Dissertation

Moonshine in Conformal Field Theory and String Theory

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines Doktors der technischen Wissenschaften

unter der Leitung von

Privatdoz. Timm Michael Wrase, PhD
E136
Institut für Theoretische Physik

eingereicht an der Technischen Universität Wien
Fakultät für Physik

von

Dipl.-Ing. **Andreas Banlaki**
Matr.Nr.: 9725099

Rückaufgasse 33/3, 1190 Wien

Wien, am 26. Mai 2020

Kurzfassung

Moonshine steht für eine überraschende Verbindung zwischen scheinbar unabhängigen Gebieten der Mathematik, nämlich modularen Objekten (Zahlentheorie) und sporadischen Gruppen (Gruppentheorie). Historisch wird John MacKays Beobachtung vom Jahr 1978 als Anfang dieses Gebiets betrachtet, in der er eine Verbindung zwischen der Kleinschen J-Funktion und der Monster Gruppe erkannte. Die genauere Untersuchung von MacKays Beobachtung und der Beweis der 'Monstrous-Moonshine-Vermutung' durch Borcherds im Jahr 1992 hat zu der Entwicklung neuer Konzept geführt und zu neuen Einblicken in der Mathematik und der Physik beigetragen. Die Entdeckung von Mathieu Moonshine durch Eguchi, Ooguri und Tachikawa im Jahr 2010, welche eine Verbindung zwischen dem elliptischen Genus von $K3$ und der größten Mathieu Gruppe herstellte, erneuerte das Interesse an der Untersuchung von Moonshine-Phänomenen. In den darauffolgenden Jahren sind viel weitere Mondschein-Phänomene entdeckt und untersucht worden, was zu neuen überraschenden Einblicken geführt hat.

In meiner Arbeit versuche ich auf verschiedene Arten neue Moonshine-Phänomene zu finden. Einerseits untersuche ich, ob es für höher dimensionale Calabi-Yau Manigfaltigketen einen Zusammenhang mit Moonshine gibt. Der Zusammenhang wird durch den elliptischen Genus der Mannigfaltigkeiten hergestellt. Die Untersuchung zeigt einige mögliche interessante Verbindungen, insbesondere von 5-dimensionalen Calabi-Yau Mannigfaltigkeiten zu Mathieu Moonshine. Bei einer genaueren Betrachtung lassen sich diese aber nicht bekräftigen. Ein anderer Zugang ergibt sich durch die Dualität zwischen heterotischem und Typ II String. Hierdurch kann Mathieu Moonshine mit topologischen Invarianten (Gromov-Witten/Gopakumar-Vafa Invarianten) von bestimmten Calabi-Yau Mannigfaltigkeiten in Zusammenhang gebracht werden. Konkret untersuche ich CHL-Orbifolds von $E_8 \times E_8$ heterotischen Kompaktifizierungen auf $K3 \times T^2$. In der effektiven 4 dimensionalen Theorie dieser Modelle stehen gewisse gravitative Kopplungen und das Prepotential der Vektormultiplet Moduli im Zusammenhang mit den 'getwisten' und 'getwinten' ellitpischen Genera von $K3$. In den dualen Typ II Kompaktifzierungen sind diese Kopplungen und das Prepotential durch die topologischen Invarianten der Calabi-Yau Mannigfaltigkeit bestimmt. Dadurch ergibt sich ein interessanter Zusammenhang zwischen Mathieu Moonshine und den Gromov-Witten/Goparkuma-Vafa Invarianten bestimmter Calabi-Yau Mannigfaltigkeiten. Des weiteren ist es möglich für bestimmte heterotische CHL Orbifolds die dualen Calabi-Yau Manigfaltigkeiten zu finden und zum Teil explizit zu konstruieren.

Abstract

Moonshine, which famously started with John MacKays observation in 1978 connecting the Klein-J-function to the Monster group, connects seemingly unrelated fields of mathematics - modular objects (number theory) and sporadic groups (group theory). The precise analysis of the initial observation and its proof by Borcherds in 1992 has led to new concepts and insights both in mathematics and physics. The discovery of Mathieu moonshine by Eguchi, Ooguri and Tachikawa in 2010, linking the elliptic genus of $K3$ to the largest Mathieu group, brought renewed interest to the research field. Since then many new moonshine phenomena have been found shedding light on ever new surprising connections.

In this thesis I search for new moonshine in different ways. I try to connect higher dimensional Calabi-Yau manifolds to moonshine by analysing their elliptic genera. I find some interesting initial results which however cease to hold under closer analysis. In a different approach I make use of heterotic-type II duality to relate Mathieu moonshine to the Gromov-Witten invariants of certain Calabi-Yau threefolds. In particular I study CHL orbifolds of $E_8 \times E_8$ heterotic string compactifications on $K3 \times T^2$. In these models the twisted twining elliptic genera of $K3$ show up in the gravitational couplings and the vector moduli prepotential of the four dimensional effective theory. In the dual type II compactifications these couplings and the prepotential are governed by the topology (Gromov-Witten/Gopakumar-Vafa invariants) of the Calabi-Yau threefolds the theory is compactified on. In this way the Calabi-Yau manifolds get connected to Mathieu moonshine. For certain CHL orbifolds of the heterotic string one is able to find the dual Calabi-Yau manifolds, in some cases by direct construction.

Acknowledgements

I want to start by thanking my adviser Timm Wrase without whom no part of this work would have been possible. Starting with suggesting the topic of my PhD and finding interesting research questions, Timm was helpful at all stages of my PhD. He was always open for questions and discussions and never failed to provide helpful answers and new ideas. Always hard working and friendly Timm was a constant source of motivation and positivity throughout my PhD.

I would also like to thank my diploma thesis adviser Manfried Faber who during the time of writing my diploma thesis brought back the joy of physics I already thought lost. He is partly responsible for my will to continue my studies and starting my PhD.

I would like to thank the members of our moonshine group, Abhiram, Abhishek and Maria for good times and discussions inside and outside of physics. Especially I have to thank Maria whom it was a joy to share the office with. I would like to thank Christoph for helpful discussions (physics and non-physics related) and for sharing the occasional beer. Further I would like to thank Harald Skarke for taking the time to listen to some of the (not always well formulated) ideas that came up during my research and his endurance in explaining some of the more mathematical aspects of physics.

It was pleasure to get to know Justin David and Araditha and I am grateful for the hospitality I experienced during my visit at IISc Bangalore. I would also like to thank the Institute of Theoretical Physics at TU Wien and all of its members for providing an excellent working and studying environment.

I would like to end by thanking my whole family for always being supportive and never asking more than the necessary questions. Especially I have to thank my sister and her family for providing me with lots of joy and distractions outside of physics.

This work was in part supported by the Austrian Science Fund (FWF): P 285552.

Contents

1 Introduction

1.1 Historic remarks and overview

This work is on Moonshine in conformal field theory and sting theory. Moonshine in general refers to surprising connections between two rather unrelated fields of mathematics namely modular functions and special finite groups (sporadic groups). In mathematics modular functions had originally mostly been considered in number theory while the classification of finite groups had been taken place completely independently ((see, e.g, [1, 2]) for some historic notes). However in 1978 John McKay observed that the coefficients of a certain modular function (the Klein-J-function [1]) correspond to sums of dimensions of irreducible representations of the largest sporadic group (the Monster group)[2]. The simplest such relation famously takes the form

$$196884 = 196883 + 1. \tag{1.1}$$

This naive observation had non the less far reaching implications for mathematics as it became clear that it was no mere coincidence but a consequence of a much deeper connection between group theory/algebra and number theory (modular objects). In the months/years following the initial observation the 'Monstrous Moonshine conjecture' was formulated by Conway and Norton [3]. The proof of this conjecture involved the construction of the 'Monster module' by Frenkel, Lepowsky, Meurman [4] with the help of vertex operator algebras, which had previously been invented by Borcherds [5]. The Monstrous Moonshine conjecture was finally proven by Borcherds in 1992 [6], for which he received the Fields medal in 1998.

The connection to physics goes back Dixon-Gispang-Harvey [7] who gave a 'physical' construction of the Monster module. More precisely they constructed a chiral conformal field theory (CFT) consisting of 24 bosons on a $\mathbb{Z}_2$ orbifold of the Leech lattice. This CFT has the Monster group as its symmetry group and its partition function is the Klein-J-function. In this setting the somehow rather abstract vertex operator algebra can be interpreted as the algebra of chiral primary fields. The connection between the formerly unconnected areas of mathematics becomes natural here, since the Monster group appears as symmetry group of a (chiral) CFT and the modular object is a counting function of states that arise as vacuum amplitude on the torus.

Another instance of moonshine was observed by Conway and Norton in [3] connecting the Conway group Co_0 to certain congruence subgroups of the modular group $SL(2,\mathbb{Z})$, which again can be linked to modular objects. A module in this setting

[1] It is uniquely defined to be the complex function invariant under the full modular group $SL(2,\mathbb{Z})$ mapping $SL(2,\mathbb{Z})\backslash\mathbb{H}\cup\{\infty\}$ bijectively to $\mathbb{C}\cup\{\infty\}$ with expansion $e^{-2\pi i z}+O(e^{2\pi iz})$ at $z\to i\infty$.

[2] Already in 1975 Andrew Ogg had noticed a connection between congruence subgroups of $\mathbb{S}L(2,\mathbb{Z})$ which later was seen to be connected to moonshine.

was found in [8, 9]. The 'physical' realization as a CFT in this case is different than for Monstrous moonshine as it also involves fermions.

Interest in moonshine was greatly renewed in 2010, when Eguchi, Ooguri and Tachikawa [10] noticed that the elliptic genus of a $\mathcal{N} = (4,4)$ non-linear sigma model on $K3$ seems to be related to the largest Mathieu group M_{24} when expanded in $\mathcal{N} = 4$ characters. Subsequent research [11–15] confirmed this connection. However the authors of [16] made it clear that M_{24} cannot appear directly as the symmetry of a $K3$ sigma model. Since the elliptic genus counts BPS states of the theory M_{24} should be a symmetry of these states. Also since it is an index and does not depend on the moduli space of the theory, M_{24} might arise as a 'sum' of symmetries at different points in moduli space. These ideas which have been pursued in [17–20] have been coined 'symmetry-surfing'.

In [21, 22] Cheng, Duncan and Harvey established a new kind of moonshine connecting all Niemeier lattices and their symmetry groups to slightly generalized modular functions (mock modular forms) which also includes Mathieu moonshine as a special case. In [15, 23] the existence of a module was shown such that the specific functions that appear are trace functions. In [24] the Umbral moonshine was related to the $K3$ sigma models. In [25] the authors showed that all umbral symmetry groups appear at different points in the moduli space of type IIA string theory on $K3 \times T^2$ which provides a natural physics starting point for its understanding.

In [26, 27] moonshine phenomena involving the Thompson group were found and proven and in [28] moonshine involving the O'Nan group was proven.

We see that the general picture that emerges is that the physical models form a natural connection between seemingly unrelated areas of modular functions and finite groups. In this setting the modular functions appear as certain functions of the states of the model. Modular symmetries acting on the moduli space of these models are then responsible for the modular properties of the counting functions. The symmetries acting on the states of the model are then the special finite groups that appear. Lately it has been tried to construct CFTs with sporadic symmetries in a more systematic way [29].

1.2 Mathieu Moonshine and Gromov-Witten invariants

In compactifications of the $E_8 \times E_8$ heterotic string on $K3 \times T^2$ the elliptic genus is connected to certain gravitational couplings of the effective theory. In this way these couplings also get related to Mathieu moonshine. Further through the duality with type II string theory compactified on Calabi-Yau threefolds it gets connected to topological invariants (Gromov-Witten/Gopakumar-Vafa invariants) of these manifolds [30, 31]. Mathieu Moonshine has also been related to the Yukawa couplings and the holomorphic 3-form in these models [32]. Further in [33] Mathieu moonshine has been shown to be related to four dimensional theories with $\mathcal{N} = 1$ supersymmetry.

So one may observe that through string dualities moonshine has also been connected to (algebraic) geometry.

1.3 Further appearances of moonshine

Since modular objects appear naturally in string compactifications potential relations to moonshine can be found in many places. As an example consider type II string theory compactified on $K3 \times T^2$. The counting function of dyons (electric and magnetic charged BPS states) is a Siegel modular form which can be obtained by a well described procedure ('Borcherds lift') from the elliptic genus of $K3$, see, e.g, [34] and reference therein. The number of dyons also gives a count of the microstates of supersymmetric black holes.In the M-theory setting this has also been explored in, e.g, [35]. The Monster module has also been related to 3 dimensional gravity by Witten in [36].

1.4 About this thesis

This thesis is aimed at finding and establishing new moonshine connections. We start by introducing the necessary mathematical building block in section 2. In section 3 we discuss some known instances of moonshine in greater depth. Then in section 4 we look for moonshine in the elliptic genera of higher dimensional Calabi-Yau manifolds. We do this by expanding them in superconformal characters and calculating their twined elliptic genera. We also analyse certain toroidal and Gepner models.This section is based on [37]. In section 5 Mathieu moonshine in CHL-orbifolds of heterotic compactifications on $K3 \times T^2$ is studied. Through duality with type II string theory it gets connected the Gromov-Witten invariants of the dual CY. We explain the models on both sides of the duality and find the dual Calabi-Yau manifolds for certain heterotic models. This section is based on [31, 38].

In the course of my PhD studies I have published the following papers

[**1**] A. Banlaki, A. Chattopadhyaya, A. Kidambi, T. Schimannek, M. Schimpf, *Heterotic strings on* $(K3 \times T^2)/Z_3$ *and their dual Calabi-Yau threefolds*, JHEP 04 (2020) 203, [1911.09697].

[**2**] A. Banlaki, A. Chowdhury, A. Kidambi, M. Schimpf, *On Mathieu Moonshine and Gromov-Witten invariants*, JHEP 02 (2020) 082,[1811.11619]

[**3**] A. Banlaki, A. Chowdhury, C. Roupec, T. Wrase, *Scaling limits of dS vacua and the swampland*, JHEP 1903 (2019) 065, [1811.07880]

[**4**] A. Banlaki, A. Chowdhury, A. Kidambi, M. Schimpf, H. Skarke, T. Wrase, *Calabi-Yau manifolds and sporadic groups*, JHEP 1802 (2018) 129, [1711.09698]

This thesis is based on [1], [2], [4] which are all connected to moonshine, which was my main research interest during my PhD. My article [3] arose from my interest in the swampland conjectures in particular in the dS swampland conjecture. My contributions to these articles has been the following:

[**1**] I Higgsed the heterotic spectrum and calculated the Gopakumar-Vafa invariants predicted from the heterotic side and checked with the results on the Type II side.

[**2**] I calculated the Higgsed spectrum of the different heterotic models. For the relevant models I searched the Kreuzer-Skarke database for matching CYs. I calculated the Gromov-Witten invariants and analysed if they match the heterotic results.

[**3**] I analysed some of scaling limits of the scalar potential and helped in analysing the scalar potential of the explicit example we discuss.

[**4**] I wrote some of the initial Mathematica code used for analysing 5-fold twined elliptic genera and calculated some of the twisted elliptic genera of the toroidal and Gepner models of chapter 5.

2 Mathematical building blocks

In this section we will introduce the basic mathematical objects that lie at the heart of moonshine. As mentioned in the introduction these are (generalized) modular forms/Jacobi forms and sporadic groups.

2.1 Modular objects

In this subsection we introduce the modular objects that appear in moonshine. We follow [39–43].

Given two complex numbers $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\mathrm{Im}(\lambda_2/\lambda_1) \neq 0$, we can define the lattice $\Lambda = \langle \lambda_1, \lambda_2 \rangle := \{n\lambda_1 + m\lambda_2 | n, m \in \mathbb{Z}\}$, and the torus

$$T^2 = \{z \in \mathbb{C} | z \sim z + \lambda, \forall \lambda \in \Lambda\}.$$

Using conformal transformations continuously connected to the identity, we can rotate and rescale the basis λ_1, λ_2 of the lattice. In particular thereby we can always set $\lambda_1 = 1$ and T^2 can hence be completely specified by the modulus $\tau := \frac{\lambda_2}{\lambda_1}$. In particular τ can be chosen to lie in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$, since we can always interchange λ_1 and λ_2. Conformal transformations which are not continuously connected to the identity correspond to a change of oriented basis of Λ. They are generated by the *modular group* $SL(2, \mathbb{Z})$ of 2×2 matrices with integer entries and unit determinant. They act on the modular parameter as [3]

$$SL(2, \mathbb{Z}) \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{H} \to \mathbb{H}\,,\ \tau \mapsto \frac{a\tau + b}{c\tau + d} \tag{2.1}$$

and are generated by the two elements [4]

$$T : \tau \to \tau + 1\,,\ S : \tau \to -\frac{1}{\tau}\,. \tag{2.2}$$

Noting that $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the only element (apart from the identity itself) that acts trivially on $\mathbb{H}$, we define the quotient $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) / \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which now acts faithfully on $\mathbb{H}$. In general we also want to add ∞ (or $i\infty$) to $\mathbb{H}$. It is then understood that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, $\gamma(\infty) := \frac{a}{c}$, so that altogether we add $\mathbb{Q} \cup \{\infty\}$ to $\mathbb{H}$. These additional points are called cusps and $SL(2, \mathbb{Z})$ acts transitively on them.

[3]Forming a subgroup of $SL(2, \mathbb{R})$ and the associated Möbius transformations with real coefficients.

[4]On the lattice this corresponds to changing to the basis: $< \lambda_1, \lambda_2 + \lambda_1 >$ and $< -\lambda_2, \lambda_1 >$ respectively.

The *fundamental domain* $SL(2,\mathbb{Z})\backslash\mathbb{H}$ may be represented by the set

$$\mathcal{F} := \{z \in \mathbb{H}| -\frac{1}{2} \leq \text{Re}z \leq \frac{1}{2} \text{ and } |z| \geq 1\} \tag{2.3}$$

together with the identification $z_1 \sim z_2$ iff Re $z_1 = \pm\frac{1}{2}$ and $z_2 = z_1 \pm 1$ or if $|z_1| = 1$ and $z_2 = -\frac{1}{z_1}$. It is known as the key-hole region. Every torus can uniquely characterised by a point in this region up to conformal transformations. Adjoining the cusps, (complex) infinity as well as $\mathbb{Q}$, to the key hole region we obtain a set of genus zero, i.e., homeomorphic to the 2-sphere S^2.

More generally for every $N \in \mathbb{N}, N \geq 1$, we define the following subgroups of the full modular group $SL(2,\mathbb{Z})$:

$$\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma | c \equiv 0 \,(\text{mod} N)\} \tag{2.4}$$

$$\Gamma_1(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma | c \equiv 0 \,(\text{mod } N) \text{ and } a, d \equiv 1 \,(\text{mod } N)\} \tag{2.5}$$

$$\Gamma(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma | b, c \equiv 0 \,(\text{mod } N) \text{ and } a, d \equiv 1 \,(\text{mod } N)\}. \tag{2.6}$$

In general we call a subgroup $\Gamma < SL(2,\mathbb{Z})$ *congruence subgroup*, if Γ contains $\Gamma(N)$ for some $N \geq 1$. The action of a congruence subgroup on the cusps, $\mathbb{Q} \cup \{\infty\}$, has finite many orbits.

We note that the points of $\Gamma_0(N)\backslash\mathbb{H}$ characterize a torus with a cyclic subgroup of fixed order N and the points of $\Gamma_1(N)\backslash\mathbb{H}$ characterize a torus with a generator of a cyclic subgroup of fixed order N [43]. The last statement can be understood in the following way [43]:

For a point in $\Gamma_1(N)\backslash\mathbb{H}$ let $\tau \in \mathcal{F}$ be the point that arises through the natural map $\Gamma_1(N)\backslash\mathbb{H} \to SL(2,\mathbb{Z})\backslash\mathbb{H}$. Then consider $1/N \in \mathbb{Z} + \tau\mathbb{Z}$ as generator of a cyclic group of order N. In general for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ the map

$$f : \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \to \mathbb{C}/\mathbb{Z} + \gamma(\tau)\mathbb{Z}\,,\; z \mapsto \frac{z}{c\tau + d} \tag{2.7}$$

is an isomorphism between the torus with complex structure τ and $\gamma(\tau)$. If we further assume $\gamma \in \Gamma_1(N)$ then $1/N$ remains fixed under a change of basis by γ, since

$$\frac{1}{N} - \frac{\frac{1}{N}}{c\tau + d} = \frac{\frac{c}{N} + \frac{d-1}{N}}{c\tau + d} \in f(\mathbb{Z} + \tau\mathbb{Z}) = \mathbb{Z} + \gamma(\tau)\mathbb{Z}. \tag{2.8}$$

So a specific point of order N is fixed. Further given a torus with complex structure τ together with a generator of a cyclic group one can find an isomorphism mapping this torus to a torus with complex structure in $\Gamma_1(N)\backslash\mathbb{H}$ and mapping the generator to $1/N$.

2.1.1 Modular forms

Having introduced the modular group above we now discuss functions which have special transformation properties under it.
We call a function $f : \mathbb{C} \to \mathbb{C}$ *modular* if it is meromorphic and invariant under the modular group, i.e.,

$$f(z) = f(\gamma z) \, , \, \forall \gamma \in SL(2, \mathbb{Z}). \tag{2.9}$$

Note that modular functions naturally form a vector space. A modular function can equivalently be seen as meromorphic function defined on the *fundamental domain* $SL(2, \mathbb{Z}) \setminus \mathbb{H}$ or as a function F on a lattices in the complex plane, satisfying $F(\Lambda) = F(\lambda\Lambda)$ for all $\lambda \in \mathbb{C}^*$, where the connection between f and F is given by: $f(z) := F(\langle 1, z\rangle)$ and $F(< \lambda_1, \lambda_2 >) := f(\frac{\lambda_1}{\lambda_2})$.
More generally for a congruence subgroup Γ of $SL(2, \mathbb{Z})$ one defines a *modular function for* Γ [5] to be a meromorphic function on $\mathbb{H}$ invariant under Γ and of exponential growth at infinity, i.e., $f(x + iy) = O(e^{Cy})$ for $y \to \infty$ and some $C > 0$.
This concept may be generalized by demanding the corresponding function F on lattices to have a certain integer *weight* k under scaling, i.e., $F(\lambda\Lambda) = \lambda^{-k}F(\Lambda)$. Translating this back to functions on $\mathbb{H}$ we define a *modular function of weight* k *for* a congruence subgroup Γ to be defined as above but with (2.9) replaced by

$$f(\gamma z) = (cz + d)^k f(z), \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{2.10}$$

Similarly we define a *modular form of weight* k *for* Γ to be a holomorphic function on $\mathbb{H}$ transforming as in (2.10) for all $\gamma \in \Gamma$ and of subexponential growth at infinity, i.e.,

$$f(x + iy) = O(e^{Cy}) \text{ for } y \to \infty \text{ and for all } C > 0. \tag{2.11}$$

The growth condition implies polynomial growth at infinity, more precisely that $f(x+iy) = O(1)$ for $y \to \infty$ and $f(\frac{1}{z}) = O(y^{-k})$ for $y \to 0$. The set of modular forms of weight k on a discrete group $\Gamma < SL(2, \mathbb{R})$ is denoted by $M_k(\Gamma)$. It forms a finite dimensional vector space and is zero for $k < 0$. The algebra $M_*(\Gamma) := \bigoplus_k M_k(\Gamma)$ consisting of modular forms of all weights on $\tilde{\Gamma}$ is finitely generated. Every modular function can be represented as a fraction of two modular forms.
Specializing property (2.10) to the case $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ we see that $f(z + 1) = f(z)$, i.e., the modular forms on a congruence group $\tilde{\Gamma}$ containing $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are periodic in z and hence may be expanded in $q := e^{2\pi i z}$,

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^n \tag{2.12}$$

[5] Sometimes also modular function on Γ is used.

The growth conditions imply that only finitely many $a_n, n < 0$ can be non-zero for a modular function and that $a_n = 0$ for $n < 0$ for a modular form. A modular form for which $a_0 = 0$ is called a *cusp form*. Further whenever $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma$ we see that $M_k(\Gamma)$ is zero for k odd (again by (2.10)).
Typical examples of modular forms of weight $2k, k > 1$, for $SL(2,\mathbb{Z})$ are the Eisenstein series given by

$$E_{2k}(z) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}, \tag{2.13}$$

where ζ is the Riemann zeta function. In particular one can show that the space of all modular forms for $SL(2,\mathbb{Z})$, $M_*(SL(2,\mathbb{Z}))$, can be generated by the Eisenstein series E_4, E_6. More explicit expressions are given in Appendix A.
Another modular function which will be of particular importance for us is the Klein-J function $J(z)$. It is the unique modular function of weight zero mapping the (enlarged) fundamental domain, $\mathcal{F} \cup \{\infty\}$, bijectively to $\mathbb{C} \cup \infty$ and with the following expansion at $z \to i\infty$ $(q \to 0)$

$$J(z) = q^{-1} + O(q). \tag{2.14}$$

In terms of the Eisenstein series $J(z)$ may be expressed as

$$J(z) = 1728 \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2} - 744. \tag{2.15}$$

Its first few expansion coefficients around $z \to i\infty$ $(q \to 0)$ are

$$J(z) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots. \tag{2.16}$$

It is a *Hauptmodul* for the function field of modular functions of weight 0 for $SL(2,\mathbb{Z})$, i.e., each function $k(z)$ of this field may be written as a rational function of $J(z)$ with complex coefficients [6] ,

$$k(z) = \frac{\sum_{i=0}^{n} a_i J(z)^i}{\sum_{j=0}^{m} b_j J(z)^j} , \ a_i, b_j \in \mathbb{C}. \tag{2.17}$$

The concept of modular forms may be generalized in different ways. The particular generalizations we will encounter are modular forms of *half-integer weight* with *multiplier system*. We simply give two examples which are important for us.

[6] Recall that meromorphic functions on the sphere (on $\mathbb{C} \cup \infty$) are in general rational functions, i.e., quotients of polynomials.

First we define the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.18}$$

It transforms under modular transformations as

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau) \ , \ \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau), \tag{2.19}$$

i.e., it is a modular form of weight $\frac{1}{2}$ with a multiplier system. As a side note, $q^{-1/24}\eta(\tau)$ is known as the Euler function $\phi(q)$ and is the multiplicative inverse of the generating functions of partitions of natural numbers, i.e.,

$$\frac{1}{\phi(q)} = q^{\frac{1}{24}} \frac{1}{\eta(\tau)} = \sum_{k=0}^{\infty} p(k) q^k \tag{2.20}$$

where $p(k)$ is the number of partitions of k. Due to the Euler identity $\eta(\tau)$ has a power series/Laurent series expansion of the form

$$\eta(\tau) = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2 - n)/2}. \tag{2.21}$$

The 24-th power of the Dedekind eta function is known as discriminant function, $\Delta(z) := \eta(z)^{24}$. It is a modular form of weight 12 for $SL(2, \mathbb{Z})$. It was originally studied by Ramanujan. Its inverse also features prominently in the chiral part of the bosonic string partition function.
As a second example we introduce the theta function

$$\theta(\tau) := \theta_3(2\tau, 0) = \sum_{n \in \mathbb{Z}} q^{n^2}. \tag{2.22}$$

It transforms as

$$\theta(z + 1) = \theta(z) \ , \ \theta(-\frac{1}{4z}) = \sqrt{\frac{2z}{\mathrm{i}}}\theta(z) \tag{2.23}$$

from which one can show that it is a weight 1/2 modular form for $\Gamma_0(4)$ with non-trivial multiplier system. If we define $N_n(k)$ to be the number of ways one can write the integer n as a sum of k squares, counting order and signs, we see that

$$\sum_{n=0}^{\infty} N_n(k) q^n = \sum_{a_1 \in \mathbb{Z}} q^{a_1^2} \times \cdots \times \sum_{a_k \in \mathbb{Z}} q^{a_k^2} = \theta(z)^k. \tag{2.24}$$

Hence $\theta(z)^k$ is the generating function for $N_n(k)$. For $k = 2$ this with some additional work can for example be used to prove that every prime p, $p \equiv 1 \mod 4$, is the sum of 2 squares [44].

We also want to mention that we can also consider $\theta_3(z,0)$ as a modular form for the full modular group $SL(2,\mathbb{Z})$. What we find is that it transforms as a component of the vector valued modular form $\Theta(z) := (\theta_2(z,0), \theta_3(z,0), \theta_4(z,0))$, which has the following transformation properties

$$\Theta\left(-\frac{1}{z}\right) = \sqrt{-\mathrm{i}z}\,\mathcal{S}\Theta(z)\,,\ \Theta(z+1) = \mathcal{T}\Theta(z) \tag{2.25}$$

where

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,,\ \mathcal{T} = \begin{pmatrix} e^{\mathrm{i}\pi/4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.26}$$

Finally we mention that there exist another generalization of modular forms, called *mock modular forms*, see, e.g., [34, 41]. Let $w \in \frac{1}{2}\mathbb{Z}$ and let h be a holomorphic function $\mathbb{H}$ with at most exponential growth at all cusps. h is called a weakly holomorphic mock modular form of weight w for a subgroup $\Gamma \leq SL(2,\mathbb{R})$ if there is a modular from of weight $2-w$ such that the sum $\hat{h} := h + g^*$ transforms like a holomorphic modular from of weight w for Γ. Here g^* is uniquely defined by g (we do not give the exact definition, since we will not need it), however g^* is not holomorphic, rather it fulfils

$$-2\mathrm{i}\Im(\tau)^w \frac{\partial}{\partial\bar{\tau}} g^*(\tau) = \overline{g(\tau)}. \tag{2.27}$$

In the physical setting mock modular forms will typically appear when the theory (its moduli space) is non-compact.

2.1.2 Jacobi forms

In this subsection we introduce a different set of modular objects, called (weak) Jacobi forms. The prime example of a Jacobi form we will encounter later will be the elliptic genus of $\mathcal{N}=2$ super conformal field theories. We start with the definition [34, 44]. A *weak Jacobi form* $\phi_{k.m}(\tau,z)$ *of weight* $k\in\mathbb{Z}$ *and index* $m\in\mathbb{Z}^+$ is a function from $\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ fulfilling the following properties

1. $\phi_{k,m}(\tau,z)$ is 'modular in τ and elliptic in z', i.e.,

$$\phi_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi_{k,m}(\tau,z), \quad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}). \tag{2.28}$$

2. $\phi_{k,m}(\tau,z)$ transforms under translations of z by $\tau\mathbb{Z}+\mathbb{Z}$ in the following way

$$\phi_{k,m}(\tau, z+\lambda\tau+\mu) = (-1)^{2m(\lambda+\mu)} e^{-2\pi i m(\lambda^2\tau+2\lambda z)} \phi_{k,m}(\tau,z), \quad \forall \lambda,\mu\in\mathbb{Z}. \tag{2.29}$$

3. From the above properties it follows that $\phi_{k,m}(\tau+1,z) = \phi_{k,m}(\tau,z+1) = \phi(\tau,z)$ which allows for a Fourier expansion

$$\phi_{k,m}(\tau,z) = \sum_{n,r} c(n,r) q^n y^r \tag{2.30}$$

where $q := e^{2\pi i\tau}, y := e^{2\pi iz}$. In particular the property (2.28) implies that $c(n,r) = C(4nm - r^2, r)$ where $C(\Delta, r)$ only depends on $r \pmod{2m}$. The quantity $\Delta = 4mn - r^2$ is called the *discriminant*. A Jacobi form $\phi_{k,m}(\tau,z)$ is called *holomorphic* if $C(\Delta,r)$ vanishes for $\Delta < 0$, i.e., if

$$c(n,r) = 0 \quad \text{unless} \quad 4nm \geq r^2, \tag{2.31}$$

it is called *a Jacobi cusp form* if $C(\Delta,r)$ vanishes for $\Delta \leq 0$, i.e., if

$$c(n,r) = 0 \quad \text{unless} \quad 4nm > r^2, \tag{2.32}$$

and it is called a *weak Jacobi form* if

$$c(n,r) = 0 \quad \text{unless} \quad n \geq 0. \tag{2.33}$$

Finally if a Jacobi form satisfies the yet weaker condition $c(n,r) = 0$ unless $n \geq n_0$ for some fixed $n_0 \in \mathbb{Z}$ it is called *weakly holomorphic Jacobi form.* The space of all holomorphic resp. cuspidal, weak, weakly holomorphic Jacobi forms of weight k and index m is denoted by $J_{k,m}, J^0_{k,m}, \tilde{J}_{k.m}, \tilde{J}^!_{k,m}$ and all are finite dimensional.

The space of weak Jacobi forms of even weight k and integer index m is generated by the Eisenstein series $E_4(\tau), E_6(\tau)$ and the functions $\phi_{-1,2}(\tau,z), \phi_{0,1}(\tau,z)$ [44] defined in Appendix A. One can easily convince oneself that the space $\tilde{J}_{0,m}$ of weak Jacobi forms of weight 0 and index m is generated by m basis elements for $m = 1,2,3,4,5$. More precisely one finds

$$\begin{aligned}
\tilde{J}_{0,1} &= \{\phi_{0,1}\} \\
\tilde{J}_{0,2} &= \{\phi_{0,1}^2, E_4\phi_{-2,1}^2\} \\
\tilde{J}_{0,3} &= \{\phi_{0,1}^3, E_4\phi_{-2,1}^2\phi_{0,1}, E_6\phi_{-2,1}^3\} \\
\tilde{J}_{0,4} &= \{\phi_{0,1}^4, E_4\phi_{-2,1}^2\phi_{0,1}^2, E_6\phi_{-2,1}^3\phi_{0,1}, E_4^2\phi_{-2,1}^4\} \\
\tilde{J}_{0,5} &= \{\phi_{0,1}^5, E_4\phi_{-2,1}^2\phi_{0,1}^3, E_6\phi_{-2,1}^3\phi_{0,1}^2, E_4^2\phi_{-2,1}^4\phi_{0,1}, E_4E_6\phi_{-2,1}^5\}.
\end{aligned}$$

The spaces of weak Jacobi forms of even weight and half integer index are related to the spaces of weak Jacobi forms of even weight and integer index through the simple relationship [45]

$$\tilde{J}_{2k,m+\frac{1}{2}} = \phi_{0,\frac{3}{2}} \tilde{J}_{2k,m-1}, \quad m \in \mathbb{Z}. \tag{2.34}$$

2.2 Sporadic groups

Apart from the modular objects discussed in the subsection above, certain special finite groups, so called sporadic groups, play an important role in Moonshine. We will introduce them and their most important properties in this subsection.
A group is a pair $(G, \cdot)$ consisting of a set G and a binary operation $\cdot : G \times G \to G$ such that the following properties are fulfilled:

(1) Identity: There exist an element $e \in G$, s.t. $e \cdot a = a \cdot e = a$ for all $a \in G$. (Such an element is unique.)

(2) Associativity: For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

(3) Inverse element: For each $a \in G$ there exists an inverse element, denoted by $a^{-1} \in G$, s.t. $a \cdot a^{-1} = a^{-1}a = e$. (The inverse element is unique.)

In the following we will mostly just write G when referring to a group $(G, \cdot)$. Also we will often use the shorthand notation ab for $a \cdot b$.
A subgroup N of a group G, denoted by $N \leq G$, is as subset $N \subseteq G$ s.t. the set N together with the restriction of the binary operation to $N \times N$ again forms a group. A subgroup N of a group G is called *normal*, denoted by $N \triangleleft G$, if $gNg^{-1} = N$ for all $g \in G$ [7], i.e., if N is invariant under conjugation. For a normal subgroup $N \triangleleft G$ we have $aN \cdot bN = abN$. Hence the set of left cosets $G/N = \{aN | a \in G\}$ form a group, called *quotient group*. A group G is called *simple* if its only normal subgroups are G itself and $\{e\}$.
Given two groups $(G, \cdot)$, $(H, \star)$ a group homomorphism $\phi : G \to H$ is a map from G to H respecting the binary operation, i.e., $\phi(a \cdot b) = \phi(a) \star \phi(b)$ for all $a, b \in G$. The kernel of ϕ, denoted by *ker* ϕ, i.e., the set of elements in G mapped to the identity of H, is a normal subgroup of G. The image of G is a a subgroup of H isomorphic to $G/ker\phi$.
A representation $R = (V, \rho)$ of a group G is tuple consisting of a vector space V and a homomorphism $\rho : G \to \mathrm{Hom}(V)$. Given a representation R, we define the *character* $\mathrm{ch}_R : G \to \mathbb{C}, g \mapsto \mathrm{Tr}_V(\rho(g))$ [8]. By the cyclic property of the trace, characters are 'class-functions', i.e., they are constant on equivalence classes of the group G. These functions form a vector space and by defining an appropriate inner product one may show that the number of linear independent characters is equal to the number of irreducible representation of the group G. Characters of finite groups may be listed

[7] We define $aN := \{ab | b \in N\}$, $Na := \{ba | b \in N\}$.

[8] By common abuse of notation, for a representation $R = (V, \rho)$ we will often use R and V interchangeably and simply write g instead of $\rho(g)$

in character tables with columns labelled by equivalence classes [9] and rows labelled by the irreducible representations. In order to classify all finite groups it is sufficient to classify all finite simple groups. This follows mainly from the *Jordan-Hölder theorem* which states that for a fixed group G and any nested sequence

$$G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_k \supset H_{k+1} = \{e\}$$

such that H_i is normal in H_{i-1} and H_{i-1}/H_i is simple, the length of the sequence is always the same and the quotient groups H_{i-1}/H_i appearing are the same up to a possible permutation.

Classification of all finite simple groups was achieved in the early 1980s by a collective effort of a large number of mathematicians (see, e.g, [1, 2]). The results have been collected in the 'ATLAS of Finite Groups' [46]. The classification can be split up into groups belonging to an 'infinite family', which are

- the cyclic groups $\mathbb{Z}_p$,(p a prime)
- the alternating groups $\mathfrak{A}_n$, for $n \geq 5$
- 16 families of Lie type

and 26 *sporadic groups* which do not belong to any infinite family. For us mostly the sporadic groups will be of interest.
The largest sporadic group is the so called (Fischer-Griess) *Monster group* (constructed by Griess in 1980 [47]), normally denoted by M. It obtained its name because of its size which is

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot \approx 8 \times 10^{53}. \quad (2.35)$$

20 of the 26 sporadic groups are subgroups of M. Figure 1 shows a digram of all the sporadic groups where a line between two groups stands for the subgroup relation. The groups which appear as subgroups of the Monster group are also referred to as the 'happy family' whereas the remaining 6 sporadic groups are sometimes called pariah.
The groups that will be of special importance to us are apart from the Monster group the Mathieu groups M_{24}, M_{23} as well as the Conway group Co_1.
Co_1 can be obtained from the Conway group Co_0 by its centre. Co_0 is the automorphism group of the Leech lattice (see Appendix B).
The largest Mathieu group M_{24} has size

$$|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23. \quad (2.36)$$

[9]These are normally labelled by the order of an element in the class followed by a capital letter, A, B..., if there is more then one equivalence class with elements of the same order.

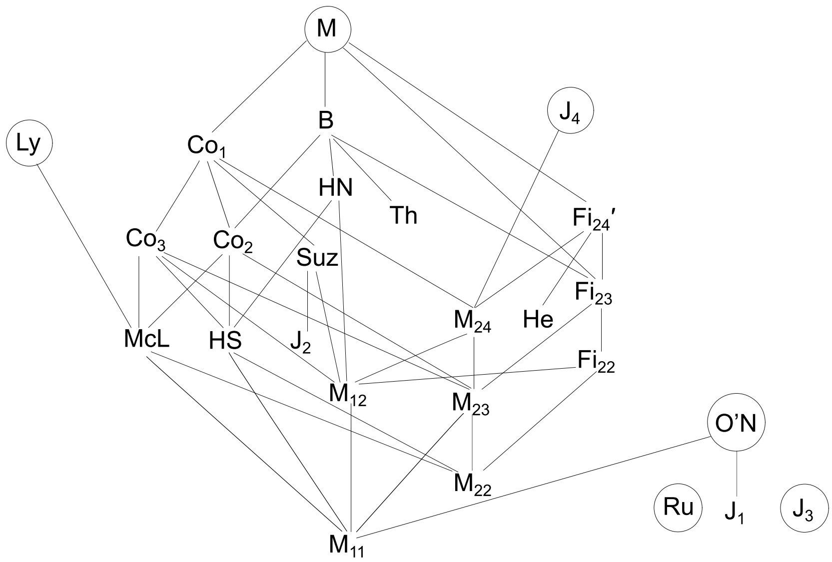

Figure 1: The sporadic simple groups. Figure from [48].

It can be thought of as subgroup of the permutation group S_{24} that is the automorphism group of the extended binary Golay code [10]. In particular M_{24} is 5-transitive[11]. The group M_{23} is of order

$$|M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23. \tag{2.37}$$

It is defined to be the subgroup of M_{24} that keeps one point fixed.

We end with some notation that will be needed in particular for the groups appearing in umbral moonshine. We call a group G the double cover of a group H, $G \simeq 2.H$, if G has a normal subgroup Z of order 2 (hence Z is central) s.t. $G/Z \simeq H$. For q a prime power, we define $\mathbb{F}_q$ to be the finite field with q elements. Then $GL_n(q)$ is defined to be the group of invertible $n \times n$ matrices with coefficients in $\mathbb{F}_q$. $AGL_n(q)$ denotes the affine linear group defined through the natural action of $GL_n(q)$ on $\mathbb{F}_q^n$ and the translation $x \mapsto x + v$ for $v \in \mathbb{F}_q^n$.

[10]The extended Golay code is an error-correcting code that encodes 12 bit of data into 24-bit of data such that any 3-bit error can be corrected. It consists of a 12-dimensional subspace W of $\{0,1\}^{24}$ of 24-bit words such that any two distinct elements of W differ in at least 8 coordinates.

[11]A permutation group G acting on n points is called *k-transitive* if for any two give sets of points, $a_1, \ldots, a_k$ and $b_1 \ldots b_k$, $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$, there exists a $g \in G$ with $g(a_i) = b_i$ for $i = 1, \ldots, k$.

3 Moonshine - first encounters and known examples

In this section we will briefly discuss known and well studied examples of Moonshine starting with Monstrous moonshine.

3.1 Monstrous Moonshine

The subject of moonshine started with the observation by John McKay in 1978 that the expansion coefficients of the Klein-J-function as in (2.16) can be expressed as sums of dimensions of irreducible representations of the Monster group. That is we may write

$$\begin{aligned} 196884 &= 192883 + 1 \\ 21493760 &= 21296876 + 196883 + 1 \\ 864299970 &= 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1 \\ &\vdots \end{aligned} \tag{3.1}$$

where the numbers on the l.h.s. are the expansion coefficients in (2.16) and the numbers on the r.h.s. are the dimensions of irreducible representations of M. Initially it was not clear if this was just some coincidence between large numbers or if it really constituted a surprising connection between previously completed unrelated parts of mathematics. Further light was shed on this subject by John Thompson [49] who proposed the existence of a graded representation V of the Monster group

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus \dots \tag{3.2}$$

with $V_{-1} = \rho_0, V_1 = \rho_1 \oplus \rho_0, V_2 = \rho_2 \oplus \rho_1 \oplus \rho_0, V_3 = \dots$ and ρ_i being the irreducible representation of M ordered by dimension such that

$$J(z) = \dim_q(V) := \dim(V_{-1})q^{-1} + \sum_{i=1}^{\infty} \dim(V_i)q^i. \tag{3.3}$$

It was then the suggestion by Thompson to consider for every $g \in M$ the more general series, now called *McKay-Thompson series*,

$$T_g(z) := \mathrm{ch}_{V,q}(g) = \mathrm{ch}_{V_{-1}}(g)q^{-1} + \sum_{i=1}^{\infty} \mathrm{ch}_{V_i}(g)q^i. \tag{3.4}$$

This was done by Conway and Norton [3] who found that various $T_g(z)$ indeed correspond to Hauptmoduls of various genus 0 modular groups. Thereby it became clear that the observations (3.1) were not mere coincidences. More precisely Conway and Norton conjectured that for each $g \in M$ the McKay-Thompson series $T_g(z)$ (as defined in (3.4)) coincides with the unique Hauptmodul with expansion $q^{-1} + O(q)$

near $z \to i\infty$ for some genus zero subgroup $\Gamma_g \leq SL_2(\mathbb{R})$, which contains $\Gamma_0(N)$ as a normal subgroup where $N \in \mathbb{N}$, $N|\Big(ord(g) \cdot gcd(24, ord(g))\Big)$. Notice that the McKay-Thompson series $T_g(z)$ only depends on the conjugacy class of g because of the cyclic property of the trace. The Monster group has 194 conjugacy classes. Moreover a character evaluated at g will be the complex conjugate of the character evaluated at g^{-1}. The total number of Hauptmoduls arising from the MacKay-Thompson series is 171.
Previously it had already been noted by Andrew Ogg in 1975 [50] that the primes p for which $\Gamma_0(p)+$[12] has genus 0 are

$$\{2, 3, 5, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\} \tag{3.5}$$

and are exactly the prime divisors of the order of the Monster group (2.35)[13]. Clearly this is somewhat explained by the Conway-Norton conjecture.
The Conway-Norton conjecture was verified by Atkin, Fong and Smith [51] numerically. An explicit construction of the envisioned graded module was finally given by Frenkel, Lepowsky and Meurman (FLM) [52]. This module is typically called the Monster module $V^\natural$. The FLM construction makes use of the concept of vertex operator algebra (which is a mathematically precise formulation of a chiral conformal field). We will not go into depth here rather we will explain the more physical approach given in [7].
This construction starts by considering 24 bosons on the Leech lattice, which is the unique self-dual, even 24-dimensional lattice with no vector of length less than 2. We follow [7] and begin the discussion with some basics notions of 2 dimensional conformal field theory (CFT). The infinitesimal conformal transformations $z \to z + \epsilon z^{n+1}$ of the complex plane are generated by the momenta

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \tag{3.6}$$

of the stress energy tensor. They satisfy the well known *Virasoro commutator algebra* relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{3.7}$$

where $c \in \mathbb{R}$ is the central charge of the theory. Representations of the Virasoro algebra may be given by *primary* fields $\phi(z)$ of a certain conformal weight h defined

[12] $\Gamma_0(p)+$ is the group generated by $\Gamma_0(p)$ together with $\frac{1}{\sqrt{p}}\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$, which are known as Atkin-Lehner involutions. It is the normaliser of $\Gamma_0(p)$ in $SL(2, \mathbb{R})$.

[13] Famously Ogg offered a bottle of Jack Daniel's whiskey to the first person to explain this coincidence.

by their operator product expansion (OPE) with the energy-momentum tensor

$$T(z_1)\phi(z_2) \sim \frac{h}{(z_1-z_2)^2}\phi(z_2) + \frac{1}{(z_1-z_2)}\partial_2\phi(z_2) + \dots \tag{3.8}$$

where on the r.h.s. we only explicitly state all the singular terms. By acting with a primary field on the vacuum one obtains a *highest weight state* , $|h\rangle = \phi(0)\,|0\rangle$. It fulfils $L_0\,|h\rangle = h\,|h\rangle\,, L_n\,|h\rangle = 0$ for all $n > 0$. By acting with the generators $L_n, n < 0$ on such a highest weight state one obtains a representation of the Virasoro algebra which for $c > 1$ and $h > 0$ will be unitary and irreducible[14]. A representation arising in such a way from a primary field of conformal weight h is in general referred to as *Verma module* and for the unitary, irreducible case denoted by $\mathbf{L}_{h,c}$. This module comes with a natural integer grading $\mathbf{L}_{h,c} = \bigoplus_n \mathbf{L}^n_{h,c}$ where

$$\mathbf{L}^n_{h,c} := \{v \in \mathbf{L}_{h,c} | L_0 v = (h+n)v\}. \tag{3.9}$$

In this context n is also called the *level*. The character of the Verma module is defined by

$$\mathrm{ch}\mathbf{L}_{h,c}(q) := q^{h-\frac{c}{24}} \sum_{n=0}^{\infty} (\mathrm{dim}\mathbf{L}^n_{q,c}) q^n = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}^{\infty}(1-q)^n}. \tag{3.10}$$

A conformal field theory consists of a left- and a right-moving (holomorphic-/anti-holomorphic) part given by conformal weight $(h,\bar{h})$. The torus partition function of the theory is defined as

$$Z_c(q,\bar{q}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \tag{3.11}$$

where the trace is taken over the Hilbert space (Verma module) of the theory and $q := \exp(2\pi i\tau)$, τ being the modular parameter of the torus. It will be of the form

$$Z_c(q,\bar{q}) = \sum_{h,\bar{h}} N_{h,\bar{h}} \mathrm{ch}\mathbf{L}_{h,c} \mathrm{ch}\mathbf{L}_{\bar{h},c} \tag{3.12}$$

where $N_{h,\bar{h}}$ counts the number of primary fields of conformal weight $(h,\bar{h})$ of the theory. $Z_c(q,\bar{q})$ should be invariant under modular transformations, (2.1), of τ. This put strong requirements on the possible form of $N_{h,\bar{h}}$, which we however will not discuss in detail here. For $c > 1$ in particular it leads to the requirement of an infinite number of primary fields. In specific situations (3.12) may factorize into left- and right-moving parts and one can consider the holomorphic part only. It is such a situation we will be interested in in the following which we will describe below.
We now consider the concrete example of 24 free bosons $X^i(z,\bar{z}), i = 1\dots 24$, on a

[14] In general unitarity restricts the values of h and c and one must factor out potential null states in order to obtain irreducible representations.

even, self-dual integer lattice $\Lambda/2 =< \mathbf{e}_1, \ldots, \mathbf{e}_{24} > /2$ of dimension 24. The action is given by

$$S = \frac{1}{2\pi} \int \partial X^i \bar{\partial} X^i + B_{ij} \partial X^i \bar{\partial} X^i = \frac{1}{2\pi} \int g_{ab} \partial X^a \bar{\partial} X^b + b_{ab} \partial X^a \bar{\partial} X^b \tag{3.13}$$

where $X^i \sim X^i + 2\pi(\lambda^i/2)$, for all $\lambda \in \Lambda$ and $X^i = (\mathbf{e}_a)^i X^a$ and hence $X^a \sim X^a + 2\pi(n^a/2)$, $n^a \in \mathbb{Z}$. The metric is defined by $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$ and $b_{ab} = B_{ij}(\mathbf{e}_a)^i(\mathbf{e}_b)^j$ are the components of the (constant) antisymmetric $B-$field w.r.t. the lattice frame. For the concrete choice

$$\mathbf{e}_a B \mathbf{e}_b = \mathbf{e}_a \cdot \mathbf{e}_b \bmod 2\,, \tag{3.14}$$

e.g., $b_{ab} = \pm g_{ab}, a \lesseqgtr b$ one can show that the Hilbert space factorizes into a holomorphic and anti-holomorphic part. That is we can split $X(z,\bar{z})$ in the following way

$$X(z,\bar{z}) = \frac{1}{2}(x(z) + x(\bar{z})). \tag{3.15}$$

where $x^i(z)$ are free holomorphic fields which fulfil $x^i(z) \sim x^i(z) + 2\pi\lambda$, for all $\lambda \in \Lambda$. Their mode expansion may be written as

$$x^i(z) = q^i - ip^i \ln z + i \sum_{n \neq 0} \frac{\alpha_n^i}{n} z^{-n} \tag{3.16}$$

with commutation relations given by

$$[q^i, p^j] = -i\delta^{ij}\,, \; [\alpha_n^i, \alpha_m^j] = n\delta^{ij}\delta_{n+m,0}. \tag{3.17}$$

For every $\beta \in \Lambda$ a highest weight state of this theory is defined by

$$p^i \left|\beta\right\rangle = \beta^i \left|\beta\right\rangle, \alpha_n \left|\beta\right\rangle = 0, \text{ for } n > 0. \tag{3.18}$$

By acting with α_{-n} on such a state the whole Fock space F_β is generated. The partition function of the theory therefore becomes

$$Z(q,\bar{q}) = \left| \frac{1}{\eta(q)^{24}} \sum_{\beta \in \Lambda} q^{\beta^2/2} \right| \tag{3.19}$$

(corresponding to the factorization $N_{h,\bar{h}} = N_h N_{\bar{h}}$ in (3.12)).
The primary fields of the theory are given by particular products of the dimension one $U(1)$ currents $j^i(z) = i\partial x^i$ and the exponentials

$$V_\beta(z) =: e^{i\beta \cdot x(z)} :, \; \beta \in \Lambda \tag{3.20}$$

of dimension $\beta^2/2$. The OPE of the vertex operators V_β as defined above is not associative. Associativity can be restored by including a factor $c(\beta)$ on the r.h.s. that satisfies, $c(\alpha)c(\beta) = \epsilon(\alpha,\beta)c(\alpha+\beta)$, where $\epsilon(\alpha,\beta)$ is a 2-cocycle[15].
The partition function of the holomorphic part of the theory is now given by

$$Z(q) = q^{-c/24}\mathrm{Tr}q^{L_0} = \frac{\Theta_\Lambda}{\eta(q)^{24}} \tag{3.22}$$

where $\Theta_\Lambda(q) = \sum_{\beta\in\Lambda} q^{\beta^2/2}$ is the theta function of the lattice Λ. For our choice of lattice - i.e., even, self-dual of dimension 24 - $Z(q)$ is modular invariant.
For the concrete lattice we are interested in, namely the Leech lattice Λ_{Leech}, which does not have points of $(\mathrm{length})^2 = 2$, we find

$$Z_{\mathrm{Leech}}(q) = \frac{\Theta_{\mathrm{Leech}}}{\eta^{24}} = \frac{1}{q} + 24 + 196884q + \cdots = J(q) + 24, \tag{3.23}$$

i.e., the modular invariant J-function which showed up in monstrous moonshine with an additional additive constant 24. The constant arises from the states generated by the 24 primary fields $i\partial x^i$ of weight 1. These may be removed by performing an additional $\mathbb{Z}_2$ orbifold, as originally considered by FLM. Concretely one simply orbifolds by the discrete symmetry $x \to -x$. This removes the unwanted states but it also enlarges the symmetry group of the theory in the following way. Before the orbifold the automorphism group of the Leech lattice is the Conway group Co_0. Because of the cocycle factors that arise in the definition of the vertex operators V_β the symmetry of the theory before the orbifold is actually an extension of Co_0 by the abelian group $(\mathbb{Z}_2)^{24}$. The symmetry group of the orbifolded Leech lattice is the first Conway group Co_1, which is the quotient of Co_0 by its centre. This again gets enlarged through the cocyle factors. This symmetry group acts on the orbifolded theory in a way that does not exchange twisted and untwisted states. The orbifolded theory however has an additional symmetry which roughly speaking boils down to exchanging twisted and untwisted states. These two symmetries together generate the Monster group which hence is a symmetry group of the orbifolded theory. In the following we will study the orbifolded theory more explicitly and show that the symmetry group is exactly the Monster group.
The lattice Λ_{Leech} has no invariant sublattices under this symmetry and the fixed points are classified by $\Lambda/2\Lambda$.
The Hilbert space $H_{(+)}$ before orbifolding is the direct product of the Fock spaces F_β (defined around (3.18)). The Hilbert space of the twisted theory will then consist

[15]For an abelian group A , $s \in \mathbb{N}$, a map $\epsilon : A\times A \to \mathbb{Z}/s\mathbb{Z}$ is called 2-cocycle if [4]

$$\epsilon(\alpha,\beta) + \epsilon(\alpha+\beta,\gamma) = \epsilon(\beta,\gamma) + \epsilon(\alpha,\beta+\gamma). \tag{3.21}$$

Cocycles naturally appear in central extensions of A.

of the invariant states of the untwisted theory, $H^+_{(+)}$ and the invariant states of the twisted theory $H^+_{(-)}$. More concrete if we denote the action the orbifold action by $g, g^2 = 1$, one defines

$$H^+_{(+)} = \{v \in H_{(+)} | gv = +v\}. \tag{3.24}$$

These states are explicitly given by

$$H^+_{(+)} = \{\alpha^{i_1}_{-n_1} \dots \alpha^{i_{2k}}_{-n_{2k}}(|\beta\rangle + |-\beta\rangle)\} \cup \{\alpha^{i_1}_{-n_1} \dots \alpha^{i_{2k+1}}_{-n_{2k+1}}(|\beta\rangle - |-\beta\rangle)\} \tag{3.25}$$

and the partition function of the untwisted sector can be calculated to be

$$Z^+_{(+)}(q) = q^{-1}\mathrm{Tr}_{H^+_{(+)}} q^{L_0} = \frac{1}{2}\left(\frac{\Theta_{\Lambda_{\mathrm{Leech}}}}{\eta^{24}} + \frac{q^{-1}}{\prod_{n=1}^{\infty}(1+q^n)^{24}}\right). \tag{3.26}$$

The twisted Hilbert space arises from fields satisfying $x^i(e^{2\pi i}z) = -x(z)(\mathrm{mod}\Lambda_{\mathrm{Leech}})$ which have the mode expansion

$$x^i(z) = \tilde{q}^i + i \sum_{m\in\mathbb{Z}+1/2} \frac{\alpha^i_n}{n} z^{-n}, \tag{3.27}$$

with α^i_n obeying commutation relations as in (3.17). The construction of the twisted Hilbert space is more subtle in particular due to the two-cocycle factors that arise in the construction of the operators (3.20). For the details we refer to [7]. For the partition function of the invariant states in this sector one finds

$$Z^+_{(-)}(q) = q^{-1}\mathrm{Tr}_{H^+_{(-)}} q^{L_0} = q^{1/2}\frac{1}{2}\left(\frac{1}{\prod_{n=1}^{\infty}(1-q^{n-1/2})^{24}} - \frac{1}{\prod_{n=1}^{\infty}(1+q^{n-1/2})^{24}}\right). \tag{3.28}$$

The complete partition function is then

$$\begin{aligned} Z_{tw}(q) = Z^+_{(+)}(q) + Z^+_{(-)}(q) &= \frac{1}{2}\frac{\Theta_{\Lambda_{Leech}}(q)}{\eta^{24}} + \frac{1}{2}2^{12}\left[\left(\frac{\eta}{\theta_2}\right)^{12} + \left(\frac{\eta}{\theta_4}\right)^{12} - \left(\frac{\eta}{\theta_3}\right)^{12}\right] \\ &= \frac{1}{2}\frac{\Theta_{\Lambda_{Leech}}(q)}{\eta^{24}} + \frac{1}{2}\frac{1}{\eta^{24}}\left[(\theta_3\theta_4)^{12} + (\theta_2\theta_3)^{12} - (\theta_1\theta_4)^{12}\right] \\ &= \frac{1}{2}(J(q) + 24(h+1)) + \frac{1}{2}(J(q) - 24) = J + 12h, \end{aligned}$$

where in the last line we have introduced the Coxeter number h of the lattice, which for our cases is $\frac{|\Lambda_2|}{24}$ and which vanishes for the Leech-lattice, i.e., $h_{\mathrm{Leech}} = 0$, so that we exactly obtain the J function as our partition function.

It remains to show that the symmetry of the theory is exactly the Monster group. We have already argued that the Monster group will be part of the symmetry group. To see that it is not larger we follow FLM and define a special product ('cross-bracket') on the Fourier components of fields of weight 2 by

$$\phi_m^i \times \phi_n^j = \frac{1}{2}([\phi_{m+1}^i, \phi_{n-1}^j] + [\phi_{n+1}^j, \phi_{m-1}^i]) \tag{3.29}$$

where i, j label the different dimension 2 fields. One can show that this cross-bracket closes on dimension 2 fields by considering the general form of the OPE of such fields [7]. As shown by FLM the infinite dimensional closed algebra given by this cross-bracket is an affinization of the Griess-algebra B. The Griess algebra itself appears as the 196884 zero mode subalgebra. It is known that the automorphism group of the Griess algebra is precisely the Monster group so we have succeeded in showing that the symmetry group of 24 bosons on the $\mathbb{Z}_2$ orbifold of the Leech lattice has exactly the Monster group as its symmetry group.

We still need that the chiral theory constructed above fulfils all the requirements of Monstrous moonshine, i.e., in particular that the traces $T_g = \mathrm{Tr}_{H^+_{(+)}+H^+_{(-)}} g q^{L_0-1}$ are indeed the genus zero Hauptmoduls conjectured. This can be done using identities fulfilled by the J-function and the T_g. The most basic one (found by Zagiers, Borcherds and others) is the relation

$$p^{-1} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} (1 - p^m q^n)^{a_{m\cdot n}} = J(\rho) - J(\tau) \tag{3.30}$$

where $p = e^{2\pi i \rho}$ and a_n are the coefficients of a q-expansion of J, i.e.,$J(q) = \sum_{n\geq -1} a_n q^n$. It implies indefinitely many relations between the a_i which fixes all a_i from just the knowledge of a_1, a_2, a_3, a_4, a_5. The structure of (3.30) is similar to denominator identities known from Lie and Kac-Moody algebras. Borcherds was able to generalize this identity to identities for the T_g's by introducing the notion of *generalised Kac-Moody algebras* [53] and in particular the Monster Lie algebra. The denominator identities for this generalized Kac-Moody algebra then leads to the necessary identities/generalizations of (3.30) [6].

In [54] Norton, based on earlier observation by himself and others, suggested a generalization of Monstrous moonshine in the following manner. For each element $g \in M$ there exists a graded projective representation $V(g) = \bigoplus_{n\in\mathbb{Q}} V(g)_n$ of the centralizer $C_M(g)$ of g in M and to each commuting pair (g, h) of elements in M there exists a holomorphic function $T_{(g,h)}$ defined on the upper half plane $\mathbb{H}$ s.t. the following holds:

(1) $T_{(g^a h^c, g^b h^d)}(\tau) = \gamma\, T_{(g,h)}(\frac{a\tau+b}{c\tau+d})$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and $\gamma \in \mathbb{C}, \gamma^{24} = 1$.

(2) $T_{(g,h)}(\tau) = T_{(k^{-1}gk, k^{-1}hk)}(\tau)$, $\forall k \in M$.

(3) There exists a lift $\tilde{h}$ of h to a linear representation on $V(g)$ s.t.

$$T_{(g,h)}(\tau) = \sum_{n\in\mathbb{Q}} \mathrm{Tr}_{V(g)_n} \tilde{h} q^{n-1}. \tag{3.31}$$

(4) $T_{(g,h)}(\tau)$ is either constant or a Hauptmodul for some genus zero congruence subgroup of $SL(2,\mathbb{Z})$.
(5) $T_{(e,h)}(\tau)$ coincides with the MacKay-Thompson series $T_h(\tau)$.
All these properties with the exception of (4) can more or less straightforwardly be understood from the construction of the Monster module as a chiral CFT as presented above. In this context $T_{(g,h)}$ are the twisted-twined partition functions which naturally fulfil the properties (1-3) and (5). The generalised moonshine conjectures have been proven in [55–57].

3.2 Conway moonshine

In the previous section we have seen that Monstrous moonshine establishes a connection between certain discrete genus zero subgroups of $SL(2,\mathbb{R})$ and the Monster group. More precisely to an element $m \in M$ (rather only its conjugacy class matters), given the knowledge of the Monster module, we associate its MacKay-Thompson series T_m and then find the appropriate genus zero group $\Gamma_m < SL(2,\mathbb{R})$ s.t. T_m induces an embedding $\Gamma_m \backslash \mathbb{H} \to \mathbb{C}$ (or alternatively s.t. Γ_m is the invariance group of T_m). In [3] Conway and Norton also described an assignment of genus zero groups $\Gamma_g < SL(2,\mathbb{R})$ to elements of the Conway group, Co_0. It can be explicitly be given as follows. For an element $g \in Co_0 = \mathrm{Aut}(\Lambda_{\mathrm{Leech}})$ that acts on $\Lambda_{\mathrm{Leech}} \otimes_{\mathbb{Z}} \mathbb{C}$ with eigenvalues $\{\varepsilon_i\}_{i=1}^{24}$ we assign $\Gamma_g < SL(2,\mathbb{R})$ to be the invariance group of the holomorphic function

$$t_g(\tau) := \prod_{n>0} \prod_{i=1}^{24} \left(1 - \varepsilon_i q^{2n-1}\right) \tag{3.32}$$

on $\mathbb{H}$. As a first step one may observe that $g = e$ the identity element, the associated function is up to a constant the function associated to the $2B$ element of the Monster group, $t_e = T_{2B} - 24$. In general it holds that

$$t_q = q^{-1} - \chi_q + O(q), \tag{3.33}$$

with $\chi_g = \sum_i^{24} \varepsilon_i$. So one defines $T_g^s(2\tau) := t_g(\tau) + \chi_g$ which then is the unique normalized principle modulus attached to the genus zero group Γ_g [9]. Also in this case one can find an infinite dimensional 'Conway module' s.t. the functions T_g^s are the corresponding McKay-Thompson series - the coefficients of T_g^s are then traces of g in this module. Such a module was sketched in [52] and more explicit constructed and studied in [8, 9] in terms of super-vertex algebras. We will review this in the more familiar language of CFTs In [4] the module was constructed as what corresponds to a chiral SCFT consisting of 8 bosons X_i and their super-partners ψ_i on the E_8 root lattice orbifolded by the $\mathbb{Z}_2$ action

$$(X_i, \psi_i) \to (-X_i, -\psi_i). \tag{3.34}$$

The theory after orbifolding is a $c = 12$ theory with no primary fields of conformal weight $h = \frac{1}{2}$ and $\mathcal{N} = 1$ supersymmetry. The partition function in the NS sector can easily be computed

$$\begin{aligned} Z_{NS,E8}(\tau)) &= \frac{1}{2}\Big(\frac{E_4(\tau)\theta_3(\tau,0)^4}{\eta^{12}} + 16\frac{\theta_4(\tau,0)^4}{\theta_2(\tau,0)^4} + 16\frac{\theta_2(\tau,0)^4}{\theta_4(\tau,0)^4}\Big) \\ &= q^{-1/2} + 0 + 276q^{1/2} + 2048q + 11202q^{3/2} + \dots . \end{aligned} \tag{3.35}$$

The first few coefficients may be split into irreducible representations of Co_1

$$276 = 276, \tag{3.36}$$

$$2048 = 1 + 276 + 1771, \tag{3.37}$$

$$\begin{aligned} 11202 &= 1 + 276 + 299 + 1771 + 8855, \\ &\vdots \end{aligned} \tag{3.38}$$

From this it seems that the SCFT has a Co_1 symmetry however how this symmetry arises is not obvious. One may use a different construction of the same SCFT by constructing it out of 24 free chiral fermions $\lambda_1, \dots, \lambda_{24}$ which are orbifolded by the $\mathbb{Z}_2$ action $\lambda_i \to -\lambda_i$. The partition function of this theory in the NS sector is simply

$$Z_{NS,ferm.}(\tau) = \frac{1}{2}\sum_{i=2}^{4}\frac{\theta_i(\tau,0)^{12}}{\eta(\tau)^{12}} \tag{3.39}$$

which is equal to (3.35) by non-trivial identities fulfilled by the theta functions. The 24 free fermions have a manifest $Spin(24)$ symmetry, but no explicit $\mathcal{N} = 1$ supersymmetry. One can construct a $\mathcal{N} = 1$ supercurrent in the following way. By an element $\mathbf{s} \in \mathbb{F}_2^{12}$, $\mathbb{F} = \{-1/2, 1/2\}$ we denote one of the $2^{12} = 4096$ linear independent ground states in the Ramond sector, created from the vacuum by the fermion zero modes $\lambda_i(0)$. Then let $\mathcal{W}_\mathbf{s}$ be the corresponding weight 3/2 spin field which implements the flow from the NS to the R sector. It was shown in [8] that there exists coefficients $c_\mathbf{s} \in \mathbb{C}$ s.t.

$$W = \sum_{\mathbf{s}\in\mathbb{F}_2^{12}} c_\mathbf{s}\mathcal{W}_\mathbf{s} \tag{3.40}$$

has the proper OPE with the energy momentum tensor to be a supercurrent. Any choice of W will break Spin(24) and it was shown in [8] that the suitable chosen $\mathcal{N} = 1$ current will break Spin(24) exactly to Co_0
As discussed in [58] this method may actually be generalized to find models with $\mathcal{N} = 2, 3$ and corresponding subgroups of Co_0 as symmetry groups.

3.3 Mathieu moonshine

In this subsection we will discuss Mathieu Moonshine, which to a large part will also be the main concern of the rest of this work.
We start by defining the *elliptic genus* $\mathcal{Z}^{ell}$ of a $\mathcal{N} = (2,2)$ superconformal field theory (SCFT) with central charges $(c, \bar{c})$ by [59]

$$\mathcal{Z}^{ell}(q,y)) = \mathrm{Tr}_{RR}\left((-1)^{F_L+F_R} y^{J_0} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \tag{3.41}$$

where $y = e^{2\pi i z}, q = e^{2\pi i \tau}$. Here J_0 is the zero mode of the third component of the $SU(2)$ R-symmetry algebra, L_0 is the zero mode of the Virasoro algebra and F_L, F_R are the respective left and right moving fermion number operators. The trace is taken over the Ramond-Ramond sector of the theory. The right moving part is just the Witten index and so for a theory with discrete spectrum [16] it will only get contributions from the right moving ground states, hence for such theories the elliptic genus will not depend on $\bar{q}$. The elliptic genus is and index that counts the BPS states of the theory and is independent of the moduli of the theory. The modularity properties of the CFT together with the spectral flow and unitarity imply that it is a weak Jacobi-form of index $m = \frac{c}{6}$ and weight 0 [61].
In [62] the elliptic genus of a $\mathcal{N} = (4,4)$ nonlinear sigma model (NLSM) on $K3$ (with central charge $(c, \bar{c}) = (6,6)$) was calculated to be

$$\mathcal{Z}^{ell}_{K3} = 8\left[\left(\frac{\theta_2(q,y)}{\theta_2(q,1)}\right)^2 + \left(\frac{\theta_3(q,y)}{\theta_3(q,1)}\right)^2 + \left(\frac{\theta_4(q,y)}{\theta_4(q,1)}\right)^2\right]. \tag{3.42}$$

Here the elliptic genus counts the number of $\frac{1}{4}$-BPS states of the theory. Contrary to the $\frac{1}{2}$-BPS states of the theory, the multiplicity of the $\frac{1}{4}$-BPS states is not completely protected when one moves around in the moduli space of the theory but the index is unchanged, i.e., two $\frac{1}{4}$-BPS states may pair to form a non-BPS state which is then no longer counted by the elliptic genus.
In [10] it was noted that when one expands the $K3$-elliptic genus in term of $\mathcal{N} = 4$ characters, the expansion coefficients can be related to dimensions of irreducible representations of the Mathieu group M_{24}. More concretely if we use the definition of $\mathcal{N} = 4$ superconformal characters given in Appendix C.2 one finds that

$$\mathcal{Z}^{ell}_{K3}(q,y) = 20\,\mathrm{ch}^{\mathcal{N}=4}_{2,0,0}(\tau,z) - 2\,\mathrm{ch}^{\mathcal{N}=4}_{2,0,\frac{1}{2}}(\tau,z) + \sum_{n=1}^{\infty} A_n\,\mathrm{ch}^{\mathcal{N}=4}_{2,n,\frac{1}{2}}(\tau,z) \tag{3.43}$$

[16] For a non-discrete spectrum the elliptic genus will in general be non-holomorphic and give rise to mock-modular forms [34, 60].

where the coefficients and their the decomposition into dimensions of irreducible representations of M_{24} are given by [17] [18]

$$\begin{aligned} A_{-1} &= 20 = 23 - 3 \cdot 1 \\ A_0 &= -2 = -2 \cdot 1 \\ A_1 &= 90 = 45 + \overline{45} \\ A_2 &= 462 = 431 + \overline{431} \\ A_3 &= 1540 = 770 + \overline{770} \\ &\vdots \end{aligned} \tag{3.44}$$

The reason the group M_{24} arises here were not completely clear. From the result of Mukai and Kondo [63, 64] it is known that the symplectic automorphisms of $K3$, i.e., automorphisms that leave the 2-form of $K3$ invariant, in general form a subgroup of M_{23}. It might seem natural to assume that this group simple gets enlarged to M_{24} through 'stringy effects'. More precisely since the elliptic genus counts $\frac{1}{4}$-BPS-states one might assume that the M_{24} acts on the sets of these BPS-states, i.e., that they form an infinite dimensional graded module [19]

$$\mathcal{H}^{BPS} = \bigoplus_{n=0}^{\infty} H_n \otimes \mathcal{H}_n^{\mathcal{N}=4} \tag{3.45}$$

where the sum runs over all $\mathcal{N} = 4$ representations that appear and H_n are the corresponding, not necessarily irreducible, M_{24}-representations that appear, with $\dim H_n = |A_n|$. In order to further strengthen the connection to M_{24}, the twined elliptic genera $\mathcal{Z}^{ell}_{K3,g}$ of $K3$ may be considered, which are defined by an insertion of an element $g \in M_{24}$ in the elliptic genus, i.e.,

$$\mathcal{Z}^{ell}_{K3,g}(\tau, z) := \mathrm{Tr}_{RR}\left(g(-1)^{F_L+F_R} y^{J_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right). \tag{3.46}$$

They form the analogue of the McKay-Thompson series in Monstrous moonshine and hence should admit an expansion similar to (3.43) but now with the coefficients A_n replaced by traces of g in the respective representations, $\tilde{A}_n = \mathrm{Tr}_{H_n}(g)$. Similar to the elliptic genus itself they are expected to transform as Jacobi forms of weight 0 and index 1 under the subgroup $\Gamma_0(N)$ of $SL(2, \mathbb{Z})$, where N is the order of the

[17] Different representations with equal dimensions have been indicated by a bar. Contrary to Monstrous moonshine also negative values appear here, since we count states with a sign depending on their statistics (bosonic/fermionic).

[18] The decomposition of 'larger' coefficients is not unique but can be fixed using the twined elliptic genera, see below.

[19] The existence of such a module has been shown in [15].

element g, possibly up to a multiplier system, i.e., (2.28) is changed to

$$\mathcal{Z}^{ell}_{K3,g}(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}) = e^{\frac{2\pi icd}{Nh_g}} e^{\frac{2\pi imcz^2}{c\tau+d}} \mathcal{Z}^{ell}_{K3,g}(\tau,z),\ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N). \tag{3.47}$$

This will lead to a non-trivial phase in the case when $h_g \neq 1$[20]. Based on these assumptions and the knowledge of the first few expansion coefficients explicit expression for all twined elliptic genera were found in [11–14] strengthening the idea that M_{24} symmetry is present. For cases where $g \in M_{23}$ and can be realized as an symplectic automorphism of $K3$ the corresponding twined elliptic genera were also calculated in [11, 65] in agreement with those results. In [15] it was proven that all expansion coefficients A_n in (3.43) can be decomposed into sums of dimensions or irreducible representations on M_{24}. In [16], by a generalization of the results of Mukai and Kondo it was shown that the group of automorphisms of a NLSM on $K3$ is in general a subgroup of the Conway group Co_1, but never M_{24} and in general also not a subgroup of M_{24}. In [66] this was expanded to include loci of singular NLSM and all possible twining genera of NLSM on $K3$ were conjectured based on the work done in [24, 67]. In [68] this conjecture was proven in a 'physical' way by demanding the absence of 'unphysical' wall crossings.
In [69] generalized Mathieu Moonshine was considered and all the twisted twining elliptic genera $\mathcal{Z}^{ell}_{K3,g,h}$ of $K3$ were calculated. For every commuting pair $g, h \in M_{24}$ these are defined by

$$\mathcal{Z}^{ell}_{K3,g,h} = \mathrm{Tr}_{RR,g}\left(h(-1)^{F_L} y^0 q^{L_0-\frac{c}{24}}(-1)^{F_R}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \tag{3.48}$$

where the trace is taken over the g-twisted Ramond sector. It is expected that these twisted twining elliptic genera transform as they were twisted twining characters of a holomorphic orbifold. In particular they are expected to fulfil the following properties:

(A) Elliptic and modular properties:

$$\mathcal{Z}^{ell}_{K3,g,h}(\tau, z+l\tau+l') = e^{-2\pi im(l^2\tau+2lz)}\mathcal{Z}^{ell}_{K3,g,h}(\tau,z), \quad \forall l,l' \in \mathbb{Z} \tag{3.49}$$

$$\mathcal{Z}^{ell}_{K3,g,h}(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}) = \chi_{g,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) e^{2\pi\mathrm{i}\frac{cz^2}{c\tau+d}}\mathcal{Z}^{ell}_{K3h,^c g^a, h^d g^d}(\tau,z),\ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}),$$

for a certain multiplier $\chi_{g,h}: SL(2,\mathbb{Z}) \to U(1)$. In particular each $\mathcal{Z}^{ell}_{K3,g,h}$ is a weak Jacobi form of weight 0 and index 1 with multiplier $\chi_{g,h}$ under a subgroup $\Gamma_{g,h}$ of $SL(2,\mathbb{Z})$.

[20] A list of h_g for Mathieu moonshine is given in [15].

(B) Invariance under conjugation of the pair $g, h \in M_{24}$:

$$\mathcal{Z}^{ell}_{K3,g,h}(\tau, z) = \xi_{g,h}(k)\mathcal{Z}^{ell}_{K3,k^{-1}gk,k^{-1}hk}(\tau, z),\ \forall k \in M_{24}. \quad (3.50)$$

where $\xi_{g,h}(k)$ is a phase.

(C) $\mathcal{Z}^{ell}_{K3,g,h}$ is supposed to have a well defined expansion in terms on $\mathcal{N} = 4$ superconformal characters which depends among other things on certain 2-cocycles c_g [69].

(D) For $g = e$, the identity of M_{24}, $\mathcal{Z}^{ell}_{K3,e,h}$ are just the twining elliptic genera considered above.

It is further postulated that these properties (in particular $\chi_{g,h}, \xi_{g,h}$ and c_g) are all controlled by a 3-cocycle α representing a class in $H^3(M_{24}, U(1))$. Under these assumptions all the twisted twining elliptic genera were calculated in [69]. Note that the set of twisted twining elliptic genera modulo modular transformations is in one to one correspondence with conjugacy classes of abelian subgroups of M_{24} generated by two (commuting) elements $g, h \in M_{24}$. There exists 55 such subgroups of which 22 correspond to cyclic subgroups. The cyclic subgroups are generated by elements of the form (e, g) and hence correspond to twining genera (and the twisted twining genera obtained from those by modular transformations). The remaining 34 twisted twining genera were calculated in [69]. Many of these twisted twining genera vanish due to obstructions, i.e., situations where the properties (A-D) above only allow for vanishing solutions. The only un-obstructed twisted twining elliptic genera, using the shorthand $\phi_{g,h} := \mathcal{Z}^{ell}_{K3,g,h}$, are

$$\phi_{2B,4A_2}, \phi_{4B,A_3}, \phi_{4B,A_4}, \phi_{2B,8A_{1,2}}, \phi_{3A,3A_3}, \phi_{3A,B_1}, \quad (3.51)$$

the last two of which vanish identically.
The cases that will be important to us are in particular where $(g, h) = (g'^r, g'^s)$ for $g' \in M_{24}$ where $(r, s) \in \{0, \ldots, N-1\}$ and $N = ord(g')$, which includes the cases where the subgroups generated by g and h are cyclic. Here compact expressions for $\mathcal{Z}^{ell}_{K3,g,h}$ are known. We define $\mathcal{Z}^{ell}_{r,s} := \mathcal{Z}^{ell}_{K3,g'^r,\tilde{g}'^s}$, then

$$\mathcal{Z}^{ell}_{r,s}(q, y) = \begin{cases} \frac{N}{8}\alpha^{(0,0)}_{\tilde{g}}\mathcal{Z}^{ell}(q, y), & (r, s) = (0, 0) \\ \frac{N}{8}\alpha^{(r,s)}_{\tilde{g}}\mathcal{Z}^{ell}(q, y) + N\beta^{(r,s)}_{\tilde{g}}(q)\frac{\theta_1^2(q,y)}{\eta^6(q)}, & (r, s) \neq (0, 0) \end{cases}, \quad (3.52)$$

where $\alpha^{(r,s)}_{g'}$ are numerical constants and $\beta^{(r,s)}_{g'}(q)$ is a weight 2 modular form under $\Gamma_0(N)$. For the cases $N = 2, 3, 5, 7$ we give the explicit expression in (5.32) and

further expressions can be found in the Appendix E of [70].
The discussion above/the existence of generalized Mathieu moonshine strengthens the case for an action of M_{24} on the $\frac{1}{4}$-BPS states of the NLSM on $K3$. Still it is not clear how exactly this action arises. At generic points of the moduli space of $K3$ the coefficients of the elliptic genus are expected to agree with graded dimensions of $\frac{1}{4}$-BPS states [20, 71, 72], but no explicit construction of the NLSM is known and the automorphism symmetry group of the NLSM is typically trivial. Since the elliptic genus does not change as one moves around in moduli space, it was suggested in [17–20] that the M_{24} action might arise by combining symmetry groups from different parts of the moduli space and it was shown how this action arises for the lowest non-trivial BPS states, i.e., for $90 = 45 + \overline{45}$. This idea has been coined 'symmetry-surfing' and was further pursued and strengthened in [20]. An other approach was taken in [25] where the authors study type IIA string theory on $K3 \times T^2$. The resulting has M_{24} and indeed all umbral symmetry groups at different points in it moduli space and so gives new insight into the appearance of Mathieu moonshine. Still it seems correct to say that Mathieu moonshine is still not completely resolved.

3.4 Umbral Moonshine

In [21, 22] a connection relating the Niemeier lattices and their symmetry group (given in table 12) to vector-valued mock modular forms was given. Thereby a Niemeier lattice with Coxeter number m will be matched to a $2m$ component mock-modular form with pole of order $q^{-\frac{1}{4m}}$ at $\tau \to i\infty$ and regular behaviour at the other cusps. The existence of appropriate module s.t. the vector valued mock modular forms arise as specific trace functions was conjectured in [22]. In [15, 23] the existence of these module was shown. In [24] the Umbral moonshine was related to the $K3$ sigma models (in particular this is done by relating the A-D-E root-systems to the A-D-E du Val singularities a $K3$ can develop).

3.5 Thompson Moonshine

We start by defining the 'Kohnen plus-space' as the set of holomorphic functions that transform like $\theta(\tau) = \theta_3(2\tau, 0)$ under $\Gamma_0(4)$ and in addition satisfy $c_n = 0$ for all $n \neq 0, 1 \pmod 4$ for their Fourier expansion at infinity $\sum_n c_n q^n$. Then let $M^!_{1/2}$ be the space of functions defined in the same way but allowed to be meromorphic at the cusps. In [73] a basis for this spaces was given. It was later noted that the Fourier coefficients of one particular basis function can be expressed as sums of dimensions of irreducible representations of the Thompson group. This led the authors of [26] to the following conjecture [27]:
There exists a $\mathbb{Z}$-graded Th-supermodule

$$W = \bigoplus_{\substack{m=-3 \\ m \equiv 0,1 \bmod 4}}^{\infty} W_m \tag{3.53}$$

where for $m \geq 0$ the graded component W_m has vanishing odd part if $m \equiv 0 \pmod 4$ and vanishing even part if $m \equiv 1 \pmod 4$, such that for all $g \in Th$ the McKay-Thompson series

$$\mathcal{T}_{[g]}(\tau) := \sum_{\substack{m=-3 \\ m\equiv 0,1 \mathrm{mod} 4}}^{\infty} \mathrm{trace}_{W_m}(g) q^m \tag{3.54}$$

is a specifically given weakly holomorphic modular form of weight 1/2 in the Kohnen plus space. This conjecture was proven in [27].

4 Searching for Moonshine in higher dimensional Calabi-Yau manifolds

In this section we will study the elliptic genera of Calabi-Yau manifolds in various dimensions. In particular we are interested in finding moonshine phenomena, similar to Mathieu moonshine studied in subsection 3.3, in CY's other than $K3$. As we will see it will in particular be natural to study CY 5-folds.

4.1 The elliptic genera of Calabi-Yau manifolds

We start by recalling some of the basic properties of elliptic genera of Calabi-Yau manifolds in various dimensions. Most of this will be based on [45, 74]. Concretely we will be interested in the elliptic genus as defined in (3.41) for an $\mathcal{N} = (2, 2)$ superconformal theory which has a CY d-fold as its target space. Such a theory has central charges $(c, \bar{c}) = (3d, 3d)$.

The elliptic genus is a weak Jacobi form of weight 0 and index $d/2$ [61]. The first term of the elliptic genus in a q-expansion is given by

$$\mathcal{Z}_{CY_d}(\tau, z) = \sum_{p=0}^{d} (-1)^p \chi_p(CY_d) y^{\frac{d}{2}-p} + \mathcal{O}(q), \tag{4.1}$$

where $\chi_p(Y_d) = \sum_{r=0}^{d} (-1)^r h^{p,r}$ [21]. For $z = 0$ $(y = 1)$ the elliptic genus reduces to the Witten index and so the higher order terms vanish in this case. Then

$$\mathcal{Z}_{CY_d}(\tau, 0) = \chi_{CY_d} = \sum_{p=0}^{d} (-1)^p \chi_p(CY_d) \tag{4.2}$$

becomes exactly the Euler number of the Calabi-Yau. For small d these properties will fix the prefactor of the generator $J_{0,\frac{d}{2}}$ in terms of the Euler number of the CY and a few of the χ_p. We proceed by analysing the elliptic genera for different dimensions.

[21] Note that for a CY d-fold that is the product of two CYs, $Y_d = Y_{d_1} \times Y_{d_2}$, $\chi_p(Y_d) = \sum_{j=0}^{i} \chi_j(Y_{d_1}) \times \chi_{i-j}(Y_{d_2})$ holds.

4.1.1 Calabi-Yau 1-folds

For $d = 1$ there exists only one Calabi-Yau (up to isomorphisms), namely the two-torus T^2. Its elliptic genus however vanishes. More generally the elliptic genus of any even dimensional $2n$-torus T^{2n} vanishes. This is due to the fermionic zero modes in the right moving Ramond sector $\mathrm{Tr}\left((-1)^{F_R}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \propto \theta_2(\bar{q}, -1) = 0$.

4.1.2 Calabi-Yau 2-folds

For $d = 2$ there are two non-trivial cases $K3$ and the Enriques surface. The elliptic genus of $K3$ shows Mathieu moonshine which already has been discussed in 3.3.
The Enriques surface, which can be obtained by a fix-point free involution of $K3$ (see, e.g, [75]), shows a moonshine phenomenon connected to M_{12} [76]. It is in a geometric way connected to M_{11} since its semi-symplectic automorphism[22] can be embedded into $M_{11} < M_{12}$ [77]. Its elliptic genus is just half that of the $K3$ and hence has an expansion in term of $\mathcal{N} = 4$ characters of the form

$$\mathcal{Z}_{\mathrm{Enr}}(\tau, z) = \phi_{0,1}(\tau, z) = 10\mathrm{ch}^{\mathcal{N}=4}_{2,0,0}(\tau, z) + \sum_{n=1}^{\infty} \frac{A_n}{2}\mathrm{ch}^{\mathcal{N}=4}_{2,n,\frac{1}{2}}(\tau, z). \tag{4.3}$$

where the A_n are as in (3.43), (3.44). Since all the A_n are even one may again interpret the expansion coefficients $A_n/2$ as dimension of some vector space. In particular they may be decomposed in term of dimensions of irreducible representations of $\mathcal{M}_{12}$ in the following way

$$\begin{aligned} 10 &= 11 - 1, \\ -1 &= -1, \\ A_1 &= 45, \\ A_2 &= 55 + 176, \\ A_3 &= 66 + 2\cdot 120 + 2\cdot 144 + 76, \\ &\vdots \end{aligned} \tag{4.4}$$

As before an expansion in terms of $\mathcal{N} = 2$ characters is also possible which leads to the same expansion with an overall minus sign.

4.1.3 Calabi-Yau 3-folds

For $d = 3$ one finds the following expansion in terms of $\mathcal{N} = 2$ characters

$$\mathcal{Z}_{CY_3}(\tau, z) = \frac{\chi_{CY_3}}{2}\phi_{0,\frac{3}{2}}(\tau, z) = \frac{\chi_{CY_3}}{2}\left(\mathrm{ch}^{\mathcal{N}=2}_{3,0,\frac{1}{2}}(\tau, z) + \mathrm{ch}^{\mathcal{N}=2}_{2,0,-\frac{1}{2}}(\tau, z)\right) \tag{4.5}$$

[22] A semi-symplectic automorphism of the Enriques surface is an automorphism whose lift to $K3$ is either symplectic or anti-symplectic.

where the normalization is fixed by the equations (4.2) and (A.15). As this expansion only includes two characters, it is not very interesting from a moonshine perspective. In section 5 we will study CY 3-folds from a different perspective by connecting their Gromov-Witten invariants to Mathieu moonshine.

4.1.4 Calabi-Yau 4-folds

For $d = 4$ the elliptic genus is an element of the space $\tilde{J}_{0,2}$ of Jacobi forms of weight 0 and index 2. This vector space is generated by the two elements as is clear from (2.34) above. Through (4.1) and (4.2) we find that the elliptic genus is uniquely fixed by the Euler number χ_{CY_4} and $\chi_0 = \sum_{r=0}^{d}(-1)^r h^{0,r}$. Using the formulas of (A.3) we find

$$\mathcal{Z}_{CY_4}(\tau, z) = \frac{\chi_{CY_4}}{144}(\phi_{0,1}(\tau, z)^2 - E_4(\tau)\phi_{-2,1}(\tau, z)^2) + \chi_0 E_4(\tau)\phi_{-2,1}(\tau, z)^2. \tag{4.6}$$

For 'genuine' CY 4-folds [23] we always have $\chi_0 = h^{0,0} + h^{0,4} = 2$.
In this situation a variety of connections to sporadic groups have already been established. In [21] the weak Jacobi form $\frac{1}{24}(\phi_{0,1}^2 - E_4\phi_{-2,1}^2)$ appeared in umbral moonshine. It was shown that it exhibits $2.M_{12}$ moonshine when expanded in terms of $\mathcal{N} = 4$ characters. In [78] the same function was expanded in terms of $\mathcal{N} = 2$ characters and shown to exhibit $L_2(11)$[24] moonshine. In addition it was shown in [58] that the weak Jacobi form $\frac{1}{6}(\phi_{0,1}^2 + 5E_4\phi_{-2,1}^2)$ exhibits M_{22} moonshine upon expansion in $\mathcal{N} = 4$ characters and shows M_{23} moonshine when expanded in $\mathcal{N} = 2$ characters. When expanded in terms of $\mathcal{N} = 1$ characters the authors of [79] showed that it also exhibits M_{24} moonshine.

4.1.5 Calabi-Yau 5-folds

For $d = 5$ the elliptic genera occurring will be a weak Jacobi forms of weight 0 and index $\frac{5}{2}$. As stated in (2.34) they can be obtained by multiplying the corresponding integer index weak Jacobi forms with the function $\phi_{0,\frac{3}{2}}$. From this it follows that the elliptic genera occurring for $d = 5$ are all proportional to $\phi_{0,\frac{3}{2}}\phi_{0,1}$. The prefactor is uniquely fixed by the Euler number of the CY 5-fold we are considering. Making use of the relations between $\mathcal{N} = 4$ characters for central charge $c = 3d$ multiplied with $\phi_{0,\frac{3}{2}}$ and $\mathcal{N} = 2$ characters for central charge $c = 3(d + 3)$ given in (C.12) in

[23] In this context we define 'genuine' to mean CY d-folds whose holonomy group is $SU(d)$ and not a subgroup thereof.

[24] Here $L_2(11)$ stands for the finite simple group $PSL(2, 11) = SL(2, \mathbb{F}_{11})/\mathbb{F}_{11}^*$, where $\mathbb{F}_{11}$ is the prime field of integers modulo 11.

Appendix C, we see that

$$\begin{aligned}\phi_{0,\frac{3}{2}}\mathrm{ch}^{\mathcal{N}=4}_{2,0,0} &= -\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}} - \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}},\\ \phi_{0,\frac{3}{2}}\mathrm{ch}^{\mathcal{N}=4}_{2,0,\frac{1}{2}} &= -\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}} - \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}} + \mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}} - \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}},\\ \phi_{0,\frac{3}{2}}\mathrm{ch}^{\mathcal{N}=4}_{2,n,\frac{1}{2}} &= -\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}} - \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}},\ \forall n\in\mathbb{N}^{\geq 1}. \end{aligned} \tag{4.7}$$

From this follows directly the following expansion of the elliptic genus of CY 5-folds in terms of $\mathcal{N}=2$ characters:

$$\begin{aligned}\mathcal{Z}_{CY_5}(\tau,z) &= \frac{\chi_{CY_5}}{24}\phi_{0,\frac{3}{2}}\phi_{0,1}\\ &= -\frac{\chi_{CY_5}}{48}\Big[22\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - 2\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &\quad+\sum_{n=1}^{\infty}A_n\left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right)\Big]. \end{aligned} \tag{4.8}$$

This expansion makes it clear that for CY 5-folds with $\chi_{CY_5}=-48$, the expansion coefficients are the same as for Mathieu moonshine, whereas for CY 5-folds with $\chi_{CY_5}=-24$ we find the coefficients of Enriques moonshine. Since the overall choice of sign of the $\mathcal{N}=2$ characters is a mere convention, the above statement also holds true for CY 5-folds with $\chi_{CY_5}=+48$ and $\chi_{CY_5}=+24$.

From these observations it is certainly not clear if there really is a concrete connection to the Mathieu (Enriques) group for CY 5-folds or even how it arises. One may of course think of obvious (and rather trivial) examples where the CY 5-fold is simple the product of a K3 (Enriques) surface with a CY 3-fold. Then the connection would just arise through the better understood case of $K3$. To understand if there is actually a deeper connection between general CY 5-folds and the Mathieu (Enriques) group we will study the twined elliptic genera for certain (genuine) CY 5-folds in section 4.2.

4.1.6 Calabi-Yau 6-folds

For $d=6$ the elliptic genus is a weak Jacobi form of weight 0 and index 3. As can be seen from (2.34) the space of such forms is 3-dimensional. The elliptic genus of a CY 6-fold may hence be uniquely fixed in terms of the Euler number χ_{CY_6} and $\chi_p=\sum_{r=0}^{6}(-1)^r h^{p,r}$, for $p=0,1$. Again by matching the first few coefficients according to formulas (4.1), (4.2) we find that the elliptic genus is given by

$$\begin{aligned}\mathcal{Z}_{CY_6} &= \frac{\chi_{CY_6}}{1728}\phi^3_{0,1}(\tau,z) - \frac{1}{567}(\chi_{CY_6}-48(\chi_1+6\chi_0))E_4(\tau)\phi^2_{-2,1}(\tau,z)\phi_{0,1}(\tau,z)\\ &\quad-\frac{1}{864}(\chi_{CY_6}-72(\chi_1-6\chi_0))E_6(\tau)\phi^3_{-2,1}(\tau,z). \end{aligned} \tag{4.9}$$

One may consider the cases where the CY 6-fold is a product of three $K3$'s, three Enriques surface or $K3$ (Enriques surface) times a CY 4-fold. In those cases a connection to (one/two) of the sporadic groups $L_2(11), M_{12}, M_{22}, M_{23}.M_{24}$ will be inherited from the connections discussed in subsections 4.1.3, 4.1.4. It certainly would be interesting to understand if in general a connection between CY 6-folds and sporadic groups does exist.
One particular example which does not obviously fall into one of the above cases may be uncovered by setting $\chi_0 = \chi_1 = 0$. Then the elliptic genus is

$$\begin{aligned}\mathcal{Z}_{CY_6} &= \frac{\chi_{CY_6}}{1728}(\phi_{0,1}^3(\tau,z) - 3E_4(\tau)\phi_{-2,1}^2(\tau,z)\phi_{0,1}(\tau,z) - 2E_6(\tau)\phi_{-2,1}^3(\tau,z))\\ &= \frac{\chi_{CY_6}}{4}\phi_{0,\frac{3}{2}}^2(\tau,z)\\ &= \chi_{CY_6}\frac{\theta_2(\tau,z)^2\theta_3(\tau,z)^2\theta_4(\tau,z)^2}{\theta_2(\tau,0)^2\theta_3(\tau,0)^2\theta_4(\tau,0)^2}\\ &= \frac{\chi_{CY_6}}{8}\Big[4\mathrm{ch}_{6,0,0}^{\mathcal{N}=4} + \Big(-2\mathrm{ch}_{6,0,\frac{1}{2}}^{\mathcal{N}=4} + 14\mathrm{ch}_{6,1,\frac{1}{2}}^{\mathcal{N}=4} + 42\mathrm{ch}_{6,2,\frac{1}{2}}^{\mathcal{N}=4} + 86\mathrm{ch}_{6,3,\frac{1}{2}}^{\mathcal{N}=4} + \dots\Big)\\ &- \left(16\mathrm{ch}_{6,1,1}^{\mathcal{N}=4} + 48\mathrm{ch}_{6,2,1}^{\mathcal{N}=4} + 112\mathrm{ch}_{6,3,1}^{\mathcal{N}=4} + \dots\right)\\ &+ \Big(6\mathrm{ch}_{6,1,\frac{3}{2}}^{\mathcal{N}=4} + 28\mathrm{ch}_{6,2,\frac{3}{2}}^{\mathcal{N}=4} + 56\mathrm{ch}_{6,3,\frac{3}{2}}^{\mathcal{N}=4} + \dots\Big)\Big].\end{aligned} \tag{4.10}$$

The expansion of this Jacobi form in terms of $\mathcal{N} = 4$ characters has appeared in [21] and can be related to $2.AGL_3(2)$ via the umbral moonshine conjecture [25]. We can find explicit examples of CY 6-folds that will acquire above elliptic genus by taking products of two CY3-folds. This manifold will automatically satisfy $\chi_0 = \chi_1 = 0$. The case that appears for umbral moonshine is $\chi_{CY_6} = 8$. Since for our examples $\chi_{CY_6} = \chi_{CY_3^{(1)}} \cdot \chi_{CY_3^{(2)}}$ holds, this maybe be achieved for any pair of CY 3-folds that have Euler number ± 2 and ± 4 respectively.

4.1.7 Calabi-Yau manifolds of dimension $d > 6$

From our discussion above we see that by taking products of lower dimensional CY manifolds one may obtain a wealth of CY d-folds ($d > 6$) with potential connection to sporadic groups. Since CY d-folds for larger d have not been systematically constructed a treatment of the general case would require a lot of work. As a start one may however consider if and when the extremal Jacobi forms of umbral moonshine [21, 22] appear as elliptic genera of CY manifolds or products thereof.

As we have already seen one example is given by the elliptic genus of $K3$. The next examples may appear for CY 4-folds, i.e., Jacobi forms of weight 0 and index 2, with $\chi_0(CY_4) = 0$. As already mentioned in 4.1.4 genuine CY 4-folds always have

[25] This conjecture was proven in [23].

$\chi_0 = 2$ which follows from the more general property that

$$\chi_0(CY_d) = \begin{cases} 0 & \text{if } d \text{ is odd} \\ 2 & \text{if } d \text{ is even} \end{cases} . \tag{4.11}$$

Alternatively $CY_d = K3 \times K3$ has $\chi_0 = 4$ and $CY_d = K3 \times T^4$ has $\chi_0 = 0$ but vanishing elliptic genus.

For weight 0 and index 3 we saw in subsection 4.1.6 that possible CY 6-folds do exist.

Next we look for CY 8-folds that can give rise to the Jacobi form of weight 0 and index 4 appearing in umbral moonshine. For this to be the case we would need $\chi_0 = \chi_1 = \chi_2 = 0$ which cannot be the case for genuine CY 8-folds but which one might be able to achieve by taking products of lower dimensional CY's. First trying with $\text{CY}_8 = \text{CY}_5 \times \text{CY}_3$ we see that $\chi_0 = \chi_1 = 0$ will hold but since $\chi_2(CY_5 \times CY_3) \propto \chi_{CY_5} \times \chi_{CY_3}$ and $\mathcal{Z}_{CY_5 \times CY_3} \propto \chi_{CY_5} \times \chi_{CY_3}$ no interesting examples exist in this case. Alternatively trying $\text{CY}_8 = K3 \times \text{CY}_3 \times \text{CY}_3$ runs into the same problem.

When one looks for CY 12-folds that give rise to the Jacobi forms that appear in umbral moonshine one finds that $\chi_0 = \chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi_6 = 0$ is required. This again is too restrictive to give rise to interesting examples.

4.2 Twined elliptic genera for specific Calabi-Yaus

In the previous subsections we have established first hints that CY d-folds for $d > 2$ may be involved in some kind of moonshine, i.e, they may via their elliptic genera be connected to certain sporadic groups. In this subsection in order to strengthen or dismiss this idea we will calculate the twined elliptic genera for specific CY d-folds. In particular the largest class of these CY d-folds will all be realized as hypersurfaces in weighted projective space.

4.2.1 Calculating twined elliptic genera for CY hypersurfaces in weighted projective ambient space

In [80] the methods to calculate elliptic genera for CY manifolds realized as hypersurfaces in weighted projective space were developed. This methods can easily be generalized to also calculate the twined genera when the symmetry element we are twining with is realized as a geometric symmetry of the CY.

Concretely we consider a CY d-fold in weighted projective space $\mathbb{CP}^{d+1}_{w_1,\ldots,w_{d+2}}$ determined as the solution of $p(\Phi_1, \ldots, \Phi_{d+2}) = 0$, where p is a transverse polynomial of degree $m = \sum_{i=1}^{d} w_i$ and Φ_i are the homogeneous coordinates of the weighted projective space (see Appendix D.2) .

Then we introduce a two-dimensional linear sigma model with $\mathcal{N} = (2,2)$ supersymmetry consisting of:

i) one abelian vector multiplet (giving rise to an $U(1)$ gauge symmetry),

ii) $d+2$ chiral multiplets Φ_i, $i=1,\dots,d+2$, with charge w_i under the $U(1)$ gauge field and zero $\mathcal{R}$-charge,

iii) one chiral multiplet X with $U(1)$ charge $-m$ and $\mathcal{R}$-charge 2.

The superpotential invariant under $U(1)$ gauge transformations is given by $W = Xp(\Phi_1,\dots,\Phi_{d+2})$ and has the correct $\mathcal{R}$ charge. The F-term equation $\partial W/\partial X = p(\Phi_1,\dots,\Phi_{d+2}) = 0$ restricts us to the CY-hypersurface we considered above.
In the next step we study the refined elliptic genus defined by

$$\mathcal{Z}_{ref}(\tau,z,u) = \mathrm{Tr}_{RR}\left((-1)^{F_L} y^{J_0} q^{L_0-\frac{d}{8}} x^Q (-1)^{F_R} \bar{q}^{\bar{L}_0-\frac{d}{8}}\right) \tag{4.12}$$

depending on an extra chemical potential $x = e^{2\pi i u}$, associated to the $U(1)$ charge Q. As one can convince oneself a chiral multiplet of $U(1)$ charge w and $\mathcal{R}$-charge R gives rise to a (multiplicative) contribution to this refined elliptic genus of the form

$$\mathcal{Z}^{\Phi}_{ref}(\tau,z,u) = \frac{\theta_1(\tau,(\frac{R}{2}-1)z+wu)}{\theta_1(\tau,\frac{R}{2}z+wu)}. \tag{4.13}$$

An abelian vector field will lead to an (u independent) factor

$$\mathcal{Z}^{vec}_{ref}(\tau,z) = \frac{i\eta(\tau)^3}{\theta_1(\tau,-z)}. \tag{4.14}$$

Altogether the theory we consider leads to

$$\mathcal{Z}_{ref}(\tau,z,u) = \frac{i\eta(\tau)^3}{\theta_1(\tau,-z)} \frac{\theta_1(\tau,-mu)}{\theta_1(\tau,z-mu)} \prod_{i=1}^{d+2} \frac{\theta_1(\tau,(\frac{R}{2}-1)z+wu)}{\theta_1(\tau,\frac{R}{2}z+wu)}. \tag{4.15}$$

From this we can obtain the standard elliptic genus by integrating over u. As shown in [80] this integral will localize to a sum of contour integrals

$$\mathcal{Z}_{ref}(\tau,z,u) = \frac{i\eta(\tau)^3}{\theta_1(\tau,-z)} \sum_{u_j\in\mathcal{M}^-_{sing}} \oint_{U=u_j} du \frac{\theta_1(\tau,-mu)}{\theta_1(\tau,z-mu)} \prod_{i=1}^{d+2} \frac{\theta_1(\tau,(\frac{R}{2}-1)z+wu)}{\theta_1(\tau,\frac{R}{2}z+wu)}, \tag{4.16}$$

where $\mathcal{M}^-_{sing}$ are the poles of the integrand where the chiral multiplets become massless. For a chiral multiplet with $U(1)$ charge Q and $\mathcal{R}$-charge R these singularities are located at

$$Qu + \frac{R}{2}z = 0, \quad \mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}. \tag{4.17}$$

The supscript in $\mathcal{M}^-_{sing}$ captures the fact that in above formula one can restrict to singularities for chiral multiplets with negative $U(1)$ charge, $Q<0$. For our theory this is only the chiral multiplet X and the singularities are the solutions of

$$-mu + z = -k - \ell\tau, \quad k, \ell \in \mathbb{Z}. \tag{4.18}$$

The integrand is periodic under $u \sim u+1 \sim u+\tau$. The solutions which are within a fundamental domain under this identification are given by

$$u = \frac{z + k + \ell\tau}{m}, \quad 0 \leq k, \ell \leq m. \tag{4.19}$$

Hence the elliptic genus (4.16) of the CY can be written as

$$\mathcal{Z}_{CY_d}(\tau, z) = \frac{i\eta(\tau)^3}{\theta_1(\tau, -z)} \sum_{k,\ell=0}^{m-1} \oint_{u=(k+\ell\tau+z)/m} du \frac{\theta_1(\tau, -mu)}{\theta_1(\tau, z - mu)} \prod_{i=1}^{d+2} \frac{\theta_1(\tau, (\frac{R}{2} - 1)z + wu)}{\theta_1(\tau, \frac{R}{2}z + wu)}. \tag{4.20}$$

By using the specific properties of the θ-function (see, e.g., Appendix B of [80]) one can further simplify this expression. Finally one obtains the following formula for the elliptic genus of CY d-fold that is a hypersurface in a weighted projective space and can be described by a transverse polynomial

$$\begin{aligned} \mathcal{Z}_{CY_d}(\tau, z) &= \sum_{k,\ell=0}^{m-1} \frac{e^{-2\pi i \ell z}}{m} \prod_{i=1}^{d+2} \frac{\theta_1(\tau, \frac{w_i}{m}(k + \ell\tau + z) - z)}{\theta_1(\tau, \frac{w_i}{m}(k + \ell\tau + z))} \\ &= \sum_{k,\ell=0}^{m-1} \frac{y^{-\ell}}{m} \prod_{i=1}^{d+2} \frac{\theta_1(q, e^{\frac{2\pi i w_i k}{m}} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m} - 1})}{\theta_1(\tau, e^{\frac{2\pi i w_i k}{m}} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m}})}. \end{aligned} \tag{4.21}$$

Now we can consider an abelian symmetry acting on the chiral multiplets as

$$g : \Phi_i \to e^{2\pi i \alpha_i} \Phi_i, \quad i = 1, 2, \ldots, d+2. \tag{4.22}$$

Twining the elliptic genus by such an symmetry will lead to a shift of the second argument of θ_1 by α_i for each of the chiral fields Φ_i. Hence we obtain for the elliptic genus twined by such a symmetry

$$\begin{aligned} \mathcal{Z}_{CY_d}(\tau, z) &= \mathrm{Tr}_{RR}\left(g(-1)^{F_L} y^{J_0} q^{L_0 - \frac{q}{8}} (-1)^{F_r} \bar{q}^{\bar{L}_0 - \frac{d}{8}}\right) \\ &= \sum_{k,\ell=0}^{m-1} \frac{e^{-2\pi i \ell z}}{m} \prod_{i=1}^{d+2} \frac{\theta_1(\tau, \alpha_i + \frac{w_i}{m}(k + \ell\tau + z) - z)}{\theta_1(\tau, \alpha_i + \frac{w_i}{m}(k + \ell\tau + z))} \\ &= \sum_{k,\ell=0}^{m-1} \frac{y^{-\ell}}{m} \prod_{i=1}^{d+2} \frac{\theta_1(q, e^{2\pi i \left(\alpha_i + \frac{w_i k}{m}\right)} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m} - 1})}{\theta_1\left(\tau, e^{2\pi i \left(\alpha_i + \frac{w_i k}{m}\right)} q^{\frac{w_i \ell}{m}} y^{\frac{w_i}{m}}\right)}. \end{aligned} \tag{4.23}$$

The case of non-abelian symmetries that permute the Φ_i may be treated by first performing a coordinate transformation that diagonalizes the permutation matrix. In the new coordinates the symmetries again appear as abelian symmetries.

4.2.2 Twisted elliptic genera of CY 5-folds

As we have seen in subsection 4.1.5 the elliptic genus of CY 5-folds seems to imply a possible connection between CY 5-folds and M_{24}. In order to get a better understanding for this we will now apply the techniques discussed in the previous subsection to CY 5-folds. A large set (more precisely there are 5 757 727) of CY 5-folds that can be described as reflexive polytopes is given on the website [81] (under `4 folds -> All files`, in the files `6dRefWH.xxx-xxx.gz`). In order to apply our methods we need to restrict this set to the ones that can be described by a transverse polynomial in weighted projective space. This then leaves us with 19 353 CY 5-folds.
The hope is that one finds CY 5-folds which twined elliptic genera show similar behaviour as we have experienced for $K3$ in Mathieu moonshine, i.e., that the coefficients that correspond to dimensions of irreducible representations of the sporadic group are replaced by the appropriate traces of the element we are twining with. For the case when the CY 5-fold is just a product of $K3$ and and a CY 3-fold (and the symmetry acts on $K3$) this will (trivially) be the case. But also for genuine CY 5-folds this may occur in certain situations. For example for the hypersurface in weighted projective space $\mathbb{CP}^6_{1,1,1,3,5,9,10}$ which has Euler number $\chi_{CY_5} = -170688 = -48 \cdot 3556$ we consider the $\mathbb{Z}_2$ symmetry given by

$$\mathbb{Z}_2 : \begin{cases} \Phi_1 \to -\Phi_1, \\ \Phi_2 \to -\Phi_2. \end{cases} \tag{4.24}$$

For this one finds the elliptic genus twisted by this symmetry is given by

$$\begin{aligned} \mathcal{Z}^{tw,2A}_{CY_5} = 14 \cdot \Big[& 2 \left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z) \right) - 2 \left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z) \right) \\ & + \sum_{n=1}^{\infty} A^{(2A)}_n \left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z) \right) \Big], \end{aligned} \tag{4.25}$$

that is the multiplicative constant at the beginning has been changed from 3556 to 14 and the rest of the expansion coefficients simply correspond to the $2A$ series of M_{24}. To explain the change in the prefactor one may decompose 3556 in various ways into irreps of M_{24} that will give rise to 14 after the twisting.
An other example that works in a similar way is the hypersurface in the weighted projective space $\mathbb{CP}^6_{1,2,2,3,4,4,8}$. Applying the same order 2 twist (4.24) one again finds the $2A$ series but now multiplied with a factor 69/2. The fractional prefactor is obviously not consistent with an interpretation in terms if M_{24} but if we limit ourselves to M_{12} also this example seems interesting.

Going one step further we can consider the order four symmetry

$$\mathbb{Z}_4 : \begin{cases} \Phi_1 \to \mathrm{i}\Phi_1, \\ \Phi_2 \to -\mathrm{i}\Phi_2. \end{cases} \tag{4.26}$$

Here one finds that twining the elliptic genus of the CY hypersurface in $\mathbb{CP}^6_{1,1,1,1,4,4,4}$ by this symmetry gives the coefficients of the 4B series of M_{24} with a prefactor of 42. More such specific examples can be found. This leads to the idea of a systematic study of the twining genera of CY 5-folds that are hypersurfaces in weighted projective spaces.
In my paper [37] this was done in the following manner:

1. From the 5 757 727 examples on the website [81] we took the 19 353 CY 5-folds that can be descried by a transverse polynomial in the homogeneous coordinates of the ambient weighted projective space.

2. Then we used by a simple code to construct a single transverse polynomial. This reduced the number of possible cases to 18 880 CY 5-folds, which to some extent may be attributed to the simplicity of the code used.

3. In 16 727 cases we were able to find a $\mathbb{Z}_2$ symmetry of the transverse polynomial.

4. For these 167272 manifolds the elliptic genus twined by such an $\mathbb{Z}_2$ symmetry was calculated. In case the calculation took too long it was aborted. By this method we arrived at 13 642 twined elliptic genera.

The elliptic genera found in this way can always be split into a linear combination of the 1A and 2A series of M_{24} with integer or half integer prefactors. As an example we may give the elliptic genus of the hypersurface in the weighted projective space $\mathbb{CP}^6_{1,1,1,1,1,1,3}$ twined by the symmetry given in (4.24)

$$\begin{aligned} \mathcal{Z}^{tw,2A}_{CY_5} = \frac{9}{2}\Big[& 22\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - 2\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right) \\ & + \sum_{n=1}^{\infty} A_n \left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right)\Big] \\ + 43 \cdot \Big[& 6\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - 2\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right) \\ & + \sum_{n=1}^{\infty} A^{(2A)}_n \left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right)\Big]. \end{aligned} \tag{4.27}$$

This behaviour may already be expected from standard CFT arguments [11, 12] since any elliptic genus twined by a group element g is a Jacobi form $\phi^g_{0,m}$ w.r.t $\Gamma_0(|g|)$ with a potentially non trivial multiplier,i.e, it transforms as in (3.47),(2.29). Twining by an element of the 1A or 2A congruence class has trivial phase while twining by a 2B element has non-trivial multiplier.

The vector space of possible twined elliptic genera for CY 5-folds is created by very few basis elements. That follows from the fact that they are simply products of $\phi_{0,\frac{3}{2}}$ and the functions that appear for $K3$ (as follows from Lemma 1.4. in [45]) and that the elements for $K3$ are created by very few basis elements [66]. In particular twining by an order two element will always give a linear combination of 1A and 2A twining genera for trivial multiplier. On the other hand for non-trivial multiplier one will always obtain something proportional to the 2B series. The fact that we in our analysis never found a 2B series follows from the fact that they will only show up for non-geometric symmetries, i.e., symmetries that treat left and right movers differently. This will then give rise to failure of level matching and which further leads to the non trivial multiplier in the twining elliptic genera.

We may choose the following two functions as basis elements for the 1A and 2A series respectively

$$\begin{aligned} f_{1a}(\tau,z) = 11 &\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - \left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right) \\ &+45\left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right) + \ldots \end{aligned} \tag{4.28}$$

$$\begin{aligned} f_{2a}(\tau,z) = 3 &\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - \left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right) \\ &-3\left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right) + \ldots . \end{aligned} \tag{4.29}$$

Out of the 13 642 twined elliptic genera one finds 927 cases that are proportional to f_{2a}. Further 811 of these the overall coefficient is an even integer which may lead one to suspect a possible M_{24} symmetry (for the remaining 116 the prefactor is odd and may agree to a 2B element of M_{12}). To further check if these manifold really have a M_{24} symmetry we proceeded as follows

1. We generated a large number of $\mathbb{Z}_2$ symmetries for each of these 927 examples and calculated the twining elliptic genera for all of them.

2. In most of the cases with multiple $\mathbb{Z}_2$ symmetry one finds one elliptic genus proportional to just the f_{2a} and the rest to be linear combinations of f_{1a} and f_{2a}.

3. For the cases with only one obvious $\mathbb{Z}_2$ symmetry we looked for higher order abelian symmetries and calculated their twining elliptic genera. In all cases one finds at least one symmetry that leads to a twined elliptic genus not consist with either M_{24} or M_{12}.

Based on these results one can conclude that none of the analysed 13 642 CY 5-folds has a strict M_{24} symmetry. The results are consistent with the idea that the CY 5-folds have small discrete symmetry groups (at certain points in moduli space). However other possibilities are also possible. For example symmetry groups that are larger than M_{24}, in particular extended by one or several $\mathbb{Z}_2$'s are not ruled out. Also theoretically one could imagine the situation where the symmetry group corresponds to multiple copies of M_{24}. Then the prefactor $\frac{\chi_{CY_5}}{48}$ could be interpreted as having $|\frac{\chi_{CY_5}}{48}|$ many different M_{24} symmetries rather than that many copies or sums of certain irreducible representations under a single M_{24}. Then the $\mathbb{Z}_2$ symmetries studied could be part of certain of those M_{24}'s corresponding to a 2A element there while for others it would simply correspond to the identity. This would then also fit with having a linear combination of f_{1a} and f_{2a}. Fractional coefficients are however still excluded.

The special cases with $\chi_{CY_5} = \pm 48(\pm 24)$ are the ones that should correspond to a single $M_{24}(M_{12})$ symmetry. The list of [81] however only contains a single such CY 5-fold which is related to the fact that all those examples have sums of weights $m = \sum_i w_i \leq 200$ which lead to mostly rather large negative Euler number. In order to generate further examples one may use PALP [82], which is however rather time consuming. Hence we chose to proceed as follows:

1. For $7 \leq m \leq 600$ partition m into 7 integer weights w_i,$i = 1, \ldots, 7$.
2. Keep those cases for which the Poincare polynomial
$$P(x) = \prod_{i=7}^{7} \frac{1 - x^{m-w_i}}{1 - x^{w_i}} \tag{4.30}$$
evaluated at $x = 1$ is an integer [26] . This is a very fast check.
3. From the remaining cases keep those for which the formula for the Euler number
$$\chi = \frac{1}{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \prod_{gcd(l,k)\cdot \frac{w_i}{m} \in \mathbb{Z}} \frac{w_i - m}{w_i} \tag{4.31}$$
gives ± 24 or ± 48.
4. For the remaining cases use PALP [82] to check the weight systems explicitly.

By taking these steps one arrives at dozens of new examples. We also calculated all Hodge numbers for those examples in order to be certain that they do correspond to different manifolds. For some of the cases we found manifolds with the same Hodge numbers so that one can not immediately conclude that they correspond to non-diffeomorphic manifolds. For all the examples we then proceeded by finding $\mathbb{Z}_2$ symmetries (table 1) and calculated the twined elliptic genera. Having calculated all

[26] This is a necessary condition for an appropriate polynomial to exit - see Appendix D.2.

Euler number	number of example	cases with $\mathbb{Z}_2$ symmetry	reflexive cases
$\chi = -48$	72 (67)	64 (59)	6 (6)
$\chi = +48$	68 (59)	51 (43)	4 (4)
$\chi = -24$	32 (29)	26 (23)	4 (4)
$\chi = +24$	27 (24)	25 (22)	4 (4)

Table 1: The number of CY 5-folds constructed as hypersurfaces in weighted projective spaces. The numbers in parenthesis give the number of CY manifolds with different Hodge numbers

the twined elliptic genera for the examples above arising from one of the geometric $\mathbb{Z}_2$ symmetries one can make two observations:

1.) Making a quantitative analysis one can observe that for $\chi = +48$ and $\chi = -48$ the twining elliptic genera are proportional to the 2A twined elliptic genus of M_{24} in 3.4% and 5.2% of the cases. Also for the coefficient of f_{1A} one observes that zero is not more likely than other small coefficients. Furthermore the is no apparent preference for even coefficients.
2.) For the cases that actually give something proportional to the $2A$ elliptic genus one observes that they have prefactors with absolute value larger than one. More specifically for the examples we considered we find the multiplicities $\{-42f_{2a}, -38f_{2a}, -22f_{2a}, -6f_{2a}, 50f_{2a}\}$. While this may seem surprising it can understood from the fact that the elliptic genus counts bosonic and fermionic states with different sign. In particular in cases with high symmetry (at special points in moduli space) we may expect a lot of states to exist which contribute to the elliptic genus but in total cancel out (at a general place in moduli space the expectation is that most of those state obtain a mass and no longer contribute). When calculating the twining elliptic genus only states that are invariant under the symmetry element inserted in the trace will be kept and hence the cancellation may not take place in the same fashion hence leading to different multiplicities. As a concrete example we may consider $\mathbb{CP}^6_{16,17,17,34,58,62,102}$ which has Euler number $\chi = -48$ and hence elliptic genus

$$\mathcal{Z}_{CY_5}^{\chi=-48} = +2f_{1a}(\tau,z) = 22\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right) - 2\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)$$
$$+90\left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right) + \dots \tag{4.32}$$

Upon twining with the $\mathbb{Z}_2$ symmetry

$$\mathbb{Z}_2 : \begin{cases} \Phi_2 \to -\Phi_2, \\ \Phi_2 \to -\Phi_2. \end{cases} \tag{4.33}$$

we obtain

$$\begin{aligned}\mathcal{Z}^{tw}_{CY_5}(\tau,z) &= 50 f_{2a}(\tau,z)\\ &= 150\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)-50\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{3}{2}}(\tau,z)\right)\\ &\quad -150\left(\mathrm{ch}^{\mathcal{N}=2}_{5,n,\frac{3}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,n,-\frac{3}{2}}(\tau,z)\right)+\ldots.\end{aligned} \tag{4.34}$$

We see that, e.g., the 22 states that belong to $\left(\mathrm{ch}^{\mathcal{N}=2}_{5,0,\frac{1}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{5,0,-\frac{1}{2}}(\tau,z)\right)$ turn into 150 such states.

One way to understand such large number is to look at the Hodge numbers of the CY. They are in general large (compared to the Euler number 48) so the have to cancel in a rather precise way. For the concrete example at hand we have

$$h^{1,1}=25,\;\; h^{1,2}=0,\;\; h^{1,3}=232,\;\; h^{1,4}=259,\;\; h^{2,2}=1692,\;\; h^{2,3}=1946. \tag{4.35}$$

Now for a general CY 5-fold it holds that (see page 6 in [45])

$$\chi_0=\chi_5=0,\chi_1=\chi_4=-\frac{1}{24}\chi_{CY_5},\chi_2=\chi_3=\frac{11}{24}\chi_{CY_5}. \tag{4.36}$$

Through twining these exact cancellations get changed and will produce larger multiplicities.

Finally for the CY manifolds with $\chi=\pm 24$ that we constructed we similarly calculated all the $\mathbb{Z}_2$ twining elliptic genera but always obtained a linear combination of f_{1a} and f_{2a} with non-zero coefficients, so never something proportional to just f_{2a}. From all these observations we can exclude a strict M_{24} or M_{12} symmetry for the cases we studied.

4.2.3 Calabi-Yau 6-folds

In subsection 4.1.6 we saw that a CY 6-fold which is a particular product of two CY 3-folds will give rise to Jacobi form that appears in umbral moonshine. However if one calculates the twined elliptic genus for a symmetry of one of the CY 3-folds it will only change by a factor relative to the elliptic genus. We can illustrate this by looking at the elliptic genus for the quintic CY, given as a hypersurfacae in $\mathbb{CP}^4_{1,1,1,1,1}$. Its elliptic genus is just

$$\mathcal{Z}_{quintic}(\tau,z)=-100\phi_{0,\frac{3}{2}}=-100\left(\mathrm{ch}^{\mathcal{N}=2}_{3,0,\frac{1}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{3,0,-\frac{1}{2}}(\tau,z)\right). \tag{4.37}$$

Twining this by a symmetry of order 5 that acts on the first two coordinates as

$$g:\;\Phi_1\to e^{2\pi\mathrm{i}/5}\Phi_1,\;\Phi_2\to e^{2\pi\mathrm{i}/5}\Phi_2 \tag{4.38}$$

yields the following twined elliptic genus

$$\mathcal{Z}^{(g)}_{quintic}(\tau,z) = -5\phi_{0,\frac{3}{2}} = -5\left(\mathrm{ch}^{\mathcal{N}=2}_{3,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{3,0,-\frac{1}{2}}(\tau,z)\right), \tag{4.39}$$

i.e., only the overall prefactor changes. So in this situation we will never obtain the twined series of $2.AGL_3(2)$.

4.3 A toroidal orbifold and two Gepner models

In the last section we analysed a large class of CY 5- folds for the existence of M_{24} symmetry without finding any evidence for it. Since there does not exist a complete list of CY 5-folds and it is not known if their number is finite, we do not know if this result is representative in any way. In particular the M_{24} symmetry might show up at special points in moduli space where the CY's have enlarged symmetry groups. In this subsection we will hence study such examples, in particular toroidal orbifold points and Gepner models.

The toroidal orbifolds we are interested in are of the form T^{10}/G where G is a discrete symmetry group of the 10-dimensional torus T^{10}. The $\mathbb{Z}_2$ toroidal orbifolds that occur in Monstrous and Conway moonshine and exchanges the signs of all coordinates are not of this type. We may however look at cases where $G = G_1 \times G_2$ and the orbifolds are of the form $T^{10}/G = T^4/G_1 \times T^6/G_2$. For $G_1 = \mathbb{Z}_m$, $m = 2,3,4,6$ we obtain the product of the singular limit of a $K3$ with a complex three dimensional space. In this case the origin of M_{24} can be thought to directly come from $K3$.

We choose to study an example where $G = \mathbb{Z}_2^4$ is generated by the following 4 elements

$$\begin{aligned}
g_1 &: \{z^1,z^2,z^3,z^4,z^5\} \to \{-z^1,-z^2,z^3,z^4,z^5\},\\
g_2 &: \{z^1,z^2,z^3,z^4,z^5\} \to \{z^1,-z^2,-z^3,z^4,z^5\},\\
g_3 &: \{z^1,z^2,z^3,z^4,z^5\} \to \{z^1,z^2,-z^3,-z^4,z^5\},\\
g_4 &: \{z^1,z^2,z^3,z^4,z^5\} \to \{z^1,z^2,z^3,-z^4,-z^5\},
\end{aligned}$$

where we have the five complex coordinates $z^1,\ldots,z^5$ on T^{10}. The elliptic genus of this particular model is

$$\begin{aligned}
\mathcal{Z}_{T^{10}/\mathbb{Z}_2^4} = 1280\Bigg[&\frac{\theta_2(\tau,z)^3\theta_3(\tau,z)\theta_4(\tau,z)}{\theta_2(\tau,0)\theta_3(\tau,0)\theta_4(\tau,0)} + \frac{\theta_2(\tau,z)^3\theta_3(\tau,z)\theta_4(\tau,z)}{\theta_2(\tau,0)\theta_3(\tau,0)\theta_4(\tau,0)}\\
&+ \frac{\theta_2(\tau,z)^3\theta_3(\tau,z)\theta_4(\tau,z)}{\theta_2(\tau,0)\theta_3(\tau,0)\theta_4(\tau,0)}\Bigg]\\
=\ & 160\phi_{0,\frac{3}{2}}\phi_{0,1}.
\end{aligned} \tag{4.40}$$

This can be easily seen to arise from the sum over all the twisted sectors (sectors with $g \in G$ insertion in the trace). Each $g \in G$ will exactly change the sign of two z^i. The untwisted sector does not contribute as we have already discussed that the elliptic genus of any torus vanishes due to the fermionic zero modes.

Now we choose to twine the elliptic genus with the symmetry h that acts on the torus as

$$h : \{z^1, z^2, z^3, z^4, z^5\} \to \{\mathrm{i}z^1, -\mathrm{i}z^2, z^3, z^4, z^5\}. \tag{4.41}$$

Since its squares to an element in G, $h^2 = g_1 \in G$, it is of order 2. Twining the elliptic genus by this element one obtains:

$$\begin{aligned}
\mathcal{Z}^{tw}_{T^{10}/\mathbb{Z}_2^4}(\tau, z) = 16\Bigg[& 6\sum_{j=1}^{4} t_j(\frac{1}{4})t_j(-\frac{1}{4})t_3(0)t_4(0) \\
& +[t_2(0)^2 + t_3(0)^2 + t_4(0)^2]\sum_{k=2}^{4}[t_1(\frac{1}{4})t_k(-\frac{1}{4}) + t_k(\frac{1}{4})t_1(-\frac{1}{4})]t_k(0) \\
& +[t_2(0)^2 + t_3(0)^2 + t_4(0)^2] \sum_{\substack{k_1\neq k_2\neq k_3\neq k_1 \\ k_1,k_2,k_3\in\{2,3,4\}}} t_{k_1}(\frac{1}{4})t_{k_2}(-\frac{1}{4})t_{k_3}(0) \\
& + \sum_{\substack{k_1\neq k_2\neq k_3\neq k_1 \\ k_1,k_2,k_3\in\{2,3,4\}}} [t_1(\frac{1}{4})t_{k_3}(-\frac{1}{4}) + t_{k_3}(\frac{1}{4})t_1(-\frac{1}{4}) + 2t_{k_1}(\frac{1}{4})t_{k_2}(-\frac{1}{4})] \\
& \cdot(t_{k_1}(0)^2 + t_{k_2}(0)^2)t_{k_3(0)}\Bigg] && (4.42) \\
= {} & 56 f_{1a} + 48 f_{2a}, && (4.43)
\end{aligned}$$

where we have introduced $t_j := \frac{\theta_j(\tau,z+x)}{\theta_j(\tau,x)}$. Since we find a linear combination of f_{1a} and f_{2a} we can again conclude that the studied symmetry element h is not related to a M_{24} symmetry of the model.

As our final example we will consider two Gepner models, namely $(1)^{15}$ and $(2)^{10}$ which are highly symmetrical. The models correspond to orbifolds by $\mathbb{Z}_{k+1}$ of tensor products of 15 resp. 10 copies of A_{k+1} for $k = 1$ and $k = 2$. In [61, 83] it was explained how to calculate the elliptic genus for such models (and this was applied to moonshine in [84]). Each minimal model A_{k+1} is obtained from a chiral multiplet Φ with a superpotential $W = \frac{\Phi^{k+2}}{k+2}$. It will contribute a multiplicative factor

$$Z_k(\tau, z) = \frac{\theta\left(\tau, \frac{k+1}{k+2}z\right)}{\theta\left(\tau, \frac{1}{k+2}z\right)} \tag{4.44}$$

to the elliptic genus. The $\mathbb{Z}_{k+2}$ orbifold of N such minimal models with central charge c ($c = 15$ in our case) gives a contribution

$$\mathcal{Z}_{\text{Gepner}}(\tau, z) = \frac{1}{k+2} \sum_{a,b=0}^{k+1} e^{\frac{\pi i c}{6}(a+b)} e^{\frac{2\pi i c}{6}(a\tau^2+2az)} (Z_k(\tau, z + a\tau + b))^N. \tag{4.45}$$

With the help of this we find the following elliptic genera for the models we want to consider

$$\mathcal{Z}_{(1)^{15}}(\tau, z) = -455\phi_{0,\frac{3}{2}}\phi_{0,1}, \tag{4.46}$$
$$\mathcal{Z}_{(2)^{10}}(\tau, z) = -615\phi_{0,\frac{3}{2}}\phi_{0,1}. \tag{4.47}$$

We now want to twine these elliptic genera by a symmetry. The simplest case is where the symmetry simply acts by multiplying the chiral multiplet with a phase $\Phi \to e^{2\pi i \alpha}\Phi$. The twined elliptic genus of a single minimal model is then

$$Z_{k,\alpha}(\tau, z) = \frac{\theta\left(\tau, \frac{k+1}{k+2}z - \alpha\right)}{\theta\left(\tau, \frac{k+1}{k+2}z + \alpha\right)}. \tag{4.48}$$

The twined elliptic genus of the Gepner model hence becomes

$$\mathcal{Z}^{tw}_{\text{Gepner}}(\tau, z) = \frac{1}{k+2} \sum_{a,b=0}^{k+1} e^{\frac{\pi i c}{6}(a+b)} e^{\frac{2\pi i c}{6}(a\tau^2+2az)} \prod_{i=1}^{N} Z_{k,\alpha_i}(\tau, z + a\tau + b). \tag{4.49}$$

We consider the $\mathbb{Z}_2$ symmetry that simply acts by multiplying the chiral multiplet with a minus sign. This results in the following twined elliptic genera

$$\mathcal{Z}^{tw}_{(1)^{15}}(\tau, z) = 77f_{1a} + 110f_{2a}, \tag{4.50}$$
$$\mathcal{Z}^{tw}_{(2)^{10}}(\tau, z) = 35f_{1a} + 100f_{2a}, \tag{4.51}$$

which are again linear combinations of f_{1a} and f_{2a}. So also in this case we conclude that the examples we studied do not lead to an outright M_{24} symmetry.

4.4 Concluding remarks

In this section we have discussed the results published in [37] on finding Moonshine in the elliptic genus in higher dimensional CY manifolds. As we have seen in particular the elliptic genus of CY 5-folds shows an interesting expansion in terms of $\mathcal{N} = 2$ characters. After calculating the twining elliptic genera for order two elements of a large class (13 642) of CY 5-folds (in particular all realized as hypersurfaces in a weighted projective ambient space) we are not able to find any examples of genuine CY 5-folds that show Mathieu moonshine. The trivial cases where the CY 5-fold is the product of $K3$ with a CY 3-fold are the only examples we are able to find. Similarly for the models that arise at special points in moduli space where we have

an extended symmetry algebra, like toroidal orbifolds and Gepner models, the only examples that show a connection to M_{24} are again the trivial cases where one factor corresponds to a $K3$. So summing up, from this analysis one cannot conclude that CY 5-folds are involved in Mathieu moonshine in any interesting kind of way.
For CY 6-folds we find connections to Jacobi forms of umbral moonshine in special cases. However also here twining the elliptic genera does not produce the appropriate twined series.
For CY4-folds connections to moonshine may still exist and may be the content of further studies.

5 Mathieu Moonshine and Gromov-Witten invariants

In this section we will present a different manner in which Mathieu Moonshine is connected to CY manifolds. In particular we will make use of the string duality to relate heterotic compactifications on $K3 \times T^2$, and orbifolds thereof, to type II compactifications on CY 3-folds. We will match heterotic and type II compactifications by comparing their spectrum, the vector moduli prepotential as well as certain gravitational couplings of the effective theory. On the heterotic side the prepotential and the gravitational couplings are related to a special index of the internal theory, called the new supersymmetric index [85–89]. As we will see this index will be connected to Mathieu Moonshine. On the type II side the prepotential and the couplings are connected to topological invariants of the CY 3-fold, called Gromov-Witten/Gopakumar-Vafa invariants, which by matching to the heterotic side will also be connected to Mathieu Moonshine.

5.1 Heterotic compactifications on $K3 \times T^2$ and CHL orbifolds

In this subsection we will discuss the compactification of the $E_8 \times E_8$ heterotic string on $K3 \times T^2$ and CHL-orbifolds. Our discussion follows [31, 38, 90–92].

When compactifying the heterotic string on a manifold X one in general needs to specify data beyond the sigma model metric on X. This is the case since the Bianchi identity for the three form field strength of the heterotic string demands $dH = \frac{1}{4}(tr(R \wedge R) - tr(F \wedge F) = 0$ (in cohomology), where R is the curvature 2-form of X and F is the fields strength of the $E_8 \times E_8$ gauge field. We write the gauge bundle associated to F as $V_1 \times V_2$ and express the condition in terms of the respective second Chern classes. Then it reads (without five-brane source)

$$c_2(X) = c_2(V_1) + c_2(V_2)\ ,\ c_1(V_1) = c_1(V_2) = 0. \tag{5.1}$$

So we see that for compactifications on a manifold with non-vanishing second Chern class we need to embed non-trivial gauge bundles (instantons) in $E_8 \times E_8$. In particular for the compactification on $K3 \times T^2$ we find

$$\int c_2(V_1) + \int c_2(V_2) = \int c_2(K3) = \chi_{K3} = 24 \tag{5.2}$$

so we need to embed in total 24 instantons in the $E_8 \times E_8$ gauge group in this situation. More precisely we need to embed $(n^{(1)}, n^{(2)})$ instantons in some subgroups $H_1 \times H_2$ of $E_8 \times E_8$, $n^{(1)} + n^{(2)} = 24$, $n^{(i)} \in \mathbb{N}$. This will break the gauge group to $G_1 \times G_2$, where G_i is the commutant of H_i in E_8. Due to the symmetry under the exchange of the two E_8 factors one can restrict to $0 \leq n_2 \leq 12$. In general this will then lead to a $\mathcal{N} = (0, 4)$ world-sheet supersymmetry and $\mathcal{N} = 2$ supersymmetry in four spacetime dimensions. The special case where one chooses $n^{(1)} = 24, n^{(2)} = 0$ and sets the gauge connection equal to the spin connection, i.e., and $H_1 = SU(2), G_1 = E_7$ is called the *standard embedding*. In this case the world-sheet supersymmetry is $\mathcal{N} = (4, 4)$ and the results presented in section 3.3 for the elliptic genus of $K3$ will hold. However as we will see in the following also for other choices of instanton embeddings a connection to M_{24} will continue to exist.

Now we turn to the structure of the internal CFT and internal Hilbert space of the theory. For the standard embedding the first E_8 factor is broken to $D_6 \times D_2$. Using the fermionic description of these latices one obtains the following structure for the internal Hilbert space [85, 92]

$$\mathcal{H}^{internal} = \mathcal{H}^{(6,6)}_{D_2 K3} \otimes \mathcal{H}^{(6,0)}_{D_6} \otimes \mathcal{H}^{(8,0)}_{E_8} \otimes \mathcal{H}^{(2,3)}_{T^2} \tag{5.3}$$

where the upper index labels the left and right central charges. $H^{(6,6)}_{D_2 K3}$ is made up of 4 left moving bosons on $K3$ together with the 4 fermions from the fermionic representation of D_2, and a right-moving $\bar{c} = 6$ supersymmetric contribution. $\mathcal{H}^{(6,0)}_{D6}$ stands for 12 fermions coming from D_6, $\mathcal{H}^{(8,0)}_{E_8}$ is the Hilbert space of the unbroken E_8 and lastly, $\mathcal{H}^{(2,3)}_{T^2}$ is the Hilbert space of two left moving bosons on T^2 and a supersymmetric $c = 3$ theory on T^2. For non-standard embeddings we in general have vector bundles with rank $r_{1,2}$ belonging to the instantons embedded in the first/second E_8. Then $2(r_1 + r_2)$ fermions from the two E_8's couple to the gauge connection, and the Hilbert space structure generalizes accordingly [93].

5.1.1 The spectrum

In this subsection we will briefly discuss how one may obtain the massless spectrum of the $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$, see, e.g., [90, 94, 95, 95–97]. One may follow either of two approaches:

i) At a a generic point in $K3$ moduli space one can obtain the number of vector and hypermultiplets of the effective 4d $\mathcal{N} = 2$ spacetime theory by dimensional reduction of the 10-dimensional $\mathcal{N} = 1$ spectrum of the heterotic string [94, 95, 97, 98], which consist of a gravity multiplet and a vector multiplet. The 10 dimensional gravity

multiplet gives an universal contribution corresponding to one gravity multiplet, 3 vector multiplets and 20 hyper multiplets.
The 10 dimensional vector multiplet is in the adjoint representation of the gauge group , i.e., $(\mathbf{248},1)\oplus(1,\mathbf{248})$. Upon embedding of instantons into subgroups $H_1\times H_2$ of $E_8\times E_8$ the adjoint representation of E_8 decomposes as

$$\begin{aligned} E_8 &\to H_i \times G_i \\ \mathbf{248} &\to (1,\mathbf{adj}G_i) + \sum_k (\mathbf{R}_k^{(H_i)},\mathbf{R}_k^{(G_i)}) + (\mathbf{adj}H_i,1) \end{aligned} \tag{5.4}$$

where **adj** stands for the adjoint representation and $\mathbf{R}_k^{(H_i)},\mathbf{R}_k^{(G_i)}$ label different representations of H_i, G_i labelled by k. The 4 dimensional gauge bosons arises from the 10 dimensional gauge boson transforming as singlets under H_i. The corresponding scalars for these vector multiplets come from the 10 dimensional gauge bosons with vector index along T^2. To obtain the number of fermions one may use index theorems [90]. Fermions charged under the the groups $H_1\times H_2$ will arrange themselves in hypermultiplets.

ii) The second method to obtain the spectrum of the heterotic string on $K3\times T^2$ is to study the theory at special points in $K3$ moduli space where $K3$ may be written as $T^4/\mathbb{Z}_M$, $M=2,3,4,6$. Here one can explicitly calculate the spectrum from the orbifolded conformal field theory, that is one compactifies the $E_8\times E_8$ heterotic string on $T^4/\mathbb{Z}_M\times T^2$ [95, 96]. Parametrising T^2 by $x_1,x_2\in[0,2\pi)$ and T^4 by $(y_1,\dots,y_4)\in[0,2\pi)$ the orbifold action by $\mathbb{Z}_M$, generated by an element g, may be written as

$$g^s : (x_1,x_2,y_1{+}iy_2,y_3{+}iy_4) \mapsto (x_1,x_2,e^{2\pi \mathrm{i}s/M}(y_1{+}iy_2),e^{-2\pi \mathrm{i}s/M}(y_3{+}iy_4)),\, s=0,\dots,3, \tag{5.5}$$

i.e., g acts diagonally on the (complex) coordinates with eigenvalues $e^{\pm 2\pi \mathrm{i} v_a}$, $v_a=(0,0,\frac{1}{M},-\frac{1}{M})$. In addition one needs to declare the action on the gauge degrees of freedom, which we will denote $\gamma(g)$. Discarding the outer automorphism (which exchange the two E_8 factors) g must act on the gauge group as an element of $E_8\times E_8$, the unbroken gauge group than being the commutant of $\gamma(g)$ in $E_8\times E_8$. For our case (where the orbifold group is generated by a single element), using the bosonic description of $E_8\times E_8$ one can always choose $\gamma(g)$ to lie in the maximal torus of the gauge group. Hence it commutes with all the Cartan currents $\mathrm{i}\bar\partial X$ and can only act by a constant shift of the chiral bosons,

$$\gamma : X^I \mapsto X^I + V^I. \tag{5.6}$$

Then $g^M = 1$ implies that MV^I needs to lie in the $E_8 \times E_8$ weight lattice. It is futher restricted by modular invariance to fulfil $M(V^2 - v^2)$ =even. Different shift vectors V correspond to the different possible instanton embeddings. There are two such embeddings for $M = 2$, five for $M = 3$, 12 for $M = 4$ and fifty-nine for $M = 6$. The spectrum can now be found by orbifold methods. States consist of combinations of left and right vertex operators, $L \otimes R$. The mass formula for such states in the sector twisted by g^n is found to be

$$m_R^2 = N_R + \frac{1}{2}(r + nv)^2 + E_n - \frac{1}{2}\,;\; m_l^2 = N_L + \frac{1}{2}(P + nV)^2 + E_n - 1. \tag{5.7}$$

Here $N_{R/L}$ are the oscillator number of the R/L part, r is a $SO(8)$ weight with $\sum_{i=1}^4 r_i$ = odd, P is an element of the $E_8 \times E_8$ weight lattice with $\sum_{I=1}^{16} P^I$ = even and E_n is the zero point energy from the twisted sector oscillators, it is give by $E_n = \frac{n(M-n)}{M^2}$. Further the multiplicities of states satisfying above mass formula are given by

$$D(g^n) = \frac{1}{M}\sum_{m=0}^{M-1} \chi(g^n, g^m)\Delta(n, m) \tag{5.8}$$

where $\Delta(n, m)$ is a phase factor given by

$$\Delta(n, m) = exp\{2\pi \mathrm{i}[(r + nv) \cdot mv - (P + nV) \cdot mV + \frac{1}{2}mn(V^2 - v^2) + m\rho]\}. \tag{5.9}$$

ρ only appears in the case of oscillators ($N_L \neq 0$), $e^{2\pi \mathrm{i}\rho}$ is the phase by which the oscillators in the T^4 are rotated by g. $\chi(g^n, g^m)$ is the number of simultaneous fixed points of g^n and g^m, $\chi(1, g^m)$ is defined to be 1. From this the gauge group and complete spectrum can be worked out [96]. A complete list of all cases can be found in [99]. We reproduce the examples that will be important to us in tables 2,3.

5.1.2 The new supersymmetric index

Having discussed the spectrum of the $E_8 \times E_8$ heterotic string on $K3 \times T^2$ in the previous section we now turn to calculating the gravitational couplings and the vector moduli prepotential. A central object that is needed to calculate these quantities [85–89] is the new supersymmetric index [100]. It is also the object that brings about the connection to Mathieu moonshine. It is defined by

$$\mathcal{Z}^{new}(q, \bar{q}) = \frac{1}{\eta(q)^2}\mathrm{Tr}_R(F(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{\bar{c}}{24}})\Big|_{c=(22,9)} \tag{5.10}$$

where the trace is taken over the Ramond sector of the internal CFT associated to the $K3 \times T^2$ and $E_8 \times E_8$ with central charge $(c, \bar{c}) = (22, 9)$. It is an natural object in $d = 4, \mathcal{N} = 2$ heterotic compactifications since it counts the number of BPS-states.

$N_h - N_v$	Gauge group, Shift vector	Twisted sectors	Hypermultiplets	$\hat{b}$
84	$E_6 \times SU(2) \times U(1)$ $\times SO(14) \times U(1)$ $(2, 1, 1, 0^5; 2, 0^7)$	g'^0 $g'^1 + g'^3$ g'^2	$(\mathbf{27}, \mathbf{2}; \mathbf{1}) + (\mathbf{1}, \mathbf{2}; \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{64})$ $+2(\mathbf{1}, \mathbf{1}; \mathbf{1})$ $12(\mathbf{1}, \mathbf{1}; \mathbf{1}) + 8(\mathbf{1}, \mathbf{2}; \mathbf{1})$ $+4(\mathbf{27}, \mathbf{1}; \mathbf{1}) + 4(\mathbf{1}, \mathbf{1}; \mathbf{14})$ $3(\mathbf{1}, \mathbf{2}; \mathbf{14}) + 10(\mathbf{1}, \mathbf{2}; \mathbf{1})$	$\frac{2}{3}$
116	$SU(8) \times SU(2)$ $\times SO(14) \times U(1)$ $(3, 1^5, 0^2; 2, 0^7)$	g'^0 $g'^1 + g'^3$ g'^2	$(\mathbf{28}, \mathbf{2}; \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{64}) + 2(\mathbf{1}, \mathbf{1}; \mathbf{1})$ $8(\mathbf{8}, \mathbf{1}; \mathbf{1}) + 4(\mathbf{8}, \mathbf{2}; \mathbf{1})$ $5(\mathbf{1}, \mathbf{2}; \mathbf{14}) + 6(\mathbf{1}, \mathbf{2}; \mathbf{1})$	$\frac{8}{9}$

Table 2: Two examples of perturbative $E_8 \times E_8$ heterotic orbifold spectra on $T^4/\mathbb{Z}_4$.

Gauge Group, shift vector	Untwisted sector	Twisted sector
$E_7 \times U(1) \times E_8$ $\frac{1}{3}(1, -1, 0^6; 0^8)$	$(\mathbf{56}; \mathbf{1}) + 2(\mathbf{1}; \mathbf{1})$ $+(\mathbf{1}; \mathbf{1})$	$9(\mathbf{56}; \mathbf{1})$ $+ 45(\mathbf{1}; \mathbf{1})$ $18(\mathbf{1}; \mathbf{1})$
$SU(9) \times E_8$ $\frac{1}{3}(2, 1^4, 0^3; 0^8)$	$(\mathbf{84}; \mathbf{1}) + 2(\mathbf{1}; \mathbf{1})$	$9(\mathbf{36}; \mathbf{1}) + 18(\mathbf{9}; \mathbf{1})$
$SO(14)^2 \times U(1)^2$ $\frac{1}{3}(2, 0^7; 2, 0^7)$	$(\mathbf{14}; \mathbf{1}) + (\mathbf{1}; \mathbf{14})$ $+(\mathbf{64}; \mathbf{1}) + (\mathbf{1}; \mathbf{64})$ $+2(\mathbf{1}; \mathbf{1})$	$9(\mathbf{14}; \mathbf{1})$ $+9(\mathbf{1}; \mathbf{14})$ $+18(\mathbf{1}; \mathbf{1})$
$E_6 \times SU(3) \times E_7 \times U(1)$ $\frac{1}{3}(2, 1^2, 0^5; 1, -1, 0^6)$	$(\mathbf{27}, \mathbf{3}; \mathbf{1})$ $+(\mathbf{1}, \mathbf{1}; \mathbf{56})$ $+2(\mathbf{1}, \mathbf{1}; \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{1})$	$9(\mathbf{27}, \mathbf{1}; \mathbf{1})$ $9(\mathbf{1}, \mathbf{3}; \mathbf{1})$ $+18(\mathbf{1}, \mathbf{3}; \mathbf{1})$
$SU(9) \times E_6 \times SU(3)$ $\frac{1}{3}(2, 1^2, 0^5; 2, 1^4, 0^3)$	$(\mathbf{84}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}; \mathbf{27}, \mathbf{3})$ $2(\mathbf{1}; \mathbf{1}, \mathbf{1})$	$9(\mathbf{9}; \mathbf{1}, \mathbf{3})$

Table 3: Hypermultiplet spectrum for different embeddings with $K3$ as $T^4/\mathbb{Z}_3$. We have not kept track of various U(1) charges.

As shown in [85], section 3, morally speaking,

$$\mathcal{Z}^{new}(q, \bar{q}) = -2i \left[\sum_{\text{BPS vectormultiplets}} q^{\Delta} q^{\bar{\Delta}} - \sum_{\text{BPS hypermultiplets}} q^{\Delta} q^{\bar{\Delta}} \right]. \tag{5.11}$$

In the case without any Wilson lines on T^2, the new supersymmetric index for the theory on $K3 \times T^2$ takes the general form [101]

$$\begin{aligned}
\mathcal{Z}^{new}(q,\bar{q}) &= Z_{K3}(q) \cdot Z_{2,2}(q,\bar{q}), \\
Z_{2,2}(q,\bar{q}) &= \sum_{p\in\Gamma_{2,2}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2} = \sum_{p\in\Gamma_{2,2}} q^{\frac{1}{2}(p_L^2-p_R^2)}e^{-2\pi\tau_2 p_R^2}, \\
p_R^2 &= \frac{1}{2T_2U_2}|-m_1U+m_2+n_1T+n_2TU|^2, \\
\frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2+m_1n_1+m_2n_2,
\end{aligned} \tag{5.12}$$

where T, U are the Kähler and complex structure moduli on T^2. We see that the new supersymmetric index factorizes into a holomorphic part $Z_{K3}(q)$ coming from $K3$ and the $E_8 \times E_8$ contributions and into a lattice sum $Z_{2,2}$. In general the new supersymmetric index will depend on the topology of the manifold one (e.g., χ_{K3}) on and on the topology of the gauge bundle, e.g., $(n^{(1)}, n^{(2)})$. However one may move in instanton moduli space, i.e., change the way the instantons are embedded in $E_8 \times E_8$. This is equivalent to moving around in the hypermultiplet moduli space by (un-)Higgsing, thereby changing the gauge group $G_1 \times G_2$. So one can easily move from standard to non-standard embeddings. Doing so however leaves the net number of vector- and hypermultiplets $N_V - N_H = 240$ as well as $(n^{(1)}, n^{(2)})$ unchanged. We note that the gauge group can be maximally Higgsed for $n^{(1)} = 0, 1, 2$ only.

For the case of $SU(2)$ bundles with instanton numbers $(n^{(1)}, n^{(2)})$ and no Wilson lines the contribution from the $K3$ part is [85]

$$Z_{K3}(q) = -2\left[\frac{n^{(1)}}{24}\frac{E_6(q)E_4(q)}{\eta(q)^{24}} + \frac{n^{(2)}}{24}\frac{E_6(q)E_4(q)}{\eta(q)^{24}}\right] = -2\frac{E_6(q)E_4(q)}{\eta(q)^{24}}. \tag{5.13}$$

By making use of the modular properties of $\mathcal{Z}^{new}$ one can make a much more general statement [30]. Namely, as will become clear from (5.41) (with $g = 1$) in subsection 5.2.2, $\tau_2\mathcal{Z}^{new}$ is a non-holomorphic modular form of weight -2 (with a pole at infinity). Given the factorisation in (5.12) one can deduce that the form of Z_{K3} as given in (5.13) is actually uniquely fixed up to a multiplicative constant. This constant can be fixed to 1 by various arguments [27].

The Hilbert space structure (5.3) implies that $Z_{K3}(q)$ admits an expansion in terms of the characters of $D6$ and the elliptic genus of $K3$ with the elliptic modulus taking

[27]Due to the gravitational anomaly 6d, $\mathcal{N} = 1$ compactifications of the heterotic string on $K3$ always satisfy $n_H - n_V = 244$, n_H, n_V here stand for the number of $\mathcal{N} = 1$ hyper-and vector multiplets. This in turn fixes the coefficient of the $q^{1/6}$ in an expansion of (5.13) to 1 [30].

special values [85]

$$\eta(q)^{12}Z_{K3}(q) =\frac{1}{2}E_4\Big[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathcal{Z}^{ell}_{K3}(q,-1) + q^{\frac{1}{4}}\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 \mathcal{Z}^{ell}_{K3}(q,-q^{1/2}) - q^{\frac{1}{4}}\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 \mathcal{Z}^{ell}_{K3}(q,q^{1/2})\Big]. \quad (5.14)$$

The factors can be understood in the following way: The E_4 factor comes from the unbroken E_8, the D_6 characters enter through the trace over the 12 free fermions coming from the broken E_8. The elliptic genus arises from taking a trace over the remaining 4 fermions, the 4 bosons in the left moving sector together with the supersymmetric right moving sector in the $(R-,R),(NS+,R)$ and $(NS-,R)$ sectors (Here + stands for the insertion of $(-1)^{F_L}$ when taking the trace and − stands for no such insertion. For the right moving part the $(-1)^{F_R}$ insertion is always there so it is left away). The $(R+,R)$ sector also contributes a term proportional to $\left(\frac{\theta_1(q)}{\eta(q)}\right)^6 \mathcal{Z}^{ell}_{K3}(q,+1)$ which we have omitted since vanishes due to $\theta_1(q)=0$. However contributions from this sector may arise from twisted sectors when studying CHL-orbifolds.
Through the presence of the $K3$ elliptic genus in (5.14) one obtains a connection between the new supersymmetric index and to M_{24}. More precisely making use of (3.43) one finds the following expansion

$$-\frac{4E_6(q)}{\eta^{12}}(q) = 20\, g_{h=1/4,l=0}(q) - 2g_{h=1/4,l=1/2}(q) + \sum_{n=1}^{\infty} A_n g_{h=n+1/4,l=1/2}(q) \quad (5.15)$$

where

$$g_{h=n+1/4,l}(q) = \left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathrm{ch}^{\mathcal{N}=4}_{3,n,l}(q,-1) + q^{\frac{1}{4}}\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 \mathrm{ch}^{\mathcal{N}=4}_{3,n,l}(q,-q^{1/2}) - q^{\frac{1}{4}}\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 \mathrm{ch}^{\mathcal{N}=4}_{3,n,l}(q,q^{1/2}) \quad (5.16)$$

and the coefficients A_n are as stated in (3.44) and are the dimensions of irreducible representation of the Mathieu group. This shows that the new supersymmetric index (for the case without Wilson lines) and thereby also the number of BPS states admits a decomposition in terms of dimensions of irreducible representations of M_{24}.

Before we turn to the connection with CY 3-folds we will discuss the generalization to CHL orbifolds to the constructions presented above. Thereby not just the elliptic genus but also the twisted twining genera of $K3$ enter the discussion.

5.2 CHL orbifolds

In this subsection we consider $\mathbb{Z}_N$ orbifolds of the $E_8\times E_8$ heterotic string compactified on $K3\times T^2$. Following [38, 70, 92, 97] we consider orbifolds where $\mathbb{Z}_N$ acts freely by an $1/N$ shift on one of the circles of T^2 together with an action on the internal CFT describing the heterotic string on $K3$. We refer to these orbifolds as CHL orbifolds [28] of order N. Orbifolding by a freely acting group is necessary in order to preserve $\mathcal{N}=2$ space time supersymmetry. Since the action involves both $K3$ and T^2 the orbifolded theories cannot be thought to be obtained from $\mathcal{N}=1$, $d=6$ vacua.

In particular we consider orbifolds where $\mathbb{Z}_N$ acts as an automorphism on $K3$. This action must preserve $SU(2)$ holonomy so it must retain the holomorphic 2-form on $K3$ and the holomorphic 1-form on T^2 [104].

So $\mathbb{Z}_N$ has to act as a symplectic automorphism (of order N) on $K3$ [29]. As mentioned in section 3.3 the symplectic automorphisms of $K3$ form a subgroup of M_{23} (which may change when moving in the moduli space of $K3$). Each automorphism corresponds to one of the following nine conjugacy classes of M_{23} [106]

$$\text{1A, 2A, 3A, 4B, 5A, 6A, 7A, 7B, 8A.} \tag{5.17}$$

We will consider examples of $\mathbb{Z}_2$ [70, 92, 97] and $\mathbb{Z}_3$ orbifolds [38].

For the $\mathbb{Z}_2$ case we follow [92] and consider a point in $K3$ moduli space, where $K3$ is expressible as $T^4/\mathbb{Z}_4$. Concretely let $(y_1, \ldots, y_4) \in [0, 2\pi)$ be coordinates on T^4 then $K3$ is obtained by the orbifold action

$$g^s : (y_1 + iy_2, y_3 + iy_4) \sim (e^{2\pi i s/4}(y_1 + iy_2), e^{-2\pi i s/4}(y_3 + iy_4)) \, , \; s = 0, \ldots, 3 \tag{5.18}$$

Parametrising the T^2 by $x_1, x_2 \in [0, 2\pi)$ the CHL action may be written as

$$g' : (x_1, x_2, y_1, y_2, y_3, y_4) \sim (x_1 + \pi, x_2, y_1 + \pi, y_2 + \pi, y_3 + \pi, y_4 + \pi). \tag{5.19}$$

For the $\mathbb{Z}_3$ case we realize $K3$ as $T^4/\mathbb{Z}_3$ [38]. We choose the coordinates on T^4 to be $(z_1, z_2) \in \mathbb{C}^2/\mathbb{Z}^2$ with periodicity

$$z_i \sim z_i + n_1 e_1 + n_2 e_2, \;\; e_1 = e^{\frac{2\pi i}{3}}, e_2 = 1, (n_1, n_2) \in \mathbb{Z}^2. \tag{5.20}$$

The orbifold limit of $K3$ is then realized by orbifolding with

$$g^s : (z_1, z_2) \mapsto (e^{\frac{2\pi i s}{3}} z_1, e^{-\frac{2\pi i s}{3}} z_2). \tag{5.21}$$

Denoting the coordinates on T^2 by $(x_1, x_2) \in [0, 2\pi)$ as above, the CHL orbifold of

[28] They are named after Chaudhuri, Hockney and Lykken who were the first to study freely acting orbifolds of heterotic string compactifications [102, 103].

[29] Since such automorphisms always have a fixed point [105] it is necessary for the action on T^2 to be fix point free, as is guaranteed by the $1/N$ shift.

order 3 can now be implemented by orbifolding with

$$g' : (x_1, x_2, z_1, z_2) \mapsto \left(x_1 + \frac{2\pi}{3}, x_2, z_2 + \frac{1}{3}e_1 + \frac{2}{3}e_2, z_2 \right). \tag{5.22}$$

5.2.1 The spectrum

As in the unorbifolded case (subsection 5.1.1), the spectrum of the CHL orbifolded theory may be again be constructed in the two ways above.
i) By dimensional reduction [97]: One needs to take into account that the number of $h^{1,1}$ forms on the orbifolded $K3$ is given by [65]

$$h^{1,1} = 2k\ ,\ ,k = \frac{24}{N+1} - 2, \quad \text{for } N = 2, 3, 5, 7. \tag{5.23}$$

Then one finds that the universal contribution from the 10 dimensional gravity multiplet after orbifolding is 1 gravity multiplet, 3 vector fields and $2k$ hypermultiplets, so only the number of hypermultiplets is changed. Similarly in the contribution coming from the 10 dimensional gauge multiplet only the number of hypermultiplets changes.
ii) As in to the unorbifolded case one can obtain the spectrum at special points in $K3$ moduli space by CFT methods [38, 92]. One thing to notice is that the sector twisted by g' only does not produce massless states since the g' action is fixed point free. The formulas of the masses of states (5.7) stay the same, only the formula for the degeneracies (5.8) changes to

$$D(g^n) = \frac{1}{M}\frac{1}{N}\sum_{m=0}^{M-1}\sum_{r=0}^{N-1} \chi(g^n, g^m g'^r)\Delta(n, m) \tag{5.24}$$

where now $\chi(g^n, g^m g'^r)$ is the number of fixed points of g^n invariant under $g^m g'^r$, so essentially just the projection onto g' invariant states has entered. Since the embedding of instantons into the gauge group has not been changed (i.e., the shift vector is the same) and the vector multiplets arise from the untwisted sector, the gauge group and number of vector multiplets is unchanged. Only the number of hypermultiplets arising from the twisted sectors may change. So by counting the common fixed points of g^n and $g^m g'^r$ for the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds the resulting spectrum can be found [38, 92]. Again we reproduce the cases that are of interest to us in tables 4 and 5. To obtain the complete spectrum the gravity multiplet and the 3 vector multiplets coming from the universal contribution from the 10 dimensional gravity multiplet still need to be added to this spectrum.

We are interested in finding the possible type II dual theories to the theories of tables 4 and 5. In practice finding the dual theory is only feasible when the rank of the gauge group is small, hence we want to Higgs the gauge group as far as possible

$N_h - N_v$	Gauge group, Shift vector	Twisted sectors	Hypermultiplets	$\hat{b}$
84	$E_6 \times SU(2) \times U(1)$ $\times SO(14) \times U(1)$ $(2, 1, 1, 0^5; 2, 0^7)$	g'^0 $g'^1 + g'^3$ g'^2	$(\mathbf{27}, \mathbf{2}; \mathbf{1}) + (\mathbf{1}, \mathbf{2}; \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{64})$ $+2(\mathbf{1}, \mathbf{1}; \mathbf{1})$ $6(\mathbf{1}, \mathbf{1}; \mathbf{1}) + 4(\mathbf{1}, \mathbf{2}; \mathbf{1})$ $+2(\mathbf{27}, \mathbf{1}; \mathbf{1}) + 2(\mathbf{1}, \mathbf{1}; \mathbf{14})$ $(\mathbf{1}, \mathbf{2}; \mathbf{14}) + 6(\mathbf{1}, \mathbf{2}; \mathbf{1})$	$\frac{2}{3}$
116	$SU(8) \times SU(2)$ $\times SO(14) \times U(1)$ $(3, 1^5, 0^2; 2, 0^7)$	g'^0 $g'^1 + g'^3$ g'^2	$(\mathbf{28}, \mathbf{2}; \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{64}) + 2(\mathbf{1}, \mathbf{1}; \mathbf{1})$ $4(\mathbf{8}, \mathbf{1}; \mathbf{1}) + 2(\mathbf{8}, \mathbf{2}; \mathbf{1})$ $3(\mathbf{1}, \mathbf{2}; \mathbf{14}) + 2(\mathbf{1}, \mathbf{2}; \mathbf{1})$	$\frac{8}{9}$

Table 4: Perturbative $E_8 \times E_8$ heterotic orbifold spectra on $T^4/\mathbb{Z}_4$ after order 2 CHL orbifolds

Gauge Group, shift shift	Untwisted sector	Twisted sector
$E_7 \times U(1) \times E_8$ $\frac{1}{3}(1, -1, 0^6; 0^8)$	$(\mathbf{56}; \mathbf{1}) + 2(\mathbf{1}; \mathbf{1})$ $+(\mathbf{1}; \mathbf{1})$	$3(\mathbf{56}; \mathbf{1})$ $+\ 15(\mathbf{1}; \mathbf{1})$ $6(\mathbf{1}; \mathbf{1})$
$SU(9) \times E_8$ $\frac{1}{3}(2, 1^4, 0^3; 0^8)$	$(\mathbf{84}; \mathbf{1}) + 2(\mathbf{1}; \mathbf{1})$	$3(\mathbf{36}; \mathbf{1}) + 6(\mathbf{9}; \mathbf{1})$
$SO(14)^2 \times U(1)^2$ $\frac{1}{3}(2, 0^7; 2, 0^7)$	$(\mathbf{14}; \mathbf{1}) + (\mathbf{1}; \mathbf{14})$ $+(\mathbf{64}; \mathbf{1}) + (\mathbf{1}; \mathbf{64})$ $+2(\mathbf{1}; \mathbf{1})$	$3(\mathbf{14}; \mathbf{1})$ $+3(\mathbf{1}; \mathbf{14})$ $+6(\mathbf{1}; \mathbf{1})$
$E_6 \times SU(3) \times E_7 \times U(1)$ $\frac{1}{3}(2, 1^2, 0^5; 1, -1, 0^6)$	$(\mathbf{27}, \mathbf{3}; \mathbf{1})$ $+(\mathbf{1}, \mathbf{1}; \mathbf{56})$ $+2(\mathbf{1}) + (\mathbf{1})$	$3(\mathbf{27}, \mathbf{1}; \mathbf{1})$ $3(\mathbf{1}, \mathbf{3}; \mathbf{1})$ $+6(\mathbf{1}, \mathbf{3}; \mathbf{1})$
$SU(9) \times E_6 \times SU(3)$ $\frac{1}{3}(2, 1^2, 0^5; 2, 1^4, 0^3)$	$(\mathbf{84}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}; \mathbf{27}, \mathbf{3})$ $2(\mathbf{1})$	$3(\mathbf{9}; \mathbf{1}, \mathbf{3})$

Table 5: Perturbative $E_8 \times E_8$ heterotic orbifold spectra on $T^4/\mathbb{Z}_3$ after order 3 CHL orbifolds. We have not kept track of various U(1) charges.

and find the dual theories of the maximally Higgsed models [30]. As illustrated in Appendix F Higgsing is done by starting at a point in moduli space where all of the charged scalars (coming from either vector or hypermultiplets) have a vanishing

[30]On the type II side Higgsing correspond to moving in the CY moduli space through conifold singularities.

VEV. Then we may Higgs the gauge group by giving VEV to scalars belonging to certain hypermultiplets, moving along the Higgs branch of moduli space. As mentioned above in section 5.1.2 this corresponds to changing the embedding of the (n_1, n_2) instantons into the $E_8 \times E_8$ gauge group while however keeping n_1, n_2 fixed. Higgsing does not change the relative number of vector and hyper multiplets $N_h - N_v$. Once we have Higgsed the gauge group in this manner we can still give a VEV to the scalars of the vector multiplets in the remaining gauge group, moving along the Coulomb of those vector multiplets. Thereby these gauge groups are broken to $U(1)$ factors, the number of those being equal to the rank of the unbroken gauge group. This will reduce the number of vector multiplets from the dimension of the gauge group to the rank of the gauge group. We give an explicit examples for how the Higgsing is done in the Appendix F. The examples of spectrum we have presented in tables 4 are all such that the gauge group can be completely Higgsed. For table 5 this is the case for the third and sixth example. That is after Higgsing we are left with only the vector multiplets coming from the universal contribution of the gravity multiplet, i.e., 3 vector multiplets, the gravity multiplet and the remaining hypermultiplets, i.e. $N_h - N_v$ hypermultiplets.

5.2.2 Gravitational couplings/threshold corrections and the new supersymmetric index

The new supersymmetric index for the CHL orbifolds of the $E_8 \times E_8$ heterotic string compactification on $K3 \times T^2$ with different instanton embeddings can be obtained by generalising the methods described in [101]. We briefly explain the results given in [38, 92]. For the case where $K3$ is realized as a $T^4/\mathbb{Z}_M$ and general non standard embedding given by shift vectors $\gamma, \tilde{\gamma}$, the new supersymmetric index of an order N CHL orbifold takes the general form

$$\mathcal{Z}^{new,N} = -\frac{1}{2\eta^{20}(q)} \sum_{a,b=0}^{M-1} \sum_{r,s=0}^{N-1} e^{-\frac{2\pi i ab}{M^2}\Gamma^2} Z_{E_8}^{(a,b)}(q) \times Z_{E_8'}^{(a,b)}(q) \times \frac{1}{2\nu} F(a,r,b,s;q) Z_{2,2}^{(r,s)}(q,\bar{q}), \tag{5.25}$$

where the partition functions of the shifted E_8 lattice are given by

$$\begin{aligned} Z_{E_8}^{(a,b)}(q) &= \sum_{\alpha,\beta=0}^{1} e^{-i\pi\beta a \sum_{I=1}^{8} \gamma^I} \prod_{I=1}^{8} \theta \begin{bmatrix} \alpha+2a\gamma^I \\ \beta+2b\gamma^I \end{bmatrix} , \\ Z_{E_8'}^{(a,b)}(q) &= \sum_{\alpha,\beta=0}^{1} e^{-i\pi\beta a \sum_{I=1}^{8} \tilde{\gamma}^I} \prod_{I=1}^{8} \theta \begin{bmatrix} \alpha+2a\tilde{\gamma}^I \\ \beta+2b\tilde{\gamma}^I \end{bmatrix} , \end{aligned} \tag{5.26}$$

$\Gamma^2 = (\gamma^2 + \tilde{\gamma}^2)$. $Z_{2,2}^{(r,s)}(q,\bar{q})$ is the twisted-twined partition function of T^2 and has the expression

$$Z_{2,2}^{(r,s)}(q,\bar{q}) = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z} \\ n_1\in\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} e^{2\pi i m_1 s/N}, \tag{5.27}$$

where the momenta are defined as in (5.12) and as before T, U are respectively the Kähler and complex structure moduli of T^2. The twist by an element of order N is reflected in the phase $e^{2\pi i m_1 s/N}$ and the fractional values of n_1. Furthermore

$$F(a,r,b,s;q) := \mathrm{Tr}_{R,g^a g'^r}\left(g^b g'^s e^{i\pi F_R^{T^4}} q^{L_0} \bar{q}^{\bar{L}_0}\right) \tag{5.28}$$

is the trace over T^4 with 4 left and 4 right moving bosons, and over right moving fermions in the twisted Ramond sector whose fermion number is $F_R^{T^4}$, g incorporates the $\mathbb{Z}_M$ action. One can work out an explicit expression

$$F(a,r,b,s;q) = k^{a,rb,s}\eta(\tau)^2 q^{\frac{-a^2}{M^2}} \frac{1}{\theta_1(\frac{a\tau+b}{3},\tau)^2} \tag{5.29}$$

where $k^{a,r,b,s}$ are complex numbers that depend on Γ^2 and their absolute value depends on the common fixed points. They can be worked out for the different situations [92, 101].
Closed expression for different situation have been obtained in [38, 92, 97]. In general one may write $\mathcal{Z}^{new,N}$ in the form

$$\mathcal{Z}^{new,N}(q,\bar{q}) = -4\sum_{r,s=0}^{N-1} Z_{2,2}^{(r,s)}(q,\bar{q}) f^{(r,s)}(q), \tag{5.30}$$

where $\eta(q)^{24} f^{(r,s)}(q)$ is a modular form of weight 10 under $\Gamma_0(N)$ [31].

We start by giving the expression $\mathcal{Z}^{new,N}$ for the standard embedding, making use of (5.30). Here one obtains [70]

$$f^{(r,s)}(q) = \frac{1}{2\eta(q)^{24}} E_4(q)\left[\frac{1}{4}\alpha_{\tilde{g}}^{(r,s)} E_6(q) - \beta_{\tilde{g}}^{(r,s)}(\tau) E_4(q)\right], \tag{5.31}$$

where $\alpha_{\tilde{g}}^{(r,s)}, \beta_{\tilde{g}}^{(r,s)}$ are the same as in (3.52) and explicit expressions can be found in

[31] The vector space of such forms for $N = 2$ is three dimensional and may be generated by E_4E_6, $E_4^2\mathcal{E}_2$ and $\mathcal{E}_2^2 E_6$, where $\mathcal{E}_2$ is defined in (5.33).

the Appendix E of [70]. For $N = 2, 3, 5, 7$ we have

$$\alpha_{\tilde{g}}^{(0,0)} = \frac{8}{N}\,,\ \alpha_{\tilde{g}}^{(r,s)} = \frac{8}{N(N+1)}\,,\ (r,s) \neq (0,0) \tag{5.32}$$
$$\beta_{\tilde{g}}^{(0,0)}(\tau) = 0\,,\ \beta_{\tilde{g}}^{(0,s)}(\tau) = -\frac{2}{N+1}\mathcal{E}_N(\tau)\,,\ \beta_{\tilde{g}}^{(r,rk)}(\tau) = \frac{2}{N(N+1)}\mathcal{E}_N\left(\frac{\tau+k}{N}\right)$$

for $1 \leq s, r, k \leq N-1$ and $\mathcal{E}_N$ is a modular form of weight 2 under $\Gamma_0(N)$ defined as

$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau \ln \frac{\eta(\tau)}{\eta(N\tau)}. \tag{5.33}$$

In this case (i.e., for the standard embedding) $\mathcal{Z}^{new,N}$ has the following expansion in terms of the twisted twining elliptic genera of $K3$ [92, 97]

$$\mathcal{Z}^{new,N}(q,\bar{q}) = \frac{1}{N}\sum_{r,s=0}^{N-1} \frac{Z_{2,2}^{(r,s)}(q,\bar{q})E_4(q)}{\eta(q)^{12}} \times \left[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathcal{Z}^{ell}_{K3,r,s}(q,-1)\right. \tag{5.34}$$
$$\left. + \left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{ell}_{K3,r,s}(q,-q^{\frac{1}{2}}) - \left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{ell}_{K3,r,s}(q,q^{\frac{1}{2}})\right],$$

where we have used the shorthand notation $\mathcal{Z}^{ell}_{K3,r,s} := \mathcal{Z}^{ell}_{K3,g'^r,g'^s}$. So by our discussion in 3.3 in this situation the new supersymmetric index becomes linked to the MacKay-Thompson series of Mathieu moonshine.

For the case of non-standard embeddings of order 2 considered in [92] one finds the following expression

$$f^{(0,0)}(q) = \frac{1}{2\eta^{24}(q)}E_4(q)E_6(q) \tag{5.35}$$
$$f^{(r,s)}(q) = \frac{3}{4\eta^{24}(q)}\left(\hat{b}\cdot\left(\frac{3}{2}\beta_{\tilde{g}}^{(r,s)}(\tau)\right)^2 + \left(\frac{2}{3}-\hat{b}\right)E_4(q)\right)$$
$$\times\left[\frac{1}{4}\alpha_{\tilde{g}}^{(r,s)}E_6(q) - \beta_{\tilde{g}}^{(r,s)}(\tau)E_4(q)\right]$$

The constant $\hat{b}$ depends on the 14 different possible instanton embeddings listed in [101] and it takes four different values

$$\hat{b} \in \left\{0, \frac{4}{9}, \frac{2}{3}, \frac{8}{9}\right\}. \tag{5.36}$$

The values for $\hat{b}$ relevant for our example are given in table 4 and the complete list of shift vectors and corresponding $\hat{b}$ values can be found in [92]. The value of $\hat{b}$ also directly determines the difference between the number of hyper and vector multiplets

Shift	$N_h - N_v$	$\tilde{a}$	$\tilde{b}$	$\tilde{c}$	$\tilde{d}$
$\frac{1}{3}(1,-1,0^6;0^8)$	-134	0	0	$\frac{1}{4}$	0
$\frac{1}{3}(2,1^4,0^3;0^8)$	-80	$\frac{1}{16}$	$-\frac{9}{16}$	$-\frac{3}{16}$	$\frac{9}{16}$
$\frac{1}{3}(2,0^7;2,0^7)$	64	$\frac{-1}{48}$	$\frac{3}{16}$	$\frac{1}{48}$	$\frac{3}{16}$
$\frac{1}{3}(1,-1,0^6;2,1^2,0^5)$	28	0	0	$\frac{-1}{32}$	$\frac{9}{32}$
$\frac{1}{3}(2,1^2,0^5;2,1^4,0^3)$	82	$\frac{-1}{32}$	$\frac{9}{32}$	$\frac{3}{64}$	$\frac{9}{64}$

Table 6: Values of $\tilde{a}, \tilde{b}, \tilde{c}$ and $N_h - N_v$ for different instanton embeddings/shift vectors with $K3$ as $T^4/\mathbb{Z}_3$ and $N = 3$ CHL orbifold.

through the relation

$$N_h - N_v = 144\,\hat{b} - 12. \tag{5.37}$$

For order 3 orbifolds one finds

$$\begin{aligned} f^{(0,1)} &= \frac{1}{3\eta^{24}}\Big(\tilde{a}E_4E_6 + \tilde{b}\mathcal{E}_3^2E_6 + \tilde{c}E_4(E_6 + 3\mathcal{E}_3E_4) + \tilde{d}\mathcal{E}_3^2(E_6 + 3\mathcal{E}_3E_4)\Big) \\ f^{(r,rk)} &= \frac{1}{3\eta^{24}}\left(\tilde{a}E_4E_6 + \frac{\tilde{b}}{9}\mathcal{E}_3'^2E_6 + \tilde{c}E_4(E_6 - \mathcal{E}_3'E_4) + \frac{\tilde{d}}{9}\mathcal{E}_3'^2(E_6 - \mathcal{E}_3'E_4)\right) \end{aligned} \tag{5.38}$$

where $1 \le r, k \le 2$, we omitted the τ dependence and $\mathcal{E}_3'(\tau) := \mathcal{E}_3(\frac{\tau+k}{3})$. The values of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ depend on the instanton embeddings and are given in table 6. It turns that for all the four non standard embeddings $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ may be expressed by one parameter in the following way

$$\begin{aligned} \tilde{a} = \frac{1}{2^7 3^3}(48+\chi)\,,\ \tilde{b} = -\frac{1}{2^7 3}(48+\chi) \\ \tilde{c} = -\frac{1}{2^8 3^3}(456+5\chi)\,,\ \tilde{d} = \frac{1}{2^8 3}(264+\chi) \end{aligned} \tag{5.39}$$

where χ will turn out to be the Euler character of the dual CY (see subsection 5.6 and 5.5.2).

5.3 Gravitational couplings and the vector moduli prepotential

As mentioned at the start of this section we want to match the vector moduli prepotential as well as certain gravitational couplings of the $d = 4$, $\mathcal{N} = 2$ effective field theory arising from the heterotic and type II compactification. Concretely in the low energy effective action the gravitational couplings F_g appear in the form

$$S = \int F_g(y,\bar{y}) \cdot F_+^{2g-2} R_+^2, \tag{5.40}$$

where F_+, R_+ are the self-dual part of the graviphoton and the Riemann tensor and $y, \bar{y}$ collective stand for the dependence on the vector moduli. The label g is chosen since on the type II side the couplings F_g arise from topological contributions of the genus g world sheet (it should hence not be confused with either the string coupling or an element of the orbifold group). The gravitational couplings on the heterotic side are perturbatively one loop level exact . Concretely for the case of the $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$ it has been shown [87, 88, 107, 108] that F_g, for $g > 1$, are given by the one-loop integral

$$\begin{aligned} F_g(y,\bar{y}) = \frac{1}{2\pi^2 (g!)^2} \int \frac{d^2\tau}{\tau_2} \Big\{ \frac{1}{\tau_2^2 \eta(\tau)^2} \mathrm{Tr}\left[(\mathrm{i}\bar{\partial} X)^{(2g-2)} (-1)^F F q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right] \\ \langle \prod_{i=1}^{g} \int d^2 x_i Z^1 \partial Z^2 (x_i) \prod_{j=1}^{g} \int d^2 \tilde{x}_i \bar{Z}^1 \partial \bar{Z}^2 (\tilde{x}_i) \rangle \Big\} \end{aligned} \tag{5.41}$$

where X is the complex coordinate on T^2 and Z^1, Z^2 are the complex coordinates on the transverse non-compact space time. The trace is taken over the Ramond sector of the internal conformal field theory. The internal trace can be calculated to be

$$\begin{aligned} \frac{1}{\eta(\tau)^2} \mathrm{Tr}\Big[\, (\mathrm{i}\bar{\partial} X)^{(2g-2)} \, (-1)^F F q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \Big] = \\ 2 \frac{1}{\eta(\tau)^{24}} \sum_{r,s} \Gamma_{2,2}^{(r,s)} \left(\frac{(p_R^{(r,s)})}{\sqrt{2 T_2 U_2}} \right)^{(2g-2)} q^{|p_L|^2/2} \bar{q}^{|p_R|^2/2} f^{(r,s)}(\tau) \end{aligned} \tag{5.42}$$

where $f^{(r,s)}(\tau)$ is defined in (5.30), $p_L^{r,s}, p_R^{r,s}$ are defined in (5.12). The integral can be evaluated by using the lattice reduction theorem by Borcherds [109], see, e.g, [70]. In order to make a connection to the type II side one need to extract the purely holomorphic part. This part can be written in term of Gopakumar-Vafa invariants of the CY on the dual type II side. For the case where we set all moduli to zero apart from (T, U), the complex structure and the complexified Kähler structure of the torus, one finds that the purely holomorphic part is given by [32]

$$\begin{aligned} \bar{F}_g^{hol} = & \frac{(-1)^{g-1}}{\pi^2} \sum_{s=0}^{N-1} \Big(\sum_{m>0} e^{-2\pi \mathrm{i} n_2 s/N} c_{g-1}^{(r,s)}(n_1 n_2/N) \mathrm{Li}_{3-3g}(e^{2\pi \mathrm{i} m \cdot y}) \\ & + \frac{1}{2} c_{g-1}^{(0,s)}(0) \zeta(3-2g) \Big) \end{aligned} \tag{5.43}$$

[32] More precisely it is the complex conjugate of the antiholomorphic part.

where ζ is the Riemann zeta function, $y = (T, U)$, $m = (n_1, n_2)$ with $n_1 \in \mathbb{Z}/N, n_2 \in \mathbb{Z}$, $r = n_1 N \bmod N$ and $m > 0$ is defined as

$$\begin{gathered} n_1, n_2 \geq 0 \text{ and } (n_1, n_2) \neq (0, 0), \\ n_1 > 0, n_2 < 0 \text{ and } n_1 |n_2| \leq 1. \end{gathered} \tag{5.44}$$

The coefficients $c_g^{(r,s)}$ are defined through

$$f^{(r,s)}(\tau)\mathcal{P}_{2g}(G_2, G_4, \ldots, G_{2g}) = \sum_{l \in \frac{\mathbb{Z}}{N}} c_{g-1}^{(r,s)}(l, 0) q^l, \tag{5.45}$$

where $G_{2k} = 2\zeta(2k)E_{2k}$, E_{2k} being the Eisenstein series of weight $2k$ and $\mathcal{P}_{2g}$ is related to the Schur polynomial $\mathcal{S}$ of order g by

$$\mathcal{P}_{2g}(x_1, x_2, \ldots, x_g) = -\mathcal{S}(x_1, \frac{1}{2}x_2, \ldots, \frac{1}{g}x_g). \tag{5.46}$$

Although (5.43) was derived under the condition $g > 1$, it may be extrapolated to $g = 1$ and $g = 0$. For $g = 0$ it correspond to 1-loop corrections to the vector-moduli prepotential. More concretely when considering the case without Wilson lines the heterotic vector moduli prepotential is of the form

$$F^{het}(S^{het}, T, U) = S^{het}TU + f^{1-loop}(T, U) + \mathcal{O}(e^{2\pi i S^{het}}) \tag{5.47}$$

where S^{het} is the heterotic dilaton and T, U are the complex structure and the complexified Kähler class as before. Then $f^{1-loop}(T, U) = p(T, U) + \bar{F}_0^{hol}$, with $p(T, U)$ being a cubic polynomial that will not be important for us.

5.4 Type II compactifications on Calabi-Yau 3-folds

In this section we briefly explain the necessary details of type II compactifications on a CY 3-fold.

5.4.1 Spectrum

This section mostly follows the discussion of chapter 14.6 in [95]. Compactifying type II string theory on a CY 3-fold leads to a $\mathcal{N} = 2$ effective theory in 4 dimensions with abelian gauge group [33]. The spectrum can be obtained by dimensional reduction of the massless 10 dimensional spectrum. For type IIA this consists of the (non-chiral) $\mathcal{N} = 2$ gravity multiplet

$$\mathcal{G}_{IIA}(10) = \{G_{MN}, \Psi_m^{(+)}, \Psi_M^{(-)}, \lambda^{(+)}, \lambda^{(-)}, B_{MN}, (C_3)_{MNP}, (C_1)_M, \Phi\}, \tag{5.48}$$

[33] Non abelian gauge symmetries can occur at specific points in moduli space where certain 2-cycles (3-cycles for type IIB) shrink to zero.

where $M, N = 0, \ldots, 9$. The superscripts on the fermions denote their chiralities. The fermions arise from the (NS, R) and (R, NS) sector. The metric G_{MN}, antisymmetric tensor B_{MN} and the dilaton are in the (NS, NS) sector. The remaining bosonic field, i.e, the vector C_1 and the 3-form C_3 come from the (R, R) sector.
Upon dimensional reduction these fields will arrange themselves in the following 4 dimensional $\mathcal{N} = 2$ multiplets:

- Gravity multiplet consisting of a graviton $g_{\mu\nu}$, a gauge boson (graviphoton) C_μ and two Majorana gravitinos $\psi_\mu^{(+)}, \psi_\mu^{(-)}$ of opposite chiralities.
- Vector multiplet consisting of gauge field A_μ, two Weyl fermions λ, ψ, and scalar ϕ all in the adjoint representation of the gauge group.
- Hypermultiplet consisting of two Weyl fermions $\psi_q, \tilde{\psi}_{\tilde{q}}^\dagger$ and two complex bosons $q, \tilde{q}^\dagger$. In $\mathcal{N} = 1$ terms (ψ_q, q) and $(\tilde{\psi}_{\tilde{q}}^\dagger, \tilde{q}^\dagger)$ make up a chiral and antichiral multiplet which are in conjugate representations.

We split the ten dimensional index in the following ($SU(3)$ covariant way) $M = (\mu, i, \bar{i})$ and denote the fluctuations of the metric around its background value by $g_{\mu\nu}$. Compactifying type IIA string theory on a CY with hodge numbers $h^{1,1}$ and $h^{2,1}$ contains one gravity multiplet, with the graviphoton originating from the ten dimensional R-R gauge field C_1. Further it contains $h^{1,1}$ vector multiplets. The complex scalars of the vector multiplets arise from metric $g_{i\bar{j}}$ and B-field $B_{i\bar{j}}$ with two internal indices. These scalars correspond to complexified Kähler moduli of the CY. The corresponding $U(1)$ gauge fields come from the R-R 3-form gauge potential $(C_3)_{\mu i\bar{j}}$. So the abelian gauge symmetry is given by $U(1)^{h^{1,1}+1}$ (which includes the graviphoton). Finally there are $h^{2,1} + 1$ hypermultiplets. The scalars of $h^{2,1}$ of these multiplets arise from metric with two internal indices g_{ij} and R-R 3-form with 3 internal indices $(C_3)_{ij\bar{k}}$. These scalars correspond to complex structure deformations. The scalars of the final multiplet arise from the R-R 3-form with three internal indices $(C_3)_{ijk}$ and the dilaton.
For type IIB compactifications the number of vector and hyper multiplets is exchanged, i.e., $N_v = h^{2,1} + 1$, $N_h = h^{1,1} + 1$. Type IIA and type IIB compactifications are related through *mirror symmetry*, i.e., a type IIA compactification on a Calabi-Yau with Hodge numbers $h^{1,1}, h^{2,1}$ is dual to a type IIB compactification on an appropriate mirror Calabi-Yau with hodge numbers $\tilde{h}^{1,1} = h^{2,1}, \tilde{h}^{2,1} = h^{1,1}$.
We end with some comments on the structure of the moduli space of the theory. Due to the $\mathcal{N} = 2$ supersymmetry no mixing between the kinetic terms of vector - and hypermultiplet scalars can occur, hence the moduli space $\mathcal{M}$ locally factorizes into the product of the two, i.e., $\mathcal{M} = \mathcal{M}_{vec} \times \mathcal{M}_{hyp}$ [34]. In particular $\mathcal{M}_{vec}$ is a (special)

[34] If we also take non-perturbative effects, i.e., D-branes, into account non-abelian gauge-symmetries may appear and some of the scalars will be charged w.r.t. to these gauge groups.

Kähler manifold and $\mathcal{M}_{hyp}$ is a quaternionic manifold due to the $\mathcal{N}=2$ supersymmetry. For type IIA and IIB compactifications the dilaton is part of a hypermultiplet which hence receives stringy corrections (perturbative and non-perturbative). The metric of vector multiplet moduli space is exact at string tree level. For type IIA the vector moduli correspond to Kähler moduli and the metric on the Kähler moduli space receives σ-model corrections at order $(\alpha'/L^2)^3$ and non-perturbative corrections (proportional to powers of $e^{-L^2/\alpha'}$) due to world-sheet instantons [35]. For the type IIB theory the vector moduli correspond to complex structure moduli and the metric on the complex structure moduli space is exact at both string tree level and σ- model tree level. We will have more to say on how to compute the prepotential of the vector moduli in the next subsection.

5.4.2 Gravitational couplings and the prepotential

The quantities we will want to compare to match heterotic and type II compactifications are couplings F_g given in (5.41), more precisely their holomorphic part defined in (5.43). As has been shown in [107] for the type II side they correspond to the genus g free energies of the topological string compactified on the appropriate dual CY M which we will denote by F_g^{GV}. We define the generating function F^{GV} for these free energies as

$$F^{GV}(g_s, y) = \sum_{g=0}^{\infty} F_g^{GV}(y) g_s^{2g-2} \tag{5.49}$$

where g_s is the string coupling and y stands for the vector multiplet moduli as before. As shown in [110, 111] it can be written in the following form

$$F^{GV}(g_s, y) = \sum_{g=0}^{\infty} \sum_{m>0} \sum_{d=1}^{\infty} n_m^g \frac{1}{d} \left(2 \sin \frac{d g_s}{2} \right)^{2g-2} e^{2\pi i d (m \cdot y)} \tag{5.50}$$

where $m = (n_1, \ldots, n_{h^{1,1}})$ labels the 2-cycles [36] and the constant n_m^g are the genus g Gopakumar-Vafa invariants. These invariants count in an exact way the number of BPS states in Calabi-Yau compactification of the type IIA theory. They are closely related to Gromov-Witten invariants, see, e.g., [112] and we will often use the Gromov-Witten and Gopakumar-Vafa invariants interchangeably although they are not strictly the same. Roughly speaking genus g Gromov-Witten invariants count the number of 'distinct ways' maps $x : \Sigma_g \to X$ from the genus g world-sheet Σ_g to the CY X. Hence they are related to the counting of curves in X. However due to multicovering (mapping Σ_g to a curve multiple times) and 'bubbling effects' (the

This will lead to potentials for those fields and to an interaction of the scalars from the two types of multiplets. The kinetic terms however still keep the product structure.

[35] Topological non-trivial embeddings of the world-sheet into the CY manifold.

[36] If we choose a basis $\{\Sigma_i\}_{i=1,\ldots,h^{1,1}}$ then a 2-cycle β can be uniquely written as $\beta = \sum_i n_i \Sigma_i$, $(n_1, \ldots, n_{h^{1,1}}) \neq (0, \ldots, 0)$.

fact that by gluing an arbitrary small handle to a genus g worldsheet maps from Σ_g to X will also contribute to the maps from Σ_{g+1} to X) Gromov-Witten invariants are in general not integers. Gopakumar-Vafa invariants in a way take care of the above effects by viewing the worldsheet as a submanifold of the Calabi-Yau (rather than a map embedding the worldsheet). Gopakumar-Vafa invariants are always integral. From the knowledge of all Gopakumar-Vafa invariants (for all genera) one can obtain all Gromov-Witten invariants and vice versa.

From (5.49), (5.50) we find the following explicit expression for F_g^{GV},

$$\begin{aligned} F_g^{GV} &= \frac{(-1)^g|B_{2g}B_{2g-2}|\chi(M)}{4g(2g-2)(2g-2)!} \\ &+ \sum_{m>0}\left[\frac{|B_{2g}|n_m^0}{2g(2g-2)!} + \frac{2(-1)^g n_m^2}{(2g-2)!} \pm \cdots - \frac{g-2}{12}n_m^{g-1} + n_m^g\right]\mathrm{Li}_{3-2g}(e^{2\pi \mathrm{i} m\cdot y}) \end{aligned} \tag{5.51}$$

where B_{2g} are the Bernoulli numbers, $\chi(M)$ is the Euler characteristic of the Calabi-Yau and the polylogarithm Li_k is defined as $\mathrm{Li}_k(x) = \sum_{i=1}^k \frac{x^n}{n^k}$.
For $g = 0$ we obtain the instanton corrections to the vector moduli prepotential, which takes the form

$$F_0^{GV} = \zeta(3)\frac{\chi(M)}{2} + \sum_{m>0} n_m^0 \mathrm{Li}_3(e^{2\pi \mathrm{i} m\cdot y}). \tag{5.52}$$

For $g = 1$ we get

$$F_1^{GV} = \sum_{m>0}\left(\frac{1}{12}n_m^0 + n_m^1\right)\mathrm{Li}_1(e^{2\pi \mathrm{i} m\cdot y}). \tag{5.53}$$

Now we want to compare this to the heterotic result (5.41). This was obtained in the weak coupling limit, $S^{het} \to \mathrm{i}\infty$ and with no Wilson lines. On the type II side this corresponds to setting $h^{1,1} - 3$ moduli (or potentially combinations thereof) to zero and sending one combination of moduli to $\mathrm{i}\infty$. Comparing constant terms one finds [70]

$$F_g^{GV} = \frac{(-1)^{g+1}}{2(2\pi)^{2g-2}}\bar{F}_g^{hol}. \tag{5.54}$$

Further one obtains the following relations between the lowest genus Gopakumar-

Vafa invariants that are related to the heterotic side through

$$\begin{aligned} n^0_{(n_1,n_2)} &= 2\sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c^{(r,s)}_{-1}, \\ n^1_{(n_1,n_2)} &= \frac{1}{2\pi^2}\sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c^{(r,s)}_{0}(m^2/2) - \frac{1}{12} n^0_{(n_1,n_2)}, \\ n^1_{(n_1,n_2)} &= \frac{1}{8\pi^2}\sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c^{(r,s)}_{1}(m^2/2) - \frac{B_4}{8} n^0_{(n_1,n_2)}, \end{aligned} \tag{5.55}$$

where $r = n_1 N \bmod N$ and $m^2 = 2n_1n_2$.

5.5 Calculating the Gromov-Witten/Gopakumar-Vafa invariants

In this section we will discuss how one can obtain the Gromov-Witten/Gopakumar Vafa invariants n^g_m showing up in the expression for the gravitational couplings on the type II side , (5.51). This can then be used to find the potentially dual CY in the heterotic-type II duality.

5.5.1 Genus zero Gromov-Witten invariants - the prepotential

The vector moduli prepotential for a type IIA compactification on a CY X in general takes the form

$$F^{IIA} = -\frac{1}{6}\kappa_{ABC}y^Ay^By^C - \frac{\chi(X)\zeta(3)}{2(2\pi)^3} + \frac{1}{(2\pi)^3}\sum_{m>0} n^0_m \mathrm{Li}_3(e^{2\pi m\cdot y}) \tag{5.56}$$

where y^A label the vector moduli, κ_{ABC} are the triple intersection numbers of the CY, as before $m = (n_1, \ldots, n_{h^{1,1}})$ labels the 2-cycles of X and n^0_m labels the Gromov-Witten invariants of multi-degree $m = (n_1, \ldots, n_{h^{1,1}})$. The prepotential (and hence the Gromov-Witten invariants) may be calculated by using mirror symmetry. That is one makes use of the fact the type IIB string theory compactified on the mirror CY, X^*, has a vector moduli prepotential that is exact at tree level, so determined by the triple intersection numbers of the dual CY X^*. This prepotential can then be mapped to the type IIA side with the help of the mirror map, that maps the IIB moduli to the respective IIA moduli.

For CY's that have a description as hypersurface or as complete intersection in an toric ambient space this has been implemented in the Mathematica package *instanton.m* [113]. It takes the generators of the Morri cone of X^* as input. Roughly speaking starting at the large complex structure point, which can be defined by the Morri cone of X^* one may calculate a particular period of the holomorphic 3-from, the so called fundamental period. From this one derives the Piccard-Fuchs system of linear differential equations which all the periods of the holomorphic three from have to fulfil. This then allows one to calculate the triple intersection numbers of X^* as well as the mirror map.

5.5.2 Conformal bootstrap

In [114] a different method was used to obtain all genus Gopakumar-Vafa invariants. As we will discuss in more detail in section 5.6, in the unorbifolded version of the heterotic - type II duality, the Calabi-Yau manifolds that appear on the type II side are fibrations of elliptic $K3$ surfaces over $\mathbb{P}^1$, which makes it possible to obtain all-genus results for the topological string [115], at least for a certain set of curve classes. More generally in [116] a special class of elliptically fibered CY 3-folds was considered, in particular where all fibers are irreducible and no fibral divisors are present. Then it was argued that the expansion coefficients $Z_\beta(\tau,\lambda)$ in an expansion of the topological string partition function $Z_{top.}$ of the form

$$Z_{top.}(\tau,\underline{t},\lambda) = Z_0(\tau,\lambda)\left(1 + \sum_{\beta\in H^{1,1}(B,\mathbb{Z})} Z_\beta(\tau,\lambda)Q^\beta\right) \tag{5.57}$$

are Jacobi forms of weight zero, where the sum is over 2-cycles in the base. The elliptic argument τ is related to the complexified volume of the fiber and the string coupling constant, which we here denote by λ, appears as elliptic argument. $Q^\beta = \exp(2\pi\mathrm{i}\sum_i \beta^i t_i)$, where $t_i, i = 1,\ldots,h^{1,1}(B)$ are shifted volumes of curves in the base. The the modular transformations acting on these arguments and under which $Z_{top.}$ is invariant, arise as part of the symplectic monodromy group acting on the integral symplectic basis of periods of the mirror of the elliptic fibered CY 3-fold. The integral BPS expansion of $Z_{top.}$ imposes a pole structure on Z_β from which it can be argued that its numerator is a weak Jacobi form and its denominator is unique.

In [114] it was argued that the CYs appearing on the type II side in the CHL orbifolded version of the heterotic-type II duality have to be genus one fibrations with N-sections, where N is the order of the CHL orbifold, for some more details see subsection 5.6. Then the arguments given above for the expansion coefficients of $Z_{top.}$ can be generalized to this situation, where now also parameters corresponding to volumes of fibral curves are included, see (E.2). One finds that for genus one fibrations with N-sections and $N \leq 4$ the $SL(2,\mathbb{Z})$ group is broken to $\Gamma_1(N)$ and the complexified Kähler parameters that correspond to the fibral curves and which we will collectively denote by $\underline{m}$ become elliptic parameters of $Z_\beta(\tau,\lambda,\underline{m})$. $Z_\beta(\tau,\lambda,\underline{m})$ are shown to be meromorphic higher degree Jacobi forms of $\Gamma_1(N)$ with additional elliptic parameters, where the numerators now are weak Jacobi forms under $\Gamma_1(N)$. We provide more details in Appendix E.

Specializing the ansatz (E.6) from the Appendix E to the examples of genus one fibered CY 3-folds with 2-sections (listed in table 9) that are dual to CHL orbifolds of order 2 of the heterotic string one finds for base degree one [114]

$$Z_1(\tau,\lambda) = \frac{\Delta_2(\tau)}{\eta(2\tau)^{24}\phi_{-2,1}(2\tau,\lambda)} \cdot \mathcal{E}_2\Big[4(6\hat{b}-5)\cdot\mathcal{E}_2^2 + 2(2-3\hat{b})\cdot E_4\Big]. \tag{5.58}$$

This can be checked against predictions from (5.55) using (5.35), (5.45) and $\hat{b} = \frac{2}{3}, \frac{4}{9}$ and one finds a perfect match.
For base degree 2 the conformal bootstrap gives the result [114]

$$Z_2(\tau,\lambda) = \left(\frac{\Delta_4(\tau)}{\eta(2\tau)^{24}} \frac{\phi_2(\tau,\lambda)}{\phi_{-2,1}(2\tau,\lambda)\phi_{-2,1}(4\tau,\lambda)}\right) \tag{5.59}$$

where ϕ_2 needs to be fixed [37] by comparing to the predictions of the heterotic side (again using (5.35), (5.45)). Once ϕ_2 has been fixed it again leads to a perfect match.

Next we turn to CY 3-folds with 3-sections, listed in table 10, that are potentially dual to CHL orbifolds of order 3 of the heterotic string on $K3\times T_2$. The Gopakumar-Vafa invariants at genus zero can be calculated using the standard techniques [117]. One finds that at degree zero w.r.t. the base of the $K3$ fibration the genus zero invariants only depend on the Euler characteristic and we list them in table 7, where d_b, d_f correspond to the degreed of the $\mathbb{P}^1$ base and the genus one fiber of the $K3$ respectively. More concretely

$$d_F = E_0\cdot\beta, d_B = \pi^{-1}(B)\cdot\beta \tag{5.60}$$

where E_0 is the divisor associated to the three section. Indeed table 7 matches with the predictions from (5.38),(5.39) and (5.45) if one identifies $d_F = n_1$, $d_B = n_1 + n_2$. In order to calculate the genus one Gopakumar-Vafa invariants one again makes use

$d_B\backslash d_F$	0	1	2	3	4
0	0	$\frac{\chi}{2}+240$	$\frac{\chi}{2}+240$	$-\chi$	$\frac{\chi}{2}+240$
1	$\frac{\chi}{2}+240$	$1962-3\chi$	$\frac{10\chi}{3}+18016$	$\frac{15\chi}{2}+95454$	$413280-30\chi$
2	0	$\frac{\chi}{2}+240$	$\frac{10\chi}{3}+18016$	$413280-30\chi$	$54\chi+5694624$
3	0	0	$-\chi$	$\frac{15\chi}{2}+95454$	$54\chi+5694624$
4	0	0	0	$\frac{\chi}{2}+240$	$413280-30\chi$
5	0	0	0	0	

Table 7: Genus zero Gopakumar-Vafa invariants of degree zero with respect to the base of the $K3$ fibration for the families of Calabi-Yau manifolds $M^{(3)}_{-1,\nu}$ and $M^{(3)}_{2,\nu}$ of table 10.

[37] We refer to (5.30), (5.31) in [114] for the explicit expression.

$d_F \backslash g$	0	1	2	3	4	5
0	$\frac{\chi}{6}+26$	0	0	0	0	0
1	$\frac{\chi}{2}+240$	0	0	0	0	0
2	$1962-3\chi$	0	0	0	0	0
3	$\frac{10\chi}{3}+18016$	$-\frac{\chi}{3}-52$	0	0	0	0
4	$\frac{15\chi}{2}+95454$	$-\chi-480$	0	0	0	0
5	$413280-30\chi$	$6\chi-3924$	0	0	0	0
6	$\frac{88\chi}{3}+1627330$	$-\frac{23\chi}{3}-36188$	$\frac{\chi}{2}+78$	0	0	0
7	$54\chi+5694624$	$-18\chi-192348$	$\frac{3\chi}{2}+720$	0	0	0
8	$18353988-195\chi$	$78\chi-838332$	$5886-9\chi$	0	0	0
9	$170\chi+55646304$	$-80\chi-3362964$	$\frac{38\chi}{3}+54464$	$-\frac{2\chi}{3}-104$	0	0
10	$291\chi+159217686$	$-157\chi-11963892$	$\frac{61\chi}{2}+290202$	$-2\chi-960$	0	0

Table 8: Gopakumar-Vafa invariants of degree zero with respect to the base of the $K3$ fibration and degree one with respect to the base of the $K3$ fiber for the families of Calabi-Yau manifolds $M^{(3)}_{-1,\nu}$ and $M^{(3)}_{2,\nu}$.

of the conformal bootstrap. Using genus zero Gopakumar-Vafa invariants to fix the ansatz one finds

$$Z_1(\tau,\lambda) = \frac{1}{48}\frac{\Delta_6(\tau)}{\eta(3\tau)^{24}\phi_{-2,1}(3\tau,\lambda)}[(120+\chi)E_3(\tau) - 9(152+\chi)\mathcal{E}_3(\tau)^2]. \tag{5.61}$$

One finds that this matches the results calculated from the heterotic side (using (5.38),(5.39),(5.45))).

5.6 Heterotic - Type II duality - finding the dual CY

In this section we will show how one may use the information gathered in the previous sections to find the dual CY manifolds corresponding to certain CHL orbifolds of the $E_8 \times E_8$ heterotic compactification on $K3 \times T^2$. We start by briefly reviewing some facts about this duality without the CHL orbifold, see, e.g, [118, 119]. The duality between the $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$ was first found in [90]. Assuming that the heterotic theory is weakly coupled in the geometric regime of the type II theory it can be shown that the CY involved in this duality always need to be an elliptic $K3$ surfaces fibered over $\mathbb{P}^1$ [120]. This may at least be motivated by using an adiabatic extension argument [121] in the following way: Starting with the well known duality between the $E_8 \times E_8$ heterotic string compactified on T^4 and type IIA compactified on $K3$ in six dimensions [122, 123], one may fiber both sides of this duality over a $\mathbb{P}^1$. The resulting geometry on the type II theory is a CY that is $K3$ fibered over $\mathbb{P}^1$. On the heterotic side one notices that in order to obtain the correct amount of unbroken spacetime supersymmetry only a T^2 inside

T^4 needs to be non-trivially fibered. The T^2 fibered over $\mathbb{P}^1$, makes up a $K3$ and the extra T^2 is trivially fibered. Under the duality the heterotic dilaton is mapped to the complexified Kähler modulus of the $\mathbb{P}^1$ that is the base of the fibration and the weak heterotic coupling limit $S^{het} \to i\infty$ corresponds to the limit of a large base. The complex structure and the complexified volume correspond to linear combinations of the complexified volume of the elliptic fiber and the $\mathbb{P}^1$ base of K_3.

In particular the maximally Higgsed heterotic theory with $(12+n, 12-n)$ instantons is dual to type II A compactified on an elliptic fibration over an Hirzebruch surface $\mathbb{F}_n$, which is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, see [90, 124] for examples and [125–127] for a treatment in F-theory. The heterotic moduli S, T, U corresponding to the dilaton and the Kähler and complex structure of T^2, are mapped to the moduli controlling the size of the two $\mathbb{P}^1$s and a modulus of the elliptic fibration.
One can observe that the unorbifolded heterotic theory has a T-duality group of the form

$$\Gamma_{het} = SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) \times \mathbb{Z}_2. \tag{5.62}$$

The first $SL(2,\mathbb{Z})$ factor correspond to the modular symmetry of the complex structure modulus of T^2. The $\mathbb{Z}_2$ symmetry comes from T-duality along one of the circles of T^2 which exchanges the complex and Kähler moduli. Combining these two symmetries then gives rise to the second $SL(2,\mathbb{Z})$ factor. On the type II side the $SL(2,\mathbb{Z})$ action arises through monodromies in the moduli space of the elliptic fibration and another monodromy creates the $\mathbb{Z}_2$ symmetry that exchanges T and U [128].
As we have discussed in subsection 5.2 the CHL orbifold of order N acts on T^2 by a $1/N$ shift along one of the circles of T^2. The group acting on the modulus of T^2 that preserves the $\mathbb{Z}_N$ action together with its generator is $\Gamma_1(N)$, which we defined in (2.5). Along the cycle not involved in the orbifold we can still perform a T-duality transformation that exchanges T and U. From this one infers that the symmetry group of the CHL orbifolded theory has the form

$$\Gamma_{CHL} = \Gamma_1(N) \times \Gamma_1(N) \times \mathbb{Z}_2. \tag{5.63}$$

On the type II side this symmetries should arise through monodromies in the quantum Kähler moduli space of the CY. This then implies that the CY manifolds appearing in the duality need to be genus 1-fibered with N-sections.
Now we turn to finding the possible dual CY manifolds. In particular this will be possible for heterotic models which can be completely Higgsed so that they only remaining vector fields are the ones coming from the compactification of the metric and the B-field on T^2 giving $N_v = 4$. For order 2 CHL orbifolds we have seen that the only models that can be completely Higgsed are the ones in 4. In [31] possible CY duals were found by searching the Kreuzer-Skarke CY-database [81] for CY manifold that have the correct hodge numbers $h^{1,1} = N_v - 1 = 3$ and $h^{2,1} = N_h - 1 = 83, 115$.

q_1	q_2	$h^{1,1}$	$h^{2,1}$	$\hat{b}_1$	$\hat{b}_2$
4	4	4	148	-	-
4	3	3	131	1	-
4	2	3	115	$\frac{8}{9}$	$\frac{8}{9}$
4	1	3	99	$\frac{7}{9}$	-
4	0	3	83	$\frac{2}{3}$	$\frac{2}{3}$

q_1	q_2	$h^{1,1}$	$h^{2,1}$	$\hat{b}_1$	$\hat{b}_2$
3	3	3	123	-	-
3	2	3	115	-	$\frac{8}{9}$
3	1	3	107	-	-
2	2	3	115	$\frac{8}{9}$	$\frac{8}{9}$

Table 9: Values of 'charges' q_1, q_2 for which the construction of [114] yields a toric variety that is a $\mathbb{P}_{112}$ fibration over $\mathbb{P}^1 \times \mathbb{P}^1$ and s.t. the generic Calabi-Yau has $h^{1,1} = 3$ (the special case $h^{1,1} = 4$ corresponds to an elliptic fibration with two independent sections). $h^{1,1}, h^{2,1}$ are the Hodge numbers of the resulting CY. $\hat{b}_1, \hat{b}_2$ are given by formula (5.64).

For all such candidate CYs the genus zero Gromov-Witten invariants were computed using the Mathematica package *instanton.m* [113] and compared with the predictions from the heterotic side (using (5.35), (5.45)). Using this method two candidate CY duals were found.

In [114] CY manifolds that are genus one fibered over $\mathbb{F}_1 = \mathbb{P}^1 \times \mathbb{P}^1$ with 2-sections were constructed systematically by a 'fiber based approach'. That is one first constructs the genus one fiber/elliptic curve as a hypersurface in a toric variety. Then the coefficients of the hypersurface equation are lifted to be sections of line bundles over some base, which in our case will be $\mathbb{F}_1$. This leads to all possible dual CYs for the cases that can (potentially) be completely Higgsed, i.e., instanton numbers $(12+n, 12-n)$, $n = 0, 1, 2$ [38]. In this way matching CYs for all such cases from [92] were found. In particular the construction made use of the fact that a genus one fibration with 2-sections can always be mapped into a fibration of degree 4 hypersurface in $\mathbb{P}_{112}$ [129]. The different solutions can be parametrized by 'charges' (q_1, q_2) which we list in table 9, together with the hodge numbers and the value of $\hat{b}$ that matches the result on the heterotic side, i.e., table 2 above. The value of $\hat{b}$ is given by

$$\hat{b} = \frac{1}{144}[160 - 16(q_1 + q_2) + 8q_1 q_2]. \tag{5.64}$$

The index of $\hat{b}$ (i.e. b_1, b_2) in table 9 corresponds to the heterotic string arising from a 5-brane that wraps the restriction of the genus one fibration to either of the $\mathbb{P}_1$'s of the base ($\mathbb{F}_1$). The single special case where $h^{1,1} = 4$ was found to correspond to an elliptic fibration with two independent sections. In particular the two CYs found in [31] also are part of the so constructed CYs.

In [38] the CYs dual to CHL orbifolds of order 3 of the heterotic string on $K3 \times T^2$

[38]The heterotic strings arises in the type IIA picture as a 5-brane wrapping a $K3$ fiber. The different geometries that possibly arise from using $\mathbb{F}_0, \mathbb{F}_2$ as base lead to the same heterotic string as argued in [114]

that are completely Higgsable were constructed by methods similar to [114]. By our discussion above we know that the dual CYs should all be genus one fibered CY 3-folds with 3-sections and $h^{1,1} = 3$ and in addition also exhibit a $K3$ fibration. We start by noting that every genus one fibration that exhibits 3-sections is birational to a hypersurface in a fibration of weighted projective space where the base is $\mathbb{P}^2$ [129]. In order for a $K3$ fibration to exist the base of the fibration needs to be a Hirzebruch surface $\mathbb{F}_n$, $n = 0, 1, 2$, so that one can restrict the construction to CY 3-folds that are hypersurfaces in a four-dimensional toric ambient space. Since the base of the $K3$ fibration has to be $\mathbb{P}^1$ the fibration has to arise from a compatible toric fibration of the ambient space. The dual CYs were then constructed by lifting $K3$ surfaces that are genus one fibered with 3 sections and have Picard number 2. Scanning all 4319 3-dimensional reflexive polytopes classified in [130] one finds that there exist 3 such $K3$ surfaces corresponding to the reflexive polytopes that are the convex hull $\Delta_n^{(3)}$ of the points

$$\begin{matrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^4 \\ \nu^5 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \\ n & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5.65}$$

where $n \in \{-1, 0, 1\}$. These can be lifted to a 4-dimensional polytope by adding the points

$$\nu'_1 = (\nu, 1)\,, \nu'_2 = (0, -1) \tag{5.66}$$

which will be reflexive when $\nu \in 2\Delta_2^{(3)}$. This way one finds 14 genus one fibered CY 3-folds that have 3-sections which are listed in table 10. Here we have also listed the Euler number and indicated the two models that match the completely Higgsable cases of table 5 by their shift vectors. For these cases one finds that the Gopakumar-Vafa invariants of various genus match (see subsection 5.5.2).

6 Conclusion and Outlook

Moonshine is fascinating subject for mathematicians and physicists alike. Since its start with McKays observation in 1978 a lot of new insight has been gained but still many questions remain open.

In this thesis I have tried to contribute to the understanding of moonshine by finding new moonshine phenomena. In section 4, based on my publication [37], I looked for moonshine phenomena connecting higher dimensional Calabi-Yau manifolds and sporadic groups. This was done by analysing their elliptic genus, which is the index that establishes the moonshine connection for $K3$ [10]. In particular

Polytope	n	ν	χ	Base B	Candidate shift
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ n & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ & \nu & & -1 \end{pmatrix}$	2	$(0,0,-1)$	-168	$\mathbb{F}_1$	
	2	$(1,0,0)$		$\mathbb{F}_0$	
	2	$(0,1,-1)$	-156	$\mathbb{F}_1$	$\frac{1}{3}(2,1^2,0^5;2,1^4,0^3)$
	-1	$(0,1,-1)$		$\mathbb{F}_1$	
	-1	$(0,2,0)$		$\mathbb{F}_0$	
	-1	$(-1,0,0)$		$\mathbb{F}_0$	
	-1	$(0,0,-1)$	-150	$\mathbb{F}_1$	
	2	$(0,0,0)$	-144	$\mathbb{F}_0$	
	-1	$(0,0,0)$		$\mathbb{F}_0$	
	-1	$(1,0,-1)$	-138	$\mathbb{F}_1$	
	2	$(0,1,0)$	-132	$\mathbb{F}_0$	
	-1	$(1,0,0)$		$\mathbb{F}_0$	
	2	$(-2,-2,0)$	-120	$\mathbb{F}_0$	$\frac{1}{3}(2,0^7;2,0^7)$
	-1	$(2,0,0)$		$\mathbb{F}_0$	

Table 10: The 14 genus one fibered Calabi-Yau threefolds $M^{(3)}_{n,\nu}$ that have 3-sections and also exhibit a compatible $K3$ fibration such that the polarization lattice is of rank two with anti-diagonal intersection form.

the twining elliptic genera of a large number (13 642) of CY 5-folds was calculated and analysed. However in all those cases no new moonshine could be found, thus strengthening the special role played by $K3$ in Mathieu moonshine. Also the elliptic genera of certain toroidal orbifolds an Gepner models was studied but also here no new moonshine phenomena could be established.

In section 5, based on my publications [31, 38], I tried to deepen the connection between $E_8 \times E_8$ heterotic string compactifications on $K3 \times T^2$ and type II compactifications on CY 3-folds, which was shown to link Mathieu moonshine to topological invariants (Gromov-Witten/Gopakumar-Vafa invariants) of the CY manifolds [30]. This was done by studying the CHL orbifolds of order 2 and 3 of the heterotic string given in [92] and [38] and finding their potentially dual CY-manifolds. Thereby one also connects the twining and twisted twining genera of $K3$ to Gromov-Witten invariants. In some cases we were able to find the dual CY manifolds and, based on the methods of [114], to also give an explicit construction for these manifolds in certain cases. Thereby also our understanding the of heterotic-type II string duality is deepened.

Going forward there are still many open problems one might address. Most imminently one might try to find the dual CY manifolds to CHL orbifolds of higher order and in general try to understand how CY manifolds 'know' about moonshine. Since $K3$ and its elliptic genus are present in many string constructions, e.g., for type II compactifications on $K3 \times T^2$ it gets connected to the microstate counting of back holes, it seems interesting also to look for new connections in such situations.

In general modular objects show up in many areas of physics and one might expect connections to moonshine to arise in some of these situations leading to potentially many further interesting connections.

A Definition of some modular objects - conventions

In this appendix we will collect the definitions of specific modular objects that are used in this work. Throughout we define $q := e^{2\pi i\tau}, y = e^{2\pi iz}$.

A.1 Eisenstein series and η-function

We start by defining the Eisenstein series E_4, E_6, the generators of modular forms on $SL(2,\mathbb{Z})$,

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1 + 240q + 2160q^2 + \dots,$$
$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1 - 504q - 16632q^2 + \dots$$

which are modular forms of weight 4 and 6 respectively.
We further define the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n). \tag{A.1}$$

It transforms under modular transformations as

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau) \tag{A.2}$$
$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau), \tag{A.3}$$

hence it is a modular form of weight $\frac{1}{2}$ with a multiplier system.
We also define the following functions

$$\Delta_4(\tau) = \frac{\eta(2\tau)^{16}}{\eta(\tau)^8}, \tag{A.4}$$
$$\Delta_6(\tau) = \frac{\eta(3\tau)^{18}}{\eta(\tau)^6}, \tag{A.5}$$
$$\Delta_8(\tau) = \frac{\eta(2\tau)^8\eta(4\tau)^{16}}{\eta(\tau)^8}. \tag{A.6}$$

Δ_{2k}, $k = 2,3,4$ is a modular form of weight $2k$ for $\Gamma_1(k)$.

A.2 Jacobi theta functions

The generalized Jacobi theta function can be written as

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(\tau, z) = \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}\left(k+\frac{a}{2}\right)^2} e^{\pi \mathrm{i}\left(k+\frac{a}{2}\right)b} e^{(2\pi iz)\left(k+\frac{a}{2}\right)}, \tag{A.7}$$

and one defines

$$\begin{aligned} \theta_1(\tau,z) =& \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\tau,z)\,, \quad \theta_2(\tau,z) = \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau,z)\,, \\ \theta_3(\tau,z) =& \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau,z)\,, \quad \theta_4(\tau,z) = \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau,z)\,. \end{aligned} \tag{A.8}$$

More explicitly one finds the following expression and product representations:

$$\begin{aligned} \theta_1(\tau,z) &= -i \sum_{n+\frac{1}{2}\in\mathbb{Z}} (-1)^{n-\frac{1}{2}} y^n q^{\frac{n^2}{2}} \\ &= -iq^{\frac{1}{8}} \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1-q^n)(1-yq^n)(1-y^{-1}q^n), \end{aligned} \tag{A.9}$$

$$\begin{aligned} \theta_2(\tau,z) &= \sum_{n+\frac{1}{2}\in\mathbb{Z}} y^n q^{\frac{n^2}{2}} \\ &= q^{\frac{1}{8}} \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1-q^n)(1+yq^n)(1+y^{-1}q^n), \end{aligned} \tag{A.10}$$

$$\begin{aligned} \theta_3(\tau,z) &= \sum_{n\in\mathbb{Z}} y^n q^{\frac{n^2}{2}} \\ &= \prod_{n=1}^{\infty} (1-q^n) \left(1+yq^{n-\frac{1}{2}}\right) \left(1+y^{-1}q^{n-\frac{1}{2}}\right), \end{aligned} \tag{A.11}$$

$$\begin{aligned} \theta_4(\tau,z) &= \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{\frac{n^2}{2}} \\ &= \prod_{n=1}^{\infty} (1-q^n) \left(1-yq^{n-\frac{1}{2}}\right) \left(1-y^{-1}q^{n-\frac{1}{2}}\right). \end{aligned} \tag{A.12}$$

Setting $z = 0, (y = 1)$ in above definitions we obtain the 'truncated' Jacobi theta functions $\theta_i(\tau) := \theta_i(\tau, z = 0)$, $i = 1, \ldots, 4$. From (A.9) we immediately see $\theta_1(\tau) = 0$.

A.3 Weak Jacobi forms

We further define the following weak Jacobi forms $\phi_{k,m}$ of weight k and index m

$$\begin{aligned}\phi_{0,1}(\tau,z) &= 4\left(\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2+\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right) \qquad \text{(A.13)}\\ &= \frac{1}{y}+10+y+\mathcal{O}(q)\\ \phi_{-2,1}(\tau,z) &= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \qquad \text{(A.14)}\\ &= -\frac{1}{y}+2-y+\mathcal{O}(q),\\ \phi_{0,\frac{3}{2}}(\tau,z) &= 2\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)} \qquad \text{(A.15)}\\ &= \frac{1}{\sqrt{y}}+\sqrt{y}+\mathcal{O}(q).\end{aligned}$$

B Some notes on lattices

We collect some basic definitions and facts about lattices following [95].
Give an n-dimensional vector space $V =< \mathbf{e}_1,\dots,\mathbf{e}_n >$, spanned by the basis vectors $\mathbf{e}_i, i=1,\dots,n$ we define a *lattice* Λ to be the set of points.

$$\Lambda = \left\{\sum_{i=1}^{n} n_i\mathbf{e}_i | n_i\in\mathbb{Z}\right\}. \qquad \text{(B.1)}$$

We will consider the cases where V is either $\mathbb{R}^n$ with Euclidean inner product or $\mathbb{R}^{p,q}, p+q=n$ with Lorentzian inner product, i.e., $\mathbf{v}\cdot\mathbf{w}=\sum_{i=1}^{p} v^I\cdot w^I-\sum_{i=p+1}^{p+q} v^I\cdot w^I$ for $\mathbf{v},\mathbf{w}\in\Lambda$. The elements of Λ have expansions $\lambda\ni v=n_i e_i^I$. Hence $g_{ij}=\mathbf{e}_i\cdot\mathbf{e}_j$ is the metric on Λ. The volume of the unit cell, containing one lattice point is given by $vol(\Lambda)=\sqrt{|\det g|}$. We further define the *dual lattice* Λ^*

$$\Lambda^* = \{\mathbf{w}\in V|\mathbf{w}\cdot\mathbf{v}\in\mathbb{Z},\forall\mathbf{v}\in\Lambda\}. \qquad \text{(B.2)}$$

The corresponding dual basis vectors then fulfil

$$\mathbf{e}_i^*\cdot\mathbf{e}_j=\delta_{ij} \qquad \text{(B.3)}$$

and the dual metric $g_{ij}^*=\mathbf{e}_i\cdot\mathbf{e}_j$ is the inverse of g_{ij}. Thereby $vol(\Lambda^*)=(vol(\Lambda))^{-1}$. A lattice is called *unimodular* if $vol(\Lambda)=1$. It is called *integral* if $\mathbf{v}\cdot\mathbf{w}\in\mathbb{Z}$, for all $\mathbf{v},\mathbf{w}\in\Lambda$ which is equivalent to $\Lambda\subset\Lambda^*$. An integral lattice is called *even* if all lattice vectors have even (length)^2. Λ is called *self-dual* if $\Lambda=\Lambda^*$. This condition is equivalent to Λ being unimodular and integral.

dimension	number of lattices
8	1
16	2
24	24
32	many

Table 11: Number of even, self-dual lattices in different dimensions.

For the study of CFTs the even, self-dual lattices are of particular interest since they lead to holomorphic CFTs with central charge c equal to the dimension of the lattice. Even, self-dual lattices only exist when the dimension is a multiple of 8. Table 11 gives the number of such lattices in up to 24 dimension (thereafter their number increases rapidly - for further details see, e.g., [131])

In 24 dimensions, which will be of particular importance to us there are 24 of such lattices, called the *Niemeier lattices*. One of these lattices, the *Leech lattice*, stands out in that its shortest vectors have $(\text{length})^2 = 4$, so in particular it contains no root vectors (vectors of $(\text{length})^2 = 2$). The remaining 23 Niemeier lattices can be labelled by the Dynkin diagram of their root system. They fall into an A-D-E classification as can be seen in table 12. The root systems arise from the A,D,E algebras by the rules that for a given lattice the Coxeter number for each factor has to be the same and the rank of the factors has to sum up to 24. The Niemeier lattices arise from modifications of the root systems including 'gluing vectors'. In table 12 the 'umbral symmetry group' G_L for each lattice L is given, which is defined to be the automorphism group of the lattice modulo the Weyl- group of the associated A-D-E system.

Niemeier root system	Umbral symmetry G
A_1^{24}	M_{24}
A_2^{12}	$2.M_{12}$
A_3^8	$2.AGL_3(2)$
A_4^6	$GL_2(5)/2$
$A_5^4 D_4$	$GL_2(3)$
A_6^4	$SL_2(3)$
$A_7^2 D_5$	Dih_4
A_8^3	Dih_6
$A_9^2 D_6$	$\mathbb{Z}_4$
$A_{11} D_7 E_6$	$\mathbb{Z}_2$
A_{12}^2	$\mathbb{Z}_4$
$A_{15} D_9$	$\mathbb{Z}_2$
$A_{17} E_7$	$\mathbb{Z}_2$
A_{24}	$\mathbb{Z}_2$
D_4^6	$3.Sym_6$
D_6^4	Sym_4
D_8^3	Sym_3
$D_{10} E_7^2$	$\mathbb{Z}_2$
D_{12}^2	$\mathbb{Z}_2$
$D_{16} E_8$	
D_{24}	
E_6^4	$GL_2(3)$
E_8^3	Sym_3

Table 12: Niemeier lattices and associated umbral symmetry.

C Superconformal algebra and characters

C.1 (Extended) $\mathcal{N}=2$ characters

For the (extended) $\mathcal{N}=2$ superconformal algebra with central charge $c=3d$, let $|\Omega\rangle$ denote a highest weight state with eigenvalues h, ℓ w.r.t. L_0 and J_0. Writing $\mathcal{H}_{h,\ell}$ for the representation belonging to $|\Omega\rangle$ we define the (graded) $\mathcal{N}=2$ characters in the Ramond sector through

$$\mathrm{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell}(\tau,z) = \mathrm{Tr}_{\mathcal{H}_{h,\ell}}\left((-1)^F q^{L_0-\frac{c}{24}} e^{2\pi i z J_0}\right), \tag{C.1}$$

where F is the fermion number and $q=e^{2\pi i\tau}$. Below we will also use $y=e^{2\pi i z}$. In the Ramond sector unitarity requires $h \geq \frac{c}{24} = \frac{d}{8}$.
The characters [132–134] are given by (using the conventions of [74][39]):

- Massles (BPS) representations exist for $h=\frac{d}{8}$; $\ell = \frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-2, \ldots, -(\frac{d}{2}-$

[39] Note that our definition of $\theta_1(\tau,z)$ differs by a minus sign from the definition used there.

1), $-\frac{d}{2}$. For $\frac{d}{2} > \ell \geq 0$ they are given by

$$\mathrm{ch}^{\mathcal{N}=2}_{d,0,\ell\geq 0}(\tau,z) = (-1)^{\ell+\frac{d}{2}}\frac{(-\mathrm{i})\theta_1(\tau,z)}{\eta(\tau)^3}y^{\ell+\frac{1}{2}}\sum_{n\in\mathbb{Z}} q^{\frac{d-1}{2}n^2+(\ell+\frac{1}{2})n}\frac{(-y)^{(d-1)n}}{1-yq^n}, \quad (C.2)$$

and for $\ell = \frac{d}{2}$ one has

$$\mathrm{ch}^{\mathcal{N}=2}_{d,0,\frac{d}{2}}(\tau,z) = (-1)^{d}\frac{(-\mathrm{i})\theta_1(\tau,z)}{\eta(\tau)^3}y^{\frac{d+1}{2}}\sum_{n\in\mathbb{Z}} q^{\frac{d-1}{2}n^2+\frac{d+1}{2}n}\frac{(1-q)\,(-y)^{(d-1)n}}{(1-yq^n)(1-yq^{n+1})} \quad (C.3)$$

- Massive (non-BPS) representations exist for $h > \frac{d}{8}$; $\ell = \frac{d}{2}, \frac{d}{2}-1, \ldots, -(\frac{d}{2}-1), -\frac{d}{2}$ and $\ell \neq 0$ for $d =$ even. For $\ell > 0$ we have

$$\mathrm{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell>0}(\tau,z) = (-1)^{\ell+\frac{d}{2}}q^{h-\frac{d}{8}}\frac{\mathrm{i}\theta_1(\tau,z)}{\eta(\tau)^3}y^{\ell-\frac{1}{2}}\sum_{n\in\mathbb{Z}} q^{\frac{d-1}{2}n^2+(\ell-\frac{1}{2})n}\,(-y)^{(d-1)n} \quad (C.4)$$

In both cases the characters for $\ell < 0$ are given by

$$\mathrm{ch}^{\mathcal{N}=2}_{d,h-\frac{c}{24},\ell<0}(\tau,z) = \mathrm{ch}^{\mathcal{N}=2}_{d,h-c/24,-\ell>0}(\tau,-z). \quad (C.5)$$

The Witten index of a massless representation is given by

$$\mathrm{ch}^{\mathcal{N}=2}_{d,0,\ell\geq 0}(\tau,z=0) = \begin{cases} (-1)^{\ell+\frac{d}{2}}, & \text{for } 0 \leq \ell < \frac{d}{2}\,, \\ 1+(-1)^d, & \text{for } \ell = \frac{d}{2}\,. \end{cases} \quad (C.6)$$

C.2 $\mathcal{N} = 4$ characters

Analogously to the $\mathcal{N} = 2$ case the (graded) characters of the $\mathcal{N} = 4$ superconformal algebra with central charge $c = 3d$, and d even, in the Ramond sector are defined as

$$\mathrm{ch}^{\mathcal{N}=4}_{d,h-\frac{c}{24},\ell}(\tau,z) = \mathrm{Tr}_{\mathcal{H}_{h,\ell}}((-1)^F q^{L_0-\frac{c}{24}}e^{4\pi\mathrm{i}zT^3_0}), \quad (C.7)$$

where h and ℓ are the eigenvalues of L_0 and T^3_0 of the highest weight state belonging to the representation $\mathcal{H}_{h,\ell}$. As in the $\mathcal{N} = 2$ case unitarity requires $h \geq \frac{d}{8}$.
The characters [135] are given by (using conventions from [136])

- Massless representation exist for $h = \frac{d}{8}$, $\ell = 0, \frac{1}{2}\ldots, \frac{d}{4}$ and are given by

$$\mathrm{ch}^{\mathcal{N}=4}_{d,0,\ell}(\tau,z) = \frac{\mathrm{i}}{\theta_1(\tau,2z)}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\sum_{\varepsilon=\pm1}\sum_{m\in\mathbb{Z}}\varepsilon\frac{e^{4\pi\mathrm{i}\varepsilon((\frac{d}{2}+1)m+\ell)(z+\frac{1}{2})}}{(1-y^{-\varepsilon}q^{-m})^2}q^{(\frac{d}{2}+1)m^2+2\ell m}. \quad (C.8)$$

In particular for $\ell = 0$ this may be written as

$$\mathrm{ch}^{\mathcal{N}=4}_{d,0,0}(\tau,z) = \frac{-\mathrm{i}}{\theta_1(\tau,2z)}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\sum_{m\in\mathbb{Z}} q^{\left(\frac{d}{2}+1\right)m^2} y^{(d+2)m}\frac{1+yq^m}{1-yq^m}. \tag{C.9}$$

- Massive representation exist for $h > \frac{d}{8}$, $\ell = \frac{1}{2}, 1, \ldots, \frac{d}{4}$ and are given by

$$\mathrm{ch}^{\mathcal{N}=4}_{d,h-\frac{c}{24},\ell}(\tau,z) = \mathrm{i}q^{h-\frac{2\ell^2}{d+2}-\frac{d}{8}}\frac{\theta_1(\tau,z)^2}{\theta_1(\tau,2z)\eta(\tau)^3}\left(\vartheta_{\frac{d}{2}+1,2\ell}(\tau,z+\frac{1}{2}) - \vartheta_{\frac{d}{2}+1,-2\ell}(\tau,z+\frac{1}{2})\right), \tag{C.10}$$

where

$$\vartheta_{P,a}(\tau,z) = \sum_{n\in\mathbb{Z}} q^{\frac{(2Pn+a)^2}{4P}} y^{2Pn+a}\,. \tag{C.11}$$

With the help of the $\mathcal{N} = 4$ characters combinations of massless $\mathcal{N} = 2$ characters which are even in z can be expressed in the following way

$$\mathrm{ch}^{\mathcal{N}=2}_{d,0,\frac{1}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{d,0,-\frac{1}{2}}(\tau,z) = (-1)^{\frac{d+1}{2}}\phi_{0,\frac{3}{2}}(\tau,z)\mathrm{ch}^{\mathcal{N}=4}_{d-3,0,0}(\tau,z)\,, \tag{C.12}$$

$$\mathrm{ch}^{\mathcal{N}=2}_{d,0,\frac{3}{2}}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{d,0,-\frac{3}{2}}(\tau,z) = (-1)^{\frac{d+1}{2}}\phi_{0,\frac{3}{2}}(\tau,z)\left(\mathrm{ch}^{\mathcal{N}=4}_{d-3,0,0}(\tau,z) + \mathrm{ch}^{\mathcal{N}=4}_{d-3,0,\frac{1}{2}}(\tau,z)\right). \tag{C.13}$$

Likewise the even-z combination of the massive $\mathcal{N} = 2$ characters can be written as

$$\mathrm{ch}^{\mathcal{N}=2}_{d,n,l}(\tau,z) + \mathrm{ch}^{\mathcal{N}=2}_{d,n,-l}(\tau,z) = (-1)^{2l+\frac{d-1}{2}}\phi_{0,\frac{3}{2}}(\tau,z)\mathrm{ch}^{\mathcal{N}=4}_{d-3,n,\frac{1}{2}(l-\frac{1}{2})}(\tau,z)\,. \tag{C.14}$$

D Calabi-Yau manifolds and toric geometry

In this appendix we will briefly summarize some properties of Calabi-Yau manifolds. In particular we will also give a short overview of their realization in weighted projective spaces and toric geometry.

D.1 Basic definitions and properties

In this subsection we give the general definitions and basic properties of CY manifolds. We follow mostly [95, 137]. A CY d-fold X is a compact Kähler manifold having d-complex dimensions and fulfilling one of the equivalent properties

i) X has vanishing first Chern class, $c_1 = \frac{1}{2\pi\mathrm{i}}[\mathcal{R}]$.

ii) X admits a Ricci flat metric.

iii) The holonomy group of X is a subgroup of $SU(d)$.[40]

[40] Mostly CY d-folds are supposed to have exactly $SU(d)$ holonomy. We will often refer to such CY d-folds as being genuine CY d-folds.

iv) X admits a covariant constant spinor.

v) X has a unique nowhere vanishing holomorphic d-form Ω .

The Hodge numbers (dimensions of the Dobeault cohomology) fulfil the following relations

- $h^{p,0} = h^{d-p,0}$ since $H^p(X) \simeq H^{d-p}(X)$, which follows by contraction with the $(d, 0)$ form Ω.
- $h^{p,q} = h^{q,p}$ by complex conjugation.
- $h^{p,q} = h^{d-q,d-p}$ by Poincare duality.
- $h^{0,0} = 1$ holds for any compact, connected Kähler complex manifold. For simply-connected Kähler manifolds, which will be the case for most CY manifolds we consider (tori are a counterexample), we further have $h^{1,0} = h^{0,1} = 0$.

The Hodge numbers can be nicely arranged in the so called Hodge diamond. For $d = 2, 3$ it takes the general form:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad , \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \ . \tag{D.1}$$

D.2 Calabi-Yau manifolds as hypersurfaces in weighted projective ambient spaces

In this section we will discuss the construction and properties of CY manifolds constructed as hypersurfaces in weighted projective spaces. We follow [95, 138] for the construction in weighted projective space.

Compact manifolds cannot be constructed as submanifolds of $\mathbb{C}^n$, however one can construct them as hypersurfaces in (weighted) projective space and generalizations thereof. We start by the defining a generalized $\mathbb{C}^*(:= \mathbb{C} \setminus \{0\})$ action on $\mathbb{C}^{n+1}$ through

$$\lambda \cdot z = \lambda \cdot (z^0, \dots, z^n) = (\lambda^{w_0} z^0, \dots, \lambda^{w_n} z^n), \tag{D.2}$$

where $\lambda \in \mathbb{C}^*$ and the non-zero integer w_i is called the weight of the homogeneous coordinate z_i. Using this action we define an equivalence relation $\sim$ on $\mathbb{C}^n$ through

$$z \sim z' \Leftrightarrow \exists \lambda \in C^* : \ z' = \lambda \cdot z \tag{D.3}$$

and the equivalence class of a point is denoted by $[z]$. Then the *weighted projective space with weights* $w_i, i = 0, \ldots, n$, is defined by $\mathbb{P}^n[w_0, \ldots, w_n] = \mathbb{P}_n[\mathbf{w}] := (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\})/\sim$. A hypersurface $X_d[\mathbf{w}]$ in weighted projective space is defined as the vanishing locus of a quasi-homogeneous polynomial $p(\lambda \cdot z) = \lambda^d p(z)$, where d is called the degree of $p(z)$,

$$X_d[\mathbf{w}] = \{[z^0 : \cdots : z^n] \in \mathbb{P}^n | p(z) = 0\}. \tag{D.4}$$

The first Chern class of $X_d[\mathbf{w}]$ vanishes if $d = \sum_{i=1}^n w_i$. In order for X_d to be non-singular, we require $dp(z) = p(z) = 0$ only to hold at $\mathbf{0}(:= [0 : \cdots : 0])$, which is not in $\mathbb{P}^n$. A necessary condition for such a polynomial to exist is that the Poincaré-polynomial, defined by

$$P(t) = \prod_{i=0}^{n} \frac{1 - t^{d-w_i}}{1 - t^{w_i}} \tag{D.5}$$

is indeed a polynomial, i.e., that the set of all multiples of $1/(d - w_i)$ includes the set of all multiples of $1/w_i$. A simple check for this condition is to see if $P(0)$ is an integer. The Euler number of $X_d[\mathbf{w}]$ is given by

$$\chi = \frac{1}{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \prod_{gcd\{l,k\} \cdot \frac{w_i}{m} \in \mathbb{Z}} \frac{w_i - m}{w_i} \tag{D.6}$$

which was proven in [138] based on [139].

D.3 Calabi-Yau manifolds as hypersurfaces in toric ambient spaces

The constructions above may be generalized in different ways. We want to consider hypersufaces in toric varieties next. We will mostly follow [140, 141] in the following discussion.

Toric varieties are generalisations of weighted projective space. Roughly speaking they are obtained by introducing additional coordinates and an appropriate number of additional equivalence relations of the form (D.2, D.3). We start by defining the necessary objects.

Let N be a n-dimensional lattice and $N_{\mathbb{R}}$ the real vector space carrying the lattice then:

-An r-dimensional cone σ_n in $N_{\mathbb{R}}$, finitely generated by $v_1, \ldots, v_s \in N_{\mathbb{R}}, s \geq r$ is defined by

$$\sigma_n = \{\lambda_1 v_1 + \cdots + \lambda_s v_s | \lambda_i \geq 0 \text{ for } 1 \leq i \leq s\} \tag{D.7}$$

An r-dimensional cone is called *simplicial* if it is generated by r (linear independent) vectors. An n-dimensional cone is called *basic* if it is generated by a basis of the lattice N.

-A *strongly convex rational polyhedral cone* is defined to be an n or lower dimensional cone, with $\mathbf{0}$ as its apex, bounded by finitely many hyperplanes ('polyhedral'), edges spanned by lattice vectors ('rational') and not containing a complete line ('strongly convex').
-A *face of a cone* σ is defined to be either σ itself or the intersection of σ with a hyperplane bounding it
-A *fan* is defined to be a finite collection of cones such that all faces of cones and all intersection of cones also belong to the fan. For a fan Σ we define $\Sigma^{(n)}$ to be the n-dimensional cones in Σ.

Let Σ be a fan consisting of a strongly convex rational polyhedral cones. For each one dimensional cone in Σ with primitive generator $v_i, i = 1, \ldots, k$ assign a homogeneous coordinate z_i. Then the generalisation of the action (D.2) and the equivalence relation (D.3) on this $\mathbb{C}^k$ is the following :

$$(z_1, \ldots, z_k) \sim (\lambda^{q_j^1} z_1, \ldots, \lambda^{q_j^k} z_k) \quad \text{if} \quad \sum_{i=1}^{k} q_j^i v_i = 0 \tag{D.8}$$

where $q_j^i \in \mathbb{Z}$ and $\gcd(q_j^i,\ i = 1, \ldots, k)$=1 (this corresponds to $k - n$ independent relations so that (locally) we have an n-dimensional manifold.)
Further similar to excluding $\mathbf{0}$ in the definition of the weighted projective space we need to exclude points where certain z_i are zero simultaneously. The set that needs to be excluded is defined by

$$Z_\Sigma := \bigcup_I \{(z_1, \ldots, z_k) : z_i = 0\ \forall i \in I\} \tag{D.9}$$

where the union $\bigcup_I$ is taken over all sets $I \subset \{1, \ldots, k\}$ for which $\{v_i : i \in I\}$ does not belong to a cone in Σ. Removing the set Z_Σ from $\mathbb{C}^k$ is the equivalent to saying that several z_i are allowed to vanish simultaneously only if the corresponding v_i belong to the same cone.

If all n-dimensional cones of Σ are basic the toric variety is given by the quotient of $\mathbb{C}^k \setminus Z_\Sigma$ by the equivalence relations (D.8). In the case that a n-dimensional cone is not basic, i.e., it is spanned by $\{v_1, \ldots, v_n\}$ that do not span the lattice N but only a sublattice, say $N(v_1, \ldots, v_n)$, additional relations will exist, which form an abelian group isomorphic to $N/N(v_1, \ldots, v_n)$. In this situation we will in addition have to factor $\mathbb{C}^k \setminus Z_\Sigma$ by this group as well. This will lead to orbifold singularities which can be removed by appropriate
subdivision of the fan [41]. In addition singularities will arise if the fan contains cones

[41]This corresponds to introducing additional coordinates and additional relations. Thereby the previous singular point is replaced by a smooth manifold. This procedure is referred to as 'blow up'.

which are not simplicial. These again can be cured by subdivision of the fan. A toric variety is smooth if and only if the fan consist of simplicial and basic cones only. It is compact if and only if the support of the fan covers the lattice.
We may think of an n-dimensional cone σ as representing coordinate patches. More concretely let σ be generated by $v_1, \ldots, v_n$, then the associated coordinate patch is $U_\sigma = \{(z_1, \ldots, z_k) | z_i \geq 0\}$. Lower dimensional cones represent regions of overlap where patches are glued together.

In the next step we want to specify the polynomials whose vanishing locus will define the CY. Starting with a lattice N and a fan Σ, $\Sigma^{(1)} = \{v_1, \ldots, v_k\}$, we consider the dual lattice of N, $M := \mathrm{Hom}(N, \mathbb{Z})$. Then we define the polyhedron Δ through

$$\Delta := \{w \in M | \langle v_i, w\rangle \geq -1, i = 1, \ldots, k\}. \tag{D.10}$$

Then consider the (Laurent) polynomial given by

$$f_\Delta := \sum_{w \in \Delta} a_w \prod_{i=1}^{k} z_i^{\langle w, v_i\rangle + 1} \tag{D.11}$$

where $a_w \in \mathbb{C}$ are parameters. Under a scaling as in (D.8) f_Δ transforms homogeneously. Now the polynomial equation $f_\Delta = 0$ is well defined. We further define the *dual polyhedron* Δ^* by

$$\Delta^* = \{v \in N_\mathbb{R} | \langle v, w\rangle \geq -1 \,, \; \forall w \in \Delta\rangle\}. \tag{D.12}$$

Δ is called *reflexive* if Δ^* is a lattice polyhedron, i.e., if the vertices of $\Delta^* \subset N_\mathbb{R}$ lie in N. Note that $(\Delta^*)^* = \Delta$.
To a pair of reflexive polyhedra (Δ, Δ^*) we can again associate the fan defined by

$$\begin{aligned}\Sigma(\Delta) = \{&\text{complete rational fan whose cones are the cones over} \\ &\text{the faces of } \Delta \text{ with apex at the origin}\}\end{aligned} \tag{D.13}$$

Since we want the fan to consist of simplicial basic cones we need a maximal star triangulation of the polyhedron.

It was shown by Bartyrev that given a pair of reflexive polyhedra (Δ, Δ^*) the hypersurface defined by the vanishing of a generic polynomial defined by Δ is a smooth CY manifold for $n \leq 4$. For $n \leq 3$ the toric variety itself is ensured to be smooth by reflexivity. For $n = 4$ the toric variety may have point like singularities, which however are missed by the generic hypersurface.

Exchanging the role of Δ and Δ^* above gives rise to the mirror CY. There exist combinatorial formulas, also found by Bartyrev [142], for the Hodge numbers of the CY as sums over lattice points of faces of the involved polyhedra, see also [141].

There are 16 reflexive 2 dimensional polyhedra. They describe elliptic curves.
All 4319 three dimensional reflexive polyhedra have been classified in [130]. They describe $K3$ surfaces.
Finally all 473,800,776 reflexive 4 dimensional polyhedra have been classified in [143] and can be accessed on [81]. They rise to 6 dimensional CY's and hence are important for string compactifications to 4 dimensions.

E Conformal bootstrap

In this appendix we will give a brief summary the conformal bootstrap, as introduced in [114], sections 2-4, for the topological string partition function on genus one fibered CY 3-folds with N-sections for $N \in \{2, 3, 4\}$.

A Calabi-Yau is said to be genus one fibered over a base B if there exists a surjective map $\pi : X \to B$ s.t. the generic fiber over B is a torus. If the projection admits a section X is called elliptically fibered. We use the convention that a genus one fibration in general does not have a section but only a N-section intersecting the generic fiber N-times, i.e., it corresponds to N points that can be identified in each fiber. Moving along closed paths in the base of the fibration these points will generically experience a monodromy (this distinguishes them from a union of N sections). In our notation a genus one fibered manifold with N-sections does not have N'-sections with $N' < N$ and in particular it has no section.
We start by considering an elliptically fibered 3-fold M with base B and projection π. By the Shioda-Tate-Wazir theorem [144] its homology group $H_4(M)$ is degenerated by the three types of divisors:
i) *vertical divisors* $D_i = \pi^{-1}\tilde{D}_i$, where $\tilde{D}_i \in H_2(B)$.
ii) *fibral divisors* consisting of rational curves fibered over a divisor in B.
iii) *sections*, which can be split into *holomorphic sections*, intersecting every fiber in a point, and *rational sections*, that intersect every smooth irreducible fiber in a point.

By convention an elliptic fibration has at least one section (which can be either holomorphic or rational). One can choose any section to be the zero section. This enables one to canonically identify every fiber with a torus, $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, using the intersection with the zero section as the origin. Addition of points on the fiber defines a group law in the fiber which may be extended to the rational sections and defines the *Mordell-Weil group* $MW(M)$. A basis of $H_4(M)$ is then given by the zero section and a set of sections that are linear independent in the Mordell-Weil group together with a basis of vertical divisors and fibral divisors.

For genus one fibered threefolds the above may be generalised [129] by considering k-sections instead of sections. One can pick a 'zero k-section' and act on it with an appropriate generalisation of the Mordell-Weil group to obtain a basis of k-sections.

For an elliptically fibered CY manifold M one can define the *Shioda map* $\sigma : MW(M) \to H_4(M,\mathbb{Z})$, which is a homomorphism from the Mordell-Weill group to $H_4(M,\mathbb{Z})$. It can uniquely be defined in terms of its intersection properties [145]. This definition may then be generalized to genus one fibrations with N-sections in the following way. Start by defining the inner product

$$\langle , \rangle : H_4(M) \times H_4(M) \to H_2(B), \ (S, S') \mapsto -\pi(S \cdot S'). \tag{E.1}$$

Then for an N-section E of a CY 3-fold one defines $\sigma(E) := E + D$ where D is the unique linear combination of the zero-N-section, vertical divisors and fibral divisors s.t. $\sigma(E)$ is orthogonal to the subspace spanned by those divisors in $H_4(M)$ w.r.t. $\langle \cdot, \cdot \rangle$.

Irreducible curves in M will arise from either curves in the base or from rational curves that are fibers of fibral divisors or from isolated rational curves over points of the base (the latter two will collectively be called fibral curves). One can expand the topological string partition function $Z_{top.} = \exp(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g)$ in the following manner

$$Z_{top.}(\tau, \underline{m}, \underline{t}, \lambda) = Z_0(\tau, \lambda) \left(1 + \sum_{\beta \in H^{1,1}(B,\mathbb{Z})} Z_\beta(\tau, \underline{m}, \lambda) Q^\beta \right) \tag{E.2}$$

where one finds that for an elliptic fibration or for a genus one fibration with N-sections, $N \in \{1, 2, 3, 4\}$, the Kähler modulus τ should be such that $N\tau$ is the complexified volume of the generic fiber. Further $\underline{m}$ are the complexified volumes of the fibral curves and $Q^\beta = \exp(2\pi \mathrm{i} \sum_i \beta^i t_i)$, where $t_i, i = 1, \ldots, h^{1,1}(B)$ are shifted volumes of curves in the base. This shift is linear in τ and was first observed for elliptic figurations in [146, 147].
Modular properties of the topological string partition function are consequences of the general transformation behaviour under monodromies in the stringy Kähler moduli space.
The automorphic properties of Z_{top} can be derived from the wave function interpretation of Z_{top} [148], which was investigated in, e.g., [149, 150].
Assume M is elliptic or genus one fibration. Parametrize the Kähler form ω as

$$\omega = \tau \cdot (E_0 + D) + \sum_{i=1}^{r} m_i \cdot \sigma(E_i) + \sum_{i=r+1}^{\mathrm{rk}(G)} m_i \cdot D_{f,i-r} + \sum_{i=1}^{b_2(B)} \tilde{t}_i \cdot D'_i \tag{E.3}$$

where $D_{f,i}$ denotes a basis of fibral divisors,$i = 1, \ldots, rk(G)$ [42], E_i, $i = 0, \ldots, r$ are independent N-sections; D'_i are vertical divisors dual to the curves $C_i = \frac{1}{n} E_0 \cdot D_i$,

[42]The notation $rk(G)$ arises from the 'F-theory-dictionary' discussed in Appendix A of [114].

with $D_i = \pi^{-1}\tilde{D}_i,\ i = 1, \dots, b_2(B)$, in the sense that

$$D_i' \cdot C_j = N\delta_{ij}. \tag{E.4}$$

The 'zero N-section' E_0 is shifted by D, the unique vertical divisor s.t. $\tilde{E}_0 = E_0 + D$ is orthogonal to all of these curves C_i. Furthermore the shifted Kähler parameters $t_i,\ i = 1, \dots, h^{1,1}(B)$ are defined to be

$$t_i = \tilde{t}_i + \frac{\tilde{a}_i}{2N}\tau, \ \text{ with } \tilde{a}_i = \int_M \tilde{E}_0^2 \cdot D_i. \tag{E.5}$$

One assumes that there are no fibral divisors at a generic point of the complex structure moduli space of M.

Then assuming an expansion as in (E.2) one can make the following ansatz for the expansion coefficients $Z_\beta(\tau, \underline{m}, \lambda)$,

$$Z_\beta(\tau, \underline{m}, \lambda) = \frac{1}{\eta(N\tau)^{12 \cdot c_1(B)\cdot\beta}} \frac{\phi_\beta(\tau, \underline{m}, \lambda)}{\prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \phi_{-2,1}(N\tau, s\lambda)}, \tag{E.6}$$

where $c_1(B)$ is the first Chern class of the base, the numerator is an element

$$\phi_\beta(\tau, \underline{m}, \lambda) \in M_*(N)[\phi_{-2,1}(N\tau, \bullet), \phi_{0,1}(N\tau, \bullet)] \cdot \Delta_{2N}(\tau)^{1-\frac{r_\beta}{N} \mod 1} \tag{E.7}$$

where $M_*(N)$ denotes the ring of modular forms for $\Gamma_1(N)$, $\bullet$ stands for any elliptic parameter $z \in \{\lambda, \underline{m}\}$ and the exponent of $\Delta_{2N}(\tau)$ is determined by the congruence relation

$$1 - \frac{r_\beta}{N} \equiv \frac{1}{2}\left[Nc_1(B) - \frac{\tilde{a}}{N}\right] \cdot \beta \mod 1. \tag{E.8}$$

The weight of ϕ_β is given by

$$w = 6c_1(B) \cdot \beta - \sum_l \beta_l \tag{E.9}$$

and the index w.r.t. the topological string coupling λ is

$$r_\beta^\lambda = \frac{1}{2N}\beta \cdot (\beta - c_1(\beta)). \tag{E.10}$$

The index matrix w.r.t. the geometric elliptic parameters $m_i,\ i = 1, \dots, \mathrm{rk}(G)$ is

$$r_{ij}^\beta = \frac{1}{N} \cdot \begin{cases} -\frac{1}{2}\pi_*(\sigma(E_i) \cdot \sigma(E_j)) \cdot \beta & \text{for } 1 \leq i, j \leq r \\ -\frac{1}{2}\pi_*(D_{f,i} \cdot D_{fi}) \cdot \beta & \text{for } r < i, j \leq \mathrm{rk}(G) \\ 0 & \text{otherwise} \end{cases}. \tag{E.11}$$

F Higgsing of the gauge group

In this appendix we will explain in some detail how the gauge group of the models of table 4 and 5 can be Higgsed by giving vacuum expectation values (vev's) to scalars in the hypermultiplets. We take the necessary branching rules from [151]. For more on Higgsing see for example [99, 152–154].

For concreteness sake we will study the third model of table 5. Before Higgsing we have the following gauge group and matter content, coming from the twisted and untwisted sector

$$\begin{aligned} &SO(14) \times SO(14) \times U(1)^2 \\ &4(\mathbf{14}, \mathbf{1}) + 4(\mathbf{1}, \mathbf{14}) + (\mathbf{64}, \mathbf{1}) + (\mathbf{1}, \mathbf{64}) + 8(\mathbf{1}, \mathbf{1})\,. \end{aligned} \tag{F.1}$$

The matter content is labelled by the representations under the two $SO(14)$ groups, i.e. we have left out the $U(1)$ charges. Counting degrees of freedom we find the numbers of vector- and hypermultiplets $N_v = 184$ and $N_h = 248$. We will start by explaining how to Higgs the first $SO(14)$ factor.

For this we notice the following branching rules

$$SO(14) \supset SO(13): \ \mathbf{14} \to \mathbf{13} + \mathbf{1}\,, \ \mathbf{64} \to \mathbf{64}. \tag{F.2}$$

Giving a vev to one **14** will break $SO(14)$ to $SO(13)$. One **13** will get 'eaten' by the broken generators of the gauge fields (turning them massive) and the result is the following gauge group and matter spectrum

$$\begin{aligned} &SO(13) \times SO(14) \times U(1)^2\,, \\ &3(\mathbf{13}, \mathbf{1}) + 4(\mathbf{1}, \mathbf{14}) + (\mathbf{64}, \mathbf{1}) + (\mathbf{1}, \mathbf{64}) + 12(\mathbf{1}, \mathbf{1}). \end{aligned} \tag{F.3}$$

We may check that $N_h - N_v = 235 - 171 = 64$ is unchanged. In the next step we give a vev to a **13** and use the branching rules

$$SO(13) \supset SO(12): \ \mathbf{13} \to \mathbf{12} + \mathbf{1}\,, \ \mathbf{64} \to \mathbf{32} + \mathbf{32}'. \tag{F.4}$$

Similar to before a **12** gets 'eaten' by the broken generators of $SO(13)$ and we find the following gauge group and matter spectrum after Higgsing

$$\begin{aligned} &SO(12) \times SO(14) \times U(1)^2\,, \\ &2(\mathbf{12}, \mathbf{1}) + 4(\mathbf{1}, \mathbf{14}) + (\mathbf{32}, \mathbf{1}) + (\mathbf{32}', \mathbf{1}) + (\mathbf{1}, \mathbf{64}) + 15(\mathbf{1}, \mathbf{1}). \end{aligned} \tag{F.5}$$

Two more similar steps lead to the following gauge group and matter spectrum

$$SO(10) \times SO(14) \times U(1)^2\,, \qquad (F.6)$$
$$4(\mathbf{1},\mathbf{14}) + 2(\mathbf{16},\mathbf{1}) + 2(\overline{\mathbf{16}},\mathbf{1}) + (\mathbf{1},\mathbf{64}) + 18(\mathbf{1},\mathbf{1}).$$

The branching rules

$$SO(10) \supset SU(5) \times U(1) : \mathbf{16} \to \mathbf{10(1)} + \bar{\mathbf{5}}(\mathbf{-3}) + \mathbf{1(5)}, \qquad (F.7)$$
$$\overline{\mathbf{16}} \to \overline{\mathbf{10}}(\mathbf{-1}) + \mathbf{5(3)} + \mathbf{1(-5)}$$

(the number in brackets give the $U(1)$ charge) indicate that we can give a vev to $\mathbf{16}$ and $\overline{\mathbf{16}}$ thereby breaking $SO(10)$ to $SU(5)$ where $\mathbf{10}, \overline{\mathbf{10}}$ and one scalar will get 'eaten' by the broken generators. The gauge group and spectrum after this step of Higgsing are thus

$$SU(5) \times SO(14) \times U(1)^2\,, \qquad (F.8)$$
$$4(\mathbf{1},\mathbf{14}) + 2(\mathbf{5},\mathbf{1}) + 2(\bar{\mathbf{5}},\mathbf{1}) + (\mathbf{10},\mathbf{1}) + (\overline{\mathbf{10}},\mathbf{1}) + (\mathbf{1},\mathbf{64}) + 21(\mathbf{1},\mathbf{1}).$$

In the next step we give a vev to $\mathbf{5}, \bar{\mathbf{5}}$ and use the branching rules

$$SU(5) \supset SU(4) \times U(1) :\ \mathbf{5} \to \mathbf{4(1)} + \mathbf{1(-4)}, \qquad (F.9)$$
$$\mathbf{10} \to \mathbf{4(-3)} + \mathbf{6(2)}$$

to obtain the gauge group and spectrum

$$SU(4) \times SO(14) \times U(1)^2\,, \qquad (F.10)$$
$$4(\mathbf{1},\mathbf{14}) + 2(\mathbf{4},\mathbf{1}) + 2(\bar{\mathbf{4}},\mathbf{1}) + (\mathbf{6},\mathbf{1}) + (\bar{\mathbf{6}},\mathbf{1}) + (\mathbf{1},\mathbf{64}) + 24(\mathbf{1},\mathbf{1}).$$

We proceed by breaking $SU(4)$ to $SU(3)$ by giving a vev to $\mathbf{4}, \bar{\mathbf{4}}$ and using the branching rules

$$SU(4) \supset SU(3) \times U(1) : \mathbf{4} \to \mathbf{3(1)} + \mathbf{1(-3)}, \qquad (F.11)$$
$$\mathbf{6} \to \mathbf{3(-2)} + \bar{\mathbf{3}}\mathbf{(2)}.$$

We obtain the following gauge group and spectrum

$$SU(3) \times SO(14) \times U(1)^2\,, \qquad (F.12)$$
$$4(\mathbf{1},\mathbf{14}) + 3(\mathbf{3},\mathbf{1}) + 3(\bar{\mathbf{3}},\mathbf{1}) + (\mathbf{1},\mathbf{64}) + 27(\mathbf{1},\mathbf{1}).$$

We can continue in similar manner using $3(\mathbf{3},\mathbf{1})$ and $3(\bar{\mathbf{3}},\mathbf{1})$ to Higgs $SU(3)$ completely and end up with

$$SO(14) \times U(1)^2\,, \qquad \text{(F.13)}$$
$$4(\mathbf{1},\mathbf{14}) + (\mathbf{1},\mathbf{64}) + 37(\mathbf{1},\mathbf{1}).$$

In the same way we may Higgs the second $SO(14)$. The two $U(1)$ factors in the gauge group may be Higgsed by any of the charged scalars. Thus we end up with a completely Higgsed gauge group and 64 neutral scalars.

G Character table of $\mathbb{M}_{24}$

Table 13: The character table of M_{24} with shorthand notation $e_n = \frac{1}{2}(-1 + \mathrm{i}\sqrt{n})$.

[g]	1a	2a	2b	3a	3b	4a	4b	4c	5a	6a	6b	7a	7b	8a	10a	11a	12a	12b	14a	14b	15a	15b	21a	21b	23a	23b
$[g^2]$	1a	1a	1a	3a	3b	2a	2a	2b	5a	3a	3b	7a	7b	4b	5a	11a	6a	6b	7a	7b	15a	15b	21a	21b	23a	23b
$[g^3]$	1a	2a	2b	1a	1a	4a	4b	4	5a	2a	2b	7b	7a	8a	10a	11a	4a	4	14b	14a	5a	5a	7b	7a	23a	23b
$[g^5]$	1a	2a	2b	3a	3b	4a	4b	4	1a	6a	6b	7b	7a	8a	2b	11a	12a	12b	14b	14a	3a	3a	21b	21a	23b	23a
$[g^7]$	1a	2a	2b	3a	3b	4a	4b	4	5a	6a	6b	1a	1a	8a	10a	11a	12a	12b	2a	2a	15b	15a	3b	3b	23b	23a
$[g^{11}]$	1a	2a	2b	3a	3b	4a	4b	4	5a	6a	6b	7a	7b	8a	10a	1a	12a	12b	14a	14b	15b	15a	21a	21b	23b	23a
$[g^{23}]$	1a	2a	2b	3a	3b	4a	4b	4	5a	6a	6b	7a	7b	8a	10a	11a	12a	12b	14a	14b	15a	15b	21a	21b	1a	1a
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
χ_3	45	-3	5	0	3	-3	1	1	0	0	-1	e_7	$\bar{e}_7$	-1	0	1	0	1	$-e_7$	$-\bar{e}_7$	0	0	e_7	$\bar{e}_7$	-1	-1
χ_4	45	-3	5	0	3	-3	1	1	0	0	-1	$\bar{e}_7$	e_7	-1	0	1	0	1	$-\bar{e}_7$	$-e_7$	0	0	$\bar{e}_7$	e_7	-1	-1
χ_5	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	e_{15}	$\bar{e}_{15}$	0	0	1	1
χ_6	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	$\bar{e}_{15}$	e_{15}	0	0	1	1
χ_7	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
χ_9	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	e_{23}	$\bar{e}_{23}$
χ_{11}	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	$\bar{e}_{23}$	e_{23}
χ_{12}	990	-18	-10	0	3	6	2	-2	0	0	-1	e_7	$\bar{e}_7$	0	0	0	0	1	e_7	$\bar{e}_7$	0	0	e_7	$\bar{e}_7$	1	1
χ_{13}	990	-18	-10	0	3	6	2	-2	0	0	-1	$\bar{e}_7$	e_7	0	0	0	0	1	$\bar{e}_7$	e_7	0	0	$\bar{e}_7$	e_7	1	1
χ_{14}	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
χ_{15}	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2e_7$	$2\bar{e}_7$	-1	0	1	0	-1	0	0	0	0	$-e_7$	$-\bar{e}_7$	0	0
χ_{16}	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\bar{e}_7$	$2e_7$	-1	0	1	0	-1	0	0	0	0	$-\bar{e}_7$	$-e_7$	0	0
χ_{17}	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0
χ_{18}	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0
χ_{19}	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
χ_{21}	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0
χ_{22}	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1
χ_{23}	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
χ_{25}	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0
χ_{26}	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

References

[1] D. Gorenstein, *Finite Simple Groups*. Springer US, 1982, 10.1007/978-1-4684-8497-7.

[2] R. Solomon, *A brief history of the classification of the finite simple groups*, Bulletin of The American Mathematical Society - BULL AMER MATH SOC **38** (07, 2001) 315–353.

[3] J. H. Conway and S. P. Norton, *Monstrous Moonshine*, *Bull. London Math. Soc.* **11** (1979) 308–339.

[4] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, vol. 134 of *Pure and Applied Mathematics*. Elsevier Science, 1989.

[5] R. E. Borcherds, *Vertex algebras, kac-moody algebras, and the monster*, *Proceedings of the National Academy of Sciences* **83** (May, 1986) 3068–3071.

[6] R. E. Borcherds, *Monstrous Moonshine and Monstrous Lie Superalgebras* , *Invent. Math.* **109** (1992) 405.

[7] L. J. Dixon, P. H. Ginsparg and J. A. Harvey, *Beauty and the Beast: Superconformal Symmetry in a Monster Module*, *Commun.Math.Phys.* **119** (1988) 221–241.

[8] J. F. Duncan, *Super-moonshine for Conway's largest sporadic group*, *ArXiv Mathematics e-prints* (Feb., 2005) , [math/0502267].

[9] J. F. R. Duncan and S. Mack-Crane, *The Moonshine Module for Conway's Group*, *SIGMA* **3** (2015) e10, [1409.3829].

[10] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group M_{24}*, *Exper.Math.* **20** (2011) 91–96, [1004.0956].

[11] M. C. Cheng, *K3 Surfaces, N=4 Dyons, and the Mathieu Group M24*, *Commun.Num.Theor.Phys.* **4** (2010) 623–658, [1005.5415].

[12] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu twining characters for K3*, *JHEP* **1009** (2010) 058, [1006.0221].

[13] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu Moonshine in the elliptic genus of K3*, *JHEP* **1010** (2010) 062, [1008.3778].

[14] T. Eguchi and K. Hikami, *Note on Twisted Elliptic Genus of K3 Surface*, *Phys.Lett.* **B694** (2011) 446–455, [1008.4924].

[15] T. Gannon, *Much ado about Mathieu*, *Adv. Math.* **301** (2016) 322–358, [1211.5531].

[16] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Symmetries of K3 sigma models*, *Commun.Num.Theor.Phys.* **6** (2012) 1–50, [1106.4315].

[17] A. Taormina and K. Wendland, *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, *JHEP* **08** (2013) 125, [1107.3834].

[18] A. Taormina and K. Wendland, *A twist in the M24 moonshine story*, 1303.3221.

[19] A. Taormina and K. Wendland, *Symmetry-surfing the moduli space of Kummer K3s*, *Proc. Symp. Pure Math.* **90** (2015) 129–154, [1303.2931].

[20] M. R. Gaberdiel, C. A. Keller and H. Paul, *Mathieu Moonshine and Symmetry Surfing*, *J. Phys.* **A50** (2017) 474002, [1609.09302].

[21] M. C. N. Cheng, J. F. R. Duncan and H. J. A., *Umbral Moonshine*, *Commun. Num. Theor. Phys.* **08** (2014) 101–242, [1204.2779].

[22] M. C. N. Cheng, J. F. R. Duncan and J. A. Harvey, *Umbral Moonshine and the Niemeier Lattices*, 1307.5793.

[23] J. F. R. Duncan, M. J. Griffin and K. Ono, *Proof of the Umbral Moonshine Conjecture*, 1503.01472.

[24] M. C. N. Cheng and S. Harrison, *Umbral Moonshine and K3 Surfaces*, *Commun. Math. Phys.* **339** (2015) 221–261, [1406.0619].

[25] S. Kachru, N. M. Paquette and R. Volpato, *3D String Theory and Umbral Moonshine*, *J. Phys.* **A50** (2017) 404003, [1603.07330].

[26] J. A. Harvey and B. C. Rayhaun, *Traces of Singular Moduli and Moonshine for the Thompson Group*, *Commun. Num. Theor. Phys.* **10** (2016) 23–62, [1504.08179].

[27] M. Griffin and M. Mertens, *A proof of the Thompson Moonshine Conjecture*, *Research in the Mathematical Sciences* **3** (07, 2016) .

[28] J. F. R. Duncan, M. H. Mertens and K. Ono, *O'nan moonshine and arithmetic*, 2017.

[29] J.-B. Bae, J. A. Harvey, K. Lee, S. Lee and B. C. Rayhaun, *Conformal Field Theories with Sporadic Group Symmetry*, 2002.02970.

[30] M. C. N. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru and T. Wrase, *Mathieu Moonshine and N=2 String Compactifications*, *JHEP* **09** (2013) 030, [1306.4981].

[31] A. Banlaki, A. Chowdhury, A. Kidambi and M. Schimpf, *On Mathieu moonshine and Gromov-Witten invariants*, *JHEP* **02** (2020) 082, [1811.11619].

[32] N. M. Paquette and T. Wrase, *Comments on M_{24} representations and CY_3 geometries*, *JHEP* **11** (2014) 155, [1409.1540].

[33] T. Wrase, *Mathieu moonshine in four dimensional $\mathcal{N} = 1$ theories*, *JHEP* **04** (2014) 069, [1402.2973].

[34] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms*, 1208.4074.

[35] M. C. Cheng, J. F. Duncan, S. M. Harrison, J. A. Harvey, S. Kachru and B. C. Rayhaun, *Attractive Strings and Five-Branes, Skew-Holomorphic Jacobi Forms and Moonshine*, *JHEP* **07** (2018) 130, [1708.07523].

[36] E. Witten, *Three-Dimensional Gravity Revisited*, 0706.3359.

[37] A. Banlaki, A. Chowdhury, A. Kidambi, M. Schimpf, H. Skarke and T. Wrase, *Calabi-Yau manifolds and sporadic groups*, *JHEP* **02** (2018) 129, [1711.09698].

[38] A. Banlaki, A. Chattopadhyaya, A. Kidambi, T. Schimannek and M. Schimpf, *Heterotic strings on* $(K3 \times T^2)/\mathbb{Z}_3$ *and their dual Calabi-Yau threefolds*, *JHEP* **04** (2020) 203, [1911.09697].

[39] T. Gannon, *Monstrous moonshine and the classification of cft*, in *Lectures given in Istanbul*, p. 9906167, 1998.

[40] S. Kachru, *Elementary introduction to Moonshine*, 1605.00697.

[41] V. Anagiannis and M. C. Cheng, *TASI Lectures on Moonshine*, *PoS* **TASI2017** (2018) 010, [1807.00723].

[42] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*. Springer New York, 1993, 10.1007/978-1-4612-0909-6.

[43] J. H. Silverman, *The Arithmetic of Elliptic Curves*. Springer New York, 2009, 10.1007/978-0-387-09494-6.

[44] M. M. Eichler and D. Zagier, *The theory of Jacobi forms*. Boston Birkhäuser, 1985.

[45] V. Gritsenko, *Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms*, math/9906190.

[46] S. N. R. P. J.H. Conway, R.T. Curtis and R. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray*. Oxford: Clarendon Press, 1985.

[47] J. Griess, Robert L., *The friendly giant.*, *Inventiones mathematicae* **69** (1982) 1–102.

[48] Figure originally created by Drschawrz, "available at English Wikipedia: https://commons.wikimedia.org/wiki/File:SporadicGroups.svg/media/File:SporadicGroups.svg."

[49] J. G. Thompson, *Some numerology between the fischer-griess monster and the elliptic modular function*, *Bulletin of the London Mathematical Society* **11** (Oct., 1979) 352–353.

[50] A. P. Ogg, *Automorphismes de courbes modulaires*, *Séminaire Delange-Pisot-Poitou. Théorie des nombres* **16** (1974-1975) .

[51] S. D. Smith, *On the head characters of the monster simple group*, *Finite Groups—Coming of Age (Montreal, Que., 1982), volume 45 of Contemp. Math., pages 303 - 313. Amer. Math. Soc., Providence, RI, 1985* .

[52] I. B. Frenkel, J. Lepowsky and A. Meurman, *A Moonshine Module for the Monster*, pp. 231–273. Springer US, New York, NY, 1985. 10.1007/978-1-4613-9550-8_12.

[53] R. Borcherds, *Generalized kac-moody algebras*, *Journal of Algebra* **115** (June, 1988) 501–512.

[54] S. P. Norton, *Generalized moonshine*, *Proc.Symp.Pure Math.* **47** (1987) 208–208.

[55] S. Carnahan, *Generalized moonshine i: Genus zero functions*, 0812.3440v3.

[56] S. Carnahan, *Generalized moonshine ii: Borcherds products*, 0908.4223v3.

[57] S. Carnahan, *Generalized moonshine iv: Monstrous lie algebras*, 1208.6254v3.

[58] M. C. N. Cheng, X. Dong, J. F. R. Duncan, S. Harrison, S. Kachru and T. Wrase, *Mock Modular Mathieu Moonshine Modules*, 1406.5502.

[59] E. Witten, *Elliptic Genera and Quantum Field Theory*, *Commun.Math.Phys.* **109** (1987) 525.

[60] J. Troost, *The non-compact elliptic genus: mock or modular*, *Journal of High Energy Physics* **2010** (Jun, 2010) .

[61] T. Kawai, Y. Yamada and S.-K. Yang, *Elliptic genera and N=2 superconformal field theory*, *Nucl. Phys.* **B414** (1994) 191–212, [hep-th/9306096].

[62] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy*, *Nucl.Phys.* **B315** (1989) 193.

[63] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, *Invent. Math.* **94** (1988) 183–221.

[64] S. Kondo, *Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of* $K3$ *surfaces*, *Duke Math. J.* **92** (04, 1998) 593–603.

[65] J. R. David, D. P. Jatkar and A. Sen, *Product representation of Dyon partition function in CHL models*, *JHEP* **06** (2006) 064, [hep-th/0602254].

[66] M. C. N. Cheng, S. M. Harrison, R. Volpato and M. Zimet, *K3 String Theory, Lattices and Moonshine*, 1612.04404.

[67] J. F. R. Duncan and S. Mack-Crane, *Derived Equivalences of K3 Surfaces and Twined Elliptic Genera*, 1506.06198.

[68] N. M. Paquette, R. Volpato and M. Zimet, *No More Walls! A Tale of Modularity, Symmetry, and Wall Crossing for 1/4 BPS Dyons*, *JHEP* **05** (2017) 047, [1702.05095].

[69] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch and R. Volpato, *Generalized Mathieu Moonshine*, *Commun. Num. Theor Phys.* **07** (2013) 145–223, [1211.7074].

[70] A. Chattopadhyaya and J. R. David, *Gravitational couplings in* $\mathcal{N} = 2$ *string compactifications and Mathieu Moonshine*, *JHEP* **05** (2018) 211, [1712.08791].

[71] C. A. Keller and I. G. Zadeh, *Lifting* $\frac{1}{4}$*-BPS States on K3 and Mathieu Moonshine*, 1905.00035.

[72] C. A. Keller and I. G. Zadeh, *Conformal perturbation theory for twisted fields*, *J.Phys.A* **53** (2020) 095401, [1907.08207].

[73] D. Zagier, *Traces of singular moduli in "Motives, polylogarithms and Hodge Theory", Lecture series 3, Intl*, 2002.

[74] T. Eguchi and K. Hikami, *N=2 Superconformal Algebra and the Entropy of Calabi-Yau Manifolds*, *Lett. Math. Phys.* **92** (2010) 269–297, [1003.1555].

[75] W. P. Barth, K. Hulek, C. A. M. Peters and A. Ven, *Compact Complex Surfaces*. Springer Berlin Heidelberg, 2004, 10.1007/978-3-642-57739-0.

[76] T. Eguchi and K. Hikami, *Enriques moonshine*, *J. Phys.* **A46** (2013) 312001, [1301.5043].

[77] S. Mukai, *Lecture notes on K3 and Enriques surfaces Notes by Slawomir Rams*, 2012.

[78] T. Eguchi and K. Hikami, *N=2 Moonshine*, *Phys.Lett.* **B717** (2012) 266–273, [1209.0610].

[79] M. C. N. Cheng, S. M. Harrison, S. Kachru and D. Whalen, *Exceptional Algebra and Sporadic Groups at c=12*, 1503.07219.

[80] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of two-dimensional N=2 gauge theories with rank-one gauge groups*, *Lett. Math. Phys.* **104** (2014) 465–493, [1305.0533].

[81] M. Kreuzer and H. Skarke, "Calabi-Yau data." http://hep.itp.tuwien.ac.at/~kreuzer/CY/.

[82] A. P. Braun, J. Knapp, E. Scheidegger, H. Skarke and N.-O. Walliser, *PALP - a User Manual*, in *Strings, gauge fields, and the geometry behind: The legacy of Maximilian Kreuzer* (A. Rebhan, L. Katzarkov, J. Knapp, R. Rashkov and E. Scheidegger, eds.), pp. 461–550. 2012. 1205.4147. DOI.

[83] E. Witten, *On the Landau-Ginzburg description of N=2 minimal models*, *Int.J.Mod.Phys.* **A9** (1994) 4783–4800, [hep-th/9304026].

[84] M. C. N. Cheng, F. Ferrari, S. M. Harrison and N. M. Paquette, *Landau-Ginzburg Orbifolds and Symmetries of K3 CFTs*, *JHEP* **01** (2017) 046, [1512.04942].

[85] J. A. Harvey and G. W. Moore, *Algebras, BPS states, and strings*, *Nucl.Phys.* **B463** (1996) 315–368, [hep-th/9510182].

[86] G. Lopes Cardoso, G. Curio and D. Lüst, *Perturbative couplings and modular forms in N=2 string models with a Wilson line*, *Nucl.Phys.* **B491** (1997) 147–183, [hep-th/9608154].

[87] I. Antoniadis, E. Gava and K. Narain, *Moduli corrections to gravitational couplings from string loops*, *Phys.Lett.* **B283** (1992) 209–212, [hep-th/9203071].

[88] I. Antoniadis, E. Gava and K. Narain, *Moduli corrections to gauge and gravitational couplings in four-dimensional superstrings*, *Nucl.Phys.* **B383** (1992) 93–109, [hep-th/9204030].

[89] S. Cecotti and C. Vafa, *Ising model and N=2 supersymmetric theories*, *Commun. Math. Phys.* **157** (1993) 139–178, [hep-th/9209085].

[90] S. Kachru and C. Vafa, *Exact results for N=2 compactifications of heterotic strings*, *Nucl.Phys.* **B450** (1995) 69–89, [hep-th/9505105].

[91] M. C. N. Cheng, X. Dong, J. F. R. Duncan, S. Harrison, S. Kachru and T. Wrase, *Mock Modular Mathieu Moonshine Modules*, 1406.5502.

[92] A. Chattopadhyaya and J. R. David, $\mathcal{N}=2$ *heterotic string compactifications on orbifolds of* $K3\times T^2$, *JHEP* **01** (2017) 037, [1611.01893].

[93] S. Harrison, S. Kachru and N. M. Paquette, *Twining Genera of (0,4) Supersymmetric Sigma Models on K3*, *JHEP* **04** (2014) 048, [1309.0510].

[94] M. A. Walton, *The Heterotic String on the Simplest Calabi-yau Manifold and Its Orbifold Limits*, *Phys. Rev.* **D37** (1988) 377.

[95] R. Blumenhagen, D. Lüst and S. Theisen, *Basic concepts of string theory*. Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013, 10.1007/978-3-642-29497-6.

[96] G. Aldazabal, A. Font, L. E. Ibanez and F. Quevedo, *Chains of N=2, D = 4 heterotic type II duals*, *Nucl. Phys.* **B461** (1996) 85–100, [hep-th/9510093].

[97] S. Datta, J. R. David and D. Lust, *Heterotic string on the CHL orbifold of K3*, *JHEP* **02** (2016) 056, [1510.05425].

[98] D. Lust, *Dual string vacua with N = 2 supersymmetry in four dimensions*, *Nucl. Phys. B Proc. Suppl.* **62** (1998) 375–403, [hep-th/9803072].

[99] G. Honecker and M. Trapletti, *Merging Heterotic Orbifolds and K3 Compactifications with Line Bundles*, *JHEP* **01** (2007) 051, [hep-th/0612030].

[100] S. Cecotti, P. Fendley, K. A. Intriligator and C. Vafa, *A New supersymmetric index*, *Nucl.Phys.* **B386** (1992) 405–452, [hep-th/9204102].

[101] S. Stieberger, *(0,2) heterotic gauge couplings and their M theory origin*, *Nucl.Phys.* **B541** (1999) 109–144, [hep-th/9807124].

[102] S. Chaudhuri, G. Hockney and J. D. Lykken, *Maximally supersymmetric string theories in* $D<10$, *Phys. Rev. Lett.* **75** (1995) 2264–2267, [hep-th/9505054].

[103] S. Chaudhuri and J. Polchinski, *Moduli space of CHL strings*, *Phys. Rev.* **D52** (1995) 7168–7173, [hep-th/9506048].

[104] P. S. Aspinwall, *Some relationships between dualities in string theory*, *Nucl. Phys. Proc. Suppl.* **46** (1996) 30–38, [hep-th/9508154].

[105] V. V. Nikulin, *Finite automorphism groups of Kählerian surfaces of type K3.*, *Tr. Mosk. Mat. O.-va* **38** (1979) 75–137.

[106] M. R. Gaberdiel and R. Volpato, *Mathieu Moonshine and Orbifold K3s*, *Contrib. Math. Comput. Sci.* **8** (2014) 109–141, [1206.5143].

[107] I. Antoniadis, E. Gava, K. Narain and T. Taylor, *Topological amplitudes in string theory*, *Nucl.Phys.* **B413** (1994) 162–184, [hep-th/9307158].

[108] I. Antoniadis, E. Gava, K. Narain and T. Taylor, *N=2 type II heterotic duality and higher derivative F terms*, Nucl.Phys. **B455** (1995) 109–130, [hep-th/9507115].

[109] R. E. Borcherds, *Automorphic forms with singularities on grassmannians*, Inventiones Mathematicae **132** (May, 1998) 491Äì562.

[110] R. Gopakumar and C. Vafa, *M theory and topological strings. 1.*, hep-th/9809187.

[111] R. Gopakumar and C. Vafa, *M theory and topological strings. 2.*, hep-th/9812127.

[112] M. Marino, *Enumerative geometry and knot invariants*, in *70th Meeting between Physicists, Theorist and Mathematicians Strasbourg, France, May 23-25, 2002*, 2002, hep-th/0210145.

[113] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, Nucl. Phys. **B433** (1995) 501–554, [hep-th/9406055].

[114] C. F. Cota, A. Klemm and T. Schimannek, *Topological strings on genus one fibered Calabi-Yau 3-folds and string dualities*, JHEP **11** (2019) 170, [1910.01988].

[115] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, *Topological string amplitudes, complete intersection Calabi-Yau spaces and threshold corrections*, JHEP **0505** (2005) 023, [hep-th/0410018].

[116] M.-x. Huang, S. Katz and A. Klemm, *Topological string on elliptic cy 3-folds and the ring of jacobi forms*, Journal of High Energy Physics **2015** (Oct, 2015) .

[117] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, Commun. Math. Phys. **167** (1995) 301–350, [hep-th/9308122].

[118] P. S. Aspinwall, *K3 surfaces and string duality, Yau, S.T. (ed.): Differential geometry inspired by string theory* (1996) 1–95, [hep-th/9611137].

[119] A. P. Braun and T. Watari, *Heterotic-Type IIA Duality and Degenerations of K3 Surfaces*, JHEP **08** (2016) 034, [1604.06437].

[120] P. S. Aspinwall and J. Louis, *On the ubiquity of K3 fibrations in string duality*, Phys.Lett. **B369** (1996) 233–242, [hep-th/9510234].

[121] C. Vafa and E. Witten, *Dual string pairs with N=1 and N=2 supersymmetry in four-dimensions*, Nucl. Phys. B Proc. Suppl. **46** (1996) 225–247, [hep-th/9507050].

[122] C. Hull and P. Townsend, *Unity of superstring dualities*, Nucl. Phys. B **438** (1995) 109–137, [hep-th/9410167].

[123] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. B **443** (1995) 85–126, [hep-th/9503124].

[124] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, Phys.Lett. **B361** (1995) 59–65, [hep-th/9505162].

[125] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 1*, Nucl.Phys. **B473** (1996) 74–92, [hep-th/9602114].

[126] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 2.*, Nucl.Phys. **B476** (1996) 437–469, [hep-th/9603161].

[127] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov et al., *Geometric singularities and enhanced gauge symmetries*, Nucl.Phys. **B481** (1996) 215–252, [hep-th/9605200].

[128] A. Klemm, W. Lerche and P. Mayr, *K3 Fibrations and heterotic type II string duality*, *Phys. Lett. B* **357** (1995) 313–322, [hep-th/9506112].

[129] V. Braun and D. R. Morrison, *F-theory on genus-one fibrations*, *Journal of High Energy Physics* **2014** (Aug, 2014) .

[130] M. Kreuzer and H. Skarke, *Classification of reflexive polyhedra in three dimensions*, *Advances in Theoretical and Mathematical Physics* **2** (1998) 853Äì871.

[131] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, vol. 290. Springer Science & Business Media, 2013.

[132] S. Odake, *Extension of $N = 2$ Superconformal Algebra and Calabi-Yau Compactification*, *Mod. Phys. Lett.* **A4** (1989) 557.

[133] S. Odake, *Character Formulas of an Extended Superconformal Algebra Relevant to String Compactification*, *Int. J. Mod. Phys.* **A5** (1990) 897.

[134] S. Odake, *$C = 3$-d Conformal Algebra With Extended Supersymmetry*, *Mod. Phys. Lett.* **A5** (1990) 561.

[135] T. Eguchi and A. Taormina, *Character Formulas for the $N = 4$ Superconformal Algebra*, *Phys. Lett.* **B200** (1988) 315.

[136] T. Eguchi and K. Hikami, *N=4 Superconformal Algebra and the Entropy of HyperKahler Manifolds*, *JHEP* **02** (2010) 019, [0909.0410].

[137] K. Becker, M. Becker and J. H. Schwarz, *String theory and M-theory: A modern introduction*. Cambridge University Press, 2006.

[138] M. Kreuzer and H. Skarke, *No mirror symmetry in Landau-Ginzburg spectra!*, *Nucl. Phys.* **B388** (1992) 113–130, [hep-th/9205004].

[139] C. Vafa, *Quantum Symmetries of String Vacua*, *Mod. Phys. Lett.* **A4** (1989) 1615.

[140] H. Skarke, *String dualities and toric geometry: An Introduction*, *Chaos Solitons Fractals* **10** (1999) 543, [hep-th/9806059].

[141] M. Kreuzer, *Toric geometry and Calabi-Yau compactifications*, *Ukr. J. Phys.* **55** (2010) 613–625, [hep-th/0612307].

[142] V. V. Batyrev, *Dual polyhedra and mirror symmetry for calabi-yau hypersurfaces in toric varieties*, 1993.

[143] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four-dimensions*, *Adv. Theor. Math. Phys.* **4** (2002) 1209–1230, [hep-th/0002240].

[144] R. Wazir, *Arithmetic on elliptic threefolds*, Compositio Mathematica **140** (May, 2004) 567Äì580.

[145] D. S. Park, *Anomaly equations and intersection theory*, Journal of High Energy Physics **2012** (Jan, 2012) .

[146] A. Klemm, J. Manschot and T. Wotschke, *Quantum geometry of elliptic Calabi-Yau manifolds*, 1205.1795.

[147] M. Alim and E. Scheidegger, *Topological Strings on Elliptic Fibrations*, 1205.1784.

[148] E. Witten, *Quantum background independence in string theory*, 1993.

[149] M. Aganagic, V. Bouchard and A. Klemm, *Topological strings and (almost) modular forms*, Communications in Mathematical Physics **277** (Nov, 2007) 771Äì819.

[150]

M. G$\sqrt{}^{\circ}$*naydin, A. NeitzkeandB. Pioline*, *Topological wave functions and heat equations*, Journal of

[151] N. Yamatsu, *Finite-dimensional lie algebras and their representations for unified model building*, 2015.

[152] M. Srednicki, *Quantum Field Theory*. Cambridge Univ. Press, Cambridge, 2007.

[153] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rept. **79** (1981) 1–128.

[154] C. Nazaroglu, *Jacobi forms of higher index and paramodular groups in* $\mathcal{N} = 2, D = 4$ *compactifications of string theory*, JHEP **12** (2013) 074, [1309.6404].

Andreas BANLAKI

Institute for Theoretical Physics, Vienna University of Technology
Wiedner Hauptstrasse 8-10/136, A-1040 Vienna, Austria
E-mail: banlaki@hep.itp.tuwien.ac.at
Telephone: +43 699 125 66 131

Personal Information

Date of Birth	August 13, 1979
Place of Birth	Vienna, Austria
Nationality	Austrian
Address	Rückaufgasse 33/3,1190 Vienna, Austria

University Education

Since September 2016	**PhD in "Theoretical Physics", Vienna University of Technology** • Topic: "Strings, Cosmology and Moonshine"; Advisor: Timm Wrase
Nov. 2012	**Vienna University of Technology, Master degree (Dipl. Ing.) in Physics, with distinction** • Specialization: Theoretical physics • Master's Thesis: "Study of Electrodynamics in the Soliton Model"
Oct. 2011	Continuation of studies in Physics, Vienna University of Technology
Oct. 2010	**Vienna University of Technology, Master degree (Dipl. Ing.) in Mathematics** • Specialization: Mathematics in Science • Master's Thesis: "Variations of the Classical Galois Connection Operations-Relations"
Oct. 2005	1st Diploma Mathematics, Vienna University of Technology
Oct. 2003	Shift study focus to Mathematics
Jan. 2000	1st Diploma in Physics, Vienna University of Technology
Oct. 1998	Start of undergraduate studies in Mathematics, Vienna University of Technology
Oct. 1997	Start of undergraduate studies in Physics, Vienna University of Technology

Work Experience

Mar. 2012 – Oct. 2013	**Consultant at ifb International AG, Austria**
May 2011 – Nov. 2011	**Internship in the Market-Risk Group of Bank Austria UniCredit**

Teaching experience

2000 – 2003	Tutor for the courses „Mathematik I“ und „Mathematik II“ at the University of Natural Resources and Life Science, Vienna

Languages

German-Danish bilingual
English - fluent

Software Skills

Mathematica, Matlab, Fortran, SQL, LaTeX good knowledge
R, C basic knowledge

PhD Schools

4-9 Jul. 2019	Pre-Strings 2019 Leuven, Irish College, Leuven, Belgium
4-8 Feb. 2019	CERN Winter School on Supergravity, Strings and Gauge Theory, CERN, Geneva, Switzerland
11-23 Jun. 2018	CARGESE Summer School, Quantum Gravity, Strings and Fields 2018, Cargese, IESC, France
12-16 Feb. 2018	CERN Winter School on Supergravity, Strings and Gauge Theory, CERN, Geneva, Switzerland
28 Jan.- 2 Feb. 2018	Winter School on Partition Functions and Automorphic Forms, BLTP JINR, Dubna, Russia
09-13 Oct. 2017	Workshop "Fields and Dualities 2017", ASC, Munich, Germany
27-31 Mar. 2017	GGI School "AdS3: theory and practice", GGI, Florence, Italy
16-24 Mar. 2017	ICTP Spring School on Superstring Theory and Related Topics, ICTP, Trieste, Italy
6-10 Feb. 2017	CERN Winterschool on Strings and Fields, CERN, Geneva, Switzerland
28 Aug.-10 Sep. 2017	Helmoltz-DIAS International Summer School "Cosmology, Strings, New Physics", BLTP JINR, Dubna, Russia

Talk at Conference

10-14 Sep. 2018	Workshop on "Moonshine", Erwin Schrödinger Institut, Vienna, Austria Talk: "Gromov-Witten Invariants and Moonshine"

Publications

1. **Heterotic strings on (K3xT^2)/Z_3 and their dual Calabi-Yau threefolds**
   A. Banlaki, A. Chattopadhyaya, A. Kidambi, T. Schimannek, M. Schimpf
   JHEP 04 (2020) 203
   doi: 10.1007/JHEP04(2020)203
   arXiv: 1911.09697 [hep-th]

2. **On Mathieu Moonshine and Gromov-Witten invariants**
   A. Banlaki, A. Chowdhury, A. Kidambi, M. Schimpf
   JHEP 02 (2020) 082
   doi: 10.1007/JHEP02(2020)082
   arXiv: 1811.11619 [hep-th]

3. **Scaling limits of dS vacua and the swampland**
   A. Banlaki, A. Chowdhury, C. Roupec, T. Wrase
   JHEP 1903 (2019) 065
   doi: 10.1007/JHEP03(2019)065
   arXiv: 1811.07880 [hep-th]

4. **Calabi-Yau manifolds and sporadic groups**
   A Banlaki, A. Chowdhury, A. Kidambi, M. Schimpf, H. Skarke, T. Wrase
   JHEP 1802 (2018) 129
   doi: 10.1007/JHEP02(2018)129
   arXiv:1711.09698 [hep-th]