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DISSERTATION

# Rearrangement and Sobolev inequalities via projection averages

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# Declaration

I hereby declare that I have written the present thesis on my own and that I have used only specified literature. The thesis has not been submitted elsewhere for the application of a scientific degree. Parts of the thesis are based on work that have been submitted by the author to scientific journals.

Vienna, June 2020

Philipp Kniefacz

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# Abstract

In this thesis, a family of sharp $L^p$ Sobolev–type inequalities is established by averaging the length of $i$-dimensional projections of the gradient of a function. Moreover, it is shown that each of these new inequalities directly implies the classical $L^p$ Sobolev inequality of Aubin and Talenti and that the strongest member of this family is the only affine invariant one among them – the affine $L^p$ Sobolev inequality of Lutwak, Yang, and Zhang. When $p = 1$, the entire family of new Sobolev inequalities is extended to functions of bounded variation to also allow for a complete classification of all extremal functions in this case.

Next, the corresponding family of Pólya–Szegö principles, associated to the aforementioned Sobolev–type inequalities, is established, as well as the Pólya–Szegö principles for a family of analytic functionals introduced recently by Haberl and Schuster. Both of these families contain the classical Pólya–Szegö principle as well as the affine Pólya–Szegö principle by Cianchi, Lutwak, Yang and Zhang as special cases and can therefore be seen as generalizations of the latter ones. Additionally, a complete characterization of the cases of equality in the sense of Brothers and Ziemer is given.

Finally, it is shown that the volume product $V(\Phi^\circ K)V(K)^{n-1}$ for each continuous, $(n-1)$–homogeneous, translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuation $\Phi$ has a full dimensional convex body as maximizer. Moreover, by the same methods, the non–polar problem for such Minkowski valuations can be dealt with as well. It is shown that the analogous non–polar volume product $V(\Phi K)V(K)^{1-n}$ exhibits full dimensional minimizers.

# Zusammenfassung

In der vorliegenden Dissertation wird eine Familie Sobolev–artiger Ungleichungen bewiesen, indem die Längen $i$–dimensionaler Projektionen des Gradienten einer Funktion passend gemittelt werden. Darüberhinaus wird gezeigt, dass jede dieser neuen Ungleichungen unmittelbar die klassische $L^p$ Sobolev Ungleichung von Aubin und Talenti impliziert und dass innerhalb dieser Familie die stärkste Ungleichung zugleich die einzig affin invariante ist, nämlich die affine $L^p$ Sobolev Ungleichung von Lutwak, Yang und Zhang. Im Fall $p = 1$ wird die gesamte Familie dieser neuartigen Ungleichungen zudem auf den Raum der Funktionen von beschränkter Variation erweitert, was eine vollständige Charakterisierung aller Gleichheitsfälle erlaubt.

Als nächstes werden die zu obigen Sobolev–artigen Ungleichungen gehörigen Pólya–Szegö Ungleichungen aufgestellt. Ebenso werden die zugehörigen Pólya–Szegö Ungleichungen einer Familie von analytischen Funktionalen, die erst kürzlich von Haberl und Schuster eingeführt wurden, bewiesen. Beide dieser neuen Familien von Ungleichungen beinhalten insbesondere das klassische sowie das von Cianchi, Lutwak, Yang und Zhang eingeführte affine Pólya–Szegö Prinzip, weswegen die neuen Ungleichungen der vorliegenden Arbeit eine Verallgemeinerung dieser klassischen Resultate darstellen. Weiters werden alle Gleichheitsfälle dieser neu erhaltenen Ungleichungen im Sinne von Brothers und Ziemer charakterisiert.

Schlussendlich wird gezeigt, dass das Produkt der Volumina $V(\Phi^\circ K)V(K)^{n-1}$ für jede stetige, $(n-1)$–homogene, translationsinvariante und $\mathrm{SO}(n)$ equivariante Minkowski–Bewertung $\Phi$ einen volldimensionalen konvexen Körper als Maximierer besitzt. Die gleichen Methoden erlauben es auch die nicht–polare Problemstellung für solche Minkowski–Bewertungen zu behandeln. In analoger Weise wird gezeigt, dass das nicht–polare Produkt der Volumina $V(\Phi K)V(K)^{1-n}$ volldimensionale Minimierer besitzt.

# Contents

# 1 Introduction

A classical research area in geometric analysis includes the search for sharp bounds for analytic functionals. Probably the best known example in this context is the classical $L^1$ Sobolev inequality, going back in its sharp form to Federer and Fleming [FF60] and Maz'ya [Maz60]. The latter being Euclidean in nature, it was a major breakthrough, when in 1999 Zhang [Zha99] proved an affine Sobolev–type inequality, that strengthened and directly implied the weaker Euclidean one. Since then a lot of effort was put into proving new affine inequalities or establishing the affine counterparts to already existing Euclidean inequalities (see for example [CLYZ09, HS09a, LYZ02]). One of the more recent results in this field is given by a family of Sobolev–type inequalities by Haberl and Schuster [HS19], which contains the classical and the affine Sobolev inequalities as special members.

A different line of research, which is closely related to the one of Sobolev–type inequalities, deals with the rearrangement of functions to minimize energy functionals. Classical results in this direction show that many such functionals are minimized by functions exhibiting enough symmetries. In this sense, a Pólya–Szegö principle refers to any kind of inequality that assures that some energy functional $\mathcal{E}$ of a function $f \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$, does not increase under a suitable rearrangement of $f$. The first and probably best known representative of such inequalities, the classical Pólya–Szegö principle (see [PS51]), proves such an inequality for the usual $L^p$ norm $\|\nabla f\|_p$ of $|\nabla f|$. Regarding the equality cases of such an inequality, Brothers and Ziemer [BZ88] provided the weakest necessary conditions on $f$, such that equality holds exactly for radially symmetric functions, that is functions whose level sets are concentric balls.

Analytic inequalities, such as Sobolev–type inequalities and Pólya–Szegö principles, are strongly related to geometric inequalities of isoperimetric type. For instance, it is very well known that the classical $L^1$ Sobolev inequality and the classical isoperimetric inequality are equivalent. Similarly, Pólya–Szegö principles are usually obtained by applying isoperimetric inequalities to the level sets of a function (see for example [CF02b, ET04, Wan13]). Thus, extending our understanding of certain analytic inequalities goes hand in hand with an increased comprehension of solutions to isoperimetric problems (and vice versa). As a result, many analytic inequalities are obtained after their geometric core is established. For instance, Zhang proved the affine Zhang–Sobolev inequality by applying (a modified version of) Petty's projection inequality [Pet71], the fundamental affine isoperimetric inequality for projection bodies. Similarly, Haberl and Schuster first established the geometric core before proving their Sobolev–type inequalities. The underlying geometry behind their result corresponds to a generalization of the Petty projection inequality, replacing the projection body by members from a large family of Minkowski valuations. Showing a Petty Projection–type inequality for all continuous, $(n-1)$–homogeneous, translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations is still an open problem, whose solution would again give rise to a large class of Sobolev–type inequalities.

In this work we will establish a family of sharp Sobolev–type inequalities, by taking suitable averages of the projection of the gradient of a function $f$ onto lower dimensional subspaces. We will achieve this, by first showing the underlying geometric inequalities and then exploit the strong connection between geometric and analytic inequalities, to prove our analytic versions. Next, we will establish the Pólya–Szegö principles for our family of energy functionals, as well as the Pólya–Szegö principles for the family of energy functionals introduced in [HS19]. In this way, we will recover the classical Pólya–Szegö principles as special cases. Finally, we will follow the classical approach of first establishing geometric inequalities in order to show analytic ones, by investigating a large family of Minkowski valuations and proving that certain associated functionals exhibit extremizers. This work is structured as follows:

Chapter 2 is a refresher on some background material from convex geometry and geometric analysis, as well as functional analysis, that we will use throughout this thesis.

In Chapter 3 we recall some classical geometric and analytic inequalities as well as some recent results, that extend the classical inequalities. Moreover, we introduce the notion of LYZ-bodies, which allows us to easily obtain analytic inequalities from geometric ones, and prove some important properties of them.

Next, in Chapter 4, we use the LYZ–body and methods similar to those introduced in [HS19] to establish a family of sharp geometric inequalities and turn those into sharp functional ones. We will see, that among our inequalities, the affine ones play an important role, by being the strongest ones. Furthermore, we provide a characterization of all cases of equality of the established Sobolev–type inequalities. Additionally, we will partly answer a question raised by Monika Ludwig, by proving our inequalities for functionals depending on other norms than the Euclidean one.

In Chapter 5 the Pólya–Szegö principles for the energy functionals introduced in Chapter 4, as well as the ones introduced by Haberl and Schuster in [HS19], will be established.

Finally, Chapter 6 is devoted to the study of Petty Projection–type inequalities for a certain class of Minkowski valuations. We show that the volume product $V(\Phi^\circ K)V(K)^{n-1}$ for any continuous, $(n-1)$–homogeneous, translation invariant and $\mathrm{SO}(n)$ equivariant Minkowksi valuation $\Phi$ exhibits a full dimensional maximizer $K$. Our method also allows us to tackle the non–polar version of this problem and enables us to prove the existence of minimizers of $V(\Phi K)V(K)^{1-n}$ for the aforementioned valuations.

Some parts of the results presented in this thesis have already been submitted to scientific journals by the author. Chapter 4 is based on joint work with Franz Schuster [KS20]. The rearrangement inequalities presented in Chapter 5 are not published yet, but are going to be published in [Kni20]. Finally, the geometric inequalities shown in Chapter 6 are work in progress jointly with Georg Hofstätter and Franz Schuster and are going to be published in [HKS20].

# 2 Background and Notation

In this chapter we collect basic notions and results from the $L^p$ Brunn–Minkowski theory of convex bodies as well as the theory of Sobolev functions and of functions of bounded variation, following essentially [KS20]. As a general reference for the material on convex geometry presented here, we recommend the books by Schneider [Sch14] and Gardner [Gar06]. A thorough introduction to the theory of Sobolev spaces, Sobolev functions and functions of bounded variation can be found in the books by Leoni [Leo17], by Evans and Gariepy [EG15] and by Ambrosio, Fusco and Pallara [AFP00].

## 2.1 Convex Bodies and the $L^p$ Brunn–Minkowski Theory

Throughout this thesis, we will work in $n$–dimensional Euclidean space $\mathbb{R}^n$, where we will always assume that $n \geq 3$. We will denote the $(n-1)$–dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. Whenever we write that some condition holds almost everywhere (or a.e.) without specifying a measure, then we mean with respect to the Lebesgue measure $\mathcal{L}$ in $\mathbb{R}$ or $\mathcal{L}^n$ in $\mathbb{R}^n$.

A non–emtpy, compact and convex set $K \subset \mathbb{R}^n$ is called a convex body. We denote the set of all convex bodies in $\mathbb{R}^n$ by $\mathcal{K}^n$. Furthermore, we denote the set of convex bodies, which have the origin in their interior, by $\mathcal{K}_0^n$. The natural metric to endow $\mathcal{K}^n$ and $\mathcal{K}_0^n$ with is the Hausdorff metric. We will denote the volume of a convex body $K$ (that is its $n$–dimensional Lebesgue measure) by $V(K)$.

A convex body $K$ is uniquely determined by its supporting hyperplanes and consequently by its support function $h(K,u) = \max\{u \cdot x : x \in K\}$ for $u \in S^{n-1}$. From this definition, it immediately follows that

$$h(\theta K, u) = h(K, \theta^{-1}u), \qquad u \in S^{n-1}, \tag{2.1}$$

for every $\theta \in \mathrm{SO}(n)$. Moreover, for $s, t \geq 0$, the *Minkowski combination*

$$sK + tL = \{sx + ty : x \in K, y \in L\}$$

of two convex bodies $K, L \in \mathcal{K}^n$ can be nicely expressed via their support functions, namely

$$h(sK + tL, \cdot) = sh(K, \cdot) + th(L, \cdot).$$

If $K \subset \mathbb{R}^n$ is a convex and closed set (not necessarily compact) containing the origin, then we can define its *polar set* $K^\circ$ by

$$K^\circ = \{x \in \mathbb{R}^n : \ x \cdot y \leq 1 \ \ \forall y \in K\}.$$

Furthermore, if $K \in \mathcal{K}_0^n$, then $K^\circ \in \mathcal{K}_0^n$. In this case, we call $K^\circ$ the *polar body* of $K$ and we have $K = K^{\circ\circ}$.

To every origin–symmetric, closed and convex set $K \subseteq \mathbb{R}^n$, we can associate its *Minkowski functional*, given by $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$. If $K \in \mathcal{K}^n$ is origin–symmetric, then $\|\cdot\|_K$ is a support function, namely

$$h(K^\circ, \cdot) = \|\cdot\|_K. \tag{2.2}$$

Additionally, if $K \in \mathcal{K}_0^n$ is origin–symmetric, then $\|\cdot\|_K$ defines a norm on $\mathbb{R}^n$ with unitball $K$. Clearly, $\|\cdot\|_{\mathbb{B}^n}$ is the usual Euclidean norm in $\mathbb{R}^n$.

Next, we denote by $\mathrm{Gr}_{n,i}$ the $i$–Grassmannian in $\mathbb{R}^n$, that is the set of all $i$–dimensional subspaces of $\mathbb{R}^n$. Let $1 \leq i \leq n-1$ and $E, F \in \mathrm{Gr}_{n,i}$ be given and choose $\vartheta \in \mathrm{SO}(n)$ such that $F = \vartheta E$. If $K \in \mathcal{K}^n$ is origin–symmetric and $i$–dimensional such that $K \subseteq E$, we write $K(F)$ instead of $\vartheta K$ for the rotated copy of $K$ contained in $F$. In this case, it is easy to see that for every $x \in \mathbb{R}^n$,

$$\|x|F\|_{K(F)^\circ} = h(\vartheta K, x). \tag{2.3}$$

For $0 \leq i \leq n$, the $i$–dimensional volume of the $i$–dimensional unitball will be denoted by $\omega_i$, its surface area is then given by $n\omega_i$. Here, the value of $\omega_p$, $p \in \mathbb{N}$, is given by

$$\omega_p = \frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2}+1\right)}, \tag{2.4}$$

where $\Gamma(x)$ denotes the Gamma function. Formula (2.4) allows to define constants for every $p \in \mathbb{R}$ and we will often make use of those.

The volume of a linear combination of convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ can be shown to be a homogeneous polynomial of degree $n$, that is

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $V(K_{i_1}, \dots, K_{i_n})$ are called the *mixed volumes* of $K_{i_1}, \dots, K_{i_n}$. They are symmetric and positively linear in their arguments, moreover

$$V(K, \dots, K) = V(K).$$

To simplify the presentation of mixed volumes, we introduce the following notation: for $0 \leq i_k \leq n$, $k = 1, \dots, m$ such that $\sum_{k=1}^m i_k = n$, the mixed volume of $i_1$ copies of $K_1$, $i_2$ copies of $K_2$ and so on, is denoted by $V(K_1[i_1], K_2[i_2], \dots, K_m[i_m])$. If only one copy of $K$ is used, we omit the brackets.

A few mixed volumes deserve special consideration: for $1 \leq i \leq n$ we denote by

$$V_i(K) = \frac{1}{\omega_{n-i}} \binom{n}{i} V(K[i], \mathbb{B}^n[n-i])$$

the $i$–th *intrinsic volume* of $K \in \mathcal{K}^n$. Moreover, we denote by

$$S(K) = nV(K[n-1], \mathbb{B}^n) = 2V_{n-1}(K)$$

the *surface area* of $K$ and by

$$w(K) = \frac{2}{\omega_n} V(K, \mathbb{B}^n[n-1]) = \frac{2\omega_{n-1}}{n\omega_n} V_1(K)$$

the *mean width* of $K$. Note that $V_n = V$.

The *surface area measure* of $K$ is the unique measure $S(K, \cdot)$ defined by

$$V(K[n-1], L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)\, dS(K, u) \tag{2.5}$$

for each $L \in \mathcal{K}^n$. The surface area measure satisfies

$$\int_{S^{n-1}} f(u)\, dS(K, u) = \int_{\partial K} f(\nu(x))\, d\mathcal{H}^{n-1}(x) \tag{2.6}$$

for each $f \in C(S^{n-1})$, where $\nu(x)$ denotes the outer unit normal of $\partial K$ at $x$. Note that this outer unit normal exists for $\mathcal{H}^{n-1}$–a.e. $x \in \partial K$. The surface area measure $S(\mathbb{B}^n, \cdot)$ of the unit ball is the spherical Lebesgue measure. To avoid writing $S(\mathbb{B}^n, \cdot)$, we will use the shorthand notation $du = dS(\mathbb{B}^n, u)$.

An important inequality for mixed volumes is the *Minkowski inequality*, stating that for two convex bodies $K, L \in \mathcal{K}^n$ the inequality

$$V(K[n-1], L)^n \geq V(K)^{n-1} V(L) \tag{2.7}$$

holds, where equality is attained if and only if $K$ and $L$ are homothetic, i.e. $K = cL + t$ for some $c > 0$ and $t \in \mathbb{R}^n$. A simple consequence of the Minkowski inequality is the following result (see [LYZ06] for a proof).

**Lemma 2.1.** *If $K, L \in \mathcal{K}_0^n$ are origin–symmetric, such that*

$$\frac{V(M, K[n-1])}{V(K)} = \frac{V(M, L[n-1])}{V(L)}$$

*for every origin–symmetric $M \in \mathcal{K}_0^n$, then $K = L$.*

Formula (2.5) implies that

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u)\, dS(K, u) \tag{2.8}$$

for any $K \in \mathcal{K}^n$. The easiest way to compute the volume of $K^\circ$ is via the polar coordinate formula: if $K \in \mathcal{K}_0^n$, then

$$V(K^\circ) = \frac{1}{n} \int_{S^{n-1}} h(K, u)^{-n}\, du. \tag{2.9}$$

While the classical Brunn–Minkowski theory of convex bodies emerges from combining the notion of volume with that of Minkowski addition, the development of its more modern $L^p$ extension, initiated by Lutwak [Lut93, Lut96], is a result of merging volume with the $L^p$ Minkowski addition of convex bodies. To make this more explicit, let $1 \leq p < \infty$ and suppose that $K, L \in \mathcal{K}_0^n$. For $t > 0$, the $L^p$ *Minkowski combination* $K +_p t \cdot L \in \mathcal{K}_0^n$ was defined in [Fir62] by

$$h(K +_p t \cdot L, \cdot)^p = h(K, \cdot)^p + t\, h(L, \cdot)^p.$$

Note that when $p = 1$, we have $K +_1 t \cdot L = K + tL$.

It was shown in [Lut93] by Lutwak that to each $K \in \mathcal{K}_0^n$ one can associate a unique Borel measure $S_p(K, \cdot)$ on $S^{n-1}$, the $L^p$ *surface area measure* of $K$, such that

$$\lim_{t \to 0^+} \frac{V(K +_p t \cdot L) - V(K)}{t} = \frac{1}{p} \int_{S^{n-1}} h(L, u)^p \, dS_p(K, u)$$

for every $L \in \mathcal{K}_0^n$. Moreover, $S_p(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S_1(K, \cdot) = S(K, \cdot)$ and its Radon–Nikodym derivative is $h(K, \cdot)^{1-p}$.

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a *Minkowski valuation* if it satisfies the *valuation property*

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L)$$

whenever $K \cup L \in \mathcal{K}^n$. The natural $L^p$ extension are the so called $L^p$ *Minkowski valuations*, where a map $\Phi_p : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is called an $L^p$ Minkowski valuation if

$$\Phi(K) +_p \Phi(L) = \Phi(K \cup L) +_p \Phi(K \cap L)$$

whenever $K \cup L \in \mathcal{K}_0^n$.

We call a Minkowski valuation *translation invariant* if

$$\Phi(K + t) = \Phi(K)$$

for each $t \in \mathbb{R}^n$ and $q$–homogeneous if

$$\Phi(\lambda K) = \lambda^q \Phi(K)$$

for each $\lambda > 0$. In the following, we will mainly be interested in translation invariant and $(n-1)$–homogeneous Minkowski valuations that are continuous in the Hausdorff metric.

Minkowski used Cauchy's projection formula

$$V_{n-1}(K|u^\perp) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS(K, v), \qquad u \in S^{n-1}, \tag{2.10}$$

for $K \in \mathcal{K}^n$, to define one of the most studied Minkowski valuations. Since the right-hand side of (2.10) is sublinear and therefore a support function for any $K \in \mathcal{K}^n$, we can define the so called *projection body of* $K$ by

$$h(\Pi K, x) = \frac{1}{2} \int_{S^{n-1}} |x \cdot v| \, dS(K, v), \qquad x \in \mathbb{R}^n. \tag{2.11}$$

A remarkable property of $\Pi$ is its $\mathrm{SL}(n)$ *contravariance*, meaning that

$$\Pi(\theta K) = \theta^{-t} \Pi K$$

for each $\theta \in \mathrm{SL}(n)$, where we denote by $\theta^{-t}$ the transpose of the inverse of $\theta$. An important characterization result due to Ludwig [Lud05] shows that each continuous, translation invariant and $\mathrm{SL}(n)$ contravariant Minkowski valuation is a multiple of $\Pi$. Schuster [Sch07] studied $\mathrm{SO}(n)$ *equivariant* Minkowski valuations $\Phi$, that is $\Phi$ satisfies

$$\Phi(\theta K) = \theta \Phi(K)$$

for each $\theta \in \mathrm{SO}(n)$ and proved that every continuous, translation invariant, $(n-1)$–homogeneous and $\mathrm{SO}(n)$ equivariant Minkowski valuation can be represented in the form

$$h(\Phi K, u) = \int_{S^{n-1}} f(u \cdot v) \, dS(K, v) \tag{2.12}$$

for some weakly positive and continuous function $f \in C(S^{n-1})$. Here, a function $f \in C(S^{n-1})$ is said to be weakly positive, if there exists $x_0 \in \mathbb{R}^n$ such that $f(u) + x_0 \cdot u \geq 0$. This characterization result will be our starting point in Chapter 6, to show that extremizers for certain isoperimetric problems for $\Phi$ exist.

Based on (2.11), a natural $L^p$ extension of the projection body operator was defined by Lutwak, Yang, and Zhang in [LYZ00]. For $1 \leq p < \infty$ and $K \in \mathcal{K}_0^n$, the $L^p$ *projection body* of $K$ is given by

$$h(\Pi_p K, x)^p = a_{n,p} \int_{S^{n-1}} |x \cdot v|^p \, dS_p(K, v), \qquad x \in \mathbb{R}^n, \tag{2.13}$$

where the normalizing constant

$$a_{n,p} = \frac{\omega_{p-1}}{2\omega_{n+p-2}} \tag{2.14}$$

was chosen such that $\Pi_p \mathbb{B}^n = \mathbb{B}^n$. Note that when $p = 1$, (2.13) is well defined for all $K \in \mathcal{K}^n$ and that, in this case, $\Pi_1 K = \omega_{n-1}^{-1} \Pi K$. Moreover, $\Pi_p$ satisfies the valuation property for $L^p$ Minkowski addition and is therefore an $L^p$ Minkowski valuation.

The range of the $L^p$ projection body map is contained in the class of $L^p$ *zonoids*. For $p \geq 1$, an origin–symmetric convex body $K \in \mathcal{K}^n$ is an $L^p$ zonoid if and only if there exists an even measure $\mu$ on $S^{n-1}$ (which is uniquely determined when $p$ is not an even integer) such that

$$h(K, x)^p = \int_{S^{n-1}} |x \cdot v|^p \, d\mu(v), \qquad x \in \mathbb{R}^n. \tag{2.15}$$

In the following, we denote by $Z_p^\mu$ the $L^p$ zonoid generated in this way by $\mu$. $L^1$ zonoids are usually just called zonoids and we simply write $Z^\mu$ instead of $Z_1^\mu$ for the zonoid generated by $\mu$.

Finally, when dealing with sequences of convex bodies, the following result, known as the *Blaschke selection theorem*, is an indispensable tool.

**Theorem 2.2.** *Every bounded sequence of convex bodies has a convergent subsequence.*

## 2.2 Sobolev Functions and Functions of Bounded Variation

We denote the set of all functions $f \in L^p(\mathbb{R}^n)$ whose weak gradient $\nabla f$ also lies in $L^p(\mathbb{R}^n)$ by $W^{1,p}(\mathbb{R}^n)$. Together with the Sobolev norm $\|f\|_{W^{1,p}(\mathbb{R}^n)} = \|f\|_p + \|\nabla f\|_p$ the space $W^{1,p}(\mathbb{R}^n)$ becomes a Banach space. Here $\|f\|_p$ denotes the usual $L^p$ norm of $f$ in $\mathbb{R}^n$ and we write $\|\nabla f\|_p$ for

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|^p \, dx,$$

where $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. Since the quantity $\frac{np}{n-p}$ appears frequently when dealing with $L^p$ Sobolev–type inequalities, we will from now on write

$$p^* := \frac{np}{n-p}.$$

It will prove useful to define the so called *homogeneous Sobolev space* $\dot{W}^{1,p}(\mathbb{R}^n)$ via

$$\dot{W}^{1,p}(\mathbb{R}^n) = \{f \in L^{p^*}(\mathbb{R}^n) : \nabla f \in L^p(\mathbb{R}^n)\}.$$

We will see later on, that $W^{1,p}(\mathbb{R}^n) \subseteq \dot{W}^{1,p}(\mathbb{R}^n)$. This extension will be necessary, since many of the extremizers for Sobolev–type inequalities (see next chapter) are not in $W^{1,p}(\mathbb{R}^n)$, but in $\dot{W}^{1,p}(\mathbb{R}^n)$.

A similar extension from functions in $W^{1,1}(\mathbb{R}^n)$ to the so called space of *functions of bounded variation* $BV(\mathbb{R}^n)$ is also necessary. A function $f \in L^1(\mathbb{R}^n)$ belongs to $BV(\mathbb{R}^n)$ if for every $1 \le i \le n$, there exists a finite signed Radon measure $D_i f$ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} f \frac{\partial \xi}{\partial x_i} \, dx = -\int_{\mathbb{R}^n} \xi \, dD_i f \tag{2.16}$$

for all compactly supported $C^1$ functions $\xi$ on $\mathbb{R}^n$.

The *variation* $|Df|$ of the vector valued measure $Df = (D_1 f, \dots, D_n f)$ on $\mathbb{R}^n$ is the non–negative Radon measure whose value at a Borel set $L \subseteq \mathbb{R}^n$ is given by

$$|Df|(L) = \sup_\pi \sum_{A \in \pi} |Df(A)|,$$

where the supremum is taken over all partitions $\pi$ of $L$ into a countable number of disjoint measurable subsets. For $f \in BV(\mathbb{R}^n)$, let $\sigma_f$ denote the Radon-Nikodym derivative of $Df$ with respect to $|Df|$. Then, by (2.16),

$$\int_{\mathbb{R}^n} f \, \mathrm{div}\phi \, dx = -\int_{\mathbb{R}^n} \phi \cdot \sigma_f \, d|Df|$$

for all continuously differentiable vector fields $\phi$ on $\mathbb{R}^n$ with compact support.

A subset $L \subseteq \mathbb{R}^n$ is called a *set of finite perimeter* if $\mathbb{1}_L \in BV(\mathbb{R}^n)$. Its *reduced boundary* $\partial^* L$ is the set of points $x \in \mathbb{R}^n$, such that the limit

$$\nu_L(x) = \lim_{r\to 0} \frac{D\mathbb{1}_L(B_r(x))}{|D\mathbb{1}_L|(B_r(x))}$$

exists and such that $|\nu_L(x)| = 1$. Here we denote by $B_r(x)$ a ball with radius $r$ around $x \in \mathbb{R}^n$. $\nu_L(x)$ is called the *generalized* or *measure theoretic outer unit normal* to $L$ at $x \in \partial^* L$. With the same notation, we define the *density* of a measurable set $L$ at $x \in \mathbb{R}^n$ by

$$D(E, x) = \lim_{r\to 0} \frac{V(E \cap B_r(x))}{V(B_r(x))}.$$

We denote the level sets of a function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$[f]_t = \{x \in \mathbb{R}^n : f(x) \geq t\}.$$

If $f \in BV(\mathbb{R}^n)$ is non–negative, then $[f]_t$ is a set of finite perimeter and $\partial^*[f]_t$ coincides $\mathcal{H}^{n-1}$–a.e. with $\{x \in \mathbb{R}^n : f(x) = t\}$ for almost every $t \geq 0$. Moreover, (as pointed out in [ET04] in the proof of Theorem 4.1), we then have $\nu_{\partial^*[f]_t}(x) = \sigma_f(x)$ for $\mathcal{H}^{n-1}$–a.e. $x \in \partial^*[f]_t$.

We say that $x \in \mathbb{R}^n$ is a point of approximate continuity of $f$, if the limits

$$f_-(x) := \inf\{t : D(\{f > t\}, x) = 0\} \qquad \text{and} \qquad f_+(x) := \sup\{t : D(\{f < t\}, x) = 0\}$$

are finite and equal. If $x$ is a point of approximate continuity of $f$, then we set $\tilde{f}(x) := f_-(x) = f_+(x)$. Moreover, if $f$ is locally integrable, then we say $f$ is approximately differentiable at a point $x$ of approximate continuity, if there exists a point $\nabla f(x) \in \mathbb{R}^n$, such that

$$\lim_{r\to 0} \frac{1}{r^{n+1}} \int_{B_r(x)} |f(y) - \tilde{f}(x) - (y - x) \cdot \nabla f(x)| \, dy = 0.$$

We call $\nabla f$ the *approximate gradient of $f$ at $x$* and we denote all those points of approximate continutity of $f$, such that $f$ admits an approximate gradient $\nabla f(x)$ at $x$ and such that $\nabla f(x) = 0$, by $D_f^0$.

The *coarea formula* relates the integral of the gradient of a function $f$ over $\mathbb{R}^n$ with an average of integrals over their level sets. A general formulation for Lipschitz functions was first established by Federer [Fed59]. We state it here in a $BV(\mathbb{R}^n)$ version due to Fleming and Rishel [FR60].

**Theorem 2.3.** *Suppose that $g : \mathbb{R}^n \to [0, \infty]$ is a Borel function. If $f \in BV(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} g(\sigma_f)\, d|Df| = \int_{-\infty}^{\infty} \int_{\partial^*[f]_t} g(\sigma_f)\, d\mathcal{H}^{n-1}\, dt. \tag{2.17}$$

Note that if $[f]_t$ is a convex body, say $K_t$, then $\partial^* K_t = \partial K_t$ and by $\nu_{\partial^*[f]_t}(x) = \sigma_f(x)$ and (2.6) the inner integral on the right-hand side of (2.17) turns into

$$\int_{\partial K_t} g(\nu_{\partial K_t}(x))\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} g(u)\, dS(K_t, u).$$

# 3 Isoperimetric, Sobolev and Rearrangement Inequalities

Geometric and analytic inequalities of isoperimetric type are a very active research field and there is a vast amount of literature and ongoing research in this direction. In this chapter we recall many of the classical isoperimetric and Sobolev inequalities and we explore the deep connection between convex bodies and Sobolev functions as well as some recent inequalities established by Haberl and Schuster in [HS19]. To characterize all equality cases in such inequalities, the use of certain rearrangement inequalities, for instance the so called Pólya–Szegö principle, has proven beneficial. Finally, we recall a result by Lutwak, Yang and Zhang, which makes the connection between convex bodies and Sobolev functions more evident, by associating to each Sobolev function $f$ an origin symmetric convex body $\langle f\rangle_p$.

The classical results appearing here are treated in any basic literature on convex bodies or Sobolev functions (we recommend the books by Schneider [Sch14] and Evans [EG15], they have also been partly summarized in [KS20]). Proofs of the affine inequalities can be found in [Pet71,Gar06,Zha99,LYZ00,LYZ02]. The result by Haberl and Schuster [HS19] provides one of the most recent developments in this area and will be the motivation for the topics treated in Chapter 4, 5 and 6. Proofs of the (affine) Pólya–Szegö principles can be found in [CLYZ09,Wan13,Ngu16,Wan15]. The results from Chapter 5 will heavily rely on this literature. Finally, the method of convexification of functions was introduced by Lutwak, Yang and Zhang in [LYZ06] and further developed by Wang in [Wan12]. Both, Chapter 4 and Chapter 5 will make use of this method.

## 3.1 Isoperimetric and Sobolev Inequalities

Probably the most prominent example highlighting the deep connection between convex bodies and Sobolev functions is given by the *isoperimetric inequality* and the *sharp Sobolev inequality*. The first one is one of the cornerstones of convex geometry. In its classical form, it states that for any $K \in \mathcal{K}^n$ which is full dimensional

$$\frac{V(K)^{n-1}}{S(K)^n} \leq \frac{V(\mathbb{B}^n)^{n-1}}{S(\mathbb{B}^n)^n}, \tag{3.1}$$

where equality is attained if and only if $K$ is a ball. As shown in [DG58], (3.1) can be extended to sets of finite perimeter.

It is well known that the isoperimetric inequality (3.1) gives rise to the sharp $L^p$ Sobolev inequality, which goes back to Federer and Fleming [FF60] and Maz'ya [Maz60] for $p = 1$ and Aubin [Aub76] and Talenti [Tal76] for $p > 1$, and which states that

$$\left(\int_{\mathbb{R}^n} \|\nabla f(x)\|^p \, dx\right)^{1/p} \geq a_{n,p} \|f\|_{\frac{np}{n-p}} \tag{3.2}$$

for any $f \in W^{1,p}(\mathbb{R}^n)$, where the optimal constant is given by

$$a_{n,p} = n^{1/p} \left(\frac{n-p}{p-1}\right)^{1-1/p} \left(\frac{\omega_n \Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)}\right)^{1/n}.$$

While the explicit knowledge of the optimal constant in (3.2) has proven beneficial in certain areas of mathematical physics, its importance is far outweighed by the classification of the extremal functions. To get a clear picture of all the extremizers in (3.2), we remark that, in analogy to the extension of (3.1) to sets of finite perimeter, one can show that for $p = 1$, (3.2) can be extended to functions of bounded variation and then reads

$$\|Df\| \geq n\omega_n^{1/n} \|f\|_{\frac{n}{n-1}}. \tag{3.3}$$

In this setting, the extremizers of (3.3) are characteristic functions of Euclidean balls. Note that such functions do not belong to $W^{1,1}(\mathbb{R}^n)$, hence there are no extremizers in $W^{1,1}(\mathbb{R}^n)$ and consequently no functions attaining equality in (3.2) when $p = 1$. A similar problem arises when trying to identify all the extremizers for $p > 1$. Apparently, it was known for some time that with the help of a rearrangement inequality of Brothers and Ziemer [BZ88] all extremizers could be identified. However, the first explicit and selfcontained proof that equality holds in (3.2) for $p > 1$ if and only if there exist $a, b > 0$, and $x_0 \in \mathbb{R}^n$ such that

$$f(x) = \pm \left(a + b\|x - x_0\|^{p/(p-1)}\right)^{1-n/p} \tag{3.4}$$

was given by Cordero-Erausquin, Nazaret and Villani [CENV04] (and in a more general form). They also pointed out the disadvantage of considering inequality (3.2) merely for functions in $W^{1,p}(\mathbb{R}^n)$, since its extremizers do not belong to that space when $p \geq \sqrt{n}$. Instead, they showed that (3.2) holds for functions $f \in \dot{W}^{1,p}(\mathbb{R}^n)$. Note all the extremizers (that is functions of the form (3.4)) lie in $\dot{W}^{1,p}(\mathbb{R}^n)$.

Another characteristic feature of geometric and analytic inequalities is the set of transformations under which these inequalities are invariant, that is the set of transformations $T : \mathbb{R}^n \to \mathbb{R}^n$, such that replacing $K$ by $TK$ in the geometric setting or $f$ by $f \circ T$ in the analytic one, does not change the left-hand and right-hand side of the inequality. For (3.1) and (3.2) (or (3.3) when $p = 1$), this set of transformations is the set of rigid motions, that is the set of transformations $T$ such that

$$T(x) = \theta x + x_0, \qquad x \in \mathbb{R}^n,$$

for $\theta \in \mathrm{SO}(n)$ and $x_0 \in \mathbb{R}^n$. Inequalities, which are invariant under this Euclidean group of rigid motions are called *Euclidean inequalities*.

Surprisingly, in 1971 Petty [Pet71] established an inequality stronger than the isoperimetric inequality (3.1), while being not only Euclidean, but even *affine* in nature, that is invariant under $\mathrm{SL}(n)$ transformations (and translations). This inequality is nowadays known as the *Petty projection inequality* and states that

$$V(\Pi^\circ K)V(K)^{n-1} \leq V(\Pi^\circ \mathbb{B}^n)V(\mathbb{B}^n)^{n-1} \tag{3.5}$$

for any $K \in \mathcal{K}^n$. In accordance with its affine invariance, equality is attained in (3.5) if and only if $K$ is an ellipsoid, that is if and only if $K$ is an affine image of a ball.

It was a major breakthrough when in 1999, Zhang [Zha99] established the first *affine invariant* $L^1$ Sobolev inequality (nowadays known as the *affine Zhang–Sobolev inequality*), by replacing the length of the gradient in (3.2) (for $p = 1$) by an average of the length of 1-dimensional projections of the gradient, which leads to a significantly stronger inequality than (3.2). More precisely, Zhang showed that

$$\left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |u \cdot \nabla f(x)|\, dx\right)^{-n} du\right)^{-1/n} \geq \frac{1}{n^{1/n}} \frac{2\omega_{n-1}}{\omega_n} \, \|f\|_{\frac{n}{n-1}} \tag{3.6}$$

whenever $f \in C^1(\mathbb{R}^n)$ has compact support. This result was recently extended by Wang [Wan12] to functions of bounded variation. In this extended form, (3.6) reads

$$\left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |u \cdot \sigma_f|\, d|Df|\right)^{-n} du\right)^{-1/n} \geq \frac{1}{n^{1/n}} \frac{2\omega_{n-1}}{\omega_n} \|f\|_{\frac{n}{n-1}} \tag{3.7}$$

whenever $f \in BV(\mathbb{R}^n)$. Similarly to (3.5), equality is attained in (3.7) if and only if $f$ is a characteristic function of an ellipsoid.

As already mentioned, the Petty projection inequality (3.5) was crucial in the first proofs of the affine Zhang–Sobolev inequality (3.6) and (3.7). Conversely, (3.7) is a functional form of (3.5) in the sense that the choice of a suitable function $f$ in the Sobolev inequality (3.7) allows to recover the Petty projection inequality (3.5). The fundamental affine isoperimetric inequality for $L^p$ projection bodies was established by Lutwak, Yang, and Zhang [LYZ00] and is known as the $L^p$ *Petty projection inequality* (see also [BÏ3, CG02, HS09b, LYZ10] for alternative proofs). It states that if $1 \leq p < \infty$ and $K \in \mathcal{K}_0^n$, then

$$V(\Pi_p^\circ K)^p V(K)^{n-p} \leq \omega_n^n \tag{3.8}$$

with equality for $p > 1$ if and only if $K$ is an ellipsoid centered at the origin. Building on [Zha99] and the $L^p$ Petty projection inequality (3.8), Lutwak, Yang and Zhang [LYZ00] established the $L^p$ extension of (3.6), by showing that for $1 \leq p < n$,

$$\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |u \cdot \nabla f(x)|^p\, dx\right)^{-n/p} du\right)^{-1/n} \geq b_{n,p}\|f\|_{p^*} \tag{3.9}$$

whenever $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, where

$$b_{n,p} = \left(\frac{2\omega_{n+p-2}}{\omega_n\omega_{p-1}}\right)^{1/p} \left(\frac{n-p}{p-1}\right)^{1-1/p} \left(\frac{\omega_n\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)}\right)^{1/n}. \tag{3.10}$$

Note that (3.9) can be expressed in terms of an integration over the Grassmannian, namely

$$\left(\int_{\mathrm{Gr}_{n,1}} \left(\int_{\mathbb{R}^n} \|\nabla f(x)|E\|^p \, dx\right)^{-n/p} dE\right)^{-1/n} \geq b_{n,p}\|f\|_{p^*}, \tag{3.11}$$

where we write $\nabla f(x)|E$ for the orthogonal projection of $\nabla f(x)$ to $E \in \mathrm{Gr}_{n,1}$. Throughout, integration over $\mathrm{Gr}_{n,j}$ is with respect to the invariant probability measure on $\mathrm{Gr}_{n,j}$. Wang [Wan13] and, independently, Nguyen [Ngu16] proved that equality holds in (3.9) for $p > 1$ if and only if

$$f(x) = \pm\left(a + \|A(x - x_0)\|^{p/(p-1)}\right)^{1-n/p}, \tag{3.12}$$

for some $a > 0$, $A \in \mathrm{GL}(n)$, and $x_0 \in \mathbb{R}^n$.

Haberl and Schuster [HS19] recently introduced a new family of $L^p$ Minkowski valuations and showed that those $L^p$ Minkowski valuations satisfy an inequality in the same vein as (3.8). Moreover, they used these new geometric inequalities to establish a family of Sobolev–type inequalities similar to (3.2) and (3.3). To state their results, we fix a point $\bar{e} \in S^{n-1}$ and denote the group of rotations $\theta$, that fix $\bar{e}$ by $\mathrm{SO}(n-1)$. Next, recall that a measure $\mu$ on $S^{n-1}$ is said to be *zonal* if $\mu \circ \theta^{-1} = \mu$ for each $\theta \in \mathrm{SO}(n-1)$. Now let $\mu$ be an even and zonal measure on $S^{n-1}$ and let $Z_p^\mu$ be the $L^p$ zonoid of revolution generated by $\mu$ (with axis of revolution $\bar{e}$), that is

$$h(Z_p^\mu, u)^p = \int_{S^{n-1}} |u \cdot v|^p \, d\mu(v).$$

We denote by $Z_p^\mu(u)$ the rotated copy of $Z_p^\mu = Z_p^\mu(\bar{e})$ such that its axis of revolution is $u$. For $K \in \mathcal{K}_0^n$, define the convex body $\Phi_p^\mu K$ by

$$h(\Phi_p^\mu K, u)^p = \int_{S^{n-1}} h(Z_p^\mu(v), u)^p \, dS_p(K, v). \tag{3.13}$$

From here on we will use the shorthand notation $\Phi_p^{\mu,\circ} K = (\Phi_p^\mu K)^\circ$.

**Remark 3.1.** *Note that when choosing $\mu$ to be discrete, the zonality forces $\mu$ to be concentrated exactly on the two antipodal points $\bar{e}$ and $-\bar{e}$. But then, it is easily verified that $\Phi_p^\mu$ coincides with $\Pi_p$ up to a factor, so the projection body operator $\Pi_p$ is just a special member of the family $\Phi_p^\mu$ of $L^p$ Minkowski valuations.*

When choosing $\mu(S^{n-1}) = a_{n,p}$ (recall that the value of $a_{n,p}$ is given by (2.14)), then we have $\Phi_p^\mu \mathbb{B}^n = \mathbb{B}^n$. The family of geometric inequalities by Haberl and Schuster can now be stated as follows.

**Theorem 3.2.** *Suppose $1 \leq p < \infty$ and that $\mu$ is an even, zonal measure on $S^{n-1}$ such that $\mu(S^{n-1}) = a_{n,p}$. If $K \in \mathcal{K}_0^n$, then*

$$V(\Phi_p^{\mu,\circ} K)^p V(K)^{n-p} \leq \omega_n^n. \tag{3.14}$$

*If $\mu$ is not discrete, then equality in* (3.14) *is attained if and only if $K$ is a Euclidean ball centered at the origin. If $\mu$ is discrete, then equality in* (3.14) *is attained if and only if $K$ is an origin–symmetric ellipsoid.*

For $p = 1$, (3.14) can be extended to sets of finite perimeter and there is no more need to require that the origin is in the interior of $K$, thus equality is attained for balls (ellipsoids respectively), not necessarily origin–symmetric.

As a consequence of Theorem 3.2, Haberl and Schuster established a family of sharp $L^p$ Sobolev–type inequalities, which can be seen as the functional forms of (3.14) (in the same sense as (3.9) is a functional form of (3.8)).

**Theorem 3.3.** *Suppose that $1 < p < n$ and that $\mu$ is an even, zonal measure on $S^{n-1}$. If $Z_p^\mu(\bar{e})$ is the $L^p$ zonoid generated by $\mu$, then, for every $f \in \dot{W}^{1,p}(\mathbb{R}^n)$,*

$$\left( \frac{1}{n\omega_n} \int_{S^{n-1}} \left( \int_{\mathbb{R}^n} \|\nabla f\|^p_{Z_p^\mu(u)^\circ} \, dx \right)^{-n/p} du \right)^{-1/n} \geq b_{n,p} \mu(S^{n-1})^{1/p} \|f\|_{p^*}. \tag{3.15}$$

*If $\mu$ is not discrete, then equality in* (3.15) *is attained if and only if $f$ is of the form* (3.4)*. If $\mu$ is discrete, then equality in* (3.15) *is attained if and only if $f$ is of the form* (3.12)*.*

For $p = 1$, Haberl and Schuster extended Theorem 3.3 to functions of bounded variation, which is the statement of the next result.

**Theorem 3.4.** *Suppose that $\mu$ is an even, zonal measure on $S^{n-1}$. If $Z^\mu(\bar{e})$ is the zonoid generated by $\mu$, then, for every $f \in BV(\mathbb{R}^n)$,*

$$\left( \int_{S^{n-1}} \left( \int_{\mathbb{R}^n} \|\sigma_f\|_{Z^\mu(u)^\circ} \, d|Df| \right)^{-n} du \right)^{-1/n} \geq \frac{\mu(S^{n-1})}{n^{1/n}} \frac{2\omega_{n-1}}{\omega_n} \|f\|_{\frac{n}{n-1}}. \tag{3.16}$$

*Equality in* (3.16) *is attained if and only if $\mu$ is discrete and $f$ is the characteristic function of an ellipsoid or $\mu$ is not discrete and $f$ is the characteristic function of a ball.*

By Remark 3.1 we see that the affine Petty projection inequality (3.8) is just a special case of Theorem 3.2 and (3.9) a special case of Theorem 3.3. Similarly, choosing $\mu$ to be the spherical Lebesgue measure yields the isoperimetric inequality (3.1) and its functional form, the sharp Sobolev inequality (3.2). The equality cases directly tell, that only if $\mu$ is discrete, then (3.15) and (3.14) turn out to be affine invariant. Indeed, whenever $\mu$ is not discrete, inequalities (3.15) and (3.14) are merely Euclidean. Finally, Haberl and Schuster showed that this unique affine member of their family of inequalities is also the strongest one amongst this family.

**Theorem 3.5.** *Suppose $1 \leq p < \infty$. If $\mu$ is an even, zonal measure on $S^{n-1}$ such that $\mu(S^{n-1}) = a_{n,p}$ and $K \in \mathcal{K}_0^n$, then*

$$V(\Phi_p^{\mu,\circ} K) \leq V(\Pi_p^\circ K) \tag{3.17}$$

*with equality if and only if $\mu$ is discrete or $\Pi_p K$ is a Euclidean ball.*

Whereas the Petty projection inequality (3.5) deals with the volume of the polar projection body $V(\Pi^\circ K)$ of $K \in \mathcal{K}^n$, it was conjectured by Petty [Pet71], that a non–polar version also holds, namely

$$\frac{V(\Pi K)}{V(K)^{n-1}} \geq \frac{V(\Pi \mathbb{B}^n)}{V(\mathbb{B}^n)^{n-1}} \tag{3.18}$$

for each $K \in \mathcal{K}^n$ with equality if and only if $K$ is an ellipsoid. Inequality (3.18) is nowadays known as the *conjectured Petty projection inequality*. In Chapter 6 we consider a family of Minkowski valuations $\Phi_g$ and show the existence of minimizers of

$$\frac{V(\Phi_g K)}{V(K)^{n-1}}.$$

Note that the conjectured Petty projection inequality directly implies the polar one, since for $K \in \mathcal{K}_0^n$, by the *Blaschke–Santaló inequality*

$$V(K)V(K^\circ) \leq \omega_n^2 \tag{3.19}$$

we have

$$V(\Pi^\circ K)V(K)^{n-1} \leq \frac{\omega_n^2 V(K)^{n-1}}{V(\Pi K)} \leq \omega_n^n.$$

A generalization of the classical isoperimetric inequality holds for the intrinsic volumes $V_i$. If $1 \leq i \leq n$ and $K \in \mathcal{K}^n$, then

$$\left(\frac{V_n(K)}{V_n(\mathbb{B}^n)}\right)^{\frac{1}{n}} \leq \left(\frac{V_i(K)}{V_i(\mathbb{B}^n)}\right)^{\frac{1}{i}} \leq \left(\frac{V_1(K)}{V_1(\mathbb{B}^n)}\right). \tag{3.20}$$

For $i = n-1$, the left-hand inequality is exactly the isoperimetric inequality, while the outermost inequality (or the left-hand inequality for $i = 1$) is called *Urysohn's inequality*. Equality in (3.20) is attained exactly for balls.

## 3.2 Symmetrization of Functions and the Pólya–Szegö Principle

Clasically, the isoperimetric inequality is proved by applying a symmetrization to a convex body that leads to a new convex body with the same volume, but smaller surface area. Similarly, many functionals on $W^{1,p}(\mathbb{R}^n)$ decrease when a function $f$ is replaced by a suitable symmetrization of $f$ that maintains the $L^p$ norm of $f$. The classical symmetrization to consider is $f^*$, the *symmetric decreasing rearrangement* (also known as *Schwarz symmetrization*, *symmetric rearrangement* or *spherical decreasing rearrangement*) of $f$. We will see that when replacing $f$ with $f^*$, many functionals satisfy a certain rearrangement inequality, the so called Pólya–Szegö principle, which states that a functional $\mathcal{E}$ does not increase under rearranging its input, that is

$$\mathcal{E}f \geq \mathcal{E}f^*.$$

To state the important results that are going to be used in Chapter 5, we begin with the definition of $f^*$. Let $f \in W^{1,p}(\mathbb{R}^n)$ (or $f \in BV(\mathbb{R}^n)$) and define its *distribution function* $\mu_f : [0, \infty) \to [0, \infty]$ by

$$\mu_f(t) = V(\{x \in \mathbb{R}^n : |f(x)| > t\}).$$

Next, the *decreasing rearrangement* $\bar{f} : [0, \infty) \to [0, \infty]$ of $f$ is defined by

$$\bar{f}(s) = \inf\{t \geq 0 : \mu_f(t) \leq s\}.$$

Now let $K \in \mathcal{K}_0^n$ be origin–symmetric and denote by $\tilde{K} \in \mathcal{K}_0^n$ a dilate of $K$, such that $V(\tilde{K}) = V(\mathbb{B}^n)$. The *convex rearrangement of $f$ with respect to $K$* is defined as the function $f^K : \mathbb{R}^n \to [0, \infty]$ given by

$$f^K(x) = \bar{f}(\omega_n \|x\|_{\tilde{K}}^n).$$

Clearly, $f^{cK} = f^K$ for any $c > 0$. Moreover, the *equimeasurability* of the convex rearrangement assures that

$$V([f]_t) = V([f^K]_t), \qquad t \geq 0,$$

and, consequently, the convex rearrangement preserves the $L^p$ norm, that is

$$\|f\|_p = \left\|f^K\right\|_p.$$

The convex rearrangement of $f$ with respect to the Euclidean unit ball $\mathbb{B}^n$ is of particular interest, it is therefore denoted by $f^* := f^{\mathbb{B}^n}$ and called the symmetric decreasing rearrangement of $f$.

The classical Pólya–Szegö principle [PS51] states that whenever $f \in W^{1,p}(\mathbb{R}^n)$, then $f^* \in W^{1,p}(\mathbb{R}^n)$ and furthermore

$$\|\nabla f\|_p \geq \|\nabla f^*\|_p. \tag{3.21}$$

Clearly there is equality in (3.21) when $f = f^*$. However, to characterize all equality conditions in (3.21), Brothers and Ziemer [BZ88] noted that there are non–symmetric functions leading to equality in (3.21). For instance, they constructed a function that has no symmetries, that is such that $f$ and $f^*$ do not coincide almost everywhere, but such that $\|\nabla f\|_p = \|\nabla f^*\|_p$. To assure, that only symmetric functions give equality in (3.21), they showed that an additional assumption is required, namely

$$V(\{|\nabla f^*| = 0\} \cap \{0 < f^* < \text{ess sup } f\}) = 0. \tag{3.22}$$

Brothers and Ziemer showed that if (3.22) holds, then there is equality in (3.21) if and only if $f = f^*$ up to some translation.

Alvino, Ferone, Trombetti and Lions [AFTL97] established a generalization of the classical Pólya–Szegö principle for general Minkowski functionals. Esposito and Trombetti [ET04] characterized the equality cases in the sense of Brothers and Ziemer. We collect all these results in the following theorem.

**Theorem 3.6.** *Let $K \in \mathcal{K}_0^n$ and $1 < p < \infty$. If $f \in W^{1,p}(\mathbb{R}^n)$ is non-negative and compactly supported, then*

$$\int_{\mathbb{R}^n} h(K, \nabla f)^p \, dx \geq \int_{\mathbb{R}^n} h(K, \nabla f^K)^p \, dx. \tag{3.23}$$

*Moreover, if $f$ is such that*

$$V(\{|\nabla f^K| = 0\} \cap \{0 < f^K < \text{ess sup } f\}) = 0,$$

*then equality in (3.23) is attained if and only if $f = f^K$ up to some translation.*

A $BV(\mathbb{R}^n)$ version of (3.21) was first established by Cianchi and Fusco [CF02b] together with a Brothers–Ziemer type result. Moreover, they showed a generalization for general Minkowski functionals. Note that the gradient of $f$ is replaced by its approximate gradient in (3.22) (see Section 2.2 for definitions and notation).

**Theorem 3.7.** *Let $K \in \mathcal{K}_0^n$. If $f \in BV(\mathbb{R}^n)$ is non–negative and compactly supported, then*

$$\int_{\mathbb{R}^n} h(K, \sigma_f) \, d|Df| \geq \int_{\mathbb{R}^n} h(K, \sigma_{f^K}) \, d|Df^K|. \tag{3.24}$$

*Equality is attained in (3.24) if and only if the level sets $[f]_t$ are homothets of $K$ for almost every $t \geq 0$. Moreover, if $f$ is such that*

$$V(D^0_{f^K} \cap \{0 < f^K < \text{ess sup } f\}) = 0,$$

*then equality in (3.24) is attained if and only if $f = f^K$ up to some translation.*

Note that the functional considered in (3.21) (that is $\|\cdot\|_p$) is exactly the one appearing on the right-hand side of the classical Sobolev inequality (3.2). A natural next step is to consider the affine energy functional

$$\mathcal{E}_p^{\text{Aff}} f := c_{n,p} \left( \int_{S^{n-1}} \left( \int_{\mathbb{R}^n} |\nabla f(x) \cdot u|^p \, dx \right)^{-n/p} du \right)^{-1/n} \tag{3.25}$$

appearing in the affine Sobolev inequality (3.9), where the constant $c_{n,p}$ is defined by

$$c_{n,p} = (n\omega_n)^{1/n} \left( \frac{n\omega_n\omega_{p-1}}{2\omega_{n+p-2}} \right)^{1/p} \tag{3.26}$$

and chosen such that

$$\mathcal{E}_p^{\mathrm{Aff}} f^* = \|\nabla f^*\|_p. \tag{3.27}$$

The resulting *affine Pólya–Szegö principle* is an achievement by Cianchi, Lutwak, Yang and Zhang [CLYZ09] (see also [Lin17, Lin19, HSX12] for generalizations).

**Theorem 3.8.** *Suppose* $p \geq 1$. *If* $f \in W^{1,p}(\mathbb{R}^n)$, *then* $f^* \in W^{1,p}(\mathbb{R}^n)$ *and*

$$\mathcal{E}_p^{\mathrm{Aff}} f \geq \mathcal{E}_p^{\mathrm{Aff}} f^*. \tag{3.28}$$

The equality cases of (3.28), in the spirit of Brothers and Ziemer, where characterized later by Wang [Wan13], who used the solution of the functional $L^p$ Minkowski problem and, independently, by Nguyen [Ngu16], who used a new approach based on the Busemann–Petty centroid inequality, inspired by a recent paper of Haddad, Jimenez and Montenegro [HJM16]. The following theorem settles the equality cases.

**Theorem 3.9.** *Let* $1 \leq p < \infty$. *If* $f \in W^{1,p}(\mathbb{R}^n)$ *is a non–negative and compactly supported function such that* (3.22) *holds, then*

$$\mathcal{E}_p^{\mathrm{Aff}} f = \mathcal{E}_p^{\mathrm{Aff}} f^* \tag{3.29}$$

*if and only if there exists* $x_0 \in \mathbb{R}^n$ *such that* $f(x) = f^E(x + x_0)$ *almost everywhere in* $\mathbb{R}^n$, *where* $E$ *is an origin symmetric ellipsoid in* $\mathbb{R}^n$.

As already carried out in [CF02b] for the $BV(\mathbb{R}^n)$ versions of the classical Pólya–Szegö principle, when $p = 1$, all the extremizers in (3.28) can be characterized even when assumption (3.22) is omitted. Note however, that the resulting extremizers are not symmetric anymore: while their level sets are still homothets of a fixed ellipsoid, they do no longer need to be concentric.

**Theorem 3.10.** *If* $f \in W^{1,1}(\mathbb{R}^n)$ *is a non–negative and compactly supported function, then*

$$\mathcal{E}_1^{\mathrm{Aff}} f = \mathcal{E}_1^{\mathrm{Aff}} f^*$$

*if and only if the level sets* $[f]_t$ *are homothetic ellipsoids up to a set of* $\mathcal{L}^n$ *measure* 0, *for almost all* $t \geq 0$.

In [LYZ02] it is also shown that

$$\|\nabla f\|_p \geq \mathcal{E}_p^{\mathrm{Aff}} f,$$

thus, by (3.27), the affine Pólya–Szegö principle strengthens the classical Pólya–Szegö principle, giving another example where an affine inequality improves its Euclidean counterpart.

## 3.3 Convexification of Functions

The correspondence between geometric and analytic inequalities was made more evident by a new more conceptualized proof of the affine $L^p$ Sobolev inequality by Lutwak, Yang and Zhang [LYZ06], where they associate to each $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ a convex body $\langle f \rangle_p$. This convexification of a Sobolev function is the content of the following theorem.

**Theorem 3.11.** *If $1 \leq p < \infty$ and $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ is not identically $0$, then there exists a unique origin–symmetric convex body $\langle f \rangle_p$ with non-empty interior such that*

$$\int_{\mathbb{R}^n} g(\nabla f(x))^p \, dx = \frac{1}{V(\langle f \rangle_p)} \int_{S^{n-1}} g(u)^p \, dS_p \left(\langle f \rangle_p, u\right)$$

*for every even continuous function $g : \mathbb{R}^n \to [0, \infty)$ that is positively $1$-homogeneous.*

We call the convex body $\langle f \rangle_p$ defined by Theorem 3.11 the *LYZ–body* of $f$. In order to see how to apply Theorem 3.11 in the context of isoperimetric inequalities, note that if $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ and we define $K = V(\langle f \rangle_p)^{-1/(n-p)} \langle f \rangle_p$, then, by (2.13) and Theorem 3.11,

$$h(\Pi_p K, y)^p = \frac{\omega_{p-1}}{2\omega_{n+p-2}} \int_{\mathbb{R}^n} |\nabla f(x) \cdot y|^p \, dx, \qquad y \in \mathbb{R}^n. \tag{3.30}$$

Hence, by the polar coordinate formula for volume, the left-hand side of (3.9) coincides up to a constant with $V(\Pi_p^\circ K)^{-1/n}$. Consequently, the $L^p$ Petty projection inequality reduces the proof of the affine $L^p$ Sobolev inequality (3.9) to a sharp estimate of the volume $V(\langle f \rangle_p)$ in terms of $\|f\|_{p^*}$ (which was established in [LYZ06]). By similar arguments, the LYZ–body allows to easily establish analytic inequalities from geometric ones. In Chapter 4 we will proceed in this way, by first establishing a geometric inequality to obtain an analytic one via the LYZ–body.

In order to deal with functions $f \in BV(\mathbb{R}^n)$ we require the following extension of Theorem 3.11 by Wang [Wan12]. Originally, Wang used an equivalent formulation to the one below without volume normalization, however in order to deal with $BV(\mathbb{R}^n)$ functions analogously to $W^{1,p}(\mathbb{R}^n)$ functions without needing to adapt the proofs (too much), we will use a volume normalized definition similar to Theorem 3.11.

**Theorem 3.12.** *If $f \in BV(\mathbb{R}^n)$ is not identically $0$, then there exists a unique origin–symmetric convex body $\langle f \rangle$ with non-empty interior such that*

$$\int_{\mathbb{R}^n} g(\sigma_f(x)) \, d|Df|(x) = \frac{1}{V(\langle f \rangle)} \int_{S^{n-1}} g(u) \, dS\left(\langle f \rangle, u\right) \tag{3.31}$$

*for every even continuous function $g : \mathbb{R}^n \to \mathbb{R}$ that is positively $1$-homogeneous.*

Note that letting $K = V(\langle f \rangle)^{\frac{1}{n-1}} \langle f \rangle$ gives the original version of Theorem 3.12 from [Wan12] without volume normalization.

In the following statement we collect a few properties of $\langle f \rangle_p$ and $\langle f \rangle$, that will be used later on. The proofs can be found in [Wan12, Wan13].

**Proposition 3.13.** *Suppose $p \geq 1$. Let $K \in \mathcal{K}_0^n$ be an origin–symmetric convex body and denote by $T_\theta : \mathbb{R}^n \to \mathbb{R}^n$ a transformation of the form*

$$T_\theta(x) = \theta x + x_0$$

*for some $\theta \in \mathrm{SL}(n)$ and $x_0 \in \mathbb{R}^n$.*

*(a) If $f \in BV(\mathbb{R}^n)$, then $\langle \mathbb{1}_K \rangle = K$.*

*(b) If $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then $\langle f \circ T_\theta^{-1} \rangle_p = \theta \langle f \rangle_p$.*

*(c) If $f \in BV(\mathbb{R}^n)$, then $\langle f \circ T_\theta^{-1} \rangle = \theta \langle f \rangle$.*

In case the level sets of $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ or $BV(\mathbb{R}^n)$ are all homothets of a fixed convex body (and concentric if $p > 1$), this body's shape is recovered by $\langle f \rangle_p$ or $\langle f \rangle$, respectively. The proofs of the next results can be found in [Wan13].

**Proposition 3.14.** *Suppose $p > 1$ and let $K \in \mathcal{K}_0^n$ be an origin–symmetric convex body. Moreover, let $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ be non–negative.*

*(a) $\langle f^K \rangle_p$ is a dilate of $K$.*

*(b) If $L \in \mathcal{K}_0^n$ is origin–symmetric, then*

$$V(\langle f^K \rangle_p) = V(\langle f^L \rangle_p). \tag{3.32}$$

*(c) $\langle f \rangle_p \subseteq \langle f^{\langle f \rangle_p} \rangle_p$ and equality holds if and only if $f = f^{\langle f \rangle_p}$ almost everywhere.*

For $f \in BV(\mathbb{R}^n)$ we can do a little better than Proposition 3.14, however the corresponding results have (as far as we know) not been treated yet. Although the proofs are very similar to those given in [Wan13], for the sake of completeness and since $\langle f \rangle$ (in contrast to $\langle f \rangle_p$) is translation invariant in each level set of $f$, which leads to a bit more relaxed results, we will collect the analogous statements to Proposition 3.14 in the next proposition and give the proofs.

**Proposition 3.15.** *Suppose $K \in \mathcal{K}_0^n$ is an origin–symmetric convex body and $f \in BV(\mathbb{R}^n)$ is non–negative.*

*(a) If the level sets $[f]_t$ are homothetic to $K$ for almost every $t \geq 0$, then $\langle f \rangle$ is a dilate of $K$.*

*(b) If $L \in \mathcal{K}_0^n$ is origin–symmetric, then*

$$V(\langle f^K \rangle) = V(\langle f^L \rangle). \tag{3.33}$$

*(c) $\langle f \rangle \subseteq \langle f^{\langle f \rangle} \rangle$ and equality holds if and only if the level sets $[f]_t$ are homothetic to $\langle f \rangle$ for almost every $t \geq 0$.*

*Proof.* First note that for $\mathcal{H}^{n-1}$–a.e. $x \in \partial^*[f]_t$, $\sigma_f(x)$ is the outer unit normal $\nu_{\partial^*[f]_t}(x)$ of $\partial^*[f]_t$ at $x$ for almost every $t \geq 0$ (see Section 2.2). By (3.31) and the coarea formula (2.17), we have

$$\begin{aligned}\frac{1}{V(\langle f\rangle)}\int_{S^{n-1}} g(u)\,dS\,(\langle f\rangle,u) &= \int_{\mathbb{R}^n} g(\sigma_f)\,d|Df| \\ &= \int_0^\infty \int_{\partial^*[f]_t} g(\sigma_f)\,d\mathcal{H}^{n-1}(x)\,dt \\ &= \int_0^\infty \int_{\partial^*[f]_t} g(\nu_{\partial^*[f]_t}(x))\,d\mathcal{H}^{n-1}(x)\,dt\end{aligned}$$

for every continuous, even and positively 1–homogeneous function $g : \mathbb{R}^n \to [0,\infty)$. If we choose $g = h(M,\cdot)$ for an arbitrary origin–symmetric convex body $M \in \mathcal{K}_0^n$, we obtain

$$\begin{aligned}\frac{1}{n}\frac{1}{V(\langle f\rangle)}\int_{S^{n-1}} h(M,u)\,dS\,(\langle f\rangle,u) &= \frac{1}{n}\int_0^\infty \int_{\partial^*[f]_t} h(M,\nu_{\partial^*[f]_t}(x))\,d\mathcal{H}^{n-1}(x)\,dt \\ &= \int_0^\infty V(M,[f]_t[n-1])\,dt,\end{aligned} \tag{3.34}$$

where the last equality is due to (2.5) and (2.6). Now let $f$ as in $(a)$. Since $V(M,[f]_t[n-1])$ is translation invariant in $[f]_t$, we can translate all the level sets $[f]_t$ such that they are concentric and therefore dilates (not only homothets) of $K$. Thus, for each level set $[f]_t$ we find a constant $c(t) > 0$ depending on $t$, such that

$$V(M,[f]_t[n-1]) = V(M,c(t)K[n-1]) = c(t)^{n-1}V(M,K[n-1]) \tag{3.35}$$

and consequently

$$\begin{aligned}\frac{V(M,\langle f\rangle[n-1])}{V(\langle f\rangle)} &= \int_0^\infty V(M,[f]_t[n-1])\,dt \\ &= V(M,K[n-1])\int_0^\infty c(t)^{n-1}\,dt \\ &= c_f V(M,K[n-1])\end{aligned} \tag{3.36}$$

where the constant $c_f > 0$ is given by $c_f = \int_0^\infty c(t)^{n-1}\,dt$. Now let $\tilde{K}$ be a suitable dilate of $K$, namely

$$\tilde{K} = \frac{K}{V(K)c_f^{n-1}}. \tag{3.37}$$

Plugging $\tilde{K}$ into (3.36) yields

$$\frac{V(M,\langle f\rangle[n-1])}{V(\langle f\rangle)} = c_f V(M,K[n-1]) = \frac{V(M,\tilde{K}[n-1])}{V(\tilde{K})}$$

for every origin–symmetric $M \in \mathcal{K}_0^n$. An application of Lemma (2.1) finally shows that $\langle f\rangle = \tilde{K}$ and proves statement $(a)$.

To prove $(b)$, we first show that the constants $c(t)$ from (3.36) (and hence $c_f$) do not depend on $K$. For instance, since $f^K = f^{cK}$ for any $c > 0$, we can freely choose the volume of $K$ and $L$ and thus assume both $K$ and $L$ to have the same volume. If we choose $c_K(t)$ such that $[f^K]_t = c_K(t)K$ and analogously $c_L(t)$ such that $[f^L]_t = c_L(t)L$, then the equimeasurability of convex rearrangement yields

$$c_K(t)^n V(K) = V([f^K]_t) = V([f^L]_t) = c_L(t)^n V(L)$$

and thus

$$c_K(t) = c_L(t) = c(t). \tag{3.38}$$

Now, denoting by $\tilde{K}$ and $\tilde{L}$ the convex bodies contructed via (3.37), which, as we have just shown, coincide with $\langle f^K \rangle$ and $\langle f^L \rangle$, we conclude

$$V(\langle f^K \rangle) = V(\tilde{K}) = \left(c_f^n V(K)\right)^{1-n} = \left(c_f^n V(L)\right)^{1-n} = V(\tilde{L}) = V(\langle f^L \rangle).$$

Finally, in order to prove $(c)$, note that, by $(a)$, there exists a constant $c > 0$, such that $\langle f^{\langle f \rangle} \rangle = c\langle f \rangle$. It remains to show, that $c \geq 1$ and $c = 1$ if and only if $[f]_t$ is a homothet of $\langle f \rangle$ for almost every $t \geq 0$. By choosing $M = \langle f \rangle$ in (3.34), we obtain

$$\begin{aligned} 1 = \frac{V(\langle f \rangle)}{V(\langle f \rangle)} &= \frac{1}{n} \frac{1}{V(\langle f \rangle)} \int_{S^{n-1}} h(\langle f \rangle, u)\, dS(\langle f \rangle, u) \\ &= \int_0^\infty V(\langle f \rangle, [f]_t[n-1])\, dt \\ &\geq \int_0^\infty V(\langle f \rangle)^{\frac{1}{n}} V([f]_t)^{\frac{n-1}{n}}\, dt, \end{aligned}$$

where the last inequality is due to Minkowskis inequality (2.7) and thus equality is attained if and only if $[f]_t$ is a homothet of $\langle f \rangle$ (up to some set of measure 0) for almost every $t \geq 0$. Hence, replacing $f$ with $f^{\langle f \rangle}$ gives equality and, by using the equimeasurability of convex symmetrization, we finish the proof by

$$\begin{aligned} 1 &\geq \int_0^\infty V(\langle f \rangle)^{\frac{1}{n}} V([f]_t)^{\frac{n-1}{n}}\, dt \\ &= \frac{1}{c} \int_0^\infty V(c\langle f \rangle)^{\frac{1}{n}} V([f^{\langle f \rangle}]_t)^{\frac{n-1}{n}}\, dt \\ &= \frac{1}{c} \int_0^\infty V(\langle f^{\langle f \rangle} \rangle)^{\frac{1}{n}} V([f^{\langle f \rangle}]_t)^{\frac{n-1}{n}}\, dt = \frac{1}{c}, \end{aligned}$$

where we have equality if and only if $[f]_t$ is a homothet of $\langle f \rangle$ (up to some set of measure 0) for almost every $t \geq 0$. □

# 4 Sobolev–type inequalities over the Grassmannian

The impact of the affine $L^p$ Sobolev inequality (3.9) is virtually unparalleled in convex geometric analysis, as it constitutes the seminal result in a rapidly evolving theory of affine analytic inequalities (see, e.g., [CLYZ09, HS09a, HSX12, DNHJM18, HJM19, Lin17, LXZ11]). Among this theory's most recent achievements is the large family of sharp $L^p$ Sobolev inequalities (3.15) by Haberl and Schuster [HS19] that had not just the classical inequality (3.2) and the affine $L^p$ Sobolev inequality (3.9) as special cases but also an $(n-1)$-dimensional counterpart to (3.9): If $1 \leq p < n$ and $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then

$$\left( \int_{\mathrm{Gr}_{n,n-1}} \left( \int_{\mathbb{R}^n} \|\nabla f(x)|E\|^p \, dx \right)^{-n/p} dE \right)^{-1/n} \geq b_{n,p} \, \|f\|_{p^*} \tag{4.1}$$

with equality for $p > 1$ if and only if $f(x)$ is of the form (3.4). The value of the optimal constant $b_{n,p}$ is given by (3.10). In the case $p = 1$, inequality (4.1) was extended to $BV(\mathbb{R}^n)$ in [HS19] and it was shown that equality holds precisely for characteristic functions of Euclidean balls. While (4.1) is *not* affine invariant, it was proved in [HS19] that it is stronger and directly implies the classical $L^p$ Sobolev inequality (3.2) in the same way as the affine $L^p$ Sobolev inequality (3.9) does. However, among these three inequalities the affine invariant one was shown in [HS19] to be the strongest one.

Comparing inequalities (3.2), (3.9), and (4.1) raises two natural questions:

- Is there a family of (sharp) $L^p$ Sobolev inequalities obtained by averaging the length of *i-dimensional* gradient projections that unifies (3.2), (3.11), and (4.1)?
- How are these gradient projection Sobolev inequalities related to each other?

In this chapter, we are going to answer both of these questions. In order to state our results it is convenient to introduce the following notation.

**Definition 4.1.** Suppose that $1 \leq i \leq n$ and $1 \leq p < n$. For $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, we define

$$\mathcal{E}_{i,p}(f) = \left( \int_{\mathrm{Gr}_{n,i}} \left( \frac{2\omega_{i+p-2}}{i\omega_i\omega_{p-1}} \int_{\mathbb{R}^n} \|\nabla f(x)|E\|^p \, dx \right)^{-n/p} dE \right)^{-1/n}. \tag{4.2}$$

In analogy to [HS19], we will extend our results for $p = 1$ to functions $f \in BV(\mathbb{R}^n)$. For this particular case, we introduce the following notation.

**Definition 4.2.** Suppose that $1 \leq i \leq n$. For $f \in BV(\mathbb{R}^n)$, we define

$$\mathcal{E}_i(f) = \left( \int_{\mathrm{Gr}_{n,i}} \left( \frac{2\omega_{i-1}}{i\omega_i} \int_{\mathbb{R}^n} \|\sigma_f|E\| \, d|Df| \right)^{-n} dE \right)^{-1/n}, \tag{4.3}$$

where $|Df|$ denotes the variation measure of $Df$ and $\sigma_f$ the Radon–Nikodym derivative of $Df$ with respect to $|Df|$.

Note that for $f \in \dot{W}^{1,1}(\mathbb{R}^n)$, $\mathcal{E}_i f$ and $\mathcal{E}_{i,1} f$ coincide. We will actually prove a Sobolev inequality for energy functionals more general than $\mathcal{E}_{i,p}$ (and $\mathcal{E}_i f$ respectively), where the Euclidean norm of the gradient projection in (4.2) can be replaced by any norm whose unit ball is a polar zonoid in $E$ (see Chapter 2 for definitions). The results presented here will appear in [KS20].

## 4.1 Auxiliary results

Here, we first recall how to lift integration of functions and measures on the homogeneous spaces $S^{n-1}$ and $\mathrm{Gr}_{n,i}$ to the Lie group $\mathrm{SO}(n)$ and use this in the second part to prove the underlying geometric inequality behind our main results in this chapter. Recall that the unit sphere $S^{n-1}$ is a homogeneous space with respect to the action of $\mathrm{SO}(n)$. Therefore, $S^{n-1}$ is diffeomorphic to $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ and there is a one-to-one correspondence between functions and measures on $S^{n-1}$ and right $\mathrm{SO}(n-1)$ invariant measures on $\mathrm{SO}(n)$. More precisely, if $\mu$ is a measure on $S^{n-1}$, then there exists a unique right $\mathrm{SO}(n-1)$ invariant measure $\breve{\mu}$ on $\mathrm{SO}(n)$ such that

$$\int_{S^{n-1}} f(u) \, d\mu(u) = \int_{\mathrm{SO}(n)} f(\phi e_n) \, d\breve{\mu}(\phi) \tag{4.4}$$

for every $f \in C(S^{n-1})$. In other words, the pushforward of $\breve{\mu}$ under the natural projection $\pi : \mathrm{SO}(n) \to S^{n-1}$, $\pi(\phi) = \phi e_n$, is $\mu$ (see, e.g., [GZ99, HS19] for more details).

From now on, let $\{e_1, \ldots, e_n\}$ denote a fixed orthonormal basis of $\mathbb{R}^n$ and for $1 \leq i \leq n$, let $E_i \in \mathrm{Gr}_{n,i}$ and $S^{i-1} \subseteq S^{n-1}$ be given by

$$E_i = \mathrm{span}\{e_1, \ldots, e_i\} \qquad \text{and} \qquad S^{i-1} = S^{n-1} \cap E_i.$$

We write $\mathrm{SO}(i)$ for the subgroup of $\mathrm{SO}(n)$ which leaves $E_i$ invariant and acts as the identity on $E_i^{\perp}$. Note that for $2 \leq i \leq n$, $\mathrm{SO}(i)$ acts transitively on $S^{i-1}$ and that

$$\mathrm{SO}(1) \subseteq \mathrm{SO}(2) \subseteq \cdots \subseteq \mathrm{SO}(n-1) \subseteq \mathrm{SO}(n).$$

Since, similarly, for $2 \leq i \leq n$, $S^{i-1}$ is diffeomorphic to $\mathrm{SO}(i)/\mathrm{SO}(i-1)$, any measure on $S^{n-1}$ whose support is concentrated on $S^{i-1}$ may be lifted either to a right $\mathrm{SO}(n-1)$ invariant measure on $\mathrm{SO}(n)$ or to a right $\mathrm{SO}(i-1)$ invariant measure on $\mathrm{SO}(i)$. In particular, we make frequent use of the fact that if $\sigma_i$ denotes the restriction of the $(i-1)$-dimensional Hausdorff measure to $S^{i-1}$, then

$$\int_{S^{n-1}} f(u)\,d\sigma_i(u) = i\omega_i \int_{\mathrm{SO}(i)} f(\phi e_i)\,d\phi \tag{4.5}$$

for every $f \in C(S^{n-1})$, where integration on the right is with respect to the Haar probability measure on $\mathrm{SO}(i)$.

Since the Lie group $\mathrm{SO}(n)$ also acts transitively on $\mathrm{Gr}_{n,i}$ for every $1 \leq i \leq n-1$, the Grassmannian $\mathrm{Gr}_{n,i}$ is diffeomorphic to $\mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$, where the subgroup $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ is the stabilizer of $E_i$ in $\mathrm{SO}(n)$. Thus, we can also lift integration with respect to measures on $\mathrm{Gr}_{n,i}$ to the group $\mathrm{SO}(n)$. Specifically, we have (as for the sphere $S^{n-1}$) that for every $f \in C(\mathrm{Gr}_{n,i})$,

$$\int_{\mathrm{Gr}_{n,i}} f(E)\,dE = \int_{\mathrm{SO}(n)} f(\phi E_i)\,d\phi. \tag{4.6}$$

Let us turn to $L^p$ zonoids. For $p \geq 1$ and an even measure $\mu$ on $S^{n-1}$, the support function of the $L^p$ zonoid $Z_p^\mu$ generated by $\mu$ can be written, by (2.15) and (4.4), as

$$h(Z_p^\mu, x)^p = \int_{\mathrm{SO}(n)} |x \cdot \phi e_n|^p \,d\breve{\mu}(\phi), \qquad x \in \mathbb{R}^n. \tag{4.7}$$

Since for $x \in \mathbb{R}^n$, we have

$$\|x\|^p = h(\mathbb{B}^n, x)^p = \frac{\omega_{p-1}}{2\omega_{n+p-2}} \int_{S^{n-1}} |x \cdot u|^p \,d\sigma_n(u) = \frac{n\omega_n\omega_{p-1}}{2\omega_{n+p-2}} \int_{\mathrm{SO}(n)} |x \cdot \phi e_n|^p \,d\phi,$$

we see that the Euclidean unit ball $\mathbb{B}^n$ is an $L^p$ zonoid for any $p \geq 1$. More general, if we denote by $D^i = \mathbb{B}^n \cap E_i$ the $i$-dimensional unit ball in $E_i$, then, by (4.5),

$$\|x|E_i\|^p = h(D^i, x)^p = \frac{\omega_{p-1}}{2\omega_{i+p-2}} \int_{S^{n-1}} |x \cdot u|^p \,d\sigma_i(u) = \frac{i\omega_i\omega_{p-1}}{2\omega_{i+p-2}} \int_{\mathrm{SO}(i)} |x \cdot \phi e_i|^p \,d\phi, \tag{4.8}$$

which shows that also $D^i$ is an $L^p$ zonoid for every $2 \leq i \leq n$ and any $p \geq 1$. When $i = 1$, we have $h(D^1, x)^p = h([-e_1, e_1], x)^p = |x \cdot e_1|^p = \|x|E_1\|^p$, that is, $D^1$ is also an $L^p$ zonoid for any $p \geq 1$.

For $1 \leq i \leq n$ and $p \geq 1$, we now define

$$q_{i,p} = \frac{2\omega_{i+p-2}}{i\omega_i\omega_{p-1}} \qquad \text{and} \qquad \nu_{i,p} = i\omega_i q_{i,p}\,\sigma_i.$$

Then the measure $\nu_{i,p}$ on $S^{n-1}$ is concentrated on $S^{i-1}$ and, by (4.8), the $L^p$ zonoid $D_p^i$ generated by $\nu_{i,p}$ satisfies for every $x \in \mathbb{R}^n$,

$$h(D_p^i, x)^p = \int_{S^{n-1}} |x \cdot u|^p \,d\nu_{i,p}(u) = \int_{\mathrm{SO}(i)} |x \cdot \phi e_i|^p \,d\phi = q_{i,p}\,\|x|E_i\|^p. \tag{4.9}$$

Noting that $D_p^1 = [-e_1, e_1]$ for every $p \geq 1$ and $h(D_p^1, x) = |x \cdot e_1| = \|x|E_1\|$, we conclude from the invariance of the Haar measure on $\mathrm{SO}(i)$ and (4.9) that for any $2 \leq i \leq n$ and every $x \in \mathbb{R}^n$,

$$q_{i,p}\,\|x|E_i\|^p = \int_{\mathrm{SO}(i)} |x \cdot \phi e_i|^p \,d\phi = \int_{\mathrm{SO}(i)} |x \cdot \phi e_1|^p \,d\phi = \int_{\mathrm{SO}(i)} \|x|\phi E_1\|^p \,d\phi. \tag{4.10}$$

With the next lemma, we generalize (4.10) by proving a useful relation between the length of $j$-dimensional projections in terms of averages of the length of $i$-dimensional projections, when $i < j$.

**Lemma 4.3.** *If $p \geq 1$ and $1 \leq i < j \leq n$, then*

$$q_{j,p}\,\|x|E_j\|^p = q_{i,p} \int_{\mathrm{SO}(j)} \|x|\phi E_i\|^p \,d\phi$$

*for every $x \in \mathbb{R}^n$.*

*Proof.* First note that, by (4.9), the desired relation is equivalent to

$$h(D_p^j, x)^p = \int_{\mathrm{SO}(j)} h(\phi D_p^i, x)^p \,d\phi. \tag{4.11}$$

In order to prove (4.11), we use (2.1) and a combination of (4.9) and (4.10) to see that

$$\int_{\mathrm{SO}(j)} h(\phi D_p^i, x)^p \,d\phi = \int_{\mathrm{SO}(j)} h(D_p^i, \phi^{-1}x)^p \,d\phi = \int_{\mathrm{SO}(j)} \int_{\mathrm{SO}(i)} h(\theta D_p^1, \phi^{-1}x)^p \,d\theta \,d\phi.$$

Thus, from an application of Fubini's theorem, (2.1), and the fact that $\mathrm{SO}(i) \subseteq \mathrm{SO}(j)$ as well as the invariance of the Haar measure on $\mathrm{SO}(j)$, we obtain

$$\int_{\mathrm{SO}(j)} h(\phi D_p^i, x)^p \,d\phi = \int_{\mathrm{SO}(i)} \int_{\mathrm{SO}(j)} h(\phi\theta D_p^1, x)^p \,d\phi \,d\theta = \int_{\mathrm{SO}(j)} h(\phi D_p^1, x)^p \,d\phi.$$

Finally, another application of (4.9) and (4.10) completes the proof of (4.11). □

## 4.2 A Grassmannian Isoperimetric Inequality

With the help of Lemma 4.3, we can now prove a geometric inequality which is critical for our proofs of the Sobolev–type inequalities over the Grassmannian in the next section.

**Theorem 4.4.** *Suppose that $1 \leq i < j \leq n$ and $p \geq 1$. If $K \in \mathcal{K}_0^n$, then*

$$\int_{\mathrm{Gr}_{n,j}} \left( q_{j,p} \int_{S^{n-1}} \|u|E\|^p \,dS_p(K,u) \right)^{-n/p} dE \leq \int_{\mathrm{Gr}_{n,i}} \left( q_{i,p} \int_{S^{n-1}} \|u|F\|^p \,dS_p(K,u) \right)^{-n/p} dF. \tag{4.12}$$

*Proof.* By Lemma 4.3 and Fubini's theorem, we have that, for any fixed $\theta \in \mathrm{SO}(n)$,

$$\left(q_{j,p}\int_{S^{n-1}}\|u|\theta E_j\|^p\,dS_p(K,u)\right)^{-n/p} = \left(q_{i,p}\int_{\mathrm{SO}(j)}\int_{S^{n-1}}\|u|\theta\phi E_i\|^p\,dS_p(K,u)\,d\phi\right)^{-n/p}.$$

Hence, by Jensen's inequality,

$$\left(q_{j,p}\int_{S^{n-1}}\|u|\theta E_j\|^p\,dS_p(K,u)\right)^{-n/p} \le \int_{\mathrm{SO}(j)}\left(q_{i,p}\int_{S^{n-1}}\|u|\theta\phi E_i\|^p\,dS_p(K,u)\right)^{-n/p} d\phi.$$

Integrating now both sides of this inequality with respect to the Haar probability measure on $\mathrm{SO}(n)$, followed by an application of Fubini's theorem and the invariance of the Haar measure on the right-hand side, we obtain

$$\int_{\mathrm{SO}(n)}\left(q_{j,p}\int_{S^{n-1}}\|u|\theta E_j\|^p\,dS_p(K,u)\right)^{-n/p} d\theta \le \int_{\mathrm{SO}(n)}\left(q_{i,p}\int_{S^{n-1}}\|u|\theta E_i\|^p\,dS_p(K,u)\right)^{-n/p} d\theta$$

which concludes the proof by (4.6). □

## 4.3 Sharp Sobolev–type Inequalities via Projection Averages

Let us start this section, by giving an answer to the second of the questions, asked in the introduction of this chapter. The next theorem shows, that the functionals $\mathcal{E}_{i,p}$ and $\mathcal{E}_i$ form a decreasing sequence.

**Theorem 4.5.** *Suppose that $1 \le i \le n$ and $1 \le p < \infty$. If $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then*

$$\mathcal{E}_{n,p}(f) \ge \mathcal{E}_{n-1,p}(f) \ge \cdots \ge \mathcal{E}_{2,p}(f) \ge \mathcal{E}_{1,p}(f). \tag{4.13}$$

*Analogously, if $f \in BV(\mathbb{R}^n)$, then*

$$\mathcal{E}_n(f) \ge \mathcal{E}_{n-1}(f) \ge \cdots \ge \mathcal{E}_2(f) \ge \mathcal{E}_1(f). \tag{4.14}$$

*Proof.* First, suppose that $1 \le p < \infty$ and that $f \in \dot{W}^{1,p}(\mathbb{R}^n)$. We may also assume that $f$ is not identically 0. Next, note that by taking

$$K = \langle f\rangle_p, \tag{4.15}$$

it follows from Theorem 3.11 that for $1 \le i \le n$ and $E \in \mathrm{Gr}_{n,i}$,

$$q_{i,p}\int_{\mathbb{R}^n}\|\nabla f(x)|E\|^p\,dx = q_{i,p}\frac{1}{V(K)}\int_{S^{n-1}}\|u|E\|^p\,dS_p(K,u).$$

Hence, definition (4.2) and Theorem 4.4 yield that for $1 \le i < j \le n$,

$$\mathcal{E}_{j,p} \ge \mathcal{E}_{i,p}$$

which proves (4.13).

The inequalities from (4.14) for $f \in BV(\mathbb{R}^n)$ not identically 0, follow from similar arguments, by taking $K = \langle f\rangle$ and applying Theorems 3.12 and 4.4. Alternatively, (4.14) can also be deduced from (4.13) by an approximation argument. □

Note that a combination of inequalities (4.13) with the affine $L^p$ Sobolev inequality (3.9), directly implies the following family of $L^p$ Sobolev inequalities: if $1 \leq i \leq n$, $1 \leq p < n$ and $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then

$$\mathcal{E}_{i,p}(f) \geq c_{n,p} \|f\|_{p^*}, \tag{4.16}$$

where

$$c_{n,p} = \left(\frac{2\omega_{n+p-2}}{\omega_n \omega_{p-1}}\right)^{1/p} \left(\frac{n-p}{p-1}\right)^{1-1/p} \left(\frac{\omega_n \Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)}\right)^{1/n}.$$

Similarly, inequalities (4.14) and Wang's extension of the affine $L^1$ Sobolev inequality to $BV(\mathbb{R}^n)$ yield the following family of Sobolev inequalities: if $1 \leq i \leq n$ and $f \in BV(\mathbb{R}^n)$, then

$$\mathcal{E}_i(f) \geq \frac{2\omega_{n-1}}{\omega_n^{1-1/n}} \|f\|_{\frac{n}{n-1}}. \tag{4.17}$$

However, in both cases the equality conditions remain to be settled.

In the following, we are not going to approach the characterization of extremal functions in (4.16) and (4.17) directly, but rather establish generalizations of these inequalities that are motivated by the fact that in the classical $L^p$ Sobolev inequality (3.2) the Euclidean norm of the gradient can be replaced by an arbitrary norm on $\mathbb{R}^n$ (see, e.g., [CENV04]). To this end, let $1 \leq i \leq n-1$, $1 \leq p < n$ and suppose that $\mu$ is an even measure on $S^{n-1}$ such that $\operatorname{span}\operatorname{supp}\mu = E_i$. For $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, we now define

$$\mathcal{E}_{i,p}^{\mu}(f) = \left(\int_{\mathrm{Gr}_{n,i}} \left(\int_{\mathbb{R}^n} \|\nabla f(x)|E\|_{Z_p^{\mu}(E)^\circ}^p \, dx\right)^{-n/p} dE\right)^{-1/n}, \tag{4.18}$$

where $Z_p^{\mu}$ denotes again the $L^p$ zonoid generated by $\mu$ (see Section 2).

The problem of whether a version of inequalities (4.16) and (4.17) also holds for more general norms than the Euclidean one was first raised by Ludwig.

**Open Problem 4.6.** *Suppose that $1 \leq i \leq n-1$, $1 \leq p < \infty$ and let $f \in \dot{W}^{1,p}(\mathbb{R}^n)$. For which $i$–dimensional and origin symmetric convex bodies $L \subset E_i$ does a sharp inequality of the form*

$$\left(\int_{\mathrm{Gr}_{n,i}} \left(\int_{\mathbb{R}^n} \|\nabla f(x)|E\|_{L(E)^\circ}^p \, dx\right)^{-n/p} dE\right)^{-1/n} \geq c_{n,i,p}(L)\, \|f\|_{p^*},$$

*where $c_{n,i,p}$ is some optimal constant, hold?*

With our next result, we establish a chain of inequalities similar to (4.13), with which we answer this question in the affirmative, when $p \geq 1$ and the unit ball of the norm is a polar $L^p$ zonoid. Later in Theorem 4.10 we will show the corresponding version for general norms for functions of bounded variation. Moreover, we characterize all equality cases in this general setting.

**Theorem 4.7.** *Suppose that $1 \leq i \leq n-1$, $1 \leq p < \infty$, and that $\mu$ is an even measure on $S^{n-1}$ such that* $\operatorname{span}\operatorname{supp}\mu = E_i$ *and $\mu(S^{n-1}) = 1$. If $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then*

$$\mathcal{E}_{n,p}(f) \geq \mathcal{E}^{\mu}_{i,p}(f) \geq \mathcal{E}_{1,p}(f) \tag{4.19}$$

*with equality in the right-hand inequality if and only if $\Pi_p\langle f\rangle_p$ is a ball.*

*Proof.* We may assume that $f$ is not identically 0. Since $1 \leq i \leq n-1$, we have, by (2.3) and (4.6),

$$\int_{\mathrm{Gr}_{n,i}} \left( \int_{\mathbb{R}^n} \|\nabla f(x)|E\|^p_{Z^{\mu}_p(E)^{\circ}}\, dx \right)^{-n/p} dE = \int_{\mathrm{SO}(n)} \left( \int_{\mathbb{R}^n} h(\phi Z^{\mu}_p, \nabla f(x))^p\, dx \right)^{-n/p} d\phi. \tag{4.20}$$

Hence, by definition (4.18) and Jensen's inequality,

$$\mathcal{E}^{\mu}_{i,p}(f) \leq \left( \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} h(\phi Z^{\mu}_p, \nabla f(x))^p\, dx\, d\phi \right)^{1/p}.$$

Thus, by (2.1) and (4.7), Fubini's theorem, the invariance of the Haar measure on $\mathrm{SO}(n)$, and the fact that $\mu(S^{n-1}) = \breve{\mu}(\mathrm{SO}(n)) = 1$, we have

$$\begin{aligned}
\mathcal{E}^{\mu}_{i,p}(f) &\leq \left( \int_{\mathbb{R}^n} \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} |\nabla f(x) \cdot \phi\psi e_n|^p\, d\phi\, d\breve{\mu}(\psi)\, dx \right)^{1/p} \\
&= \left( \int_{\mathbb{R}^n} \int_{\mathrm{SO}(n)} |\nabla f(x) \cdot \phi e_n|^p\, d\phi\, dx \right)^{1/p} \\
&= \left( q_{n,p} \int_{\mathbb{R}^n} \|\nabla f(x)\|^p\, dx \right)^{1/p} \\
&= \mathcal{E}_{n,p}(f),
\end{aligned}$$

which proves the left-hand inequality in (4.19).

In order to prove the right-hand inequality in (4.19), we use as before (4.20), (2.1), and (4.7), followed by Fubini's theorem, to see that

$$\mathcal{E}^{\mu}_{i,p}(f) = \left( \int_{\mathrm{SO}(n)} \left( \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} |\nabla f(x) \cdot \phi\psi e_n|^p\, dx\, d\breve{\mu}(\psi) \right)^{-n/p} d\phi \right)^{-1/n}. \tag{4.21}$$

Since $\breve{\mu}(\mathrm{SO}(n)) = 1$, an application of Jensen's inequality therefore shows that

$$\mathcal{E}^{\mu}_{i,p}(f) \geq \left( \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} \left( \int_{\mathbb{R}^n} |\nabla f(x) \cdot \phi\psi e_n|^p\, dx \right)^{-n/p} d\breve{\mu}(\psi)\, d\phi \right)^{-1/n}. \tag{4.22}$$

Finally, Fubini's theorem, the invariance of the Haar measure on SO($n$), (4.6), and the fact that $q_{1,p}=1$, yield the desired inequality

$$\begin{aligned}\mathcal{E}^{\mu}_{i,p}(f) &\geq \left(\int_{\mathrm{SO}(n)}\int_{\mathrm{SO}(n)}\left(\int_{\mathbb{R}^n}|\nabla f(x)\cdot\phi\psi e_n|^p\,dx\right)^{-n/p}d\phi\,d\breve{\mu}(\psi)\right)^{-1/n}\\ &= \left(\int_{\mathrm{SO}(n)}\left(\int_{\mathbb{R}^n}\|\nabla f(x)|\phi E_1\|^p\,dx\right)^{-n/p}d\phi\right)^{-1/n}\\ &= \mathcal{E}_{1,p}(f).\end{aligned}$$

Note that for $1\leq p<n$, $h(Z^{\mu}_p,x)^p$ is a positive multiple of $\|x|E_1\|^p$ when $i=1$, thus $\mathcal{E}^{\mu}_{1,p}f$ and $\mathcal{E}_{1,p}f$ coincide in this case.

It remains to show that equality in the second inequality in (4.19) implies $\Pi_p\langle f\rangle_p$ is a ball, when $2\leq i\leq n-1$. In order to see this, let $K=\langle f\rangle_p$ and note that to have equality, we must have equality in (4.22), or, equivalently, by (2.13),

$$\left(\int_{S^{n-1}}h(\Pi_pK,\phi v)^p\,d\mu(v)\right)^{-n/p}=\int_{S^{n-1}}h(\Pi_pK,\phi v)^{-n}\,d\mu(v) \tag{4.23}$$

for every $\phi\in\mathrm{SO}(n)$. By the equality conditions of Jensen's inequality, (4.23) holds if and only if for each $\phi\in\mathrm{SO}(n)$, there exists a $c_\phi>0$ such that

$$h(\Pi_pK,\phi v)=c_\phi \text{ for } \mu\text{-a.e. } v\in S^{n-1}.$$

Thus, since $2\leq i\leq n-1$ and $\operatorname{span}\operatorname{supp}\mu=E_i$, there exist (at least) two linearly independent unit vectors $u_1,u_2\in S^{n-1}$ such that for each $\phi\in\mathrm{SO}(n)$,

$$h(\Pi_pK,\phi u_1)=h(\Pi_pK,\phi u_2)=c_\phi. \tag{4.24}$$

Let $t=u_1\cdot u_2$ and denote by $H_{u_1,t}=\{x\in\mathbb{R}^n: u_1\cdot x=t\}$. Then, $-1<t<1$ and for $w\in S^{n-1}\cap H_{u_1,t}$, there exists $\vartheta\in\mathrm{SO}(n)$ such that $\vartheta u_1=u_1$ and $\vartheta u_2=w$. Replacing $\phi$ by $\phi\vartheta$ in (4.24), thus yields

$$c_{\phi\vartheta}=h(\Pi_pK,\phi\vartheta u_1)=h(\Pi_pK,\phi\vartheta u_2)=h(\Pi_pK,\phi u_1)=h(\Pi_pK,\phi w)=c_\phi.$$

Since $w\in S^{n-1}\cap H_{u_1,t}$ was arbitrary, we see that for each $\phi\in\mathrm{SO}(n)$, there exists $c_\phi>0$ such that

$$h(\Pi_pK,\phi v)=c_\phi \text{ for all } v\in S^{n-1}\cap H_{u_1,t}$$

or, equivalently,

$$h(\Pi_pK,u)=c_\phi \text{ for all } u\in S^{n-1}\cap H_{\phi u_1,t}. \tag{4.25}$$

In particular, by choosing $\phi$ to be the identity, we obtain

$$h(\Pi_pK,u)=c_{\mathrm{id}} \text{ for all } u\in S^{n-1}\cap H_{u_1,t}.$$

Now, if we choose $\phi$ in (4.25) such that $S^{n-1}\cap H_{u_1,t}$ and $S^{n-1}\cap H_{\phi u_1,t}$ have non-empty intersection, then it follows that $c_{\mathrm{id}}=c_\phi$. But, since we can reach any point on $S^{n-1}$ by finitely many iterations of this procedure, we obtain $h(\Pi_pK,u)=c_{\mathrm{id}}$ for all $u\in S^{n-1}$, that is, $\Pi_pK$ is a ball as desired. $\square$

The extension of (4.16) to general norms and the characterization of the equality conditions is now a consequence of the above theorem.

**Theorem 4.8.** *Suppose that $1 \leq i \leq n-1$, $1 \leq p < n$, and that $\mu$ is an even measure on $S^{n-1}$ such that* $\operatorname{span}\operatorname{supp}\mu = E_i$. *If $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then*

$$\mathcal{E}^{\mu}_{i,p}(f) \geq \mu(S^{n-1})^{1/p} c_{n,p} \|f\|_{p^*} \tag{4.26}$$

*with equality for $p > 1$, if and only if $f(x)$ has the form* (3.4) *when $i > 1$, and if and only if $f(x)$ has the form* (3.12) *when $i = 1$.*

*Proof.* Note that for $1 \leq p < n$, the homogeneity of $\mathcal{E}^{\mu}_{i,p}$ and a combination of the right-hand inequality in (4.19) with the affine $L^p$ Sobolev inequality (3.9), yield inequality (4.26). Moreover, since $h(Z^{\mu}_p, x)^p$ is a positive multiple of $\|x|E_1\|^p$ when $i = 1$, we see that (4.26) reduces to (3.9) in this case. In particular, equality holds in (4.26) when $i = 1$ if and only if $f(x)$ has the form (3.12). It remains to settle the equality conditions for (4.26) when $2 \leq i \leq n-1$. To this end, note that by (4.19), any function of the form (3.4) must be an extremizer of (4.26). In order to show the converse, we may assume again that $f$ is not identically 0 and that $\mu(S^{n-1}) = 1$. Now, since, equality in (4.26) implies equality in the right-hand inequality of (4.19), we must have that $\Pi_p\langle f\rangle_p$ is a Euclidean ball and equality must also hold in (3.9), that is, $f(x)$ is of the form (3.12). The latter implies that there exists an origin–symmetric ellipsoid $E \subseteq \mathbb{R}^n$ and $x_1 \in \mathbb{R}^n$ such that $f(x) = f^E(x + x_1)$ for a.e. $x \in \mathbb{R}^n$ (cf. proof of Corollary 4.1 in [Ngu16]). Hence, by Proposition 3.14 (a), $\langle f\rangle_p$ is a dilate of $E = A\mathbb{B}^n$ for suitable $A \in \mathrm{GL}(n)$. However, since

$$\Pi_p(A\mathbb{B}^n) = |\det A|^{1/p} A^{-\mathrm{T}}\Pi_p\mathbb{B}^n = |\det A|^{1/p} A^{-\mathrm{T}}\mathbb{B}^n$$

for any $A \in \mathrm{GL}(n)$ (cf. [LYZ00]) and $\Pi_p\langle f\rangle_p$ is a ball, $E$ must be a Euclidean ball as well. This implies that $f$ is an extremizer of (3.2) (see, e.g., [Wan13]). □

Let us note that the special case of Theorem 4.8, where $\mu$ is taken to be the $(i-1)$-dimensional Hausdorff measure on $S^{i-1}$ normalized such that $\mu(S^{i-1}) = q_{i,p}^{-1}$, gives the following Sobolev–type inequality together with its equality cases, which answers the first question asked in the introduction of this chapter.

**Theorem 4.9.** *Suppose that $1 \leq i \leq n$ and $1 \leq p < n$. If $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then*

$$\mathcal{E}_{i,p}(f) \geq c_{n,p}\|f\|_{p^*}, \tag{4.27}$$

*where*

$$c_{n,p} = \left(\frac{2\omega_{n+p-2}}{\omega_n\omega_{p-1}}\right)^{1/p}\left(\frac{n-p}{p-1}\right)^{1-1/p}\left(\frac{\omega_n\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)}\right)^{1/n}.$$

*For $p > 1$, equality holds if and only if $f(x)$ has the form* (3.4) *when $i > 1$, and if and only if $f(x)$ has the form* (3.12) *when $i = 1$.*

Next, we want to emphasize that by (4.19), each of the $L^p$ Sobolev inequalities from (4.26) is stronger than the classical inequality (3.2) and that, in turn, the strongest one among them is the affine $L^p$ Sobolev inequality (3.9). Finally, let us remark that it is an open problem whether an inequality like (4.26) still holds, when the $L^p$ zonoids $Z_p^\mu$ in the definition of $\mathcal{E}_{i,p}^\mu$ are replaced by more general convex bodies.

With our next result, we establish an extension of Theorem 4.8 for the case $p = 1$ to functions of bounded variations.

**Theorem 4.10.** *Suppose that* $1 \leq i \leq n-1$ *and that* $\mu$ *is an even measure on* $S^{n-1}$ *such that* $\operatorname{span}\operatorname{supp}\mu = E_i$*. If* $f \in BV(\mathbb{R}^n)$*, then*

$$\mathcal{E}_i^\mu(f) := \left( \int_{\mathrm{Gr}_{n,i}} \left( \int_{\mathbb{R}^n} \|\sigma_f|E\|_{Z^\mu(E)^\circ}^p \, d|Df| \right)^{-n} dE \right)^{-1/n} \geq \frac{2\omega_{n-1}\mu(S^{n-1})}{\omega_n^{1-1/n}} \|f\|_{\frac{n}{n-1}} \quad (4.28)$$

*with equality if and only if* $f$ *is the multiple of a characteristic function of a ball when* $i > 1$ *and that of an ellipsoid when* $i = 1$*. Moreover, if* $1 \leq p < \infty$ *and* $\mu(S^{n-1}) = 1$*, then*

$$\mathcal{E}_n(f) \geq \mathcal{E}_i^\mu(f) \geq \mathcal{E}_1(f) \quad (4.29)$$

*where equality is attained in the right-hand inequality if and only if* $\Pi\langle f\rangle$ *is a ball.*

*Proof.* Since the proof of (4.29) is almost verbatim the same as that of (4.19) (basically, by replacing $\nabla f$ by $\sigma_f$), we will not repeat it here. Having established (4.29), the Sobolev inequalities (4.28) follow from Wang's extension of the affine Zhang–Sobolev inequality. Moreover, when $i = 1$, inequality (4.28) reduces to the affine Zhang–Sobolev inequality on $BV(\mathbb{R}^n)$.

In order to settle the equality conditions for (4.28) when $2 \leq i \leq n-1$, note that by (4.29) and the (extended) affine Zhang–Sobolev inequality, $f$ must be a multiple of the characteristic function of an ellipsoid $E = A\mathbb{B}^n$ for suitable $A \in \mathrm{GL}(n)$. Since, by Proposition 3.13 (a), we have $\langle \mathbb{1}_E \rangle = E$, we infer that $\Pi E$ must be a ball. But, since

$$\Pi(A\mathbb{B}^n) = |\det A| A^{-\mathrm{T}} \Pi \mathbb{B}^n = \omega_{n-1} |\det A| A^{-\mathrm{T}} \mathbb{B}^n$$

for any $A \in \mathrm{GL}(n)$ (cf. [Gar06, Theorem 4.1.5]), $f$ must actually be a multiple of the characteristic function of a ball. □

Our last result of this section is a sharp Sobolev–type inequality together with its equality cases, that is an extension of Theorem 4.9 for the case $p = 1$ to functions of bounded variations. It is a direct consequence of Wang's $BV(\mathbb{R}^n)$ extension of the affine Zhang–Sobolev inequality, together with Theorem 4.10, where, again as in the proof of Theorem 4.9, $\mu$ is chosen to be the $(i-1)$–dimensional Hausdorff measure on $S^{i-1}$, normalized such that $\mu(S^{i-1}) = q_{i,1}^{-1}$.

**Theorem 4.11.** *Suppose that $1 \leq i \leq n$. If $f \in BV(\mathbb{R}^n)$, then*

$$\mathcal{E}_i(f) \geq \frac{2\omega_{n-1}}{\omega_n^{1-1/n}} \|f\|_{\frac{n}{n-1}} \tag{4.30}$$

*with equality if and only if $f$ is the multiple of a characteristic function of a ball when $i > 1$ and that of an ellipsoid when $i = 1$.*

## 4.4 Moser–Trudinger and Morrey–Sobolev type Inequalities over the Grassmannian

In the last sections we established Sobolev–type inequalities for $\mathcal{E}_{i,p}$ when $1 \leq p < n$. A natural next step is to consider the limiting value $p = n$ and the superlimiting values $p > n$ and, similarly to the classical Sobolev inequalities, consider bounds for the norms $\|\nabla f\|_p$. In the first case, the resulting inequalities are so called Moser–Trudinger inequalities, whereas the second case leads to so called Morrey–Sobolev inequalities.

In this section we will establish the Moser–Trudinger and Morrey–Sobolev inequalities for the energies $\mathcal{E}_{i,p}$ and $\mathcal{E}_{i,p}^{\mu}$. All the inequalities appearing in this section are a direct consequence of (4.19).

*Moser–Trudinger type inequalities*

Let us start with the limiting case $p = n$ of the classical $L^p$ Sobolev inequality. A classical result by Moser and Trudinger [Tru67,Mos71] (see also [CLYZ09,HSX12,Wan15] for extensions) states that there exists a constant $c(n)$, such that

$$\frac{1}{\operatorname{supp}(f)} \int_{\operatorname{supp}(f)} \exp\left(\frac{n\omega_n^{1/n}|f(x)|}{\|\nabla f\|_n}\right) dx \leq c(n) \tag{4.31}$$

for every $f \in W^{1,n}(\mathbb{R}^n)$ such that $0 < V(\operatorname{supp}(f)) < \infty$. Here, the constant $n\omega_n^{1/n}$ is optimal in the sense that inequality (4.31) would fail for any real number $\tilde{c}(n)$ if $n\omega_n^{1/n}$ were to be replaced by a larger number.

If we denote by $\mathcal{G}$ the set of all nondecreasing and locally absolutely continuous functions $g$ on $[0, 1]$, then the optimal constant $c(n)$ can be computed via

$$c(n) = \sup_{g \in \mathcal{G}} \int_0^\infty \exp(g(t)^{n/(n-1)} - t)\, dt.$$

The affine Moser-Trudinger inequality by Cianchi, Lutwak Yang and Zhang [CLYZ09] is a stronger version of (4.31) and states that

$$\frac{1}{\operatorname{supp}(f)} \int_{\operatorname{supp}(f)} \exp\left(\frac{n\omega_n^{1/n}|f(x)|}{\mathcal{E}_p^{\mathrm{Aff}} f}\right) dx \leq c(n) \tag{4.32}$$

for every $f \in W^{1,n}(\mathbb{R}^n)$ such that $0 < V(\mathrm{supp}(f)) < \infty$. By applying Theorem 4.7, we can prove a Moser–Trudinger type inequality over the Grassmannian.

**Corollary 4.12.** *Suppose* $1 \leq i \leq n$ *and let* $f \in W^{1,n}(\mathbb{R}^n)$, *such that* $0 < V(\mathrm{supp}(f)) < \infty$. *With the same constant* $c(n)$ *as above, the inequality*

$$\frac{1}{\mathrm{supp}(f)} \int_{\mathrm{supp}(f)} \exp\left(\frac{n\omega_n^{1/n}|f(x)|}{\mathcal{E}_{i,p}f}\right) dx \leq c(n)$$

*holds. The constant* $n\omega_n^{1/n}$ *is optimal in the sense that this inequality would fail for any real number* $\tilde{c}(n)$ *if* $n\omega_n^{1/n}$ *were to be replaced by a larger number.*

*Proof.* The Moser–Trudinger inequality for the affine energy $\mathcal{E}_p^{\mathrm{Aff}}$ together with (4.19) gives

$$\begin{aligned} c(n) &\geq \frac{1}{\mathrm{supp}(f)} \int_{\mathrm{supp}(f)} \exp\left(\frac{n\omega_n^{1/n}|f(x)|}{\mathcal{E}_p^{\mathrm{Aff}}f}\right) dx \\ &\geq \frac{1}{\mathrm{supp}(f)} \int_{\mathrm{supp}(f)} \exp\left(\frac{n\omega_n^{1/n}|f(x)|}{\mathcal{E}_{i,p}f}\right) dx. \end{aligned}$$

□

*Morrey–Sobolev type inequalities*

For $p > n$, the sharp Morrey–Sobolev embedding theorem [Tal94] states that every function in $W^{1,p}(\mathbb{R}^n)$ is essentially bounded, where the optimal bound is given by

$$\|f\|_\infty \leq d_{n,p} V(\mathrm{supp}(f))^{1/n-1/p} \|\nabla f\|_p \tag{4.33}$$

for any $f \in W^{1,p}(\mathbb{R}^n)$ such that $V(\mathrm{supp}(f)) < \infty$. Here, the optimal constant $d_{n,p}$ is given by

$$d_{n,p} = n^{-1/p} \omega_n^{-1/n} \left(\frac{p-1}{p-n}\right)^{1/p^*},$$

where, as usual, $p^* = \frac{np}{n-p}$. The affine version of (4.33) was established by Cianchi, Lutwak, Yang and Zhang [CLYZ09] and states that

$$\|f\|_\infty \leq d_{n,p} V(\mathrm{supp}(f))^{1/n-1/p} \mathcal{E}_p^{\mathrm{Aff}} f \tag{4.34}$$

for any $f \in W^{1,p}(\mathbb{R}^n)$ such that $V(\mathrm{supp}(f)) < \infty$, with the same optimal constant $d_{n,p}$ as in (4.33). Equality in (4.34) holds whenever

$$f(x) = a(1 - |A(x - x_x)^{\frac{p-n}{p-1}}|)_+, \tag{4.35}$$

for some $a \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $A \in \mathrm{GL}(n)$. Here, the subscript "+" denotes the positive part. The affine Morrey–Sobolev inequality immediately implies a Grasmann version.

**Corollary 4.13.** *Let* $1 \leq i \leq n$, $p > n$ *and* $f \in W^{1,p}(\mathbb{R}^n)$. *If* $V(\text{supp}(f)) < \infty$, *then*

$$\|f\|_\infty \leq d_{n,p} V(\text{supp}(f))^{1/n-1/p} \mathcal{E}_{i,p} f. \tag{4.36}$$

*Equality in* (4.36) *holds whenever*

$$f(x) = a(1 - |b(x - x_0)|^{\frac{n-p}{p-1}})_+ \tag{4.37}$$

*for some* $a, b \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$.

*Proof.* The inequality is a direct consequence of (4.34) and Theorem 4.7. To obtain the desired equality cases, note that for any function $f$ of the form (4.37), $\Pi_p\langle f\rangle_p$ is a ball. Thus, combining the equality cases in Theorem 4.7 with (4.35) finishes the proof. □

# 5 Generalized Pólya–Szegö Principles and other Rearrangement Inequalities

A Pólya–Szegö principle refers to any kind of inequality that assures that some energy functional $\mathcal{E}$ of a function $f \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$, does not increase under symmetric decreasing rearrangement of $f$. In the last chapter, we established Moser–Trudinger and Morrey–Sobolev inequalities for the family $\mathcal{E}_{i,p}$ of energy functionals. Such inequalities are traditionally proved by establishing a Pólya–Szegö principle for the energy functionals first, and then using this principle and the fact that the $L^p$ norm of $f$ does not change under rearrangement, to obtain a large class of further Sobolev–type or rearrangement inequalities. Moreover, Pólya–Szegö principles are used to solve a wide range of variational problems (see [Kaw85, Kaw86, Kes06, Tal93] and the references therein).

We have already seen in Chapter 3 how energy functionals exhibiting a Sobolev–type inequality are natural candidates for a Pólya–Szegö principle, such as the classical energy $\|\nabla f\|_p$, giving rise to the classical Pólya–Szegö principle, or the energy functional $\mathcal{E}_p^{\mathrm{Aff}} f$, leading to the affine Pólya–Szegö principle. Here we will consider two families of energy functionals. The first family consists of the energy functionals

$$\mathcal{E}_p^\mu f := \left( \frac{1}{n} \int_{S^{n-1}} \left( \int_{\mathbb{R}^n} \|\nabla f\|^p_{Z_p^\mu(u)^\circ} \right)^{-n/p} du \right)^{-1/n}, \tag{5.1}$$

appearing in the Sobolev–type inequalities (3.15) and (3.16) by Haberl and Schuster [HS19]. The second one,

$$\mathcal{E}_{i,p}^\mu f = \left( \int_{\mathrm{Gr}_{n,i}} \left( \int_{\mathbb{R}^n} \|\nabla f(x)|E\|^p_{Z_p^\mu(E)^\circ} \, dx \right)^{-n/p} dE \right)^{-1/n},$$

was introduced in the last Chapter.

In this chapter, we are going to establish the Pólya–Szegö principles for $\mathcal{E}_p^\mu$ and $\mathcal{E}_{i,p}^\mu$, namely

$$\mathcal{E}_p^\mu f \geq \mathcal{E}_p^\mu f^* \qquad \text{and} \qquad \mathcal{E}_{i,p}^\mu f \geq \mathcal{E}_{i,p}^\mu f^*, \tag{5.2}$$

for $f \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$, and $f \in BV(\mathbb{R}^n)$, together with the characterization of extremal functions in (5.2) in the sense of Brothers and Ziemer, that is we will show that, assuming

$$V(\{|\nabla f^*| = 0\} \cap \{0 < f^* < \text{ess sup} \, f\}) = 0, \tag{5.3}$$

equality in (5.2) holds if and only if $f$ satisfies certain symmetries.

We will achieve this via two different approaches, one exploiting directly the underlying isoperimetric inequality

$$V(\Phi_p^{\mu,\circ}K) \leq V(\Phi_p^{\mu,\circ}\mathbb{B}^n) \tag{5.4}$$

if $V(K) = V(\mathbb{B}^n)$, the other one relying on the affine bound

$$\mathcal{E}_{i,p}^{\mu} f \geq \mathcal{E}_p^{\mathrm{Aff}} f. \tag{5.5}$$

While the second proof is a bit easier, using the already known affine Pólya–Szegö principle and its corresponding Brothers–Ziemer type characterization of the equality cases, the first proof does not assume any kind of relation as in (5.5). We will introduce a family of Minkowski valuations in Chapter 6, for which the corresponding functional energy does not necessarily satisfy relation (5.5) and thus for which the second approach is more beneficial.

All the results presented here are not published yet, but are going to appear in [Kni20].

## 5.1 The Pólya–Szegö Principle for $\mathcal{E}_p^{\mu}$

In this section we will establish the Pólya–Szegö principle for the energy functionals $\mathcal{E}_p^{\mu}$. Note that we could use the same proof as for Theorem 5.5 given in the next section. However, we want to give a proof here which does not depend on the bound via the affine energy, that is

$$\mathcal{E}_p^{\mu} f \geq \mathcal{E}_p^{\mathrm{Aff}} f,$$

or on its geometric counterpart, namely

$$V(\Phi_p^{\mu,\circ}K) \leq V(\Pi_p^{\mu,\circ}K).$$

Moreover, since for $p = 1$ the affine Pólya–Szegö principle was shown only for functions $f \in W^{1,1}(\mathbb{R}^n)$ [CLYZ09, Wan13], this procedure allows us to prove it for $f \in BV(\mathbb{R}^n)$ (although the proof given in [Wan13] can be adapted easily for $f \in BV(\mathbb{R}^n)$). To this end, we are going to use a method introduced by Wang in [Wan13] and [Wan15]. Note that the results from this section can also be shown via a different approach, based on the method by Nguyen [Ngu16] (as was pointed out to us in a private communication with V.H. Nguyen).

Let us start by showing how to write $\mathcal{E}_p^{\mu} f$ in terms of $V(\Phi_p^{\mu,\circ}K)$.

**Lemma 5.1.** *Suppose that $1 \leq p < \infty$ and that $\mu$ is an even, zonal measure on $S^{n-1}$. If $f \in W^{1,p}(\mathbb{R}^n)$, then*

$$\mathcal{E}_p^{\mu} f = \frac{V(\Phi_p^{\mu,\circ}\langle f\rangle_p)^{-1/n}}{V(\langle f\rangle_p)^{1/p}}.$$

*Proof.* By the polar coordinate formula (2.9) and the definition of $\Phi_p^\mu$ (3.13), we have

$$\begin{aligned} V(\Phi_p^{\mu,\circ}\langle f\rangle_p) &= \frac{1}{n}\int_{S^{n-1}} h(\Phi_p^\mu\langle f\rangle_p, u)^{-n}\, du \\ &= \frac{1}{n}\int_{S^{n-1}} \left(\int_{S^{n-1}} h(Z_p^\mu(u), v)^p\, dS_p(\langle f\rangle_p, v)\right)^{-n/p} du. \end{aligned}$$

Hence, Theorem 3.11 yields

$$\begin{aligned} V(\Phi_p^{\mu,\circ}\langle f\rangle_p)^{-1/n} &= \left(\frac{1}{n}\int_{S^{n-1}} \left(V(\langle f\rangle_p)\int_{\mathbb{R}^n} h(Z_p^\mu(u), \nabla f)^p\, dx\right)^{-n/p} du\right)^{-1/n} \\ &= V(\langle f\rangle_p)^{1/p}\, \mathcal{E}_p^\mu f. \end{aligned}$$

□

We are now in a position to prove the Pólya–Szegö principle and a Brothers–Ziemer type result for $\mathcal{E}_p^\mu$.

**Theorem 5.2.** *Suppose that $1 \le p < \infty$ and that $\mu$ is an even, zonal measure on $S^{n-1}$. If $f \in W^{1,p}(\mathbb{R}^n)$ is non–negative and compactly supported, then*

$$\mathcal{E}_p^\mu f \ge \mathcal{E}_p^\mu f^*. \tag{5.6}$$

*Furthermore, if (5.3) holds, then there is equality in (5.6) if and only if $\mu$ is discrete and*

$$f(x) = f^E(x + x_0) \qquad a.e.$$

*for some $x_0 \in \mathbb{R}^n$ and some ellipsoid $E \subset \mathbb{R}^n$, or if $\mu$ is not discrete and*

$$f(x) = f^*(x + x_0) \qquad a.e.$$

*for some $x_0 \in \mathbb{R}^n$.*

*Proof.* Let us first note, that the map

$$K \mapsto \frac{V(\Phi_p^{\mu,\circ}K)^{-1/n}}{V(K)^{1/p}}$$

is $(-1)$–homogeneous, thus, by Proposition 3.14 (a), we have

$$\frac{V(\Phi_p^{\mu,\circ}\langle f\rangle_p)^{-1/n}}{V(\langle f\rangle_p)^{1/p}} = \frac{V(\Phi_p^{\mu,\circ}\frac{1}{c}\langle f^{\langle f\rangle_p}\rangle_p)^{-1/n}}{V(\frac{1}{c}\langle f^{\langle f\rangle_p}\rangle_p)^{1/p}} = c\,\frac{V(\Phi_p^{\mu,\circ}\langle f^{\langle f\rangle_p}\rangle_p)^{-1/n}}{V(\langle f^{\langle f\rangle_p}\rangle_p)^{1/p}}$$

for some $c \ge 1$. Consequently,

$$\begin{aligned} \mathcal{E}_p^\mu f &= \frac{V(\Phi_p^{\mu,\circ}\langle f\rangle_p)^{-1/n}}{V(\langle f\rangle_p)^{1/p}} \\ &\ge \frac{V(\Phi_p^{\mu,\circ}\langle f^{\langle f\rangle_p}\rangle_p)^{-1/n}}{V(\langle f^{\langle f\rangle_p}\rangle_p)^{1/p}} \end{aligned} \tag{5.7}$$

with equality if and only if $f = f^{\langle f \rangle_p}$ almost everywhere. Now, by Proposition 3.14 (b), we have $V(\langle f^{\mathbb{B}^n} \rangle_p) = V(\langle f^{\langle f \rangle_p} \rangle_p)$, and by applying Theorem 3.2 and Proposition 3.14 (a), we obtain

$$\frac{V(\Phi_p^{\mu,\circ} \langle f^{\langle f \rangle_p} \rangle_p)^{-1/n}}{V(\langle f^{\langle f \rangle_p} \rangle_p)^{1/p}} \geq \frac{V(\Phi_p^{\mu,\circ} \langle f^{\mathbb{B}^n} \rangle_p)^{-1/n}}{V(\langle f^{\mathbb{B}^n} \rangle_p)^{1/p}} = \mathcal{E}_p^\mu f^{\mathbb{B}^n}. \tag{5.8}$$

Since $f^{\mathbb{B}^n} = f^*$, the inequality follows.

Now let $f \in W^{1,p}(\mathbb{R}^n)$ be non–negative, such that

$$V(\{|\nabla f^*(x)| = 0\} \cap \{0 < f^*(x) < \text{ess sup} f\}) = 0.$$

To have equality in (5.6), we must have equality in (5.7) and (5.8), which forces

$$f = f^{\langle f \rangle_p} \tag{5.9}$$

almost everywhere and

$$\langle f^{\langle f \rangle_p} \rangle_p = E \tag{5.10}$$

for some ellipsoid $E \subset \mathbb{R}^n$ if $\mu$ is discrete, and

$$\langle f^{\langle f \rangle_p} \rangle_p = c\mathbb{B}^n \tag{5.11}$$

for some constant $c > 0$ otherwise. Finally, since by Proposition 3.14 (a) $\langle f^{\langle f \rangle_p} \rangle_p$ is a multiple of $\langle f \rangle_p$, (5.10) and (5.11) show that $\langle f \rangle_p$ is an ellipsoid and therefore, by (5.9), $f = f^E$ up to translation if $\mu$ is discrete, and that $\langle f \rangle_p$ is a ball and, thus, $f = f^*$, up to translation, otherwise. □

Using the same proof, but replacing $\langle f \rangle_p$ with $\langle f \rangle$ for $f \in BV(\mathbb{R}^n)$, yields the $BV(\mathbb{R}^n)$ version of Theorem 5.2.

**Theorem 5.3.** *Suppose that $\mu$ is an even, zonal measure on $S^{n-1}$. If $f \in BV(\mathbb{R}^n)$ is non–negative and compactly supported, then*

$$\mathcal{E}^\mu f \geq \mathcal{E}^\mu f^*. \tag{5.12}$$

*Furthermore, if*

$$V(D^0_{f^*} \cap \{0 < f^* < \text{ess sup} f\}) = 0, \tag{5.13}$$

*then there is equality in* (5.12) *if and only if $\mu$ is discrete and*

$$f(x) = f^E(x + x_0) \qquad a.e.$$

*for some $x_0 \in \mathbb{R}^n$ and some ellipsoid $E \subset \mathbb{R}^n$, or if $\mu$ is not discrete and*

$$f(x) = f^*(x + x_0) \qquad a.e.$$

*for some $x_0 \in \mathbb{R}^n$.*

For $f \in BV(\mathbb{R}^n)$ we can utilize Proposition 3.15 to characterize all equality cases in (5.12) even without the additional assumption (5.13). Note however, that while the resulting functions will still have level sets homothetic to a ball (or ellipsoid if $\mu$ is discrete), those level sets need no longer be concentric.

**Theorem 5.4.** *Suppose that $\mu$ is an even, zonal measure on $S^{n-1}$. If $f \in BV(\mathbb{R}^n)$ is non–negative and compactly supported, then*

$$\mathcal{E}^\mu f = \mathcal{E}^\mu f^*.$$

*if and only if $\mu$ is discrete and the level sets $[f]_t$ are homothets of an ellipsoid up to some set of $\mathcal{L}^n$ measure 0 for almost every $t \geq 0$, or if $\mu$ is not discrete and the level sets $[f]_t$ are homothets of a Euclidean ball up to some set of $\mathcal{L}^n$ measure 0 for almost every $t \geq 0$.*

*Proof.* To have equality, we need to have equality in (5.7) and (5.8) (where $p = 1$ and $\langle f \rangle_p$ has to be replaced by $\langle f \rangle$). Hence, by (3.24), $[f]_t$ has to be a homothet of $\langle f \rangle$ for almost every $t \geq 0$ and furthermore

$$\langle f^{\langle f \rangle} \rangle = E \tag{5.14}$$

for some ellipsoid $E \subset \mathbb{R}^n$ if $\mu$ is discrete and

$$\langle f^{\langle f \rangle} \rangle = c\mathbb{B}^n \tag{5.15}$$

for some $c > 0$ otherwise. Now (5.14) and (5.15) force $\langle f \rangle$ to be an ellipsoid or a ball, thus, the level sets $[f]_t$ need to be homothets of an ellipsoid or a ball as stated. □

Note that taking $\mu$ to be discrete, recovers the affine Pólya–Szegö principle by Cianchi, Lutwak, Yang and Zhang as well as the equality cases by Wang for functions $f \in BV(\mathbb{R}^n)$.

## 5.2 The Pólya–Szegö Principles for $\mathcal{E}^\mu_{i,p}$ and $\mathcal{E}_{i,p}$

As in the last section, let us start by proving the Pólya–Szegö principle for $\mathcal{E}^\mu_{i,p}$ when $f \in W^{1,p}(\mathbb{R}^n)$ for $p \geq 1$ and characterize the equality conditions in the sense of Brothers and Ziemer.

**Theorem 5.5.** *Suppose that $1 \leq i \leq n-1$, $1 \leq p < \infty$, and that $\mu$ is an even measure on $S^{n-1}$ such that* $\operatorname{span}\operatorname{supp}\mu = E_i$ *and $\mu(S^{n-1}) = 1$. If $f \in W^{1,p}(\mathbb{R}^n)$ is non–negative and compactly supported, then*

$$\mathcal{E}^\mu_{i,p} f \geq \mathcal{E}^\mu_{i,p} f^*. \tag{5.16}$$

*Furthermore, if (5.3) holds, then there is equality in (5.16) if and only if $i = 1$ and*

$$f(x) = f^E(x + x_0) \qquad a.e. \tag{5.17}$$

*for some $x_0 \in \mathbb{R}^n$ and some ellipsoid $E \subset \mathbb{R}^n$, or if $i > 1$ and*

$$f(x) = f^*(x + x_0) \qquad a.e.$$

*for some $x_0 \in \mathbb{R}^n$.*

*Proof.* Since the case $i = 1$ is already settled (this is just the special case of Theorem 5.2 when $\mu$ is discrete), we will assume $i > 1$ throughout this proof.

For $f \in W^{1,p}(\mathbb{R}^n)$, let us first show that plugging in $f^*$ in (5.5) gives equality. To this end, since $f^* = f^{\mathbb{B}^n}$, note that we have $\langle f^* \rangle_p = c\mathbb{B}^n$ for some constant $c > 0$. Applying the $p$–projection body yields

$$\Pi_p \langle f^* \rangle_p = c^{n-1} \Pi_p \mathbb{B}^n = c^{n-1} \mathbb{B}^n.$$

Thus, $\Pi_p \langle f^* \rangle_p$ is a ball, which, according to Theorem 4.7, implies

$$\mathcal{E}^{\mu}_{i,p} f^* = \mathcal{E}^{\mathrm{Aff}}_p f^*. \tag{5.18}$$

The claim is now a simple consequence of (5.5), the affine Pólya–Szegö principle and (5.18), since

$$\mathcal{E}^{\mu}_{i,p} f \geq \mathcal{E}^{\mathrm{Aff}}_p f \geq \mathcal{E}^{\mathrm{Aff}}_p f^* = \mathcal{E}^{\mu}_{i,p} f^*. \tag{5.19}$$

To have equality in Theorem 5.16, we must have equality in (5.19), which implies that $\Pi_p \langle f \rangle_p$ is a ball. The Brothers–Ziemer type characterization of the equality cases of the affine Pólya–Szegö principle [Wan13] states that if (5.3) holds, then there is equality in the second inequality if and only if

$$f(x) = f^E(x + x_0) \qquad \text{a.e.}$$

for some $x_0 \in \mathbb{R}^n$ and some ellipsoid $E$. Together, the equality conditions yield that if (5.3) holds, then $\Pi_p \langle f \rangle_p$ is a ball and

$$\Pi_p \langle f \rangle_p = \Pi_p \langle f^E \rangle_p = c \Pi_p E$$

for some constant $c > 0$. However, $\Pi_p E$ is a ball if and only if $E$ is a multiple of the Euclidean unitball $\mathbb{B}^n$. Therefore $f(x) = f^{\mathbb{B}^n}(x + x_0)$ for some $x_0 \in \mathbb{R}^n$ which finishes the proof. □

Note that (5.19) shows that the affine Pólya–Szegö principle is stronger than (5.16), which in turn is stronger than the classical Pólya–Szegö principle, since

$$\|\nabla f\|_p \geq \mathcal{E}^{\mu}_{i,p} f \geq \mathcal{E}^{\mu}_{i,p} f^* = \mathcal{E}^{\mathrm{Aff}}_p f^* = \|\nabla f^*\|_p.$$

An analogous argument as in the proof of Theorem 5.5, using $\langle f \rangle$ for $f \in BV(\mathbb{R}^n)$ gives the following version of Theorem 5.5 for functions $f \in BV(\mathbb{R}^n)$.

**Theorem 5.6.** *Suppose that $1 \leq i \leq n-1$ and that $\mu$ is an even measure on $S^{n-1}$ such that* $\operatorname{span}\operatorname{supp}\mu = E_i$. *If $f \in BV(\mathbb{R}^n)$ is non–negative and compactly supported, then*

$$\mathcal{E}^{\mu}_i f \geq \mathcal{E}^{\mu}_i f^*. \tag{5.20}$$

*Furthermore, if*

$$V(D^0_{f^*} \cap \{0 < f^* < \operatorname{ess\,sup} f\}) = 0,$$

*then there is equality in* (5.16) *if and only if* $i = 1$ *and*

$$f(x) = f^E(x + x_0) \qquad a.e.$$

*for some* $x_0 \in \mathbb{R}^n$ *and some ellipsoid* $E \subset \mathbb{R}^n$*, or if* $i > 1$ *and*

$$f(x) = f^*(x + x_0) \qquad a.e.$$

*for some* $x_0 \in \mathbb{R}^n$*.*

Similar to Theorem 5.4, we can get rid of assumption (5.13).

**Theorem 5.7.** *Suppose that* $1 \leq i \leq n-1$ *and that* $\mu$ *is an even measure on* $S^{n-1}$ *such that* $\operatorname{span}\operatorname{supp}\mu = E_i$*. If* $f \in BV(\mathbb{R}^n)$ *is non–negative and compactly supported, then*

$$\mathcal{E}_i^\mu f = \mathcal{E}_i^\mu f^*$$

*if and only if* $i = 1$ *and the level sets* $[f]_t$ *are homothets of an ellipsoid up to some set of* $\mathcal{L}^n$ *measure* 0 *for almost every* $t \geq 0$*, or if* $i > 1$ *and the level sets* $[f]_t$ *are homothets of a Euclidean ball up to some set of* $\mathcal{L}^n$ *measure* 0 *for almost every* $t \geq 0$*.*

*Proof.* If $i = 1$, the statement is the same as Theorem 5.4. So it remains to show the statement when $i > 1$. To have equality we need to have equality in (5.19) (for $p = 1$ and $f \in BV(\mathbb{R}^n)$), which, by Theorem 5.4 and Proposition 3.15 (a), forces $[f]_t$ to be homothetic to $\mathbb{B}^n$ for almost every $t \geq 0$. □

By choosing $\mu$ to be the $(i-1)$–dimensional Hausdorff measure on $S^{i-1}$ normalized such that $\mu(S^{i-1}) = q_{i,p}^{-1}$ (see last chapter for precise definitions and values of the constants), gives a Pólya–Szegö principle for $\mathcal{E}_{i,p}$.

# 6 Generalized Petty Projection–type Inequalities

Our approach towards the functional inequalities so far has been to prove an underlying geometric inequality first and then to exploit the strong connection to functional inequalities given via the LYZ–body $\langle f \rangle_p$ of functions $f \in W^{1,p}(\mathbb{R}^n)$ (or $\langle f \rangle$ for $f \in BV(\mathbb{R}^n)$ respectively). For instance, the underlying geometry behind the Sobolev–type inequalities (3.15) established by Haberl and Schuster was a Petty Projection–type inequality for the Minkowski valuations $\Phi^\mu$ (see Chapter 3 for the defintion of $\Phi^\mu$). The following representation theorem for continuous, $(n-1)$–homogeneous, translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations was shown by Schuster [Sch07] and is the basis for our investigation of a generalization of the Petty Projection–type inequalities (3.14) by Haberl and Schuster [HS19]. Recall that we say a function is zonal, if it is invariant under rotations $\theta \in \mathrm{SO}(n-1)$ about a fixed axis $\bar{e}$, and that by $\theta_u \in \mathrm{SO}(n-1)$ we denote an arbitrary rotation such that $\theta_u \bar{e} = u$.

**Theorem 6.1.** *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, $(n-1)$–homogeneous, translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuation, then there is a weakly positive and zonal function $g \in C(S^{n-1})$, unique up to addition of a linear function, such that*

$$h(\Phi_{n-1}K, u) = \int_{S^{n-1}} g(\theta_u^{-1} v)\, dS(K, v). \tag{6.1}$$

Note that $\theta_u$ in (6.1) is not uniquely defined, but the zonality of $g$ makes (6.1) well defined for any choice of $\theta_u$. Moreover, note that Theorem 6.1 was stated in [Sch07] for so called *Blaschke–Minkowski homomorphisms*. However, as shown in [SW18] (where also a representation for support functions of Minkowski valuations without assumption on its degree is shown), every Minkowski valuation as in Theorem 6.1 is a Blaschke–Minkowski homomorphism.

The geometric inequality established for $\Phi^\mu$ is a first step at proving a Petty projection–type inequality for Minkowski valuations given by (6.1), by proving this type of inequality when $g$ is the support function of a zonoid of revolution. A generalization of this result could give valuable insights into an answer to Open Problem 4.6 in full generality and thus could be used to establish Sobolev–type inequalities (over the Grassmannian) for arbitrary norms.

We will partially extend the result by Haberl and Schuster in this chapter, by showing, that the volume product

$$P_g(K) := V(\Phi_g^\circ K) V(K)^{n-1} \tag{6.2}$$

has maximizers in $\mathcal{K}_0^n$, where $g$ is a continuous, positive and zonal function. Here, we denote an $(n-1)$–homogeneous Minkowski valuation that is generated by a function $g$ in the sense of (6.1) by $\Phi_g$. Next, we prove the existence of minimizers for the non–polar version, that is we prove that

$$R_g(K) := \frac{V(\Phi_g K)}{V(K)^{n-1}} \tag{6.3}$$

has minimizers in $\mathcal{K}_0^n$. In both cases we do not know the exact extremizers, but we conjecture those extremizers to be balls. If this is the case, than we could choose $g = h(L, \cdot)$ for some origin symmetric convex body of revolution $L \in \mathcal{K}_0^n$ and (6.2) would directly yield $BV(\mathbb{R}^n)$ Sobolev–type inequalities for general norms.

The classical approach by Lutwak,Yang and Zhang [LYZ00] to show the Petty projection inequality (3.8) for $\Pi_p$, was to establish the relation

$$S_u \Pi_p^* K \subseteq \Pi_p^* S_u K, \tag{6.4}$$

for smooth origin symmetric convex bodies $K$, where we denote by $S_u K$ the Steiner symmetrization of $K \in \mathcal{K}^n$ in direction $u$. Applying the volume on both sides of (6.4) and using the facts that Steiner symmetrization is volume preserving and that one can find a sequence of directions $(u_i)_{i\in\mathbb{N}}$, $u_i \in S^{n-1}$ for $i \in \mathbb{N}$, such that the consecutive application $S_{u_k} \cdots S_{u_1} K$ converges to a Euclidean ball as $k \to \infty$, directly gives Petty's projection inequality (3.8). However, such an approach does not work for our general setting, as can be seen by taking $g = h(D, \cdot)$, where $D = \mathbb{B}^n \cap \bar{e}^\perp$ is the unit disk in $\mathbb{R}^n$. Haberl and Schuster could circumvent this problem when $g$ is the support function of a zonoid of revolution by establishing the strong connection

$$V(\Phi_g^\circ K) \leq V(\Pi^\circ K).$$

Since it is not clear, if such an inequality holds for every continuous, translation invariant, $(n-1)$–homogeneous and $\mathrm{SO}(n)$ equivariant Minkowski valuation, we will approach the problem in a different way, which will allow us to also prove the existence of extremizers in the non–polar version. The main idea behind our proof is to show that the polar volume product (or the non–polar ratio) can be bounded by a constant multiple of the isoperimetric ratio

$$I(K) := \frac{V(K)^{n-1}}{S(K)^n}$$

(by the reciprocal of this ratio, respectively). However, by showing that $I(K)$ tends to zero if the convex body $K$ tends to something lower dimensional (but not to a point), we conclude that a converging and maximizing sequence of convex bodies (minimizing sequence respectively) must converge to a full dimensional convex body. Hence, an extremizer exists.

## 6.1 Auxiliary Results

From now on, we say $\Phi_g$ is generated by $g$, if $\Phi_g$ and $g$ satisfy relation (6.1). Furthermore, note that by the SO($n$)–invariance of the spherical Lebesgue measure we have

$$h(\Phi_g \mathbb{B}^n, v) = \int_{S^{n-1}} g(\theta_v^{-1}u)\, du = \int_{S^{n-1}} g(u)\, du.$$

Thus, in order to make sure that $\Phi_g \mathbb{B}^n = \mathbb{B}^n$, we may always assume that $g$ is normalized such that

$$\int_{S^{n-1}} g(u)\, du = 1. \tag{6.5}$$

We will make use of the fact that $g(\theta_u^{-1}v)$ depends only on the scalar product $u \cdot v$, therefore $g(\theta_u^{-1}v) = g(\theta_v^{-1}u)$.

Let us first recall a lemma by Esposito, Fusco and Trombetti (Lemma 4.1 in [EFT05], see also [DFMS17]), which lets us estimate the isoperimetric ratio $I(K)$ by the diameter $\mathrm{diam}(K)$ of $K \in \mathcal{K}^n$.

**Lemma 6.2.** *Suppose $n \geq 2$. If $K \in \mathcal{K}_0^n$, then there exists a constant $c_n$ depending only on the dimension $n$, such that*

$$\mathrm{diam}(K) \leq c_n \frac{S(K)^{n-1}}{V(K)^{n-2}}. \tag{6.6}$$

Next we show that if $(K_l)_{l\in\mathbb{N}}$ is a sequence of convex bodies with the origin in their interior, converging to something lower dimensional, then the isoperimetric ratio $I(K_l)$ tends to zero as $l \to \infty$.

**Lemma 6.3.** *Let $(K_l)_{l\in\mathbb{N}}$ be a sequence of convex bodies with the origin in their interior, which converges to some convex body $K$ of dimension $\mathrm{dim}K = i$. If $1 < i < n$, then*

$$I(K_l) \to 0$$

*as $l \to \infty$.*

*Proof.* First note that since $\mathrm{dim}K \geq 1$, the diameter $\mathrm{diam}(K)$ is positive. By (6.6) and the isoperimetric inequality (3.1) we have

$$I(K_l) = \frac{V(K_l)^{n-2}}{S(K_l)^{n-1}} \frac{V(K_l)}{S(K_l)} \leq \frac{c_n}{\mathrm{diam}(K_l)} \frac{V(K_l)}{S(K_l)} \leq \tilde{c}_n \frac{V(K_l)^{\frac{1}{n}}}{\mathrm{diam}(K_l)}$$

for some constant $\tilde{c}_n$ depending only on $n$. Since $\mathrm{diam}(K_l) \to \mathrm{diam}(K) > 0$ as $l \to \infty$, the claim follows. □

The key ingredients for the proofs are the following two estimates for

$$\frac{P_g(K)}{I(K)} = V(\Phi_g^\circ K) S(K)^n$$

and

$$R_g(K)\,I(K) = \frac{V(\Phi_g K)}{S(K)^n}. \tag{6.7}$$

**Lemma 6.4.** *Let $\Phi_g$ be a continuous, translation invariant, $(n-1)$–homogeneous and* SO$(n)$ *equivariant Minkowski valuation with generating function $g$. If $K \in \mathcal{K}^n$, then*

$$c_{n,g} \leq V(\Phi_g^\circ K) S(K)^n \leq d_{n,g}, \tag{6.8}$$

*where the constants $c_{n,g}$ and $d_{n,g}$ are given by*

$$c_{n,g} = \omega_n (n\omega_n)^n.$$

*and*

$$d_{n,g} = \frac{1}{n}\int_{S^{n-1}} g(u)^{-n}\,du$$

*Proof.* By the polar coordinate formula, (6.1), Jensen's inequality, Fubini's theorem and the SO$(n)$–invariance of the spherical Lebesgue measure, we obtain

$$\begin{aligned}
V(\Phi_g^\circ K) &= \frac{1}{n}\int_{S^{n-1}} h(\Phi_g K, u)^{-n}\,du \\
&= S(K)^{-n}\frac{1}{n}\int_{S^{n-1}} \left(\frac{h(\Phi_g K, u)}{S(K)}\right)^{-n} du \\
&= S(K)^{-n}\frac{1}{n}\int_{S^{n-1}} \left(\int_{S^{n-1}} g(\theta_u^{-1} v)\frac{dS(K,v)}{S(K)}\right)^{-n} du \\
&\leq S(K)^{-n}\frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}} g(\theta_u^{-1} v)^{-n}\frac{dS(K,v)}{S(K)}\,du \\
&= S(K)^{-n}\frac{1}{n}\int_{S^{n-1}} g(u)^{-n}\,du,
\end{aligned}$$

which shows the right-hand inequality.

To get the left-hand inequality in (6.8), we apply the polar coordinate formula, Jensen's inequality, (6.1) and Fubini's theorem to obtain

$$\begin{aligned}
V(\Phi_g^\circ K) &= \frac{1}{n}\int_{S^{n-1}} h(\Phi_g K, u)^{-n}\,du \\
&\geq \omega_n \left(\int_{S^{n-1}} h(\Phi_g K, u)\frac{du}{n\omega_n}\right)^{-n} \\
&= \omega_n \left(\int_{S^{n-1}}\int_{S^{n-1}} g(\theta_u^{-1} v)\,dS(K,v)\frac{du}{n\omega_n}\right)^{-n} \\
&= \omega_n \left(\frac{S(K)}{n\omega_n}\right)^{-n},
\end{aligned}$$

where the last equality is due to the chosen normalization (6.5) for $g$. □

Note, that if $g$ is the support function of a full dimensional convex body of revolution, say $L$, then

$$d_{n,g} = V(L^\circ).$$

**Lemma 6.5.** *Let $\Phi_g$ be a continuous, translation invariant, $(n-1)$–homogeneous and* SO$(n)$ *equivariant Minkowski valuation with generating function $g$. If $K \in \mathcal{K}^n$, then*

$$\tilde{c}_{n,g} \leq \frac{V(\Phi_g K)}{S(K)^n} \leq \tilde{d}_{n,g} \tag{6.9}$$

*where the constants $\tilde{c}_{n,g}$ and $\tilde{d}_{n,g}$ are given by*

$$\tilde{c}_{n,g} = \left( \min_{w \in S^{n-1}} g(w) \right)^n \omega_n$$

*and*

$$\tilde{d}_{n,g} = \frac{\omega_n^{1-n}}{n^n}.$$

*Proof.* Observe that by (2.8), (6.1), Fubini's theorem and (2.5) we obtain

$$\begin{aligned} V(\Phi_g K) &= \frac{1}{n} \int_{S^{n-1}} h(\Phi_g K, u) \, dS(\Phi_g K, u) \\ &= \frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} g(\theta_u^{-1} v) \, dS(K, v) \, dS(\Phi_g K, u) \\ &\geq \min_{w \in S^{n-1}} g(w) \frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} dS(\Phi_g K, u) \, dS(K, v) \\ &= \min_{w \in S^{n-1}} g(w) \int_{S^{n-1}} V(B, \Phi_g K[n-1]) \, dS(K, v). \end{aligned} \tag{6.10}$$

Now, an application of Minkowski's inequality (2.7) yields

$$\begin{aligned} V(\Phi_g K) &\geq \min_{w \in S^{n-1}} g(w) \int_{S^{n-1}} V(B)^{\frac{1}{n}} V(\Phi_g K)^{\frac{n-1}{n}} \, dS(K, u) \\ &= \min_{w \in S^{n-1}} g(w) \omega_n^{\frac{1}{n}} V(\Phi_g K)^{\frac{n-1}{n}} S(K), \end{aligned}$$

which, after dividing by $V(\Phi_g K)^{\frac{n-1}{n}}$, gives

$$V(\Phi_g K)^{\frac{1}{n}} \geq \min_{w \in S^{n-1}} g(w) \omega_n^{\frac{1}{n}} S(K).$$

Taking this inequality to the power $n$ and dividing by $S(K)^n$ gives the left-hand inequality in (6.9).

To prove the upper bound, we apply Urysohn's inequality (that is $i = 1$ in (3.20)) to $V(\Phi_g K)$ to obtain

$$\begin{aligned} V(\Phi_g K) &\leq \left( \frac{nV(\Phi_g K, \mathbb{B}^n[n-1])}{n\omega_n} \right)^n \omega_n \\ &= \frac{\omega_n^{1-n}}{n^n} \left( \int_{S^{n-1}} h(\Phi_g K, u)\, du \right)^n \\ &= \frac{\omega_n^{1-n}}{n^n} \left( \int_{S^{n-1}} \int_{S^{n-1}} g(\theta_u^{-1} v)\, dS(K, v)\, du \right)^n \\ &= \frac{\omega_n^{1-n}}{n^n} S(K)^n \end{aligned}$$

where the last equality is due to Fubini's theorem and the chosen normalization of $g$. $\square$

By restricting ourselves to support functions, say $g = h(L, \cdot)$ for some origin–symmetric $L \in \mathcal{K}_0^n$, we can obtain a better lower bound by directly applying Minkowski's inequality in (6.10). This leads to the constant

$$\tilde{c}_{n,L} = V(L)$$

and consequently, by (6.9), to the inequalities

$$V(L) \leq \frac{V(\Phi_g K)}{S(K)^n} \leq \frac{\omega_n^{1-n}}{n^n}. \tag{6.11}$$

Note that now the outermost inequality in (6.11) is exactly Urysohn's inequality for $L$, since the normalization $\int_{S^{n-1}} g(u)\, du = 1$ corresponds to the mean width normalization $\frac{w(L)}{2} = nV(L, \mathbb{B}^n[n-1]) = 1$.

## 6.2 Existence of Maximizers for the Polar Volume Product

We are now in a position to prove the existence of maximizers of

$$V(\Phi_g^\circ K) V(K)^{n-1}.$$

**Theorem 6.6.** *Suppose $\Phi_g$ is a continuous, translation invariant, $(n-1)$–homogeneous and $\mathrm{SO}(n)$ equivariant Minkowski valuation with generating function $g > 0$. There exists a full dimensional convex body $K_g \in \mathcal{K}^n$, such that*

$$V(\Phi_g^\circ K) V(K)^{n-1} \leq V(\Phi_g^\circ K_g) V(K_g)^{n-1}$$

*for each full dimensional $K \in \mathcal{K}^n$.*

*Proof.* Let $K \in \mathcal{K}^n$ be full dimensional. We start by showing that $K \mapsto V(\Phi_g^\circ K)V(K)^{n-1}$ is bounded. By Lemma 6.4 and the classical isoperimetric inequality (3.1), we have

$$\begin{aligned} V(\Phi_g^\circ K)V(K)^{n-1} &\leq d_{n,g}\frac{V(K)^{n-1}}{S(K)^n} \\ &\leq d_{n,g}\frac{V(\mathbb{B}^n)^{n-1}}{S(\mathbb{B}^n)^n} \end{aligned}$$

for some constant $d_{n,g}$ depending on $n$ and $g$ only. Thus, the supremum

$$S := \sup_{K \in \mathcal{K}^n} V(\Phi_g^\circ K)V(K)^{n-1} < \infty$$

exists. Since the product $V(\Phi_g^\circ K)V(K)^{n-1}$ is 0–homogeneous and translation invariant, we can restrict ourselves to convex bodies $K \subseteq \mathbb{B}^n$. Now let $(K_l)_{l\in\mathbb{N}}$ be a maximizing sequence of full dimensional convex bodies, that is such that $V(\Phi_g^\circ K_l)V(K_l)^{n-1}$ converges to $S$ for $l \to \infty$. Again, by the scale invariance of $V(\Phi_g^\circ K)V(K)^{n-1}$, we can assume that $\operatorname{diam}(K_l) = 1$ for all $l \in \mathbb{N}$. Now, since $(K_l)_{l\in\mathbb{N}}$ is bounded by $\mathbb{B}^n$, Blaschke's selection theorem (Theorem 2.2) guarantees the existence of a convergent subsequence of $(K_l)_{l\in\mathbb{N}}$. W.l.o.g. we will denote this subsequence again by $(K_l)_{l\in\mathbb{N}}$ and its limit by $K$. If $K$ is full dimensional, the claim is proved.

Let us therefore assume $K$ is lower dimensional. By the normalization $\operatorname{diam}(K_l) = 1$ and since $K_l \to K$ for $l \to \infty$, we have $\operatorname{diam}(K) = 1$. Thus, $K$ can not be a point and consequently $1 < \dim K < n$. What remains to show is that $K$ can not be a maximizer of $V(\Phi_g^\circ K)V(K)^{n-1}$, which contradicts our construction of $K$ and proves that the assumption $\dim K < n$ is wrong. To prove this, observe that by Lemma 6.4 we have

$$V(\Phi_g^\circ K_l)V(K_l)^{n-1} \leq d_{n,g}\frac{V(K_l)^{n-1}}{S(K_l)^n}.$$

for all $l \in \mathbb{N}$. But since by our assumption $(K_l)_{l\in\mathbb{N}}$ converges to something lower dimensional, Lemma 6.3 shows that

$$V(\Phi_g^\circ K_l)V(K_l)^{n-1} \to 0$$

as $l \to \infty$, which contradicts the fact that $(K_l)_{l\in\mathbb{N}}$ was a maximizing (sub–)sequence. □

Note that this proof gives no insight into how the maximizer $K_g$ looks like. We conjecture however, that $K_g$ is a ball and that balls are the only maximizers.

## 6.3 Existence of Minimizers for the Volume Ratio

Similar to the existence proof in the last section, we will show here the existence of minimzers of the volume ratio

$$\frac{V(\Phi_g K)}{V(K)^{n-1}}.$$

**Theorem 6.7.** *Suppose $\Phi_g$ is a continuous, translation invariant, $(n-1)$–homogeneous and $\mathrm{SO}(n)$ equivariant Minkowski valuation with generating function $g > 0$. There exists a full dimensional convex body $K_g \in \mathcal{K}^n$, such that*

$$\frac{V(\Phi_g K)}{V(K)^{n-1}} \geq \frac{V(\Phi_g K_g)}{V(K_g)^{n-1}}$$

*for all full dimensional $K \in \mathcal{K}^n$.*

*Proof.* Let $K \in \mathcal{K}^n$ be full dimensional. We start by showing that $K \mapsto \frac{V(\Phi_g K)}{V(K)^{n-1}}$ is bounded from below. By Lemma 6.5 and the classical isoperimetric inequality (3.1), we have

$$\begin{aligned}\frac{V(\Phi_g K)}{V(K)^{n-1}} &\geq \tilde{c}_{n,g}\frac{S(K)^n}{V(K)^{n-1}} \\ &\geq \tilde{c}_{n,g}\frac{S(\mathbb{B}^n)^n}{V(\mathbb{B}^n)^{n-1}}\end{aligned}$$

for some constant $\tilde{c}_{n,g}$ depending on $n$ and $g$ only. Thus, the infimum

$$\inf_{K\in\mathcal{K}^n}\frac{V(\Phi_g K)}{V(K)^{n-1}} \tag{6.12}$$

is strictly positive. By the same arguments as in the proof of Theorem 6.6, we find a convergent sequence $(K_l)_{l\in\mathbb{N}}$ of full dimensional convex bodies, such that $\frac{V(\Phi_g K_l)}{V(K_l)^{n-1}}$ converges to the infimum (6.12). If we denote the limit of $K_l$ by $K$, it remains to show that $K$ is full dimensional. Let us therefore assume $K$ is lower dimensional. By the normalization $\operatorname{diam}(K_l) = 1$ and since $K_l \to K$ for $l \to \infty$, we have $\operatorname{diam}(K) = 1$. Thus, $K$ can not be a point and consequently $1 < \dim K < n$. Now observe that by Lemma 6.5 we have

$$\frac{V(\Phi_g K_l)}{V(K_l)^{n-1}} \geq \tilde{c}_{n,g}\frac{S(K_l)^n}{V(K_l)^{n-1}}$$

for all $l \in \mathbb{N}$. But since by our assumption $(K_l)_{l\in\mathbb{N}}$ converges to something lower dimensional, Lemma 6.3 shows that

$$\frac{V(\Phi_g K_l)}{V(K_l)^{n-1}} \to \infty$$

as $l \to \infty$, which contradicts our construction of $(K_l)_{l\in\mathbb{N}}$ as minimizing (sub–)sequence. □

The exact minimizers (or their uniqueness) are not known, but the next result shows that $K_g$ and $\Phi_g^2 K_g$ have to be homothets, where $\Phi_g^2 K_g = \Phi_g \Phi_g K_g$. This result can already be found in [Sch07].

**Proposition 6.8.** *If $K_g$ is a minimizer of* (6.3)*, then it is homothetic to $\Phi_g^2 K_g$.*

*Proof.* First, let $K \in \mathcal{K}^n$ and note that by (2.5),(6.1) and Fubini's theorem we have

$$\begin{aligned} V(\Phi_g K, L[n-1]) &= \frac{1}{n}\int_{S^{n-1}} h(\Phi_g K, u)\, dS(L,u) \\ &= \frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}} g(\theta_v^{n-1} u)\, dS(K,v)\, dS(L,u) \\ &= \frac{1}{n}\int_{S^{n-1}} h(\Phi_g L, v)\, dS(K,v) \\ &= V(\Phi_g L, K[n-1]). \end{aligned}$$

If we choose $L = \Phi_g K$ we obtain

$$V(\Phi_g K) = V(\Phi_g K, \Phi_g K[n-1]) = V(\Phi_g^2 K, K[n-1]). \tag{6.13}$$

Now, Minkowskis inequality (2.7) gives

$$V(\Phi_g^2 K, K[n-1])^n \geq V(\Phi_g^2 K) V(K)^{n-1} \tag{6.14}$$

with equality if and only if $K$ and $\Phi_g^2 K$ are homothetic. Together, (6.13) and (6.14) yield

$$V(\Phi_g K)^n \geq V(\Phi_g^2 K) V(K)^{n-1}$$

and consequently

$$\frac{V(\Phi_g K)}{V(K)^{n-1}} \geq \frac{V(\Phi_g^2 K)}{V(\Phi_g K)^{n-1}} \tag{6.15}$$

where equality is attained if and only if $K$ is homothetic to $\Phi_g^2 K$. Setting $K = K_g$ in (6.15), we must have equality since $K_g$ is a minimizer and the claim follows. □

Just like the conjectured Petty projection inequality implies the polar Petty projection inequality (3.5), Theorem 6.7 implies Theorem 6.6 if we can show that Euclidean balls are minimizers of Theorem 6.7.

**Proposition 6.9.** *If Euclidean balls are minimizers of $R_g$, then Theorem 6.7 implies Theorem 6.6 and Euclidean balls are maximizers of $P_g$.*

*Proof.* Let $K_g$ be a maximizer of $P_g$. By the Blaschke–Santaló inequality (3.19) we obtain

$$V(\Phi_g^\circ K_g) V(K_g)^{n-1} \leq \omega_n^2 \frac{V(K_g)^{n-1}}{V(\Phi_g K_g)} \leq \omega_n^2 \frac{V(\mathbb{B}^n)^{n-1}}{V(\mathbb{B}^n)} = \omega_n^n = V(\Phi_g^\circ \mathbb{B}^n) V(\mathbb{B}^n)^{n-1},$$

thus $K_g = \mathbb{B}^n$. □

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*Curriculum Vitae*

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## Education

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2008–2017 **Student of mathematics**, *Vienna University of Technology*, Master Thesis: *“Otto Calculus or The Weak Riemannian Structure of* $(\mathcal{P}_2(M), \mathcal{W}_2)$*”*, Advisors: Prof. Monika Ludwig, Dr. Gabriel Maresch.

2007 **Matura**, *Akademisches Gymnasium Wien*, Beethovenplatz 1, 1010 Wien.

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