CONSTANT CONGESTION BRAMBLES IN DIRECTED GRAPHS[*]

TOMÁŠ MASAŘÍK[†], MARCIN PILIPCZUK[‡], PAWEŁ RZĄŻEWSKI[§], AND
MANUEL SORGE[¶]

Abstract. The Directed Grid Theorem, stating that there is a function f such that a directed graph of directed treewidth at least $f(k)$ contains a directed grid of size at least k as a butterfly minor, after being a conjecture for nearly 20 years, was proved in 2015 by Kawarabayashi and Kreutzer. However, the function f obtained in the proof is very fast growing. In this work, we show that if one relaxes *directed grid* to *bramble of constant congestion*, one can obtain a polynomial bound. More precisely, we show that for every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of congestion at most 8 and size at least k.

1. Introduction. The Grid Minor Theorem, proved by Robertson and Seymour [19], is arguably one of the most important structural characterizations of treewidth. Informally speaking, it asserts that a *grid minor* is a canonical obstacle to small treewidth: a graph of large treewidth necessarily contains a big grid as a minor. The relation of "large" and "big" in this statement, being nonelementary in the original proof, after a series of improvements has been proved to be a polynomial of relatively small degree.

THEOREM 1.1 (Chuzhoy, Tan [4]). *For every $k \geq 1$ there exists $t = \mathcal{O}(k^9 \mathrm{polylog} k)$ such that every graph of treewidth at least t contains a $k \times k$ grid as a minor.*

In the mid-1990s, Johnson et al. [9] proposed an analogue of treewidth for directed graphs, called *directed treewidth*, and conjectured an analogous statement (with the appropriate notion of a directed grid). After nearly 20 years, the Directed Grid Theorem was proved in 2015 by Kawarabayashi and Kreutzer [12]. However, their proof yields a very high dependency between the required directed treewidth bound and the promised size of the directed grid.

[†]Institute of Informatics, University of Warsaw, Warsaw, 02-097, Poland, and Department of Mathematics, Simon Fraser University, BC, Canada (masarik@mimuw.edu.pl).

[‡]Institute of Informatics, University of Warsaw, Warsaw, 02-097, Poland (malcin@mimuw.edu.pl).

[§]Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, 00-662, Poland, and Institute of Informatics, University of Warsaw, Warsaw, 02-097, Poland (pawel.rzazewski@pw.edu.pl).

[¶]Institute of Informatics, University of Warsaw, Warsaw, 02-097, Poland, and Faculty of Informatics, TU Wien, Vienna, Austria (manuel.sorge@ac.tuwien.ac.at).

While searching for better and better bounds for the (undirected) Grid Minor Theorem, researchers investigated relaxed notions of a grid (e.g., Kawarabayashi, Kobayashi, and Kreutzer [10]). In some sense, the "most relaxed" notion of a grid is a *bramble*: a family $\mathcal{B}$ of connected subgraphs of a given graph such that every two $B_1, B_2 \in \mathcal{B}$ intersect in at least one vertex or there exists an edge with one endpoint in B_1 and one endpoint in B_2. Brambles can be large; the notion of complexity of a bramble is its *order*—the minimum size of a vertex set that intersects every element of a bramble. We also refer to the *size* of a bramble as the number of its elements and the *congestion* of a bramble as a maximum number of elements that contain a single vertex; note that the size of the bramble is bounded by the product of its order and congestion.

To link brambles and grids, first note that a $k \times k$ grid contains a simple bramble of order k and size k^2: the elements of a bramble are the subgraphs that consist of the ith row and the jth column of the grid for $1 \leq i, j \leq k$. If one wants a bramble of congestion 2, a bramble of size k whose elements are subgraphs consisting of the ith row and ith column of the grid for every $1 \leq i \leq k$ is of order $\lceil k/2 \rceil$. In the other direction, brambles of small congestion can replace grids if one wants to use a grid as an object that allows arbitrary interconnections of small congestion between different pairs of vertices on its boundary. Such a usage appears, e.g., in arguments for the Disjoint Paths problem (cf. [1, 2, 3, 5]).

Surprisingly, as proved by Seymour and Thomas [20], brambles form a dual object tightly linked to treewidth: the maximum order of a bramble in a graph is *exactly* the treewidth of the graph plus one. However, as shown by Grohe and Marx [6] and sharpened by Hatzel et al. [8], brambles of high order may need to have exponential size: while a graph of treewidth k necessarily contains a bramble of order $\widetilde{\Omega}(\sqrt{k})$ of congestion 2 (and thus of size linear in their order), there are classes of graphs (e.g., constant-degree expanders) where for every $0 < \delta < 1/2$ any bramble of order $\widetilde{\Omega}(k^{0.5+\delta})$ requires size exponential in roughly $k^{2\delta}$. Here, the notation $\widetilde{\Omega}$ and $\widetilde{\mathcal{O}}$ omits polylogarithmic factors.

A slightly more organized bramble of congestion 2, namely two families of vertex-disjoint paths with an intersection graph containing a large clique minor (with size bound of quartic dependence on the treewidth), has been shown to exist in undirected graphs by Reed and Wood [18].

In directed graphs, the notion of a bramble naturally generalizes to the notion of a *directed bramble*, a family of strongly connected subgraphs such that every two subgraphs intersect in a vertex, or the graph contains an arc with a tail in the first subgraph and a head in the second and an arc with a tail in the second subgraph and a head in the first. The order of a directed bramble is defined in the same way as in undirected graphs. For brevity we usually shorten "directed bramble" to "bramble" below. While we no longer have a tight relation between the directed treewidth and the maximum order of a directed bramble, these two graph parameters are within a constant factor of each other, as shown by Reed [17]. However, the lower bound of Grohe and Marx [6] also applies to directed graphs: There are digraph families where a graph of directed treewidth k contains only brambles of order $k^{0.5+\delta}$ of exponential size for any $0 < \delta < 0.5$.

Hence, it is natural to ask what order of a bramble of constant congestion we can expect in a directed graph of directed treewidth t. The lower bound of Grohe and Marx shows that we cannot hope for a better answer than $\widetilde{\mathcal{O}}(\sqrt{t})$. Since a directed grid contains a bramble of congestion 2 and order linear in the size of the grid, the Directed

Grid Theorem implies that for every $k \geq 1$ there exists $t = t(k)$ such that directed treewidth at least t guarantees the existence of a bramble of order k and congestion 2. However, the function $t = t(k)$ stemming from the proof of Kawarabayashi and Kreutzer [12] is very fast-growing. Similarly, a half-integral variant of the Directed Grid Theorem [10] could be used to obtain a bramble of order k and congestion 4, but also there the function $t = t(k)$ is very fast-growing.

In this work, we show that this dependency can be made polynomial if we are satisfied with slightly larger congestion.

THEOREM 1.2. *For every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of congestion at most* 8 *and size at least k.*

So far, similar bounds were known only for planar graphs, where Hatzel, Kawarabayashi, and Kreutzer showed a polynomial bound (with degree 6 of the polynomial) for the Directed Grid Theorem [7]. Decreasing the congestion in Theorem 1.2, ideally to 2, even at the cost of higher polynomial dependency of t on k, remains an interesting open problem. Optimizing the parameters in the other direction would also be interesting; for all we know, obtaining the dependency $t = \widetilde{\mathcal{O}}(k^2)$ for constant congestion may be possible.

On the technical level, the proof of Theorem 1.2 borrows a number of tools from previous works. From Reed and Wood [18], we borrow the idea of using Kostochka–Thomason degeneracy bounds for graphs excluding a minor [13, 21] to ensure the existence of a large clique minor in an intersection graph of a family of strongly connected subgraphs, if it turns out to be dense (which immediately gives a desired bramble). We also use their Lovász Local Lemma–based argument to find a large independent set in a multipartite graph of low degeneracy. Similarly as in the proof of the Directed Grid Theorem [12] and in its planar variant [7], we start from the notion of a path system and its existence (with appropriate parameters) in graphs of high directed treewidth. Finally, from our recent proof of the half- and quarter-integral directed Erdős–Pósa property [14, 15], we reuse the Partitioning Lemma, allowing us to find a large number of closed walks with small congestion. On top of the above, compared to [7] and [14, 15], the proof of Theorem 1.2 offers a much more elaborate analysis of the studied path system, allowing us to find the desired bramble.

Organization. We collect the formal statements of results from previous work in section 2. In section 3 we gather tools that show how to obtain a low congestion bramble in various special situations, and we show how to obtain some intermediate structures. In section 4 we then show how to combine all the tools to prove Theorem 1.2.

2. Preliminaries and notation. For integers $n \in \mathbb{N}$ we use $[n]$ to denote $\{1, 2, \ldots, n\}$.

Basics. Throughout we use D to denote directed graphs and G, H for undirected graphs. Let D be a directed graph. A *walk* in D is a sequence W of vertices such that for each pair of consecutive vertices u, v in W there is an arc (u, v) in D. A walk is *closed* if it starts and finishes with the same vertex. Let W be a walk. We denote by $V(W)$ the set of vertices which occur in the sequence W. A *subwalk* of W is a segment of W, that is, a subsequence of consecutive elements. The number of *occurrences* of a vertex v in W, denoted by $\mathrm{oc}(v, W)$, is the number of times it occurs in the sequence W; if W is closed and v is its starting vertex, then it is the number of times v occurs in the sequence W minus one. The *length* of a walk W is the sum

of the numbers of occurrences of the vertices in $V(W)$.

DEFINITION 2.1 (congestion, overlap). *Let $\mathcal{W}$ be a family of walks in D and $\mathcal{S}$ be a family of subsets of $V(D)$. We define*

$$overlap(\mathcal{W}) := \max_{v \in V(D)} \sum_{W \in \mathcal{W}} \mathrm{oc}(v, W),$$

$$congestion(\mathcal{S}) := \max_{v \in V(D)} |\{S \in \mathcal{S} \mid v \in S\}|.$$

For a set of walks $\mathcal{W}$, its congestion is the congestion of the family $\{V(W) \mid W \in \mathcal{W}\}$.

Linkages, path systems, minors. Let D be an undirected graph. For $A, B \subseteq V(D)$, such that $|A| = |B|$, a *linkage* from A to B in D is a set of $|A|$ pairwise vertex-disjoint paths in D, each with a starting vertex in A and an ending vertex in B. A set $X \subseteq V(D)$ is *well linked* if for every $A, B \subseteq X$, s.t. $|A| = |B|$ there are $|A|$ vertex-disjoint A-B-paths in $D - (X \setminus (A \cup B))$.

DEFINITION 2.2 (path system). *Let $a, b \in \mathbb{N}$. An (a,b)-path system $(P_i, A_i, B_i)_{i=1}^a$ consists of*
- *vertex-disjoint paths $P_1, P_2, \ldots, P_a$, and*
- *for every $i \in [a]$, two sets $A_i, B_i \subseteq V(P_i)$, each of size b, such that every vertex of B_i appears on P_i later than all vertices of A_i,*

such that $\bigcup_{i=1}^a A_i \cup B_i$ is well linked in D.

In this work, we do not need the exact (and involved) definition of directed treewidth; instead, we immediately jump to path systems via the following lemma.

LEMMA 2.3 (Kawarabayashi, Kreutzer [12, 11] (implicit); see also [14, Lemma 7]). *There exists a constant c_{KK} such that for every two integers $a, b \geq 1$ every directed graph D of directed treewidth at least $c_{KK} \cdot a^2 b^2$ contains an (a,b)-path system.*

The *average degree* of an undirected graph with n vertices and m edges is $2m/n$. An undirected graph G is *d-degenerate* if every subgraph of G contains a vertex of degree at most d. The *degeneracy* of an undirected graph G is the minimum integer $d \in \mathbb{N}$ such that G is d-degenerate. Observe that for every undirected graph G we have $\Delta_a(G) \leq 2d(G)$, where $\Delta_a(G)$ is G's average degree and $d(G)$ is G's degeneracy. The degeneracy is linked to clique minors. Let G be an undirected graph. An undirected graph H is called a *minor* of G if it results from a subgraph G' of G by partitioning the vertices of G' into sets $V_1, V_2, \ldots, V_\ell$ that each induce a connected subgraph of G' and then contracting each set V_i, $i \in [\ell]$, into a single vertex. The sets $V_1, V_2, \ldots, V_\ell$ are also called the *branch sets* of H.

THEOREM 2.4 (Kostochka [13, Theorem 1], Thomason [21, Theorem] (restated)). *There exists a constant c_{KT}, such that for every $a \geq 2$, every undirected graph G with degeneracy at least $c_{KT} \cdot a \cdot \sqrt{\log a}$ contains K_a as a minor.*

LEMMA 2.5 (Reed and Wood [18, Lemma 4.3]). *Let r be an integer with $r \geq 2$, d be a positive real, and H be an r-colored undirected graph with the color classes $V_1, \ldots, V_r$. Assume that for every $i \in [r]$ it holds that $|V_i| \geq 4e(r-1)d$ and for every $i \neq j$ the graph $H[V_i \cup V_j]$ is d-degenerate. Then there exists an independent set $\{x_1, \ldots, x_r\}$ such that $x_i \in V_i$ for every $i \in [r]$.*

For a family $\mathcal{S}$ of sets, its *intersection graph*, denoted by $\mathrm{Int}(\mathcal{S})$, is an undirected graph with vertex set $\mathcal{S}$, and two distinct sets $S_1, S_2 \in \mathcal{S}$ are adjacent in $\mathrm{Int}(\mathcal{S})$ if $S_1 \cap S_2 \neq \emptyset$. If $\mathcal{W}$ is a set of walks, then the *intersection graph* $\mathrm{Int}(\mathcal{W})$ is defined as $\mathrm{Int}(\{V(W) \mid W \in \mathcal{W}\})$.

3. Tools. We now gather the new tools that we need in the main proof. The general setting is that, if the directed treewidth of our graph is large enough, then there is a path system (see Definition 2.2) containing a large number of sets A_i, B_i and large linkages between them. We then distinguish several cases for sets of pairs of linkages and the densities of the intersection graphs of the paths in these linkages. We end up with three fundamental scenarios, each of which allows us to define a desired bramble. How the brambles are obtained in these scenarios is shown in subsection 3.1.

In subsection 3.2 we derive a set of tools that allow us to partition paths in the sets of linkages mentioned above in such a way as to keep both the congestion low and the intersection graphs of the parts sufficiently dense.

3.1. Extracting a bramble. The first tool allows us to extract a desired bramble if our directed graph contains a set of walks whose intersection graph is sufficiently dense.

LEMMA 3.1 (dense winning scenario). *Let c_{KT} be the constant from Theorem 2.4. If a directed graph D contains a family $\mathcal{W}$ of closed walks of congestion α, whose intersection graph is not $c_{\mathsf{KT}} \cdot d \cdot \sqrt{\log d}$-degenerate, then D contains a bramble of congestion α and size d.*

Proof. Since $\mathrm{Int}(\mathcal{W})$ is not $c_{\mathsf{KT}} \cdot d \cdot \sqrt{\log d}$-degenerate, by Theorem 2.4, it contains a K_d minor $\mathcal{K}$. Since each branch set of $\mathcal{K}$ induces a connected subgraph of $\mathrm{Int}(\mathcal{W})$, and each vertex of $\mathrm{Int}(\mathcal{W})$ is a closed walk in D, we obtain that each branch set of $\mathcal{K}$ corresponds to a strongly connected subgraph of D. Furthermore, since between any two branch sets of $\mathcal{K}$ there is an edge in $\mathrm{Int}(\mathcal{W})$, we conclude that the subsets of $V(D)$ corresponding to branch sets of $\mathcal{K}$ form a bramble in D. The congestion bound follows from the fact that the congestion of $\mathcal{W}$ is α. $\quad\square$

Using the second tool, we obtain a desired bramble if our directed graph contains a path system that is large enough and there is a subset of paths with low congestion.

LEMMA 3.2 (sparse winning scenario). *There is an absolute constant c with the following property. Let D be a directed graph, let $a > 1$, $b \geq 1$, and let $(P_i, A_i, B_i)_{i=1}^{a}$ be an (a, b)-path system in D. Let $\mathcal{I}$ be a subset of $[a] \times [a] \setminus \{(i, i) \mid i \in [a]\}$, such that $|\mathcal{I}| \geq 0.6 \cdot a(a - 1)$. Assume that for every $(i, j) \in \mathcal{I}$ we have a path $P_{i,j}$ from B_i to A_j such that $\{P_{i,j} \mid (i, j) \in \mathcal{I}\}$ is of congestion at most α. Then D contains a bramble of congestion at most $2 + 2\alpha$ and size at least $c \cdot \left(\frac{a^{1/2}}{\log^{1/4} a}\right)$.*

Proof. Consider an undirected graph H with vertex set $[a]$ and $\{i, j\} \in E(H)$ if both $(i, j) \in \mathcal{I}$ and $(j, i) \in \mathcal{I}$. Since $|\mathcal{I}| \geq 0.6 \cdot a(a - 1)$, we have $|E(H)| \geq 0.1 \cdot \binom{a}{2}$. By Theorem 2.4, H contains a clique minor of size $p \geq c' \cdot a/\sqrt{\log a}$, where c' is an absolute constant. Without loss of generality, assume that $p = \binom{q}{2}$ for some integer q such that $q \geq c \cdot a^{1/2}/\log^{1/4} a$ for some constant c. Let $(B_{x,y})_{\{x,y\} \in \binom{[q]}{2}}$ be the family of branch sets of a clique minor of size p in H. Observe that for every $x \in [q]$, the subgraph of H induced by $\bigcup_{y \in [q] \setminus \{x\}} B_{x,y}$ is connected; let T_x be an arbitrary spanning tree of that subgraph. Note that, for every two distinct $x, y \in [q]$, the trees T_x and T_y intersect in $B_{x,y}$. On the other hand, every vertex and every edge of H are contained in at most two trees T_x.

Now we transfer the set of trees above to a bramble in D. For every $i \in [a]$, let e_i be the last edge of P_i, whose tail is in A_i. Note that e_i is well defined, as the set B_i follows A_i on P_i; see Figure 1 (left). For every edge $e = ij \in E(H)$, let W_e be a closed walk in D obtained as follows. We start with $P_{i,j}$, and then we follow P_j until we arrive at the starting vertex of $P_{j,i}$. Then we follow $P_{j,i}$, and then P_i until we close the
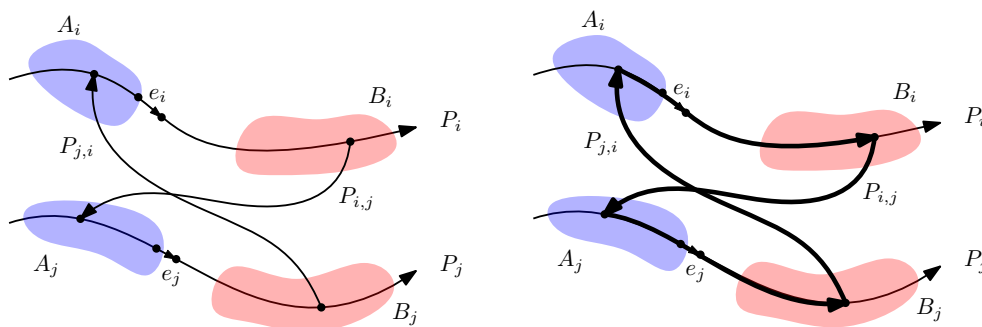
FIG. 1. *The walk W_e constructed in the proof of Lemma* 3.2.

walk; see Figure 1 (right). Note that W_e contains both e_i and e_j. For every $x \in [q]$, define a subgraph D_x of D as the union of all walks W_e for all $e \in E(T_x)$. Since for every $e = \{i, j\} \in E(H)$, the walk W_e contains e_i and e_j, and T_x is connected, the graph D_x is strongly connected and contains all edges e_i for $i \in V(T_x)$. Thus, since every two trees T_x and T_y intersect in $B_{x,y}$, the family $(D_x)_{x \in [q]}$ is a bramble of size $q \geq c \cdot \left(\frac{a^{1/2}}{\log^{1/4} a} \right)$ in D.

Now we argue that the congestion of the constructed bramble is at most $2\alpha + 2$. Each vertex is in at most α paths $P_{i,j}$ and in at most one path P_i. Thus each vertex appears in at most $\alpha + 1$ walks W_e. Each walk W_e might appear in at most two sets of the bramble, so the overall congestion is at most $2\alpha + 2$. □

Now we make Lemma 3.2 more easily accessible by making it applicable directly to a path system whose linkages have intersection graphs that are sufficiently sparse.

LEMMA 3.3 (sparse winning scenario, wrapped).   *Let c be the constant from Lemma 3.2. Let D be a directed graph, let $a > 1$, $b \geq 1$, and let $(P_i, A_i, B_i)_{i=1}^a$ be an (a, b)-path system in D. Let $\mathcal{I}$ be a subset of $[a] \times [a] \setminus \{(i, i) \mid i \in [a]\}$, such that $|\mathcal{I}| \geq 0.6 \cdot a(a-1)$. For $(i, j) \in \mathcal{I}$, let $\mathcal{L}_{i,j}$ be a linkage of size b from B_i to A_j. Assume that there is an integer d such that for every distinct $(i, j), (i', j') \in \mathcal{I}$ the intersection graph of $\mathcal{L}_{i,j}$ and $\mathcal{L}_{i',j'}$ is d-degenerate. If $b > 4 \cdot e \cdot a^2 \cdot d$, then D contains a bramble of congestion at most 4 and size at least $c \cdot \left(\frac{a^{1/2}}{\log^{1/4} a} \right)$.*

Proof. Construct an auxiliary undirected graph H, whose vertices are the paths in $\bigcup_{(i,j) \in \mathcal{I}} \mathcal{L}_{i,j}$. Two paths are adjacent in H if they contain a common vertex. Note that each set $\mathcal{L}_{i,j}$ is independent in H, so H is $|\mathcal{I}|$-partite. Furthermore, as for each $(i, j), (i', j') \in \mathcal{I}$, the graph $H[\mathcal{L}_{i,j} \cup \mathcal{L}_{i',j'}]$ is precisely the intersection graph of $\mathcal{L}_{i,j}$ and $\mathcal{L}_{i',j'}$, it is d-degenerate. Finally, since $\mathcal{I} \subseteq [a] \times [a]$, we have

$$b \geq 4 \cdot e \cdot a^2 \cdot d \geq 4 \cdot e \cdot (|\mathcal{I}| - 1) \cdot d.$$

Thus, applying Lemma 2.5 to H yields a single path $P_{i,j} \in \mathcal{L}_{i,j}$ for each $(i, j) \in \mathcal{I}$, such that the paths in $\{P_{i,j}\}_{(i,j) \in \mathcal{I}}$ are pairwise disjoint.

Now we observe that the set $\mathcal{I}$ and the family $\{P_{i,j}\}_{(i,j) \in \mathcal{I}}$ of paths satisfy the assumptions of Lemma 3.2 with $\alpha = 1$. Thus, the application of Lemma 3.2 yields the desired bramble. □

3.2. Closed walks and threaded linkages. In this section, we introduce a key object: the threaded linkage, which will be a main building block in our main

proof. Informally, later on we want to order and connect paths within a given linkage. In order to achieve this, we construct one long walk that contains all the paths from the linkage, interconnected by walks which we denote as threads. The long walk then naturally orders the paths in the linkage.

Our goal in this section is then, given a threaded linkage, to find a collection of closed walks, each containing a path from a linkage or, in case of two threaded linkages, a collection of closed walks, each containing a path from both linkages. The latter outcome is provided by another basic tool: the Bowtie Lemma, which might be useful on its own. This concept was essentially proved and used in [15, 14] in a slightly different setting. We describe the differences below.

DEFINITION 3.4 (threaded linkage). *Let D be a directed graph. A* threaded linkage *in D is a pair $(W, \mathcal{L})$ where $\mathcal{L} = \{L_1, L_2, \ldots, L_\ell\}$ is a linkage in D and W is a walk such that there exist $\ell - 1$ paths $Q_1, Q_2, \ldots, Q_{\ell-1}$ in D such that W is the concatenation of $L_1, Q_1, L_2, Q_2, \ldots, Q_{\ell-1}, L_\ell$ in that order. The paths Q_i are called* threads. *A threaded linkage $(W, \mathcal{L})$ with $W = (L_1, Q_1, \ldots, Q_{\ell-1}, L_\ell)$ is* untangled *if for every i, the thread Q_i may only intersect the rest of W in L_i or L_{i+1}.*

The *size* of an (untangled) threaded linkage $(W, \mathcal{L})$ is the size of the linkage $\mathcal{L}$, and its *overlap* is the overlap of the walk W.

Next, we show how to construct threaded linkages from a path system.

LEMMA 3.5 (construction of threaded linkages). *Let $(P_i, A_i, B_i)_{i=1}^a$ be an (a, b)-path system for $a, b \in \mathbb{N}$. Then, for all $i, j \in [a]$, there exist a linkage $\mathcal{L}_{i,j}$ from B_i to A_j and a threaded linkage $(W_{i,j}, \mathcal{L}_{i,j})$ of size b and overlap at most 3.*

Proof. We construct a threaded linkage for each $i, j \in [a]$ separately. For every $i, j \in [a]$, we fix a linkage $\mathcal{L}_{i,j}$ from B_i to A_j and a linkage $\overleftarrow{\mathcal{L}}_{i,j}$ from A_j to B_i; these linkages exist by the well-linkedness of $\bigcup_{i=1}^a A_i \cup B_i$.

For every $P \in \mathcal{L}_{i,j}$, let $\rho_{i,j}(P)$ be the path of $\overleftarrow{\mathcal{L}}_{i,j}$ that starts at the ending point of P, and let $\pi_{i,j}(P)$ be the path of $\mathcal{L}_{i,j}$ that starts at the ending point of $\rho_{i,j}(P)$. Note that $\pi_{i,j}$ is a permutation of $\mathcal{L}_{i,j}$. Let $\mathcal{C}_{i,j}$ be the family of cycles of the permutation $\pi_{i,j}$. Observe that every such cycle corresponds to a closed walk composed of the paths in $\mathcal{L}_{i,j}$ and $\overleftarrow{\mathcal{L}}_{i,j}$.

From every cycle $C \in \mathcal{C}_{i,j}$ we arbitrarily select one path from $\mathcal{L}_{i,j}$; we call it the *representative* of C. Let $C_1, C_2, \ldots, C_r$ be the elements of $\mathcal{C}_{i,j}$ in order of the appearance of the starting points of their representatives along P_i. Define the walk $W_{i,j}$ as follows: follow P_i, and for every $\ell \in [r]$, when we encounter the starting point of the representative of C_ℓ, follow the respective closed walk corresponding to C_ℓ, returning back to the starting point of the representative of C_ℓ, and then continue going along P_i. Finally, trim $W_{i,j}$ so that it starts and ends with a path of $\mathcal{L}_{i,j}$, as required by the definition of a threaded linkage.

Recall that the size of $(W_{i,j}, \mathcal{L}_{i,j})$ is the size of the linkage $\mathcal{L}_{i,j}$, i.e., b. Now let us argue about the overlap. The walk $W_{i,j}$ consists of the following subwalks: (1) paths of $\mathcal{L}_{i,j}$ (each path is used exactly once), (2) paths of $\overleftarrow{\mathcal{L}}_{i,j}$ (each path is used at most once), and (3) some pairwise vertex-disjoint subpaths of P_i. Note that the subwalks within each of these three groups are vertex-disjoint. Thus the overlap of $(W_{i,j}, \mathcal{L}_{i,j})$ is at most 3. $\square$

Now, we refine a given threaded linkage to get at least one good outcome: either a collection of closed walks, each containing a path from the linkage, or an untangled threaded linkage.

LEMMA 3.6 (construction of closed walks or untangled threaded linkages). *Let* $(W, \mathcal{L})$ *be a threaded linkage of size* b *and of overlap* α. *Let* $x, d \in \mathbb{N}$ *such that* $b \geq xd + (d - 1)$. *Then one of the following exists:*

1. *A family* $\mathcal{Z}$ *of* d *closed walks, such that for every walk* $W \in \mathcal{Z}$ *there exists a distinct path* $P(W) \in \mathcal{L}$ *that is a subwalk of* W, *and* $\mathcal{Z}$ *has overlap* α.
2. *An untangled threaded linkage* $(W', \mathcal{L}')$ *where* W' *is a subwalk of* W *and* $\mathcal{L}' \subseteq \mathcal{L}$ *is of size at least* x. *In particular,* $(W', \mathcal{L}')$ *is of overlap* α.

Proof. Let z be the length of W (i.e., the number of occurrences of vertices). For $1 \leq p \leq q \leq z$, by $W[p]$ we will denote the pth vertex of W, and by $W[p, q]$ we will denote the subwalk $W[p], W[p+1], \ldots, W[q]$.

A *useful walk of* W is a subwalk $W[p, q]$ of W, such that $W[p] = W[q]$ and $W[p, q]$ contains at least one path of $\mathcal{L}$ as a subwalk. The pair of indices (p, q) is called a *useful intersection*.

We greedily construct a sequence $I_1, I_2, \ldots, I_\ell$ of useful walks as follows: $I_1 = W[p_1, q_1]$ is a useful walk of W such that q_1 is the smallest possible, and subsequently $I_{\xi+1} = W[p_{\xi+1}, q_{\xi+1}]$ is a useful walk of W such that $p_{\xi+1} > q_\xi$ and $q_{\xi+1}$ is the smallest possible. The greedy construction stops when there are no useful walks starting after q_ℓ.

First, consider the case that $\ell \geq d$. Then every useful walk I_ξ is a closed walk and, as W is of overlap α, the family $\{I_\xi \mid \xi \in [\ell]\}$ is of overlap α. Furthermore, by the definition of a useful walk, every I_ξ contains a distinct path from $\mathcal{L}$, so we obtain the first desired outcome.

So now consider the case that $\ell < d$. We select $\ell + 1$ subwalks $I'_1, I'_2, \ldots, I'_{\ell+1}$ in W as follows. The subwalk I'_1 is defined as $W[1, q_1 - 1]$. Then, for $2 \leq \xi \leq \ell$, we define I'_ξ as $W[q_{\xi-1} + 1, q_\xi - 1]$. Finally, we define $I'_{\ell+1} := W[q_\ell + 1, z]$.

By the construction of the walks I_ξ, no I'_ξ contains a useful walk. Furthermore, the union of all walks I'_ξ covers W, except for $q_1, q_2, \ldots, q_\ell$. Hence, for at least $|\mathcal{L}| - \ell$ paths $P \in \mathcal{L}$ it holds that there is $\xi \in [\ell - 1]$ such that P is fully contained in I'_ξ. So there is some $\xi \in [\ell + 1]$, such that I'_ξ contains at least

$$\frac{|\mathcal{L}| - \ell}{\ell + 1} \geq \frac{b - (d-1)}{d} \geq x$$

paths of $\mathcal{L}$, where the last inequality holds because $b \geq xd + (d - 1)$ by precondition. Let $\mathcal{L}' \subseteq \mathcal{L}$ be the set of paths contained in I'_ξ, and let W' be the walk I'_ξ, trimmed so that it starts and ends with a path of $\mathcal{L}'$. We note that $(W', \mathcal{L}')$ is an untangled threaded linkage as I'_ξ contains no useful intersection. Thus, in this case, we obtain the second desired outcome. $\square$

LEMMA 3.7 (Bowtie Lemma). *For* $d \geq 1$, *let* $(W_1, \mathcal{L}_1)$ *and* $(W_2, \mathcal{L}_2)$ *be two threaded linkages of overlap* α *and* β, *respectively, such that the intersection graph* $I(\mathcal{L}_1, \mathcal{L}_2)$ *of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *is not* $(2^9 \cdot 5 \cdot d)$-*degenerate. Then there is a family* $\mathcal{Z}$ *of* d *closed walks such that every walk in* $\mathcal{Z}$ *contains at least one distinct path of* $\mathcal{L}_1$ *and one distinct path of* $\mathcal{L}_2$ *as a subwalk, and the congestion of* $\mathcal{Z}$ *is at most* $\alpha + \beta$.

Furthermore, if $(W_1, \mathcal{L}_1)$ *(*$(W_2, \mathcal{L}_2)$, *respectively) is untangled, then* $\mathcal{Z}$ *is of congestion at most* $\beta + 1$ *(*$\alpha + 1$, *respectively), and if both* $(W_1, \mathcal{L}_1)$, $(W_2, \mathcal{L}_2)$ *are untangled, then* $\mathcal{Z}$ *is of congestion at most* 2.

The main difference with the version proved implicitly in [15, 14] is that, there, only the containment of a subpath of a path in $\mathcal{L}_1$ and of a subpath of a path in $\mathcal{L}_2$ was guaranteed as opposed to the whole path as provided in the statement above. For the proof, we make use of the following Partitioning Lemma.
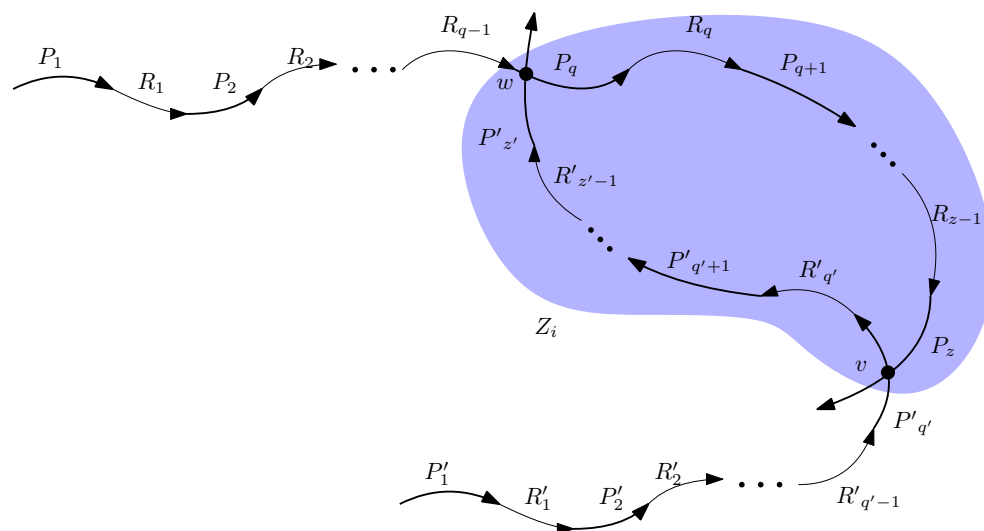
FIG. 2. *The walk Z_i constructed in the proof of Lemma* 3.7.

LEMMA 3.8 (Partitioning Lemma [14, Lemma 11]). *Let $k, r \geq 1$ be two integers, and let G be a bipartite graph with bipartition classes $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ and minimum degree at least $2^9 \cdot r \cdot k$. Then there are k sets $U_1, U_2, \ldots, U_k$, and k sets $W_1, W_2, \ldots, W_k$, such that*

1. *for each $i \in [k]$ the set U_i is a segment of X and the set W_i is a segment of Y, i.e., there are integers $\alpha_{i,U}, \omega_{i,U}, \alpha_{i,W}, \omega_{i,W}$ with $\alpha_{i,U} < \omega_{i,U}$ and $\alpha_{i,W} < \omega_{i,W}$ such that $U_i = \{x_{\alpha_{i,U}}, x_{\alpha_{i,U}+1}, \ldots, x_{\omega_{i,U}}\}$ and $W_i = \{y_{\alpha_{i,W}}, y_{\alpha_{i,W}+1}, \ldots, y_{\omega_{i,W}}\}$,*
2. *for each distinct $i, j \in [k]$ we have $U_i \cap U_j = \emptyset$ and $W_i \cap W_j = \emptyset$, and*
3. *for every $i \in [k]$, the average degree of the graph $G[U_i \cup W_i]$ is at least r.*

Now we are ready to prove the Bowtie Lemma.

Proof of Lemma 3.7. First, we invoke Lemma 3.8 for $k = d$, $r = 5$, and the intersection graph $I(\mathcal{L}_1, \mathcal{L}_2)$, where the order of paths in each linkage is naturally determined by the ordering of their appearance in the walks W_1, W_2. We obtain a sequence of subsets $\mathcal{L}_1^1, \mathcal{L}_1^2, \ldots, \mathcal{L}_1^d$ of $\mathcal{L}_1$ and a sequence of subsets $\mathcal{L}_2^1, \mathcal{L}_2^2, \ldots, \mathcal{L}_2^d$ of $\mathcal{L}_2$, satisfying the conditions given in the lemma. In particular, $\mathcal{L}_1^1, \mathcal{L}_1^2, \ldots, \mathcal{L}_1^d$ are pairwise disjoint, and $\mathcal{L}_2^1, \mathcal{L}_2^2, \ldots, \mathcal{L}_2^d$ are pairwise disjoint.

Fix some $i \in [d]$. Observe that $I(\mathcal{L}_1^i \cup \mathcal{L}_2^i)$ is of average degree at least 5. Thus, there are $\widetilde{\mathcal{L}}_1^i \subseteq \mathcal{L}_1^i$ and $\widetilde{\mathcal{L}}_2^i \subseteq \mathcal{L}_2^i$, such that the graph $I(\widetilde{\mathcal{L}}_1^i \cup \widetilde{\mathcal{L}}_2^i)$ is of minimum degree at least 3. Indeed, after a vertex of degree at most 2 is removed, the average degree is still at least 5.

Let $P_1, P_2, \ldots, P_z$ be the paths of $\widetilde{\mathcal{L}}_1^i$, ordered by their appearance on W_1. Let $R_1, R_2, \ldots, R_{z-1}$ be the walks, such that R_i is the subwalk of W_1 between P_i and P_{i+1}. Note that R_i might be either a single thread of $(W_1, \mathcal{L}_1)$ or a subwalk consisting of alternating threads and paths of $\mathcal{L}^i$ that were not included in $\widetilde{\mathcal{L}}^i$.

Similarly we define $P_1', P_2', \ldots, P_{z'}'$ as the paths of $\widetilde{\mathcal{L}}_2^i$ and $R_1', R_2', \ldots, R_{z'-1}'$ as the corresponding subwalks of W_2. Finally, let us name the concatenations of the

subwalks defined above:

$$W_1^i := P_1, R_1, P_2, R_2, \ldots, R_{z-1}, P_z, \text{ and}$$
$$W_2^i := P_1', R_1', P_2', R_2', \ldots, R_{z'-1}', P_{z'}'.$$

Since P_z is of degree at least 3 in $I(\widetilde{\mathcal{L}}_1^i, \widetilde{\mathcal{L}}_2^i)$, there exists a common vertex v of P_z and some $P_{q'}'$ for $q' < z' - 1$. Symmetrically, there is a common vertex w of $P_{z'}'$ and P_q for some $q < z - 1$. Consequently, we may construct a closed walk as follows; see also Figure 2:

$$Z_i := v, \text{the rest of } P_{q'}', R_{q'}', P_{q'+1}', \ldots, w, \text{the rest of } P_q, R_q, P_{q+1}, \ldots, v.$$

Note that Z_i fully contains at least one path from each linkage, i.e., $P_{q+1} \in \mathcal{L}_1$ and $P_{q'+1}' \in \mathcal{L}_2$. Furthermore, since $\mathcal{L}_1^i$ is disjoint from $\mathcal{L}_1^j$, $j \neq i$, path P_{q+1} is distinct to Z_i and analogously for $P_{q'+1}'$.

To see that the family $\mathcal{Z} = \{Z_i \mid i \in [d]\}$ satisfies the statement of the lemma, it remains to discuss the congestion. By condition 2 of Lemma 3.8 each vertex can appear at most α times on W_1; thus it contributes as a part of at most α walks W_1^i. Similarly, each vertex contributes as a part of at most β walks W_2^i. Summing up, each vertex may appear in at most $\alpha + \beta$ elements of $\mathcal{Z}$.

Now, suppose that one of the input threaded linkages, say, $(W_1, \mathcal{L}_1)$, is untangled. Let us enumerate $\mathcal{L}_1 = \{L_1, L_2, \ldots, L_\ell\}$ and let $Q_1, Q_2, \ldots, Q_{\ell-1}$ be the threads of $(W_1, \mathcal{L}_1)$ as in Definition 3.4. Recall that the paths from $\mathcal{L}_1$ are vertex-disjoint. Now consider a thread Q_j. Note that at most one walk W_1^i might contain Q_j as a subwalk. Furthermore, if Q_j is a subwalk of W_1^i, then so are L_j and L_{j+1}. Thus, since $(W_1, \mathcal{L}_1)$ is untangled, each vertex from W_1 might contribute to the congestion of at most one walk in $Z \in \mathcal{Z}$. ◻

4. Main proof. Using the tools from section 3, we now prove Theorem 1.2. As mentioned before, the starting point is a path system (see Definition 2.2) containing a large number of sets A_i, B_i, $i \in [a]$, and large linkages $\mathcal{L}_{i,j}$ between pairs of such sets. The basic goal is to exploit the interplay between the dense and sparse winning scenarios from subsection 3.1. The dense winning scenario (Lemma 3.1) is applicable if there is a pair of linkages whose intersection graph has large degeneracy. The crux is how to apply the sparse winning scenarios (Lemmas 3.2 and 3.3) because they need a fraction of pairs of linkages that is slightly larger than half of the available pairs of linkages. Our goal is thus to distinguish three subsets of the set of pairs of linkages such that (i) one of the subsets will be larger than half the available pairs of linkages and (ii) each of the subsets can be used to define a bramble of large size. We obtain these three distinguished subsets by using two partially overlaying matchings in auxiliary graphs whose vertex sets are the available pairs of linkages and whose edges represent large degeneracy of the corresponding intersection graphs.

Some complications arise from the aim of keeping the congestion low. To achieve low congestion, instead of applying the sparse winning scenarios directly to the linkages from the path set system, we use first the tools from subsection 3.2 to obtain subsets of pairs of linkages of lower congestion.

Setup. Let $k \in \mathbb{N}$ with $k > 1$, and let D be a directed graph of directed treewidth t. We show that if $t \geq c_t \cdot k^{48} \log^{13} k$, then D contains a bramble of congestion at most 8 and size k. Herein, c_t is a constant that we specify below.

We start by fixing the following parameters. Let c be the constant in Lemma 3.3. Without loss of generality we can assume that $c \leq \frac{1}{3^{1/4}}$ (note that if Lemma 3.3 holds

for a constant c, then it also holds for all smaller constants). Thus $c^{-4} \geq 3$; this will be used in the case analysis later in the proof. We put $c_a = c^{-4} \geq 3$. Let $c_{d_3} := c_{\mathsf{KT}}$ be the constant from Theorem 2.4. We define

$$
\begin{aligned}
a &= \left\lceil c_a \cdot k^2 (1 + \log k)^{1/2} \right\rceil, \\
d_3 &= \left\lceil c_{d_3} \cdot k \sqrt{\log k} \right\rceil, \\
d_2 &= \left\lceil 2^{11} 5 e \cdot a^2 d_3 \right\rceil = \mathcal{O}(k^5 \log^{3/2} k), \\
d_1 &= \left\lceil 2^{11} 5 e \cdot a^2 d_2 \right\rceil = \mathcal{O}(k^9 \log^{5/2} k), \text{ and} \\
b &= \left\lceil 4 e \cdot a^2 d_1^2 \right\rceil = \mathcal{O}(k^{22} \log^6 k).
\end{aligned}
$$

Let c_{KK} be the constant in Lemma 2.3. Choose the constant c_t such that $t \geq c_{\mathsf{KK}} a^2 b^2$. To see that this is possible, observe that

$$
\begin{aligned}
c_{\mathsf{KK}} a^2 b^2 &\leq 2^5 e^2 \cdot c_{\mathsf{KK}} \cdot a^6 d_1^4 \\
&\leq 2^{50} 5^4 e^6 \cdot c_{\mathsf{KK}} \cdot a^{14} d_2^4 \\
&\leq 2^{95} 5^8 e^{10} \cdot c_{\mathsf{KK}} \cdot a^{22} d_3^4 \\
&\leq 2^{96} 5^8 e^{10} \cdot c_{\mathsf{KK}} \cdot c_{d_3}^4 \cdot a^{22} k^4 \log^2 k \\
&\leq 2^{97} 5^8 e^{10} \cdot c_{\mathsf{KK}} \cdot c_{d_3}^4 \cdot c_a^{22} \cdot k^{48} \log^2 k (1 + \log k)^{11} \\
&\leq 2^{97} 5^8 e^{10} \cdot c_{\mathsf{KK}} \cdot c_{d_3}^4 \cdot c_a^{22} \cdot k^{48} \log^2 k (2 \log k)^{11} \\
&\leq 2^{108} 5^8 e^{10} \cdot c_{\mathsf{KK}} \cdot c_{d_3}^4 \cdot c_a^{22} \cdot k^{48} \log^{13} k.
\end{aligned}
$$

Thus, putting $c_t = 2^{108} 5^8 e^{10} \cdot c_{\mathsf{KK}} \cdot c_{d_3}^4 c_a^{22}$, we have $t \geq c_{\mathsf{KK}} a^2 b^2$. Hence, by Lemma 2.3, there is an (a, b)-path system $(P_i, A_i, B_i)_{i=1}^a$ in D.

Large subsets of pairs of linkages and sets of closed walks of low congestion. Let $V = \{(i, j) \mid i, j \in [a] \wedge i \neq j\}$. Our aim is now to find the three subsets of V such that one of them will be larger than $|V|/2$ mentioned above and to take subsets of some linkages in order to achieve low congestion. We achieve the second aim by using sets of closed walks derived from some of the linkages. See Table 1 for the various linkages and sets of walks that we define below and their properties.

We apply Lemma 3.5 to $(P_i, A_i, B_i)_{i=1}^a$, obtaining for every $(i, j) \in V$ a threaded linkage $(W_{i,j}, \mathcal{L}_{i,j})$ of size b and overlap at most 3. Then, we apply Lemma 3.6 to $(W_{i,j}, \mathcal{L}_{i,j})$ with $x = 4ea^2 d_1 + 1$ and $d = d_1/(2^9 \cdot 5)$. Note that $xd + (d-1) \leq 4e \cdot a^2 d_1^2/(2^9 \cdot 5) + d_1/(2^8 \cdot 5) \leq 4e \cdot a^2 d_1^2 \leq b$. Hence, the preconditions of Lemma 3.6 are satisfied. Let $Z \subseteq V$ be the set of pairs (i, j) for which the application of Lemma 3.6 results in the first outcome.

For each $(i, j) \in V \setminus Z$, let $(W'_{i,j}, \mathcal{L}'_{i,j})$ be the *untangled* threaded linkage resulting from the application of Lemma 3.6. Note that $|\mathcal{L}'_{i,j}| \geq x = 4e \cdot a^2 d_1 + 1$.

For each $(i, j) \in Z$, that is, for each (i, j) where applying Lemma 3.6 results in the first outcome, let $\mathcal{Z}_{i,j}$ be the family of closed walks resulting from the application of Lemma 3.6. Observe that $\mathcal{Z}_{i,j}$ is of size at least $d_1/(2^9 \cdot 5)$ and of overlap at most 3. By the definition of $\mathcal{Z}_{i,j}$, for each walk $W \in \mathcal{Z}_{i,j}$ there is a distinct path $P(W) \in \mathcal{L}_{i,j}$ such that $P(W)$ is a subwalk of W. Define the linkage $\mathcal{L}'_{i,j} := \{P(W) \mid W \in \mathcal{Z}_{i,j}\}$. For convenience, we denote also $W'_{i,j} := W_{i,j}$. Observe that $(W'_{i,j}, \mathcal{L}'_{i,j})$ is a threaded linkage (but not necessarily untangled).

For both $\ell = 1, 2$, let $E_\ell \subseteq \binom{V}{2}$ be the set of those pairs $\{(i, j), (i', j')\} \in \binom{V}{2}$ for which both the intersection graphs of $\mathcal{L}'_{i,j}$ and of $\mathcal{L}'_{i',j'}$ are not d_ℓ-degenerate. Define

TABLE 1
Important linkages and families of closed walks. Here, "o" stands for overlap, and "c" stands for congestion.

| Index	Family	o/c	Size	Comments
$(i,j) \in V$	$(W_{i,j}, \mathcal{L}_{i,j})$	o ≤ 3	b	threaded linkage
$(i,j) \in Z$	$\mathcal{Z}_{i,j}$	o ≤ 3	$d_1/(2^9 \cdot 5)$	closed walks each $W \in \mathcal{Z}_{i,j}$ contains $P_{i,j}(W) \in \mathcal{L}_{i,j}$
$(i,j) \in V \setminus Z$	$(W'_{i,j}, \mathcal{L}'_{i,j})$	o ≤ 3	$4ea^2 d_1 + 1$	untangled threaded linkage
$e \in M_1$	$\mathcal{Z}_e$	c ≤ 2	$d_1/(2^9 \cdot 5)$	closed walks, for $(i,j) \in e$ each $W \in \mathcal{Z}_e$ contains $P_{i,j}(W) \in \mathcal{L}_{i,j}$
$e \in M_2$	$\mathcal{Z}_e$	c ≤ 4	$d_2/(2^9 \cdot 5)$	closed walks, for $(i,j) \in e$ each $W \in \mathcal{Z}_e$ contains $P_{i,j}(W) \in \mathcal{L}_{i,j}$
$F \subseteq Z \cup M_1 \cup M_2,$ $\quad 1 \leq |F| \leq 2$	$\mathcal{Z}_F$	c ≤ 8	$d_2/(2^9 \cdot 5)$	intersection graph d_3-degenerate
$(i,j) \in Z$	$\mathcal{L}_{i,j}^{\mathcal{Z}}$	c ≤ 1	$d_1/(2^9 \cdot 5)$	linkage $\mathcal{L}_{i,j}^{\mathcal{Z}} = \{P_{i,j}(W) \mid W \in \mathcal{Z}_{i,j}\}$
$(i,j) \in V(M_1)$	$\mathcal{L}_{i,j}^{\mathcal{Z}}$	c ≤ 1	$d_1/(2^9 \cdot 5)$	linkage, for $e \in M_1$ with $(i,j) \in e$ $\mathcal{L}_{i,j}^{\mathcal{Z}} = \{P_{i,j}(W) \mid W \in \mathcal{Z}_e\}$
$(i,j) \in V(M_2)$ $\quad \setminus (Z \cup V(M_1))$	$\mathcal{L}_{i,j}^{\mathcal{Z}}$	c ≤ 1	$d_2/(2^9 \cdot 5)$	linkage, for $e \in M_2$ with $(i,j) \in e$ $\mathcal{L}_{i,j}^{\mathcal{Z}} = \{P_{i,j}(W) \mid W \in \mathcal{Z}_e\}$
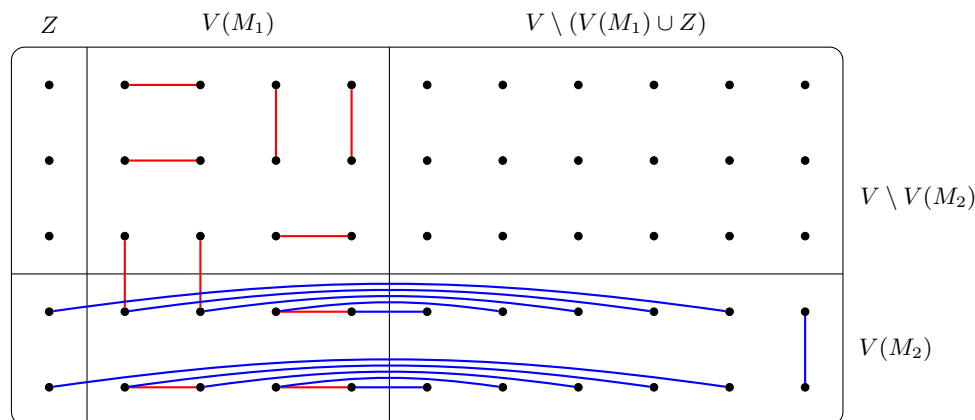
FIG. 3. *The relation of V, Z, and the matchings M_1 (red) and M_2 (blue). (Color available online.)*

an undirected graph $H_\ell = (V, E_\ell)$. Since $d_1 \geq d_2$, we have $E_1 \subseteq E_2$, and thus H_1 is a subgraph of H_2.

Let M_1 be a maximum matching in $H_1 - Z$. Let M_2 be a maximum matching in the undirected graph $(V, E(H_2) \setminus \binom{V(M_1) \cup Z}{2})$, that is, in the graph that results from H_2 by removing all edges with both endpoints in $V(M_1) \cup Z$; see Figure 3.

We are now ready to define the promised three vertex subsets of V such that one of them is guaranteed to be sufficiently large for our purposes.

CLAIM 4.1. *At least one of the following cases occurs:*
Case 1. $|V \setminus (V(M_1) \cup Z)| \geq 0.6|V|$.

FIG. 4. *Proof of Claim* 4.1*: The number of stars in each region corresponds to the number of times the set is counted in the proof of Claim* 4.1.

Case 2. $|V(M_1) \cup V(M_2) \cup Z| \geq 0.6|V|$.
Case 3. $|V \setminus V(M_2)| \geq 0.6|V|$.

Proof. It suffices to show that

$$(4.1) \qquad 2|V \setminus (V(M_1) \cup Z)| + 2|V(M_1) \cup V(M_2) \cup Z| + |V \setminus V(M_2)| \geq 3|V|.$$

Consider how often vertices in the following vertex subsets are counted in the left-hand side of (4.1); consult also Figure 4:

- Each vertex in $V \setminus (V(M_1) \cup V(M_2) \cup Z)$ is counted thrice.
- Each vertex in $V(M_2) \setminus (V(M_1) \cup Z)$ is counted four times.
- Each vertex in $V(M_2) \cap (V(M_1) \cup Z)$ is counted twice.
- Each vertex in $V(M_1) \setminus V(M_2)$ is counted thrice.
- Each vertex in $Z \setminus V(M_2)$ is counted thrice.

Now note that no vertex of V occurs in two or more of the above sets. Hence, the left-hand side of (4.1) is at least

$$3|V \setminus (V(M_1) \cup V(M_2) \cup Z)| + 4|V(M_2) \setminus (V(M_1) \cup Z)|$$
$$+ 2|V(M_2) \cap (V(M_1) \cup Z)| + 3|V(M_1) \setminus V(M_2)| + 3|Z \setminus V(M_2)|.$$

Observe that $|(V(M_1) \cup Z) \cap V(M_2)| \leq |V(M_2) \setminus (V(M_1) \cup Z)|$ because every edge of M_2 has at most one endpoint in $V(M_1) \cup Z$. Thus, the left-hand side of (4.1) is at least

$$3|V \setminus (V(M_1) \cup V(M_2) \cup Z)| + 3|V(M_2) \setminus (V(M_1) \cup Z)|$$
$$+ 3|V(M_2) \cap (V(M_1) \cup Z)| + 3|V(M_1) \setminus V(M_2)| + 3|Z \setminus V(M_2)|,$$

which is equal to $3|V|$ (recall here that $V(M_1) \cap Z = \emptyset$), as claimed. $\qquad \square$

We now continue with definitions of sets of closed walks and sublinkages that we need to derive a bramble in each of the cases above. Refer again to Table 1 for a summary of all important linkages and families of closed walks defined here.

For both $\ell = 1, 2$ and for each $e = \{(i,j),(i',j')\} \in M_\ell$, apply Lemma 3.7 to $(W'_{i,j}, \mathcal{L}'_{i,j})$ and $(W'_{i',j'}, \mathcal{L}'_{i',j'})$ to find a family $\mathcal{Z}_e$ of $d_\ell/(2^9 \cdot 5)$ closed walks, such that

every walk in $\mathcal{Z}_e$ contains at least one distinct path from $\mathcal{L}'_{i,j}$ and at least one distinct path from $\mathcal{L}'_{i',j'}$. Since the matchings M_1 and M_2 are disjoint, we obtain $|M_1| + |M_2|$ families of closed walks $\mathcal{Z}_e$ for $e \in M_1 \cup M_2$.

Let us now analyze the congestion of the families $\mathcal{Z}_e$. Recall that for all $(i,j) \notin Z$, the threaded linkage $(W'_{i,j}, \mathcal{L}'_{i,j})$ is untangled. Thus for each $e \in M_1$, both its endpoints correspond to untangled linkages, so by Lemma 3.7, the congestion of $\mathcal{Z}_e$ is at most 2. Now consider the family Z_e for an edge $e \in M_2$. At least one endpoint (i,j) of e is in $V \setminus Z$, and hence $(W'_{i,j}, \mathcal{L}'_{i,j})$ is untangled. Furthermore, for the other endpoint (i',j') of e we have that $\mathcal{L}'_{i',j'}$ is of congestion at most 3. So by Lemma 3.7, the congestion of $\mathcal{Z}_e$ is at most 4.

Now suppose that for some $F \subseteq M_1 \cup M_2 \cup Z$ of size 1 or 2, the intersection graph of $\mathcal{Z}_F := \bigcup_{g \in F} \mathcal{Z}_g$ is not d_3-degenerate. Recall that for each $(i,j) \in Z$, the congestion of $\mathcal{Z}_{i,j}$ is at most 3 (since it resulted from the first outcome of Lemma 3.6 and because the congestion is at most the overlap), and for each $e \in M_1 \cup M_2$ the congestion of $\mathcal{Z}_e$ is at most 4. Thus the congestion of $\mathcal{Z}_F$ is at most 8. Applying Lemma 3.1 to $\mathcal{Z}_F$ thus yields a bramble of size k and congestion at most 8, finishing the proof in this case. Thus, henceforth the following claim holds.

CLAIM 4.2. *For each $F \subseteq M_1 \cup M_2 \cup Z$ of size 1 or 2 the intersection graph of $\mathcal{Z}_F$ is d_3-degenerate.*

For every $e \in M_1$ and endpoint $(i,j) \in e$ proceed as follows. Recall that every $W \in \mathcal{Z}_e$ contains a distinct path in $\mathcal{L}'_{i,j}$ by Lemma 3.7. For every $W \in \mathcal{Z}_e$ let $P_{i,j}(W) \in \mathcal{L}'_{i,j}$ be this path. Let $\mathcal{L}^{\mathcal{Z}}_{i,j} = \{P_{i,j}(W) \mid W \in \mathcal{Z}_e\}$. Note that $|\mathcal{L}^{\mathcal{Z}}_{i,j}| \geq |\mathcal{Z}_e| \geq d_1/(2^9 \cdot 5)$.

Similarly, for every $e \in M_2$ and endpoint $(i,j) \in e \setminus (V(M_1) \cup Z)$, proceed as follows. For every $W \in \mathcal{Z}_e$, pick a distinct path $P_{i,j}(W) \in \mathcal{L}'_{i,j}$ on W (again guaranteed to exist by Lemma 3.7). Let $\mathcal{L}^{\mathcal{Z}}_{i,j} = \{P_{i,j}(W) \mid W \in \mathcal{Z}_e\}$. Note that $|\mathcal{L}^{\mathcal{Z}}_{i,j}| \geq |\mathcal{Z}_e| \geq d_2/(2^9 \cdot 5)$.

Furthermore, for every $(i,j) \in Z$ set $\mathcal{L}^{\mathcal{Z}}_{i,j} = \mathcal{L}'_{i,j}$. Note that $|\mathcal{L}^{\mathcal{Z}}_{i,j}| \geq |\mathcal{Z}_{i,j}| \geq d_1/(2^9 \cdot 5)$. Recall that the application of Lemma 3.6 for $(W_{i,j}, \mathcal{L}_{i,j})$ in the beginning resulted in the first outcome. Thus, for every $W \in \mathcal{Z}_{i,j}$ there is a distinct path $P(W) \in \mathcal{L}_{i,j} \subseteq \mathcal{L}'_{i,j} = \mathcal{L}^{\mathcal{Z}}_{i,j}$ such that $P(W)$ is a subwalk of W. For every $W \in \mathcal{Z}_{i,j}$ denote $P_{i,j}(W) = P(W) \in \mathcal{L}'_{i,j}$.

Case distinction. We are now ready to deal with the possible outcomes of Claim 4.1 one by one. Refer to Table 1 to recall the properties of the linkages and families of closed walks used below.

Case 1: $|V \setminus (V(M_1) \cup Z)| \geq 0.6|V|$ *(large independent set in H_1).* We would like to apply Lemma 3.3 to the path system $(P_i, A_i, B_i)_{i=1}^a$ with $\mathcal{I} := V \setminus (V(M_1) \cup Z)$. To apply Lemma 3.3 we check that (i) $|\mathcal{I}| \geq 0.6 \cdot a \cdot (a-1)$, which is true since $|\mathcal{I}| \geq 0.6|V| = 0.6 \cdot a \cdot (a-1)$, that (ii) for every $(i,j) \in \mathcal{I}$ there is a linkage of size at least $4e \cdot a^2 d_1 + 1$ between points in A_i and B_j, that (iii) for every two $(i,j),(i',j') \in \mathcal{I}$ the intersection graph of the two linkages is d_1-degenerate, and that (iv) the size of the linkages is strictly larger than $4e \cdot a^2 d_1$, which clearly holds by definition.

For the linkages in point (ii) we choose the linkages $\mathcal{L}'_{i,j}$. Observe that, since each $(i,j) \in \mathcal{I}$ is not in Z, we have $|\mathcal{L}'_{i,j}| > 4e \cdot a^2 d_1$, as required by point (ii). Since M_1 is a maximum matching in $H_1 - Z$, it follows that $\mathcal{I}$ is an independent set in H_1. By the definition of H_1 and M_1, for every two distinct pairs $(i,j),(i',j') \in \mathcal{I}$, the intersection graph of $\mathcal{L}'_{i,j}$ and $\mathcal{L}'_{i',j'}$ is thus d_1-degenerate. Thus, point (iii) holds as well, meaning that Lemma 3.3 is applicable.

Recall that c is the constant in Lemma 3.3, and we have $c_a = c^{-4} \geq 3$. The application of Lemma 3.3 yields a bramble of congestion at most 4 and size at least

$$c \cdot \frac{a^{1/2}}{\log^{1/4} a} \geq c \cdot \frac{(c_a \cdot k^2(1 + \log k)^{1/2})^{1/2}}{\log^{1/4}(c_a k^2(1 + \log k)^{1/2})} = k \cdot \frac{cc_a^{1/2}(1 + \log k)^{1/4}}{\log^{1/4}(c_a k^2(1 + \log k)^{1/2})}$$

$$\geq k \cdot \frac{cc_a^{1/2}(1 + \log k)^{1/4}}{\log^{1/4}(c_a k^3)} = k \cdot \frac{cc_a^{1/2}(1 + \log k)^{1/4}}{(\log c_a + 3 \log k)^{1/4}}$$

$$(4.2) \qquad \geq k \cdot \frac{cc_a^{1/4} \cdot (c_a + c_a \log k)^{1/4}}{(\log c_a + 3 \log k)^{1/4}} = k \cdot \frac{(c_a + c_a \log k)^{1/4}}{(\log c_a + 3 \log k)^{1/4}}.$$

As $c_a \geq 3$, the right-hand side of inequality (4.2) is at least k, finishing the proof in this case.

Case 2: $|V(M_1) \cup V(M_2) \cup Z| \geq 0.6|V|$ *(large matchings).* We show that we may apply Lemma 3.2 to a set of paths obtained from the families $\mathcal{Z}_g$ for $g \in Z \cup M_1 \cup M_2$. Let J be the undirected, $(|Z| + |M_1| + |M_2|)$-partite graph whose vertex set is the disjoint union of the *color classes* $\{\mathcal{Z}_g \mid g \in Z \cup M_1 \cup M_2\}$ such that, for every pair of distinct $g, g' \in Z \cup M_1 \cup M_2$, the graph $J[\mathcal{Z}_g \cup \mathcal{Z}_{g'}]$ is the intersection graph of $\mathcal{Z}_g$ and $\mathcal{Z}_{g'}$ (and J contains no further edges). Hence we obtain a graph properly colored with $|Z| + |M_1| + |M_2| \leq |V| = a(a-1)$ colors. We claim that by Lemma 2.5 we may thus select a walk $W_h \in \mathcal{Z}_h$ for every $h \in M_1 \cup M_2 \cup Z$ such that the walks W_h are pairwise vertex-disjoint. To that end, it suffices to show the required relation on the number of colors of J, the size of the color classes, and the degeneracy of all subgraphs induced by two color classes. By Claim 4.2 each subgraph of J that is induced by two different colors is d_3-degenerate. Moreover, for every $g \in Z \cup M_1 \cup M_2$ we have $|\mathcal{Z}_g| \geq d_2/(2^9 \cdot 5)$ by definition of $\mathcal{Z}_g$ and since $d_1 > d_2$. By definition of d_2, we have $d_2/(2^9 \cdot 5) \geq 4e \cdot a^2 d_3 \geq 4e(a(a-1)-1)d_3$, as required by Lemma 2.5. Thus, indeed, we may choose the walks W_h as specified.

Let $\mathcal{I}$ be a subset of $Z \cup V(M_1) \cup V(M_2)$ of size exactly $\lceil 0.6a(a-1) \rceil$. The set of paths to which we apply Lemma 3.2 is derived from the walks W_h as follows. If $h = \{(i,j),(i',j')\} \in M_1 \cup M_2$, then, by definition of $\mathcal{Z}_h$, walk W_h contains a path $P_{i,j}(W_h) \in \mathcal{L}_{i,j}$ and a path $P_{i',j'}(W_h) \in \mathcal{L}_{i',j'}$. If $h = (i,j) \in Z$, then, by definition of $\mathcal{Z}_h$, walk W_h contains a path $P_{i,j}(W_h) \in \mathcal{L}_{i,j}^{\mathcal{Z}}$. Construct a family $\mathcal{Q} = \{P_{i,j} \mid (i,j) \in \mathcal{I}\}$ by, for each $(i,j) \in \mathcal{I}$, choosing an arbitrary walk W_h such that $(i,j) = h$ or $(i,j) \in h$ and putting $P_{i,j} = P_{i,j}(W_h)$. Note that for each $(i,j) \in \mathcal{I}$ we have $P_{i,j} \in \mathcal{L}_{i,j}$. Since two paths $P_{i,j}, P_{i',j'}$ may only share vertices if they stem from the same walk W_h, we have that $\mathcal{Q}$ has congestion at most 2. By applying Lemma 3.2 to $\mathcal{Q}$, we obtain a bramble of congestion at most $2 + 2 \cdot 2 = 6$ and of size at least $c \cdot \frac{a^{1/2}}{\log^{1/4} a}$. By the same calculation as in inequality (4.2), this bramble has size at least k, finishing the proof in this case.

Case 3: $|V \setminus V(M_2)| \geq 0.6|V|$ *(large matching anti-adjacent to an independent set).* We now would like to apply Lemma 3.3 with $\mathcal{I} := V \setminus V(M_2)$ to obtain a bramble of size k and low congestion. To apply Lemma 3.3 we check that (i) $|\mathcal{I}| \geq 0.6 \cdot a \cdot (a-1)$, which is true since $|\mathcal{I}| = |V \setminus V(M_2)| \geq 0.6|V| = 0.6 \cdot a \cdot (a-1)$, that (ii) for every $(i,j) \in \mathcal{I}$ there is a linkage of size $d_1/(2^9 \cdot 5)$ between points in A_i and B_j, that (iii) for every two $(i,j),(i',j') \in \mathcal{I}$ the intersection graph of the two linkages is d_2-degenerate, and that (iv) $d_1/(2^9 \cdot 5) > 4 \cdot e \cdot a^2 \cdot d_2$, which clearly holds by definition.

As linkages for point (ii), for each $(i,j) \in \mathcal{I} \cap (Z \cup V(M_1))$ we take the linkage $\mathcal{L}_{i,j}^{\mathcal{Z}}$. Note that $|\mathcal{L}_{i,j}^{\mathcal{Z}}| \geq d_1/(2^9 \cdot 5)$. For each $(i,j) \in \mathcal{I} \setminus (Z \cup V(M_1))$ we take the linkage $\mathcal{L}'_{i,j}$. For convenience we denote $\mathcal{L}_{i,j}^{\mathcal{Z}} := \mathcal{L}'_{i,j}$. Note that $|\mathcal{L}_{i,j}^{\mathcal{Z}}| \geq d_1/(2^9 \cdot 5)$ as

well.

For point (iii), observe that, for each $(i,j), (i',j') \in Z \cup V(M_1)$, the intersection graph of the linkages $\mathcal{L}_{i,j}^{\mathcal{Z}}, \mathcal{L}_{i',j'}^{\mathcal{Z}}$ is d_3-degenerate (and thus d_2-degenerate) because the paths in these linkages are contained as subwalks in $\mathcal{Z}_{\{(i,j),(i',j')\}}$ and by Claim 4.2. For each $(i,j) \in \mathcal{I}$ and $(i',j') \in \mathcal{I} \setminus (Z \cup V(M_1))$ the intersection graph between $\mathcal{L}_{i,j}^{\mathcal{Z}}$ and $\mathcal{L}_{i',j'}^{\mathcal{Z}}$ is d_2-degenerate because otherwise M_2 would not be maximum. Hence also point (iii) holds, and it follows that Lemma 3.3 is applicable.

From Lemma 3.3 we obtain a bramble of congestion at most 4 and size at least $c \cdot \frac{a^{1/2}}{\log^{1/4} a}$, which is at least k by inequality (4.2).

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