TECHNISCHE UNIVERSITÄT WIEN

DIPLOMARBEIT

Pricing of simple and path-dependent European Options in the Jacobi Stochastic Volatility Model

zur Erlangung des akademisches Grades

Diplom-Ingenieurin

im Rahmen des Studiums

Finanz- und Versicherungsmathematik

eingereicht von

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ausgeführt am Institut für Stochastik und Wirtschaftsmathematik
der Fakultät für Mathematik und Geoinformation der Technischen Universität Wien

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Wien, Mai 2020

(Unterschrift Verfasserin) (Unterschrift Betreuer)

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IN MEMORIAM TO MY DAD
WHO ENCOURAGED ME TO FINISH MY STUDIES.
HIS LOVE AND TRUST CARRIED ME WITHIN THIS WORK.

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Abstract

We discuss a stochastic volatility model in which the squared volatility is modeled as an affine transformed Jacobi process. In this model, which contains the Heston model as limiting case, the log-price density as well as the density of the finite dimensional distributions of the log-returns admit a closed-form series representation with respect to the generalized Hermite polynomials, known as Gram-Charlier series expansion. We use this to derive series representations for option prices and we find explicit formulas for European call, put and binary options. The pricing technique is expanded to path-dependent European options whose payoff functions depend on finitely many monitoring dates. Approximation errors, which occur by truncation of the series at some finite order, are also studied and illustrated by some numerical examples.

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Chapter 0

Introduction

The aim of this thesis is to discuss the Jacobi stochastic volatility model and the price approximation technique for simple and path-dependent options as proposed and introduced in [1].

In this stochastic volatility model the variance process is modeled as a transformed Jacobi process from which is known that it has a bounded state interval. The boundedness of the state interval first allows a series representation of the log-price density in terms of generalized Hermite polynomials, which is known as the Gram-Charlier series expansion of type A [6, p.222], and second it is also used to define the state-dependent correlation of the stock price and variance process which is in contrast to the assumed constant correlation in other popular stochastic volatility models such as the model by Hull and White in [18], the Stein-Stein model [30] or the famous Heston model [16].

The model for the stock price process is described by a two-dimensional diffusion process under a risk-neutral measure which has the property that its infinitesimal generator maps polynomials into polynomials. Such processes are referred to as polynomial diffusions and are treated e.g. in [12], [7] and [23]. Polynomial diffusions have the nice feature that their moments are tractable, which turns out to be essential for the representation of the log-price density. Formulas for the computation of the finite moments in the one-dimensional case are given e.g. in [13] and in [9] for the square-root process, which is used in the Heston model. Here we derive and use a multidimensional version for the calculation of the finite joint moments.

Option pricing is done under a Gaussian reference measure and the likelihood ratio as well as the discounted payoff function are considered as elements of the L^2 space with respect to the auxiliary measure. Together with the L^2 inner product with respect to the Gaussian measure it becomes a Hilbert space and allows Fourier series representation of the functions. We find series representations of option prices which are interpreted as the inner product of the discounted payoff and the likelihood ratio function in the underlying Hilbert space. The Fourier series representation of the likelihood ratio function from the risk-neutral to the reference measure is related to the Gram-Charlier Type A series expansion of the unknown log-price density which is given in closed-form due to the polynomial property of the model. For the discounted payoff function only the square-integrability with respect to the density of a normal distribution is required for the interpretation of the option price as inner product. Closed-form series representations of option prices for European call and put options as well as for binary options are found. However, if the payoff function of an European claim is not square integrable or no explicit formulas for the Fourier coefficients in the Fourier series representation of the payoff function are available, the computation of

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the option price can be done by Gauss-Hermite quadrature.

Path-dependent options with discounted payoff functions dependent on finitely many observation points are treated as the multidimensional extension of the foregoing pricing technique. The joint density of the log-returns also allows a Gram-Charlier Type A series expansion in terms of products of generalized Hermite polynomials with closed-form coefficients. As reference measure the multivariate Gaussian measure will be taken and even if the Fourier coefficients cannot be calculated explicitly, the option price can be computed by multivariate Gauss-Hermite quadrature.

The thesis is organized as follows: In the first chapter the theory of polynomial diffusions and mathematical tools which are needed later on are presented (compare [5], [22], [21], [26], [27] and [17]). The second chapter is intended to give an introduction to stochastic volatility modeling and focuses on financial application of the mathematical theory, based on [3], [14] and [32]. The Jacobi stochastic volatility model is introduced and discussed in the third chapter and the option valuation in this model is presented in the fourth chapter. In the appendix a documented code for the implementation as well as the source code of the studied examples written in the programming language `R` are given.

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Chapter 1

Preliminaries

This chapter collects several definitions and results on which this thesis is built on. It is divided into two sections: In the first section polynomial diffusions are introduced. In order to show afterwards that the stochastic volatility model in discussion is well-defined, existence and uniqueness results are given. The key property of polynomial diffusions, namely that the conditional expectation of a polynomial is a polynomial of the same or lower degree, is then derived and termed as 'moment formula'.

The second section covers the mathematical tools, which will be used for a series representation of the log-price density and of the payoff function for pricing options in the model.

1.1 Introduction to Polynomial Diffusions

Typically the stochastic processes used in financial applications are described as solution of stochastic differential equations (SDE). In this thesis we consider stochastic differential equations for a d-dimensional continuous stochastic process $X = (X_t)_{t\geq 0}$ of the following form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0, \tag{1.1}$$

where $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are continuous Borel-measurable functions and $W = (W_t)_{t\geq 0}$ denotes a d-dimensional standard Brownian motion. The vector μ represents the short-term growth of X and is referred to as drift vector, whereas the $d \times d$-matrix σ describes the short-term variability of the process, which is also known as dispersion matrix. The diffusion matrix a is given by $a = \sigma\sigma^\top$.

Definition 1.1. By a solution of (1.1) we mean a d-dimensional stochastic process X defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ of sub-σ algebras of $\mathscr{F}$ satisfying the usual conditions of completeness and right-continuity such that

(i) there exists a d-dimensional $\mathbb{F}$-standard Brownian motion $W = (W_t)_{t\geq 0}$ and

(ii) X is a d-dimensional continuous process adapted to $\mathbb{F}$ and has the Ito representation (integral representation)

$$X_t(\omega) = x_0 + \int_0^t \mu(X_s(\omega))ds + \int_0^t \sigma(X_s(\omega))dW_s \quad \mathbb{P} - a.s. \tag{1.2}$$

with $\mathbb{P}(\int_0^t |\mu_i(X_s(\omega))| + \sigma_{ij}^2(X_s(\omega))ds < \infty) = 1$ for all $0 \leq t < \infty$ and $1 \leq i, j \leq d$.

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We refer to a solution X of (1.1) to be unique, if any another solution X' has the same law as X.

Remark 1.2. This concept of solution is known as weak solution of the SDE, since neither the probability space nor the underlying Brownian motion are given in advance, but both are part of the construction of the solution.

For $f \in C_b^2(\mathbb{R}^d)$[1] we define the second order partial differential operator

$$\mathscr{G}f(x) := \sum_{i=1}^{d} \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i x_j}(x), \tag{1.3}$$

where $a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x)$ and which can be written in vector form as [2]

$$\mathscr{G}f(x) := \mu(x)^\top \nabla f(x) + \frac{1}{2} \operatorname{tr}\left(a(x)\nabla^2 f(x)\right).$$

Note, that the operator $\mathscr{G}$ corresponds to the infinitesimal drift term in Ito's formula, if f is applied on X.

The next proposition relates the existence of a solution of the SDE with the action of $\mathscr{G}$ on $C_b^2(\mathbb{R}^d)$. With Ito's formula and the representation theorem of locally square-integrable martingales (see [19, Theorem 7.1', p.90]) we have the following equivalent characterization:

Proposition 1.3. *If X is an $\mathbb{F}$-adapted solution of (1.1), then*

$$f(X_t) - f(x_0) - \int_0^t \mathscr{G}f(X_s)ds, \quad t \geq 0, \tag{1.4}$$

is a continuous locally square-integrable $\mathbb{F}$-martingale for every $f \in C_b^2(\mathbb{R}^d)$.

Conversely, if X is an $\mathbb{F}$-adapted d-dimensional process such that (1.4) is a continuous locally square-integrable martingale for all $f \in C_b^2(\mathbb{R}^d)$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, then on an extension $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathscr{F}, \mathbb{P})$ we find a d-dimensional $\tilde{\mathbb{F}}$-standard Brownian motion such that X is a solution of (1.1), where $\tilde{\mathbb{F}}$ is as well an extension of $\mathbb{F}$.

Proof. See [19, Proposition 2.1, p.169]. □

1.1.1 Polynomial diffusions

In the following we define a class of stochastic processes, which are characterized by the action of $\mathscr{G}$ on the ring of polynomials over $\mathbb{R}$, denoted by $\mathrm{Pol}(\mathbb{R}^d)$. The next definition and lemma are mainly a restatement of Definition 2.1 and Lemma 2.2 in [12].

[1] Notation: $C_b^2(\mathbb{R}^d)$ denotes the set of two times continuously differentiable bounded functions with bounded derivatives up to second order.

[2] Notation: ∇f denotes the gradient of f, $\nabla^2 f$ its Hessian matrix, $\operatorname{tr}(A)$ denotes the trace of the matrix A and by $^\top$ the transpose is meant.

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Definition 1.4. The operator $\mathscr{G}$ from (1.3) is called polynomial preserving, if it is an endomorphism of the subset $\mathrm{Pol}_n(\mathbb{R}^d)$ for each $n \in \mathbb{N}$, where $\mathrm{Pol}_n(\mathbb{R}^d)$ denotes the ring of all polynomials on $\mathbb{R}^d$ of degree at most n.

Let $E \subseteq \mathbb{R}^d$. We say that $\mathscr{G}$ is well-defined on $\mathrm{Pol}(E)$, the space of all polynomials restricted on E, if for any $f \in \mathrm{Pol}(E)$:

$$\mathscr{G}f(x) = 0 \ \forall x \in E \Rightarrow f(x) = 0 \ \forall x \in E.$$

The operator $\mathscr{G}$ is called polynomial on E, if it is well-defined on $\mathrm{Pol}(E)$ and polynomial preserving.

Lemma 1.5. *Let $\mathscr{G}$ be well-defined on $Pol(E)$. The operator $\mathscr{G}$ maps $Pol_n(E)$ into itself for each $n \in \mathbb{N}$ if and only if for the SDE given by (1.1) the components of the drift vector μ lie in $Pol_1(E)$ and those of the diffusion matrix $a = (a_{ij})_{1\leq i,j\leq d}$ in $Pol_2(E)$.*

Proof. If $\mathscr{G}$ is polynomial on E, then in particular for any monomial in $\mathrm{Pol}_1(E)$ we have $\mathrm{Pol}_1(E) \ni \mathscr{G}x_i = \mu_i$, $1 \leq i \leq d$. Considering the monomials x_ix_j, $1 \leq i,j \leq d$ for which by assumption $\mathscr{G}(x_ix_j)$ lie in $\mathrm{Pol}_2(E)$, we have $\mathscr{G}(x_ix_j) = a_{ij} + \mu_ix_i + \mu_jx_j$, hence $a_{ij} \in \mathrm{Pol}_2(E)$.

For the reverse direction let $p \in \mathrm{Pol}_N(E)$ with $\deg(p) = k$, $0 \leq k \leq N$. The components of the gradient ∇p are each polynomials of degree $\max(k-1, 0)$ and those of the Hessian matrix $\nabla^2 p$ are polynomials of degree $\max(k-2, 0)$. Since the entries of μ and a are also polynomials of degree one resp. at most two, we certainly have $\mathscr{G}p \in \mathrm{Pol}(E)$ having degree $\deg(\mathscr{G}p) = \max\{\deg(b^\top \nabla p), \deg(\mathrm{tr}(a\nabla^2 p))\} = k$. Thus, $\mathscr{G}Pol_N(E) \subseteq \mathrm{Pol}_N(E)$ for arbitrary $N \in \mathbb{N}$. □

Definition 1.6. Let $E \subseteq \mathbb{R}^d$. A d-dimensional polynomial diffusion X is a continuous E-valued stochastic process, which is a solution of the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad \mu_i \in \mathrm{Pol}_1(\mathbb{R}^d),\ a_{ij} \in Pol_2(\mathbb{R}^d),\ 1 \leq i,j \leq d, \tag{1.5}$$

where $a = \sigma\sigma^\top$ is a positive semi-definite $d \times d$- matrix on E and for which the associated operator $\mathscr{G}$ is well-defined on $\mathrm{Pol}(E)$.

Remarks 1.7. Note, that in Definition 1.6 neither the uniqueness of solutions nor that the solution is a Markov process on E is required.

A polynomial diffusion which is also a Markov process is termed as 'polynomial process' in [7]. More precisely, a time-homogeneous Markov process with state space E is a n-polynomial process with state space E, if and only if the operator $\mathscr{G}$ leaves $\mathrm{Pol}_n(E)$ invariant for some $n \in \mathbb{N}$, i.e. $\mathscr{G}Pol_m(E) \subset Pol_m(E)$ holds for all $m \leq n$ (see Theorem 2.7 in [7]). For Markov processes the operator $\mathscr{G}$ is then referred to as (infinitesimal) generator of X.

1.1.2 State space and Existence of solutions

As it is done in [12], we consider as state space a semi-algebraic set: Let $\mathscr{P} \subset \mathrm{Pol}(\mathbb{R}^d)$ denote a finite collection of polynomials on $\mathbb{R}^d$. We define the state space E as

$$E = \{x \in \mathbb{R}^d : p(x) \geq 0 \quad \forall p \in \mathscr{P}\},$$

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i.e. as the finite intersection of subsets of $\mathbb{R}^d$ for which each $p \in \mathscr{P}$ is positive.

The positive maximum principle guarantees, that for any E-valued solution of (1.5) with initial value $X_0 \in E$ the operator $\mathscr{G}$ is well-defined on $\mathrm{Pol}(E)$ as the next lemma shows.

Lemma 1.8. *Let $p \in Pol(\mathbb{R}^d)$ and $\bar{x} \in E$ be a maximizer of p over E, i.e. $p(\bar{x}) = \sup_{x \in E} p(x)$. For an E-valued solution of (1.5) with initial value $X_0 = \bar{x}$ we have $\mathscr{G}p(\bar{x}) \leq 0$.*

We cite the proof of Lemma 2.3 in [12].

Proof. We prove the lemma by contradiction. Let X be an E-valued solution of (1.5) and assume $\mathscr{G}p(\bar{x}) > 0$. Since $\bar{x}$ is a maximizer over E we have $p(x) \leq p(\bar{x})$ for all $x \in E$. Let $\tau := \inf\{t \geq 0 : \mathscr{G}p(X_t) \leq 0\}$ denote the first time where the image of $\mathscr{G}p$ is not strictly positive. For $t \in (0, \tau)$ we have $p(X_t) \leq p(\bar{x})$ and $\mathscr{G}p(X_t) > 0$, which implies for the stopped process

$$p(X_{t\wedge\tau}) - p(\bar{x}) - \int_0^{t\wedge\tau} \mathscr{G}p(X_s)ds < 0 \quad \forall t > 0.$$

Since X is a solution, the left hand side is a local martingale starting from zero and strictly negative for all $t > 0$, which yields the desired contradiction. Therefore we must have $\mathscr{G}p(\bar{x}) \leq 0$. □

We outline the main results for the existence of an E-valued solution of (1.5) in the next theorem. The first part gives necessary conditions for the invariance of E, the second summarizes, using the geometric and algebraic properties of the state space, the sufficient conditions for a SDE of the form (1.5) to have an E-valued solution for every initial value in E. The formulation is a special case of Theorem 5.1 and Theorem 5.3 in [12].

Theorem 1.9. *(i) Let X be an E-valued solution of (1.5) for any $X_0 = x_0 \in E$. Then*

$$a(x)\nabla p(x) = 0 \text{ and } \mathscr{G}p(x) \geq 0 \text{ on the boundary } E \cap \{p = 0\} \ \forall p \in \mathscr{P}.$$

(ii) Let E be defined as above by a finite collection of polynomials $\mathscr{P}$ such that for each $p \in \mathscr{P}$ the ideal of p of the polynomial ring $Pol(\mathbb{R}^d)$ denoted by (p) equals the ideal generated by the zero set $Z = \{x \in \mathbb{R}^d : p(x) = 0\}$ of p, i.e.

$$\mathscr{I}(Z) := \{q \in Pol(\mathbb{R}^d) : \ q(x) = 0 \quad \forall x \in Z\}.$$

Further let the $d \times d$-matrix a be positive semi-definite on E as well as

$$a(x)\nabla p(x) = 0 \text{ on } \{p = 0\} \text{ and } \mathscr{G}p(x) > 0 \text{ on } E \cap \{p = 0\} \ \forall p \in \mathscr{P}.$$

Then $\mathscr{G}$ is polynomial on E and (1.5) admits an E-valued solution for every initial value in E.

Remarks 1.10. The property $a\nabla p = 0$ on $\bigcap_{p \in \mathscr{P}} \{p = 0\}$ says that there is no fluctuation orthogonal to the boundary.

The condition $\mathscr{G}p \geq 0$ is interpreted as inward pointing drift at the boundary and this condition is strengthened in (ii) to ensure that X has no absorbing state at the boundary of E.

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In the next theorem the boundary attainment of E-valued solutions of (1.5) is considered. As before we give necessary and sufficient conditions for a solution to stay at the interior $E \setminus \bigcup_{p\in\mathscr{P}}\{p=0\}$. The results are derived from Theorem 5.7 in [12] in conjunction with Proposition 2.2 in [23]. For a proof we refer to [12].

Theorem 1.11. *Let X be an E-valued solution of (1.5). For each $p \in \mathscr{P}$ let h_p be the vector of polynomials satisfying $a(x)\nabla p(x) = h_p(x)p(x)$.*

(i) If there exists a neighborhood U of $E \cap \{p=0\}$ such that

$$2\mathscr{G}p(x) - h_p(x)^\top \nabla p(x) \geq 0 \text{ on } E\cap U \quad \text{or} \quad 2\mathscr{G}p(x) - h_p(x)^\top \nabla p(x) = 0 \text{ on } \{p=0\},$$

then $p(X_t) > 0$ for all $t > 0$.

(ii) Let $\bar{x} \in E \cap \{p=0\}$ be a boundary point. If

$$\mathscr{G}p(\bar{x}) \geq 0 \quad \text{and} \quad 2\mathscr{G}p(\bar{x}) - h_p(\bar{x})^\top \nabla p(\bar{x}) < 0,$$

then $p(X_t) = 0$ for some finite t with positive probability.

Corollary 1.12. *The interior $E \setminus \bigcup_{p\in\mathscr{P}}\{p=0\}$ is invariant for X if and only if*

$$2\mathscr{G}p(x) - h_p(x)^\top \nabla p(x) \geq 0 \quad \text{for all } x \in \bigcup_{p\in\mathscr{P}}\{p=0\}$$

and for all $p \in \mathscr{P}$ with corresponding h_p as in Theorem 1.11.

1.1.3 Uniqueness and Moment formula

We make use of the following proposition to prove uniqueness of E-valued solutions of (1.5). The imposed condition ensures that the moment generating function of X exists in some neighborhood of the origin for all $t \geq 0$ which in turn uniquely determines the distribution of X ([8, Theorem 1, p.430]). The result is taken from Lemma 4.1 and Theorem 3.3 in [12].

Proposition 1.13. *Let X be an E-valued solution of (1.5) with deterministic initial value $X_0 = x_0$. If for each $t > 0$ there exists an $\epsilon > 0$ such that $\mathbb{E}_{\mathbb{P}}\left[\exp\left(\epsilon\|X_t\|\right)\right] < \infty$ and X' is another E-valued solution starting at x_0, then X and X' have the same law.*

In particular, if the linear growth condition

$$\|a(x)\| \leq C(1+\|x\|) \quad \text{for all } x \in E \tag{1.6}$$

for some constant C is satisfied, then X admits finite exponential moments for all $t \geq 0$.

From the polynomial property of the associated operator $\mathscr{G}$ of the polynomial diffusion $(X_t)_{t\geq 0}$ we finally show that the conditional expectation of a polynomial in X_t given the information $\mathscr{F}_s$, $s < t$, is a again a polynomial of the same or lower degree evaluated at the current state. The result will be referred to as 'moment formula' and contains as a special case a formula for the finite joint conditional moments of X.

For the d-dimensional polynomial diffusion $(X_t)_{t\geq 0}$ with state space E given by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0,$$

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and the associated operator

$$\mathscr{G}p(x) := \mu(x)^\top \nabla p(x) + \frac{1}{2}\operatorname{tr}\left(a(x)\nabla^2 p(x)\right), \quad p \in \operatorname{Pol}(\mathbb{R}^d),$$

let $\operatorname{Pol}(E) = \{p \in \operatorname{Pol}(\mathbb{R}^d) |\ p : E \to \mathbb{R}\}$ denote (as before) the set of polynomials restricted on $E \subseteq \mathbb{R}^d$, which is not only a polynomial ring but also a real vector space with subspace $\operatorname{Pol}_N(E)$ consisting of polynomials with degree at most $N \in \mathbb{N}$. Let M denote the dimension of $\operatorname{Pol}_N(E)$ and $B = \{b_1(x), \dots, b_M(x)\}$ a basis for $\operatorname{Pol}_N(E)$. We write $B(x)^\top = (b_1(x), \dots, b_M(x))^\top$ for the ordered basis vector such that each polynomial $p \in \operatorname{Pol}_N(E)$ can be represented in the form $p(x) = B(x)^\top \vec{p}$ with the coordinate vector $\vec{p} \in \mathbb{R}^M$.

Since $\mathscr{G}$ is a linear mapping of $\operatorname{Pol}_N(E)$ into itself, on the one hand we can find a constant $C > 0$ such that $|\mathscr{G}p(x)| \le Cp(x)$ holds for every $p \in \operatorname{Pol}_N(E)$ and on the other hand the action of $\mathscr{G}$ on $\operatorname{Pol}_N(E)$ can be represented by $\mathscr{G}p(x) = B(x)^\top G\vec{p}$, for any $p \in \operatorname{Pol}_N(E)$ with the coordinate representation $\vec{p}$ and the representation matrix G of $\mathscr{G}$ with respect B.

Further Ito's formula applied to X_t for $p \in \operatorname{Pol}_N(E)$ gives

$$\begin{aligned} p(X_t) &= p(X_0) + \int_0^t \mathscr{G}p(X_s)ds + \int_0^t \sigma(X_s)^\top \nabla p(X_s)dW_s \\ &= \vec{p}^\top B(X_0) + \vec{p}^\top G^\top \int_0^t B(X_s)ds + \int_0^t \sigma(X_s)^\top \nabla p(X_s)dW_s. \end{aligned} \tag{1.7}$$

In the next lemma we establish an inequality, which firstly gives that the stochastic integral in (1.7) is a $\mathbb{P}$-martingale and secondly allows to interchange the order of integration and taking expectation. We make use of Grownwall's inequality, which is given before.

Lemma 1.14 (Grownwall's inequality)**.** *Suppose for a continuous real-valued function h there exists an integrable function $g : [0, T] \to \mathbb{R}$ and a constant $C \ge 0$ such that*

$$0 \le h(t) \le g(t) + C\int_0^t h(s)ds, \quad 0 \le t \le T,$$

holds. Then the inequality

$$h(t) \le g(t) + C\int_0^t g(s)e^{C(t-s)}ds$$

is satisfied for $0 \le t \le T$.
In particular for $g(t) \equiv k$, we have $h(t) \le ke^{Ct}$ for $0 \le t \le T$.

Lemma 1.15. *For every $N \in \mathbb{N}$ and any deterministic initial value $X_0 \in E$ there exists a constant $C > 0$ such that*

$$\mathbb{E}_{\mathbb{P}}[1 + \|X_t\|^{2N}] \le \left(1 + \|X_0\|^{2N}\right)e^{Ct}, \quad t \ge 0.$$

Proof. The proof is an extended version of the proof of Lemma B.1 in [12, p.953] and follows the same idea as in the proof of [7, Theorem 2.10].

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We set $p(x) = 1 + \|x\|^{2N}$ and since $p \in \mathrm{Pol}_{2N}(E)$ we find a constant $C > 0$ such that $|\mathscr{G}p(x)| \leq Cp(x)$ $\forall x \in E$ holds. Let τ_m denote the first exit time of X from the ball $\{x \in \mathbb{R}^d : \|x\| < m\}$. Then, by Ito's lemma we have

$$p(X_{t\wedge\tau_m}) = p(X_0) + \int_0^{t\wedge\tau_m} \mathscr{G}p(X_s)ds + \int_0^{t\wedge\tau_m} \sigma(X_s)^\top \nabla p(X_s)dW_s.$$

Since the stochastic integral is bounded, taking expectations gives

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[p(X_{t\wedge\tau_m})] &= p(X_0) + \mathbb{E}_{\mathbb{P}}[\int_0^{t\wedge\tau_m} \underbrace{\mathscr{G}p(X_s)}_{\leq Cp(X_s)} ds] \\ &\leq p(X_0) + C\ \mathbb{E}_{\mathbb{P}}[\int_0^{t\wedge\tau_m} p(X_s)ds] \\ &\leq p(X_0) + C\ \mathbb{E}_{\mathbb{P}}[\int_0^{t} p(X_{s\wedge\tau_m})ds]. \end{aligned} \tag{1.8}$$

The last inequality follows by the distinction of the cases $\{\tau_m \leq t\}$ and $\{\tau_m > t\}$: On $\{\tau_m > t\}$ we have certainly equality and for $\{\tau_m \leq t\}$ we use

$$\int_0^{t\wedge\tau_m} p(X_s)ds = \int_0^{\tau_m} p(X_{s\wedge\tau_m})ds \leq \int_0^{\tau_m} p(X_{s\wedge\tau_m})ds + \int_{\tau_m}^{t} \underbrace{p(X_{s\wedge\tau_m})}_{>0} ds.$$

Since $p \geq 0$, by Fubini's theorem for positive integrands the above inequality (1.8) becomes

$$\mathbb{E}_{\mathbb{P}}[p(X_{t\wedge\tau_m})] \leq p(X_0) + C\int_0^t \mathbb{E}_{\mathbb{P}}[p(X_{s\wedge\tau_m})]ds.$$

By Grownwall's inequality (Lemma 1.14) for $h(t) := \mathbb{E}_{\mathbb{P}}[p(X_{t\wedge\tau_m})]$ we obtain

$$\mathbb{E}_{\mathbb{P}}[p(X_{t\wedge\tau_m})] \leq p(X_0)\exp(Ct),$$

now, sending $m \to \infty$ and by Fatou's lemma we have

$$\begin{aligned} p(X_0)\exp(Ct) &\geq \lim_{m\to\infty} \mathbb{E}_{\mathbb{P}}[p(X_{t\wedge\tau_m})] \\ &\geq \mathbb{E}_{\mathbb{P}}[\lim_{m\to\infty} p(X_{t\wedge\tau_m})] = \mathbb{E}_{\mathbb{P}}[p(X_t)], \end{aligned}$$

consequently, $\left(1 + \|X_0\|^{2N}\right) e^{Ct} \geq \mathbb{E}_{\mathbb{P}}[1 + \|X_t\|^{2N}]$. □

Now we can prove the main result:

Theorem 1.16 (Moment formula)**.** *For every $p \in Pol_N(E)$ with coordinate representation $\vec{p}$ with respect to the basis B and $0 \leq s \leq t$ we have*

$$\mathbb{E}_{\mathbb{P}}[p(X_t)|\mathscr{F}_s] = B(X_s)^\top e^{(t-s)G}\vec{p},$$

where G is the matrix representation of the linear map $\mathscr{G} : Pol_N(E) \to Pol_N(E)$.

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The proof is a refinement of the proof of Theorem 3.1 in [12, p.953].

Proof. We already know from (1.7), that every polynomial $p \in \mathrm{Pol}_N(E)$ can be represented as

$$\vec{p}^\top B(X_t) = \vec{p}^\top B(X_s) + \vec{p}^\top G^\top \int_s^t B(X_u)du + \int_s^t \sigma(X_u)^\top \nabla p(X_u) dW_u, \tag{1.9}$$

where the quadratic variation of the stochastic integral is given by

$$\int_s^t \Big(\sigma(X_u)^\top \nabla p(X_u)\Big)^\top \Big(\sigma(X_u)^\top \nabla p(X_u)\Big)\, du = \int_s^t \nabla p(X_u)^\top a(X_u) \nabla p(X_u) du. \tag{1.10}$$

The integrand of (1.10) is a polynomial of degree at most $2N$ and we find a constant $K > 0$ such that $\nabla p(X_u)^\top a(X_u)\nabla p(X_u) \leq K(1+\|X_u\|^{2N})$ for all $u \in [s,t]$ holds. By the preceding lemma we find a constant $C > 0$ such that

$$\mathbb{E}_{\mathbb{P}}[\int_s^t \nabla p(X_u)^\top a(X_u)\nabla p(X_u)du] \leq K \int_s^t \left(1 + \|X_0\|^{2N}\right) e^{Cu} du < \infty,$$

which shows that the above stochastic integral is a martingale. Taking conditional expectation of (1.9) and using its linearity yields

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[p(X_t)|\mathscr{F}_s] &= \vec{p}^\top \mathbb{E}_{\mathbb{P}}[B(X_t)|\mathscr{F}_s] = \vec{p}^\top \left(B(X_s) + G^\top \mathbb{E}_{\mathbb{P}}[\int_s^t B(X_u)du|\mathscr{F}_s]\right) \\ &= \vec{p}^\top \left(B(X_s) + G^\top \int_s^t \mathbb{E}_{\mathbb{P}}[B(X_u)|\mathscr{F}_s]du\right).\end{aligned}$$

In the second line we used Fubini's theorem, which is also justified by Lemma 1.15 and the same argument as before. Setting $F(t) := \mathbb{E}_{\mathbb{P}}[B(X_t)|\mathscr{F}_s]$, from the above equation we obtain the vector valued integral equation

$$F(t) = F(s) + G^\top \int_s^t F(u)du, \quad \forall s \leq t < \infty,$$

which has the unique solution $F(t) = e^{(t-s)G^\top} F(s) = (e^{(t-s)G})^\top F(s)$. Finally,

$$\mathbb{E}_{\mathbb{P}}[p(X_t)|\mathscr{F}_s] = \vec{p}^\top (e^{(t-s)G})^\top B(X_s) = (e^{(t-s)G}\vec{p})^\top B(X_s) = B(X_s)^\top e^{(t-s)G}\vec{p}.$$

□

Remarks 1.17. (1) The conditional expectation of any polynomial depends only on the last known state X_s and not on the whole information $\mathscr{F}_s$ up to time s. We can therefore write

$$\mathbb{E}_{\mathbb{P}}[p(X_t)|X_s] = B(X_s)^\top e^{(t-s)G}\vec{p}. \tag{1.11}$$

We further note that the matrix exponential $e^{(t-s)G}$ only depends on the time difference (lag) $t-s$, i.e. for all $0 < s \leq t$ we have

$$\mathbb{E}_{\mathbb{P}}[p(X_t)|X_s = x] = B(x)^\top e^{(t-s)G}\vec{p} = \mathbb{E}_{\mathbb{P}}[p(X_{t-s})|X_0 = x] \ \forall x \in E. \tag{1.12}$$

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(2) The conditional joint moments $\mathbb{E}_{\mathbb{P}}[X_{t,1}^{n_1} X_{t,2}^{n_2} \cdot \ldots \cdot X_{t,d}^{n_d} | X_s]$, where n_i represents the exponent of the i-th coordinate of X_t, are calculated by taking the standard basis of $\mathrm{Pol}_N(E)$, where $N \geq \sum_{i=1}^{d} n_i$, and projection on the desired component, i.e. we choose the k-th standard basis vector as coordinate vector $\vec{p} = \vec{e}_k$, where k is the index of the basis element $x_1^{n_1} x_2^{n_2} \cdot \ldots \cdot x_d^{n_d}$ of the ordered basis vector.

1.2 L^2 Theory and its Applications

This section gives a short introduction to Hilbert spaces and its fundamental results and applications needed for the proceeding work. The general L^2 theory part primarily bases on [2, p.241–251] and [29, p.280–284]. The proofs are taken from [29], partially adapted to give a consistent notation throughout this thesis.

As application of the L^2 theory we find the orthogonal polynomials which arise if the underlying (probability) measure of the Hilbert space is described by a distribution function on $\mathbb{R}$. They serve as a polynomial approximation for the elements of the Hilbert space and are typically used for numerical integration of functions where no analytical solution is available.

1.2.1 Hilbert space

For a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$ we consider the real vector space

$$\mathscr{L}^2(\Omega, \mathscr{F}, \mathbb{P}) = \mathscr{L}^2(\mathrm{d}\mathbb{P}) = \{f : \Omega \to \mathbb{R} \text{ measurable } \mid \int_\Omega f^2(\omega)\mathrm{d}\mathbb{P}(\omega) < \infty\}$$

of square integrable functions and for $f \in \mathscr{L}^2$ we define

$$\|f\| = \left(\int_\Omega f^2(\omega)\mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{2}}.$$

We have that $\|.\|$ is a seminorm on $\mathscr{L}^2(\mathrm{d}\mathbb{P})$ and it becomes a norm on its quotient space

$$L^2(\mathrm{d}\mathbb{P}) = \mathscr{L}^2(\mathrm{d}\mathbb{P})/N,$$

where $N = \{f \in \mathscr{L}^2(\mathrm{d}\mathbb{P}) : \|f\| = 0\}$. Hence $L^2(\mathrm{d}\mathbb{P})$ consists of all equivalence classes $[f]$ with a given equivalence relation $f \sim g :\Leftrightarrow \|f - g\| = 0$, meaning that two functions belong to the same class, if the set on which they differ has measure zero.

The given norm induces a metric $d(f, g) = \|f - g\|\ \forall f, g \in L^2(\mathrm{d}\mathbb{P})$, and we have that $(L^2(\mathrm{d}\mathbb{P}), \|.\|)$ is a complete normed vector space, i.e. for every sequence $\{f_n\}_{n\in\mathbb{N}} \in L^2(\mathrm{d}\mathbb{P})$ for which we have $\lim_{m,n\to\infty} d(f_n, f_m) = \lim_{m,n\to\infty}\|f_n - f_m\| = 0$, there exists a $f \in L^2(\mathrm{d}\mathbb{P})$ such that $\lim_{n\to\infty} d(f_n, f) = \lim_{n\to\infty}\|f_n - f\| = 0$ holds. In that case it is said, that the sequence $\{f_n\}_{n\in\mathbb{N}}$ convergences in mean square to its limit f.

The map $\langle ., . \rangle : L^2(\mathrm{d}\mathbb{P}) \times L^2(\mathrm{d}\mathbb{P}) \to \mathbb{R}$ given by

$$\langle f, g \rangle = \int_\Omega f(\omega)g(\omega)\mathrm{d}\mathbb{P}(\omega) = \frac{1}{2}\left(\|f + g\|^2 - \|f\|^2 - \|g\|^2\right)$$

defines an inner product. Alternatively, Cauchy-Schwarz inequality $|\langle f, g\rangle| \leq \|f\|\|g\|$ for all $f, g \in L^2(\mathrm{d}\mathbb{P})$ can be used to show that $\langle ., . \rangle$ is well-defined. We have that $(L^2(\mathrm{d}\mathbb{P}), \langle ., . \rangle)$ is a real Hilbert space.

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1.2.2 Fourier series expansion

Definition 1.18. We say that $f, g \in L^2(\mathrm{d}\mathbb{P})$ are orthogonal, if $\langle f, g\rangle = 0$.

A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is called orthonormal, if $\langle \varphi_n, \varphi_m\rangle = \delta_{mn}$[3] holds for all $m, n \in \mathbb{N}$.

As a consequence from Cauchy-Schwarz inequality we have the following

Lemma 1.19 (Bessel's inequality)**.** *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence. For all $f \in L^2(\mathrm{d}\mathbb{P})$ the following inequality holds*

$$\sum_{n=0}^{\infty} f_n^2 \leq \|f\|^2, \quad f_n = \langle f, \varphi_n\rangle, \ n \in \mathbb{N}.$$

Proof. We have

$$\begin{aligned}
0 &\leq \|f - \sum_{n=0}^{N} f_n\varphi_n\|^2 \\
&= \int_\Omega \left(f(\omega) - \sum_{n=0}^{N} f_n\varphi_n(\omega)\right)^2 \mathrm{d}\mathbb{P}(\omega) \\
&= \int_\Omega f^2(\omega)\mathrm{d}\mathbb{P}(\omega) - 2\sum_{n=0}^{N} f_n \int_\Omega f(\omega)\varphi_n(\omega)\mathrm{d}\mathbb{P}(\omega) + \int_\Omega \left(\sum_{n=0}^{N} f_n\varphi_n(\omega)\right)^2 \mathrm{d}\mathbb{P}(\omega) \\
&= \|f\|^2 - 2\sum_{n=0}^{N} f_n\langle f, \varphi_n\rangle + \sum_{n=0}^{N} f_n^2 \\
&= \|f\|^2 - \sum_{n=0}^{N} f_n^2.
\end{aligned}$$

In the third line we used for the last expression that the mixed terms vanish. Since N was arbitrary, we can conclude $\sum_{n=0}^{\infty} f_n^2 \leq \|f\|^2$. □

Bessel's inequality implies, if $\sum_{n=0}^{\infty} f_n^2 = \lim_{N\to\infty} \sum_{n=0}^{N} f_n^2$ converges, then so does $\sum_{n=0}^{\infty} f_n\varphi_n$. As the first series is increasing in N, the partial sums of the latter constitute an approximating sequence of f. In the case of equality the approximating sequence converges to f in mean square, thus we get a series representation of f,

$$f(\omega) = \sum_{n=0}^{\infty} f_n\varphi_n(\omega) = \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle\varphi_n(\omega),$$

which we refer to as (generalized) Fourier series expansion of f and f_n as the corresponding (generalized) Fourier coefficients. Note, that this representation holds in the L^2 sense[4] but not almost surely.

[3] $\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$ denotes the Kronecker-Delta symbol.

[4] $f(\omega) = \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle\varphi_n(\omega) :\Leftrightarrow \|f - \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle\varphi_n\| = 0$.

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In the next theorem equivalent conditions for equality in the Bessel's inequality are given.

Theorem 1.20 (Parseval's theorem)**.** *For any orthonormal sequence $\{\varphi_n\}_{n\in\mathbb{N}} \in L^2(\mathrm{d}\mathbb{P})$ the following statements are equivalent:*

(i) The finite linear combinations of $\{\varphi_n\}_{n\in\mathbb{N}}$ form a dense subset of $L^2(\mathrm{d}\mathbb{P})$, i.e. $\forall\epsilon > 0$ and $f \in L^2(\mathrm{d}\mathbb{P})$ there exists an integer N and real numbers $a_0, \dots, a_N$ such that

$$\|f - \sum_{n=0}^{N} a_n\varphi_n\| \leq \epsilon.$$

(ii) If $f \in L^2(\mathrm{d}\mathbb{P})$ satisfies $\langle f, \varphi_n\rangle = 0\ \forall n \in \mathbb{N}$, then $f = 0$.

(iii) For all $f \in L^2(\mathrm{d}\mathbb{P})$ the Parseval's identity

$$\|f\|^2 = \sum_{n=0}^{\infty} f_n^2 \quad \textit{where} \quad f_n = \langle f, \varphi_n\rangle$$

is satisfied.

Proof.

(i) $\Rightarrow$ (ii) Let $f \in L^2(\mathrm{d}\mathbb{P})$ satisfy $\langle f, \varphi_n\rangle = 0 \quad \forall n \in \mathbb{N}$. By assumption we have

$$f(\omega) = \sum_{n=0}^{N} a_n\varphi_n(\omega) + R_N(\omega) \tag{1.13}$$

with R_N denoting the remaining term with $\|R_N\| \leq \epsilon$. Computing the squared norm of f using orthogonality yields

$$\|f\|^2 = \sum_{n=0}^{N} a_n^2 + 2\sum_{n=0}^{N} a_n\langle\varphi_n, R_N\rangle + \|R_N\|^2.$$

By (1.13) we have $\langle\varphi_n, R_N\rangle = \langle\varphi_n, f - \sum_{n=0}^{N} a_n\varphi_n\rangle = -a_n$ and obtain

$$\|f\|^2 = \sum_{n=0}^{N} a_n^2 - 2\sum_{n=0}^{N} a_n^2 + \|R_N\|^2 < \epsilon^2,$$

since ϵ was arbitrary, we have $f = 0$.

(ii) $\Rightarrow$ (iii) For $f \in L^2(\mathrm{d}\mathbb{P})$ we have by Bessel's inequality and the concluding remark that

$$g(\omega) := \sum_{n=0}^{\infty} f_n\varphi_n(\omega) < \infty$$

holds with $f_n = \langle f, \varphi_n\rangle$. It remains to show, that $\|f - g\| = 0$ is satisfied. This follows immediately using $\langle f - g, \varphi_n\rangle = f_n - f_n = 0$ for all $n \in \mathbb{N}$, and by assumption we get $f - g = 0$.

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(iii) $\Rightarrow$ (i) From $\|f\|^2 = \sum_{n=0}^{\infty} f_n^2 = \sum_{n=0}^{\infty} \langle f_n\varphi_n, f_n\varphi_n\rangle = \sum_{n=0}^{\infty} \|f_n\varphi_n\|^2$, we conclude

$$\|f - \sum_{n=0}^{N} f_n\varphi_n\|^2 = \sum_{n=N+1}^{\infty} f_n^2.$$

Since the right hand side is finite by Bessel's inequality, this sum tends to zero as $N \to \infty$, which shows that f can be arbitrarily close approximated by a linear combination of the first N terms of the orthonormal sequence with $\epsilon := \sum_{n=N+1}^{\infty} f_n^2$. □

Definition 1.21. An orthonormal sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ for which its linear span is dense in $L^2(\mathrm{d}\mathbb{P})$ is said to be a complete orthonormal sequence.

1.2.3 Orthogonal polynomials

For a real-valued random variable $X : \Omega \to \mathbb{R}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ with distribution function F let

$$L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), dF) = L^2(dF) = \{f : \mathbb{R} \to \mathbb{R} | \int_{\mathbb{R}} f^2(x)dF(x) < \infty\}$$

denote the Hilbert space of square integrable Borel-measurable functions of X, i.e. the set of functions f for which $\mathbb{E}_{\mathbb{P}}[f^2(X)] < \infty$ holds with inner product

$$\mathbb{E}_{\mathbb{P}}[f(X)g(X)] = \int_{\mathbb{R}} f(x)g(x)dF(x), \quad f, g \in L^2(dF).$$

If the distribution function F is absolutely continuous with density w, we denote this space as weighted L^2 space with weight w and write

$$L^2_w = \{f : \mathbb{R} \to \mathbb{R} | \ \|f\|_w^2 := \int_{\mathbb{R}} f^2(x)w(x)dx < \infty\},$$
$$\langle f, g\rangle_w = \int_{\mathbb{R}} f(x)g(x)w(x)dx, \quad f, g \in L^2_w.$$

For a distribution function F having infinitely many points of increase and finite moments $\alpha_n = \int_{\mathbb{R}} x^n dF(x) = \mathbb{E}_{\mathbb{P}}[X^n]$, $n \in \mathbb{N}$, of all orders, the real vector space of polynomials $\mathrm{Pol}(\mathbb{R})$ is a subspace of $L^2(dF)$. It is shown in [24, Proposition, p.45], that $\mathrm{Pol}(\mathbb{R})$ is even a dense subset of $L^2(dF)$, if there exists $\epsilon > 0$ such that $\int_{\mathbb{R}} e^{\epsilon|x|}dF(x) < \infty$.

Note, that if F has infinitely many points of increase, the set $\{x^n\}_{n\in\mathbb{N}}$ is linear independent: For any arbitrary $p \in \mathrm{Pol}(\mathbb{R})$ the relation $\int_{\mathbb{R}} p^2(x)dF(x) = 0$ is possible, only if p vanishes at all points of increase of F. Since there are infinitely many of those and by continuity of p we conclude that p vanishes identically.

For the standard basis $\{x^n\}_{n\in\mathbb{N}}$ of $\mathrm{Pol}(\mathbb{R})$ an orthonormal set can be constructed using either Gram-Schmidt procedure or the next result, which is taken from [6, p.131]. An alternative formulation is given in [31, Theorem 2.1.1, p.23].

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Proposition 1.22. *Given a distribution function F with infinitely many points of increase and for which all moments exist, there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ uniquely determined by the following conditions:*

(i) The polynomial p_n has degree n with a positive leading coefficient. In particular we have $p_0(x) = 1$.

(ii) The p_n satisfy the orthogonality conditions $\int_{\mathbb{R}} p_n(x)p_m(x)dF(x) = \delta_{mn}$.

Proof. Let $\alpha_n = \int_{\mathbb{R}} x^n dF(x)$ denote the nth moment of F. Consider the quadratic form in the variables $c_0, c_1, \ldots, c_n \in \mathbb{R}$, given by

$$0 \leq \int_{\mathbb{R}} (c_0 + c_1 x + \ldots c_n x^n)^2 dF(x) = \sum_{i,j=0}^{n} c_i c_j \alpha_{i+j} =: c^\top Q_n c$$

for $c^\top = (c_0, \ldots, c_n)^\top$ and Q_n is the representation matrix of the quadratic form.

Note, that we have a strict inequality if the distribution function has at least $(n+1)$ points of increase and therefore we have for the determinant

$$|Q_n| = \begin{vmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_n & \alpha_{n+1} & \ldots & \alpha_{2n} \end{vmatrix} > 0.$$

In the following we construct a sequence of polynomials such that (i) and (ii) are satisfied. For $n = 0$ we have $p_0(x) = c_0 = 1$, since $\displaystyle\int_{\mathbb{R}} c_0^2 dF(x) = 1$. We use that x^k is a linear combination of $p_0, p_1, \ldots, p_k$ for the construction of a polynomial p_k of degree k. For (ii) to hold, necessarily the conditions

$$\int_{\mathbb{R}} x^k p_n(x) dF(x) = 0, \quad k = 0, 1, \ldots, n-1, \tag{1.14}$$

must be satisfied. Rewriting (1.14) for $p_n(x) = c_0 + c_1 x + \ldots c_n x^n$ the coefficients $c_0, \ldots c_n$ are obtained by the n conditions

$$\int_{\mathbb{R}} x^k (c_0 + c_1 x + \ldots c_n x^n) dF(x) = \sum_{i=0}^{n} c_i \alpha_{i+k} = 0, \quad 0 \leq k \leq n-1,$$

and by

$$\int_{\mathbb{R}} (c_0 + c_1 x + \ldots c_n x^n)^2 dF(x) = 1.$$

The latter condition simplifies to

$$\int_{\mathbb{R}} p_n(x)(c_0 + c_1 x + \ldots c_n x^n) dF(x) \overset{(1.14)}{=} c_n \int_{\mathbb{R}} p_n(x) x^n dF(x) = c_n \sum_{i=0}^{n} c_i \alpha_{i+n} = 1. \tag{1.15}$$

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Hence the coefficients $c_0, \dots, c_n$ must satisfy

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{n-1} & \alpha_n & \dots & \alpha_{2n-1} \\ c_n\alpha_n & c_n\alpha_{n+1} & \dots & c_n\alpha_{2n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{1.16}$$

The linear system (1.16) has a unique solution under the constraint $c_n > 0$, since Q_n is invertible for $c_n \neq 0$, the identity $c_n = 0$ would contradict (1.15), so the proof is finished. □

Definition 1.23. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials. The elements of $\{p_n\}_{n\in\mathbb{N}}$ are called orthogonal polynomials[5] associated with the distribution function F, if

(i) p_n is a polynomial of degree n with positive leading coefficient and

(ii) $\displaystyle\int_{\mathbb{R}} p_n(x)p_m(x)dF(x) = 0, \ \forall m \neq n.$

Note, that by linearity of the integral any orthogonal polynomial p_n is orthogonal over $\mathbb{R}$ to all polynomials of degree less than n.

1.2.4 (Generalized) Hermite polynomials

The orthogonal polynomials associated with the normal distribution are of particular interest in this thesis. We use that the (probabilistic) Hermite polynomials $\{\mathrm{He}_n\}_{n\in\mathbb{N}}$ defined by the relation

$$(-1)^n e^{-\frac{x^2}{2}} \mathrm{He}_n(x) = \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \ n \in \mathbb{N}, \tag{1.17}$$

form a complete orthogonal set on L^2_w with $w(x) = e^{-\frac{x^2}{2}}$, and we have

$$\mathrm{He}_0(x) = 1, \quad \mathrm{He}_1(x) = x, \quad \mathrm{He}_2(x) = x^2 - 1, \quad \dots .$$

From these orthogonal polynomials an orthonormal basis for the Hilbert space of square integrable functions with respect to $\mathscr{N}(\mu_w, \sigma_w^2)$ is derived in the following theorem.

Theorem 1.24. *The polynomials*

$$H_n(x) := \frac{1}{\sqrt{n!}} \mathrm{He}_n\left(\frac{x-\mu_w}{\sigma_w}\right), \ n \in \mathbb{N}, \tag{1.18}$$

are orthonormal with respect to the inner product $\langle .,. \rangle_w$ with w denoting the density of the normal distribution $\mathscr{N}(\mu_w, \sigma_w^2)$.

[5] A collection of orthogonal polynomials and their properties is found in [25].

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Proof. We show, that the conditions of Proposition 1.22 are satisfied. First, observe that

$$\begin{aligned}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} &= \frac{d^{n-1}}{dx^{n-1}}\left(-xe^{-\frac{x^2}{2}}\right) = \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d^k}{dx^k}(-x)\frac{d^{n-k-1}}{dx^{n-k-1}}e^{-\frac{x^2}{2}}\\ &= -x\frac{d^{n-1}}{dx^{n-1}}e^{-\frac{x^2}{2}} - (n-1)\frac{d^{n-2}}{dx^{n-2}}e^{-\frac{x^2}{2}}\end{aligned}$$

yields by (1.17)

$$e^{-\frac{x^2}{2}}\mathrm{He}_n(x) = e^{-\frac{x^2}{2}}x\mathrm{He}_{n-1}(x) - e^{-\frac{x^2}{2}}(n-1)\mathrm{He}_{n-2}(x),$$

and consequently we have the three term recurrence formula

$$\mathrm{He}_n(x) = x\mathrm{He}_{n-1}(x) - (n-1)\mathrm{He}_{n-2}(x),\ n \geq 2. \tag{1.19}$$

Since $\mathrm{He}_0(x) = 1$ and $\mathrm{He}_1(x) = x$, the above recursion shows, that the degree of the polynomial on the left hand side increases by one in every step. Multiplication of the monic He_n with $\frac{1}{\sqrt{n!}}$ and the affine transformation $x \mapsto \frac{x-\mu_w}{\sigma_w}$ do not affect the positivity of the leading coefficient. Therefore H_n is a polynomial of degree n with positive leading coefficient.

To show, that the orthogonality conditions are satisfied, we notice

$$\begin{aligned}\int_{\mathbb{R}}\mathrm{He}_n^2(x)e^{-\frac{x^2}{2}}dx &\overset{(1.17)}{=} \int_{\mathbb{R}}\mathrm{He}_n(x)(-1)^n\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}dx\\ &= \underbrace{(-1)^n\mathrm{He}_n(x)\frac{d^{n-1}}{dx^{n-1}}e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty}}_{=0} + (-1)^{n+1}\int_{\mathbb{R}}\frac{d}{dx}\mathrm{He}_n(x)\frac{d^{n-1}}{dx^{n-1}}e^{-\frac{x^2}{2}}dx.\end{aligned}$$

Using integration by parts another $(n-1)$ times gives the relation

$$\int_{\mathbb{R}}\mathrm{He}_n^2(x)e^{-\frac{x^2}{2}}dx = (-1)^{2n}\int_{\mathbb{R}}\frac{d^n}{dx^n}\mathrm{He}_n(x)e^{-\frac{x^2}{2}}dx = \int_{\mathbb{R}}\frac{d^n}{dx^n}\mathrm{He}_n(x)e^{-\frac{x^2}{2}}dx. \tag{1.20}$$

Consequently, by a change of variable $z = \frac{x-\mu_w}{\sigma_w}$ we have

$$\begin{aligned}\langle H_n, H_n\rangle_w &= \frac{1}{\sqrt{2\pi}\sigma_w}\int_{\mathbb{R}}H_n^2(x)e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}}dx \overset{(1.17)}{=} \frac{1}{n!\sqrt{2\pi}\sigma_w}\int_{\mathbb{R}}\mathrm{He}_n^2\left(\frac{x-\mu_w}{\sigma_w}\right)e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}}dx\\ &= \frac{1}{n!\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{He}_n^2(z)e^{-\frac{z^2}{2}}dz \overset{(1.20)}{=} \frac{1}{n!\sqrt{2\pi}}\int_{\mathbb{R}}\underbrace{\frac{d^n}{dz^n}\mathrm{He}_n(z)}_{=n!}e^{-\frac{z^2}{2}}dz = 1.\end{aligned}$$

Integration by parts n times gives $\int_{\mathbb{R}}\mathrm{He}_n(x)\mathrm{He}_m(x)e^{-\frac{x^2}{2}}dx = 0$ for $m \neq n$, hence,

$$\begin{aligned}\langle H_n, H_m\rangle_w &= \frac{1}{\sqrt{n!}\sqrt{m!}\sqrt{2\pi\sigma_w^2}}\int_{\mathbb{R}}\mathrm{He}_n\left(\frac{x-\mu_w}{\sigma_w}\right)\mathrm{He}_m\left(\frac{x-\mu_w}{\sigma_w}\right)e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}}dx\\ &= \frac{1}{\sqrt{n!}\sqrt{m!}\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{He}_n(z)\mathrm{He}_m(z)e^{-\frac{z^2}{2}}dz = 0.\end{aligned}$$

□

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Definition 1.25. The polynomials H_n, $n \in \mathbb{N}$, from (1.18) are called generalized Hermite polynomials.

As a consequence of the above theorem every square-integrable function f with respect to the density w of the normal distribution $\mathcal{N}(\mu_w, \sigma_w^2)$ has the Fourier series representation

$$f(x) = \sum_{n=0}^{\infty} f_n H_n(x), \tag{1.21}$$

with respect to the generalized Hermite polynomials and where the Fourier coefficients are given by

$$f_n = \frac{1}{\sqrt{2\pi\sigma_w^2}} \int_{\mathbb{R}} f(x) H_n(x) e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx, \quad n \in \mathbb{N}.$$

The infinite series (1.21) is identified as f, since $\|f - \sum_{n=0}^{\infty} f_n H_n\|_w = 0$. Any finite series $\sum_{n=0}^{N} f_n H_n(x)$ truncated at some order $N \in \mathbb{N}$ is the best polynomial L_w^2 approximation for f in the sense, that the polynomial of degree N minimizes the distance $\|f - \sum_{n=0}^{N} f_n H_n\|_w$.

Gram-Charlier series

A particular case of series representation in terms of orthogonal polynomials is considered in [6]: Motivated by the Central Limit Theorem, which states, that a standardized random variable

$$Z = \frac{X - \mathbb{E}[X]}{\mathbb{V}(X)},$$

where X is the sum of n independent random variables, converges under some conditions in distribution to a standard normal random variable as $n \to \infty$, the density f of Z might be written as the sum

$$f(z) = \varphi(z) + r(z), \tag{1.22}$$

where the density φ of the standard normal distribution is regarded as first approximation to f and the remainder term r represents its deviation from the true density. Then a series representation of the form

$$f(z) = \sum_{k=0}^{\infty} \frac{c_k}{k!} \frac{d^k}{dz^k} \varphi(z), \quad c_k \in \mathbb{R}$$

for the density f is considered, which rewrites, using the definition of the Hermite polynomials (1.17) to

$$f(z) = \sum_{k=0}^{\infty} \frac{c_k}{k!} (-1)^k \mathrm{He}_k(z) \varphi(z) = \varphi(z) \sum_{k=0}^{\infty} \frac{c_k}{k!} (-1)^k \mathrm{He}_k(z), \tag{1.23}$$

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with coefficients c_n, which are computed through the relation

$$(-1)^n \int_{\mathbb{R}} f(z)\mathrm{He}_n(z)dz \overset{(1.23)}{=} \sum_{k=0}^{\infty} \frac{c_k}{k!}(-1)^{n+k} \underbrace{\int_{\mathbb{R}} \mathrm{He}_k(z)\mathrm{He}_n(z)\varphi(z)dz}_{=\langle \mathrm{He}_k, \mathrm{He}_n\rangle_\varphi = n!\delta_{kn}} = c_n.$$

A series of the form (1.23) is termed as Gram-Charlier series of Type A [6, Chapter 17.6, p.222] and since f is assumed to be a density, we obtain $c_0 = 1$ and consequently, that f is of the form (1.22).

Further, for the likelihood ratio $\ell(z) = \frac{f(z)}{\varphi(z)}$ we obtain from the above the Fourier series representation

$$\ell(z) = \sum_{k=0}^{\infty} \frac{c_k}{k!}(-1)^k \mathrm{He}_k(z)$$

in terms of Hermite polynomials with coefficients given by

$$c_k = (-1)^k \int_{\mathbb{R}} \ell(z)\mathrm{He}_k(z)\varphi(z)dz = (-1)^k \langle \ell, \mathrm{He}_k\rangle_\varphi.$$

Note, that if the Central Limit Theorem applies for the sum X, then X is asymptotically normal and the above Gram-Charlier series expansion can be generalized to a series representation of the density g of X of the form

$$g(x) = \sum_{k=0}^{\infty} \frac{c_k}{k!} \frac{d^k}{dx^k} w(x), \quad c_k \in \mathbb{R},$$

where w denotes the density of $\mathcal{N}(\mu_w, \sigma_w^2)$ for some $\mu_w \in \mathbb{R}$ and $\sigma_w > 0$. Using a change of variable $x = \mu_w + \sigma_w z$ twice, we derive

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} \frac{c_k}{k!} \frac{d^k}{dx^k} w(x) = \sum_{k=0}^{\infty} \frac{c_k}{k!} \frac{d^k}{dz^k} \left(\frac{1}{\sigma_w} w(\mu_w + \sigma_w z) \right) = \sum_{k=0}^{\infty} \frac{c_k}{k!} \frac{d^k}{dz^k} \varphi(z) \\ &\overset{(1.17)}{=} \sum_{k=0}^{\infty} \frac{c_k}{k!}(-1)^k \mathrm{He}_k(z)\varphi(z) = \sum_{k=0}^{\infty} \frac{c_k}{k!}(-1)^k \mathrm{He}_k\left(\frac{x-\mu_w}{\sigma_w}\right) \sigma_w w(x) \\ &\overset{(1.18)}{=} w(x) \sum_{k=0}^{\infty} \frac{c_k \sigma_w}{\sqrt{k!}}(-1)^k H_k(x), \end{aligned}$$

this is an expansion of the density with respect to the generalized Hermite polynomials, where the coefficients c_k are given by

$$c_k = \frac{(-1)^k \sqrt{k!}}{\sigma_w} \int_{\mathbb{R}} g(x) H_k(x) dx.$$

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Gauss-Hermite quadrature

The Hermite polynomials arise in numerical analysis in connection with Gaussian quadrature. For the approximation of an integral of a real-valued function over $(-\infty, \infty)$ the generalized Hermite polynomials can as well be used as the next proposition shows.

Proposition 1.26 (Gauss-Hermite quadrature formula)**.** *Let f be an integrable function with respect to the density of a normal distribution $\mathcal{N}(\mu_w, \sigma_w^2)$. Then its integral can be approximated by*

$$\int_{\mathbb{R}} f(x) \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx \approx \sum_{i=1}^{m} w_i f(x_i), \tag{1.24}$$

where $x_1, \ldots, x_m$ denote the roots of the m-th generalized Hermite polynomial H_m and the weights $w_1, \ldots, w_m$ are given by

$$w_i = \frac{1}{\sqrt{2\pi\sigma_w^2}(H_m'(x_i))^2} \int_{\mathbb{R}} \frac{H_m^2(x)}{(x-x_i)^2} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx, \quad 1 \le i \le m. \tag{1.25}$$

The m-point approximation has degree of precision $2m-1$.

Proof. The proof is done in two steps: First a polynomial p_{2m-1} of degree $2m-1$ is constructed, which interpolates the data $(x_i, f(x_i))$, $1 \le i \le m$, where $x_1, \ldots, x_m$ denote the zeros of the m-th generalized Hermite polynomial H_m and has the additional property that its first derivative p_{2m-1}' agrees with f' at the given values. In the second step it is shown, that the integral of p_{2m-1} with respect to the density of the normal distribution takes the form (1.24).

Step 1: Let $x_1, \ldots, x_m$ denote the roots of the generalized Hermite polynomial H_m and define the functions

$$l_i(x) := \frac{H_m(x)}{(x-x_i)H_m'(x_i)}, \quad 1 \le i \le m. \tag{1.26}$$

Note, l_i is a polynomial of degree $m-1$ satisfying $l_i(x_j) = \delta_{ij}$.

To see that the latter property holds, we use that H_m has m real distinct roots (see [31, Theorem 3.3.1, p.44]) and consequently $H_m'(x_i) \neq 0$ $\forall i \in \{1, \ldots, m\}$. For $i \neq j$ we have $l_i(x_j) = 0$ and for $i = j$ the de l'Hôpital's rule yields

$$l_i(x_i) = \lim_{x \to x_i} \frac{H_m(x)}{(x-x_i)H_m'(x_i)} = \lim_{x \to x_i} \frac{H_m'(x)}{H_m'(x_i)} = 1.$$

We further define the interpolation polynomial p_{2m-1} as

$$p_{2m-1}(x) := \sum_{i=1}^{m} v_i(x) f(x_i) + \sum_{i=1}^{m} \bar{v}_i(x) f'(x_i) \tag{1.27}$$

where v_i and $\bar{v}_i$ are polynomials of degree $2m-1$ given by

$$v_i(x) = \left[1 - 2l_i'(x_i)(x-x_i)\right] l_i^2(x), \quad \bar{v}_i(x) = (x-x_i) l_i^2(x).$$

We claim that the polynomial p_{2m-1} satisfies for $1 \le i \le m$

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(i) $p_{2m-1}(x_i) = f(x_i)$,

(ii) $p'_{2m-1}(x_i) = f'(x_i)$.

For (i) we immediately get $v_i(x_i) = l_i^2(x_i) = 1$ and $v_i(x_j) = 0$ for all $i \in \{1, \dots, m\}$ and $i \neq j$ as well as $\bar{v}_i(x_j) \equiv 0$ for $1 \leq i, j \leq m$. This leads to the desired result

$$p_{2m-1}(x_i) = \sum_{i=1}^{m} v_i(x_i) f(x_i) + \sum_{i=1}^{m} \bar{v}_i(x_i) f'(x_i) = f(x_i).$$

The derivatives appearing in (1.27) are calculated as

$$v_i'(x) = -2l_i'(x_i)l_i^2(x) + [1 - 2l_i'(x_i)(x - x_i)]2l_i(x)l_i'(x), \quad \bar{v}_i'(x) = l_i^2(x) + (x - x_i)2l_i(x)l_i'(x).$$

Hence, evaluated at the zeros we obtain

$$v_i'(x_j) \equiv 0, \quad \bar{v}_i'(x_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Therefore, $p'_{2m-1}(x_i) = \sum_{i=1}^{m} v_i'(x_i) f(x_i) + \sum_{i=1}^{m} \bar{v}_i'(x_i) f'(x_i) = f'(x_i)$, which gives (ii).

Step 2: Integration of (1.27) with respect to the density of $\mathscr{N}(\mu_w, \sigma_w^2)$ gives

$$\int_{\mathbb{R}} p_{2m-1}(x) \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx = \sum_{i=1}^{m} w_i f(x_i) + \sum_{i=1}^{m} \bar{w}_i f'(x_i)$$

where the weights w_i and $\bar{w}_i$ are given by

$$w_i = \int_{\mathbb{R}} v_i(x) \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx, \quad \bar{w}_i = \int_{\mathbb{R}} \bar{v}_i(x) \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx.$$

We show, that $\bar{w}_i = 0$ for $1 \leq i \leq m$ holds to establish (1.24):

$$\begin{aligned} \bar{w}_i &= \int_{\mathbb{R}} (x - x_i) l_i^2(x) \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx \overset{(1.26)}{=} \int_{\mathbb{R}} l_i(x) \frac{H_m(x)}{H_m'(x_i)} \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx \\ &= \frac{1}{H_m'(x_i)} \langle l_i, H_m \rangle_w = 0, \end{aligned}$$

since H_m is orthogonal to all polynomials of degree less then m, consequently also to l_i which has degree $m - 1$.

The weights w_i take after substitution and using $\bar{w}_i = 0$ the form

$$\begin{aligned} w_i &= \int_{\mathbb{R}} \left[1 - 2l_i'(x_i)(x - x_i)\right] l_i^2(x) \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx \\ &= \int_{\mathbb{R}} l_i^2(x) \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx - 2l_i'(x_i)\bar{w}_i \overset{(1.26)}{=} \frac{(H_m'(x_i))^{-2}}{\sqrt{2\pi\sigma_w^2}} \int_{\mathbb{R}} \frac{H_m^2(x)}{(x - x_i)^2} e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}} dx, \end{aligned}$$

which shows (1.25) and finishes the proof.

□

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Remark 1.27. The Gauss-Hermite quadrature formula becomes with respect to the Hermite polynomials

$$\int_{\mathbb{R}} f(x)\frac{1}{\sqrt{2\pi\sigma_w^2}}e^{-\frac{(x-\mu_w)^2}{2\sigma_w^2}}dx \overset{x=\mu_w+\sigma_w z}{=} \int_{-\infty}^{\infty} f(\mu_w+\sigma_w z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \approx \sum_{i=1}^{m} w_i f(\mu_w+\sigma_w z_i),$$

where $z_1, \ldots, z_m$ are the roots of He_m and the weights are given by

$$w_i = \frac{\sigma_w^2}{\sqrt{2\pi}(\mathrm{He}_m'(z_i))^2}\int_{\mathbb{R}} \frac{\mathrm{He}_m^2(z)}{(z-z_i)^2}e^{-\frac{z^2}{2}}dz, \quad 1 \le i \le m.$$

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Chapter 2

Introduction to Stochastic Volatility Models

Financial market models, in which the volatility of the stock is modeled separately by a stochastic process, are referred to as stochastic volatility models (SV models). They should be considered as an extension of the Black Scholes model to explain observed market phenomena by relaxing the assumption of constant volatility of the stock price.

We start with a review of the Black Scholes model and its shortcomings as motivation for the introduction of stochastic volatility models. The price one has to pay to let volatility vary randomly, is that the models become incomplete which is discussed afterwards. At the end of the chapter some examples of stochastic volatility models are given.

2.1 Brief discussion of the Black Scholes Model

In the Black-Scholes framework [4] the stock price process $S = (S_t)_{t\in\mathbb{R}_+}$ is modeled as a geometric Brownian motion with constant drift μ and volatility σ, i.e., as the solution of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

on a suitable filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{P}$ denoting the actual probability measure and $W = (W_t)_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion, which induces the Brownian filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$, i.e. the natural filtration for a Brownian motion augmented by the set containing all null sets of $\mathscr{F}$.

Thus, for all $t \in \mathbb{R}_+$ the stock price process $S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right)$ is lognormally distributed with mean $\log S_0 + (\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$ or equivalently for its log-returns we have

$$\log\left(\frac{S_u}{S_t}\right) \sim \mathscr{N}\left((\mu - \frac{\sigma^2}{2})(u-t), \sigma^2(u-t)\right), \ \forall u > t.$$

Furthermore it is assumed, that the interest rate r of the riskless bond $B = (B_t)_{t\geq 0}$ with $B_t = e^{rt}$ is known and constant over time.

Pricing of an European option with maturity T is done under the unique martingale measure

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$\mathbb{Q}$, which is obtained by the measure change

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathscr{E}\left(-\frac{\mu - r}{\sigma}W\right)_T = \exp\left(-\frac{\mu - r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right).$$

Under the risk-neutral measure $\mathbb{Q}$ the Black-Scholes model is of the form

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}, \end{aligned}$$

where $W_t^{\mathbb{Q}} := W_t + \frac{\mu - r}{\sigma}t$ is a $\mathbb{Q}$-Brownian motion according to Girsanov's theorem. The value of a simple European option with payoff function $f(S_T)$ is given, using the Markov property of S, by the function

$$v(t,s) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[f(S_T)|S_t = s], \ t \in [0,T],$$

and its fair price (at $t = 0$) corresponds to

$$\pi_f = e^{-rT}\mathbb{E}_{\mathbb{Q}}[f(S_T)].$$

In particular the value functions of an European call option C^{BS} and an European put option P^{BS} with maturity T and strike price K are given by the famous Black-Scholes formulas

$$\begin{aligned} C^{BS}(t,s,K,T;\sigma) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+|S_t = s] = s\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \\ P^{BS}(t,s,K,T;\sigma) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(K - S_T)^+|S_t = s] = Ke^{-r(T-t)}\Phi(-d_2) - s\Phi(-d_1), \end{aligned}$$

where

$$d_1 = \frac{\log(s/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and Φ denotes the distribution function of a standard normal random variable. The latter formula follows by put-call parity (the discounted stock price is a true martingale)

$$\begin{aligned} s - Ke^{-r(T-t)} &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[S_T - K|S_t = s] \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+|S_t = s] - e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(K - S_T)^+|S_t = s] \\ &= C^{BS}(t,s,K,T;\sigma) - P^{BS}(t,s,K,T;\sigma). \end{aligned}$$

The valuation of options in the Black-Scholes model for real market data only requires to estimate the volatility σ of the underlying stock S, since the other parameters can be observed directly. This can be done using either the historical volatility or the implied volatility.

The historical volatility

$$\hat{\sigma}^H = \sqrt{\frac{1}{\Delta t}\frac{1}{n-1}\sum_{i=1}^{n}(X_{t_i} - \bar{X})^2}.$$

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results from the values of the log-returns $X_{t_i} := \log S_{t_i} - \log S_{t_{i-1}}$ at equidistant observation points $t_0 < t_1 < \ldots < t_n$ with $\Delta t := t_i - t_{i-1}$, $1 \leq i \leq n$. Using that the obtained sequence $(X_{t_i})_{1 \leq i \leq n}$ consists of i.i.d. normally distributed random variables with variance $\sigma^2 \Delta t$ and equating with the sample variance of the historical log-returns $\frac{1}{n-1} \sum_{i=1}^{n} (X_{t_i} - \bar{X})^2$ with $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_{t_i}$ gives the historical volatility as a consistent and unbiased estimator for σ.

On the contrary, the implied volatility σ^I denotes the computed value for σ, such that the calculated value by the Black-Scholes formulas matches the observed market price. For observed market prices of a call option C^{obs} or a put option P^{obs} the implied volatility is uniquely determined through the relation

$$C^{BS}(t, s, K, T; \sigma^I) = C^{obs} \text{ or } P^{BS}(t, s, K, T; \sigma^I) = P^{obs}.$$

In practice the latter approach is used to obtain a value for the volatility of the underlying stock price. It is reasoned, that historical volatility just reflects the volatility of the past whereas the implied volatility gives the market expectation of volatility for the future.

An evidence that the assumptions of the Black-Scholes model are too restrictive to match accurately the observed market data is given by the so-called recognized volatility smile curves: If one plots the strike prices of options of the same type against its implied volatility the graph of the mapping $K \mapsto \sigma^I(K)$ describes a curve rather than a straight line. The appearance of the curve give rise to name it volatility smile.

Empirical studies of the time series of log-returns of the stock price also show, that 'large moves follow large moves and small moves follow small moves' [15, p.2]. This result named as volatility clustering, implies that volatility is auto-correlated. In addition, volatility has a tendency to move towards a long-term mean, thus volatility has the mean-reverting property.

Further observations of the market data suggest, that the stock prices and their volatility are negatively correlated, which is known as 'leverage effect', and that the empirical distribution of the log-returns of stock prices exhibits 'fat-tails' and is 'highly peaked'. That is, in comparison to the assumed normal distribution of the log-returns in the Black-Scholes model extreme stock prices values occur more often than predicted by this model. 'FHighly peaked' simply says, that the mean of the log-returns is higher than those of the assumed normal distribution.

2.2 Description of Stochastic Volatility Models

In stochastic volatility models the local dynamics of the stock price are typically of the form

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t,$$

where μ denotes the rate of expected return and $(\sigma_t)_{t \in \mathbb{R}_+}$ constitutes a strictly positive process whose stochastic behavior is explained by a new source of randomness different to the given one of the stock price process.

The introduction of a new stochastic process $(\sigma_t)_{t \geq 0} > 0$, $\forall t \in \mathbb{R}_+$, for modeling the volatility of the stock price shall explain the observed market phenomena in a way that

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properties based on historical data for volatility as well as the Black Scholes volatility smile curves are taken into account. More precisely it shall be possible to reproduce the pattern of implied volatility with the new process.

One usually uses tractable processes such as Ito processes satisfying an SDE or jump diffusions in continuous time as a building block for $(\sigma_t)_{t\geq 0}$. It is also common to model volatility as a stochastic process in discrete time, since the realized volatility is typically given at discrete observation points. To keep it simple, we restrict our considerations on one Ito process, which shall explain the behavior of observed volatility.

We consider stochastic volatility models as financial market models consisting of a bond B, a stock S and a stochastic process V on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with dynamics under the real-world measure $\mathbb{P}$ given by

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= \mu S_t dt + p(V_t) S_t dW_t, \\ dV_t &= \alpha(V_t)dt + \beta(V_t)dZ_t, \\ d[W,Z]_t &= \rho(V_t)dt. \end{aligned}$$

W and Z are two $\mathbb{P}$-Brownian motions with instantaneous correlation $\rho(v)$ depending on the realization of V. The volatility $p(V_t) := \sigma_t$ is defined as a positive function p of the stochastic process V, which has time-homogeneous drift $\alpha(v)$ and volatility $\beta(v)$. The other remaining parameters are the same as in the Black Scholes framework.

A decomposition of the correlated Brownian motions into

$$Z_t := W_{1t}, \quad W_t := \rho(V_t)W_{1t} + \sqrt{1-\rho^2(V_t)}W_{2t},$$

where $W = (W_1, W_2)$ denotes a two dimensional Brownian motion, allows to write the model in the form

$$\begin{aligned} dB_t &= rB_t dt \\ dV_t &= \alpha(V_t)dt + \beta(V_t)dW_{1t} \\ dS_t &= \mu S_t dt + p(V_t)S_t \left(\rho(V_t)dW_{1t} + \sqrt{1-\rho^2(V_t)}dW_{2t}\right). \end{aligned} \tag{2.1}$$

The available information up to time t is given by the σ-algebra

$$\mathscr{F}_t = \sigma((W_{1s}, W_{2s}) : s \leq t) \vee \mathscr{N} \subset \mathscr{F}$$

augmented by the set $\mathscr{N}$ containing all null sets of $\mathscr{F}$ and by $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ we denote the Brownian filtration (satisfying the usual conditions of completeness and right-continuity).

In the following we shall exclude the case that $|\rho(V_t)| = 1$ for any $t \in \mathbb{R}_+$. Otherwise there exists a moment, where the Brownian motions are perfectly correlated, hence any change in value of the volatility process can be explained by the Brownian motion, which drives the stock price process. This restriction is not far reaching, in the most popular SV models, which are discussed below, it is either assumed that the instantaneous correlation is constant over time with absolute value less than one or even zero.

Note, that in the case of perfect constant correlation the stochastic processes can each be described by the same one-dimensional Brownian motion and the stochastic volatility model can be reduced to a local volatility model (compare e.g. [11]).

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2.3 Incompleteness of Stochastic Volatility Models

Adding a new source of randomness may lead to incompleteness of a model, especially if its corresponding risk cannot be hedged directly as in the case of stochastic volatility. If option valuation is done via hedging, typically traded derivatives of the same type with longer maturity than the one which shall be priced are used as hedging instruments. It is implicitly assumed, that the traded hedging instruments are valued in the same way as the hedging object. This means, that any change in the volatility of the stock affects both derivatives simultaneously. Since the dynamics of the volatility process depend on the underlying measure, it is reasonable to introduce a stochastic process $(\phi_t)_{t\in[0,T]}$, with T denoting the maturity of the hedging object to parametrize all possible pricing measures.

Proposition 2.1. *Let a stochastic volatility model with the dynamics under the real-world measure $\mathbb{P}$ of the form*

$$\begin{aligned} dB_t &= rB_t dt \\ dV_t &= \alpha(V_t)dt + \beta(V_t)dW_{1t} \\ dS_t &= \mu S_t dt + p(V_t)S_t\left(\rho(V_t)dW_{1t} + \sqrt{1-\rho^2(V_t)}dW_{2t}\right) \end{aligned}$$

be given and $|\rho(V_t)| \neq 1$ for all $t \in \mathbb{R}_+$. For arbitrary fixed $T > 0$ the risk neutral measures $\mathbb{Q}^{(\phi)}$ are obtained by the measure change

$$\frac{d\mathbb{Q}^{(\phi)}}{d\mathbb{P}} := \mathscr{E}\left(-\int_0^{\cdot} \phi_s dW_{1s}\right)_T \mathscr{E}\left(-\int_0^{\cdot} \gamma_s dW_{2s}\right)_T,$$

where $(\phi_t)_{t\in[0,T]}$ is a progressively measurable stochastic process satisfying

$$\mathbb{E}_{\mathbb{P}}[\int_0^T \phi_t^2 dt] < \infty \tag{2.2}$$

and $(\gamma_t)_{t\in[0,T]}$ is given by

$$\gamma_t = \frac{1}{\sqrt{1-\rho^2(V_t)}}\left(\frac{\mu - r}{p(V_t)} - \rho(V_t)\phi_t\right). \tag{2.3}$$

Furthermore the dynamics under $\mathbb{Q}^{(\phi)}$ for a $\mathbb{Q}^{(\phi)}$-Brownian motion $W^\phi = (W_1^\phi, W_2^\phi)$ become

$$\begin{aligned} dV_t &= (\alpha(V_t) - \beta(V_t)\phi_t)dt + \beta(V_t)dW_{1t}^\phi, \\ dS_t &= rS_t dt + p(V_t)S_t\left(\rho(V_t)dW_{1t}^\phi + \sqrt{1-\rho^2(V_t)}dW_{2t}^\phi\right). \end{aligned}$$

Proof. Note, that (2.2) and (2.3) imply, that $\mathscr{E}\left(-\int_0^{\cdot} \gamma_s dW_{2s}\right)_T$ and $\mathscr{E}\left(-\int_0^{\cdot} \phi_s dW_{1s}\right)_T$ are strictly positive true $\mathbb{P}$-martingales with mean one. If we perform a measure change to $\mathbb{Q}^{(\phi)}$ given by

$$\frac{d\mathbb{Q}^{(\phi)}}{d\mathbb{P}} := \mathscr{E}\left(-\int_0^{\cdot} \phi_s dW_{1s}\right)_T \mathscr{E}\left(-\int_0^{\cdot} \gamma_s dW_{2s}\right)_T,$$

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we find that, according to Girsanov's theorem,

$$W_{1t}^{\phi} := W_{1t} + \int_0^t \phi_s ds, \quad W_{2t}^{\phi} := W_{2t} + \int_0^t \gamma_s ds$$

are again Brownian motions under $\mathbb{Q}^{(\phi)}$. Since the integrals are of finite variation, their correlation does not change under $\mathbb{Q}^{(\phi)}$. Hence, under the new measure we have

$$\begin{aligned} dV_t &= (\alpha(V_t) - \beta(V_t)\phi_t)dt + \beta(V_t)dW_{1t}^{\phi}, \\ dS_t &= \left(\mu - p(V_t)\left[\rho(V_t)\phi_t + \sqrt{1-\rho^2(V_t)}\gamma_t\right]\right) S_t dt \\ &\quad + p(V_t)S_t\left(\rho(V_t)dW_{1t}^{\phi} + \sqrt{1-\rho^2(V_t)}dW_{2t}^{\phi}\right). \end{aligned}$$

For $\mathbb{Q}^{(\phi)}$ to be a risk-neutral measure, the drift-term of the stock price relative to the stock has to agree with the interest rate r, i.e. solving

$$\mu - p(V_t)\left[\rho(V_t)\phi_t + \sqrt{1-\rho^2(V_t)}\gamma_t\right] = r$$

for γ_t gives

$$\gamma_t = \frac{1}{\sqrt{1-\rho^2(V_t)}}\left(\frac{\mu - r}{p(V_t)} - \rho(V_t)\phi_t\right),$$

provided that $p(V_t) \neq 0$ and $|\rho(V_t)| \neq 1$. □

Since there exist more than one risk-neutral measure, it is not clear which one shall be chosen for option valuation. This uncertainty is reflected by the stochastic process $(\phi_t)_{t\in[0,T]}$ and we refer to it as the market price of volatility risk. It can be considered as a risk premium the investor has to pay in the case of an investment. This upcoming idea connects the choice of the valuation measure with the investor's risk preferences and leads to utility based pricing. We will not follow this approach in this thesis, we assume in some sense utility-indifference valuation, which holds if we assume that the market price of volatility risk is always zero. As a consequence of canceling risk preferences the selected risk-neutral measure $\mathbb{Q}^{(0)} =: \mathbb{Q}$ is unique and well-defined, as long as $p(V_t) \neq 0$ and $|\rho(V_t)| \neq 1$ $\forall t \in [0,T]$ hold.

Henceforth the description of a stochastic volatility model is given under the selected risk-neutral measure $\mathbb{Q}$ under which the dynamics of the stock price process S and the volatility process V become

$$\begin{aligned} dV_t &= \alpha(V_t)dt + \beta(V_t)dW_{1t}, \\ dS_t &= rS_t dt + p(V_t)S_t\left(\rho(V_t)dW_{1t} + \sqrt{1-\rho^2(V_t)}dW_{2t}\right) \end{aligned} \tag{2.4}$$

where $W = (W_1, W_2)$ denotes a standard Brownian motion modeled on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ with $\mathbb{F}$ as standard Brownian filtration.

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2.4 Examples of Stochastic Volatility Models

We close this section by giving some typical examples of SV models of the form (2.4).

In the model by Hull & White [18] stochastic volatility is modeled by the variance process V which is of the same form as the stock price process in the Black-Scholes model. Constant correlation over time is assumed and the model is written as

$$\begin{aligned} dV_t &= \theta V_t dt + \sigma V_t dW_{1t}, \\ dS_t &= rS_t dt + \sqrt{V_t} S_t \left(\rho dW_{1t} + \sqrt{1-\rho^2} dW_{2t} \right). \end{aligned}$$

The model proposed in [30] which is often termed as 'Stein-Stein model' has the form

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \sigma dW_{1t}, \\ dS_t &= rS_t dt + |V_t| S_t dW_{2t}. \end{aligned}$$

Here the process V models the volatility, which is assumed to be uncorrelated with S, and a reflecting barrier at zero is added to ensure, that no 'negative volatility' is used for the dynamics of the stock price. The process V is known as Ornstein-Uhlenbeck process. It is the simplest mean-reverting Ito process with constant volatility (of volatility). Observe that for $V_t > \theta$ the drift term becomes negative as long as V_t stays above the level θ and for lower values than θ the process increases until the level θ is reached. The parameter θ is referred to as the long-term mean and the speed of movement towards the long-term mean is determined by κ.

In the model by Heston [16] the variance V is modeled from the Ornstein-Uhlenbeck process (OU) for volatility,

$$d\sqrt{V_t} = -\tilde{\kappa}\sqrt{V_t}dt + \tilde{\sigma} dW_{1t},$$

which becomes by Ito's formula

$$dV_t = \left(\tilde{\sigma}^2 - 2\tilde{\kappa}V_t\right) dt + 2\tilde{\sigma}\sqrt{V_t} dW_{1t},$$

and takes (by setting $\sigma = 2\tilde{\sigma}$, $\kappa = 2\tilde{\kappa}$ and $\theta = \frac{\tilde{\sigma}^2}{2\tilde{\kappa}}$) the form of the square-root process or Cox-Ingersoll-Ross process (CIR)

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{1t}.$$

The resulting model is known as Heston model and is given by

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{1t}, \\ dS_t &= rS_t dt + \sqrt{V_t} S_t \left(\rho dW_{1t} + \sqrt{1-\rho^2} dW_{2t} \right). \end{aligned}$$

In the Heston model it is assumed, that the instantaneous correlation ρ is independent of V and constant over time, i.e. $\frac{d[S,V]_t}{\sqrt{d[V]_t d[S]_t}} = \rho$. The square-root process is mean-reverting

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and takes values in $\mathbb{R}_+$. The zero bound is not attained in finite time a.s., if and only if the Feller condition $2\kappa\theta \geq \sigma^2$ holds.

It is also notable, that the OU-process as well as the CIR-process are polynomial diffusions. They also belong to the class of Pearson diffusions: For the unique ergodic solution of the SDE the corresponding probability density of the invariant distribution solves the Pearson differential equation (compare [13, p.440], [33, p.264]).

All SV models have in common that the process V tunes the kurtosis of the resulting log-return distribution (thinner/fatter tails compared to the normal distribution) whereas the instantaneous correlation between V and S determines the skewness of the log-return distribution: Positive correlation results in a more positively (right-)skewed distribution than the normal distribution, i.e the mass of the distribution is concentrated on the left and the right tails are made longer. Conversely, negative correlation yields a negative (left-)skewed distribution with drown-out left tail and more mass on the right. The intuition behind these effects can be explained as follows: For positive correlation very high stock prices become more probable than for fixed volatility since high stock prices are associated with high volatilities and small log-returns occur more often. The converse also holds and therefore it is less likely that large changes take place for low stock prices. For negative correlation the reverse is true: Since a price decrease comes along with an increase of volatility, the chance of large positive price changes is increasing. It is unlikely that very high stock prices result from negative correlation because a price increase reduces volatility.

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Chapter 3

The Jacobi Stochastic Volatility Model

In this chapter we discuss the stochastic volatility model introduced in [1], which is described by a polynomial diffusion. The first section deals with the model specification, which is followed by a discussion of the process for modeling volatility.

In the second section it is shown, that the log-price density admits a series representation with respect to the generalized Hermite polynomials and how the moment formula applies for the computation of the conditional moments, which appear in the series representation.

3.1 Model description

Let

$$Q(v) = \frac{(v_{\max} - v)(v - v_{\min})}{(\sqrt{v_{\max}} - \sqrt{v_{\min}})^2} \tag{3.1}$$

be a quadratic function on $\mathbb{R}$ with $0 \leq v_{\min} < v_{\max}$.

We consider a stochastic volatility model in which the stock price $(S_t)_{t\geq 0}$ is given as stochastic exponential, i.e. it satisfies the stochastic differential equation

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \tag{3.2}$$

with variance process (squared volatility process) $(V_t)_{t\geq 0}$ given as solution of

$$dV_t = \kappa\,(\theta - V_t)\,dt + \sigma\sqrt{Q(V_t)}dB_t \tag{3.3}$$

and instantaneous correlation

$$d[W, B]_t = \rho\sqrt{\frac{Q(V_t)}{V_t}}dt,\ \ \rho \in [-1, 1] \tag{3.4}$$

of the standard Brownian motions $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$. For the stock price r denotes the interest rate, for the latter process $\theta \in [v_{\min}, v_{\max}]$ the long-term mean, $\kappa > 0$ the speed of mean-reversion towards the long-term mean and $\sigma > 0$ the scaling factor for the volatility (of the squared volatility).

First observe that the scaling factor $\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^{-2}$ in (3.1) is chosen such that $Q(v) \leq v$ holds for all $v \in \mathbb{R}$:

$$\begin{aligned} Q(v) \leq v \ &\Leftrightarrow\ (v_{\max} - v)(v - v_{\min}) \leq v\,(\sqrt{v_{\max}} - \sqrt{v_{\min}})^2 \\ &\Leftrightarrow\ 0 \leq v^2 - 2\sqrt{v_{\max}v_{\min}}v + v_{\max}v_{\min} = (v - \sqrt{v_{\max}v_{\min}})^2. \end{aligned}$$

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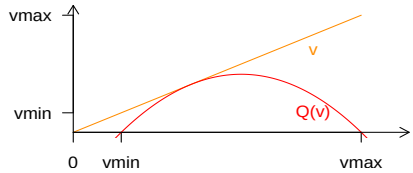
Restricted on $[v_{\min}, v_{\max}]$ the quadratic function is positive and (3.4) is well-defined as well as the square root in (3.3), if $(V_t)_{t\geq 0}$ stays in $\{v \in \mathbb{R} : Q(v) \geq 0\}$.



The variance process defined by (3.3) belongs to the class of polynomial diffusions and has the associated second order differential operator

$$\mathscr{G}f(v) = \kappa(\theta - v)f'(v) + \frac{1}{2}\sigma^2 Q(v)f''(v).$$

For the compact state interval $E = \{v \in \mathbb{R} : Q(v) \geq 0\} = [v_{\min}, v_{\max}]$ the next theorem shows, that $(V_t)_{t\geq 0}$ is well-defined and conditions for the parameters are given such that no boundary attainment occurs.

Theorem 3.1. *For any initial value $V_0 \in [v_{\min}, v_{\max}]$ the SDE (3.3) admits a unique $[v_{\min}, v_{\max}]$-valued solution.*

In particular $(V_t)_{t\geq 0}$ stays in $(v_{\min}, v_{\max})$ $\forall t \geq 0$, if and only if $V_0 \in (v_{\min}, v_{\max})$ and

$$2\kappa \min\{\theta - v_{\min}, v_{\max} - \theta\} \geq \frac{\sigma^2 (v_{\max} - v_{\min})}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}. \tag{3.5}$$

Proof. For the existence of a $[v_{\min}, v_{\max}]$-valued solution we show, that the conditions of Theorem 1.9 (ii) are satisfied. Observe, that we have equality for the ideals

$$\big((v_{\max} - v)(v - v_{\min})\big) = \mathscr{I}(\{v_{\min}, v_{\max}\})$$

and that the diffusion term $a(v) = \sigma^2 Q(v)$ is positive on $[v_{\min}, v_{\max}]$. We compute

$$a(v)Q'(v) = \frac{\sigma^2 Q(v)\left((v_{\max} - v) - (v - v_{\min})\right)}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2} = 0 \quad \text{for } v \in \{v_{\min}, v_{\max}\}$$

and

$$\mathscr{G}Q(v) = \frac{\kappa(\theta - v)\left((v_{\max} - v) - (v - v_{\min})\right)}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2} - \sigma^2 Q(v) > 0 \quad \text{for } v \in \{v_{\min}, v_{\max}\}.$$

Hence (3.3) has an $[v_{\min}, v_{\max}]$-valued solution and by the compactness of the state interval uniqueness follows by Proposition 1.13.

By Corollary 1.12 the boundaries are not reached in finite time if and only if for $V_t \in \{v_{\min}, v_{\max}\}$ the condition

$$2\mathscr{G}Q(V_t) - h(V_t)Q'(V_t) \geq 0 \quad \text{where} \quad h(v) = \frac{\sigma^2\left((v_{\max} - v) - (v - v_{\min})\right)}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}$$

holds for all $t \geq 0$. A straightforward calculation yields the inequalities

$$\frac{\sigma^2(v_{\max} - v_{\min})}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2} \leq 2\kappa(\theta - v_{\min}) \quad \text{and} \quad \frac{\sigma^2(v_{\max} - v_{\min})}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2} \leq 2\kappa(v_{\max} - \theta),$$

which implies (3.5). □

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Remark 3.2. The solution of (3.3) is an affine transformation of the Jacobi process[6] $(X_t)_{t\geq 0}$, which is the unique $[0,1]$-valued solution of

$$dX_t = \kappa\left(\tilde{\theta} - X_t\right)dt + \tilde{\sigma}\sqrt{(1-X_t)X_t}dW_t, \quad X_0 = x \quad \forall x \in (0,1).$$

Indeed, Ito's formula applied on $X_t := \frac{V_t - v_{\min}}{v_{\max} - v_{\min}}$ gives

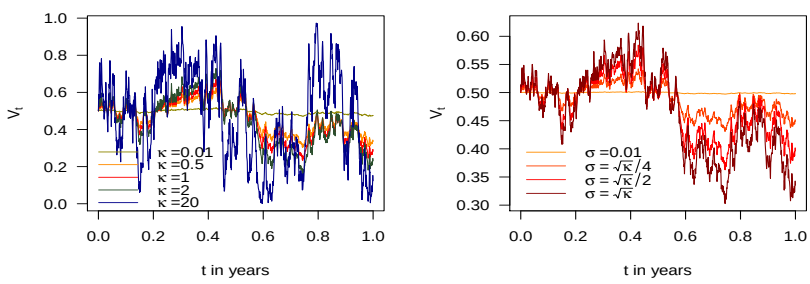
$$\begin{aligned} dX_t = \frac{dV_t}{v_{\max} - v_{\min}} &= \kappa\frac{\theta - V_t}{v_{\max} - v_{\min}}dt + \frac{\sigma}{\sqrt{v_{\max}} - \sqrt{v_{\min}}}\sqrt{\frac{(v_{\max} - V_t)(V_t - v_{\min})}{(v_{\max} - v_{\min})^2}}dW_t \\ &= \kappa\left(\frac{\theta - v_{\min}}{v_{\max} - v_{\min}} - X_t\right)dt + \frac{\sigma}{\sqrt{v_{\max}} - \sqrt{v_{\min}}}\sqrt{(1-X_t)X_t}dW_{1t} \\ &= \kappa\left(\tilde{\theta} - X_t\right)dt + \tilde{\sigma}\sqrt{(1-X_t)X_t}dW_t, \end{aligned}$$

where $\tilde{\theta} = \frac{\theta - v_{\min}}{v_{\max} - v_{\min}}$ denotes the transformed long-term mean and $\tilde{\sigma} = \frac{\sigma}{\sqrt{v_{\max}} - \sqrt{v_{\min}}}$.

To get an idea how the parameter values influence the shape of the variance process (3.3) we simulate one path out of one randomly chosen fixed Brownian path for different choices of parameters κ, σ and θ using Euler discretization combined with Accept-Reject method.

Since the path properties do not change for any affine transformation of the state interval, we restrict our considerations on the unit interval, i.e. $v_{\min} = 0$ and $v_{\max} = 1$. In this case the quadratic function simplifies to $Q(v) = (1-v)v$ and the 'no boundary attainment condition' (3.5) becomes $2\kappa\min\{1-\theta, \theta\} \geq \sigma^2$. By a symmetry argument it suffices to consider $\theta \in (0, \frac{1}{2}]$ and therefore this leads to the restriction $\frac{\sigma^2}{2\kappa} \in (0, \frac{1}{2}]$. For a fixed $\kappa > 0$ the possible choice for σ is $\sigma \in (0, \sqrt{\kappa}]$ and consequently for θ we require that $\frac{\sigma^2}{2\kappa} \leq \theta \leq 1 - \frac{\sigma^2}{2\kappa}$ is satisfied. Observe that for $\theta = \frac{1}{2}$ the last condition holds $\forall \sigma \in (0, \sqrt{\kappa})$. For comparable results we therefore fix $\theta = \frac{1}{2}$ and consider $(\kappa, \sigma) \in \mathbb{R}_+ \times (0, \sqrt{\kappa}]$.

The next figures display the simulation results for step size $\Delta t = 0.001$. It is important to note, that the path of Brownian motion was chosen so that no rejection occurred in the simulation. The left panel shows for different values for speed of mean-reversion κ and for maximal possible volatility factor σ the paths within one year, starting at $V_0 = \theta$. On the right for fixed $\kappa = 1$ the fluctuations in dependence of the volatility factor are displayed.



[6]The generator of X admits a spectral decomposition for which the eigenfunctions are Jacobi polynomials and therefore it is often referred to as Jacobi process (see [13, p.442]).

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We can define the model using (3.1) – (3.5) and by setting

$$W_{1t} := B_t,\ W_{2t} := \sqrt{\frac{V_t}{V_t - \rho^2 Q(V_t)}} W_t - \rho \sqrt{\frac{Q(V_t)}{V_t - \rho^2 Q(V_t)}} B_t,$$

as follows[7]:

Definition 3.3. The Jacobi stochastic volatility model consists of a two-dimensional stochastic process (V, S) on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ which is the unique solution of

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{Q(V_t)}dW_{1t}, & V_0 &= v, \\ dS_t &= rS_t dt + \rho\sqrt{Q(V_t)}S_t dW_{1t} + \sqrt{V_t - \rho^2 Q(V_t)}S_t dW_{2t}, & S_0 &= s, \end{aligned} \tag{3.6}$$

for any deterministic initial value $(v, s) \in (v_{\min}, v_{\max}) \times \mathbb{R}_+$, where $W = (W_1, W_2)$ denotes a two-dimensional standard Brownian motion and $\mathbb{F}$ the Brownian filtration. The parameters for squared volatility $\kappa, \sigma > 0$, θ are chosen such that (3.5) is satisfied, the quadratic function Q is given by (3.1), r denotes the interest rate and for the correlation factor we take $\rho \in (-1, 1)$.

As an immediate consequence of the model specification we have that the discounted stock price process

$$\begin{aligned} e^{-rt}S_t &= S_0 \exp\left(-\frac{V_t}{2}t + \rho\int_0^t \sqrt{Q(V_s)}dW_{1s} + \int_0^t \sqrt{V_s - \rho^2 Q(V_s)}dW_{2s}\right) \\ &= S_0 \mathscr{E}\left(\int_0^t \rho\sqrt{Q(V_s)}dW_{1s}\right) \mathscr{E}\left(\int_0^t \sqrt{V_s - \rho^2 Q(V_s)}dW_{2s}\right) \end{aligned}$$

is a true $\mathbb{Q}$-martingale. This follows by the boundedness of the variance process V for which both stochastic exponentials constitute true independent martingales whose product is again a true martingale.

Formally, for $V_0 = \theta \in \{v_{\min}, v_{\max}\}$, we have constant volatility for all $t \geq 0$ and obtain the Black Scholes model

$$dS_t = rS_t dt + \sqrt{V_0}S_t dW_{2t}.$$

Note also, that for the parameters $v_{\min} = 0$ and $v_{\max} \to \infty$ one gets the Heston model

$$\begin{aligned} dV_t &= \kappa\left(\theta - V_t\right)dt + \sigma\sqrt{V_t}dW_t, \\ dS_t &= rS_t dt + \sqrt{V_t}S_t\left(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}\right), \end{aligned}$$

as limiting case, since the quadratic function Q becomes

$$\lim_{v_{\max}\to\infty} Q(v) = \lim_{v_{\max}\to\infty} \frac{(v_{\max} - v)v}{v_{\max}} = \lim_{v_{\max}\to\infty} v - \frac{v^2}{v_{\max}} = v.$$

[7] Our model specification slightly differs from the one in [1], in which the 'Jacobi model' need not to satisfy (3.5) and ρ is allowed to take the values ± 1. Additionaly there the stock is assumed to pay dividends with constant continuous dividend yield.

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3.2 Properties of the Log-price process

In the following it is more convenient to express the stock price process $(S_t)_{t\geq 0}$ in terms of the log-price process $(X_t)_{t\geq 0}$. By Ito's formula applied on $\log S_t = X_t$ we obtain under the risk-neutral measure $\mathbb{Q}$ the representation of the stochastic volatility model as

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{Q(V_t)}dW_{1t}, && V_0 \in (v_{\min}, v_{\max}), \\ dX_t &= \left(r - \frac{V_t}{2}\right)dt + \rho\sqrt{Q(V_t)}dW_{1t} + \sqrt{V_t - \rho^2 Q(V_t)}dW_{2t}, && X_0 \in \mathbb{R}, \end{aligned} \tag{3.7}$$

or equivalently as two-dimensional polynomial diffusion with drift vector μ and diffusion matrix a given by

$$\mu(v) = \begin{pmatrix} \kappa(\theta - v) \\ r - \frac{v}{2} \end{pmatrix}, \quad a(v) = \begin{pmatrix} \sigma^2 Q(v) & \rho\sigma Q(v) \\ \rho\sigma Q(v) & v \end{pmatrix}. \tag{3.8}$$

The next result shows, that for every path of V the log-price of the stock and the finite dimensional distributions of the log-returns follow a normal distribution. The idea follows [28, p. 173 ff.]

Theorem 3.4. *In the stochastic volatility model specified by (3.7) the log-price process X conditioned on any path of V is a Gaussian process.*

In particular we have $X_t|(V_s)_{s\in[0,t]} \sim \mathcal{N}(M_t, C_t)$ with

$$\begin{aligned} M_t &= X_0 + \int_0^t \left(r - \frac{V_s}{2}\right)ds + \frac{\rho}{\sigma}\left(V_t - V_0 - \int_0^t \kappa(\theta - V_s)ds\right), \\ C_t &= \int_0^t V_s - \rho^2 Q(V_s)ds, \end{aligned} \tag{3.9}$$

and $(X_{t_1} - X_{t_0}, \ldots, X_{t_d} - X_{t_{d-1}})|(V_s)_{s\in[0,t_d]} \sim \mathcal{N}(\boldsymbol{M}_{t_0,t_1,\ldots,t_d}, \boldsymbol{C}_{t_0,t_1,\ldots,t_d})$ with mean vector and covariance matrix given by

$$\begin{aligned} \boldsymbol{M}_{t_0,t_1,\ldots,t_d} &= \left(M_{t_1} - M_{t_0}, M_{t_2} - M_{t_1}, \ldots, M_{t_d} - M_{t_{d-1}}\right), \\ \boldsymbol{C}_{t_0,t_1,\ldots,t_d} &= \operatorname{diag}\left(C_{t_1} - C_{t_0}, C_{t_2} - C_{t_1}, \ldots, C_{t_d} - C_{t_{d-1}}\right) \end{aligned} \tag{3.10}$$

for $0 \leq t_0 < t_1 < \ldots < t_d$.

Proof. The log-price process given by

$$X_t = X_0 + \int_0^t \left(r - \frac{V_s}{2}\right)ds + \int_0^t \rho\sqrt{Q(V_s)}dW_{1s} + \int_0^t \sqrt{V_s - \rho^2 Q(V_s)}dW_{2s}$$

decomposes for given $\{V_s\}_{s\in[0,t]}$ with $V_t - V_0 - \int_0^t \kappa(\theta - V_s)ds = \sigma\int_0^t \sqrt{Q(V_s)}dW_{1s}$ into a pure deterministic part

$$M_t = X_0 + \int_0^t \left(r - \frac{V_s}{2}\right)ds + \frac{\rho}{\sigma}\left(V_t - V_0 - \int_0^t \kappa(\theta - V_s)ds\right)$$

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and a stochastic integral with deterministic integrands and corresponding quadratic variation

$$X_t - M_t = \int_0^t \sqrt{V_s - \rho^2 Q(V_s)} dW_{2s}, \quad C_t = [X - M]_t = \int_0^t V_s - \rho^2 Q(V_s) ds.$$

For $\lambda \in \mathbb{R}$ we define

$$\mathscr{E}\left(\lambda(X - M)\right)_t = \exp\left(\lambda(X_t - M_t) - \frac{\lambda^2}{2} C_t\right), \tag{3.11}$$

which is a $\mathbb{Q}$-martingale and hence we obtain

$$1 = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\lambda(X_t - M_t) - \frac{\lambda^2}{2} C_t\right)\right] = e^{-\lambda M_t - \frac{\lambda^2}{2} C_t} \mathbb{E}_{\mathbb{Q}}[e^{\lambda X_t}].$$

Thus, the moment generating function of X_t is given by

$$\mathbb{E}_{\mathbb{Q}}[e^{\lambda X_t}] = e^{\lambda M_t + \frac{\lambda^2}{2} C_t}.$$

The right hand side corresponds to the moment generating function of a normally distributed random variable with mean M_t and variance C_t, and therefore we conclude $X_t | (V_s)_{s \in [0,t]} \sim \mathscr{N}(M_t, C_t)$.

In order for the conditional log-price process to be a Gaussian process, we have to verify, that all conditional finite dimensional distributions are jointly normal, i.e. the vector $(X_{t_0}, \ldots, X_{t_d})$ is conditional normally distributed for $0 \leq t_0 < t_1 < \ldots < t_d$ or equivalently that the increments $\left(X_{t_{j+1}} - X_{t_j}\right)_{0 \leq j \leq d-1}$ conditioned on any path of V are mutually independent and normally distributed. We show the latter condition in the case of two increments, the rest follows by an induction argument.

For $s < t$ we consider $X_s - X_0$ and $X_t - X_s$. By the martingale property of (3.11) we have

$$\mathbb{E}_{\mathbb{Q}}[e^{\lambda(X_t - M_t) - \frac{\lambda^2}{2} C_t} | \mathscr{F}_s] = e^{\lambda(X_s - M_s) - \frac{\lambda^2}{2} C_s}$$

and therefore

$$\mathbb{E}_{\mathbb{Q}}[e^{\lambda(X_t - X_s)} | \mathscr{F}_s] = e^{\lambda(M_t - M_s) + \frac{\lambda^2}{2}(C_t - C_s)}, \tag{3.12}$$

which gives $X_t - X_s \sim \mathscr{N}(M_t - M_s, C_t - C_s)$. Multiplication of the above expression with the $\mathscr{F}_s$-measurable term $e^{\mu(X_s - X_0)}$, $\mu \in \mathbb{R}$, on both sides and taking expectation yields

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[e^{\lambda(X_t - X_s)} e^{\mu(X_s - X_0)}] &= \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[e^{\lambda(X_t - X_s)} e^{\mu(X_s - X_0)} | \mathscr{F}_s]] \\
&= \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[e^{\lambda(X_t - X_s)} | \mathscr{F}_s] e^{\mu(X_s - X_0)}] \\
&\overset{(3.12)}{=} e^{\lambda(M_t - M_s) + \frac{\lambda^2}{2}(C_t - C_s)} \mathbb{E}_{\mathbb{Q}}[e^{\mu(X_s - X_0)}] \\
&\overset{(3.12)}{=} e^{\lambda(M_t - M_s) + \frac{\lambda^2}{2}(C_t - C_s)} e^{\mu(M_s - M_0) + \frac{\mu^2}{2}(C_s - C_0)} \\
&= \mathbb{E}_{\mathbb{Q}}[e^{\lambda(X_t - X_s)}] \mathbb{E}_{\mathbb{Q}}[e^{\mu(X_s - X_0)}],
\end{aligned}$$

which shows the independence of the increments. □

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From the pathwise distributions of the log-price process the (unconditional) density g_T for the log-price X_T, $T > 0$, is obtained by averaging over all sample paths of $(V_s)_{s\in[0,T]}$, i.e.

$$g_T(x) = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\sqrt{2\pi C_T}}\exp\left(\frac{(x-M_T)^2}{2C_T}\right)\right]. \tag{3.13}$$

3.2.1 Series expansion of the Log-price density

Our next goal is to establish the existence of a convergent series expansion of (3.13) in terms of generalized Hermite polynomials; more specifically a representation of the form

$$g_T(x) = w(x)\sum_{k=0}^{\infty}\langle\frac{g_T}{w}, H_k\rangle_w H_k(x), \tag{3.14}$$

where w denotes the density of a normal distribution $\mathcal{N}(\mu_w, \sigma_w^2)$, $\langle .,.\rangle_w$ the inner product in L^2_w and $\{H_k\}_{k\in\mathbb{N}}$ the sequence of generalized Hermite polynomials.

From Section 1.2.4 we already know, that this representation is related to the Fourier series expansion

$$\ell(x) = \sum_{k=0}^{\infty}\langle \ell, H_k\rangle_w H_k(x), \tag{3.15}$$

where $\ell(x) := g_T(x)/w(x)$. The right hand side of (3.15) converges by Bessel's inequality (Lemma 1.19) if $\ell \in L^2_w$, and equality in L^2_w holds by Parseval's theorem (Theorem 1.20) for the orthonormal basis $\{H_k\}_{k\in\mathbb{N}}$ of L^2_w. For $\ell \in L^2_w$ to hold, i.e. $\int_{\mathbb{R}} g_T^2(x)/w(x)dx < \infty$, it suffices to show

$$\int_{\mathbb{R}} g_T^2(x)e^{\epsilon x^2}dx < \infty \quad \text{for some } \epsilon > 0. \tag{3.16}$$

We see in the next theorem, that (3.16) is satisfied for $\epsilon < \frac{1}{2v_{\max}T}$.

Theorem 3.5. *For all $\epsilon < \frac{1}{2v_{\max}T}$ the density g_T of the log-price X_T satisfies*

$$\int_{\mathbb{R}} e^{\epsilon x^2} g_T(x)dx < \infty.$$

Furthermore $g_T(x)$ and $e^{\epsilon x^2}g_T(x)$ are uniformly bounded on $\mathbb{R}$.

Proof. We use the same idea as in the proof of [1, Theorem 4.1].

Let

$$G_T(x) := \frac{1}{\sqrt{2\pi C_T}}\exp\left(-\frac{(x-M_T)^2}{2C_T}\right) \tag{3.17}$$

denote the density of X_T conditioned on any path V with mean M_T and variance C_T given by (3.9). Since $G_T(x) \le \frac{1}{\sqrt{2\pi C_T}}$ and $\mathbb{E}[C_T^{-\frac{1}{2}}] < \infty$ we get that

$$g_T(x) = \mathbb{E}_{\mathbb{Q}}[G_T(x)] \le \frac{1}{\sqrt{2\pi}}\mathbb{E}_{\mathbb{Q}}[C_T^{-\frac{1}{2}}]$$

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is uniformly bounded.

For the conditional mean M_T, the conditional variance C_T and by

$$X_0 - v_{\max}T - \frac{\rho}{\sigma}v_{\max}(1+\kappa T) \leq M_T \leq X_0 + rT + \frac{\rho}{\sigma}v_{\max}(1+\kappa T),$$

we have the uniform bounds $0 < C_T \leq v_{\max}T$ and $|M_T| \leq K$, where K is a constant depending on X_0 and T. Then, using $C_T \leq v_{\max}T$, further gives

$$e^{\epsilon x^2}G_T(x) = \frac{1}{\sqrt{2\pi C_T}}\exp\left(\epsilon x^2 - \frac{(x-M_T)^2}{2C_T}\right) \leq \frac{1}{\sqrt{2\pi C_T}}\exp\left(\epsilon x^2 - \frac{(x-M_T)^2}{2v_{\max}T}\right).$$

We compute $\epsilon x^2 - \frac{(x-M_T)^2}{2v_{\max}T} = \left(\epsilon - \frac{1}{2v_{\max}T}\right)x^2 + \frac{M_T}{v_{\max}T}x - \frac{M_T^2}{2v_{\max}T}$ and by setting

$$\delta := \frac{1}{2v_{\max}T} - \epsilon > 0 \tag{3.18}$$

the exponent becomes

$$\begin{aligned}\epsilon x^2 - \frac{(x-M_T)^2}{2v_{\max}T} &= -\delta\left(x - \frac{M_T}{2\delta v_{\max}T}\right)^2 + \frac{M_T^2}{2v_{\max}T}\left(\frac{1}{2\delta v_{\max}T} - 1\right)\\ &= -\frac{1}{2}\frac{\left(x - \frac{M_T}{2\delta v_{\max}T}\right)^2}{\frac{1}{2\delta}} + \frac{M_T^2\epsilon}{2v_{\max}T\delta}.\end{aligned}$$

Consequently, for $|M_T| \leq K$, one gets

$$\begin{aligned}e^{\epsilon x^2}G_T(x) &\leq \underbrace{\frac{1}{\sqrt{2\pi\frac{1}{2\delta}}}\exp\left(-\frac{1}{2}\frac{(x-\frac{M_T}{2\delta v_{\max}T})^2}{\frac{1}{2\delta}}\right)}_{\text{density of } \mathcal{N}(\frac{M_T}{2\delta v_{\max}T},\frac{1}{2\delta})}\sqrt{\frac{\frac{1}{2\delta}}{C_T}}\exp\left(\frac{M_T^2\epsilon}{2\delta v_{\max}T}\right)\\ &\leq \frac{1}{\sqrt{2\pi\frac{1}{2\delta}}}\exp\left(-\frac{1}{2}\frac{(x-\frac{M_T}{2\delta v_{\max}T})^2}{\frac{1}{2\delta}}\right)\sqrt{\frac{\frac{1}{2\delta}}{C_T}}\exp\left(\frac{K^2\epsilon}{2\delta v_{\max}T}\right).\end{aligned} \tag{3.19}$$

Thus, we find the uniform bound

$$e^{\epsilon x^2}g_T(x) = \mathbb{E}_{\mathbb{Q}}[e^{\epsilon x^2}G_T(x)] \leq \frac{1}{\sqrt{2\pi}}\exp\left(\frac{K^2\epsilon}{2\delta v_{\max}T}\right)\mathbb{E}_{\mathbb{Q}}[C_T^{-\frac{1}{2}}] < \infty.$$

Integration of (3.19) gives

$$\int_{\mathbb{R}} e^{\epsilon x^2}G_T(x)dx \leq \sqrt{\frac{\frac{1}{2\delta}}{C_T}}\exp\left(\frac{K^2\epsilon}{2\delta v_{\max}T}\right),$$

taking expectation and using Fubini's theorem we finally obtain

$$\mathbb{E}_{\mathbb{Q}}[\int_{\mathbb{R}} e^{\epsilon x^2}G_T(x)dx] = \int_{\mathbb{R}} e^{\epsilon x^2}g_T(x)dx < \infty$$

for ϵ from (3.18). □

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As corollary we obtain the following conditions for the mean μ_w and the variance σ_w^2 for the series representation of g_T to be convergent:

Corollary 3.6. *For* $\sigma_w^2 > \frac{v_{\max}T}{2}$ *and* $\mu_w \in \mathbb{R}$ *we have* $\ell \in L_w^2$ *i.e.* $\int_{\mathbb{R}} \frac{g_T^2(x)}{w(x)} dx < \infty$.

Proof. Since $e^{\epsilon x^2} g_T(x)$ is uniformly bounded on $\mathbb{R}$ and integrable $\forall \epsilon < \frac{1}{2v_{\max}T}$, we have for $\mu_w = 0$ and σ_w^2 satisfying $\frac{1}{2\sigma_w^2} \leq 2\epsilon < \frac{1}{v_{\max}T}$ the pointwise bound

$$\frac{1}{w(x)} g_T^2(x) \leq e^{2\epsilon x^2} g_T^2(x)$$

and therefore

$$\int_{\mathbb{R}} \frac{1}{w(x)} g_T^2(x) dx \leq \int_{\mathbb{R}} e^{2\epsilon x^2} g_T^2(x) dx = \int_{\mathbb{R}} \left(e^{\epsilon x^2} g_T(x)\right)^2 dx < \infty.$$

For $\mu_w \neq 0$ another inspection of the proof of Theorem 3.5 shows, that $e^{\epsilon x^2} g_T(x + \mu_w)$ is also uniformly bounded for all $\epsilon < \frac{1}{2v_{\max}T}$ (replace M_T by $M_T + \mu_w$ and its uniform bound becomes $K + \mu_w$). Thus, we recover the first case by a change of variable $x = z - \mu_w$,

$$\int_{\mathbb{R}} e^{\frac{1}{2\sigma_w^2}(z-\mu_w)^2} g_T^2(z) dz = \int_{\mathbb{R}} e^{\frac{1}{2\sigma_w^2}x^2} g_T^2(x + \mu_w) dx.$$

□

Remark 3.7. For $\epsilon \in (0, \frac{1}{2v_{\max}T})$ we have $g_T(x) \sim e^{-\epsilon x^2}$, meaning that the asymptotic behavior of the log-price distribution is of the same order as in the Black-Scholes model. Note also, that for $v_{\max} \to \infty$ the above theorem is only valid for $\epsilon = 0$, which implies that the Heston model allows fatter tails than the Jacobi stochastic volatility model. However, in the Heston model the density g_T does not admit a convergent series representation of the above form for any choice of the parameters μ_w and σ_w^2.

Summing up, the density g_T of the log-price X_T can be expanded as series with respect to generalized Hermite polynomials,

$$g_T(x) = w(x) \sum_{k=0}^{\infty} \ell_k H_k(x),$$

where the parameters for the density of the normal distribution w satisfy

$$\sigma_w^2 > \frac{v_{\max}T}{2}, \quad \mu_w \in \mathbb{R}, \tag{3.20}$$

and the coefficients ℓ_k, which henceforth we refer to as Hermite moments, are given by

$$\ell_k = \langle \frac{g_T}{w}, H_k \rangle_w = \int_{\mathbb{R}} \frac{g_T(x)}{w(x)} H_k(x) w(x) dx = \int_{\mathbb{R}} g_T(x) H_k(x) dx = \mathbb{E}_{\mathbb{Q}}[H_k(X_T)].$$

The (Hermite-) moments can be explicitly calculated as we will see in the next section. We make use, that the Jacobi stochastic volatility model is described by a two-dimensional polynomial diffusion.

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3.2.2 Computation of the Moments

The Jacobi stochastic volatility model now considered as two-dimensional polynomial diffusion with drift vector and diffusion matrix from (3.8) allows a closed-form of its finite conditional moments as we have already seen in section 1.1.3. In our model the moment formula is stated as follows:

Theorem 3.8 (Moment formula)**.** *For every $p \in Pol_N((v_{\min}, v_{\max}) \times \mathbb{R})$ with coordinate representation $\vec{p}$ with respect to an ordered basis vector B of $Pol_N((v_{\min}, v_{\max}) \times \mathbb{R})$ and $0 \leq s \leq t$ we have*

$$\mathbb{E}_{\mathbb{Q}}[p(V_t, X_t)|\mathscr{F}_s] = B(V_s, X_s)^\top e^{(t-s)G}\vec{p},$$

where G is the matrix representation of the linear map $\mathscr{G} : Pol_N(\mathbb{R}^2) \to Pol_N(\mathbb{R}^2)$,

$$p(v,x) \mapsto \begin{pmatrix} \kappa(\theta - v) \\ r - \frac{v}{2} \end{pmatrix}^\top \begin{pmatrix} p_v(v,x) \\ p_x(v,x) \end{pmatrix} + \operatorname{tr}\left(\begin{pmatrix} \sigma^2 Q(v) & \rho\sigma Q(v) \\ \rho\sigma Q(v) & v \end{pmatrix} \begin{pmatrix} p_{vv}(v,x) & p_{vx}(v,x) \\ p_{xv}(v,x) & p_{xx}(v,x) \end{pmatrix} \right).$$

In the following we consider two particular bases for the $M = \frac{(N+1)(N+2)}{2}$-dimensional vector space $\mathrm{Pol}_N(\mathbb{R}^2)$, namely

(1) the standard basis $\{b_k(v,x) = v^m x^n : \ m,n \in \mathbb{N}, \ m+n \leq N, \ 1 \leq k \leq M\}$ and

(2) the basis $\{h_k(v,x) = v^m H_n(x) : \ m,n \in \mathbb{N}, \ m+n \leq N, \ 1 \leq k \leq M\}$ where the generalized Hermite polynomials are involved.

The first basis works well for the computation of the log-price moments and the second will be used to calculate the Hermite moments.

We choose in any case for ordering the basis elements the enumeration[8]

$$\begin{aligned} \epsilon : \{0,1,\dots,N\} \times \{0,1,\dots,N\} &\to \{1,2,\dots,M\}, \\ \epsilon(m,n) &= n + 1 + \sum_{i=0}^{m-1} (N+1-i), \end{aligned} \tag{3.21}$$

with inverse $\epsilon^{-1}(k) = (m,n), \ 1 \leq k \leq M$, where

$$\begin{aligned} m &= \max\{i \in \mathbb{N} : \ i(N+1) - \frac{i(i-1)}{2} < k\}, \\ n &= k - 1 - m(N+1) + \frac{m(m-1)}{2}. \end{aligned}$$

For the standard basis (1) we obtain the ordered basis vector

$$\begin{aligned} B(v,x)^\top &= \left(1, x, x^2, \dots, x^{N-1}, x^N, v, vx, vx^2, \dots, vx^{N-1}, \dots, v^{N-1}, v^{N-1}x, v^N\right) \\ &= \left(b_{\epsilon(0,0)}, b_{\epsilon(0,1)}, b_{\epsilon(0,2)}, \dots b_{\epsilon(0,N-1)}, b_{\epsilon(0,N)}, b_{\epsilon(1,0)}, \dots, b_{\epsilon(1,N-1)}, \dots, b_{\epsilon(N,0)}\right), \end{aligned}$$

[8] We specify the enumeration for convenience. The results are valid for arbitrary enumerations.

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and for (2), where x^n is replaced by $H_n(x)$, we get

$$\begin{aligned} B(v,x)^\top &= \big(1, H_1(x), H_2(x), \dots, H_{n-1}(x), H_n(x), v, vH_1(x), vH_2(x), \dots, vH_{n-1}(x), \dots, v^N\big) \\ &= \big(h_{\epsilon(0,0)}, h_{\epsilon(0,1)}, h_{\epsilon(0,2)}, \dots, h_{\epsilon(0,N-1)}, h_{\epsilon(0,N)}, h_{\epsilon(1,0)}, \dots, h_{\epsilon(1,N-1)}, \dots, h_{\epsilon(N,0)}\big). \end{aligned}$$

The mapping ϵ assigns each pair of exponents (m,n), considered as index of a two dimensional array of basis elements (see the scheme below), the index of the ordered vector, which is given by appending the elements of each row of the scheme in ascending order.

	$n=0$	$n=1$	$n=2$	$\dots$	$n=N-1$	$n=N$
$m=0$	1	x	x^2	$\dots$	x^{N-1}	x^N
$m=1$	v	vx	vx^2	$\dots$	vx^{N-1}	
$m=2$	v^2	v^2x	v^2x^2	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$m=N-1$	v^{N-1}	$v^{N-1}x$				
$m=N$	v^N					

Table 3.1: Upper triangle scheme for enumeration

We demonstrate how the representation matrix G is computed with respect to the standard basis and how the calculation of the log-price moments is done. The computation of the Hermite moments follows the same idea and the results are presented afterwards.

Log-price moments

For each basis element $b_{\epsilon(m,n)}(v,x) = v^m x^n$, $0 \le m,n \le N$, the image under $\mathscr{G}$ is given by

$$\mathscr{G}(b_{\epsilon(m,n)}) = \sum_{0\le i,j\le N} G_{\epsilon(i,j),\epsilon(m,n)} b_{\epsilon(i,j)},$$

where $G_{\epsilon(i,j),\epsilon(m,n)}$ denotes the entry in the $\epsilon(i,j)$-th row and $\epsilon(m,n)$-th column of the representation matrix $G = (G_{\epsilon(i,j),\epsilon(m,n)})_{0\le i,j,m,n\le N}$. A straightforward computation gives

$$\begin{aligned} \mathscr{G}(b_{\epsilon(m,n)}) &= \mathscr{G}(v^m x^n) \\ &= \kappa(\theta - v) m v^{m-1} x^n + \left(r - \frac{v}{2}\right) n v^m x^{n-1} \\ &\quad + \frac{1}{2}\left[\sigma^2 Q(v) m(m-1) v^{m-2} x^n + 2\sigma\rho Q(v) m n v^{m-1} x^{n-1} + v n(n-1) v^m x^{n-2}\right], \end{aligned}$$

which after rearranging the terms yields

$$\begin{aligned} \mathscr{G}(b_{\epsilon(m,n)}) = {} & G_{\epsilon(m-2,n),\epsilon(m,n)} b_{\epsilon(m-2,n)} + G_{\epsilon(m-1,n-1),\epsilon(m,n)} b_{\epsilon(m-1,n-1)} \\ & + G_{\epsilon(m-1,n),\epsilon(m,n)} b_{\epsilon(m-1,n)} + G_{\epsilon(m,n-1),\epsilon(m,n)} b_{\epsilon(m,n-1)} \\ & + G_{\epsilon(m,n),\epsilon(m,n)} b_{\epsilon(m,n)} + G_{\epsilon(m+1,n-2),\epsilon(m,n)} b_{\epsilon(m+1,n-2)} \\ & + G_{\epsilon(m+1,n-1),\epsilon(m,n)} b_{\epsilon(m+1,n-1)}, \end{aligned}$$

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where the coefficients are given by

$$
\begin{aligned}
G_{\epsilon(m-2,n),\epsilon(m,n)} &= -\frac{m(m-1)\sigma^2 v_{\min} v_{\max}}{2\left(\sqrt{v_{\max}}-\sqrt{v_{\min}}\right)^2}, \quad m \geq 2,\\
G_{\epsilon(m-1,n-1),\epsilon(m,n)} &= -\frac{mn\rho\sigma v_{\min} v_{\max}}{\left(\sqrt{v_{\max}}-\sqrt{v_{\min}}\right)^2}, \; m,n \geq 1,\\
G_{\epsilon(m-1,n),\epsilon(m,n)} &= m\left(\kappa\theta + \frac{(m-1)\sigma^2(v_{\min}+v_{\max})}{2\left(\sqrt{v_{\max}}-\sqrt{v_{\min}}\right)^2}\right), \quad m \geq 1,\\
G_{\epsilon(m,n-1),\epsilon(m,n)} &= n\left(r + \frac{m\rho\sigma(v_{\min}+v_{\max})}{\left(\sqrt{v_{\max}}-\sqrt{v_{\min}}\right)^2}\right), \quad n \geq 1,\\
G_{\epsilon(m,n),\epsilon(m,n)} &= -m\left(\kappa + \frac{(m-1)\sigma^2}{2\left(\sqrt{v_{\max}}-\sqrt{v_{\min}}\right)^2}\right), \quad m,n \geq 0,\\
G_{\epsilon(m+1,n-2),\epsilon(m,n)} &= \frac{n(n-1)}{2}, \quad n \geq 2,\\
G_{\epsilon(m+1,n-1),\epsilon(m,n)} &= -n\left(\frac{1}{2} + \frac{mn\rho\sigma}{\left(\sqrt{v_{\max}}-\sqrt{v_{\min}}\right)^2}\right), \quad n \geq 1.
\end{aligned}
$$

Thus, each column of G has at most seven coefficients different from zero.

The n-th log-price moment is identified as the polynomial

$$
p(v,x) = x^n = b_{\epsilon(0,n)}(v,x) = \Big(b_{\epsilon(0,0)}(v,x), \ldots, b_{\epsilon(N,0)}(v,x)\Big)\vec{e}_{\epsilon(0,n)},
$$

and the moment formula (Theorem 3.8) then becomes

$$
\mathbb{E}_{\mathbb{Q}}[X_T^n] = \Big(b_{\epsilon(0,0)}(V_0,X_0), \ldots, b_{\epsilon(N,0)}(V_0,X_0)\Big)e^{T\cdot G}\vec{e}_{\epsilon(0,n)}, \; 1 \leq n \leq N.
$$

Example 3.9 (First two log-price moments)**.** For the variance process we choose the parameters $v_{\min} = 0$, $v_{\max} = 1$, $\kappa = 1$, $\sigma = \frac{1}{4}$, $\theta = V_0 = \frac{1}{2}$, for the log-price process $X_0 = 0$, $r = 0$ and the correlation factor is given by $\rho = -\frac{1}{2}$. For $T = 1$ (one year) and $N = 2$ we have with respect to the ordered standard basis vector $B(v,x)^\top = (1, x, x^2, v, vx, v^2)$ the representation matrix G and its matrix exponential e^G given by

$$
G = \begin{pmatrix}
0 & 0 & 0 & \frac{1}{2} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{2} & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{1}{2} & 1 & -1 & -\frac{1}{8} & \frac{17}{16}\\
0 & 0 & -1 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & -\frac{3}{8} & -\frac{33}{16}
\end{pmatrix}, e^G = \begin{pmatrix}
1 & -0.09 & 0.20 & 0.32 & -0.04 & 0.10\\
0 & 1 & -0.18 & 0 & 0.32 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & -0.32 & 0.72 & 0.37 & -0.17 & 0.24\\
0 & 0 & -0.63 & 0 & 0.37 & 0\\
0 & 0 & 0.07 & 0 & -0.08 & 0.13
\end{pmatrix}.
$$

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For the first two moments we compute

$$\mathbb{E}_{\mathbb{Q}}[X_1] = \left(1, 0, 0, \frac{1}{2}, 0, \frac{1}{4}\right) \cdot \begin{pmatrix} 1 & -0.09 & 0.20 & 0.32 & -0.04 & 0.10 \\ 0 & 1 & -0.18 & 0 & 0.32 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.32 & 0.72 & 0.37 & -0.17 & 0.24 \\ 0 & 0 & -0.63 & 0 & 0.37 & 0 \\ 0 & 0 & 0.07 & 0 & -0.08 & 0.13 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -0.2500,$$

$$\mathbb{E}_{\mathbb{Q}}[X_1^2] = \left(1, 0, 0, \frac{1}{2}, 0, \frac{1}{4}\right) \cdot \begin{pmatrix} 1 & -0.09 & 0.20 & 0.32 & -0.04 & 0.10 \\ 0 & 1 & -0.18 & 0 & 0.32 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.32 & 0.72 & 0.37 & -0.17 & 0.24 \\ 0 & 0 & -0.63 & 0 & 0.37 & 0 \\ 0 & 0 & 0.07 & 0 & -0.08 & 0.13 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0.5745.$$

Hermite moments

The representation matrix G with respect to $h_{\epsilon(0,0)}, h_{\epsilon(0,1)}, \ldots, h_{\epsilon(0,N)}, \ldots, h_{\epsilon(N,0)}$ using the elementary property

$$H_n'(x) \overset{(1.18)}{=} \frac{1}{\sigma_w} \frac{1}{\sqrt{n!}} \mathrm{He}_n' \left(\frac{x-\mu_w}{\sigma_w}\right) \underset{(1.19)}{\overset{(1.17)}{=}} \frac{1}{\sigma_w} \frac{n}{\sqrt{n!}} \mathrm{He}_{n-1} \left(\frac{x-\mu_w}{\sigma_w}\right) \overset{(1.18)}{=} \frac{\sqrt{n}}{\sigma_w} H_{n-1}(x),$$

has the non-zero entries (see [1, p.688])

$$\begin{aligned}
G_{\epsilon(m-2,n),\epsilon(m,n)} &= -\frac{m(m-1)\sigma^2 v_{\min} v_{\max}}{2\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}, \quad m \geq 2, \\
G_{\epsilon(m-1,n-1),\epsilon(m,n)} &= -\frac{m\sqrt{n}\rho\sigma v_{\min} v_{\max}}{\sigma_w\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}, \quad m, n \geq 1, \\
G_{\epsilon(m-1,n),\epsilon(m,n)} &= m\left(\kappa\theta + \frac{(m-1)\sigma^2(v_{\min} + v_{\max})}{2\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}\right), \quad m \geq 1, \\
G_{\epsilon(m,n-1),\epsilon(m,n)} &= \frac{\sqrt{n}}{\sigma_w}\left(r + \frac{m\rho\sigma(v_{\min} + v_{\max})}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}\right), \quad n \geq 1, \\
G_{\epsilon(m,n),\epsilon(m,n)} &= -m\left(\kappa + \frac{(m-1)\sigma^2}{2\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}\right), \quad m, n \geq 0, \\
G_{\epsilon(m+1,n-2),\epsilon(m,n)} &= \frac{\sqrt{n(n-1)}}{2\sigma_w^2}, \quad n \geq 2, \\
G_{\epsilon(m+1,n-1),\epsilon(m,n)} &= -\frac{\sqrt{n}}{\sigma_w}\left(\frac{1}{2} + \frac{m\rho\sigma}{\left(\sqrt{v_{\max}} - \sqrt{v_{\min}}\right)^2}\right), \quad n \geq 1,
\end{aligned}$$

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and the moment formula for the Hermite moments $\ell_n = \mathbb{E}_{\mathbb{Q}}[H_n(X_T)],\ 1 \leq n \leq N$, writes as

$$\ell_n = \Big(h_{\epsilon(0,0)}(V_0, X_0), \ldots, h_{\epsilon(N,0)}(V_0, X_0)\Big) e^{TG} \vec{e}_{\epsilon(0,n)}, \quad 0 \leq n \leq N.$$

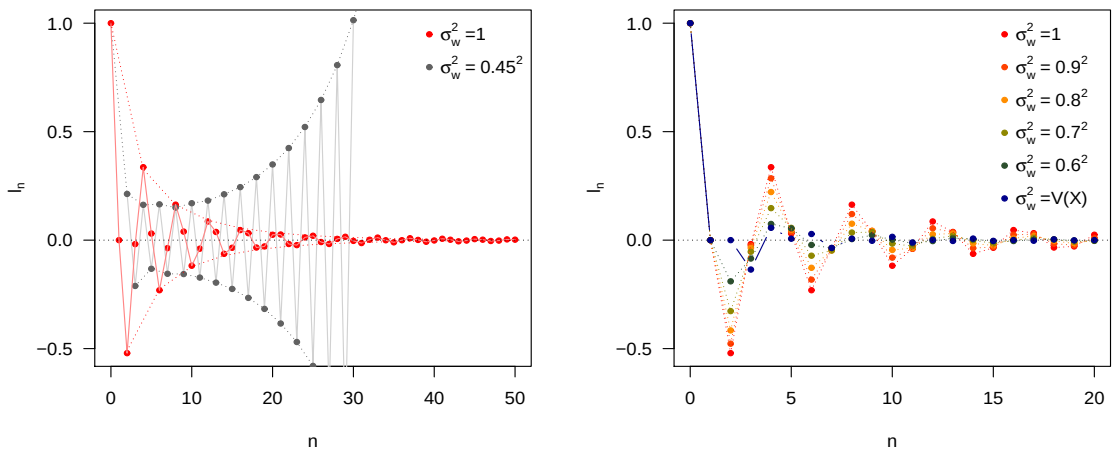
In practical applications we can only use a finite series representation of the log-price density up to a certain order N, which needs to be given in advance for the computation of the Hermite moments. Therefore it can be argued, that convergence properties of the series expansion are not relevant in that case. If we know, that the first N terms explain accurately the significant pattern of the series, we do not care whether the series is convergent or divergent.

The study of the asymptotic properties of the series is related to the question of how much better the approximation becomes by considering further terms in the expansion. As pointed out in [6], the order of magnitude of the coefficients is not steadily decreasing as N increases for Gram-Charlier Type A series (compare [6, p.227]) and therefore the requirement, that we find parameters μ_w and σ_w^2 such that the likelihood ratio $\ell = g_T/w$ lies in L_w^2 is important to guarantee convergence of the series expansion.

Example 3.10 (Order of magnitude of Hermite moments)**.** The next figure displays the Hermite moments of the log-price density for fixed mean $\mu_w = \mathbb{E}_{\mathbb{Q}}[X_T] = -0.125$ and different variances σ_w^2. The model parameters are given by $X_0 = 0$, $T = 0.5$, $v_{\min} = 0$, $v_{\max} = 1$, $\kappa = 1$, $\theta = V_0 = 0.5$, $\sigma = 1$ and $\rho = -0.5$.

Both panels (Figure 3.1) show the oscillating behavior of the Hermite moments. The amplitudes are decreasing as n increases, if $\sigma_w^2 > \frac{v_{\max} T}{2}$ is satisfied. If the condition does not hold, the absolute values of the Hermite moments will rapidly increase for increasing n (gray points). Further it is shown on the right panel, that the smaller σ_w the faster the decline of the absolute values of ℓ_n. The best result is obtained by the choice $\sigma_w^2 = \mathbb{V}_{\mathbb{Q}}(X_T) = 0.2633$ (blue points).

Figure 3.1: Hermite moments of the log-price density



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For the above model parameters the resulting series expansions of the log-price density g_T with respect to w (solid lines) up to different truncation orders $N \in \{5, 10, 20, 30\}$ are considered below in Figure 3.2. The dotted lines represent the corresponding auxiliary density w of the normal distribution with $\mu_w = \mathbb{E}_{\mathbb{Q}}[X_T]$, $\sigma_w^2 = 1$ (red) and $\sigma_w^2 = \mathbb{V}_{\mathbb{Q}}(X_T)$ (blue).

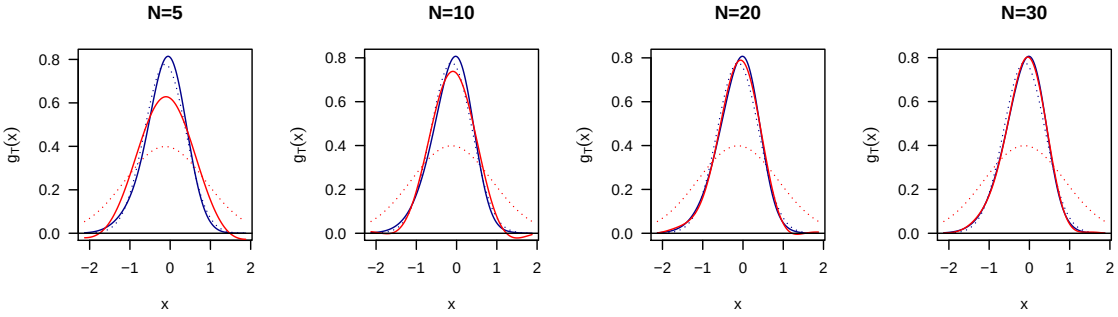

Figure 3.2: Series expansion of the log-price density

At order $N = 30$ the approximations of g_T are almost identical and it is readily seen, that small values for the Hermite moments do not significantly change the shape of the approximation. A small number of terms seems to be sufficient for a good approximation of g_T by w if the parameters of the auxiliary density are given by the mean and the variance of the log-price.

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Chapter 4

European Option pricing in the Jacobi Stochastic Volatility Model

In this chapter we derive series representations of option prices in the Jacobi stochastic volatility model. Further we study approximation errors, which occur by truncation of the series at some finite order and we finally show that the pricing technique can be extended to path-dependent options.

4.1 Series representation of European claims

Let f be the discounted payoff function of a contingent claim at maturity $T > 0$ with underlying $S = e^X$ and let

$$\pi_f = \mathbb{E}_{\mathbb{Q}}[f(X_T)] = \int_{\mathbb{R}} f(x) g_T(x) dx$$

denote the price of the claim. From the last chapter we already know, that the log-price density admits a series expansion in terms of generalized Hermite polynomials

$$g_T(x) = w(x) \sum_{k=0}^{\infty} \mathbb{E}_{\mathbb{Q}}[H_k(X_T)] H_k(x), \tag{4.1}$$

where w is the density of a normal distribution with mean μ_w and variance σ_w^2, and corresponding Hermite moments $\mathbb{E}_{\mathbb{Q}}[H_k(X_T)]$, $k \in \mathbb{N}$. Further we know the Fourier series expansion of the likelihood ratio $\ell(x) = g_T(x)/w(x)$ which is given by

$$\ell(x) = \sum_{k=0}^{\infty} \ell_k H_k(x), \quad \ell_k = \mathbb{E}_{\mathbb{Q}}[H_k(X_T)]. \tag{4.2}$$

Using the representation (4.1) the option price is given as infinite series

$$\pi_f = \sum_{k=0}^{\infty} \mathbb{E}_{\mathbb{Q}}[H_k(X_T)] \int_{\mathbb{R}} f(x) H_k(x) w(x) dx, \tag{4.3}$$

and the integrals can be computed by the Gauss-Hermite quadrature formula (Proposition 1.26) if there no analytic expressions exist.

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If f is square-integrable with respect to w, i.e. $f \in L^2_w$, we find the Fourier series representation

$$f(x) = \sum_{k=0}^{\infty} f_k H_k(x), \quad f_k = \langle f, H_k \rangle_w, \tag{4.4}$$

with respect to the orthonormal set $\{H_k\}_{k\in\mathbb{N}}$. We use the representation of the likelihood ratio ℓ from (4.2) to write the option price as

$$\pi_f = \mathbb{E}_{\mathbb{Q}}[f(X_T)] = \int_{\mathbb{R}} f(x)\ell(x)w(x)dx = \langle f, \ell \rangle_w, \tag{4.5}$$

where the inner product becomes

$$\begin{aligned}\langle f, \ell \rangle_w &= \langle \sum_{n=0}^{\infty} f_n H_n, \sum_{m=0}^{\infty} \ell_m H_m \rangle_w = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle f_n H_n, \ell_m H_m \rangle_w \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_n \ell_m \underbrace{\langle H_n, H_m \rangle_w}_{=\delta_{mn}} = \sum_{n=0}^{\infty} f_n \ell_n.\end{aligned}$$

We interpret the option price in (4.5) as the inner product of the discounted payoff and likelihood ratio function with respect to a Gaussian reference measure and ℓ as the likelihood ratio (Radon-Nikodym derivative) of the measure change from the risk-neutral measure $\mathbb{Q}$ to a Gaussian reference measure with density w.

If we match the first moments of X_T with respect to g_T and w such that

$$\mu_w = \mathbb{E}_{\mathbb{Q}}[X_T] = \int_{\mathbb{R}} x g_T(x)dx = \int_{\mathbb{R}} x w(x)dx, \tag{4.6}$$

$$\sigma_w^2 = \mathbb{V}_{\mathbb{Q}}(X_T) = \int_{\mathbb{R}} (x-\mu_w)^2 g_T(x)dx = \int_{\mathbb{R}} (x-\mu_w)^2 w(x)dx, \tag{4.7}$$

the first two Hermite moments become zero. Indeed, using that g_T is a density and from (4.7) the equality $\int_{\mathbb{R}} \left(\frac{x-\mu_w}{\sigma_w}\right)^2 g_T(x)dx = 1$ gives

$$\begin{aligned}\ell_1 &= \int_{\mathbb{R}} H_1(x) g_T(x)dx = \int_{\mathbb{R}} \frac{x-\mu_w}{\sigma_w} g_T(x)dx \overset{(4.6)}{=} \frac{\mu_w}{\sigma_w} - \frac{\mu_w}{\sigma_w}\int_{\mathbb{R}} g_T(x)dx = 0, \\ \ell_2 &= \int_{\mathbb{R}} H_2(x) g_T(x)dx = \int_{\mathbb{R}} \left(\frac{(x-\mu_w)^2}{\sigma_w^2} - 1\right) g_T(x)dx \overset{(4.7)}{=} 0.\end{aligned}$$

As a consequence the option price becomes $\pi_f = f_0 + \sum_{n=3}^{\infty} f_n \ell_n$ and decomposes into

(i) the Black Scholes price $f_0 = \langle f, H_0 \rangle_w = \int_{\mathbb{R}} f(x)w(x)dx$,
i.e. $X_T = X_0 + (r - \sigma_{BS}^2/2)T + \sigma_{BS} W_T$ with $\sigma_{BS} = \sigma_w/\sqrt{T}$, and

(ii) the sum $\sum_{n=3}^{\infty} f_n \ell_n$, which is regarded as the correction of the Black Scholes price due to stochastic volatility.

Remark 4.1. Note, that we earlier imposed the condition $\sigma_w^2 > v_{\max} T/2$, i.e. matching the second moment might not always be feasible.

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4.2 Examples

Formulas for the Fourier coefficients in (4.4) of plain vanilla and binary options are presented in this section. As usual w denotes the density of the normal distribution with mean μ_w and variance σ_w^2 and $\{H_n\}_{n\in\mathbb{N}}$ the generalized Hermite polynomials, which form an orthonormal basis of L_w^2. The formulas will be stated in terms of the probabilistic Hermite polynomials $\{\mathrm{He}_n\}_{n\in\mathbb{N}}$. Recall, that we have

$$H_n(x) = \frac{1}{\sqrt{n!}}\mathrm{He}_n\left(\frac{x-\mu_w}{\sigma_w}\right), \ n \in \mathbb{N}.$$

4.2.1 Call option

Proposition 4.2. *For the discounted payoff function of a call option with log-strike K,*

$$f(x) = e^{-rT}(e^x - e^K)^+,$$

the Fourier coefficients are given by

$$\begin{aligned} f_0 &= e^{-rT+\mu_w} I_0\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right) - e^{-rT+K}\Phi\left(\frac{\mu_w - K}{\sigma_w}\right), \\ f_n &= \frac{\sigma_w}{\sqrt{n!}} e^{-rT+\mu_w} I_{n-1}\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right), \quad n \geq 1. \end{aligned}$$

The functions $I_n(c;\sigma)$ are defined by the recursion

$$\begin{aligned} I_n(c;\sigma) &= \mathrm{He}_{n-1}(c)e^{c\sigma}\varphi(c) + \sigma I_{n-1}(c;\sigma), \quad n \geq 1, \\ I_0(c;\sigma) &= e^{\frac{\sigma^2}{2}}\Phi(\sigma - c), \end{aligned} \tag{4.8}$$

where Φ denotes the distribution function of $\mathcal{N}(0,1)$, φ its density and He_n the n-th Hermite polynomial.

The proof is a refinement of the proof of Theorem 3.7 in [1, p.692].

Proof. For the Fourier coefficients of the representation $f(x) = \sum_{n=0}^{\infty} f_n H_n(x)$ by a change of variable $x = \sigma_w z + \mu_w$ we obtain

$$\begin{aligned} f_n &= \int_{\mathbb{R}} f(x)H_n(x)w(x)dx = e^{-rT}\int_K^{\infty}\left(e^x - e^K\right)H_n(x)w(x)dx \\ &= e^{-rT}\int_{\frac{K-\mu_w}{\sigma_w}}^{\infty}\left(e^{\sigma_w z+\mu_w} - e^K\right)\underbrace{H_n(\sigma_w z+\mu_w)}_{=\frac{1}{\sqrt{n!}}\mathrm{He}_n(z)}\underbrace{w(\sigma_w z+\mu_w)\sigma_w}_{=\varphi(z)}dz \\ &= \frac{e^{-rT+\mu_w}}{\sqrt{n!}}\int_{\frac{K-\mu_w}{\sigma_w}}^{\infty} e^{\sigma_w z}\mathrm{He}_n(z)\varphi(z)dz - \frac{e^{-rT+K}}{\sqrt{n!}}\int_{\frac{K-\mu_w}{\sigma_w}}^{\infty}\mathrm{He}_n(z)\varphi(z)dz. \end{aligned} \tag{4.9}$$

We define

$$I_n(c;\sigma) := \int_c^{\infty} e^{\sigma z}\mathrm{He}_n(z)\varphi(z)dz \tag{4.10}$$

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and show, that they satisfy the above recursion formula (4.8). For $n = 0$ we have

$$I_0(c;\sigma) = \int_c^\infty e^{\sigma z}\varphi(z)dz = e^{\frac{\sigma^2}{2}}\int_c^\infty \varphi(z-\sigma)dz = e^{\frac{\sigma^2}{2}}\int_{c-\sigma}^\infty \varphi(z)dz = e^{\frac{\sigma^2}{2}}\Phi(\sigma - c).$$

For $n > 0$ we use the identity $\mathrm{He}_n(z) = z\mathrm{He}_{n-1}(z) - \mathrm{He}'_{n-1}(z)$ and compute

$$\begin{aligned} I_n(c;\sigma) &= \int_c^\infty e^{\sigma z} z\mathrm{He}_{n-1}(z)\varphi(z)dz - \int_c^\infty e^{\sigma z}\mathrm{He}'_{n-1}(z)\varphi(z)dz \\ &= \int_c^\infty e^{\sigma z} z\mathrm{He}_{n-1}(z)\varphi(z)dz - e^{\sigma z}\mathrm{He}_{n-1}(z)\varphi(z)\Big|_{z=c}^\infty \\ &\qquad + \int_c^\infty \left(\sigma e^{\sigma z} - z e^{\sigma z}\right)\mathrm{He}_{n-1}(z)\varphi(z)dz \\ &= e^{\sigma c}\mathrm{He}_{n-1}(c)\varphi(c) + \sigma I_{n-1}(c;\sigma). \end{aligned}$$

Writing (4.9) in terms of the auxiliary functions I from (4.10) gives

$$f_n = \frac{e^{-rT+\mu_w}}{\sqrt{n!}} I_n\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right) - \frac{e^{-rT+K}}{\sqrt{n!}} I_n\left(\frac{K-\mu_w}{\sigma_w};0\right),$$

which simplifies for $n = 0$ to

$$f_0 = e^{-rT+\mu_w} I_0\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right) - e^{-rT+K}\Phi\left(\frac{\mu_w - K}{\sigma_w}\right),$$

and in any other case to

$$\begin{aligned} f_n &= \frac{e^{-rT+\mu_w}}{\sqrt{n!}}\left[\mathrm{He}_{n-1}\left(\frac{K-\mu_w}{\sigma_w}\right)e^{K-\mu_w}\varphi\left(\frac{K-\mu_w}{\sigma_w}\right) + \sigma_w I_{n-1}\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right)\right] \\ &\qquad - \frac{e^{-rT+K}}{\sqrt{n!}}\mathrm{He}_{n-1}\left(\frac{K-\mu_w}{\sigma_w}\right)\varphi\left(\frac{K-\mu_w}{\sigma_w}\right) \\ &= \frac{e^{-rT+\mu_w}}{\sqrt{n!}}\sigma_w I_{n-1}\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right). \end{aligned}$$

□

4.2.2 Put option

Proposition 4.3. *For the discounted payoff function of a put option with log-strike K,*

$$f(x) = e^{-rT}(e^K - e^x)^+,$$

the corresponding Fourier coefficients are given by

$$\begin{aligned} f_0 &= e^{-rT+\mu_w}\left[I_0\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right) - \bar{I}_0(\sigma_w)\right] + e^{-rT+K}\Phi\left(\frac{K-\mu_w}{\sigma_w}\right), \\ f_n &= \frac{\sigma_w}{\sqrt{n!}}e^{-rT+\mu_w}\left[I_{n-1}\left(\frac{K-\mu_w}{\sigma_w};\sigma_w\right) - \bar{I}_{n-1}(\sigma_w)\right], \quad n \geq 1, \end{aligned}$$

where I,Φ and φ are as in Proposition 4.2 and

$$\bar{I}_n(\sigma) = \sigma^n e^{\frac{\sigma^2}{2}}, \quad n \geq 0. \tag{4.11}$$

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Proof. First note, that the Fourier coefficients of the put option rewrites to

$$\begin{aligned} f_n &= e^{-rT} \int_{-\infty}^{\infty} \left(e^K - e^x\right)^+ H_n(x) w(x) dx \\ &= e^{-rT} \int_K^{\infty} \left(e^x - e^K\right) H_n(x) w(x) dx - e^{-rT} \int_{-\infty}^{\infty} e^x H_n(x) w(x) dx \\ &\qquad + e^{-rT+K} \int_{-\infty}^{\infty} H_n(x) w(x) dx. \end{aligned} \tag{4.12}$$

For the first term we have by Proposition 4.2

$$\begin{aligned} & e^{-rT} \int_K^{\infty} \left(e^x - e^K\right) H_n(x) w(x) dx \\ &\qquad = \begin{cases} e^{-rT+\mu_w} I_0\left(\frac{K-\mu_w}{\sigma_w}; \sigma_w\right) - e^{-rT+K} \Phi\left(\frac{\mu_w - K}{\sigma_w}\right), & n = 0, \\ \frac{\sigma_w}{\sqrt{n!}} e^{-rT+\mu_w} I_{n-1}\left(\frac{K-\mu_w}{\sigma_w}; \sigma_w\right), & n > 0, \end{cases} \end{aligned}$$

and for the other we introduce

$$\bar{I}_n(\sigma) := \lim_{c \to -\infty} I_n(c; \sigma), \quad n \in \mathbb{N}, \tag{4.13}$$

which yields to the recursion

$$\bar{I}_0(\sigma) = e^{\frac{\sigma^2}{2}}, \quad \bar{I}_n(\sigma) = \sigma \bar{I}_{n-1}(\sigma), \quad n \geq 1,$$

and consequently to

$$\bar{I}_n(\sigma) = \sigma^n e^{\frac{\sigma^2}{2}}, \quad n \geq 0.$$

Hence, by the same arguments as in the last proposition we get for the second term

$$e^{-rT} \int_{-\infty}^{\infty} e^x H_n(x) w(x) dx = \begin{cases} e^{-rT+\mu_w} \bar{I}_0\left(\sigma_w\right) - e^{rT+K}, & n = 0, \\ \frac{\sigma_w}{\sqrt{n!}} e^{-rT+\mu_w} \bar{I}_{n-1}\left(\sigma_w\right), & n > 0, \end{cases}$$

and for the third

$$e^{-rT+K} \int_{-\infty}^{\infty} H_n(x) w(x) dx = \frac{e^{-rT+K}}{\sqrt{n!}} \bar{I}_n(0) \overset{(4.11)}{=} \begin{cases} e^{-rT+K}, & n = 0, \\ 0, & n > 0. \end{cases}$$

Substituting the above expressions in (4.12) gives the result. □

Since the Fourier coefficients of binary options can be derived from the above formulas, we state the results as corollaries of the last two propositions.

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4.2.3 Binary call option

Corollary 4.4. *The Fourier coefficients of the digital call option with payoff function*

$$f(x) = e^{-rT}\mathbb{1}_{[K;\infty)}(x),\ \ K \in \mathbb{R},$$

are given by

$$\begin{aligned} f_0 &= e^{-rT}\Phi\left(\frac{\mu_w - K}{\sigma_w}\right), \\ f_n &= \frac{e^{-rT}}{\sqrt{n!}}\mathrm{He}_{n-1}\left(\frac{K-\mu_w}{\sigma_w}\right)\varphi\left(\frac{K-\mu_w}{\sigma_w}\right), \quad n \geq 1. \end{aligned}$$

Proof. With the functions I from (4.10) and (4.8) we get

$$\begin{aligned} f_n &= e^{-rT}\int_K^\infty H_n(x)w(x)dx = \frac{e^{-rT}}{\sqrt{n!}}\int_{\frac{K-\mu_w}{\sigma_w}}^\infty \mathrm{He}_n(z)\varphi(z)dz = \frac{e^{-rT}}{\sqrt{n!}}I_n\left(\frac{K-\mu_w}{\sigma_w};0\right) \\ &= \begin{cases} e^{-rT}\Phi\left(\frac{\mu_w-K}{\sigma_w}\right), & n=0, \\ \frac{e^{-rT}}{\sqrt{n!}}\mathrm{He}_{n-1}\left(\frac{K-\mu_w}{\sigma_w}\right)\varphi\left(\frac{K-\mu_w}{\sigma_w}\right), & n>0. \end{cases} \end{aligned}$$

□

Remark 4.5. For a generic binary option with a payoff function of the form $f(x) = \mathbb{1}_{[K_1,K_2)}(x)$ its Fourier coefficients are derived by $\mathbb{1}_{[K_1,K_2)}(x) = \mathbb{1}_{[K_1,\infty)}(x) - \mathbb{1}_{[K_2\infty)}(x)$ and the above corollary.

4.2.4 Binary put option

Corollary 4.6. *For a digital put option with discounted payoff function*

$$f(x) = e^{-rT}\mathbb{1}_{(-\infty;K]}(x),\ \ K \in \mathbb{R},$$

we have

$$\begin{aligned} f_0 &= e^{-rT}\Phi\left(\frac{K-\mu_w}{\sigma_w}\right), \\ f_n &= -\frac{e^{-rT}}{\sqrt{n!}}\mathrm{He}_{n-1}\left(\frac{K-\mu_w}{\sigma_w}\right)\varphi\left(\frac{K-\mu_w}{\sigma_w}\right), \quad n \geq 1. \end{aligned}$$

Proof. Use the definitions of $\bar{I}$ and I from (4.11) and (4.10) for

$$\begin{aligned} f_n = e^{-rT}\int_{-\infty}^K H_n(x)w(x)dx &= e^{-rT}\left(\int_{-\infty}^\infty H_n(x)w(x)dx - \int_K^\infty H_n(x)w(x)dx\right) \\ &= \frac{e^{-rT}}{\sqrt{n!}}\left(\bar{I}_n(0) - I_n\left(\frac{K-\mu_w}{\sigma_w};0\right)\right). \end{aligned}$$

□

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4.3 Error analysis

In practice the option price π_f is approximated by the finite series

$$\pi_f^{(N)} = \sum_{n=0}^{N} f_n \ell_n, \tag{4.14}$$

which is obtained by truncation of π_f at some finite order N. Since the summands of the series are given as the Fourier coefficients of functions in L^2_w the price approximation $\pi_f^{(N)}$ converges to π_f as $N \to \infty$ and induces an efficient price approximation scheme. If we introduce

$$f^{(N)}(x) = \sum_{n=0}^{N} f_n H_n(x), \quad \ell^{(N)} = \sum_{n=0}^{N} \ell_n H_n(x),$$

and denote the approximation error by

$$\epsilon^{(N)} := \pi_f - \pi_f^{(N)} = \sum_{n=N+1}^{\infty} f_n \ell_n = \langle f - f^{(N)}, \ell - \ell^{(N)} \rangle_w,$$

the Cauchy Schwarz inequality implies the error bound

$$|\epsilon^{(N)}| \leq \|f - f^{(N)}\|_w \|\ell - \ell^{(N)}\|_w = \left(\|f\|_w^2 - \|f^{(N)}\|_w^2\right)^{\frac{1}{2}} \left(\|\ell\|_w^2 - \|\ell^{(N)}\|_w^2\right)^{\frac{1}{2}}. \tag{4.15}$$

The equality on the right hand side follows by the orthogonality of the generalized Hermite polynomials, which gives

$$\begin{aligned} \|f - f^{(N)}\|_w^2 &= \int_{\mathbb{R}} (f(x) - f^{(N)}(x))^2 w(x) dx = \|f\|_w^2 - 2\langle f, f^{(N)} \rangle_w + \|f^{(N)}\|_w^2 \\ &= \|f\|_w^2 - \|f^{(N)}\|_w^2, \end{aligned}$$

using

$$\langle f, f^{(N)} \rangle_w = \sum_{m=0}^{\infty} \sum_{n=0}^{N} f_m f_n \int_{\mathbb{R}} H_m(x) H_n(x) w(x) dx = \sum_{n=0}^{N} f_n^2 = \|f^{(N)}\|_w^2.$$

In (4.15) the expressions $\|f^{(N)}\|_w^2$ and $\|\ell^{(N)}\|_w^2$ are known to us, and $\|f\|_w^2 = \int_{\mathbb{R}} f^2(x) w(x) dx$ can be computed via Gauss-Hermite quadrature (Proposition 1.26). For the remaining term we have

$$\|\ell\|_w^2 = \int_{\mathbb{R}} \frac{g_T^2(x)}{w(x)} dx = \int_{\mathbb{R}} \frac{g_T(x)}{w(x)} g_T(x) dx = \mathbb{E}_{\mathbb{Q}}[\frac{g_T(X_T)}{w(X_T)}],$$

where $g_T(x) = \mathbb{E}_{\mathbb{Q}}[G_T(x)]$ is given as expectation over all sample paths from V. For a fixed path $\{V_t,\ t \in [0, T]\}$ we have according to Theorem 3.4

$$G_T(x) = \frac{1}{\sqrt{2\pi C_T}} \exp\left(-\frac{(x - M_T)^2}{2C_T}\right).$$

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The random variables M_T and C_T are independent of X_T and we can compute

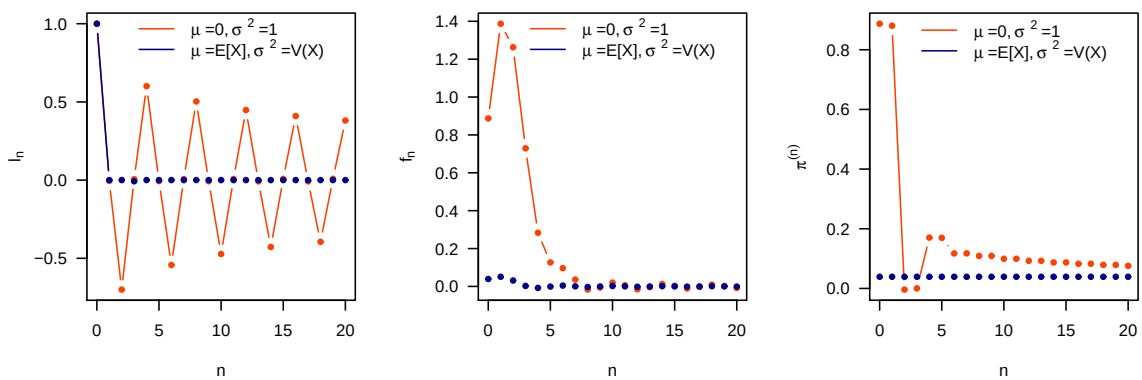
$$\|\ell\|_w^2 = \mathbb{E}_{\mathbb{Q}}\left[\frac{\varphi(X_T, M_T, C_T)}{\varphi(X_T, \mu_w, \sigma_w^2)}\right],$$

with $\varphi(x, M_T, C_T) = G_T(x)$ and $\varphi(x, \mu_w, \sigma_w^2) = w(x)$ by Monte Carlo simulation of the triple (X_T, M_T, C_T).

Thus, the approximation error depends in first line on the model parameters and in second line on the parameters μ_w and σ_w which in turn influence the values of the Fourier coefficients and the Hermite moments. Dependent on these values, we find in each case the truncation order N such that the approximation error $\epsilon^{(N)}$ does not exceed the right hand side of (4.15).

Note, that the time to maturity T does also influence the parameters of w especially if moment matching of w and g_T is done. If one is interested in keeping N as small as possible while retaining good approximation results, the parameters for the auxiliary density of the log-price density series representation need to be adapted for different maturities to enforce the fast decrease of the amplitudes of the Hermite moments.

Example 4.7 (ATM call option with one week maturity)**.** As model parameters we choose for the variance process $\kappa = 1$, $\theta = V_0 = 0.5$, $v_{\min} = 0$, $v_{\max} = 1$ and $\sigma = 0.25$. The correlation factor between variance and log-price process is given by $\rho = -0.5$ and as initial log-price we set $X_0 = 0$. The figures below show the impact of two different auxiliary densities for pricing a call option with maturity $T = \frac{1}{52}$ and log-strike $K = 0$. It demonstrates, that matching the mean and variance of the log-price (blue points) improves the convergence to the true option price and a low truncation order N suffices to obtain an accurate price. It is also notable, that if the truncation order is not adjusted to the parameters of the auxiliary density, then low expansion orders cause mispricings.



4.4 Path-dependent Options

In this section we extend the pricing technique to path-dependent options whose payoff functions depend on finitely many monitoring dates. We make use of the fact, that the

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log-returns conditioned on any path of the variance process V are multivariate normally distributed and that the log-prices can be written in terms of the log-returns.

Let $f(X_{t_1}, \dots, X_{t_d})$ denote the discounted payoff function of an exotic option with discrete observation dates $0 = t_0 < t_1 < t_2 \dots < t_d = T$. Further let $Y_{t_i} := X_{t_i} - X_{t_{i-1}}$ be the log-return from the time interval $[t_{i-1}, t_i]$ with length $\Delta t_i = t_i - t_{i-1}$ for all $1 \leq i \leq d$. Through the relation $X_{t_i} = X_0 + \sum_{j=1}^{i} Y_{t_j}$ we can write the option price π_f as

$$\pi_f = \mathbb{E}_{\mathbb{Q}}[f(X_{t_1}, \dots, X_{t_d})] = \int_{\mathbb{R}^d} f(X_0 + y_1, \dots, X_0 + \sum_{j=1}^{d} y_j) g_{t_1,\dots,t_d}(y_1, \dots, y_d) dy_d \dots dy_1,$$

where

$$g_{t_1,\dots,t_d}(y_1, \dots, y_d) = g_{t_1,\dots,t_d}(\mathbf{y}) = \mathbb{E}_{\mathbb{Q}}\left[\prod_{i=1}^{d} \frac{1}{\sqrt{2\pi \Delta C_{t_i}}} \exp\left(\frac{(y_i - \Delta M_{t_i})^2}{2\Delta C_{t_i}}\right)\right],$$

and

$$\Delta C_{t_i} := C_{t_i} - C_{t_{i-1}} = \int_{t_{i-1}}^{t_i} V_s - \rho^2 Q(V_s) ds,$$
$$\Delta M_{t_i} := M_{t_i} - M_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \left(r - \frac{V_s}{2}\right) ds + \frac{\rho}{\sigma}\left(V_{t_i} - V_{t_{i-1}} - \int_{t_{i-1}}^{t_i} \kappa(\theta - V_s) ds\right)$$

are the same as in Theorem 3.4. We introduce the discounted payoff function with respect to the log-returns

$$\tilde{f}(\mathbf{y}) := f(X_0 + y_1, \dots, X_0 + \sum_{j=1}^{d} y_j),$$

such that the option price simplifies to

$$\pi_f = \int_{\mathbb{R}^d} \tilde{f}(\mathbf{y}) g_{t_1,\dots,t_d}(\mathbf{y}) d\mathbf{y}.$$

Using the conditional independence and conditional normal distribution of the log-returns we can consider the pricing problem as multidimensional extension of the one-dimensional case. The density of the log-returns is approximated by the product

$$w(\mathbf{y}) = \prod_{i=1}^{d} w_i(y_i),$$

of marginal densities w_i of univariate normal distributions with mean μ_{w_i} and variance $\sigma^2_{w_i}$ for the time interval $[t_{i-1}, t_i]$ of two consecutive monitoring dates. The intuition behind is to split the time to maturity into the time intervals of observations for which we already know how to price them: For each w_i we use, that the generalized Hermite polynomials $\{H^{(i)}_{n_i}\}_{n_i \in \mathbb{N}}$

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form an orthonormal set of $L^2_{w_i}$ and that for the product space $L^2_w = L^2_{w_1} \times \ldots \times L^2_{w_d}$ the products

$$H_{n_1,\ldots,n_d}(y_1,\ldots,y_d) = \prod_{i=1}^{d} H^{(i)}_{n_i}(y_i),\ \ n_1,\ldots,n_d \in \mathbb{N}, \tag{4.16}$$

constitute an orthonormal basis for L^2_w [10, Theorem 3.2.18, p.69]. The series representation of the d-dimensional log-return density becomes

$$g_{t_1,\ldots t_d}(\mathbf{y}) = w(\mathbf{y}) \sum_{n_1,\ldots,n_d \geq 0} \ell_{n_1,\ldots,n_d} H_{n_1,\ldots,n_d}(\mathbf{y}) \tag{4.17}$$

with multidimensional Hermite moments

$$\ell_{n_1,\ldots,n_d} = \mathbb{E}_{\mathbb{Q}}[H^{(1)}_{n_1}(Y_{t_1}) \cdots H^{(d)}_{n_d}(Y_{t_d})]. \tag{4.18}$$

The option price for $\tilde{f} \in L^2_w$ writes as

$$\pi_f = \int_{\mathbb{R}^d} \tilde{f}(\mathbf{y}) g_{t_1,\ldots t_d}(\mathbf{y}) d\mathbf{y} = \sum_{n_1,\ldots,n_d \geq 0} \tilde{f}_{n_1,\ldots,n_d} \ell_{n_1,\ldots,n_d}, \tag{4.19}$$

where $\tilde{f}_{n_1,\ldots,n_d}$ are the Fourier coefficients of the representation

$$\tilde{f}(y_1,\ldots,y_d) = \sum_{n_1+\ldots+n_d \geq 0}^{\infty} \tilde{f}_{n_1,\ldots,n_d} H_{n_1,\ldots,n_d}(y_1,\ldots,y_d).$$

The price approximation is given in analogy to the univariate case by

$$\pi^{(N)}_f = \sum_{n_1+\ldots+n_d=0}^{N} \tilde{f}_{n_1,\ldots,n_d} \ell_{n_1,\ldots,n_d}, \tag{4.20}$$

and it converges as well in weighted mean square to the exact price as $N \to \infty$.

If $\tilde{f}$ is only integrable or if there is no closed-form of the Fourier coefficients available with respect to the product density w, then (4.19) is computed by standard multivariate Gauss-Hermite quadrature (compare [20]), i.e. by a change of variable for $(Y_{t_1},\ldots,Y_{t_d}) \sim \mathcal{N}(\mu,\Sigma)$ with $\mu = (\mu_{w_1},\ldots,\mu_{w_d})^\top$ and $\Sigma = \mathrm{diag}(\sigma_{w_1},\ldots,\sigma_{w_d})$ we approximate

$$\begin{aligned}
\pi_f &\overset{(4.17)}{=} \int_{\mathbb{R}^d} \tilde{f}(\mathbf{y}) \left(\sum_{n_1,\ldots,n_d \geq 0} \ell_{n_1,\ldots,n_d} H_{n_1,\ldots,n_d}(\mathbf{y}) \right) w(\mathbf{y}) d\mathbf{y} \\
&\underset{(1.18)}{\overset{(4.16)}{=}} \int_{\mathbb{R}^d} \tilde{f}(\mu + \Sigma \mathbf{z}) \left(\sum_{n_1,\ldots,n_d \geq 0} \ell_{n_1,\ldots,n_d} \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} \mathrm{He}_{n_i}(z_i) \right) \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|\mathbf{z}\|^2}{2}} d\mathbf{z}
\end{aligned}$$

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by

$$\pi_f^{(N)} = \int_{\mathbb{R}^d} \tilde{f}(\mu + \Sigma\mathbf{z}) \left(\sum_{n_1+\ldots+n_d=0}^{N} \ell_{n_1,\ldots,n_d} \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} \mathrm{He}_{n_i}(z_i) \right) \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|\mathbf{z}\|^2}{2}} d\mathbf{z}$$
$$\approx \sum_{m=1}^{M} w_m \tilde{f}(\mu + \Sigma\mathbf{z}_m) \left(\sum_{n_1+\ldots+n_d=0}^{N} \ell_{n_1,\ldots,n_d} \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} \mathrm{He}_{n_i}(z_{m,i}) \right),$$

with M points $\mathbf{z}_m = (z_{m,1}, \ldots, z_{m,d}) \in \mathbb{R}^d$ and corresponding weights $w_m \in (0,1)$.

We proceed analogously as in the case of pricing simple options. First we show the existence of a convergent series representation of the density of the log-returns (4.17), then we concentrate on the computation of the Hermite moments (4.18) and afterwards examples of exotic options are given.

The next theorem ensures, that an auxiliary product density w with parameters μ_{w_i} and σ_{w_i} for $1 \leq i \leq d$ exists such that the likelihood ratio $g_{t_1,\ldots t_d}/w$ lies in L^2_w. The formulation and its proof are an extension to the multidimensional case of Theorem 3.5.

Theorem 4.8. *For any $\epsilon_i < \frac{1}{2v_{\max}\Delta t_i}$, $i = 1, \ldots, d$ we have that*

$$\int_{\mathbb{R}^d} e^{\sum_{i=1}^d \epsilon_i y_i^2} g_{t_1,\ldots,t_d}(y_1, \ldots, y_d) dy_1 \ldots dy_d < \infty$$

and that $g_{t_1,\ldots,t_d}(\boldsymbol{y})$ as well as $e^{\sum_{i=1}^d \epsilon_i y_i^2} g_{t_1,\ldots,t_d}(\boldsymbol{y})$ are uniformly bounded on $\mathbb{R}^d$.

Proof. For the conditional density of the log-returns

$$G_{t_1,\ldots,t_d}(\mathbf{y}) = \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi\Delta C_{t_i}}} \exp\left(\frac{(y_i - \Delta M_{t_i})^2}{2\Delta C_{t_i}} \right)$$

we find the uniform bound $K \geq |\Delta M_{t_i}|\ \forall i$ and for each i we see that $0 < \Delta C_{t_i} \leq v_{\max}\Delta t_i$ holds. Further,

$$g_{t_1,\ldots,t_d}(\mathbf{y}) = \mathbb{E}_{\mathbb{Q}}[G_{t_1,\ldots,t_d}(\mathbf{y})] \leq \mathbb{E}_{\mathbb{Q}}[\prod_{i=1}^{d} (\Delta C_{t_i})^{-1}] < \infty,$$

since each ΔC_{t_i} is uniformly bounded with lower bound strictly greater than zero.

Again we consider the exponent of $e^{\sum_{i=1}^d \epsilon_i y_i^2} G_{t_1,\ldots,t_d}(\mathbf{y})$, which becomes by completing the square

$$\sum_{i=1}^{d} \epsilon_i y_i^2 - \frac{(y_i - \Delta M_{t_i})^2}{2\Delta C_{t_i}} = \sum_{i=1}^{d} \left(\epsilon_i - \frac{1}{2\Delta C_{t_i}} \right) y_i^2 + \frac{\Delta M_{t_i}}{\Delta C_{t_i}} y_i - \frac{(\Delta M_{t_i})^2}{2\Delta C_{t_i}}.$$

Setting $\delta_i := \frac{1}{2\Delta C_{t_i}} - \epsilon_i > 0$ we obtain for the exponent

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$$-\sum_{i=1}^{d}\delta_i\left(y_i-\frac{\Delta M_{t_i}}{2\delta_i\Delta C_{t_i}}\right)^2+\frac{(\Delta M_{t_i})^2}{2\Delta C_{t_i}}\left(\frac{1}{2\delta_i\Delta C_{t_i}}-1\right)=-\sum_{i=1}^{d}\delta_i\left(y_i-\frac{\Delta M_{t_i}}{2\delta_i\Delta C_{t_i}}\right)^2+\frac{(\Delta M_{t_i})^2\epsilon_i}{2\delta_i\Delta C_{t_i}}.$$

Hence,

$$e^{\sum_{i=1}^{d}\epsilon_i y_i^2}G_{t_1,\dots,t_d}(\mathbf{y})=\prod_{i=1}^{d}\frac{1}{\sqrt{2\pi\frac{1}{2\delta_i}}}\exp\left(-\frac{1}{2}\frac{\left(y_i-\frac{\Delta M_{t_i}}{2\delta_i\Delta C_{t_i}}\right)^2}{\frac{1}{2\delta_i}}\right)\sqrt{\frac{\frac{1}{2\delta_i}}{\Delta C_{t_i}}}\exp\left(\frac{(\Delta M_{t_i})^2\epsilon_i}{2\delta_i\Delta C_{t_i}}\right) \tag{4.21}$$

and using that $K\geq|\Delta M_{t_i}|$, for all i and $|\Delta C_{t_i}|\leq v_{\max}\Delta t_i$ gives

$$\begin{aligned}\int_{\mathbb{R}^d}e^{\sum_{i=1}^{d}\epsilon_i y_i^2}G_{t_1,\dots,t_d}(\mathbf{y})d\mathbf{y}&=\prod_{i=1}^{d}\frac{1}{\sqrt{2\delta_i\Delta C_{t_i}}}\exp\left(\frac{(\Delta M_{t_i})^2\epsilon_i}{2\delta_i\Delta C_{t_i}}\right)\\&\leq\prod_{i=1}^{d}\frac{1}{\sqrt{1-2v_{\max}\Delta t_i\epsilon_i}}\exp\left(\frac{K^2\epsilon_i}{1-2v_{\max}\Delta t_i\epsilon_i}\right),\end{aligned}$$

which is finite, if $\frac{1}{2v_{\max}\Delta t_i}>\epsilon_i$ holds for all $1\leq i\leq d$. The last inequality results from

$$2\delta_i\Delta C_{t_i}=2\Delta C_{t_i}\left(\frac{1}{2\Delta C_{t_i}}-\epsilon_i\right)=1-2\Delta C_{t_i}\epsilon_i\geq 1-2v_{\max}\Delta t_i\epsilon_i.$$

Taking expectation and afterwards Fubini's theorem shows

$$\mathbb{E}_{\mathbb{Q}}[\int_{\mathbb{R}^d}e^{\sum_{i=1}^{d}\epsilon_i y_i^2}G_{t_1,\dots,t_d}(\mathbf{y})d\mathbf{y}]=\int_{\mathbb{R}^d}e^{\sum_{i=1}^{d}\epsilon_i y_i^2}g_{t_1,\dots,t_d}(\mathbf{y})d\mathbf{y}<\infty.$$

From (4.21) we also get for $\frac{1}{2v_{\max}\Delta t_i}>\epsilon_i$ the uniform boundedness since

$$\begin{aligned}e^{\sum_{i=1}^{d}\epsilon_i y_i^2}g_{t_1,\dots,t_d}(\mathbf{y})&=e^{\sum_{i=1}^{d}\epsilon_i y_i^2}\mathbb{E}_{\mathbb{Q}}[G_{t_1,\dots,t_d}(\mathbf{y})]\\&\leq(2\pi)^{-\frac{d}{2}}\mathbb{E}_{\mathbb{Q}}[\prod_{i=1}^{d}(\Delta C_{t_i})^{-1}]\prod_{i=1}^{d}\exp\left(\frac{K^2\epsilon_i}{1-2v_{\max}\Delta t_i\epsilon_i}\right)<\infty.\end{aligned}$$

□

On the same line as in the one-dimensional case, namely that every bounded integrable function on $\mathbb{R}^d$ is square-integrable on $\mathbb{R}^d$, we state as immediate consequence:

Corollary 4.9. *For $\sigma_{w_i}^2>\frac{v_{\max}\Delta t_i}{2}$ and arbitrarily chosen $\mu_{w_i}\in\mathbb{R}$, $1\leq i\leq d$, we have*

$$\int_{\mathbb{R}^d}\frac{g_{t_1,\dots,t_d}(\boldsymbol{y})}{w(\boldsymbol{y})}d\boldsymbol{y}<\infty.$$

Thus, for each interval of the monitoring dates $[t_{i-1},t_i]$ we obtain the same condition (3.20) as in the one-dimensional case.

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4.4.1 Multidimensional Hermite moments

For each time interval $[t_{i-1}, t_i]$ and auxiliary density w_i with parameters μ_{w_i} and σ_{w_i} let

$$\{h^{(i)}_{\epsilon(m,n)}(v,x) = v^m H^{(i)}_n(x),\ 0 \le m+n \le N\}$$

denote the set of basis vectors of $\mathrm{Pol}_N(\mathbb{R}^2)$ with $\{H^{(i)}_n\}_{0\le n\le N}$ being the sequence of generalized Hermite polynomials with respect to μ_{w_i} and σ_{w_i}. Its ordered basis vector is given by

$$B^{(i)}(v,x)^\top = (h^{(i)}_1(v,x), \dots, h^{(i)}_M(v,x))^\top$$

with $M = \dim Pol_N(\mathbb{R}^2)$ under the enumeration ϵ. The $M \times M$ representation matrix of $\mathscr{G}$ for $[t_{i-1}, t_i]$ with respect to $B^{(i)}$ is denoted by $G^{(i)}$.

We define the $M \times M$ matrix $A^{(n_i,i+1)}$ as the transformation matrix for the basis change from the basis $B^{(i+1)}$ for the time interval $[t_i, t_{i+1}]$ to the previous one $B^{(i)}$ for $[t_{i-1}, t_i]$. The index n_i denotes the degree of the generalized Hermite polynomial at time step i used in the series representation of (4.17) up to order $N = \sum_{i=1}^d n_i$.

The next theorem shows, how the Hermite moments up to order $N = \sum_{i=1}^d n_i$ can be efficiently computed. The proof follows the same idea as given in [1, Theorem A.3, p.689]. Our version provides an explicit description of the transformation matrix for the enumeration (3.21) introduced in Section 3.2.2.

Theorem 4.10. *For any $n_1, \dots, n_d \in \mathbb{N}$ and $N \ge \sum_{i=1}^d n_i$ the multidimensional Hermite moments $\mathbb{E}_\mathbb{Q}[H^{(1)}_{n_1}(Y_{t_1}) \cdots H^{(d)}_{n_d}(Y_{t_d})] = \ell_{n_1,\dots,n_d}$ can be computed by*

$$\ell_{n_1,\dots,n_d} = B^{(1)}(V_0, 0)^\top \left(\prod_{i=1}^{d-1} e^{\Delta t_i G^{(i)}} A^{(n_i,i+1)} \right) e^{\Delta t_d G^{(d)}} \vec{e}_{\epsilon(0,n_d)}, \tag{4.22}$$

where the non-zero entries of the transformation matrix for $0 \le k \le n_{i+1}$ are given by

$$A^{(n_i,i+1)}_{\epsilon(0,k),\epsilon(k-1,N-k+1)+1+j} = H^{(i+1)}_j(0), \quad 0 \le j \le n_{i+1} - k,$$
$$\epsilon(-1, N+1) := 0.$$

Proof. We show that the formula holds for $d = 2$, the rest follows by an inductive argument. For $0 = t_0 < t_1 < t_2$ the Hermite moments become, using the law of iterated expectation

$$\begin{aligned}\ell_{n_1,n_2} &= \mathbb{E}_\mathbb{Q}[H^{(1)}_{n_1}(Y_{t_1}) H^{(2)}_{n_2}(Y_{t_2})] = \mathbb{E}_\mathbb{Q}[H^{(1)}_{n_1}(X_{t_1} - X_0) H^{(2)}_{n_2}(X_{t_2} - X_{t_1})] \\ &= \mathbb{E}_\mathbb{Q}[H^{(1)}_{n_1}(X_{t_1} - X_0) \mathbb{E}_\mathbb{Q}[H^{(2)}_{n_2}(X_{t_2} - X_{t_1}) | \mathscr{F}_{t_1}]].\end{aligned}$$

Since the conditional expectation only depends on the last known state and not on the whole path, and by the independence of $X_{t_2} - X_{t_1}$ from X_{t_1} which follows from (3.6) we obtain for the conditional expectation

$$\begin{aligned}\mathbb{E}_\mathbb{Q}[H^{(2)}_{n_2}(X_{t_2} - X_{t_1}) | \mathscr{F}_{t_1}] &= \mathbb{E}_\mathbb{Q}[H^{(2)}_{n_2}(X_{t_2} - X_{t_1}) | V_{t_1}, X_{t_1}] = \mathbb{E}_\mathbb{Q}[H^{(2)}_{n_2}(X_{t_2}) | V_{t_1}, X_{t_1} = 0] \\ &= \mathbb{E}_\mathbb{Q}[H^{(2)}_{n_2}(X_{\Delta t_2}) | V_0 = V_{t_1}, X_0 = 0],\end{aligned}$$

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where we used as well the time-homogeneity in the last step. By the moment formula (Theorem 3.8) we get

$$\mathbb{E}_{\mathbb{Q}}[H_{n_2}^{(2)}(X_{\Delta t_2})|V_0 = V_{t_1}, X_0 = 0] = B^{(2)}(V_{t_1}, 0)^\top e^{\Delta t_2 G^{(2)}} \vec{e}_{\epsilon(0,n_2)} \tag{4.23}$$

with $B^{(2)}(v,0)^\top = (1, H_1^{(2)}(0), H_2^{(2)}(0), \ldots, H_N^{(2)}(0), v, vH_1^{(2)}(0), \ldots, vH_{N-1}^{(2)}(0), \ldots, v^N)^\top$ and

$$\vec{u}^{(n_2,2)} := e^{\Delta t_2 G^{(2)}} \vec{e}_{\epsilon(0,n_2)} \tag{4.24}$$

represents the $\epsilon(0,n_2)$-th column of $e^{\Delta t_2 G^{(2)}}$. Thus,

$$B^{(2)}(v,0)^\top \vec{u}^{(n_2,2)} = \sum_{m=0}^{n_2} a_m v^m \tag{4.25}$$

is a polynomial solely in v where $a_m = \sum_{j=0}^{n_2-m} H_j^{(2)}(0) u_{\epsilon(m,j)}^{(n_2,2)}$. Writing out the coefficients

$$\begin{aligned}
a_0 &= H_0^{(2)}(0) u_{\epsilon(0,0)}^{(n_2,2)} + H_1^{(2)}(0) u_{\epsilon(0,1)}^{(n_2,2)} + \ldots + H_{n_2-1}^{(2)}(0) u_{\epsilon(0,n_2-1)}^{(n_2,2)} + H_{n_2}^{(2)}(0) u_{\epsilon(0,n_2)}^{(n_2,2)} \\
a_1 &= H_0^{(2)}(0) u_{\epsilon(1,0)}^{(n_2,2)} + H_1^{(2)}(0) u_{\epsilon(1,1)}^{(n_2,2)} + \ldots + H_{n_2-1}^{(2)}(0) u_{\epsilon(1,n_2-1)}^{(n_2,2)} \\
&\vdots \\
a_{n_2} &= H_0^{(2)}(0) u_{\epsilon(n_2,0)}^{(n_2,2)} \\
a_m &= 0, \quad \forall m: \ n_2 < m \leq M
\end{aligned}$$

and arranging as column vector $(a_0, a_1, \ldots, a_{n_2}, a_{n_2+1}, \ldots, a_M)^\top = A^{(n_1,2)} \vec{u}^{(n_2,2)}$ the transformation matrix $A^{(n_1,2)}$ has the form

$$\left(\begin{array}{cccc|ccc|c|cc|c|ccc}
H_0^{(2)}(0) & H_1^{(2)}(0) & \ldots & H_{n_2}^{(2)}(0) & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & H_0^{(2)}(0) & \ldots & H_{n_2-1}^{(2)}(0) & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & H_0^{(2)}(0) & H_1^{(2)}(0) & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & H_0^{(2)}(0) & 0 & \ldots & 0 \\
\hline
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0
\end{array}\right).$$

We obtain for the Hermite moment

$$\begin{aligned}
\ell_{n_1,n_2} &= \mathbb{E}_{\mathbb{Q}}[H_{n_1}^{(1)}(X_{t_1}) B^{(2)}(V_{t_1},0)^\top e^{\Delta t_2 G^{(2)}} \vec{e}_{\epsilon(0,n_2)}] = \mathbb{E}_{\mathbb{Q}}[H_{n_1}^{(1)}(X_{t_1}) B^{(2)}(V_{t_1},0)^\top \vec{u}^{(n_2,2)}] \\
&\overset{(4.25)}{=} \mathbb{E}_{\mathbb{Q}}[\sum_{m=0}^{n_2} a_m \underbrace{V_{t_1}^m H_{n_1}^{(1)}(X_{t_1})}_{=h_{\epsilon(m,n_1)}^{(1)}(V_{t_1},X_{t_1})}] = \mathbb{E}_{\mathbb{Q}}[B^{(1)}(V_{t_1},X_{t_1})^\top \vec{p}],
\end{aligned}$$

where $\vec{p} = A^{(n_1,2)} \vec{u}^{(n_2,2)}$, and again by the moment formula

$$\ell_{n_1,n_2} = B^{(1)}(V_0,0)^\top e^{\Delta t_1 G^{(1)}} \vec{p} \overset{(4.24)}{=} B^{(1)}(V_0,0)^\top \left(e^{\Delta t_1 G^{(1)}} A^{(n_1,2)}\right) e^{\Delta t_2 G^{(2)}} \vec{e}_{\epsilon(0,n_2)}.$$

□

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Remark 4.11. For fixed truncation order N the number of d-dimensional Hermite moments up to order N, i.e. the cardinality of the set

$$|\{(n_1,\dots,n_d):\ 0\le\sum_{i=1}^{d}n_i\le N,\ n_i\in\{0,1,\dots,N\},\ 1\le i\le d\}|$$

corresponds to the total number of possibilities to partition all sums $n=\sum_{i=1}^{d}n_i$ where $n\in\{0,1,\dots,N\}$ in d blocks allowing zeros where order matters. The number of all distinct d-dimensional Hermite moments up to order N also equals the number of possibilities to choose d elements of the set of all possible sums $n\in\{0,1,\dots,N\}$ where repetitions are allowed and order doesn't matter. Thus, it is given by $\binom{N+d}{N}$ (compare [1, p.686]).

The additional effort for the calculation of any multidimensional Hermite moment compared to the one-dimensional case is given by the products of the matrix exponentials and the sparse transformation matrices $A^{(n_i,i+1)}$ for $1\le i\le d-1$ and $0\le n_i\le N$ in the brackets of (4.22).

Example 4.12 (Bivariate Hermite moments)**.** We demonstrate the computation of the Hermite moments up to order $N=2$ for the monitoring dates $t_1=1/2$ and $t_2=1$ and the model parameters $\kappa=1$, $\theta=V_0=1/2$, $v_{\min}=0$, $v_{\max}=1$ and $\sigma=1/4$ for the variance process and $r=0$, and $X_0=0$ for the log-price process with correlation factor $\rho=-1/2$. First we compute the parameters for the product density as

$$\mu=(\mu_1,\mu_2)^\top=(-0.125,-0.25)^\top\quad\text{and}\quad\Sigma=\operatorname{diag}(\sigma_1,\sigma_2)=\operatorname{diag}(0.5034,0.7155)$$

by matching the means and variances of the log-prices X_{t_1} and X_{t_2}. For the Hermite moments

$$\begin{aligned}\ell_{n_1,n_2}&=\mathbb{E}_{\mathbb{Q}}[H^{(1)}_{n_1}(Y_{t_1})H^{(2)}_{n_2}(Y_{t_2})]\\&=(1,0,0,\frac{1}{2},0,\frac{1}{4})\left(e^{\Delta t_1G^{(1)}}A^{(n_1,2)}\right)e^{\Delta t_2G^{(2)}}\vec{e}_{\epsilon(0,n_2)},\end{aligned}$$

we have the matrix exponentials and transformation matrices

$$e^{0.5G^{(1)}}=\begin{pmatrix}1&-0.05&0.15&0.20&-0.02&0.04\\0&1&-0.07&0&0.20&0\\0&0&1&0&0&0\\0&-0.39&1.15&0.61&-0.17&0.25\\0&0&-0.55&0&0.61&0\\0&0&0.08&0&-0.18&0.36\end{pmatrix},$$

$$e^{0.5G^{(2)}}=\begin{pmatrix}1&-0.04&0.08&0.20&-0.01&0.04\\0&1&-0.05&0&0.20&0\\0&0&1&0&0&0\\0&-0.27&0.57&0.61&-0.12&0.25\\0&0&-0.39&0&0.61&0\\0&0&0.04&0&-0.12&0.36\end{pmatrix},$$

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$$A^{(n_1,2)} = \begin{pmatrix} 1 & a_1 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \left\{ \begin{array}{ll} a_1 = a_2 = b = c_1 = c_2 = 0, & n_1 = 0 \\ a_1 = 0.2, c_1 = 1, b = a_2 = c_2 = 0, & n_1 = 1 \\ a_1 = a_2 = 0.2, b = -0.7, c_1 = c_2 = 1, & n_2 = 2 \end{array} \right\}.$$

Under the enumeration ϵ the coordinate vector $\vec{e}_{\epsilon(0,n_2)}$ corresponds to the first three standard basis vectors $\vec{e}_i$, $1 \leq i \leq 3$ and the Hermite moments are computed as

$$\ell_{0,0} = 1,\ \ell_{1,0} = 1,\ \ell_{0,1} = -0.0372,\ \ell_{2,0} = 1,\ \ell_{1,1} = 0.2111,\ \ell_{0,2} = 0.0758.$$

4.4.2 Examples

In contrast to simple European options there might be no closed-form available for the computation of the Fourier coefficients. We give here two examples covering both cases: Forward call options for which no numerical integration is needed and Asian call options which are computed using standard multivariate Gauss-Hermite quadrature.

Forward start call options

We consider two sorts of forward start call options: one with fixed strike and one with proportional strike to a future underlying price. In the first case the discounted payoff-function is given by

$$f(x_1, x_2) = e^{-rT}(e^{x_2 - x_1} - K)^+,$$

where x_1 resp. x_2 denotes the future log-price at time t resp. T. With respect to the log-returns $y_i = x_i - x_{i-1}$ the discounted payoff function becomes

$$\tilde{f}(y_1, y_2) = e^{-rT}(e^{y_2} - K)^+,$$

which is independent from the log-returns at time t and its price coincides with those of a call option with strike K.

For proportional strike we have the discounted payoff-function

$$f(x_1, x_2) = e^{-rT}(e^{x_2} - Ke^{x_1})^+,$$

and the corresponding function expressed by the log-returns writes as

$$\tilde{f}(y_1, y_2) = e^{-rT}(e^{X_0 + y_1 + y_2} - Ke^{X_0 + y_1})^+ = e^{X_0 - rT}(e^{y_1 + y_2} - Ke^{y_1})^+.$$

The Fourier coefficients $\tilde{f}_{n_1,n_2}$ are computed using

$$\begin{aligned} \tilde{f}_{n_1,n_2} &= \int_{\mathbb{R}^2} \tilde{f}(y_1, y_2) H_{n_1}^{(1)}(y_1) H_{n_2}^{(2)}(y_2) w_1(y_1) w_2(y_2) dy_1 dy_2 \\ &= e^{X_0 - rT} \underbrace{\int_{\mathbb{R}} e^{y_1} H_{n_1}^{(1)}(y_1) w_1(y_1) dy_1}_{= f_{n_1}^{(0,-\infty)}} \underbrace{\int_K^{\infty} (e^{y_2} - K) H_{n_2}^{(2)}(y_2) dy_2}_{= f_{n_2}^{(0,\log(K))}}, \end{aligned}$$

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where $f_n^{(r,K)}$ denotes the n-th Fourier coefficient of a call option (Proposition 4.2) with interest rate r and log-strike K. Thus, the option price expanded up to order N is given by

$$\pi_f^{(N)} = \sum_{n=0}^{N} \sum_{n_1+n_2=0}^{n} \tilde{f}_{n_1,n_2} \ell_{n_1,n_2}.$$

Asian call options

The discounted payoff-function of a discrete arithmetic Asian call option with fixed strike K is given by

$$\begin{aligned} f(x_1,\ldots,x_d) &= e^{-rT}\left(\frac{1}{d}\sum_{i=1}^{d} e^{x_i} - K\right)^+ \\ &= e^{-rT}\left(\frac{1}{d}\sum_{i=1}^{d} e^{X_0+\sum_{j=1}^{i} y_j} - K\right)^+ = \tilde{f}(y_1,\ldots,y_d), \end{aligned}$$

with log-returns $y_k = x_k - x_{k-1}$, $1 \le k \le d$, where x_k is the log-price at time t_k and $0 = t_0 < t_1 < \ldots < t_d = T$.

The discounted payoff function of a discrete arithmetic Asian call option with floating strike with respect to the log-returns becomes

$$\tilde{f}(y_1,\ldots,y_d) = e^{-rT}\left(e^{X_0+\sum_{j=1}^{d} y_j} - \frac{K}{d}\sum_{i=1}^{d} e^{X_0+\sum_{j=1}^{i} y_j}\right)^+.$$

In both cases we use for the calculation of the approximated option prices

$$\begin{aligned} \pi_f^{(N)} &= \int_{\mathbb{R}^d} \tilde{f}(\mathbf{y}) g_{t_1,\ldots,t_d}^{(N)}(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^d} \tilde{f}(\mathbf{y}) \left(\sum_{n_1+\ldots+n_d=0}^{N} \ell_{n_1,\ldots,n_d} H_{n_1,\ldots,n_d}(\mathbf{y})\right) \frac{1}{(2\pi\det(\Sigma))^{d/2}} e^{-\frac{1}{2}(\mathbf{y}-\mu)^\top \Sigma^{-1}(\mathbf{y}-\mu)} d\mathbf{y} \end{aligned}$$

the known finite series expansion of the density of the log-price returns $g_{t_1,\ldots,t_d}^{(N)}$ in terms of multidimensional Hermite polynomials $H_{n_1,\ldots,n_d}$ (4.16) and Hermite moments $\ell_{n_1,\ldots,n_d}$ for the auxiliary density of the normal distribution $\mathcal{N}(\mu,\Sigma)$ with $\mu = (\mu_{w_1},\ldots,\mu_{w_d})^\top$ and $\Sigma = \mathrm{diag}(\sigma_{w_1},\ldots,\sigma_{w_d})$. By a change of variable for each coordinate $y_k = \mu_{w_k} + \sigma_{w_k} z_k$, $1 \le k \le d$ the above becomes

$$\pi_f^{(N)} = \int_{\mathbb{R}^d} \tilde{f}(\mu + \Sigma\mathbf{z}) \left(\sum_{n_1+\ldots+n_d=0}^{N} \ell_{n_1,\ldots,n_d} \prod_{k=1}^{d} \frac{1}{\sqrt{n_k!}} \mathrm{He}_{n_k}(z_k)\right) \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}\|\mathbf{z}\|^2} d\mathbf{z},$$

and can be computed via d-dimensional Gauss-Hermite quadrature.

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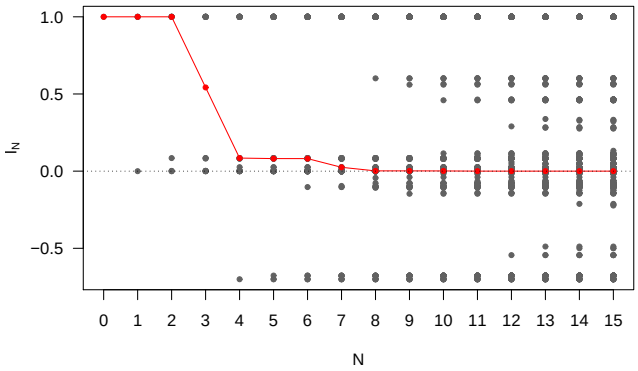
Example 4.13 (ATM arithmetic Asian call option with fixed strike and maturity in four weeks and weekly average of the underlying)**.** The parameters for the payoff function are given by $K = 1$, $t_i = i/52$, $i = 1, 2, 3, 4$ and for the model parameters we choose $\kappa = 1$, $\theta = V_0 = 1/2$, $v_{\min} = 0$, $v_{\max} = 1$ and $\sigma = 1/4$ for the variance process, and $r = 0$ and $X_0 = 0$ for the log-price process with correlation factor $\rho = -1/2$.

The four dimensional Hermite moments are computed up to order $N = 15$ and Figure 4.1 displays on the left panel the computed values (gray points). The red points are the medians for each N and on the right in the table the number of compositions $c(N)$ for each N are given.



N	$c(N)$	N	$c(N)$	N	$c(N)$
1	4	6	84	11	364
2	10	7	120	12	455
3	20	8	165	13	560
4	35	9	220	14	680
5	56	10	286	15	816

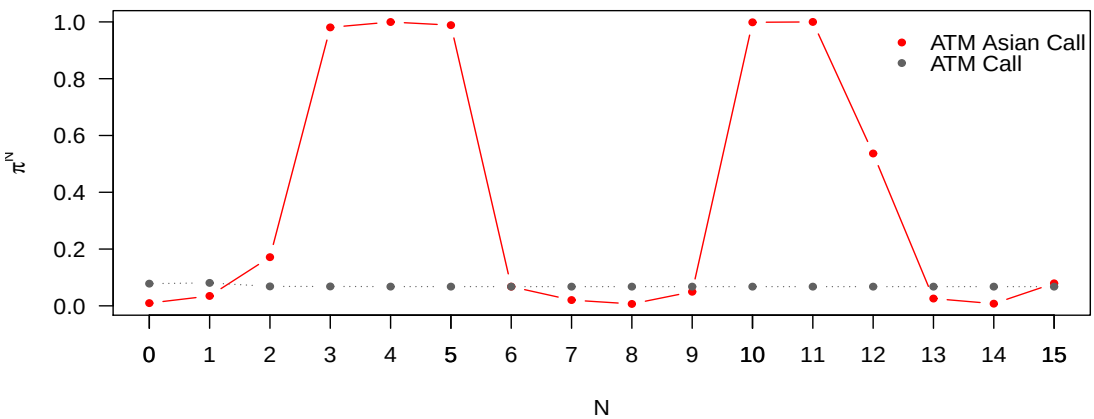
$$\sum_{N=0}^{15} c(N) = 3876 = \binom{19}{15}.$$

Figure 4.1: Four dimensional Hermite moments and the number of their compositions

It is seen, that the large part of the Hermite moments are of small order of magnitude and therefore their contributions to the option price is negligible.

The computational result of the Asian call option price for different truncation orders is displayed in Figure 4.2 below in comparison to the call price with same strike and maturity $T = t_4$ (gray points). The figure points out, that a truncation order of $N = 15$ is far from being sufficient to get an accurate price of an Asian call option.

Figure 4.2: Comparison of ATM Asian call option price and ATM Call option price with same maturity



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4.5 Conclusio

Due to the compact state interval of the variance process the Jacobi stochastic volatility model allows a convergent series representation of the log-prices and log-returns with respect to an auxiliary density function, which is chosen as the density of a normal distribution. The expansion is done with respect to generalized Hermite polynomials and the coefficients in the series expansion are given as the expectations of generalized Hermite polynomials under the chosen risk-neutral measure, reasoning the name Hermite moments. The Hermite moments are in closed-form, since the model is described by a two dimensional polynomial diffusion, and measure the deviation of the log-price resp. log-returns distribution from a normal distribution. It turns out that matching mean and variance of the log-price with the parameters of the auxiliary density leads to good approximation results of the series expansion already for small expansion orders.

For simple European claims we found explicit series representation for the option prices and we saw that the truncation of the expansions at some finite order gives an efficient price approximation; especially if moment matching of the log-price distribution and the auxiliary normal distribution is feasible, the first term gives the Black-Scholes price of the claim and the remaining terms serve as price correction due to stochastic volatility. Not only the Black-Scholes model can be considered as special case of the Jacobi stochastic volatility model, but also the famous Heston model appears as its limiting case. Our pricing technique is not applicable in the limiting case, since the Heston model does not allow a convergent series expansion of the log-price density.

The extension to path-dependent options in this model follows the same idea as for simple options, if we consider the log-returns instead of the log-prices and split the time interval up to maturity into subintervals, bounded by the monitoring dates. Although the underlying idea is simple, the implementation of computational efficient price approximations can be cumbersome, especially if numerical integration is needed. Accurate price approximations and low computational times do not come together without any further improvement of implementation strategies.

The seven model parameters κ, θ, σ, $v_{\min}$, $v_{\max}$, V_0 and ρ and those needed for the auxiliary distributions make the model versatile at the cost of possible mispricings if the truncation order N is not adjusted to the used parameters. The calibration of the Jacobi stochastic volatility model turns out to be a challenging task. Not only the amount of parameters, which needs to be fitted, requires a huge number of computations such that the implied volatility surface is matched as best as possible, but also the truncation orders N have a great impact on the computational time of the calibration procedure. Since the variance of the auxiliary distribution cannot always be matched, a high value[9] for N is necessary to avoid abnormal specifications. Hence for the calibration of the model to given market data in a reasonable time processors with high CPU frequencies are required[10].

[9] It is experienced, that the order does not need to exceed $N = 100$ ([1, Version 1 (2016), p.16]).
[10] For some selected reference values compare [1, Version 1 (2016), p.19].

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Appendix

```
##########################################################################
###       C A L C U L A T I O N   O F   T H E   M O M E N T S          ###
##########################################################################

# dependency on other packages:
require(expm)
require(partitions)
require(Deducer)

#############################
# 1. Enumeration:           #
#############################

# Function index
# @ description:  returns the corresponding number of a basis element
# @ return:       numeric
# @ parameters:   m: power of v
#                 n: power of x
#                 N: highest degree of a basis element

index<-function(m,n,N){
  if(m==0){return(n+1);}
  else{
    NumPrevious<-(N+1)*m-sum(seq(0,m-1)); # counts previous elements up to
        the last row
    return(NumPrevious+(n+1));
  }
}

# Function inverseIndex
# @ description:  inverse function of index,
#                 for a given index of the basis vector the powers of
#                 v and m are returned
# @ return:       numeric (m,n)
# @ parameters:   k: index of basis vector
#                 N: highest degree of a basis element

inverseIndex<-function(k,N){
  m<-0
  while(m*(N+1)-m*(m-1)/2<k){
    m<-m+1;
  }
  if(m!=0){
    maxM<-m-1;
    m<-maxM; #corresponding row
  }
```

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```
  n<-k-1-m*(N+1)+m*(m-1)/2; #corresponding column
  return(c(m,n));
};

#############################
# 2. Basis:                 #
#############################

# Function StandardBasis
# @ description:  creates an ordered basis vector with elements of
#                 the form V^m*X^n
#                 at the initial values V0 and X0
# @ return:       numeric vector of length (N+1)(N+2)/2
# @ parameters:   V0: initial value of variance process
#                 X0: initial value of the log price process
#                 N: highest degree of a basis element

StandardBasis<-function(V0,X0,N){
  v<-numeric((N+1)*(N+2)/2);

  for (m in seq(0,N)){
    for(n in seq(0,N-m)){
      value<-V0^m*X0^n
      i<-index(m,n,N);
      v[i]<-value
    }
  }
  return(v);
}

# Function HermiteBasis
# @ description:  creates an ordered basis vector with elements of
#                 the form V^m*H_n(X)
#                 at the initial values V0 and X0
# @ return:       numeric vector of length (N+1)(N+2)/2
# @ parameters:   V0: initial value of variance process
#                 X0: initial value of the log price process
#                 N: highest degree of a basis element
#                 mu: expected value of the auxiliary density
#                 sigma: volatility of the auxiliary density

HermiteBasis<-function(V0,X0,N,mu,sigma){
  v<-numeric((N+1)*(N+2)/2);
  H<-GeneralizedHermite(N,X0,mu,sigma);
  for (m in seq(0,N)){
    for(n in seq(0,N-m)){
      value<-V0^m*H[n+1];
      i<-index(m,n,N);
      v[i]<-value
    }
  }
  return(v);
}

# Function Hermite
```


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```
# @ description:  gives all Hermite polynomials evaluated at x
#                 up to order n
# @ return:       numeric vector of length n+1
# @ parameters:   n: highest degree of the Hermite polynomial
#                 x: value, at which the polynomials are evaluated

Hermite<-function(n,x){
  He<-numeric(n+1);
  He[1]<-1; #H0
  if(n>0) He[2]<-x; #H1
  if(n>1){
    for(k in seq(2,n)){
      He[k+1]<-x*He[k]-(k-1)*He[k-1];
    }
  }
  return(He);
}

# Function GeneralizedHermite
# @ description:  gives all generalized Hermite polynomials evaluated at x
#                 up to order n
# @ return:       numeric vector of length n+1
# @ parameters:   n: highest degree of the Hermite polynomial
#                 x: value, at which the polynomials are evaluated
#                 mu: expected value of the auxiliary distribution
#                 sigma: volatility of the auxiliary distribution

GeneralizedHermite<-function(n,x,mu,sigma){
  H<-numeric(n+1);
  z<-(x-mu)/sigma;
  H<-Hermite(n,z);
  for(k in seq(1,length(H))){ #normalizing
    H[k]<-H[k]/sqrt(factorial(k-1));
  }
  return(H);
}

#############################
# 3. Representationmatrix:  #
#############################

# Function StandardRepMatrix
# @ description:  computes the representation matrix of the action
#                 of G on the standard basis
# @ return:       matrix of dimension (N+1)(N+2)/2 times (N+1)(N+2)/2
# @ parameters:   kappa: mean-reverting speed of the variance process
#                 theta: long-term mean of the variance process
#                 vmin: lower bound of variance process
#                 vmax: upper bound of variance process
#                 sigma: volatility (factor) of variance process
#                 r: interest rate
#                 rho: correlation factor of variance and log price process
#                 N: highest degree of a basis element

StandardRepMatrix<-function(kappa,theta,vmin,vmax,sigma,r,rho,N){
```

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```
  G<-matrix(nrow=(N+1)*(N+2)/2, ncol=(N+1)*(N+2)/2);

  for (m in seq(0,N)){ # m exponent of v
    for(n in seq(0,N-m)){# n exponent of x
      v<-getCol(m,n,N,kappa,theta,vmin,vmax,sigma,r,rho) #image of basis
          vector as column
      i<-index(m,n,N);
      G[,i]<-v
    }
  }
  return(G);
};

# Function getCol
# @ description:  computes the linear combination of basis elements of
#                 the image of a basis vector (m,n);
#                 helping function for the function StandardRepMatrix
# @ return:       numeric vector of length (N+1)(N+2)/2
# @ parameters:   m: power of V
#                 n: power of X
#                 N: highest degree of a basis element
#                 kappa: mean-reverting speed of the variance process
#                 theta: long-term mean of the variance process
#                 vmin: lower bound of variance process
#                 vmax: upper bound of variance process
#                 sigma: volatility (factor) of variance process
#                 r: interest rate
#                 rho: correlation factor of variance and log price process

getCol<-function(m,n,N,kappa,theta,vmin,vmax,sigma,r,rho){
  a<-1/(sqrt(vmax)-sqrt(vmin))^2;
  v<-numeric((N+1)*(N+2)/2);
  #calculate the 7 entries:

  #(m-2,n):
  if(m>=2 && n>=0){
    i<-index(m-2,n,N);
    v[i]<- -sigma^2*vmin*vmax*m*(m-1)*a/2;
  }

  #(m-1,n-1):
  if(m>=1 && n>=1){
    i<-index(m-1,n-1,N);
    v[i]<- -rho*sigma*vmin*vmax*m*n*a;
  }

  #(m-1,n):
  if(m>=1 && n>=0){
    i<-index(m-1,n,N);
    v[i]<-kappa*theta*m+sigma^2*a*(vmin+vmax)*m*(m-1)/2;
  }

  #(m,n-1):
  if(m>=0 && n>=1){
    i<-index(m,n-1,N);
```


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```
    v[i]<-r*n+rho*sigma*(vmin+vmax)*m*n*a;
  }

  #(m,n):
  if(m>=0 && n>=0){
    i<-index(m,n,N);
    v[i]<- -(kappa*m+sigma^2*a*m*(m-1)/2);
  }

  #(m+1,n-2):
  if(m>=0 && n>=2){
    i<-index(m+1,n-2,N);
    v[i]<- n*(n-1)/2;
  }

  #(m+1,n-1):
  if(m>=0 && n>=1){
    i<-index(m+1,n-1,N);
    v[i]<- -(n/2+rho*sigma*a*m*n);
  }

  return(v);
}


# Function HermiteRepMatrix
# @ description:  computes the representation matrix of the action
#                 of G on the  Hermite basis
# @ return:       matrix of dimension (N+1)(N+2)/2 times (N+1)(N+2)/2
# @ parameters:   kappa: mean-reverting speed of the variance process
#                 theta: long-term mean of the variance process
#                 vmin: lower bound of variance process
#                 vmax: upper bound of variance process
#                 sigma: volatility (factor) of variance process
#                 r: interest rate
#                 rho: correlation factor of variance and log price process
#                 N: highest degree of a basis element
#                 sigma_w: volatility of the auxiliary density

HermiteRepMatrix<-function(kappa,theta,vmin,vmax,sigma,r,rho,N,sigma_w){
  G<-matrix(nrow=(N+1)*(N+2)/2, ncol=(N+1)*(N+2)/2);

  for (m in seq(0,N)){ # m exponent of v
    for(n in seq(0,N-m)){ # n degree of Hermite polynomial in x
      v<-getHermiteCol(m,n,N,kappa,theta,vmin,vmax,sigma,r,rho,sigma_w) # image of basis vector as column
      i<-index(m,n,N);
      G[,i]<-v
    }
  }
  return(G);
};

# Function getHermiteCol
# @ description:  computes the linear combination of basis elements of
```

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```
#                    the image of a Hermite basis vector (m,n);
#                    helping function for the function HermiteRepMatrix
# @ return:          numeric vector of length (N+1)(N+2)/2
# @ parameters:      m: power of V
#                    n: degree of generalized Hermite polynomial in X
#                    N: highest degree of a basis element
#                    kappa: mean-reverting speed of the variance process
#                    theta: long-term mean of the variance process
#                    vmin: lower bound of variance process
#                    vmax: upper bound of variance process
#                    sigma: volatility (factor) of variance process
#                    r: interest rate
#                    rho: correlation factor of variance and log price process
#                    sigma_w: volatility of the auxiliary density

getHermiteCol<-function(m,n,N,kappa,theta,vmin,vmax,sigma,r,rho,sigma_w){
  a<-1/(sqrt(vmax)-sqrt(vmin))^2;
  v<-numeric((N+1)*(N+2)/2);
  #calculate the 7 entries:

  #(m-2,n):
  if(m>=2 && n>=0){
    i<-index(m-2,n,N);
    v[i]<- -sigma^2*vmin*vmax*m*(m-1)*a/2; #no change
  }

  #(m-1,n-1):
  if(m>=1 && n>=1){
    i<-index(m-1,n-1,N);
    v[i]<- -rho*sigma*vmin*vmax*m*a*sqrt(n)/sigma_w;
  }

  #(m-1,n):
  if(m>=1 && n>=0){
    i<-index(m-1,n,N);
    v[i]<-kappa*theta*m+sigma^2*a*(vmin+vmax)*m*(m-1)/2; #no change
  }

  #(m,n-1):
  if(m>=0 && n>=1){
    i<-index(m,n-1,N);
    v[i]<-r*sqrt(n)/sigma_w+rho*sigma*(vmin+vmax)*m*sqrt(n)/sigma_w*a;
  }

  #(m,n):
  if(m>=0 && n>=0){
    i<-index(m,n,N);
    v[i]<- -(kappa*m+sigma^2*a*m*(m-1)/2); #no change
  }

  #(m+1,n-2):
  if(m>=0 && n>=2){
    i<-index(m+1,n-2,N);
    v[i]<- sqrt(n*(n-1))/2/sigma_w^2;
  }
```


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```
  #(m+1,n-1):
  if(m>=0 && n>=1){
    i<-index(m+1,n-1,N);
    v[i]<- -(sqrt(n)/2/sigma_w+rho*sigma*a*m*sqrt(n)/sigma_w);
  }

  return(v);
}

#############################
# 4. Moments                #
#############################

# Function Moments
# @ description:  gives all joint moments for given basis of order N and
#                 and representation matrix at time (difference) t
# @ return:       numeric vector of length (N+1)(N+2)/2
# @ parameters:   B: vector of ordered basis elements
#                 G: representation matrix
#                 t: time or time difference in years

Moments<-function(B,G,t){
  M<-nrow(G)
  MM<-numeric(M)
  #compute matrix exponential for each action
  V<-diag(M)
  for (i in seq(1,M)){
    MM[i]<-sum(B*expAtv(G, V[i,],t)$eAtv)
  }
  return(MM)
}

# Function Moment
# @ description:  gives the searched moment for given basis of order N and
#                 and representation matrix at time (difference) t
# @ return:       double
# @ parameters:   B: vector of ordered basis elements
#                 G: representation matrix
#                 t: time or time difference in years
#                 m: exponent of V
#                 n: exponent of X or H_n(X)
#                 N: highest degree of a basis element

Moment<-function(B,G,t,m,n,N){
  ind<-index(m,n,N)
  M<-nrow(G)
  v<-numeric(M)
  v[ind]<-1
 return(B%*%expAtv(G,v,t)$eAtv)
}

# Function HermiteMoments
# @ description:  gives all Hermite moments for given basis of order N and
#                 and representation matrix at time (difference) t
```

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```
# @ return:        vector of length N+1
# @ parameters:    H: vector of ordered Hermite basis elements
#                  G: representation matrix
#                  t: time or time difference in years
#                  N: highest degree of a basis element

HermiteMoments<-function(H,G,t,N){
  lN<-numeric(N+1)
  M<-nrow(G)
  for(n in seq(0,N)){
    v<-numeric(M)
    v[index(0,n,N)]<-1
    lN[n+1]<-H%*%expAtv(G,v,t)$eAtv
  }
  return(lN)
}

# Function params_w
# @ description:  gives the best parameters of the auxiliary density w
#                  by matching mean and variance
# @ return:        vector (mu_w,sigma_w)
# @ parameters:    B: vector of ordered basis elements
#                  G: representation matrix
#                  t: time or time difference in years
#                  N: highest degree of a basis element
#                  vmax: upper bound for state of variance process

params_w<-function(B,G,t,N,vmax){
  mu_w<-Moment(B,G,t,0,1,N)
  M2<- Moment(B,G,t,0,2,N)
  sigma_w<-sqrt(M2-mu_w^2)
  #check if allowed
  if(sigma_w^2<=vmax*t/2){
    sigma_w<-sqrt(vmax*t/2)+0.01
  }
  return(c(mu_w,sigma_w))
}

#######################################
# 5. Multidimensional Hermite moments #
#######################################

# Function ATransformMatrix
# @ description:  gives the basis transformation matrix from one time
#                  interval to the previous one for current Hermite Basis up
#                  to order N;
#                  used for the computation of the n-th Hermite moment
#                  of the previous time step
# @ return:        (N+1)(N+2)/2 x (N+1)(N+2)/2 matrix
# @ parameters:    n: degree of the previous Hermite moment
#                  mu_w: expectation of the auxiliary density for current
#                  time interval
#                  sigma_w: volatility of the auxiliary density for current
#                  time interval
#                  N: highest degree of a basis element
```


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```
ATransformMatrix<-function(n,mu_w,sigma_w,N){
  M<-(N+1)*(N+2)/2
  A<-matrix(0,M,M)
  H<-GeneralizedHermite(n,0,mu_w,sigma_w)
  #entries:
  for(k in seq(0,n)){              #row
    for(j in seq(0,n-k)){          #column
      A[k+1,index(k,j,N)]<-H[j+1]
    }
  }
  return(A)
}

# Function multiHermiteMoments
# @ description:  returns a matrix with all multidimensional Hermite moments
#                 up to total order N for d time intervals
# @ return:       matrix with col for n1,...,nd and last col l_n1,...,nd
# @ parameters:   H: vector of ordered Hermite basis elements of first time
#                 interval
#                 N: total order of Hermite moments
#                 dates: vector of ascending d+1 dates in years, first is 0
#                 muW: expectation vector of multivar. normal distribution
#                 sigmaW: covariance matrix of multivar. normal distribution
#                 GG: MxdM matrix of cbinded representation matrices G(i)
#                 for d time intervals

multiHermiteMoments<-function(H,N,dates,muW,sigmaW,GG){
  #get dimension/ number of time intervals
  d<-length(dates)-1
  M<-(N+1)*(N+2)/2 #dimension matrices

  #create matrix with all partitions of N in d blocks
  indicesMatrix<-NULL
  indicesMatrix<-rbind(indicesMatrix, numeric(d)) # total degree 0
  for(k in seq(1,N)){ #total degree of n1,...nd,
    partitions<-t(restrictedparts(k,d))
    n<-nrow(partitions)
    for(i in seq(1,n)){
      indicesMatrix<-rbind(indicesMatrix,perm(partitions[i,]))
    }
  }

  #parameters for Hermite moments, which are computed once
  nrows<-nrow(indicesMatrix)
  lastn<-ncol(indicesMatrix)
  lastG<-GG[1:M,((d-1)*M+1):(d*M)] #extract repMatrix of last time interval

  midExpG<-NULL #matrix exponentials of all time intervals inbetween
  for(i in seq(1,d-1)){ #compute matrix exponentials
    G<-GG[1:M, ((i-1)*M+1):(i*M)]
    midExpG<-cbind(midExpG,expm((dates[i+1]-dates[i])*G))
  }

  #create matrix for multidimensional Hermite moments and indices
```

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```
  multHerm<-NULL
  #calculate each Hermite moment
  for(i in seq(1,nrows)){ #for each combination of indicesMatrix
    #last factor:
    v_End<-numeric(M)  #final coordinate vector
    v_End[index(0,indicesMatrix[i,lastn],N)]<-1
    last<-expAtv(lastG,v_End,dates[d+1]-dates[d])$eAtv

    #middle matrix products:
    matprod<-diag(M)
    for(j in seq(1,lastn-1)){ #columns of indicesMatrix
      currentValue<-indicesMatrix[i,j]
      A<-ATransformMatrix(currentValue,muW[j],sigmaW[j,j],N)
      matprod<-matprod%*%midExpG[1:M,((j-1)*M+1):(j*M)]%*%A
    }
    #evaluate for given basis and add entry
    currentHermiteMoment<-indicesMatrix[i,] #indices
    currentHermiteMoment[lastn+1]<-t(H)%*%matprod%*%last #value
    multHerm<-rbind(multHerm,currentHermiteMoment)
  }

return(as.matrix.data.frame(multHerm))

}
```


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```
##########################################################################
## C A L C U L A T I O N   O F   F O U R I E R   C O E F F I C I E N T S ##
##########################################################################

# Function FC_Call
# @ description:  gives the Fourier coefficients of a call option
#                 up to a given order
#                 uses function I for computation
# @ return:       numeric vector n+1
# @ parameters:   r: interest rate
#                 K: log-strike
#                 T: maturity
#                 mu: expectation of the auxiliary distribution
#                 sigma: volatility of the auxiliary distribution
#                 n: highest degree for which the computation is done

FC_Call<-function(r,K,T,mu,sigma,n){
  f<-numeric(n+1); #array for Fourier coefficients
  c<-(K-mu)/sigma;
  alpha<-sigma;
  I1<-I(c,alpha,n);
  I2<-I(c,0,n);
  for(j in seq(1,n+1)){
    f[j]<-exp(-r*T+mu)*I1[j]/sqrt(factorial(j-1))-
      exp(-r*T+K)*I2[j]/sqrt(factorial(j-1));
  }
  return(f);
}

# Function I
# @ description:  computes the inner product of exp(alpha*z) and He_k(z)
#                 w.r.t. the density of standard normal distribution
#                 over the interval (c,\infty) for all k up to n
# @ return:       numeric vector n+1
# @ parameters:   c: lower bound of the interval
#                 alpha: given number
#                 n: highest degree for which the computation is done

I<-function(c,alpha,n){
  I<-numeric(n+1);
  H<-Hermite(n,c);
  I[1]<-exp(alpha^2/2)*pnorm(alpha-c); #I0
  if(n>0){
    for(k in seq(1,n)){
      I[k+1]<-H[k]*exp(alpha*c)*dnorm(c)+alpha*I[k];
    }
  }
  return(I);
}

# Function Hermite
# @ description:  gives all Hermite polynomials evaluated at x
#                 up to order n
# @ return:       numeric vector of length n+1
```

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```
# @ parameters:   n: highest degree of the Hermite polynomial
#                 x: value, at which the polynomials are evaluated

Hermite<-function(n,x){
  He<-numeric(n+1);
  He[1]<-1; #H0
  if(n>0) He[2]<-x; #H1
  if(n>1){
    for(k in seq(2,n)){
      He[k+1]<-x*He[k]-(k-1)*He[k-1];
    }
  }
  return(He);
}

# Function FC_Put
# @ description:  gives the Fourier coefficients of a put option
#                 up to a given order
#                 uses function I and J for computation
# @ return:       numeric vector n+1
# @ parameters:   r: interest rate
#                 K: log-strike
#                 T: maturity
#                 mu: expectation of the auxiliary distribution
#                 sigma: volatility of the auxiliary distribution
#                 n: highest degree for which the computation is done

FC_Put<-function(r,K,T,mu,sigma,n){
  f<-numeric(n+1); #array for Fourier coefficients
  c<-(K-mu)/sigma;
  alpha<-sigma;
  I1<-I(c,alpha,n);
  I2<-J(alpha,n);
  f[1]<-exp(-r*T+mu)*(I1[1]-I2[1])+exp(-r*T+K)*pnorm(c); #f0
  for(j in seq(1,n)){
    f[j+1]<-sigma*exp(-r*T+mu)/sqrt(factorial(j+1))*(I1[j]-I2[j]);
  }
  return(f);
}

# Function J
# @ description:  special case of function I, where the lower limit changed
#                 from c to -\infty;
# @ return:       numeric vector n+1
# @ parameters:   alpha: given number
#                 n: highest degree for which the computation is done

J<-function(alpha,n){
  J<-numeric(n+1);

    for(k in seq(1,n+1)){
      J[k]<-alpha^(k-1)*exp(alpha^2/2);
    }
   return(J);
}
```


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```
# Function FC_BinCall
# @ description:  gives the Fourier coefficients of a binary call option
#                 up to a given order
#                 uses Hermite polynomials for computation
# @ return:       numeric vector n+1
# @ parameters:   r: interest rate
#                 K: lower bound
#                 T: maturity
#                 mu: expectation of the auxiliary distribution
#                 sigma: volatility of the auxiliary distribution
#                 n: highest degree for which the computation is done

FC_BinCall<-function(r,K,T,mu,sigma,n){
  f<-numeric(n+1); #array for Fourier coefficients
  c<-(K-mu)/sigma;
  H<-Hermite(n,c);

  f[1]<-exp(-r*T)*pnorm(-c); #f0
  for(j in seq(1,n)){
    f[j+1]<-exp(-r*T)/sqrt(factorial(j))*H[j]*dnorm(c);
  }
  return(f);
}

# Function FC_BinPut
# @ description:  gives the Fourier coefficients of a binary put option
#                 up to a given order
#                 uses Hermite polynomials for computation
# @ return:       numeric vector n+1
# @ parameters:   r: interest rate
#                 K: upper bound
#                 T: maturity
#                 mu: expectation of the auxiliary distribution
#                 sigma: volatility of the auxiliary distribution
#                 n: highest degree for which the computation is done

FC_BinPut<-function(r,K,T,mu,sigma,n){
  f<-numeric(n+1); #array for Fourier coefficients
  c<-(K-mu)/sigma;
  H<-Hermite(n,c);

  f[1]<-exp(-r*T)*pnorm(c); #f0
  for(j in seq(1,n)){
    f[j+1]<- -exp(-r*T)/sqrt(factorial(j))*H[j]*dnorm(c);
  }
  return(f);
}
```

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```
##########################################################################
###                 E X A M P L E S   A N D   F I G U R E S              ###
##########################################################################

# Example 3.9 (First two log-price Moments):
vmin<-0
vmax<-1
kappa<-1
sigma<-0.25
theta<-1/2
V0<-1/2
X0<-0
r<-0
rho<--0.5
T<-1
N<-2

B<-StandardBasis(V0,X0,N)
G<-StandardRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N)
expG<-expm(G)
Moment(B,G,T,0,1,N)
Moment(B,G,T,0,2,N)

#######################################################

# Example 3.10 (Order of magnitude of Hermite moments):

vmin<-0
vmax<-1
kappa<-1
sigma<-1
theta<-1/2
V0<-1/2
X0<-0
r<-0
rho<--0.5
T<-1/2
N<-50

B<-StandardBasis(V0,X0,N)
G<-StandardRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N)
params<-params_w(B,G,T,N,vmax)
mu_w<-params[1]
sigma_w<-params[2]

## left figure:
SIG<-c(1,0.45)
H1<-HermiteBasis(V0,X0,N,mu_w,SIG[1])
G1<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,SIG[1])
l1<-HermiteMoments(H1,G1,T,N)

H2<-HermiteBasis(V0,X0,N,mu_w,SIG[2])
G2<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,SIG[2])
l2<-HermiteMoments(H2,G2,T,N)
```


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```
H<-HermiteBasis(V0,X0,N,mu_w,sigma_w)
GH<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,sigma_w)
lbest<-HermiteMoments(H,GH,T,N)

plot(seq(0,N),l1,type="p",pch=20, col="#ff0707",xlab="n",
     ylab=expression(l[n]), las=1)
lines(seq(0,N),l1,type="l",pch=20, col="#ff8888")
lines(seq(0,N,4),l1[seq(1,N+1,4)], col="#ff0707", type="l", lty="dotted")
lines(seq(2,N,4),l1[seq(3,N+1,4)], col="#ff0707", type="l", lty="dotted")
lines(seq(2,30),l2[seq(3,31)], col="#636363", type="p",pch=20)
lines(seq(2,30),l2[seq(3,31)], col="#d1d1d1", type="l",pch=20)
lines(seq(0,N,2),l2[seq(1,N+1,2)], col="#636363", type="l", lty="dotted" )
lines(seq(3,N,2),l2[seq(4,N+1,2)], col="#636363", type="l", lty="dotted" )
legend(38,1.1,legend=c(expression(sigma[w]^2~"=1"),
                       expression(sigma[w]^2~"="~0.45^2)),
       pch = 20,  y.intersp=1, col=c("#ff0707","#636363"),
       box.lty=0,bg="transparent")
abline(h=0, lty="dotted")

## right figure:
SIG2<-c(sigma_w,seq(0.6,1,0.1))
N<-20
GNsigma_w<-function(s){
            HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,s)}
HNsigma_w<-function(s){HermiteBasis(V0,X0,N,mu_w,s)}
lSIG<-NULL

for(s in SIG2){
  GN<-GNsigma_w(s)
  HN<-HNsigma_w(s)
  l<-HermiteMoments(HN,GN,T,N)
  lSIG<-cbind(lSIG,l)
}

plot(seq(0,N),lSIG[,6],type="p",pch=20, col="#ff0707",xlab="n",
     ylab=expression(l[n]), las=1)
lines(seq(0,N),lSIG[,5], col="#ff4400", type="p",pch=20)
lines(seq(0,N),lSIG[,4], col="#ff8e06", type="p",pch=20)
lines(seq(0,N),lSIG[,3], col="#908800", type="p",pch=20)
lines(seq(0,N),lSIG[,2], col="#2e5032", type="p",pch=20)
lines(seq(0,N),lSIG[,6], col="#ff0707", type="l",pch=20, lty="dotted")
lines(seq(0,N),lSIG[,5], col="#ff4400", type="l",pch=20, lty="dotted")
lines(seq(0,N),lSIG[,4], col="#ff8e06", type="l",pch=20, lty="dotted")
lines(seq(0,N),lSIG[,3], col="#908800", type="l",pch=20, lty="dotted")
lines(seq(0,N),lSIG[,2], col="#2e5032", type="l",pch=20, lty="dotted")
lines(seq(0,N),lSIG[,1], col="#01018b", type="b",pch=20)
abline(h=0, lty="dotted")
legend(15,1.1,legend=c(expression(sigma[w]^2~"=1"),
                       expression(sigma[w]^2~"="~0.9^2),
                       expression(sigma[w]^2~"="~0.8^2),
                       expression(sigma[w]^2~"="~0.7^2),
                       expression(sigma[w]^2~"="~0.6^2),
                       expression(sigma[w]^2~"=V(X)")),
       pch = 20,  y.intersp=1, col=c("#ff0707","#ff4400",
```

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```
                                        "#ff8e06","#908800",
                                        "#2e5032","#01018b"),
        box.lty=0,bg="transparent")

## below figure: Series expansion of the log-price density
x<-seq(mu_w-2*1,mu_w+2*1,0.01)
gT<-function(x,n){dnorm(x,mu_w,sigma_w)*
    sum(GeneralizedHermite(n,x,mu_w,sigma_w)*lbest[1:(n+1)])}
gT2<-function(x,n){dnorm(x,mu_w,1)*
    sum(GeneralizedHermite(n,x,mu_w,1)*l1[1:(n+1)])}

par(mfrow=c(1,4))
val<-c(5,10,20,30)
for(v in val){
plot(x,sapply(x,n=v,gT), type="l",col="#01018b", xlab=expression(x),
     ylab=expression(g[T](x)), las=1, main=paste("N=", v))
lines(x,dnorm(x,mu_w,sigma_w), lty="dotted",col="#01018b")
lines(x,sapply(x,n=v,gT2), col="#ff0707")
lines(x,dnorm(x,mu_w,1), lty="dotted",col="#ff0707")
abline(h=0)
}

#######################################################

# Example 4.7 (ATM call option with one week maturity):

vmin<-0
vmax<-1
kappa<-1
sigma<-0.25
theta<-1/2
V0<-1/2
X0<-0
r<-0
rho<--0.5
T<-1/52
N<-20
K<-0

B<-StandardBasis(V0,X0,N)
G<-StandardRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N)
params<-params_w(B,G,T,N,vmax)
mu_w<-params[1]
sigma_w<-params[2]

H1<-HermiteBasis(V0,X0,N,mu_w,sigma_w)
GH1<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,sigma_w)
a1<-HermiteMoments(H1,GH1,T,N)

H2<-HermiteBasis(V0,X0,N,0,1)
GH2<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,1)
a2<-HermiteMoments(H2,GH2,T,N)

b1<-FC_Call(r,K,T,mu_w,sigma_w,N)
b2<-FC_Call(r,K,T,0,1,N)
```


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```
c1<-cumsum(a1*b1)
c2<-cumsum(a2*b2)

par(mfrow=c(1,3))
plot(seq(0,N),a2, type="b",pch=20,lty=1,xlab="n",
     ylab=expression(l[n]), las=1, col="#ff4400")
lines(seq(0,N),a1, type="b",pch=20,lty=1, col="#01018b")
legend(2,1.1,legend=c(expression(mu~"=0,\ "~sigma^2~"=1"),
                      expression(mu~"=E[X],\ "~sigma^2~"=V(X)")),
       lty = 1,   col=c("#ff4400","#01018b"), box.lty=0,bg="transparent")


plot(seq(0,N),b2, type="b",pch=20,lty=1, xlab="n",
     ylab=expression(f[n]), las=1, col="#ff4400")
lines(seq(0,N),b1, type="b",pch=20,lty=1, col="#01018b")
legend(2,1.47,legend=c(expression(mu~"=0,\ "~sigma^2~"=1"),
                       expression(mu~"=E[X],\ "~sigma^2~"=V(X)")),
       lty = 1,   col=c("#ff4400","#01018b"), box.lty=0,bg="transparent")

plot(seq(0,N),c2, type="b",pch=20,lty=1,xlab="n",
     ylab=expression(pi^(n)), las=1, col="#ff4400")
lines(seq(0,N),c1, type="b",pch=20,lty=1, col="#01018b")
legend(2,0.935,legend=c(expression(mu~"=0,\ "~sigma^2~"=1"),
                        expression(mu~"=E[X],\ "~sigma^2~"=V(X)")),
       lty = 1,   col=c("#ff4400","#01018b"), box.lty=0,bg="transparent")

#######################################################

# Example 4.12 (Bivariate Hermite moments):

vmin<-0
vmax<-1
kappa<-1
sigma<-0.25
theta<-1/2
V0<-1/2
X0<-0
r<-0
rho<--0.5
N<-2
dates<-c(0,1/2,1)

B<-StandardBasis(V0,X0,N)
G<-StandardRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N)
muW<-numeric(length(dates)-1)
sigmaW<-diag(length(dates)-1)
GG<-NULL
for(i in seq(2,length(dates))){
  current<-params_w(B,G,dates[i],N,vmax)
  muW[i-1]<-current[1]
  sigmaW[i-1,i-1]<-current[2]
  GH<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,current[2])
  GG<-cbind(GG,GH)
}
```

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```
H<-HermiteBasis(V0,X0,N,muW[1],sigmaW[1,1])
M<-nrow(GG)
expG1<-expm((dates[2]-dates[1])*GG[1:M,1:M])
expG2<-expm((dates[3]-dates[2])*GG[1:M,(M+1):(2*M)])
HM<-multiHermiteMoments(H,N,dates,muW,sigmaW,GG)

#####################################################

#Example 4.13 (ATM Asian fixed strike call option with maturity in four
    weeks and weekly average of underlying):

require(MultiGHQuad)

vmin<-0
vmax<-1
kappa<-1
sigma<-0.25
theta<-0.5
V0<-0.5
X0<-0
r<-0
rho<--0.5
N<-15
dates<-c(0,1/52,2/52,3/52,4/52)

B<-StandardBasis(V0,X0,N)
G<-StandardRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N)
muW<-numeric(length(dates)-1)
sigmaW<-diag(length(dates)-1)
GG<-NULL #representation matrices for each time interval

# calculate for each time interval the params of auxiliary density
# and then the representation matrices wrt this params
for(i in seq(2,length(dates))){
  current<-params_w(B,G,dates[i],N,vmax)
  muW[i-1]<-current[1]
  sigmaW[i-1,i-1]<-current[2]
  GH<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,r,rho,N,current[2])
  GG<-cbind(GG,GH)
}

H<-HermiteBasis(V0,X0,N,muW[1],sigmaW[1,1])
HM<-multiHermiteMoments(H,N,dates,muW,sigmaW,GG)

multHermDF<-as.data.frame(HM)
names(multHermDF)<-c("n1","n2","n3","n4","l")

# create df summarizing the results
dfmultH<-data.frame(N=integer(), firstIndex=integer(), lastIndex=integer(),
                    minL=double(), maxL=double(), meanL=double()
                    )
for(k in seq(0,N)){
  rowsK<-rownames(subset(multHermDF, n1+n2+n3+n4==k))
  first<-rowsK[1]
  last<-rowsK[length(rowsK)]
```


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```
  newEntry<-data.frame(N=k,firstIndex=first,lastIndex=last,
                       minL=min(multHermDF$l[first:last]),
                       maxL=max(multHermDF$l[first:last]),
                       meanL=median(multHermDF$l[first:last])
                       )
  dfmultH<-rbind(dfmultH,newEntry)
}

#plot multHermite moments
plot(seq(0,N),dfmultH$meanL, ylim=c(min(dfmultH$minL),1),type="l",pch=20,
     col="#ff0707",xlab="N",xaxt="n", ylab=expression(l[N]), las=1)
#add points
for(k in seq(1,N+1)){
  y<-multHermDF$l[dfmultH$firstIndex[k]:dfmultH$lastIndex[k]]
  points(rep(k-1,length(y)),y,pch=20, col="#636363")
}
lines(seq(0,N),dfmultH$meanL,type="p",pch=20, col="#ff0707")
abline(h=0,lty="dotted")
axis(1, at=seq(0,N),labels=seq(0,N))

#number of compositions:
comp<-numeric(N+1)
comp<-as.integer(dfmultH$lastIndex)-as.integer(dfmultH$firstIndex)+1

## computation of the option price:
AsianC<-function(y,X0,K,r,T){ #discounted payoff function
  priceS<-exp(cumsum(y)+X0)
  return(exp(-r*T)*max(mean(priceS)-K,0))
}
gmult<-function(y,MuW,SigmaW,df,n){ #log-return density, n... row number
  return(sum(normalizedHermite(df$n1[1:n],(y[1]-MuW[1])/SigmaW[1,1])*
               normalizedHermite(df$n2[1:n],(y[2]-MuW[2])/SigmaW[2,2])*
               normalizedHermite(df$n3[1:n],(y[3]-MuW[3])/SigmaW[3,3])*
               normalizedHermite(df$n4[1:n],(y[4]-MuW[4])/SigmaW[4,4])*
               df$l[1:n]))
}
normalizedHermite<-function(n,z){ #n... degree of Hermite polynomial
  values<-numeric(length(n))
  for(k in seq(1,length(values))){
    values[k]<-1/sqrt(factorial(n[k]))*Hermite(n[k],z)[n[k]+1]
  }
  return(values)
}
# resulting function for 4-dim Gauss-Hermite quadrature:
fgmult<-function(z,MuW=muW,SigmaW=sigmaW,n,df,K,X0,r,T){
  return(AsianC(MuW+z*SigmaW,X0,K,r,T)*gmult(MuW+z*SigmaW,MuW,SigmaW,df,n))
}
#Gauss-Hermite quadrature:
grid <- init.quad(Q = 4) #points and weights
integralDensity<-eval.quad(gmult,grid, n=15,MuW=muW,SigmaW=sigmaW,
                           df=multHermDF)[1]
prices<-numeric(N+1)
for(N in seq(0,length(prices)-1)){
  k<-dfmultH$lastIndex[N+1] #find highest index for which total sum is N
  prices[N+1]<-eval.quad(fgmult, grid, n=k,df=multHermDF,K=1,X0=0,r=0,
```

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```
                                T=dates[4])[1]/integralDensity #price ATM
}
# call price with log-strike 0
fc<-FC_Call(r,0,dates[4],muW[4],sigmaW[4,4],N)
Hc<-HermiteBasis(V0,X0,N,muW[4],sigmaW[4,4])
Gc<-HermiteRepMatrix(kappa,theta,vmin,vmax,sigma,0,rho,N,sigmaW[4,4])
lc<-HermiteMoments(Hc,Gc,dates[4],N)
cprices<-cumsum(fc*lc)

# plot results:
plot(seq(0,N),prices,type="b",pch=20, col="#ff0707", las=1,
     xlab="N",ylab=expression(pi^N))
lines(seq(0,N),cprices,type="b",pch=20,lty="dotted", col="#636363")
axis(1, at=seq(0,N),labels=seq(0,N))
legend(12,1.0,legend=c("ATM Asian Call", "ATM Call"),  pch = 20,
          col=c("#ff0707","#636363"),box.lty=0,bg="transparent")
```


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