Entanglement quantification in atomic ensembles

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Witnessed entanglement

Classes of entanglement measures are defined as [1]

$$\mathcal{E}_{\mathcal{M}}(\rho) := \max\{0, -\min_{W\in\mathcal{M}} \mathrm{Tr}[W\rho]\} \,, \tag{1}$$

where $\mathcal{M}$ is a subset of entanglement witnesses $\mathcal{M} \subset \mathcal{W}$.

- $\mathcal{M}_{\rm BSA} = \{W \in \mathcal{W} | \mathbf{1} + W \geq 0\}$ $\Rightarrow$ *best separable approx.* $\mathcal{E}_{\rm BSA}$

$$\min t \in [0,1] \quad \text{such that} \quad \rho = (1-t)\,\sigma + t\,\delta\rho \,, \tag{2}$$

- $\mathcal{M}_{\rm GR} = \{W \in \mathcal{W} | \mathbf{1} - W \geq 0\}$ $\Rightarrow$ *generalized robustness* $\mathcal{E}_{\rm GR}$

$$\min s \in [0,\infty) \quad \text{such that} \quad \frac{1}{1+s}\rho + \frac{s}{1+s}\rho' \quad \text{is separable}\,, \tag{3}$$

Bounds from linear uncertainty relations

Let us consider entanglement criteria in the form

$$\mathcal{S}(\rho) := \sum_k \Delta^2(O_k) - \langle B\rangle \geq 0 \tag{4}$$

with $\Delta^2(O_k) = \langle O_k^2\rangle - \langle O_k\rangle^2$. We have $n^* \leq \mathcal{S}(\rho) \leq m^*$ for all ρ with

$$-n^* \leq n := \lambda_{\max}(B), \tag{5}$$

$$m^* \leq m := \sum_k \lambda_{\max}(O_k)^2 - \lambda_{\min}(B) \,, \tag{6}$$

where $\lambda_{\min(\max)}(A)$ denoting the minimal (maximal) eigenvalue of A

Lemma 2. *Every entanglement criterion that can be written in the form of Eq. (4) provides a lower bound on the best separable approximation $\mathcal{E}_{BSA} \geq -\mathcal{S}(\rho)/n$, and to the generalized robustness $\mathcal{E}_{GR} \geq -\mathcal{S}(\rho)/m$.*

Proof: We write $\Delta^2(O_k)_\rho = \min_{s_k} \langle (O_k - s_k\mathbf{1})^2\rangle$, with $s_k \in \mathbb{R}$. Thus:

$$\mathcal{S}(\rho) = \min_{\mathbf{s}} \langle W(\mathbf{s})\rangle \qquad \text{with} \quad W(\mathbf{s}) := \sum_k (O_k - s_k\mathbf{1})^2 - B. \tag{7}$$

Since $W(\mathbf{s})/n \in \mathcal{M}_{\rm BSA}$ we have the bound

$$\mathcal{E}_{\rm BSA} \geq -\min_{\mathbf{s}} \langle W(\mathbf{s})/n\rangle = -\mathcal{S}(\rho)/n \tag{8}$$

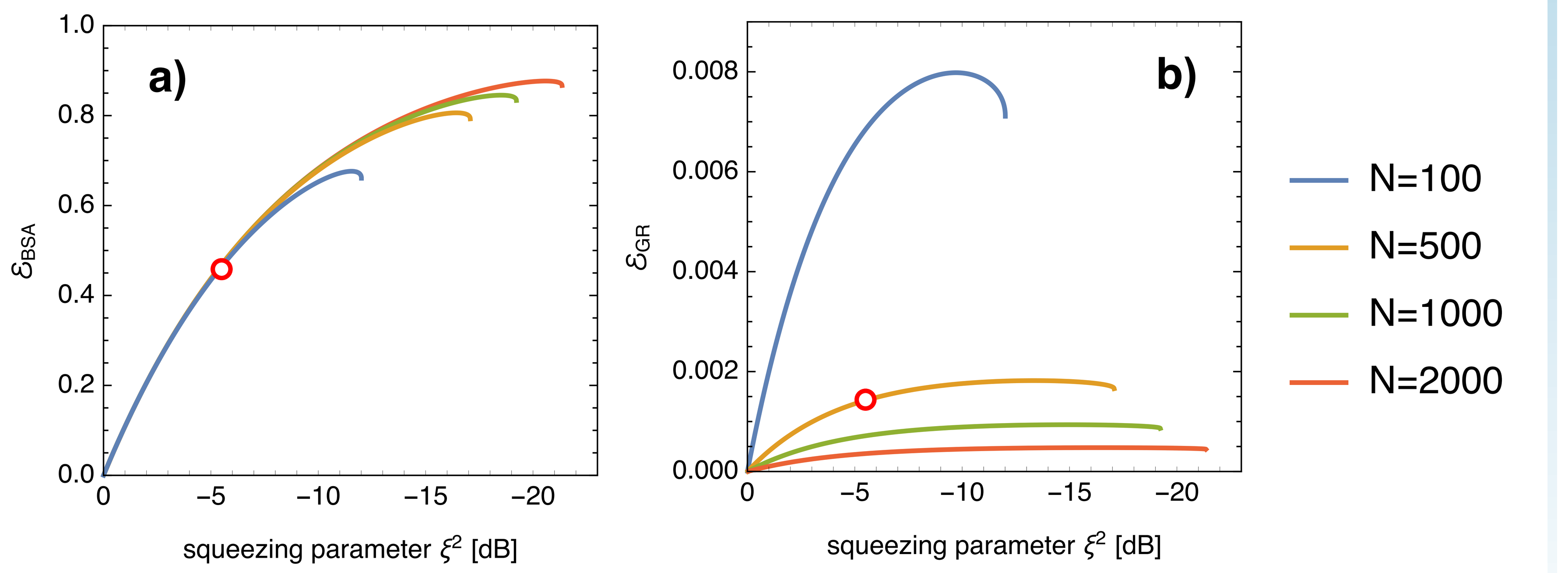
Similarly, to bound $\mathcal{E}_{\rm GR}$, one notices that

$$\Delta^2(O_k) \leq \lambda_{\max}(O_k)^2 \;\Rightarrow\; W(\mathbf{s})/m \in \mathcal{M}_{\rm GR} \;\Rightarrow\; \mathcal{E}_{\rm GR} \geq -\mathcal{S}(\rho)/m \tag{9}$$

Application I: single spin squeezed state

Consider the Wineland parameter [3] $\xi^2 := N\Delta^2(J_z)/\langle J_x\rangle^2$. We have:

$$\mathcal{E}_{\rm BSA} \geq \mathcal{C}\left(1 - \sqrt{\xi^2}\right) \quad \text{with} \quad \mathcal{C} := \langle J_x\rangle/(N/2)\,,$$
$$\mathcal{E}_{\rm GR} \geq \frac{\mathcal{C}^2}{N}(1-\xi^2) + O(N^{-2})\,. \tag{10}$$



Lower bounds on the BSA, panel **a)**, and on the GR, panel **b)**, as per Eqs. (10). The red circles is data from [4] where $N = 476$

Bounds from product uncertainty relations

Let us consider entanglement criteria in the form

$$\mathcal{U}^2(\rho) := \frac{\Delta^2(O_1)\Delta^2(O_2)}{\langle B\rangle^2} \geq 1 \tag{11}$$

Lemma 3. *Every entanglement criterion that can be written in the form of Eq. (11) provides a lower bound on the best separable approximation $\mathcal{E}_{BSA} \geq \frac{\langle B\rangle}{n}[1 - \mathcal{U}(\rho)]$, and to the generalized robustness.*

Proof: Eq. (11) implies that for all separable states

$$\mathcal{P}(\rho) := \Delta^2(O_1)\Delta^2(O_2) - \langle B\rangle^2 \geq 0\,, \tag{12}$$

which can be written as [2]

$$\Delta^2(O_1) = -4\inf_{t\in\mathbb{R}} \left[t^2\Delta^2(O_2) - |t|\,\langle B\rangle\right]\,. \tag{13}$$

To summarise, Eq. (13) implies that for any $t \in \mathbb{R}$, all separable states satisfy the inequality

$$\mathcal{S}_t(\rho) := \Delta^2(O_1) + 4t^2\Delta^2(O_2) - 4\,|t|\,\langle B\rangle \geq 0, \tag{14}$$

which takes the form of Eq. (4) with $O_2 \mapsto 2tO_2$ and $B \mapsto 4\,|t|\,B$. Thus, we find

$$\mathcal{P}(\rho) = \min_{\mathbf{s},t} 4\,|t|\,n\,\langle W(\mathbf{s},t)\rangle \quad \text{with} \quad W(\mathbf{s},t) \in \mathcal{M}_{\rm BSA}\,. \tag{15}$$

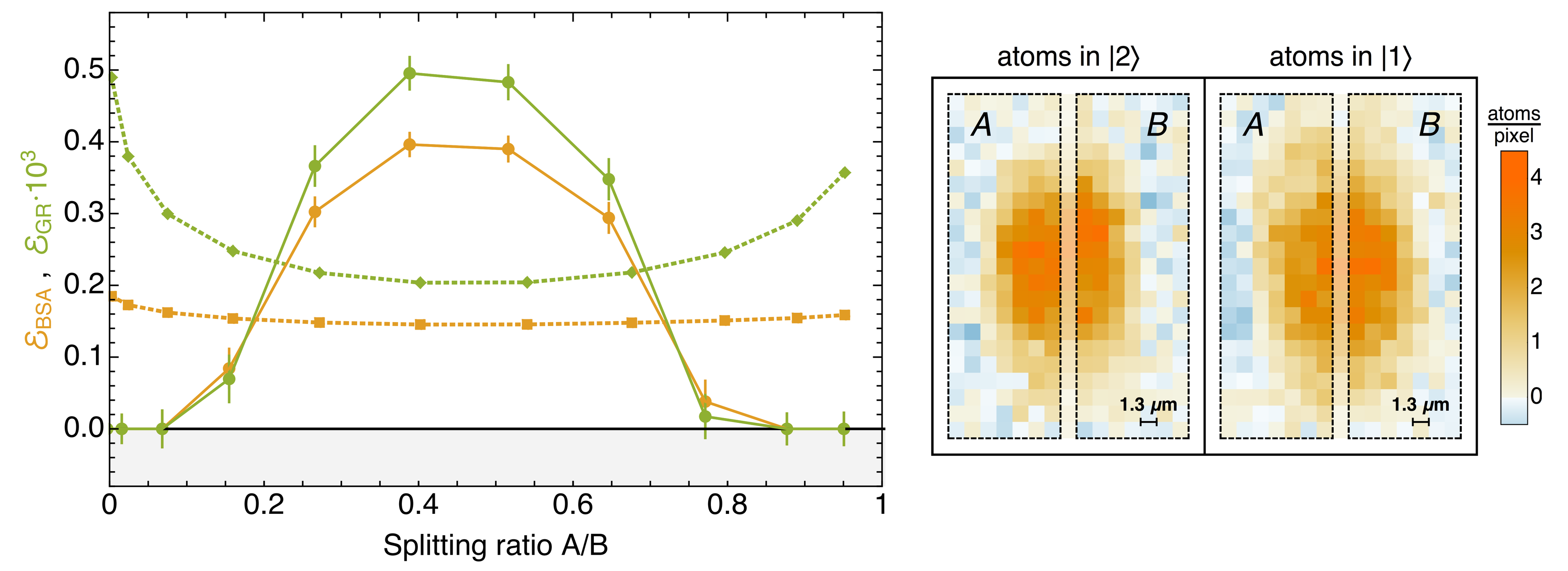
Analytically optimizing over t proves the claim. Similar reasoning can be done for the generalized robustness. ■

Application II: Split spin-squeezed state

For a split spin squeezed state we can use a criterion from Giovannetti *et al.* [2]:

$$\mathcal{G}^2 := \frac{\Delta^2(g_z J_z^A + J_z^B)\Delta^2(g_y J_y^A + J_y^B)}{\left(|g_z g_y|\,|\langle J_x^A\rangle| + |\langle J_x^B\rangle|\right)^2/4} \geq 1 \tag{16}$$

for $g_z, g_y \in \mathbb{R}$. The largest lower bound on $\mathcal{E}_{\rm BSA}$ and $\mathcal{E}_{\rm GR}$ arises from a minimization over g_z and g_y.



Left: Lower bounds on the BSA and the GR, as obtained from Eq. (16) according to Lemma 3. Data from [5] with $N = 590$ atoms. The dotted lines show the maximum amount of entanglement that could be explained by detection cross-talk [5]. Right: single-shot absorption images of the atomic densities for the two internal degrees of freedom, with an example of regions A and B.

References

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