_w^2(I)} f(x) Lg(x)\, N_c(dx)$,

(iii) $D(T_t f) = e^{-t}\, T_t(Df)$.

Proof. We first fix some Notation. For functions $f, g \in \mathcal{F}C_c^\infty(\ell_w^2(I);\mathbb{R})$ let $u_f, u_g \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ and $k_1, \dots, k_n \in I$ be such that $f(x) = u_f(x_{k_1}, \dots, x_{k_n})$ and $g(x) = u_g(x_{k_1}, \dots, x_{k_n})$ for $x \in \ell_w^2(I)$. We shall denote $\bar{x} = (x_{k_1}, \dots, x_{k_n})$. Furthermore, define $C := \operatorname{diag}(c_{k_1}^2, \dots, c_{k_n}^2) \in \mathbb{R}^{n \times n}$ and the probability measure ν_C on the Borel sets $\mathcal{B}(\mathbb{R}^n)$ by

$$\nu_C(B) = \frac{1}{\sqrt{(2\pi)^n \det C}} \int_B e^{-\frac{1}{2}(C^{-1}y, y)}\, dy, \quad B \in \mathcal{B}(\mathbb{R}^n).$$

(i). Since

$$T_t f(x) = \int_{\ell_w^2(I)} f(e^{-t}x + \sqrt{1 - e^{-2t}} y)\, N_c(dy) = \int_{\mathbb{R}^n} u_f(e^{-t}\bar{x} + \sqrt{1 - e^{-2t}} \bar{y})\, \nu_C(d\bar{y})$$

and $u_f \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$, the map $t \mapsto T_t f(x)$ is differentiable. Thus

$$\frac{d}{dt} T_t f = \lim_{s \to 0} \frac{T_{s+t} f - T_t f}{s} = \lim_{s \to 0} \frac{T_s T_t f - T_t f}{s} = L(T_t f).$$

(ii). Since $D_k f(x) = \frac{\partial}{\partial x_k} u_f(\bar{x})$, we have

$$(Df(x), Dg(x))_{\ell_c^2(I)} = \sum_{k \in I} c_k^2 D_k f(x) D_k g(x) = \big(C \nabla u_f(\bar{x})\big) \cdot \nabla u_g(\bar{x}).$$

Furthermore,

$$\begin{aligned}
Lf(x) &= \frac{d}{dt} T_t f(x)\Big|_{t=0} \\
&= \frac{d}{dt} \int_{\mathbb{R}^n} u_f(e^{-t}\bar{x} + \sqrt{1 - e^{-2t}}\, \bar{y})\, \nu_C(d\bar{y})\Big|_{t=0} \\
&= \int_{\mathbb{R}^n} \nabla u_f(e^{-t}\bar{x} + \sqrt{1 - e^{-2t}}\, \bar{y}) \cdot (-e^{-t}\bar{x} + (1 - e^{-2t})^{-1/2} e^{-2t}\, \bar{y})\, \nu_C(d\bar{y})\Big|_{t=0} \\
&= -\bar{x} \cdot \nabla u_f(\bar{x}) + \int_{\mathbb{R}^n} \nabla u_f(e^{-t}\bar{x} + \sqrt{1 - e^{-2t}}\, \bar{y}) \cdot ((1 - e^{-2t})^{-1/2} e^{-2t}\, \bar{y})\, \nu_C(d\bar{y})\Big|_{t=0}.
\end{aligned}$$

Using the definition of ν_C and integrating once by parts, we get

$$\begin{aligned}
&(1-e^{-2t})^{-1/2}e^{-2t}\int_{\mathbb{R}^n}\nabla u_f(e^{-t}\bar{x}+\sqrt{1-e^{-2t}}\,\bar{y})\cdot\bar{y}\,\nu_C(d\bar{y})\\
&=\frac{(1-e^{-2t})^{-1/2}e^{-2t}}{\sqrt{(2\pi)^n\det C}}\int_{\mathbb{R}^n}\nabla u_f(e^{-t}\bar{x}+\sqrt{1-e^{-2t}}\,\bar{y})\cdot\underbrace{\bar{y}\,e^{-1/2(C^{-1}\bar{y},\bar{y})}}_{=-C\nabla\exp(-1/2(C^{-1}\bar{y},\bar{y}))}d\bar{y}\\
&=\frac{(1-e^{-2t})^{-1/2}e^{-2t}}{\sqrt{(2\pi)^n\det C}}\int_{\mathbb{R}^n}\operatorname{div}\Big(C\nabla u_f(e^{-t}\bar{x}+\sqrt{1-e^{-2t}}\,\bar{y})\Big)\sqrt{1-e^{-2t}}\,e^{-1/2(C^{-1}\bar{y},\bar{y})}d\bar{y}\\
&=e^{-2t}\int_{\mathbb{R}^n}\operatorname{div}\Big(C\nabla u_f(e^{-t}\bar{x}+\sqrt{1-e^{-2t}}\,\bar{y})\Big)\,\nu_C(d\bar{y})
\end{aligned}$$

and hence

$$Lf(x)=-\bar{x}\cdot\nabla u_f(\bar{x})+\operatorname{div}\big(C\nabla u_f(\bar{x})\big).$$

Thus

$$\begin{aligned}
&\int_{\ell^2_w(I)}(Df(x),Dg(x))_{\ell^2_c(I)}\,N_c(dx)\\
&=\int_{\mathbb{R}^n}\big(C\nabla u_f(\bar{x})\big)\cdot\nabla u_g(\bar{x})\,\nu_C(d\bar{x})\\
&=((2\pi)^n\det C)^{-1/2}\int_{\mathbb{R}^n}\big(C\nabla u_f(\bar{x})\big)\cdot\nabla u_g(\bar{x})\,e^{-1/2(C^{-1}\bar{x},\bar{x})}\,d\bar{x}\\
&=((2\pi)^n\det C)^{-1/2}\int_{\mathbb{R}^n}\operatorname{div}\big(C\nabla u_f(\bar{x})\,e^{-1/2(C^{-1}\bar{x},\bar{x})}\big)u_g(\bar{x})\,d\bar{x}\\
&=((2\pi)^n\det C)^{-1/2}\int_{\mathbb{R}^n}\Big(\operatorname{div}\big(C\nabla u_f(\bar{x})\big)-C\,\nabla u_f(\bar{x})\,C^{-1}\,\bar{x}\Big)\,e^{-1/2(C^{-1}\bar{x},\bar{x})}\,u_g(\bar{x})\,d\bar{x}\\
&=\int_{\mathbb{R}^n}\big(\operatorname{div}\big(C\nabla u_f(\bar{x})\big)-\bar{x}\cdot\nabla u_f(\bar{x})\big)\,u_g(\bar{x})\,\nu_C(d\bar{x})\\
&=\int_{\ell^2_w(I)}g(x)Lf(x)\,N_c(dx)
\end{aligned}$$

(iii). By the computation in the proof of (i) and chainrule we have

$$\begin{aligned}
D(T_tf)(x)&=\nabla\int_{\mathbb{R}^n}u_f(e^{-t}\bar{x}+\sqrt{1-e^{-2t}}\,\bar{y})\,\nu_C(d\bar{y})\\
&=\int_{\mathbb{R}^n}\nabla u_f(e^{-t}\bar{x}+\sqrt{1-e^{-2t}}\,\bar{y})\,e^{-t}\,\nu_C(d\bar{y})\\
&=e^{-t}\,T_t(\nabla u_f)(\bar{x})\\
&=e^{-t}\,T_t(Df)(x).
\end{aligned}$$

❑

5.2 Wiener chaos

5.15 Definition (Wiener Chaos)**.** Consider the closed linear subspace $E_n\subseteq L^2(\ell^2_w(I),N_c)$ given by

$$E_n=\operatorname{span}\big\{H_{k_1}(l_1)\cdots H_{k_j}(l_j)\,:\,j\in\mathbb{N},l_1,\dots,l_j\in\ell^2_w(I)',k_1+\cdots+k_j\le n\big\},$$

where H_{k_i} is the k_i-th Hermite Polynomial from Definition 5.2.

Define $\mathcal{X}_0 \subseteq L^2(\ell^2_w(I), N_c)$ as the space of constants and for $n \in \mathbb{N}$ define $\mathcal{X}_n \subseteq L^2(\ell^2_w(I), N_c)$ as the orthogonal complement of E_{n-1} in E_n, i.e. $\mathcal{X}_n = E_n \cap E_{n-1}^\perp$. Then $\mathcal{X}_n$ is called the *n-th Wiener Chaos*.

Furthermore we define I_n as the orthogonal projection from $L^2(\ell^2_w(I), N_c)$ to $\mathcal{X}_n$.

5.16 Lemma. *Let $f \in L^2(\ell^2_w(I), N_c)$ and $t \geq 0$. Then*

$$T_t f = \sum_{n=0}^{\infty} e^{-nt} I_n(f).$$

Proof. See [5, Theorem 2.9.2]. ❑

5.17 Theorem (Log-Sobolev Inequality). *For $f \in \mathcal{F}C_c^\infty(\ell^2_w(I); \mathbb{R})$ with $f \geq a > 0$ we have*

$$\int_{\ell^2_w(I)} \|Df(x)\|^2_{\ell^2_c(I)} \, N_c(dx) \geq \frac{1}{2} \operatorname{Ent}(f^2), \tag{8}$$

where

$$\operatorname{Ent}(f^2) := \int_{\ell^2_w(I)} f^2(x) \log f^2(x) \, N_c(dx) - \int_{\ell^2_w(I)} f^2(x) \, N_c(dx) \, \log \Big(\int_{\ell^2_w(I)} f^2(x) \, N_c(dx) \Big).$$

Proof. For $f \in \mathcal{F}C_c^\infty(\ell^2_w(I); \mathbb{R})$ let $u_f \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$ and $k_1, \dots, k_n \in I$ be such that $f(x) = u_f(x_{k_1}, \dots, x_{k_n})$ for $x \in \ell^2_w(I)$. We shall denote $\bar{x} = (x_{k_1}, \dots, x_{k_n})$. Furthermore, define $C := \operatorname{diag}(c^2_{k_1}, \dots, c^2_{k_n}) \in \mathbb{R}^{n \times n}$ and the probability measure ν_C on the Borel sets $\mathcal{B}(\mathbb{R}^n)$ by

$$\nu_C(B) = \frac{1}{\sqrt{(2\pi)^n \det C}} \int_B e^{-\frac{1}{2}(C^{-1}y, y)} \, dy, \quad B \in \mathcal{B}(\mathbb{R}^n).$$

Then (8) is equivalent to

$$\int_{\mathbb{R}^n} \big(C \, \nabla u_f(\bar{x})\big) \cdot \nabla u_f(\bar{x}) \, \nu_C(d\bar{x}) \geq \frac{1}{2} \operatorname{Ent}(u_f^2).$$

We define $g(x) := f^2(x)$. Then $\nabla u_f(\bar{x}) = 1/2 \, \nabla u_g(\bar{x}) / \sqrt{u_g(\bar{x})}$. Hence we have to show that

$$\frac{1}{2} \int_{\mathbb{R}^n} u_g(\bar{x})^{-1} \big(C \, \nabla u_g(\bar{x})\big) \cdot \nabla u_g(\bar{x}) \, \nu_C(d\bar{x}) \geq \operatorname{Ent}(u_g).$$

From Lemma 5.12 (iv), we have

$$\operatorname{Ent}(u_g) = -\int_0^\infty \frac{d}{dt} \int_{\mathbb{R}^n} T_t u_g(\bar{x}) \log T_t u_g(\bar{x}) \, \nu_C(d\bar{x}) \, dt.$$

Since $u_g \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$ we have that $T_t u_g$ is bounded above and from $f \geq a$ we get $T_t u_g \geq a^2$. Thus we can change integration and differentiation and use Lemma 5.14 (i) to get

$$\begin{aligned}
\operatorname{Ent}(u_g) &= -\int_0^\infty \int_{\mathbb{R}^n} \frac{d}{dt} \big(T_t u_g(\bar{x}) \log T_t u_g(\bar{x})\big) \, \nu_C(d\bar{x}) \, dt \\
&= -\int_0^\infty \int_{\mathbb{R}^n} \Big(L T_t u_g(\bar{x}) \log T_t u_g(\bar{x}) + \frac{d}{dt} T_t u_g(\bar{x}) \Big) \, \nu_C(d\bar{x}) \, dt \\
&= -\int_0^\infty \int_{\mathbb{R}^n} L T_t u_g(\bar{x}) \log T_t u_g(\bar{x}) \, \nu_C(d\bar{x}) \, dt,
\end{aligned}$$

since by Fubini and Lemma 5.12 (iv) we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} \frac{d}{dt} T_t u_g(\bar{x})\, \nu_C(d\bar{x})\, dt &= \int_{\mathbb{R}^n} \int_0^\infty \frac{d}{dt} T_t u_g(\bar{x})\, dt\, \nu_C(d\bar{x}) \\ &= \int_{\mathbb{R}^n} \lim_{t\to\infty} T_t u_g(\bar{x}) \nu_C(d\bar{x}) - \int_{\mathbb{R}^n} u_g(\bar{x})\, \nu_C(d\bar{x}) \\ &= \int_{\mathbb{R}^n} u_g(\bar{x})\, \nu_C(d\bar{x}) - \int_{\mathbb{R}^n} u_g(\bar{x})\, \nu_C(d\bar{x}) = 0.\end{aligned}$$

Thus by Lemma 5.14 (ii) and (iii) we have

$$\begin{aligned}\mathrm{Ent}(u_g) &= \int_0^\infty \int_{\mathbb{R}^n} \big(C\, \nabla(T_t u_g(\bar{x}))\big) \cdot \nabla(\log T_t u_g(\bar{x}))\, \nu_C(d\bar{x})\, dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} (T_t u_g(\bar{x}))^{-1} \big(C\, \nabla(T_t u_g(\bar{x}))\big) \cdot \nabla(T_t u_g(\bar{x}))\, \nu_C(d\bar{x})\, dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} (T_t u_g(\bar{x}))^{-1}\, e^{-2t} \big(C\, T_t(\nabla u_g(\bar{x}))\big) \cdot T_t(\nabla u_g(\bar{x}))\, \nu_C(d\bar{x})\, dt.\end{aligned}$$

Using Lemma 5.14 (ii) and (iii), we finally get

$$\begin{aligned}\mathrm{Ent}(u_g) &= \int_0^\infty \int_{\mathbb{R}^n} (T_t u_g(\bar{x}))^{-1}\, e^{-2t} \left(\sum_{i=1}^n c_{k_i}^2 T_t\big(u_g(\bar{x})^{-1/2} u_g(\bar{x})^{1/2} \partial x_{k_i} u_g(\bar{x})\big)^2 \right) \nu_C(d\bar{x})\, dt \\ &\le \int_0^\infty \int_{\mathbb{R}^n} (T_t u_g(\bar{x}))^{-1}\, e^{-2t} \left(\sum_{i=1}^n c_{k_i}^2 T_t\big(u_g(\bar{x})\big) T_t\big(u_g(\bar{x})^{-1} (\partial x_{k_i} u_g(\bar{x}))^2\big) \right) \nu_C(d\bar{x})\, dt \\ &= \int_0^\infty e^{-2t} \sum_{i=1}^n c_{k_i}^2 \int_{\mathbb{R}^n} T_t\big(u_g(\bar{x})^{-1} (\partial x_{k_i} u_g(\bar{x}))^2\big) \nu_C(d\bar{x})\, dt \\ &= \int_0^\infty e^{-2t} \sum_{i=1}^n c_{k_i}^2 \int_{\mathbb{R}^n} u_g(\bar{x})^{-1} (\partial x_{k_i} u_g(\bar{x}))^2\, \nu_C(d\bar{x})\, dt \\ &= \int_0^\infty e^{-2t}\, dt \int_{\mathbb{R}^n} u_g(\bar{x})^{-1} \big(C \nabla u_g(\bar{x})\big) \cdot \nabla u_g(\bar{x})\, \nu_C(d\bar{x}) \\ &= \frac{1}{2} \int_{\mathbb{R}^n} u_g(\bar{x})^{-1} \big(C \nabla u_g(\bar{x})\big) \cdot \nabla u_g(\bar{x})\, \nu_C(d\bar{x}).\end{aligned}$$

❑

5.18 Remark. The statements of Lemma 5.14 and Theorem 5.17 can be generalized to functions f, where Df exists in a weak sense, see e.g. [5]. However, we only formulated the results as strong as we need them, since we only need this results to prove the next Theorem, a hypercontractivity estimate for the Ornstein-Uhlenbeck semigroup. //

5.19 Theorem (Hypercontractivity estimate)**.** *Let $t \ge 0$, $p \ge 2$, $f \in L^p(\ell_w^2(I), N_c)$ and define $q(t) := 1 + (p-1)\, e^{2t}$. Then*

$$\|T_t f\|_{L^{q(t)}} \le \|f\|_{L^p}\,.$$

Proof. At first consider $f \in \mathcal{F}C_c^\infty(\ell^2_w(I);\mathbb{R})$ with $f \geq a > 0$. We define a function $G : [0,\infty) \to [0,\infty)$ by $G(t) := \|T_t f\|_{L^{q(t)}}$. Since f is smooth, this function is differentiable and

$$G'(t) = G(t)\left(-\frac{q'(t)}{q(t)^2}\log F(t) + \frac{F'(t)}{q(t)F(t)}\right),$$

where $F(t) := G(t)^{q(t)}$. We show that $G'(t) \leq 0$ for $t \geq 0$, since then we have $\|T_t f\|_{L^{q(t)}} = G(t) \leq G(0) = \|f\|_{L^p}$.

Since $G(t), q(t), q'(t), F(t) \geq 0$, the inequality $G'(t) \leq 0$ is equivalent to

$$-\frac{F(t)\log F(t)}{q(t)} + \frac{F'(t)}{q'(t)} \leq 0.$$

Since f is smooth and $T_t f \geq a$, we can change differentiation and integration to get

$$F'(t) = \int_{\ell^2_w(I)} T_t f(x)^{q(t)} \left(q'(t)\log T_t f(x) + q(t)\frac{LT_t f(x)}{T_t f(x)}\right) N_c(dx).$$

Thus using $q'(t) = 2(p-1)e2t = 2(q(t)-1)$ and denoting $g := T_t f$ we have

$$\begin{aligned}
&-\frac{F(t)\log F(t)}{q(t)} + \frac{F'(t)}{q'(t)} \\
&= -\frac{F(t)\log F(t)}{q(t)} + \int_{\ell^2_w(I)} g(x)^{q(t)} \log g(x)\, N_c(dx) + \frac{q(t)}{q'(t)} \int_{\ell^2_w(I)} g(x)^{q(t)-1}\, Lg(x)\, N_c(dx) \\
&= -q(t)^{-1} \int_{\ell^2_w(I)} g(x)^{q(t)}\, N_c(dx) \log\left(\int_{\ell^2_w(I)} g(x)^{q(t)}\, N_c(dx)\right) \\
&\qquad + \int_{\ell^2_w(I)} g(x)^{q(t)} \log g(x)\, N_c(dx) + \frac{q(t)}{q'(t)} \int_{\ell^2_w(I)} g(x)^{q(t)-1}\, Lg(x)\, N_c(dx) \\
&= -q(t)^{-1} \int_{\ell^2_w(I)} g(x)^{q(t)}\, N_c(dx) \log\left(\int_{\ell^2_w(I)} g(x)^{q(t)}\, N_c(dx)\right) \\
&\qquad + \int_{\ell^2_w(I)} g(x)^{q(t)} \log g(x)\, N_c(dx) \\
&\qquad - \frac{q(t)}{q'(t)} \int_{\ell^2_w(I)} (q(t)-1) g(x)^{q(t)-2} (Dg(x), Dg(x))_{\ell^2_c(I)}\, N_c(dx) \\
&= -q(t)^{-1} \int_{\ell^2_w(I)} g(x)^{q(t)}\, N_c(dx) \log\left(\int_{\ell^2_w(I)} g(x)^{q(t)}\, N_c(dx)\right) \\
&\qquad + \int_{\ell^2_w(I)} g(x)^{q(t)} \log g(x)\, N_c(dx) \\
&\qquad - \frac{q(t)}{2} \int_{\ell^2_w(I)} g(x)^{q(t)-2} (Dg(x), Dg(x))_{\ell^2_c(I)}\, N_c(dx) \\
&= q(t)^{-1}\left(\mathrm{Ent}(g^{q(t)}) - 2\int_{\ell^2_w(I)} \left(D(g(x)^{q(t)/2}), D(g(x)^{q(t)/2})\right)_{\ell^2_c(I)} N_c(dx)\right).
\end{aligned}$$

Finally, by the Log-Sobolev inequality, Theorem 5.17, the last term is non positive, which concludes the proof for testfunctions f with $f \geq a > 0$.

For $f \in \mathcal{F}C_c^\infty(\ell^2_w(I);\mathbb{R})$ with $f \geq 0$ we take a sequence f_n of testfunctions with $f_n \geq 1/n$ and $f_n \to f$ in $L^{q(t)}$ for $n \to \infty$. Then we have by Lemma 5.12 (i) that

$$\|T_t f_n - T_t f\|_{L^{q(t)}} \leq \|f_n - f\|_{L^{q(t)}} \to 0$$

and since $p \leq q(t)$ we also have

$$\|f_n - f\|_{L^p} \leq \|f_n - f\|_{L^{q(t)}} \to 0$$

for $n \to \infty$. Since for every $n \in \mathbb{N}$ we can apply what we already showed, we have $\|T_t f_n\|_{L^{q(t)}} \leq \|f_n\|_{L^p}$ and by taking the limit on both sides also $\|T_t f\|_{L^{q(t)}} \leq \|f\|_{L^p}$.

For $f \in L^p(\ell^2_w(I), N_c)$ with $f \geq 0$ we take a sequence f_n of testfunctions with $f_n \geq 0$ and $f_n \to f$ in L^p for $n \to \infty$. From what we already showed, we get $\|T_t f_n\|_{L^{q(t)}} \leq \|f_n\|_{L^p} \to \|f\|_{L^p}$, hence $\|T_t f_n\|_{L^{q(t)}}$ is bounded. Since $L^q(t)$ is a reflexive Banach space, there exists a weakly convergent subsequence $T_t f_{n_k} \rightharpoonup u \in L^{q(t)}$. Since $p \leq q(t)$, we have $L^{q(t)} \subseteq L^p$ and thus $(L^p)' \subseteq (L^{q(t)})'$. Hence $T_t f_{n_k} \rightharpoonup u \in L^p$. But

$$\|T_t f_{n_k} - T_t f\|_{L^p} \leq \|f_{n_k} - f\|_{L^p} \overset{k\to\infty}{\longrightarrow} 0$$

by Lemma 5.12, thus $u = T_t f$. So we have $T_t f_{n_k} \rightharpoonup T_t f \in L^{q(t)}$ and by the lower semicontinuity of norms

$$\|T_t f\|_{L^{q(t)}} \leq \liminf_{k\to\infty} \|T_t f_{n_k}\|_{L^{q(t)}} \leq \liminf_{k\to\infty} \|f_{n_k}\|_{L^p} = \|f\|_{L^p}\,.$$

For arbitrary $f \in L^p(\ell^2_w(I), N_c)$, note that $|T_t f| \leq T_t |f|$. Since $|f| \in L^p(\ell^2_w(I), N_c)$ with $|f| \geq 0$, we get

$$\|T_t f\|_{L^{q(t)}} \leq \|T_t |f|\|_{L^{q(t)}} \leq \||f|\|_{L^p} = \|f\|_{L^p}\,.$$

❑

5.20 Theorem (Wiener Chaos estimate)**.** *Let $n \in \mathbb{N}$, $q \geq 2$ and $f \in L^2(\ell^2_w(I), N_c)$. Then*

$$\|I_n(f)\|_{L^q(\ell^2_w(I),N_c)} \leq (q-1)^{n/2}\,\|f\|_{L^2(\ell^2_w(I),N_c)}\,.$$

Proof. For $t := \ln(q-1)/2 \geq 0$ we have $q = 1 + e^{2t}$. Applying Theorem 5.19 with $q(t) = q$ and $p = 2$ to $f \in L^2(\ell^2_w(I), N_c)$ yields

$$\|T_t f\|_{L^q} \leq \|f\|_{L^2}\,.$$

Since $I_n : L^2(\ell^2_w(I), N_c) \to L^2(\ell^2_w(I), N_c)$ is an orthogonal projection, we have $\|I_n f\|_{L^2} \leq \|f\|_{L^2}$ for $f \in L^2(\ell^2_w(I), N_c)$, and thus

$$\|T_t(I_n f)\|_{L^q} \leq \|I_n f\|_{L^2} \leq \|f\|_{L^2}\,.$$

From Lemma 5.16 it follows, that $T_t(I_n f) = e^{-nt}\, I_n f$, and hence

$$\left\|e^{-nt}\, I_n f\right\|_{L^q} = \|T_t(I_n f)\|_{L^q} \leq \|f\|_{L^2}\,.$$

Finally, this is equivalent to

$$\|I_n f\|_{L^q} \leq e^{nt}\,\|f\|_{L^2} = e^{n\ln(q-1)/2}\,\|f\|_{L^2} = (q-1)^{n/2}\,\|f\|_{L^2}\,.$$

❑

In the following we give a formulation of the Wiener Chaos estimate on $H^{-\epsilon}(\mathbb{T}^2) \simeq \ell^2_w(\mathbb{Z}^2)$.

In terms of $H^{-\epsilon}(\mathbb{T}^2)$ Definition 5.15 reads

$$E_n = \operatorname{span}\left\{H_{k_1}(l_1)\cdots H_{k_j}(l_j) \,:\, j \in \mathbb{N}, l_1, \ldots, l_j \in H^{-\epsilon}(\mathbb{T}^2)', k_1 + \cdots + k_j \leq n\right\},$$

where w.l.o.g. we can assume that $\|l_i\|_{H^{-1}(\mathbb{T}^2)} = 1$ for $i = 1, \ldots, j$. Then $\mathcal{X}_0$ is the space of constants and $\mathcal{X}_n = E_n \cap E_{n-1}^{\perp}$, where the orthogonality means orthogonal in $L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma)$. We still denote the orthogonal projection from $L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma)$ to $\mathcal{X}_n$ by I_n.

5.21 Corollary. *Let $n \in \mathbb{N}$, $q \geq 2$ and $f \in L^2(H^{-\epsilon}(\mathbb{T}^2), \gamma)$. Then*

$$\|I_n(f)\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq (q-1)^{n/2}\, \|f\|_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)}\,.$$

5.22 Lemma. *Let $n, N \in \mathbb{N}$ and $U_{n,N} = \int_{\mathbb{T}^2} \sqrt{n!}\rho_N^n H_n(\varphi_N(x)/\rho_N)\, dx$ defined in Lemma 5.8. Then $U_{n,N} \in \mathcal{X}_n$.*

Proof. Since by Lemma 5.1 $T_{x,N}$ is bounded and linear with

$$\|T_{x,N}\|^2_{H^{-1}(\mathbb{T}^2)} = \sum_{|k|\leq N} (m + 4\pi^2|k|^2)^{-1} = \rho_N^2,$$

we have $\|T_{x,N}/\rho_N\|_{H^{-1}(\mathbb{T}^2)} = 1$. Thus $H_n(T_{x,N}/\rho_N) \in E_n$. Hence $\sqrt{n!}\rho_N^n H_n(T_{x,N}/\rho_N) \in E_n$ and $U_{n,N} \in E_n$. It remains to show that $(H_n(l_1), H_k(l_2))_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)} = 0$ for $n > k$ and $l_1, l_2 \in H^{-\epsilon}(\mathbb{T}^2)'$ with $\|l_{1,2}\|_{H^{-1}(\mathbb{T}^2)} = 1$. Indeed, we have with the Generating function $G(t,x) = e^{-t^2/2+tx}$ that

$$\begin{aligned}\int_{H^{-\epsilon}(\mathbb{T}^2)} G(t, l_1(\varphi))G(s, l_2(\varphi))\,\gamma(d\varphi) &= e^{-(t^2+s^2)/2}\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{tl_1(\varphi)+sl_2(\varphi)}\,\gamma(d\varphi)\\ &= e^{-(t^2+s^2)/2}\, e^{1/2\|tl_1+sl_2\|^2_{H^{-1}(\mathbb{T}^2)}}\\ &= e^{-(t^2+s^2)/2}\, e^{1/2\|tl_1\|^2_{H^{-1}(\mathbb{T}^2)}+1/2\|sl_2\|^2_{H^{-1}(\mathbb{T}^2)}+st(l_1,l_2)_{H^{-1}(\mathbb{T}^2)}}\\ &= e^{st(l_1,l_2)_{H^{-1}(\mathbb{T}^2)}}.\end{aligned}$$

On the other hand, Lemma 5.3 yields

$$\begin{aligned}\int_{H^{-\epsilon}(\mathbb{T}^2)} G(t, l_1(\varphi))G(s, l_2(\varphi))\,\gamma(d\varphi) &= \int_{H^{-\epsilon}(\mathbb{T}^2)} \sum_{n=0}^{\infty}\frac{t^n}{\sqrt{n!}}H_n(l_1(\varphi))\sum_{k=0}^{\infty}\frac{s^k}{\sqrt{k!}}H_k(l_2(\varphi))\,\gamma(d\varphi)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^n s^k}{\sqrt{n!k!}}\int_{H^{-\epsilon}(\mathbb{T}^2)} H_n(l_1(\varphi))H_k(l_2(\varphi))\,\gamma(d\varphi).\end{aligned}$$

Comparing coefficients yields

$$\int_{H^{-\epsilon}(\mathbb{T}^2)} H_n(l_1(\varphi))H_k(l_2(\varphi))\,\gamma(d\varphi) = \delta_{n,k}(l_1, l_2)_{H^{-1}(\mathbb{T}^2)}.$$

❑

Together with the Wiener Chaos estimate, Corollary 5.21, the next result will be the main ingredient to show that $e^{-U_n} \in L^p(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$ for every $p \geq 1$.

5.23 Theorem (Nelson estimate)**.** *Let $(X_N)_{N\in\mathbb{N}}$ be a sequence in $L^q(H^{-\epsilon}(\mathbb{T}^2), \gamma)$ for every $q \geq 2$. Assume that there exist constants $C_1, C_2 > 0$ and $l \in \mathbb{N}$ s.t.*

(i) $\|X_N - X_M\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq C_1\, q^l\, N^{-1/2}$ for every $N < M$ and $q \geq 2$, in particular there exists $X \in L^q(H^{-\epsilon}(\mathbb{T}^2), \gamma)$ s.t. $\|X - X_N\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \to 0$ for $N \to \infty$,

(ii) $X_N(\varphi) \geq -C_2 \ln(N)^l$ for every $N \in \mathbb{N}$ and $\varphi \in H^{-\epsilon}(\mathbb{T}^2)$.

Then for every $p \geq 1$ there exists a constant $C_p > 0$ s.t.

(I) $\|e^{-X_N}\|_{L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq C_p$ for every $N \in \mathbb{N}$,

(II) $\|e^{-X} - e^{-X_N}\|_{L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \to 0$ for $N \to \infty$.

Proof. Proof of (I): We only consider the case $p = 1$. For $p > 1$ we can replace X_N by pX_N.

By the Layer cake formula, we have

$$\begin{aligned}\|e^{-X_N}\|_{L^1} &= \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-X_N(\varphi)}\,\gamma(d\varphi) = \int_0^\infty \gamma([e^{-X_N} > t])\,dt = \int_0^\infty \gamma([X_N < -\ln t])\,dt \\ &= \int_0^1 \underbrace{\gamma([X_N < -\ln t])}_{\leq 1}\,dt + \int_1^\infty \gamma([X_N < -\ln t])\,dt \\ &= 1 + \int_1^\infty \gamma([X_N < -\ln t])\,dt.\end{aligned}$$

In the following, we will show that there exist some $C, \lambda > 0$ independent of N, s.t. $\gamma([X_N < -\ln t]) \leq C\, e^{-\lambda \ln(t)^2}$. Since the latter one is integrable on $[1, \infty)$, this will then imply *(I)*.

Step 1: For $t \geq 1$ we define $N_0(t) := \lfloor \exp\big((\ln(t)/(2\,C_2))^{1/l}\big)\rfloor$. The goal is to show that

$$\gamma([X_N < -\ln(t)]) \leq \exp\Big(-q\,\ln\big(\frac{\sqrt{N_0(t)}\,\ln(t)}{C_1\,q^l\,2}\big)\Big) \tag{9}$$

for every $q \geq 2$ and $N \in \mathbb{N}$. For $N \leq N_0(t)$ we have by assumption *(ii)*

$$X_N \geq -C_2\,\ln(N)^l \geq -C_2\,\ln(N_0(t))^l \geq -\frac{\ln(t)}{2}.$$

Hence $\gamma([X_N < -\ln(t)]) = 0$ and in particular we have (9). Assume now $N > N_0(t)$. If $X_N < -\ln(t)$, then by the above computation we see

$$X_N - X_{N_0(t)} \leq -\ln(t) - X_{N_0(t)} \leq -\ln(t)/2.$$

Using this, the Markov inequality and assumption *(i)* yield

$$\begin{aligned}\gamma([X_N < -\ln(t)]) &\leq \gamma([|X_N - X_{N_0(t)}| > \ln(t)/2]) \\ &\leq (2/\ln(t))^q\, \|X_N - X_{N_0(t)}\|_{L^q}^q \\ &\leq (2/\ln(t))^q\, q^{lq}\, C_1^q\, N_0(t)^{-q/2} \\ &= \exp\Big(-q\,\ln\big(\frac{\sqrt{N_0(t)}\,\ln(t)}{C_1\,q^l\,2}\big)\Big).\end{aligned}$$

Step 2: We show that there exists $C, \delta > 0$ s.t.

$$\gamma([X_N < -\ln(t)]) \le C\, e^{-\delta(\sqrt{N_0(t)}\,\ln(t)/2)^{1/l}}$$

for $t \ge 1$ and $N \in \mathbb{N}$. Define $x_0 := 2^{l+1} C_1$, $\delta := \ln(2)/(2\,C_1)^{1/l}$, $\widetilde{q} := (\sqrt{N_0(t)}\,\ln(t)/(4\,C_1))^{1/l}$ and $C := \exp(\delta\, x_0^{1/l})$. Since γ is a probability measure, for $\ln(t)/2 < x_0/\sqrt{N_0(t)}$ we have

$$\gamma([X_N < -\ln(t)]) \le 1 \le C\, e^{-\delta(\sqrt{N_0(t)}\,\ln(t)/2)^{1/l}}.$$

If $\ln(t)/2 \ge x_0/\sqrt{N_0(t)}$, we have

$$\widetilde{q} \ge \Big(\frac{\sqrt{N_0(t)}}{2\,C1}\,\frac{x_0}{\sqrt{N_0(t)}}\Big)^{1/l} = \Big(\frac{2^{l+1}\,C_1}{2\,C1}\Big)^{1/l} = 2$$

and

$$\ln\Big(\frac{\sqrt{N_0(t)}\,\ln(t)}{C_1\,\widetilde{q}^l\,2}\Big) = \ln\Big(\frac{\sqrt{N_0(t)}\,\ln(t)\,4\,C_1}{C_1\,\sqrt{N_0(t)}\,\ln(t)\,2}\Big) = \ln(2).$$

Therefore we can apply step 1 with $q = \widetilde{q}$ and get

$$\begin{aligned}\gamma([X_N < -\ln(t)]) &\le \exp\Big(-\widetilde{q}\,\ln\big(\frac{\sqrt{N_0(t)}\,\ln(t)}{C_1\,\widetilde{q}^l\,2}\big)\Big) = \exp(-\widetilde{q}\,\ln(2))\\ &= \exp\Big(-\big(\frac{\sqrt{N_0(t)}\,\ln(t)}{4\,C_1}\big)^{1/l}\,\ln(2)\Big)\\ &= \exp\big(-\delta(\sqrt{N_0(t)}\,\ln(t)/2)^{1/l}\big).\end{aligned}$$

Step 3: We show that there exists $\lambda > 0$ s.t.

$$\gamma([X_N < -\ln(t)]) \le C\, e^{-\lambda\,\ln(t)^2}$$

for $t \ge 1$ and $N \in \mathbb{N}$. First note that $N_0(t) \ge \exp\big((\ln(t)/(2\,C_2))^{1/l}\big) - 1$ and $4l^2 - 2l \ge 1$ since $l \ge 1$. Using the inequality $e^y - 1 \ge y^k/k!$, valid for every $y \ge 0$ and $k \in \mathbb{N}$, with $y := (\ln(t)/(2\,C_2))^{1/l}$ and $k := 4l^2 - 2l$, we get

$$\begin{aligned}\Big(\frac{\ln(t)}{2}\,\sqrt{N_0(t)}\Big)^{1/l} &\ge \Big(\frac{\ln(t)}{2}\,\sqrt{e^y - 1}\Big)^{1/l} \ge \Big(\frac{\ln(t)}{2}\,\sqrt{y^k/k!}\Big)^{1/l} = \Big(\frac{\ln(t)}{2}\,\frac{\ln(t)^{2l-1}}{\sqrt{k!}\,(2\,C_2)^{2l-1}}\Big)^{1/l}\\ &= \ln(t)^2\,(4\,\sqrt{k!}^{1/l}\,C_2^{2-1/l})^{-1}.\end{aligned}$$

Together with step 2, we see that the desired estimate holds true with

$$\lambda := \delta\,(4\sqrt{(4l^2 - 2l)!}^{1/l}\,C_2^{2-1/l})^{-1}.$$

Proof of (II): From assumption *(i)* we see that $\|X - X_N\|_{L^q} \to 0$ for $N \to \infty$. This implies $X_N \xrightarrow{\gamma} X$ and as composition of continuous functions also $e^{-pX_N} \xrightarrow{\gamma} e^{-pX}$ for every $p \ge 1$. For $\epsilon > 0$ and $N \in \mathbb{N}$ we define

$$B_{N,\epsilon} := \big[|e^{-X} - e^{-X_N}| < \epsilon\big].$$

Note that $e^{-X_N} \xrightarrow{\gamma} e^{-X}$ is equivalent to $\gamma(B^c_{N,\epsilon}) \to 0$ for $N \to \infty$. By the triangle inequality and Cauchy-Schwarz, we get

$$\begin{aligned}\|e^{-X} - e^{-X_N}\|_{L^p} &\le \Big\|(e^{-X} - e^{-X_N})\,\mathbb{1}_{B_{N,\epsilon}}\Big\|_{L^p} + \Big\|(e^{-X} - e^{-X_N})\,\mathbb{1}_{B^c_{N,\epsilon}}\Big\|_{L^p}\\ &\le \epsilon\,\gamma(B_{N,\epsilon})^{1/p} + \|e^{-X} - e^{-X_N}\|_{L^{2p}}\,\gamma(B^c_{N,\epsilon})^{1/(2p)}.\end{aligned} \tag{10}$$

Note that by Fatou's Lemma and *(I)* we have

$$\|e^{-X}\|_{L^{2p}} \leq \liminf_{N\to\infty} \|e^{-X_N}\|_{L^{2p}} \leq C_{2p}.$$

Thus

$$\begin{aligned}\|e^{-X} - e^{-X_N}\|_{L^p} &\leq \epsilon + \big(\|e^{-X}\|_{L^{2p}} + \|e^{-X_N}\|_{L^{2p}}\big)\,\gamma(B^c_{N,\epsilon})^{1/(2p)} \\ &\leq \epsilon + 2\,C_{2p}\,\gamma(B^c_{N,\epsilon})^{1/(2p)}.\end{aligned}$$

Taking the lim sup in N on both sides finally concludes the proof, since $\gamma(B^c_{N,\epsilon}) \to 0$ and $\epsilon > 0$ was arbitrary. ❏

5.24 Corollary. *Let $p \geq 1$, $n \in 2\mathbb{N}$ and $U_{n,N}$, U_n the potentials defined in* (6) *and* (7)*, respectively. Then there exists a constant $C_p > 0$ s.t.*

$$\|e^{-U_{n,N}}\|_{L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq C_p$$

for every $N \in \mathbb{N}$. Moreover

$$\lim_{N\to\infty} \|e^{-U_n} - e^{-U_{n,N}}\|_{L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma)} = 0.$$

In particular we have $e^{-U_n} \in L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma)$.

Proof. We only need to show that the assumptions of Theorem 5.23 are satisfied for $X_N := U_{n,N}$ and $X := U_n$.

Step 1: We show that

$$\|U_{n,M} - U_{n,N}\|_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq C_1/\sqrt{N} \tag{11}$$

for $M, N \in \mathbb{N}$, $N < M$, where C_1 depends only on n. Indeed, from the proof of Theorem 5.9 we know

$$\|U_{n,M} - U_{n,N}\|^2_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq C_1\Big(\sum_{N<|k|\leq M} (m + 4\pi^2|k|^2)^{-2}\Big)^{1/2}$$

for $N < M$, where C_1 depends only on n. Bounding the sum by an integral yields

$$\begin{aligned}\sum_{N<|k|\leq M} (m+4\pi^2|k|^2)^{-2} \leq \sum_{N<|k|} (m+4\pi^2|k|^2)^{-2} &\leq 2\int_{N-1<|x|} (m+4\pi^2|x|^2)^{-2}\,dx \\ &= 2\int_0^{2\pi}\int_{N-1}^{\infty} \frac{r}{(m+4\pi^2r^2)^2}\,dr\,d\phi \\ &= 4\pi\,\frac{-1}{8\pi^2(m+4\pi^2r^2)}\Big|_{N-1}^{\infty} \\ &= \frac{1}{2\pi(m+4\pi^2(N-1)^2)} \leq \frac{1}{N^2}.\end{aligned}$$

Thus $\|U_{n,M} - U_{n,N}\|_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq C_1/\sqrt{N}$ for $N < M$.

Step 2: We show that

$$\|U_{n,M} - U_{n,N}\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq (q-1)^{n/2}\,C_1/\sqrt{N} \tag{12}$$

for $M, N \in \mathbb{N}$, $N < M$ and arbitrary $q \geq 2$. Since by Lemma 5.22, $U_{n,M}$ and $U_{n,N}$ are elements of the n-th Wiener Chaos, we have $I_n(U_{n,M}) = U_{n,M}$ and $I_n(U_{n,N}) = U_{n,N}$. Therefore Corollary 5.21 yields

$$\begin{aligned}\|U_{n,M} - U_{n,N}\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} &= \|I_n(U_{n,M} - U_{n,N})\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \\ &\leq (q-1)^{n/2} \, \|U_{n,M} - U_{n,N}\|_{L^2(H^{-\epsilon}(\mathbb{T}^2),\gamma)}\end{aligned}$$

for $q \geq 2$ and together with (11) we deduce (12).

Step 3: We show that

$$U_{n,N}(\varphi) \geq -C_2 \, (\ln N)^{n/2} \tag{13}$$

for $N \in \mathbb{N}$, $N \geq 2$ and $\varphi \in H^{-\epsilon}(\mathbb{T}^2)$, where C_2 depends only on n. Since the n-th Hermite polynomial H_n is a polynomial of even degree n with positive leading coefficient, there exists a constant $C > 0$ depending on n, s.t. $H_n > -C$. Thus

$$U_{n,N}(\varphi) = \int_{\mathbb{T}^2} :\varphi_N^n:(x) \, dx = \int_{\mathbb{T}^2} \sqrt{n!} \, \rho_N^n \, H_n(\varphi_N(x)/\rho_N) \, dx \geq -C \sqrt{n!} \, \rho_N^n.$$

Together with

$$\begin{aligned}\rho_N^2 &= \sum_{|k| \leq N} (m + 4\pi^2 |k|^2)^{-1} \leq 2 \int_{|x| \leq N} (m + 4\pi^2 |x|^2)^{-1} \, dx \\ &= 2 \int_0^{2\pi} \int_0^N \frac{r}{m + 4\pi^2 r^2} \, dr \, d\phi \\ &= 4\pi \, \frac{1}{8\pi^2} \, \ln(m + 4\pi^2 r^2) \Big|_0^N \\ &= \frac{1}{2\pi} \, \ln(m + 4\pi^2 N^2) \leq 2 \ln N,\end{aligned}$$

where the last inequality holds for $N \geq \max\{2, m\}$, we get (13).

❑

5.25 Corollary. *Let $p \geq 1$, $n \in \mathbb{N}$ and $U_{n,N}$, U_n the potentials defined in (6) and (7), respectively. We define potentials U_N and U by*

$$U_N := \sum_{k=1}^{2n} a_k \, U_{k,N}$$

and

$$U := \sum_{k=1}^{2n} a_k \, U_k,$$

where $a_i \in \mathbb{R}$ for $i = 1, \ldots, 2n$ and $a_{2n} > 0$.

Then there exists a constant $C_p > 0$ s.t.

$$\left\| e^{-U_N} \right\|_{L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq C_p$$

for every $N \in \mathbb{N}$. Moreover

$$\lim_{N \to \infty} \left\| e^{-U} - e^{-U_N} \right\|_{L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma)} = 0.$$

In particular we have $e^{-U} \in L^p(H^{-\epsilon}(\mathbb{T}^2), \gamma)$*.*

Proof. We only need to show that the assumptions of Theorem 5.23 are satisfied for $X_N := U_N$ and $X := U$.

From the proof of Corollary 5.24 we already know that

$$\|U_{n,M} - U_{n,N}\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \leq (q-1)^{n/2}\, C_{1,n}/\sqrt{N}$$

for $N < M$, $n \in \mathbb{N}$ and $q \geq 2$. Hence by the triangle inequality we get

$$\begin{aligned}
\|U_M - U_N\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} &\leq \sum_{k=1}^{2n} |a_k|\ \|U_{k,M} - U_{k,N}\|_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)} \\
&\leq \sum_{k=1}^{2n} |a_k|\,(q-1)^{k/2}\, C_{1,k}\, N^{-1/2} \\
&\leq q^n\, N^{-1/2} \underbrace{\sum_{k=1}^{2n} |a_k|\, C_{1,k}}_{=:C_1}
\end{aligned}$$

and assumption *(i)* is satisfied.

To show assumption *(ii)*, first note that by definition we have

$$U_N(\varphi) = \sum_{k=1}^{2n} a_k\, U_{k,N} = \sum_{k=1}^{2n} a_k \int_{\mathbb{T}^2} \sqrt{k!}\, \rho_N^k\, H_k(\varphi_N(x)/\rho_N)\, dx$$

for $\varphi \in H^{-\epsilon}(\mathbb{T}^2)$. We define a polynomial P by $P(y) := \sum_{k=1}^{2n} a_k \sqrt{k!}\, \rho_N^k\, H_k(y)$. Then

$$\begin{aligned}
P(y) &= \rho_N^{2n}\left(a_{2n}\sqrt{(2n)!}\, H_{2n}(y) + \sum_{k=1}^{2n-1} a_k \sqrt{k!}\, \rho_N^{k-2n}\, H_k(y)\right) \\
&\geq \rho_N^{2n}\left(a_{2n}\sqrt{(2n)!}\, H_{2n}(y) - \sum_{k=1}^{2n-1} |a_k| \sqrt{k!}\, \underbrace{\rho_N^{k-2n}}_{\leq 1}\, |H_k(y)|\right) \\
&\geq \rho_N^{2n}\left(a_{2n}\sqrt{(2n)!}\, H_{2n}(y) - \sum_{k=1}^{2n-1} |a_k| \sqrt{k!}\, |H_k(y)|\right).
\end{aligned}$$

Since the latter one is bounded from below, we get $P(y) \geq -C\rho_N^{2n}$ for some constant $C > 0$ independent of N and hence

$$U_N(\varphi) = \int_{\mathbb{T}^2} P(\varphi_N(x)/\rho_N)\, dx \geq -C\rho_N^{2n}.$$

From step 3 in the proof of Corollary 5.24 we get that

$$U_N(\varphi) \geq -C\, \ln(N)^n,$$

and the proof is complete. ❏

5.26 Remark. For $n \in 2\mathbb{N}$ and $p \geq 1$ we also have $e^{-U_n} \in L^p(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$, since for $N \in \mathbb{N}$ we have

$$\begin{aligned}\|e^{-U_{n,N}}\|^p_{L^p(H^{-\epsilon}(\mathbb{T}^2),\gamma_\beta)} &= \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-pU_{n,N}(\varphi)}\,\gamma_\beta(d\varphi) = \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-pU_{n,N}(\varphi/\sqrt{\beta})}\,\gamma(d\varphi)\\ &= \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-p/\sqrt{\beta}^n U_{n,N}(\varphi)}\,\gamma(d\varphi) = \|e^{-U_{n,N}}\|^{p/\sqrt{\beta}^n}_{L^{p/\sqrt{\beta}^n}(H^{-\epsilon}(\mathbb{T}^2),\gamma)}\\ &\leq C^{p/\sqrt{\beta}^n}_{p/\sqrt{\beta}^n}.\end{aligned}$$

Thus the proof of Theorem 5.23 (II) implies $e^{-U_n} \in L^p(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$.

In addition, for U from Corollary 5.25 we have $e^{-U} \in L^p(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$, since for $N, M \in \mathbb{N}$ and $q \geq 2$ we have

$$\begin{aligned}\|U_{n,N} - U_{n,M}\|^q_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma_\beta)} &= \int_{H^{-\epsilon}(\mathbb{T}^2)} |U_{n,N}(\varphi) - U_{n,M}(\varphi)|^q\,\gamma_\beta(d\varphi)\\ &= \int_{H^{-\epsilon}(\mathbb{T}^2)} |U_{n,N}(\varphi/\sqrt{\beta}) - U_{n,M}(\varphi/\sqrt{\beta})|^q\,\gamma(d\varphi)\\ &= \int_{H^{-\epsilon}(\mathbb{T}^2)} \beta^{-nq/2}|U_{n,N}(\varphi) - U_{n,M}(\varphi)|^q\,\gamma(d\varphi)\\ &= \beta^{-nq/2}\,\|U_{n,N} - U_{n,M}\|^q_{L^q(H^{-\epsilon}(\mathbb{T}^2),\gamma)}.\end{aligned}$$

Thus the proof of Corollary 5.25 also yields $e^{-U} \in L^p(H^{-\epsilon}(\mathbb{T}^2), \gamma_\beta)$. //

5.27 Remark. Note that this strategy fails in dimension $d > 2$. Another construction of the measure ν_β is achieved in [4]. Their method allows also to construct the measure in $d = 3$. //

6 Laplace Asymptotics on $H^{-\epsilon}(\mathbb{T}^2)$

Following [2], the goal is to study the asymptotic behavior of

$$\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi)$$

in the limit $\beta \to \infty$ for

$$U(\varphi) = \sum_{k=3}^{2n} a_k \int_{\mathbb{T}^2} :\varphi^k:(x)\,dx = \sum_{k=3}^{2n} a_k U_k(\varphi),$$

defined in Corollary 5.25. Since U is not an element of $C^2(H^{-\epsilon}(\mathbb{T}^2), \gamma)$, we cannot use the results of Section 2.

6.1 Theorem. *Consider $U_4(\varphi) = \int_{\mathbb{T}^2} :\varphi^4:(x)\,dx$, defined in (7). Then*

$$\lim_{\beta\to\infty} \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\beta U_4(\varphi)}\,\gamma_\beta(d\varphi) = 1.$$

Proof. Since $U_4(\varphi/\sqrt{\beta}) = \beta^{-2}\,U_4(\varphi)$, we have for $\beta > 1$ that

$$\begin{aligned} e^{-\beta^{-1}U_4(\varphi)} &= e^{-\beta^{-1}U_4(\varphi)}\,\mathbb{1}_{[U_4\geq 0]}(\varphi) + e^{-\beta^{-1}U_4(\varphi)}\,\mathbb{1}_{[U_4<0]}(\varphi) \\ &\leq 1 + e^{-U_4(\varphi)}\,\mathbb{1}_{[U_4<0]}(\varphi) \\ &\leq 1 + e^{-U_4(\varphi)}, \end{aligned}$$

and the latter one is integrable by Corollary 5.24. Thus we can change limit and integral to see

$$\begin{aligned} \lim_{\beta\to\infty} \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\beta U_4(\varphi)}\,\gamma_\beta(d\varphi) &= \lim_{\beta\to\infty} \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\beta U_4(\varphi/\sqrt{\beta})}\,\gamma(d\varphi) \\ &= \lim_{\beta\to\infty} \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\beta^{-1}U_4(\varphi)}\,\gamma(d\varphi) \\ &= \int_{H^{-\epsilon}(\mathbb{T}^2)} \lim_{\beta\to\infty} e^{-\beta^{-1}U_4(\varphi)}\,\gamma(d\varphi) \\ &= 1. \end{aligned}$$

❑

The next more general result is taken from [2]. We shall give here the details of some aspects of the proof.

6.2 Theorem. *Let $n \in \mathbb{N}$, $n \geq 2$, $a_3, \ldots, a_{2n} \in \mathbb{R}$, $a_{2n} > 0$ and define the potential*

$$U(\varphi) := \sum_{k=3}^{2n} a_k \int_{\mathbb{T}^2} :\varphi^k:(x)\,dx = \sum_{k=3}^{2n} a_k U_k(\varphi),$$

defined in Corollary 5.25.

Then there exists $\alpha > 0$ s.t.

$$\lim_{\beta\to\infty} \int_{B_\alpha} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi) = 1,$$

where $B_\alpha = \left\{\varphi \in H^{-\epsilon}(\mathbb{T}^2) \,:\, \|\varphi\|_{H^{-\epsilon}(\mathbb{T}^2)} \le \alpha\right\}$.

Proof. We define

$$V(\varphi) := \sum_{k=3}^{2n} |a_k U_k(\varphi)|^{1/k}$$

and observe that

$$\underbrace{\int_{B_\alpha} e^{-\beta U(\varphi)}\,\gamma_\beta(d\varphi)}_{=:I(\beta)} = \underbrace{\int_{B_\alpha} e^{-\beta U(\varphi)}\,\mathbb{1}_{[V(\varphi)>\delta]}(\varphi)\,\gamma_\beta(d\varphi)}_{=:I_1(\beta)} + \underbrace{\int_{B_\alpha} e^{-\beta U(\varphi)}\,\mathbb{1}_{[V(\varphi)\le\delta]}(\varphi)\,\gamma_\beta(d\varphi)}_{=:I_2(\beta)}$$

for $\delta \in (0,1)$.

Step 1: One shows that $I_1(\beta) \to 0$ if $\beta \to \infty$ for $\alpha > 0$ sufficiently small. The main tool is a Large Deviation estimate, see [2, Theorem 2.15, Lemma 2.16] for details.

Step 2: We show that

$$\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\delta V(\varphi)^2}\gamma(d\varphi) \le \liminf_{\beta\to\infty} I_2(\beta) \le \limsup_{\beta\to\infty} I_2(\beta) \le \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{\delta V(\varphi)^2}\gamma(d\varphi).$$

First note that by Lemma 2.2 we have

$$I_2(\beta) = \int_{B_{\alpha\sqrt{\beta}}} e^{-\beta U(\varphi/\sqrt{\beta})}\,\mathbb{1}_{V(\varphi/\sqrt{\beta})\le\delta}(\varphi)\,\gamma(d\varphi).$$

For $V(\varphi/\sqrt{\beta}) \le \delta$ we have $|a_k U_k(\varphi/\sqrt{\beta})|^{1/k} \le V(\varphi/\sqrt{\beta}) \le \delta < 1$, hence

$$\begin{aligned}
|U(\varphi/\sqrt{\beta})| &= \left|\sum_{k=3}^{2n} a_k U_k(\varphi/\sqrt{\beta})\right| \\
&\le \sum_{k=3}^{2n} |a_k U_k(\varphi/\sqrt{\beta})|^{k/k} \\
&\le \sum_{k=3}^{2n} |a_k U_k(\varphi/\sqrt{\beta})|^{3/k} \\
&\le \left(\sum_{k=3}^{2n} |a_k U_k(\varphi/\sqrt{\beta})|^{1/k}\right)^3 \\
&= V(\varphi/\sqrt{\beta})^3.
\end{aligned}$$

Thus

$$-\delta V(\varphi/\sqrt{\beta})^2 \le -V(\varphi/\sqrt{\beta})^3 \le -U(\varphi/\sqrt{\beta}) \le V(\varphi/\sqrt{\beta})^3 \le \delta V(\varphi/\sqrt{\beta})^2.$$

Since $V(\varphi/\sqrt{\beta}) = 1/\sqrt{\beta}\,V(\varphi)$ we get

$$\begin{aligned}
e^{-\delta V(\varphi)^2}\,\mathbb{1}_{[V(\varphi)\le\delta\sqrt{\beta}]}(\varphi) &= e^{-\delta\beta V(\varphi/\sqrt{\beta})^2}\,\mathbb{1}_{[V(\varphi/\sqrt{\beta})\le\delta]}(\varphi) \\
&\le e^{-\beta U(\varphi/\sqrt{\beta})}\,\mathbb{1}_{[V(\varphi/\sqrt{\beta})\le\delta]}(\varphi) \\
&\le e^{\delta\beta V(\varphi/\sqrt{\beta})^2}\,\mathbb{1}_{[V(\varphi/\sqrt{\beta})\le\delta]}(\varphi) \\
&= e^{\delta V(\varphi)^2}\,\mathbb{1}_{[V(\varphi)\le\delta\sqrt{\beta}]}(\varphi).
\end{aligned}$$

Finally Fatou's Lemma yields

$$\begin{aligned}\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\delta V(\varphi)^2}\gamma(d\varphi) &\le \liminf_{\beta\to\infty} \int_{B_{\sqrt{\alpha\beta}}} e^{-\delta V(\varphi)^2}\, \mathbb{1}_{[V(\varphi)\le\delta\sqrt{\beta}]}(\varphi)\,\gamma(d\varphi)\\ &\le \liminf_{\beta\to\infty} \int_{B_{\sqrt{\alpha\beta}}} e^{-\beta U(\varphi/\sqrt{\beta})}\, \mathbb{1}_{[V(\varphi/\sqrt{\beta})\le\delta]}(\varphi)\,\gamma(d\varphi)\\ &\le \limsup_{\beta\to\infty} \int_{B_{\sqrt{\alpha\beta}}} e^{-\beta U(\varphi/\sqrt{\beta})}\, \mathbb{1}_{[V(\varphi/\sqrt{\beta})\le\delta]}(\varphi)\,\gamma(d\varphi)\\ &\le \limsup_{\beta\to\infty} \int_{B_{\sqrt{\alpha\beta}}} e^{\delta V(\varphi)^2}\, \mathbb{1}_{[V(\varphi)\le\delta\sqrt{\beta}]}(\varphi)\,\gamma(d\varphi)\\ &\le \int_{H^{-\epsilon}(\mathbb{T}^2)} e^{\delta V(\varphi)^2}\gamma(d\varphi).\end{aligned}$$

Step 3: One shows that $\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{\delta V(\varphi)^2}\,\gamma(d\varphi) < \infty$ for $\delta > 0$ sufficiently small. See [2, Theorem 2.15, Lemma 2.16] for details.

Step 4: We finally show that $1 \le \liminf I(\beta) \le \limsup I(\beta) \le 1$. By the previous steps we have

$$\begin{aligned}\limsup_{\beta\to\infty} I(\beta) &= \limsup_{\delta\to 0}\left(\limsup_{\beta\to\infty} I(\beta)\right)\\ &\le \limsup_{\delta\to 0}\Big(\underbrace{\limsup_{\beta\to\infty} I_1(\beta)}_{=0} + \limsup_{\beta\to\infty} I_2(\beta)\Big)\\ &= \limsup_{\delta\to 0}\left(\limsup_{\beta\to\infty} I_2(\beta)\right)\\ &\le \limsup_{\delta\to 0}\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{\delta V(\varphi)^2}\,\gamma(d\varphi)\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\liminf_{\beta\to\infty} I(\beta) &= \liminf_{\delta\to 0}\left(\liminf_{\beta\to\infty} I(\beta)\right)\\ &\ge \liminf_{\delta\to 0}\Big(\underbrace{\liminf_{\beta\to\infty} I_1(\beta)}_{=0} + \liminf_{\beta\to\infty} I_2(\beta)\Big)\\ &= \liminf_{\delta\to 0}\left(\liminf_{\beta\to\infty} I_2(\beta)\right)\\ &\ge \liminf_{\delta\to 0}\int_{H^{-\epsilon}(\mathbb{T}^2)} e^{-\delta V(\varphi)^2}\,\gamma(d\varphi)\\ &= 1.\end{aligned}$$

❑

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