TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Diplomarbeit

Spectral Multiplicity for Operators on a Hilbert Space

zur Erlangung des akademischen Grades

Diplom-Ingenieur

im Rahmen des Studiums

Technische Mathematik

eingereicht von

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ausgeführt am Institut für Analysis und Scientific Computing
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Wien, 15.02.2023

(Verfasser) (Betreuer)

Acknowledgements

First, I want to thank my supervisor Univ. Prof. Dr. Harald Woracek for kindling my interest in functional analysis during his excellent classes, and for his continuous support in all matters concerning this thesis.
Further, I would like to thank all of my friends and companions at university. You provided invigorating discussions that strengthened my appreciation of mathematics, and you convinced me to start and finish a second master.
Finally, I also want to thank my family, whose support allowed me to study in the first place. Without you, this wouldn't have been possible.

Abstract

As an underlying theory, functional analysis is central to many fields in mathematics and theoretical physics, such as variational calculus or quantum mechanics, and (bounded) linear operators often play a major role. For example in quantum theory, the states can be described as elements of a suitable Hilbert Space, with hermitian linear operators taking the role of measurements. Similarly, linear differential equations can often be understood as operator equations with a linear differential operator, and solutions can be found by applying functional analysis. This means there is a large benefit in better understanding such operators, and maybe even classifying them.

Unfortunately, a unifying theory for general linear operators on Banach spaces has yet to be discovered. However, for the reduced problem of normal bounded linear operators on separable Hilbert spaces this is actually possible. The so called *Multiplicity Theory* gives a complete classification of such operators according to their spectra and spectral multiplicity, as it extends the finite-dimensional idea of classifying normal matrices by their spectra and spectral multiplicity.

This thesis aims to give a (relatively) self-contained introduction to Multiplicity Theory for an interested reader with a basic university education in mathematics. As a capstone, the proof of the multiplicity theorem will be presented, which describes the aforementioned classification of normal bounded linear operators on separable Hilbert spaces.

Kurzfassung

Funktionalanalysis ist ein wichtiges Fundament für viele Gebiete der Mathematik und theoretischen Physik, wobei (beschränkte) lineare Operatoren oft eine zentrale Rolle spielen. Beispielsweise werden in der Quantentheorie die Quantenzustände als Elemente eines geeigneten Hilbertraumes interpretiert, während Messungen durch hermitesche Operatoren dargestellt werden. Weiters werden lineare Differentialgleichungen oft als Operatorgleichungen mit einem linearen Differentialoperator verstanden, und Lösungen können mit Hilfe der Funktionalanalysis ermittelt werden. Aufgrund dieser Tatsachen ist ein tieferes Verständnis von solchen Operatoren von großem Nutzen.

Leider gibt es bis dato keine allgemeine Theorie, die sämtliche linearen Operatoren auf Banachräumen klassifiziert. Das reduzierte Problem für normale, beschränkte lineare Operatoren auf separablen Hilberträumen hat jedoch in der Tat eine derartige Klassifizierung. Die sogenannte *Multiplizitätstheorie* beschreibt sämtliche solche Operatoren eindeutig mit Hilfe ihres Spektrums und ihrer spektralen Vielfachheit. Dabei ist die Theorie eine Erweiterung der endlich-dimensionalen Klassifizierung von normalen Matrizen nach ihrem Spektrum und der spektralen Vielfachheit.

Das Ziel dieser Arbeit ist es, eine (soweit möglich) in sich geschlossene Einführung in das Gebiet der Multiplizitätstheorie zu geben, wobei am Ende der Beweis des Satzes über Multiplizitätstheorie steht. Dabei richtet sich diese Arbeit an interessierte Leser_innen mit einer Grundausbildung in Hochschulmathematik.

Contents

Chapter 1

Introduction

The goal of this thesis is to provide the reader with an understandable and focused introduction to multiplicity theory on separable Hilbert spaces. To this end, it provides a reworked form of [2] with many proofs added and the structure rearranged to provide a more concise experience. This text is suited for students who have just completed their BSc. in mathematics, as it goes through all relevant steps without requiring in depth knowledge of topics such as functional analysis or measure theory. Of course, it will hopefully be appealing to people further down their educational path as well, if they desire to familiarize themselves with the topic. The text states all used lemmata, propositions and theorems clearly, so it is possible to skip parts and only go back when something is unclear.

This thesis consists of four chapters, with the first chapter (evidently) being the introduction. Afterwards, the second chapter concerns itself with the notation used in this work. The third chapter contains the core of this thesis, and it consists of the development of multiplicity theory on separable Hilbert spaces. It is divided into several sections, each containing a coherent step in developing the theory. Finally, the fourth chapter holds the appendix, where all prerequisite theorems and propositions are collected for reference, in addition to sources.

For ease of writing and reading in the subsequent text, the author has chosen to employ the pronoun "we" and written the thesis in first person plural.

1.1 Understanding Multiplicity Theory

The idea of multiplicity theory is to classify all normal operators on separable Hilbert spaces according to the multiplicity of their spectrum. To understand how this might work, we first look at the finite-dimensional case. We know that each normal matrix M is diagonalizable with the form

$$M = U^{-1}DU = U^{-1}\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & \lambda_n \end{pmatrix} U \quad .$$

Since U is unitary we know that D and M are the same modulo a Hilbert space homomorphism, and we denote $M \simeq D$. If we reorder the eigenvalues to obtain a new diagonal matrix $\tilde{D}$, we can still find a unitary $\tilde{U}$ so that $M \simeq \tilde{D}$. Therefore it only matters which eigenvalues appear and how often they appear, and we can classify all normal matrices in $\mathbb{R}^{n\times n}$ up to a unitary transformation just by their eigenvalues.

The Central Multiplicity Theorem We now want to generalize this result, and during the course of this thesis we will prove that this is indeed possible for separable Hilbert spaces! According to Theorem 3.10.7, a normal operator N on a separable Hilbert space is (up to a unitary transformation) uniquely defined by the following objects:

- A measure μ with $\mathrm{supp}(\mu) = \sigma(N)$.
- A multiplicity function $\# : \mathbb{C} \mapsto \{0, 1, 2, ..., \infty\}$ for which we find $\# \geq 1$ μ-almost everywhere.

There are of course some differences to the finite-dimensional case, since we now deal with infinitely large vector spaces. First, the discrete spectrum of eigenvalues is replaced by the possibly continuous spectrum $\sigma(N)$. Second, we can find that certain sets of the spectrum "appear" and infinite number of times, hence the possibility for $\#$ to take on the value ∞. Further, we can understand the measure μ as the weight of our continuous "eigenvalues", and the concept should be familiar from measure theory and L^2 spaces. Taking this all into account, we still get a function that counts the "number of appearances" for our "eigenvalues" and which (together with μ) uniquely defines N.

1.2 A Short Overview of our Approach

The insights we discussed above will only be the end point of our foray into multiplicity theory in Section 3.10, and on our way there we will actually prove a set of such theorems. The reason for this is that they build on each other, so we have to start with a slightly different classification. Therefore we will give here a quick overlook of the three theorems that will occupy our attention in Sections 3.8 and 3.9.

First Multiplicity Theorem To understand the first theorem, we go back to the finite-dimensional case and inspect a normal matrix $M \simeq D$. As an example, we set $D = \mathrm{diag}(1, -3, 1, 5, 2, 2, -3, 2)$. We can now rearrange the eigenvalues to get

$$M \simeq \tilde{D} = \mathrm{diag}(5, -3, 1, 2, -3, 1, 2, 2) \quad . \tag{1.1}$$

Although it might not be obvious at first, we have just ordered the eigenvalues into the sets $(5, -3, 1, 2)$, $(-3, 1, 2)$ and (2). In this way, each set contains a certain eigenvalue only once. We can now deconstruct

$$\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^3 \oplus \mathbb{R} \quad .$$

The matrix $\tilde{D}$ also lends itself to such a deconstruction, and we see

$$M \simeq \tilde{D} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \oplus \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \oplus (2) \quad . \tag{1.2}$$

Again, the normal matrix M can be (up to a unitary transformation) uniquely defined by this form, and we will see in Theorem 3.8.8 that a similar fact holds for a normal operator N on a separable Hilbert space.The theorem gices us a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ so that $\operatorname{supp}(\mu_1) = \sigma(N)$, $\mu_{n+1} \ll \mu_n$ for all $n \in \mathbb{N}$ and

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu_n} \quad .$$

Here, the operators N_{μ_n} are operators on $L^2(\mu_n)$ that mirror the diagonal matrices above. On the other hand, the fact that $\mu_{n+1} \ll \mu_n$ for all $n \in \mathbb{N}$ is similar to how we ordered the diagonal matrices by decreasing size, with each set being a subset of the previous set.

Second Multiplicity Theorem The second Theorem 3.9.2 will have a very similar form to the first. The key difference is that we will replace the measures $(\mu_n)_{n\in\mathbb{N}}$ by a single measure μ and a sequence of Borel sets $(\Delta_n)_{n\in\mathbb{N}}$ with $\Delta_1 = \sigma(N)$, $\Delta_{n+1} \subseteq \Delta_n$ for all $n \in \mathbb{N}$ and

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}} \quad .$$

When compared to our matrix, this corresponds to Equation 1.2 as well.

Third Multiplicity Theorem For the third theorem, we will go back to the matrix $M \simeq D$ in Equation 1.1. This time, we rearrange the eigenvalues in another way to obtain

$$M \simeq D' = \operatorname{diag}(5, -3, -3, 1, 1, 2, 2, 2) \quad .$$

Further, we divide $\mathbb{R}^8$ differently this time and get

$$\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^3 \quad .$$

Similarly as above, we can now deconstruct

$$M \simeq \tilde{D} = (5) \oplus \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad . \tag{1.3}$$

In this case, we have taken all similar eigenvalues together and "put them" into separate subspaces. Again, we find a corresponding generalization for a normal operator N on a separable Hilbert space in the form of Theorem 3.9.5. The theorem tells us that there are measures μ_∞ and $(\mu_n)_{n\in\mathbb{N}}$ so that all of them are mutually singular and

$$N \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{n=1}^{\infty} (N_{\mu_n})^n \quad .$$

We see that the measure μ_n corresponds to $(N_{\mu_n})^n$, that is the direct so of n times N_{μ_n}, which is similar to the diagonal matrices in Equation 1.3 consisting of n-times the same value. Further, the fact that μ_∞ and the $(\mu_n)_{n\in\mathbb{N}}$ are mutually singular is akin to the property that the matrices don't share any eigenvalues between them. The big difference to the finite-dimensional case here is the addition of $(N_{\mu_\infty})^\infty$ to account for all parts of the spectrum that "appear" infinitely often.

Now this last theorem already has a striking similarity to our initial considerations. This is especially apparent for the finite-dimensional case in Equation 1.3, where we basically have already ordered the eigenvalues by their multiplicity. Similarly for the infinite-dimensional case, we can understand $\mathrm{supp}(\mu_n)$ as the set of spectral values with multiplicity n and $\mathrm{supp}(\mu_\infty)$ as the set of spectral values with multiplicity ∞. Although the actual proof is unfortunately a bit more complex, this will be our guiding idea in Section 3.10 when we prove the central multiplicity Theorem 3.10.7.

Before we head off, it is important to note that we have just named these four theorems (First -, Second -, Third -, Central Multiplicity Theorem) here to emphasize their role and order within this thesis. Outside of this work, they bear no special label.

Chapter 2

Notation

In this chapter we will introduce the notation used throughout the thesis, along with some preliminary definitions. Each conceptual group is listed separately, so that we have a better overview.

2.1 Analysis

- Let $(X, \mathscr{T})$ be a topological space, and let $A \subseteq X$. Then we denote the *closure of* A by $\mathrm{cl}(A)$ and the *interior of* A by $\mathrm{int}(A)$.
- We denote nets with $(x_i)_{i \in I}$ indexed by I. If the net converges to x, we write $x_i \to x$. As long as no explicit topology is stated, we always take this convergence with respect to the norm topology.
- Let $f : X \mapsto Y$ be a function and let $X_s \subset X$. Then we denote the restriction of f to X_s as $f|_{X_s} : X_s \mapsto Y$.
- Let $x, y \in \mathbb{R}$. We denote the open interval between them by (x, y) and the closed interval by $[x, y]$. In this vein we also denote the half-open intervals $(x, y]$ and $[x, y)$.
- We denote the complex conjugate of $z \in \mathbb{C}$ by $\overline{z}$, with a similar notation for functions.

2.2 Hilbert Spaces

- Unless explicitly stated otherwise, $\mathscr{H}$ is always a Hilbert space.
- We denote the scalar product of $x, y \in \mathscr{H}$ as $\langle x, y \rangle_{\mathscr{H}}$. If the base Hilbert space is obvious, we may omit the subscript and write $\langle x, y \rangle = \langle x, y \rangle_{\mathscr{H}}$.
- We denote the topological dual space of $\mathscr{H}$ by $\mathscr{H}^*$.
- We denote orthogonal vectors x, y by $x \perp y$. A similar notation applies for orthogonal subspaces.
- For a set $\mathscr{K} \subseteq \mathscr{H}$ we denote the orthogonal complement of $\mathscr{K}$ by $\mathscr{K}^\perp$.
- We denote the specific Hilbert space of all absolute square summable sequences in $\mathbb{C}$ as l^2.

- If there exists a unitary operator $U : \mathscr{H} \mapsto \mathscr{K}$, then we call $\mathscr{H}$ and $\mathscr{K}$ unitarily equivalent and write $\mathscr{H} \simeq \mathscr{K}$.

2.3 Linear Bounded Operators

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces.

- We write $\mathscr{B}(\mathscr{H}, \mathscr{K})$ for the set of all bounded linear functions that map $\mathscr{H}$ to $\mathscr{K}$. If $\mathscr{K} = \mathscr{H}$, we simplify the notation to $\mathscr{B}(\mathscr{H})$.
- Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$. Then we denote the kernel by $\ker A$ and the range by $\operatorname{ran} A$.
- Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$. Then we denote by $A^* \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ the adjoint operator of A.
- Let $\lambda \in \mathbb{C}$. Then we denote by $\lambda_{\mathscr{H}} \in \mathscr{B}(\mathscr{H})$ the multiplication by λ on $\mathscr{H}$.
- Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ and $h \in \mathscr{H}$. Then we denote $\mathscr{A}h$ for the set $\{Ah : A \in \mathscr{A}\}$.
- For $A \in \mathscr{B}(\mathscr{H})$ and $\mathscr{H}_s \subseteq \mathscr{H}$, we have similar notations for $A\mathscr{H}_s$ and $\mathscr{A}\mathscr{H}_s$.
- Let $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{H})$. If there exists a unitary operator $U : \mathscr{H} \mapsto \mathscr{K}$ and we have $UAU^{-1} = B$, then we call A and B unitarily equivalent and we write $A \simeq B$.

In addition, we recall the following definitions:

Definition 2.3.1. Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$. Then we call $\mathscr{A}$ an *algebra* if for all $\lambda \in \mathbb{C}$ and $A, B \in \mathscr{A}$ we find

(a) $\lambda A \in \mathscr{A}$.

(b) $A + B \in \mathscr{A}$.

(c) $AB \in \mathscr{A}$.

Definition 2.3.2. Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$. Then we call $\mathscr{A}$ a C^**-algebra* if and only if $\mathscr{A}$ is an algebra, closed and for all $A \in \mathscr{A}$ we find $A^* \in \mathscr{A}$.

- Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$. Then we denote the smallest C^*-algebra containing $\mathscr{A}$ as $C^*(\mathscr{A})$.

2.4 Direct Sum

- We denote the direct sum of two Hilbert spaces, vectors or operators by $\oplus$, e.g. $\mathscr{H} \oplus \mathscr{K}$.
- If $\mathscr{H}_1, \mathscr{H}_2 \subseteq \mathscr{H}$ and $\mathscr{H}_1 \perp \mathscr{H}_2$, we will identify $\mathscr{H}_1 \oplus \mathscr{H}_2 = \mathscr{H}_1 + \mathscr{H}_2$. The same holds true for vectors $h_1 \in \mathscr{H}_1$ and $h_2 \in \mathscr{H}_2$ with $h_1 \oplus h_2 = h_1 + h_2$.
- We denote the direct sum of a sequence of vectors, operators or Hilbert spaces by $\bigoplus_{n=1}^{N}$, with $N \in \mathbb{N}$, e.g. $\bigoplus_{n=1}^{N} \mathscr{H}_n$. To denote countable direct sums, we replace N by ∞, e.g. $\bigoplus_{n=1}^{\infty} \mathscr{H}_n$.

- We denote $\mathscr{H}^N = \bigoplus_{n=1}^{N} \mathscr{H}$, and for $A \in \mathscr{B}(\mathscr{H})$ we denote $A^N = \bigoplus_{n=1}^{N} A$. Again we denote countable powers by replacing N by ∞, that is $\mathscr{H}^\infty$ and A^∞.
- Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$. Similarly as above we denote $\mathscr{A}^N = \{A^N : A \in \mathscr{A}\}$ and $\mathscr{A}^\infty = \{A^\infty : A \in \mathscr{A}\}$.

2.5 Measure Theory

Let (X, Ω) be a measurable space, let μ, ν be measures on (X, Ω) and let $\Delta \in \Omega$.

- We denote by $B^\Omega(X, \mathbb{C})$ the set of complex-valued, bounded functions $X \mapsto \mathbb{C}$ which are measurable with respect to Ω. If X is a subset of $\mathbb{C}$ and Ω is the Borel algebra, we abbreviate the notation to $B(X)$.
- We denote the complement of Δ as $\Delta^c = X \setminus \Delta$.
- We denote the indicator function of Δ by $\chi_\Delta : X \mapsto \{0, 1\}$, that is $\chi_\Delta(x) = 1$ for $x \in \Delta$ and $\chi_\Delta(x) = 0$ else.
- If μ is absolutely continuous with respect to ν we write $\mu \ll \nu$. If $\nu \ll \mu$ holds as well, we write $[\mu] = [\nu]$.
- We denote the restriction of μ to Δ as $\mu|_\Delta$. That means for $\omega \in \Omega$ we have $\mu|_\Delta(\omega) = \mu(\Delta \cap \omega)$.

In addition, we remember the following definitions:

Definition 2.5.1. The measures μ and ν are *mutually singular* if and only if there exists a set $\omega \in \Omega$ so that $\mu(\omega^c) = 0$ and $\nu(\omega) = 0$.

Definition 2.5.2. A *Radon measure* μ is a Borel measure on $\mathbb{C}$ with the following properties

(a) μ is locally finite. That means for every $z \in \mathbb{C}$ there exists a neighbourhood U_z so that $\mu(U_z) < \infty$.

(b) μ is inner regular. That means for every open set U we find that $\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ compact}\}$.

Remark. Radon measures can be defined generally for topological spaces, but in this thesis we restrict ourselves to $\mathbb{C}$.

2.6 Spectral Measures

- Let $A \in \mathscr{B}(\mathscr{H})$. Then we denote by $\sigma(A)$ the spectrum of A and $\rho(A) = \mathbb{C} \setminus \sigma(A)$ as the resolvent of A.

The last piece of notation requires an understanding of spectral measures and the spectral Theorem, which we find in the Appendix under Definition 4.6.1, Theorem 4.6.4 and Theorem 4.6.3.

- Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator, let E be the spectral measure and let $\phi \in B(\sigma(N))$. Then we denote the spectral integral of ϕ by $\phi(N) = \int \phi(z) dE(z)$.

Chapter 3

Multiplicity Theory

3.1 Topologies on $\mathscr{B}(\mathscr{H})$

We start by investigating two topologies on $\mathscr{B}(\mathscr{H})$, namely the *weak operator topology* (WOT) and the *strong operator topology* (SOT). On a first glance, they don't have much to do with normal operators or multiplicity theory, but we need certain properties for Sections 3.2 and 3.4 when we will be grappling with specific subalgebras of $\mathscr{B}(\mathscr{H})$. Our crowning achievements in this section are Corollary 3.1.5 and Proposition 3.1.6, which will help us formulate Theorem 3.2.6 later on. We start now by defining WOT and SOT and proving some straightforward propositions.

Definition 3.1.1. Let $\mathscr{H}$ be a Hilbert space. We take the *weak operator topology* (WOT) on $\mathscr{B}(\mathscr{H})$ as the locally convex topology defined by the family of seminorms $\{p_{h,k} : h, k \in \mathscr{H}\}$, where $p_{h,k}(A) := |\langle Ah, k\rangle|$ for all $A \in \mathscr{B}(\mathscr{H})$. Similarly, we take the *strong operator topology* (SOT) on $\mathscr{B}(\mathscr{H})$ as the locally convex topology defined by the family of seminorms $\{p_h : h \in \mathscr{H}\}$, where $p_h(A) := ||Ah||$ for all $A \in \mathscr{B}(\mathscr{H})$.

Proposition 3.1.1. *Let $(A_i)_{i\in I}$ be a net in $\mathscr{B}(\mathscr{H})$. We have $A_i \xrightarrow{\text{WOT}} A$ if and only if $A_i h \xrightarrow{\text{weak}} Ah$ for all $h, k \in \mathscr{H}$. Similarly, we also have $A_i \xrightarrow{\text{SOT}} A$ if and only if $A_i h \to Ah$ for all $h \in \mathscr{H}$.*

Proof. The proposition follows directly from Theorem 4.3.2. □

Proposition 3.1.2. *Let $\mathscr{H}$ be a separable Hilbert space. Then the* WOT *and* SOT *are metrizable on bounded subsets of $\mathscr{B}(\mathscr{H})$*

Proof. We will show the proposition for WOT. The proof for SOT is almost identical, only changing the expressions $|\langle (A-B)b_m, b_n\rangle|$ to $||(A-B)b_n||$ and just summing over n.

Let $(b_n)_{n\in\mathbb{N}}$ be a dense, countable subset of the unit ball of $\mathscr{H}$, which exists since $\mathscr{H}$ is separable. For $A, B \in \mathscr{B}(\mathscr{H})$, we define

$$d_W(A,B) := \sum_{m,n=1}^{\infty} 2^{-(m+n)} |\langle (A-B)b_m, b_n\rangle| \quad .$$

Since $||b_n|| \leq 1$, the sum can be estimated by $2^{-(m+n)}||A-B||$ and thus converges. By using the triangle inequality for norms, we get $d_W(A,B) \leq d_W(A,C) + d_W(C,B)$. Further, if $d_W(A,B) = 0$, we have $A - B = 0$ on a dense subset of of the unit ball of $\mathscr{H}$. Because $A - B$ is continuous, this means $A - B = 0$ on the unit ball and thus on $\mathscr{H}$. Therefore

d_W is a metric on $\mathscr{B}(\mathscr{H})$. Let us now consider a bounded net $(A_i)_{i\in I}$ in $\mathscr{B}(\mathscr{H})$ with $||A_i|| < C$. First we assume that $A_i \to A$ with respect to d_W, that is $d_W(A_i, A) \to 0$. Since $(b_n)_{n\in\mathbb{N}}$ is dense in the unit ball of $\mathscr{H}$, for any $h, k \in \mathscr{H}$ we find subsequences $b_{n_h} \to e_h := h/||h||$ and $b_{n_k} \to e_k := k/||k||$. We remember that $||b_n|| \leq 1$ and now look at the term $\langle (A_i - A)h, k\rangle$ to get

$$|\langle (A_i - A)e_h, e_k\rangle| =$$

$$|\langle A_i(e_h - b_{n_h}), e_k\rangle + \langle A_i b_{n_h}, (e_k - b_{n_k})\rangle + \langle (A_i - A)b_{n_h}, b_{n_k}\rangle - \langle A b_{n_h}, (e_k - b_{n_k})\rangle - \langle A(e_h - b_{n_h}), e_k\rangle|$$
$$\leq (C + ||A||)(||e_h - b_{n_h}|| + ||e_k - b_{n_k}||) + |\langle (A_i - A)b_{n_h}, b_{n_k}\rangle| \quad .$$

For a given $\epsilon > 0$, we can choose b_{n_h} and b_{n_k} such that the first term becomes $< \frac{\epsilon}{2}$. Because of $d_W(A_i, A) \to 0$, we also find

$$|\langle (A_i - A)b_{n_h}, b_{n_k}\rangle| \leq 2^{n_h + n_k} d_W(A_i, A) \to 0 \quad .$$

This means $\langle A_i e_h, e_k\rangle \to \langle A e_h, e_k\rangle$ and therefore also $\langle A_i h, k\rangle \to \langle Ah, k\rangle$ for any $h, k \in \mathscr{H}$.

Conversely, assume that $A_i \to A$ with respect to the WOT. Since $|\langle (A_i - A)b_m, b_n\rangle| \leq C + ||A||$, for a given $\epsilon > 0$, we can choose m_0, n_0 such that

$$d_W(A_i, A) < \sum_{m\leq m_0, n\leq n_0} 2^{-(m+n)} |\langle (A_i - A)b_m, b_n\rangle| + \frac{\epsilon}{2} \quad .$$

$A_i \to A$ in WOT implies that $\langle A_i b_m, b_n\rangle \to \langle A b_m, b_n\rangle$ for all m, n. Because the sum consists of a finite amount of terms, we have $d_W(A_i, A) \to 0$.

Together this means that the bounded net $(A_i)_{i\in I}$ converges with respect to d_W if and only if it converges in the WOT. Thus, restricted to a bounded subset of $\mathscr{B}(\mathscr{H})$, the closed sets of both topologies are the same and therefore the topologies are identical. □

Lemma 3.1.3. *Let X be the locally convex topological vector space generated by the family of seminorms M, and let $f : X \mapsto \mathbb{C}$ be a continuous linear functional. Then there are $p_1, ..., p_n \in M$ and $\alpha \in \mathbb{R}$ such that $|f(x)| \leq \alpha \sum_{k=1}^{n} p_k(x)$ for all $x \in X$.*

Proof. Let us assume the contraposition, that is that for every $\alpha \in \mathbb{R}$ and for every finite $P \subseteq M$ there is an $x_{\alpha,P} \in X$ such that we have

$$|f(x_{\alpha,P})| > \alpha \sum_{p\in P} p(x_{\alpha,P}) \quad .$$

We then rescale $\tilde{x}_{\alpha,P} := x_{\alpha,P}/|f(x_{\alpha,P})|$, which gives us for any $p \in P$ the inequality

$$\frac{1}{\alpha} > \sum_{p\in P} p(\tilde{x}_{\alpha,P}) \geq p(\tilde{x}_{\alpha,P}) \quad .$$

We make $\tilde{x}_{\alpha,P}$ into a net by defining $(\alpha, P) \preceq (\tilde{\alpha}, \tilde{P})$ if $\alpha \leq \tilde{\alpha}$ and $P \subseteq \tilde{P}$. Because of the previous inequality, we have that $p(\tilde{x}_{\alpha,P}) \to 0$ for all $p \in M$. According to Theorem 4.3.2, this means that $\tilde{x}_{\alpha,P} \to 0$. However, we also find that $|f(\tilde{x}_{\alpha,P})| \equiv 1$. This is a contradiction to the fact that f is linear and continuous. □

Proposition 3.1.4. *Let $\mathscr{H}$ be a Hilbert space. If $L : \mathscr{B}(\mathscr{H}) \to \mathbb{C}$ is a linear functional, then the following statements are equivalent:*

(a) L is SOT *continuous*

(b) L is WOT *continuous*

(c) There are vectors $g_1, ..., g_n$ and $h_1, ..., h_n$ in $\mathscr{H}$ such that for every $A \in \mathscr{B}(\mathscr{H})$ we have $L(A) = \sum_{k=1}^{n} \langle Ag_k, h_k \rangle$

Proof. We will proove the proposition by showing $(c) \Rightarrow (b) \Rightarrow (a) \Rightarrow (c)$.

$(c) \Rightarrow (b)$ Since the terms $\langle Ag_k, h_k \rangle$ are explicitly WOT continuous, so is their sum.

$(b) \Rightarrow (a)$ From Proposition 3.1.2 we see that $A_i \xrightarrow{\text{SOT}} A$ implies $A_i \xrightarrow{\text{WOT}} A$, and thus WOT is coarser than SOT. This means that WOT continuity implies SOT continuity.

$(a) \Rightarrow (c)$ By Lemma 3.1.3 and the fact that the SOT is generated by the seminorms $p_h(A) = ||Ah||$, there exist vectors $g_1, ..., g_n \in \mathscr{H}$ such that

$$|L(A)| \leq \sum_{k=1}^{n} ||Ag_k|| \leq \sqrt{n} \left(\sum_{k=1}^{n} ||Ag_k|| \right)^{\frac{1}{2}} \quad .$$

We rescale $\tilde{g}_k := \sqrt{n} g_k$ and take a look at $\mathscr{K} := \{A\tilde{g}_1 \oplus ... \oplus A\tilde{g}_n : A \in \mathscr{B}(\mathscr{H})\}$ as a subset of $\mathscr{H}^n$. $F(A\tilde{g}_1 \oplus ... \oplus A\tilde{g}_n) := L(A)$ is then a linear functional on $\mathscr{K}$ with

$$F(A\tilde{g}_1 \oplus ... \oplus A\tilde{g}_n) \leq \left(\sum_{k=1}^{n} ||A\tilde{g}_k|| \right)^{\frac{1}{2}} = ||A\tilde{g}_1 \oplus ... \oplus A\tilde{g}_n|| \quad .$$

Using Theorem 4.3.3 and setting $||.||$ as the required seminorm, we can extend F to a continuous linear functional $\tilde{F}$ on $\mathscr{H}^n$. Theorem 4.4.1 now gives us a vector $h_1 \oplus ... \oplus h_n \in \mathscr{H}^n$ such that for all $f_1 \oplus ... \oplus f_n \in \mathscr{H}^n$ we have

$$\tilde{F}(f_1 \oplus ... \oplus f_n) = \langle f_1 \oplus ... \oplus f_n, h_1 \oplus ... \oplus h_n \rangle = \sum_{k=1}^{n} \langle f_k, h_k \rangle \quad .$$

The fact that $\tilde{F}(A\tilde{g}_1 \oplus ... \oplus A\tilde{g}_n) = L(A)$ concludes the proof. □

Proposition 3.1.4 tells us that sets defined as the SOT-closure with an SOT-continuous functional are automatically also WOT-closed. If we apply the Hahn-Banach Theorem 4.3.4 to the SOT-Topology, we get exactly such a case, and we can leverage this in the following Corollary.

Corollary 3.1.5. *If $\mathscr{C}$ is a convex subset of $\mathscr{B}(\mathscr{H})$, the* WOT *closure of $\mathscr{C}$ equals the* SOT *closure of $\mathscr{C}$.*

Proof. Proposition 3.1.2 shows that WOT is coarser than SOT and thus SOT-cl$(\mathscr{C}) \subseteq$ WOT-cl$(\mathscr{C})$. Take now an $A \notin$ SOT-cl$(\mathscr{C})$. Because of Theorem 4.3.4, we find a SOT-continuous linear functional L and $\gamma \in \mathbb{R}$ such that Re $L(A) > \gamma$ and

$$\mathscr{C} \subseteq K := \{B \in \mathscr{B}(\mathscr{H}) : \text{Re } L(B) \leq \gamma\} \quad .$$

Because of Proposition 3.1.4, L is also WOT continuous and therefore WOT-cl$(\mathscr{C}) \subseteq$ WOT-cl$(K) = K$. Therefore we have $A \notin$ WOT-cl$(\mathscr{C})$, which gives us WOT-cl$(\mathscr{C}) \subseteq$ SOT-cl$(\mathscr{C})$ and thus WOT-cl$(\mathscr{C}) =$ SOT-cl$(\mathscr{C})$. □

Finally, we want to find another formulation for the SOT-closure of subalgebras of $\mathscr{B}(\mathscr{H})$. To this end we need the notion of invariant subspaces.

Definition 3.1.2. Let $\mathscr{H}$ be a Hilbert space and let $A \in \mathscr{B}(\mathscr{H})$. We define an *invariant subspace* $\mathscr{M} \subseteq \mathscr{H}$ for A as a closed linear subspace such that $A\mathscr{M} \subseteq \mathscr{M}$, and we call the collection of all such invariant subspaces Lat$A$. For a set $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ we denote Lat$\mathscr{S} := \bigcap_{S\in\mathscr{S}}$ Lat$S$.

Proposition 3.1.6. *If $\mathscr{A}$ is a subalgebra of $\mathscr{B}(\mathscr{H})$ containing 1, then we have*

$$\text{SOT-cl}(\mathscr{A}) = \{B \in \mathscr{B}(\mathscr{H}) : \text{Lat}\mathscr{A}^n \subseteq \text{Lat}B^n \ \text{ for all } n \in \mathbb{N}\} \quad .$$

Proof. Let us denote the set on the right hand side by $\mathscr{K}$. First we show that SOT-cl$(\mathscr{A}) \subseteq \mathscr{K}$. Assume that $A_i \xrightarrow{\text{SOT}} A$ and $(A_i)_{i\in I} \subseteq \mathscr{A}$. Take $n \in \mathbb{N}$ and we get $(A_i^n)_{i\in I} \subseteq \mathscr{A}^n$. Now for any $h = h_1 \oplus ... \oplus h_n \in \mathscr{H}^n$, we find that

$$A_i^n h = A_i h_1 \oplus ... \oplus A_i h_n \to A h_1 \oplus ... \oplus A h_n = A^n h \quad .$$

This means we also have $A_i^n \xrightarrow{\text{SOT}} A^n$. Now take $\mathscr{M}_n \in$ Lat$\mathscr{A}^n$. As an invariant subspace for all operators in $\mathscr{A}^n$, it satisfies $A_i^n \mathscr{M}_n \subseteq \mathscr{M}_n$. We then look at $A^n h$ for $h \in \mathscr{M}_n$, and we saw earlier that $A_i^n h \to A^n h$. But $(A_i^n h)_{i\in I} \in \mathscr{M}_n$, and $\mathscr{M}_n$ is closed, so $A^n h \in \mathscr{M}_n$ and thus $A^n \mathscr{M}_n \subseteq \mathscr{M}_n$. Taken for all $n \in \mathbb{N}$, this means that $A \in \mathscr{K}$.

Now we show that $\mathscr{K} \subseteq$ SOT-cl$(\mathscr{A})$. For $B \in \mathscr{K}$, we will construct a net $(A_i)_{i\in I} \subseteq \mathscr{A}$ so that $A_i \xrightarrow{\text{SOT}} B$. As indices, we take (ϵ, H), where $\epsilon > 0$ and $H = \{h_1, ..., h_n\}$ is a finite set of vectors in $\mathscr{H}$, and we define $(\epsilon, H) \preceq (\tilde{\epsilon}, \tilde{H})$ if $\epsilon > \tilde{\epsilon}$ and $H \subseteq \tilde{H}$. We now want to find $A_{\epsilon,H} \in \mathscr{A}$ so that for any finite set H and any vector $h \in H$, we have $p_h(A_{\epsilon,H} - B) = ||(A_{\epsilon,H} - B)h|| < \epsilon$. Since the p_h are just the generating seminorms of SOT, we have by Theorem 4.3.2 that $A_{\epsilon,H} \xrightarrow{\text{SOT}} B$ and thus $B \in$ SOT-cl$(\mathscr{A})$.

Let $H = \{h_1, ..., h_n\}$ and $\epsilon > 0$ be fixed. Now let us inspect $S_H := \{Ah_1 \oplus ... \oplus Ah_n : A \in \mathscr{A}\}$. Because $\mathscr{A}$ is an algebra, S_H is a linear subspace invariant for all $A^n \in \mathscr{A}^n$ and thus cl$(S_H) \in$ Lat$\mathscr{A}^n \subseteq$ LatB^n. This means for all $x \in$ cl(S_H) we have $B^n x \in$ cl(S_H). Since $1 \in \mathscr{A}$, we have $1 \in \mathscr{A}^n$, which leads to $h_1 \oplus ... \oplus h_n \in S_H$ and therefore to $B^n(h_1 \oplus ... \oplus h_n) = Bh_1 \oplus ... \oplus Bh_n \in$ cl(S_H). Since S_H is dense in cl(S_H), we can choose an $A \in \mathscr{A}$, so that $||(A^n - B^n)(h_1 \oplus ... \oplus h_n)|| < \epsilon$. This means in particular that $||(A - B)h_j|| < \epsilon$ for all $h_j \in H$, and we can take the chosen A as $A_{\epsilon,H}$. □

3.2 The Commutant

In this section, we concern ourselves with the commutant of subsets of $\mathscr{B}(\mathscr{H})$. We will need this theory for the definition of von Neumann algebras in Section 3.4. The big star this time will be the Double Commutant Theorem 3.2.6, which links the properties of the double commutant to the topologies we discussed in the last section. In addition, we will discuss the commutant of algebras of the type $\mathscr{A}_\mu \subseteq \mathscr{B}(L^2(\mu))$ in Theorem 3.2.8. We will see in Section 3.5 that we can represent certain subspaces of $\mathscr{H}$ with $L^2(\mu)$ with specific measures μ, and thus $\mathscr{A}_\mu$ will become important later on. We once more start with a definition, and afterwards we will obtain some fundamental properties of the commutant.

Definition 3.2.1. If $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$, we define

$$\mathscr{S}' := \{A \in \mathscr{B}(\mathscr{H}) : AS = SA \quad \forall S \in \mathscr{S}\} \quad .$$

We call the set $\mathscr{S}'$ the *commutant* of $\mathscr{S}$. In a similar fashion, we define $\mathscr{S}'' := (\mathscr{S}')'$ as the *double commutant*.

Proposition 3.2.1. *Let $\mathscr{S}, \tilde{\mathscr{S}} \subseteq \mathscr{B}(\mathscr{H})$, and let $\mathscr{S} \subseteq \tilde{\mathscr{S}}$. Then we have $\tilde{\mathscr{S}}' \subseteq \mathscr{S}'$.*

Proof. We can express $\mathscr{S}'$ and $\tilde{\mathscr{S}}'$ differently by taking

$$\mathscr{S}' = \bigcap_{A \in \mathscr{S}} \{B \in \mathscr{B}(\mathscr{H}) : AB = BA\}$$

$$\tilde{\mathscr{S}}' = \bigcap_{A \in \tilde{\mathscr{S}}} \{B \in \mathscr{B}(\mathscr{H}) : AB = BA\} \quad .$$

Since $\mathscr{S} \subseteq \tilde{\mathscr{S}}$, we see that $\tilde{\mathscr{S}}' \subseteq \mathscr{S}'$. □

Proposition 3.2.2. *Let $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$. Then we have $(\mathscr{S}'')' = \mathscr{S}'$.*

Proof. For $A \in \mathscr{S}$, we have for all $B \in \mathscr{S}'$ that $AB = BA$ and thus $A \in \mathscr{S}''$ and $\mathscr{S} \subseteq \mathscr{S}''$. Similarly, we have $\mathscr{S}' \subseteq (\mathscr{S}'')'$. On the other hand, because $\mathscr{S} \subseteq \mathscr{S}''$, we can use Proposition 3.2.1 to get $(\mathscr{S}'')' \subseteq \mathscr{S}'$. □

Proposition 3.2.3. *Let $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$. Then $\mathscr{S}'$ is* WOT *closed.*

Proof. We take $A \in \mathscr{S}$ and the net $(B_i)_{i\in I} \subseteq \mathscr{S}'$ with $B_i \xrightarrow{\text{WOT}} B$. This means we have $B_iA = AB_i$ for all $i \in I$. For any $h, k \in \mathscr{H}$, we find

$$|\langle (AB_i - AB)h, k\rangle| = |\langle (B_i - B)h, A^*k\rangle| = |\langle (B_i - B)h, \tilde{k}\rangle| \to 0$$

$$|\langle (B_iA - BA)h, k\rangle| = |\langle (B_i - B)(Ah), k\rangle| = |\langle (B_i - B)\tilde{h}, k\rangle| \to 0 \quad .$$

Because of Theorem 4.3.2, this means that $B_iA \xrightarrow{\text{WOT}} BA$ and $AB_i \xrightarrow{\text{WOT}} AB$. So altogether we ca take the WOT limit of $B_iA = AB_i$ on both sides to get $BA = AB$ and thus find that $\mathscr{S}'$ is WOT-closed. □

Proposition 3.2.4. *Let $\mathscr{H}$ be a Hilbert space and $H \subseteq \mathscr{H}$ be a closed linear subspace. Let further P_H be the projection onto H and $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$. Then $P_H \in \mathscr{S}'$ if and only if $AH \subseteq H$ and $A^*H \subseteq H$ for every $A \in \mathscr{S}$.*

Proof. Let $h, h' \in \mathscr{H}$, take $A \in \mathscr{S}$ and assume that $AH \subseteq H$ and $A^*H \subseteq H$. We then have

$$\langle AP_Hh, h'\rangle = \langle P_HAP_Hh, h'\rangle = \langle P_Hh, A^*P_Hh'\rangle$$
$$= \langle h, P_HA^*P_Hh'\rangle = \langle h, A^*P_Hh'\rangle = \langle P_HAh, h'\rangle \quad .$$

This means that $P_H \in \mathscr{S}'$.

Now let us consider the reverse implication and assume $P_H \in \mathscr{S}'$. This means that also $(1 - P_H) \in \mathscr{S}'$. We take $A \in \mathscr{S}$, $h \in H$ and $k \in H^\perp$ and get

$$\langle Ah, k\rangle = \langle AP_H h, k\rangle = \langle P_H Ah, k\rangle = \langle Ah, P_H k\rangle = 0$$

$$\langle A^* h, k\rangle = \langle h, A(1 - P_H)k\rangle = \langle h, (1 - P_H)Ak\rangle = \langle (1 - P_H)h, Ak\rangle = 0 \quad .$$

This means that $AH \subseteq (H^\perp)^\perp = H$ and $A^*H \subseteq (H^\perp)^\perp = H$ for all $A \in \mathscr{S}$. □

Next, we will prove another short proposition and then fuse our previous insights with the properties about the SOT and WOT we have derived in Section 3.1 to obtain the Double Commutant Theorem.

Proposition 3.2.5. *Let $A_n \in \mathscr{B}(\mathscr{H}_n)$ for $n \in \mathbb{N}$, and let us define $A := \bigoplus_{n=1}^\infty A_n$. Then we have*

$$A^* = \bigoplus_{n=1}^\infty A_n^* \quad .$$

Remark. This means especially that A is normal if all $(A_n)_{n\in\mathbb{N}}$ are normal.

Proof. Let us designate $\mathscr{H} := \bigoplus_{n=1}^\infty \mathscr{H}_n$ and take $x, y \in \mathscr{H}$. We can write both as $x = \bigoplus_{n=1}^\infty x_n$ and $y = \bigoplus_{n=1}^\infty y_n$ with $x_n, x_n \in \mathscr{H}_n$ for all $n \in \mathbb{N}$. Now we calculate $\langle Ax, y\rangle_{\mathscr{H}}$ to see

$$\langle Ax, y\rangle_{\mathscr{H}} = \sum_{n=1}^\infty \langle A_n x_n, y_n\rangle_{\mathscr{H}_n} = \sum_{n=1}^\infty \langle x_n, A_n^* y_n\rangle_{\mathscr{H}_n} = \langle x, A^* y\rangle_{\mathscr{H}} \quad .$$

This concludes the proof. □

Theorem 3.2.6 (Double Commutant)**.** *Let $\mathscr{A}$ be a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ and $1 \in \mathscr{A}$. Then $\mathscr{A}''$ is the* SOT *(and also the* WOT*) closure of $\mathscr{A}$.*

Proof. By Proposition 3.2.3 we see that $\mathscr{A}'' = (\mathscr{A}')'$ is WOT-closed. Further, we see that $\mathscr{A}''$ is convex, and also $\mathscr{A} \subseteq \mathscr{A}''$. With Corollary 3.1.5, we can now deduce that SOT-cl$\mathscr{A}$ = WOT-cl$\mathscr{A} \subseteq \mathscr{A}''$.

For the other inclusion, we use Proposition 3.1.6 to show that $\mathscr{A}'' \subseteq$ SOT-cl$\mathscr{A}$. For $C \in \mathscr{A}''$ and $n \in \mathbb{N}$, we take $\mathscr{M} \in \mathrm{Lat}\mathscr{A}^n$ and must show that $C^n\mathscr{M} \subseteq \mathscr{M}$. Because $\mathscr{M} \in \mathrm{Lat}\mathscr{A}^n$, we get $A^n\mathscr{M} \subseteq \mathscr{M}$. Additionally, we have for every $A \in \mathscr{A}$ that $A^* \in \mathscr{A}$ and because of Proposition 3.2.5 we have $(A^*)^n = (A^n)^*$ and thus find $(A^n)^*\mathscr{M} \subseteq \mathscr{M}$. Now let $P_{\mathscr{M}}$ be the projection onto $\mathscr{M}$. Proposition 3.2.4 tells us that $P_{\mathscr{M}} \in (\mathscr{A}^n)'$. This means that $C^n P_{\mathscr{M}} = P_{\mathscr{M}} C^n$, which implies $C^n\mathscr{M} \subseteq \mathscr{M}$. Thus we have $C^n \in$ SOT-cl$\mathscr{A}$, which means $\mathscr{A}'' \subseteq$ SOT-cl$\mathscr{A}$ = WOT-cl$\mathscr{A}$. Together with our first calculations we find $\mathscr{A}''$ = SOT-cl$\mathscr{A}$ = WOT-cl$\mathscr{A}$ □

The Double Commutant Theorem is a powerful theorem that lets us calculate the commutant without actually investigating commuting properties, just by using the SOT- or WOT-closure. This will help us immensely in Section 3.4 when we get to actually calculating specific commutants.

Now we redirect our attention to the functions $L^\infty(\mu) \subseteq \mathscr{B}(L^2(\mu))$ and show the properties of $(L^\infty(\mu))'$.

Definition 3.2.2. Let (X, Ω, μ) be a measure space and $\phi \in L^\infty(\mu)$. We define $M_\phi \in \mathscr{B}(L^2(\mu))$ by $M_\phi f = \phi f$ as the *multiplication operator* of ϕ. Further, we denote the set of all such operators by $\mathscr{A}_\mu := \{M_\phi : \phi \in L^\infty(\mu)\}$.

Proposition 3.2.7. *Let (X, Ω, μ) be a σ-finite measure space $M_\phi \in \mathscr{A}_\mu$. Then $||M_\phi|| = ||\phi||_\infty$.*

Proof. For any $f \in L^2(\mu)$, we have the inequality

$$||M_\phi(f)||^2 = \int_X |\phi|^2 |f|^2 d\mu \leq ||\phi||_\infty^2 ||f||^2 \quad .$$

This means that $||M_\phi|| \leq ||\phi||_\infty$.

To prove $||\phi||_\infty \leq ||M_\phi||$, we consider that (X, Ω, μ) is σ-finite. Therefore we can take a sequence $\{\Delta_n\} \subseteq \Omega$ so that $0 < \mu(\Delta_n) < \infty$, $\bigcup_{n=1}^\infty \Delta_n = X$ and for $i \neq j$ we have $\Delta_i \cap \Delta_j = \emptyset$. Now for $\alpha \in \mathbb{R}_+$ we define the set $\omega_\alpha := \{x \in X : |\phi(x)| \geq \alpha\}$. If $\mu(\omega_\alpha) = \infty$, because of $\omega_\alpha = \bigcup_{n=1}^\infty (\Delta_n \cap \omega_\alpha)$ and $\mu(\Delta_n) < \infty$, we see that there is at least one n so that $0 < \mu(\Delta_n \cap \omega_\alpha) < \infty$. Therefore if $0 < \mu(\omega_\alpha)$, we can always find a $\tilde{\omega}_\alpha \subseteq \omega_\alpha$ with $0 < \mu(\tilde{\omega}_\alpha) < \infty$. Now for an $\alpha < ||\phi||_\infty$, we see that $0 < \mu(\omega_\alpha)$ and therefore we have

$$||M_\phi(\chi_{\tilde{\omega}_\alpha})||^2 = \int_{\tilde{\omega}_\alpha} |\phi|^2 d\mu \geq \alpha \mu(\tilde{\omega}_\alpha) = \alpha ||\chi_{\tilde{\omega}_\alpha}||^2 \quad .$$

Therefore $||M_\phi|| \geq \alpha$ for all $\alpha < ||\phi||_\infty$ and thus $||M_\phi|| \geq ||\phi||_\infty$. □

Theorem 3.2.8. *Let (X, Ω, μ) be a finite measure space. Then $\mathscr{A}_\mu = \mathscr{A}_\mu' = \mathscr{A}_\mu''$.*

Proof. Since the multiplication of functions in $L^\infty(\mu)$ is abelian, $\mathscr{A}_\mu$ is also abelian and so $\mathscr{A}_\mu \subseteq \mathscr{A}_\mu'$. We now prove the other inclusion $\mathscr{A}_\mu' \subseteq \mathscr{A}_\mu$. Since $\mu(X) < \infty$, we have $1 \in L^2(\mu)$. We take $A \in \mathscr{A}_\mu' \subseteq \mathscr{B}(L^2(\mu))$ and we can set $\phi := A(1)$ and get $\phi \in L^2(\mu)$. Let $\psi \in L^\infty(\mu)$, then because of $\mu(X) < \infty$ we have $L^\infty(\mu) \subseteq L^2(\mu)$ and $A(\psi) = AM_\psi(1) = M_\psi A(1) = M_\psi \phi = \psi\phi$. We now define $\omega_n := \{x \in X : |\phi(x)| \geq n\}$. Since $\chi_{\omega_n} \in L^\infty(\mu)$, we can set $\psi = \chi_{\omega_n}$ and get the inequality

$$||A||^2 \mu(\omega_n) = ||A||^2 ||\chi_{\omega_n}||^2 \geq ||A(\chi_{\omega_n})||^2 = ||\phi\chi_{\omega_n}||^2 = \int_{\omega_n} |\phi|^2 d\mu \geq n^2 \mu(\omega_n) \quad .$$

The inequality tells us that if $\mu(\omega_n) > 0$, we have $||A|| \geq n$. Because A is a bounded operator, there has to be an $\tilde{n}$ so that $\mu(\omega_{\tilde{n}}) = 0$ and therefore $||\phi||_\infty \leq \tilde{n}$, which gives $\phi \in L^\infty(\mu)$. The equality $A(\psi) = \psi\phi$ tells us that we have $A|_{L^\infty(\mu)} = M_\phi$. According to Theorem 4.7.7, $L^\infty(\mu) \cap L^2(\mu) = L^\infty(\mu)$ is dense in $L^2(\mu)$ and therefore $A = M_\phi$, which means that $\mathscr{A}_\mu' \subseteq \mathscr{A}_\mu$. □

Remark. We can generalize Theorem 3.2.8 for σ-finite measures (see [2] chapter IX.6 Theorem 6.6), but we will not prove it since it is not required for the remainder of this work.

Without knowing it yet, we have just proven that $\mathscr{A}_\mu$ is a maximal abelian von Neumann algebra. This might not impress us much at the moment, but we will use the very convenient property $\mathscr{A}_\mu = \mathscr{A}_\mu' = \mathscr{A}_\mu''$ in Section 3.6 to show that $\mathscr{A}_\mu$ is the double commutant of

$\{M_z\}$, that is the multiplication operator by z which we will call N_μ. By then, we will have established the importance of this double commutant $\{N_\mu\}''$, and we will be well on the way to prove our first multiplicity theorem.

3.3 Pseudo-Commuting Normal Operators

In this section we will look at normal operators that "commute" with maps from one Hilbert space to another, that is $NX = XM$ with $X \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$. We will investigate the relationship between them in two parts, of which the Fuglede-Putnam Theorem 3.3.2 is the first major conclusion, and the the equivalence relation in Proposition 3.3.5 is the second major conclusion. This section is a bit of a tangent to our larger goal, but we will need to use Theorem 3.3.2 and Proposition 3.3.5 at key points in this thesis to progress. To this end, we start right away with a lemma and then directly the Fuglede-Putnam Theorem.

Lemma 3.3.1. *Let $A \in \mathscr{B}(\mathscr{H})$ be a herimitian operator, that is $A^* = A$. Then* $\exp(iA)$ *is a unitary operator.*

Proof. We use Proposition 4.5.1 to show that $(\exp(iA))^* \exp(iA) = 1$. First, we know that $(\exp(iA))^* = \exp(-iA^*)$. Second, because of $A = A^*$ and thus $A^*A = AA^*$, we know that $\exp(-iA^*) \exp(iA) = \exp(i(A - A^*)) = \exp(0) = 1$. □

Theorem 3.3.2 (Fuglede-Putnam)**.** *Let $N \in \mathscr{B}(\mathscr{H}_1)$ and $M \in \mathscr{B}(\mathscr{H}_2)$ be normal operators, and let $X \in \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$. If $NX = XM$ holds, then $N^*X = XM^*$ holds as well.*

Remark. We can, in particular, set $\mathscr{H}_1 = \mathscr{H}_2$ and also $N = M$ to obtain that $NX = XN$ implies $N^*X = XN^*$ for a normal operator N and any $X \in \mathscr{B}(\mathscr{H}_1)$.

Proof. We can deduce immediately from $NX = XM$ that for $k \in \mathbb{N}$ we have $N^kX = XM^k$ and thus for any polynomial $p(N)$ we have $p(N)X = Xp(M)$. For any bounded linear operator X, Definition 4.5.1 tells us that the exponential $\exp(X)$ is the limit of polynomials. Thus we can fix a $z \in \mathbb{C}$ and see that

$$\exp(izN)X = \lim_{n\to\infty} p_n(N)X = \lim_{n\to\infty} p_n(M)X = X\exp(izM) \quad .$$

We then see with Proposition 4.5.1 that $X = \exp(-izN)X\exp(izM)$. We can now define the operator valued function $f(z) : \mathscr{H}_2 \mapsto \mathscr{H}_1$ in the following way

$$f(z) := e^{-izN^*}Xe^{izM^*} = e^{-izN^*}e^{-i\bar{z}N}Xe^{i\bar{z}M}e^{izM^*} \quad .$$

Again with Proposition and the fact that N, M are normal operators, we get

$$f(z) = e^{-izN^*-i\bar{z}N}Xe^{i\bar{z}M+izM^*} \quad .$$

Now $zN^* + \bar{z}N$ and $zM^* + \bar{z}M$ are both herimitian operators, so by Lemma 3.3.1, $\exp(-izN^* - i\bar{z}N)$ and $\exp(izM^* + i\bar{z}M)$ are unitary. This means that $||\exp(-i\bar{z}N^* -$

$izN)|| = 1$ and $||\exp(izM^* + i\bar{z}N)|| = 1$, which means that $||f(z)|| \leq ||X||$. On the other hand, by Proposition 4.5.2 we get that $f(z)$ is differentiable with the first derivative

$$f'(z) = -iN^* f(z) + f(z)iM^* \quad .$$

Since $||f(z)||$ is bounded by $||X||$, we can use Theorem 4.5.3 to see that $f(z)$ is constant and thus $f'(z) = 0$. We remind ourselves that $f(0) = X$ and get $0 = f'(0) = i(-N^*X + XM^*)$, which leads to the theorem. □

The Fuglede-Putnam Theorem is a very powerful tool, since it automatically gives us properties of the adjoints N^*, M^* from an equation containg just N and M. As mentioned in the remark, we also get $N^*X = XN^*$ from $NX = XN$ and thus also $NX^* = X^*N$. This means that the commutant of a set consisting of normal operators also contains all adjoints. We will use this property especially when investigating the double commutant $\{N\}''$ of a normal operator in Proposition 3.4.4. First, however, we will focus on $X \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and show that that on certain subspaces we actually get $N \simeq M$. To this end we will show a generalized Polar Decomposition Theorem 3.3.4, which we will use for the subsequent proof of $N \simeq M$.

Definition 3.3.1. Let $X \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$. Then we define the *absolute operator of* X as

$$|X| := \int \sqrt{z} dE(z) \quad .$$

Here, E is the spectral measure for X^*X.

Proposition 3.3.3. *For $X \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ the operator $|X| \in \mathscr{B}(\mathscr{H}_1)$ is well defined and the following properties hold*

(a) $|X|^* = |X|$.

(b) For $h, k \in \mathscr{H}_1$ we find $\langle Xh, Xk\rangle_{\mathscr{H}_2} = \langle |X|h, |X|k\rangle_{\mathscr{H}_1}$.

(c) $\ker |X| = \ker X$ *and* $\mathrm{cl}(\mathrm{ran}|X|) = (\ker X)^\perp$.

Proof. We see that $X^*X \in \mathscr{B}(\mathscr{H}_1)$ is a self-adjoint operator, and thus according to Theorem 4.6.4 it has a spectral measure E with

$$\int z dE(z) = X^*X \quad .$$

Further, we see that

$$\langle X^*Xh, h\rangle_{\mathscr{H}_1} = ||Xh||^2_{\mathscr{H}_1} \quad .$$

Since X^*X is self-adjoint, we therefore know by Proposition 4.6.2 that $\sigma(X^*X) \geq 0$. Since $\mathrm{supp}(E) = \sigma(X^*X)$, we see that $\sqrt{z} \in B(\sigma(X^*X))$, so $|X| = \int \sqrt{z} dE(z)$ is well defined. Theorem 4.6.3 then tells us that

$$|X|^* = \int \overline{\sqrt{z}} dE(z) = \int \sqrt{z} dE(z) = |X| \quad .$$

Now we consider $h, k \in \mathscr{H}_1$ and see

$$\langle |X|h, |X|k\rangle_{\mathscr{H}_1} = \langle |X|^2 h, k\rangle_{\mathscr{H}_1} = \int z dE_{h,k}(z) = \langle X^* X h, k\rangle_{\mathscr{H}_2} = \langle Xh, Xk\rangle_{\mathscr{H}_2} \quad .$$

To show the last statement, we remember the second statement and see for $h \in \mathscr{H}_1$

$$||Xh||_{\mathscr{H}_2} = ||(|X|h)||_{\mathscr{H}_1} \quad .$$

Thus we find $\ker |X| = \ker X$. Now we take $f \in \ker |X|$ and $h \in \operatorname{ran}|X|$, which means $h = |X|k$ for some $k \in \mathscr{H}_1$, and we calculate

$$\langle h, f\rangle_{\mathscr{H}_1} = \langle |X|k, f\rangle_{\mathscr{H}_1} = \langle k, |X|f\rangle_{\mathscr{H}_1} = 0 \quad .$$

Since $f \in \ker |X| = \ker X$, we find that $h \in (\ker X)^\perp$ and thus $\operatorname{ran}|X| \subseteq (\ker X)^\perp$. Since $(\ker X)^\perp$ is closed, this automatically implies $\operatorname{cl}(\operatorname{ran}|X|) \subseteq (\ker X)^\perp$. On the other hand, we inspect $g \in (\operatorname{ran}|X|)^\perp$ and see

$$\langle |X|g, |X|g\rangle_{\mathscr{H}_1} = \langle |X|(|X|g), g\rangle_{\mathscr{H}_1} = 0 \quad .$$

This means $|X|g = 0$ and so we have $(\operatorname{ran}|X|)^\perp \subseteq \ker |X|$. We can then form the orthogonal complement and get

$$(\ker X)^\perp = (\ker |X|)^\perp \subseteq ((\operatorname{ran}|X|)^\perp)^\perp = \operatorname{cl}(\operatorname{ran}|X|) \quad .$$

Ultimately we therefore find $\operatorname{cl}(\operatorname{ran}|X|) = (\ker X)^\perp$. □

Theorem 3.3.4 (Polar Decomposition)**.** *For $X \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ we have $X = W|X|$ with $W \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$, and $W|_{(\ker X)^\perp}$ is unitary with*

$$\operatorname{ran} W|_{(\ker X)^\perp} = \operatorname{cl}(\operatorname{ran} X) \quad .$$

Proof. Proposition 3.3.3 tells us that for any $h \in \mathscr{H}_1$ and $k \in \mathscr{H}_2$ we have $\langle Xh, Xk\rangle_{\mathscr{H}_2} = \langle |X|h, |X|k\rangle_{\mathscr{H}_1}$. This means angle and norm are the same under X and $|X|$, so we can define the unitary operator $\tilde{W} : \operatorname{ran}|X| \mapsto \operatorname{ran} X$ via $\tilde{W}(|X|h) = Xh$. Since $\tilde{W}$ is an isometry, we can extend it to $\tilde{W} : \operatorname{cl}(\operatorname{ran}|X|) \mapsto \operatorname{cl}(\operatorname{ran} X)$. Now we remember that $\mathscr{H}_1 = \ker X \oplus (\ker X)^\perp$. Proposition 3.3.3 states that $\operatorname{cl}(\operatorname{ran}|X|) = (\ker X)^\perp$, so we can define $W \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ as $W = 0 \oplus \tilde{W}$ and see that by definition $W|_{(\ker X)^\perp} = \tilde{W}$ is unitary, and that

$$\operatorname{ran} W|_{(\ker X)^\perp} = \operatorname{cl}(\operatorname{ran} X) \quad .$$

Now it remains to be shown that $X = W|X|$. We take $h \in \mathscr{H}_1$, remember the definition of $\tilde{W}$ and calculate

$$W|X|h = W(|X|h) = \tilde{W}(|X|h) = Xh \quad .$$

This proves the theorem. □

Proposition 3.3.5. *Let $N \in \mathscr{B}(\mathscr{H}_1)$ and $M \in \mathscr{B}(\mathscr{H}_2)$ be normal operators, and let $X \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ so that $XN = MX$. Let us further denote $\tilde{\mathscr{H}_1} := (\ker X)^\perp$ and $\tilde{\mathscr{H}_2} := \operatorname{cl}(\operatorname{ran} X)$. Then $N|_{\tilde{\mathscr{H}_1}} \in \mathscr{B}(\tilde{\mathscr{H}_1})$ and $M|_{\tilde{\mathscr{H}_2}} \in \mathscr{B}(\tilde{\mathscr{H}_2})$, and we have $N|_{\tilde{\mathscr{H}_1}} \simeq M|_{\tilde{\mathscr{H}_2}}$.*

Proof. First we will prove $M|_{\tilde{\mathscr{H}}_2} \in \mathscr{B}(\tilde{\mathscr{H}}_2)$. For any $h \in \operatorname{ran} X$, we have $h = Xk$ for some $k \in \mathscr{H}_1$. We can now calculate

$$Mh = MXk = XNk \quad .$$

Therefore $Mh \in \operatorname{ran} X$. Since M is continuous, we see that also $M\tilde{\mathscr{H}}_2 \subseteq \tilde{\mathscr{H}}_2$. To show $N|_{\tilde{\mathscr{H}}_1} \in \mathscr{B}(\tilde{\mathscr{H}}_1)$, we consider Theorem 3.3.2 and get $XN^* = M^*X$. Now we take $h \in \ker X$ and see

$$XN^*h = M^*Xh = 0 \quad .$$

Therefore we have $N^* \ker X \subseteq \ker X$. We can then inspect $k \in (\ker X)^\perp$ and again $h \in \ker X$ and get

$$\langle Nk, h\rangle_{\mathscr{H}_1} = \langle k, N^*h\rangle_{\mathscr{H}_1} = 0 \quad .$$

The last equation is due to the fact that $N^*h \in \ker X$. Therefore $Nk \in (\ker X)^\perp$ and so we have $N|_{\tilde{\mathscr{H}}_1} \in \mathscr{B}(\tilde{\mathscr{H}}_1)$.

Now we want to show $N|_{\tilde{\mathscr{H}}_1} \simeq M|_{\tilde{\mathscr{H}}_2}$. We take the adjoint of $XN^* = M^*X$ and get

$$NX^* = X^*M \quad .$$

Together with the initial equation $XN = MX$ this gives

$$X^*XN = X^*MX = NX^*X \quad .$$

Therefore we see that $X^*X \in \{N\}'$. Theorem 4.6.3 and Theorem 4.6.4 now tell us that $\phi(X^*X) \in \{N\}'$ for all $\phi \in B(\sigma(X^*X))$, and from Proposition 3.3.3 we see that $|X| = \phi_{\sqrt{\cdot}}(X^*X)$ with $\phi_{\sqrt{\cdot}} \in B(\sigma(X^*X))$. Altogether this means that $|X| \in \{N\}'$. We now use Theorem 3.3.4 to get $X = W|X|$ and calculate

$$MW|X| = MX = XN = W|X|N = WN|X| \quad .$$

This means on $\operatorname{cl}(\operatorname{ran}|X|)$ the following equation holds

$$MW|_{\operatorname{cl}(\operatorname{ran}|X|)} = WN|_{\operatorname{cl}(\operatorname{ran}|X|)} \quad .$$

Now we remember from Proposition 3.3.3 that $\operatorname{cl}(\operatorname{ran}|X|) = (\ker X)^\perp = \tilde{\mathscr{H}}_1$, and from Theorem 3.3.4 that $\operatorname{ran} W|_{(\ker X)^\perp} = \operatorname{cl}(\operatorname{ran} X) = \tilde{\mathscr{H}}_2$. Thus we can deduce

$$M|_{\tilde{\mathscr{H}}_2} W|_{\tilde{\mathscr{H}}_1} = W|_{\tilde{\mathscr{H}}_1} N|_{\tilde{\mathscr{H}}_1} \quad .$$

Since Theorem 3.3.4 also states that $W|_{\tilde{\mathscr{H}}_1}$ is a unitary operator, we therefore get $M|_{\tilde{\mathscr{H}}_2} \simeq N|_{\tilde{\mathscr{H}}_1}$. □

3.4 Abelian von Neumann Algebras

In this section we will focus on a certain type of subalgebras of $\mathscr{B}(\mathscr{H})$, namely the titular von Neumann algebras. Especially the von Neumann algebra $W^*(N)$ generated by the normal operator N is important, as it will serve as one of the foundations of multiplicity theory. In conjunction, we will introduce separating and cyclic vectors, which help describe sets of operators by their action on a (separating or cyclic) vector, and they will appear hand in hand with $W^*(N)$ later on.

We go medias in res and define von Neumann algebras and generated von Neumann algebras first, and we will prove a few lemmata that will help us calculate with them. Our final goal in this section will be Corollary 3.4.9, which tells us for separable Hilbert spaces that every abelian C^* algebra (and thus also every abelian von Neumann algebra, especially $W^*(N)$) has a separating vector.

Definition 3.4.1. We define a *von Neumann algebra* $\mathscr{A}$ as a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ with the property $\mathscr{A} = \mathscr{A}''$.

Definition 3.4.2. Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$. We define $W^*(\mathscr{A})$ as the smallest von Neumann algebra containing $\mathscr{A}$ and call it the *von Neumann algebra generated by* $\mathscr{A}$. For $\mathscr{A} = \{A\}$, we write $W^*(\{A\}) = W^*(A)$.

Lemma 3.4.1. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ be an algebra closed under the $*$-operation. Then we have $\mathrm{cl}\mathscr{A} = C^*(\mathscr{A})$. Additionally, if $\mathscr{A}$ is abelian, then $C^*(\mathscr{A})$ is also abelian.*

Proof. First we consider $A, B \in \mathrm{cl}\mathscr{A}$ and $\lambda \in \mathbb{C}$ with nets $(A_i)_{i\in I}, (B_i)_{i\in I} \subseteq \mathscr{A}$ so that $A_i \to A$ and $B_i \to B$. Since addition and multiplication are continuous with respect to the operator norm, we have $(A_i + \lambda B_i) \in \mathrm{cl}\mathscr{A}$ and thus $(A + \lambda B) \in \mathrm{cl}\mathscr{A}$. We also know for a fixed $k \in I$ that

$$||A_k B - A_k B_i|| \leq ||A_k|| \cdot ||B - B_i|| \to 0 \quad .$$

This means that $A_k B_i \to A_k B \in \mathrm{cl}\mathscr{A}$. Now we take the other limit and similarly get $A_k B \to AB \in \mathrm{cl}\mathscr{A}$. Finally we consider that for any $C \in \mathscr{B}(\mathscr{H})$ we have $||C|| = ||C^*||$. Thus we see

$$||A^* - A_i^*|| = ||A - A_i|| \to 0 \quad .$$

This means that $A_i^* \to A^* \in \mathrm{cl}\mathscr{A}$. Altogether we can deduce that $\mathrm{cl}\mathscr{A}$ is an algebra closed under the $*$-operation. Because it is also norm-closed, $\mathrm{cl}\mathscr{A}$ is a C^*-algebra.

Since $\mathscr{A} \subseteq \mathrm{cl}\mathscr{A}$ and $\mathrm{cl}\mathscr{A}$ is a C^*-algebra, we have $C^*(\mathscr{A}) \subseteq \mathrm{cl}\mathscr{A}$. Conversely, because $\mathscr{A} \subseteq C^*(\mathscr{A})$, and $C^*(\mathscr{A})$ is closed, we have $\mathrm{cl}\mathscr{A} \subseteq C^*(\mathscr{A})$. Together this means $\mathrm{cl}\mathscr{A} = C^*(\mathscr{A})$.

Now assume that $\mathscr{A}$ is abelian and take $A, B \in \mathrm{cl}\mathscr{A}$ with $(A_i)_{i\in I}, (B_i)_{i\in I} \subseteq \mathscr{A}$ so that $A_i \to A$ and $B_i \to B$. For any fixed $k \in I$ we then have

$$||A_k B - B A_k|| \leq ||A_k|| \cdot ||B - B_i|| + ||A_k B_i - B_i A_k|| + ||B - B_i|| \cdot ||A_k||$$

$$= 2||A_k|| \cdot ||B - B_i|| \to 0 \quad .$$

That means $\{A_k B - B A_k\} = 0$ for all $k \in I$. As discussed, multiplication and addition are continuous and therefore we can take the limit and obtain $AB = BA$. □

Lemma 3.4.2. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ be an abelian C^*-algebra and let $B \in \mathscr{A}'$ be a normal operator. Then $C^*(\mathscr{A} \cup \{B\})$ is also an abelian C^*-algebra.*

Proof. According to Lemma 3.4.1, a good start is to find a minimal algebra closed under the $*$-operation that contains $\mathscr{A} \cup \{B\}$. We start by defining

$$\mathscr{P} := \{ \sum_{j,k=0}^{m,n} (A_{j,k} + \lambda_{j,k}) B^j (B^*)^k : m,n \in \mathbb{N} \ , \ \lambda_{j,k} \in \mathbb{C} \ , \ \lambda_{0,0} = 0 \text{ and } A_{j,k} \in \mathscr{A} \} \quad .$$

The $\lambda_{0,0} = 0$ is needed so that $1 \in \mathscr{P}$ if and only if $1 \in \mathscr{A}$.

We first prove that $\mathscr{P}$ is an abelian algebra closed under the $*$-operation. For $A \in \mathscr{A}$, we have $A^* \in \mathscr{A}$ and thus $A^*B = BA^*$. We can take the adjoint of this equation to get $B^*A = AB^*$, which means $B^* \in \mathscr{A}'$. Since $\mathscr{A}$ itself commutes, we can directly calculate that $\mathscr{P}$ is an algebra. Further, $\mathscr{P}$ is symmetric in B and B^*, and since $\mathscr{A}$ was closed under the $*$-operation, we find that for $C \in \mathscr{P}$ we have $C^* \in \mathscr{P}$. Thus $\mathscr{P}$ is an abelian algebra closed with respect to the $*$-operation.

Because $\mathscr{A} \cup \{B\} \subseteq \mathscr{P}$, we have $C^*(\mathscr{A} \cup \{B\}) \subseteq C^*(\mathscr{P})$. Conversely, since $C^*(\mathscr{A} \cup \{B\})$ is an algebra containing $\mathscr{A}$, B and B^*, we see that $\mathscr{P} \subseteq C^*(\mathscr{A} \cup \{B\})$ and therefore $C^*(\mathscr{P}) \subseteq C^*(\mathscr{A} \cup \{B\})$ and thus $C^*(\mathscr{P}) = C^*(\mathscr{A} \cup \{B\})$. Lemma 3.4.1 now tells us that $C^*(\mathscr{A} \cup \{B\}) = C^*(\mathscr{P})$ is equal to $\mathrm{cl}\mathscr{P}$ and that $\mathrm{cl}\mathscr{P}$ is abelian. □

Lemma 3.4.3. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ be a C^*-algebra with $1 \in \mathscr{A}$. Then we find*

$$(\mathrm{SOT})\ \mathrm{cl}(\mathscr{A}) = (\mathrm{WOT})\ \mathrm{cl}(\mathscr{A}) = \mathscr{A}'' = W^*(\mathscr{A}) \quad .$$

Additionally, if $\mathscr{A}$ is abelian, then $\mathscr{A}''$ is also abelian.

Proof. We start by showing that $\mathscr{A}''$ is a von Neumann algebra. For $S \in \mathscr{A}'$, $A, B \in \mathscr{A}''$ and $\lambda \in \mathbb{C}$ we see that

$$(A + \lambda B)S = S(A + \lambda B) \quad .$$

Therefore $\mathscr{A}''$ is an algebra. Further, we know that for $A \in \mathscr{A}$ we have $A^* \in \mathscr{A}$, and thus for $S \in \mathscr{A}'$ we find $A^*S = SA^*$ and thus $S^* \in \mathscr{A}'$. Similarly we can see that for $B \in \mathscr{A}''$ we have $B^* \in \mathscr{A}''$. Finally,since $1 \in \mathscr{A}$ we can use Theorem 3.2.6 to see that $\mathscr{A}'' = \text{SOT-cl}(\mathscr{A}) = \text{WOT-cl}(\mathscr{A})$, and thus $\mathscr{A}''$ is especially norm closed. Altogether we find that $\mathscr{A}''$ is a C^*-algebra, and by Proposition 3.2.2 we see that $(\mathscr{A}'')'' = \mathscr{A}''$. Therefore $\mathscr{A}''$ is a von Neumann algebra.

Since $A \subseteq \text{SOT-cl}\mathscr{A} = \mathscr{A}''$, we find that $W^*(\mathscr{A}) \subseteq \mathscr{A}''$. On the other hand, Proposition 3.2.1 tells us that $(W^*(\mathscr{A}))' \subseteq \mathscr{A}'$. This in turn gives us $\mathscr{A}'' \subseteq (W^*(\mathscr{A}))'' = W^*(\mathscr{A})$, since $W^*(\mathscr{A})$ is a von Neumann algebra. Therefore we get $\mathscr{A}'' = W^*(\mathscr{A})$.

Finally, we assume that $\mathscr{A}$ is abelian and take again $A, B \in \mathscr{A}''$ with $(A_i), (B_i) \subseteq \mathscr{A}$ and $A_i \xrightarrow{\text{SOT}} A$ and $B_i \xrightarrow{\text{SOT}} B$. For a $h \in \mathscr{H}$ and a fixed k, we have

$$\begin{aligned} ||(A_kB - BA_k)h|| &\le ||(A_kB - A_kB_i)h|| + ||(A_kB_i - B_iA_k)h|| + ||(BA_k - B_iA_k)h|| \\ &= ||(A_kB - A_kB_i)h|| + ||(BA_k - B_iA_k)h|| \to 0 \quad . \end{aligned}$$

This means $A_iB - BA_i = 0$ for all i. Similarly as before, we can now take the limit and get $AB = BA$, which means that $\mathscr{A}''$ is abelian. □

Now that we have a certain grip on how to calculate $C^*(\mathscr{A})$ and $W^*(\mathscr{A})$, we can move on to investigate $W^*(N)$ specifically for a normal operator N. We will find a close connection to the set of all polynomials in N and N^*, so we first make the following definition.

Definition 3.4.3. Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then we define $\mathscr{P}_N$ as the *algebra of all polynomials in* N, N^*, that is

$$\mathscr{P}_N := \{ \sum_{j,k=0}^{m,n} \lambda_{j,k} N^j (N^*)^k : m,n \in \mathbb{N} \ , \ \lambda_{j,k} \in \mathbb{C} \} \quad .$$

Proposition 3.4.4. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then $W^*(N) = \{N\}'' =$ SOT-cl$(\mathscr{P}_N) =$ WOT-cl$(\mathscr{P}_N)$, and $W^*(N)$ is abelian.*

Proof. First we acknowledge that by $\{N\} \subseteq W^*(N)$, and thus by Proposition 3.2.1 we have $W^*(N)' \subseteq \{N\}'$ and further $\{N\}'' \subseteq W^*(N)''$. Since $W^*(N)$ is a von Neumann algebra, we have $W^*(N)'' = W^*(N)$ and thus $\{N\}'' \subseteq W^*(N)$.

To prove $W^*(N) \subseteq \{N\}''$, we start by taking $B \in \{N\}'$. Theorem 3.3.2 tells us that, because N is normal, we have $BN^* = N^*B$ and thus $N^* \in \{N\}''$. Also, for $\lambda \in \mathbb{C}$ we find $B\lambda = \lambda B$, and thus $\mathbb{C} \subseteq \{N\}''$. Since all $\lambda \in \mathbb{C}$ and N, N^* are in $\{N\}''$, we find that $\mathscr{P}_N \subseteq \{N\}''$. Now we take $S \in \{N\}'$ and $A \in$ WOT-cl$(\{N\}'')$ with $(A_i)_{i\in I}$ so that $A_i \xrightarrow{WOT} A$. We see that $A_iS = SA_i$ implies $AS = SA$ and thus $A \in \{N\}''$ and $\{N\}''$ is WOT-closed. Therefore it is especially norm closed, and we find cl$(\mathscr{P}_N) \subseteq \{N\}''$. By Lemma 3.4.1 we see that cl$(\mathscr{P}_N)$ is an abelian $C*$-algebra. Further, we know that $1 \in \mathscr{P}_N \subseteq$ cl$(\mathscr{P}_N)$, and we remember that the WOT-closure of the norm closure is just the WOT closure. Now we can use Lemma 3.4.3 to get

$$W^*(\text{cl}(\mathscr{P}_N)) = \text{WOT-cl}(\text{cl}(\mathscr{P}_N)) = \text{WOT-cl}(\mathscr{P}_N) \subseteq \text{WOT-cl}(\{N\}'') = \{N\}'' \quad .$$

Since we have especially $N \in \mathscr{P}_N$, this means $W^*(N) \subseteq \{N\}''$, and together with our initial insight we find $W^*(N) = \{N\}''$. This inclusion chain also tells us $\{N\}'' =$ WOT-cl$(\mathscr{P}_N)$, and from Lemma 3.4.3 we get that WOT-cl$(\mathscr{P}_N) =$ SOT-cl$(\mathscr{P}_N)$ and that $\{N\}'' = W^*(N)$ is abelian. □

Next we define cyclic and separating vectors. Both concepts will accompany us throughout the rest of this thesis.

Definition 3.4.4. Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ and $e_c, e_s \in \mathscr{H}$. We call the vector e_c a *cyclic vector* for $\mathscr{A}$ if $\{Ae_c : A \in \mathscr{A}\}$ is dense in $\mathscr{H}$, and the vector e_s a *separating vector* for $\mathscr{A}$ if for all $A \in \mathscr{A}$ it holds that $Ae_s = 0 \Leftrightarrow A = 0$.

The importance of cyclic vectors comes from the fact that they describe (in a way) whether $\mathscr{A}$ is dense in $\mathscr{H}$, as they let us map $\mathscr{A}$ densely into $\mathscr{H}$. This is especially useful if vectors of the type $\{Ae_c : A \in \mathscr{A}\}$ have some special property, or if we want to leverage certain special properties of $\mathscr{A}$. On the other hand, if $\mathscr{A}$ is an algebra, a separating vector allows for the inverse mapping. We can see that then, each element in $\{Ae_s : A \in \mathscr{A}\}$ uniquely corresponds to an element in $\mathscr{A}$. This allows us to leverage our knowledge about $\mathscr{H}$ when making calculations with $\mathscr{A}$. For the moment, we keep these insights in the back of our head as we continue on towards Theorem 3.4.6 and our considerations about maximal abelian von Neumann algebras. To this end, we first need the definition of a maximal abelian von Neumann algebra.

Definition 3.4.5. Let $\mathscr{A}_m \subseteq \mathscr{B}(\mathscr{H})$ be an abelian von Neumann algebra. We call $\mathscr{A}_m$ a *maximal abelian von Neumann algebra* if there is no abelian von Neumann algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ so that $\mathscr{A}_m \subsetneq \mathscr{A}$.

Lemma 3.4.5. *Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then there exists a normed sequence $(e_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}$ so that for $i \neq j$ it holds $\mathrm{cl}\{Ae_i : A \in \mathscr{A}\} \perp \mathrm{cl}\{Ae_j : A \in \mathscr{A}\}$ and $\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathrm{cl}\{Ae_n : A \in \mathscr{A}\}$.*

Proof. Since $\mathscr{H}$ is separable, it admits a countable orthonormal basis $(e_n)_{n\in\mathbb{N}}$. For each n define $E_n := \mathrm{cl}\{Ae_n : A \in \mathscr{A}\}$, and let P_n be the projection onto E_n. Now we inductively create a new (possibly finite) normed sequence $(\tilde{e_n})_{n\in\mathbb{N}}$ by starting with $\tilde{e}_1 := e_1$ and thus $n_1 = 1$, and then setting $\tilde{e}_{j+1} = \frac{1}{\alpha}\sum_{k=1}^{j}(1 - P_{n_k})e_{n_{j+1}}$.Here, $n_{j+1} > n_j$ is the next index so that $e_{n_{j+1}} \notin \bigoplus_{k=1}^{j} E_{n_k}$ and α is a constant so that $\tilde{e}_{j+1}$ is normed. Now we define $\tilde{E}_n := \mathrm{cl}\{A\tilde{e}_n : A \in \mathscr{A}\}$ and take $h \in \tilde{E}_n$ and $n < n'$. From the construction of $\tilde{e}_{n'}$, we see that $\tilde{e}_{n'} \perp \tilde{E}_n$ and thus $\langle \tilde{e}_{n'}, h\rangle = 0$. Now for any $A \in \mathscr{A}$, we know that $A^* \in \mathscr{A}$ and thus $A^*h \in \tilde{E}_n$. Therefore we have $\langle A\tilde{e}_{n'}, h\rangle = \langle \tilde{e}_{n'}, A^*h\rangle = 0$, which means $\tilde{E}_{n'} \perp \tilde{E}_n$, and therefore we can also look at $\bigoplus_{n=1}^{N} \tilde{E}_n$. But first, we remember that 1 commutes with every operator, so $1 \in \mathscr{A}'' = \mathscr{A}$, and thus $e_n \in \tilde{E}_n$. The construction of $\tilde{E}_n$ shows that $\bigoplus_{n=1}^{N} \tilde{E}_n = \mathrm{cl}(\mathrm{span}(\bigcup_{n=1}^{n_N} E_n)) \supseteq \bigoplus_{n=1}^{n_N} \mathrm{span}(\{e_n\})$. The $(e_n)_{n\in\mathbb{N}}$ form an orthonormal basis of $\mathscr{H}$, so we can conclude $\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathrm{span}(\{e_n\}) = \bigoplus_{n=1}^{\infty} \tilde{E}_n$. □

Theorem 3.4.6. *Let $\mathscr{A}$ be an abelian C^*-subalgebra of $\mathscr{B}(\mathscr{H})$. Then the following statements are equivalent:*

(a) $\mathscr{A}$ is a maximal abelian von Neumann algebra

(b) $\mathscr{A} = \mathscr{A}'$

If $\mathscr{H}$ is separable, then these statements also imply:

(c) $\mathscr{A}$ has a cyclic vector, contains 1 and is SOT *closed*

Proof. (a) $\Rightarrow$ (b) Since 1 commutes with every element in $\mathscr{B}(\mathscr{H})$, we have $1 \in \mathscr{A}$. Assume $\mathscr{A} \neq \mathscr{A}'$ and take $B \in \mathscr{A}' \setminus \mathscr{A}$. If $BB^* \neq B^*B$, we have $B(B + B^*) \neq (B + B^*)B$ and because $B \in \mathscr{A}'$, it follows that $(B + B^*) \notin \mathscr{A}$. We also have $(B + B^*)$ normal and as discussed in the proof of Lemma 3.4.2 that $B^* \in \mathscr{A}'$, and thus $(B + B^*) \in \mathscr{A}'$. Therefore we can always choose $B \in \mathscr{A}' \setminus \mathscr{A}$ to be normal. Now we consider $\tilde{\mathscr{A}} := C^*(\mathscr{A} \cup \{B\})$, and we have $\mathscr{A} \subsetneq \tilde{\mathscr{A}} \subseteq \tilde{\mathscr{A}}''$. Lemma 3.4.2 tells us that $\tilde{\mathscr{A}}$ is abelian, and therefore Lemma 3.4.3 then tells us that $\tilde{\mathscr{A}}''$ is an abelian von Neumann algebra. This, however, is a contradiction to the maximality of $\mathscr{A}$.

(b) $\Rightarrow$ (a) Assume for $B \notin \mathscr{A}$ that $W^*(\mathscr{A} \cup \{B\})$ is abelian. Since it is abelian, we have $B \in \mathscr{A}'$ and thus $B \in \mathscr{A}$, which is a contradiction.

(b) $\Rightarrow$ (c) By Lemma 3.4.5, we have a normed sequence $(e_n)_{n\in\mathbb{N}}$ so that if we define $E_n := \mathrm{cl}\{Ae_n : A \in \mathscr{A}\}$, we have $\mathscr{H} = \bigoplus_{n=1}^{\infty} E_n$ and $E_i \perp E_j$ for $i \neq j$. Now look at the projection $P_n : \mathscr{H} \to E_n$. $\mathscr{A}$ is a C^*-algebra, so for all $A \in \mathscr{A}$ it holds that $AE_n \subseteq E_n$,

$A^* \in \mathscr{A}$ and therefore also $A^*E_n \subseteq E_n$. Proposition 3.2.4 thus tells us that $P_n \in \mathscr{A}' = \mathscr{A}$. We construct the vector $e_0 := \sum_{n=1}^{\infty} \frac{e_n}{\sqrt{2}^n}$ and define $E_0 := \mathrm{cl}\{Ae_0 : A \in \mathscr{A}\}$. We see that $\mathscr{A} P_n \subseteq \mathscr{A}$ and so $E_n = \mathrm{cl}\{AP_n e_0 : A \in \mathscr{A}\} \subseteq \mathrm{cl}\{Ae_0 : A \in \mathscr{A}\} = E_0$. We can deduce from the previous inclusion that $\mathscr{H} = \bigoplus_{n=1}^{\infty} E_n \subseteq E_0$ and thus e_0 is a cyclic vector. Further we have $1 \in \mathscr{A}' = \mathscr{A}$ and because of Theorem 3.2.6 we know that $\mathscr{A} = \mathscr{A}''$ is SOT-closed. □

Remark. For separable Hilbert spaces, statement (c) actually also implies (a) and (b). Since the proof requires Gelfand Theory, we will omit it and instead only do a special case later on with Corollary 3.7.10. The full proof can be found in [2] chapter IX.7 under Theorem 7.8.

Although this insight into maximal abelian von Neumann algebras is highly interesting, we will only use it to prove the following Corollary 3.4.9. However, under the surface, a certain maximal abelian von Neumann algebra will accompany us for some time, namely $\mathscr{A}_\mu$ for Radon measures μ with compact support. We have already proven that $\mathscr{A}_\mu = \mathscr{A}_\mu' = \mathscr{A}_\mu''$, so we know that it is a maximal abelian von Neumann algebra, and in Section 3.6 we will find out that $\mathscr{A}_\mu = W(N_\mu)$. Further, since $L^2(\mu) \cap L^\infty(\mu)$ is dense in $L^2(\mu)$ (see Theorem 4.7.7), we can infer that the cyclic vector given by Theorem 3.4.6 is just the constant 1 function. But before we delve too deep into this, we focus again on our subject of abelian von Neumann algebras, and we will prove the final corollary of this section.

Lemma 3.4.7. *Let $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ be an abelian von Neumann algebra. Then there exists a maximal abelian von Neumann algebra $\mathscr{A}_m \subseteq \mathscr{B}(\mathscr{H})$ so that $\mathscr{A} \subseteq \mathscr{A}_m$.*

Proof. Let $\mathscr{N}_{\mathscr{A}}$ be the set of all abelian von Neumann algebras in $\mathscr{B}(\mathscr{H})$ containing $\mathscr{A}$. The set $\mathscr{N}_{\mathscr{A}}$ is partially ordered by the inclusion relation $\subseteq$. Now let $(N_i)_{i\in I} \subseteq \mathscr{N}_{\mathscr{A}}$ be a chain, and we define $N := \bigcup_i N_i$. For $A, B \in N$, there are i_A and i_B so that $A \in N_{i_A}$ and $B \in N_{i_B}$. Since $(N_i)_{i\in I}$ is a chain, we can assume without loss of generality $i_A \leq i_B$ and thus $A, B \in N_{i_B}$. Because N_{i_B} is an abelian algebra, we have $AB \in N_{i_B}$, $(A + \lambda B) \in N_{i_B}$, $A^*, B^* \in N_{i_B}$ and $AB = BA$. We have $N_{i_B} \subseteq N$, so N is an abelian algebra closed with respect to the $*$-operation. Lemma 3.4.1 now tells us that $C^*(N)$ is an abelian C^*-algebra, Lemma 3.4.2 tells us that $C^*(C^*(N) \cup \{1\})$ is also an abelian C^*-algebra and finally Lemma 3.4.3 tells us that $\tilde{N} := C^*(C^*(N) \cup \{1\})''$ is an abelian von Neuman algebra. For all i we have:

$$N_i \subseteq N \subseteq C^*(N) \subseteq C^*(C^*(N) \cup \{1\}) \subseteq \tilde{N} \quad .$$

Since $\tilde{N}$ is an abelian von Neumann algebra, we also have $\tilde{N} \in \mathscr{N}_{\mathscr{A}}$ and therefore the chain $(N_i)_{i\in I}$ has an upper bound. According to Theorem 4.2.1 this means that $\mathscr{N}_{\mathscr{A}}$ has a maximal element, which is a maximal abelian von Neumann algebra. □

Lemma 3.4.8. *Let $e_c \in \mathscr{H}$ be a cyclic vector for $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$. Then e_c is a separating vector for $\mathscr{A}'$.*

Proof. Take $S \in \mathscr{A}'$ and assume that $Se_c = 0$. For $A \in \mathscr{A}$, we see that $SAe_c = ASe_c = 0$. But since $\mathrm{cl}\{Ae_c : A \in \mathscr{A}\} = \mathscr{H}$ and S continuous, this means that $S\mathscr{H} = \{0\}$ and thus $S = 0$. □

Corollary 3.4.9. *Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{A}$ be an abelian C^*-subalgebra of $\mathscr{B}(\mathscr{H})$. Then we find that $\mathscr{A}$ has a separating vector.*

Remark. Since every abelian von Neumann algebra is by definition also an abelian C^*-algebra, this corollary holds for abelian von Neumann algebra as well.

Proof. Lemma 3.4.2 tells us that $\tilde{\mathscr{A}} := C^*(\mathscr{A} \cup \{1\})$ is an abelian C^*-algebra and Lemma 3.4.3 tells us that $\tilde{\mathscr{A}}''$ is an abelian von Neumann algebra. Because of Lemma 3.4.7, there is a maximal abelian von Neumann algebra $\tilde{\mathscr{A}_m} \subseteq \mathscr{B}(\mathscr{H})$ so that $\tilde{\mathscr{A}}'' \subseteq \tilde{\mathscr{A}_m}$. Because of Theorem 3.4.6, there exists a cyclic vector $e_c \in \mathscr{H}$ for $\tilde{\mathscr{A}_m}$. Because $\tilde{\mathscr{A}_m}' = \tilde{\mathscr{A}_m}$, Lemma 3.4.8 tells us that e_c is also a separating vector for $\tilde{\mathscr{A}_m}$, and because of $\mathscr{A} \subseteq \tilde{\mathscr{A}}'' \subseteq \tilde{\mathscr{A}_m}$, we see that e_c is also a separating vector for $\mathscr{A}$. □

3.5 Vector-Associated Measures

In this section we will look at the connection between $W^*(N)$ and certain measures associated with vectors from $\mathscr{H}$. Our main goal is to show Theorem 3.5.8, which will tell us that certain subspaces $\mathscr{H}_h$, generated by a vector h and $W^*(N)$, are unitarily equivalent to $L^2(\mu_h)$ with vector-associated measures μ_h. Even more, we get a unitary equivalence for N and functions $\phi(N)$ of the operator N. This connection is the fundamental reason how the von Neumann algebras $W^*(N)$ tie into multiplicity theory, as we hinted at earlier. To understand this, we imagine that we fully separate (a separable) $\mathscr{H}$ into a direct sum of orthogonal subspaces created by $W^*(N)$, as we have done in Lemma 3.4.5. We can then identify this partition via Theorem 3.5.8 to get

$$\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathscr{H}_n \simeq \bigoplus_{n=1}^{\infty} L^2(\mu_n) \quad .$$

With this unitary equivalence, we will have achieved the first step of the first multiplicity theorem. However, before we can get there, we first need to prove Theorem 3.5.8.

Definition 3.5.1. Let μ be a Radon measure with compact support. Then we define the *canonical multiplication operator* $N_\mu \in \mathscr{B}(L^2(\mu))$ by $N_\mu : f(z) \mapsto zf(z)$.

We will use the operator N_μ throughout the rest of this thesis, as the fundamental idea of multiplicity theory is to reduce any normal operator to a direct sum of canonical multiplication operators for different measure spaces. Now we will prove some of its properties.

Proposition 3.5.1. *Let μ be a Radon measure with compact support. Then N_μ is a normal operator.*

Proof. Because N_μ^* is just multiplication by $\bar{z}$, and since z and $\bar{z}$ commute multiplicatively, we see that $N_\mu N_\mu^* = N_\mu^* N_\mu$. □

Proposition 3.5.2. *The spectrum of N_μ is then given by $\sigma(N_\mu) = \operatorname{supp}(\mu)$.*

Proof. Let us denote $S = \text{supp}(\mu)$ for the rest of this proof. First we show that $\sigma(N_\mu) \subseteq S$ by picking a $\lambda \notin S$ and showing $\lambda \notin \sigma(N_\mu)$. By Definition 3.5.1 μ is a Radon measure, so we find that $\mu(\mathbb{C} \setminus S) = 0$ and thus the function $(z-\lambda)^{-1}$ is defined μ-almost everywhere. In addition S is compact, so $(z-\lambda)^{-1}$ is bounded on S and thus $(z-\lambda)^{-1} \in L^\infty(\mu)$. We can now define the multiplication operator $\psi_\lambda : f \mapsto (z-\lambda)^{-1} f$ with $\psi_\lambda \in \mathscr{B}(L^2(\mu))$. By definition of N_μ, we then have $(N_\mu - \lambda)\psi_\lambda(f) = (z-\lambda)(z-\lambda)^{-1} f = f$, and thus $N_\mu - \lambda$ is invertible and $\lambda \notin \sigma(N_\mu)$.

Now let us prove that $S \subseteq \sigma(N_\mu)$. We choose $\lambda \in S$ and denote by D_ϵ the open circle around λ with radius $\epsilon > 0$. Since S is the support of μ and D_ϵ is a neighbourhood of λ, we find that $\mu(D_\epsilon) > 0$ for all $\epsilon > 0$. S is also compact, so $\mu(D_\epsilon) < \mu(S) < \infty$ and therefore $\chi_{D_\epsilon} \in L^2(\mu)$. We look at the function $(z-\lambda)\chi_{D_\epsilon} = (N_\mu - \lambda)\chi_{D_\epsilon}$ and assume that $\psi_\lambda \in \mathscr{B}(L^2(\mu))$ is the inverse of $N_\mu - \lambda$. By definition we have $|z-\lambda| < \epsilon$ for $z \in D_\epsilon$ and thus $||(N_\mu - \lambda)\chi_{D_\epsilon}|| < \epsilon ||\chi_{D_\epsilon}||$. Further, we find that $||\chi_{D_\epsilon}||^2 = \mu(D_\epsilon) > 0$, and so $\chi_{D_\epsilon} \neq 0$. Since $\psi_\lambda(N-\lambda)(\chi_{D_\epsilon}) = \chi_{D_\epsilon}$, this also implies that $(N-\lambda)(\chi_{D_\epsilon}) \neq 0$. Therefore we have

$$||\psi_\lambda|| = \sup\left\{\frac{||\psi_\lambda(f)||}{||f||} : f \in L^2(\mu)\ ,\ f \neq 0\right\} \geq \frac{||\psi_\lambda(N-\lambda)(\chi_{D_\epsilon})||}{||(N-\lambda)(\chi_{D_\epsilon})||} = \frac{||\chi_{D_\epsilon}||}{\epsilon||\chi_{D_\epsilon}||} = \frac{1}{\epsilon} \quad .$$

Taking the limit $\epsilon \to 0$ gives us a contradiction, because $\psi_\lambda \in \mathscr{B}(L^2(\mu))$ implies that $||\psi_\lambda|| < \infty$. Thus we find that no such ψ_λ can exist and therefore $\lambda \in \sigma(N_\mu)$. □

Proposition 3.5.3. *For two Radon measures μ_1 and μ_2 with compact support on $\mathbb{C}$, we have $N_{\mu_1} \simeq N_{\mu_2}$ if and only if $[\mu_1] = [\mu_2]$.*

Proof. First we assume $[\mu_1] = [\mu_2]$ and put $\phi := \frac{d\mu_1}{d\mu_2}$. According to Theorem 4.7.3, $\phi \geq 0$ and we can take $\sqrt{\phi}$. Now we consider $f \in L^2(\mu_1)$ and see with Proposition 4.7.4 that

$$\int |f|^2 d\mu_1 = \int |f|^2 \phi d\mu_2 = \int |f\sqrt{\phi}|^2 d\mu_2 \quad .$$

This means that $f\sqrt{\phi} \in L^2(\mu_2)$ and also that the mapping $U : L^2(\mu_1) \mapsto L^2(\mu_2)$ defined by $U : f \mapsto \sqrt{\phi} f$ is an isometry. Now let us denote $\psi := \frac{d\mu_2}{d\mu_1}$, and we can go through the same steps to get an isometry $\tilde{U} : L^2(\mu_2) \mapsto L^2(\mu_1)$. Proposition 4.7.4 also tells us that $\psi = \phi^{-1}$ and thus $\tilde{U} = U^{-1}$, which means $L^2(\mu_1) \simeq L^2(\mu_2)$. Additionally we have for any $g \in L^2(\mu_2)$

$$UN_{\mu_1}U^{-1}(g) = UN_{\mu_2}(\phi^{-1}g) = U(\phi^{-1}zf) = UU^{-1}(zf) = zf = N_{\mu_2} f \quad .$$

Thus we find that $N_{\mu_1} \simeq N_{\mu_2}$.

Now let us assume that $V : L^2(\mu_1) \mapsto L^2(\mu_2)$ is unitary with the property $VN_{\mu_1}V^{-1} = N_{\mu_2}$. This implies that $\sigma(N_{\mu_1}) = \sigma(N_{\mu_2})$, which together with Proposition 3.5.2 means that we can define $S := \text{supp}(\mu_1) = \text{supp}(\mu_2)$, and we restrict our proof to the space S. Now we inspect $N^n_{\mu_1}$ and we see that

$$VN^n_{\mu_1}V^{-1} = N^n_{\mu_2} \quad .$$

Further, we have proven in Proposition 3.5.1 that N_{μ_1} and N_{μ_2} are normal, so we get from Theorem 3.3.2 that

$$VN_{\mu_1}^*V^{-1} = N_{\mu_2}^* \quad .$$

Since V, V^{-1} are linear, we can extend this argumentation to any polynomial $p(N_{\mu_1}, N_{\mu_1}^*)$ so that we find

$$Vp(N_{\mu_1}, N_{\mu_1}^*)V^{-1} = p(N_{\mu_2}, N_{\mu_2}^*) \quad .$$

We now remember that $N_{\mu_1} = M_z$ on $L^2(\mu_1)$. Theorem 4.1.1 tells us that the polynomials in z and $\bar{z}$ are dense in $C(S)$ endowed with the $||.||_\infty$ norm. Further, Proposition 3.2.7 gives $||M_g^i|| = ||g||_\infty$ for $g \in L^\infty(\mu_i)$. This means that for any $u \in C(S)$, if a sequence of polynomials $(p_{u,n})_{n\in\mathbb{N}}$ converges to u with respect to the $||.||_\infty$ norm, we have $p_{u,n}(N_{\mu_1}, N_{\mu_1}^*) \to M_u$ with respect to the operator norm, and we have the same for $p_{u,n}(N_{\mu_2}, N_{\mu_2}^*) \to M_u$ on $L^2(\mu_2)$. Since V is bounded, we find

$$VM_u = \lim_{n\to\infty} Vp_{u,n}(N_{\mu_1}, N_{\mu_1}^*) = \lim_{n\to\infty} p_{u,n}(N_{\mu_2}, N_{\mu_2}^*)V = M_uV \quad .$$

This means we have $VM_uV^{-1} = M_u$ for all $u \in C(S)$. We also know that $C(S) \subseteq L^2(\mu_1)$, because μ_1 is a Radon measure and S is compact, and therefore according to Proposition 4.7.6 we have $\mu_1(S) < \infty$. This means we can define $\phi := V(1)$, and for $u \in C(S)$ we have

$$V(u) = VM_u^1(1) = VM_u^1V^{-1}(\phi) = M_u^2(\phi) = \phi u \quad .$$

Our goal is now to prove that ϕ is related to $\frac{d\mu_1}{d\mu_2}$ via Theorem 4.7.9. Therefore we choose any non-negative function $u_+ \in C(S)$ with $u_+ \geq 0$ and see that also $\sqrt{u_+} \in C(S)$. Now we remember that V is an isometry and get

$$\int u_+ d\mu_1 = ||\sqrt{u_+}||_{L^2(\mu_1)}^2 = ||V(\sqrt{u_+})||_{L^2(\mu_2)}^2 = ||\phi\sqrt{u_+}||_{L^2(\mu_2)}^2 = \int |\phi|^2 u_+ d\mu_2 \quad .$$

We can split any $u \in C(S)$ into the sum of four non-negative functions with pre-factors $1, -1, i, -i$. By the additivity of the integral, we therefore see that the previous equality can be expanded to $\int ud\mu_1 = \int |\phi|^2 ud\mu_2$. We can now interpret the previous integral as a linear functional $\zeta(u) := \int ud\mu_1$ acting on $u \in C(S)$. Theorem 4.7.9 then tells us that there is a unique Radon measure μ so that $\zeta(u) = \int ud\mu$, and thus $\mu = \mu_1$. On the other hand, we can define the measure $\nu(A) := \int_A |\phi|^2 d\mu_2$. Since μ_2 is a Radon measure, we see by Proposition 4.7.6 that ν is also a Radon measure. Further, we have $\zeta(u) = \int ud\mu_1 = \int |\phi|^2 ud\mu_2$ and can thus again use Theorem 4.7.9 to get $\nu = \mu_1 = \mu$. We have $d\mu_1 = d\nu = |\phi|^2 d\mu_2$, so we see that $\mu_1 \ll \mu_2$ and also $\frac{d\mu_1}{d\mu_2} = |\phi|^2$. Repeating the previous steps with μ_1 and μ_2 switched gives us $\mu_2 \ll \mu_1$. $\square$

Next, we define the subspace $\mathscr{H}_h$. They will be the main method of how we investigate $\mathscr{H}$ from now on.

Definition 3.5.2. Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator, let $A \in W^*(N)$ and let $h \in \mathscr{H}$. We now define $\mathscr{H}_h := \mathrm{cl}(W^*(N)h)$, and we further define $A_h := A|\mathscr{H}_h$, that is $A_h x \mapsto Ax$ for all $x \in \mathscr{H}_h$.

Proposition 3.5.4. *Let A_h and $\mathscr{H}_h$ be as in Definition 3.5.2. Then we find that $\mathscr{H}_h$ is a closed subspace, and that $A_h\mathscr{H}_h \subseteq \mathscr{H}_h$ and $A_h^*\mathscr{H}_h \subseteq \mathscr{H}_h$.*

Proof. Let $x, y \in W^*(N)h$ so that $T_x h = x$ and $T_y h = y$. Since $W^*(N)$ is an algebra, we see for $\lambda \in \mathbb{C}$ that also $T_x + \lambda T_y \in W^*(N)$ and thus we have

$$x + \lambda y = T_x h + \lambda T_y h = (T_x + \lambda T_y)h \in W^*(N)h \quad .$$

Therefore $W^*(N)h$ is a linear subspace and thus $\mathscr{H}_h = \mathrm{cl}(W^*(N)h)$ is a closed linear subspace. Now we take $A \in W^*(N)$ and $x \in \mathscr{H}_h$ and want to show that $Ax \in \mathscr{H}_h$. Let therefore $(T_{x,i})_{i\in I} \subseteq W^*(N)$ be a net so that $T_{x,i}h \to x$. Since $A \in \mathscr{B}(\mathscr{H})$, we see that $AT_{x,i}h \to Ax$, but we also know that $AT_{x,i} \in W^*(N)$ because $W^*(N)$ is an algebra. This means $AT_{x,i}h \in \mathscr{H}_h$, and since $\mathscr{H}_h$ is closed also $Ax \in \mathscr{H}_h$. The same holds for A^*, since $W^*(N)$ is closed with respect to the $*$-operation and thus $A^* \in W^*(N)$. □

Lemma 3.5.5. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator and let $h \in \mathscr{H}$. Then we have $\mathrm{cl}(\mathscr{P}_N h) = \mathscr{H}_h$.*

Proof. Proposition 3.4.4 tells us that $\mathscr{P}_N \subseteq W^*(N)$ and thus we have $\mathrm{cl}(\mathscr{P}_N h) \subseteq \mathrm{cl}(W^*(N)h) = \mathscr{H}_h$.

For the other inclusion, we again refer to Proposition 3.4.4 to see that $W^*(N) = \text{SOT-cl}\mathscr{P}_N$, with $\mathscr{P}_N$ being the algebra of polynomials in N, N^*. Thus, for any $T \in W^*(N)$, we have a net of polynomials $(p_i(N,N^*))_{i\in I} \subseteq \mathscr{P}_N$ so that $p_i(N,N^*) \xrightarrow{\text{SOT}} T$ and thus $p_i(N,N^*)h \to Th$. Therefore we find that $W^*(N)h \subseteq \mathrm{cl}(\mathscr{P}_N h)$, which implies $\mathscr{H}_h = \mathrm{cl}(W^*(N)h) \subseteq \mathrm{cl}(\mathscr{P}_N h)$. □

Next, we define vector-associated measures for a normal operator N. As discussed above, we will see that $\mathscr{H}_h \simeq L^2(\mu_h)$ and $N|_{\mathscr{H}_h} \simeq N_{\mu_h}$. Therefore we can imagine such a vector-associated measure μ_h as capturing all relevant information about N (and in a wider sense $W^*(N)$) on the subspace $\mathscr{H}_h$.

Definition 3.5.3. Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator with spectral measure E and let $h \in \mathscr{H}$. Then we define the *associated measure of h with respect to N* as $\mu_h(\Delta) := ||E(\Delta)h||^2 = E_{h,h}(\Delta)$.

Proposition 3.5.6. *μ_h is a Radon measure with compact support.*

Proof. This is a direct consequence of Proposition 4.7.6 and the fact that $\mu_h = E_{h,h}$. □

Finally, we come close to proving Theorem 3.5.8, where we will use the concepts introduced so far in this section. Before that, however, we will show a quick Proposition.

Proposition 3.5.7. *For a normal operator $N \in \mathscr{B}(\mathscr{H})$ we find $\{\phi(N) : \phi \in B(\sigma(N))\} \subseteq W^*(N)$.*

Remark. We will actually see in Theorem 3.7.7 that $\{\phi(N) : \phi \in B(\sigma(N))\} = W^*(N)$.

Proof. We start by showing $\{\phi(N) : \phi \in B(\sigma(N))\} \subseteq \{N\}''$. For any $T \in \{N\}'$, we have $TN = NT$ and thus by Theorem 3.3.2 also $TN^* = N^*T$. We take E as the spectral measure for N, and Theorem 4.6.4 then tells us that for any Borel set Δ we have $TE(\Delta) = E(\Delta)T$. Now for any $\phi(N)$ and $h, g \in \mathscr{H}$, we see that

$$\langle T\phi(N)h, g\rangle = \langle \phi(N)h, T^*g\rangle = \int \phi(z)dE_{h,T^*g}(z) \quad .$$

Now we can use $TE(\Delta) = E(\Delta)T$ to see

$$\langle E(\Delta)h, T^*g\rangle = \langle TE(\Delta)h, g\rangle = \langle E(\Delta)Th, g\rangle \quad .$$

Thus we can continue the previous equation

$$\int \phi(z)dE_{h,T^*g}(z) = \int \phi(z)dE_{Th,g}(z) = \langle \phi(N)Th, g\rangle \quad .$$

Since h, g were arbitrary, we find that $T\phi(N) = \phi(N)T$ for all $T \in \{N\}'$ and thus $\phi(N) \in \{N\}''$. By Proposition 3.4.4, we have $\{N\}'' = W^*(N)$ and so the lemma follows. $\square$

Theorem 3.5.8. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator, let μ be some Radon measure with bounded support and let $\mathscr{K} \subseteq \mathscr{H}$ be a closed subspace with the properties $N\mathscr{K} \subseteq \mathscr{K}$ and $N^*\mathscr{K} \subseteq \mathscr{K}$. Then $N|_{\mathscr{K}} \simeq N_\mu$ if and only if $\mathscr{K} = \mathscr{H}_h$ and $\mu = \mu_h$ for some $h \in \mathscr{H}$. In this case, there exists a unique unitary operator $V : \mathscr{H}_h \mapsto L^2(\mu_h)$ with the following properties:*

(a) $VN_hV^{-1} = N_{\mu_h}$.

(b) $Vh = 1$.

These properties further imply that for $\phi \in B(\sigma(N))$ we have $V\phi(N)|_{\mathscr{H}_h}V^{-1} = M_\phi$.

Proof "⇒". First, we consider the case $N|_{\mathscr{K}} \simeq N_\mu$ via the unitary operator $V : \mathscr{K} \mapsto L^2(\mu)$, and we start by showing $\mathscr{K} = \mathscr{H}_h$ for some $h \in \mathscr{H}$. By Proposition 3.4.4, we see that the polynomials $\mathscr{P}_{N_\mu}$ in N_μ, N_μ^* are a subset of $W^*(N_\mu)$. Since N_μ is the multiplication operator by z, the operator $p(N_\mu, N_\mu^*)$ is just the multiplication operator by $p(z, \bar{z})$. Proposition 4.7.6 tells us that μ is finite, so we can consider $1 \in L^2(\mu)$. Because we have $p(z, \bar{z}) = p(N_\mu, N_\mu^*)1$, we see that $\mathscr{P}_{N_\mu}1 \subseteq L^2(\mu)$ is the set of all polynomials. Theorem 4.1.1 tells us that $\mathscr{P}_{N_\mu}1$ is dense in $C(\mathrm{supp}(\mu))$ with respect to the $||.||_\infty$ norm. Since $\mu(\mathbb{C}) < \infty$, we see that convergence with respect to $||.||_\infty$ implies convergence with respect to $||.||_2$. This means that $\mathscr{P}_{N_\mu}1$ is dense in $C(X)$ with respect to the $||.||_2$ norm. Together with Theorem 4.7.8 we see that $\mathscr{P}_{N_\mu}1$ is dense in $L^2(\mu)$, that is $\mathrm{cl}\mathscr{P}_{N_\mu}1 = L^2(\mu)$.

We now remind ourselves that $N|_{\mathscr{K}}x = Nx$ and $N^*|_{\mathscr{K}}x = N^*x$ for $x \in \mathscr{K}$, so from now on we will omit the subscript $|_{\mathscr{K}}$. Similarly to the proof of Proposition 3.5.3, we can see that for any polynomial $p(N, N^*)$ we have

$$Vp(N, N^*)V^{-1} = p(N_\mu, N_\mu^*) \quad .$$

This implies in turn that $\mathscr{P}_N(V^{-1}1) = V^{-1}(\mathscr{P}_{N_\mu}1)$, and thus

$$\mathrm{cl}(\mathscr{P}_N(V^{-1}1)) = V^{-1}(\mathrm{cl}(\mathscr{P}_{N_\mu}1)) = V^{-1}(L^2(\mu)) \quad .$$

We can set $V^{-1}1 = h$ and we get with Lemma 3.5.5 that

$$V^{-1}(L^2(\mu)) = \mathrm{cl}(\mathscr{P}_N h) = \mathscr{H}_h \quad .$$

Since $V^{-1}L^2(\mu) = \mathscr{K}$, we have $\mathscr{K} = \mathscr{H}_h$.

Now it remains for us to show that $\mu = \mu_h$. We have already proven that for any polynomial p we have $Vp(N,N^*)V^{-1} = p(N_\mu, N_\mu^*) = M_p$. In addition, we see that $p^*(N,N^*) = p(N^*,N)$. Together with the fact that V is unitary we thus get

$$\int |p|^2 d\mu_h = \int |p|^2 dE_{h,h} = \langle p(N,N^*)p^*(N,N^*)h, h\rangle = ||p(N,N^*)p^*(N,N^*)h||^2_{\mathscr{H}_h}$$

$$= ||V^{-1}Vp(N,N^*)V^{-1}Vp^*(N,N^*)V^{-1}Vh||^2_{\mathscr{H}_h} = ||p(z,\bar{z})\bar{p}(z,\bar{z})1||^2_{L^2(\mu)} = \int |p|^2 d\mu \quad .$$

Now let $K = \mathrm{supp}(\mu) \cup \mathrm{supp}(\mu_h)$. We know that both supports are compact, so K is compact as well. Theorem 4.1.1 tells us that $\mathscr{P}$ is dense in $C(K)$ with respect to the $||.||_\infty$ norm. This means that for any $f \in C(K)$ there is a net $(p_i)_{i\in I} \subseteq \mathscr{P}$ so that $p_i \xrightarrow{||.||_\infty} f$. We thus get

$$\left|\int |p_i|^2 d\mu - \int |f|^2 d\mu\right| \leq \int \left||p_i|^2 - |f|^2\right| d\mu \leq ||(|f|^2 - |p_i|^2)||_\infty \cdot \mu(K) \to 0 \quad .$$

The last step is due to the fact that μ is a Radon measure and K is compact, so $\mu(K) < \infty$. The same limit holds for μ_h. Therefore, for any positive function $g \in C(K)$ we have

$$\int g^2 d\mu = \lim_{i\in I} \int |p_i|^2 d\mu = \lim_{i \in I} \int |p_i|^2 d\mu_h = \int g^2 d\mu_h$$

We can now proceed in the same way as for Proposition 3.5.3 and we obtain $\mu = \mu_h$.

Proof "$\Leftarrow$". Now we assume that $\mathscr{K} = \mathscr{H}_h$ and $\mu = \mu_h$ and show that $N|_{\mathscr{K}} = N_h \simeq N_{\mu_h}$. First, let us remark that Proposition 3.5.6 tells us that μ_h is indeed a Radon measure with bounded support. We start by taking $\phi \in B(\sigma(N))$ and see

$$||\phi(N)h||^2 = \langle \phi(N)h, \phi(N)h\rangle = \langle \phi(N)\phi(N)^* h, h\rangle = \int \phi\bar{\phi} dE_{h,h} = \int |\phi|^2 d\mu_h \quad .$$

Now we consider $B(\sigma(N)) \subseteq L^2(\mu_h)$, which is possible since $\sigma(N)$ is compact and thus $\mu_h(\sigma(N)) < \infty$ according to Proposition 4.7.6. We know from Proposition 3.5.7 that $\phi(N) \in W^*(N)$ and thus $\phi(N)h \in \mathscr{H}_h$. Therefore we can define $U : B(\sigma(N)) \mapsto \mathscr{H}_h$ by $U\phi = \phi(N)h$. The previous calculation shows that U is an isometry. Since $\sigma(N)$ is bounded, we find that all polynomials in $z, \bar{z}$ lie in $B(\sigma(N))$. Now for any polynomial $p(z,\bar{z})$ we have $Up(z,\bar{z}) = [\int p(z,\bar{z})dE(z)]h$. To evaluate $\int p(z,\bar{z})dE(z)$, we first remind ourselves that Theorem 4.6.4 gives $\int z dE(z) = N$, and further that for $\lambda \in \mathbb{C}$ we have $\int \lambda dE(z) = \lambda$. Theorem 4.6.3 tells us that the mapping $\phi \mapsto \int \phi dE$ is linear and compatible with multiplication and the $*$-operation, so we find that $\int p(z,\bar{z})dE(z) = p(N,N^*)$ and thus $p(N,N^*)h \in U(B(\sigma(N)))$. Altogether this means that $\mathscr{P}_N h \subseteq U(B(\sigma(N)))$. By Lemma 3.5.5 we now see that $\mathscr{H}_h = \mathrm{cl}(\mathscr{P}_N h) \subseteq \mathrm{cl}(U(B(\sigma(N))))$. Further, since $\mathscr{H}_h$ is closed and $U(B(\sigma(N))) \subseteq \mathscr{H}_h$, we have $\mathrm{cl}(U(B(\sigma(N)))) \subseteq \mathscr{H}_h$. On the other hand we can identify $B(\sigma(N))$ with $L^\infty(\mu_h)$ by taking a bounded representative from the equivalence classes in $L^\infty(\mu_h)$. Theorem 4.7.7 now tells us that $\mathrm{cl}(B(\sigma(N))) = \mathrm{cl}(L^\infty(\mu_h)) = L^2(\mu_h)$ with respect to the $||.||_2$ norm. Since U is an isometry, we can extend it to an isometry $\tilde{U} : \mathrm{cl}(B(\sigma(N))) \mapsto \mathrm{cl}(U(B(\sigma(N))))$. With the previous insight, this becomes $\tilde{U} : L^2(\mu_h) \mapsto \mathscr{H}_h$. Clearly $\tilde{U}$ is also linear, so it is an isomorphism, and we can take $V := \tilde{U}^{-1}$. We see that V is linear and unitary. Further, by definition of $\tilde{U}$ we have $V^{-1}1 = \tilde{U}1 = h$, and thus $Vh = 1$. We now show that $V\phi(N)|_{\mathscr{H}_h}V^{-1} = M_\phi$ for $\phi \in B(\sigma(N))$. Since we only investigate elements $x \in \mathscr{H}_h$, we remind ourselves that $\phi(N)x = \phi(N)|_{\mathscr{H}_h}x$ and omit the subscript from hereon. We take $\psi \in B(\sigma(N))$ and look at

$$V^{-1}M_\phi V\psi(N)h = V^{-1}M_\phi\psi(z) = V^{-1}[\phi\psi](z)$$

$$= [\int \phi(z)\psi(z)dE(z)]h = [\int \phi(z)dE(z)][\int \psi(z)dE(z)]h = \phi(N)\psi(N)h \quad .$$

Here we used Theorem 4.6.3 for the multiplicativity of the spectral integral, the fact $\phi\psi \in B(\sigma(N))$ and the explicit definition of $V^{-1} = U$ on the subset $B(\sigma(N))$. So $V\phi(N)V^{-1} = M_\phi$ holds on $B(\sigma(N)) \subset L^2(\mu_h)$. However, as we discussed before $B(\sigma(N))$ is dense in $L^2(\mu_h)$, so $V\phi(N)V^{-1} = M_\phi$ holds everywhere. Since $M_z = N_{\mu_h}$, this means especially $N_h \simeq N_{\mu_h}$.

Proof of Uniqueness. To conclude the proof of Theorem 3.5.8, we show the uniqueness of V, which requires only $VN_hV^{-1} = N_{\mu_h}$ and $Vh = 1$. Let us assume that the unitary operator I fulfils $Ih = 1$ and $INI^{-1} = N_{\mu_h}$. We can again extend this relation to polynomials and see

$$Ip(N, N^*)I^{-1} = p(N_\mu, N_\mu^*) = Vp(N, N^*)V^{-1} \quad .$$

Since $I^{-1}1 = h = V^{-1}1$, we see that for all $p(N, N^*)h$ we have $V = I$. Now Lemma 3.5.5 tells us that $\mathrm{cl}(\mathscr{P}_N h) = \mathscr{H}_h$, so $I = V$ on a dense subset and thus $I = V$ everywhere. □

3.6 The von Neumann algebra $W^*(N_\mu)$

One of our intermediate goals is to fully classify $W^*(N)$ in Section 3.7, as this will help us with handling not only the von Neumann algebra itself but also objects like $\mathscr{H}_h$. In this section, we will talk about the special case $W^*(N_\mu)$, and we will finally prove that $W^*(N_\mu) = \mathscr{A}_\mu$. Further, we will show some extra properties of the spectral integral for N_μ which provide a well rounded picture of how spectral integration works for N_μ, and which we need for proofs later on.

Lemma 3.6.1. *Let μ be a Radon measure with compact support on $\mathbb{C}$. Then we find* WOT-$\mathrm{cl}(\mathscr{P}_{N_\mu}) = \mathscr{A}_\mu$.

Proof. First, we remember that $N_\mu f = zf$ for $f \in L^2(\mu)$, and therefore we find for any $p(N_\mu, N_\mu^*) \in \mathscr{P}_{N_\mu}$ that $p(N_\mu, N_\mu^*)f = p(z, \bar{z})f$ which means $p(N_\mu, N_\mu^*) = M_p$. We further remark that μ has compact support and thus for every continuous function $c \in C(\mathbb{C})$ we have $||c||_\infty < \infty$ and therefore $c \in L^\infty$- This especially means that the polynomials are bounded and thus $\mathscr{P}_{N_\mu} \subseteq \mathscr{A}_\mu$. Now let us rephrase the lemma by taking $f, g \in L^2(\mu)$ and $\phi \in L^\infty(\mu)$. Our goal is to prove that there exists a net of polynomials $(p_i)_{i\in I}$ so that

$$\langle p_i(N_\mu, N_\mu^*)f, g\rangle = \langle M_{p_i} f, g\rangle = \int p_i f\bar{g} d\mu \to \int \phi f \bar{g} d\mu = \langle M_\phi f, g\rangle \quad .$$

We will first reduce our proof to $\phi = c \in C(\mathbb{C})$. Theorem 4.1.1 tells us that the polynomials are dense in $C(\mathrm{supp}(\mu))$ with respect to the $||.||_\infty$ norm. Since μ is a Radon measure, we know that $\mu(\mathbb{C}\setminus\mathrm{supp}(\mu)) = 0$ and therefore the polynomials are dense in $C(\mathbb{C})$ with respect to the $||.||_\infty$ norm. This means we find a net of polynomials $(p_i)_{i\in I}$ so that $p_i \xrightarrow{||\cdot||_\infty} c$, which leads to

$$\left|\int_{\mathbb{C}} p_i f\bar{g} d\mu - \int_{\mathbb{C}} c f\bar{g} d\mu\right| = \left|\int (p_i - c) f\bar{g} d\mu\right| \le ||p_i - c||_\infty \left|\int f\bar{g} d\mu\right| \to 0 \quad .$$

Therefore $\mathscr{P}_{N_\mu}$ is WOT dense in the subset of multiplication operators for continuous functions, that is $\{M_c : c \in C(\mathbb{C})\} \subseteq$ WOT-cl$(\mathscr{P}_{N_\mu})$. Therefore our goal is now to show that $\mathscr{A}_\mu \subseteq$ WOT-cl$(\{M_c : c \in C(\mathbb{C})\})$. To this end, for $f, g \in L^2(\mu)$ and $\phi \in L^\infty(\mu)$ we have to find a net $(c_i)_{i\in I} \subseteq C(\mathbb{C})$ so that

$$\int c_i f\bar{g} d\mu \to \int \phi f\bar{g} d\mu \quad .$$

We will now do this using Theorem 4.7.10. Since μ is a Radon measure and it has compact support, we know by Proposition 4.7.6 that $\mu(\mathbb{C}) < \infty$. Let us now denote $D := \{z : |\phi(z)| \le ||\phi||_\infty\}$. We know that $\mu(D) = \mu(\mathbb{C}) < \infty$. According to Theorem 4.7.10, there exists a compact set K_n so that $\mu(D \setminus K_n) < \frac{1}{n}$ and $\phi|_{K_n}$ is continuous. Further we know that $\mathbb{C}$ is topologically normal, and so according to Theorem 4.1.2 there exists a continuous extension $c_n : \mathbb{C} \mapsto \mathbb{C}$ so that $c_n|_{K_n} = \phi|_{K_n}$ and $\sup\{|c_n(z)| : z \in \mathbb{C}\} = \sup\{|\phi(z)| : z \in K_n\} \le ||\phi||_\infty$. We remember $\mu(D) = \mu(\mathbb{C})$ and thus we have

$$\left|\int_{\mathbb{C}} c_n f\bar{g} d\mu - \int_{\mathbb{C}} \phi f\bar{g} d\mu\right| = \left|\int_D (c_n - \phi) f\bar{g} d\mu\right| = \left|\int_{D\setminus K_n} (c_n - \phi) f\bar{g} d\mu\right|$$

$$\le (||c_n||_\infty + ||\phi||_\infty)\left|\int_{D\setminus K_n} f\bar{g} d\mu\right| \le 2||\phi||_\infty \left|\int_{D\setminus K_n} f\bar{g} d\mu\right| \to 0 \quad .$$

The limit at the end is due to the fact that $\mu(D \setminus K_n) < \frac{1}{n} \to 0$. Therefore we know $\mathscr{A}_\mu \subseteq$ WOT-cl$(\{M_c : c \in C(\mathbb{C})\})$, and by the previous calculations we know WOT-cl$(\{M_c : c \in C(\mathbb{C})\}) \subseteq$ WOT-cl$(\mathscr{P}_{N_\mu})$. In addition we have $\mathscr{P}_{N_\mu} \subseteq \mathscr{A}_\mu$ and we know by Theorem 3.2.8 together with Theorem 3.2.6 that $\mathscr{A}_\mu$ is WOT-closed. Thus we obtain in the end $\mathscr{A}_\mu =$ WOT-cl$(\mathscr{P}_{N_\mu})$. □

Theorem 3.6.2. *Let μ be a Radon measure with compact support on $\mathbb{C}$. Then $\{N_\mu\}' = \mathscr{A}_\mu = W^*(N_\mu)$.*

Proof. By Proposition 3.4.4, we have $W^*(N_\mu) = \{N_\mu\}'' =$ WOT-cl$\mathscr{P}_{N_\mu}$, and now Lemma 3.6.1 gives us that WOT-cl$\mathscr{P}_{N_\mu} = \mathscr{A}_\mu$. Theorem 3.2.8 then shows that $\mathscr{A}_\mu = (\mathscr{A}_\mu)'$, which concludes the proof. □

The next theorem and corollary can be deduced without the technical apparatus we have gathered so far. By simply supposing the intuitive fact that the spectral measure $E(\Delta)$ for N_μ is given by multiplication with χ_Δ, we could go through the check-list in Theorem 4.6.4 and prove that this is true. However, this would be a very lengthy process, and by using our previous calculations we can not only abridge the proofs but also gain the chance to see our insights at work.

Theorem 3.6.3. *Let μ be a Radon measure with bounded support, and let $\phi \in B(\operatorname{supp}(\mu))$. Then $\phi(N_\mu) = M_\phi$.*

Proof. We obviously have $N_\mu \simeq N_\mu$ via the identity, and thus we can use Theorem 3.5.8 to get $h \in L^2(\mu)$ so that $L^2(\mu) = \mathscr{H}_h$ and $\mu = \mu_h$. Further, we have a unique unitary operator $V : \mathscr{H}_h \mapsto L^2(\mu)$ with $VN_hV^{-1} = N_\mu$ and $Vh = 1$. Since $L^2(\mu) = \mathscr{H}_h$ we see that $V \in \mathscr{B}(L^2(\mu))$, and we also find that $N_h = N_\mu$. Together we obtain that $VN_\mu = N_\mu V$ and thus $V \in \{N_\mu\}'$, and we can see the same for V^{-1}. According to Theorem 3.6.2 this means that there exist $\psi, \tilde{\psi} \in L^\infty(\mu)$ so that $V = M_\psi$ and $V^{-1} = M_{\tilde{\psi}}$. Further, we know from Theorem 3.5.8 that for $\phi \in B(\sigma(N_\mu))$ we have $V\phi(N_\mu)|_{\mathscr{H}_h}V^{-1} = M_\phi$ and, using again $\mathscr{H}_h = L^2(\mu)$, we thus find $V\phi(N_\mu)V^{-1} = M_\phi$. We can now insert our previous insights to get

$$\phi(N_\mu) = V^{-1}M_\phi V = M_{\tilde{\psi}}M_\phi M_\psi = M_{\tilde{\psi}}M_\psi M_\phi = M_\phi \quad .$$

The second to last equality is due to the fact that the multiplication operators commute and it gives us the theorem. □

Corollary 3.6.4. *Let μ be a Radon measure with bounded support and let E be the spectral measure for N_μ. Then for a Borel set Δ we find that $E(\Delta) = M_{\chi_\Delta}$. In particular, this means for $h \in L^2(\mu)$ that*

$$\mu_h(\Delta) = \int_\Delta |h(z)|^2 d\mu(z) \quad .$$

Remark. This means especially that $\mu_h = \mu$ if we take $h = 1 \in L^2(\mu)$.

Proof. We know from Theorem 4.6.3 that for a Borel set Δ and the function $\phi_\Delta := \chi_\Delta$ we have $\phi_\Delta(N) = E(\Delta)$. Together with Theorem 3.6.3 this gives us now

$$E(\Delta) = \phi_\Delta(N) = M_{\phi_\Delta} = M_{\chi_\Delta} \quad .$$

The second statement now follows easily by inserting into the definition of μ_h. We see

$$\mu_h(\Delta) = \langle E(\Delta)h, h\rangle = \langle \chi_\Delta h, h\rangle = \int_\Delta |h(z)|^2 d\mu(z) \quad .$$

This proves the corollary. □

3.7 Scalar-Valued Measures and $W^*(N)$

In this last section before we tackle multiplicity theory proper, we introduce the final important concept in the form of scalar-valued measures, and we will round out our knowledge about the more general von Neumann algebra $W^*(N)$. Unfortunately, insights from both areas feed into each other, which is why we combine them into one section. One big goal in this section will be Theorem 3.7.7 and the subsequent corollaries. There, we will derive an easier description of $W^*(N)$, and understand the changes of $W^*(N)$ between different spaces and subspaces (there won't be a significant change as $W^*(N)$ stays the same under restrictions and unitary transformations). The other goal is the classification of scalar-valued measures in Theorem 3.7.11, where we will show that scalar-valued measures and separating vectors are two sides of the same coin.

Definition 3.7.1. Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator with spectral measure E, and let μ be a Radon measure. We call μ a *scalar-valued spectral measure for N* if for all Borel sets Δ we have $\mu(\Delta) = 0$ if and only if $E(\Delta) = 0$.

Proposition 3.7.1. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator, and let μ be a scalar-valued spectral measure for N. Then we have*

- $[\mu] = [\nu]$, *if ν is another scalar-valued measure for N.*
- $\mu_h \ll \mu$ *for $h \in \mathscr{H}$.*
- *supp*$(\mu) = \sigma(N)$.

Proof. Let E be the spectral measure for N. First, we take a Borel set Δ so that $\mu(\Delta) = 0$. By the definition of scalar-valued measures, this means $E(\Delta) = 0$, which in turn means $\nu(\Delta) = 0$. Therefore we find $\nu \ll \mu$. The same calculation works the other way around, so we find $[\mu] = [\nu]$.

Now we continue to the second part and take again a Borel set Δ so that $\mu(\Delta) = 0$. We therefore have $E(\Delta) = 0$ and thus

$$\mu_h(\Delta) = ||E(\Delta)h||^2 = 0 \quad .$$

For the third part, we remember that according to Theorem 4.6.4 we have $\operatorname{supp}(E) = \sigma(N)$. On the other hand, for a point $z \in \operatorname{supp}(E)$ we find for every neighbourhood U_z that $E(U_z) \neq 0$. However, μ is a scalar-valued measure for N and thus $E(\Delta) = 0$ is equivalent to $\mu(\Delta) = 0$ for any Borel set Δ. Therefore we get $\mu(U_z) \neq 0$ and thus $z \in \operatorname{supp}(\mu)$. The same argumentation works the other way round and we get $\operatorname{supp}(\mu) = \operatorname{supp}(E)$. Together with our previous insight this means

$$\operatorname{supp}(\mu) = \operatorname{supp}(E) = \sigma(N) \quad .$$

This concludes the proof. □

Lemma 3.7.2. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator, and let $h \in \mathscr{H}$ be a separating vector for $W^*(N)$. Then we find that μ_h is a scalar-valued measure for N.*

Remark. We will prove further down in Theorem 3.7.11 that the inverse is also true, that is for every scalar-valued spectral measure μ for N there is a $h \in \mathscr{H}$ so that $\mu = \mu_h$ and h is a separating vector for $W^*(N)$.

Proof. Let E be the spectral measure of N. We have already shown in Proposition 3.7.1 that $\mu_h \ll E$. To see the inverse, we take a Borel set Δ with $\mu_h(\Delta) = 0$. This means that $E(\Delta)h = 0$. However, we know that $E(\Delta) = \int \chi_\Delta(z)dE(z)$, which means by Proposition 3.5.7 that $E(\Delta) \in W^*(N)$. Since h is a separating vector for $W^*(N)$, we can thus infer that $E(\Delta) = 0$. □

After establishing scalar-valued measures and some facts about them, we now turn our attention back towards $W^*(N)$. The following lemmata are partly very technical, but they all lead up to Theorem 3.7.7, where we fully describe $W^*(N)$.

Lemma 3.7.3. *Let $T \in \mathscr{B}(\mathscr{H})$ be an operator, and let $\mathscr{K} \subseteq \mathscr{H}$ be a closed subspace. If we have $T\mathscr{K} \subseteq \mathscr{K}$ and $T^*\mathscr{K} \subseteq \mathscr{K}$, then $(T|_\mathscr{K})^* = T^*|_\mathscr{K}$. In this case, $T|_\mathscr{K}$ is normal if T is normal.*

Proof. First, because $T\mathscr{K} \subseteq \mathscr{K}$, we remark that $T|_\mathscr{K} \in \mathscr{B}(\mathscr{K})$. For $x, y \in \mathscr{K}$ we see

$$\langle (T|_\mathscr{K})^* x, y\rangle_\mathscr{K} = \langle x, T|_\mathscr{K} y\rangle_\mathscr{K} = \langle x, Ty\rangle_\mathscr{H} = \langle T^*x, y\rangle_\mathscr{H} = \langle T^*|_\mathscr{K} x, y\rangle_\mathscr{K} \quad .$$

Therefore we have $(T|_\mathscr{K})^* = T^*|_\mathscr{K}$. In addition, we see for a normal T that

$$T|_\mathscr{K}(T|_\mathscr{K})^*x = T|_\mathscr{K}T^*|_\mathscr{K}x = TT^*x = T^*Tx = \ldots = (T|_\mathscr{K})^*T|_\mathscr{K}x \quad .$$

This means that $T|_\mathscr{K}$ is normal. □

Lemma 3.7.4. *Let $N \in \mathscr{B}(\mathscr{H})$ be normal, and let $\mathscr{K} \subseteq \mathscr{H}$ be a closed subspace so that $T\mathscr{K} \subseteq \mathscr{K}$ for all $T \in W^*(N)$. We define $\rho_\mathscr{K}$ on $W^*(N)$ by $\rho_\mathscr{K}(T) = T|_\mathscr{K}$. Then $\rho_\mathscr{K}(W^*(N)) = W^*(N|_\mathscr{K})$, it is* WOT*-continuous and a $*$-epimorphism (that is surjective, linear and compatible with multiplication and the $*$-operation).*

Proof. First we remind ourselves that because $T\mathscr{K} \subseteq \mathscr{K}$ for all $T \in W^*(N)$ and $N \in W^*(N)$, we know $N|_\mathscr{K} \in \mathscr{B}(\mathscr{K})$. Further we find by $N^* \in W^*(N)$ and Lemma 3.7.3 that $N|_\mathscr{K}$ is normal and thus $W^*(N|_\mathscr{K})$ is well defined. Now we prove that $\rho_\mathscr{K}$ is linear and compatible with multiplication and the $*$-operation. We inspect $A, B \in W^*(N)$, $\lambda \in \mathbb{C}$ and $x, y \in \mathscr{K}$. The linearity comes straightforward by taking

$$\rho_\mathscr{K}(A + \lambda B)x = (A + \lambda B)x = Ax + \lambda Bx = (\rho_\mathscr{K}(A) + \lambda\rho_\mathscr{K}(B))x \quad .$$

Therefore $\rho_\mathscr{K}(A + \lambda B) = \rho_\mathscr{K}(A) + \lambda\rho_\mathscr{K}(B)$. Now for the multiplication, we remind ourselves that $Bx \in \mathscr{K}$ and see

$$\rho_\mathscr{K}(AB)x = ABx = \rho_\mathscr{K}(A)Bx = \rho_\mathscr{K}(A)\rho_\mathscr{K}(B)x \quad .$$

Thus we have $\rho_\mathscr{K}(AB) = \rho_\mathscr{K}(A)\rho_\mathscr{K}(B)$. Finally for the adjoint, we remember that $A^* \in W^*(N)$ and thus also $A^*\mathscr{K} \subseteq \mathscr{K}$. Now we again take Lemma 3.7.3 and see that

$$\rho_\mathscr{K}(A^*) = A^*|_\mathscr{K} = (A|_\mathscr{K})^* = \rho_\mathscr{K}(A)^* \quad .$$

Now we show that $\rho_{\mathscr{K}}$ is WOT-continuous. We take a net $(A_i)_{i\in I} \subseteq W^*(N)$ with $A_i \xrightarrow{\text{WOT}} A$ and $A \in W^*(N)$. Now let $x, y \in \mathscr{K}$. By looking at the definitions for $\rho_{\mathscr{K}}$, we see that

$$\langle \rho_{\mathscr{K}}(A_i)x, y\rangle = \langle A_i x, y\rangle \to \langle Ax, y\rangle = \langle \rho_{\mathscr{K}}(A)x, y\rangle \quad .$$

Thus we get that $\rho_{\mathscr{K}}(A_i) \xrightarrow{\text{WOT}} \rho_{\mathscr{K}}(A)$ and therefore $\rho_{\mathscr{K}}$ is WOT continuous.

In the next step, we prove that $\rho_{\mathscr{K}}$ maps $W^*(N)$ surjectively to $W^*(N|_{\mathscr{K}})$. We consider that $\rho_{\mathscr{K}}(N) = N|_{\mathscr{K}}$, $\rho_{\mathscr{K}}(N^*) = N^*|_{\mathscr{K}}$ and for $\lambda \in \mathbb{C}$ that $\rho_{\mathscr{K}}(\lambda) = \lambda$. Further, because $\rho_{\mathscr{K}}$ is linear and compatible with multiplication, we find that for any polynomial $p(N, N^*) \in \mathscr{P}_N$ we have $\rho_{\mathscr{K}}(p(N, N^*)) = p(N|_{\mathscr{K}}, N^*|_{\mathscr{K}})$. This means that $\rho_{\mathscr{K}}(\mathscr{P}_N) = \mathscr{P}_{N|_{\mathscr{K}}}$. Now Proposition 3.4.4 tells us that $W^*(N) = \text{WOT-cl}\mathscr{P}_N$ and $W^*(N|_{\mathscr{K}}) = \text{WOT-cl}\mathscr{P}_{N|_{\mathscr{K}}}$, and therefore the equation becomes

$$\rho_{\mathscr{K}}(W^*(N)) = \rho_{\mathscr{K}}(\text{WOT-cl}\mathscr{P}_N) = \text{WOT-cl}\mathscr{P}_{N|_{\mathscr{K}}} = W^*(N|_{\mathscr{K}}) \quad .$$

This means $\rho_{\mathscr{K}}$ is surjective. □

Lemma 3.7.5. *Let $N \in \mathscr{B}(\mathscr{H})$ be normal. Then the spectral measure $\tilde{E}$ for N_h is just E_h, that is $\tilde{E}(\Delta) = E(\Delta)|\mathscr{H}_h$.*

Proof. Theorem 4.6.4 tells us that $\tilde{E}$ is unique, and thus we only have to show that E_h is a well defined spectral measure and that $N_h = \int z dE_h(z)$. Proposition 3.5.7 tells us that $E(\Delta) = \chi_\Delta(N) \in W^*(N)$, and therefore according to Proposition 3.5.4 we have $E_h(\Delta)\mathscr{H}_h \subseteq \mathscr{H}_h$ for all Borel sets Δ. Further, for $x \in \mathscr{H}_h$ we have $E_h(\Delta)x = E(\Delta)x$ and thus E_h is σ-additive, E_h is a projection and $E(\mathbb{C}) = 1$, which all means that E_h is a spectral measure. Further, we see that for $x, y \in \mathscr{H}_h$ we have $\langle E_h(\Delta)x, y\rangle = \langle E(\Delta)x, y\rangle$, and therefore

$$\langle N_h x, y\rangle = \langle Nx, y\rangle = \int z dE_{x,y}(z) = \int z d(E_h)_{x,y}(z) \quad .$$

Thus we find $N_h = \int z dE_h(z)$. Theorem 4.6.4 now gives us the uniqueness of the spectral measure for a normal operator, which means that E_h is the spectral measure for N_h. □

Lemma 3.7.6. *We define $\rho_h = \rho_{\mathscr{H}_h}$ as in Lemma 3.7.4. Then for any $\phi \in B(\sigma(N))$, we find that $\rho_h(\phi(N)) = \phi(N_h)$, and for all $A \in W^*(N)$ there exists a $\phi_A \in B(\sigma(N_h))$ so that $\rho_h(A) = \phi_A(N_h)$.*

Proof. Let E be the spectral measure for N, then Lemma 3.7.5 tells us that the spectral measure for N_h is just given by E_h. We take $x, y \in \mathscr{H}_h$ and remind ourselves that $\langle E(\Delta)x, y\rangle = \langle E_h(\Delta)x, y\rangle$. This gives us

$$\langle \rho_h(\phi(N))x, y\rangle = \langle \phi(N)x, y\rangle = \int \phi(z) dE_{x,y}(z) = \int \phi(z) d(E_h)_{x,y}(z) = \langle \phi(N_h)x, y\rangle \quad .$$

Therefore we have $\rho_h(\phi(N)) = \phi(N_h)$.

For the second part, we take $A \in W^*(N)$ and want to show that there exists a $\phi_A \in B(\sigma(N_h))$ so that $\rho_h(A) = \phi_A(N_h)$. Theorem 3.5.8 tells us that $N_h \simeq N_\mu$ for some Borel measure μ. Now let $V : \mathscr{H}_h \mapsto L^2(\mu)$ be the isomorphism so that $VN_hV^{-1} = N_\mu$

and $Vh = 1$, and we define $A_\mu := V\rho_h(A)V^{-1} \in \mathscr{B}(L^2(\mu))$. Because of Proposition 3.4.4, we know that $W^*(N)$ is abelian, so we see

$$A_\mu N_\mu = V\rho_h(A)N_h V^{-1} = V\rho_h(A)\rho_h(N)V^{-1} = V\rho_h(AN)V^{-1}$$
$$= V\rho_h(NA)V^{-1} = ... = N_\mu A_\mu \quad .$$

Therefore $A_\mu \in \{N_\mu\}'$, and Theorem 3.6.2 tells us $\{N_\mu\}' = \mathscr{A}_\mu$, so there is a function $\phi_A \in B(\sigma(N_h))$ so that $A_\mu = M_{\phi_A}$. Now we remind ourselves that $\mathrm{cl}(W^*(N_h)h) = \mathscr{H}_h$, and by Lemma 3.5.5 we have $\mathrm{cl}(\mathscr{P}_{N_h}h) = \mathrm{cl}(W^*(N_h)h) = \mathscr{H}_h$. Therefore it suffices to show that $\rho_h(A) = \int \phi_A dE_h = \phi_A(N_h)$ on the dense set $\mathscr{P}_{N_h}h$. Theorem 3.5.8 shows that for $\phi \in B(\sigma(N_h))$ we have $\phi(N_h)h = V^{-1}M_\phi 1$. This means for $p(N_h, N_h^*) \in \mathscr{P}_{N_h}$ we get

$$\rho_h(A)p(N_h, N_h^*)h = V^{-1}V\rho_h(A)V^{-1}Vp(N_h, N_h^*)h = V^{-1}A_\mu M_p 1$$
$$= V^{-1}M_{\phi_A}M_p 1 = V^{-1}M_{\phi_A p}1 = (\phi_A p)(N_h)h = \phi_A(N_h)p(N_h, N_h^*)h \quad .$$

The last equality is due to the fact that the spectral integral is multiplicative, as shown by Theorem 4.6.3. We can now wrap up and remind ourselves that $\rho_h(A) = \phi_A(N_h)$ on $\mathscr{P}_{N_h}h$ which is dense in $\mathscr{H}_h$, so we have $\rho_h(A) = \phi_A(N_h)$ on $\mathscr{H}_h$. $\square$

Theorem 3.7.7. *Let $\mathscr{H}$ be a separable Hilbert space and let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then we have $\{\phi(N) : \phi \in B(\sigma(N))\} = W^*(N)$.*

Proof. Let us denote $\mathscr{A} = \{\phi(N) : \phi \in B(\sigma(N))\}$. Proposition 3.5.7 gives us $\mathscr{A} \subseteq \{N\}'' = W^*(N)$. Thus we need only to show that $W^*(N) \subseteq \mathscr{A}$. We have already elaborated further up that $p(N, N^*) = \int p(z, \bar{z})dE(z)$ for any $p \in \mathscr{P}_N$, which means that $\mathscr{P}_N \subseteq \mathscr{A}$. If we now show that $\mathscr{A} = \text{WOT-cl}\mathscr{A}$, we can use Proposition 3.4.4 to get $W^*(N) = \text{WOT-cl}\mathscr{P}_N \subseteq \text{WOT-cl}\mathscr{A} = \mathscr{A}$.

To show that $\mathscr{A}$ is indeed WOT-closed, we consider a net $(\phi_i)_{i\in I} \subseteq B(\sigma(N))$ so that $\phi_i(N) \xrightarrow{\text{WOT}} T$. Proposition 3.4.4 tells us that $W^*(N)$ is WOT-closed, and since $\mathscr{A} \subseteq W^*(N)$, we know that $T \in W^*(N)$. Now for any $h \in \mathscr{H}$, we know by Lemma 3.7.6 that $\phi_i(N_h) = \rho_h(\phi_i(N)) \to \rho_h(T) = T_h$, and also that there is a $\phi_{T,h} \in B(\sigma(N_h))$ so that $T_h = \phi_{T,h}(N_h)$. Further, Theorem 3.5.8 tells us that there exists a unitary operator $V : \mathscr{H}_h \mapsto L^2(\mu_h)$ with $V\phi(N)|_{\mathscr{H}_h}V^{-1} = M_\phi$, and Lemma 3.7.6 tells us $\phi(N)|_{\mathscr{H}_h} = \phi(N_h)$. Now we take any $f, g \in L^2(\mu_h)$ and find

$$\int \phi_i f g d\mu_h = \langle \phi_i(N_h)V^{-1}h, V^{-1}g\rangle \to \langle \phi_{T,h}(N_h)V^{-1}h, V^{-1}g\rangle = \int \phi_{T,h} f, g d\mu_h \quad .$$

Therefore we see that $\phi_i \xrightarrow{\text{WOT}} \phi_{T,h}$ in $\mathscr{B}(L^2(\mu_h))$. Now we remember Corollary 3.4.9, and since $\mathscr{H}$ is separable we can take a separating vector e_s for $W^*(N)$, and we can prove in a similar vein that $\phi_i \xrightarrow{\text{WOT}} \phi_{T,e_s}$ in $\mathscr{B}(L^2(\mu_{e_s}))$. Our goal is now to show that $\phi_{T,e_s} = \phi_{T,h}$ μ_h-almost everywhere. Lemma 3.7.2 tells us that μ_{e_s} is a scalar-valued measure, and Proposition 3.7.1 gives $\mu_h \ll \mu_{e_s}$. Further, we obtain by Theorem 4.7.3 a positive function $\psi = \frac{d\mu_h}{d\mu_{e_s}}$. Now for any Borel set Δ we see

$$\int_\Delta d\mu_h = \int \chi_\Delta \psi d\mu_{e_s} = \int (\chi_\Delta\psi)^{\frac{1}{2}} (\chi_\Delta\psi)^{\frac{1}{2}} d\mu_{e_s} \quad .$$

We know that μ_h is a Radon measure with compact support, and thus according to Proposition 4.7.6 it is finite. Therefore we see that $(\chi_\Delta \psi)^{\frac{1}{2}} \in L^2(\mu_{e_s})$ and we have

$$\int_\Delta \phi_i d\mu_h = \int \phi_i \, (\chi_\Delta \psi)^{\frac{1}{2}} \, (\chi_\Delta \psi)^{\frac{1}{2}} \, d\mu_{e_s} \to \int \phi_{T,e_s} \, (\chi_\Delta \psi)^{\frac{1}{2}} \, (\chi_\Delta \psi)^{\frac{1}{2}} \, d\mu_{e_s} = \int \phi_{T,e_s} d\mu_h \quad .$$

On the other hand, we have $\chi_\Delta \in L^2(\mu_h)$ and thus

$$\int_\Delta \phi_i d\mu_h = \int \phi_i (\chi_\Delta)^{\frac{1}{2}} (\chi_\Delta)^{\frac{1}{2}} d\mu_h \to \int \phi_{T,h} (\chi_\Delta)^{\frac{1}{2}} (\chi_\Delta)^{\frac{1}{2}} d\mu_h = \int_\Delta \phi_{T,h} d\mu_h \quad .$$

Therefore we see that $\phi_{T,e_s} = \phi_{T,h}$ μ_h-almost everywhere. Going back and using again the unitary operator from Theorem 3.5.8, we see

$$Th = \phi_{T,h}(N)|_{\mathscr{H}_h} h = V^{-1} M_{\phi_{T,h}} V h = V^{-1} M_{\phi_{T,e_s}} V h = \phi_{T,e_s}(N)|_{\mathscr{H}_h} h = \phi_{T,e_s}(N) h \quad .$$

Since h was arbitrary, we have shown that $T = \phi_{T,e_s}(N)$ and thus $T \in \mathscr{A}$ which means $W^*(N) \subseteq \mathscr{A}$. □

From Theorem 3.7.7 we can derive a set of corollaries, since we now have an expression for $W^*(N)$ that is rather easy to deal with. Corollaries 3.7.8 and 3.7.9 tell us how $W^*(N)$ behaves on subspaces and under unitary transformations respectively, which will be important for multiplicity theory when we deal a lot with unitary equivalences and partitions into subspaces. Lastly, Corollary 3.7.10 completes Theorem 3.4.6 for the special case of $W^*(N)$, which is why we feature it more prominently rather than relegating it to some lemma when we need it.

Corollary 3.7.8. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator, and let $\mathscr{K} \subseteq \mathscr{H}$ be a closed subspace with $N\mathscr{K} \subseteq \mathscr{K}$ and $N^*\mathscr{K} \subseteq \mathscr{K}$. Then we have $\phi(N|_{\mathscr{K}}) = \phi(N)|_{\mathscr{K}}$ for $\phi \in B(\sigma(N))$. In particular if $\mathscr{H}$ is separable this means $W^*(N|_{\mathscr{K}}) = W^*(N)|_{\mathscr{K}}$.*

Proof. Since N is normal, we know according to Lemma 3.7.3 that $N|_{\mathscr{K}}$ is also normal. Now we take E as the spectral measure for N and $E_{\mathscr{K}}$ as the spectral measure for $N|_{\mathscr{K}}$. First, we define the function $\tilde{E}(\Delta) := E(\Delta)|_{\mathscr{K}}$. We can now show that $E_{\mathscr{K}} = \tilde{E}$ in a similar way as we did in Lemma 3.7.5.

Let now $\phi \in B(\sigma(N))$. Since $\sigma(N|_{\mathscr{K}}) \subseteq \sigma(N)$, we know that $\phi \in B(\sigma(N|_{\mathscr{K}}))$. We inspect $k_1, k_2 \in \mathscr{K}$ and get

$$\langle \phi(N|_{\mathscr{K}}) k_1, k_2 \rangle_{\mathscr{K}} = \int \phi(z) d\tilde{E}_{k_1,k_2}(z) = \int \phi(z) dE_{k_1,k_2}(z)$$

$$= \langle \phi(N) k_1, k_2 \rangle_{\mathscr{H}} = \langle \phi(N)|_{\mathscr{K}} k_1, k_2 \rangle_{\mathscr{K}} \quad .$$

This proves the first part $\phi(N|_{\mathscr{K}}) = \phi(N)|_{\mathscr{K}}$ of the theorem.

It remains for us to show that for every $T \in W^*(N|_{\mathscr{K}})$ we find a $\tilde{T} \in W^*(N)$ so that $T = \tilde{T}|_{\mathscr{K}}$. Due to Theorem 3.7.7 we know that $W^*(N) = \{\phi(N) : \phi \in B(\sigma(N))\}$ and a similar equality for $W^*(N|_{\mathscr{K}})$. Thus we have $T = \phi(N|_{\mathscr{K}})$. We know that $\sigma(N|_{\mathscr{K}}) \subseteq \sigma(N)$. We define a function $\tilde{\phi} \in B(\sigma(N))$ so that $\tilde{\phi}(z) = \phi(z)$ for $z \in \sigma(N|_{\mathscr{K}})$ and $\tilde{\phi} = 0$ else. Now know that $\tilde{\phi}(N)|_{\mathscr{K}} = \tilde{\phi}(N|_{\mathscr{K}})$. Further, Theorem 4.6.4 tells us that $\mathrm{supp}(E_{\mathscr{K}}) = \sigma(N|_{\mathscr{K}})$ and therefore $\tilde{\phi}(N|_{\mathscr{K}}) = \phi(N|_{\mathscr{K}})$. Now we pick $\tilde{T} = \tilde{\phi}(N)$ and see

$$\tilde{T}|_{\mathscr{K}} = \tilde{\phi}(N)|_{\mathscr{K}} = \tilde{\phi}(N|_{\mathscr{K}}) = \phi(N|_{\mathscr{K}}) = T \quad .$$

Since $T \in W^*(N|_{\mathscr{K}})$ was arbitrary we see that $W^*(N|_{\mathscr{K}}) = W^*(N)|_{\mathscr{K}}$. □

Corollary 3.7.9. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and let $N \in \mathscr{B}(\mathscr{H})$ and $M \in \mathscr{B}(\mathscr{K})$ be normal operators. Let further $V : \mathscr{H} \mapsto \mathscr{K}$ be an unitary operator with the property $VNV^{-1} = M$. Then we have $V\phi(N)V^{-1} = \phi(M)$ for $\phi \in B(\sigma(N))$. This means in particular for separable $\mathscr{H}$ and $\mathscr{K}$ that $VW^*(N)V^{-1} = W^*(M)$.*

Remark. The proof of Corollary 3.7.9 is very similar to the proof of Corollary 3.7.8, with the difference being that $\mathscr{K}$ now is not a subspace but an isomorphic Hilbert space.

Proof. Let E_N be the spectral measure associated with N and let E_M be the spectral measure associated with M. Because $N \simeq M$, we know that $\sigma(N) = \sigma(M)$, and therefore $\mathrm{supp}(E_N) = \sigma(N) = \sigma(M) = \mathrm{supp}(E_M)$. In our first step, we now want to show that $VE_NV^{-1} = E_M$. We begin by proving that $\tilde{E} = VE_NV^{-1}$ is indeed a spectral measure. For $k_1, k_2 \in \mathscr{K}$ we find that

$$\tilde{E}_{k_1,k_2} = \langle \tilde{E}k_1, k_2 \rangle_{\mathscr{K}} = \langle E_N V^{-1}k_1, V^{-1}k_2 \rangle_{\mathscr{H}} = E_{N,V^{-1}k_1,V^{-1}k_2} \quad .$$

Since E_N is a spectral measure, we know that $E_{N,V^{-1}k_1,V^{-1}k_2}$ is a complex measure and thus so is $\tilde{E}_{k_1,k_2}$. Further, since V^{-1} is an isometry, we have

$$\langle \tilde{E}(\sigma(M))k_1, k_2 \rangle_{\mathscr{K}} = \langle E_N(\sigma(N))V^{-1}k_1, V^{-1}k_2 \rangle_{\mathscr{H}} = \langle V^{-1}k_1, V^{-1}k_2 \rangle_{\mathscr{H}} = \langle k_1, k_2 \rangle_{\mathscr{K}} \quad .$$

Therefore $\tilde{E}(\sigma(M))$ is the identity. Finally, because $E_N(\Delta)$ is a projection we have

$$\tilde{E}(\Delta)^2 = VE_N(\Delta)V^{-1}VE_N(\Delta)V^{-1} = VE_N(\Delta)V^{-1} = \tilde{E}(\Delta) \quad .$$

Together, this means that according to Definition 4.6.1 we see that $\tilde{E}$ is indeed a spectral measure.

Now we want to show $M = \int z d\tilde{E}(z)$, which by the uniqueness statement of Theorem 4.6.4 means $\tilde{E} = E_M$. We see for $k_1, k_2 \in \mathscr{K}$ that

$$\begin{aligned}\langle \left[\int z d\tilde{E}(z)\right] k_1, k_2 \rangle &= \int z d\tilde{E}_{k_1,k_2}(z) = \int z dE_{N,V^{-1}k_1,V^{-1}k_2}(z) \\ &= \langle NV^{-1}k_1, V^{-1}k_2 \rangle_{\mathscr{H}} = \langle VNV^{-1}k_1, k_2 \rangle_{\mathscr{K}} = \langle Mk_1, k_2 \rangle_{\mathscr{K}} \quad .\end{aligned}$$

Here we used the facts that E_N is the spectral measure associated with N, that V is an isometry and that $VNV^{-1} = M$. Therefore we have $M = \int z d\tilde{E}(z)$ which by Theorem 4.6.4 means $\tilde{E} = E_M$.

To finally prove the Corollary, we turn to $\phi \in B(\sigma(N))$. We now have for $k_1, k_2 \in \mathscr{K}$

$$\langle V\phi(N)V^{-1}k_1, k_2 \rangle_{\mathscr{K}} = \langle \phi(N)V^{-1}k_1, V^{-1}k_2 \rangle_{\mathscr{H}} = \int \phi(z) dE_{N,V^{-1}k_1,V^{-1}k_2}(z)$$

$$= \int \phi(z) d\tilde{E}_{k_1,k_2}(z) = \int \phi(z) dE_{M,k_1,k_2}(z) = \langle \phi(M) k_1, k_2 \rangle_{\mathscr{K}} \quad .$$

This proves $V\phi(N)V^{-1} = \phi(M)$. Now we have proven in Theorem 3.7.7 that $W^*(N) = \{\phi(N) : \phi \in B(\sigma(N))\}$, and a similar equality holds for $W^*(M)$. Thus we have the previous equation for all elements of $W^*(N)$ and $W^*(M)$, that is we can write in a sense that $VW^*(N)V^{-1} = W^*(M)$. □

Corollary 3.7.10. *Let $\mathscr{H}$ be separable and let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. If there exists a cyclic vector $h \in \mathscr{H}$ for $W^*(N)$, then $\{N\}' = W^*(N)' = W^*(N)$, that is $W^*(N)$ is a maximal abelian von Neumann algebra.*

Remark. The fact that $W^*(N)$ has a cyclic vector implies that $\mathscr{H}$ is separable. Further, $1 \in W^*(N)$ and $W^*(N)$ is SOT-closed according to Proposition 3.4.4. Therefore we can see Corollary 3.7.10 as a special case for the inverse implication in Theorem 3.4.6.

Proof. First, we see that $N_h = N$ since $\mathscr{H}_h = \mathscr{H}$, and therefore Theorem 3.5.8 gives us a unitary operator $V : \mathscr{H} \mapsto L^2(\mu_h)$ so that $N \simeq N_{\mu_h}$. For ease of writing, we will also substitute $\mu = \mu_h$. Now we take $K \in \{N\}'$ and denote $K_\mu = VKV^{-1}$

$$K_\mu N_\mu = VKNV^{-1} = VNKV^{-1} = N_\mu K_\mu \quad .$$

Thus we see that $K_\mu \in \{N_\mu\}'$ and therefore according to Theorem 3.6.2 we have $K_\mu \in W^*(N_\mu)$. Here we can use Corollary 3.7.9 to see that $V^{-1}W^*(N_\mu)V = W^*(N)$, and therefore $V^{-1}K_\mu V \in W^*(N)$, and since $K = V^{-1}K_\mu V$ we have $\{N\}' \subseteq W^*(N)$. Now we remember that $W^*(N)$ is abelian, so we have $W^*(N) \subseteq W^*(N)'$. Finally we combine our previous insights with Proposition 3.2.1 and $\{N\} \subseteq W^*(N)$ to get

$$W^*(N) \subseteq W^*(N)' \subseteq \{N\}' \subseteq W^*(N) \quad .$$

This implies that $\{N\}' = W^*(N)' = W^*(N)$, and thus by Theorem 3.4.6 we get that $W^*(N)$ is a maximal abelian von Neumann algebra. □

Now we come to the final step before we plunge into the actual first multiplicity theorem in the next section. This theorem consolidates our knowledge about scalar-valued measures and we will show their correspondence to separating vectors.

Theorem 3.7.11. *Let $\mathscr{H}$ be separable and let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then we find*

(a) There exists a scalar-valued measure μ for N.

(b) For every $h \in \mathscr{H}$ with h being a separating vector for $W^*(N)$, the measure μ_h is a scalar-valued measure for N.

(c) For every scalar-valued measure μ for N, there exists a $h \in \mathscr{H}$ so that h is a separating vector for $W^*(N)$ and $\mu = \mu_h$.

Proof. First, we prove *(a)* and *(b)*. Lemma 3.7.2 gives us *(b)*, and by Corollary 3.4.9 we find that $W^*(N)$ has a separating vector h and thus μ_h is also a scalar-valued measure for N.

Now we move on to *(c)*. We take the vector h from our previous considerations and see that μ_h is a scalar-valued measure. Proposition 3.7.1 thus gives us $[\mu] = [\mu_h]$. Proposition 3.5.3 then tells us that $N_\mu \simeq N_{\mu_h}$ and Theorem 3.5.8 tells us that $N_{\mu_h} \simeq N_h$, so altogether we get $N_h \simeq N_\mu$. Theorem 3.5.8 now tells us that there exists a vector $k \in \mathscr{H}_h$ so that $\mathscr{H}_k = \mathscr{H}_h$ and $\mu_k = \mu$. Further we get a unitary operator $V : \mathscr{H}_h \mapsto L^2(\mu)$ so that $Vk = 1$ and for any $\phi(N)$ it holds $V\phi(N)V^{-1} = M_\phi$. Let now $\phi(N) \in W^*(N)$. Corollary 3.7.8 tells us that $\phi(N)|_{\mathscr{H}_k} = \phi(N_k)$, so we get

$$||\phi(N)k||^2_{\mathscr{H}} = ||V\phi(N)V^{-1}Vk||^2_{\mathscr{H}} = ||\phi \cdot 1||^2_{L^2(\mu)} = \int |\phi(z)|^2 d\mu(z) \quad .$$

Thus if we have $\phi(N)k = 0$ if and only if $\phi(z) = 0$ μ-almost everywhere. Since $N_h \simeq N_{\mu_h}$, we can do the same calculation for h and see that $\phi(N)h = 0$ if and only if $\phi(z) = 0$ μ_h-almost everywhere. Further, we have that $[\mu] = [\mu_h]$ and so $\phi(z) = 0$ μ-almost everywhere is equivalent to $\phi(z) = 0$ μ_h-almost everywhere. Therefore, if $\phi(N)k = 0$, we get that also $\phi(N)h = 0$. However, since h is a separating vector for $W^*(N)$, this means that $\phi(N) = 0$. We know from Theorem 3.7.7 that $\{\phi(N) : \phi \in B(\sigma(N))\} = W^*(N)$, and so we have proven that k is a separating vector for $W^*(N)$. Finally we remember that $\mu = \mu_k$, which concludes the theorem. □

3.8 Multiplicity Theory on Seperable Hilbert Spaces

In the last sections, we have diligently laid out the groundwork for this section. Now it is finally time to reap the rewards and prove the First Multiplicity Theorem 3.8.8. The basic idea is rather straightforward. Theorem 3.5.8 tells us that for every $h \in \mathscr{H}$ we have $\mathscr{H}_h \simeq L^2(\mu_h)$ and $N|_{\mathscr{H}_h} \simeq N_{\mu_h}$, and we remember from way back in Lemma 3.4.5 that each separable Hilbert space $\mathscr{H}$ has a partition $\mathscr{H} = \bigoplus_{n=1}^\infty \mathscr{H}_{e_n}$. We can combine both facts and after a quick calculation we get

$$\mathscr{H} \simeq \bigoplus_{n=1}^\infty L^2(\mu_{e_n}) \quad , \quad N \simeq \bigoplus_{n=1}^\infty N_{\mu_{e_n}} \quad .$$

However, if we remember our initial goals for our Multiplicity Theorem, we did not only want to deconstruct our operator into a direct sum of more simple operators. We also required that the spectra of these simple operators to form a decreasing sequence, which is roughly equivalent to $\mu_{e_{n+1}} \ll \mu_{e_n}$. This is where our theory about scalar-valued measures comes into play. As we remember from Proposition 3.7.1, for a scalar valued measure μ we have $\mu_h \ll \mu$ for all $h \in \mathscr{H}$. On the other hand, Theorem 3.7.11 tells us that scalar-valued measures are equivalent to separating vectors for $W^*(N)$. Thus the solution is to inductively choose the $(e_n)_{n\in\mathbb{N}}$ so that e_{n+1} is a separating vector on the orthogonal complement of the space spanned by the $\mathscr{H}_i$ with $1 \leq i \leq n$. We will now show some lemmata that formalize what we have just discussed, before we plunge into the proof of Theorem 3.8.8 at the end of this section.

Lemma 3.8.1. *Let $(\mathscr{H}_n)_{n\in\mathbb{N}}$ and $(\mathscr{K}_n)_{n\in\mathbb{N}}$ be sequences of Hilbert spaces, and let further $(V_n)_{n\in\mathbb{N}}$ be a sequence of unitary operators with $V_n : \mathscr{H}_n \mapsto \mathscr{K}_n$ for all $n \in \mathbb{N}$. Then there exists a unique unitary operator with*

$$V : \bigoplus_{n\in\mathbb{N}} \mathscr{H}_n \mapsto \bigoplus_{n\in\mathbb{N}} \mathscr{K}_n \quad ; \quad V|_{\mathscr{H}_n} = V_n \quad .$$

Further, let $N_n \in \mathscr{B}(\mathscr{H}_n)$ and $M_n \in \mathscr{B}(\mathscr{K}_n)$ for $n \in \mathbb{N}$. If $V_n N_n V_n^{-1} = M_n$ holds for all $n \in \mathbb{N}$, then we find

$$V \bigoplus_{n=1}^{\infty} N_n V^{-1} = \bigoplus_{n=1}^{\infty} M_n \quad .$$

Proof. This Lemma follows immediately from setting $V : \mathscr{H} \mapsto \mathscr{K}$ as

$$V := \bigoplus_{n=1}^{\infty} V_n \quad .$$

For uniqueness, we take $h \in \bigoplus_{n=1}^{\infty} \mathscr{H}_n$ and see that $h = \bigoplus_{n=1}^{\infty} h_n$, with $h_n \in \mathscr{H}_n$ for $n \in \mathbb{N}$. Let now I be an operator that fulfils the same conditions as V. Then we see

$$Vh - Ih = \bigoplus_{n=1}^{\infty} V_n h_n - \bigoplus_{n=1}^{\infty} V_n h_n = 0 \quad .$$

Thus we get $V = I$. □

Lemma 3.8.2. *Let μ be a Radon measure, and let Δ be a Borel set. Then $\mu|_\Delta$ is also a Radon measure, and we find* $\operatorname{supp}(\mu|_\Delta) \subseteq \operatorname{supp}(\mu) \cap \operatorname{cl}(\Delta)$.

Proof. First we have to show that $\mu|_\Delta$ is both inner regular and locally finite. We take $z \in \mathbb{C}$. Since μ is a Radon measure, we know that there exists a neighbourhood U_z so that $\mu(U_z) < \infty$. This in turn means that

$$\mu|_\Delta(U_z) = \mu(U_z \cap \Delta) \leq \mu(U_z) < \infty \quad .$$

Thus we see that $\mu|_\Delta$ is also locally finite. Now to show inner regularity, let U be an open set. We know that μ is inner regular, so

$$\mu(U) = \sup\{\mu(K) : K \text{ is compact and } K \subseteq U\} \quad .$$

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence so that all $K_n \subseteq U$ and $\mu(K_n) \to \mu(U)$. This means especially that

$$\mu(U \setminus K_n) = \mu(U) - \mu(K_n) \to 0 \quad .$$

We can intersect the sets on the left hand side with Δ and get

$$\begin{aligned} \mu|_\Delta(U) - \mu|_\Delta(K_n) &= \mu(U \cap \Delta) - \mu(K_n \cap \Delta) \\ &= \mu((U \cap \Delta) \setminus (K_n \cap \Delta)) \leq \mu(U \setminus K_n) \to 0 \quad . \end{aligned}$$

Since all $(K_n)_{n\in\mathbb{N}}$ were compact subsets of U, we see

$$\mu|_\Delta(U) = \sup\{\mu|_\Delta(K) : K \text{ is compact and } K \subseteq U\} \quad .$$

This means $\mu|_\Delta$ is inner regular, and thus a Radon measure.

Now we proceed to show the second part of the lemma and take $z \in \operatorname{supp}(\mu|_\Delta)$. This means that for each neighbourhood U_z of z, we have $\mu|_\Delta(U_z) > 0$. Since $\mu|_\Delta(U_z) \leq \mu(U_z)$, this means that $z \in \operatorname{supp}(\mu)$. Now we assume that $z \in \operatorname{cl}(\Delta)^c$. Then there exists a neighbourhood U_z^Δ of z so that $U_z^\Delta \cap \Delta = \emptyset$. This leads to $\mu|_\Delta(U_z^\Delta) = 0$, which is a contradiction to the fact that $z \in \operatorname{supp}(\mu|_\Delta)$. Altogether we get

$$\operatorname{supp}(\mu|_\Delta) \subseteq \operatorname{supp}(\mu) \cap \operatorname{cl}(\Delta) \quad .$$

This proves the lemma. □

Lemma 3.8.3. *Let μ, ν be two finite measures on the measurable space (X, Ω), and let $\nu \ll \mu$. Then there exists a $\Delta \in \Omega$ so that the measure $\mu|_\Delta$ is mutually absolutely continuous with ν, that is $[\nu] = [\mu|_\Delta]$.*

Proof. We denote by $\mathscr{N}_\nu \subseteq \Omega$ the set of sets with ν measure zero, that is $\mathscr{N}_\nu := \{A \in \Omega : \nu(A) = 0\}$. Now we consider $\mu : \mathscr{N}_\nu \mapsto \mathbb{R}_+$, and since μ is a finite measure, we know that $\mu(\Omega)$ is bounded and thus $\mu|_{\mathscr{N}_\nu}$ is also bounded. Therefore we can define $t := \sup\{\mu(A) : A \in \mathscr{N}_\nu\}$ and get a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{N}_\nu$ so that $\mu(A_n) \to t$. We now define $\tilde{A}_n := \bigcup_{i\leq n} A_i$ and $\tilde{A} := \bigcup_{n=1}^\infty A_n$. Since ν is a measure and thus σ-additive, we have $\nu(\tilde{A}_n) = 0$ for all $n \in \mathbb{N}$ and also $\nu(\tilde{A}) = 0$, and therefore $\tilde{A}_n, \tilde{A} \in \mathscr{N}_\nu$. In addition, we note that $(\tilde{A}_n)_{n\in\mathbb{N}}$ is a monotone series of sets and thus, due to σ-additivity of μ, we have $\mu(\tilde{A}_n) \to \mu(A)$. Therefore we get $\mu(\tilde{A}) = t$.

We now define $\Delta := \tilde{A}^c$. Since $\nu(\tilde{A}) = 0$, we know that for any $B \in \Omega$ we have

$$\nu(B) = \nu(B \cap \tilde{A}) + \nu(B \cap \Delta) = \nu(B \cap \Delta) \quad .$$

Let us now assume that $\mu|_\Delta(B) = \mu(B \cap \Delta) = 0$. Because $\nu \ll \mu$, this means that $\nu(B \cap \Delta) = 0$ and thus also $\nu(B) = 0$. Therefore we find that $\nu \ll \mu|_\Delta$. On the other hand, let us assume $\nu(B) = 0$. If $\mu|_\Delta(B) > 0$, then we have $\mu(B \cap \Delta) > 0$ and therefore find

$$\mu((\tilde{A}) \cup (B \cap \Delta)) = \mu((\tilde{A}) \cup (B \cap \tilde{A}^c)) = \mu(\tilde{A}) + \mu(B \cap \tilde{A}^c) = t + \mu|_\Delta(B) > t \quad .$$

However, $\nu((\tilde{A}) \cup (B \cap \Delta)) = 0$ and so this is in contradiction to the fact that $t = \sup \mu(\mathscr{N}_\nu)$. Therefore we find that $\mu|_\Delta(B) > 0$ and thus $\mu|_\Delta \ll \nu$. □

Lemma 3.8.4. *Let $h_1, h_2 \in \mathscr{H}$ and $h_2 \perp \mathscr{H}_{h_1}$. Then we find that $\mathscr{H}_{h_1} \perp \mathscr{H}_{h_2}$. Further let $h = h_1 + h_2$. Then we find that $\mathscr{H}_h \subseteq \mathscr{H}_{h_1} \oplus \mathscr{H}_{h_2}$.*

Proof. We assume $N \in \mathscr{B}(\mathscr{H})$ so that $\mathscr{H}_h = \operatorname{cl}(W^*(N)h)$, and the same goes for $\mathscr{H}_{h_1}$ and $\mathscr{H}_{h_2}$. First we will prove $\mathscr{H}_{h_1} \perp \mathscr{H}_{h_2}$, and for this we take $x_1 \in \mathscr{H}_{h_1}$ and $x_2 \in \mathscr{H}_{h_2}$. We know that there exists a sequence $(A_{x_2,n})_{n\in\mathbb{N}} \subseteq W^*(N)$ so that $A_{x_2,n}h_2 \to x_2$. Further, we remind ourselves that by Proposition 3.5.4 we know that $\mathscr{H}_{h_1}$ is closed under action from $W^*(N)$. We also know that $W^*(N)$ contains all adjoints, so we have $A_{x_2,n}^* x_1 \in \mathscr{H}_{h_1}$ for all

$n \in \mathbb{N}$. Together with the facts that $h_2 \perp \mathscr{H}_{h_1}$ and that the scalar product is continuous, this gives us

$$\langle x_1, x_2 \rangle = \lim_{n\to\infty} \langle x_1, A_{x_2,n} h_2 \rangle = \lim_{n\to\infty} \langle A^*_{x_2,n} x_1, h_2 \rangle = 0 \quad .$$

Thus we find that $\mathscr{H}_{h_1} \perp \mathscr{H}_{h_2}$.

Now we move on to show that $\mathscr{H}_h \subseteq \mathscr{H}_{h_1} \oplus \mathscr{H}_{h_2}$. Let us take $x \in \mathscr{H}_h$ with $x_n := A_{x,n} h$ and $x_n \to x$ in a similar notation as above. We see for all $n \in \mathbb{N}$ that

$$x_n = A_{x,n} h = A_{x,n} h_1 + A_{x,n} h_2 \in \mathscr{H}_{h_1} \oplus \mathscr{H}_{h_2} \quad .$$

As the direct sum of two closed subspaces, $\mathscr{H}_{h_1} \oplus \mathscr{H}_{h_2}$ is again closed and therefore we have $x \in \mathscr{H}_{h_1} \oplus \mathscr{H}_{h_2}$. Thus we find that $\mathscr{H}_h \subseteq \mathscr{H}_{h_1} \oplus \mathscr{H}_{h_2}$. □

Lemma 3.8.5. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator and let $h \in \mathscr{H}$. Then we find a separating vector $e_h \in \mathscr{H}$ for $W^*(N)$ so that $h \in \mathscr{H}_{e_h}$.*

Proof. This proof will consist of three steps. First we want to reduce the Hilbert space for our problem to the direct sum $\mathscr{H}_{red} \simeq L^2(\mu) \oplus L^2(\mu|_\Delta)$, and we will show that for $\phi(N) \in W^*(N)$ we have

$$\phi(N)|_{\mathscr{H}_{red}} \simeq M_\phi \oplus M_\phi \in \mathscr{B}(L^2(\mu) \oplus L^2(\mu|_\Delta)) \quad .$$

Afterwards we will construct a vector in $L^2(\mu) \oplus L^2(\mu|_\Delta)$ that will become our candidate for e_h. In the second step, we then show that e_h is indeed separating, and in the third step we will show $h \in \mathscr{H}_{e_h}$.

To commence with the first step, we remember that Corollary 3.4.9 tells us that the C^*-algebra $W^*(N)$ has a separating vector e_s, which will be a good starting point of our investigation. We can then split $h = h_\parallel + h_\perp$ where $h_\parallel \in \mathscr{H}_{e_s}$ and $h_\perp \in \mathscr{H}_{e_s}^\perp$. According to Lemma 3.8.4, we know that $\mathscr{H}_{e_s} \perp \mathscr{H}_{h_\perp}$, so we can consider $h \in \mathscr{H}_{e_s} \oplus \mathscr{H}_{h_\perp}$ and reduce our calculations to this subspace. By Theorem 3.5.8 and Lemma 3.8.1 we know that

$$\mathscr{H}_{e_s} \oplus \mathscr{H}_{h_\perp} \simeq L^2(\mu_{e_s}) \oplus L^2(\mu_{h_\perp})$$

$$N_{e_s} \oplus N_{h_\perp} \simeq N_{\mu_{e_s}} \oplus N_{\mu_{h_\perp}} \quad .$$

Since e_s is a separating vector for $W^*(N)$ we know by Theorem 3.7.11 that μ_{e_s} is a scalar-valued spectral measure for N and thus by Proposition 3.7.1 we know that $\mu_{h_\perp} \ll \mu_{e_s}$. We now designate $\mu := \mu_{e_s}$, and Lemma 3.8.3 tells us that there exists a Borel set Δ so that $[\mu|_\Delta] = [\mu_{h_\perp}]$. With Proposition 3.5.3 this means $N_{\mu|_\Delta} \simeq N_{\mu_{h_\perp}}$ and so we get

$$\mathscr{H}_{e_s} \oplus \mathscr{H}_{h_\perp} \simeq L^2(\mu) \oplus L^2(\mu|_\Delta)$$

$$N_{e_s} \oplus N_{h_\perp} \simeq N_\mu \oplus N_{\mu|_\Delta} \quad .$$

Since $\mu = \mu_{e_s}$, Theorem 3.7.11 now tells us that there exists a unique unitary $V_1 : \mathscr{H}_{e_s} \mapsto L^2(\mu)$ so that $Ve_s = 1$ and $V_1 \phi(N_{e_s}) V_1^{-1} = M_\phi$ for all $\phi \in B(\sigma(N))$. When it comes to $N_{\mu|_\Delta} \simeq N_{h_\perp}$, we get a similar relation, however we have to remember that $\mu|_\Delta \neq \mu_{h_\perp}$. Thus Theorem 3.7.11 only tells us that there exists a vector $x_\perp \in \mathscr{H}_{h_\perp}$ so that $\mathscr{H}_{x_\perp} = \mathscr{H}_{h_\perp}$ and $\mu|_\Delta = \mu_{x_\perp}$. Additionally we get a unique unitary $V_2 : \mathscr{H}_{e_s} \mapsto L^2(\mu|_\Delta)$ so that $V_2 x_\perp = 1$

and $V_2\phi(N_{x_\perp})V_2^{-1} = M_\phi$ for all $\phi \in B(\sigma(N))$. Restating our previous equations with this new insight we now see

$$\mathscr{H}_{e_s} \oplus \mathscr{H}_{x_\perp} \simeq L^2(\mu) \oplus L^2(\mu|_\Delta)$$
$$N_{e_s} \oplus N_{x_\perp} \simeq N_\mu \oplus N_{\mu|_\Delta} \quad .$$

By Theorem 3.7.7 we have $W^*(N) = \{\phi(N) : \phi \in B(\sigma(N))\}$. According to Corollary 3.7.8 for $\phi \in B(\sigma(N))$ this gives

$$\phi(N)|_{\mathscr{H}_{e_s}\oplus\mathscr{H}_{x_\perp}} = \phi(N_{e_s}) \oplus \phi(N_{x_\perp}) \quad .$$

We now label $V := V_1 \oplus V_2$, and by Lemma 3.8.1 we see that $V : \mathscr{H}_{e_s} \oplus \mathscr{H}_{x_\perp} \mapsto L^2(\mu) \oplus L^2(\mu|_\Delta)$ is a unitary operator with $V^{-1} = V_1^{-1} \oplus V_2^{-1}$. By combining this with our previous considerations and taking $\phi \in B(\sigma(N))$ we get

$$V\phi(N)|_{\mathscr{H}_{e_s}\oplus\mathscr{H}_{x_\perp}} V^{-1}$$
$$= V_1\phi(N_{e_s})V_1^{-1} \oplus V_2\phi(N_{x_\perp})V_2^{-1}$$
$$= M_\phi \oplus M_\phi \quad .$$

We now remember that $h = h_\parallel + h_\perp$ with $h_\parallel \in \mathscr{H}_{e_s}$ and $h_\perp \in \mathscr{H}_{x_\perp}$, so we can denote

$$g_1 \oplus g_2 := Vh = V_1 h_\parallel \oplus V_2 h_\perp \quad .$$

This helps us define

$$f_1(z) := \begin{cases} 1 & \text{if } z \in \Delta^c \\ g_1(z) & \text{if } z \in \Delta \end{cases}$$
$$f_2(z) := g_2(z) \quad .$$

With this definition in hand, we take $e_h := V^{-1}(f_1 \oplus f_2)$. This definition might seem arbitrary at first, but the subsequent calculations will prove that e_h is the separating vector required by this lemma. It is important to remember that $e_h \in \mathscr{H}_{e_s} \oplus \mathscr{H}_{x_\perp} \subseteq \mathscr{H}$.

We now proceed to the second part of this proof, where we want to show that e_h is separating for $W^*(N)$. Let us now assume $M_\phi f_1 \oplus M_\phi f_2 = 0$. This means especially that $M_\phi f_1 = 0$ μ-almost everywhere on Δ^c. Since $f_1 = 1$ on Δ^c, we can conclude that $M_\phi 1 = \phi(z) = 0$ μ-almost everywhere on Δ^c. On the other hand, let us look at $M_\phi f_2 = 0$. We remember that $f_2(z) = V_2 h_\perp$, and we now want to show that $|f_2(z)| > 0$ μ-almost everywhere on Δ. Let now E be the spectral measure for N and ω be a Borel set. According to Theorem 4.6.3 we have $\phi_\omega(N) = E(\omega)$ with $\phi_\omega(z) = \chi_\omega(z)$. This means we get

$$\mu_{h_\perp}(\omega) = \langle E(\omega)h_\perp, h_\perp\rangle_{\mathscr{H}} = \langle E(\omega)|_{\mathscr{H}_{x_\perp}} h_\perp, h_\perp\rangle_{\mathscr{H}_{x_\perp}}$$
$$= \langle \chi_\omega f_2, f_2\rangle_{L^2(\mu|_\Delta)} = \int_\Delta \chi_\omega |f_2(z)|^2 d\mu(z) \quad .$$

In the last equality we used the fact that $\frac{d\mu|_\Delta}{d\mu} = \chi_\Delta$, as given by Proposition 4.7.5. Further we see that according to Corollary 3.6.4 the last expression is precisely μ_{f_2}, and so altogether we obtain $\mu_{f_2} = \mu_{h_\perp}$. Since by definition of Δ we had $[\mu|_\Delta] = [\mu_{h_\perp}]$, this means that also $[\mu|_\Delta] = [\mu_{f_2}]$. However, since $\mu_{f_2}(\omega) = \int_\Delta \chi_\omega |f_2(z)|^2 d\mu(z)$, this is only possible if $|f_2| > 0$ μ-almost everywhere on Δ. Therefore we see that $M_\phi f_2 = 0$ only if $\phi(z) = 0$ μ-almost everywhere on Δ. Together we have found that $M_\phi f_1 \oplus M_\phi f_2 = 0$ implies both $\phi(z) = 0$ μ-almost everywhere on Δ^c and $\phi(z) = 0$ μ-almost everywhere on Δ, which means that $\phi(z) = 0$ μ-almost everywhere. Now we remember that $\mu = \mu_{e_s}$ and e_s is a separating vector for $W^*(N)$, so according to Theorem 3.7.11 we know that μ is a scalar-valued measure for N. This means that $\phi(z) = 0$ E-almost everywhere as well, and so $\phi(N) = 0$. Therefore we have $\phi(N) = 0$ if and only if

$$\phi(N)V^{-1}(f_1 \oplus f_2) = V^{-1}(M_\phi f_1 \oplus M_\phi f_2) = 0 \quad .$$

We remind ourselves that by Theorem 3.7.7 we have $W^*(N) = \{\phi(N) : \phi \in B(\sigma(N))\}$, so we have shown altogether that $V^{-1}(f_1 \oplus f_2)$ is a separating vector for $W^*(N)$. Thus we can set $e_h := V^{-1}(f_1 \oplus f_2)$.

In the third part of the proof, we will show that $h \in \mathscr{H}_{e_h}$. To this end, we will split h differently into $h = E(\Delta)h + E(\Delta^c)h$. We immediately see that

$$VE(\Delta)h = \chi_\Delta g_1 \oplus \chi_\Delta g_2 \quad .$$

On the other hand, since $\chi_\Delta f_1 = \chi_\Delta g_1$ and $f_2 = g_2$, we also get

$$VE(\Delta)e_h = VE(\Delta)V^{-1}f_1 \oplus f_2 = \chi_\Delta g_1 \oplus \chi_\Delta g_2 \quad .$$

Because V is unitary, we find $E(\Delta)h = E(\Delta)e_h$. The other half is a bit more complicated. Since $\mu|_\Delta(\Delta^c) = 0$, we see that $\chi_{\Delta^c} = 0$ $\mu|_\Delta$-almost everywhere and so we have

$$VE(\Delta^c)h = \chi_{\Delta^c} g_1 \oplus \chi_{\Delta^c} g_2 = \chi_{\Delta^c} g_1 \oplus 0 \quad .$$

Theorem 4.7.7 tells us that $L^\infty(\mu) \cap L^2(\mu)$ is dense in $L^2(\mu)$, so there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ with $\phi_n \to \chi_{\Delta^c} g_1$ in $L^2(\mu)$. We can now find representatives $(\tilde{\phi}_n)_{n\in\mathbb{N}}$ so that $\sup |\tilde{\phi}_n(z)| \leq ||\phi_n||_{L^\infty(\mu)}$, and thus $(\tilde{\phi}_n)_{n\in\mathbb{N}} \subseteq B(\sigma(N))$. Before we make the computation for $E(\Delta^c)h$, we remind ourselves that $f_1 = 1$ on Δ^c, so we get

$$VE(\Delta^c)e_h = VE(\Delta^c)V^{-1}f_1 \oplus f_2 = \chi_{\Delta^c} \oplus 0 \quad .$$

Now we can apply our recent insights to see

$$V\tilde{\phi}_n(N)E(\Delta^c)e_h = V\tilde{\phi}_n(N)V^{-1}VE(\Delta^c)V^{-1}(f_1 \oplus f_2)$$

$$= V\tilde{\phi}_n(N)V^{-1}(\chi_{\Delta^c} \oplus 0) = \chi_{\Delta^c}\tilde{\phi}_n \oplus 0 \quad .$$

Since $\tilde{\phi}_n \to \chi_{\Delta^c} g_1$, we see that

$$\chi_{\Delta^c}\tilde{\phi}_n \to \chi^2_{\Delta^c} g_1 = \chi_{\Delta^c} g_1 \quad .$$

Again using the fact that V is unitary, we find that $\tilde{\phi}_n(N)E(\Delta^c)e_h \to E(\Delta^c)h$. We now tie everything together to define $x_n := (E(\Delta) + \tilde{\phi}_n(N)E(\Delta^c))e_h$ for $n \in \mathbb{N}$ and we see

$$x_n = (E(\Delta) + \tilde{\phi}_n(N)E(\Delta^c))e_h = E(\Delta)h + \tilde{\phi}_n(N)E(\Delta^c)e_h$$

$$\to E(\Delta)h + E(\Delta^c)h = h \quad .$$

To conclude our proof, we now remind ourselves that $E(\Delta), E(\Delta^c) \in W^*(N)$. Since $W^*(N)$ is an algebra, we also have $E(\Delta) + \tilde{\phi}_n(N)E(\Delta^c)$ for all $n \in \mathbb{N}$. Thus we get $x_n \in \mathscr{H}_{e_h}$, and since $x_n \to h$ and $\mathscr{H}_{e_h}$ is closed, we also have $h \in \mathscr{H}_{e_h}$. We have already proven that e_h is separating for $W^*(N)$, so this concludes the lemma. □

Lemma 3.8.6. *Let $A_n \in \mathscr{B}(\mathscr{H}_n)$ with $n \in \mathbb{N}$ and let $A \simeq \bigoplus_{n=1}^{\infty} A_n$. Then we have* $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \sigma(A_n)\right) = \sigma(A)$.

Proof. First let us denote $\mathscr{H}$ so that $A \in \mathscr{B}(\mathscr{H})$, $\mathscr{H}_\oplus = \bigoplus_{n=1}^{\infty} \mathscr{H}_n$ and $A_\oplus = \bigoplus_{n=1}^{\infty} A_n$. We want to show that $\sigma(A_\oplus) = \sigma(A)$. We know that there exists a unitary operator $V : \mathscr{H}_\oplus \mapsto \mathscr{H}$ with

$$V^{-1}AV = A_\oplus \quad .$$

Next, we take $\lambda \in \rho(A)$, and we define $B_\lambda = (A - \lambda_{\mathscr{H}})^{-1}$. Further, we remark that $V^{-1}\lambda_{\mathscr{H}}V = \lambda_{\mathscr{H}_\oplus}$. Altogether, we can show

$$V^{-1}B_\lambda V(A_\oplus - \lambda_{\mathscr{H}_\oplus}) = V^{-1}B_\lambda V V^{-1}(A - \lambda_{\mathscr{H}})V = V^{-1}\mathbf{1}_{\mathscr{H}}V = \mathbf{1}_{\mathscr{H}_\oplus} \quad .$$

The exact same calculation holds for

$$(A_\oplus - \lambda_{\mathscr{H}_\oplus})V^{-1}B_\lambda V = ... = \mathbf{1}_{\mathscr{H}_\oplus} \quad .$$

This means we have $\lambda \in \rho(A_\oplus)$ and thus $\rho(A_\oplus) \subseteq \rho(A)$. We can now just swap out A and $A_\oplus$ in the previous calculation to also get $\rho(A) \subseteq \rho(A_\oplus)$. Since $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and $\rho(A_\oplus) = \mathbb{C} \setminus \sigma(A_\oplus)$, this gives us $\sigma(A_\oplus) = \sigma(A)$. This means we can now move on to show $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \sigma(A_n)\right) = \sigma(A_\oplus)$.

First, we want to prove $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \sigma(A_n)\right) \subseteq \sigma(A_\oplus)$ by seeing that $\rho(A_\oplus) \subseteq \rho(A_n)$ for all $n \in \mathbb{N}$. We take $\lambda \in \rho(A_\oplus)$ and know that $B_{\lambda_\oplus} := (A_\oplus - \lambda_{\mathscr{H}_\oplus})^{-1}$ exists. Now let $P_{\mathscr{H}_n}$ be the projection from $\mathscr{H}_\oplus$ onto $\mathscr{H}_n$ for $n \in \mathbb{N}$. Since $A_\oplus$ is a direct sum and $\lambda_{\mathscr{H}_\oplus}$ commutes with every other linear operator, we get

$$P_{\mathscr{H}_n}(A_\oplus - \lambda_{\mathscr{H}_\oplus}) = (A_\oplus - \lambda_{\mathscr{H}_\oplus})P_{\mathscr{H}_n} = P_{\mathscr{H}_n}(A_\oplus - \lambda_{\mathscr{H}_\oplus})P_{\mathscr{H}_n} \quad .$$

This knowledge now helps us to make the following calculation

$$\begin{aligned} P_{\mathscr{H}_n} &= P_{\mathscr{H}_n}\mathbf{1}_{\mathscr{H}_\oplus}P_{\mathscr{H}_n} = P_{\mathscr{H}_n}(A_\oplus - \lambda_{\mathscr{H}_\oplus})B_{\lambda_\oplus}P_{\mathscr{H}_n} \\ &= \left(P_{\mathscr{H}_n}(A_\oplus - \lambda_{\mathscr{H}_\oplus})P_{\mathscr{H}_n}\right)\left(P_{\mathscr{H}_n}B_{\lambda_\oplus}P_{\mathscr{H}_n}\right) \quad . \end{aligned}$$

We can now interpret both sides as operators on $\mathscr{H}_n$. From the definition of $A_\oplus$ we get

$$\left(P_{\mathscr{H}_n}(A_\oplus - \lambda_{\mathscr{H}_\oplus})P_{\mathscr{H}_n}\right)|_{\mathscr{H}_n} = A_n - \lambda_{\mathscr{H}_n} \quad .$$

Further, we have $P_{\mathscr{H}_n}|_{\mathscr{H}_n} = \mathbf{1}_{\mathscr{H}_n}$. We denote $\left(P_{\mathscr{H}_n}B_{\lambda_\oplus}P_{\mathscr{H}_n}\right)|_{\mathscr{H}_n} = B_{\lambda_n}$, and if we combine this with our previous insight, we get

$$\mathbf{1}_{\mathscr{H}_n} = (A_n - \lambda_{\mathscr{H}_n})B_{\lambda_n} \quad .$$

We can similarly show that B_{λ_n} is the left-inverse, and thus $A_n - \lambda_{\mathscr{H}_n}$ is invertible. Thich means that $\lambda \in \rho(A_n)$ and therefore $\rho(A_\oplus) \subseteq \rho(A_n)$. Similar as before, this gives us $\sigma(A_n) \subseteq \sigma(A_\oplus)$ for all $n \in \mathbb{N}$. We now remember that $\sigma(A_\oplus)$ is closed to get

$$\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \sigma(A_n)\right) \subseteq \sigma(A_\oplus) \quad .$$

This proves the first inclusion.

Now we proceed to show $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \sigma(A_n)\right) \supseteq \sigma(A_\oplus)$. To this end we take $\lambda \in \mathrm{int}(\bigcap_{n\in\mathbb{N}} \rho(A_n))$, which means we find an $\epsilon > 0$ so that the open ball of size ϵ centered around λ is contained within all $\rho(A_n)$. This especially implies that $\mathrm{dist}(\sigma(A_n), \lambda) \geq \epsilon$ and thus we find by Proposition 4.6.1 that

$$||(A_n - \lambda_{\mathscr{H}_n})^{-1}|| \leq \frac{1}{\epsilon} \quad .$$

Therefore, the family $((A_n - \lambda_{\mathscr{H}_n})^{-1})_{n\in\mathbb{N}}$ has bounded norm and thus we can define the linear operator

$$B_{\lambda_\oplus} := \bigoplus_{n=1}^{\infty} (A_n - \lambda_{\mathscr{H}_n})^{-1} \quad .$$

The calculations above tell us that $||B_{\lambda_\oplus}|| \leq \frac{1}{\epsilon}$, and so $B_{\lambda_\oplus} \in \mathscr{B}(\mathscr{H}_\oplus)$. Now find

$$B_{\lambda_\oplus}(A_\oplus - \lambda_{\mathscr{H}_\oplus}) = B_{\lambda_\oplus}\left(\bigoplus_{n=1}^{\infty}(A_n - \lambda_{\mathscr{H}_n})\right) = \bigoplus_{n=1}^{\infty} \mathbf{1}_{\mathscr{H}_n} = \mathbf{1}_{\mathscr{H}_\oplus} \quad .$$

In a similar manner we get $(A_\oplus - \lambda_{\mathscr{H}_\oplus})B_{\lambda_\oplus} = \mathbf{1}_{\mathscr{H}_\oplus}$, and we see that $\lambda \in \rho(A_\oplus)$. This means we get

$$\mathrm{int}\left(\bigcap_{n\in\mathbb{N}} \rho(A_n)\right) \subseteq \rho(A_\oplus) \quad .$$

We remember that $\rho(A_n) = \mathbb{C} \setminus \sigma(A_n)$, and by taking the complement on both sides we find

$$\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \sigma(A_n)\right) \supseteq \sigma(A_\oplus)$$

This proves the second inclusion and thus the lemma. □

Note. For the next lemma we need some additional notation to facilitate our calculations. If we split a Hilbert space $\mathscr{H}$ into $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, we can introduce a matrix notation for all vectors and operators. For example with $h \in \mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$, we have $h = h_1 \oplus h_2$ and we can make the following identification

$$Ah \sim \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} A_{11}h_1 + A_{12}h_2 \\ A_{21}h_1 + A_{22}h_2 \end{pmatrix} \quad .$$

Further, for $B \in \mathscr{B}(\mathscr{H})$ we can represent AB as

$$AB \sim \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} \quad .$$

Additionally, with some quick calculation, we can get matrix rules for the adjoint of A. This means

$$A^* \sim \begin{pmatrix} A^*_{11} & A^*_{12} \\ A^*_{21} & A^*_{22} \end{pmatrix} = \begin{pmatrix} (A_{11})^* & (A_{21})^* \\ (A_{12})^* & (A_{22})^* \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}^* \quad .$$

It is important to remember that here A^*_{12} is the $\mathscr{H}_2 \mapsto \mathscr{H}_1$ component the adjoint of A, while $(A_{12})^*$ is the adjoint of the $\mathscr{H}_2 \mapsto \mathscr{H}_1$ component of A. The previous equality tells us especially that $A^*_{12} = (A_{21})^*$.

Lemma 3.8.7. *Let $N_1 \in \mathscr{B}(\mathscr{H}_1)$, $A \in \mathscr{B}(\mathscr{H}_2)$, $N_2 \in \mathscr{B}(\mathscr{K}_1)$ and $B \in \mathscr{B}(\mathscr{K}_2)$ be normal operators. Further, let $h \in \mathscr{H}_1$ so that h is cyclic for $W^*(N_1)$. Then if $N_1 \simeq N_2$ and $N_1 \oplus A \simeq N_2 \oplus B$ we have $A \simeq B$.*

Proof. The core issue here is that we have a unitary map giving us $N_1 \oplus A \simeq N_2 \oplus B$, but that doesn't mean this unitary map preserves $N_1 \simeq N_2$ and $A \simeq B$. Therefore we want to use the additional information of $N_1 \simeq N_2$ and the cyclic vector $h \in \mathscr{H}_1$ to obtain $A \simeq B$.

To do this, we first simplify by taking $N_1 = N$ and $\mathscr{H}_1 = \mathscr{H}$ and removing N_2 from the problem. Since $N = N_1 \simeq N_2$, we can take a unitary operator $I : \mathscr{H} \mapsto \mathscr{K}_1$ so that $INI^{-1} = N_2$. Now we define the operator $\tilde{I} : \mathscr{H} \oplus \mathscr{K}_2 \mapsto \mathscr{K}_1 \oplus \mathscr{K}_2$ by setting $\tilde{I} = I \oplus \mathbf{1}_{\mathscr{K}_2}$. Lemma 3.8.1 tells us that $\tilde{I}$ is unitary and that $\tilde{I}^{-1} = I^{-1} \oplus \mathbf{1}_{\mathscr{K}_2}$. Now we can derive

$$\tilde{I}(N \oplus B)\tilde{I}^{-1} = (I \oplus \mathbf{1}_{\mathscr{K}_2})(N \oplus B)(I^{-1} \oplus \mathbf{1}_{\mathscr{K}_2}) = INI^{-1} \oplus B = N_2 \oplus B \quad .$$

This means we have $N_2 \oplus B \simeq N \oplus B$, and because of $N_1 \oplus A \simeq N_2 \oplus B$ and $N_1 = N$ we get $N \oplus A \simeq N \oplus B$. This reduces the complexity of the problem a bit, since we can now just focus on A, B, N on the Hilbert spaces $\mathscr{H}_2, \mathscr{K}_2, \mathscr{H}$ respectively.

Now we turn to the actual proof. We take $U : \mathscr{H} \oplus \mathscr{H}_2 \mapsto \mathscr{H} \oplus \mathscr{K}_2$ to be the unitary operator with the property

$$U(N \oplus A)U^{-1} = N \oplus B \quad .$$

We want to proceed in two steps, where first we establish the core matrix equations for U, N, A, B, which will make our subsequent calculations easier. In the second step, we use these equations, in conjunction with Proposition 3.3.5, to obtain $A \simeq B$.

We see by Proposition 3.2.5 that $(X \oplus Y)^* = X^* \oplus Y^*$ for any bounded linear operators X, Y. Further, since U is unitary, we have $U^{-1} = U^*$, and thus we get

$$\begin{gathered} U(N^* \oplus A^*)U^{-1} = \\ = U(N \oplus A)^*U^* = (U(N \oplus A)U^*)^* = (N \oplus B)^* = \\ = N^* \oplus B^* \quad . \end{gathered}$$

We can then rewrite these equations to obtain

$$\begin{gathered} U(N \oplus A) = (N \oplus B)U \\ U(N^* \oplus A^*) = (N^* \oplus B^*)U \quad . \end{gathered} \tag{3.1}$$

To better understand their interactions, we now want to split the operators U, $N \oplus A$ and $N \oplus B$ into 2×2 matrices like discussed above. First, we see

$$N \oplus A \sim \begin{pmatrix} N & 0 \\ 0 & A \end{pmatrix} \qquad N \oplus B \sim \begin{pmatrix} N & 0 \\ 0 & B \end{pmatrix} \quad .$$

Further, although it is not an automorphism, we can obtain a similar matrix representation for U and see[1]

$$U \sim \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \quad .$$

Now we translate equations (3.1) intro matrix form and get

$$\begin{pmatrix} U_{11}N & U_{12}A \\ U_{21}N & U_{22}A \end{pmatrix} = \begin{pmatrix} NU_{11} & NU_{12} \\ BU_{21} & BU_{22} \end{pmatrix}$$

$$\begin{pmatrix} U_{11}N^* & U_{12}A^* \\ U_{21}N^* & U_{22}A^* \end{pmatrix} = \begin{pmatrix} N^*U_{11} & N^*U_{12} \\ B^*U_{21} & B^*U_{22} \end{pmatrix} \quad . \tag{3.2}$$

From now on, we will refer to the respective equations by either $(3.2)_{ij}$ or $(3.2)^*_{ij}$.[2] Further, we get from $\mathbf{1}_{\mathscr{H}\oplus\mathscr{H}_2} = U^*U$ and $\mathbf{1}_{\mathscr{H}\oplus\mathscr{K}_2} = UU^*$ the following two matrix equations

$$\begin{pmatrix} \mathbf{1}_{\mathscr{H}} & 0 \\ 0 & \mathbf{1}_{\mathscr{H}_2} \end{pmatrix} = \begin{pmatrix} (U_{11})^*U_{11} + (U_{21})^*U_{21} & (U_{11})^*U_{12} + (U_{21})^*U_{22} \\ (U_{12})^*U_{11} + (U_{22})^*U_{21} & (U_{12})^*U_{12} + (U_{22})^*U_{22} \end{pmatrix} \tag{3.3}$$

$$\begin{pmatrix} \mathbf{1}_{\mathscr{H}} & 0 \\ 0 & \mathbf{1}_{\mathscr{K}_2} \end{pmatrix} = \begin{pmatrix} U_{11}(U_{11})^* + U_{12}(U_{12})^* & U_{11}(U_{21})^* + U_{12}(U_{22})^* \\ U_{21}(U_{11})^* + U_{22}(U_{12})^* & U_{21}(U_{21})^* + U_{22}(U_{22})^* \end{pmatrix} \quad . \tag{3.4}$$

Here, we apply the same enumeration scheme as with Equations 3.2.

Now we remember that our main goal is to show $A \simeq B$. Equations $(3.2)_{22}$ and $(3.2)^*_{22}$ are a good start, although the problem is that U_{22} is not necessarily unitary or an isomorphism. Our approach will therefore be to separate $\mathscr{H}_2$ into $\ker U_{22}$ and $(\ker U_{22})^\perp$, and to show $A \simeq B$ for both subspaces separately.

We remember that A, B are normal operators and start by using Proposition 3.3.5 together with $U_{22}A = BU_{22}$, that is Equation $(3.2)_{22}$, to get $A|_{(\ker U_{22})^\perp} \simeq B|_{\mathrm{cl}(\mathrm{ran} U_{22})}$. Propositions 4.4.4 and 4.4.2 now tell us

$$\mathrm{cl}(\mathrm{ran} U_{22}) = ((\mathrm{ran} U_{22})^\perp)^\perp = (\ker(U_{22})^*)^\perp \quad .$$

Combining this with our previous insight, we get

$$A|_{(\ker U_{22})^\perp} \simeq B|_{(\ker(U_{22})^*)^\perp} \quad .$$

Now we turn our attention to $A|_{\ker U_{22}}$. Our next goal will be to use Proposition 3.3.5 again to get

$$A|_{\ker U_{22}} \simeq N|_{U_{12}(\ker U_{22})}$$

[1] It is important for us to keep in mind that here we have $U_{11} : \mathscr{H} \mapsto \mathscr{H}$, $U_{12} : \mathscr{H}_2 \mapsto \mathscr{H}$, $U_{21} : \mathscr{H} \mapsto \mathscr{K}_2$ and $U_{22} : \mathscr{H}_2 \mapsto \mathscr{K}_2$.

[2] For example, Equation $(3.2)^*_{21}$ designates $U_{12}N^* = B^*U_{12}$.

$$B|_{\ker(U_{22})^*} \simeq N|_{(U_{21})^*(\ker(U_{22})^*)} \quad .$$

Then, we will show that $U_{12}(\ker U_{22}) = (U_{21})^*(\ker(U_{22})^*)$, which will give us $A|_{\ker U_{22}} \simeq B|_{\ker(U_{22})^*}$. To this end we take $U_{12}A = NU_{12}$, that is Equation $(3.2)_{12}$. However, before using Proposition 3.3.5, we want to reduce the domain to $\ker U_{22}$, which means we want to show that $A|_{\ker U_{22}} \in \mathscr{B}(\ker U_{22})$. Using $U_{22}A^* = B^*U_{22}$, that is Equation $(3.2)^*_{22}$, and the exact same line of reasoning as before, we get $A^*|_{(\ker U_{22})^\perp} \simeq B^*|_{(\ker(U_{22})^*)^\perp}$ and thus $A^*|_{(\ker U_{22})^\perp} \in \mathscr{B}((\ker U_{22})^\perp)$. This means $A^*(\ker U_{22})^\perp \subseteq (\ker U_{22})^\perp$, so for $x \in \ker U_{22}$ and $y \in (\ker U_{22})^\perp$ we find

$$\langle Ax, y\rangle = \langle x, A^*y\rangle = 0 \quad .$$

Therefore we have $Ax \in ((\ker U_{22})^\perp)^\perp = \ker U_{22}$ and so we get $A \in \mathscr{B}(\ker U_{22})$. Now we are ready to use Equation $(3.2)_{12}$ with a reduced domain which yields

$$NU_{12}|_{\ker U_{22}} = U_{12}|_{\ker U_{22}} A|_{\ker U_{22}} \quad .$$

Using Proposition 3.3.5 now gives us the rather clumsy equation

$$N|_{\mathrm{cl}(\mathrm{ran} U_{12}|_{\ker U_{22}})} \simeq (A|_{\ker U_{22}})|_{(\ker U_{12}|_{\ker U_{22}})^\perp} \quad .$$

However, this will get substantially simpler as we now show that

$$\mathrm{cl}(\mathrm{ran} U_{12}|_{\ker U_{22}}) = U_{12} \ker U_{22}$$

$$(\ker U_{12}|_{\ker U_{22}})^\perp = \ker U_{22} \quad .$$

We take $h \in \ker U_{22}$ and we see that

$$Uh \sim \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} U_{12}h \\ 0 \end{pmatrix}$$

Since U is unitary, the previous equation shows us that $U_{12}|_{\ker U_{22}}$ preserves the norm on $\ker U_{22}$ and thus $U_{12}|_{\ker U_{22}}$ is also unitary in the sense that $U_{12}|_{\ker U_{22}} : \ker U_{22} \mapsto U_{12} \ker U_{22}$. Therefore $U_{12}|_{\ker U_{22}} h = 0$ if and only if $h = 0$ and thus we find $\ker U_{12}|_{\ker U_{22}} = \{0\}$, which in turn means

$$(\ker U_{12}|_{\ker U_{22}})^\perp = \ker U_{22} \quad .$$

Further, we know that $\ker U_{22}$ as a kernel is closed, and since $U_{12}|_{\ker U_{22}}$ is unitary and thus isometric, we see that $U_{12}|_{\ker U_{22}} \ker U_{22}$ is also closed. Therefore we find

$$\mathrm{cl}(\mathrm{ran} U_{12}|_{\ker U_{22}}) = \mathrm{cl}(U_{12} \ker U_{22}) = \mathrm{cl}(U_{12}|_{\ker U_{22}} \ker U_{22})$$

$$= U_{12}|_{\ker U_{22}} \ker U_{22} = U_{12} \ker U_{22} \quad .$$

These two insights now enable us to rewrite the clumsy equation above into

$$N|_{U_{12} \ker U_{22}} \simeq A|_{\ker U_{22}} \quad .$$

Now we want to show the remaining unitary similarity for $B^*|_{\ker U_{22}^*}$. We take the adjoint of Equations $(3.2)^*$ and see[3]

$$\begin{pmatrix} (U_{11})^*N & (U_{21})^*B \\ (U_{12})^*N & (U_{22})^*B \end{pmatrix} = \begin{pmatrix} N(U_{11})^* & N(U_{21})^* \\ A(U_{12})^* & A(U_{22})^* \end{pmatrix} \quad .$$

Because U is unitary, U^* is also unitary and we can perform the same steps as before to get

$$N|_{(U_{21})^*(\ker(U_{22})^*)} \simeq B|_{\ker(U_{22})^*} \quad .$$

This leaves us to prove $U_{12} \ker U_{22} = (U_{21})^*(\ker(U_{22})^*)$ so that we get $A|_{\ker U_{22}} \simeq B|_{\ker(U_{22})^*}$.

Let us now denote $\mathscr{M}_1 = U_{12}(\ker U_{22})$ and $\mathscr{M}_2 = (U_{21})^*(\ker(U_{22})^*)$. We start by showing the intermediate step of $\mathscr{M}_1 = \ker(U_{11})^*$. For this we take $h \in \ker U_{22}$ and use $0 = (U_{11})^*U_{12} + (U_{21})^*U_{22}$, that is Equation $(3.3)_{12}$, to see that

$$0 = (U_{11})^*U_{12}h + (U_{21})^*U_{22}h = (U_{11})^*U_{12}h \quad .$$

Therefore $U_{12}h \in \ker(U_{11})^*$, so we can deduce that

$$\mathscr{M}_1 = U_{12}(\ker U_{22}) \subseteq \ker(U_{11})^* \quad .$$

Now on the other hand, for $h' \in \ker(U_{11})^*$ we see by $0 = U_{21}(U_{11})^* + U_{22}(U_{12})^*$ (Equation $(3.4)_{21}$) that

$$0 = U_{21}(U_{11})^*h' + U_{22}(U_{12})^*h' = U_{22}(U_{12})^*h' \quad .$$

We can now deduce $(U_{12})^*(\ker(U_{11})^*) \subseteq \ker(U_{22})$. Further, we use $\mathbf{1}_{\mathscr{H}} = U_{11}(U_{11})^* + U_{12}(U_{12})^*$, that is Equation $(3.4)_{11}$, to get

$$h' = U_{11}(U_{11})^*h' + U_{12}(U_{12})^*h' = U_{12}(U_{12})^*h' \quad .$$

This means we have $\ker(U_{11})^* = U_{12}(U_{12})^* \ker(U_{11})^*$, and we can write the chain of inclusion

$$\ker(U_{11})^* = U_{12}(U_{12})^*(\ker(U_{11})^*) \subseteq U_{12} \ker(U_{22}) = \mathscr{M}_1 \quad .$$

Altogether we have shown

$$\mathscr{M}_1 = \ker(U_{11})^* \quad .$$

By replacing U with U^*, we can also show that

$$\mathscr{M}_2 = \ker U_{11} \quad .$$

For the last step towards $\mathscr{M}_1 = \mathscr{M}_2$, we now want to show $\ker U_{11} = \ker(U_{11})^*$, and for this, we will finally use the fact that $W^*(N)$ has a cyclic vector. We know $U_{11}N = NU_{11}$ (Equation $(3.2)_{11}$), so we have $U_{11} \in \{N\}'$. Corollary 3.7.10 then implies $U_{11} \in W^*(N)$, which in turn means $(U_{11})^* \in W^*(N)$. Since $W^*(N)$ is abelian, we thus get that U_{11} is normal and together with Proposition 4.4.3 this implies $\ker U_{11} = \ker(U_{11})^*$ and thus $\mathscr{M}_1 = \mathscr{M}_2$.

[3] This calculation is just the matrix adjoint of Equations $(3.2)^*$.

To tie everything so far together, we remind ourselves that at the start, we proved

$$A|_{(\ker U_{22})^\perp} \simeq B|_{(\ker (U_{22})^*)^\perp} \quad .$$

Next, we have shown that

$$A|_{\ker U_{22}} \simeq N|_{U_{12}\ker U_{22}}$$
$$B|_{\ker (U_{22})^*} \simeq N|_{(U_{21})^*(\ker (U_{22})^*)} \quad .$$

Further, we found that

$$U_{12}\ker U_{22} = \ker(U_{11})^* = \ker U_{11} = (U_{21})^*(\ker(U_{22})^*) \quad .$$

This means we have

$$A|_{\ker U_{22}} \simeq N|_{\ker U_{11}} \simeq B|_{\ker(U_{22})^*} \quad .$$

For the last step we consider

$$\mathscr{H}_2 = \ker U_{22} \oplus (\ker U_{22})^\perp \quad , \quad \mathscr{K}_2 = \ker(U_{22})^* \oplus (\ker(U_{22})^*)^\perp \quad .$$

Additionally we can also separate A and B into

$$A = A|_{\ker U_{22}} \oplus A|_{(\ker U_{22})^\perp} \quad , \quad B = B|_{\ker(U_{22})^*} \oplus B|_{(\ker(U_{22})^*)^\perp}$$

Together with the equivalences $A|_{\ker U_{22}} \simeq B|_{\ker(U_{22})^*}$ and $A|_{(\ker U_{22})^\perp} \simeq B|_{(\ker(U_{22})^*)^\perp}$, we can use Lemma 3.8.1 to see that indeed $A \simeq B$. □

Theorem 3.8.8 (First Multiplicity Theorem)**.** *Let $\mathscr{H}$ be a separable Hilbert space and $N \in \mathscr{B}(\mathscr{H})$, be a normal operator.*
(a) There is a (possibly finite) sequence of Radon measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{C}$ with compact support such that $\mu_{n+1} \ll \mu_n$ for all $n \in \mathbb{N}$ and

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu_n} \quad .$$

(b) For each such representation as given in (a) we find that μ_1 is a scalar-valued measure for N.
(c) Let further $M \in \mathscr{B}(\mathscr{K})$ be a normal operator with respective measures $(\nu_n)_{n\in\mathbb{N}}$ as given in (a). Then we find that $N \simeq M$ if and only if $[\mu_n] = [\nu_n]$ for all n.

Proof of Theorem 3.8.8 (a). We remind ourselves that $W^*(N)$ is a von Neumann algebra and $\mathscr{H}$ is separable, so we can use Lemma 3.4.5 to obtain a sequence $(e_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}$ so that

$$\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathrm{cl}(W^*(N)e_n) = \bigoplus_{n=1}^{\infty} \mathscr{H}_{e_n} \quad .$$

Further, we know according to Theorem 3.5.8 that $N|_{\mathscr{H}_{e_n}} \simeq N_{\mu_{e_n}}$ and thus we get

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu_{e_n}} \quad .$$

As mentioned before, this is a good starting point, however we do not have any control about the various μ_{e_n} and their behaviour. To capture the "maximum amount of information" about N with each subspace, we don't want the e_n to be arbitrary, but instead we need them to be separating for $W^*(N)$ (or some subset of it), since then $W^*(N)e_n$ has a bijection to $W^*(N)$. For this endeavour we will use Lemma 3.8.5. This way we can assure that actually $\mu_{n+1} \ll \mu_n$ for all $n \in \mathbb{N}$.

For a start, we label s_1 the separating vector with $e_1 \in \mathscr{H}_{s_1}$, which is given to us by Lemma 3.8.5. We will now inductively define the $(s_n)_{n\in\mathbb{N}}$ as follows. Let us assume that $\mathscr{K}_n = \bigoplus_{k=1}^n \mathscr{H}_{s_k}$ is well defined with $\mathscr{H}_{s_i} \perp \mathscr{H}_{s_j}$ for $i \neq j$. Then we will choose $s_{n+1} \in \mathscr{K}_n^\perp$, which ensures via Lemma 3.8.4 that $\mathscr{H}_{s_{n+1}} \perp \mathscr{H}_{s_k}$ for all $k \leq n$ and thus $\mathscr{K}_{n+1} = \bigoplus_{k=1}^{n+1} \mathscr{H}_{s_k}$ is well defined. However, we will not take any vector as s_{n+1} but want to make sure that it is a separating vector for $W^*(N|_{\mathscr{K}_n^\perp})$ and that $\mathscr{H}_{e_{n+1}} \subseteq \mathscr{K}_{n+1}$. To this end we now want to show that $N|_{\mathscr{K}_n^\perp} \in \mathscr{B}(\mathscr{K}_n^\perp)$ and that $N|_{\mathscr{K}_n^\perp}$ is normal, and afterwards we will use Lemma 3.8.5 to find a suitable s_{n+1}.

First, we see from Proposition 3.5.4 that for all $T \in W^*(N)$ we have $T\mathscr{H}_{s_k} \subseteq \mathscr{H}_{s_k}$, and thus $T\mathscr{K}_n \subseteq \mathscr{K}_n$. Since $T^* \in W^*(N)$, we can take $x \in \mathscr{K}_n$ and $y \in \mathscr{K}_n^\perp$ and see $T^*x \in \mathscr{K}_n$ and thus $0 = \langle T^*x, y\rangle = \langle x, Ty\rangle$. This means $T\mathscr{K}_n^\perp \subseteq \mathscr{K}_n^\perp$. Since both $N, N^* \in W^*(N)$, we can use Corollary 3.7.8 to see that $W^*(N|_{\mathscr{K}_n^\perp}) = W^*(N)|_{\mathscr{K}_n^\perp}$, which especially implies that $N|_{\mathscr{K}_n^\perp} \in \mathscr{B}(\mathscr{K}_n^\perp)$ and that $N|_{\mathscr{K}_n^\perp}$ is a normal operator with $(N|_{\mathscr{K}_n^\perp})^* = N^*|_{\mathscr{K}_n^\perp}$. Similarly, we see that $W^*(N|_{\mathscr{K}_n}) = W^*(N)|_{\mathscr{K}_n}$. Now we return to the partition $\mathscr{H} = \bigoplus_{n=1}^\infty \mathscr{H}_{e_n}$ (from Lemma 3.4.5) and separate $e_{n+1} = e_{n+1}^\parallel + e_{n+1}^\perp$, where $e_{n+1}^\parallel \in \mathscr{K}_n$ and $e_{n+1}^\perp \in \mathscr{K}_n^\perp$. Next, we invoke Lemma 3.8.5 to find a separating vector $s_{n+1} \in \mathscr{K}_n^\perp$ for $W^*(N|_{\mathscr{K}_n^\perp})$ so that

$$\mathrm{cl}\left(W^*(N|_{\mathscr{K}_n^\perp})e_{n+1}^\perp\right) \subseteq \mathrm{cl}\left(W^*(N|_{\mathscr{K}_n^\perp})s_{n+1}\right) \quad .$$

Since $W^*(N|_{\mathscr{K}_n^\perp}) = W^*(N)|_{\mathscr{K}_n^\perp}$ we have for $k \in \mathscr{K}_n^\perp$ that

$$\mathrm{cl}\left(W^*(N|_{\mathscr{K}_n^\perp})k\right) = \mathrm{cl}\left(W^*(N)|_{\mathscr{K}_n^\perp}k\right) = \mathrm{cl}\left(W^*(N)k\right) = \mathscr{H}_k \quad .$$

Thus we can rewrite the above equation to $\mathscr{H}_{e_{n+1}^\perp} \subseteq \mathscr{H}_{s_{n+1}}$. On the other hand we know that $e_{n+1}^\parallel \in \mathscr{K}_n$ and $W^*(N|_{\mathscr{K}_n}) = W^*(N)|_{\mathscr{K}_n}$, which means especially that

$$\mathscr{H}_{e_{n+1}^\parallel} = \mathrm{cl}\left(W^*(N)|_{\mathscr{K}_n}e_{n+1}^\parallel\right) = \mathrm{cl}\left(W^*(N|_{\mathscr{K}_n})e_{n+1}^\parallel\right) \subseteq \mathscr{K}_n \quad .$$

This means that $\mathscr{H}_{e_{n+1}^\parallel} \subseteq \mathscr{K}_n$. On another note, we see that $\mathscr{H}_{s_{n+1}} \subseteq \mathscr{K}_n^\perp$, and so $\mathscr{K}_{n+1} = \mathscr{K}_n \oplus \mathscr{H}_{s_{n+1}}$ is a well defined subspace of $\mathscr{H}$. Because $e_{n+1} = e_{n+1}^\parallel + e_{n+1}^\perp$ with $e_{n+1}^\parallel \perp e_{n+1}^\perp$, we can use Lemma 3.8.4 to get $\mathscr{H}_{e_{n+1}} \subseteq \mathscr{H}_{e_{n+1}^\parallel} \oplus \mathscr{H}_{e_{n+1}^\perp}$. Combining this with the insights that $\mathscr{H}_{e_{n+1}^\parallel} \subseteq \mathscr{K}_n$, that $\mathscr{H}_{e_{n+1}^\perp} \subseteq \mathscr{H}_{s_{n+1}}$ and $\mathscr{K}_{n+1} = \mathscr{K}_n \oplus \mathscr{H}_{s_{n+1}}$, we finally get $\mathscr{H}_{e_{n+1}} \subseteq \mathscr{K}_{n+1}$. This construction ensures the following inclusion

$$\bigoplus_{n=1}^N \mathscr{H}_{e_n} \subseteq \mathscr{K}_N = \bigoplus_{n=1}^N \mathscr{H}_{s_n} \quad .$$

This means we get

$$\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathscr{H}_{e_n} \subseteq \bigoplus_{n=1}^{\infty} \mathscr{H}_{s_n} \subseteq \mathscr{H} \quad .$$

Therefore we have $\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathscr{H}_{s_n}$ and also

$$N = \bigoplus_{n=1}^{\infty} N|_{\mathscr{H}_{s_n}} \quad .$$

As discussed above, Theorem 3.5.8 tells us that $N|_{\mathscr{H}_{s_n}} \simeq N_{\mu_{s_n}}$ and thus together with Lemma 3.8.1 we get

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu_{s_n}} \quad .$$

The previous construction would have been possible with just Lemma 3.4.5, that is with a simple deconstruction $\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathscr{H}_{e_n}$ and no special requirements for $(e_n)_{n\in\mathbb{N}}$. In this case however, there would be no special relation between the μ_{e_n}. Thus, to prove that $\mu_{s_{n+1}} \ll \mu_{s_n}$, we want to use Theorem 3.7.11 in conjunction with Proposition 3.7.1, which tells us that if $x \in \mathscr{H}$ is separating for $W^*(N)$, then μ_x is a scalar-valued measure and thus for any $h \in \mathscr{H}$ we have $\mu_h \ll \mu_x$. We now remember that s_1 is separating for $W^*(N)$ and s_{n+1} is separating for $W^*(N|_{\mathscr{K}_n^\perp})$ with $\mathscr{K}_n = \bigoplus_{k=1}^{n} \mathscr{H}_{s_k}$. We therefore obtain right away that $\mu_{s_2} \ll \mu_{s_1}$. Further, we see that $\mathscr{K}_{n-1} \subseteq \mathscr{K}_n$ and since $s_{n+1} \in \mathscr{K}_n^\perp$ we therefore have $s_{n+1} \in \mathscr{K}_{n-1}^\perp$. However, s_n is separating for $W^*(N|_{\mathscr{K}_{n-1}^\perp})$ and thus we find $\mu_{s_{n+1}} \ll \mu_{s_n}$.

One important caveat here is that we defined μ_{s_n} with respect to $\mathscr{H}$ and N. This means for the spectral measure E for N and a Borel set Δ that

$$\mu_{s_n}(\Delta) = \langle E(\Delta)s_n, s_n\rangle_{\mathscr{H}} \quad .$$

There might be a problem for us as we now take μ_{s_n} with $\mathscr{K}_n^\perp$ as underlying vector space and $N|_{\mathscr{K}_n^\perp}$ as normal operator. However, we have already shown during the proof of Corollary 3.7.8 that $E|_{\mathscr{K}_n^\perp}$ is the spectral measure for $N|_{\mathscr{K}_n^\perp}$, and thus we get for a Borel set Δ that

$$\mu_{s_n}(\Delta) = \langle E(\Delta)s_n, s_n\rangle_{\mathscr{H}} = \langle E(\Delta)|_{\mathscr{K}_n^\perp} s_n, s_n\rangle_{\mathscr{K}_n^\perp} \quad .$$

Therefore we can take μ_{s_n} with respect to either $\mathscr{H}$ or $\mathscr{K}_n^\perp$.

For the last step we have to define the actual Radon measures μ_n. We just set $\mu_{s_n} = \mu_n$ and we remember Proposition 3.5.6, which states that μ_{s_n} is a Radon measure with compact support. We now combine this with our previous insights to arrive at statement *(a)* of our Theorem. $\square$

Proof of Theorem 3.8.8 (b). Let us denote $N_\oplus := \bigoplus_{n=1}^{\infty} N_{\mu_n}$ and $\mathscr{H}_\oplus := \bigoplus_{n=1}^{\infty} L^2(\mu_n)$. Since $\mu_n \ll \mu_1$ for $n \in \mathbb{N}$, we have $\operatorname{supp}(\mu_n) \subseteq \operatorname{supp}(\mu_1)$, which is equivalent to $\sigma(N_{\mu_n}) \subseteq \sigma(N_{\mu_1})$ according to Proposition 3.5.2. Together with Lemma 3.8.6 and the fact that spectra are always closed this gives us

$$\sigma(N_\oplus) = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} \sigma(N_{\mu_n})\right) = \operatorname{cl}(\sigma(N_{\mu_1})) = \sigma(N_{\mu_1}) = \operatorname{supp}(\mu_1) \quad .$$

Now we remember that the $(N_{\mu_n})_{n\in\mathbb{N}}$ are normal and thus by Proposition 3.2.5 $N_\oplus$ is normal and $N_\oplus^* := \bigoplus_{n=1}^\infty N_{\mu_n}^*$. For any $n \in \mathbb{N}$ we understand that $L^2(\mu_n) \subseteq \mathscr{H}_\oplus$ is a closed subspace[4] with

$$N_\oplus L^2(\mu_n) = N_{\mu_n} L^2(\mu_n) \subseteq L^2(\mu_n) \quad .$$

We can produce the same equation for $N_\oplus^*$, and since $N_\oplus$ is normal we can use Corollary 3.7.8 to get $\phi(N_\oplus)|_{L^2(\mu_n)} = \phi(N_{\mu_n})$ for any $\phi \in B(\sigma(N_\oplus))$. Now we see from Theorem 3.6.3 that $\phi(N_{\mu_n}) = M_\phi$ on $L^2(\mu_n)$. Since $\mu_n \ll \mu_1$, we see that $M_\phi = 0$ on $L^2(\mu_1)$ implies $M_\phi = 0$ on $L^2(\mu_n)$. Thus $\phi(N_\oplus)|_{L^2(\mu_1)} = 0$ implies $\phi(N_\oplus)|_{L^2(\mu_n)} = 0$ for all $n \in \mathbb{N}$ and therefore $\phi(N_\oplus) = 0$. Now we take $h_\oplus := 1 \in L^2(\mu_1) \subseteq \mathscr{H}_\oplus$ and see that $M_\phi h = \phi$, and thus $M_\phi h_\oplus = 0$ if and only if $\phi = 0$ with respect to $L^2(\mu_1)$. This is equivalent to $M_\phi = 0$ on $L^2(\mu_1)$, which we have shown to be equivalent to $\phi(N_\oplus) = 0$. Now Theorem 3.7.7 tells us that all elements of $W^*(N_\oplus)$ can be written as some $\phi(N_\oplus)$, which means we have just proven that $h_\oplus$ is a separating vector for $W^*(N_\oplus)$.

Now we take $V : \mathscr{H} \mapsto \mathscr{H}_\oplus$ as the unitary operator with $VNV^{-1} = N_\oplus$ and define $h = V^{-1}h_\oplus$. Because of Corollary 3.7.9, we see that $VW^*(N)V^{-1} = W^*(N_\oplus)$ and thus h is also a separating vector for $W^*(N)$. We can therefore use Theorem 3.7.11 to see that μ_h is a scalar-valued spectral measure for N.

Our final goal is to show that $\mu_h = \mu_1$. We remember that for a Borel set Δ we have $E(\Delta) = \phi_\Delta(N)$ with E being the spectral measure for N and $\phi_\Delta = \chi_\Delta$. Collecting all our previous knowledge, we can now obtain the following equality

$$\begin{aligned}\mu_h(\Delta) &= \langle E(\Delta)h, h\rangle_{\mathscr{H}} = \langle \phi_\Delta(N)h, h\rangle_{\mathscr{H}} \\ &= \langle V\phi_\Delta(N)V^{-1}h_\oplus, h_\oplus\rangle_{\mathscr{H}_\oplus} = \langle \phi_\Delta(N_\oplus)h_\oplus, h_\oplus\rangle_{\mathscr{H}_\oplus} \\ &= \langle \phi_\Delta(N_\oplus)|_{L^2(\mu_1)}1, 1\rangle_{L^2(\mu_1)} = \langle M_{\phi_\Delta}1, 1\rangle_{L^2(\mu_1)} \\ &= \int \phi_\Delta(z)d\mu_1(z) = \int \chi_\Delta(z)d\mu_1(z) = \mu_1(\Delta) \quad .\end{aligned}$$

Therefore we have $\mu_h = \mu_1$, and since μ_h is a scalar-valued spectral measure for N so is μ_1. □

Proof of Theorem 3.8.8 (c). First we examine the case where $[\mu_n] = [\nu_n]$ for all $n \in \mathbb{N}$. Proposition 3.5.3 then tells us that $N_{\mu_n} \simeq N_{\nu_n}$, which, together with Lemma 3.8.1 gives

$$N \simeq \bigoplus_{n=1}^\infty N_{\mu_n} \simeq \bigoplus_{n=1}^\infty N_{\nu_n} \simeq M \quad .$$

For the other implication, we take a look at the case where $N \simeq M$, and will produce our proof in two steps. As a start, we will show that $[\mu_1] = [\nu_1]$. Next, we inductively reduce the problem for the measures $(\mu_i)_{i\geq n+1}$ and $(\mu_i)_{i\geq n+1}$ to the initial case and subsequently obtain $[\mu_{n+1}] = [\nu_{n+1}]$.

Let $V : \mathscr{H} \mapsto \mathscr{K}$ be a unitary operator so that $VNV^{-1} = M$, and let E_N, E_M be the spectral measures for N and M respectively. We know by Theorem 4.6.3 that for a

[4] We write this inclusion to emphasize that we don't consider $L^2(\mu_n)$ on its own but rather as a subspace of $\mathscr{H}_\oplus$.

Borel set Δ and the function $\phi_\Delta(z) := \chi_\Delta(z)$ we have $\phi_\Delta(N) = E_N(\Delta)$, and similarly $\phi_\Delta(M) = E_M(\Delta)$. Since $\phi_\Delta \in B(\sigma(N))$, we know by Corollary 3.7.9 that

$$VE_N(\Delta)V^{-1} = V\phi_\Delta(N)V^{-1} = \phi_\Delta(M) = E_M(\Delta) \quad .$$

However, V is unitary, so we see that $E_N(\Delta) = 0$ if and only if $E_M(\Delta) = 0$. Now, according to part *(b)* of this theorem, μ_1 and ν_1 are scalar-valued spectral measures for N and M respectively. Therefore we see that

$$\mu_1(\Delta) = 0 \quad \Leftrightarrow \quad E_N(\Delta) = 0 \quad \Leftrightarrow \quad E_M(\Delta) = 0 \quad \Leftrightarrow \quad \nu_1(\Delta) = 0 \quad .$$

Altogether we can conclude $[\mu_1] = [\nu_1]$.

For the next step, we want to prove $[\mu_{n+1}] = [\nu_{n+1}]$ by induction. Our induction assumptions are twofold. First, we assume $[\mu_n] = [\nu_n]$. For the second, we define $N_n := \bigoplus_{i=n}^\infty N_{\mu_i}$ and $M_n := \bigoplus_{i=n}^\infty N_{\nu_i}$, and we assume that

$$N_n \simeq M_n \quad .$$

As the induction start we have just proven $[\mu_1] = [\nu_1]$, and from $N \simeq M$ together with our assumptions about $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ we know that

$$N_1 = \bigoplus_{i=1}^\infty N_{\mu_i} \simeq N \simeq M \simeq \bigoplus_{i=1}^\infty N_{\nu_i} = M_1 \quad .$$

Now we consider the assumptions true for n and want to deduce them for $n+1$. We get from our second assumption that

$$N_{\mu_n} \oplus N_{n+1} = N_n \simeq M_n = N_{\nu_n} \oplus M_{n+1} \quad .$$

Now we want to use Lemma 3.8.7 to obtain $N_{n+1} \simeq M_{n+1}$, but first we have to show that all involved operators are normal and that $W^*(N_{\mu_n})$ and $W^*(N_{\nu_n})$ have a cyclic vector. For the first fact, we know for a Radon measure μ with compact support by Proposition 3.5.1 that N_μ is normal. Further, Proposition 3.2.5 gives us that a direct sum of normal operators is normal again. The second fact follows from the proof of Theorem 3.5.8 There we have shown that for a Radon measure μ with compact support, we find $1 \in L^2(\mu)$ and $\mathrm{cl}(\mathscr{P}_{N_\mu}1) = L^2(\mu)$, and finally $\mathscr{P}_{N_\mu} \subseteq W^*(N_\mu)$. This means $\mathrm{cl}(W^*(N_\mu)1) = L^2(\mu)$ and thus we can do the same for μ_n and ν_n to see that $W^*(N_{\mu_n})$ and $W^*(N_{\nu_n})$ both have a cyclic vector. Altogether we can use Lemma 3.8.7 to obtain

$$\bigoplus_{i=n+1}^\infty N_{\mu_i} = N_{n+1} \simeq M_{n+1} = \bigoplus_{i=n+1}^\infty N_{\nu_i} \quad .$$

We now have normal operators $N_{n+1} \simeq M_{n+1}$ with partitions $(\mu_i)_{i\geq n+1}$ and $(\nu_i)_{i\geq n+1}$ respectively, where the measures fulfil all properties given in part *(a)* of this theorem. Therefore we can use our reasoning from earlier to obtain $[\mu_{n+1}] = [\nu_{n+1}]$, which concludes our induction. $\square$

3.9 Alternative Formulations of Multiplicity Theory

After having proven the First Multiplicity Theorem, we can now investigate further. We will restate Theorem 3.8.8 in two different ways in Theorems 3.9.2 and 3.9.5, with each theorem building upon the previous one. For the Second Multiplicity Theorem, we simply want to switch the measures $(\mu_n)_{n\in\mathbb{N}}$ with just one scalar-valued measure μ and a decreasing sequence of Borel sets $(\Delta_n)_{n\in\mathbb{N}}$ with $\Delta_{n+1} \subseteq \Delta_n$ so that

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}} \quad .$$

Therefore, we want to construct the sets so that $[\mu_n] = [\mu|_{\Delta_n}]$ for all $n \in \mathbb{N}$. The reason why we want to prefer this restatement is that we can extract the multiplicities much easier from the sets $(\Delta_n)_{n\in\mathbb{N}}$, since we see that the set $\Delta_n \setminus \Delta_{n+1}$ "appears" exactly n times in the sum above. We will use this fact when further developing multiplicity theory, but before we get ahead of ourselves, let us prove two small lemmata, after which we will come to the Second Multiplicity Theorem.

Lemma 3.9.1. *Let μ be a Radon measure, and let Δ be a Borel set so that $\mu(\Delta^c) = 0$. Then we have $\mu = \mu|_\Delta$. In particular this means $\mu = \mu|_{\operatorname{supp}(\mu)}$.*

Proof. For a Borel set A we find

$$\mu(A) = \mu(A \cap \Delta) + \mu(A \cap \Delta^c) = \mu(A \cap \Delta) = \mu|_\Delta(A) \quad .$$

Thus we get $\mu = \mu|_\Delta$. For the second point, we remember that by Proposition 4.7.6 we have $\mu((\operatorname{supp}(\mu))^c) = 0$. □

Theorem 3.9.2 (Second Multiplicity Theorem)**.** *Let $\mathscr{H}$ be a separable Hilbert space and let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator.*
(a) Let μ be a scalar-valued measure for N. Then we find a decreasing sequence $(\Delta_n)_{n\in\mathbb{N}}$ of Borel subsets of $\sigma(N)$ so that $\Delta_1 = \sigma(N)$ and

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}} \quad .$$

(b) Let $\mathscr{K}$ be a separable Hilbert space and let $M \in \mathscr{B}(\mathscr{K})$ be normal with a scalar-valued measure ν. Let further $(\Sigma_n)_{n\in\mathbb{N}}$ be the corresponding sequence of Borel sets for M, as detailed in (a). Then $N \simeq M$ if and only if $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$.

Proof of Theorem 3.9.2 (a). We will prove this by using the partition of N as given by *(a)* in Theorem 3.8.8, that is

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\tilde{\mu}_n} \quad .$$

Our goal is to define the Δ_n in such a way that $[\mu|_{\Delta_n}] = [\tilde{\mu}_n]$ for all $n \in \mathbb{N}$. Proposition 3.5.3 then tells us that $N_{\tilde{\mu}_n} \simeq N_{\mu|_{\Delta_n}}$, which means we can get the proof by using Lemma 3.8.1.

To this end, we will first define $\Delta_1 := \sigma(N)$ and show that $[\mu|_{\Delta_1}] = [\tilde{\mu}_1]$. Part *(b)* of Theorem 3.8.8 tells us that $\tilde{\mu}_1$ is a scalar-valued spectral measure for N, and so Proposition 3.7.1 gives us $[\mu] = [\tilde{\mu}_1]$. By Lemma 3.9.1 we further get that $\mu = \mu|_{\sigma(N)}$, and so we see that

$$[\mu|_{\Delta_1}] = [\mu] = [\tilde{\mu}_1] \quad .$$

Now we move on to inductively define Δ_{n+1}. We assume that we already have a decreasing sequence $(\Delta_i)_{i\leq n}$ with $[\mu|_{\Delta_i}] = [\tilde{\mu}_i]$ for all $i \leq n$. Part *(a)* of Theorem 3.8.8 now tells us that $\tilde{\mu}_{n+1} \ll \tilde{\mu}_n$, and so we have $\tilde{\mu}_{n+1} \ll \mu|_{\Delta_n}$. This means we can use Lemma 3.8.3 to obtain a Borel set $\tilde{\Delta}_{n+1}$ so that $[\tilde{\mu}_{n+1}] = [(\mu|_{\Delta_n})|_{\tilde{\Delta}_{n+1}}]$. By the definition of restricted measures, we see for a Borel set ω that

$$(\mu|_{\Delta_n})|_{\tilde{\Delta}_{n+1}}(\omega) = \mu|_{\Delta_n}(\omega \cap \tilde{\Delta}_{n+1}) = \mu(\omega \cap \tilde{\Delta}_{n+1} \cap \Delta_n) = \mu|_{\tilde{\Delta}_{n+1}\cap\Delta_n}(\omega) \quad .$$

We now define $\Delta_{n+1} := \tilde{\Delta}_{n+1} \cap \Delta_n$ and see that $\Delta_{n+1} \subseteq \Delta_n$ and $[\mu|_{\Delta_{n+1}}] = [\tilde{\mu}_{n+1}]$. This means our induction is successful and we get a decreasing sequence of Borel sets $(\Delta_n)_{n\in\mathbb{N}}$ with $[\mu|_{\Delta_n}] = [\tilde{\mu}_n]$ for all $n \in \mathbb{N}$.

Finally, we can return to our initial quest. As discussed, Proposition 3.5.3 tells us that $N_{\tilde{\mu}_n} \simeq N_{\mu|_{\Delta_n}}$ for all $n \in \mathbb{N}$ and thus we get by Lemma 3.8.1 that

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\tilde{\mu}_n} \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}} \quad .$$

This concludes the proof. □

Proof of Theorem 3.9.2 (b). We will prove part *(b)* in two steps. First we show that $N \simeq M$ if and only if $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$ for all $n \in \mathbb{N}$ by making use of part *(c)* of Theorem 3.8.8. Then we will show that $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$ for all $n \in \mathbb{N}$ is equivalent to $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$.

For the first part, we want to use the uniqueness given in part *(c)* of Theorem 3.8.8. We know that

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}} \quad , \quad M \simeq \bigoplus_{n=1}^{\infty} N_{\nu|_{\Sigma_n}} \quad .$$

This means we have to show that the $\mu|_{\Delta_n}$ are Radon measures with compact support and that $\mu|_{\Delta_{n+1}} \ll \mu|_{\Delta_n}$ for all $n \in \mathbb{N}$. We remember that μ is a scalar-valued spectral measure for N, and so by Theorem 3.7.11 there exists a vector $h \in \mathscr{H}$ so that $\mu = \mu_h$. Proposition 4.6.5 then tells us that μ_h is a Radon measure with compact support, and we get from Lemma 3.8.2 that $\mu|_{\Delta_n}$ is also a Radon measure with compact support for all $n \in \mathbb{N}$. Because for $n \in \mathbb{N}$ we have $\Delta_{n+1} \subseteq \Delta_n$, we see that $\mu|_{\Delta_{n+1}} \ll \mu|_{\Delta_n}$. This means our partition $N \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}}$ is exactly as described in Theorem 3.8.8, and we can deduce the same for $M \simeq \bigoplus_{n=1}^{\infty} N_{\nu|_{\Sigma_n}}$. This means $N \simeq M$ if and only if $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$ for all $n \in \mathbb{N}$.

We will now turn to show that this is equivalent to $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$. First, we assume that we have $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$ for all $n \in \mathbb{N}$. Since μ is a scalar-valued spectral measure for N, we know by Proposition 3.7.1 that $\operatorname{supp}(\mu) = \sigma(N)$ and thus Lemma 3.9.1 tells us that $\mu = \mu|_{\sigma(N)}$. On the other hand, we have by definition that $\Delta_1 = \sigma(N)$ and so we see $\mu = \mu|_{\Delta_1}$, and can show the same for $\nu = \nu|_{\Sigma_1}$. By our assumption, we also know that $[\mu|_{\Delta_1}] = [\nu|_{\Sigma_1}]$, which altogether gives us

$$[\mu] = [\mu|_{\Delta_1}] = [\nu|_{\Sigma_1}] = [\nu] \quad .$$

Next, we take $n \in \mathbb{N}$ and inspect the following

$$\nu|_{\Sigma_n}(\Delta_n \setminus \Sigma_n) = \nu(\Sigma_n \cap (\Delta_n \setminus \Sigma_n)) = \nu(\emptyset) = 0 \quad .$$

Since we know from our assumption that $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$, we see that

$$0 = \mu|_{\Delta_n}(\Delta_n \setminus \Sigma_n) = \mu(\Delta_n \cap (\Delta_n \setminus \Sigma_n)) = \mu(\Delta_n \setminus \Sigma_n) \quad .$$

In a similar fashion we see $\nu(\Sigma_n \setminus \Delta_n) = 0$, and since we have shown that $[\mu] = [\nu]$ we get $\mu(\Sigma_n \setminus \Delta_n) = 0$. Altogether we see that $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$ for all $n \in \mathbb{N}$ implies $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$.

Now we will prove the other implication and assume $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and ω be a Borel set so that $\mu|_{\Delta_n}(\omega) = 0$. We use the additivity of measures and our assumption to see

$$\begin{aligned} 0 = \mu|_{\Delta_n}(\omega) &= \mu(\Delta_n \cap \omega) \\ &= \mu((\Delta_n \setminus \Sigma_n) \cap \omega) + \mu(\Sigma_n \cap \omega) - \mu((\Sigma_n \setminus \Delta_n) \cap \omega) \\ &= 0 + \mu(\Sigma_n \cap \omega) - 0 \quad . \end{aligned}$$

Since $[\mu] = [\nu]$, we thus know that

$$0 = \nu(\Sigma_n \cap \omega) = \nu|_{\Sigma_n}(\omega) \quad .$$

This means $\nu|_{\Sigma_n} \ll \mu|_{\Delta_n}$, and similarly we can show that $\mu|_{\Delta_n} \ll \nu|_{\Sigma_n}$. Therefore we see that $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$ implies $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$ for all $n \in \mathbb{N}$.

Finally, we have shown that $N \simeq M$ if and only if $[\mu|_{\Delta_n}] = [\nu|_{\Sigma_n}]$ for all $n \in \mathbb{N}$, which is in turn equivalent to $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$. This proves the theorem. □

Now we can set our sights towards the Third Multiplicity Theorem 3.9.5. As discussed above, we will use the sets $(\Delta_n)_{n\in\mathbb{N}}$ from Theorem 3.9.2 to define $\omega_n := \Delta_n \setminus \Delta_{n+1}$ and $\omega_\infty := \bigcap_{i=1}^\infty \Delta_i$. Each ω_n will be "contained" in the direct sum in Theorem 3.9.2 exactly n times, so with a bit of clever rearranging we find

$$N \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}} \simeq (N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{n=1}^{\infty} (N_{\mu|_{\omega_n}})^n \quad .$$

We also notice that $\mu|_{\omega_\infty}$ and the $(\mu|_{\omega_n})_{n\in\mathbb{N}}$ are mutually singular, and we have already arrived at Theorem 3.9.5. Again, we will quickly prove two lemmatas and then flesh out the thoughts above for the proof of the Third Multiplicity Theorem.

Lemma 3.9.3. *Let μ be Radon measure with compact support and let $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of Borel sets so that $\mu(\Delta_i \cap \Delta_j) = 0$ for all $i \neq j$ and $\mu((\bigcup_{n=1}^{\infty} \Delta_n)^c) = 0$. Then we find:*
(a)

$$\mu = \sum_{n=1}^{\infty} \mu|_{\Delta_n} \quad .$$

(b)

$$N_\mu \simeq \bigoplus_{n=1}^{\infty} N_{\mu|_{\Delta_n}} \quad .$$

Proof. The proof of *(a)* is rather easy. We define $\Delta := \bigcup_{n=1}^{\infty} \Delta_n$ and find $\mu(\Delta^c) = 0$, so Lemma 3.9.1 tells us that $\mu = \mu|_\Delta$. Now we take a Borel set ω and see

$$\mu(\omega) = \mu|_\Delta(\omega) = \mu\left(\omega \cap \bigcup_{n=1}^{\infty} \Delta_n\right) = \mu\left(\bigcup_{n=1}^{\infty} (\Delta_n \cap \omega)\right) \quad .$$

Since we have $\mu(\Delta_i \cap \Delta_j) = 0$ for all $i \neq j$, we can split this expression into a sum and get

$$\mu(\omega) = \mu\left(\bigcup_{n=1}^{\infty} (\Delta_n \cap \omega)\right) = \sum_{n=1}^{\infty} \mu(\Delta_n \cap \omega) = \sum_{n=1}^{\infty} \mu|_{\Delta_n}(\omega) \quad .$$

This means we find $\mu = \sum_{n=1}^{\infty} \mu|_{\Delta_n}$.

Now we turn to show *(b)*. First we remark that according to Proposition 4.7.5 we have for $n \in \mathbb{N}$ that $\frac{d\mu|_{\Delta_n}}{d\mu} = \chi_{\Delta_n}$. Thus, for $f \in L^2(\mu)$ according to Proposition 4.7.6 we find

$$||f||^2_{L^2(\mu|_{\Delta_n})} = \int \chi_{\Delta_n} |f|^2 d\mu \leq ||f||^2_{L^2(\mu)} < \infty \quad .$$

This means that $f \in L^2(\mu|_{\Delta_n})$. Therefore we can define the function

$$V : L^2(\mu) \mapsto \bigoplus_{n=1}^{\infty} L^2(\mu|_{\Delta_n}) \quad , \quad V(f) = \bigoplus_{n=1}^{\infty} f \quad .$$

This function is linear, and we can show that it is also an isometry. For this, we remember $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$ and $\mu(\Delta_i \cap \Delta_j) = 0$ for all $i \neq j$. Therefore there is at most one χ_{Δ_n} with non-zero value μ-almost everywhere. This means we have

$$\chi_\Delta = \sum_{n=1}^{\infty} \chi_{\Delta_n} \quad .$$

We now take $f \in L^2(\mu)$, label $L^2_\oplus := \bigoplus_{n=1}^{\infty} L^2(\mu|_{\Delta_n})$ and see

$$||Vf||^2_{L^2_\oplus} = \sum_{n=1}^{\infty} \int |f|^2 d\mu|_{\Delta_n} = \sum_{n=1}^{\infty} \int \chi_{\Delta_n} |f|^2 d\mu = \int \chi_\Delta |f|^2 d\mu \quad .$$

The fact that $\mu = \mu|_\Delta$ means we can use Proposition 4.7.5 to get $d\mu = \chi_\Delta d\mu$, and we obtain

$$||Vf||^2_{L^2_\oplus} = \int \chi_\Delta |f|^2 d\mu = \int |f|^2 d\mu = ||f||^2_{L^2(\mu)} \quad .$$

Thus, V is an isometry and thus injective. We now want to prove that V is also surjective, which gives us altogether that V is unitary. Let thus $\tilde{f} \in L^2_\oplus$ with $\tilde{f} = \bigoplus_{i=1}^\infty f_i$. Further we define $f = \sum_{i=1}^\infty \chi_{\Delta_i} f_i$. We want to show that $Vf = \tilde{f}$, but we don't yet know if $f \in L^2(\mu)$. Since $\mu|_{\Delta_i}(\Delta_i^c) = 0$ for all $i \in \mathbb{N}$, we see that f_i and $\chi_{\Delta_i} f_i$ are the same $\mu|_{\Delta_i}$-almost everywhere, and thus

$$\tilde{f} = \bigoplus_{i=1}^\infty \chi_{\Delta_i} f_i \quad .$$

We can now reverse the calculations above and see that $f \in L^2(\mu)$ and $Vf = \tilde{f}$. Thus V is surjective and therefore unitary.

Now we turn to canonical multiplication operators and their equivalence. According to Lemma 3.8.2 we see that $\mu|_{\Delta_n}$ is again a Radon measure with $\mathrm{supp}(\mu|_{\Delta_n}) \subseteq \mathrm{supp}(\mu)$. Since μ has compact support and $\mathrm{supp}(\mu|_{\Delta_n})$ is closed we see that it is also compact. Therefore $N_{\mu|_{\Delta_n}}$ is well defined for all $n \in \mathbb{N}$. We take $f \in L^2(\mu)$ and see

$$VN_\mu f = Vzf = \bigoplus_{i=1}^\infty zf = \bigoplus_{i=1}^\infty N_{\mu|_{\Delta_i}} f = \left(\bigoplus_{i=1}^\infty N_{\mu|_{\Delta_i}}\right) \bigoplus_{i=1}^\infty f = \left(\bigoplus_{i=1}^\infty N_{\mu|_{\Delta_i}}\right) Vf \quad .$$

First, this means that $\bigoplus_{i=1}^\infty N_{\mu|_{\Delta_i}}$ is a continuous linear operator, and further that $VN_\mu = \bigoplus_{i=1}^\infty N_{\mu|_{\Delta_i}} V$. Since V is unitary, we find $N_\mu \simeq \bigoplus_{i=1}^\infty N_{\mu|_{\Delta_i}}$. □

Lemma 3.9.4. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures. Then the $(\mu_n)_{n\in\mathbb{N}}$ are mutually singular if and only if there exists a sequence of sets $(\omega_n)_{n\in\mathbb{N}}$ so that*

$$\mu_i = \mu_i|_{\omega_i} \quad , \quad \omega_i \cap \omega_j = \emptyset$$

$$i, j \in \mathbb{N} \quad , \quad i \neq j \quad .$$

Proof. We start by reminding ourselves that two measures μ and ν are mutually singular if there exists a Borel set Δ so that $\mu(\Delta) = 0$ and $\nu(\Delta) = 0$. Let us now assume that we have a sequence of sets $(\omega_n)_{n\in\mathbb{N}}$ so that

$$\mu_i = \mu_i|_{\omega_i} \quad , \quad \omega_i \cap \omega_j = \emptyset$$

$$i, j \in \mathbb{N} \quad , \quad i \neq j \quad .$$

Thus for $i, j \in \mathbb{N}$ and $i \neq j$ we find

$$\mu_i(\omega_i^c) = \mu_i|_{\omega_i}(\omega_i^c) = \mu_i(\omega_i^c \cap \omega_i) = 0$$

$$\mu_j(\omega_i) = \mu_j|_{\omega_j}(\omega_i) = \mu_j(\omega_i \cap \omega_j) = 0 \quad .$$

Therefore we find that μ_i and μ_j are mutually singular. Since i, j were chosen arbitrarily, this means the $(\mu_n)_{n\in\mathbb{N}}$ are mutually singular.

Next we assume that the $(\mu_n)_{n\in\mathbb{N}}$ are mutually singular and we want to construct the sets $(\omega_n)_{n\in\mathbb{N}}$ to fulfil the conditions stated in the lemma. We know for $i, j \in \mathbb{N}$ and $i \neq j$ that there exists a set Δ_{ij} so that

$$\mu_i(\Delta_{ij}^c) = 0 \quad , \quad \mu_j(\Delta_{ij}) = 0 \quad .$$

We now take $i \in \mathbb{N}$ and define $\omega_i := \bigcap_{j\neq i} \Delta_{ij}$, and we see that

$$\mu_i(\omega_i^c) = \mu_i\left(\bigcup_{j\neq i} \Delta_{ij}^c\right) \leq \sum_{j\neq i} \mu_i(\Delta_{ij}^c) = 0 \quad .$$

Therefore we can use Lemma 3.9.1 to obtain $\mu_i = \mu_i|_{\omega_i}$. On the other hand, we see for $j \neq i$ that $\omega_i \subseteq \Delta_{ij}$, which means

$$\mu_j(\omega_i) \leq \mu_j(\Delta_{ij}) = 0 \quad .$$

This means we have found suitable sets $(\mu_n)_{n\in\mathbb{N}}$ so that the conditions of the lemma are fulfilled. □

Theorem 3.9.5 (Third Multiplicity Theorem)**.** *Let $\mathscr{H}$ be a separable Hilbert space and let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator.*
(a) Let μ be a scalar-valued spectral measure for N. Then we find mutually singular Radon measures μ_∞ and $\mu_1, \mu_2, \ldots$ with compact support so that $\mu = \mu_\infty + \sum_{n=1}^\infty \mu_n$ and

$$N \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{n=1}^\infty (N_{\mu_n})^n \quad .$$

(b) If $\mathscr{K}$ is a separable Hilbert space and $M \in \mathscr{B}(\mathscr{K})$ is normal with corresponding measures $\nu_\infty, \nu_1, \nu_2, \ldots$, then we have $N \simeq M$ if and only if $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$ and $[\mu_\infty] = [\nu_\infty]$.

Remark. It is actually not necessary to assume that μ is a scalar-valued measure, or that $\mu_\infty + \sum_{n=1}^\infty \mu_n$ is a measure at all. The proof for this is based on *(b)* from Theorem 3.8.8, but we won't need it in this thesis.

Proof of Theorem 3.9.5 (a). For this proof, we start with the deconstruction of N given by Theorem 3.9.2, that is the decreasing sequence $(\Delta_n)_{n\in\mathbb{N}}$ of Borel sets, with the property

$$N \simeq \bigoplus_{n=1}^\infty N_{\mu|_{\Delta_n}} \quad .$$

We now want to partition the sets $(\Delta_n)_{n\in\mathbb{N}}$ further into disjoint sets ω_∞ and $(\omega_n)_{n\in\mathbb{N}}$. Afterwards, we will use Lemma 3.9.3 to get

$$\mu|_{\Delta_n} = \mu|_{\omega_\infty} + \sum_{i=n}^\infty \mu|_{\omega_i}$$

$$N_{\mu|_{\Delta_n}} \simeq N_{\mu|_{\omega_\infty}} \oplus \bigoplus_{i=n}^\infty N_{\mu_{\omega_i}} \quad .$$

We can then insert this in our first decomposition obtained by Theorem 3.9.2 and rearrange the terms to obtain the desired expression. Finally we will show that $\mu = \mu|_{\omega_\infty} + \sum_{i=1}^{\infty} \mu|_{\omega_i}$, and that $\mu|_{\omega_\infty}$ and the $(\mu|_{\omega_n})_{n\in\mathbb{N}}$ are mutually singular.

As discussed we start with the deconstruction of N given by Theorem 3.9.2 and define the sets

$$\omega_n := \Delta_n \setminus \Delta_{n+1} \quad , \quad \omega_\infty := \bigcap_{n=1}^{\infty} \Delta_n \quad .$$

From this definition we see that $\Delta_n = \omega_\infty \cup \bigcup_{i=n}^{\infty} \omega_i$. This means we have

$$\mu|_{\Delta_n}\left(\left(\omega_\infty \cup \bigcup_{i=n}^{\infty} \omega_i\right)^c\right) = \mu|_{\Delta_n}(\Delta_n^c) = 0 \quad .$$

Further we find for $i \neq j$ that $\omega_i \cap \omega_j = \emptyset$ and $\omega_i \cap \omega_\infty = \emptyset$, and therefore we get

$$\mu|_{\Delta_n}(\omega_i \cap \omega_j) = 0 \quad , \quad \mu|_{\Delta_n}(\omega_i \cap \omega_\infty) = 0 \quad .$$

This means the preconditions for Lemma 3.9.3 are fulfilled and we get

$$\mu|_{\Delta_n} = (\mu|_{\Delta_n})|_{\omega_\infty} + \sum_{i=n}^{\infty} (\mu|_{\Delta_n})|_{\omega_i}$$

$$N_{\mu|_{\Delta_n}} \simeq N_{(\mu|_{\Delta_n})|_{\omega_\infty}} \oplus \bigoplus_{i=n}^{\infty} N_{(\mu|_{\Delta_n})|_{\omega_i}} \quad .$$

However, since $\omega_i \subseteq \Delta_i \subseteq \Delta_n$ for $i \geq n$, we get that $(\mu|_{\Delta_n})|_{\omega_i} = \mu|_{\omega_i}$ for $i \geq n$. Similarly we have $\omega_\infty \subseteq \Delta_n$ and thus $(\mu|_{\Delta_n})|_{\omega_\infty} = \mu|_{\omega_\infty}$. Altogether we get

$$\mu|_{\Delta_n} = \mu|_{\omega_\infty} + \sum_{i=n}^{\infty} \mu|_{\omega_i}$$

$$N_{\mu|_{\Delta_n}} \simeq N_{\mu|_{\omega_\infty}} \oplus \bigoplus_{i=n}^{\infty} N_{\mu_{\omega_i}} \quad .$$

We can insert this into the deconstruction of N given by Theorem 3.9.2 and get

$$N \simeq \bigoplus_{n=1}^{\infty}\left(N_{\mu|_{\omega_\infty}} \oplus \bigoplus_{k\geq n} N_{\mu|_{\omega_k}}\right) = \bigoplus_{n=1}^{\infty}\left(N_{\mu|_{\omega_\infty}} \oplus N_{\mu|_{\omega_n}} \oplus N_{\mu_{\omega_{n+1}}} \oplus \ldots\right) \quad .$$

When taking the direct sum of vector spaces, their ordering is irrelevant up to a unitary transformation, so we can rearrange the terms above and get

$$N \simeq (N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{n=1}^{\infty} (N_{\mu|_{\omega_n}})^n \quad .$$

This is already the form required by the theorem, so we only have to prove that the measures fulfil all necessary properties.

It remains to be shown that $\mu = \mu|_{\omega_\infty} + \sum_{n=1}^{\infty} \mu|_{\omega_n}$ and that $\mu|_{\omega_\infty}$ and the $(\mu|_{\omega_n})_{n\in\mathbb{N}}$ are mutually singular. For one, we have seen that $\mu|_{\Delta_1} = \mu|_{\omega_\infty} + \sum_{i=1}^{\infty} \mu|_{\omega_i}$, and we know from Theorem 3.9.2 that $\Delta_1 = \sigma(N)$ and that μ is a scalar-valued spectral measure. By Proposition 3.7.1 we further get $\sigma(N) = \operatorname{supp}(\mu)$, and Lemma 3.9.1 tells us that $\mu = \mu|_{\operatorname{supp}(\mu)}$. Altogether we find

$$\mu = \mu|_{\operatorname{supp}(\mu)} = \mu|_{\sigma(N)} = \mu|_{\Delta_1} = \mu|_{\omega_\infty} + \sum_{i=1}^{\infty} \mu|_{\omega_i} \quad .$$

Now let us inspect whether the measures are mutually singular. For $i \neq j$ we know that $\omega_j \cap \omega_i = \emptyset$ and also $\omega_\infty \cap \omega_i = \emptyset$. Therefore we can use Lemma 3.9.4 and see that the measures are mutually singular. We now label $\mu_i := \mu|_{\omega_i}$ for $i \in \mathbb{N}$ and $\mu_\infty := \mu|_{\omega_\infty}$ and obtain the theorem. □

Proof of Theorem 3.9.5 (b). We will prove the statement by inspecting both directions of the implication. First we assume that $[\mu_\infty] = [\nu_\infty]$ and $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$. We now want to show that this implies $N \simeq M$. By Proposition 3.5.3 we know that $N_{\mu_i} \simeq N_{\nu_i}$ for all $i \in \mathbb{N}$ and $N_{\mu_\infty} \simeq N_{\nu_\infty}$. Using Lemma 3.8.1 we therefore get

$$N \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{n=1}^{\infty} (N_{\mu_n})^n \simeq (N_{\nu_\infty})^\infty \oplus \bigoplus_{n=1}^{\infty} (N_{\nu_n})^n \simeq M \quad .$$

This proves one implication.

Now we turn to the other implication and assume $N \simeq M$, and we want to obtain $[\mu_\infty] = [\nu_\infty]$ and $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$ from this assumption. This implication is unfortunately much harder to show, and we will go at it in several steps. First we will reverse-engineer the proof of *(a)* to get measures μ and ν together with decreasing sequences of sets $(\Delta_n)_{n\in\mathbb{N}}$ and $(\Sigma_n)_{n\in\mathbb{N}}$ so that we can use part *(b)* of Theorem 3.9.2. From this we will obtain that $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$. In the second step, we will show that this in turn implies $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$ and finally $[\mu_\infty] = [\nu_\infty]$.

For the first step we take the well defined scalar-valued spectral measure

$$\mu := \mu_\infty + \sum_{n=1}^{\infty} \mu_n \quad .$$

According to Theorem 3.7.11 there exists a vector $h \in \mathscr{H}$ so that $\mu = \mu_h$, and Proposition 4.6.5 then tells us that μ_h is a Radon measure with compact support. Our goal is to use μ as a springboard to apply Theorem 3.9.2. We know that the $(\mu_i)_{i\in\mathbb{N}}$ and μ_∞ are mutually singular, so we can use Lemma 3.9.4 to get sets $(\omega_i)_{i\in\mathbb{N}}$ and ω_∞ with the properties

$$\mu_i = \mu_i|_{\omega_i} \quad , \quad \omega_i \cap \omega_j = \emptyset$$

$$\mu_\infty = \mu_\infty|_{\omega_\infty} \quad , \quad \omega_\infty \cap \omega_j = \emptyset$$
$$i, j \in \mathbb{N} \quad , \quad i \neq j \quad .$$

We can now reverse-engineer $(\Delta_n)_{n\in\mathbb{N}}$ by defining $n \geq 2$ the sets

$$\Delta_1 := \sigma(N) \quad , \quad \Delta_n := \omega_\infty \cup \bigcup_{i=n}^{\infty} \omega_i \quad .$$

Before we get into the meat of it all, we will prove some properties relating μ to the $(\mu_i)_{i\in\mathbb{N}}$.

We start by showing that $\mu|_{\Delta_n} = \mu_\infty + \sum_{i=n}^{\infty} \mu_i$. First, we deal with the general case and afterwards we will turn towards the special case $n = 1$. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $i < n$. We remember that the $(\omega_i)_{i\in\mathbb{N}}$ and ω_∞ are disjoint and that $\mu_j(\omega_i) = \mu_j(\omega_j \cap \omega_i) = 0$ for $i \neq j$. Similarly we see $\mu_\infty(\omega_i) = 0$. This means we get

$$\mu_i(\Delta_n) = \mu_i\left(\omega_\infty \cup \bigcup_{j=n}^{\infty} \omega_j\right) = \mu_i(\omega_\infty) + \sum_{j=n}^{\infty} \mu_i(\omega_j) = 0 \quad .$$

Therefore we find that for any Borel set Δ we have $\mu_i(\Delta \cap \Delta_n) = 0$. On the other hand we take $i \geq n$ and see that $\omega_i \subseteq \Delta_n$. We remember that $\mu_i = \mu_i|_{\omega_i}$ and we calculate

$$\begin{aligned}\mu_i(\Delta \cap \Delta_n) &= \mu_i|_{\omega_i}(\Delta \cap \Delta_n) = \mu_i(\Delta \cap \Delta_n \cap \omega_i)\\ &= \mu_i(\Delta \cap \omega_i) = \mu_i|_{\omega_i}(\Delta) = \mu_i(\Delta) \quad .\end{aligned}$$

Taken together, this means we have $\mu_i(\Delta \cap \Delta_n) = \mu_i(\Delta)$ if $i \geq n$ and $\mu_i(\Delta \cap \Delta_n) = 0$ if $i < n$. Since $\omega_\infty \subseteq \Delta_n$, we can show similarly that $\mu_\infty(\Delta \cap \Delta_n) = \mu_\infty(\Delta)$. Therefore we can now calculate

$$\mu|_{\Delta_n}(\Delta) = \mu(\Delta \cap \Delta_n) = \mu_\infty(\Delta \cap \Delta_n) + \sum_{i=1}^{\infty} \mu_i(\Delta \cap \Delta_n) = \mu_\infty(\Delta) + \sum_{i=n}^{\infty} \mu_i(\Delta) \quad .$$

This means we get for $n \geq 2$ that

$$\mu|_{\Delta_n} = \mu_\infty + \sum_{i=n}^{\infty} \mu_i \quad .$$

For the case $n = 1$, we know that μ is a scalar-valued spectral measure and Proposition 3.7.1 tells us that $\operatorname{supp}(\mu) = \sigma(N)$. Further, we have $\Delta_1 = \sigma(N)$ and Lemma 3.9.1 tells us that $\mu = \mu|_{\operatorname{supp}(\mu)}$, so altogether we get

$$\mu|_{\Delta_1} = \mu = \mu_\infty + \sum_{n=1}^{\infty} \mu_n \quad .$$

Now we move on to inspect $(\mu|_{\Delta_n})|_{\omega_i}$ for $n, i \in \mathbb{N}$. We remember that $\mu_i(\omega_j) = 0$ for $i \neq j$ and $\mu_\infty(\omega_i) = \mu_i(\omega_\infty) = 0$, and we can use a similar argumentation as above to obtain

$$(\mu|_{\Delta_n})|_{\omega_i} = \begin{cases} \mu_i & \text{if } i \geq n \\ 0 & \text{if } i < n \end{cases} \quad , \quad (\mu|_{\Delta_n})|_{\omega_\infty} = \mu_\infty$$

Now we want to use these insights together with Lemma 3.9.3 and show

$$N_{\mu|_{\Delta_n}} \simeq N_{(\mu|_{\Delta_n})|_{\omega_\infty}} \oplus \bigoplus_{i=n}^{\infty} N_{(\mu|_{\Delta_n})|_{\omega_i}} = N_{\mu_\infty} \oplus \bigoplus_{i=n}^{\infty} N_{\mu_i} \quad . \tag{3.5}$$

To use the lemma we have to prove for $n \in \mathbb{N}$ that $\mu|_{\Delta_n}$ is a Radon measure with compact support, that $\mu|_{\Delta_n}(\omega_i \cap \omega_j) = 0$ and $\mu|_{\Delta_n}(\omega_i \cap \omega_\infty) = 0$ for $i \neq j$ and $i \geq n$. Finally, we also have to prove that $\mu|_{\Delta_n}((\omega_\infty \cup \bigcup_{i=n}^\infty \omega_i)^c) = 0$. For the first condition, we have already stated that μ is a Radon measure with compact support, so according to Lemma 3.8.2 we know that $\mu|_{\Delta_n}$ is also a Radon measure with compact support. Coming to the second condition, we know that the $(\omega_i)_{i\in\mathbb{N}}$ and ω_∞ are disjoint, so we get $\mu|_{\Delta_n}(\omega_i \cap \omega_j) = \mu|_{\Delta_n}(\omega_i \cap \omega_\infty) = 0$ with $i \geq n$ and $i \neq j$. Now for the third condition we see

$$\mu|_{\Delta_n}\left(\left(\omega_\infty \cup \bigcup_{i=n}^\infty \omega_i\right)^c\right) = \mu|_{\Delta_n}(\Delta_n^c) = \mu(\Delta_n^c \cap \Delta_n) = \mu(\emptyset) = 0 \quad .$$

This means we can use Lemma 3.9.3 and obtain Equation 3.5. Now we take Lemma 3.8.1 to form the direct sum and then rearrange the terms, so we get

$$\bigoplus_{n=1}^\infty N_{\mu|_{\Delta_n}} \simeq \bigoplus_{n=1}^\infty \left(N_{\mu_\infty} \bigoplus_{i=n}^\infty N_{\mu_i}\right) \simeq N_{\mu_\infty}^\infty \oplus \bigoplus_{n=1}^\infty N_{\mu_n}^n \simeq N \quad .$$

Therefore we have found a scalar-valued spectral measure μ and a sequence of Borel sets $(\Delta_n)_{n\in\mathbb{N}}$ with $\Delta_1 = \sigma(N)$ so that $\bigoplus_{n=1}^\infty N_{\mu|_{\Delta_n}} \simeq N$, which are the exact conditions for Theorem 3.9.2.

Next we turn our attention to M, ν_∞ and $(\nu_i)_{i\in\mathbb{N}}$. We know that we have the scalar-valued spectral measure

$$\nu := \nu_\infty + \sum_{n=1}^\infty \nu_n \quad .$$

We can now define in the same manner as before the sets $(\gamma_i)_{i\in\mathbb{N}}$ with

$$\nu_i = \nu_i|_{\gamma_i} \quad , \quad \gamma_i \cap \gamma_j = \emptyset$$

$$\nu_\infty = \nu_\infty|_{\gamma_\infty} \quad , \quad \gamma_\infty \cap \gamma_j = \emptyset$$

$$i, j \in \mathbb{N} \quad , \quad i \neq j \quad .$$

We further define for $n \geq 2$ the sets

$$\Sigma_1 := \sigma(M) \quad , \quad \Sigma_n := \gamma_\infty \cup \bigcup_{i=n}^\infty \gamma_i \quad .$$

As we have shown before, we get $\bigoplus_{n=1}^\infty N_{\nu|_{\Sigma_n}} \simeq M$ with the same conditions as given in Theorem 3.9.2. To tie everything up, we remember that our assumption was $N \simeq M$, and so part *(b)* of Theorem 3.9.2 tells us that $[\mu] = [\nu]$ and $\mu(\Delta_n \setminus \Sigma_n) = 0 = \mu(\Sigma_n \setminus \Delta_n)$ for all $n \in \mathbb{N}$. This concludes the first step of our proof.

In the second step of this proof we show that this leads to $[\mu_\infty] = [\nu_\infty]$ and $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$. We start with $i \in \mathbb{N}$ and want to prove $[\mu_i] = [\nu_i]$. To this end we will show some equalities that will help us along the way. First we see that

$$\mu|_{\Delta_i} - \mu|_{\Delta_{i+1}} = \mu_\infty + \sum_{j=i}^\infty \mu_j - \mu_\infty - \sum_{j=i+1}^\infty \mu_j = \mu_i \quad .$$

We then remember that $\mu(\Delta_i \setminus \Sigma_i) = 0 = \mu(\Sigma_i \setminus \Delta_i)$ and get for a Borel set Δ that

$$\mu|_{\Delta_i}(\Delta) = \mu(\Delta \cap \Delta_i)$$
$$= \mu(\Delta \cap \Sigma_i) - \mu(\Delta \cap (\Sigma_i \setminus \Delta_i)) + \mu(\Delta \cap (\Delta_i \setminus \Sigma_i)) = \mu(\Delta \cap \Sigma_i) \quad .$$

Similarly we get $\mu|_{\Delta_{i+1}}(\Delta) = \mu(\Delta \cap \Sigma_{i+1})$. Finally, we know that $\Sigma_{i+1} \subseteq \Sigma_i$, so we get

$$\mu(\Delta \cap \Sigma_i) - \mu(\Delta \cap \Sigma_{i+1}) = \mu(\Delta \cap (\Sigma_i \setminus \Sigma_{i+1})) \quad .$$

Next we take Δ so that $\mu_i(\Delta) = 0$ and see

$$0 = \mu_i(\Delta) = \mu|_{\Delta_i}(\Delta) - \mu|_{\Delta_{i+1}}(\Delta) = \mu(\Delta \cap \Delta_i) - \mu(\Delta \cap \Delta_{i+1})$$
$$= \mu(\Delta \cap \Sigma_i) - \mu(\Delta \cap \Sigma_{i+1}) = \mu(\Delta \cap (\Sigma_i \setminus \Sigma_{i+1})) \quad .$$

Now we use the fact that $[\mu] = [\nu]$ and a chain of similar equalities as above to get

$$0 = \nu(\Delta \cap (\Sigma_i \setminus \Sigma_{i+1})) = \nu|_{\Sigma_i}(\Delta) - \nu|_{\Sigma_{i+1}}(\Delta) = \nu_i(\Delta) \quad .$$

This means we have found that $\nu_i \ll \mu_i$. We can show $\mu_i \ll \nu_i$ by switching μ_i with ν_i and Δ_i with Σ_i, so altogether we obtain $[\mu_i] = [\nu_i]$.

Unfortunately the deductions above don't hold true for $[\mu_\infty] = [\nu_\infty]$, so we need some additional considerations. To this end, we will now define for $i \in \mathbb{N}$ the sets

$$\eta_i := \omega_i \cap \gamma_i \quad , \quad \Xi_\infty := \left(\bigcup_{i=1}^{\infty} \eta_i \right)^c$$

We will now show that $\mu_i(\Xi_\infty) = 0$ for $i \in \mathbb{N}$ and $\mu_\infty(\Xi_\infty^c) = 0$ to obtain $\mu|_{\Xi_\infty} = \mu_\infty$. To this end we take $i \in \mathbb{N}$ and consider $[\mu_i] = [\nu_i]$. Since we know $\nu_i(\gamma_i^c) = \nu_i(\gamma_i \cap \gamma_i^c) = 0$, we get $\mu_i(\gamma_i^c) = 0$. Further we similarly have $\mu_i(\omega_i^c) = 0$, so together we find

$$\mu_i(\eta_i^c) = \mu_i(\omega_i^c \cup \gamma_i^c) \leq \mu_i(\omega_i^c) + \mu_i(\gamma_i^c) = 0 \quad .$$

Because $\Xi_\infty \subseteq \eta_i^c$, we see that $\mu_i(\Xi_\infty) \leq \mu_i(\eta_i^c) = 0$. On the other hand we remember that $\mu_\infty(\omega_i) = \mu_\infty(\omega_\infty \cap \omega_i) = 0$ for $i \in \mathbb{N}$, so we get

$$\mu_\infty(\Xi_\infty^c) = \mu_\infty \left(\bigcup_{i=1}^{\infty} \eta_i \right) \leq \sum_{i=1}^{\infty} \mu_\infty(\omega_i \cap \gamma_i) = 0 \quad .$$

According to Lemma 3.9.1 this means $\mu_\infty|_{\Xi_\infty} = \mu_\infty$. Now we take a Borel set Δ and calculate

$$\mu|_{\Xi_\infty}(\Delta) = \mu_\infty(\Delta \cap \Xi_\infty) + \sum_{i=1}^{\infty} \mu_i(\Delta \cap \Xi_\infty) = \mu_\infty(\Delta \cap \Xi_\infty) = \mu_\infty|_{\Xi_\infty}(\Delta) = \mu_\infty(\Delta) \quad .$$

Thus we get $\mu|_{\Xi_\infty} = \mu_\infty$. Since the $(\eta_i)_{i\in\mathbb{N}}$ are defined symmetrically in ω and γ, we can show $\nu|_{\Xi_\infty} = \nu_\infty$ in the same way. Now we take a Borel set Δ so that $\mu_\infty(\Delta) = 0$, which means

$$0 = \mu_\infty(\Delta) = \mu|_{\Xi_\infty}(\Delta) = \mu(\Delta \cap \Xi_\infty) \quad .$$

Since $[\mu] = [\nu]$, this means we also get

$$0 = \nu(\Delta \cap \Xi_\infty) = \nu|_{\Xi_\infty}(\Delta) = \nu_\infty(\Delta) \quad .$$

Therefore we have $\nu_\infty \ll \mu_\infty$, and we can similarly show $\nu_\infty \ll \mu_\infty$. This means we have $[\mu_\infty] = [\nu_\infty]$, which concludes the proof. □

3.10 Multiplicity Functions

We have finally reached the crowning section of this thesis, in which we will state our Central Multiplicity Theorem and obtain the multiplicity function $\#$ for N. Before we can do that however, we must introduce some definitions to understand where and how the multiplicity function operates, and how we can construct a relevant operator $N_\#$ so that we actually find $N \simeq N_\#$.

Definition 3.10.1. Let $\mathscr{H}$ be a Hilbert space, let (X, Ω, μ) be a measure space, and let $f : X \mapsto \mathscr{H}$. Then we call f *measurable* if the function $f_g : X \mapsto \mathbb{C}$ defined by $f_g(x) := \langle f(x), g\rangle$ is measurable for each $g \in \mathscr{H}$.

Proposition 3.10.1. *Let $\mathscr{H}$ be a separable Hilbert space, let (X, Ω, μ) be a measure space, and let $f : X \mapsto \mathscr{H}$ be a measurable function. Then the function $||f||_{\mathscr{H}} : X \mapsto \mathbb{C}$ defined by $||f||_{\mathscr{H}}(x) = ||f(x)||_{\mathscr{H}}$ is a measurable function.*

Proof. Since $\mathscr{H}$ is separable, we can find a countable orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathscr{H}$. We know that f_{e_n} is measurable for $n \in \mathbb{N}$, and we know that we have the point-wise convergence

$$\lim_{N\to\infty} \sum_{n=1}^{N} |f_{e_n}(x)|^2 = \lim_{N\to\infty} \sum_{n=1}^{N} |\langle f(x), e_n\rangle|^2 = ||f(x)||^2_{\mathscr{H}} \quad .$$

Therefore $||f(x)||^2_{\mathscr{H}}$ is also a measurable function, and because $\sqrt{.} : \mathbb{C} \mapsto \mathbb{C}$ is measurable we see that $||f(x)||_{\mathscr{H}}$ is measurable as well. □

Definition 3.10.2. Let $\mathscr{H}$ be a separable Hilbert space, and let (X, Ω, μ) be a measure space. We define the set

$$\mathscr{L}^2(\mu, \mathscr{H}) := \{f : f \text{ is measurable and } \int ||f(x)||^2_{\mathscr{H}} d\mu(x) < \infty\} \quad .$$

Further we define the inner product on $\mathscr{L}^2(\mu, \mathscr{H})$ as

$$\langle f, g\rangle_{\mathscr{L}^2(\mu,\mathscr{H})} := \int \langle f(x), g(x)\rangle_{\mathscr{H}} d\mu(x) \quad .$$

Proposition 3.10.2. *The set $\mathscr{L}^2(\mu, \mathscr{H})$ with the pointwise vector space operations of $\mathscr{H}$ is a vector space. Further, the inner product from Definition 3.10.2 is well defined, and the equivalence classes $L^2(\mu, \mathscr{H})$ of $\mathscr{L}^2(\mu, \mathscr{H})$ form a Hilbert space.*

Proof. First we will investigate whether $\mathscr{L}^2(\mu,\mathscr{H})$ is a vector space. For $f\in\mathscr{L}^2(\mu,\mathscr{H})$ $\lambda\in\mathbb{C}$ we find

$$\int ||\lambda f(x)||^2_{\mathscr{H}}d\mu(x)=\int|\lambda|^2||f(x)||^2_{\mathscr{H}}d\mu(x)<\infty \quad .$$

Now we remember that for $a,b\in\mathbb{R}$ we have $(a+b)^2\leq 2(a^2+b^2)$. Therefore we see for $g\in\mathscr{L}^2(\mu,\mathscr{H})$ that

$$\int ||f(x)+g(x)||^2_{\mathscr{H}}d\mu(x)\leq\int 2(||f||^2_{\mathscr{H}}+||g||^2_{\mathscr{H}})d\mu(x)<\infty \quad .$$

Altogether we find that summation and multiplication maps to $\mathscr{L}^2(\mu,\mathscr{H})$ again, so it is indeed a vector space.

Now we proceed to discuss the inner product. By similar considerations as above, we see that it is sesquilinear. Now we take $f,g\in\mathscr{L}^2(\mu,\mathscr{H})$ and see that $||f||_{\mathscr{H}},||g||_{\mathscr{H}}\in L^2(\mu)$. This leads us to calculate

$$\left|\int\langle f(x),g(x)\rangle_{\mathscr{H}}d\mu(x)\right|\leq\int ||f(x)||_{\mathscr{H}}||g(x)||_{\mathscr{H}}d\mu(x)$$

$$\leq ||(||f||_{\mathscr{H}})||_{L^2(\mu)}||(||g||_{\mathscr{H}})||_{L^2(\mu)}<\infty \quad .$$

Therefore the scalar product is well defined.

Finally we want to show that the equivalence classes of $\mathscr{L}^2(\mu,\mathscr{H})$ form a Hilbert space. For $f\in\mathscr{L}^2(\mu,\mathscr{H})$ we find

$$||f||^2_{\mathscr{L}^2(\mu,\mathscr{H})}=\langle f,f\rangle_{\mathscr{L}^2(\mu,\mathscr{H})}=\int ||f(x)||^2_{\mathscr{H}}d\mu(x) \quad .$$

It now suffices to show that $\mathscr{L}^2(\mu,\mathscr{H})$ is closed with respect to $||.||^2_{\mathscr{L}^2(\mu,\mathscr{H})}$. To this end we take a Cauchy sequence $(f_k)_{k\in\mathbb{N}}\subseteq\mathscr{L}^2(\mu,\mathscr{H})$ and we want to show that there exists a $f\in\mathscr{L}^2(\mu,\mathscr{H})$ so that $||f_k-f||_{\mathscr{L}^2(\mu,\mathscr{H})}\to 0$. Since $\mathscr{H}$ is separable, we can further find an orthonormal base $(e_n)_{n\in\mathbb{N}}$. Further, for $g\in\mathscr{L}^2(\mu,\mathscr{H})$ we have a monotonously increasing convergence

$$\sum_{n=1}^{N}|g_{e_n}(x)|^2\to\sum_{n=1}^{\infty}|g_{e_n}(x)|^2=||g(x)||^2_{\mathscr{H}} \quad .$$

Therefore we can use Theorem 4.7.1 to take the limit out of the integral and see

$$||g||^2_{\mathscr{L}^2(\mu,\mathscr{H})}=\int\sum_{n=1}^{\infty}|g_{e_n}(x)|^2d\mu(x)=\sum_{n=1}^{\infty}\int|g_{e_n}(x)|^2d\mu(x)=\sum_{n=1}^{\infty}||g_{e_n}||^2_{L^2(\mu)} \quad .$$

This means especially that $(f_{k,e_n})_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu)$ for $n\in\mathbb{N}$, and therefore there exists a sequence of limit functions $(h_n)_{n\in\mathbb{N}}\subseteq L^2(\mu)$ with $||f_{k,e_n}-h_n||_{L^2(\mu)}\to 0$ for $n\in\mathbb{N}$. Without concerning ourselves about the convergence, we now define

$$f=\sum_{n=1}^{\infty}h_ne_n \quad .$$

One issue is whether $f\in\mathscr{L}^2(\mu,\mathscr{H})$, but the $(h_n)_{n\in\mathbb{N}}$ are measurable, so we have only to think about $||f||_{\mathscr{L}^2(\mu,\mathscr{H})}<\infty$. Since the $(f_n)_n$ form a Cauchy sequence, this follows automatically if we manage to show $||f_n - f||_{\mathscr{L}^2(\mu,\mathscr{H})} \to 0$. To this end we choose $\epsilon > 0$ and $k \in \mathbb{N}$ so that $||f_k - f_j||_{\mathscr{L}^2(\mu,\mathscr{H})} < \epsilon$ for all $j \geq k$. Now we can take $N \in \mathbb{N}$ and calculate

$$\sum_{n=1}^{N} ||h_n - f_{k,e_n}||^2_{L^2(\mu)} \leq 2\sum_{n=1}^{N}(||h_n - f_{j,e_n}||^2_{L^2(\mu)} + ||f_{j,e_n} - f_{k,e_n}||^2_{L^2(\mu)})$$

$$\leq 2||f_j - f_k||_{\mathscr{L}^2(\mu,\mathscr{H})} + 2\sum_{n=1}^{N} ||h_n - f_{j,e_n}||^2_{L^2(\mu)} < 2\epsilon + 2\sum_{n=1}^{N} ||h_n - f_{j,e_n}||^2_{L^2(\mu)} \quad .$$

We know that $||f_{j,e_n} - h_n||_{L^2(\mu)} \to 0$, and since we have a finite sum and the only requirement for j was $j \geq k$, we can take the limit $j \to \infty$ to get $\sum_{n=1}^{N} ||h_n - f_{k,e_n}||^2_{L^2(\mu)} \leq 2\epsilon$. Now we do the same thing for $N \to \infty$ to obtain

$$||f - f_k||^2_{\mathscr{L}^2(\mu,\mathscr{H})} = \sum_{n=1}^{\infty} ||h_n - f_{k,e_n}||^2_{L^2(\mu)} \leq 2\epsilon \quad .$$

Additionally, for $j \geq k$ we find

$$||f - f_j||^2_{\mathscr{L}^2(\mu,\mathscr{H})} \leq 2(||f - f_k||^2_{\mathscr{L}^2(\mu,\mathscr{H})} + ||f_k - f_j||^2_{\mathscr{L}^2(\mu,\mathscr{H})}) \leq 6\epsilon \quad .$$

Therefore we find that $||f - f_n||^2_{\mathscr{L}^2(\mu,\mathscr{H})} \to 0$, which means that $(f_n)_{n\in\mathbb{N}}$ has a limit value in $\mathscr{L}^2(\mu,\mathscr{H})$. Finally, the proof that $L^2(\mu,\mathscr{H})$ is a Hilbert space is almost the same proof as for the fact that the equivalence classes of square-integrable functions over μ called $L^2(\mu)$ is a Hilbert space. $\square$

We now take l^2 to be the Hilbert space of all absolute square summable complex sequences, and we will turn our attention to $L^2(\mu, l^2)$. This space is essentially the space of sequences in $L^2(\mu)$ with the added condition that their L^2-norms are square-summable. The following two propositions elaborate on this.

Proposition 3.10.3. *Let μ be a measure, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^2(\mu)$ so that*

$$\sum_{n=1}^{\infty} ||f_n||^2_{L^2(\mu)} < \infty \quad .$$

Then we define $f(z) := (f_n(z))_{n\in\mathbb{N}}$ and see that $f \in L^2(\mu, l^2)$. Additionally, every element of $L^2(\mu, l^2)$ has the aforementioned form.

Proof. Since there might be some $z \in \mathbb{C}$ where $(f_n(z))_{n\in\mathbb{N}}$ is not in l^2, we first have to find representatives of $(f_n)_{n\in\mathbb{N}}$ so that $f(z) \in l^2$ for all $z \in \mathbb{C}$. We define the set

$$A := \{z : ||f(z)||_{l^2} < \infty\} \quad .$$

We will prove that $\mu(A^c) = 0$ and thus we can just set $f(z) = (0)_{n\in\mathbb{N}}$ for $z \in A^c$. To this end we calculate

$$\sum_{n=1}^{\infty} \int |f_n(z)|^2 d\mu(z) = \sum_{n=1}^{\infty} ||f_n||^2_{L^2(\mu)} < \infty \quad .$$

We can now use Theorem 4.7.1 to switch the integral with the summation and obtain

$$\int ||f(z)||_{l^2}^2 d\mu(z) = \int \sum_{n=1}^{\infty} |f_n(z)|^2 d\mu(z) < \infty \quad .$$

Thus we find $||f(z)||_{l^2}^2 < \infty$ μ-almost everywhere, and we find a sequence of representatives so that $f(z) = (f_n(z))_{n\in\mathbb{N}}$ on A and $f(z) = (0)_{n\in\mathbb{N}}$ for $z \in A^c$. Since $(0)_{n\in\mathbb{N}} \in l^2$, this means $f(z) \in l^2$ for all $z \in \mathbb{C}$.

Now we want to show that f is measurable according to Definition 3.10.1. We take $g = (g_n)_{n\in\mathbb{N}} \in l^2$ and see that

$$|\langle f(z), g\rangle_{l^2}| \leq ||f(z)||_{l^2}^2 ||g||_{l^2} \quad .$$

On the other hand we also have

$$\langle f(z), g\rangle_{l^2} \leq \sum_{n=1}^{\infty} f_n(z) g_n \quad .$$

Since the sum converges, f_n is measurable and g_n is just a constant for $n \in \mathbb{N}$, we see that the sum is measurable again. Thus we find that f is measurable.

Now we discuss whether $||f||_{L^2(\mu,l^2)}^2 < \infty$. We have seen that $\sum_{n=1}^{\infty} \int |f_n(z)|^2 d\mu(z) < \infty$, and since all functions are positive, we can use Theorem 4.7.1 to swap integral and sum to get

$$\int ||f(z)||_{l^2} d\mu(z) = \int \sum_{n=1}^{\infty} |f_n(z)|^2 d\mu(z) = \sum_{n=1}^{\infty} \int |f_n(z)|^2 d\mu(z) < \infty \quad .$$

By Definition 3.10.2 the left hand side is exactly $||f||_{L^2(\mu,l^2)}$ and therefore we find $||f||_{L^2(\mu,l^2)} < \infty$. This means $f \in L^2(\mu, l^2)$.

Now let $f \in L^2(\mu, l^2)$. This means $f(z) \in l^2$ for all $z \in \mathbb{C}$ and we can write $f(z) = (f_n(z))_{n\in\mathbb{C}}$. We then reverse the steps above and see

$$\sum_{n=1}^{\infty} ||f_n||_{L^2(\mu)}^2 = \sum_{n=1}^{\infty} \int |f_n(z)|^2 d\mu(z)$$

$$= \int \sum_{n=1}^{\infty} |f_n(z)|^2 d\mu(z) = \int ||f(z)||_{l^2}^2 d\mu(z) = ||f||_{L^2(\mu,l^2)}^2 < \infty \quad .$$

This means especially $f_n \in L^2(\mu)$ for $n \in \mathbb{N}$, and we have proven that f is indeed of the form described above. □

Proposition 3.10.4. *Let $f \in L^2(\mu, l^2)$ with $f = (f_n)_{n\in\mathbb{N}}$. Then we find*

$$||f||_{L^2(\mu,l^2)}^2 = \sum_{n=1}^{\infty} ||f_n||_{L^2(\mu)}^2 \quad .$$

Proof. We have shown a more general version of this during the proof of Proposition 3.10.2, where we have shown for $g \in L^2(\mu, \mathscr{H})$ and an orthonormal base $(e_n)_{n\in\mathbb{N}} \subseteq \mathscr{H}$ that

$$||g||^2_{L^2(\mu,\mathscr{H})} = \sum_{n=1}^{\infty} |\langle g, e_n\rangle_{\mathscr{H}}|^2 \quad .$$

We now take the orthonormal base of l^2 that is formed by the $(\delta_{n,i})_{n\in\mathbb{N}}$ with is $\delta_{n,i} = 1$ if $n = i$ and $\delta_{n,i} = 0$ otherwise. This leads directly to

$$||f||^2_{l^2} = \sum_{n=1}^{\infty} |\langle f, (\delta_{n,i})_{n\in\mathbb{N}}\rangle_{l^2}|^2 = \sum_{n=1}^{\infty} ||f_n||^2_{L^2(\mu)} \quad .$$

□

Now we are ready to define the multiplicity function $\#$, together with the canonical multiplication operator $N_\#$ associated with a certain multiplicity function and the Hilbert space it operates on.

Definition 3.10.3. Let μ be a Radon measure with compact support. The we define a *multiplicity function for* μ as a function $\# : \mathbb{C} \mapsto \{0, 1, 2, ..., \infty\}$ so that $\#(z) \geq 1$ μ-almost everywhere.

Definition 3.10.4. We define for $n \in \mathbb{N}$ the subspace $l^2_n \subseteq l^2$ as

$$l^2_n := \{(a_k)_{k\in\mathbb{N}} : (a_k)_{k\in\mathbb{N}} \in l^2 \text{ and } a_k = 0 \text{ for } k > n\}$$

We further define $l^2_\infty := l^2$.

Definition 3.10.5. Let μ be a Radon measure with compact support, and let $\#$ be a multiplicity function for μ. Then we define the subspace $\mathscr{D}_\#$ of $L^2(\mu, l^2)$ as

$$\mathscr{D}_\# := \{f : f \in L^2(\mu, l^2) \text{ and } f(z) \in l^2_{\#(z)} \ \mu\text{-almost everywhere}\}$$

Further we define the *canonical multiplication operator* $N_\# : \mathscr{D}_\# \mapsto \mathscr{D}_\#$ as the operator

$$N_\# f(z) := z f(z) \quad .$$

Proposition 3.10.5. *Let μ be a Radon measure with compact support and let $\#$ be a multiplicity function for μ. Then the space $\mathscr{D}_\#$ is a closed subspace of $L^2(\mu, l^2)$ and thus a Hilbert space. Further we find that $N_\# \in \mathscr{B}(\mathscr{D}_\#)$.*

Proof. By Proposition 3.10.2 we already know that $L^2(\mu, l^2)$ is a Hilbert space, so it suffices to show that $\mathscr{D}_\#$ is closed. Now let $(g_k)_{k\in\mathbb{N}} \subseteq \mathscr{D}_\#$ be a sequence so that $||g_k - f||_{L^2(\mu,l^2)} \to 0$ for some $f \in L^2(\mu, l^2)$. By Proposition 3.10.4 this means especially for $n \in \mathbb{N}$ that $||(g_k)_n - f_n||_{L^2(\mu)} \to 0$. Now we define the set $A_n := \{z : \#(z) < n\}$. Since $(g_k)_n \in \mathscr{D}_\#$, we know that $(g_k)_n = 0$ μ-almost everywhere on A_n for $k \in \mathbb{N}$. Thus we find

$$\int_{A_n} |f_n(z)|^2 d\mu(z) = \int_{A_n} |f_n(z) - (g_k)_n(z)|^2 d\mu(z) \leq \int |f_n - (g_k)_n|^2 d\mu(z)$$

$$= ||f_n - (g_k)_n||_{L^2(\mu)} \to 0 \quad .$$

This means $f_n = 0$ μ-almost everywhere on A_n, which in turn implies $f(z) \in l^2_{\#(z)}$ μ-almost everywhere and thus we find $f \in \mathscr{D}_\#$.

Now we turn towards showing that $N_\# \in \mathscr{B}(\mathscr{D}_\#)$. We can see by the definition that $N_\#$ is linear, so it remains to prove that it is bounded and $N_\#\mathscr{D}_\# \subseteq \mathscr{D}_\#$. We start with the latter fact and take $n \in \mathbb{N}$ and $f \in \mathscr{D}_\#$, and we know that for $n \in \mathbb{N}$ we have $f_n(z) = 0$ μ-almost everywhere on A_n. Since $N_\# f(z) = zf(z)$, we know that also μ-almost everywhere on A_n we have

$$(N_\# f)_n(z) = zf_n(z) = 0 \quad .$$

This holds for all $n \in \mathbb{N}$, so we get $N_\# f \in \mathscr{D}_\#$ and thus $N_\#\mathscr{D}_\# \subseteq \mathscr{D}_\#$. For the boundedness of $N_\# f$ we consider $r_\mu := \sup\{|z| : z \in \operatorname{supp}(\mu)\}$. Since $\operatorname{supp}(\mu)$ is compact, this supremum is finite. Further, we remind ourselves that according to Proposition 4.7.6 we have $\mu(\operatorname{supp}(\mu)^c) = 0$. Thus for $n \in \mathbb{N}$ and $f \in \mathscr{D}_\#$ we have μ-almost everywhere that

$$|(N_\# f)_n(z)| = |zf_n(z)| \le r_\mu |f_n(z)| \quad .$$

This holds for all $n \in \mathbb{N}$, so we can calculate

$$||N_\# f||^2_{\mathscr{D}_\#} = \int \sum_{n=1}^{\infty} |f_n(z)|^2 d\mu(z) \le r_\mu^2 \int \sum_{n=1}^{\infty} |f_n(z)|^2 d\mu(z) = r_\mu^2 ||f||^2_{\mathscr{D}_\#} \quad .$$

Therefore we find that $||N_\#|| \le r_\mu$ and thus $N_\# \in \mathscr{B}(\mathscr{D}_\#)$. □

Now all the pieces are set up and we are ready to tackle the Central Multiplicity Theorem. Hopefully we can already see the similarity bewteen $\mathscr{D}_\#$ and $(L^2(\mu_\infty))^\infty \oplus \bigoplus_{n=1}^\infty (L^2(\mu_n))^n$, so from here on we basically just have to work through the technical details to show their unitary equivalence and subsequently

$$N_\# \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{n=1}^{\infty} (N_{\mu_n})^n \simeq N \quad .$$

We start our final push with a lemma, after which we prove Theorem 3.10.7.

Lemma 3.10.6. *Let μ be a Radon measure with compact support and let $\#$ be a multiplicity function for μ. We denote for $i \in \mathbb{N}$ the sets $\omega_i := \#^{-1}(i)$ and $\omega_\infty := \#^{-1}(\infty)$. Then we find $\mu = \mu|_{\omega_\infty} + \sum_{i=1}^\infty \mu|_{\omega_i}$ and*

$$N_\# \simeq (N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\mu|_{\omega_i}})^i \quad .$$

Proof. To facilitate our notation, we first set

$$L^2_\oplus := L^2(\mu|_{\omega_\infty}) \oplus \bigoplus_{i=1}^{\infty} L^2(\mu|_{\omega_i}) \quad .$$

Although we need to go through a lot of technical detail to prove it, this lemma is relatively intuitive. First, we will quickly show that $\mu = \mu|_{\omega_\infty} + \sum_{i=1}^{\infty} \mu|_{\omega_i}$. Next, we will try to rearrange the spaces $L^2(\mu|_{\omega_i})$ and $L^2(\mu|_{\omega_\infty})$ so that

$$L^2_{\oplus} \simeq \bigoplus_{k=1}^{\infty} L^2(\mu|_{\Delta_k}) \quad .$$

Here, the $(\Delta_k)_{k\in\mathbb{N}}$ will be akin to the sets given in Theorem 3.9.2 so that $\Delta_{k+1} \subseteq \Delta_k$ for all $k \in \mathbb{N}$. For $f \in \bigoplus_{k=1}^{\infty} L^2(\mu|_{\Delta_k})$ we can then write $f = \bigoplus_{k=1}^{\infty} \chi_{\Delta_k} f_k$, and we see that e.g. on Δ_1 only $\chi_{\Delta_1} f_1$ is potentially non-zero. Therefore $(\chi_{\Delta_k} f_k)_{k\in\mathbb{N}} \in l_j^2$ on Δ_j^c, which already looks a lot like $\mathscr{D}_{\#}$ (and we will confirm in our derivation that $(\chi_{\Delta_k} f_k)_{k\in\mathbb{N}} \in \mathscr{D}_{\#}$). With these considerations as a basis, we will define an operator

$$V : L^2_{\oplus} \mapsto \mathscr{D}_{\#} \quad .$$

We can then show that V is unitary, and that

$$N_{\#} = V\left((N_{\mu|_{\omega_\infty}})^{\infty} \oplus \bigoplus_{i=1}^{\infty} (N_{\mu|_{\omega_i}})^i \right) V^{-1} \quad .$$

It is important to note, however, that we will not directly use the aforementioned sets $(\Delta_k)_{k\in\mathbb{N}}$, and we just mentioned them to gain a better understanding of the proof. Instead, we will define functions $(f_k)_{k\in\mathbb{N}}$ with the relevant properties directly.

First we see that $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$ and $\omega_i \cap \omega_\infty = \emptyset$ for $i \in \mathbb{N}$. Further we define

$$\Omega := \omega_\infty \cup \bigcup_{i=1}^{\infty} \omega_i = \#^{-1}(\{1, 2, ..., \infty\}) \quad .$$

This means we find $\Omega^c = \#^{-1}(0)$. Since $\#$ is a multiplicity function for μ, we know that $\mu(\#^{-1}(0)) = 0$, and thus we have $\mu(\Omega^c) = 0$. Therefore we can use Lemma 3.9.3 to get $\mu = \mu|_{\omega_\infty} + \sum_{i=1}^{\infty} \mu|_{\omega_i}$. To facilitate the future notation, we further relabel $\mu_i := \mu|_{\omega_i}$ for $i \in \mathbb{N}$ and $\mu_\infty := \mu|_{\omega_\infty}$.

Now we proceed to take $f \in L^2_{\oplus}$ and we want to construct a sequence of functions $(f_k)_{k\in\mathbb{N}}$ as discussed in the introduction. We start by labelling the components of f in the following way

$$f = \left(\bigoplus_{k=1}^{\infty} f_{\infty,k} \right) \oplus \left(\bigoplus_{i=1}^{\infty} \bigoplus_{k=1}^{i} f_{i,k} \right) \quad .$$

Here we have $f_{\infty,k} \in L^2(\mu_\infty)$ for all $k \in \mathbb{N}$ and $f_{i,k} \in L^2(\mu_i)$ for all $k \leq i$ and $i \in \mathbb{N}$. Now we can we define for $k \in \mathbb{N}$ the function

$$f_k := \chi_{\omega_\infty} f_{\infty,k} + \sum_{i=k}^{\infty} \chi_{\omega_i} f_{i,k} \quad .$$

At the moment this definition is just pointwise, but we will show in the following that $f_k \in L^2(\mu)$. We know that

$$||f||^2_{L^2_\oplus} = \left(\sum_{k=1}^{\infty} \int |f_{\infty,k}(z)|^2 d\mu_\infty(z)\right) + \left(\sum_{i=1}^{\infty}\sum_{k=1}^{i} \int |f_{i,k}(z)|^2 d\mu_i(z)\right) \quad .$$

All terms in the second sum are positive, so we can rearrange it. In addition, Proposition 4.7.5 tells us that $\frac{d\mu_\infty}{d\mu} = \chi_{\omega_\infty}$ and for $i \in \mathbb{N}$ that $\frac{d\mu_i}{d\mu} = \chi_{\omega_i}$. Together we find

$$||f||^2_{L^2_\oplus} = \left(\sum_{k=1}^{\infty} \int \chi_{\omega_\infty}(z)|f_{\infty,k}(z)|^2 d\mu(z)\right) + \left(\sum_{k=1}^{\infty}\sum_{i=k}^{\infty} \int \chi_{\omega_\infty}(z)|f_{i,k}(z)|^2 d\mu(z)\right) \quad .$$

For $i \in \mathbb{N}$ and $i \neq j$ we know that $\omega_i \cap \omega_j = \emptyset$ and $\omega_i \cap \omega_\infty = \emptyset$, so we find

$$\chi_{\omega_i}\chi_{\omega_j} = 0 \quad , \quad \chi_{\omega_i}\chi_{\omega_\infty} = 0 \quad .$$

This means we get

$$|f_k|^2 = \chi_{\omega_\infty}|f_{\infty,k}|^2 + \sum_{i=k}^{\infty} \chi_{\omega_i}|f_{i,k}|^2 \quad .$$

Because all entries in the following integral are positive, we can use Theorem 4.7.1 to obtain

$$\int |f_k(z)|^2 d\mu(z) = \int \chi_{\omega_\infty}(z)|f_{\infty,k}(z)|^2 d\mu(z) + \sum_{i=k}^{\infty} \int \chi_{\omega_i}(z)|f_{i,k}(z)|^2 d\mu(z) \quad .$$

This means we find that

$$\sum_{k=1}^{\infty} ||f_k||^2_{L^2(\mu)} = \sum_{k=1}^{\infty} \int |f_k(z)|^2 d\mu(z) = ||f||^2_{L^2_\oplus} < \infty \quad .$$

Therefore we especially get $f_k \in L^2(\mu)$.

Our next goal is to show that $(f_k)_{k\in\mathbb{N}} \in \mathscr{D}_\#$. We have just shown that $\sum_{k=1}^{\infty} ||f_k||^2_{L^2(\mu)} < \infty$, and so Proposition 3.10.3 tells us that $(f_k)_{k\in\mathbb{N}} \in L^2(\mu, l^2)$. Now we remember again for $i \in \mathbb{N}$ and $i \neq j$ that we have $\omega_i \cap \omega_j = \emptyset$ and $\omega_i \cap \omega_\infty = \emptyset$. Next we take $i < k \leq j$ and see that for $z \in \omega_i$ we get $\chi_{\omega_j}(z) = 0$ and $\chi_{\omega_\infty}(z) = 0$. This means we also get

$$f_k(z) = \chi_{\omega_\infty}(z) f_{\infty,k}(z) + \sum_{j=k}^{\infty} \chi_{\omega_i}(z) f_{j,k}(z) = 0 \quad .$$

Since $\omega_i = \#^{-1}(i)$, this means we have $(f_k(z))_{k\in\mathbb{N}} \in l^2_i$ for $z \in \#^{-1}(i)$ and therefore $(f_k)_{k\in\mathbb{N}} \in \mathscr{D}_\#$.

Altogether we can now take our whole previous calculation and define the operator $V : L^2_{\oplus} \mapsto \mathscr{D}_{\#}$ as $Vf := (f_k)_{k\in\mathbb{N}}$. Our next step is to prove that V is unitary. Previously we have shown that $\sum_{k=1}^{\infty} ||f_k||^2_{L^2(\mu)} = ||f||^2_{L^2_{\oplus}}$, and Proposition 3.10.4 tells us

$$||f||^2_{L^2_{\oplus}} = \sum_{k=1}^{\infty} ||f_k||^2_{L^2(\mu)} = ||Vf||^2_{L^2(\mu,l^2)} = ||Vf||^2_{\mathscr{D}_{\#}} \quad .$$

Therefore we know that V is unitary as a function onto its image.

This means we have to prove that $VL^2_{\oplus} = \mathscr{D}_{\#}$. To this end, let $(f_k)_{k\in\mathbb{N}} \in \mathscr{D}_{\#}$, and we now want to find $f \in L^2_{\oplus}$ so that $Vf = (f_k)_{k\in\mathbb{N}}$. We will reverse our previous deduction and define $f_{i,k} := \chi_{\omega_i} f_k$ for $i \geq k$ and $i, k \in \mathbb{N}$ and $f_{\infty,k} := \chi_{\omega_\infty} f_k$. First we set $\Omega_k := \omega_\infty \cup \bigcup_{i=k}^{\infty} \omega_i$. Since $(f_k)_{k\in\mathbb{N}} \in \mathscr{D}_{\#}$, we know that $f_k(z) = 0$ μ-almost everywhere for $\{z : \#(z) < k\}$. Further, we remember that $\omega_i = \#^{-1}(i)$ for $i \in \mathbb{N}$ and $\omega_\infty = \#^{-1}(\infty)$. Together we find that $\{z : \#(z) < k\} = \Omega_k^c$ and thus $\mu(\Omega_k^c) = 0$. Additionally we have $\omega_i \cap \omega_j = \emptyset$ and $\omega_i \cap \omega_\infty = \emptyset$ for $i \neq j$ and $i, j \in \mathbb{N}$. If we combine these insights, we get

$$f_{\infty,k} + \sum_{i=k}^{\infty} f_{i,k} = f_k \chi_{\omega_\infty} + \sum_{i=k}^{\infty} f_k \chi_{\omega_i} = f_k \chi_{\Omega_k} = f_k \quad \mu\text{-almost everywhere.}$$

By squaring the previous equation and using the fact that $\chi_{\omega_i}\chi_{\omega_j} = 0$ and $\chi_{\omega_i}\chi_{\omega_\infty} = 0$ for $i \neq j$ and $i, j \in \mathbb{N}$, we then get

$$|f_{\infty,k}|^2 + \sum_{i=k}^{\infty} |f_{i,k}|^2 = \left| f_{\infty,k} + \sum_{i=k}^{\infty} f_{i,k} \right|^2 = |f_k|^2 \quad \mu\text{-almost everywhere.}$$

This means we have

$$\int |f_{\infty,k}(z)|^2 d\mu(z) + \int \sum_{i=k}^{\infty} |f_{i,k}(z)|^2 d\mu(z) = \int |f_k(z)|^2 d\mu(z) = ||f_k||^2_{L^2(\mu)} \quad .$$

Since all entries in the sum are positive, we can use Theorem 4.7.1 to switch sum and integral. Further, we know that for $i \in \mathbb{N}$ we have $\frac{d\mu_i}{d\mu} = \chi_{\omega_i}$ and $f_{i,k}\chi_{\omega_i} = f_{i,k}$, and the same goes for $\frac{d\mu_\infty}{d\mu} = \chi_{\omega_\infty}$ and $f_{\infty,k}\chi_{\omega_i} = f_{\infty,k}$. This means altogether we get

$$\begin{aligned} ||f_{\infty,k}||^2_{L^2(\mu_\infty)} + \sum_{i=k}^{\infty} ||f_{i,k}||^2_{L^2(\mu_i)} &= \int |f_{\infty,k}(z)|^2 d\mu_\infty(z) + \sum_{i=k}^{\infty} \int |f_{i,k}(z)|^2 d\mu_i(z) \\ &= \int \chi_{\omega_\infty}(z) |f_{\infty,k}(z)|^2 d\mu(z) + \sum_{i=k}^{\infty} \int \chi_{\omega_i}(z) |f_{i,k}(z)|^2 d\mu(z) \\ &= \int |f_{\infty,k}(z)|^2 d\mu(z) + \int \sum_{i=k}^{\infty} |f_{i,k}(z)|^2 d\mu(z) = ||f_k||^2_{L^2(\mu)} \quad . \end{aligned}$$

Therefore we especially have $||f_{i,k}||^2_{L^2(\mu_i)} < \infty$ and thus $f_{i,k} \in L^2(\mu_i)$ for $i \in \mathbb{N}$, and similarly we find $f_{\infty,k} \in L^2(\mu_\infty)$. We can now sum over all k and obtain

$$\left(\sum_{k=1}^{\infty} ||f_{\infty,k}||^2_{L^2(\mu_\infty)}\right) + \left(\sum_{k=1}^{\infty}\sum_{i=k}^{\infty} ||f_{i,k}||^2_{L^2(\mu_i)}\right) = \sum_{k=1}^{\infty} ||f_k||^2_{L^2(\mu)} \quad .$$

Proposition 3.10.4 tells us that the right hand side is exactly $||(f_k)_{k\in\mathbb{N}}||^2_{\mathscr{D}_\#}$, and since all elements of the sums on the left hand side are positive, we can rearrange them to get

$$\left(\sum_{k=1}^{\infty} ||f_{\infty,k}||^2_{L^2(\mu_\infty)}\right) + \left(\sum_{i=1}^{\infty}\sum_{k=1}^{i} ||f_{i,k}||^2_{L^2(\mu_i)}\right) = ||(f_k)_{k\in\mathbb{N}}||^2_{\mathscr{D}_\#} \quad .$$

This means we can define

$$f := \left(\bigoplus_{k=1}^{\infty} f_{\infty,k}\right) \oplus \left(\bigoplus_{i=1}^{\infty}\bigoplus_{k=1}^{i} f_{i,k}\right) \quad .$$

The equation above has shown us that $||f||_{L^2(\oplus)} = ||(f_k)_{k\in\mathbb{N}}||_{\mathscr{D}_\#}$, so $f \in L^2_\oplus$.

Now we only need to confirm that $Vf = (f_k)_{k\in\mathbb{N}}$. For $k \in \mathbb{N}$ we have defined $(Vf)_k = f_{\infty,k} + \sum_{i=k}^{\infty} f_{i,k}$, and we have shown that $f_{\infty,k} + \sum_{i=k}^{\infty} f_{i,k} = f_k$ μ-almost everywhere. By using Proposition 3.10.4 again to express the norm on $\mathscr{D}_\#$, we see

$$||Vf - (f_k)_{k\in\mathbb{N}}||^2_{\mathscr{D}_\#} = ||Vf - (f_k)_{k\in\mathbb{N}}||^2_{L^2(\mu,l^2)} = \sum_{k=1}^{\infty} ||(Vf)_k - f_k||^2_{L^2(\mu)} = 0 \quad .$$

This means we have found an $f \in L^2_\oplus$ so that $Vf = (f_k)_{k\in\mathbb{N}}$ and thus $VL^2_\oplus = \mathscr{D}_\#$.

For the final part, we inspect the unitary equivalence. We remember that N_μ is just the multiplication by z on $L^2(\mu)$ and $N_\#$ is just the multiplication by z on $\mathscr{D}_\#$. We take $f \in L^2_\oplus$, use the same notation as before and get

$$\begin{aligned}(N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{i=1}^{\infty}(N_{\mu|_{\omega_i}})^i f &= \left(\bigoplus_{k=1}^{\infty} zf_{\infty,k}\right) \oplus \left(\bigoplus_{i=1}^{\infty}\bigoplus_{k=1}^{i} zf_{i,k}\right) \\ &= V^{-1}(zf_k)_{k\in\mathbb{N}} = V^{-1}N_\#(f_k)_{k\in\mathbb{N}} = V^{-1}N_\# Vf \quad .\end{aligned}$$

Since V is unitary, this means we have $N_\# \simeq (N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{i=1}^{\infty}(N_{\mu|_{\omega_i}})^i$. □

Theorem 3.10.7 (Final Multiplicity Theorem)**.** *Let $\mathscr{H}$ be a separable Hilbert space and let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator.*
(a) Let μ be a scalar-valued measure for N. Then we find a multiplicity function $\#_N$ for μ so that

$$N \simeq N_{\#_N} \quad .$$

(b) Let $\mathscr{K}$ be a separable Hilbert space and let $M \in \mathscr{B}(\mathscr{K})$ be normal with corresponding scalar-valued measure ν and multiplicity function $\#_M$. Then we find that $N \simeq M$ if and only if $[\mu] = [\nu]$ and $\#_N = \#_M$ μ-almost everywhere.

Remark. We remind ourselves that according to Corollary 3.4.9 we can always find a separating vector $h \in \mathscr{H}$ for $W^*(N)$ and thus by Theorem 3.7.11 we have that μ_h is a scalar-valued measure for N. Therefore the representation above is always valid.

Proof of Theorem 3.10.7 (a). We start by taking the partition from Theorem 3.9.5 to get measures μ_∞ and $(\mu_i)_{i\in\mathbb{N}}$ so that $\mu = \mu_\infty + \sum_{i=1}^\infty \mu_i$ and

$$N \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{i=1}^\infty (N_{\mu_i})^i \quad . \tag{3.6}$$

From these measures we will construct a multiplicity function $\#_N$ for μ by using the fact that they are all mutually singular. Then we will prove that it is indeed a multiplicity function, and further that $N_{\#_N} \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{i=1}^\infty N_{\mu_i}^i$.

Since μ_∞ and the $(\mu_i)_{i\in\mathbb{N}}$ are mutually singular, we can use Lemma 3.9.4 to get sets ω_∞ and $(\omega_i)_{i\in\mathbb{N}}$ so that

$$\begin{aligned} \mu_i = \mu_i|_{\omega_i} \quad &, \quad \omega_i \cap \omega_j = \emptyset \\ \mu_\infty = \mu_\infty|_{\omega_\infty} \quad &, \quad \omega_\infty \cap \omega_j = \emptyset \\ i, j \in \mathbb{N} \quad &, \quad i \neq j \quad . \end{aligned}$$

We now take $\Omega := \omega_\infty \cup \bigcup_{i=1}^\infty \omega_i$. This lets us define the function $\#_N : \mathbb{C} \mapsto \{0, 1, ..., \infty\}$ as

$$\#_N(z) := \begin{cases} i & \text{if } z \in \omega_i \\ \infty & \text{if } z \in \omega_\infty \\ 0 & \text{if } z \in \Omega^c \end{cases} \quad .$$

To check whether $\#_N$ is a multiplicity function for μ, we need to verify that $\mu(\Omega^c) = 0$, and that μ is a Radon measure with compact support. For the first condition, we take $i \in \mathbb{N}$. We remark that $\Omega^c = \omega_\infty^c \cap \bigcap_{j=1}^\infty \omega_j^c$ and thus $\Omega^c \subseteq \omega_\infty^c$ and $\Omega^c \subseteq \omega_i^c$. In addition, we remember that $\mu_i = \mu_i|_{\omega_i}$, so we have

$$\mu_i(\Omega^c) = \mu_i|_{\omega_i}(\Omega^c) = \mu_i(\Omega^c \cap \omega_i) \leq \mu_i(\omega_i^c \cap \omega_i) = 0 \quad .$$

Similarly we obtain $\mu_\infty(\Omega^c) = 0$. Taking this together, we find that

$$\mu(\Omega^c) = \mu_\infty(\Omega^c) + \sum_{i=1}^\infty \mu_i(\Omega^c) = 0 \quad .$$

For the second condition, we know that μ is a scalar-valued measure for μ. Theorem 3.7.11 then tells us that there is a $h \in \mathscr{H}$ so that $\mu = \mu_h$ and Proposition 4.6.5 ensures that $\mu_h = \mu$ is a Radon measure with compact support. Therefore $\#_N$ is a multiplicity function for μ and we can use Lemma 3.10.6 to obtain

$$N_{\#_N} \simeq (N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{i=1}^\infty (N_{\mu|_{\omega_i}})^i \quad . \tag{3.7}$$

Now we want to show that $\mu|_{\omega_i} = \mu_i$ for $i \in \mathbb{N}$ and $\mu|_{\omega_\infty} = \mu_\infty$. We remember that $\omega_i \cap \omega_j = \emptyset$ and $\omega_i \cap \omega_\infty = \emptyset$ for $i \neq j$. We further remember that $\mu_i|_{\omega_i} = \mu_i$ for $i \in \mathbb{N}$ and $\mu_\infty = \mu_\infty|_{\omega_\infty}$. Together, this means we can take a Borel set Δ and calculate

$$\mu|_{\omega_i}(\Delta) = \mu(\Delta \cap \omega_i) = \mu_\infty|_{\omega_\infty}(\Delta \cap \omega_i) + \sum_{j=1}^{\infty} \mu_j|_{\omega_j}(\Delta \cap \omega_i)$$

$$= \mu_\infty(\Delta \cap \omega_i \cap \omega_\infty) + \sum_{j=1}^{\infty} \mu_j(\Delta \cap \omega_i \cap \omega_j) = \mu_i(\Delta \cap \omega_i) = \mu_i|_{\omega_i}(\Delta) = \mu_i(\Delta) \quad .$$

Thus we have $\mu|_{\omega_i} = \mu_i$ for $i \in \mathbb{N}$, and in a similar way we get $\mu|_\infty = \mu_\infty$. Taken together with equations 3.6 and 3.7 we obtain

$$N_{\#_N} \simeq (N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\mu|_{\omega_i}})^i = (N_{\mu_\infty})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\mu_i})^i \simeq N \quad .$$

This concludes the proof. □

Proof of Theorem 3.10.7 (b). We will conduct the proof in two steps. First, we reverse-engineer the proof of part *(a)* to obtain a representation of N and M as given in Theorem 3.9.5, that is

$$N \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\mu_i})^i \quad , \quad M \simeq (N_{\nu_\infty})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\nu_i})^i$$

The theorem then tells us that $N \simeq M$ is equivalent to $[\mu_\infty] = [\nu_\infty]$ and $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$. In the second step we will show that this is, in turn, equivalent to $[\mu] = [\nu]$ and $\#_N = \#_M$ μ-almost everywhere.

To start the first step, we use Lemma 3.10.6 to see that

$$N_{\#_N} \simeq (N_{\mu|_{\omega_\infty}})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\mu|_{\omega_i}})^i \quad .$$

Here we have $\omega_i = \#_N^{-1}(i)$ for $i \in \mathbb{N}$ and $\omega_\infty = \#_N^{-1}(\infty)$. We know that $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$ and $\omega_i \cap \omega_\infty = \emptyset$ for $i \in \mathbb{N}$. Additionally, we define $\Omega := \omega_\infty \cup \bigcup_{i=1}^{\infty} \omega_i$ and we see that $\Omega^c = \#_N^{-1}(0)$. Since $\#_N$ is a multiplicity function for μ, this means that $\mu(\Omega^c) = 0$, and therefore we can apply Lemma 3.9.3 to obtain

$$\mu = \mu_{\omega_\infty} + \sum_{i=1}^{\infty} \mu_{\omega_i} \quad .$$

Now we remind ourselves that μ is a scalar-valued spectral measure and thus we can use Theorem 3.7.11 together with Proposition 4.6.5 to see that μ is a Radon measure with compact support. This means according to Lemma 3.8.2 that $\mu|_{\omega_\infty}$ and the $(\mu|_{\omega_i})_{i\in\mathbb{N}}$ are Radon measures with compact support as well. Further, since $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$ and $\omega_i \cap \omega_\infty = \emptyset$ for $i \in \mathbb{N}$ we can use Lemma 3.9.4 to see that the $(\mu|_{\omega_i})_{i\in\mathbb{N}}$ and $\mu|_{\omega_\infty}$

are mutually singular. We then relabel $\mu_i := \mu|_{\omega_i}$ for $i \in \mathbb{N}$ and $\mu_\infty := \mu|_{\omega_\infty}$. Because $N \simeq N_{\#_N}$, we have

$$N \simeq N_{\#_N} \simeq (N_{\mu_\infty})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\mu_i})^i \quad .$$

Our previous insights show that this representation is of the form given in Theorem 3.9.5. We can do the same for $M \simeq N_{\#_M}$ and get

$$M \simeq N_{\#_M} \simeq (N_{\nu_\infty})^\infty \oplus \bigoplus_{i=1}^{\infty} (N_{\nu_i})^i \quad .$$

Here we have $\nu_i = \nu|_{\gamma_i}$ with $\gamma_i = \#_M^{-1}(i)$ for $i \in \mathbb{N}$, and $\nu_\infty = \nu|_{\gamma_\infty}$ with $\gamma_\infty = \#_M^{-1}(\infty)$. To cap off this part of the proof, Theorem 3.9.5 now tells us that $N \simeq M$ if and only if $[\mu_\infty] = [\nu_\infty]$ and $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$.

Now we will prove that this condition is equivalent to $[\mu] = [\nu]$ and $\#_N = \#_M$ μ-almost everywhere. We start by assuming that $[\mu] = [\nu]$ and $\#_N = \#_M$ μ-almost everywhere, and we take $i \in \mathbb{N}$. Since $\omega_i = \#_N^{-1}(i)$ and that $\gamma_i = \#_M^{-1}(i)$, this means that

$$\mu(\omega_i \setminus \gamma_i) = 0 \quad , \quad \mu(\gamma_i \setminus \omega_i) = 0 \quad .$$

Let now Δ be a Borel set so that $\mu_i(\Delta) = 0$. We know that $\mu_i = \mu|_{\omega_i}$, so we get

$$\begin{aligned} 0 &= \mu_i(\Delta) = \mu(\Delta \cap \omega_i) \\ &= \mu(\Delta \cap \gamma_i) + \mu(\Delta \cap (\omega_i \setminus \gamma_i)) - \mu(\Delta \cap (\gamma_i \setminus \omega_i)) = \mu(\Delta \cap \gamma_i) \quad . \end{aligned}$$

Since $[\mu] = [\nu]$ and $\nu_i = \nu|_{\gamma_i}$, we therefore see

$$0 = \nu(\Delta \cap \gamma_i) = \nu|_{\gamma_i}(\Delta) = \nu_i(\Delta) \quad .$$

Thus we obtain $\nu_i \ll \mu_i$, and by switching μ_i and ν_i in the previous calculations we can also get $\mu_i \ll \nu_i$. This means $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$, and we can show similarly that $[\mu_\infty] = [\nu_\infty]$. This concludes one implication.

For the other implication, we assume that $[\mu_i] = [\nu_i]$ for all $i \in \mathbb{N}$ and $[\mu_\infty] = [\nu_\infty]$, and we want to show this leads to $[\mu] = [\nu]$ and $\#_N = \#_M$ μ-almost everywhere. First, we take a Borel set Δ so that $\mu(\Delta) = 0$. This means

$$0 = \mu(\Delta) = \mu_\infty(\Delta) + \sum_{i=1}^{\infty} \mu_i(\Delta) \quad .$$

Therefore we can deduce $\mu_\infty(\Delta) = 0$ and $\mu_i(\Delta) = 0$ for all $i \in \mathbb{N}$. With our assumption, this means we also find $\nu_\infty(\Delta) = 0$ and $\nu_i(\Delta) = 0$ for all $i \in \mathbb{N}$. Thus we can calculate

$$0 = \nu_\infty(\Delta) + \sum_{i=1}^{\infty} \nu_i(\Delta) = \nu(\Delta) \quad .$$

Therefore we have shown $\nu \ll \mu$, and similarly we can show $\mu \ll \nu$ to obtain $[\mu] = [\nu]$. Now turn towards $\#_N$ and $\#_M$ and define the sets

$$S_i := \{z : \#_N(z) = \#_M(z) = i\} \quad \text{for } i \in \mathbb{N} \quad , \quad S_\infty := \{z : \#_N(z) = \#_M(z) = \infty\}$$

$$S := \{z : \#_N(z) = \#_M(z)\} = S_\infty \cup \bigcup_{i=1}^{\infty} S_i \quad .$$

Next, we remember that $\omega_i = \#_N^{-1}(i)$ and $\gamma_i = \#_M^{-1}(i)$ for $i \in \mathbb{N}$ and that $\omega_\infty = \#_N^{-1}(\infty)$ and $\gamma_\infty = \#_M^{-1}(\infty)$. This means we get

$$S_i = \omega_i \cap \gamma_i \quad \text{for } i \in \mathbb{N} \quad , \quad S_\infty = \omega_\infty \cap \gamma_\infty \quad .$$

We now consider $i \in \mathbb{N}$ and find

$$\mu_i(\omega_i^c) = \mu|_{\omega_i}(\omega_i^c) = \mu(\omega_i \cap \omega_i^c) = 0 \quad .$$

Similarly, we get $\nu_i(\gamma_i^c) = 0$, and since $[\mu_i] = [\nu_i]$ this means $\mu_i(\gamma_i^c) = 0$ Together we find

$$\mu_i(S_i^c) = \mu_i(\omega_i^c \cup \gamma_i^c) \leq \mu_i(\omega_i^c) + \mu_i(\gamma_i^c) = 0 \quad .$$

Since $S^c = S_\infty^c \cap \bigcap_{i=1}^\infty S_i^c$, we find that $S^c \subseteq S_i^c$. Therefore we get $\mu_i(S^c) \leq \mu_i(S_i^c) = 0$, and we can deduce $\mu_\infty(S^c) = 0$ in the same way. Thus we have

$$\mu(S^c) = \mu_\infty(S^c) + \sum_{i=1}^{\infty} \mu_i(S^c) = 0 \quad .$$

We now remember that $S = \{z : \#_N(z) = \#_M(z)\}$, and thus we find that $\#_N = \#_M$ μ-almost everywhere. This concludes the second implication and thus the proof. $\square$

Chapter 4

Appendix

In this appendix, we provide definitions, propositions and theorems that are required for the main text which we won't prove. Many of them should be familiar to maths students with at least a bachelor's degree, and we will provide references for further details on them. They are divided into conceptual groups so that we might have a better overview.

4.1 Analysis & Topology

Theorem 4.1.1 (Stone-Weierstrass)**.** *Let $X \subseteq \mathbb{C}$ be a compact space and let $C(X)$ be the algebra of continuous functions on X endowed with the $||.||_\infty$ norm. Now let $\mathscr{A}$ be a subalgebra of $C(X)$ with the following properties*

(a) For $f \in \mathscr{A}$ we also find $\overline{f} \in \mathscr{A}$.

(b) For every $z \in \mathbb{C}$ there exists a $f_z \in \mathscr{A}$ so that $f_z(z) \neq 0$.

(c) For every pair $z_1 \neq z_2$ there exists a function $f_{z_1,z_2} \in \mathscr{A}$ so that $f_{z_1,z_2}(z_1) \neq f_{z_1,z_2}(z_2)$.

Then we find that $\mathscr{A}$ is dense in $C(X)$.

Note. See Corollary 12.18.9 in [5].

Theorem 4.1.2 (Tietze)**.** *Let X be a normal space, $A \subseteq X$ be a closed subset and $f : A \mapsto \mathbb{C}$ be a continuous function. Then there exists a continuous function $F : X \mapsto \mathbb{C}$ so that $F|_A = f$ and $\sup\{|f(x)| : a \in A\} = \sup\{F(x) : x \in X\}$.*

Note. See page 83 in [9].

4.2 Zorn's Lemma

Theorem 4.2.1 (Zorn's lemma)**.** *Let P be a partially ordered set. If every chain in P has an upper bound in P, then P contains a maximal element.*

Note. See Theorem 13.0.7 in [6].

4.3 Locally Convex Vector Spaces

Lemma 4.3.1. *Let X be a vector space and p be a seminorm on X. Then $N_p := \{x \in X : p(x) = 0\}$ is a subspace and the function $[x + N_p] \mapsto p(x)$ is a norm on the vector space $X_p := X/N_p$.*

Definition 4.3.1. Let X be a vector space and M be a family of seminorms on X. We call M *separating* if $\bigcap_{p\in M} N_p = \{0\}$.

Theorem 4.3.2. *Let X be a vector space and M a separating family of seminorms on X. Further for all $p \in M$ let X_p be defined as in Lemma 4.3.1 and let $\pi_p : X \mapsto X_p$ be the canonical projection. Then the initial topology T_M generated by the projections $\pi_p, p \in M$ makes (X, T_M) into a locally convex topological vector space. In addition, a net $(x_i)_{i\in I}$ in X converges to x if and only if $p(x_j - x) \to 0$ for all $p \in M$.*

Note. See Theorem 5.1.4 in [1].

Theorem 4.3.3 (Hahn-Banach)**.** *Let X be a vector space, M be a linear subspace of X and $f : M \to \mathbb{C}$ be linear. Further, let p be a seminorm on X with $|f(x)| \leq p(x)$ for all $x \in M$. Then there exists a linear $F : X \to \mathbb{C}$ with $F|_M = f$ and $|F(x)| \leq p(x)$ for all $x \in X$.*

Note. See Theorem 5.2.3 in [1] or Corollary III.6.4 in [2].

Theorem 4.3.4 (Hahn-Banach Separation Theorem)**.** *Let X be a locally convex topological vector space, and let $A, B \subseteq X$ be disjoint, nonempty and convex subsets of X. In addition, let A be compact and B be closed. Then there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ and $f \in X^*$ such that for all $x \in A$ and $y \in B$ it holds*

$$\mathrm{Re} f(x) \leq \gamma_1 < \gamma_2 \leq \mathrm{Re} f(y) \quad .$$

Note. See Theorem 5.2.5 in [1].

4.4 Hilbert Spaces

Theorem 4.4.1 (Riesz-Fischer)**.** *The mapping $\Psi : \mathscr{H} \mapsto \mathscr{H}^*$ with $\Psi(y)(x) = \langle x, y\rangle$ is an isometric and conjugate linear bijection.*

Note. See Proposition 3.2.5 in [1] or Theorem I.3.4 in [2].

Proposition 4.4.2. *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$. Then we find $(\mathrm{ran} A)^\perp = \ker A^*$.*

Note. See Proposition 6.6.2 in [1].

Proposition 4.4.3. *Let $A \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then $\ker A = \ker A^*$.*

Note. See Proposition 6.1(b) in [7].

Proposition 4.4.4. *Let $\mathscr{K} \subseteq \mathscr{H}$ be a subspace. Then we have $(\mathscr{K}^\perp)^\perp = \mathrm{cl}(\mathscr{K})$.*

Note. See Corollary 3.2.4 in [1].

4.5 Operator-Valued Functions

Definition 4.5.1. Let $\mathscr{S}$ be a Banach space, and let $B \in \mathscr{B}(\mathscr{S})$. Then the *exponential of* B is defined as $\exp(B) := \sum_{n=1}^{\infty} \frac{1}{n!} B^n$. Sometimes we will also write $\exp(B) = e^B$.

Proposition 4.5.1. *Let $\mathscr{S}$ be a Banach space, and let $B, \tilde{B} \in \mathscr{B}(\mathscr{S})$. Then, the following two rules for calculation apply:*

1. $(\exp(B))^* = \exp(B^*)$
2. *If $B\tilde{B} = \tilde{B}B$ then we have $\exp(B)\exp(\tilde{B}) = \exp(B + \tilde{B})$.*

Note. Follows from Definition 4.5.1.

Proposition 4.5.2. *Let B be defined as in Definition 4.5.1, and let $z \in \mathbb{C}$. The function $f : z \mapsto \exp(zB)$ is differentiable for all $z \in \mathbb{C}$ and the first derivative is given by $f'(z) = B\exp(zB)$.*

Note. Similar to Example 9.3.20 in [6], we just replace $\mathbb{R}^{p \times p}$ with $\mathscr{B}(\mathscr{H})$.

Theorem 4.5.3 (Liouvilles Theorem for Operators)**.** *Let $\mathscr{H}$ be a Hilbert space and let $f : \mathbb{C} \mapsto \mathscr{B}(\mathscr{H})$ be differentiable for all $z \in \mathbb{C}$. If there exists a $K \in \mathbb{R}_+$ such that $||f(z)|| < K$ for all $z \in \mathbb{C}$, then f is constant.*

Note. See Problem V.2.2 in [10].

4.6 Spectrum & Spectral Theorem

Proposition 4.6.1. *Let $A \in \mathscr{B}(\mathscr{H})$ and let $\lambda \in \rho(A)$. Then we find*

$$||(A - \lambda_{\mathscr{H}})^{-1}|| \leq \frac{1}{\mathrm{dist}(\sigma(A), \lambda)} \quad .$$

Note. See Lemma 6.4.10 in [1].

Proposition 4.6.2. *A self adjoint operator $A \in \mathscr{B}(\mathscr{H})$ has spectrum $\sigma(A) \subseteq \mathbb{R}$. Further, we find $\sigma(A) \geq 0$ if and only if $\langle Ah, h\rangle \geq 0$ for all $h \in \mathscr{H}$.*

Note. See Corollary 6.6.13 in [1].

Definition 4.6.1. Let (X, Ω) be a measurable space, let $\mathscr{H}$ be a Hilbert space and let $E : \Omega \mapsto \mathscr{B}(\mathscr{H})$ be a Hilbert space. The function E is called a *spectral measure* if it fulfils the following three properties

1. For all $h_1, h_2 \in \mathscr{H}$ the function $E_{h_1,h_2} : \Omega \mapsto \mathbb{C}$ defined by $E_{h_1,h_2}(\Delta) = \langle E(\Delta)h_1, h_2\rangle$ is a complex measure.
2. $E(X)$ is the identity.
3. For all $\Delta \in \Omega$ we have that $E(\Delta)$ is a projection.

Theorem 4.6.3. *Let E be a spectral measure on the measurable space (X, Ω) for the Hilbert space $\mathscr{H}$. We consider the mapping $\Phi_E : B^\Omega(X, \mathbb{C}) \mapsto \mathscr{B}(\mathscr{H})$ defined by $\Phi_E : \phi \mapsto \int \phi dE$. Then Φ_E has the following properties:*

1. *For all $\Delta \in \Omega$ we have $\Phi_E(\chi_\Delta) = E(\Delta)$.*
2. *Φ_E is an algebra-homeomorphism compatible with the $*$-operation.*
3. *Let now $A \in \mathscr{B}(\mathscr{H})$ and $f \in B^\Omega(X, \mathbb{C})$. Then we have the implication* $\forall\, \Delta \in \Omega \;\; AE(\Delta) = E(\Delta)A \quad \Rightarrow \quad A\Phi_E(f) = \Phi_E(f)A$

Note. Follows from the definition of a spectral measure and Proposition IX.1.12 in [2].

Theorem 4.6.4 (Spectral Theorem)**.** *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then there exists a unique spectral measure E on $\sigma(N)$ such that*

$$T = \int z dE(z) \quad .$$

Let $A \in \mathscr{B}(\mathscr{H})$ and let $S(\sigma(N))$ be the Borel sets of $\sigma(N)$. Then, the following additional properties hold for E and T:

1. $\operatorname{supp} E = \sigma(T)$
2. $\forall\, \Delta \in S(\sigma(N)) \;\; AE(\Delta) = E(\Delta)A \quad \Leftrightarrow \quad AN = NA$ *and* $A^*N = NA^*$

Note. See Theorem IX.2.2 in [2].

Proposition 4.6.5. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator and E be the associated spectral measure. Then the complex measure $E_{g,h}$ is a Radon measure with compact support.*

Note. See Theorem 2.18 in [8].

4.7 Measure Theory

Theorem 4.7.1. *Let (X, Ω, μ) be a measure space and let $(f_n)_{n\in\mathbb{N}}$ be a series of functions $f_n : X \mapsto [0, \infty]$. Let further $f_n \to$ be monotonically increasing and convergent μ-almost everywhere. Then we find*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu \quad .$$

Note. See Theorem IV.2.7 in [3].

Theorem 4.7.2. *Let (X, Ω, μ) be a measure space and let $(f_n)_{n\in\mathbb{N}}$ be a series of functions $f_n : X \mapsto \mathbb{R}$. Let further $f_n \to f$ μ-almost everywhere. If there exists a function $g : X \mapsto \mathbb{R}$ so that $|f_n| \leq g$ μ-almost everywhere for all $n \in \mathbb{N}$ and $\int g d\mu < \infty$, then we find*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu \quad .$$

Note. See Theorem IV.5.2 in [3].

Theorem 4.7.3 (Radon-Nikodym). *Let (X, Ω) be a measurable space and let μ, ν be two σ-finite measures on this measurable space. If $\nu \ll \mu$, then there exists a Ω-measurable function $f : X \mapsto [0, \infty)$ so that for $A \in \Omega$ we have*

$$\nu(A) = \int_A f d\mu \quad .$$

This function f is also denoted by $\frac{d\nu}{d\mu}$.

Note. See Theorem VII.2.3 in [3].

Proposition 4.7.4. *Let (X, Ω) and μ, ν be as in Theorem 4.7.3 with $\nu \ll \mu$, and let f be a ν-integrable function. Then the following statements are true:*

1. $\int f d\nu = \int f \frac{d\nu}{d\mu} d\mu$.
2. *If also $\mu \ll \nu$, then we have $\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\mu}\right)^{-1}$.*

Note. A combination of Theorem IV.2.12, Theorem VII.2.3 and Exercise VII.2.4 in [3].

Proposition 4.7.5. *Let (X, Ω, μ) be a measure space and let $\Delta \in \Omega$. Then the measure $\mu|_\Delta \ll \mu$ and we have*

$$\frac{d\mu|_\Delta}{d\mu} = \chi_\Delta \quad .$$

Note. Follows from Theorem 4.7.3.

Proposition 4.7.6. *Let μ be a Radon measure, let $K := \operatorname{supp}(\mu)$ and let $f \in L^1(\mu)$. Then the following statements hold:*

1. *K^c is measurable and $\mu(K^c) = 0$.*
2. *If K is compact, then μ is finite.*
3. *The measure ν defined by $\nu(\Delta) = \int_\Delta |f| d\mu$ is also a Radon measure.*

Note. See §256 in [4].

Theorem 4.7.7. *Let μ be a σ-finite measure. Then we have that $L^\infty(\mu) \cap L^2(\mu)$ is dense in $L^2(\mu)$ with respect to the $||.||_{L^2}$ norm.*

Note. Is a corollary to Theorem VI.2.28 in [3].

Theorem 4.7.8. *Let μ be a Radon measure with compact support X. Then $C(X)$ is dense in $L^2(\mu)$ with respect to the $||.||_{L^2}$ norm.*

Note. Follows from Theorem VI.2.31 in [3] if we substitute $\mathbb{R}^2$ with $\mathbb{C}$.

Theorem 4.7.9 (Riesz Representation Theorem). *Let X be a locally compact Hausdorff space, and let $C_c(X)$ be the space of continuous functions with compact support on X. Then for any positive linear functional ψ on $C_c(X)$, there is a unique Radon measure μ on X such that for all $f \in C_c(X)$ we have*

$$\psi(f) = \int f d\mu$$

Remark. This means especially that for a Radon measure μ the functional $\zeta(f) = \int f d\mu$ is uniquely defined by μ.

Note. See Theorem VIII.2.5 in [3].

Theorem 4.7.10 (Lusin)**.** *Let X be a Hausdorff space and (X, Ω, μ) be a measure space with μ a Radon measure. Let further $f : X \mapsto \mathbb{C}$ be a Ω-measurable function. Then for every $A \in \Omega$ with $\mu(A) < \infty$ and $\epsilon > 0$, we find a compact set K with $\mu(A \setminus K) < \epsilon$ so that $f|_K$ is continuous.*

Note. See Theorem VIII.1.18 in [3].

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